MODULES OF GENERALIZED FRACTIONS, DIRECT SYSTEMS OF DETERMINANTAL MAPS AND OTHER TOPICS IN COMMUTATIVE ALGEBRA

by

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Dedicated

to

my parents

The following record of research work is submitted as a thesis in support of an application for the degree of Doctor of Philosophy at the University of Edinburgh, having been submitted for no other degree. Except where acknowledgement is made, the work is original.

CONTENTS

ACKNOWLEDGEMENTS ... i

SUMMARY ... iii

CHAPTER I

1. Preliminaries ... 1
2. Triangular subsets of A^n and modules of generalized fractions ... 2
3. Saturation and restriction ... 8
4. Complexes of modules of generalized fractions ... 12

CHAPTER II

1. Matrices and modules of generalized fractions ... 17
2. Poor M–sequences and determinantal maps ... 27

CHAPTER III

1. Denominator systems and chains of triangular subsets ... 45
2. Denominator system complexes and complexes of modules of generalized fractions ... 53
3. Complexes of Cousin type and complexes of modules of generalized fractions ... 63
4. Direct limits and flat dimension of generalized fractions ... 69

CHAPTER IV

1. Vanishing of modules of generalized fractions ... 77
2. Modules of generalized fractions and local cohomology ... 86
3. Generalized Cohen–Macaulay rings and lengths of generalized fractions ... 90

CHAPTER V

1. Seminormality and F–purity in local rings ... 108

APPENDIX 1 – On direct limits ... 117

REFERENCES ... 124

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In Chapter II, Dr. O'Carroll suggested the rôle of matrices in a module of generalized fractions as a topic worthy of further investigation and also suggested that 2.1.3. might be relevant. Concerning the injectivity of the determinantal map (2.2.6. and 2.2.12.) it was Dr. O'Carroll who suggested reducing to the situation of a finitely generated module over a Noetherian ring, which approach ultimately yielded 2.2.6., and employing homological methods to obtain a proof of the result in its full generality, 2.2.12..

In Chapter III he suggested that I investigate the connections between the theory of Kersken's denominator system complexes and the theory of complexes of modules of generalized fractions. In addition he provided the proof of 3.2.5. appearing herein, which was more succinct than my own version.

It was Dr. O'Carroll's suggestion that I examine criteria for the vanishing of modules of generalized fractions in Chapter IV and he also proposed that I investigate connections between local cohomology and modules of generalized fractions by considering the Čech complex.

The topics considered in the final chapter were also suggested by Dr. O'Carroll and in particular he provided the argument in the final paragraph of the proof of 5.1.11..

Several of the results appearing in this thesis have been submitted for publication elsewhere. In particular 2.1.2., 2.1.4. and 2.1.5. appear in [**8**]. We are grateful to the referee of [**8**] for suggesting the method employed in the proof of 4.1.8., a result which appeared in the original version of [**8**] with a somewhat longer proof than that related in this thesis. Of the results contained in Chapter III, 3.2.4., 3.2.5., 3.2.8. and 3.3.3.-3.3.5. appear in [**10**], whilst the final section of the chapter appears in its entirety in [**9**].

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Summary

In this thesis we are mainly concerned with the theory of the modules of generalized fractions of Sharp and Zakeri, which is a generalization of the well-known concept of localization in Commutative Algebra. In Chapter I we give a brief description of the formation and properties of modules of generalized fractions, and we summarise the known results concerning such modules which we shall require for the later work of the thesis.

In Chapter II we focus our attention on the rôle of matrices in modules of generalized fractions. We show that it is not necessary to consider only lower triangular matrices when making identifications in such modules and we identify a larger set of matrices from which we are free to choose. Moreover, in this chapter we consider the situation where M is an A-module and $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are M-sequences with the property that $xA \supseteq yA$. We demonstrate that, for this situation, the map $M/xM \rightarrow M/yM$ induced by Cramer's rule is injective, thereby dispensing with the finiteness conditions present in previous versions of the result.

Chapter III is concerned with the connections between Kersken's

theory of denominator system complexes and complexes of modules of generalized fractions. We show that the two concepts are equivalent and, by making use of some ideas present in Kersken's theory, we obtain a description of an arbitrary module of generalized fractions as the direct limit of a system of localized quotient modules. Using this approach, we investigate the connections between Cousin complexes and generalized fractions and obtain results concerning the flat dimensions of certain modules of generalized fractions.

In Chapter IV we investigate conditions necessary and/or sufficient for the vanishing of certain modules of generalized fractions and show how modules of this form can be expressed as a homomorphic image of an ordinary module of fractions. We then employ this description to express certain cohomology modules as modules of generalized fractions. The final part of this chapter relates the length of certain quotient rings to the lengths of cyclic submodules of generalized fractions, for the situation where the base ring is a generalized Cohen-Macaulay ring. We then apply this approach to the situation of an arbitrary Noetherian ring of low dimension.

The final chapter, Chapter V, deals with the relationship between the properties of seminormality and F-purity in 1-dimensional Noetherian local rings, and we investigate conditions under which the two properties are equivalent in the 1-dimensional case.

Whilst chapters are numbered by upper case Roman numerals in this thesis we use Arabic numerals when referring to an individual section of a chapter. Thus, for example, "3.2." refers to the second section of Chapter III.

CHAPTER I

1. Preliminaries

Throughout this thesis all rings considered will be commutative with a non-zero multiplicative identity, but will not be assumed to be either Noetherian or local unless explicitly stated. A will always denote such a ring. Ideals of A will be denoted by lower case Gothic letters such as $\mathfrak{a}$, and the set of all prime ideals of A will be denoted by Spec A. The set of all maximal ideals of A will be denoted by Maxspec A.

Let M be an A-module. The support of M, i.e. the set of all primes $\mathfrak{p} \in$ Spec A for which $M_{\mathfrak{p}} \neq 0$, will be denoted by $\mathrm{Supp}_A M$, and (in Noetherian situations) $\mathrm{Ass}_A M$ will denote the set of associated primes of M. For a ring A, the *(Krull) dimension* of A, written dim A, is defined to be the length of the longest chain of primes in Spec A, if this exists, and ∞ otherwise. For a non-zero A-module M, the *(Krull) dimension* of M, written $\dim_A M$, is defined to be the length of the longest chain of primes in $\mathrm{Supp}_A M$, if this exists, and ∞ otherwise. We shall adopt the convention whereby the dimension of the zero module is defined to be -1. For $\mathfrak{p} \in$ Spec A, the *height* of $\mathfrak{p}$, written ht$\mathfrak{p}$, is defined to be $\dim A_{\mathfrak{p}}$.

The set of all integers will be denoted by the symbol $\mathbb{Z}$, the set of all positive integers by $\mathbb{N}$, and the set of all nonnegative integers by $\mathbb{N}_0$. For $n \in \mathbb{N}$, A^n will denote the Cartesian product of n factors of A, $M_n(A)$ will denote the set of all n $\times$ n matrices with entries in A, and $D_n(A)$ will denote the set of all n $\times$ n lower triangular matrices with entries in A. For $H \in M_n(A)$, the determinant of H will be written $|H|$, and matrix transpose will be denoted by ^{T}. Whenever we can do so without ambiguity, we will denote $(u_1,\dots,u_n)$ by u and the matrix $[u_1 \dots u_n]^T$ by u^T, together with obvious extensions of this notation. For $n \in \mathbb{N}$, a sequence $x_1,\dots,x_n$ of elements of A will be called an *M-sequence* if, for each $i=1,\dots,n$

$$\text{(i)}\ ((\sum_{1}^{i-1} x_jM):x_i) = \sum_{1}^{i-1} x_jM$$

with the obvious interpretation when i=1, and

$$\text{(ii)}\ \sum_{1}^{n} x_jM \neq M.$$

A sequence which satisfies condition (i) will be referred to as a *poor* M-sequence.

2. Triangular subsets of A^n and modules of generalized fractions

The localization of an A-module M with respect to a multiplicatively closed subset S of A is a fundamental concept in Commutative Algebra. In [35] Sharp and Zakeri describe a process which generalizes this idea whereby they construct modules known as *modules of generalized fractions*, in their terminology. Moreover, in subsequent papers [36,37] they have demonstrated that this concept has various wide-ranging applications in Commutative Algebra. In this section we give a brief description of the construction of modules of generalized fractions and outline some of their important properties, which we shall frequently employ throughout this thesis.

Let $n \in \mathbb{N}$. A non-empty subset U of A^n is a *triangular* subset if

(i) for all $(u_1,\dots,u_n) \in U$ and $\alpha_1,\dots,\alpha_n \in \mathbb{N}$, $(u_1^{\alpha_1},\dots,u_n^{\alpha_n}) \in U$, and

(ii) for all $u=(u_1,\dots,u_n)$ and $v=(v_1,\dots,v_n) \in U$, there exist $w=(w_1,\dots,w_n) \in U$, and $H,K \in D_n(A)$ such that

$$Hu^T = w^T = Kv^T.$$

For an A-module M and a triangular subset U of A^n, we can construct the module of generalized fractions as follows. Let U be a triangular subset of A^n, and let M be an A-module. We define a relation $\sim$ on $M \times U$ as follows: for $a,b \in M$ and $u,v \in U$, we write $(a,u) \sim (b,v)$ if and only if there exist $w \in U$ and $H,K \in D_n(A)$ such that $Hu^T = w^T = Kv^T$, and $|H|a-|K|b \in \sum_1^{n-1} w_i M$. In **[35]** it is shown that $\sim$ is an equivalence relation on $M \times U$ and from now on we shall denote by a/u the equivalence class of (a,u), and let $U^{-n}M$ stand for the set of equivalence classes of $\sim$. Furthermore, $U^{-n}M$ can be furnished with an A-module structure under the following operations:

$$a/s + b/t = (|H|a + |K|b)/u$$

for $a,b \in M$, $s,t \in U$ and any choice of $u \in U$ and $H,K \in D_n(A)$ such that $Hs^T = u^T = Kv^T$, and

$$r.(a/s) = ra/s$$

for $a \in M$, $s \in U$ and $r \in A$. The A-module $U^{-n}M$ is known as a *module of generalized fractions*.

The main difficulties in proving that $\sim$ is an equivalence relation and that $U^{-n}M$ has an A-module structure lie in demonstrating that $\sim$ is transitive and that the addition defined above is unambiguous. Whilst we do not give explicit proofs of these properties in this thesis, we include the following two results from [35] which are of central importance therein, and which will be employed elsewhere in our work.

1.2.1. **Lemma** [35, 2.2]. *Let $u,v\in U$ and suppose that there exists $H\in D_n(A)$ such that $Hu^T = v^T$. Then*

$$|H|u_i \in \sum_1^i Av_i \quad \text{for all } i = 1,\dots,n.$$

Proof. The result follows in a straightforward manner from the fact that $\operatorname{adj}(K).K = |K|I_n$ for any $K\in M_n(A)$ (Cramer's rule).

1.2.2. **Lemma** [35, 2.3]. *Let $u,v\in U$ and suppose that there exist $H,K\in D_n(A)$ such that $Hu^T = v^T = Ku^T$. Then*

$$|DH| - |DK| \in \sum_1^{n-1} Av_i^2$$

where D is the diagonal matrix $\operatorname{diag}(v_1,\dots,v_n)$.

Proof. Let $H = [h_{ij}]$ and $K = [k_{ij}]$. Set

$$H_i = \begin{cases} 1 & \text{if } i = 0 \\ \prod_1^i h_{jj} & \text{if } 1 \le i \le n, \end{cases} \qquad K_i = \begin{cases} \prod_{i+1}^n k_{jj}, & \text{if } 0 \le i < n \\ 1, & \text{if } i = n. \end{cases}$$

Let $1 \leq i \leq n$. Then by hypothesis, $(h_{ii} - k_{ii})u_i \in \sum_1^{i-1} Au_j$; thus

$$(h_{ii}-k_{ii})(v_i - \sum_{j=1}^{i-1} h_{ij}u_j) \in \sum_1^{i-1} Au_j.$$

It follows from 1.2.1 that

$$H_{i-1}(h_{ii} - k_{ii})(v_i - \sum_{j=1}^{i-1} h_{ij}u_j) \in \sum_1^{i-1} Av_j.$$

Therefore $|D|H_{i-1}(h_{ii} - k_{ii})K_i \in \sum_1^{i-1} Av_j^2$, that is

$$|D|(H_iK_i - H_{i-1}K_{i-1}) \in \sum_1^{i-1} Av_j^2.$$

Hence $|D|\sum_1^n (H_iK_i - H_{i-1}K_{i-1}) \in \sum_1^{n-1} Av_j^2$, which gives the required result.

In the following proposition we list some of the most important properties of modules of generalized fractions which are necessary for many of our calculations.

1.2.3. **Proposition** [35, 36]. *Let $m \in M$, and let $u,v \in U$ be such that $v^T = Hu^T$ for some $H \in D_n(A)$. Then, in $U^{-n}M$,*

(i) $m/u = |H|m/v$;

(ii) if $m \in \sum_1^{n-1} u_i M$ then $m/u = 0$;

(iii) [36, 2.1] if $u_n m/u = 0$ then $m/u = 0$.

Proof. Parts (i) and (ii) are immediate from the construction of

$U^{-n}M$. To prove (iii), suppose that $u_n m/u = 0$ in $U^{-n}M$. Then there exist $w \in U$ and $H = [h_{ij}] \in D_n(A)$ such that $Hu^T = w^T$ and $|H|u_n m \in \sum_1^{n-1} w_i M$. Hence

$$(\prod_1^{n-1} h_{ii})(w_n - \sum_1^{n-1} h_{ni}u_i)m \in \sum_1^{n-1} w_i M.$$

It follows from 1.2.1 that

$$(\prod_1^{n-1} h_{ii})w_n m \in \sum_1^{n-1} w_i M \ ;$$

hence by (ii)

$$(\prod_1^{n} h_{ii})w_n m/(w_1,\dots,w_{n-1},w_n^2) = 0$$

in $U^{-n}M$. It follows from (i) that

$$(\prod_1^{n} h_{ii})m/(w_1,\dots,w_{n-1},w_n) = 0 \ ,$$

and the result follows on applying (i), since $Hu^T = w^T$.

Suppose now that $x = (x_1,\dots,x_n) \in A^n$ and set

$$U(x)_n = \{ (x_1^{\alpha_1},\dots,x_n^{\alpha_n}) \mid \alpha_1,\dots,\alpha_n \in \mathbb{N}_0, \ 1 \le i \le n \}$$

where $x_i^{\alpha_i} = 1$ if $\alpha_i = 0$. Then it is easily seen that $U(x)_n$ is a triangular subset of A^n, and given an A-module M we can form the module of generalized fractions $U(x)_n^{-n}M$. For simplicity of notation we will denote a typical element $m/(x_1^{\alpha_1},\dots,x_n^{\alpha_n})$ by m/x^{α} where α stands for $(\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$ along with obvious extensions of this notation. We have the following result due to Zakeri, which we state without proof.

1.2.4. **Proposition** [39, Chap.II, 2.2]. *In $U(x)_n^{-n}M$, $a/x^{\alpha} = b/x^{\beta}$ if and only if there exists a positive integer $\gamma \geq \alpha_i, \beta_i$, $1 \leq i \leq n$, such that*

$$x_1^{\gamma-\alpha_1} \ldots x_n^{\gamma-\alpha_n} a - x_1^{\gamma-\beta_1} \ldots x_n^{\gamma-\beta_n} b \in \sum_{1}^{n-1} x_i^{\gamma} M.$$

(ii) The endomorphism of $U(x)_n^{-n}M$ induced by multiplication by x_n is an automorphism.

Modules of generalized fractions of this form play an important rôle in Chapter IV where the connections between generalized fractions and local cohomology are examined. Furthermore, the following results demonstrate that the study of such modules is central to the theory of arbitrary modules of generalized fractions.

Let U be an arbitrary triangular subset of A^n. We can define a relation $\leq$ on U in the following manner. For $u, v \in U$ we say that $u \leq v$ if and only if there exists $H \in D_n(A)$ such that $v^T = Hu^T$. Clearly $\leq$ is a quasi-order on U and $(U, \leq)$ is a directed set.

1.2.5. **Proposition** [35]. *Let $x, y \in U$ be such that $Hx^T = y^T$ for some $H \in D_n(A)$. Then given $x^{\alpha} \in U(x)_n$, there exist $y^{\beta} \in U(y)_n$ and $K \in D_n(A)$ such that $(y^{\beta})^T = K(x^{\alpha})^T$. Furthermore, there is induced an A-homomorphism $U(x)_n^{-n}M \to U(y)_n^{-n}M$ under which m/x^{α} is mapped to $|K|m/y^{\beta}$ and m/x to $|H|m/y$ for all $m \in M$.*

It is easily seen that $\{U(x)_n^{-n}M \mid x \in U\}$ becomes a direct system under these homomorphisms and we have the following result.

1.2.6. **Theorem** [35, 3.5]. $\varinjlim_{x \in U} U(x)_n^{-n} M \cong U^{-n} M$ *under the canonical map.*

3. Saturation and restriction

A familiar concept in the usual theory of the localization of an A-module M with respect to a multiplicatively closed subset S of A is the notion of saturation. For such an S, we define the saturation of S to be

$$\tilde{S} = \{ x \in A \mid xy \in S \text{ for some } y \in A \}.$$

We say that S is saturated precisely when $S = \tilde{S}$, and we recall that there is a natural equivalence of functors $S^{-1} \to \tilde{S}^{-1}$. In [28] Riley introduced the idea of the saturation of an arbitrary triangular subset U of A^n, which has properties analogous to the situation described above and which coincides with the usual notion of saturation when n=1 and U is a multiplicatively closed subset of A. Whilst Riley gives several equivalent characterisations of saturation, the following definition is the one which we shall use most frequently throughout this thesis.

1.3.1. **Definition** [28, Chap.I, 2.2]. Let $n \in \mathbb{N}$ and let U be a triangular subset of A^n. Then the *saturation* of U, denoted $\tilde{U}$, is defined to be the set

$$\{ v \in A^n \mid \text{there exist } H \in D_n(A) \text{ and } u \in U \text{ such that } Hv^T = u^T \}.$$

U will be called *saturated* if $U = \tilde{U}$.

It is a straightforward matter to show that $\tilde{U}$ is itself a saturated triangular subset of A^n and that, in the case where n=1 and U is a multiplicatively closed subset of A, $\widetilde{U}$ coincides with the usual definition. The following result due to Riley will be employed on numerous occasions throughout this thesis.

1.3.2. **Theorem** [28, Chap.I, 2.9]. *Let $n \in \mathbb{N}$ and let U be a triangular subset of A^n. Let M be an A-module. Then the natural homomorphism $\phi_M : U^{-n}M \to \widetilde{U}^{-n}M$, such that*

$$\phi_M(m/u) = m/u$$

for all $m \in M$, $u \in U$, is an isomorphism.

Proof. It is straightforward to verify that ϕ_M is a well-defined homomorphism. Suppose now that $m \in M$ and $u \in \widetilde{U}$. From the definition of $\widetilde{U}$ there exist $H \in D_n(A)$ and $v \in U$ such that $Hu^T = v^T$. Applying 1.2.3.(i) we have

$$m/u = |H|m/v = \phi_M(|H|m/v) \quad .$$

and so ϕ_M is surjective. Suppose now that $\phi_M(m/u) = 0$. Then there exist $v \in \widetilde{U}$ and $H \in D_n(A)$ such that $|H|m \in \sum_1^{n-1} v_i M$. Since $v \in \widetilde{U}$, there exist $w \in U$ and $K \in D_n(A)$ such that $Kv^T = w^T$, and it follows from 1.2.1. that $|K||H|m \in \sum_1^{n-1} w_i M$. Therefore $|KH|m/w = 0$ in $U^{-n}M$ by 1.2.3.(ii), and hence $m/u = 0$ in $U^{-n}M$ by 1.2.3.(i), showing that ϕ_M is injective.

The significance of 1.3.2. is that it allows us to assume that the triangular subset with which we are working is saturated without any loss of generality. This assumption will be particularly valuable in Chapter II, where the rôle of matrices in a module of generalized fractions is examined in more detail, and also in Chapter III, where Kersken's denominator system theory is recast in the setting of the theory expounded in this chapter. On the other hand, if U is a triangular subset of A^n and m is a positive integer such that $1 \leq m \leq n$, then the set

$$\{(u_1,\dots,u_m) \mid (u_1,\dots,u_n) \in U, \text{ for some } u_{m+1},\dots,u_n \in A\}$$

is a triangular subset of A^n which we will call the *restriction* of U to A^m. If U is saturated then it is a simple matter to show that the restriction of U to A^m is also saturated, and this is left to the reader as an exercise.

For a triangular subset U of A^n we set

$$U[1] = \{ (u_1,\dots,u_n,1) \mid (u_1,\dots,u_n) \in U \}.$$

Clearly U[1] is a triangular subset of A^{n+1}, and we shall denote a typical element $(u_1,\dots,u_n,1)$ by $(u,1)$. As will become apparent in Chapter IV, triangular subsets of this form play an important role in the theory of local cohomology, and we shall encounter them in various settings throughout this thesis. In [26], O'Carroll shows that for a triangular subset U of A^n and an A-module M, the module of generalized fractions $U[1]^{-n-1}M$ can be exhibited as the direct limit of a system of quotients of M in the following fashion. Let $u,v \in U$ be such that $Hu^T = v^T$ for some $H \in D_n(A)$. Then by 1.2.1. there is a homomorphism $\alpha_H : M/\sum_1^n u_iM \to M/\sum_1^n v_iM$ such that

$$\alpha_H(m + \sum_1^n u_iM) = |H|m + \sum_1^n v_iM.$$

For simplicity we will abbreviate $\sum_{i}^{n} u_i M$ to uM, and we shall use obvious extensions of this notation. The map α_H will be known as the determinantal map induced by H and will be studied in some depth in Chapter II. Under these maps the set $\{M/xM \mid x \in U\}$ forms a direct system (see Appendix I) and we have the following result due to O'Carroll.

1.3.3. **Theorem** [26, 2.4.]. $\varinjlim_{x \in U} M/xM = U[1]^{-n-1}M.$

Proof. Let L denote $\varinjlim_{x \in U} M/xM$, and for each $x \in U$, let $\psi_x : M/xM \to U[1]^{-n-1}M$ be the map such that

$$\psi_x(m+xM) = m/(x,1).$$

It is clear from 1.2.3.(ii) that ψ_x is a well-defined homomorphism. Given $x, y \in U$ with $y^T = Hx^T$ where $H \in D_n(A)$, we have that

$$[y\ 1]^T = \begin{bmatrix} H & 0 \\ 0 & 1 \end{bmatrix} [x\ 1]^T .$$

Hence applying 1.2.3.(i) to $U[1]^{-n-1}M$, it follows that $m/(x,1) = |H|m/(y,1)$ so that $\psi_x = \psi_y \alpha_H$ where α_H is the determinantal map induced by H, and hence the family of maps $\{ \psi_x \mid x \in U\}$ induces a homomorphism $\psi: L \to U[1]^{-n-1}M$. It is straightforward to verify that ψ is surjective and so it remains to prove that $\mathrm{Ker}\psi = 0$.

Let $k \in \mathrm{Ker}\psi$. Then there exist $y \in U$ and $m+yM \in M/yM$ such that $\theta_y(m + yM) = k$ † and $\psi_y(m + yM) = 0$. Therefore $m/(y,1) = 0$ in $U[1]^{-n-1}M$, so that there exist $K \in D_{n+1}(A)$ and $(z,1) \in U[1]$ such that $K[y\ 1]^T = [z\ 1]^T$ and $|K|m \in zM$. Now if H is the top n x n-submatrix of K, it follows from Cramer's

† where $\theta_y : M/yM \to L$ is the natural map

rule that

$$|K|.1 \in |H|.1 + zA,$$

so that $|H|m \in zM$. Since $z^T = Hy^T$, $\alpha_H(m + yM) = 0$. Hence

$$k = \theta_y(m + yM) = \theta_z(\alpha_H(m + yM)) = 0.$$

In Chapter III we will give a proof of a generalization of 1.3.3., again due to O'Carroll, which extends the ideas of 1.3.3. to modules of generalized fractions with respect to arbitrary triangular subsets.

4. Complexes of modules of generalized fractions

Some of the most important applications of modules of generalized fractions in Commutative Algebra involve certain complexes of such modules. For example, in [39] Zakeri demonstrates that the minimal injective resolution of a Gorenstein module can be exhibited as a complex of modules of generalized fractions. Furthermore, certain types of modules such as balanced big Cohen-Macaulay modules can be characterised by the exactness of such complexes. Our chief interest in these complexes in this thesis stems from their connections with complexes of Cousin type and the denominator system complexes of Kersken, both of which are examined in Chapter III. We first review the construction of a complex of modules of generalized fractions.

The symbol $\mathcal{U}$ will denote a family of sets $\{ U_i \mid i \in \mathbb{N} \}$ such that

(i) U_i is a triangular subset of A^i for all $i \in \mathbb{N}$;

(ii) whenever $(u_1,\dots,u_i) \in U_i$ with $i > 1$, then $(u_1,\dots,u_{i-1}) \in U_{i-1}$;

(iii) whenever $(u_1,\dots,u_i) \in U$, then $(u_1,\dots,u_i,1) \in U_{i+1}$;

(iv) $(1) \in U_1$.

In view of 1.3.2., which allows us to assume (iii) and (iv) with no loss of generality, we can replace (ii), (iii) and (iv) with the single condition

(v) for all $i \in \mathbb{N}$, U_i is the restriction of U_{i+1} to A^i.

Let $\mathcal{U}$ be such a family of triangular subsets on A, and let M be an A-module. We can construct a complex $\mathcal{C}(\mathcal{U},M)$ of A-modules and A-module homomorphisms, given by

$$\mathcal{C}(\mathcal{U},M) : 0 \to M \xrightarrow{d^0} U_1^{-1}M \xrightarrow{d^1} \dots \to U_i^{-i}M \xrightarrow{d^i} U_{i+1}^{-i-1}M \to \dots$$

where $d^0(m) = m/(1)$ and $d^i(m/(u_1,\dots,u_i)) = m/(u_1,\dots,u_i,1)$. The family $\mathcal{U}$ is called a *chain of triangular subsets* on A and $\mathcal{C}(\mathcal{U},M)$ is known as the *associated complex.*

Let $\{x_i \mid i \in \mathbb{N}\}$ be a sequence of elements of A. Then we can form a chain of triangular subsets $\{U(x)_i \mid i \in \mathbb{N}\}$, denoted by $\mathcal{U}(x)$, where $U(x)_n$ is formed from the truncated sequence $x_1,\dots,x_n$ in the manner described after

1.2.3.. Given an A-module M, we may then form the complex $\mathcal{C}(\mathcal{U}(x),M)$ associated with $\mathcal{U}(x)$.

If $\mathcal{U}$ is an arbitrary chain of triangular subsets on A we can define T to be the set of all sequences $x = \{x_i \mid i \in \mathbb{N}\}$ of elements of A such that

(i) $x_i = 1$ for all sufficiently large n, and

(ii) $(x_1,\dots,x_n) \in U_n$ for all $n \geq 1$.

Now T is a directed set under the following relation: for $x,y \in T$ we say that $x \leq y$ precisely when $y^T = Hx^T$ for some infinite lower triangular matrix H, and given x and y in T with $x \leq y$, there is induced a morphism of complexes $\mathcal{C}(\mathcal{U}(x),M) \to \mathcal{C}(\mathcal{U}(y),M)$ which restricts to the corresponding homomorphism in the direct system described in 1.2.5.. Analogously to 1.2.6., we have the following result.

1.4.1. **Proposition** [25, 2.1]. $\varinjlim_{x \in T} \mathcal{C}(\mathcal{U}(x),M) = \mathcal{C}(\mathcal{U},M)$.

In [25] O'Carroll gives a universal characterisation of the complex $\mathcal{C}(\mathcal{U}(x),M)$ in the manner described below.

Let x be an infinite sequence of elements of A and define another associated complex $\mathcal{C}(x,M) = \{M^{(n)} \mid n \geq -1\}$ of A-modules and A-module homomorphisms as follows. Set $M^{(-1)} = 0$, $M^{(0)} = M$ and let $f^{-1}: M^{(-1)} \to M^{(0)}$ be the canonical map. Suppose that for $i \geq 0$, the A-modules $M^{(i-1)}$ and $M^{(i)}$ and the map f^{i-1} have been defined. Let $N^{(i+1)} = \operatorname{Coker} f^{i-1}$, let $M^{(i+1)} = N^{(i+1)}_{x_{i+1}}$ (localization with respect to x_{i+1}), and let $f^i: M^{(i)} \to M^{(i+1)}$ be the canonical homomorphism. It is clear that $f^{i+1} \circ f^i = 0$, so that $\mathcal{C}(x,M)$ is indeed a complex. We have the following important result.

1.4.2. **Theorem** [25, 2.2]. *There is an isomorphism of complexes $\Theta : \mathcal{C}(x,M) \to \mathcal{C}(\mathcal{U}(x),M)$ with $\Theta = \{\theta^n \mid n \geq -1\}$ such that θ^0 is the identity map on M.*

Proof. The definition of θ^{-1} is obvious. Now let θ^0 be the identity map on M, and define θ^1: $M_{x_1} \to U(x)_1^{-1}M$ by $\theta^1(a/x_1^{\alpha}) = a/(x_1^{\alpha})$. It is a simple matter to show that θ^1 is a well-defined isomorphism and that $d^0 \circ \theta^0 = \theta^1 \circ f^0$. Now let $i \geq 1$, and suppose that we have defined, for $-1 \leq j \leq i$, A-isomorphisms $\theta^j : M^{(j)} \to U(x)_j^{-j}M$ such that $d^{j-1} \circ \theta^{j-1} = \theta^j \circ f^{j-1}$. Now

$$\theta^i(\mathrm{Im}\ f^{i-1}) = \mathrm{Im}\ d^{i-1} \subseteq \mathrm{Ker}\ d^i,$$

so that there is induced a surjective A-homomorphism $\chi^{i+1}: N^{(i+1)} \to \mathrm{Im}\ d^i$, and by 1.2.4.(ii), χ^{i+1} in turn induces a surjective A-homomorphism

$$\theta^{i+1}: M^{(i+1)} \to U(x)_{i+1}^{-i-1}M,$$

such that $d^i \circ \theta^i = \theta^{i+1} \circ f^i$. It remains to prove that Ker $\theta^{i+1} = 0$.

For this it suffices to consider an element in Ker θ^{i+1} of the form $\bar{b}/1$, where $\bar{b}$ is the canonical image of b in $N^{(i+1)}$. Now $\theta^i(b) = a/(x_1^{\alpha_1},\ldots,x_i^{\alpha_i})$, say, and $a/(x_1^{\alpha_1},\ldots,x_i^{\alpha_i},1) = 0$ in $U(x)_{i+1}^{-i-1}M$. By 1.2.4.(i), there exists a positive integer $\gamma \geq \alpha_j$, $1 \leq j \leq i$, such that

$$x_1^{\gamma-\alpha_1}\ldots x_i^{\gamma-\alpha_i}x_{i+1}^{\gamma}a \in \sum_1^i x_j^{\gamma}M.$$

Let $c = x_1^{\gamma-\alpha_1} \ldots x_i^{\gamma-\alpha_i} a$. Then

$$x_{i+1}^{\gamma} c \in \sum_1^i x_j^{\gamma} M$$

and $\theta^i(b) = a/(x_1^{\alpha_1}, \ldots, x_i^{\alpha_i}) = c/(x_1^{\gamma}, \ldots, x_i^{\gamma})$ in $U(x)_i^{-i}M$, by 1.2.3.(i). If we write

$$x_{i+1}^{\gamma} c = \sum_1^i x_j^{\gamma} m_j$$

where $m_j \in M$, $1 \leq j \leq i$, then

$$\begin{aligned}\theta^i(x_{i+1}^{\gamma} b) &= x_{i+1}^{\gamma} c/(x_1^{\gamma}, \ldots, x_i^{\gamma}) \\ &= x_i^{\gamma} m_i/(x_1^{\gamma}, \ldots, x_i^{\gamma}) \quad \text{by 1.2.3.(ii)} \\ &= m_i/(x_1^{\gamma}, \ldots, x_{i-1}^{\gamma}, 1) \text{ by 1.2.3.(i)} \\ &\in \operatorname{Im} d^{i-1}.\end{aligned}$$

It follows that $x_{i+1}^{\gamma} b \in \operatorname{Im} f^{i-1}$, hence $x_{i+1}^{\gamma} . \bar{b} = 0$ in $N^{(i+1)}$. Therefore $\bar{b}/1 = 0$ in $M^{i+1} = N^{(i+1)}_{x_{i+1}}$, and the result follows.

Because $- \otimes_A M$ commutes with ordinary localization, the taking of quotients and direct limits, a direct consequence of 1.4.1. and 1.4.2. is the following result, originally proved by Zakeri by direct calculation.

1.4.3. **Corollary** [39, Chap.II, 2.12]. *Let U be an arbitrary triangular subset of A^n and let M be an A-module. Then $U^{-n}M \cong U^{-n}A \otimes_A M$ under the obvious map.*

CHAPTER II

1. Matrices and modules of generalized fractions

In Chapter I we have seen how the module of generalized fractions $U^{-n}M$ can be constructed from an A-module M and a triangular subset U of A^n. In the first section of this chapter, we focus our attention on a particular property of such modules, of which we have already made considerable use in the previous chapter, namely 1.2.1.(i), which we now recall: if $u,v \in U$ and $H \in D_n(A)$ are such that $Hu^T = v^T$ then, in $U^{-n}M$,

$$m/u = |H|m/v \text{ , for all } m \in M. \qquad (*)$$

As $(*)$ provides a rule frequently employed in calculations involving modules of generalized fractions, it is clearly of interest to investigate the extent to which $(*)$ can be generalized to encompass a wider class of matrices than $D_n(A)$. The motivation for this programme of research is provided by the following result, originally proved by Riley by a computational method, and which is stated here without proof. In view of the fact that $M_n(A)$ and $D_n(A)$ coincide when n=1 we restrict our attention to triangular subsets of A^{n+1}, $n \in \mathbb{N}$.

2.1.1. **Proposition** [28, Chap.I, 3.1]. *Let $n \in \mathbb{N}$ and let U be a triangular subset of A^{n+1}. Suppose that for any permutation σ of $\{1,\dots,n\}$, whenever $(u_1,\dots,u_{n+1}) \in U$, then $(u_{\sigma(1)},\dots,u_{\sigma(n)},u_{n+1}) \in U$. Let M be an A-module, $m \in M$, $(u_1,\dots,u_{n+1}) \in U$, and let σ be a permutation of $\{1,\dots,n\}$. Then in $U^{-n-1}M$,*

$m/(u_1,...,u_{n+1}) = (sgn\,\sigma)/(u_{\sigma(1)},...,u_{\sigma(n)},u_{n+1}).$

Triangular subsets with the properties of that considered in 2.1.1. occur frequently in the known examples (see, for example, 2.1.6.). We remark that for a given permutation of $\{1,...,n\}$, σ, $sgn\sigma = |P|$, where $P \in M_n(A)$ is the permutation matrix associated with σ. Furthermore, if σ is non-trivial, then $P \notin D_n(A)$, and likewise $P' \notin D_{n+1}(A)$, where

$$P' = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}.$$

Now $P'[u_1 \ldots u_{n+1}]^T = [u_{\sigma(1)} \ldots u_{\sigma(n)}\ u_{n+1}]^T$, and it follows that P' satisfies $(*)$ since $|P'| = |P|$.

Our primary aim is to obtain a generalization of 2.1.1. which allows us to replace the matrix P by any element of $M_n(A)$. In order to achieve this we shall require the following two preliminary lemmas.

2.1.2. **Lemma.** *Let U be a triangular subset of* A^{n+1} *and let* $u,v \in U$ *be such that* $u_{n+1} = v_{n+1}$, *and* $\sum_1^n Av_i \subseteq \sum_1^n Au_i$. *Then there exist* $w \in U$, $H = [h_{ij}] \in D_{n+1}(A)$ *and* $K = [k_{ij}] \in D_{n+1}(A)$ *such that* $h_{n+1\ n+1} = k_{n+1\ n+1}$ *and* $Hu^T = w^T = Kv^T$.

Proof. By the properties of triangular subsets there exist $w \in U$, $H' = [h'_{ij}] \in D_{n+1}(A)$ and $K' = [k'_{ij}] \in D_{n+1}(A)$ such that $H'u^T = w^T = K'v^T$. This implies that

$$w_{n+1} = \sum_{j=1}^{n+1} k'_{n+1\ j} v_j$$

$$= \sum_{j=1}^{n} k'_{n+1\ j} v_j + k'_{n+1\ n+1} u_{n+1}$$

$$= \sum_{j=1}^{n} h_j u_j + k'_{n+1\ n+1} u_{n+1}$$

for $h_j \in A$, $1 \leq j \leq n$, since $u_{n+1} = v_{n+1}$ and $\sum_1^n Av_i \subseteq \sum_1^n Au_i$.

Now let $K = K'$, and let H be the matrix obtained from H' by replacing $h'_{n+1\ j}$ with h_j, $1 \leq j \leq n$, $h'_{n+1\ n+1}$ with $k'_{n+1\ n+1}$, and leaving the first n rows unchanged. Then $Hu^T = w^T = Kv^T$ and $h_{n+1\ n+1} = k'_{n+1\ n+1} = k_{n+1\ n+1}$ as required.

The following result is of considerable importance both here and at other places in this thesis, as it provides a relationship between the action of arbitrary matrices and the rôle played by lower triangular matrices in a module of generalized fractions.

2.1.3. **Lemma** [O'Carroll, **26**, 3.3]. *Let* $u = (u_1, \ldots, u_n) \in A^n$ *and* $v = (v_1, \ldots, v_n) \in A^n$ *and let* $K \in M_n(A)$, $H \in D_n(A)$ *satisfy* $Ku^T = v^T = Hu^T$. *Set* $v_0 = \prod_1^n v_i$. *Then*

$$v_0(|H| - |K|) \in \sum_1^n Av_i^2.$$

Proof. Let $H = [h_{ij}]$. If $t = 0$ let $H_t = [1]$, and if $1 \leq t \leq n$ let H_t be the top left $t \times t$-submatrix of H. Let $K = [k_{ij}]$. If $t = n$ let $K_t = [1]$, and if $0 \leq t < n$ let K_t be the bottom right $(n-t) \times (n-t)$-submatrix of K. Let us further suppose that i is a fixed integer such that $1 \leq i \leq n$. Then

$$v_i = \sum_{j=1}^{i} h_{ij} u_j = \sum_{j=1}^{n} k_{ij} u_j\ ,$$

so that

$$(h_{ii} - k_{ii})v_i \in \sum_1^{i-1} Au_j + h_{ii}\left(\sum_{j=i+1}^{n} k_{ij}u_j\right). \qquad (\dagger)$$

Now $H_{i-1}[u_1 \ldots u_{i-1}]^T = [v_1 \ldots v_{i-1}]^T$, so by Cramer's Rule and ($\dagger$), and noting that $|H_{i-1}|(h_{ii}) = |H_i|$, it follows that

$$|H_{i-1}|(h_{ii} - k_{ii})v_i \in \sum_1^{i-1} Av_j + |H_i|\left(\sum_{j=i+1}^{n} k_{ij}u_j\right).$$

(We will use a similar argument below.) Thus

$$(|H_i||K_i| - |H_{i-1}|k_{ii}|K_i|)v_i \in \sum_1^{i-1} Av_j + |H_i||K_i|\left(\sum_{j=i+1}^{n} k_{ij}u_j\right). \qquad (\dagger\dagger)$$

Now $K_i[u_{i+1} \ldots u_n]^T = [w_{i+1} \ldots w_n]^T$ where

$$w_j = v_j - \sum_{t=1}^{i} k_{jt}u_t\ ,\ i+1 \le j \le n,$$

so by Cramer's Rule, for $i+1 \le j \le n$,

$$|K_i|u_j \in \sum_{i+1}^{n} Av_t + \sum_1^{i-1} Au_t - u_i|K_i^{(j)}|$$

where $K_i^{(j)}$ is K_i with its $(j-i)^{th}$ column replaced by $[k_{i+1\ i} \ldots k_{ni}]^T$.

As before, it follows that for $i+1 \le j \le n$,

$$|H_i||K_i|u_j \in \sum_{t \ne i} Av_t - |H_{i-1}|h_{ii}u_i \cdot |K_i^{(j)}|.$$

Since $h_{ii}u_i \in v_i + \sum_1^{i-1} Au_t$ and $|H_{i-1}|u_t \in \sum_1^{i-1} Av_s$, $1 \le t \le i-1$, we deduce that for

$i+1 \leq j \leq n$,

$$|H_i||K_i|u_j \in \sum_{t\neq i} Av_t - |H_{i-1}||K_i^{(j)}|v_i \, . \qquad (\dagger\dagger\dagger)$$

Taking $(\dagger\dagger\dagger)$ together with $(\dagger\dagger)$, it follows that

$$(\, |H_i||K_i| - |H_{i-1}|(k_{ii}|K_i| - \sum_{j=i+1}^{n} k_{ij}|K_i^{(j)}|\,)\,)v_i \in \sum_{t\neq i} Av_t;$$

so that

$$(|H_i||K_i| - |H_{i-1}||K_{i-1}|)v_i \in \sum_{t\neq i} Av_t \, .$$

Hence

$$(|H_i||K_i| - |H_{i-1}||K_{i-1}|)v_0 \in \sum_{1}^{n} Av_t^2 \, .$$

The result follows on summing over i .

There is an obvious similarity between 2.1.3. and 1.2.2. and it is reasonable to ask whether 2.1.3. can be "strengthened" by replacing $\sum_1^n$ with $\sum_1^{n-1}$ in its statement. However the following elementary example demonstrates that this cannot be done in general.

Suppose that $A = \mathbb{Z}$, $n=2$, $u=v=(3,2)$, and let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} .$$

Then $Hu^T = v^T = Ku^T$, but $v_1v_2(|H|-|K|) = 6 \notin 9\mathbb{Z}$.

Equipped with the preceding two results, we are now in a position to present our first main result, which is a generalization of 2.1.1..

2.1.4. **Theorem.** *Let U be a triangular subset of A^{n+1} and let $u,v \in U$ be such that $u_{n+1} = v_{n+1}$ and $[v_1 \ \dots \ v_n]^T = Q[u_1 \ \dots \ u_n]^T$ for some $Q \in M_n(A)$. Then in $U^{-n-1}M$,*

$$m/u = |Q|m/v \text{ , for all } m \in M.$$

Proof. We first note that $[v_1 \ \dots \ v_n]^T = Q[u_1 \ \dots \ u_n]^T$ for some $Q \in M_n(A)$ if and only if $\sum_1^n Av_i \subseteq \sum_1^n Au_i$, so that the conditions of 2.1.2. are satisfied. Thus there exist $w \in U$, $H = [h_{ij}] \in D_{n+1}(A)$ and $K = [k_{ij}] \in D_{n+1}(A)$ such that $h_{n+1\ n+1} = k_{n+1\ n+1}$ and $Hu^T = w^T = Kv^T$.

Let H' and K' be the top left $n \times n$-submatrices of H and K respectively, let $w_0 = \prod_1^n w_i$, and let $D \in D_{n+1}(A)$ be the diagonal matrix $\mathrm{diag}(w_1,\dots,w_{n+1})$. We note that $|D| = w_0w_{n+1}$. In $U^{-n-1}M$,

$$m/(u_1,\dots,u_{n+1}) - |Q|m/(v_1,\dots,v_{n+1})$$

$$= |H|m/(w_1,\dots,w_{n+1}) - |K||Q|m/(w_1,\dots,w_{n+1}) \text{ , by 1.2.3.(i)}$$

$$= (|H| - |K||Q|)m/(w_1,\dots,w_{n+1})$$

$$= |D|(|H| - |K||Q|)m/(w_1^2,w_2^2,\dots,w_{n+1}^2) \qquad \text{by 1.2.3.(i)}$$

$$= h_{n+1\ n+1}w_{n+1}w_0(|H'| - |K'||Q|)m/(w_1^2,w_2^2,\dots,w_{n+1}^2) \text{ .}$$

Since $H'[u_1 \ldots u_n]^T = [w_1 \ldots w_n]^T = K'Q[u_1 \ldots u_n]^T$, and $H' \in D_n(A)$,

$$w_0(|H'| - |K'||Q|) \in \sum_1^n Aw_i^2 \text{ , by 2.1.3..}$$

Therefore, in $U^{-n-1}M$,

$$h_{n+1\ n+1}w_{n+1}w_0(|H'| - |K'||Q|)m/(w_1^2,\ldots,w_{n+1}^2) = 0 \text{ ,}$$

by 1.2.3.(ii), so that

$$m/(u_1,\ldots,u_{n+1}) = |Q|m/(v_1,\ldots,v_{n+1}).$$

We can see from Theorem 2.1.4. that if $u,v \in U$ and $H \in D_{n+1}(A)$ are such that $v^T = Hu^T$, then (*) will hold whenever the matrix H is of the form

$$\begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{where } K \in M_n(A) .$$

Indeed our next result, by making use of the notion of saturation discussed in Chapter I, demonstrates that (*) continues to hold for a still larger class of matrices.

2.1.5. **Theorem.** *Let U be a triangular subset of* A^{n+1} *and let M be an A-module. Suppose that there exist* $u,v \in U$ *and* $K = [k_{ij}] \in M_{n+1}(A)$ *such that* $k_{i\ n+1} = 0$, $1 \le i \le n$, *and* $Ku^T = v^T$. *Then in* $U^{-n-1}M$,

$$m/(u_1,\ldots,u_{n+1}) = |K|m/(v_1,\ldots,v_{n+1}),$$

for all $m \in M$.

Proof. Let K′ denote the top left $n \times n$-submatrix of K and let I denote the $n \times n$ identity matrix. Then

$$[v_1 \dots v_{n+1}]^T = \begin{bmatrix} K' & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & & 0 \\ k_{n+1\,1} & \dots & k_{n+1\,n+1} \end{bmatrix} [u_1 \dots u_{n+1}]^T$$

$$= \begin{bmatrix} K' & 0 \\ 0 & 1 \end{bmatrix} [u_1 \dots u_n \; v_{n+1}]^T .$$

By the properties of triangular subsets there exist $w \in U$, $H \in D_{n+1}(A)$ and $J \in D_{n+1}(A)$ such that $Hu^T = w^T = Jv^T$, so that

$$w_{n+1} \in \sum_1^n Av_i + Av_{n+1} .$$

This implies that

$$w_{n+1} \in \sum_1^n Au_i + Av_{n+1} ,$$

since $\sum_1^n Av_i \subseteq \sum_1^n Au_i$, so that

$$w_{n+1} = \sum_1^n t_i u_i + t_{n+1} v_{n+1} , \quad t_i \in A , \quad 1 \leq i \leq n+1 .$$

Let H′ be the matrix obtained from H by substituting t_j for $h_{n+1\,j}$, $1 \leq j \leq n+1$, and leaving the first n rows unchanged. Then

$$[w_1 \ldots w_{n+1}]^T = H'[u_1 \ldots u_n \; v_{n+1}]^T .$$

At this point we recall the remarks made in Chapter I following 1.3.2., which allow us to assume without loss of generality that U is a saturated triangular subset of A^{n+1}. Since $H' \in D_{n+1}(A)$, it follows that $(u_1,\ldots,u_n,v_{n+1}) \in U$. Thus

$$\begin{aligned} m/(u_1,\ldots,u_{n+1}) &= k_{n+1\;n+1} m/(u_1,\ldots,u_n,v_{n+1}) \quad \text{, by 1.2.3.(i) ,} \\ &= |K'| k_{n+1\;n+1} m/(v_1,\ldots,v_n,v_{n+1}) \quad \text{, by 2.1.4.,} \\ &= |K| m/(v_1,\ldots,v_{n+1}). \end{aligned}$$

We have now reached the stage when we can say that (*) holds whenever $H \in M_n(A)$ $(n>1)$ is of the form

$$\begin{bmatrix} H' & 0 \\ h_{n1} \ldots & h_{nn} \end{bmatrix},$$

where the top left $(n-1) \times (n-1)$-submatrix H' can be any element of $M_{n-1}(A)$. However the following example demonstrates that (*) need not hold if we do not enforce this restriction on the matrix H.

2.1.6. **Example.** Let A be a Noetherian local ring of dimension $n>1$ with maximal ideal $\mathfrak{m}$. We let the abbreviation 's.o.p.' stand for 'system of parameters'. Now let

$$W = \{(u_1,...,u_n) \in A^n \mid u_1,...,u_n \text{ form an s.o.p.}\}.$$

In [**36**, §3] it is shown that W is a triangular subset of A^n, and that

$$W[1]^{-n-1}A = H^n_{\mathfrak{m}}(A) \neq 0 ,$$

where $H^n_{\mathfrak{m}}(A)$ is the n^{th} local cohomology module of A with respect to the maximal ideal $\mathfrak{m}$. It is straightforward to show that

$$m/(u_1,...,u_n) = 0 \text{ in } W^{-n}A \Rightarrow m/(u_1,...,u_n,1) = 0 \text{ in } W[1]^{-n-1}A ,$$

and so we deduce that $W^{-n}A \neq 0$. Suppose now that

$$m/(u_1,...,u_n) \neq 0 \text{ in } W^{-n}A .$$

Then by 1.2.3.(iii) ,

$$u_n m/(u_1,...,u_n) \neq 0 .$$

Now let σ be the permutation of $\{1,...,n\}$ which interchanges 1 and n, and let P be the $n \times n$ permutation matrix associated with σ . We remark that P does not belong to the class of matrices described after 2.1.5.. Then

$$P[u_1 \dots u_n]^T = [u_n \ u_2 \dots u_{n-1} \ u_1]^T$$

and $(u_n,u_2,...,u_{n-1},u_1) \in W$ since it is an s.o.p.. However

$$|P|u_n m/(u_n,u_2,\dots,u_{n-1},u_1) = 0 \text{ , by 1.2.3.(ii) ,}$$

so that ($*$) does not hold in this case.

2. Poor M-sequences and determinantal maps

Throughout this section we shall be concerned with the determinantal maps which were first discussed in Chapter I and whose definition we now recall. Let M be an A-module and let $x=(x_1,\dots,x_n)\in A^n$ and $y=(y_1,\dots,y_n)\in A^n$ be such that

$$[y_1 \dots y_n]^T = H[x_1 \dots x_n]^T$$

for some $H\in M_n(A)$. It follows from Cramer's Rule that $|H|(\sum_1^n Ax_i) \subseteq \sum_1^n Ay_i$, so that, in the notation of 1.3.3., there is a well-defined homomorphism $\alpha_H: M/xM \to M/yM$, such that

$$\alpha_H(m+xM) = |H|m+yM \text{ .}$$

In particular we shall be concerned with the situation where $x_1,\dots,x_n$ and $y_1,\dots,y_n$ are poor M-sequences, and it is our ultimate aim to prove that the map α_H is injective in this case. Indeed, it is already known that α_H is injective under certain restrictions on the ring A, the module M, and the matrix H. We first give a brief summary of the various situations where α_H is known to be a monomorphism.

2.2.1. **Theorem** [25, 3.2]. *Let A be a ring, let M be an A-module and* *let* $x_1,\ldots,x_n$ *and* $y_1,\ldots,y_n$ *(denoted* x *and* y *respectively) be poor M-sequences such that* $y^T = Hx^T$ *for some* $H \in D_n(A)$. *Then the determinantal map* $\alpha_H : M/xM \to M/yM$ *such that*

$$\alpha_H(m+xM) = |H|m+yM$$

is a monomorphism.

The next result gives an important property of poor M-sequences, and is required for the proof of 2.2.3..

2.2.2. **Lemma** [39, Chap.II, 3.11]. *Let M be an A-module and let* $x_1,\ldots,x_n$ *be a poor M-sequence. Then* $x_1^{\alpha_1},\ldots,x_n^{\alpha_n}$ *is a poor M-sequence for all positive integers* $\alpha_1,\ldots,\alpha_n$.

2.2.3. **Theorem** (cf.[26, 3.7]) *Let M be an A-module, and let* $x_1,\ldots,x_n$ *and* $y_1,\ldots,y_n$ *be poor M-sequences such that* $y^T = Hx^T$ *for some* $H \in M_n(A)$. *Suppose further that there exists a sequence* $z = z_1,\ldots,z_n$ *of elements of A, and* $J,K \in D_n(A)$, *such that* $z_1,\ldots,z_n$ *is a poor M-sequence and* $Jx^T = z^T = Ky^T$. *Then the determinantal map* $\alpha_H : M/xM \to M/yM$ *is a monomorphism.*

Proof. Let $D \in D_n(A)$ be the matrix $\mathrm{diag}(z_1,\ldots,z_n)$ and let z^2 denote the sequence $z_1^2,\ldots,z_n^2$, so that z^2 is a poor M-sequence by 2.2.2.. It follows from 2.1.3. that

$$|D|(|J| - |H||K|) \in \sum_1^n Az_i^2$$

so that $\alpha_{DJ} = \alpha_{DKH}$ where $\alpha_{DJ}, \alpha_{DKH} : M/xM \to M/z^2M$ are the determinantal maps associated with DJ and DKH respectively.

Now α_{DJ} is injective, by 2.2.1. so that α_{DKH} is likewise injective. Expressing α_{DKH} as the composition $\alpha_{DK}\alpha_{H}$, it follows that α_H is a monomorphism.

An immediate corollary of 2.2.3. is the following result due to O'Carroll.

2.2.4. **Corollary** [26, 3.7]. *Let A be a Noetherian ring and let M be a finitely generated A-module. Suppose that $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are poor M-sequences and that $y^T = Hx^T$ for some $H \in M_n(A)$. Then the determinantal map $\alpha_H : M/xM \to M/yM$ is a monomorphism.*

Proof. By [39, Chap.II, 3.15] and 2.2.2. above, the poor M-sequences of length n form a triangular subset of A^n so that the conditions of 2.2.3. are satisfied; the result is then immediate.

Let us now consider the situation where A is a ring (not necessarily Noetherian), and M=A. Pertaining to this situation, we have the following result from [14]. The method of proof employed in [14] is adapted in this thesis to prove the main result of this section, 2.2.12., (which generalises 2.2.4. to an arbitrary A-module).

2.2.5. **Theorem** [14, p.690]. *Let A be a ring and let $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ be A-sequences such that $y^T = Hx^T$ for some $H \in M_n(A)$. Then the determinantal map $\alpha_H : A/xA \to A/yA$ is a monomorphism.*

In 2.2.1. and 2.2.3.-2.2.5. we see that the map α_H is known to be injective in a variety of situations. Indeed, in this section we shall prove that α_H continues to be a monomorphism in the absence of all the conditions on ring, module or matrix, required in the proofs of these previous results. This will be achieved in Theorem 2.2.12..

Our original intention was to obtain a "direct" proof of 2.2.12. by reducing the general situation, where A is an arbitrary ring and M is any A-module, to the case where A is Noetherian and M is finitely generated, and then to use 2.2.4. to show that α_H is injective. Unfortunately, this approach, although initially appealing, did not yield a proof of the general result. However, the case where n=2 is tractable by this method, and we include a proof of this special case (2.2.6.) as it involves some interesting ideas concerning poor M-sequences, and identifies the problems encountered if we attempt to generalise the approach to higher values of n.

2.2.6. **Theorem.** *Let A be a ring and let M be an A-module. Suppose that x_1,x_2 and y_1,y_2 are poor M-sequences such that $y^T = Hx^T$ for some $H=[h_{ij}] \in M_2(A)$. Then the determinantal map $\alpha_H:M/xM \to M/yM$ is a monomorphism.*

Before giving the proof of 2.2.6. we shall require a few preliminary results which are of some interest independent of their role in the solution of this problem. In addition we must make the following simplification in the statement of 2.2.6..

Let R be the Noetherian ring $\mathbb{Z}[X_1,X_2,H_{11},H_{12},H_{21},H_{22}]$ where X_i,H_{ij}, i,j=1,2 are indeterminates. Now M can be given an R-module structure by restriction of the scalars, where the restricting ring homomorphism $f:R \to A$ is such that $f(X_i)=x_i$ and $f(H_{ij})=h_{ij}$. Let $H' \in M_2(R)$ be the matrix of indeterminates $[H_{ij}]$, and let

$$H'[X_1\ X_2]^T = [Y_1\ Y_2]^T.$$

It is clear that $f(Y_i) = y_i$, that $X=X_1,X_2$ and $Y=Y_1,Y_2$ are poor M-sequences, and that the determinantal (R-module) map $\alpha_{H'}:M/XM \to M/YM$ is injective if and only if the corresponding A-module map α_H is injective. Therefore, in the statement of 2.2.6., we can assume that the ring A is Noetherian without any loss of generality.

Unfortunately, in general we cannot reduce the statement of 2.2.6. to the case where M is finitely generated. However, as will be apparent later, the following result does enable us to replace M with a module possessing the properties required for our proof of 2.2.6..

2.2.7. **Proposition.** *Let A be a ring, not necessarily Noetherian, let M be an A-module and let x_1,x_2 be a poor M-sequence.*

(i) For any submodule $M' \subseteq M$, the unique smallest submodule N such that $M' \subseteq N$ and x_1,x_2 is a poor N-sequence is given by

$$N = [\bigcup_{k=1}^{\infty}(M':x_1^k)] \cap [\bigcup_{k=1}^{\infty}(M':x_2^k)] = \bigcup_{k=1}^{\infty}(M':I^k)$$

where $I=(x_1,x_2)A$.

(ii) If z_1,z_2 is a poor M-sequence such that $x^T = Hz^T$ for some $H \in M_2(A)$, then z_1,z_2 is a poor N-sequence where N is the submodule defined in (i).

Proof. (i) We first show that x_1,x_2 is a poor N-sequence. Since x_1 acts as a non-zerodivisor (n.z.d.) on M, it clearly acts as a n.z.d. on N. Suppose now that

$$x_2 m \in x_1 N ,$$

for some $m \in N$. Since $x_1N \subseteq x_1M$ and x_1,x_2 is a poor M–sequence, it follows that $m \in x_1M$ so that $m = x_1t$ for some $t \in M$. Now $m \in (M':x_1^p)$ for some $p \in \mathbb{N}$, from the definition of N, so that $x_1^p m \in M'$. It follows that $x_1^{p+1}t \in M'$, which implies that

$$t \in (M':x_1^{p+1}) . \qquad (*)$$

Now

$$x_2m \in x_1N$$

$$\Rightarrow x_2(x_1t) \in x_1N$$

$$\Rightarrow x_1(x_2t) \in x_1N$$

$$\Rightarrow x_2t \in N \quad \text{since } x_1 \text{ acts as a n.z.d. on } M,$$

$$\Rightarrow x_2t \in (M':x_2^q) \text{ for some } q \in \mathbb{N},$$

$$\Rightarrow t \in (M':x_2^{q+1}). \qquad (\dagger)$$

It follows from $(*)$, $(\dagger)$ and the definition of N, that $t \in N$. This implies that $m \in x_1N$, so that x_1,x_2 must be a poor N–sequence.

Suppose now that L is a submodule of M which contains M′ but which does not contain N. Since $N \nsubseteq L$, there exist $m \in N$ such that $m \notin L$, and $a,b \in \mathbb{N}$ such that $x_1^a m \in M'$ and $x_2^b m \in M'$ from the definition of N. Since x_1 acts as a n.z.d. on M, x_1^a acts as a n.z.d. on M, and it follows that $x_1^a m \notin x_1^a L$ since $m \notin L$.

However

$$x_2^b(x_1^a m) = x_1^a(x_2^b m) \in x_1^a L .$$

This implies that x_1^a, x_2^b is not a poor L-sequence, and it follows from 2.2.2 that x_1, x_2 is not a poor L-sequence. Therefore N is the unique smallest submodule of M containing M′ such that x_1, x_2 is a poor N-sequence.

It remains to show that $N = \bigcup_{k=1}^{\infty}(M':I^k)$, where $I = (x_1, x_2)A$.

Let $m \in N$. Then there exist $a, b \in \mathbb{N}$ such that $x_1^a m \in M'$ and $x_2^b m \in M'$. Let $t \in I^{a+b}$. Then it is an easy exercise to verify that t can be written as a sum of terms in x_1 and x_2, each of which contains either x_1^a or x_2^b. It follows that $tm \in M'$, and hence $m \in (M':I^{a+b})$, so that $N \subseteq \bigcup_{k=1}^{\infty}(M':I^k)$.

Now suppose that $m \in (M':I^r)$ for some $r \in \mathbb{N}$. Then $m \in (M':x_1^r)$ and $m \in (M':x_2^r)$, so that $m \in N$. This completes the proof of (i).

(ii) As before it is clear that z_1 acts as a n.z.d. on N. It therefore remains to show that

$$z_2 m \in z_1 N,\ m \in N \Rightarrow m \in z_1 N.$$

Let $m \in N$ be such that $z_2 m \in z_1 N$. Then $m \in z_1 M$, since z_1, z_2 is a poor M-sequence, so that $m = z_1 t$, for some $t \in M$.

Since $m \in N$, it follows that $I^k m \subseteq M'$, for some $k \in \mathbb{N}$, so that $I^k z_1 t \subseteq M'$.

Now z_1 acts as a n.z.d. on M, so that $z_2 m = z_2 z_1 t \in z_1 N$ implies that $z_2 t \in N$. It follows that $I^n z_2 t \subseteq M'$ for some $n \in \mathbb{N}$. Let $p = \max\{k, n\}$. Then

$$I^p z_2 t \subseteq M' \text{ and } I^p z_1 t \subseteq M',$$

so that

$$x_1 I^p t \subseteq M' \text{ and } x_2 I^p t \subseteq M',$$

since $x^T = Hz^T$. Therefore

$$(c_1 x_1 + c_2 x_2) I^p t \in M',$$

for any choice of $c_1, c_2 \in A$, and it follows that

$$I^{p+1} t \subseteq M'.$$

This implies that $t \in N$. Thus $m = z_1 t \in z_1 N$, so that z_1, z_2 is a poor N-sequence.

In addition, we shall require the following result, due to Zakeri, for our proof of 2.2.6..

2.2.8. **Lemma** [39, Chap.II, 3.14] *Let A be Noetherian and let M be an A-module. Suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_{n+1}$ are poor M-sequences. Then*

$$\sum_{1}^{n+1} Ay_i \not\subseteq \mathfrak{p}, \textit{ for all } \mathfrak{p} \in Ass_A(M/\sum_{1}^{n} x_i M).$$

Proof of 2.2.6.. Let M, x_1, x_2, y_1, y_2, $H \in M_2(A)$ and α_H be as defined in the statement of Theorem 2.2.6.. We assume without loss of generality that

A is Noetherian. Suppose that

$$\alpha_H(m + xM) = 0 \quad , \text{ for some } m \in M,$$

so that

$$|H|m = y_1m_1 + y_2m_2 \text{ , for some } m_1,m_2 \in M \text{ .}$$

Let $M' = Am + Am_1 + Am_2$. Then M' is a Noetherian A-submodule of M. Now consider the submodule $N = \bigcup_{k=1}^{\infty}(M':I^k)$, where $I = (y_1,y_2)A$. Then x_1,x_2 and y_1,y_2 are poor N-sequences by 2.2.7.. Applying 2.2.8. to the poor N-sequences x_1y_1 and y_1,y_2 it follows that, for each $\mathfrak{p} \in \text{Ass}_A(N/x_1y_1N)$, there exists $w \in I$ such that $w \notin \mathfrak{p}$.

Let $\mathfrak{p} \in \text{Ass}_A(N/x_1y_1N)$. Then $\mathfrak{p} = (x_1y_1N : t)$, for some $t \in N \setminus x_1y_1N$. Now consider $r \in \mathfrak{p}$. Then $rt = x_1y_1n$ for some $n \in N$. Since $w \in I$, it follows that there exists a positive integer k_r such that $w^{k_r}t \in M'$ and $w^{k_r}n \in M'$. From the choice of w, it follows that $w^{k_r}t \notin x_1y_1N$, hence $w^{k_r}t \notin x_1y_1M'$.

Now

$$r(w^{k_r}t) = w^{k_r}x_1y_1n \in x_1y_1M'$$

since $w^{k_r}n \in M'$, so that $r \in (x_1y_1M' : w^{k_r}t)$.

At this point we recall our assumption that A is Noetherian, so that the ascending chain of ideals $\{(x_1y_1M' : w^qt) \mid q \in \mathbb{N}\}$ has an upper bound, $(x_1y_1M' : w^kt)$, for some positive integer k. It therefore follows that $\mathfrak{p} \subseteq (x_1y_1M' : w^kt)$.

Now let $r' \in (x_1y_1M':w^kt)$. Then

$$r'w^kt \in x_1y_1M' ,$$

so that

$$r'w^kt \in x_1y_1N \ .$$

It follows that $r'w^k \in \mathfrak{p}$, which implies that $r' \in \mathfrak{p}$, since $w \notin \mathfrak{p}$. Therefore

$$\mathfrak{p} = (x_1y_1M':w^kt) \in Ass_A(M'/x_1y_1M') ,$$

which implies that

$$Ass_A(N/x_1y_1N) \subseteq Ass_A(M'/x_1y_1M') \ .$$

Hence $Ass_A(N/x_1y_1N)$ is a finite set of primes, since M' is Noetherian. Since, by 2.2.8., $I \not\subseteq \mathfrak{p}'$, for all $\mathfrak{p}' \in Ass_A(N/x_1y_1N)$, it follows from the above that

$$I \not\subseteq \bigcup \mathfrak{p}', \ \mathfrak{p}' \in Ass_A(N/x_1y_1N).$$

Hence there exists $z \in I$ such that x_1y_1,z forms a poor N-sequence. Now $z \in I = (y_1,y_2)A \subseteq (x_1,x_2)A$, so that we can construct $J,K \in D_2(A)$, such that

$$J[x_1 \ x_2]^T = [x_1y_1 \ z]^T = K[y_1 \ y_2]^T \ .$$

It follows from 2.2.3. that $\alpha'_H:N/xN \rightarrow N/yN$ is a monomorphism, where α'_H is the determinantal map associated with H. By assumption,

$$|H|m = y_1m_1 + y_2m_2 ,$$

which implies that $m \in xN \subseteq xM$. Therefore $\alpha_H:M/xM \to M/yM$ must be injective, and the proof is complete.

If we attempt to generalise our proof of 2.2.6. to the situation where the M-sequences are of arbitrary length, we quickly experience difficulties, which we now describe.

A crucial step in the proof of 2.2.6. involves the identification of the smallest submodule N, containing a given submodule, for which y_1,y_2 is a poor N-sequence. This is effected by 2.2.7.(i), and it follows that any poor M-sequence x_1,x_2 for which $(y_1,y_2)A \subseteq (x_1,x_2)A$, is automatically a poor N-sequence, by 2.2.7.(ii).

Unfortunately we are unable to obtain a result corresponding to 2.2.7.(i) for the case where the poor M-sequences in question are of length 3. Indeed, even if it were possible to identify the smallest submodule N, containing a given submodule, for which a poor M-sequence y_1,y_2,y_3 is also a poor N-sequence, the following example demonstrates that the analogue of 2.2.7.(ii) need not be true.

Example [20, p.102, Ex.7]. Let k be a field and let M=k[x,y,z], where x,y and z are indeterminates over k. Now let u=x, v=y(1-x) and w=z(1-x). If we set $Q = M_{(x,y,z)}$ (localization at the maximal ideal (x,y,z)), and let R=k[u,v,w], then $R \subset M \subset Q$, so that M and Q are naturally R-modules.

Now it can be shown that u,v,w is a poor Q-sequence. Since Q is a

Noetherian local ring, and u,v and w are contained in its maximal ideal, it follows from [20, Theorem 119] that w,v,u is also a poor Q-sequence. In addition, it can be shown that u,v,w is a poor M-sequence. If the analogue of 2.2.7.(ii) were true for sequences of length 3, we should expect v,w,u to be a poor M-sequence also, since (u,v,w)R=(v,w,u)R. However,

$$yw = yz(1-x) = zv ,$$

but $y \notin vM$. It follows that v,w,u is not a poor M-sequence.

This inability to identify a submodule with suitable properties leads to a breakdown of the proof of 2.2.6. when the poor M-sequences considered have length greater than 2. Moreover, efforts to generalize 2.2.6. by the technique of induction, in the manner of [25, 3.2] where lower triangular matrices are considered, have proved fruitless, so that we are forced to adopt a quite different approach in order to prove the result in its full generality.

In [14], Theorem 2.2.5. is proved by way of an argument which involves the use of the Koszul complex and the Ext functor, and we modify these ideas in this thesis to prove the main result of this section, Theorem 2.2.12.. Before presenting 2.2.12., we require the following three results, all of which play a central part in its proof.

2.2.9. **Lemma** [20, p.100]. *Let A be a ring, let C and D be A-modules and suppose that there exists an element $x \in A$ such that x acts as a n.z.d. on D and $xC = 0$. Then $Hom_A(C,D) = 0$.*

2.2.10. **Theorem** [20, p.101]. *Let S and T be A-modules. Suppose that* *the elements* $x_1,\dots,x_n$ *constitute an S-sequence and that* $\sum_1^n x_i T = 0$. *Then*

$$Ext_A^n(T,S) \cong Hom_A(T,S/\sum_1^n x_i S) .$$

2.2.11. **Lemma** [17, p.1038, Prop.21]. *Let X be a Noetherian ring and let* $s_1,\dots,s_n$ *be elements of X. Let* $grade(s_1,\dots,s_n) = g$ *and let K be an* $m \times n$ *matrix of indeterminates* $[k_{ij}]$ *over X with* $m \leq g$. *Then* $t_1,\dots,t_m$ *is an* $X[k_{ij}]$*-sequence, where*

$$[t_1 \ \dots \ t_m]^T = K\,[s_1 \ \dots \ s_n]^T .$$

We are now in a position to present the main result of this section.

2.2.12. **Theorem**. *Let A be a ring and let M be an A-module. Suppose that there exist M-sequences* $x_1,\dots,x_n$ *and* $y_1,\dots,y_n$, *and* $H = [h_{ij}] \in M_n(A)$, *such that*

$$y^T = Hx^T .$$

Then the determinantal map $\alpha_H : M/xM \to M/yM$ *defined by*

$$\alpha_H(m+xM) = |H|m+yM$$

is a monomorphism.

Proof. Let $R = \mathbb{Z}[X_1,\dots,X_n,H_{11},H_{12},\dots,H_{ij},\dots,H_{nn}]$, where the elements X_i

and H_{ij} are all indeterminates over $\mathbb{Z}$. Let $f : R \to A$ be the ring homomorphism such that

$$f(X_i) = x_i \ , \ 1 \leq i \leq n \ ,$$

$$f(H_{ij}) = h_{ij} \ , \ 1 \leq i \leq n \ , \ 1 \leq j \leq n \ .$$

Then M can be given an R-module structure by defining r.m to be f(r)m where $r \in R$, $m \in M$.

Let $[Y_1 \ \dots \ Y_n]^T = [H_{ij}][X_1 \ \dots \ X_n]^T$. For any $m \in M$,

$$\begin{aligned} Y_i.m &= f(\sum_{j=1}^{n} H_{ij}X_j) \ . \ m \\ &= (\sum_{j=1}^{n} h_{ij}x_j)m \\ &= y_i m \ , \ 1 \leq i \leq n \ . \end{aligned}$$

Similarly, $|[H_{ij}]| \ . \ m = |H|m$. Therefore $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$ are M-sequences. In addition, $X_1,\dots,X_n$ is an R-sequence since the elements X_i are all indeterminates over $\mathbb{Z}$. We now apply 2.2.11. to the situation where $X = \mathbb{Z}[X_1,\dots,X_n]$, $s_i = X_i$, $K = [H_{ij}]$ and $g = n = m$, to show that $Y_1,\dots,Y_n$ is also an R-sequence.

For simplicity of notation we identify elements of R with their images in A and relabel X_i as x_i, Y_i as y_i and H_{ij} as h_{ij}, $1 \leq i,j \leq n$. From this point forward, $\alpha : M/xM \to M/yM$ will denote the R-module determinantal map induced by the matrix $H \in M_n(R)$, which is injective if and only if the corresponding A-module determinantal map, α_H, is injective.

Now consider the short exact sequence of R-modules

$$0 \to xR/yR \xrightarrow{i} R/yR \xrightarrow{\pi} R/xR \to 0$$

where i is the inclusion map and π the canonical projection map. This yields the exact sequence

$$\mathrm{Ext}_R^{n-1}(xR/yR,M) \to \mathrm{Ext}_R^n(R/xR,M) \to \mathrm{Ext}_R^n(R/yR,M)\ .$$

By 2.2.10.

$$\mathrm{Ext}_R^{n-1}(xR/yR,M) \cong \mathrm{Hom}_R(xR/yR,M/\sum_1^{n-1} y_iM),$$

and

$$\mathrm{Hom}_R(xR/yR,M/\sum_1^{n-1} y_iM) = 0$$

by 2.2.9.. Thus the map from $\mathrm{Ext}_R^n(R/xR,M)$ to $\mathrm{Ext}_R^n(R/yR,M)$ induced by the projection map π is injective. By making use of the Koszul complex we now calculate this induced map. The reader is referred to [**14**, pp.687-692].

Since $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are R-sequences, the Koszul complexes $K^\bullet(x,R)$ and $K^\bullet(y,R)$ provide projective resolutions of R/xR and R/yR respectively.

As in [**14**], we obtain the following morphism of exact complexes;

$$
\begin{array}{ccccccccccccccc}
K^{\bullet}(x,R): 0 & \to & R & \xrightarrow{\partial_n} & \bigoplus_1^n R & \xrightarrow{\partial_{n-1}} & \cdots & \to & \bigoplus_1^n R & \xrightarrow{\partial_1} & R & \to & R/xR & \to & 0 \\
 & & \uparrow \phi_n & & \uparrow \phi_{n-1} & & & & \uparrow \phi_1 & & \uparrow \phi_0 & & \uparrow \pi & & \\
K^{\bullet}(y,R): 0 & \to & R & \xrightarrow{\partial_n} & \bigoplus_1^n R & \xrightarrow{\partial_{n-1}} & \cdots & \to & \bigoplus_1^n R & \xrightarrow{\partial_1} & R & \to & R/yR & \to & 0
\end{array}
$$

where ϕ_0 is the identity map and $\phi_n(r) = |H|r$.

Applying $\mathrm{Hom}_R(-,M)$ gives, in particular,

$$
\begin{array}{ccccc}
\to \mathrm{Hom}_R(\bigoplus_1^n R,M) & \xrightarrow{\partial_n^*} & \mathrm{Hom}_R(R,M) & \to & 0 \\
\downarrow \phi_{n-1}^* & & \downarrow \phi_n^* & & \\
\to \mathrm{Hom}_R(\bigoplus_1^n R,M) & \xrightarrow{\partial_n^*} & \mathrm{Hom}_R(R,M) & \to & 0 .
\end{array}
$$

We note that there is an isomorphism β: $\mathrm{Hom}_R(R,M) \to M$ such that

$\beta(f) = f(1)$.

Now let $f \in \mathrm{Hom}_R(\bigoplus_1^n R,M)$. Then, by the definition of ∂_n^*,

$$
\begin{aligned}
(\partial_n^* f)(1) &= f\partial_n(1) \\
&= f((x_1,-x_2,\ldots,(-1)^{n-1}x_n)) \\
&= \sum_1^n (-1)^{i-1} x_i f(e_i), \\
&= \sum_1^n x_i f_i(1) ,
\end{aligned}
$$

where e_i is the i^{th} basis vector of $\overset{n}{\underset{1}{\oplus}}R$ and $f_i(1) = (-1)^{i-1}f(e_i)$. It therefore follows that

$$\partial_n^*(f) \in \sum_1^n x_i \mathrm{Hom}_R(R,M) = x\mathrm{Hom}_R(R,M)$$

so that $\mathrm{Im}\,\partial_n^* \subseteq x\mathrm{Hom}_R(R,M)$.

The argument can be reversed to show that $x\mathrm{Hom}_R(R,M) \subseteq \mathrm{Im}\,\partial_n^*$ so that $\mathrm{Im}\,\partial_n^* = x\mathrm{Hom}_R(R,M)$ in $K^\bullet(x,R)$. The same proof shows that $\mathrm{Im}\,\partial_n^* = y\mathrm{Hom}_R(R,M)$ in $K^\bullet(y,R)$.

We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(R,M) & \overset{\beta}{\cong} & M \\ \phi_n^* \big\downarrow & & \big\downarrow \psi \\ \mathrm{Hom}_R(R,M) & \overset{\beta}{\cong} & M \end{array}$$

where ψ is the homomorphism induced by ϕ_n^*. Now $\phi_n^*(f) = f\phi_n = |H|f$, so that ψ must likewise be multiplication by $|H|$. Since $\mathrm{Im}\ \partial_n^* = x\mathrm{Hom}_R(R,M)$ (resp. $y\mathrm{Hom}_R(R,M)$), we have the following commutative diagram, where α is the map induced by multiplication by $|H|$ and ϕ the map induced by π:

$$\begin{array}{ccc} \mathrm{Ext}_R^n(R/xR,M) & \cong & M/xM \\ \phi \big\downarrow & & \big\downarrow \alpha \\ \mathrm{Ext}_R^n(R/yR,M) & \cong & M/yM \end{array} .$$

It has already been shown that ϕ is injective, so that the map α must be injective also, and the result follows.

1. Denominator systems and chains of triangular subsets

Our purpose in this section is to investigate the connection between chains of triangular subsets on a ring A, as described in 1.4., and the denominator systems described by Kersken in [21]. In the following section we shall show that the denominator system complexes of Kersken are identical to the complexes of modules of generalized fractions of 1.4.. It follows that, in this respect, the notions of a denominator system and a chain of triangular subsets are essentially equivalent. We begin by recalling from [21] the definition of a denominator system over a ring A.

3.1.1. **Definition** [21]. A set $\mathcal{J} \subseteq \bigcup_{0}^{\infty} A^i$ of sequences in A (where $A^0 = \{\emptyset\}$) is a *denominator system over A* if it satisfies the following conditions:

(i) $\mathcal{J} \neq \emptyset$;

(ii) if $(f_1,\ldots,f_p) \in \mathcal{J}^p = \mathcal{J} \cap A^p$, then $(f_1,\ldots,f_j) \in \mathcal{J}$ for all j such that $0 \leq j \leq p$;

(iii) if $f = (f_1,\ldots,f_p) \in \mathcal{J}^p$, then

$$S(f) = \{f_{p+1} \in A \mid (f_1,\ldots,f_p,f_{p+1}) \in \mathcal{J}\}$$

is a multiplicatively closed subset (m.c.s.) of A;

(iv) if $f,g \in \mathcal{J}^p$ are such that $gA \subseteq fA$, then

$$S(g) \subseteq S(f) \subseteq (S(g) + fA)^{\sim},$$

where $(S(g) + fA)^{\sim}$ is the saturation of the m.c.s. $(S(g) + fA)$. We further assume that $\mathcal{S}$ contains the empty sequence $\emptyset$ and sequences of 1's of arbitrary length.

A direct consequence of condition (iv) is the following result, which will be used on numerous occasions throughout this chapter.

3.1.2. **Proposition [21].** *Let $\mathcal{S}$ be a denominator system over A, let M be an A-module and let $p \in \mathbb{N}$. Suppose that $f, g \in \mathcal{S}^p$ are such that $gA \subseteq fA$. Then*

$$(M/fM)_{S(g)} \cong (M/fM)_{S(f)}$$

under the canonical homomorphism.

Proof. It is straightforward to show that

$$(A/fA)_{S(g)} = (A/fA)_{(S(g)+fA)} = (A/fA)_{(S(g)+fA)^{\sim}},$$

so that $(A/fA)_{S(g)} = (A/fA)_{S(f)}$ by 3.1.1.(iv). The result then follows on applying $-\otimes_A M$.

We recall from Chapter I that if U is a triangular subset of A^p, and $u, v \in U$, then, by definition, there exists $w \in U$ such that $w_j \in (u_1,\dots,u_j)A \cap (v_1,\dots,v_j)A$, $1 \le j \le p$. In view of this fact, the next result concerning denominator systems is strongly suggestive of a connection with triangular subsets.

3.1.3. **Lemma [21, (1.1)].** *Let $\mathcal{S}$ be a denominator system over A and let f,g be sequences in $\mathcal{S}$ of length $p \in \mathbb{N}$. Then there is a sequence $h \in \mathcal{S}^p$*

such that $h_j \in (f_1,\dots,f_j)A.(g_1,\dots,g_j)A,\ 1 \le j \le p.$

Proof. By 3.1.1.(iii), the lemma is true for sequences of length 1. Suppose now that $j > 1$, $f,g \in \mathcal{S}^j$ and that the lemma has been proved for sequences of length j-1. It follows that there exists $(h_1,\dots,h_{j-1}) \in \mathcal{S}^{j-1}$ such that $h_k \in (f_1,\dots,f_k)A.(g_1,\dots,g_k)A$, $1 \le k \le j-1$. Now

$$f_j \in S(f_1,\dots,f_{j-1}) \subseteq (S(h_1,\dots,h_{j-1}) + (f_1,\dots,f_{j-1})A)^{\sim},$$

by 3.1.1.(iv), so that there exist $a_1 \in A$ and $t_1 \in S(h_1,\dots,h_{j-1})$ such that $a_1 f_j \equiv t_1 \mathrm{mod}(f_1,\dots,f_{j-1})A$, which implies that $t_1 \in (f_1,\dots,f_j)A$.

Similarly there exists $t_2 \in S(h_1,\dots,h_{j-1}) \cap (g_1,\dots,g_j)A$. If we let $h_j = t_1 t_2$, it follows from 3.1.1.(iii) that $h_j \in S(h_1,\dots,h_{j-1})$. Clearly

$$(h_1,\dots,h_j)A \subseteq (f_1,\dots,f_j)A.(g_1,\dots,g_j)A,$$

and the result follows by induction.

Let $\mathfrak{a}$ be an ideal of A and let M be an A-module. The $\mathcal{S}$*-height* of $\mathfrak{a}$, denoted $\mathcal{S}$-ht$\mathfrak{a}$ is given by

$$\mathcal{S}\text{-ht}\,\mathfrak{a} = \sup\{\, i \in \mathbb{N}_0 \mid \mathfrak{a} \text{ contains a sequence in } \mathcal{S} \text{ of length } i \,\}.$$

The $\mathcal{S}$*-height* of M, denoted $\mathcal{S}$-htm M, is given by

$$\mathcal{S}\text{-htm}\, M = \inf\{\, \mathcal{S}\text{-ht}(\mathrm{Ann}_A x) \mid x \in M \,\}.$$

The following result summarises several important properties of the $\mathcal{S}$-ht of an ideal, all of which are easily verifiable from 3.1.3..

3.1.4. **Corollary** [21, (1.2)]. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of A. Then*

(i) $\mathcal{S}$*-ht*$(\mathfrak{a}\cap\mathfrak{b})$ = $\mathcal{S}$*-ht* $\mathfrak{a}.\mathfrak{b}$ = *inf{* $\mathcal{S}$*-ht* $\mathfrak{a}$, $\mathcal{S}$*-ht* $\mathfrak{b}$ *};*

(ii) if $(f_1,...,f_p) \in \mathcal{S}^p$, *then* $\mathcal{S}$*-ht*$(f_1^{r_1},...,f_p^{r_p}) \geq p$, *for all* $r_1,...,r_p \in \mathbb{N}$;

(iii) let $\mathcal{S}$*-ht* $\mathfrak{a} = p$, *let* $k \leq p$ *and suppose that* $(f_1,...,f_k) \in \mathcal{S}^k$ *is a sequence of elements of* $\mathfrak{a}$. *Then there exist* $f_{k+1},...,f_p \in \mathfrak{a}$, *such that* $(f_1,...,f_p) \in \mathcal{S}^p$.

We recall from Chapter II the advantages offered by working with saturated triangular subsets. There exists a similar notion concerning denominator systems which we now describe.

A denominator system $\mathcal{S}$ is said to be *saturated* if for all $f \in \mathcal{S}$,

$$S(f) = (S(f) + fA)^{\sim}.$$

The following result provides a useful characterisation of saturated denominator systems.

3.1.5. **Corollary** [21, (1.3)]. *A denominator system* $\mathcal{S}$ *is saturated if and only if whenever* $(f_1,...,f_p) \in A^p$ *is such that* $\mathcal{S}$*-ht* $(f_1,...,f_j)A \geq j$, $1 \leq j \leq p$, *then* $(f_1,...,f_p) \in \mathcal{S}$.

Proof. Suppose that $\mathcal{S}$ is saturated and $(f_1,...,f_p) \in A^p$ is such that $\mathcal{S}$-ht $(f_1,...,f_j)A \geq j$, $1 \leq j \leq p$. Now it follows from the definition of a saturated denominator system that $(f_1) \in \mathcal{S}$. Suppose that $j>1$ and $(f_1,...,f_{j-1}) \in \mathcal{S}^{j-1}$. By 3.1.4.(iii), there exists $g \in (f_1,...,f_j)A$ such that $g \in S(f_1,...,f_{j-1})$.

It follows that

$$f_j \in (S(f_1,...,f_{j-1}) + (f_1,...,f_{j-1})A)^{\sim} = S(f_1,...,f_{j-1}),$$

so that $(f_1,...,f_j) \in \mathcal{S}^j$.

Conversely, suppose that $\mathcal{S}$ contains all $(f_1,...,f_p) \in A^p$, $p \in \mathbb{N}$, such that $\mathcal{S}\text{-ht}(f_1,...,f_j)A \geq j$, $1 \leq j \leq p$. Let $f \in \mathcal{S}^p$, where we now fix p, and let $f' \in (S(f) + fA)^{\sim}$. Then there exists $g \in S(f)$ such that $g \in (f_1,...,f_p,f')A$, so that $\mathcal{S}\text{-ht}\ (f_1,...,f_p,f')A \geq p+1$. It follows that $f' \in S(f)$, so that $\mathcal{S}$ must be saturated.

For a denominator system $\mathcal{S}$, we define the *saturation* of $\mathcal{S}$, denoted $\mathcal{S}^{\sim}$, to be the set of sequences $(f_1,...,f_p) \in \bigcup_{0}^{\infty} A^i$, for which $\mathcal{S}\text{-ht}\ (f_1,...,f_j)A \geq j$, $1 \leq j \leq p$. It can easily be shown that $\mathcal{S}^{\sim}$ is the smallest saturated denominator system which contains $\mathcal{S}$.

3.1.6. **Example** [21, (1.7)(a)]. Let A be a Noetherian ring and let M be a finitely generated A-module. Then the set of all poor M-sequences forms a saturated denominator system over A.

Now, in the situation of 3.1.6., it is known that the poor M-sequences form a chain of triangular subsets on A so that the above example provides further evidence of the tie-up between denominator systems and chains of triangular subsets. The next result investigates this connection explicitly.

3.1.7. **Proposition.** *Let $\mathcal{S}$ be a saturated denominator system over A and let $U_i = \mathcal{S}^i$, $i \in \mathbb{N}$. Then $\mathcal{U} = \{ U_i \mid i \in \mathbb{N} \}$ forms a chain of saturated triangular subsets over A.*

Proof Let $f = (f_1,\dots,f_p) \in U_p$. It follows from 3.1.4.(ii) that $\mathcal{J}$-ht$(f_1^{r_1},\dots,f_j^{r_j})A \geq j$, $1 \leq j \leq p$, for all choices of positive integers $r_1,\dots,r_p$, so that $(f_1^{r_1},\dots,f_p^{r_p}) \in U^p$ by 3.1.5..

Now suppose that $g = (g_1,\dots,g_p) \in U_p$. By 3.1.3. there exists $(h_1,\dots,h_p) \in U_p$ such that $h_j \in (f_1,\dots,f_j)A.(g_1,\dots,g_j)A$, $1 \leq j \leq p$, and since $(f_1,\dots,f_j)A.(g_1,\dots,g_j)A$ is contained in $(f_1,\dots,f_j)A \cap (g_1,\dots,g_j)A$ we can clearly construct two lower triangular matrices H,K over A such that

$$H[f_1 \dots f_p]^T = [h_1 \dots h_p]^T = K[g_1 \dots g_p]^T.$$

Hence U_p is a triangular subset of A^p, for all $p \in \mathbb{N}$. It remains to show that U_p is a saturated triangular subset and that U_p is the restriction of U_{p+1} to A^p, $p \in \mathbb{N}$.

Suppose that $v \in A^p$ is such that $Hv^T = u^T$, for some $u \in U_p$ and $H \in D_p(A)$. Since $(u_1,\dots,u_j)A \subseteq (v_1,\dots,v_j)A$, $1 \leq j \leq p$, it follows that $\mathcal{J}$-ht $(v_1,\dots,v_j)A \geq j$, $1 \leq j \leq p$. Therefore $v \in U_p$, since $\mathcal{J}$ is a saturated denominator system, and it follows that U_p is a saturated triangular subset of A^p.

By 3.1.1.(ii) it is clear that the restriction of U_{p+1} to A^p is contained in U_p. Now consider $(u_1,\dots,u_p) \in U_p$. It follows from 3.1.5. that $(u_1,\dots,u_p,1) \in U_{p+1}$, so that U_p is indeed the restriction of U_{p+1} to A^p, and the result follows.

We now prove the converse to 3.1.7.. This result has been proved independently by Hamieh and Zakeri in [**16**, 2.5] by a method essentially similar to that which is employed below.

3.1.8. **Proposition.** *Let* $\mathcal{U} = \{U_i \mid i \in \mathbb{N}\}$ *be a chain of saturated* *triangular subsets over A. Then* $\mathcal{S} = \bigcup_{1}^{\infty} U_i$ *is a saturated denominator system over A.*

Proof. We first show that $\mathcal{S}$ is a denominator system by verifying 3.1.1.(i)–(iv). Clearly $\mathcal{S} \neq \emptyset$, so that (i) is true. Property (ii) is obviously true since $\mathcal{U}$ is a chain of triangular subsets.

Now let $f = (f_1,...,f_p) \in U_p$, and let $S(f) = \{f_{p+1} \mid (f_1,...,f_{p+1}) \in U_{p+1}\}$. In order to verify (iii), we must show that S(f) is a m.c.s. of A.

Suppose that $f_{p+1}, f'_{p+1} \in S(f)$. Since U_{p+1} is a triangular subset of A^{p+1}, there exist $(g_1,...,g_{p+1}) \in U_{p+1}$ and $H,K \in D_{p+1}(A)$, such that

$$H[f_1 \ \dots \ f_{p+1}]^T = [g_1 \ \dots \ g_{p+1}]^T = K[f_1 \ \dots \ f'_{p+1}]^T .$$

In addition, $(g_1,...,g_{p+1}^2) \in U_{p+1}$. Now, in an obvious notation,

$$\begin{aligned} g_{p+1}^2 &= (\sum_{j=1}^{p} h_{p+1\ j} f_j + h_{p+1\ p+1} f_{p+1})(\sum_{j=1}^{p} k_{p+1\ j} f_j + k_{p+1\ p+1} f'_{p+1}) \\ &= \sum_{j=1}^{p} m_j f_j + m_{p+1}(f_{p+1} f'_{p+1}) , \end{aligned}$$

for some $m_1,...,m_{p+1} \in A$. If we denote by $H' \in D_{p+1}(A)$ the matrix formed from H by replacing the $(p+1)^{th}$ row by $(m_1,...,m_{p+1})$ and leaving the other rows unchanged, then

$$H'[f_1 \ \dots \ f_p \ f_{p+1} f'_{p+1}]^T = [g_1 \ \dots \ g^2_{p+1}]^T,$$

which implies that $(f_1,...,f_p, f_{p+1} f'_{p+1}) \in U_{p+1} = \mathcal{S}^{p+1}$, since U_{p+1} is a saturated triangular subset of A^{p+1}. It therefore follows that $f_{p+1} f'_{p+1} \in S(f_1,...,f_p)$, so that

$S(f_1,\dots,f_p)$ must be a m.c.s. of A as required.

It now remains to verify 3.1.1.(iv). Suppose that $f = (f_1,\dots,f_p)$ and $g = (g_1,\dots,g_p)$ are elements of U_p such that $gA \subsetneq fA$. If $(g_1,\dots,g_{p+1}) \in U_{p+1}$ we can use the argument of the proof of 2.1.5. to show that $(f_1,\dots,f_p,g_{p+1}) \in U_{p+1}$ also. It follows that $S(g) \subseteq S(f)$.

Now suppose that $f_{p+1} \in S(f)$, so that $(f_1,\dots,f_{p+1}) \in U_{p+1}$. Since, in addition, $(g_1,\dots,g_{p+1}) \in U_{p+1}$, there exist $(h_1,\dots,h_{p+1}) \in U_{p+1}$ and $S,K \in D_{p+1}(A)$ such that

$$S[f_1 \dots f_{p+1}]^T = [h_1 \dots h_{p+1}]^T = K[g_1 \dots g_{p+1}]^T.$$

It is easily seen that $(g_1,\dots,g_p,h_{p+1}) \in U_{p+1}$. Now, in an obvious notation,

$$h_{p+1} = \sum_{j=1}^{p+1} s_{p+1\ j} f_j \in fA + s_{p+1\ p+1} f_{p+1},$$

and it follows that $s_{p+1\ p+1} f_{p+1} \in S(g) + fA$, since $h_{p+1} \in S(g)$. Therefore $f_{p+1} \in (S(g) + fA)^{\sim}$, which implies that

$$S(f) \subseteq (S(g) + fA)^{\sim} ,$$

as required.

Thus, 3.1.1.(i)-(iv) are verified for $\mathcal{S} = \bigcup_{i}^{\infty} U_i$ and it follows that $\mathcal{S}$ is a denominator system over A. In order to complete the proof we must show that $\mathcal{S}$ is a saturated denominator system.

Let $p \in \mathbb{N}$ and let $f \in \mathcal{S}^p$. Suppose that $f' \in (S(f)+fA)^{\sim}$, so that

$$h'f' = f_{p+1} + \sum_{1}^{p} h_j f_j$$

for some $f_{p+1} \in S(f)$, $h', h_1, \dots, h_p \in A$. It follows that

$$f_{p+1} = -\sum_{1}^{p} h_j f_j + h'f' \quad ,$$

so that it is a simple matter to construct a matrix $H \in D_{p+1}(A)$ such that

$$H[f_1 \dots f_p \; f']^T = [f_1 \dots f_p \; f_{p+1}]^T.$$

Since $\mathcal{U}$ is a chain of saturated triangular subsets, it follows that $(f_1, \dots, f_p, f') \in \mathcal{J}$, so that $f' \in S(f)$. Therefore $S(f) = (S(f)+fA)^{\sim}$, so that $\mathcal{J}$ is a saturated denominator system.

It is clear from 3.1.7. and 3.1.8. that there exists a 1-1 correspondence between saturated denominator systems and chains of saturated triangular subsets. In the following section we will investigate the connections between the respective complexes constructed using these objects, namely denominator system complexes and complexes of modules of generalized fractions.

2. Denominator system complexes and complexes of modules of generalized fractions

Let A be a ring. In [21] Kersken constructs, from an A-module M and a denominator system $\mathcal{J}$, a complex $\tilde{C}^{\bullet}(\mathcal{J};M)$. We remark that in [21], $\tilde{C}^{\bullet}(\mathcal{J};M)$ is referred to as a *Cousin complex*, a term which is used in a different sense in this thesis. For this reason, we shall refer to $\tilde{C}^{\bullet}(\mathcal{J};M)$ as a *denominator system complex.* In this section, we give a brief description of the construction of denominator system complexes, and we show that these complexes are identical to the complexes of modules of generalized fractions which were discussed in 1.4.. The equivalence of these two structures has also been demonstrated in [16], by Hamieh and Zakeri, whose method of proof differs from that employed in this thesis.

Before describing the construction of the denominator system complex $\tilde{C}^{\bullet}(\mathcal{J};M)$, we require some preliminary results and definitions.

3.2.1.**Lemma** [21, (2.1)]. *Let $\mathcal{J}$ be a denominator system, let $p \in \mathbb{N}$ and let M be an A-module such that $\mathcal{J}$-ht($Ann_A M$) $\geq p$. Then, if $f, g \in \mathcal{J}^p$ are sequences in $Ann_A M$, $M_{S(f)} \cong M_{S(g)}$.*

Proof. By 3.1.3. there exists $h \in \mathcal{J}^p$ such that $hA \subseteq fA \cap gA$. It follows from 3.1.2. that $(M/fM)_{S(f)} \cong (M/fM)_{S(h)}$ and $(M/gM)_{S(g)} \cong (M/gM)_{S(h)}$. The result now follows since fM = 0 = gM.

The above result leads to the following definition. Let $\mathcal{J}$ and p be as in 3.2.1.. For a finitely generated A-module N, such that $\mathcal{J}$-htm N $\geq$ p, we define $C(\mathcal{J}^p;N)$ to be the module $N_{S(f)}$, for any sequence $f \in \mathcal{J}^p$ such that $fA \subseteq Ann_A N$. By 3.2.1. this is uniquely specified up to isomorphism.

Suppose now that M is an A-module such that $\mathcal{J}$-htm M $\geq$ p. Then it is left to the reader as a simple exercise to show that $\{ C(\mathcal{J}^p;N) \mid N \subseteq M, \text{ N finitely generated} \}$ is a direct system under the maps induced by inclusion and localization at an appropriate S(f), and we define $C(\mathcal{J}^p;M)$ to be the direct limit of this system. Furthermore, since M is the direct limit of its finitely generated submodules, the system of canonical maps $\{ \varepsilon_N : N \to C(\mathcal{J}^p;N) \}$ induces a map $\varepsilon_M : M \to C(\mathcal{J}^p;M)$. It is shown in

[21] that $\mathcal{S}$-htm (Coker ε_M) $\geq$ p+1, a fact which is important for the construction of the denominator system complex. We now give this construction in the following theorem, which we state without proof.

3.2.2. **Theorem** [21, (2.4)]. *Let M be an A-module and let $\mathcal{S}$ be a denominator system over A. There is, up to isomorphism, precisely one complex*

$$\tilde{C}^{\bullet}(\mathcal{S};M): \qquad \ldots \rightarrow \tilde{C}^{i}(\mathcal{S};M) \overset{\delta^i}{\rightarrow} \tilde{C}^{i+1}(\mathcal{S};M) \rightarrow \ldots$$

with the following properties:

(i) $\tilde{C}^{i}(\mathcal{S};M) = 0$, $\delta^i = 0$, *for* $i < -1$;

(ii) $\tilde{C}^{-1}(\mathcal{S};M) = M$;

(iii) $\tilde{C}^{p}(\mathcal{S};M) = C(\mathcal{S}^p;\ \mathrm{Coker}\,\delta^{p-2})$, *and* δ^{p-1} *is the composition of the canonical homomorphisms*

$$\tilde{C}^{p-1}(\mathcal{S};M) \rightarrow \mathrm{Coker}\,\delta^{p-2} \rightarrow \tilde{C}^{p}(\mathcal{S},M),\ p \geq 0.$$

The following result is important for our purposes, as it allows us to work with saturated denominator systems without any loss of generality.

3.2.3. **Proposition** [21]. *Let M and $\mathcal{S}$ be as in 3.2.2., and let $\mathcal{S}^{\sim}$ denote the saturation of $\mathcal{S}$. Then there is an isomorphism of complexes $\tilde{C}^{\bullet}(\mathcal{S};M) \rightarrow \tilde{C}^{\bullet}(\mathcal{S}^{\sim};M)$ which restricts to the identity map on $\tilde{C}^{-1}(\mathcal{S};M)$.*

Proof. For $f \in \mathcal{S}^p$, let $\tilde{S}(f) = \{f_{p+1} \mid (f_1,\ldots,f_{p+1}) \in \mathcal{S}^{\sim}\}$. It is clear that

$(S(f)+fA)^{\sim} \subseteq \tilde{S}(f)$. Now suppose that $f' \in \tilde{S}(f)$. Then $(f_1,\ldots,f_p,f') \in \mathcal{S}^{\sim}$, so that $\mathcal{S}$-ht $(f_1,\ldots,f_p,f')A \geq p+1$, by the definition of saturation. By 3.1.4.(iii), there exists $f_{p+1} \in (f_1,\ldots,f_p,f')A \cap S(f)$. It follows that $f' \in (S(f)+fA)^{\sim}$, so that $\tilde{S}(f) = (S(f)+fA)^{\sim}$.

Now let X be an A-module such that $\mathcal{S}^{\sim}$-htm $X \geq p$. It is easily seen that this condition is equivalent to $\mathcal{S}$-htm $X \geq p$. For a finitely generated submodule $N \subseteq X$ there therefore exists $h \in \mathcal{S}^p$ such that $hA \subseteq \mathrm{Ann}_A N$. Since $h \in \mathcal{S}^{\sim}$, it follows that

$$C(\mathcal{S}^{\sim p};N) = N_{\tilde{S}(h)} = N_{(S(h)+hA)^{\sim}} = N_{S(h)} = C(\mathcal{S}^{p};N) .$$

Therefore $C(\mathcal{S}^{\sim p};X) = C(\mathcal{S}^{p};X)$, and the result follows immediately from the construction of the denominator system complex.

As a consequence of 3.2.3., we can work exclusively with saturated denominator systems, in a fashion similar to that in which 1.3.2. allows us to consider only chains of saturated triangular subsets without any loss of generality. The reader will recall from the previous section that saturated denominator systems are identical to chains of saturated triangular subsets in an obvious way, and we shall now demonstrate that the respective complexes associated with these objects are also identical. Before proceeding with the main result of this section, 3.2.7., we require some preliminary results concerning complexes of modules of generalized fractions.

3.2.4. **Proposition.** *Let $\mathcal{U} = \{U_i \mid i \in \mathbb{N}\}$ be a chain of triangular subsets on A. Let M be an A-module and let $\mathcal{C}(\mathcal{U},M)$ be the complex*

$$0 \to M \overset{e^0}{\to} U_1^{-1}M \overset{e^1}{\to} \dots \to U_i^{-i}M \overset{e^i}{\to} U_{i+1}^{-i-1}M \to \dots ,$$

as was described in 1.4.. Then Coker $e^{n-1} \cong U_n[1]^{-n-1}M$ *under the natural map, for all* $n \in \mathbb{N}$.

Proof. The complex $\mathcal{C}(\mathcal{U},M)$ is the direct limit of the direct system of complexes $\{\mathcal{C}(\mathcal{U}(x),M)\}$, (with corresponding maps $e_x^{\cdot}$), as described prior to 1.4.1.. It is an easily verifiable consequence of 1.4.2. that Coker $e_x^{n-1} \cong U_n(x)[1]^{-n-1}M$, under the natural map. The result follows on passing to the direct limit and applying 1.2.6. to the direct system $\{ U(x)_n[1]^{-n-1}M \mid x \in U_n\}$.

3.2.5. **Lemma.** *Let* $\mathcal{U}$ *and* M *be as in 3.2.4. and let* $n \in \mathbb{N}$. *Suppose that* $m/(u_1,\dots,u_n,1) = 0$ *in* $U_{n+1}^{-n-1}M$. *Then there exist* $(v_1,\dots,v_{n+1}) \in U_{n+1}$ *and* $H \in D_n(A)$ *such that* $H\,[u_1 \dots u_n]^T = [v_1 \dots v_n]^T$ *and* $v_{n+1}|H|m \in \sum_1^n v_iM$.

Proof. Let $m/(u_1,\dots,u_n,1) = 0$ in $U_{n+1}^{-n-1}M$. Then there exist $(w_1,\dots,w_{n+1}) \in U_{n+1}$ and $K \in D_{n+1}(A)$ such that $K\,[u_1 \dots u_n\ 1\,]^T = [w_1 \dots w_{n+1}]^T$ and $|K|m \in \sum_1^n w_iM$. Let H' be the top left $n\times n$-submatrix of K. By Cramer's Rule,

$$|K|.1 \in |H'|w_{n+1} + \sum_1^n w_iA,$$

so that $|H'|w_{n+1}m \in \sum_1^n w_iM$, since $|K|m \in \sum_1^n w_iM$. Setting $H = H'$ and $v_i = w_i$, $1 \le i \le n+1$, the result follows.

3.2.6. **Proposition.** *Let* $\mathcal{U}$ *be a chain of saturated triangular subsets of A, let M be an A-module and let* $n \in \mathbb{N}$. *Then, for any submodule* $N \subseteq M$ *and* $u = (u_1,\dots,u_n) \in U_n$,

$$N/(u,1) \;=\; \{\, m/(u_1,\dots,u_n,1) \mid m \in N \,\} \text{ , and}$$

$$N/(u,S(u)) = \{\, m/(u_1,\dots,u_n,s) \mid m \in N \text{ , } s \in S(u) \,\}$$

are submodules of $U_n[1]^{-n-1}M$ *and* $U_{n+1}^{-n-1}M$ *respectively, where S(u) has the same meaning as its denominator system analogue. Furthermore,*

$$[N/(u,1)]_{S(u)} \cong N/(u,S(u))$$

under the natural map.

Proof. It is straightforward to show that N/(u,1) and N/(u,S(u)) are submodules of $U_n[1]^{-n-1}M$ and $U_{n+1}^{-n-1}M$ respectively and this is left to the reader. For simplicity of notation, we denote typical elements of N/(u,1) and N/(u,S(u)) by m/(u,1) and m/(u,s) respectively.

Now let $\phi : [\, N/(u,1)]_{S(u)} \to N/(u,S(u))$ be defined by

$$\phi\{(m/(u,1))/s\} = m/(u,s) \text{ .}$$

We must first show that ϕ is well-defined.

Suppose that $(m_1/(u,1))/s_1 = (m_2/(u,1))/s_2$ in $[N/(u,1)]_{S(u)}$. Hence there exists $t \in S(u)$ such that

$$t(s_2m_1/(u,1) - s_1m_2/(u,1)) = 0 \quad \text{in } U_n[1]^{-n-1}M,$$

$$\Rightarrow \quad t(s_2m_1 - s_1m_2)/(u,1) \quad = 0 \quad \text{in } U_n[1]^{-n-1}M,$$

$$\Rightarrow \quad t(s_2m_1 - s_1m_2)/(u,1) \quad = 0 \quad \text{in } U_{n+1}^{-n-1}M \text{ .}$$

By 1.2.3.(iii), it follows that

$$t(s_2m_1 - s_1m_2)/(u,ts_1s_2) = 0 \text{ in } U_{n+1}^{-n-1}M ,$$

which implies that $m_1/(u,s_1) = m_2/(u,s_2)$ by 1.2.3.(i). Therefore ϕ is well-defined.

It is clear that ϕ is a surjective A-homomorphism, so that the proof is complete on verifying that ϕ is injective. Suppose that $m/(u,s) = 0$ in $N/(u,S(u))$. Then $m/(u,1) = 0$ in $N/(u,S(u))$ by 1.2.3.(iii), and it follows from 3.2.5. that there exist $v \in U_{n+1}$ and $H \in D_n(A)$ such that $H[u_1 \ldots u_n]^T = [v_1 \ldots v_n]^T$ and $v_{n+1}|H|m \in \sum_1^n v_iM$. This implies that $v_{n+1}m/(u_1,\ldots,u_n,1) = 0$ in $U_n[1]^{-n-1}M$, by 1.2.3.(i). Moreover, $v_{n+1} \in S(u)$ so that $m/(u,1) = 0$ in $[N/(u,1)]_{S(u)}$. It now follows that ϕ is injective.

We are now in a position to present the main result of this section which has also been proved by Hamieh and Zakeri in **[16]** by a computational method.

3.2.7. **Theorem**. *Let $\mathcal{U} = \{ U_i \mid i \in \mathbb{N} \}$ be a chain of saturated triangular subsets on A, let $\mathcal{J} = \bigcup_1^\infty U_i$ be the corresponding saturated denominator system and let M be an A-module. Then there is a degree 1 isomorphism of complexes $\phi^\bullet : \tilde{C}^\bullet(\mathcal{J};M) \to \mathcal{C}(\mathcal{U},M)$ such that $\phi^{-1} : M \to M$ is the identity map.*

Proof. Let $p \in \mathbb{N}_0$ and suppose that there already exist isomorphisms ϕ^i, $i \le p-1$, such that ϕ^{-1} is the identity map and the following diagram is commutative.

$$\begin{array}{ccccccccccc} 0 & \to & M & \xrightarrow{e^0} & U_1^{-1}M & \xrightarrow{e^1} & \cdots & \xrightarrow{e^{p-1}} & U_p^{-p}M & & \\ \uparrow\phi^{-2} & & \uparrow\phi^{-1} & & \uparrow\phi^{0} & & & & \uparrow\phi^{p-1} & & \\ 0 & \to & M & \xrightarrow{\delta^{-1}} & \tilde{C}^0(\mathcal{Y};M) & \xrightarrow{\delta^{0}} \cdots \xrightarrow{\delta^{p-2}} & & & \tilde{C}^{p-1}(\mathcal{Y};M) & \xrightarrow{\delta^{p-1}} & \tilde{C}^{p}(\mathcal{Y};M)\ . \end{array}$$

By 3.2.2., $\tilde{C}^p(\mathcal{Y};M) = C(\mathcal{Y}^p;\mathrm{Coker}\ \delta^{p-2})$. Now ϕ^{p-1} induces an isomorphism ψ: Coker $\delta^{p-2} \to$ Coker e^{p-1}, and Coker $e^{p-1} \cong U_p[1]^{-p-1}M$ by 3.2.4.. It follows that ψ in turn induces an isomorphism $C(\mathcal{Y}^p;\mathrm{Coker}\ \delta^{p-2}) \to C(\mathcal{Y}^p;U_p[1]^{-p-1}M)$. Any finitely generated submodule of $U_p[1]^{-p-1}M$ can be written, in the notation of 3.2.6., as $N/(u,1)$, where N is a finitely generated submodule of M and $u \in U_p$. Now u is clearly a sequence in $\mathrm{Ann}_A(N/(u,1))$ by 1.2.3.(ii), so that $C(\mathcal{Y}^p;N/(u,1)) = [N/(u,1)]_{S(u)}$. It therefore follows that

$$C(\mathcal{Y}^p;U_p[1]^{-p-1}M) = \varinjlim\{[N/(u,1)]_{S(u)} \mid N/(u,1) \text{ a f.g. submodule of } U_p[1]^{-p-1}M\},$$

where the map $[N_1/(u_1,1)]_{S(u_1)} \to [N_2/(u_2,1)]_{S(u_2)}$, with $N_1/(u_1,1) \subseteq N_2/(u_2,1)$, is the homomorphism induced by localization at $S(u_2)$ composed with the isomorphism of 3.2.1.. By 3.2.6., we can identify each $[N/(u,1)]_{S(u)}$ with its isomorphic image $N/(u,S(u))$ in $U_{p+1}^{-p-1}M$. It is left as an exercise to verify that the map $[N_1/(u_1,1)]_{S(u_1)} \to [N_2/(u_2,1)]_{S(u_2)}$, where $N_1/(u_1,1) \subseteq N_2/(u_2,1)$, induces the inclusion map $N_1/(u_1,S(u_1)) \subseteq N_2/(u_2,S(u_2))$ under this identification. It follows that

$$C(\mathcal{Y}^p;U_p[1]^{-p-1}M) = \varinjlim\{N/(u,S(u)) \mid N/(u,1) \text{ a f.g. submodule of } U_p[1]^{-p-1}M\},$$

a direct system of submodules of $U_{p+1}^{-p-1}M$ under inclusion maps. It is now easily seen that

$$C(\mathcal{Y}^p; U_p[1]^{-p-1}M) \cong U_{p+1}^{-p-1}M \quad ,$$

so that ψ induces an isomorphism $\phi^p : C(\mathcal{Y}^p; \mathrm{Coker}\ \delta^{p-2}) \to U_{p+1}^{-p-1}M$. It is clear from the above construction that $e^p\phi^{p-1} = \phi^p\delta^{p-1}$, and the result follows by induction.

It follows from 3.2.7. that the concepts of a denominator system complex and a complex of modules of generalized fractions are equivalent. We conclude this section by employing some of the ideas encountered in the proof of 3.2.7. to prove a generalisation of 1.3.3.. This result was originally proved by O'Carroll in [**10**] by an alternative method.

Let U be a saturated triangular subset of A^{n+1}, let U_n be the restriction of U to A^n and let M be an A-module. We recall from 1.3. that for $x, y \in U_n$ such that $y^T = Hx^T$, $H \in D_n(A)$, there is a homomorphism $M/xM \to M/yM$ which is induced by multiplication by $|H|$. If we subsequently localize at $S(y)$ and compose the resultant map with the isomorphism of 3.1.2., we obtain a map

$$\phi_{xy}: (M/xM)_{S(x)} \to (M/yM)_{S(y)} \ .$$

Now under these these homomorphisms $\{ (M/xM)_{S(x)} \mid x \in U_n \}$ forms a direct system (see Appendix I), and we have the following generalisation of 1.3.3., originally due to O'Carroll.

3.2.8. **Theorem** . *Let U and M be as above. Then*

$$\varinjlim_{x \in U_n} \{ (M/xM)_{S(x)} \} \cong U^{-n-1} M .$$

Proof. Let $L = \varinjlim_{x \in U_n} \{(M/xM)_{S(x)}\}$ and let $\{ \mu_x:(M/xM)_{S(x)} \to L \}$ be the corresponding natural maps. Now it is known from 1.3.3. that $\varinjlim_{x \in U_n} \{ M/xM \} = U_n[1]^{-n-1}M$, with natural maps λ_x, say. Let $x \in U_n$. We first show that $\text{Ker } \mu_x = (\text{Ker } \lambda_x)_{S(x)}$.

Suppose that $(m+xM)/s \in \text{Ker } \mu_x$, so that there exist $y \in U_n$ and $H \in D_n(A)$ such that $y^T = Hx^T$ and $\phi_{xy}((m+xM)/s) = 0$. Since $(m+xM)/s = (m'+xM)/s'$, where $m' \in M$ and $s' \in S(y)$, by 3.1.2., it follows that there exists $v \in S(y)$ such that $v|H|m' \in yM$, so that $vm'+xM \in \text{Ker } \lambda_x$. Therefore

$$(m+xM)/s = (m'+xM)/s' = (vm'+xM)/s'v \in (\text{Ker } \lambda_x)_{S(x)}.$$

Similarly it can be shown that $(\text{Ker } \lambda_x)_{S(x)} \subseteq \text{Ker } \mu_x$.

It therefore follows that

$$\text{Im } \mu_x \cong [(M/xM)/\text{Ker } \lambda_x]_{S(x)} \cong [M/(x,1)]_{S(x)} ,$$

in the notation of 3.2.6.. Now $[(M/(x,1)]_{S(x)} \cong M/(x,S(x))$, by 3.2.6.. If $y \in U_n$ and $H \in D_n(A)$ are such that $Hx^T = y^T$ it can easily be shown that the inclusion map $\text{Im } \mu_x \subseteq \text{Im } \mu_y$ induces the inclusion map $M/(x,S(x)) \subseteq M/(y,S(y))$ under this identification. Therefore

$$L = \bigcup_{x \in U_n} \text{Im } \mu_x \cong \bigcup_{x \in U_n} (M/(x,S(x)) = U^{-n-1}M ,$$

as required.

3. Complexes of Cousin type and complexes of modules of generalized fractions

In this section, we make use of some of the ideas and results of the previous section to give a simplified account of the connection between complexes of Cousin type and complexes of modules of generalized fractions, originally investigated by Riley, Sharp and Zakeri in **[28]** and **[29]**. We begin by giving some definitions. Throughout this section, A is a Noetherian ring.

3.3.1. **Definition** [33, 1.1]. A sequence $\mathcal{F} = \{ F_i \mid i \in \mathbb{N}_0\}$ of subsets of Spec A is called a *filtration* of Spec A if, for each $i \in \mathbb{N}_0$, $F_i \supsetneq F_{i+1}$ and if each member of $F_i \setminus F_{i+1}$, which set is denoted ∂F_i, is a minimal member of F_i with respect to inclusion. Furthermore, we say that a filtration $\mathcal{F}$ *admits* M if $\mathrm{Supp}_A M \subseteq F_0$.

3.3.2. **Definition** [28, Chapter III, 2.1.]. Let $\mathcal{F} = \{ F_i \mid i \in \mathbb{N}_0\}$ be a filtration of Spec A which admits M. A complex $X^\bullet = \{ X^i \mid i \geq -2 \}$ of A-modules and A-homomorphisms is said to be of *Cousin type with respect to* $\mathcal{F}$ if it has the form

$$0 \xrightarrow{d^{-2}} M \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} \cdots \to X^i \xrightarrow{d^i} X^{i+1} \to \cdots$$

and satisfies the following conditions, for each $n \in \mathbb{N}_0$:

(i) $\mathrm{Supp}_A X^n \subseteq F_n$;

(ii) $\mathrm{Supp}_A(\mathrm{Coker}\ d^{n-2}) \subseteq F_n$;

(iii) $\mathrm{Supp}_A(\mathrm{Ker}\ d^{n-1} / \mathrm{Im}\ d^{n-2}) \subseteq F_{n+1}$; and

(iv) the natural A-homomorphism $\xi^n : X^n \to \bigoplus_{\mathfrak{p} \in \partial F_n} (X^n)_{\mathfrak{p}}$, such that for $x \in X^n$ and $\mathfrak{p} \in \partial F_n$, the component of $\xi^n(x)$ in the summand $(X^n)_{\mathfrak{p}}$ is $x/1$ (it follows from (i) and [**32**, 2.2 and 2.3] that such an A-homomorphism exists), is an isomorphism.

Suppose now that $\mathcal{U} = \{ U_i \mid i \in \mathbb{N} \}$ is a chain of triangular subsets on A. Given an A-module M, we can form the complex

$$\mathcal{C}(\mathcal{U},M) : 0 \to M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} \dots \to U_i^{-i}M \xrightarrow{e^i} U_{i+1}^{-i-1}M \to \dots .$$

Put $G_0 = \mathrm{Supp}_A\ M$ and, for $i \in \mathbb{N}$, define

$$G_i = \{ \mathfrak{p} \in \mathrm{Supp}_A\ M \mid \text{there exists } (u_1,\dots,u_i) \in U_i \text{ with } \sum_{1}^{i} Au_j \subseteq \mathfrak{p} \}.$$

The family $\mathcal{G} = \{ G_i \mid i \in \mathbb{N}_0\}$ of sets of primes of A is called *the sequence (of sets of primes) induced by* $\mathcal{U}$ *and M.* It is a straightforward matter to show that the chain of saturated triangular subsets consisting of the saturations of the U_i induces the same sequence as $\mathcal{U}$, and so we may assume without loss of generality that $\mathcal{U}$ is a chain of saturated triangular subsets.

Let us suppose that $\mathcal{U}$ is a chain of saturated triangular subsets on A, M is an A-module and that the sequence $\mathcal{G}$ induced by $\mathcal{U}$ and M is a filtration of Spec A. It is shown in [**28**, Chapter III, §1] that, when this is the case, the complex $\mathcal{C}(\mathcal{U},M)$ is of Cousin type with respect to $\mathcal{G}$. In this section, we give a proof of this result which makes use of the material of the previous section to avoid some of the technical manipulations of generalized fractions present in [28]. Before proceeding with the main result, 3.3.4., we make some observations and give a preliminary result.

Consider $f = (f_1,\dots,f_n) \in U_n$ and, as in the previous section, let $S(f) = \{ f_{n+1} \in A \mid (f_1,\dots,f_{n+1}) \in U_{n+1} \}$. Then $S(f)$ is a saturated m.c.s. of A and its complement in A is therefore the union of the prime ideals which do not intersect it. The following proposition provides a useful relationship between this set of primes and the primes of $\mathcal{G}$. For simplicity of notation, Spec $A_{S(f)}$ is identified with its homeomorphic image in Spec A, whenever the context demands it.

3.3.3. **Proposition.** *Let $\mathcal{U}$, $\mathcal{G}$ and f be as above. Then*

(i) $\mathrm{Supp}_A(M/fM) \subseteq G_n$;

(ii) Spec $A_{S(f)} \cap \mathrm{Supp}_A(M/fM) \subseteq \partial G_n$;

(iii) Spec $A_{S(f)} \cap \mathrm{Supp}_A(M/fM)$ consists of minimal members of $\mathrm{Supp}_A(M/fM)$.

Proof. (i) If $\mathfrak{p} \in \mathrm{Supp}_A(M/fM)$, then $fA \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \mathrm{Supp}_A M$. Therefore $\mathfrak{p} \in G_n$.

(ii) Let $\mathfrak{p} \in G_{n+1}$. Then there exists $(g_1,\dots,g_{n+1}) \in U_{n+1}$ such that $\sum_1^{n+1} g_i A \subseteq \mathfrak{p}$. For any $f_{n+1} \in S(f)$, there exist $H, K \in D_{n+1}(A)$ and $(w_1,\dots,w_{n+1}) \in U_{n+1}$ such that

$$H[f_1 \dots f_{n+1}]^T = [w_1 \dots w_{n+1}]^T = K[g_1 \dots g_{n+1}]^T ,$$

so in particular $w_{n+1} \in \mathfrak{p}$. Denoting $(w_1,...,w_n)$ by w, we have $wA \subseteq fA$ so that, by 3.1.1.(iv), $S(w) \subseteq S(f)$. Thus $w_{n+1} \in S(f)$, so that $\mathfrak{p} \cap S(f) \neq \emptyset$. Hence $\mathfrak{p} \notin \text{Spec } A_{S(f)}$, and the result follows from (i).

(iii) This is immediate from (i) and (ii) since $\mathcal{G}$ is a filtration of Spec A.

We can now present the main result of this section which was originally proved by Riley in **[28]**.

3.3.4. **Theorem** [**28**, Chapter III, 1.8 and 1.11]. *Let $\mathcal{U} = \{ U_i \mid i \in \mathbb{N} \}$ be a chain of saturated triangular subsets on A, let M be an A-module and suppose that the sequence of sets of primes $\mathcal{G} = \{ G_i \mid i \in \mathbb{N}_0 \}$, induced by $\mathcal{U}$ and M, is a filtration of Spec A which admits M. Then the complex $\mathcal{C}(\mathcal{U},M)$ is of Cousin type for M with respect to $\mathcal{G}$ (where, in the notations given above, X^i and d^i correspond to $U_{i+1}^{-i-1}M$ and e^{i+1} respectively).*

Proof. We must verify properties (i)–(iv) of 3.3.2. (for all $n \in \mathbb{N}_0$), taking into account the various small changes in notation given above.

Let $n \in \mathbb{N}_0$, and consider $\mathfrak{p} \in \text{Supp}_A M$ where $\mathfrak{p} \notin G_n$. Let $m/(u_1,...,u_{n+1}) \in U_{n+1}^{-n-1}M$. Since $\mathfrak{p} \notin G_n$, $\sum_{1}^{n} u_iA \nsubseteq \mathfrak{p}$; hence $u_i \notin \mathfrak{p}$ for some $i \in \{1,...n\}$. Now $u_i m/(u_1,...,u_{n+1}) = 0$, by 1.2.3.(ii), and it follows that $(U_{n+1}^{-n-1}M)_{\mathfrak{p}} = 0$. Hence $\text{Supp}_A(U_{n+1}^{-n-1}M) \subseteq G_n$, so property (i) of 3.3.2. is verified.

By 3.2.4., $\text{Coker } d^{n-2} \cong U_n[1]^{-n-1}M$. Property (ii) of 3.3.2. now follows by the preceding argument.

As for property (iii) we first remark that it is a straightforward consequence of 3.2.4. that

$$\operatorname{Ker} e^n / \operatorname{Im} e^{n-1} \cong \operatorname{Ker} \phi ,$$

where $\phi : U_n[1]^{-n-1}M \to U_{n+1}^{-n-1}M$ is the natural map. Consider $\mathfrak{p} \in \operatorname{Supp}_A M$, where $\mathfrak{p} \notin G_{n+1}$, and consider an arbitrary element $m/(u_1,\ldots,u_n,1)$ of $\operatorname{Ker} \phi$. Then $m/(u_1,\ldots,u_n,1) = 0$ in $U_{n+1}^{-n-1}M$. Hence, by 3.2.5., there exist $H \in D_n(A)$ and $(v_1,\ldots,v_{n+1}) \in U_{n+1}$ such that

$$H[u_1 \ldots u_n]^T = [v_1 \ldots v_n]^T,\ v_{n+1}|H|m \in \sum_1^n v_i M .$$

Thus in $U_n[1]^{-n-1}M$, $|H|m/(v_1,\ldots,v_n,1)$ is annihilated by v_{n+1}. In addition it is annihilated by v_i, $1 \le i \le n$. Since $\mathfrak{p} \notin G_{n+1}$, there exists j such that $1 \le j \le n+1$, with $v_j \notin \mathfrak{p}$. Hence $|H|m/(v_1,\ldots,v_n,1) = m/(u_1,\ldots,u_n,1)$ has the zero element as image in $(\operatorname{Ker} \phi)_{\mathfrak{p}}$, so $(\operatorname{Ker} \phi)_{\mathfrak{p}} = 0$. Thus property (iii) of 3.2.2. holds for the complex $\mathcal{C}(\mathcal{U},M)$.

In order to prove that the final property holds, we recall that, by 3.2.8., $U_{n+1}^{-n-1}M$ is the direct limit of the family $\{ (M/uM)_{S(u)} \mid u \in U_n \}$, under homomorphisms obtained from localization of determinantal maps. Fix $f = (f_1,\ldots,f_n) \in U_n$, and consider the module $(M/fM)_{S(f)}$; if $\mathfrak{p}$ is in its support then $fA \subseteq \mathfrak{p}$, so $\mathfrak{p} \in G_n$. In this situation [32, 2.2 and 2.3] guarantee the existence of the natural homomorphism

$$\psi_f : (M/fM)_{S(f)} \to \bigoplus_{\mathfrak{p} \in \partial G_n} [(M/fM)_{S(f)}]_{\mathfrak{p}} .$$

Since $fA \subseteq \mathfrak{p}$ and $\mathfrak{p} \in \partial G_n$ imply that $S(f) \cap \mathfrak{p} = \emptyset$, it is clear that there is a natural isomorphism

$$\bigoplus_{\mathfrak{p} \in \partial G_n} [(M/fM)_{S(f)}]_{\mathfrak{p}} \cong \bigoplus_{\mathfrak{p} \in \partial G_n \cap \operatorname{Supp}_A M/fM} (M/fM)_{\mathfrak{p}} . \qquad (*)$$

Now consider $\mathfrak{q} \in \operatorname{Spec} A_{S(f)}$. If $\mathfrak{q} \notin \operatorname{Supp}_A(M/fM)$, then the localization of ψ_f at $\mathfrak{q}$ is trivially an isomorphism. If $\mathfrak{q} \in \operatorname{Supp}_A(M/fM)$, then by 3.3.3.(ii), $\mathfrak{q} \in \partial G_n \cap \operatorname{Supp}_A(M/fM)$. Now for $\mathfrak{p} \in \partial G_n \cap \operatorname{Supp}_A(M/fM)$

$$[(M/fM)_{\mathfrak{p}}]_{\mathfrak{q}} = \begin{cases} (M/fM)_{\mathfrak{q}} & \text{, if } \mathfrak{p} = \mathfrak{q} \\ 0 & \text{, if } \mathfrak{p} \neq \mathfrak{q} \text{ , by 3.3.3.(iii).} \end{cases}$$

Therefore it follows immediately from (*) that the localization of ψ_f at such a $\mathfrak{q}$ is an isomorphism. Thus, ψ_f is an $A_{S(f)}$-isomorphism (and therefore an A-isomorphism), and the result follows on passing to the direct limit (see Appendix I).

Now in [29], Riley, Sharp and Zakeri examine the situation where M is an A-module with the property that $\operatorname{Ass}_A M$ has only finitely many members and $\mathcal{F} = \{ F_i \mid i \in \mathbb{N}_0\}$ is a filtration of Spec A which admits M. In this case it was shown [29, 2.3] that the family $\mathcal{U} = \{ U_n \mid n \in \mathbb{N} \}$, where

$$U_n = \{ (u_1,\ldots,u_n) \in A^n \mid \text{for each } i = 1,\ldots,n, \quad \sum_{1}^{i} Au_j \nsubseteq \mathfrak{p} \text{ for all } \mathfrak{p} \in \partial F_{i-1} \cap \operatorname{Supp}_A M \},$$

is in fact a chain of triangular subsets on A.

We now show that the main result of [29] can be derived in a direct way from 3.3.4..

3.3.5. **Theorem** [29, 2.5]. *Let M be an A-module such that $\operatorname{Ass}_A M$ has only finitely many members, let $\mathcal{F}$ be a filtration of Spec A which admits M, and let $\mathcal{U} = \{ U_n \mid n \in \mathbb{N} \}$ be the chain of triangular subsets defined above.*

Then the complex $\mathcal{C}(\mathcal{U},M)$ is of Cousin type for M with respect to $\mathcal{F}$.

Proof. Consider the filtration $\mathcal{G} = \{G_i \mid i \in \mathbb{N}_0\}$ where $G_i = F_i \cap \mathrm{Supp}_A M$. Clearly it suffices to prove the result for $\mathcal{G}$ in place of $\mathcal{F}$. Now let $\mathcal{H} = \{ H_i \mid i \in \mathbb{N}_0\}$ be the sequence of sets of primes induced by $\mathcal{U}$ and M. In view of Theorem 3.3.4., it is enough to show that $\mathcal{H} = \mathcal{G}$.

Now $H_0 = G_0 = \mathrm{Supp}_A M$. Assume that $n \in \mathbb{N}$ and that $H_i = G_i$ for $i = 0,\dots,n-1$. Let $\mathfrak{p} \in H_n$. Then there exists $(u_1,\dots,u_n) \in U_n$ such that $\sum_1^n Au_i \subseteq \mathfrak{p}$, so $\mathfrak{p} \notin \partial G_i$ for $i = 0,\dots,n-1$. Hence $\mathfrak{p} \in G_n$, so $H_n \subseteq G_n$.

Conversely let $\mathfrak{p} \in G_n$. Then $\mathfrak{p} \in G_{n-1}$, since $G_n \subseteq G_{n-1}$, so $\mathfrak{p} \in H_{n-1}$ by the induction hypothesis. Therefore there exists $(u_1,\dots,u_{n-1}) \in U_{n-1}$ such that $\sum_1^{n-1} Au_i \subseteq \mathfrak{p}$. Now by [29, 2.1] there are only finitely many members of ∂G_{n-1} which contain $\sum_1^{n-1} Au_i$; denote these by $\mathfrak{q}_1,\dots, \mathfrak{q}_r$, say. If $\mathfrak{p} \subseteq \bigcup \mathfrak{q}_j$ then $\mathfrak{p} \subseteq \mathfrak{q}_m$ for some m, which gives a contradiction since $\mathfrak{p} \in G_n$. Therefore there exists $u_n \in \mathfrak{p}$ such that $u_n \notin \mathfrak{q}_j$ for $j = 1,\dots,r$. Hence $\sum_1^n Au_i \not\subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \partial G_{n-1}$. This implies that $(u_1,\dots,u_n) \in U_n$, so $\mathfrak{p} \in H_n$. Thus $H_n = G_n$, so $\mathcal{H} = \mathcal{G}$ by induction, and the result follows.

4. Direct limits and flat dimension of generalized fractions

In this section we return to the situation where A is an arbitrary ring. For an A-module M, we define the *flat dimension* of M, denoted flatdim.M, to be the largest integer k such that there exists an A-module X with $\mathrm{Tor}_k(M,X) \neq 0$, if such a k exists, and ∞ otherwise. Now if S is an m.c.s. of A, then it is well-known that

$$S^{-1}Tor_m(M,X) = Tor_m(S^{-1}M,X)$$

for any A-module X and $m \in \mathbb{N}_0$, and from this it easily follows that

$$\text{flatdim}.S^{-1}M \leq \text{flatdim}.M$$

for any m.c.s. $S \subset A$. In view of the fact that the formation of modules of generalized fractions with respect to triangular subsets of A^1 is equivalent to ordinary localization, it is of obvious interest to investigate whether any analogous relationship exists between flatdim.M and flatdim.$U^{-n}M$, where $n \geq 2$. This has already been done by Riley in [28], and de Chela in [7] for the case where M is an A-module and U is a triangular subset of A^n such that $u_1,\ldots,u_{n-1}$ is a poor M-sequence for all $(u_1,\ldots,u_n) \in U$, and it is this situation with which we are concerned in this section.

Now Theorem 3.2.8. provides us with a description of an arbitrary module of generalized fractions $U^{-n}M$ as the direct limit of a system of localized quotients of M. By adopting this approach to the calculation of flat dimension, we prove a generalization of a result of Riley, from which one of the main results of [7] follows in a straightforward manner. Indeed the main result of this section 3.4.5., demonstrates that a very concrete relationship exists between the flat dimension of $U^{-n}M$ and the flat dimension of the modules in the direct system for the situation described above.

We shall make considerable use of the following result which involves direct systems and the Tor functor.

3.4.1. **Proposition.** [28, Chapter IV, 5.4; see also Appendix I]. *Let* $\{ M_x , \phi_{xy} \}_{x,y \in \Lambda}$ *be a direct system of A-modules and A-homomorphisms.*

Let $k \in \mathbb{N}_0$ and let X be an A-module. Then

$$\{ Tor_k(M_x, X), \phi'_{xy} \}_{x,y \in \Lambda}$$

forms a direct system where $\phi'_{xy} : Tor_k(M_x, X) \to Tor_k(M_y, X)$, $x, y \in \Lambda$, is the map induced from ϕ_{xy} by $Tor_k(-, X)$. Furthermore, under these homomorphisms,

$$Tor_k(\varinjlim_{x \in \Lambda} M_x, X) \cong \varinjlim_{x \in \Lambda} Tor_k(M_x, X).$$

Until further notice, let M be an A-module such that flatdim.M = k, $k \in \mathbb{N}_0$, and U is a triangular subset of A^{n+1}, $n \in \mathbb{N}$, with the property that $u_1, \dots, u_n$ is a poor M-sequence for all $(u_1, \dots, u_{n+1}) \in U$. As before, U_n will denote the restriction of U to A^n. We have the following corollary to 3.4.1..

3.4.2. **Corollary**. *Let $m \in \mathbb{N}_0$ and let X be an A-module. Then*

$$Tor_m(U^{-n-1} M, X) = \varinjlim_{x \in U_n} Tor_m([M/xM]_{S(x)}, X)$$

under the maps induced by $Tor_m(-, X)$ from those in the direct system of 3.2.8..

Proof. This immediate from 3.2.8. and 3.4.1..

Our first aim is to give a simplified proof of a result of Riley, for which we require the following lemma from folklore.

3.4.3. **Lemma**. *Let $x_1, \dots, x_n$ be an M-sequence. Then flatdim.$M/xM \leq n+k$ (and hence flatdim.$(M/xM)_S \leq n+k$ for any m.c.s. $S \subset A$).*

Proof. By induction it suffices to prove the result for an M-sequence of length one. The proof in this case is entirely standard.

3.4.4. **Proposition** [Riley, **28**, Chapter IV, 5.5]. *Let U, n, M, and k be as described above. Then flatdim.* $U^{-n-1}M \leq k+n$.

Proof. This is immediate from 3.4.2. and 3.4.3..

We now present the main result of this section which examines in more detail the relationship between flatdim.$U^{-n-1}M$ and the flat dimensions of the individual modules in the direct system $\{ (M/uM)_{S(u)} \mid u \in U_n \}$.

3.4.5. **Theorem**. *Let U, M, n and k be as before. Then*

$$\text{flatdim.}U^{-n-1}M = \sup_{u \in U_n} \{ \text{flatdim.}(M/uM)_{S(u)} \}.$$

Proof. We deal first with the case when U is saturated. By 3.4.3., $\sup_{u \in U_n} \{ \text{flatdim.}(M/uM)_{S(u)} \} = s$ for some $s \in \mathbb{N}$, so that there exists $x = (x_1,\dots,x_n) \in U_n$ such that flatdim.$(M/xM)_{S(x)} = s$. It is clear from 3.4.2. that flatdim.$U^{-n-1}M \leq s$. Now suppose that $H = [h_{ij}] \in D_n(A)$ and $y \in U_n$ are such that $y^T = Hx^T$ and let $\phi : (M/xM)_{S(x)} \to (M/yM)_{S(y)}$ be the corresponding homomorphism in the direct system of 3.2.8.. Since flatdim.$(M/xM)_{S(x)} = s$, there exists an A-module X such that $\text{Tor}_s([M/xM]_{S(x)},X) \neq 0$. Furthermore we have a map

$$\phi' : \text{Tor}_s([M/xM]_{S(x)},X) \to \text{Tor}_s([M/yM]_{S(y)},X)$$

induced by ϕ. By 3.4.2. and the properties of direct systems, it suffices to show that ϕ' is injective in order to show that $\text{Tor}_s(U^{-n-1}M,X) \neq 0$.

We will denote by $H_i \in D_n(A)$, $1 \leq i \leq n$, the matrix whose i^{th} row is the i^{th} row of H and whose other rows are those of the nxn identity matrix. In addition, $(x_1,\dots,x_{i-1},y_i,\dots,y_n) \in A^n$ will be denoted by $(x,y)_i$, $1 \leq i \leq n$. Since U_n is saturated, it follows that $(x,y)_i \in U_n$, $1 \leq i \leq n$, and we also note that $H_i[x\ y]^T_{i+1} = [x\ y]^T_i$ and that $|H_i| = h_{ii}$.

The map ϕ can be expressed as the composition $\phi_1 \circ \phi_2 \circ \dots \circ \phi_n$ where

$$\phi_i : [M/(x,y)_{i+1}M]_{S(x,y)_{i+1}} \to [M/(x,y)_iM]_{S(x,y)_i}$$

is the map from the direct system of 3.2.8.. Therefore ϕ' can be expressed as the composition $\phi_1' \circ \phi_2' \circ \dots \circ \phi_n'$ where

$$\phi_i' : Tor_s([M/(x,y)_{i+1}M]_{S(x,y)_{i+1}}, X) \to Tor_s([M/(x,y)_iM]_{S(x,y)_i}, X)$$

is the map induced by ϕ_i. To show that ϕ' is injective it suffices to demonstrate that ϕ_i' is injective, $1 \leq i \leq n$.

Since $x_1,\dots,x_i,y_{i+1},\dots,y_n$ and $x_1,\dots,x_{i-1},y_i,\dots,y_n$ are both poor M-sequences by [25, 3.2], it follows from 2.2.1. that the determinantal map

$$\psi : M/(x,y)_{i+1}M \to M/(x,y)_iM$$

is injective. Now Coker $\psi = M/(x,y)_iM + h_{ii}M$, and since $y_iM \subseteq \sum_1^{i-1} x_jM + h_{ii}M$, Coker $\psi = M/(x_1,\dots,x_{i-1},h_{ii},y_{i+1},\dots,y_n)M$, here denoted $M/(x,h_{ii},y)M$. It is a simple exercise to construct a matrix $K \in D_n(A)$ such that $K[x\ h_{ii}\ y]^T = y^T$, so that $(x,h_{ii},y) \in U_n$ since U_n is saturated. We therefore have the exact sequence

$$0 \to M/(x,y)_{i+1}M \to M/(x,y)_iM \to M/(x,h_{ii},y)M \to 0 .$$

Now localizing at S(y) and employing 3.1.2. gives the exact sequence

$$0 \to [M/(x,y)_{i+1}M]_{S(x,y)_{i+1}} \to [M/(x,y)_iM]_{S(x,y)_i}$$

$$\to [M/(x,h_{ii},y)M]_{S(x,h_{ii},y)} \to 0$$

In the long exact sequence induced by Tor(−,X) we have

$$\ldots \to Tor_{s+1}([M/(x,h_{ii},y)M]_{S(x,h_{ii},y)},X) \to Tor_s([M/(x,y)_{i+1}M]_{S(x,y)_{i+1}},X)$$

$$\xrightarrow{\phi_i'} Tor_s([M/(x,y)_iM]_{S(x,y)_i},X) \to \ldots .$$

Since $(x,h_{ii},y) \in U_n$, $Tor_{s+1}([M/(x,h_{ii},y)M]_{S(x,h_{ii},y)},X) = 0$ by the definition of s. Therefore the map ϕ_i' is injective, $1 \leq i \leq n$, and thus ϕ' is injective. Hence

$$Tor_s(U^{-n-1}M,X) = \varinjlim_{x \in U_n} Tor_s([M/uM]_{S(u)},X) \neq 0 ,$$

since the map from $Tor_s([M/xM]_{S(x)},X)$ to the direct limit must also be injective, and so flatdim.$U^{-n-1}M = s$.

The general case can be reduced to the saturated case by use of 1.3.2 and a repetition of the above argument.

We can now deduce in a straightforward manner one of the main results of [7].

3.4.6. **Proposition** [Flores de Chela, 7, 3.9]. *Let U be a triangular subset of A^{n+1} such that $u_1,\ldots,u_n$ is a poor A-sequence for all $(u_1,\ldots,u_{n+1}) \in U$. If $U^{-n-1}A \neq 0$ then flatdim.$U^{-n-1}A = n$.*

Proof. Suppose that $U^{-n-1}A \neq 0$, so that by 3.2.8. there exists $x \in U_n$

such that $(A/xA)_{S(x)} \neq 0$. We show that $\text{flatdim.}(A/xA)_{S(x)} = n$, and the result follows from 3.4.5.. Now 3.4.3., with $k = 0$, shows that $\text{flatdim.}(A/xA)_{S(x)} \leq n$. To show that equality holds we consider the Koszul complex $K^{\bullet}(A;x_1,\dots,x_n)$ which provides a free resolution of A/xA:

$$K^{\bullet}(A;x_1,\dots,x_n) : 0 \to A \xrightarrow{\partial_n} \bigoplus_{1}^{\binom{n}{n-1}} A \xrightarrow{\partial_{n-1}} \dots \to \bigoplus_{1}^{\binom{n}{1}} A \xrightarrow{\partial_1} A \to 0 .$$

Tensoring with $(A/xA)_{S(x)}$ gives in particular

$$0 \to (A/xA)_{S(x)} \xrightarrow{\partial_n \otimes 1} \bigoplus_{1}^{\binom{n}{n-1}} (A/xA)_{S(x)} \to .$$

Now $\text{Tor}_n(A/xA,(A/xA)_{S(x)}) = \text{Ker}\,\partial_n \otimes 1 = (A/xA)_{S(x)} \neq 0$. It therefore follows that $\text{flatdim.}(A/xA)_{S(x)} = n$, and hence $\text{flatdim.}U^{-n-1}A = n$, by 3.4.5..

Our final result of this section demonstrates another situation in which 3.4.5. can be used to give a precise value for the flat dimension of a module of generalized fractions.

3.4.7. **Proposition.** *Let A be a Noetherian ring, let M be a finitely generated A-module of flat dimension $k < \infty$ and let U be a triangular subset of A^n consisting of poor M-sequences. If $U[1]^{-n-1}M \neq 0$, then $\text{flatdim.}U[1]^{-n-1}M = n+k$.*

Proof. Consider $x \in U$ such that $M/xM \neq 0$. Since M/xM is finitely generated, $\text{flatdim.}M/xM = \text{proj.dim.}(M/xM)$ by [3, p.122, Ex. 3(b)]. By [22, p.129, Lemma 5], $\text{proj.dim.}M/xM$ is equal to the supremum of $\text{proj.dim.}(M/xM)_{\mathfrak{m}}$ (as an $A_{\mathfrak{m}}$ -module) for the maximal ideals $\mathfrak{m}$ of A. Since $M/xM \neq 0$, there exists $\mathfrak{m} \in \text{Maxspec}\,A$ such that $(M/xM)_{\mathfrak{m}} \neq 0$. In this case $x_1,\dots,x_n$ considered as elements of $A_{\mathfrak{m}}$, form an $M_{\mathfrak{m}}$ -sequence. Therefore applying [22, p.131, Lemma 6] and induction we have that

$$\text{proj.dim.}_{A_{\mathfrak{m}}}(M/xM)_{\mathfrak{m}} = n+k .$$

Hence flatdim.M/xM = n+k for all $x \in U$ such that $M/xM \neq 0$, and the result follows from 3.4.5..

CHAPTER IV

1. Vanishing of modules of generalized fractions

In this section we include some miscellaneous results concerning modules of generalized fractions and, in some particular cases, we investigate conditions necessary and/or sufficient for a module of generalized fractions to vanish. This problem has previously been investigated by Hamieh and Sharp in [**15**], the main result of which we now recall.

4.1.1. **Theorem** [**15**, 3.2]. *Let M be an A-module such that $\dim_A M = n$. Then, if $k \geq n+2$, $U^{-k}M = 0$ for any triangular subset $U \subset A^k$.*

The following result, due to Zakeri, is of importance in the proof of 4.1.1., which we do not include in this thesis, and also in the proofs of results which appear later in this section.

4.1.2. **Proposition** [**39**, Chapter III, 4.5]. *Let $n \in \mathbb{N}$, let U be a triangular subset of A^n, and let S be a multiplicatively closed subset of A. Let $\phi : A \to A_S$ denote the natural ring homomorphism, and set*

$$U_S = \{ (\phi(u_1),\dots,\phi(u_n)) \mid (u_1,\dots,u_n) \in U \}.$$

Then U_S is a triangular subset of $(A_S)^n$ and there is an isomorphism of A_S-modules $\psi : (U^{-n}M)_S \to U_S^{-n}M_S$ which is such that, for $m \in M$, $(u_1,\dots,u_n) \in U$ and $s \in S$,

$$\psi \left([m/(u_1,\dots,u_n)]/s \right) = [m/s] / (\phi(u_1),\dots,\phi(u_n)).$$

We now focus our attention on triangular subsets of the form $U(x)_n$, previously discussed in 1.2., where $x = (x_1,\dots,x_n) \in A^n$, and M is an A-module, and we examine conditions sufficient for the vanishing of $U(x)_n^{-n}M$. The following result will prove useful when dealing with modules of generalized fractions of this form.

4.1.3. **Proposition** *Let $x = \{ x_i \mid i \in \mathbb{N} \}$ be a sequence of elements of A, let M be an A-module and let $k \in \mathbb{N}$. Denote by $y = \{ y_i \mid i \in \mathbb{N} \}$ the sequence of elements of A such that $y_i = x_{i+k}$, $i \in \mathbb{N}$. Then for all $n \geq k$,*

$$U(x)_n^{-n}M = U(y)_{n-k}^{-n+k}(U(x)_k[1]^{-k-1}M),$$

under the canonical map.

Proof. Let X denote the A-module $U(x)_k[1]^{-k-1}M$, and consider the complex $\mathcal{C}(\mathcal{U}(x),M)$:

$$0 \to M \xrightarrow{d_x^0} U(x)_1^{-1}M \xrightarrow{d_x^1} \dots \to U(x)_{k-1}^{-k+1}M \xrightarrow{d_x^{k-1}} U(x)_k^{-k}M \xrightarrow{d_x^k} U(x)_{k+1}^{-k-1}M \xrightarrow{d_x^{k+1}} \dots .$$

It follows from 3.2.4. that X is naturally isomorphic to Coker d_x^{k-1}, so that localization at $x_{k+1} = y_1$ induces an isomorphism between $U(y)_1^{-1}X$ and $U(x)_{k+1}^{-k-1}M$ by 1.4.2.. Furthermore, since d_x^k can be expressed as the composition

$$U(x)_k^{-k}M \to \text{Coker } d_x^{k-1} \to (\text{Coker } d_x^{k-1})_{x_{k+1}},$$

again by 1.4.2., we have that Coker $d_y^0 \cong$ Coker d_x^k where $d_y^0: X \to X_{y_1}$ is the natural map. On localizing this isomorphism at $y_2 = x_{k+2}$ and applying 1.4.2., it follows $U(y)_2^{-2}X$ and $U(x)_{k+2}^{-k-2}M$ are naturally isomorphic. We therefore have the following commutative diagram

$$\begin{array}{ccc} U(x)_{k+1}^{-k-1}M & \xrightarrow{d_x^{k+1}} & U(x)_{k+2}^{-k-2}M \\ \wr\| & & \wr\| \\ U(y)_1^{-1}X & \xrightarrow{d_y'} & U(y)_2^{-2}X \end{array},$$

and the result follows from 1.4.2., following the repeated formation of cokernels and localizations.

The next result now follows in a straightforward manner from 4.1.2. and 4.1.3..

4.1.4. **Theorem.** *Let $x = (x_1,\ldots,x_n) \in A^n$ and let M be an A-module. Then*

$$U(x)_n^{-n} M \cong \left(\bigotimes_{1\,A}^{n-1} U(x_i, 1)_2^{-2} A \right) \otimes_A A_{x_n} \otimes_A M .$$

Proof. By 1.4.3. the result is true when $n=1$, with the obvious interpretation in this case. Assume now that $n>1$ and that the result holds for all triangular subsets of the form $U(y)_{n-1}$ where $y = (y_1,\ldots,y_{n-1}) \in A^{n-1}$. By 4.1.3.,

$$\begin{aligned} U(x)_n^{-n}M &= U(x_2,\ldots,x_n)_{n-1}^{-n+1}(U(x_1,1)_2^{-2}M) \\ &= U(x_1,1)_2^{-2}A \otimes_A U(x_2,\ldots,x_n)_{n-1}^{-n+1}A \otimes_A M, \text{ by 1.4.3.,} \\ &= \left(\bigotimes_{1\,A}^{n-1} U(x_i,1)_2^{-2}A \right) \otimes_A A_{x_n} \otimes_A M \end{aligned}$$

by the induction hypothesis.

An immediate consequence of 4.1.4. is the following corollary which has also been verified independently by R.Y.Sharp.

4.1.5. **Corollary.** *Let σ be a permutation of $\{1,\ldots, n-1\}$ and let $x = (x_1,\ldots,x_n) \in A^n$ and $y = (x_{\sigma(1)},\ldots,x_{\sigma(n-1)},x_n) \in A^n$. Then*

$$U(x)_n^{-n} M = U(y)_n^{-n} M .$$

Proof. The result follows from 4.1.4. and the commutativity of the tensor product.

We remark at this point that 4.1.5. can be obtained in a straightforward way from 1.2.4.(i), from which result it follows that $U(x)_n^{-n}M$ is the direct limit of the direct system

$$\{ (M/ \sum_{i}^{n-1} x_i^{\alpha_i} M)_{x_n} \mid \alpha_1,\ldots,\alpha_{n-1} \in \mathbb{N} \}$$

under the determinantal maps induced by matrices of the form $\operatorname{diag}(x_1^{\beta_1},\ldots,x_{n-1}^{\beta_{n-1}})$, $\beta_1,\ldots,\beta_{n-1} \in \mathbb{N}_0$.

We now investigate conditions on the module M and the elements $x_1,\ldots,x_n$ which are sufficient for the vanishing of $U(x)_n^{-n}M$.

4.1.6. **Proposition.** *Let $x = (x_1,\ldots,x_n) \in A^n$ and let M be an A-module. Suppose that $\dim_{A_{x_n}} M_{x_n} < n-1$. Then $U(x)_n^{-n} M = 0$.*

Proof. By 3.2.4. and 1.4.2.,

$$U(x)_n^{-n}M = (U(x)_{n-1}[1]^{-n}M)_{x_n}$$

$$= (U(x)_{n-1}[1])_{x_n}^{-n}M_{x_n} \quad \text{by 4.1.2.,}$$

$$= 0 \quad \text{by 4.1.1..}$$

In the following corollary we make use of 4.1.5. to give a further condition sufficient for the vanishing of $U(x)_n^{-n}M$, which involves the elements $x_1,\dots,x_{n-1}$.

4.1.7. **Corollary.** *Let x and M be as in 4.1.6.. Suppose that for some $k < n$, $\dim_{A_{x_k}} M_{x_k} < n-2$. Then $U(x)_n^{-n}M = 0$.*

Proof. Let y be the element of A^n obtained from x by interchanging x_k and x_{n-1}.

Since $\dim_{A_{x_k}} M_{x_k} < n-2$, it follows from 4.1.6. that $U(y)_{n-1}^{-n+1}M = 0$. This implies that $U(y)_n^{-n}M = 0$, and the result follows from 4.1.5..

Our next result provides a description of a module of generalized fractions of the form $U(x)_n^{-n}M$ which will prove most useful in the following section, where connections between modules of generalized fractions and local cohomology modules are investigated.

4.1.8. **Theorem.** *Let M be an A-module and let $x=(x_1,\dots,x_n)\in A^n$. Then*

$$U(x)_n^{-n}M \cong M_{x_1\dots x_n} \Big/ \sum_{j=1}^{n-1} \overline{M}_{x_1\dots\hat{x}_j\dots x_n},$$

where $\overline{M}_{x_1\dots\hat{x}_j\dots x_n}$ denotes the natural image of $M_{x_1\dots\hat{x}_j\dots x_n}$ in $M_{x_1\dots x_n}$.

Proof. Let $\phi: M_{x_1 \ldots x_n} \to U(x)_n^{-n}M$ be defined by

$$\phi(m/x_1^{\alpha_1} \ldots x_n^{\alpha_n}) = m/(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) .$$

It is a simple matter to show that ϕ is a well-defined, surjective A-homomorphism. Let us now fix j, where $1 \leq j \leq n-1$, and let $m/x_1^{\alpha_1} .. x_j^0 .. x_n^{\alpha_n}$ denote a typical element of $\overline{M}_{x_1 .. \hat{x}_j .. x_n}$. Then

$$\phi(m/x_1^{\alpha_1} .. x_j^0 .. x_n^{\alpha_n}) = m/(x_1^{\alpha_1}, \ldots, 1, \ldots, x_n^{\alpha_n}) = 0,$$

by 1.2.3.(ii). It is therefore clear that $\sum_{j=1}^{n-1} \overline{M}_{x_1 .. \hat{x}_j .. x_n} \subseteq \operatorname{Ker} \phi$.

Conversely suppose that $m/(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) = 0$ in $U(x)_n^{-n}M$, so that, by 1.2.4.(i), there exists $\gamma \geq \alpha_1, \ldots, \alpha_n$ such that

$$x_1^{\gamma-\alpha_1} \ldots x_n^{\gamma-\alpha_n} m \in \sum_1^{n-1} x_i^{\gamma} M. \qquad (*)$$

Now it follows from (*) that there exist $m_1, \ldots, m_{n-1} \in M$ such that

$$\begin{aligned} m/x_1^{\alpha_1} .. x_n^{\alpha_n} &= x_1^{\gamma-\alpha_1} .. x_n^{\gamma-\alpha_n} m/(x_1 \ldots x_n)^{\gamma} \\ &= \sum_{j=1}^{n-1} x_j^{\gamma} m_j/(x_1 \ldots x_n)^{\gamma} \\ &= \sum_{j=1}^{n-1} m_j/(x_1 .. \hat{x}_j .. x_n)^{\gamma} \\ &\in \sum_{j=1}^{n-1} \overline{M}_{x_1 .. \hat{x}_j .. x_n}. \end{aligned}$$

Therefore $\operatorname{Ker}\phi = \sum_{j=1}^{n-1} \overline{M}_{x_1 .. \hat{x}_j .. x_n}$, and the result follows.

In the final part of this section we make use of 4.1.8. to determine necessary and sufficient conditions for the vanishing of a module of generalized fractions in some particular cases. The following result has also been verified by H. Zakeri, by a method which avoids the technical calculations of this account.

4.1.9. **Proposition.** *Let $x = (x_1, x_2) \in A^2$ and suppose that x_1 is not nilpotent in A. Then*

$$U(x)_2^{-2}A = 0 \iff x_2 \text{ is nilpotent in } B/x_1 B,$$

where B is the ring $A/\bigcup_1^{\infty}(0:x_1^{\,k})$ and x_1, x_2 are identified with their canonical images in B.

Proof. We begin by noting that if x_1 is nilpotent in A, then $A_{x_1} = 0$ and hence $U(x)_2^{-2}A = 0$ trivially.

Suppose that $U(x)_2^{-2}A = 0$. By 4.1.8., this implies that $A_{x_1x_2}/\bar{A}_{x_2} = 0$, which in turn implies that $1/x_1x_2 \in \bar{A}_{x_2}$. Therefore there exist $a \in A$ and $\alpha \in \mathbb{N}$ such that

$$1/x_1x_2 = a/x_2^{\alpha} \text{ in } A_{x_1x_2}.$$

We therefore have a positive integer n such that

$$(x_1x_2)^n(ax_1x_2 - x_2^{\alpha}) = 0 \text{ in } A,$$

which implies that

$$x_2^n\,(ax_1x_2 - x_2^{\alpha}) = 0 \text{ in } B.$$

It now follows that $x_2^{n+\alpha} = ax_2^{n+1}x_1$ in B, so that x_2 is nilpotent in B/x_1B. A similar argument proves the reverse implication and the details of this are left to the reader.

By making use of 4.1.8. in the manner of the previous result and insisting that the ring A be local we obtain the final result of this section.

4.1.10. **Proposition.** *Let A be a local ring and let $x = (x_1, x_2) \in A^2$, where x_2 is not a unit in A. Suppose that $U(x)_2^{-2}A \neq 0$ and $U(x)_2[1]^{-3}A = 0$. Then x_2 is a zero-divisor in B/x_1B, where $B = A/\bigcup_1^{\infty}(0:x_1^k)$.*

Proof. By 4.1.8., $U(x)_2[1]^{-3}A$ can be expressed, in the usual notation, as the quotient module $A_{x_1x_2}/\bar{A}_{x_1}+\bar{A}_{x_2}$, and it follows that $U(x)_2[1]^{-3}A = 0$ only if

$$1/x_1x_2 = a/x_1^{\alpha} + b/x_2^{\beta} \quad \text{in } A_{x_1x_2},$$

for some $a,b \in A$ and $\alpha, \beta \in \mathbb{N}_0$. Furthermore, since $U(x)_2^{-2}A \neq 0$, it follows from the proof of 4.1.9. that $\alpha > 0$. Let us further suppose that α is the smallest integer for which $1/x_1x_2$ can be expressed in the above manner, and assume without loss of generality that $\beta > 1$. We first consider the case where $\alpha > 1$. In $A_{x_1x_2}$,

$$1/x_1x_2 = a/x_1^{\alpha} + b/x_2^{\beta} = (ax_2^{\beta}+bx_1^{\alpha})/x_1^{\alpha}x_2^{\beta} .$$

There therefore exists a positive integer n such that

$$(x_1x_2)^n(x_1^{\alpha}x_2^{\beta} - x_1x_2(ax_2^{\beta} + bx_1^{\alpha})) = 0 .$$

This implies that

$$(x_1x_2)^{n+1}(x_1^{\alpha-1}x_2^{\beta-1} - ax_2^{\beta} - bx_1^{\alpha}) = 0$$

$$\Rightarrow \quad x_2^{n+1}(x_1^{\alpha-1}x_2^{\beta-1} - ax_2^{\beta} - bx_1^{\alpha}) = 0 \text{ in } B,$$

$$\Rightarrow \quad -ax_2^{\beta+n+1} = 0 \text{ in } B/x_1B .$$

Since $U(x)_2^{-2}A \neq 0$, x_2 is not nilpotent in B/x_1B by 4.1.9.. Suppose now that $a = 0$ in B/x_1B. Then $a = cx_1 + d$, where $c \in A$ and $d \in \bigcup_{l}^{\infty}(0:x_1^{k})$. It now follows that

$$a/x_1^{\alpha} = (cx_1+d)/x_1^{\alpha} = c/x_1^{\alpha-1},$$

since $d/x_1^{\alpha} = 0$. This clearly contradicts the minimality of α and so we deduce that $a \neq 0$ in B/x_1B so that x_2 is indeed a zero-divisor in B/x_1B.

Finally, we consider the case where $\alpha=1$. By the argument used above, there exists a positive integer n such that

$$(x_1x_2)^{n+1}(x_2^{\beta-1} - ax_2^{\beta} - bx_1) = 0$$

in A, from which it follows in a straightforward manner that

$$x_2^{n+\beta}(1 - ax_2) = 0$$

in B/x_1B. This implies that x_2 is either a zero-divisor or a unit in B/x_1B, since it cannot be nilpotent. Now, as previously stated, x_2 is not a unit in A so that its image is likewise not a unit in B/x_1B, since A is local and B/x_1B is non-trivial. Therefore we can exclude the latter possibility and the result follows.

2. Modules of generalized fractions and local cohomology

In this section we shall concern ourselves with the connection between top local cohomology modules and modules of generalized fractions. If A is a Noetherian local ring, such that dim A = n, $x_1,...,x_n$ is a system of parameters (henceforth denoted 's.o.p.') for A and M is an A-module, then it has been shown by Sharp and Zakeri in [**36**, 3.5] that the top local cohomology module $H_{\mathfrak{m}}^n(M)$ is isomorphic to the module of generalized fractions $U(x)_n[1]^{-n-1}M$, where $x = (x_1,...,x_n) \in A^n$. The main result of this section, 4.2.2., is a generalization of this known theory which dispenses with the requirements that A is local and $x_1,...,x_n$ form an s.o.p. for A. We firstly give an account of the calculation of local cohomology modules using the Čech complex. The main elements of the following description are to be found in [**30**, Chapter 3, pp.75-79].

Let A be a Noetherian ring and let $x_1,...,x_n$ be elements of A. We consider the following complex, known as the *Čech complex*:

$$C^{\bullet}: 0 \rightarrow C^0 \overset{d^0}{\rightarrow} \dots \overset{d^{n-1}}{\rightarrow} C^n \overset{d^n}{\rightarrow} 0$$

where

$$C^k = \bigoplus_{1 \le i_1 < \dots < i_k \le n} A_{x_{i_1} \dots x_{i_k}} ,$$

and where

$$d^k\colon C^k \to C^{k+1}$$

is defined on the component

$$A_{x_{i_1}\dots x_{i_k}} \to A_{x_{j_1}\dots x_{j_{k+1}}}$$

to be

$$(-1)^s \text{ . the natural map: } A_{x_{i_1}\dots x_{i_k}} \to A_{x_{j_1}\dots x_{j_{k+1}}},$$

if $\{i_1,\dots,i_k\} = \{j_1,\dots,\hat{j}_s,\dots,j_{k+1}\}$, and 0 otherwise. It is a simple exercise to verify that $C^\bullet$ is indeed a complex.

4.2.1. **Proposition** cf.[30, Chapter 3, 2.3]. *Let M be an A-module and let* $\mathfrak{a} = (x_1,\dots,x_n)A$. *Then*

$$H^i_{\mathfrak{a}}(M) \cong H^i(M \otimes C^\bullet)$$

for all $i \in \mathbb{N}_0$.

Proof. By the corollary to Theorem 10 of Chapter 6 of [23], it suffices to show that

(i) the isomorphism holds when i=0;

(ii) a short exact sequence of A-modules gives rise to a long exact sequence of modules $H^i(- \otimes C^\bullet)$;

(iii) $H^i(Q \otimes C^\bullet) = 0$ for $i > 0$ if Q is injective.

Now

$$\begin{aligned} H^0(M \otimes C^\bullet) &= \operatorname{Ker}(M \otimes A \to M \otimes (\bigoplus_1^n A_{x_i})) \\ &= \operatorname{Ker}(M \to \bigoplus_1^n M_{x_i}) \\ &= \{ m \in M \mid \text{there exists } k \in \mathbb{N} \text{ such that} \\ &\qquad x_i^k m = 0,\ 1 \le i \le n \}. \end{aligned}$$

It is now straightforward to show that $H^0(M \otimes C^\bullet) = H^0_{\mathfrak{a}}(M)$, and (i) follows.

In order to verify (ii), we first of all note that C^k is a flat A-module for all k, being a finite direct sum of localizations of A. Therefore, if $0 \to L' \to L \to L'' \to 0$ is a short exact sequence of A-modules, we have a short exact sequence of complexes

$$0 \to L' \otimes C^\bullet \to L \otimes C^\bullet \to L'' \otimes C^\bullet \to 0 .$$

This gives rise to the required long exact sequence.

To demonstrate that property (iii) holds, we make use of the structure theorem for injective modules (see [**38**, (2.32)]) and assume that $Q = E(A/\mathfrak{p})$ for some prime $\mathfrak{p}$, where E(−) denotes the injective hull. If $\mathfrak{a} \subseteq \mathfrak{p}$, then for each j, $1 \le j \le n$, $E(A/\mathfrak{p})$ is annihilated by some power of x_j by [**30**, Chapter 1, 3.4], so that $E(A/\mathfrak{p})_{x_j} = 0$. This implies that $E(A/\mathfrak{p}) \otimes C^i = 0$, and therefore $H^i(E(A/\mathfrak{p}) \otimes C^\bullet) = 0$, for all $i > 0$.

Alternatively suppose that $\mathfrak{a} \not\subseteq \mathfrak{p}$, so that there exists x_j such that

$x_j \notin \mathfrak{p}$. Appealing once more to the proof of [30, Chapter 1, 3.4], it follows that multiplication by x_j is a bijection on $E(A/\mathfrak{p})$. For $k \geq 1$ we can therefore define a homotopy $s_k : E(A/\mathfrak{p}) \otimes C^k \to E(A/\mathfrak{p}) \otimes C^{k-1}$, by the following relation:

$$e \in E(A/\mathfrak{p})_{x_{i_1}\ldots x_{i_k}} \to \begin{cases} (-1)^r e \in E(A/\mathfrak{p})_{x_{i_1}\ldots\hat{x}_{i_r}\ldots x_{i_k}} & \text{if } j=i_r \\ 0 \text{ if } j \notin \{ i_1,\ldots,i_k \}. \end{cases}$$

It is now a straightforward matter to verify that $s_k d^{k-1} + d^k s_{k+1}$ is the identity mapping, so that $H^i(E(A/\mathfrak{p}) \otimes C^\bullet) = 0$, for all $i \geq 1$, and the proof is complete.

We now give the main result of this section.

4.2.2. **Theorem** *Let $x_1,\ldots,x_n$, $\mathfrak{a}$, M and $C^\bullet$ be as in 4.2.1. and let $x = (x_1,\ldots,x_n) \in A^n$. Then*

$$H^n_{\mathfrak{a}}(M) \cong U(x)_n[1]^{-n-1} M.$$

Proof. By 4.2.1., $H^n_{\mathfrak{a}}(A) = H^n(M \otimes C^\bullet) = \text{Coker } d^{n-1}$. It is easily seen that

$$\text{Im } d^{n-1} = \sum_{j=1}^{n} \overline{M}_{x_1\ldots x_j\ldots x_n},$$

in the notation of 4.1.8., so that

$$H_{\mathfrak{a}}^n(M) = M_{x_1\ldots x_n} / \sum_1^n \overline{M}_{x_1\ldots\hat{x}_j\ldots x_n}.$$

The result now follows from 4.1.8..

3. Generalized Cohen–Macaulay rings and lengths of generalized fractions

Until further notice, we shall assume that A is a Noetherian local ring of dimension d with maximal ideal $\mathfrak{m}$. For such a ring A, we shall denote by $U(A) \subset A^{d+1}$ the set

$$\{(x_1,\ldots,x_d,1) \mid x_1,\ldots,x_d \text{ form an s.o.p. for } A\}.$$

We recall from **[36]** that U(A) is a triangular subset of A^{d+1} and that the top local cohomology module $H^d_{\mathfrak{m}}(A)$ is isomorphic to $U(A)^{-d-1}A$. In this section we shall be mainly concerned with the situation where A is a generalized Cohen–Macaulay ring, henceforth denoted g.c.m. ring, which we now define. For a more extensive study of g.c.m. rings see [31, **12**].

4.3.1. **Definition** [31, 3.3] *A ring A is a g.c.m. ring if it satisfies the following equivalent conditions:*

(i) for each $i=0,\ldots,d-1$, the local cohomology module $H^i_{\mathfrak{m}}(A)$ has finite length;

(ii) there exists a positive integer n such that, for each s.o.p. $x_1,\ldots,x_d$,

$$\left(\sum_{1}^{i-1} x_jA\right) : x_i \subseteq \left(\sum_{1}^{i-1} x_jA\right) : \mathfrak{m}^n ,$$

for each i, $1 \le i \le d$, with the obvious interpretation when $i = 1$.

Suppose now that A is a g.c.m. ring. In [34], Sharp and Hamieh obtain a formula for the length of a cyclic submodule of $U(A)^{-d-1}A$ of the form $A/(x_1^{\alpha_1},\ldots,x_d^{\alpha_d},1)$ (in the notation of 3.2.6.), which is valid whenever $\alpha_1,\ldots,\alpha_d$ exceed a certain constant which they calculate. Furthermore, in [31], Schenzel, Ngô Viet Trung and Nguyen Tu Cuong derive a formula for the length of the quotient ring $A/\sum_1^d x_j^{n_j}A$, which holds for all sufficiently large values of the positive integers n_j. Our aim in this section is to investigate the connections between these two areas of research, and by using the results of [34] in conjunction with known results concerning generalized fractions, to shed light on the range of values of the n_j for which the latter formula is valid.

Before proceeding with the results, we require some preliminary definitions and description of notation, and we give some important properties of g.c.m. rings.

For a Noetherian local ring A of dimension d, and an $\mathfrak{m}$-primary ideal $\mathfrak{q}$, the *multiplicity* of $\mathfrak{q}$, denoted $e(\mathfrak{q})$, is an integer given by the formula

$$e(\mathfrak{q}) = \lim_{n\to\infty} d!\, l(A/\mathfrak{q}^n)/n^d ,$$

where $l(-)$ denotes length. Whenever $x_1,\ldots,x_d$ form an s.o.p. for A, then $e(x_1,\ldots,x_d)$ denotes the multiplicity of the ideal $(x_1,\ldots,x_d)A$. For a comprehensive account of the theory of multiplicities, see [24].

Let L be an Artinian A-module. It is known that there exists a unique smallest submodule $L_0 \subseteq L$ such that L/L_0 has finite length. Now $\mathfrak{m}^i L \subseteq L_0$ for some $i \in \mathbb{N}$, and we define the *stability index* of L, denoted s(L), to be the least integer with this property. We remark that if L is itself of finite length, then s = s(L) is the least integer for which $\mathfrak{m}^s L = 0$. The *residual length* of the A-module L, denoted $l'(L)$, is given by the integer $l(L/L_0)$.

We now return to the situation where A is a g.c.m. ring and U(A) is the triangular subset described before 4.3.1.. Pertaining to this situation we have the following two results, due to Hamieh and Sharp.

4.3.2. **Proposition [34**, 3.6.] *Let A be a g.c.m. ring and let $x_1,\dots,x_d$ form an s.o.p. for A. Set*

$$t = \sum_{i=1}^{d-1} \binom{d-1}{i-1} s(H^i_{\mathfrak{m}}(A)) .$$

Let $r \in \mathbb{N}$. Then

$$(0 :_{U(A)^{-d-1}A} \mathfrak{m}^r) \subseteq A/(x_1^{r+t},\dots,x_d^{r+t},1) .$$

4.3.3. **Theorem** [**34**, 3.7.] *Let A, $x_1,\dots,x_d$ and t be as described in 4.3.2.. Then for all positive integers $n_1,\dots,n_d \geq t$, we have*

$$l(A/(x_1^{n_1},\dots,x_d^{n_d},1)) = e(x_1,\dots,x_d)n_1\dots n_d - \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H^i_{\mathfrak{m}}(A)) .$$

The reader will notice a similarity between 4.3.3. and the following result from [**31**], and it is the connection between these two results which concerns us in this section.

4.3.4. **Theorem [31**, 3.3 & 3.7] *Let A be a g.c.m. ring and let $x_1,\dots,x_d$ form an s.o.p. for A. Then there exists a positive integer n such that*

$$l(A/\sum_{1}^{d} x_i^{n_i} A) = e(x_1,\dots,x_d)n_1\dots n_d + \sum_{i=0}^{d-1} \binom{d-1}{i} l(H_{\mathfrak{m}}^{i}(A))$$

whenever $n_1,\dots,n_d \geq n$. *Furthermore, the value of* n *is independent of the choice of the s.o.p.* $x_1,\dots,x_d$.

At this point, we recall from 1.3.3. that the module of generalized fractions $U(A)^{-d-1}A$ can be exhibited as the direct limit of the direct system $\{ A/\sum_1^d u_iA \mid u_1,\dots,u_d$ form an s.o.p. for $A \}$, under determinantal maps. If $x_1,\dots,x_d$ form an s.o.p. for A, $n_1,\dots,n_d$ are positive integers and $\phi : A/\sum_1^d x_i^{n_i} A \to U(A)^{-d-1}A$ is the natural map, then it is a simple matter to see that Im ϕ is the the cyclic submodule $A/(x_1^{n_1},\dots x_d^{n_d}, 1) \subseteq U(A)^{-d-1}A$. From this it follows that

$$l(A/\sum_1^d x_i^{n_i}A) = l(A/(x_1^{n_1},\dots,x_d^{n_d},1)) + l(\mathrm{Ker}\ \phi) .$$

In view of this fact, we shall concern ourselves with the following two questions. Can l(Ker ϕ) be computed in the absence of 4.3.4., so enabling us to deduce 4.3.4. from 4.3.3.? Does this approach shed any light on the range of values of the integers n_i for which the formula of 4.3.4. is valid? As we shall demonstrate, both these questions can be answered in the affirmative. However, before this can be achieved we require further auxiliary results.

In some of the later proofs we will wish to reduce to the situation where the ring A has non-zero depth. To this end, the following proposition proves very useful.

4.3.5. **Proposition** [34, 2.1]. *Let A be a Noetherian local ring of dimension d with maximal ideal* $\mathfrak{m}$, *let* $B = A/H_{\mathfrak{m}}^0(A)$ *and let* $^{-}:A \to B$ *denote the natural map. Then* *(i) B again has dimension d and, as A-modules,*

$H^i_{\mathfrak{m}}(A) \cong H^i_{\mathfrak{m}}(B)$ *for all* $i \in \mathbb{N}$ *;*

(ii) $x_1,\dots,x_d$ *form an s.o.p. for A if and only if* $\bar{x}_1,\dots,\bar{x}_d$ *form an s.o.p. for B. Furthermore* $e_A(x_1,\dots,x_d) = e_B(\bar{x}_1,\dots,\bar{x}_d)$.

(iii) The relation $\psi : U(A)^{-d-1}(A) \to U(B)^{-d-1}(B)$ *defined by*

$$\psi\ (\ a/(u_1,\dots,u_d,1)) = \bar{a}/(\bar{u}_1,\dots,\bar{u}_d,\bar{1})$$

is an isomorphism.

In view of the notation which appears later in this section we shall henceforth identify $x \in A$ with its image $\bar{x} \in B = A/H^0_{\mathfrak{m}}(A)$. An immediate corollary to 4.3.5. is the following result.

4.3.6. **Corollary** [34]. *Let A be a g.c.m. ring of dimension d and let* $B = A/H^0_{\mathfrak{m}}(A)$*. Then*

(i) B is a g.c.m. ring of dimension d;

(ii) if $x_1,\dots,x_d$ *form an s.o.p. for A, then* x_i *is a n.z.d. in B,* $1 \le i \le d$.

Proof. Both assertions follow in a straightforward manner from 4.3.1. and 4.3.5..

In later proofs, we shall wish to use the technique of induction on the dimension of the ring in question. We shall therefore require the following result which provides very useful means of passing to a suitable ring of lower dimension.

4.3.7. **Proposition** [12, 34]. *Let A be a g.c.m. ring of dimension d and let a be a subset of a system of parameters (s.s.o.p.) for A. Then*

*(i) [**12**, (2.6)(2)] there is an exact sequence*

$$0 \to 0{:}a \to H^0_{\mathfrak{m}}(A) \xrightarrow{a} H^0_{\mathfrak{m}}(A) \to H^0_{\mathfrak{m}}(A/aA)$$

$$\to H^1_{\mathfrak{m}}(A) \xrightarrow{a} H^1_{\mathfrak{m}}(A) \to H^1_{\mathfrak{m}}(A/aA)$$

$$\to H^i_{\mathfrak{m}}(A) \xrightarrow{a} H^i_{\mathfrak{m}}(A) \to H^i_{\mathfrak{m}}(A/aA) \ \ldots \ .$$

*(ii) [**12**, (2.6)(3)] A/aA is a g.c.m. ring of dimension d−1;*

(iii) $s(H^i_{\mathfrak{m}}(A/aA) \leq s(H^i_{\mathfrak{m}}(A)) + s(H^{i+1}_{\mathfrak{m}}(A)),\ 0 \leq i \leq d-2.$

Proof. Assertions (i) and (ii) are proved in [**12**]. Assertion (iii) follows from [**34**, 3.4] and (i) above.

As previously noted, 4.3.5. enables us to reduce to the situation of a ring of non-zero depth. Before we can make effective use of this, we must first deduce some facts concerning the ideal structure of the ring produced in this manner. This is achieved in 4.3.9.. The following lemma, due to O'Carroll, is needed for the proof of 4.3.9..

4.3.8. **Lemma**. *Let A be a g.c.m. ring and let $x \in A$ be an s.s.o.p. for A. Then, for all $n \in \mathbb{N}$,*

$$x^n A \cap H^0_{\mathfrak{m}}(A) = x^n H^0_{\mathfrak{m}}(A) .$$

Proof. It is clear that $x^nH^0_{\mathfrak{m}}(A) \subseteq x^nA \cap H^0_{\mathfrak{m}}(A)$.

Since A is Noetherian, $H^0_{\mathfrak{m}}(A) = (0:\mathfrak{m}^z)$ for some positive integer z. Suppose now that $x^na \in H^0_{\mathfrak{m}}(A)$, $a \in A$. Then $x^na\,\mathfrak{m}^z = 0$, so that $a\mathfrak{m}^z \subseteq 0:x^n$. It now follows from 4.3.1.(ii) with i = 1, that there exists $r \in \mathbb{N}$ such that $0:x^n \subseteq 0:\mathfrak{m}^r$. Therefore $a\mathfrak{m}^z \subseteq 0:\mathfrak{m}^r$ which implies that $a\mathfrak{m}^{z+r} = 0$, from which we deduce that $a \in H^0_{\mathfrak{m}}(A)$. The result now follows.

The next result is of importance in the proof of the main theorem of this section, 4.3.11., and is also of some independent interest.

4.3.9. **Proposition.** *Let A be a g.c.m. ring of dimension d and let $x_1,\dots,x_d$ form an s.o.p. for A. Set $t' = \sum_{i=0}^{d-1}\binom{d}{i}\, s(H^i_{\mathfrak{m}}(A))$ and suppose that $n_1,\dots,n_d \geq t'$. Then*

$$\left(\sum_{1}^{d} x_i^{n_i}A\right) \cap H^0_{\mathfrak{m}}(A) = 0 .$$

Proof. We first of all note that the result holds when the ring A is of dimension 0, trivially, and of dimension 1, by 4.3.8.. Suppose now that d > 1 and that the result has been proved for all g.c.m. rings of dimension less than d. Since $t' \geq s(H^0_{\mathfrak{m}}(A))$, it follows from 4.3.8. that $x_1^{n_1}A \cap H^0_{\mathfrak{m}}(A) = 0$. Therefore $H^0_{\mathfrak{m}}(A)$ is naturally contained in $\bar{A} = A/x_1^{n_1}A$ under the canonical projection map. Furthermore, under this map, the image of $H^0_{\mathfrak{m}}(A)$ lies in $H^0_{\mathfrak{m}}(\bar{A})$. (Due to the frequent changes of ring employed in the work of this section, the symbol $\mathfrak{m}$ will be used to denote the maximal ideal of all rings considered. We find this notation both efficient and unambiguous as all rings share a common residue field.)

Now $A/x_1^{n_1}A$ is a g.c.m. ring of dimension d−1, by 4.3.7.(ii), and it follows from 4.3.7.(iii) in a straightforward manner that

$$\sum_{i=0}^{d-2}\binom{d-1}{i} s(H^i_{\mathfrak{m}}(\bar{A})) \leq \sum_{i=0}^{d-1}\binom{d}{i} s(H^i_{\mathfrak{m}}(A)) = t'.$$

Since $\bar{x}_2,\ldots\bar{x}_d$ form an s.o.p. for $\bar{A}$ it follows from the induction hypothesis that

$$(\sum_2^d \bar{x}_i^{n_i}\bar{A}) \cap H^0_{\mathfrak{m}}(\bar{A}) = 0. \qquad (*)$$

Let us now suppose that $y \in (\sum_1^d x_i^{n_i}A) \cap H^0_{\mathfrak{m}}(A)$. If $y = \sum_1^d a_i x_i^{n_i}$, then by the above argument, the image of y under the canonical projection map, $\bar{y} = \sum_2^d \bar{a}_i\bar{x}_i^n$, is contained in $H^0_{\mathfrak{m}}(\bar{A})$. It now follows from (*) that $\bar{y} = 0$ in $\bar{A}$, which implies that $y \in x_1^{n_1}A$. Therefore $y = 0$, by 4.3.8., since $n_1 \geq t' \geq s(H^0_{\mathfrak{m}}(A))$, and the proof is complete.

4.3.10. **Proposition** [37, 2.4 & 2.7]. *Let A be a Noetherian local ring of dimension d and let $x_1 \in A$ form an s.s.o.p. for A. Suppose that x_1 is a n.z.d. in A, and let $\bar{A} = A/x_1A$. Then there exists an A-module homomorphism $\eta: U(\bar{A})^{-d}\bar{A} \to U(A)^{-d-1}A$ defined by*

$$\eta\,(\bar{a}/(\bar{y}_2,\ldots,\bar{y}_d)) = a/(x_1,y_2,\ldots,y_d).$$

Furthermore, there is a commutative diagram

$$\begin{array}{ccc} U(\bar{A})^{-d}\bar{A} & \overset{\eta}{\to} & U(A)^{-d-1}A \\ \|\wr & & \|\wr \\ H^{d-1}_{\mathfrak{m}}(\bar{A}) & \overset{f}{\to} & H^{d}_{\mathfrak{m}}(A), \end{array}$$

where f is the connecting homomorphism in the long exact sequence of local cohomology modules induced from the short exact sequence

$$0 \to A \overset{x_1}{\to} A \to A/x_1A \to 0.$$

Consequently $\operatorname{Ker}\eta \cong H^{d-1}_{\mathfrak{m}}(A) / x_1 H^{d-1}_{\mathfrak{m}}(A)$.

We are now in a position to give the main result of this section.

4.3.11. **Theorem.** *Let A be a g.c.m. ring of dimension d and let* $x_1,\dots,x_d$ *form an s.o.p. for A. Set* $t' = \sum_{i=0}^{d-1}\binom{d}{i} s(H^{i}_{\mathfrak{m}}(A))$, *and suppose that* $n_1,\dots,n_d \geq t'$. *Then*

$$l(\operatorname{Ker}\phi) = \sum_{i=0}^{d-1}\binom{d}{i} l(H^{i}_{\mathfrak{m}}(A)),$$

where $\phi : A/\sum_{1}^{d} x_i^{n_i}A \to U(A)^{-d-1}A$ *is the natural map.*

Proof. The proof proceeds by induction on d. The result is trivial when d=0. Suppose that d=1 and x is an s.o.p. for A. Let $t' = s(H^0_{\mathfrak{m}}(A))$ and assume that $n \geq t'$. Now $x^nH^0_{\mathfrak{m}}(A) = 0$, so that $x^nA \cap H^0_{\mathfrak{m}}(A) = 0$ by 4.3.8.. Setting $B = A/H^0_{\mathfrak{m}}(A)$ and applying 4.3.5.(iii) we can exhibit ϕ as the composition

$$A/x^nA \overset{\pi}{\to} B/x^nB \overset{\phi_B}{\to} U(B)^{-d-1}B \cong U(A)^{-d-1}A,$$

where ϕ_B is the natural map. Now x is a n.z.d. in B, which is therefore a Cohen-Macaulay ring. It follows from 1.3.3. and 2.2.3. that ϕ_B is injective so that

$$l(\text{Ker } \phi) = l(\text{Ker } \pi) = l(H^0_{\mathfrak{m}}(A)) ,$$

since $x^n A \cap H^0_{\mathfrak{m}}(A) = 0$. Therefore the result holds when d = 1.

Now suppose that d > 1 and that the theorem has been proved for all g.c.m. rings of dimension less than d. As before, let $B = A/H^0_{\mathfrak{m}}(A)$, let $t' = \sum_{i=0}^{d-1} \binom{d}{i} l(H^i_{\mathfrak{m}}(A))$ and assume that $n_1,\dots,n_d \geq t'$. Once again, by 4.3.5.(iii), we express ϕ as the composition

$$A/\sum_1^d x_i^{n_i}A \xrightarrow{\pi} B/\sum_1^d x_i^{n_i}B \xrightarrow{\phi_B} U(B)^{-d-1}B \cong U(A)^{-d-1}A .$$

By 4.3.9., $l(\text{Ker } \pi) = l(H^0_{\mathfrak{m}}(A))$, so that

$$l(\text{Ker } \phi) = l(\text{Ker } \phi_B) + l(H^0_{\mathfrak{m}}(A)) . \qquad (*)$$

We therefore focus our attention on the ring B and the map ϕ_B. By 4.3.6., B is itself a g.c.m. ring and $x_1,\dots,x_d$ form an s.o.p. for B, with each x_i a n.z.d. in B. Let $\bar{B} = B/x_1^{n_1}B$ and let $^-:B \to \bar{B}$ denote the natural map. Since $x_1^{n_1}$ is a n.z.d. in B, there exists, by 4.3.10., a commutative diagram

$$\begin{array}{ccc} U(\bar{B})^{-d}\bar{B} & \xrightarrow{\eta} & U(B)^{-d-1}B \\ \wr\| & & \wr\| \\ H^{d-1}_{\mathfrak{m}}(\bar{B}) & \rightarrow & H^d_{\mathfrak{m}}(B), \end{array}$$

where $\operatorname{Ker}\eta \cong H^{d-1}_{\mathfrak{m}}(B)/x_1^{n_1} H^{d-1}_{\mathfrak{m}}(B)$. In addition, we have the following commutative diagram:

$$\begin{array}{ccc} \bar{B}/\sum_{2}^{d} x_i^{n_i}\bar{B} & \cong & B/\sum_{1}^{d} x_i^{n_i} B \\ \phi_{\bar{B}} \downarrow & \swarrow \psi & \downarrow \phi \\ U(\bar{B})^{-d}\bar{B} & \xrightarrow{\eta} & U(B)^{-d-1}B \end{array} \qquad (\dagger)$$

Now, by 4.3.5.(i), $H^{d-1}_{\mathfrak{m}}(B) \cong H^{d-1}_{\mathfrak{m}}(A)$ so that $x_1^{n_1}H^{d-1}_{\mathfrak{m}}(B) = 0$, from the definition of t'. It therefore follows that $\operatorname{Ker}\eta \cong H^{d-1}_{\mathfrak{m}}(B)$.

By 4.3.7.(ii), $\bar{B}$ is a g.c.m. ring of dimension d−1 and, by 4.3.7.(iii), we have that $s(H^{i}_{\mathfrak{m}}(\bar{B})) \leq s(H^{i}_{\mathfrak{m}}(B)) + s(H^{i+1}_{\mathfrak{m}}(B))$, for all $1 \leq i \leq d-2$. This implies that

$$\begin{aligned} \sum_{i=1}^{d-2} \binom{d-2}{i-1} s(H^{i}_{\mathfrak{m}}(\bar{B})) &\leq \sum_{i=1}^{d-2} \binom{d-1}{i} s(H^{i}_{\mathfrak{m}}(\bar{B})) \\ &\leq \sum_{i=1}^{d-2} \binom{d-1}{i} \{s(H^{i}_{\mathfrak{m}}(B)) + s(H^{i+1}_{\mathfrak{m}}(B))\} \\ &\leq \sum_{i=1}^{d-2} \binom{d}{i} s(H^{i}_{\mathfrak{m}}(B)) + (d-1)s(H^{d-1}_{\mathfrak{m}}(B)) \\ &\leq t' - s(H^{d-1}_{\mathfrak{m}}(B)) . \end{aligned}$$

It therefore follows that

$$\sum_{i=1}^{d-2} \binom{d-2}{i-1} s(H^{i}_{\mathfrak{m}}(\bar{B})) + s(H^{d-1}_{\mathfrak{m}}(B)) \leq t' \qquad (**)$$

Now $\operatorname{Ker}\eta$ is annihilated by $\mathfrak{m}^{s(H^{d-1}_{\mathfrak{m}}(B))}$, so that, by 4.3.2. and $(**)$,

$$\text{Ker } \eta \subseteq \bar{B}/(\bar{x}_2^{n_2},\ldots,\bar{x}_d^{n_d}, 1) = \text{Im } \phi_{\bar{B}} .$$

We therefore have in (†) the exact sequence (with the obvious maps)

$$0 \to \text{Ker } \psi \to \text{Ker } \phi_B \to \text{Ker } \eta \to 0 ,$$

so that

$$\begin{aligned} l(\text{Ker } \phi_B) &= l(\text{Ker } \psi) + l(\text{Ker } \eta) \\ &= l(\text{Ker } \phi_{\bar{B}}) + l(\text{Ker } \eta) \\ &= \sum_{i=0}^{d-2} \binom{d-1}{i} \, l(H_{\mathfrak{m}}^i(\bar{B})) + l(H_{\mathfrak{m}}^{d-1}(B)) , \end{aligned}$$

by (**) and the induction hypothesis.

Now consider the short exact sequence

$$0 \to B \xrightarrow{x_1^{n_1}} B \to B/x_1^{n_1}B \to 0 .$$

This yields the long exact sequence in homology

$$\begin{aligned} 0 &\to H_{\mathfrak{m}}^0(B) \xrightarrow{x_1^{n_1}} H_{\mathfrak{m}}^0(B) \to H_{\mathfrak{m}}^0(\bar{B}) \\ &\to H_{\mathfrak{m}}^1(B) \xrightarrow{x_1^{n_1}} H_{\mathfrak{m}}^1(B) \to H_{\mathfrak{m}}^1(\bar{B}) \\ \ldots &\to H_{\mathfrak{m}}^i(B) \xrightarrow{x_1^{n_1}} H_{\mathfrak{m}}^i(B) \to H_{\mathfrak{m}}^i(\bar{B}) \to \ldots . \end{aligned}$$

Since $n_1 \geq t'$, all the maps for $i=0,\ldots,d-1$ induced by multiplication by $x_1^{n_1}$ are zero, and it follows that

$$l(H^i_{\mathfrak{m}}(\overline{B})) = l(H^i_{\mathfrak{m}}(B)) + l(H^{i+1}_{\mathfrak{m}}(B)),\ 0 \le i \le d-2 .$$

Thus

$$\begin{aligned} l(\text{Ker } \phi_B) &= \sum_{i=0}^{d-2} \binom{d-1}{i} l(H^i_{\mathfrak{m}}(\overline{B})) + l(H^{d-1}_{\mathfrak{m}}(B)) \\ &= \sum_{i=0}^{d-2} \binom{d-1}{i} \{l(H^i_{\mathfrak{m}}(B)) + l(H^{i+1}_{\mathfrak{m}}(B))\} + l(H^{d-1}_{\mathfrak{m}}(B)) \\ &= \sum_{i=0}^{d-2} \binom{d}{i} l(H^i_{\mathfrak{m}}(B)) + d.l(H^{d-1}_{\mathfrak{m}}(B)) \\ &= \sum_{i=1}^{d-1} \binom{d}{i} l(H^i_{\mathfrak{m}}(A)) \end{aligned}$$

since $H^i_{\mathfrak{m}}(B) = H^i_{\mathfrak{m}}(A)$, $i \ge 1$, and $H^0_{\mathfrak{m}}(B) = 0$. We recall from (*) that $l(\text{Ker } \phi) = l(\text{Ker } \phi_B) + l(H^0_{\mathfrak{m}}(A))$, so that

$$l(\text{Ker } \phi) = \sum_{i=0}^{d-1} \binom{d}{i} l(H^i_{\mathfrak{m}}(A))$$

as required.

We can now restate 4.3.4. as a corollary to 4.3.11.. In addition, we can give information concerning the constant n which appears in the statement of 4.3.4..

4.3.12. **Corollary**. *Let A, $x_1,\dots,x_d$ and t' be as in 4.3.11., and suppose that $n_1,\dots,n_d \ge t'$. Then*

$$l(A/\sum_1^d x_i^{n_i} A) = e(x_1,\dots,x_d)n_1\dots n_d + \sum_{i=0}^{d-1} \binom{d-1}{i} l(H^i_{\mathfrak{m}}(A)) .$$

Proof. Let $\phi: A/\sum_1^d x_i^{n_i} A \to U(A)^{-d-1}A$ be the natural map. Then

$l(A/\sum_{1}^{d} x_i^{n_i} A) = l(\mathrm{Ker}\ \phi) + l(A/(x_1^{n_1},\ldots,x_d^{n_d}; 1))$. It is a simple matter to verify that $t' \geq \sum_{i=1}^{d-1} \binom{d-1}{i-1} s(H_{\mathfrak{m}}^{i}(A))$, so that

$$l(A/(x_1^{n_1},\ldots,x_d^{n_d}; 1)) = e(x_1,\ldots,x_d)n_1\ldots n_d - \sum_{i=1}^{d-1} \binom{d-1}{i-1} l(H_{\mathfrak{m}}^{i}(A)),$$

by 4.3.3.. The result now follows in a straightforward manner from 4.3.11..

From now on we shall no longer insist that A is a g.c.m. ring. For the situations where A is a Noetherian local ring of dimension 1 and 2 respectively we have the following two results.

4.3.13. **Proposition** [34, 3.1.] *Let dim A = 1 and suppose that x is an s.o.p. for A. Then for all $n \in \mathbb{N}$,*

$$l(A/(x^n, 1)) = e(x)n .$$

4.3.14. **Theorem** [34, 3.2.] *Let dim A = 2 and suppose that x_1, x_2 form an s.o.p. for A. Let $l'(H_{\mathfrak{m}}^{1}(A))$ be the residual length of the Artinian module $H_{\mathfrak{m}}^{1}(A)$, and let s be the stability index of $H_{\mathfrak{m}}^{1}(A)$. Then, for all $n_1, n_2 \geq s$,*

$$l(A/(x_1^{n_1}, x_2^{n_2}, 1)) = e(x_1, x_2)n_1 n_2 - l'(H_{\mathfrak{m}}^{1}(A)) .$$

It is our intention to apply the ideas of the proof of 4.3.11. to the above result in order to investigate conditions under which $l(A/(x_1^{n_1}, x_2^{n_2})A)$ can be calculated, where A is a ring of dimension 2 and x_1, x_2 is an s.o.p. for A. The following lemma will be of use.

4.3.15. **Lemma.** *Let A be a Noetherian local ring. Then for all sufficiently large* $n \in \mathbb{N}$,

$$\mathfrak{m}^n \cap H^0_{\mathfrak{m}}(A) = 0 .$$

Proof. Since $l(H^0_{\mathfrak{m}}(A))$ is finite, the descending chain of ideals $\{\mathfrak{m}^n \cap H^0_{\mathfrak{m}}(A) \mid n \in \mathbb{N}\}$ becomes stationary. By the Krull Intersection Theorem, [1, 10.20], $\bigcap_{1}^{\infty} \mathfrak{m}^n = 0$, and the result follows.

For the situation where dim A = 1, we have the following proposition.

4.3.16. **Proposition.** *Let* $\dim A = 1$, *and let* $x \in A$ *form an s.o.p. for A. Suppose that k is the least integer such that* $\mathfrak{m}^k \cap H^0_{\mathfrak{m}}(A) = 0$. *Then, whenever* $n \geq k$,

$$l(A/x^n A) = e(x)n + l(H^0_{\mathfrak{m}}(A)) .$$

Proof. Let B denote the ring $A/H^0_{\mathfrak{m}}(A)$. If $n \geq k$, then $x^n A \cap H^0_{\mathfrak{m}}(A) = 0$, so that

$$l(A/x^n A) = l(B/x^n B) + l(H^0_{\mathfrak{m}}(A)).$$

Now B is a Cohen-Macaulay ring so that, by [**24**, p.311] and 4.3.5.(ii), $l(B/x^n B) = e(x)n$ and the result follows.

We complete this section with the following result which deals with the case where dimA = 2.

4.3.17. **Theorem.** *Suppose that* $\dim A = 2$, *and let* x_1, x_2 *form an s.o.p. for* A. *Let* k *be the least integer such that* $\mathfrak{m}^k \cap H^0_{\mathfrak{m}}(A) = 0$ *and let* $s = \max\{ k, s(H^1_{\mathfrak{m}}(A))\}$. *Suppose that* $n_1 \geq s$ *is a fixed positive integer. Then there exists* $z \in A$ *such that, for all sufficiently large values of the integer* n_2, $(x_1^{n_1}, x_2^{n_2})A = (z, x_2^{n_2})A$ *and*

$$l(A/(x_1^{n_1}, x_2^{n_2})A) = e(x_1, x_2)n_1 n_2 + l(H^0_{\mathfrak{m}}(A/zA)) .$$

Proof. Let $B = A/H^0_{\mathfrak{m}}(A)$, and let $\phi_B : B/(x_1^{n_1}, x_2^{n_2})B \to U(B)^{-3}B$ be the natural map. It follows from the proof of [**34**, 3.2.] (with the roles of x_1 and x_2 therein interchanged) that there exists an element $y \in B$ with the following properties:

(i) y is a n.z.d. in B;

(ii) $l(H^1_{\mathfrak{m}}(B)/y^{n_1} H^1_{\mathfrak{m}}(B)) = l'(H^1_{\mathfrak{m}}(B))$, the residual length of $H^1_{\mathfrak{m}}(B)$;

(iii) $(y^{n_1}, x_2^{n_2})B = (x_1^{n}, x_2^{n_2})B$ and $B/(x_1^{n_1}, x_2^{n_2}; 1) = B/(y^{n_1}, x_2^{n_2}; 1)B$;

(iv) $e(y, x_2) = e(x_1, x_2)$.

Now let $\bar{B} = B/y^{n_1}B$. By 4.3.10. we have a commutative diagram

$$\begin{array}{ccc} \bar{B}/x_2^{n_2}\bar{B} & \cong & B/(y^{n_1}, x_2^{n_2})B \\ \phi_{\bar{B}} \downarrow & & \phi_B \downarrow \\ U(\bar{B})^{-2}\bar{B} & \stackrel{\eta}{\to} & U(B)^{-3}B \end{array} ,$$

where $\operatorname{Ker} \eta \cong H^1_{\mathfrak{m}}(B)/y^{n_1} H^1_{\mathfrak{m}}(B)$.

From our choice of y it follows that $x_2^{n_2}$ Ker $\eta = 0$ whenever $n_2 \geq s(H^1_{\mathfrak{m}}(B))$. Therefore, for sufficiently large values of n_2, Ker $\eta \subseteq$ Im $\phi_{\bar{B}}$, by [**34**, 2.8], and we have the following exact sequence of finite length modules:

$$0 \to \text{Ker } \phi_{\bar{B}} \to \text{Ker } \phi_B \to \text{Ker } \eta \to 0 .$$

This implies that

$$l(\text{Ker } \phi_B) = l(\text{Ker } \phi_{\bar{B}}) + l(\text{Ker } \eta) . \quad (\dagger)$$

For all sufficiently large values of n_2, $l(\text{Ker } \phi_{\bar{B}}) = l(H^0_{\mathfrak{m}}(\bar{B}))$, by 4.3.13. and 4.3.16., so that, by ($\dagger$) and our choice of y,

$$l(\text{Ker } \phi_B) = l(H^0_{\mathfrak{m}}(\bar{B})) + l'(H^1_{\mathfrak{m}}(B)) .$$

It now follows from 4.3.14. that

$$l(B/(y^{n_1},x_2^{n_2})B) = e(y,x_2)n_1n_2 + l(H^0_{\mathfrak{m}}(\bar{B})).$$

By 4.3.15., for sufficiently large values of n_2, $(x_1^{n_1},x_2^{n_2})A \cap H^0_{\mathfrak{m}}(A) = 0$, so that

$$l(A/(x_1^{n_1},x_2^{n_2})A) = e(x_1,x_2)n_1n_2 + l(H^0_{\mathfrak{m}}(\bar{B})) + l(H^0_{\mathfrak{m}}(A))$$

by 4.3.5.(ii) and our choice of y. For such a value of n_2, $y^{n_1} \in B$ is the image of a unique element $z \in (x_1^{n_1},x_2^{n_2})A$, and it easily follows that $(z,x_2^{n_2})A = (x_1^{n_1},x_2^{n_2})A$. It now remains for us to show that

$$l(H_{\mathfrak{m}}^0(A/zA)) = l(H_{\mathfrak{m}}^0(A)) + l(H_{\mathfrak{m}}^0(\overline{B})).$$

Now $zA \cap H_{\mathfrak{m}}^0(A) = 0$ and it follows that $H_{\mathfrak{m}}^0(A)$ is naturally contained in $H_{\mathfrak{m}}^0(A/zA)$. It therefore suffices to demonstrate that $H_{\mathfrak{m}}^0(A/zA)$ is mapped onto $H_{\mathfrak{m}}^0(\overline{B})$ under the canonical projection map. It is clear that the image of $H_{\mathfrak{m}}^0(A/zA)$ lies in $H_{\mathfrak{m}}^0(\overline{B})$. Now suppose that x is an element of A whose image in $\overline{B}$, $\overline{x}$, lies in $H_{\mathfrak{m}}^0(\overline{B})$, so that $x\mathfrak{m}^r \subseteq zA + H_{\mathfrak{m}}^0(A)$, for some $r \in \mathbb{N}$. But $H_{\mathfrak{m}}^0(A) = 0:\mathfrak{m}^t$ for some $t \in \mathbb{N}$, which implies that $x\mathfrak{m}^{t+r} \subseteq zA$. It now follows that $\overline{x}$ is the image of an element of $H_{\mathfrak{m}}^0(A/zA)$. Therefore

$$l(H_{\mathfrak{m}}^0(A/zA)) = l(H_{\mathfrak{m}}^0(A)) + l(H_{\mathfrak{m}}^0(\overline{B})),$$

so that

$$l(A/(x_1^{n_1}, x_2^{n_2})A) = e(x_1, x_2)n_1n_2 + l(H_{\mathfrak{m}}^0(A/zA))$$

as required.

CHAPTER V

1. Seminormality and F–purity in local rings.

In this final chapter we consider the properties of seminormality and F–purity in Noetherian local rings, two properties which are closely related, especially in the 1–dimensional case. By making use of this relationship, we obtain a simplified proof of a result due Goto and Watanabe **[13]**, which describes the structure of a certain class of F–pure 1–dimensional rings. Finally, we investigate conditions under which the two properties are equivalent in the 1–dimensional case.

From this point on, A will denote a reduced Noetherian local ring with maximal ideal $\mathfrak{m}$. The classical ring of quotients of A will be denoted by Q(A) and $\bar{A}$ will denote the integral closure of A in Q(A).

5.1.1. **Definition** (see [**11**, 1.1]). A ring A is *seminormal* if it satisfies the following equivalent conditions:

(i) if $a \in Q(A)$ and $a^2, a^3 \in A$, then $a \in A$;

(ii) if $a \in Q(A)$ and there exists $k \in \mathbb{N}$ such that $a^t \in A$ whenever $t \geq k$, then $a \in A$.

In the particular case where dim A = 1, we have a further characterization of seminormality, namely:

(iii) A is seminormal if $\mathfrak{m} = J(\bar{A})$, where $J(\bar{A})$ denotes the Jacobson radical of $\bar{A}$.

5.1.2. **Definition.** Suppose that A is a ring of characteristic p, p a prime, and let $F:A \to A$ be the Frobenius endomorphism of A. Let A^F denote A when regarded as a A-module by F. Then A is said to be *F-pure* if, for all A-modules E, the map $h_E:E \to A^F \otimes_A E$, defined by $h_E(x) = 1 \otimes_A x$ for all $x \in E$, is injective. We say that F is *finite* when A^F is a finite A-module.

It is evident from results already known that a connection exists between these two properties, and the following proposition from **[13]** enables us to examine this relationship more closely. Although we do not impose the same restrictions on the ring A that appear in [**13**], the proof which appears therein applies equally well to the more general situation which we consider in this thesis.

5.1.3. **Proposition** [**13**, (2.2)]. *Let A be an F-pure ring, let Q = Q(A), and let* $f_{Q/A}:Q/A \to Q/A$ *be defined by the relation*

$$f_{Q/A}(x \bmod A) = x^p \bmod A,$$

for all $x \in Q$. *Then* $f_{Q/A}$ *is injective.*

An immediate consequence of 5.1.3. is the following result, which provides a generalization of [**18**, 5.31].

5.1.4. **Proposition**. *Let A be a ring of characteristic p. If A is F-pure, then A is seminormal.*

Proof. Suppose that $a \in Q(A)$ and $a^2, a^3 \in A$. Then it easily seen that $a^p \in A$, so that, by 5.1.3., $a \in A$ and the result follows.

From now on we focus our attention on the situation where A is a 1-dimensional ring. Concerning this situation we have the following result.

5.1.5. **Theorem** (cf.[**13**, (1.1)]). *Let A be a seminormal ring of dimension 1, and let $\mathfrak{p}_1,\ldots,\mathfrak{p}_n$ be the minimal primes of A. Then, for each i, $1 \leq i \leq n$,*

(i) $\mathfrak{m} = \mathfrak{p}_i \oplus \bigcap_{j \neq i} \mathfrak{p}_j$, *and*

(ii) $\mathfrak{m} = \bigoplus_{i=1}^{n} (\bigcap_{j \neq i} \mathfrak{p}_j)$.

Proof. Since A is reduced, the natural map $A \to \bigoplus_{1}^{n} A/\mathfrak{p}_i$ is injective, so that A can be regarded as a subring of $\bigoplus_{1}^{n} A/\mathfrak{p}_i \subset Q(A)$ under this map. Fix i, $1 \leq i \leq n$, let $x \in \mathfrak{m}$, and consider $(0,\ldots,\bar{x},\ldots,0) \in \bigoplus_{1}^{n} A/\mathfrak{p}_j$, where $\bar{x} \in A/\mathfrak{p}_i$. Now $r(\mathfrak{p}_i \oplus \bigcap_{j \neq i} \mathfrak{p}_j) = \mathfrak{m}$, so that there exists a positive integer k, such that $\mathfrak{m}^t \subseteq \mathfrak{p}_i \oplus \bigcap_{j \neq i} \mathfrak{p}_j$ whenever $t \geq k$. It follows that for each $t \geq k$, there exists $u \in \bigcap_{j \neq i} \mathfrak{p}_j$ such that $x^t - u \in \mathfrak{p}_i$. This implies that $(0,\ldots,\bar{x}^t,\ldots,0) \in A$ for all $t \geq k$, so that $(0,\ldots,\bar{x},\ldots,0) \in A$ by 5.1.1.(ii). There therefore exists an element $v \in \bigcap_{j \neq i} \mathfrak{p}_j$ such that $x - v \in \mathfrak{p}_i$, so that $x \in \mathfrak{p}_i \oplus \bigcap_{j \neq i} \mathfrak{p}_j$, and (i) follows.

To see that (ii) holds, observe that the above argument shows that $\mathfrak{m} = \bigoplus_{1}^{n} \mathfrak{m}_i$, where $\mathfrak{m}_i$ is the image of $\mathfrak{m}$ in $A/\mathfrak{p}_i$, $1 \leq i \leq n$. It follows from (i) that $\mathfrak{m}_i = \mathfrak{m}/\mathfrak{p}_i \cong \bigcap_{j \neq i} \mathfrak{p}_j$, $1 \leq i \leq n$, so that $\mathfrak{m} = \bigoplus_{i=1}^{n} \bigcap_{j \neq i} \mathfrak{p}_j$ as required.

We now make use of 5.1.4. and 5.1.5. to prove a structure theorem for a certain class of 1-dimensional F-pure rings, originally proved by Goto and Watanabe.

5.1.6. **Theorem** [**13**, (1.1)]. *Let A be a 1-dimensional ring of prime characteristic p. Suppose that the field $k \cong A/\mathfrak{m}$ is algebraically closed. Then A is F-pure if and only if*

$$\hat{A} \cong k[[X_1,\ldots,X_n]]/(\ldots,X_iX_j,\ldots)_{i \neq j},$$

where $\hat{A}$ denotes the completion of A with respect to $\mathfrak{m}$.

Proof. We begin by noting that A is F-pure if and only if $\hat{A}$ is F-pure by the argument on p.466 of [6]. Now $\hat{A}$ satisfies the conditions of the statement of the theorem, so that we can assume with no loss of generality that A is complete. It is is a straightforward consequence of [6, 1.12] that rings of the form $k[[X_1,\dots,X_n]]/(\dots,X_iX_j,\dots)_{i\neq j}$ are F-pure.

Let us now suppose that A is F-pure and that $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ are the minimal primes of A. As in the proof of 5.1.5., we can consider A to be a subring of $\bigoplus_1^n A/\mathfrak{p}_i$. Furthermore , by [2, Chap.V, 1.2], $\bar{A} = \bigoplus_1^n \overline{A/\mathfrak{p}_i}$ so that we have

$$A \subseteq \bigoplus_1^n A/\mathfrak{p}_i \subseteq \bigoplus_1^n \overline{A/\mathfrak{p}_i} \ .$$

Since A is complete, it follows that $A/\mathfrak{p}_i$ is complete, $1 \leq i \leq n$, so that each $\overline{A/\mathfrak{p}_i}$ is a local ring with maximal ideal $\bar{\mathfrak{m}}_i$, say. By 5.1.4., A is seminormal so that, by 5.1.1.(iii),

$$\mathfrak{m} = J(\bar{A}) = \bigoplus_1^n \bar{\mathfrak{m}}_i \ ,$$

so that the natural image of $\mathfrak{m}$ in $\overline{A/\mathfrak{p}_i}$ is $\bar{\mathfrak{m}}_i$. Moreover, since k is algebraically closed, it follows that $(\overline{A/\mathfrak{p}_i})/\bar{\mathfrak{m}}_i = k$, $1 \leq i \leq n$, so that $\overline{A/\mathfrak{p}_i} = A/\mathfrak{p}_i$, and thus each $A/\mathfrak{p}_i$ is integrally closed. It now follows from [1, 9.2] that $\bar{\mathfrak{m}}_i$ is a principal ideal, $1 \leq i \leq n$, so that, by 5.1.5., there exist elements $x_i \in \bigcap_{j\neq i} \mathfrak{p}_j$, $1 \leq i \leq n$, such that

$$\mathfrak{m} = \bigoplus_1^n Ax_i \ .$$

Since A is complete, it is possible to define a surjective k-algebra homomorphism

$$f: k[[X_1,\dots,X_n]] \to A,$$

such that $f(X_i) = x_i$, $1 \le i \le n$. It can easily be verified that $\mathrm{Ker}\, f = (\dots,X_iX_j,\dots)_{i\neq j}$, so that

$$A \cong k[[X_1,\dots,X_n]]/(\dots,X_iX_j,\dots)_{i\neq j},$$

as required.

The following result demonstrates the close connection between seminormality and F-purity in the 1-dimensional case. At this point we remark that if A is a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite, then it follows from [**22**, Th.108 and Th.78] that $\bar{A}$ is a finite A-module. We are therefore justified in replacing the latter condition, which appears in the original statements of 5.1.7. and 5.1.9., with the former condition which is more appropriate to the work of this chapter.

5.1.7. **Theorem** [**4**, §3]. *Let A be a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite. Suppose that the field $k \cong A/\mathfrak{m}$ is algebraically closed. Then A is seminormal if and only if*

$$\hat{A} \cong k[[X_1,\dots,X_n]]/(\dots,X_iX_j,\dots)_{i\neq j}.$$

It is clear from 5.1.6. and 5.1.7. that a ring satisfying the hypotheses of 5.1.7. is seminormal if and only if it is F-pure. It is our aim in the final part of this chapter to find weaker conditions under which the two properties continue to be equivalent. To this end we shall require the following auxiliary results.

5.1.8. **Lemma** [**27**, 1.2]. *Let k be a field and let L be a reduced Noetherian k-algebra of dimension 0. Then $k + TL[T]$ is seminormal, where T is an indeterminate over L, and $k + TL[T]$ is identified with a subring of $L[T]$ in the natural way.*

5.1.9. **Proposition** [**4**, Cor.2, **27**, 1.5]. *Let A be a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite. Then the following conditions are equivalent:*

(i) A is seminormal;

(ii) $\hat{A}$ is seminormal;

(iii) $Gr(A)$ is k-isomorphic to $k + TK[T]$, where $Gr(A)$ denotes the associated graded ring of A with respect to $\mathfrak{m}$ and $K = \overline{A}/J(\overline{A})$.

(iv) $Gr(A)$ is reduced and seminormal.

5.1.10. **Lemma** (cf[**18**, 4.6]). *Let $k \subset k'$ be fields of non-zero characteristic p, and suppose that k' is separable over k. Let R be a k-algebra such that $R \otimes_k k'$ is F-pure. Then R is F-pure.*

Proof. This follows in a straightforward manner from [**18**, 4.6] and 5.1.2..

We now give the main result of this chapter, which shows that the properties of seminormality and F-purity are equivalent for a wider class of rings than that considered in 5.1.7..

5.1.11. **Theorem**. *Let A be a 1-dimensional ring of prime characteristic p, whose Frobenius endomorphism is finite. Suppose that $k \cong A/\mathfrak{m}$ is a perfect field. Then A is F-pure if and only if A is seminormal.*

Proof. If A is F-pure then A is seminormal, by 5.1.4.. Let us now suppose that A is seminormal so that, by 5.1.9.,

$$Gr(A) \quad \cong_k \quad k + TK[T],$$

where $K = \bar{A}/J(\bar{A})$. Moreover, by 5.1.9, we can assume that A is complete, so that $k \subset A$, by [22, (28)P]. Now, if $\bar{k}$ denotes the algebraic closure of k, then

$$(k + TK[T]) \otimes_k \bar{k} = (k \otimes_k \bar{k}) \oplus (\bar{k} \otimes_k TK[T])$$

$$= \bar{k} \oplus T(K \otimes_k \bar{k})[T] .$$

We claim that, as $\bar{k}$-algebras, $\bar{k} \oplus T(K \otimes_k \bar{k})[T] \cong Gr(A \otimes_k \bar{k})$.

Since $\bar{k}$ is integral and flat over k and since $\bar{k} \subset A \otimes_k \bar{k}$, it easily follows that $\bar{k} \otimes_k \mathfrak{m} = \bar{\mathfrak{m}}$ is the unique maximal ideal of $A \otimes_k \bar{k}$. We have the exact sequence of k-vector spaces

$$0 \to \mathfrak{m}^{n+1} \to \mathfrak{m}^n \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to 0,$$

which, on applying $-\otimes_k \bar{k}$, yields the following exact sequence,

$$0 \to \bar{k} \otimes_k \mathfrak{m}^{n+1} \to \bar{k} \otimes_k \mathfrak{m}^n \to \bar{k} \otimes_k \mathfrak{m}^n/\mathfrak{m}^{n+1} \to 0.$$

It now follows that, as $\bar{k}$-modules,

$$\bar{k} \otimes_k \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong (\bar{k} \otimes_k \mathfrak{m}^n)/(\bar{k} \otimes_k \mathfrak{m}^{n+1}) = \bar{\mathfrak{m}}^n/\bar{\mathfrak{m}}^{n+1} .$$

Therefore

$$\bar{k} \oplus T(K \otimes_k \bar{k})[T] \cong_{\bar{k}} \bar{k} \otimes_{\bar{k}} Gr(A) \cong_{\bar{k}} Gr(\bar{k} \otimes_k A) . \qquad (*)$$

Now K is a finite product of fields, each of which is a finite separable extension of k, since $\bar{A}$ is a finite A-module, by the remarks preceding 5.1.7., and since k is perfect. It follows from [19, 3.3.(iv)] that the ring $K \otimes_k \bar{k}$ is reduced. In addition, $K \otimes_k \bar{k}$ is Noetherian and zero dimensional, so that by 5.1.8. and (*), $Gr(A \otimes_k \bar{k})$ is reduced, as is easily seen, and seminormal.

Now $k \subset A$ so that, by a straightforward adaption of the proof of [22, p.212, Cor.2], A is a finite k[[x]]-module, where x is an indeterminate over k. Hence $A \otimes_k \bar{k}$ is a finite module over $k[[x]] \otimes_k \bar{k}$. We now show that $k[[x]] \otimes_k \bar{k}$ is Noetherian.

Let us first consider the domain k[[x]]. Since an element $\sum_0^\infty k_i x^i \in k[[x]]$, with $k_0 \neq 0$ is a unit in k[[x]] it is easily seen that the quotient field of k[[x]], denoted Q(k[[x]]), consists of elements of the form $\sum_d^\infty k_i x^i$, where $d \in \mathbb{Z}$, and it is now a simple matter to verify that k is algebraically closed in Q(k[[x]]). This means that k is *maximally algebraic* in Q(k[[x]]), in the notation of [40, p.196] so that, by [40, p.198, Cor.2], $Q(k[[x]]) \otimes_k \bar{k}$ is a domain which in turn implies that $k[[x]] \otimes_k \bar{k}$ is also a domain, as $\bar{k}$ is flat over k. In addition, $\bar{k}$ is integral over k and it follows that $k[[x]] \otimes_k \bar{k}$ is a 1-dimensional local domain whose maximal ideal is generated by the single element $x \otimes_k 1$, so that, by [1, p.84, Ex.1], $k[[x]] \otimes_k \bar{k}$ is Noetherian. This implies that $A \otimes_k \bar{k}$, a finite $k[[x]] \otimes_k \bar{k}$-module, is itself Noetherian. We now have that $A \otimes_k \bar{k}$ is a 1-dimensional Noetherian local ring of prime characteristic p, whose Frobenius endomorphism is easily seen to be finite, so that we can deduce from 5.1.9. that $A \otimes_k \bar{k}$ is seminormal. Furthermore, since the residue field of $A \otimes_k \bar{k}$ is algebraically closed, it follows from 5.1.6. and 5.1.7. that $A \otimes_k \bar{k}$ is F-pure. Now k is perfect, so that $\bar{k}$ is separable over k, and we deduce from 5.1.10. that A itself is F-pure. This completes the proof.

APPENDIX 1

On direct limits

We begin this appendix by recalling the definition of a direct system. Let I be a directed set, let $\{M_i\}_{i \in I}$ be a family of A-modules indexed by I, and for each pair $i,j \in I$ with $i \leq j$ let $\mu_{ij}:M_i \to M_j$ be an A-module homomorphism. Then we say that the modules M_i and the maps μ_{ij} form a *direct system* $\{M_i,\mu_{ij}\}$ if

(i) μ_{ii} is the identity map on M_i for all $i \in I$ and

(ii) $\mu_{ik} = \mu_{jk}\mu_{ij}$, whenever $i \leq j \leq k$.

The *direct limit* of the system $\{M_i,\mu_{ij}\}$ is defined to be the A-module $M = \bigoplus_{i \in I} M_i/D$, where D is the submodule of $\bigoplus_{i \in I} M_i$ generated by all elements of the form $\mu_{ij}(x_i) - x_i$, where $i,j \in I$, with $i \leq j$, and $x_i \in M_i$. Let $\mu: \bigoplus_{i \in I} M_i \to M$ denote the natural projection map and, for all $j \in I$, let $\mu_j : M_j \to M$ be the restriction of μ to M_j. Then it is known that if $\{M_i,\mu_{ij}\}$ is a direct system then its direct limit M satisfies the following properties (see [1, Chap.2, Exs.14-16]).

(1) Every element $x \in M$ can be written as $\mu_j(x_j)$ for some $j \in I$ and $x_j \in M_j$.

(2) If $x_i \in M_i$ is such that $\mu_i(x_i) = 0$ in M then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

(3) If N is an A-module and $\{ \alpha_i:M_i \to N, i \in I \}$ are A-module homomorphisms such that $\alpha_i = \alpha_j\mu_{ij}$ whenever $i \leq j$, then there exists a unique A-module homomorphism $\alpha:M \to N$ such that $\alpha_i = \alpha\mu_i$ for all $i \in I$.

Now let U be a triangular subset of A^{n+1}, $n \in \mathbb{N}$, let U_n be the restriction of U to A^n and let M be an A-module. Then U_n is a directed set and, as was seen in Chapter III, if $u \leq v$ so that $v^T = Hu^T$ for some $H \in D_n(A)$, there exists a map

$$\phi_{uv}:(M/uM)_{S(u)} \rightarrow (M/vM)_{S(v)},$$

induced by the matrix H. We have described the system

$$\{ (M/uM)_{S(u)}, \phi_{uv} \} \qquad (*)$$

as a direct system and, at various places throughout this thesis, we have appealed to properties (1)–(3) listed above when considering the system (*). However, as the reader may have noted, the maps ϕ_{uv} described above are not uniquely determined by $u,v \in U_n$, but depend on the particular choice of matrix H. With this in mind, we now relabel the system (*)

$$\{ (M/uM)_{S(u)} , \Phi_{uv} \} , \qquad (**)$$

where Φ_{uv} denotes the set of all maps $(M/uM)_{S(u)} \rightarrow (M/vM)_{S(v)}$ which can be induced in the above manner. We shall now demonstrate that the direct limit of the system (**) possesses properties analogous to (1)–(3) of the direct limit of an ordinary direct system.

Proposition 1. *Let I be a directed set and let $\{ M_i \}$ be a set of A-modules indexed by I. Whenever $i \leq j$ let Φ_{ij} denote a non-empty set of A-homomorphisms $M_i \rightarrow M_j$, and suppose that the following conditions are satisfied:*

(i) Φ_{ii} *contains the identity map on* M_i, *for all* $i \in I$;

(ii) if $i \leq j \leq k$ *then for all* $\phi_{ij} \in \Phi_{ij}$ *and* $\phi_{jk} \in \Phi_{jk}$, $\phi_{jk}\phi_{ij} \in \Phi_{ik}$;

(iii) for all $i \in I$, *there exists* $j \geq i$ *and* $\psi_{ij} \in \Phi_{ij}$ *such that whenever* $k \leq i$ *and* $\phi_{ki}, \phi'_{ki} \in \Phi_{ki}$, *then* $\psi_{ij}\phi_{ki} = \psi_{ij}\phi'_{ki}$.

Let D be the submodule of $\bigoplus_{i \in I} M_i$ *generated by all elements of the form* $\phi_{ij}(x_i) - x_i$, *where* $i,j \in I$ *and* $x_i \in M_i$. *Then the direct limit* $M = \bigoplus_{i \in I} M_i/D$ *of the system* $\{ M_i, \Phi_{ij} \}$ *(with natural maps* μ_i, $i \in I$*) possesses property (1) above, and properties (2′) and (3′) described below.*

(2′) If $x_i \in M_i$ *is such that* $\mu_i(x_i) = 0$ *in M then there exists* $j \geq i$ *and* $\phi_{ij} \in \Phi_{ij}$ *such that* $\phi_{ij}(x_i) = 0$ *in* M_j.

(3′) If N is an A-module and $\{ \alpha_i : M_i \to N, i \in I \}$ *are A-module homomorphisms such that* $\alpha_i = \alpha_j\phi_{ij}$ *whenever* $i \leq j$ *and* $\phi_{ij} \in \Phi_{ij}$, *then there exists a unique A-module homomorphism* $\alpha : M \to N$ *such that* $\alpha_i = \alpha\mu_i$ *for all* $i \in I$.

Proof (cf. [3, VIII, 4.3 and 4.4]). If $x \in M$ then there exists a finite sum $\sum_i x_i \in \bigoplus_{k \in I} M_k$, where each $x_i \in M_i$, such that $x = \mu(\sum_i x_i)$. Choose $j \in I$ such that $j \geq i$ for all i appearing in the above sum and, for each such i, select a map $\phi_{ij} \in \Phi_{ij}$. Then

$$x = \mu(\sum_i x_i) = \mu(\sum_i \phi_{ij}(x_i)) = \mu_j(x_j)$$

where $x_j = \sum_i \phi_{ij}(x_i) \in M_j$, as required. Thus property (1) is verified for M.

Suppose now that $x_i \in M_i$ and $\mu_i(x_i) = 0$ in M. Then x_i is a finite sum of the form

$$\sum (\phi_{jk}(x_j) - x_j) .$$

Now choose $t \in I$ such that t exceeds i and all $j,k \in I$ appearing in the above sum. Then by (iii), there exists $m \geq t$ and $\psi_{tm} \in \Phi_{tm}$ such that, whenever $n \leq t$ and $\phi_{nt}, \phi'_{nt} \in \Phi_{nt}$, then $\psi_{tm}\phi_{nt} = \psi_{tm}\phi'_{nt}$. Now

$$\psi_{tm}\phi_{it}(x_i) = \psi_{tm}\phi_{it}(x_i) - x_i + x_i$$

$$= \psi_{tm}\phi_{it}(x_i) - x_i + \sum (\phi_{jk}(x_j) - x_j) .$$

Furthermore, for each term of the sum on the right,

$$\phi_{jk}(x_j) - x_j = \psi_{tm}\phi_{kt}\phi_{jk}(x_j) - x_j - (\psi_{tm}\phi_{kt}(\phi_{jk}(x_j)) - \phi_{jk}(x_j)) .$$

where $\phi_{kt} \in \Phi_{kt}$. It follows that $\psi_{tm}\phi_{it}(x_i)$ can be written as a finite sum of the form

$$\sum (\psi_{tm}\phi_{st}(x_s) - x_s)$$

where each $s \leq t \leq m$. From the choice of ψ_{tm}, we can consider all the terms with a common s to be grouped into a single term. Since any relation on a direct sum is a consequence of relations on individual summands,

$$\psi_{tm}\phi_{it}(x_i) = \sum (\psi_{tm}\phi_{st}(x_s) - x_s)$$

implies that $x_s = 0$ if $s \neq m$. If $s = m$, then $s = t = m$ and we have that

$$\psi_{tm}\phi_{it}(x_i) = \psi_{tm}\phi_{st}(x_s) - x_s.$$

From our choice of ψ_{tm} and the fact that Φ_{ss} contains the identity mapping, it now follows that

$$\psi_{tm}\psi_{tm}\phi_{it}(x_i) = 0,$$

as required, and so property (2′) is verified.

Suppose now that N is an A-module and let $\{\alpha_i:M_i \to N\}_{i\in I}$ be a set of homomorphisms such that $\alpha_j\phi_{ij} = \alpha_i$, for all $\phi_{ij}\in\Phi_{ij}$, $i,j\in I$ with $i \leq j$. Define a map $\alpha:M \to N$ as follows. For $x\in M$ let $\alpha(x) = \alpha_j(x_j)$ for any choice of $j\in I$ and $x_j\in M_j$ such that $\mu_j(x_j) = x$. We now show that α is well-defined.

Suppose that $\mu_i(x_i) = \mu_j(x_j)$, and choose $k \geq i,j$ $\phi_{ik}\in\Phi_{ik}$ and $\phi_{jk}\in\Phi_{jk}$. Then $\mu_k(\phi_{ik}(x_i)) = \mu_k(\phi_{jk}(x_j))$ so that, by property (2′), there exists $t \geq k$ and $\phi_{kt}\in\Phi_{kt}$ such that $\phi_{kt}\phi_{ik}(x_i) = \phi_{kt}\phi_{jk}(x_j)$. It now follows that $\alpha_i(x_i) = \alpha_j(x_j)$, so that α is a well-defined homomorphism. From the above construction, $\alpha\mu_i = \alpha_i$ for all $i\in I$, so that property (3′) holds for M, since uniqueness is clear.

We shall use the term *generalized direct system* to describe a system which satisfies conditions (i)-(iii) in the statement of Theorem 1. Let us consider once more the system

$$\{ (M/uM)_{S(u)} , \Phi_{uv} \}. \qquad (\dagger)$$

Then (†) is a generalized direct system, as we now demonstrate. That (†) satisfies (i) and (ii) is obvious. If we consider $u = (u_1,\ldots,u_n)\in U_n$, and let u^2

denote $(u_1^2,\dots,u_n^2)\in U_n$ and $\psi:(M/uM)_{S(u)} \to (M/u^2M)_{S(u^2)}$ denote the map induced by the matrix $\mathrm{diag}(u_1,\dots,u_n)$, then it follows from 1.2.2. that

$$\psi\phi_{tu} = \psi\phi'_{tu} ,$$

whenever $t \leq u$ and $\phi_{tu},\phi'_{tu}\in \Phi_{tu}$. It follows that the system (†) satisfies condition (iii) of Theorem 1, and is therefore a generalized direct system. We can now employ properties (1),(2′) and(3′) when dealing with its direct limit. This justifies the proofs of 1.3.3. and 3.2.8..

The final part of the proof of 3.3.4. assumes (employing an obvious notation) that, for a generalized direct system $\{ M_i , \Phi_{ij} \}$ with direct limit M,

$$\varinjlim_{i\in I} \{ \bigoplus_{\lambda\in\Lambda}(M_i)_{S_\lambda} , \bigoplus_{\lambda\in I} (\Phi_{ij})_{S_\lambda}\} = \bigoplus_{\lambda\in\Lambda} M_{S_\lambda} ,$$

where the S_λ are multiplicatively closed subsets of A and $\bigoplus_{\lambda\in\Lambda}(\Phi_{ij})_{S_\lambda}$ consists of mappings of the form $\bigoplus_{\lambda\in\Lambda}(\phi_{ij})_{S_\lambda}$, where $\phi_{ij}\in \Phi_{ij}$. This can be verified in a standard fashion after noting that $\{ \bigoplus_{\lambda\in\Lambda}(M_i)_{S_\lambda} , \bigoplus_{\lambda\in\Lambda}(\Phi_{ij})_{S_\lambda}\}$ is itself a generalized direct system whose limit therefore satisfies (1), (2′) and (3′).

In 3.4. we consider the situation of a triangular subset $U\subset A^{n+1}$ and an A-module M, such that $u_1,\dots,u_n$ is a poor M-sequence for all $u = (u_1,\dots,u_n)\in U_n$. Consider now the generalized direct system

$$\{ (M/uM)_{S(u)} , \Phi_{uv} \} ,$$

and suppose that $u,v\in U_n$ are such that $u \leq v$. Let $\phi_{uv},\phi'_{uv}\in \Phi_{uv}$ and, employing the usual notation, let $\psi : (M/vM)_{S(v)} \to (M/v^2M)_{S(v^2)}$ be the map induced by the matrix $\mathrm{diag}(v_1,\ldots,v_n)$. Then, by 1.2.2.,

$$\psi\phi_{uv} = \psi\phi'_{uv} .$$

Since ψ is injective by 2.2.1., it follows that $\phi_{uv} = \phi'_{uv}$, so that the set Φ_{uv} consists of a unique homomorphism. Therefore the generalized direct system above is a direct system of the ordinary type, so that we require no special analysis in this case.

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