SOME TOPICS IN THE UNITARY SYMMETRY OF ELEMENTARY PARTICLE INTERACTIONS

Thesis

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INTRODUCTION

After outlining the SU(3) symmetry of strong interactions, which is well satisfied experimentally and becoming famous, we describe its application to weak interactions, and concentrate on Cabibbo's theory which assumes special properties of weak interaction currents with respect to SU(3). Cabibbo's theory seems well satisfied experimentally, though some of his assumptions are on not very secure grounds. For the purpose of checking these assumptions separately, we derive sum rules between experimentally observed branching ratios and decay coupling constants. At present, these sum rules are satisfied within experimental errors, which are large, but may hopefully be reduced in the future.

Pomeranchuk's theorems have been of great interest in high energy physics, and group theoretical approaches to particles extend these theorems, and sometimes enable one to derive stronger results. It is shown in Chapter 6 what information may be obtained using SU(6), and appears that although additional non-trivial results follow compared with SU(3), we may not prove anything definitive, such as the derivation of the conjecture of zero quantum number exchange dominance, from the theorem. I am very grateful to Professor E.J. Squires of Durham University for correspondence and discussion about this.

Following confirmation of SU(3) as a group suitable for the description of the purely "internal" quantum numbers of elementary particles, an attempt was made to try to combine this symmetry with a group which accounts for spin conservation.

The first proposal along these lines was that the group SU(6) be a good symmetry; this group contains both SU(3) and SU(2) (the spin group) as subgroups. At first sight, however, this scheme appears nonrelativistic, and this resulted in a number of further proposals for relativistic "extension" of SU(6). This has resulted in an avalanche of literature during the first half of 1965. Of these schemes, we choose just one, which is due to Feza Gürsey, and is, in a way, the most simple-minded of all the schemes, since it consists only in finding a relativistically covariant spin operator, which is a non-local momentum-dependent operator. We describe the derivation of this in Chapters IV and V. In Chapter VII we show that the magnetic moment operator is simply related to W_μ, the Bargmann-Wigner operator. Hence, according to SU(6), the ratio of proton and neutron magnetic moments is $-3/2$, a result well-known in the static limit. Finally, we consider the forms of interaction Lagrangians in SU(6) and show that the conventional ones, taken in the right combinations, lead to the desired σ, τ symmetry in the static limit. For reference, an appendix on Young's diagrams is added.

CHAPTER I

OUTLINE OF SU (3) SYMMETRY OF STRONG INTERACTIONS.

1. INTRODUCTION

It is perhaps true to say that unitary symmetry is now even more well-known amongst physicists who do not work "professionally" with it, than Regge poles were three years ago. This is due to two factors; the experimental verification of SU(3) has been superb, and the theory is easy; especially now that so many people are engaged in a search for higher symmetries than SU(3), which leads them to excursions into SU(6), U(12), non-compact $\tilde{U}(12)$, $U(6) \otimes U(6)$, U(6, 6) and even non-unitary groups such as SL(6, C) etc. Compared with this sophisticated group theory SU(3) appears like the alphabet, (It is in fact the easiest non trivial group there is -- SU(2) is regarded by mathematicians as trivial), and I think it is not either worthwhile or called for to go rigorously from A to Z of SU(3). Most of this is well-known, and in general most of what is not well-known is not worth knowing. Of course, there are several different approaches to group theory in physics, differing generally in the amount of mathematics they contain, and this makes any "complete" coverage of the field virtually impossible. Instead, I shall just mention the most well-tried and successful references to the subject and content myself thereafter with a quick review of SU(3), stopping only to emphasize some points and review others - most of these being because of their later importance in SU(6).

Stated briefly, the idea of any symmetry is to group particles into families. In the hypothetical limit in which the symmetry is exact, all the members of a family become identical in their physical properties. Thus if electromagnetism is "turned off", the proton and neutron become identical (the nucleon), the sigma triplet also become identical, etc. SU(3) just extends isospin symmetry to include strange particles. So, in the SU(3) limit, the $\frac{1}{2}^+$ baryons p, n, Σ^+, Σ^0, Σ^-, Λ, Ξ^0, Ξ^- are identical, and so are the mesons π^+, π^0, π^-, η, K^+, K^-, K^0, $\overline{K}^0$. This limit, of course, is a good deal more hypothetical, as it were, than that in which electromagnetism is absent, since for example, the K particle is about 4 times as massive as the π. So, in the theory of SU(3), a good deal of attention is paid to the breaking of the symmetry. In particular, if one assumes that the symmetry breaking term transforms in a particular way under the group, one may derive the famous Gell-Mann - Okubo mass formula, so mysteriously well satisfied experimentally.

The original approach to SU(3) was through vector currents in weak interactions, and Gell-Mann's (1962) paper is a superb account of this. Ne'eman's (1961) paper is on the same subject. We deal with the subject of weak interaction currents and their conservation in Chapters II & III. Briefly, all vector currents are conserved in the limit in which all mass differences within a multiplet are zero. Following this, an entirely different approach to the subject began, with the accent on strong interactions "only", i.e. with no reference to weak interaction currents. The problem here is that we must conserve two quantum numbers, T_3 and Y (hypercharge = B + S) and so mathematically speaking, we want to look for a group of rank 2. Amongst the semi-simple groups, the choice fortunately is fairly restricted being only SU(3), B_2, C_2 and G_2. G_2 (an exceptional group in Cartan's classification) was the only serious rival to SU(3), and being wise after the event, one can say that what success it had was due to the fact that it contained SU(3) as a subgroup. For the aesthetically-minded, it had the additional attraction of being the group associated with quaternion algebras. However, it had no 8 dimensional representation, but only a 7, so the Λ had to be singlet, and this was considered a disadvantage. An excellent and comprehensive review of the mathematics of group theory and of the above four groups as candidates for higher symmetries is by Behrends, Dreitlein, Fronsdal and B.W. Lee (1962). See also Fronsdal (1962).

These two approaches, of Gell-Mann on the one hand, and of Behrends et al. on the other, were compared and reviewed by d'Espagnat (1962). Speiser and Tarski (1963) reviewed all possible groups containing 8 dimensional representations, and their paper has a good treatment of global properties of groups. We mention also de Swart's paper (1963). The problem of breaking and of the mass formula is treated by Gell-Mann (1961 and 1962), Okubo (1962) and Gürsey, T.D. Lee and Nauenberg (1964). Other reviews are by Sakurai (1963), Gell-Mann and Ne'eman (1965), Gasiorowitz and Glashow (1965), Lipkin (1965) and Cutkosky (1964). It is interesting as well as historically accurate, to note that unitary symmetry was first proposed in connexion with the Sakata model by Ikeda, Oyawa and Ohnuki (1959). The experimental consequences and their comparison with the facts is outlined also by Glashow and Rosenfeld (1963).

2. Symmetric Sakata Model and Quarks.

Apart from the trivial one, the lowest dimensional representation of SU(3) is the 3 dimensional one. In the version of SU(3) based on the Sakata model (this version is now abandoned), p, n and Λ were assigned to this representation. In the eightfold way, in which all 8 baryons belong to the same representation, it appears that no particles belong to the fundamental representation. If there were such "fundamental" particles, the success of SU(3) would be a lot easier to understand and Gell-Mann (1964b) and Zweig (unpublished) suggested that perhaps such particles do exist. It is now beginning to be thought that they are fictitious, the experimental lower limit on their mass being $\approx$ 3 Bev. On the other hand, the existence of heavy triplets would explain the mass formula success (Gürsey et al., 1964). Whether or not they exist anyway, it is instructive to work with them and regard them as mathematical entities. Moreover, we shall label them, p, n, and Λ. We can now work with three basic fermion fields.

Let us first note some facts concerning the representations of SU(3):-

(i) The simplest representation of transformations on a three dimensional vector looks like

$$\phi_a \rightarrow U_{ab}\,\phi_b$$ (Einstein summation convention)

$$U^{-1}_{ab} = U^{+}_{ab}$$

We denote this by 3. Its adjoint

$$\phi^a \rightarrow U^{+}_{ab}\,\phi^b$$

also forms a representation, called $\bar{3}$.

(ii) The general representation of SU(3) may be constructed from these simplest representations by forming tensors. The irreducible tensorial set is symmetric in all upper indices and in all lower indices. All its contractions involving one upper and one lower index vanish. An irreducible tensorial set with p upper and q lower indices yields a representation labelled D(p, q) and of dimension

$$\tfrac{1}{2}(p + 1)(q + 1)(p + q + 2) \quad .$$

(iii) It is often required to take the direct product of several representations and to extract the irreducible components. This procedure is facilitated with the use of Young's diagrams (see appendix). We quote the following results:

$$3 \times 3 = \bar{3} + 6$$
$$\bar{3} \times \bar{3} = 3 + \bar{6}$$
$$3 \times \bar{3} = 1 + 8$$
$$3 \times 8 = 3 + \bar{6} + 15 \; (q = 2,\ p = 1)$$
$$8 \times 8 = 1 + 8 + 8 + 10 + \overline{10} + 27.$$

(iv) A representation with p = q is self-adjoint and of dimension $(p + 1)^3$. Mesons are generally believed to belong to self-adjoint representations, and in SU(3) these are 1, 8, 27 etc. Otherwise $\overline{D}(p, q) = D(q, p)$.

Let us represent p, n and Λ by a three-component vector

$$\psi = \begin{pmatrix} \Lambda \\ p \\ n \end{pmatrix}$$

and first rewrite the classical symmetries as infinitesimal unitary operations on this vector.

I. Baryon number conservation

$$\psi \rightarrow \left[1 + i\varepsilon \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] \psi$$

II. Hypercharge conservation

$$\psi \rightarrow \left[1 + i\varepsilon' \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right] \psi$$

III. Charge independence - isospin invariance

$$\psi \rightarrow \left[1 + \tfrac{1}{2} \vec{\theta} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\vec{\tau}} \\ 0 & & \end{pmatrix} \right] \psi \quad .$$

These are all <u>exact</u> symmetries of strong interactions. The jump to SU(3) is made by hypothesizing that <u>any</u> 3 × 3 unitary matrix is an approximate symmetry of the Lagrangian. In fact, we will get U_3, since I and II involve the trace-baryon number is included in the group, if we assign it to the quarks.

So we assume that

$$\psi \rightarrow (1 + i \sum_{\alpha} \theta_{\alpha} T_{\alpha})\psi$$

is an approximate symmetry, where θ_{α} are nine real infinitesimals and T_{α} are a complete set of nine 3×3 hermitian matrices. Besides the five conserved currents corresponding to I, II and III, there are now four additional strangeness-bearing partially (or approximately) conserved currents

(a) $\bar{\Lambda} \gamma_{\mu} n + \ldots.$

(b) $\bar{\Lambda} \gamma_{\mu} p + \ldots.$

h.c. of (a)

h.c. of (b)

whose divergences are proportional to $(m_n - m_{\Lambda})$, $(m_p - m_{\Lambda})$, and so they are conserved in the limit of degenerate baryon mass. As the Λ - N mass difference, however, is much greater than the p - n mass difference, so are the medium-strong symmetry-breaking interactions much stronger than electromagnetism.

The jump to unitary symmetry is made less abrupt by the following observation of Matthews and Salam (1962). If to the classical symmetries I, II and III is adjoined the <u>discrete</u> operation

$$\psi \rightarrow S\psi = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \psi$$

or

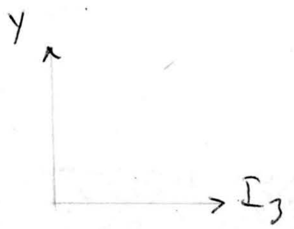
$$\left\{\begin{array}{lll} p & \rightarrow & p \\ n & \rightarrow & \Lambda \\ \Lambda & \rightarrow & n \end{array}\right.$$

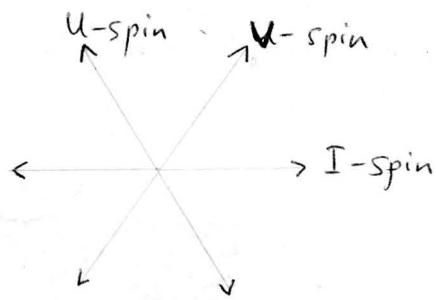
which is assumed to be an approximate symmetry of the Lagrangian, then it follows that all of U_3 must be a higher symmetry. The proof is easy:-

$$\psi \rightarrow S\left[1 + \tfrac{i}{2}\,\vec{\theta}\cdot\begin{pmatrix} 0 & 0 & 0 \\ 0 & \vec{\tau} & \\ 0 & & \end{pmatrix}\right]S\psi$$

$$= \left[1 + \tfrac{i}{2}\,\vec{\theta}\cdot\begin{pmatrix} \vec{\tau} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\psi$$

which is a sort of isospin, in Λ - p space. It is called by Levinson, Lipkin and Meshkov (1962) U-spin, in analogy with I-spin and appears on the root diagram as follows.





The similarity of "I-spin, U-spin, V-spin" with childhood memories of verb conjugation, is due to the fact that Lipkin is one of the authors!

3. Mesons

The quark-antiquark pseudoscalar sources

$$\overline{\psi}^{\alpha} \gamma_5 \psi_{\beta}$$

transform according to $3 \times \overline{3} = 1 + 8$, and so may be invariantly coupled to either a singlet meson

$$\phi \overline{\psi}^{\alpha} \gamma_5 \psi_{\alpha} \quad , \qquad \phi^{+} = \phi$$

or to an octet of mesons

$$\overline{\psi}^{\alpha} \gamma_5 \phi_{\alpha}^{\beta} \psi_{\beta}$$

$$\phi_{\alpha}^{\beta +} = \phi_{\beta}^{\alpha} \quad , \qquad \phi_{\alpha}^{\alpha} = 0$$

or to both. Explicitly the above expansion may be written as

$$\phi_{11} \bar{\Lambda} \gamma_5 \Lambda + \phi_{12} \bar{\Lambda} \gamma_5 p + \phi_{13} \bar{\Lambda} \gamma_5 p$$
$$+ \phi_{21} \bar{p} \gamma_5 \Lambda + \phi_{22} \bar{p} \gamma_5 p + \phi_{23} \bar{p} \gamma_5 n$$
$$+ \phi_{31} \bar{n} \gamma_5 \Lambda + \phi_{32} \bar{n} \gamma_5 p + \phi_{33} \bar{n} \gamma_5 n$$

So we may identify ϕ_{21} and ϕ_{23} as an isotopic doublet with hypercharge (= strangness here) one

$$\phi_{21} = K^{+}$$
$$\phi_{23} = K^{o}$$

and their hermitian adjoints

$$\phi_{12} = K^{-}$$
$$\phi_{32} = \overline{K}^{o}$$

Similarly, ϕ_{23} and ϕ_{32} comprise two components of an isotriplet:

$$\phi_{23} = \pi^+$$

$$\phi_{32} = \pi^-$$

The identification of π^0 follows from demanding that π^+, π^- and π^0 form an isotriplet, as do $\bar{p}n$, $\bar{n}p$ and $(\bar{p}p - \bar{n}n)/\sqrt{2}$, so

$$\frac{\phi_{22} - \phi_{33}}{\sqrt{2}} = \pi^0 \ .$$

This gives seven of the mesons. The remaining one, now called η, corresponds to

$$\frac{1}{\sqrt{6}}(-2\phi_{11} + \phi_{22} + \phi_{33}) = \eta$$

and was predicted by SU(3). It has been discovered at 550 Mev.

We may now write out the meson matrix explicitly

$$\phi = \begin{pmatrix} -\frac{2}{\sqrt{6}}\eta & K^- & K^0 \\ K^+ & \frac{\eta}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \bar{K}^0 & \pi^- & \frac{\eta}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} \end{pmatrix}$$

and it is traceless, as required for SU(3).

In matrix notation, then, the invariant interactions becomes

$$\bar{\psi}\, \gamma_5\, \phi\, \psi$$

and the symmetry operations become

$$\psi \rightarrow U\psi$$
$$\bar{\psi} \rightarrow \bar{\psi}U^{-1}$$
$$\phi \rightarrow U\phi U^{-1}$$

for finite transformation U, and for infinitesimal transformations

$$\psi \rightarrow \psi + i\varepsilon T\psi$$
$$\psi \rightarrow \bar{\psi} - i\varepsilon\bar{\psi}T$$
$$\phi \rightarrow \phi + i\varepsilon[T, \phi]$$

where T is an arbitrary hermitian matrix, and ε is infinitesimal. Note that transformation I acts trivially on ϕ - the mesons have no baryon number.

4. Baryons and Resonances

As far as the Sakata model is concerned, all is well up to this point. But now, one has to ask, where do Σ and Ξ go? We can get more baryons by looking for bound states of meson and Sakaton, which will transform like

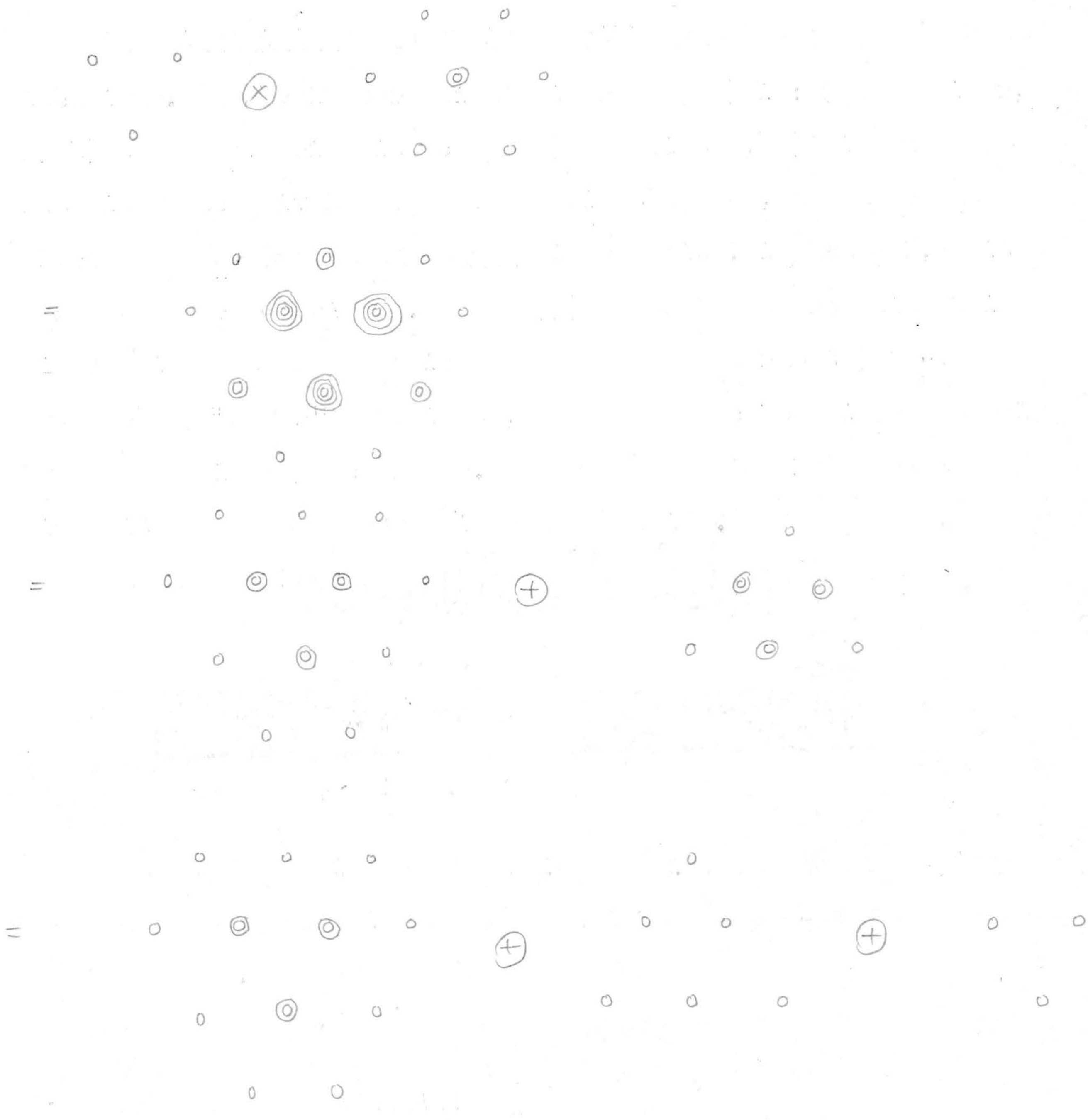
$$3 \times 8 = 3 + \bar{6} + 15 .$$

A word is called for here about how to determine the isospin and strangeness content of an irreducible representation:

the <u>dimensionalities</u> we can fine easily from the Young diagram technique. The rule for weight diagrams (WD) is: (see Speiser, 1964) take WD_1, bring its centre (the point 0, 0) without rotation successively over every weight of WD_2, and mark the places of WD_1. We then get <u>one</u> diagram with, in general, quite a few degenerate weights. In SU(3) the rule for reducing this diagram is that as we go inwards from the outermost (non-degenerate) "layer" of weights, the degeneracy can increase only by steps of 1 for each layer. We can then finally check the dimensions from the Young diagram reduction.

As an example

and also



Σ fits into 6 or 15, whereas Ξ fits only into 15. But the remaining resonances, which should go into these representations, have spin $^3/2$, and clearly all the particles in one representation have the same spin and parity. For these and other reasons, the Sakata model was abandoned, and its place taken by the eightfold way, in which all the baryons are placed together in an eight-dimensional representation. So we have the baryon octet

$$\psi = \begin{pmatrix} -\frac{2}{\sqrt{6}}\Lambda & \Xi^- & \Xi^0 \\ p & \frac{\Lambda}{\sqrt{6}} + \frac{\Sigma^0}{\sqrt{2}} & \Sigma^+ \\ n & \Sigma^- & \frac{\Lambda}{\sqrt{6}} - \frac{\Sigma^0}{\sqrt{2}} \end{pmatrix}$$

and its adjoint

$$\overline{\psi} = \begin{pmatrix} -\frac{2}{\sqrt{6}}\overline{\Lambda} & \overline{p} & \overline{n} \\ \overline{\Xi^-} & \frac{\overline{\Lambda}}{\sqrt{6}} + \frac{\overline{\Sigma^0}}{\sqrt{2}} & \overline{\Sigma^-} \\ \overline{\Xi^0} & \overline{\Sigma^+} & \frac{\overline{\Lambda}}{\sqrt{6}} - \frac{\overline{\Sigma^0}}{\sqrt{2}} \end{pmatrix}$$

and the free Lagrangian is

$$\mathcal{L}_0 = -i\,\mathrm{Tr}\left\{\overline{\psi}\gamma_\mu\partial_\mu\psi + m\overline{\psi}\psi\right\} + \mathrm{Tr}\left\{\phi(\Box + m^2)\phi\right\}$$

where

$$\mathrm{Tr}(\bar{\Psi}\Psi) = \bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0 + \bar{p}p + \bar{n}n + \bar{\Lambda}\Lambda + \bar{\Sigma}^+\Sigma^+ + \bar{\Sigma}^0\Sigma^0 + \bar{\Sigma}^-\Sigma^-$$

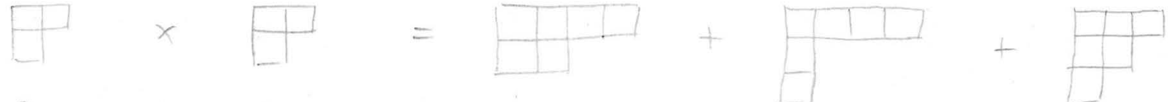
where, of course, the trace is an invariant in SU_3.

To construct invariant trilinear interactions in ψ, $\bar{\psi}$ and ϕ, since there are two independent traces of three matrices, $\mathrm{Tr}\,(\bar{\psi}\,\gamma_5\,\psi\,\phi)$ and $\mathrm{Tr}\,(\bar{\psi}\,\gamma_5\,\phi\,\psi)$, in general we take a linear combination suggested by Gell-Mann, and write the interaction Lagrangian as

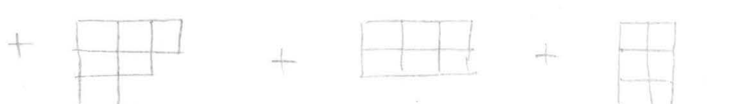
$$\mathcal{L}_{int} = g_f\,\mathrm{Tr}(\bar{\Psi}\gamma_5[\Psi,\phi]) + g_d\,\mathrm{Tr}(\bar{\Psi}\gamma_5\{\Psi,\phi\})$$

and g_f/g_d is called the f/d ratio. Experimentally (see e.g. Martin and Wali (1963) and (1964) and Cutkosky (1963)), the f parameter was determined to be between 0.25 and 0.45, where $f + d = 1$. SU(6) in fact fixes the f/d ratio to be $2/3$ (see e.g. Gürsey, Pais and Radicati (1964)), giving $f = 0.4$.

In the eightfold way, baryon-meson resonances have to be assigned to one of the representations appearing in the product decomposition



$$8 \times 8 = 27 + 10 + 8$$



$$+\ 8 + \overline{10} + 1$$

To accommodate the $N_{3/2,\,3/2}$ resonance, we must use either the 27 or 10 representations, as only these have $I = {}^3/2$ states. The problem was really to decide between these. It was made a lot easier by the fact that, in the representation 10 the weight diagram is triangular, as shown below, and in such representations, the mass formula follows an equal spacing rule with respect to isotopic multiplets (see Glashow and Sakurai, 1962):

N_{33}	1238 Mev.	
Y_1	1385 Mev	
Ξ^*	1532 Mev	(predicted and found)
Ω^-	1679	(do.)

Thus Ξ^* was predicted at 1532 Mev, which agrees with the measured mass, and Ω^- was then predicted at 1679 Mev. It was found (V.E. Barnes et al., 1964) as is well known, at 1676 $\pm$ 20 Mev. This brilliant success confirmed the eightfold way as against the Sakata model (10 does not appear in 3×8) and also 10 as against 27 for fermion resonances.

The other predictions and successes (and failures) of SU(3) will not be discussed here, but they are to be found in the numerous reviews mentioned earlier.

CHAPTER II

SYMMETRY PROPERTIES OF WEAK INTERACTIONS

1. Introductory remarks on weak interactions.

Weak interactions may more or less uniquely be defined as those interactions which cause the decay of spin $\frac{1}{2}^+$ baryons and 0^- mesons with parity violation, and those interactions involving neutrinos. They may be divided into 3 classes:-

(a) Purely leptonic. There is only one example of this type,

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

(b) Leptonic (or semi-leptonic). These involve both strongly interacting particles (hadrons) and leptons. The most well-known example is β-decay:-

$$n \rightarrow p + e^- + \bar{\nu}_e \qquad (1)$$

Strange particle decays also come into this category:-

$$\Lambda \rightarrow p + e^- + \bar{\nu}_e \qquad (2)$$

(c) Non-leptonic. These involve no leptons:-

$$\Lambda \rightarrow p + \pi^- \ .$$

Since we are eventually going to consider what unitary symmetry has to say about weak interactions, and unitary symmetry is concerned only with hadrons, we do not consider class (a) at all. Since, in addition, there are extra difficulties associated with non-leptonic processes, we shall not consider class (c) either. Let us say in passing, though, that unitary symmetry has been applied to non-leptonic processes, with some good success. (Cabibbo (1964), Sugawara (1964), B.W. Lee (1964), and for SU_6, Babu (1965)).

So we consider class (b). Examples (1) and (2) are archetypal as far as selection rules are concerned, since (1) has $\Delta I = 1$, $\Delta S = 0$ and (2) has $\Delta I = \frac{1}{2}$, $\Delta S = 1$, for the participating hadrons. All leptonic decays can be described by one of these two selection rules. In principle, the selection rule $\Delta S = 1$, $\Delta I = 3/2$ is possible, for example in

$$\Sigma^+ \rightarrow n + e^+ + \nu_e$$

$$I_3 = 1 \qquad I_3 = -\tfrac{1}{2}$$

and in 1963 it looked as if this reaction might actually be taking place, but the evidence now is consistent with no such reaction. Also, reactions with $\Delta S = 2$, such as

$$\Xi^- \rightarrow n + e^- + \bar{\nu}_e$$

do not occur, we think. So we are left with the types

$$\Delta I = 1, \quad \Delta S = 0 \qquad (1)$$

$$\Delta I = \tfrac{1}{2}, \quad \Delta S = 1 \qquad (2)$$

In the language of field theory, if we want to write down the simplest hamiltonian, describing say, β-decay, we put

$$H_\beta \sim \bar{\psi}_p \psi_n \bar{\psi}_e \psi_{\nu_e}$$

The next simplest one we put down by analogy with the **vector** nature of electromagnetic interactions, and so, following Fermi, we put

$$H_\beta \sim \bar{\psi}_p \gamma_\mu \psi_n \bar{\psi}_e \gamma_\mu \gamma_5 \psi_\nu \quad + \text{h.c.} \tag{3}$$

The γ_5 is put in to get the right behaviour in the static limit.

2. **(V - A) Fermi interaction.**

This is almost good enough, but not quite. We know that weak interactions violate parity conservation, so, since the **observable** is proportional to $|H|^2$, to get a pseudoscalar observable we must have H composed of the sum of two terms of opposite parity. The interaction

$$H = g_V \bar{\psi}_p \gamma_\mu \psi_n \bar{\psi}_e \gamma_\mu (1+\gamma_5) \psi_\nu + g_A \bar{\psi} \gamma_\mu \gamma_5 \psi_n \bar{\psi}_e \gamma_\mu (1+\gamma_5) \psi_\nu \tag{4}$$

accomplishes this. This is very close to Fermi's interaction, but was arrived at 23 years later! In eq. (4), V and A stand for vector and axial vector, as describing the way the proton and neutron enter the interaction. It would be nice if $g_V = g_A$, and we would then get the so-called V - A interaction. This is not exactly the case, but will be explained later: in fact we have V - 1.2 A approximately.

We call the objects

$\bar{\psi}\, O\, \psi$, where $O = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$ currents,

and we now have the important results that (a) weak interactions are described by a hamiltonian of the form

(hadron current) × (lepton current)

and each current is a combination of vector (γ_μ) and axial vector ($\gamma_\mu\gamma_5$) terms only. In the case of the lepton current the form is exactly V - A, but for the hadrons is V - xA where $x \neq 1$ is caused by renormalisation of A.

Originally one believed x = 1, and the V - A form is famous. At the time (c. 1958, 1959) various theoretical arguments were put forward which lead uniquely to this form of interaction. The most important are (i) chirality invariance (Marshak and Sudarshan, 1958) and (ii) the two component formulation of Dirac spinors by Feynman and Gell-Mann (1958). There is an excellent discussion of these theories in the recent review by Wu (1964). The reason that we mention them here is that there is an interesting new proposal by Zachariasen and Zweig (1964), in connexion with (ii) and the Christenson-Cronin-Fitch-Turlay (1964) observation of $K_2^0 \rightarrow 2\pi$, which seems to imply the breakdown of CP invariance. The point is that the V - A hamiltonian respects CP invariance, but if we include other currents (S, T, P) we may succeed in violating both parity and CP. Now Gell-Mann and Feynman arrived at (V - A) with the reasoning that derivative coupling of two-component spinor fields is forbidden. This is some sort of criterion of elementarity of the particles participating. But if we believe in quarks, we may abandon elementarity at the level of so-far-observed particles, and include S, T, P currents, thus getting CP violation. This is also the basis of Gell-Mann's* U(12) symmetry, based on commutation relations of such quark-currents.

3. <u>The Conserved Vector Current Hypothesis</u>.

(a) Description.

Let us consider again the decays

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

and

$$n \rightarrow p + e^- + \bar{\nu}_e$$

and let them be described by coupling constants g_μ and (g_V, g_A) where $g_{V,A}$ is the coefficient of the (V, A) current. Now, from experiments on the $0^+ \rightarrow 0^+$ transition in O^{14}, g_V in the beta-decay is found to be within 2 per cent of the Fermi constant g_μ of μ-decay. This excellent agreement is not really a blessing, but a puzzle, since in beta decay we expect strong renormalisation effects from the virtual emission and reabsorption of pions and baryons. This renormalisation, of course, is not present in μ decay. To explain this unexpected agreement, Feynman and Gell-Mann (1958) and earlier Gershtein and Zeldovich (1955) proposed the conserved vector current (CVC) theory.

The idea behind this is magnificently simple. Let us think in terms of physical processes. Nucleons can emit and absorb virtual pions such as $n \leftrightarrow n + \pi^0 \longleftrightarrow p + \pi^- \leftrightarrow n + \pi^+ + \pi^- + \ldots$ Therefore, a neutron exists for only a fraction of its lifetime

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* Feynman et al. 1964.

as a bare neutron; the rest of its life it exists as a proton surrounded by a negatively charged pion cloud or as a neutron surrounded by a neutral pion cloud, etc. The neutron in the latter state is called a dressed or physical neutron to differentiate it from a bare neutron. In the old beta decay theory, only the bare nucleon, and not the dressed nucleon, is assumed to undergo beta decay. Therefore, a nucleon undergoes beta decay only for a fraction of its lifetime, and the effective coupling strength of the nucleon must be proportionately reduced or renormalised by the fraction of time spent as a dressed nucleon.

On the other hand, a muon does not have strong interactions. Its Fermi interaction strength needs no renormalisation. Therefore the effective coupling constant in muon decay should equal the intrinsic one. So it was a mystery why the effective strength of the vector couplings in both beta and muon decay were found to be equal within 2 per cent. Feynman and Gell-Mann said that the answer is that the pions carry with them the beta interaction strength when they are virtually emitted from the nucleons

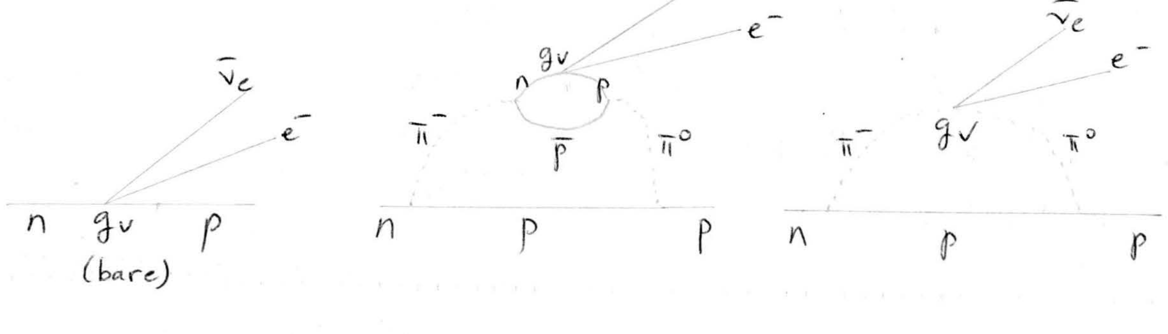


Fig. 1. Feynman diagram of the decay of the physical neutron.

(see Fig. 1) and so the vector part of the nuclear beta interaction is so arranged as to have no renormalisation effect.

(b) Analogy with electromagnetism.

To think up how to actually write down this condition, we realise that there is a direct electromagnetic analogy. Consider the electron and the proton. The electron is a simple object - a Dirac particle with a point charge (except for small radiative corrections) - whereas the proton is a very complicated object containing a meson cloud surrounding a bare nucleon core. Yet the total charge of the proton, which one measures in low energy electron-proton scattering, is the same as the proton charge one would measure if there were no pion interaction. In fact, <u>all</u> interactions are arranged in such a way that the electric charge of the proton is the same whether it is bare or dressed.

How do we achieve this equality in electromagnetism? First, electric charge conservation holds in the process

$$p \longleftrightarrow n + \pi^+$$

i.e. the π^+ has the same charge as the proton. Second, even when the proton is in the "dissociated" state, the interaction of the π^+ with the electromagnetic field is the <u>same</u> as that of the proton. (Fig. 2). Mathematically, the vector potential A_μ couples to the <u>conserved charge current</u> which consists of the <u>sum</u> of the p and π^+ currents. Of course, if the pion interaction with the electromagnetic field were <u>different</u> from the proton interaction, such as happens for the magnetic moment, this conservation law would not hold. So the magnetic moment of the physical proton differs from that of the

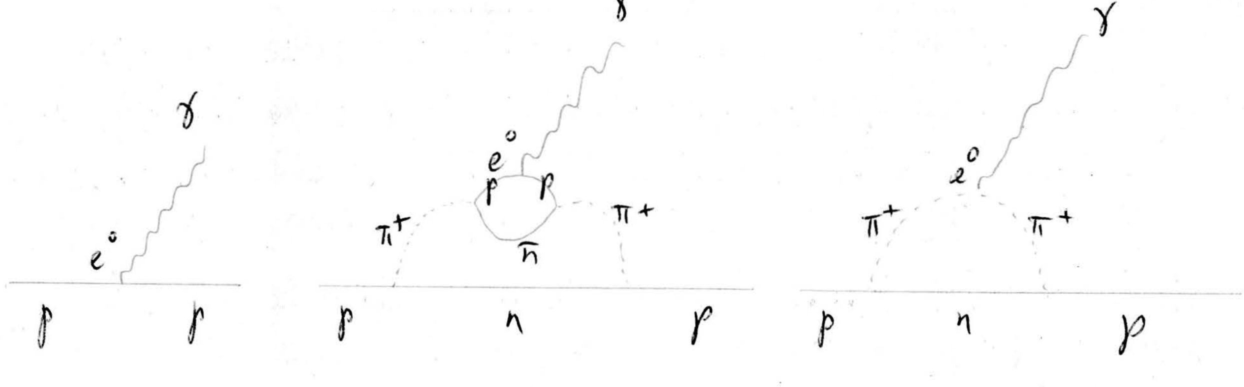


Fig. 2. Feynman diagram of the e - m interaction of the physical proton.

bare proton.

Let us now write all this down in terms of equations. The charge current for a proton is given by the space and time components

$$j_\mu = \{\vec{j}, j_4\} = \{\psi_p^\dagger \vec{\alpha} \psi_p , i\psi_p^\dagger \psi_p\}$$

$$= \{e\frac{\vec{v}}{c} , ie\}$$

$$= \bar{\psi}_p \gamma_\mu \psi_p$$

A neutron, of course, has no charge current. Since $\tau_3 = +1$ for proton and $\tau_3 = -1$ for neutron, we may write the nucleon charge current as

$$j_\mu = \bar{\psi}_N \gamma_\mu \frac{1+\tau_3}{2} \psi_N$$

$$= \frac{1}{2}\bar{\psi}_N\gamma_\mu\psi_N + \frac{1}{2}\bar{\psi}_N\gamma_\mu\tau_3\psi_N$$

$$= j_\mu^S + j_\mu^V$$

decomposing into isoscalar and isovector parts.

Now conservation of this current implies conservation of j_μ^S and j_μ^V separately. Conservation of j_μ^S implies conservation of the number of nucleons. But j_μ^V is not conserved by itself unless we add the pion contribution, i.e.

$$J_\mu^3 = \frac{1}{2}\bar{\psi}_N\gamma_\mu\tau_3\psi_N + (\vec{\pi}\times\partial_\mu\vec{\pi})_3$$

and this corresponds to the Feynman diagram of fig. 2. Note that if we are to derive the conserved current from a Lagrangian and action principle, we must therefore include a pion term in the Lagrangian. Hence we may not regard the pion as just a nucleon-antinucleon bound state (as in the Fermi-Yang model) but must regard it as in some sense "elementary".

(c) Formulation of CVC theory.

For a conventional vector beta interaction, the nucleon current is given by

$$J_\mu^+ = \frac{1}{\sqrt{2}}\bar{\psi}_N\gamma_\mu\tau_+\psi_N \qquad \text{for } \beta^- \text{ decay}$$

where
$$\tau_+ \psi_n = \frac{\tau_1 + i\tau_2}{\sqrt{2}} \psi_n = \sqrt{2}\, \psi_p$$

$$\tau_+ \psi_p = 0 \quad ,$$

and similarly

$$J_\mu^- = \frac{1}{\sqrt{2}} \bar{\psi}_N \gamma_\mu \tau_- \psi_N \qquad \text{for } \beta^+ \text{ decay.}$$

These currents are very similar to the electromagnetic isovector current. In fact, $J_\mu^{V_3}$, J_μ^+ and J_μ^- are the three components of the <u>same</u> isotopic spin current J_μ.

Feynman and Gell-Mann (1958) suggested that, just as for electromagnetism, we must supplement the nucleon current by a pion term, so that we have

$$J_\mu^+ = \frac{1}{\sqrt{2}} \bar{\psi}_N \gamma_\mu \tau_+ \psi_N + (\vec{\pi} \times \partial_\mu \vec{\pi})_+$$

Physically this is equivalent to attributing the same beta-interaction strength to the direct pion-lepton as to the baryon-lepton vertex, as in Fig. 1. Since the strong interactions are charge independent, we have conservation of isotopic spin $\vec{I}$, a generalisation of conservation of charge, i,e, of I_3. So the Feynman - Gell-Mann hypothesis amounts to the assumption that the total isotopic spin current, including nucleon and pion (and any other particles that happen to be around and that have isospin) terms, is conserved.

The analogy between the beta interaction and electromagnetism may be illustrated by the following table (see Wu, 1964).

	Electrodynamics	Vector β interaction
Coupling constant	e	$\frac{1}{\sqrt{2}} g_V$
Current	$J_\mu^{\ 3}$	J_μ^+
Field Potential	A_μ	$\bar{\psi}_e \gamma_\mu (1 + \gamma_5)\psi$
Interaction Hamiltonian	$eJ_\mu^{\ 3}\ A_\mu$	$\frac{1}{\sqrt{2}} g_V J_\mu^+\ \bar{\psi}_e \gamma_\mu (1 + \gamma_5)\psi_{\nu_e}$

4. <u>Can Other Current be Conserved</u>?

(a) Strange vector currents.

The thing we must be careful of is that we always know what we <u>mean</u> when we advocate conserving a particular current. We have just seen that the CVC hypothesis, when applied to the current in beta decay, is equivalent to conservation of the isospin current, and is therefore equivalent to conservation of isospin in <u>strong interactions</u>. This is a beautiful and remarkable connection between strong and weak interactions, and is the approach used by Gell-Mann (1962) in his classic paper on unitary symmetry. If we adopt this philosophy, then it is quite meaningful to talk about conservation of the strange vector current, since strange virtual mesons surrounding, say a Λ particle, will be $\bar{K}^o$, and in analogy with the previous case, CVC will now say that $\bar{K}^o$ has the same coupling to $(e^- \bar{\nu}_e)$ as Λ has. In other words, the $(\Lambda \bar{p})(e^- \bar{\nu}_e)$ coupling constant is renormalised. This, of course, need not be experimentally borne out by the facts, but this is what we mean by a conserved strange vector current.

(b) Axial vector current.

It is a natural question to ask whether we may also conserve the axial vector current. There was, in fact, a lot of talk about this round about 1959.

Let us first notice that in beta decay, the axial coupling constant is larger than the vector coupling constant

$$\frac{g_A}{g_V} \approx 1.2$$

so g_A is not renormalised, but it is tempting to think it almost is, i.e., it is in the limit of something having zero mass.

The condition for its conservation is

$$\langle p' | \partial_\lambda j_\lambda^A(x) | p \rangle = 0$$

or

$$q_\lambda \langle p' | j_\lambda^A(x) | p \rangle = 0$$

$$q = p' - p$$

(i) We can easily show that, if j^A is conserved, pion decay is forbidden. The decay is determined by the matrix element

$$\langle \mu\bar{\nu} | j_\lambda^\dagger j_\lambda^{(L)} | \pi \rangle = \langle \mu\bar{\nu} | j_\lambda^{(L)} | 0 \rangle \langle 0 | j_\lambda^{A\dagger} | \pi \rangle$$

$j_\lambda^{(L)}$ = lepton current

because π is a pseudoscalar.

From Lorentz invariance

$$\langle 0 | j_\lambda^A | \pi \rangle = C q_\lambda$$

where q_λ is the pion momentum. Current conservation then demands

$$0 = \langle 0 | q_\lambda j_\lambda^A | \pi \rangle = C q^2 = -\mu^2 C$$

μ = pion mass

so $C = 0$

or $\mu = 0$.

So we still could have conservation of the axial current in the limit in which the pion mass vanishes. This is the supposed SU(3) symmetry limit of Gürsey, Lee and Nauenberg (1964).

(ii) Splitting up the axial vector current into isoscalar and isovector parts, as we did previously, if we conserve each part, we get conservation of chirality (i.e. "handedness") and

and "isotopic chirality" (see e.g. Nambu, 1962). Then we can "double" our groups and get, for example $SU(3)_L \times SU(3)_R$ where L and R stand for right and left handed spinors. This was proposed by Gell-Mann (1964a), but leads to a doubling in the number of particles predicted - for each positive (negative) parity particle, we predict a negative (positive) parity one. This is not seen experimentally, so the scheme is generally discredited. Also, for the current to be conserved, we require the baryon mass = 0

(iii) What we do in practice is to postulate a "partially conserved axial current" (PCAC), embodied in the famous Goldberger-Treiman relation, which relates the divergence of the axial current to the pion mass. But let us mention that, to achieve an unrenormalised g_A, i.e. $g_A = g_V$, it is not sufficient to assume a conserved axial current naively; we must also have zero nucleon mass (or baryon mass, in general). This was pointed out by Treiman (see Pais and Treiman (1964)).

CHAPTER III.

WEAK INTERACTIONS AND UNITARY SYMMETRY

1. The Idea of Universality.

(a) Discussion.

In the last chapter we considered what sort of interaction Hamiltonian to write down for the weak interactions, and, improving on Fermi's original version, to account for parity violation, we arrived at the form

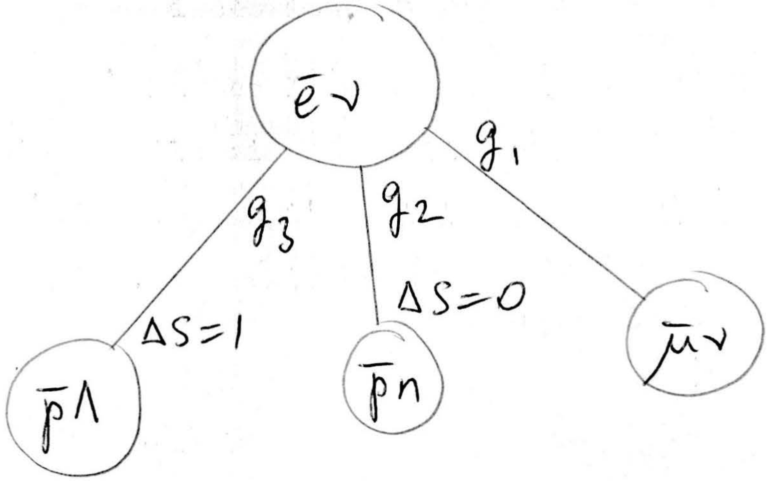
$$\mathcal{H} = g\, J_\mu\, j_\mu^{(L)}$$

where $j_\mu^{(L)} = \bar{\psi}_e\, \gamma_\mu\, (1 + \gamma_5)\psi_\nu$ = (V - A) lepton current

and J_μ = V - xA hadron current.

This is called a current-current interaction. We believe that all weak interactions may be described by a current-current interaction, even if it is only an "effective" one. By this we mean that probably the "true" Hamiltonian (if we believe in Hamiltonians) is not of this form, which becomes very singular at high energies, and so we consider for instance a form $\mathcal{H} = g\, J_\mu\, W_\mu$ where W_μ is an intermediate boson. But if the boson has high mass (which it has - the lower limit is now ~ 1.5 BeV) then we get an effective interaction as above. So now that all interactions are of the same form, we may ask, are they all of the same strength? That is, are all the coupling constants equal? A relation between the coupling constants, the simpler the better, is then equivalent to some sort of <u>universality</u> of the weak interactions.

Let us draw pictorially all the weak interactions we may have



where the things in the circles are schematic, e.g. $\bar{p}n$ means any non-strange combination of particles (current) and $\bar{p}\Lambda$ means a current of strangeness $= \pm 1$. The relevant coupling constants are g_1, g_2 and g_3. g_1 is the purely leptonic coupling constant for μ decay.

Let us summarise what we may say about the vector and axial vector parts of g_1, g_2 and g_3.

(i) Vector part:-

What is g_2 ? -- If CVC holds, then measured g_2 = "real" (unrenormalised) g_2.

(ii) What is g_3 ? -- There is approximate CVC for $\Delta S = 1$ (it is broken because of the large mass differences between strange and non-strange particles). So in an approximate sense we can meaningfully talk about $g_3 = g_1$, etc.

(ii) Axial Vector Part:- Since there is no CVC, we cannot really talk about unrenormalised couplings.

(b) Search for principle that gives relations between g_1, g_2 and g_3.

Let us first note that in practice $g_2^V \sim g_1^V$ as we saw above (it is just this that leads to CVC). Also $\Delta S = 1$ decays are weaker than $S = 0$ decays by a factor of about 5,

i,e, $\dfrac{g_3}{g_2} \sim \dfrac{1}{5}$.

Let us start from the simplest possibility and continue until we reach something which is physically reasonable.

(i) Naive Universal Fermi Interaction (UFI)

$$g_1 = g_2 = g_3 \ .$$

This is flatly contradicted by experiment, but $g_1 \approx g_2$ experimentally.

(ii) Less naive UFI

$S = 1$ processes are outside the universality scheme, and $g_1 = g_2$, g_3 unrelated.

Experimentally, $\sim$ 2 per cent discrepancy between g_1 and g_2.

(iii) SU_3 tells us that $\Delta S = 1$ and $\Delta S = 0$ processes are related, since it relates strange and non-strange particles.

Gell-Mann - Cabibbo hypothesis for UFI:-

(Gell-Mann & Levy, 1958, Cabibbo, 1963)

$$g_1 = \sqrt{g_2^2 + g_3^2}$$

Then if we put $\begin{cases} g_2 = g_1 \cos\theta \\ g_3 = g_1 \sin\theta \end{cases}$

then g_1 and θ are unrelated, and we can determine θ from experiment. We discuss this in the next section.

2. Cabibbo's Theory.

We shall discuss Cabibbo's assumptions and their implications one at a time.

(a) Weak interaction selection rules.

It is observed that all experimentally observed weak interactions are consistent with the selection rules

$$\Delta S = 0 \quad , \quad \Delta I = 1$$

$$\Delta S = \Delta Q = 1, \quad \therefore \quad \Delta I = \tfrac{1}{2}$$

and that

$$\Delta S = - \Delta Q \, , \quad \therefore \quad \Delta I = 3/2$$

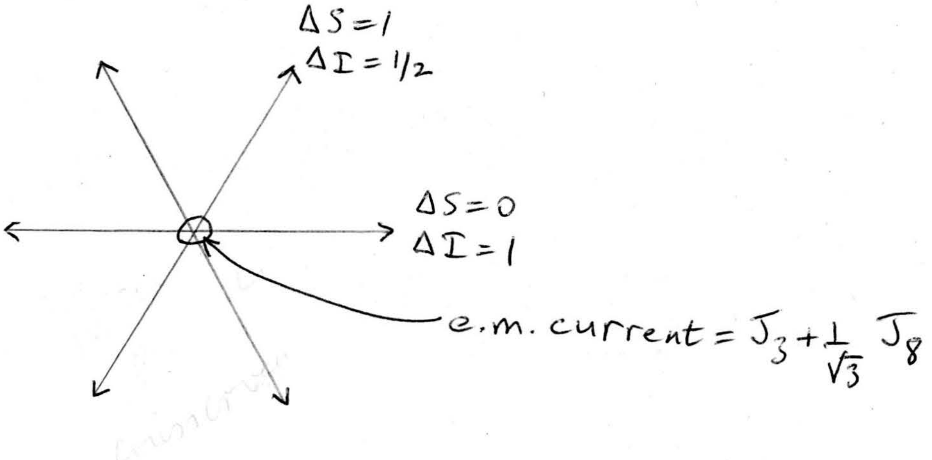
and

$$\Delta S = 2$$

decays do <u>not</u> occur.

Therefore there are two sorts of currents, both charged, with $S = 0$, $I = 1$ and $S = 1$, $I = \frac{1}{2}$. Cabibbo assumes that these currents transform like an <u>octet</u> under SU_3. This assumption is very convenient, since an octet will contain no more charged currents. If $\Delta S = - \Delta Q$ or $\Delta S = 2$ decays do appear, then these currents would have to be assigned to a 10 or 27 representation.

(b) Cabibbo assumes that the vector part of J_μ is in the same octet as the electromagnetic current. This then automatically implies CVC for both $\Delta S = 0$ and $\Delta S = 1$ processes. The vector current octet with them looks like



SU_3 has a very advantageous feature, the famous f/d ratio, which often enables one to fix things up by using the arbitrariness of this ratio. This is precisely what we do here, for determining g_A/g_V.

The point is that a current is made up of two particles, both of which belong to an octet, and so the current may belong to any one of the two octets in the resulting decomposition

$$8 \times 8 = 27 + 10 + \overline{10} + 8_S + 8_a + 1$$

and in general will belong to a linear combination. The two octets may be arranged so that they are respectively symmetric and anti-symmetric in the component particles, as indicated above. So in general we have an amount f of 8_a and an amount d of 8_{symm}. Hence the arbitrary f/d ratio. However, the vector current octet must be consierved, i.e.

$$\partial_\mu J^V_\mu = 0$$

and hence must be an f-type octet. For the axial current, we still have both f and d types. We shall see later that for β-decay

$$\frac{g_A}{g_V} = \frac{f+d}{f} = 1 + \frac{d}{f} .$$

SU(6) determines the f/d ratio and gives $g_A/g_V = 1.67$, which is, of course, too large. One argues that symmetry breaking reduces this, but this suggestion has some of the properties of a pious hope.

(c) The relative strength of $\Delta S = 1$ and $\Delta S = 0$ decays is now determined by an angle θ. Cabibbo assumes that this angle is the <u>same</u> for j^A as for j^V. It is possible to verify this, and the indications are certainly that this is so. If so, it is a remarkable thing, that axial and vector currents point in the same direction in SU(3) space, as it were.

We may find θ_A from comparing the decays $\pi \rightarrow \mu\nu$ and $K \rightarrow \mu\nu$, both of which take place with axial currents only ($0^- \rightarrow$ vac.). In fact

$$\frac{\Gamma(K^+ \to \mu\nu)}{\Gamma(\pi^+ \to \mu\nu)} = \tan^2\theta_A \frac{m_K \left(1 - \frac{m_\mu^2}{m_K^2}\right)^2}{m_\pi \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}$$

and hence $\theta_A = 0.257$.

θ_V is found from the decays $K^+ \rightarrow \pi^0 + e^+ + \nu$ and $\pi^+ \rightarrow \pi^0 e^+ \nu$ both of which are $0^- \rightarrow 0^-$ (Fermi) transitions and so are vector interactions only. So we get (see the next section for details)

$$\theta_V = 0.26$$

and the two angles coincide to within experimental errors.

3. <u>Discussion of Cabibbo's Theory</u>.

(a) The quantitative results of Cabibbo's theory rest on the assumption that the $\Delta S = 1$ currents are not renormalised at all. This would in fact be the case if unitary symmetry were exact. But it is broken, and in particular there is an appreciable $\pi - K$ and $N - \Sigma$ mass difference. Sakurai (1964) has estimated the effect of the breaking, and derived a corrected value of θ, which is smaller than Cabibbo's θ.

(b) In terms of θ, one can now express g_2 ($\Delta S = 0$, $\Delta I = 1$) and g_3 ($\Delta S = 1$, $\Delta I = \frac{1}{2}$) in terms of g_1, and one gets coupling constants of the right order of magnitude. The most important result is

$$g_2 = g_1 \cos \theta$$

which then gives, using Cabibbo's value of θ, a 6 per cent discrepancy between the coupling constants of beta decay and μ decay. This has overshot however, since the actual discrepancy is about 2 or 3 per cent. Using Sakurai's corrected θ (see above) we get the right value for g_2.

In any case, it is extremely interesting that we have a relation between the strangeness conserving and strangeness violating decays, on the one hand, and the purely leptonic μ decay which is not at all related to the strong interactions, on the other hand. Needless to say, the reason for this is not at all understood.

(c) It is important to test separately the vector and axial vector part of the Cabibbo theory, since the assumptions about the axial vector are on less secure grounds. The assumptions are two - (i) that the axial currents really do belong to the same octet. Stated equivalently, that the f/d ratio is the same for the $S = 0$ and $|S| = 1$ axial currents; (ii) that the angle θ is the same for the axial as for the vector octet. Making these assumptions separately, we derive sum rules for already existing experimental branching ratios and coupling constants, to help to decide how good the assumptions are. We discuss this in the next section.

(d) It is interesting that θ is small. Various speculations have been made about what θ would be if unitary symmetry were exact. Of course, there is no guiding principle here, you have to invent your own.

The first thing that comes to mind is the possibility that $\theta = 0$ in the SU(3) limit. This would mean that strange particle decays are forbidden (Oehme and Segré, 1964).

Assuming that neutral currents do not play a part in the octet (they are absent in leptonic decays, but not of course in non-leptonic ones), and also that the photon be placed in the same octet as the hypothetical intermediate boson, Matthews and Salam (1964) have shown that in the exact symmetry limit, we should have $\cos\theta = (1 + \sqrt{3})/2\sqrt{2}$, which agrees with Cabibbo's value.

Using different reasoning, Oehme (1964) makes a case for

$$\theta \sim \frac{m_\pi}{m_k}$$

which is numerically about right. Here, one assumption is that the direction of the current in unitary space is determined by the SU(3) breaking term in the strong Hamiltonian.

4. Tests of Cabibbo's Theory.

For convenience, let us first proceed according to Cabibbo's assumptions. In SU(3), the baryon and antibaryon octets are described by the following matrices

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & -\Xi^0 & \frac{2\Lambda}{\sqrt{6}} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \frac{\bar{\Sigma}^0}{\sqrt{2}} - \frac{\bar{\Lambda}}{\sqrt{6}} & \bar{\Sigma}^- & \bar{\Xi}^- \\ \bar{\Sigma}^+ & -\frac{\bar{\Sigma}^0}{\sqrt{2}} - \frac{\bar{\Lambda}}{\sqrt{6}} & \bar{\Xi}^0 \\ \bar{p} & -\bar{n} & \frac{2\bar{\Lambda}}{\sqrt{6}} \end{pmatrix}$$

The strangeness conserving weak current has $Y = 0$, $I = 1$ and so transforms like Σ^+, i.e. the element J^1_2 of the current octet. Similarly J^1_3 transforms like p with $y = 1$, $I = \frac{1}{2}$ and so is the strangeness changing current. So, according to Cabibbo, the matrix element of the weak current taken between baryon states C and D is

$$\langle D | J_\mu | C \rangle = \langle D | \cos\theta \left\{ (J_\mu^V)^1_2 + (J_\mu^A)^1_2 \right\} + \sin\theta \left\{ (J_\mu^V)^1_3 + (J_\mu^A)^1_3 \right\} | C \rangle$$

We assume that the currents J_μ belong to an octet and so the matrix elements are related to each other by SU(3) Clebsch-Gordan coefficients:

$$\langle B^\alpha_\beta | J^i_j | B^\sigma_\rho \rangle = \lambda\, C^{\alpha i \sigma}_{\beta j \rho} \tag{1}$$

where the C's are the coefficients required to make

$$\left(C^{\alpha i \sigma}_{\beta j \rho}\, J^j_i\, \overline{B}^\beta_\alpha B^\rho_\sigma \right)$$

a unitary scalar and λ is the so-called reduced matrix element. Equation (1) is the Wigner-Eckart theorem. The C-coefficients can be obtained by reference to tables (e.g. de Swart, 1963). Here, however, we shall derive these in a simple way.

We want to contract the indices so as to make $\overline{B}{}^\beta_\alpha B^\rho_\sigma J^i_j$ an invariant. Since the trace of matrices is invariant under SU(3) (this is what S stands for!) we form the traces

$$\mathrm{Tr}\,(J\overline{B}B) \quad \text{and} \quad \mathrm{Tr}\,(JB\overline{B}) \; .$$

Since there are three matrices involved it is clear that these are the only two independent traces. It is more usual to form the following symmetric and antisymmetric combinations:

$$\mathrm{Tr}\,\{J(B\overline{B} \pm \overline{B}B)\}$$

These are called D and F type respectively and the reduced matrix element for these two cases will be denoted by D and F.

For the strangeness conserving decays, we want the matrix element of J^1_2 which goes with $\{(D - F)(B\overline{B})^2 + (D + F)(\overline{B}B)^2\}$, where the explicit expressions for $(\bar{B}B)^2_1$, and $(B\bar{B})^2_1$ are given below:

$$(\bar{B}B)^2_1 = \left\{ \bar{\Sigma}^+\left(\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}}\right) - \left(\frac{\bar{\Sigma}^0}{\sqrt{2}} + \frac{\bar{\Lambda}}{\sqrt{6}}\right)\Sigma^- + \bar{\Xi}^0 \Xi^- \right\}$$

$$(B\bar{B})^2_1 = \left\{ \left(\frac{\bar{\Sigma}^0}{\sqrt{2}} - \frac{\bar{\Lambda}}{\sqrt{6}}\right)\Sigma^- + \bar{\Sigma}^+\left(-\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}}\right) + \bar{p}\, n \right\}$$

Similarly for the strangeness changing current we form

$$\left\{ (D-F)(B\bar{B})^3_1 + (D+F)(\bar{B}B)^3_1 \right\}$$

where

$$(\bar{B}B)^3_1 = \left\{ \bar{p}\left(\frac{\Sigma^0}{\sqrt{2}} - \frac{\Lambda}{\sqrt{6}}\right) - \bar{n}\Sigma^- + \frac{2\bar{\Lambda}}{\sqrt{6}}\Xi^- \right\}$$

$$(B\bar{B})^3_1 = \left\{ \left(\frac{\bar{\Sigma}^0}{\sqrt{2}} - \frac{\bar{\Lambda}}{\sqrt{6}}\right)\Xi^- - \bar{\Sigma}^+\Xi^0 + \frac{2}{\sqrt{6}}\bar{p}\Lambda \right\}$$

So it is a simple matter to write down the following matrix elements:

1. Strangeness-conserving decays

$$\langle p | J^1_2 | n \rangle \sim D + F$$

$$\langle \Lambda | J^1_2 | \Sigma^- \rangle \sim -\frac{2D}{\sqrt{6}}$$

$$\left(\sim \langle \Sigma^+ | J^1_2 | \Lambda \rangle\right)$$

$$\langle \Sigma^0 | J^1_2 | \Sigma^- \rangle \sim \sqrt{2}\, F$$

2. Strangeness-changing decays.

$$\langle p | J_3' | \Lambda \rangle \sim \frac{D + 3F}{\sqrt{6}}$$

$$\langle n | J_3' | \Sigma^- \rangle \sim -D + F$$

$$\langle \Lambda | J_3' | \Xi^- \rangle \sim \frac{D - 3F}{\sqrt{6}}$$

Cabibbo assumes that J^V belongs to the same octet as the electromagnetic current, so is conserved, and so is pure F type. From above, we can see that the coefficient D should be zero, for the isospin current J_2' which is just the step-up operator for isospin cannot induce $\Sigma^- \leftrightarrow \Lambda$ transition which is, from above, pure D type. Also following Cabibbo we introduce the coefficients cos Θ for J_2' and sin Θ for J_3'. Thus we have the following table

Decay	V	A	Branching ratio according to UFI
$n \to p+\ell+\nu$	$\cos\Theta$	$(D+F)\cos\Theta$	
$\Sigma^+ \to \Lambda + \ell^+ + \nu$	0	$-\sqrt{\frac{2}{3}}D\cos\Theta$	
$\Lambda \to p+\ell^- + \bar{\nu}$	$\sqrt{\frac{3}{2}}\sin\Theta$	$\frac{1}{\sqrt{6}}(D+3F)\sin\Theta$	1.4 %
$\Sigma^- \to n+\ell^- + \bar{\nu}$	$\sin\Theta$	$(-D+F)\sin\Theta$	5.1 %
$\Xi^- \to \Lambda + \ell^- + \bar{\nu}$	$-\sqrt{\frac{3}{2}}\sin\Theta$	$\frac{1}{\sqrt{6}}(D-3F)\sin\Theta$	1.4 %

Now let us make more general assumptions, and not assume that the angle θ is the same for vector and axial vector currents, and also not assume that the f/d ratio is the same for the strange as for the non-strange current (i.e. that these currents transform like members of the same octet).

Let us call the coefficients of the symmetric and anti-symmetric terms respectively d and f for the strangeness conserving current, and D and F for the strangeness changing current. Let us in addition represent the relative strength of the strangeness conserving and strangeness changing currents by C_V and S_V for the vector current and C_A and S_A for the axial vector current. Then we may draw up the following table for the leptonic decay matrix elements.

$M^2 \sim |V|^2 + a|A|^2$

Decay	V	A	a
$\Lambda \to p + e^- + \bar{\nu}$	$-\sqrt{\frac{3}{2}}\, S_V$	$-\sqrt{\frac{3}{2}}\left(F + \frac{D}{3}\right) S_A$	2.98
$\Sigma^- \to n + e^- + \bar{\nu}$	$-S_V$	$(D - F)\, S_A$	2.95
$\Sigma^- \to \Lambda + e^- + \bar{\nu}$	0	$\sqrt{\frac{2}{3}}\, d\, C_A$	3.00
$\Xi^- \to \Lambda + e^- + \bar{\nu}$	$\sqrt{\frac{3}{2}}\, S_V$	$\sqrt{\frac{3}{2}}\left(F - \frac{D}{3}\right) S_A$	2.98

In addition

$$\left(\frac{G_V^{n\to p}}{G_V^{\mu}}\right) = C_V \; ; \quad \left(\frac{G_A}{G_V}\right) = (f+d)\,\frac{C_A}{C_V}$$

The decays $K^+ \rightarrow \mu\nu$ and $\pi^+ \rightarrow \mu\nu$ are both axial vector decays, and we may write

$$\frac{\Gamma(K^+ \rightarrow \mu\nu)}{\Gamma(\pi^+ \rightarrow \mu\nu)} = \frac{S_A^2}{C_A^2} \cdot \frac{m_K}{m_\pi} \left(\frac{1 - \frac{m_\mu^2}{m_K^2}}{1 - \frac{m_\mu^2}{m_\pi^2}} \right)^2$$

The decays $K^+ \rightarrow \pi^0 e^+ + \nu$ and $\pi^+ \rightarrow \pi^0 e^+ + \nu$ are both $0^- \rightarrow 0^-$ transitions, and so are due to vector interactions only. Hence we may write the amplitudes as (see, e.g., Dalitz, 1964)

$$\pi^+ \rightarrow \pi^0 e^+ \nu : \quad (p_+ + p_0)_\mu \, j_\mu \langle \pi^+ | J^V_{\Delta S=0} | \pi^0 \rangle C_V$$

$$K^+ \rightarrow \pi^0 e^+ \nu : \quad (p_+ + p_0)_\mu \, j_\mu \langle K^+ | J^V_{\Delta S=1} | \pi^0 \rangle S_V$$

(In the SU(3) limit there is only one form factor in Ke_3 decay).

Now $\langle \pi^+ | J^V_{\Delta S=0} | \pi^0 \rangle = \sqrt{2}\, G^V_\beta / C_V = \sqrt{2}\, G$

and $\langle K^+ | J^V_{\Delta S=1} | \pi^0 \rangle = \frac{\sqrt{2}}{2} G^V_\beta / C_V = \frac{1}{\sqrt{2}} G$

The relative factor of ½ is from the SU(3) Clebsch-Gordan coefficients. Using these matrix elements, we have (see, e.g. Jackson, 1962)

$$\Gamma(\pi^+ \rightarrow \pi^0 e \nu) = \frac{2G^2}{30\pi^3} \Delta^5 \cdot R_0\left(\frac{m_e}{\Delta}\right) \cdot C_V^2$$

where $\Delta = (m_{\pi^+} - m_{\pi^0})$ and R_0 is a recoil factor, here having the value 0.94. Similarly

$$\Gamma(K^+ \to \pi^0 e^+ \nu) = \frac{4G^2}{3072\pi^3} m_K^5 \cdot R_1\left(\frac{m_\pi}{m_K}\right) \cdot S_V^2$$

and the recoil factor R_1 has the value $R_1 = 0.571$.

So

$$\frac{\Gamma(K^+ \to \pi^0 e^+ \nu)}{\Gamma(\pi^+ \to \pi^0 e^+ \nu)} = \frac{S_V^2}{C_V^2}\left(\frac{m_K}{m_{\pi^+} - m_{\pi^0}}\right)^5 \times \frac{5 \times 0.571}{256 \times 0.94}$$

Thus we may compile the following:-

$$(\Lambda) = \frac{\Lambda \to p e \bar{\nu}}{\text{all } \Lambda} = 0.55 \times 10^{-2}\left[S_V^2 + 2.98\left(F + \frac{D}{3}\right)^2 S_A^2\right]$$

$$(\Sigma_1^-) = \frac{\Sigma^- \to n e \bar{\nu}}{\text{all } \Sigma^-} = 1.52 \times 10^{-2}\left[S_V^2 + 2.95(D - F)^2 S_A^2\right]$$

$$(\Sigma_0^-) = \frac{\Sigma^- \to \Lambda e \bar{\nu}}{\text{all } \Sigma^-} = 0.40 \times 10^{-4}\left[3\, d\, C_A^2\right]$$

$$(\Xi^-) = \frac{\Xi^- \to \Lambda e \bar{\nu}}{\text{all } \Xi^-} = 8.55 \times 10^{-3}\left[S_V^2 + 2.98\left(F - \frac{D}{3}\right)^2 S_A^2\right]$$

$$g_1^2 = \left(\frac{G_V^{n \to p}}{G_V^{\mu}}\right)^2 = C_V^2$$

$$g_2^2 = \left(\frac{G_A}{G_V}\right)^2_{n \to p} = (f + d)^2 \frac{C_A^2}{C_V^2}$$

$$\left(\frac{K}{\pi}\right)_1 = \frac{\Gamma(K\to\mu\nu)}{\Gamma(\pi\to\mu\nu)} = \frac{S_A^2}{C_A^2}\left(\frac{m_K}{m_\pi}\right)\cdot\left(\frac{1-\frac{m_\mu^2}{m_K^2}}{1-\frac{m_\mu^2}{m_\pi^2}}\right)^2$$

$$\left(\frac{K}{\pi}\right)_2 = \frac{\Gamma(K^+\to\pi^0 e\nu)}{\Gamma(\pi^+\to\pi^0 e\nu)} = \frac{S_V^2}{C_V^2}\left(\frac{m_K}{m_{\pi^+}-m_{\pi^0}}\right)^5 \times 0.119 \qquad (1)$$

Where the first 4 branching ratios have been derived from the universal Fermi interaction (see e.g. Feynman and Gell-Mann, 1958).

Cabibbo's assumptions are

(i) $C_V^2 + S_V^2 = 1$

(ii) $C_A^2 + S_A^2 = 1$

(iii) $C_V = C_A$

(iv) $f/d = F/D$

In order to derive any relations between the equations (1) we must use at least 3 assumptions. We therefore choose the sets (i), (ii) and (iii), and (i), (ii) and (iv). In other words, we always have an "angle" characterising the relative strength of strangeness conserving and strangeness changing processes. Our two sets of assumptions are (a) the same angle for vector and axial vector currents (b) the same f/d ratio for strangeness-changing and strangeness conserving axial vector currents.

(a) <u>Same angle</u>. We get the following sum rules:-

$$\sqrt{\frac{\dfrac{(\Sigma_1^-)}{q\,s_v^2}-1}{m}}+2\sqrt{\frac{\dfrac{(\Xi^-)}{s\,s_v^2}-1}{l}}=\sqrt{\frac{\dfrac{(\Lambda)}{p\,s_v^2}-1}{l}} \qquad (2)$$

$$\frac{\left(\frac{K}{\pi}\right)_1}{f_1(m)}=\frac{\left(\frac{K}{\pi}\right)_2}{f_2(m)}=\frac{1-g_1^2}{g_1^2} \qquad (3)$$

using the notation of equation (1) and

$$p = 0.55\times10^{-2}$$
$$q = 1.52\times10^{-2}$$
$$s = 8.55\times10^{-3}$$
$$s_v^2 = 1-g_1^2$$

$$f_1(m)=\left(\frac{m_K}{m_\pi}\right)\left(\frac{1-\frac{m_\mu^2}{m_K^2}}{1-\frac{m_\mu^2}{m_\pi^2}}\right)^2=17.60$$

$$f_2(m)=\left(\frac{m_K}{m_{\pi^+}-m_{\pi^0}}\right)^5\times0.0119$$
$$=1.693\times10^8$$

Using Willis's results (1964), equations (2) and (3) give

$$LHS = 3.06\,[1.18], \quad RHS = 0.95\,[1.04] \qquad (2)$$

$$7.08\begin{pmatrix}7.86\\6.50\end{pmatrix}\times10^{-2}=5.73\begin{pmatrix}6.69\\4.98\end{pmatrix}\times10^{-2}=5.15\begin{pmatrix}7.30\\3.09\end{pmatrix}\times10^{-2} \qquad (3)$$

where the figures in brackets are consistent with the experimental errors. The square roots of the first two terms of equation (3) are the tangents of the Cabibbo angles, giving

$$\tan\theta_V = 0.27\ , \quad \tan\theta_A = 0.24$$

(For equation (3), the lifetimes and branching ratios were taken from Roos's tables (1963) and the branching ratio for $\pi^+ \rightarrow \pi^0 e^+ \nu$ from Wu (1964)).

(b) <u>Same f/d ratio</u>. We get the sum rules

$$4lk\left[\frac{(\Sigma_1^-)}{q} - g_1^2 - 1\right] = 3m(1-k)^2\left[\frac{(\Lambda)}{p} - \frac{(\Xi^-)}{s}\right] \qquad (4)$$

$$g_1^2 = \frac{f_2(m)}{f_2(m) + \left(\frac{K}{\pi}\right)_2} \qquad (5)$$

where the symbols are as before and

$$(1+k) = g_1 g_2 \sqrt{\frac{1.20 \times 10^{-4}}{(\Sigma_0^-)}}$$

Experimentally, these give

$$LHS = 0.21\,[0.02]\,, \quad RHS = -0.24\,[+0.54] \qquad (4)$$

$$LHS = 0.951 \pm 0.023\,, \quad RHS = 0.946 \pm 0.008 \qquad (5)$$

Where again the figures in brackets are those consistent with the experimental errors. The large errors arise in general because of the occurrence of the term $1 - g_1^2$, whose proportionate error is very large.

CHAPTER IV.

WIGNER'S SUPERMULTIPLET THEORY AND ITS EXTENSION TO PARTICLE PHYSICS.

1. Supermultiplet Theory.

The physics of elementary particles is a natural extension of atomic and nuclear physics, and started from the study of the mesons which glue nucleons tightly together to form nuclei. So it is indeed fitting that SU(6), the new much-talked-about group of elementary particles, should be a simple extension of Wigner's group SU(4) for nuclear supermultiplets; Wigner's theory, of course, is highly non-trivial, and a very great idea.

We shall discuss the physical motivations and content of SU(4), and for this we go backwards in time about 30 years. The very first step in the direction of so-called "internal" symmetries was taken by Heisenberg, who suggested that the neutron and proton are just two states of the same particle, the nucleon. One said that in "isospin" space, the neutron has third component $-\frac{1}{2}$ and the proton $+\frac{1}{2}$, and the nucleon defines an isotopic multiplet. Now, in nuclear forces, there existed (and still exists!) the property of charge independence, which says

$$V_{pp} = V_{nn} = V_{pn} \quad \text{in the same state.}$$

For instance, for the ^1S state, this relation applies, whereas for the ^3S state it is meaningless, since the Pauli exclusion principle does not permit two identical nucleons to be in a ^3S state. In our modern group theoretical language, the principle of charge independence can be stated by saying that the nuclear force is isospin independent, i.e. does not depend on the "orientation" of the nucleon in isospin space.

Now if we jump ahead and assume, as Yukawa suggested, that nuclear forces are due to exchange of "heavy" mesons, then we can state the assumption of isospin independence in terms of our mesons, and their interaction at the nucleon vertex. We can then attempt to verify charge independence by studying the pion-nucleon vertex by scattering at high energy. This has been done by now, of course, but in those days was not possible.

So what was wanted was some way of verifying charge independence <u>without</u> studying meson interactions. It was at this point that Wigner (1937) suggested that we can verify whether or not isospin is conserved in the nuclear multiplets. This is independent of any model of strong coupling, and in particular of the meson hypothesis. The point is that we assume isospin independence, and we already have spin independence of nuclear interactions, so $I(I + 1)$, $J(J + 1)$, I_3 and J_3 are all good quantum numbers, and the nucleon levels we observe are eigenstates of these operators. (Of course, isospin invariance is broken by the Coulomb interaction, just as spin invariance is broken by spin-orbit coupling.)

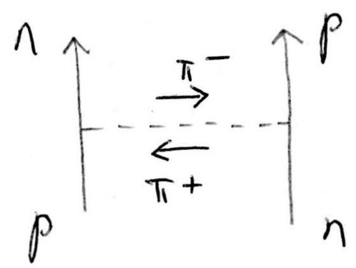
Let us now study the levels of the nucleon ($I = \frac{1}{2}$, $J = \frac{1}{2}$). We label the states by (I_3, J_3)

$p\uparrow$	$p\downarrow$	$n\uparrow$	$n\downarrow$
$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(-\frac{1}{2},\frac{1}{2})$	$(-\frac{1}{2},-\frac{1}{2})$

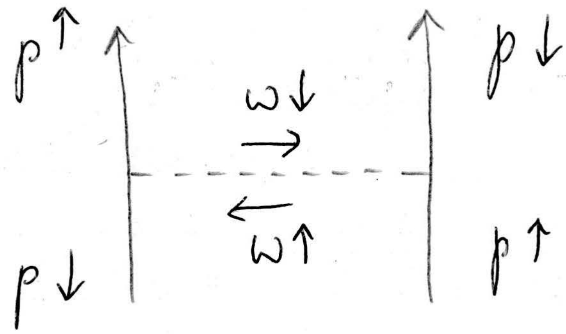
The brackets mark I-spin and J-spin multiplets and we have invariance under the group $(SU(2))_I \times (SU(2))_J$. But if we did not know what was going on and just looked at the nucleon, we would <u>observe</u> 4 (almost) degenerate states, and the group of invariance we would guess at would be SU(4), the group of the 4 dimensional harmonic oscillator. We should then classify our states according to the irreducible representations of SU(4), and see what success we have. Recent evidence on this is reported by Franzini and Radicati (1963).

If the group of invariance <u>is</u> SU(4), this corresponds to <u>more</u> than conservation of I-spin and J-spin separately. To see this, let us state the case in terms of mesons.

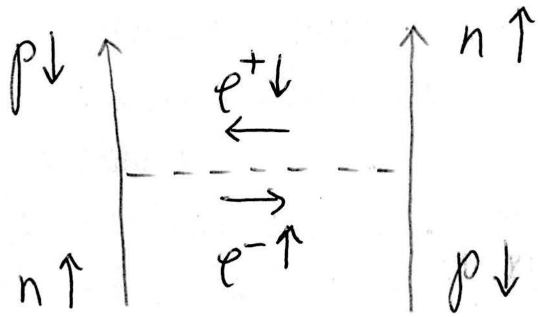
I-invariance will be achieved by a meson which changes a neutron into a proton, but leaves the spin invariant, as shown. This is a π-meson.



J-invariance will be achieved by a meson which flips the spin but does not change the charge, i.e. a neutral spin one meson



If π and ω mesons both contribute to the nuclear force in an invariant way, we have invariance under$(SU(2)_I) \times (SU(2))_J$. But suppose now that we have a meson which has both charge and spin, and thus can change a spin up neutron into a spin down proton:-



then this operation will belong to the group SU(4) only, which can interchange all possible combinations of spin and isospin, but <u>not</u> to $(SU(2))_I \times (SU(2))_J$. If all these mesons are coupled with equal strength, we have invariance under SU(4). We may draw a table of the different nucleon forces, and the exchanged particle to which they correspond.

Force	Spin flip	Charge flip	Meson	(I, J)
Wigner	No	No	η	(0, 0)
Bartlett	Yes	No	ω	(0, 1)
Heisenberg	No	Yes	π	(1, 0)
Majorana	Yes	Yes	ρ	(1, 1)

2. Extension to Elementary Particles

These considerations are readily extended to include strange particles. The isospin group SU(2) is then extended to SU(3). This was first realised by Gürsey & Radicati (1964) and independently by Sakita (1964a). Instead of proton and neutron, above, visualise 3 "basic" particles, an isodoublet and isosinglet, with $J = \frac{1}{2}$

$$\begin{pmatrix} D^+ \\ D^0 \\ S^0 \end{pmatrix}$$

There are now 6 degenerate states and our group is SU(6).

3. Mesons in SU(4) and SU(6).

We have already discussed the role of mesons in SU(4). Let us now count the number of states the mesons occupy. For a meson of spin J and isospin I, this is $(2J + 1)(2I + 1)$. So we have

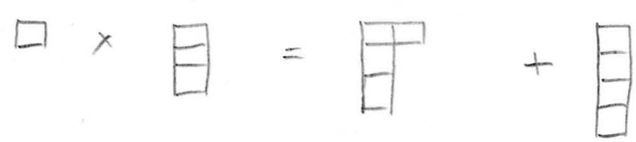
		J	I	N = (2J+1)(2I+1)	
Bartlett	ω	1	0	3	
Heisenberg	π	0	1	3	
Majorana	ρ	1	1	9	
				15	dim.
Wigner	η	0	0	1	dim.

There are 16 ($=4^2$) meson states, which split into 15 and 1 under SU(4). We may see this also by forming nucleon-antinucleon combinations and lebelling the $SU(2)_I \times SU(2)_J$ multiplicity (2I+1, 2J+1).

I-spin	2×2	=	$3 + 1$
J-spin	2×2	=	$3 + 1$

$$\therefore \quad SU(4) \qquad (2,2) \times (2,2) = (3,1) + (1,3) + (3,3) + (1,1)$$
$$= 3 + 3 + 9 + 1$$
$$= 15 + 1 \ .$$

The decomposition may also be done with Young diagrams (see appendix for details of these)



$4 \times \bar{4} = 15 + 1$

For SU(6) we proceed exactly similarly, using SU(3) decompositions in baryon-antibaryon states:

$$3 \times \bar{3} = 8 + 1$$

$$2 \times 2 = 1 + 3$$

$$\therefore \qquad 6 \times \bar{6} = (1,1) + (8,1) + (1,3) + (8,3)$$

so we get two octets and two singlets of spin 0 and 1. As usual, the (1,1) state will be a singlet (trace) under SU(6) and will belong to a different irreducible representation. So we have

$$6 \times \bar{6} = 35 + 1 \; .$$

The parity must be - (baryon-antibaryon in S-wave state) so 35 is composed of

1^- nonet

0^- octet

and we get a natural explanation of the well-known 9 vector and 8 pseudo-scalar mesons. The pseudoscalar singlet χ (960 Mev) belongs to a different representation of SU(6), and this perhaps explains why it doesn't mix with η, as ϕ does with ω, a problem well-known in SU(3) decays.

4. Symmetry Breaking for the Mesons.

The usual mass formula for mesons in SU(3) is

$$\mu^2 = \mu_o^2 + a \; I(I+1) - \frac{Y^2}{4}$$

Now let us introduce Heisenberg forces which split the π and K from ρ and K^*. Let us adjust the mass formula correspondingly by adding a Casimir operator for spin, so

$$\mu^2 = \mu_o^2 + a \quad I(I+1) - \frac{y^2}{4} \quad + bJ(J+1)$$

Consider $\mu^2_{K^*}$ and μ^2_K Same I, Y, different J

μ^2_ρ and μ^2_π do. do.

Subtract,

$$\mu^2_K - \mu^2_\rho = \mu^2_K - \mu^2_\pi$$

and this equation is satisfied to 1 per cent. It was known to be true in SU(3) days, but was regarded as mysterious.

5. Baryons in SU(6).

If we believe that the 6 dimensional representation of SU(6) is a quark state, then to get a physical baryon, we must combine 3 quarks together. In SU(3)

$$3 \times 3 \times 3 = (6 + \bar{3}) \times 3$$

$$= 10 + 8 + 8 + 1$$

and we see that 10 is totally symmetric in the 3 threes, 1 totally antisymmetric. In SU(6), we must decompose $6 \times 6 \times 6$. The totally symmetric state contains

$$N_S = \frac{n(n+1)(n+2)}{3!} \quad \text{states}$$

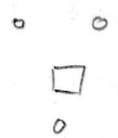
where $n = 6$, $N_S = 56$

Similarly $N_A = \dfrac{n(n-1)(n-2)}{3!} = \dfrac{n!}{3!(n-3)!}$

$$N_A = 20 \ .$$

Which do we choose?

By the Pauli exclusion principle, S is antisymmetric, i.e. repulsive, in space coordinates, and A is symmetric, i.e. attractive. So A seems the obvious choice. But for this, the quarks would need to be fractionally charged. We can avoid this (which is a feature of Gell-Mann's model (1964)) if we have a boson "core" as in the appendix to Gürsey, Lee and Nauenberg (1964) - see also Lee (1965). This core can also provide attraction for the mutually repelling quarks and allow us to have the 56 representation. Then a physical baryon has the appearance



o — o repulsive

o — □ attractive

and we are back to Bohr's atomic model! Ultimately, we choose 56 because it gives the better SU(3) and SU(2) content:-

56	=	(10, 4)	+	(8, 2)
		spin $^3/_2$		spin $\frac{1}{2}$
		decuplet		octet

which fits exactly with the observed particles. 20 gives

20	=	(8, 2)	+	(1, 4)
		spin $\frac{1}{2}$		spin $^3/_2$
		octet		singlet.

The symmetry breaking for the baryons is again represented by a simple generalisation of the SU(3) mass formula to

$$M = M_o + a \quad I(I+1) - \frac{y^2}{4} \quad + 6y \; + cJ(J+1)$$

which gives

$$\Xi^* - \Xi = Y_1^* - \Sigma$$

which is correct to 2 per cent. Of course, these forms for the mass formula are by no means unique. The mass formula is discussed further by Pais (1964), Kuo and Yao (1964) and Beg and Singh (1964a, b).

6. Resonances in SU(6).

For the product decomposition of SU(6) representations, we refer to the appendix. Meson baryon resonances will belong to

$$56 \times 35 = 70 + 56 + 700 + 1134$$

$$\left.\begin{array}{ll} \downarrow & \\ (8,\ 4) & - \text{ found} \\ (10,\ 2) & \\ (8,\ 2) & \\ (1,\ 2) & - \text{ found} \end{array}\right\} \text{parity } -$$

Some of the states belonging to 70 have been found, as indicated. These are discussed by Pais (1964).

Dyson and Xuong (1964) have classified Y = 2 states. From

$$56 \times 56 = 462 + 1050 + 1134 + 490$$

and the fact that Y = 2 fermions must belong to an antisymmetric state, the deuteron must belong to 490 or 1050. Dyson & Xuong choose 490 and from the mass formula, identify other members of the multiplet. They also consider Y = 0 states. In general, resonances can be fitted well into SU(6), but we refer to the literature for details.

7. Concluding Remarks.

The spin part of SU(6), which is SU(2), is usually thought of as being represented by the Pauli σ matrices, but this in fact only represents the spin in the rest frame of the particle. The problem of finding a covariant spin operator which reduces to σ in the rest frame of the particle is a difficult one. There have been various attempts, but the one we shall adopt is Gürsey's solution, which is dealt with in the next chapter. Using this definition of spin, the free Lagrangian is invariant, and part of the interaction Lagrangian is invariant also, so that SU(6) is a good symmetry in the conventional sense of the word symmetry. In, for example, Salam's scheme, the free Lagrangian is not invariant and so, on boosting, particles change from one representation to another. This can be used perhaps to explain the appearance of more particles with increasing energy, but is not our philosophy.

CHAPTER V

THE POINCARE GROUP AND THE COVARIANT DEFINITION OF SPIN.

1. The Poincaré Group

Whatever may be the fate of symmetries such as SU(3), or any other "internal" symmetry scheme, the Poincaré (inhomogeneous Lorentz) group is, we believe, of fundamental importance, since its operators are Lorentz transformations, and we believe that all laws of physics must be Lorentz covariant, i.e. they must have the same form in different frames which can be reached one from the other by a Lorentz transformation. The Poincaré group defines a symmetry, which is a purely kinematic one. While it is true, we believe, that the kinematic symmetries continue to play a role of great importance, it is not true that they form a closed book; the amount of new results relating to the Poincaré group which have been obtained in the past few years is evidence of this. Also, it is not absolutely certain that it is the Poincaré group which plays the fundamental role, and Wigner and Philips (1962) and Gürsey (1963, 1964), (Gürsey & Lee, 1963) have done some very interesting work on the de Sitter group, which breaks down to the Poincaré group in the limit of flat space-time. Also the interpretation of reflection generators which "extend" the Poincaré group has raised a number of questions relating to measurability (Wigner, 1964). It is also worth mentioning that Wigner's theory of types under these operations(1939, 1964) has lead Tarimer (1965) to a space-time theory of quarks, and thence of internal quantum numbers.

It will be in order to make some remarks about the group we are considering. First, we are not including reflections of space and time, that is, we treat the restricted Poincaré group, and secondly, the fact that the group is inhomogeneous means that we also consider, apart from pure Lorentz ("boost") transformations and rotations, translations of the origin. It is only by virtue of including translation operators (which are generated by momentum) that we can define spin. Also, from the point of view of interactions, translation invariance is only compatible with local interactions. For instance the Coulomb interaction e^2/r_{12} is not invariant under $r_{12} \rightarrow r_{12} + a$, and we must replace this interaction by two vertices, each of which is translationally invariant, and between which is propagated a photon.

Finally, in the way of introductory remarks, a very relevant one is that the Lorentz group, both in its homogeneous and inhomogeneous version, is non-compact. This is a mathematical term which corresponds to the physical observation that by doing a series of Lorentz transformations along the hyperbola in $(\vec{x}, t)$ space, you never arrive back at your starting point, since you keep on speeding up. This is to be contrasted, for example, with rotations in a plane, where after rotating through 2π, you are again at the starting point. Rotation groups are compact, whereas Lorentz groups are non-compact. The very important consequence of this is that unitary representations are infinite dimensional. Thus they cannot be represented by finite dimensional matrices. There are representations, of course, using finite dimensional matrices, but they are not unitary. We shall note the importance of these representations further on.

2. The Wigner Representation of the Poincaré Group.

This representation is the most well-known to physicists, and indeed leads to the greatest physical information. We shall not discuss it in detail here, for it is a task which must be done either with complete thoroughness or not at all. We will merely sketch the results and for details refer to the original monumental work of Wigner (1939) and also to his lectures at Istanbul in 1962 (1964), which contain a summary of the 1939 paper. Other summaries of the Poincaré group are to be found in Wightman (1960), Wigner (1963) and Macfarlane (1963).

We must first distinguish between the two groups we shall deal with - the "actual" Poincaré (i.e. transformations of 4-vectors etc., in space-time) and the quantum mechanical Poincare group, which induces (unitary) transformations on state vectors in Hilbert-space (i.e. "wave functions"). Let us deal with these one at a time.

(a) The restricted Poincaré group.

Consider coordinate space, with real coordinates $x_0 \ldots x_3$, $x_0 = ct$, $g = +1,-1,-1,-1$. A pure Lorentz transformation along the x-axis is now

$$\left.\begin{aligned} x_0' &= \gamma x_0 - \beta\gamma x_1 \\ x_1' &= \gamma x_1 - \beta\gamma x_0 \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}\right\} \quad \beta = v/c, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{1}$$

From (1), $\gamma^2 - \gamma^2\beta^2 = 1$

But $\cosh^2 u - \sinh^2 u = 1.$

So we may put
$$\cosh u = \gamma$$
$$\sinh u = \gamma\beta$$
$$\tanh u = \beta = \frac{v}{c} = \text{"rapidity"}$$

and we have

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \cosh u & -\sinh u & & \\ -\sinh u & \cosh u & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x' = \Lambda \qquad x$$

and it will be noted that for this pure (boost) Lorentz transformation, Λ is symmetric. A general Lorentz transformation contains rotations, for which the matrix is not symmetric, but whose off-diagonal elements are $\sin\theta$ and $-\sin\theta$, for rotation through an angle θ. A boost transformation may be regarded as a rotation through an imaginary angle u, and so a general (homogeneous) Lorentz transformation may be viewed as a rotation through a complex angle $(\theta + iu)$.

The Poincare group includes translations, so let us denote the product of the displacement by the 4-vector a and the Lorentz transformation Λ by (a, Λ). The group equation then reads

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2) \tag{2}$$

whose physical interpretation is easy enough.

Now the four dimensional vector a can be uniquely characterised also by the hermitian matrix $h(a)$.

$$h(a) = \begin{pmatrix} a_t - a_z & a_x + ia_y \\ a_x - ia_y & a_t + a_z \end{pmatrix}$$

and the determinant of h is then just the Lorentz length of the vector a.

Now consider the transformation

$$Bh(a)B^+ = h(a')$$

If $\det B = 1$, then B transforms the vector a linearly into a vector a', leaving the length of a unchanged. This defines a Lorentz transformation, and corresponds to

$$\Lambda a = a'.$$

In fact, B defines the group $SL(2, C)$ where S = special = unimodular, L = linear, 2 = 2 dimensional, C = complex transformations, and $SL(2, C)$ bears a 2 to 1 homomorphism with the Lorentz group, since B and $-B$ transform a into the same a'. In terms of h and B, equation (2) now reads

$$(h_1, B_1)(h_2, B_2) = (h_1 + B_1 h_2 B_1^+, B_1 B_2) \qquad (3)$$

and we write equation (3) in symbolic form, using a set of equations. We write

$$(h, B) = [h]\,(B)$$

then

$$
\begin{aligned}
[h_1][h_2] &= [h_1 + h_2] \\
(B_1)(B_2) &= (B_1 B_2) \qquad (4) \\
(B)[h] &= [BhB^+](B)
\end{aligned}
$$

Also we note that for a boost transformation

$$B(\Lambda^T) = B(\Lambda)^+$$

(b) <u>The Quantum Mechanical Restricted Poincaré Group</u>.

Corresponding to the symmetry transformation (a, Λ) there is a unitary or antiunitary $O(a,\Lambda)$ which transforms the quantum mechanical description of any state ψ into the description $\psi' = O(a,\Lambda)\psi$. The fact that O is unitary or anti-unitary comes from conservation of transition probability:-

$$(\psi, \phi)^2 = (\psi', \phi')^2$$

and if we do not include discrete reflections, O can always be made unitary. Actually, O is only determined up to a phase, but this phase can be proved to be ± 1.

So we have

$$O(a_1, \Lambda_1)O(a_2, \Lambda_2) = \pm O(a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2)$$

and corresponding to equation (4) we have

$$
\left.
\begin{aligned}
T_{h_1} T_{h_2} &= T_{h_1 + h_2} \\
L_{B_1} L_{B_2} &= L_{B_1 B_2} \\
L_B T_h &= T_{BhB^+} L_B
\end{aligned}
\right\} \qquad (5)
$$

where L stands for Lorentz transformation and T for translation.

(c) Representations

We now find explicit forms for L and T. They will be unitary, and so infinite dimensional, as mentioned before.

As we introduced a 2 x 2 matrix corresponding to translations, let us now introduce a similar matrix for momentum

$$p = \begin{pmatrix} p_t + p_z & -p_x - ip_y \\ -p_x + ip_y & p_t - p_z \end{pmatrix} \qquad (6)$$

It will be noticed that the signs are reversed from the definition of h(a). p is the h which corresponds to the contragradient vector $G\rho$.

$$p = h(G\rho) \quad .$$

The wave functions will now depend on p plus a discrete variable.

The mathematical meaning of p is connected with the operators T_h which correspond to pure displacements. These form an abelian invariant subgroup. We put

$$T_{h(a)}\,\psi(p) = \exp\left[\, i(a_t p_t - a_x p_x - a_z p_z - a_z p_z)\right]\psi(p)$$

This formula is intuitively obvious, when we observe that $p_x \sim \frac{\partial}{\partial x}$ etc., but its rigorous establishment requires some careful justification (Wigner, 1939) which we omit here.

Instead of $\psi(p)$, let us introduce the notation $|p, \zeta\rangle$, where ζ is a necessary variable, since there may be several (in fact there may be an infinite number of) state vectors which transform the same way under displacements. We have

$$T_{h(a)} | p, \zeta > \ = \ \exp\left[i(p_t a_t - \vec{p}.\vec{a})\right] | p, \zeta >$$

Now

$$\mathrm{Trace}\left[h(a)p \right] \ = \ 2(p_t a_t \ - \ \vec{p}.\vec{a})$$

So introducing the notation

$$< hp > \ = \ \tfrac{1}{2}\, \mathrm{Trace}\,(hp)$$

we have

$$T_h | p, \zeta > \ = \ \exp(i < hp >) | p, \zeta > \qquad (7)$$

(This was of course why we introduced a contravariant, and not a covariant, p).

We have now dealt with translations. Their effect on a state vector is expressed by equation (7). All we need to do now is find out what happens to $| p, \zeta >$ under a homogeneous Lorentz transformation, L_B - then we have solved the Poincare group! The way to proceed is easily seen, principally because it is the only thing we can do! We use equations (5), especially the third one. Let us do this, and apply T_h to $L_B | p, \zeta >$ to see how it transforms

$$\begin{aligned} T_h (L_B | p, \zeta >) &= T_h L_B | p, \zeta > \\ &= L_B T_{B^{-1} h B^{t-1}} | p, \zeta > \\ &= L_B \exp(i < B^{-1} h B^{t-1} p >) | p, \zeta > \\ &= \exp(i < B^{-1} h B^{t-1} p >) L_B | p, \zeta > \end{aligned}$$

where the last step is valid because the exponential is a coefficient and L_B is linear. Since the symbol $\langle \ldots \rangle$ indicates a trace, we may shift the first factor to the end, and get

$$T_h (L_B | p, \varsigma\rangle) = \exp(i\langle h B^{+^{-1}} p B^{-1}\rangle) L_B | p, \varsigma\rangle \tag{8}$$

So we have solved our problem already - or almost! Comparing equations (6) and (7), we see that $L_B | p, \varsigma\rangle$ transforms under T_h as the vectors $| B^{+^{-1}} p B^{-1}, \varsigma\rangle$, and is therefore a linear combination of these:

$$L_B | p, \varsigma\rangle = \sum_{\eta} A_\eta | B^{+^{-1}} p B^{-1}, \eta\rangle \tag{9}$$

It is probably no exaggeration to say that this equation is the most important in the whole of physics, and why, we shall see in a few lines. It is essentially at the bottom of Feynman rules for calculating graphs, and we know that these lay the foundations of particle physics.

The key to the problem is to ask the question, what are the A_η? They are not difficult to find, and the best derivation is in Wigner (1964). The outcome is that they are <u>rotation matrices</u>. This is remarkable! To know how states transform under the Poincare group, we do not need to know anything except the group of 3 dimensional rotations, SU_2 - which we know anyway! Let us add a word here - the A_η are <u>only</u> rotation matrices for states with <u>positive time-like momenta</u>. This is important, as we shall see later.

This rotation group is called the little group. Physically It is the subgroup of the Lorentz group that leaves a particular momentum invariant. For states with positive time-like momenta, it is SU_2. For states with light-like momenta it is the Euclidean group E_2, i.e. the group of all translations and rotations in 2 dimensions. For states of space-like momenta, it is the 2 + 1 Lorentz group (i.e. with metric + + -).

It is easy to verify explicitly that the little groups are these. We want to find what further restrictions our matrices B have, in order that a particular momentum component be invariant under the group.

Apart from an overall metric, p of equation (6) may be written

$$p_0 + \vec{p}\cdot\vec{\tau} = p_0 + p_1\tau_1 + p_2\tau_2 + p_3\tau_3 .$$

We may easily prove the following:-

(a) $\quad (p_0 + \vec{p}\cdot\vec{\tau})^{\sim} = p_0 + p_1\tau_1 - p_2\tau_2 + p_3\tau_3$

(b) $\quad (p_0 + \vec{p}\cdot\vec{\tau})^{-1} = p_0 - p_1\tau_1 - p_2\tau_2 - p_3\tau_3$

(c) $\quad \tau_2(p_0 + \vec{p}\cdot\vec{\tau})\tau_2^{-1} = p_0 - p_1\tau_1 + p_2\tau_2 - p_3\tau_3$

since (a) $\tilde{\tau}_1 = \tau_1, \quad \tilde{\tau}_2 = -\tau_2, \quad \tilde{\tau}_3 = \tau_3,$

(b) $p_0^2 - \vec{p}\cdot\vec{p} = 1$ by hypothesis.

Hence if we put $p_0 + \vec{p}\cdot\vec{\tau} = B$, a general 2 x 2 complex matrix, it obeys

$$(\tilde{B})^{-1} = \tau_2 B \tau_2 \qquad (\text{since } \tau_2 = \tau_2^{-1}) \qquad (10)$$

(i) Time-like momentum.

Put $p = p_o$, $\vec{p} = 0$

then $$C_1 \, p_o \, C_1^+ = p_o$$

where C_1 is the particular Lorentz matrix which leaves p_o invariant

$$\therefore \quad C_1 C_1^+ = 1 \tag{11}$$

C_1 is already unimodular, so now it is unitary also and C_1 belongs to SU(2).

(ii) Space-like momenta

Put $p_o = p_1 = p_3 = 0$

then $C_2 p_2 \tau_2 C_2^+ = p_2 \tau_2$

by hypothesis.

$$\therefore \quad \tau_2 C_2 \tau_2 C_2^+ = 1$$

and substituting from (10)

$$C_2^+ = C_2$$

$$\text{i.e. } C_2 = C_2^* \tag{12}$$

i.e. we have the group of real 2 x 2 linear transformations. Looking at our original matrix B, it is just the group that leaves, in this case, $p_o^2 - p_1^2 - p_3^2$ invariant, i.e. the 2 + 1 Lorentz group.

3. <u>Spin-noncovariant definition</u>.

We must now write down what are the generators of the group, and their commutation relations. To find these we consider infinitesimal transformations.

Let us first consider translations:-

$$x'_\mu = x_\mu + a_\mu$$

so adopting the convention of writing a ket in configuration space,

$$|x_\mu + a_\mu\rangle = |x_\mu\rangle + a_\mu \frac{\partial}{\partial x_\mu} |x_\mu\rangle$$

Putting

$$\frac{\partial}{\partial x_\mu} = i p_\mu \quad (\hbar = 1)$$

then

$$|x_\mu + a_\mu\rangle = |x_\mu\rangle + i a_\nu p_\nu |x_\mu\rangle$$

$$= (1 + i a_\nu p_\nu)|x_\mu\rangle$$

and p_μ is the generator of translations. For a finite transformation,

$$|x_\mu + a_\mu\rangle = e^{i a_\nu p_\nu} |x_\mu\rangle$$

Changes in p_μ are exactly analogous. We put

$$p_\mu \rightarrow \lambda_{\mu\nu} p_\nu$$

$$\lambda_{\mu\nu} = g_{\mu\nu} + \omega_{\mu\nu}$$

and

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

So the transformation of Hilbert space vectors is

$$|p_\mu + \omega_{\mu\nu} p_\nu\rangle = |p_\mu\rangle + \omega_{\alpha\beta}\, p_\alpha \frac{\partial}{\partial p_\beta} |p_\mu\rangle$$

$$= \left[1 + \tfrac{1}{2}\omega_{\alpha\beta}\left(p_\alpha \frac{\partial}{\partial p_\beta} - p_\beta \frac{\partial}{\partial p_\alpha}\right)\right] |p_\mu\rangle$$

and the generator (for the case of no spin, as above) of 4-momentum transformations is

$$J_{\mu\nu} = \tfrac{1}{2}\left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}\right)$$

and in general one adds a term $\Sigma_{\mu\nu}\ (= -\Sigma_{\nu\mu})$ corresponding to the existence of spin. A finite transformation is

$$|p'_\mu\rangle = e^{\frac{1}{2}\Omega_{\mu\nu}\left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}\right)} |p_\mu\rangle$$

$$= e^{\frac{1}{2}\Omega_{\mu\nu} J_{\mu\nu}} |p_\mu\rangle$$

in general. $J_{\mu\nu}$ is an angular momentum.

We now write down commutation relations between these generators. Physically speaking, these relations express what happens when we perform two operations in succession.

The commutation relations are

$$\begin{aligned}
[J_{\kappa\lambda}, J_{\mu\nu}] &= i\left(\delta_{\lambda\mu} J_{\kappa\nu} - \delta_{\kappa\mu} J_{\lambda\nu} + \delta_{\kappa\nu} J_{\lambda\mu} - \delta_{\lambda\nu} J_{\kappa\mu}\right)\\
[J_{\kappa\lambda}, P_\mu] &= i\left(\delta_{\lambda\mu} P_\kappa - \delta_{\kappa\mu} P_\lambda\right)\\
[P_\lambda, P_\mu] &= 0
\end{aligned} \tag{12}$$

We have changed our notation, mainly because of the worry of upper and lower indices. Here the metric tensor $g_{\mu\nu} = \delta_{\mu\nu}$, $\mu, \nu = 1, \dots 4$ and $x_4 = i x_0$.

Let us now split these up. We put

$$P_4 = iP_o \qquad P_o \text{ hermitian} = \text{energy}$$

$$J_{4n} = i\,K_n \quad n = 1,2,3 \qquad \text{pure Lorentz transformation (boost).}$$

$$J_{\ell m} = \varepsilon_{\ell mn}\, J_n \quad \ell, m, n = 1,2,3 \qquad \text{pure rotation.}$$

The commutation relations now become

	Comment
$[J_k, J_\ell] = i\,\varepsilon_{k\ell m}\, J_m$	Pure rotations form a subgroup.
$[J_k, K_\ell] = i\,\varepsilon_{k\ell m}\, K_m$	'Boost' transformations form a vector under rotations.
$[K_k, K_\ell] = -i\,\varepsilon_{k\ell m}\, J_m$	2 boost transformations = 1 rotation (Thomas precession)
$[J_k, P_\ell] = i\,\varepsilon_{k\ell m}\, P_m$	$\vec{p}$ behaves as a vector under rotations.
$[J_k, P_\ell] = 0$	P_o behaves as a scalar.
$\left.\begin{array}{l}[K_k, P_\ell] = -i\,P_o\,\delta_{k\ell} \\ [K_k, P_o] = -i\,P_k\end{array}\right\}$	By a pure Lorentz transformation, $\vec{P} \leftrightarrow P_o$.

Let us now introduce the operator *

$$W_\kappa = \frac{1}{2i}\,\varepsilon_{\kappa\lambda\mu\nu}\, J_{\lambda\mu}\, P_\nu \tag{13A}$$

---

* See Bargmann and Wigner (1948). The definition of this operator is attributed to Pauli by Lubański. For a recent review see Fradkin and Good (1961).

We may now easily convince ourselves that

$$W_\lambda P_\lambda = 0 \tag{14}$$

$$[W_\lambda, P_\mu] = 0 \tag{15}$$

Also the operators

$$\left.\begin{aligned} M^2 &= -P_\lambda P_\lambda \\ \text{and} \qquad C^2 &= W_\lambda W_\lambda \end{aligned}\right\} \tag{16}$$

<u>commute</u> with <u>all</u> the operators J_μ and P_μ. They are the <u>Casimir operators</u>. In addition

$$[W_\rho, W_\sigma] = \varepsilon_{\rho\sigma\lambda\mu} W_\lambda P_\mu \tag{17}$$

All this is quite general. Let us now make a few distinctions. M^2 from equation (16) is not positive definite, so in general M (its square root) is not hermitian. Physically, this means that a particle may not have a <u>real</u> mass - we know from our childhood days that exchanged particles have imaginary mass. According to the values of M^2 and P_o, we now divide Hilbert space into 3 parts, as follows.

<u>$\mathcal{H}^I$</u>. $\mathcal{H}^I$ is that subspace of Hilbert space which contains states for which M^2 is positive definite (i.e. M is hermitian) and P_o is positive definite* (i.e. momenta are time-like). This corresponds clearly, to states for "real" particles with mass.

---

* i.e. $P_o \neq 0$. Solutions to the wave equation for which $P_o < 0$ may be transformed to those with $P_o > 0$ by a PCT transformation.

$\mathcal{H}^{II}$. Is the subspace for which $M^2 = 0$ and $P_o > 0$ (positive definite). So momenta are light-like. This clearly corresponds to massless particles.

$\mathcal{H}^{III}$. is the subspace for which M^2 is negative definite. This corresponds to virtual particles.

Note that the 3 different little groups we obtained above, are just the little groups corresponding to these 3 subspaces of Hilbert space.

We will now consider $\mathcal{H}^{I}$ further. The analysis that follows, leading to the definition of spin, is taken from Gursey (1965c).

In $\mathcal{H}^{I}$

M^{-1} and P_o^{-1} are both defined. $(M + P_o)^{-1}$ is also well defined.

Let $L_{\mu\nu}$ be a set of quantities that commute with P_μ and W_λ, and let $L_{\mu\nu}$ form an orthogonal matrix:-

$$L_{\mu\nu} L_{\rho\nu} = \delta_{\mu\rho}$$

$L_{\mu\nu}$ is the Lorentz matrix that takes P_μ to a particular frame, in this case the rest frame.

Introduce

$$\begin{aligned} P_\lambda' &= L_{\lambda\mu} P_\mu \\ W_\lambda' &= L_{\lambda\mu} P_\mu \end{aligned} \tag{18}$$

Comparing with equations (14) - (17) we now have

$$[P_\lambda', P_\mu'] = 0 \tag{19}$$

$$[P_\lambda', W_\mu'] = 0 \tag{20}$$

$$[W_\rho', W_\sigma'] = \epsilon_{\rho\sigma\lambda\mu} W_\lambda' P_\mu' \tag{21}$$

$$P_\lambda' W_\lambda' = 0 \tag{22}$$

since commutation relations are independent of any particular frame.

Also
$$\begin{aligned} M^2 &= - P_\lambda' P_\lambda' \\ C^2 &= W_\lambda' W_\lambda' \end{aligned} \tag{23}$$

In general, $L_{\mu\nu}$ is, of course, a function of P_λ,

$$L_{\mu\nu} = L_{\mu\nu}(P_\lambda) \quad .$$

Since we are in $\mathcal{H}^{I}$, we can choose

$$\vec{P'} = 0 \tag{24}$$

so from (24), (22) and (23) we see

$$W_o' = 0 \tag{25}$$

$$P_o' = M \tag{26}$$

$$C^2 = W_\lambda' W_\lambda' = \vec{W}'^2 - W_o'^2 = \vec{W}'^2 \tag{27}$$

Define $$\vec{S} = M^{-1} \vec{W}' \tag{28}$$

then $$C^2 = M^2 \vec{S}^2 \tag{29}$$

Let us find the commutation of $\vec{S}$, using equations (28), (21) and (25):-

$$\begin{aligned} [W_1', W_2'] &= \epsilon_{12\mu\nu} W_\mu' P_\nu' \\ &= \epsilon_{1234} (W_3' P_4' - W_4' P_3' \\ &= MW_3' \\ &M^2 S_3' \end{aligned}$$

But L.H.S. $=$ $M^2 [S_1', S_2']$

$$\therefore \quad [S_1, S_2] = i S_3 \quad \text{and cyclic} \tag{30}$$

and $\vec{S}$ defines the <u>spin</u>.

The operators S_i from an SU(2) group, which is the same little group that we arrived at before for time-like momenta. Since $\vec{S}$ generate SU(2), then

$$\vec{S} \cdot \vec{S} = s(s+1) \quad \text{where } s = \tfrac{1}{2}, 1, {}^3/2, \ldots.$$

So from (29)

$$\underline{C^2 = M^2 s(s+1)} \tag{31}$$

So the two Casimir operators define mass and spin. Note also that

$$[S_k, P_\lambda] = 0 \tag{32}$$

so S_k is translationally invariant.

We must now find an explicit expression for S_i.

We write the elements of $L_{\mu\nu}$ as

$$\left.\begin{aligned} L_{k\ell} &= \delta_{k\ell} + \frac{P_k P_\ell}{M(M+P_o)} \\ L_{n4} &= -L_{4n} = i\frac{P_n}{M} \\ L_{44} &= -i\frac{P_4}{M} = \frac{P_o}{M} \end{aligned}\right\} \tag{33}$$

So

$$P_k' = L_{k\mu} P_\mu = L_{kn} P_n + L_{k4} P_4 = 0$$

$$P_4' = L_{4n} P_n + L_{44} P_4 = iM .$$

We may find S_k from W_k' as follows:-

$$\begin{aligned} S_k &= M^{-1} W'_k \\ &= M^{-1} L_k \; W \\ &= M^{-1} (L_{k\ell} W_\ell + L_{k4} W_4) \\ &= M^{-1} \left(W_k + \frac{P_k(\vec{P}.\vec{W})}{M(M+P_o)} - \frac{P_k W_o}{M}\right) \end{aligned}$$

and now from equation (14) written as $\vec{W}.\vec{P} = W_o P_o$, we get

$$S = \frac{1}{M}\left(\vec{W} - \frac{W_o \vec{P}}{M+P_o}\right) \tag{34}$$

We may also express $\vec{W}$ in terms of $\vec{J}$ and $\vec{K}$, and so get, using $W_i = P_o J_i + \varepsilon_{ijk} K_j P_k$,

$$\vec{S} = \vec{J}\,\frac{P_o}{M} - \frac{\vec{K} \times \vec{P}}{M} - \frac{(\vec{J}.\vec{P})\,\vec{P}}{M(M+P_o)} \tag{35}$$

Equation (34) is the usual form for the relativistic spin operator (see, e.g. Macfarlane (1963)).

4. Spin and the Little Groups for Arbitrary Momentum.

Let us now sum up the situation. First, we restrict ourselves to that subspace of Hilbert space where M^2 is positive definite and P_o is positive, i.e., in terms of particles, to positive energy particles of time-like momentum. Having noted this restriction, we can find a spin operator $\vec{S}$ defined by equation (34), which obeys the commutation relations of SU(2), equation (30), and is translationally invariant, equation (32). These are all necessary requirements for a good spin operator. In addition, our two Casimir operators, which are multiples of the identity for a fixed representation of the group, just define mass and spin (or, more accurately, mass and mass x spin) as we see from equations (16) and (31).

Let us now define more exactly the relation to the little groups. The S_i of equation (34) generate the spin group that induces the little group transformation on each state with definite time-like momentum. Note that this little group definition holds for each momentum separately. Normally the little group is defined as a subgroup of L which leaves a <u>definite</u> momentum invariant, so that other momenta are <u>not</u> invariant; this transformation leaves <u>each one</u> invariant.

We shall now deal briefly with space-like momenta.

(a) <u>Little Groups for Space-like Momenta</u>.

Of course, we know what to expect - the 2 + 1 Lorentz group.

For a space-like momentum we can choose $P_o = P_1 = P_2 = 0$ and the remaining component is non-zero.

Proceeding in the same spirit as before, we postulate the existence of an $L_{\mu\nu}(P_\lambda)$ such that

$$\begin{aligned} P'_4 &= P'_1 = P'_2 = 0 \\ P'_3 &= -iM \qquad (>0) \end{aligned} \tag{36}$$

Now, as in equation (22),

$$P'_\lambda W_\lambda' = 0 \quad \therefore \quad W_3' = 0 \tag{37}$$

So W_λ' has non-zero components

W_4' antihermitian

W_1' hermitian

W_2' hermitian

and is time-like.

Now $M^2 < 0$ $\therefore$ M is antihermitian, so iM is hermitian.

Let us define 3 hermitian operators

N_1, N_2, R_3 such that

$$\left\{\begin{aligned} W_1' &= iMN_1 \\ W_2' &= iMN_2 \\ W_4' &= MR_3 \end{aligned}\right. \tag{38}$$

using equation (21), i.e.

$$[W_k', W_\lambda'] = \varepsilon_{K\lambda\mu\nu} W'_\mu P'_\nu$$

we get

$$\begin{aligned} [W_1', W_2'] &= \varepsilon_{12\mu\nu} W'_\mu P'_\nu \\ &= \varepsilon_{1243} W'_4 P'_3 \\ &= - W_4' (-iM) \end{aligned} \tag{38a}$$

using equations (36) and (37), so (38a) is

$$-M^2 [N_1, N_2] = i M^2 R_3 .$$

By a similar process with the other W''s, we find the commutation relations

$$\left.\begin{aligned} [N_1, N_2] &= - i R_3 \\ [R_3, N_1] &= i N_2 \\ [R_3, N_2] &= - i N_1 \end{aligned}\right\} \qquad (39)$$

So N_1, N_2 and R_3 form a closed algebra under commutation. They therefore can be made generators of a group. Further, since

$$[W'_K, P_\lambda] = 0 \qquad (20) \& (18)$$

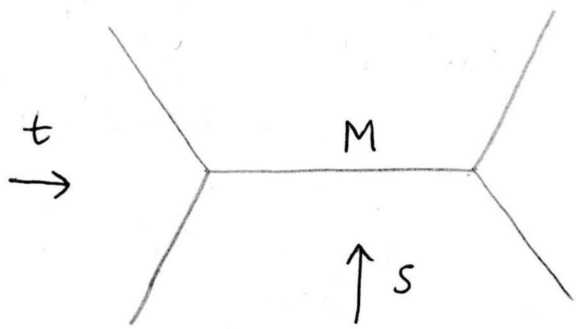
then

$$\left.\begin{aligned} [N_1, P_\lambda] &= 0 \\ [N_2, P_\lambda] &= 0 \\ [R_3, P_\lambda] &= 0 \end{aligned}\right\} \qquad (40)$$

and N_1, N_2 and R_3 generate the little group for space-like momenta. By comparing the commutation relations of N and R, e.g. (39) with those of K and J, equation (13), we see that N_1 and N_2 are boosts along the axes 1 and 2, and R_3 is a rotation in the 1-2 plane. They therefore generate the 2 + 1 Lorentz group, as we saw before.

We have now arrived at a curious conclusion, that the little group for space-like momenta is the 2 + 1 Lorentz group, and is therefore non-compact. This means that its unitarity representations are infinite dimensional, and therefore we cannot associate spin with these momenta; or we can say that this is a case of "infinite spin", in the usual terminology. This term gives us no physical insight, though!

Having obtained the little groups for states characterized by time-like and space-like momenta, let us consider a scattering process, in particular the simplest prototype whose Feynman graph is drawn below.



If we look in the t-channel at physical energy, the particle M is "real" and has time-like momentum. Its little group is therefore SU(2) and corresponds to the fact that the particle is in an eigenstate of $s(s+1)$ and S_3 - i.e. we may measure its spin. This we all know - as the particle is time-like, so of course we may measure its spin. But let us now cross to the s-channel. Here, for physical energy, the particle M has space-like momentum and since its little group, the 2 + 1 Lorentz group, is non-compact, we may not associate any discrete eigenvalue with the particle, so we cannot measure its spin. Or, rather, it is meaningless to talk about its spin as belonging to an irreducible representation of SU(2). In general it will belong to a mixture of representations of SU(2) - it can have any spin.

What do we mean, then, when we say that an exchanged particle has spin 0, or 1, or whatever else? We mean that by analytic continuation from the s to the t channel, where the little group now becomes SU(2), we may measure its spin. Crossing changes the momentum of M from time-like to space-like (and vice versa); and the corresponding little group from SU(2) to L(2,1). If we look at the commutation relations (39) for L(2,1), we see that they differ from SU(2) commutation relations (e.g. equation (30)) only by a sign. This indicates that SU(2) and L(2,1) have the same complex algebra, and this complex little group will be the little group of the complex Lorentz group, which we get from the Lorentz group by imposing locality, as we do in field theory. Taking the real and imaginary parts of the little group, we get SU(2) and L(2,1) and so from the complex Lorentz group we obtain time- and space-like momenta on the same footing. We may obtain the complex Lorentz group, for example, by letting the four-momentum p_λ take on complex values, as in S-matrix theory and $p_\lambda \, p_\lambda = -m^2$ will then give energy and momentum conservation for the intermediate states.

5. Spin-covariant Definition.

The operator S of equation (34) is hermitian, obeys SU(2) commutation relations, is translationally invariant, but is not covariant. This may be seen from the fact that its transformation under pure Lorentz transformations is given by

$$[K_j, S_\ell] = i \, \frac{\delta_{j\ell} \, \vec{S}.\vec{P} - S_j \, P_\ell}{M + P_0} \qquad .$$

But also the states $|\vec{p}, s_3\rangle$, corresponding to the eigenvalues of $\vec{P}$ and S_3 do not transform simply under the homogeneous

Lorentz group, since the K (boost) operators take us from one eigenvalue of S_3 to another. The states which do transform simply are those transforming like the representations (S, 0) or (0, S) of the homogeneous Lorentz group. These are obtained from the former states as follows.

$$\psi_L(\vec{p}) = e^{\vec{S}\cdot\vec{\lambda}(\vec{p})} \psi_W(\vec{p}) \qquad (41)$$

where
$$\psi_W(\vec{p}) = \begin{pmatrix} |\vec{p}, s\rangle \\ |\vec{p}, s-1\rangle \\ \vdots \\ |\vec{p}, -s\rangle \end{pmatrix}$$

are the Wigner states which transform simply (irreducibly) under the little group (see Shaw 1964). The states ψ_L (p) (L = left) transform like the (S, 0) representation of the homogeneous Lorentz group, while

$$\psi_R(\vec{p}) = e^{-\vec{S}\cdot\vec{\lambda}(\vec{p})} \psi_W(\vec{p})$$

$$= e^{-2\vec{S}\cdot\vec{\lambda}(\vec{p})} \psi_L(\vec{p})$$

transform like (0, S). $\vec{\lambda}(\vec{p})$ is given by

$$\vec{\lambda}(\vec{p}) = \frac{\vec{p}}{|\vec{p}|} \tanh^{-1} \frac{|\vec{p}|}{p_0} \qquad (42)$$

and so we see from the equations following equation (1) that $\exp(\vec{s}.\vec{\lambda}\ (p))$ is a pure Lorentz transformation.

To $\vec{S}\,\psi_W(p)$ corresponds the covariant state

$$e^{\vec{S}.\vec{\lambda}(\vec{p})}\,\vec{S}\,\psi_W(\vec{p}) = \vec{X}\,\psi_L(\vec{p})$$

where

$$\vec{X}(\vec{p}) = e^{\vec{S}.\vec{\lambda}(\vec{p})}\,\vec{S}\,e^{-\vec{S}.\vec{\lambda}(\vec{p})} \tag{43}$$

It follows that

$$(\vec{X})^{\dagger} = e^{-\vec{S}.\vec{\lambda}(\vec{p})}\,\vec{S}\,e^{\vec{S}.\vec{\lambda}(\vec{p})} \tag{44}$$

So we now have an operator $\vec{X}$ which trasnforms covariantly, obeys SU(2) commutation relations, is translationally invariant, but is not hermitian, equations (43) and (44), at least not at first sight. But we must be more careful, since we must always define a scalar product with respect to which an operator is hermitian, and although $\vec{X}$, eq. (43) is not hermitian with respect to the usual (Wigner) scalar product, it is hermitian with respect to the scalar product written below in equivalent forms

$$(\psi,\psi) = \int \psi_W^{\dagger}(p)\,\psi_W(p)\,d^3p$$

$$= \int \psi_L^{\dagger}(p)\,e^{-2\vec{S}.\vec{\lambda}(\vec{p})}\,\psi_L(\vec{p})\,d^3p$$

$$= \int \psi_R^\dagger(p)\, e^{+2\vec{S}\cdot\vec{\lambda}(\vec{p})}\, \psi_R(p)\, d^3p$$

$$= -\frac{1}{2} \int (\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)\, d^3p$$

(45)

This was shown by Shaw (1964). (See also Weinberg (1964) and Joos (1962)).

Thus we have, for the expectation value of spin

$$(\psi, \vec{S}\psi) = \int \psi_W^\dagger \vec{S} \psi_W\, d^3p$$
$$= \int \psi_L^\dagger e^{-\vec{S}\cdot\vec{\lambda}} \vec{S} e^{-\vec{S}\cdot\vec{\lambda}} \psi_L\, d^3p$$

$$= \frac{1}{2} \int (\psi_R^\dagger \vec{X} \psi_L + \psi_L^\dagger \vec{X} \psi_R)\, d^3p \qquad (46)$$

which is hermitian with respect to the scalar product defined by (45). Also the mean values $(\psi, \vec{S}\,\psi)$ are constants of the motion.

Let us now evaluate $\vec{X}$ from equation (43) explicitly. First, from equation (13A), the orbital part of J does not contribute to W, so

$$W_K = \frac{1}{2i}\, \varepsilon_{K\lambda\mu}\, S_{\lambda\mu}\, P$$

$$W_4 = \frac{1}{2i}\, \varepsilon_{4\lambda\mu}\; S_{\lambda\mu}\; P$$

$$= -\frac{1}{2i}\, \varepsilon_{ijk}\, S_{ij}\, P_k$$

$$= -\frac{2}{2i}\,(S.P) \;=\; -\frac{1}{i}\,(S.P) \qquad (47)$$

$$\therefore \quad W_o = \frac{1}{i} W_4 = (\vec{S}.\vec{P}) \qquad (48)$$

where (47) follows because if $J_{\ell m} = \varepsilon_{\ell mn} J_n$, then $\varepsilon_{\ell mk} J_{\ell m} = 2J_k$.

Let us put $\vec{\lambda} = \frac{\vec{P}}{P} \tanh^{-1} \frac{P}{P_o}$

$$\vec{\lambda} = \frac{\vec{P}}{P} \cdot \theta \qquad (49)$$

where $P = |\vec{P}|$, $\theta = \tanh^{-1} P/P_o$

$$\therefore \quad \sinh\theta = P/M \;, \quad \cosh\theta = P_o/M \qquad (50)$$

$$\therefore \quad \vec{S} \cdot \vec{\lambda} = \frac{\vec{S}.\vec{P}}{P}\,\theta = \frac{W_o}{P}\,\theta \qquad (51)$$

So, from equation (43)

$$X_i = \exp\left(\frac{W_o}{P}\theta\right) S_i \exp\left(-\frac{W_o}{P}\theta\right)$$

$$= \left(1 + \frac{W_o}{P}\theta + \frac{1}{2!}\frac{W_o^2}{P^2}\theta^2 + \cdots\right) S_i \left(1 - \frac{W_o}{P}\theta + \frac{1}{2!}\frac{W_o^2}{P^2}\theta^2 + \cdots\right)$$

$$= S_i + \frac{\theta}{P}[W_o, S_i] + \frac{1}{2!}\frac{\theta^2}{P^2}[W_o,[W_o,S_i]]$$

$$+ \frac{1}{3!}\frac{\theta^3}{P^3}[W_o,[W_o,[W_o,S_i]]] \qquad (52)$$

We must now evaluate the commutators.

$$C_1 = [W_o, S_i] = \frac{1}{M}[W_o, W_i] \qquad \text{from equations (34) \& (15)}$$

$$= \frac{-i}{M}[W_4, W_i]$$

$$= \frac{-i}{M}\varepsilon_{4ijk} W_j P_k \qquad \text{from equation (17)}$$

$$= \frac{1}{M}\varepsilon_{ijk} W_j P_k \tag{53}$$

$$C_2 = [W_o, C_1] = \frac{i}{M}[W_o, \varepsilon_{ijk} W_j P_k]$$

$$= \frac{i\varepsilon_{ijk}}{M}[W_o, W_j] P_k \qquad \text{from equation (15)}$$

$$= \frac{i\varepsilon_{ijk}}{M} i\varepsilon_{jmn} W_m P_n P_k \qquad \text{from above}$$

$$= -\frac{1}{M}\left[(\vec{W} \cdot \vec{P}) P_i - (P)^2 W_i\right]$$

$$= \frac{1}{M}\left[P^2 W_i - W_o P_o P_i\right] \qquad \text{from equation (14)} \tag{54}$$

$$C_3 = [W_o, C_2]$$

$$= \frac{P^2}{M}[W_o, W_i]$$

$$= P^2 C_1 \tag{55}$$

Similarly,

$$C_4 = P^2 C_2, \quad C_5 = P^2 C_3 = P^4 C_1, \quad \text{etc.} \tag{56}$$

Substituting (53) - (56) into equation (52) we get

$$X_i = S_i + \frac{\theta}{P} C_1 + \frac{\theta^2}{2!P^2} C_2 + \frac{\theta^3}{3!P^3} C_3 + \frac{\theta^4}{4!P^4} C_4 + \cdots\cdots$$

$$= S_i + \frac{C_1}{P} \sinh \theta + \frac{C_2}{P^2} (\cosh \theta - 1)$$

$$= S_i + \frac{1}{M} C_1 + \frac{1}{M(P_o+M)} C_2$$

from equation (50),

$$= \frac{1}{M^2}\Big[W_i P_o - W_o P_i + [W_o, W_i]\Big] \tag{57}$$

using (53) and (54) .

Now let us evaluate

$$\varepsilon_{ijk}[W_j, W_k] .$$

From (17)

$$[W_j, W_k] = \varepsilon_{jk\mu} W_\mu P$$

$$= \varepsilon_{jk\ell 4} (W_\ell P_4 - W_4 P_\ell)$$

$$= -i\, \varepsilon_{jk\ell} (W_\ell P_o - W_o P_\ell)$$

$$\text{so } \varepsilon_{ijk}[W_j, W_k] = i\, \varepsilon_{ijk}\, \varepsilon_{jk\ell} (W_\ell P_o - W_o P_\ell)$$

$$= -2i (W_i P_o - W_o P_i)$$

$$\therefore (W_i P_o - W_o P_o) = \frac{i}{2} \varepsilon_{ijk}[W_j, W_k] \tag{58}$$

Substituting (58) into (57) we get

$$X_i = \frac{i}{M^2}\Big\{ \frac{i}{2} \varepsilon_{ijk}[W_j, W_k] + i[W_i, W_4] \Big\} \tag{59}$$

$$X_i = \frac{1}{M^2}\Big\{ \frac{1}{2} \varepsilon_{ijk} \; W_j, W_k + [W_i, W_4] \Big\}$$

We may now relate this to the self-dual part of $[W_\mu, W_\nu]$, as follows;

Put
$$X_{\mu\nu} = W_{\mu\nu} + W_{\mu\nu}^D$$
$$= W_{\mu\nu} + \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} W_{\rho\sigma} \tag{60}$$

where
$$W_{\mu\nu} = \frac{i}{M^2}[W_\mu, W_\nu] \tag{61}$$

then, in analogy with the electromagnetic field tensor $F_{\mu\nu}$, $X_{\mu\nu}$ will only have 3 independent components; the self-dual and anti-self-dual parts of $F_{\mu\nu}$ are the electric and magnetic fields, each three-vectors.

$$X_{jk} = \frac{i}{M^2}\left\{[W_j, W_k] + \tfrac{1}{2}\varepsilon_{jk\rho\sigma}[W_\rho, W_\sigma]\right\}$$
$$= \frac{i}{M^2}\left\{[W_j, W_k] + \varepsilon_{jk\ell}[W_\ell, W_4]\right\}$$

so

$$X_i = \tfrac{1}{2}\varepsilon_{ijk} X_{jk}$$
$$= \frac{i}{M^2}\left\{\frac{1}{2}\varepsilon_{ijk}[W_j, W_k] + [W_i, W_4]\right\} \tag{62}$$

and
$$X_i^+ = \frac{-i}{M^2}\left\{-\frac{1}{2}\varepsilon_{ijk}[W_j, W_k] + [W_i, W_4]\right\} \tag{63}$$

So from (59) and (62) we see that $\vec{X}$ is associated with the self-dual combination $[W_\mu, W_\nu] + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}[W_\rho, W_\sigma]$ while $\vec{X}^+$ corresponds to the antiself-dual combination.

It is tedious but straightforward to verify the commutation relations

$$\left[W_{\alpha\beta}, W_{\rho\sigma}\right] = -\frac{1}{4M^2}\left(\varepsilon_{\alpha\beta\sigma\delta} W_\rho - \varepsilon_{\alpha\beta\rho\delta} W_\sigma\right) P_\delta \tag{64}$$

where $W_{\alpha\beta}$ is defined by equation (61), and equation (17) holds, and for both $X_{\mu\nu}$ and $X^+_{\mu\nu}$ we have

$$\left[X_{K\lambda}, X_{\mu\nu}\right] = i\left(\delta_{K\nu} X_{\lambda\mu} - \delta_{K\mu} X_{\lambda\nu} + \delta_{\lambda\mu} X_{K\nu} - \delta_{\lambda\nu} X_{K\mu}\right) \tag{65}$$

and, for X_i and X_i^+,

$$\left[X_i, X_j\right] = i\,\varepsilon_{ijk} X_k \tag{66}$$

so that X does indeed define a covariant spin operator which generates the little group, for time-like momenta. X transforms covariantly like the (1, 0) representation of the homogeneous Lorentz group, and X^+ like (0, 1), while W_μ transforms like $(\frac{1}{2}, \frac{1}{2})$.

6. Equivalent forms of the spin operator

Equation (59) is hardly in an immediately recognizable form, and we will now evaluate X for the case of spin $\frac{1}{2}$ particles, and will notice that we have seen X around before. We have seen that equation (59) is equivalent to equation (43), and for convenience of calculation, we evaluate this expression.

We first substitute

$$e^{\vec{S}\cdot\vec{\lambda}(p)} = \left(\frac{\not{p}}{m}\gamma_4\right)^{1/2} = \frac{m + p_0 - \gamma_5\vec{\sigma}\cdot\vec{p}}{\sqrt{2m(m+p_0)}} \tag{67}$$

so if $\vec{S} = \vec{\sigma}/2$, then $\vec{X}$ is given by

$$\vec{X} = \frac{1}{2m(m+p_o)} (m + p_o - \gamma_5 \vec{\sigma}.\vec{p}) \frac{\vec{\sigma}}{2}(m + p_o + \gamma_5 \vec{\sigma} . \vec{p}) \tag{68}$$

which eventually and straightforwardly becomes

$$\vec{X} = \frac{\vec{\sigma}}{2} \cdot \frac{p_o}{m} - \vec{p}\frac{\gamma_5}{2m} + \frac{\gamma_5}{2m}\vec{\sigma}(\vec{\sigma} . p) - \frac{1}{2m(m+p_o)} \vec{p} .(\vec{\sigma} .\vec{p}) \tag{69}$$

Now we find the form of X when it acts on positive energy states only, i.e. for which

$$\vec{\alpha}. \vec{p} + \gamma_4 m = p_o$$

$$\text{i.e.} \quad \gamma_5 \vec{\sigma}.\vec{p} = \gamma_4 m - p_o \quad . \tag{70}$$

We find

$$\vec{X} = \frac{\vec{\sigma}}{2}\gamma_4 - \frac{\gamma_5 \vec{p}}{2(m+p_o)} (1 + \gamma_4) \tag{71}$$

We note that this is the same as the Foldy-Wouthuysen (1950) mean-spin operator, defined by

$$\begin{aligned} \vec{X} &= U \frac{\vec{\sigma}}{2} U^+ \\ U &= \left(\frac{m+i\vec{\gamma}.\vec{p}}{p_o}\right)^{\frac{1}{2}} = \frac{m+p_o + i\vec{\gamma}.\vec{p}}{\sqrt{2p_o(p_o+m)}} \end{aligned} \tag{72}$$

This was pointed out by Gürsey (1965a). In fact eq. (72) is (Foldy-and Wouthuysen, 1950),

$$\vec{X} = \frac{\vec{\sigma}}{2} - i\gamma_4\gamma_5 \frac{\vec{\sigma} \times \vec{p}}{2p_o} - \frac{\vec{p} \times(\vec{\sigma} \times \vec{p})}{2p_o(p_o+m)} \tag{73}$$

$$= \frac{\vec{\sigma}}{2}\,\frac{m}{p_o} - \frac{\gamma_4\gamma_5}{2p_o}\vec{p} + \frac{\gamma_4\gamma_5}{2p_o}\,\vec{\sigma}(\vec{\sigma}\cdot\vec{p}) + \frac{\vec{p}(\vec{\sigma}\cdot\vec{p})}{2p_o(p_o+m)} \qquad (74)$$

Again, if we use the positive energy condition (70), (74) becomes

$$\vec{X} = \gamma_4\frac{\vec{\sigma}}{2} - \frac{\gamma_5\vec{p}}{2(m+p_o)}(1+\gamma_4) \qquad (75)$$

which is the same as (71).

$\vec{X}$ is also equal to the spin operator used in several places in the literature, and which is discussed, for example, in Rose's book (1961), p. 72.

$$\vec{S} = \gamma_4\left[(\vec{\sigma}\cdot\vec{e}_1)\vec{e}_1 + (\vec{\sigma}\cdot\vec{e}_2)\vec{e}_2\right] + (\vec{\sigma}\cdot\hat{p})\hat{p} \qquad (76)$$

where $\vec{e}_1$, $\vec{e}_2$ and $\hat{p}$ form a right-handed orthonormal triad, and $\hat{p} = \vec{p}/|\vec{p}|$. Using the relation

$$\vec{\sigma} = (\vec{\sigma}\cdot\hat{p})\hat{p} + (\vec{\sigma}\cdot\vec{e}_1)\vec{e}_1 + (\vec{\sigma}\cdot\vec{e}_2)\vec{e}_2 \qquad (77)$$

equation (76) becomes

$$\vec{S} = \gamma_4\vec{\sigma} + (1-\gamma_4)(\vec{\sigma}\cdot\hat{p})\hat{p} \qquad (78)$$

Now, using the positive energy condition (70), equation (78) is

$$\vec{S} = \gamma_4\vec{\sigma} + (1-\gamma_4)\,\gamma_5\,\frac{\gamma_4 m - p_o}{(p_o^2 - m^2)}\,\vec{p}$$

$$= \gamma_4\vec{\sigma} - (1-\gamma_4)\frac{\gamma_5\vec{p}}{(m+p_o)} \qquad (79)$$

$$= 2\vec{X}\ .$$

The operator in equation (76) is used by Lipkin to define "W-spin" (Lipkin and Meshkov, 1965), though the curious selection rules they get are not due to this operator, but due to the fact that γ_4 is an operator of $\tilde{U}(12)$, which it is not of SU(6), so their selection rules do not apply to SU(6). Equation (76) is also used by Barnes et al. (1965).

It is easily seen, for example from equation (75), that in the rest frame, where $\vec{p} = 0$ and $\gamma_4 = 1$, that $\vec{X}$ reduces to $\vec{\sigma}/2$, as it should.

7. General Remarks Concerning SU(6).

Collecting together our results of this chapter, we conclude by reasserting their relevance to SU(6). This group arises from the combination of the approximate internal symmetry group SU(3) with a "spin group" SU(2) associated with the Poincaré group. Its generators are the generalised operators for the little group in the subspace $\mathcal{H}^I$ of Hilbert space for which momenta are time-like with positive energy and non-zero rest mass. For more general momenta (such as those belonging to virtual particles in the perturbation treatment of field theory) the little group is not compact in general, being isomorphic with the (2 + 1) Lorentz group for space-like momenta or the two dimensional Euclidean group for light-like momenta and its combination with SU(3) would lead to various subgroups of a non-compact group admitting as its subgroup SU(3) and a group isomorphic with the homogeneous Lorentz group. This group G, a possible generalization of SU(6) formally resembles U(12) of Salam (1964, 5) and Gell-Mann (1965a, b) and Feynman et al. (1964), but unlike U(12) admits the translationally invariant SU(6) as a subgroup.

Thus our philosophy of non-compact groups is different from that of Gell-Mann and Salam. Our group G_6 (see footnote 5 of Gursey & Radicati (1964) has as generators A_λ, J_i, $A_\lambda J_i$, K_i, P_μ where $\lambda = 1,\ldots,8$, $i = 1,\ldots, 3$, $\mu = 1,\ldots, 4$ and A generate SU(3). Thus on "boosting" and on translation, our SU(3) multiplets <u>preserve</u> their form in the irreducible representation. This philosophy is opposed to that of Dothan, Gell-Mann & Ne'eman (1965). Also, the exact symmetry limit is the same as the exact SU(3) symmetry limit, i.e. equal masses for all members of a representation. We do not attempt to derive mass splitting from an exact symmetry.

The SU(2) operators S_i for single particle states are constants of motion like the SU(3) operators, since they commute with P_ν. However, if we consider two particle states, unlike the sum of SU(3) generators $A_\lambda = A_\lambda^{(1)} + A_\lambda^{(2)}$, the sum $\vec{\mathcal{S}} = \vec{S}^{(1)} + \vec{S}^{(2)}$ commutes with $P_\mu = P_\mu^{(1)} + P_\mu^{(2)}$ for free particles, but not in general if the two particles interact. What does commute with P_μ is the operator $\vec{S} = \vec{\mathbf{S}} + \vec{L}_{12}$ where $\vec{L}_{12}$ is a certain function of the Poincaré generators $P_\mu^{(1)}$, $J_{\mu\nu}^{(1)}$, $P_\mu^{(2)}$, $J_{\mu\nu}^{(2)}$ and is the relativistic version of the relative orbital angular momentum of the two particles. So in general, two representations of SU(2) cannot be multiplied; $\vec{L}_{12}$ which has no counterpart in the internal symmetry space for exact SU(3) limit, acts like a spurion. Thus, for example, in the processes $\rho \to 2\pi$ and $N_{33} \to N + \pi$, we must insert a relativistic spurion and assign it to a representation of SU(6). This is one way in which SU(6) symmetry is violated.

It is violated also in another, more fundamental, way. We saw that crossing changes the little group from SU(2) to L(2, 1).

Thus SU(6) does not allow crossing symmetry. But any <u>local</u> interaction Hamiltonian contains crossing symmetry insofar as it leads to different processes, some of which are related by crossing. It follows that the locality of the interaction is another source of the breaking of SU(6). We shall discuss this further in Chapter 8.

There is much to be said about SU(6), since it is a non-trivial combination of SU(3) and spin. We leave most of this unsaid, since there is already a wide and increasing literature on the subject. Let us conclude this chapter, however, with the remark, already made, but sufficiently important to repeat, that S U(6) is <u>more</u> than SU(3) x SU(2). It is because of this that its success is mysterious.

Let the generators of U(3) be T^i, $i = 1, \ldots, 9$ and of U(2) be σ_μ in the rest frame, $\mu = 1, \ldots, 4$, $\sigma_4 = 1$, $\vec{\sigma}$ = Pauli matrix, then the generators of U(6) are $T^i \sigma_\mu$ and their commutation relations are

$$\begin{aligned} \left[T^i \sigma_\mu, T^j \sigma_\nu\right] &= T^i T^j \sigma_\mu \sigma_\nu - T^j T^i \sigma_\nu \sigma_\mu \\ &= \tfrac{1}{2}\left\{T^i, T^j\right\}\left[\sigma_\mu, \sigma_\nu\right] \\ &\quad + \tfrac{1}{2}\left[T^i, T^j\right]\left\{\sigma_\mu, \sigma_\nu\right\} \end{aligned} \tag{80}$$

and it is just the presence of the anticommutator in equation (80) that marks the difference between SU(6) and SU(3) x SU(2). By means of these, which go <u>outside</u> the SU(3) and SU(2) algebra respectively, we may make a transition involving a change of spin and unitary spin with <u>one</u> generator, in <u>one</u> step, involving <u>one</u> meson, in diagram language. As far as the other generators (K_a) of Lorentz transformations are concerned, they commute with T^i, and so from

$$[J_a, K_b] = i\, \varepsilon_{abc} K_c$$

$$(a,b,c = 1,2,3)$$

we have

$$\begin{aligned}[T^i J_a, K_b] &= T^i [J_a, K_b] \\ &= i\, T^i \varepsilon_{abc} K_c \qquad (81)\end{aligned}$$

and so SU(3) multiplets preserve their form on boosting, as remarked above.

CHAPTER VI

SU(6) AND THE POMERANCHUK THEOREMS.

1. The Pomeranchuk Theorems

The theoretical study of high energy scattering processes has long been a field of great activity, since some of the readiest data we have available are those of cross sections at high energy. Also, some theoretical considerations are simpler at high energy, for instance in group theory, the mass differences between particles become small compared with the total energy, and one hopes that the relevant group, if a good one, will display its virtues at higher energies. The Pomeranchuk theorems are both relations at high energy between cross-sections for different processes.

The Pomeranchuk conjecture, sometimes called Pomeranchuk's first theorem, was suggested by Pomeranchuk and Okun (1956). They suggested that in forward direction at very high energies, exchange amplitudes, and in particular charge exchange amplitudes, become negligible compared with the non-exchange amplitude. This hypothesis was justified in physical terms by arguing that exchange scattering is to be regarded as a special case of inelastic collision which at high energies is in competition with all other inelastic processes. Non-exchange scattering $a + b \rightarrow a + b$, on the contrary, is truly elastic in the sense that each of the two incident particles remains identical to itself, so that interference between incident and scattered waves is possible, and scattering is bound to occur as the shadow of all inelastic processes.

The Pomeranchuk <u>theorem</u> (1958), (sometimes called his second theorem), is simpler and, in general, not as restrictive. It states that at high energies, the total cross section for ab collision is equal to that for $\bar{a}b$ collision. By the optical theorem, this is the same as saying that the imaginary parts of the forward scattering amplitudes for the two processes, are equal. Pomeranchuk considered the example of pp and $p\bar{p}$ scattering, and the conditions under which $\sigma_{pp}(\infty) = \sigma_{p\bar{p}}(\infty)$ are (i) the forward scattering amplitudes f_{pp} and $f_{p\bar{p}}$ satisfy once subtracted dispersion relations, and (ii) the total cross sections approach constant values rapidly at high energy. This theorem has very recently been proved rigorously by Martin (1965).

We wish now to investigate the consequences of both the theorem and the conjecture, when higher symmetries are invoked which place particles in multiplets. In general we shall find that our result depends on whether a particle and its antiparticle belong to the same or to different multiplets.

2. <u>Examples of Symmetry Schemes and the Pomeranchuk Theorems.</u>

(a) πN scattering.

According to the <u>theorem</u>

$$\sigma_{tot}(\pi^- p) = \sigma_{tot}(\pi^+ p) \qquad (1)$$

π transforms as $\{1\}$ under the isospin group, and N as $\{\frac{1}{2}\}$. So the direct product produces two invariant amplitudes $A_{1/2}$ and $A_{3/2}$ according to which state of total isospin the πN is in. Let A stand for the imaginary part of the forward scattering amplitude. So from (1)

$$\tfrac{1}{2}\,(A_{1/2} + A_{3/2}) \quad = \quad A_{3/2}$$

$$\text{i.e.} \quad A_{1/2} \quad = \quad A_{3/2} \qquad (2)$$

According to the conjecture

$$A(p + \pi^- \rightarrow n + \pi^+) \quad = \quad 0$$

$$\therefore \quad A_{1/2} \quad = \quad A_{3/2} \qquad (3)$$

the same as (2). In general the two statements are not equivalent, but we shall see from what follows that where one particle and its antiparticle (here π) belong to the same multiplet, then the conjecture implies the theorem, but not necessarily vice versa.

(b) NK Scattering.

Here N and $\overline{N}$, and K and $\overline{K}$, all belong to different $I = \tfrac{1}{2}$ multiplets. For NK scattering, we have amplitudes A_1 and A_o, whereas for $N\overline{K}$ scattering, we have amplitudes A_1' and A_o'.

(i) Conjecture

$$A(pK^o \mid nK^+) \quad = \quad 0$$

$$\therefore \quad A_1 \quad = \quad A_o \qquad (4)$$

Similarly for $N\overline{K}$ scattering

$$A_1' \quad = \quad A_o' \qquad (5)$$

(ii) Theorem

$$(pK^+ \mid pK^+) \quad = \quad (pK^- \mid pK^-)$$

$$\therefore \quad A_1 \quad = \quad \tfrac{1}{2}(A_1' + A_o') \qquad (6)$$

also $(pK^o \;\; pK^o) = (p\overline{K^o} \mid p\overline{K^o})$

$$\therefore \quad A_1' \quad = \quad \tfrac{1}{2}(A_1 + A_o) \qquad (7)$$

These relations (4) and (5), and (6) and (7), are different. since K and $\bar{K}$ belong to different SU(2) multiplets. The theorem and conjecture together give

$$A_o = A_o' = A_1 = A_1'$$

(c) A and $\bar{A}$ belong to the same multiplet.

Let us now generalise our πN result to any symmetry. Let us for convenience consider meson-baryon scattering (MB), and assume that it is the mesons that belong to a self-conjugate representation, i.e. M and $\bar{M}$ belong to the same representation. In this case, the Pomeranchuk theorem takes on a very convenient form, which was first pointed out by Amati, Prentki and Stanghellini (1962). Referring to the absorption forward amplitude, we write the Pomeranchuk theorem as

$$\langle MB | T | MB \rangle = \langle \bar{M}B | T | \bar{M} B \rangle \tag{8}$$

Now we cross to the t channel, and the above relation becomes

$$\langle M\bar{M} | T | \bar{B}B \rangle = \langle \bar{M}M | T | \bar{B}B \rangle \tag{9}$$

i.e.

$$\langle (M\bar{M} - \bar{M}M) | T | \bar{B}B \rangle = 0 \tag{10}$$

i.e. only amplitudes connecting the baryons to a symmetric combination of mesons contribute to the scattering. Since the crossing matrix is square and non-singular, the number of independent amplitudes is the same in the s and t channels. In terms of group theory, this means we reduce $M\bar{M}$ and $B\bar{B}$ and select only the terms arising from the product decomposition of $M\bar{M}$ which are symmetric with respect to interchange of $M\bar{M}$. In general, this symmetry is seen from the Young diagram.

e.g. 1. <u>isospin</u>

For $B \times \bar{B}$ we have

$$2 \times 2 = 3 + 1$$

and for $M\bar{M}$:- $3 \times 3 = 5 + 3 + 1$

so in general there are two independent amplitudes $A_{3 \to 3}$ and $A_{1 \to 1}$, or A_3 and A_1 (denoted above as A_1 and A_0). To see how many independent amplitudes remain after imposition of the Pomeranchuk theorem, we must investigate the symmetry properties of the meson decomposition. Call the 2 boxes for each $\{3\}$ representation, a and b, and c and d. Now perform the interchanges $(a \leftrightarrow c)$ $(b \leftrightarrow d)$, remembering that boxes in the same row are symmetric under interchange, those in the same column antisymmetric.

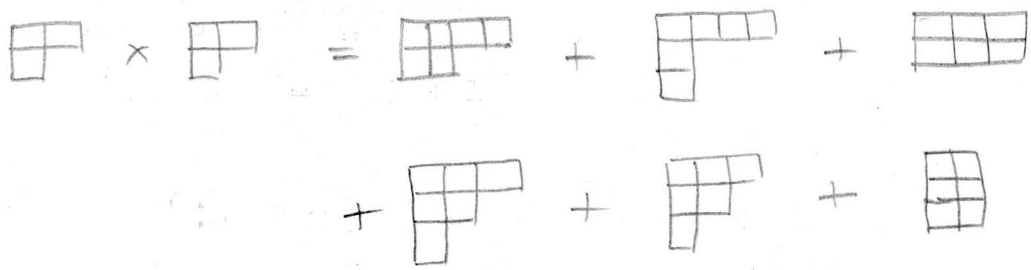
So we have

$$\begin{array}{lcl} 5 : & (S)\,(S) & = (S) \\ 3 : & (S)\,(A) & = (A) \\ 1 : & (A)\,(A) & = (S) \end{array}$$

in an obvious notation, showing that 1 is symmetric, 3 is antisymmetric. So from equation (10), only 1 contributes to the scattering; there is now only 1 independent amplitude, and moreover it corresponds to the exchange of a singlet, i.e. zero quantum number, so we have also proved the conjecture. We saw this before from equations (2) and (3).

e.g. 2 SU(3)

Here the mesons and baryons both belong to $\{8\}$, and so for the decomposition we have:-



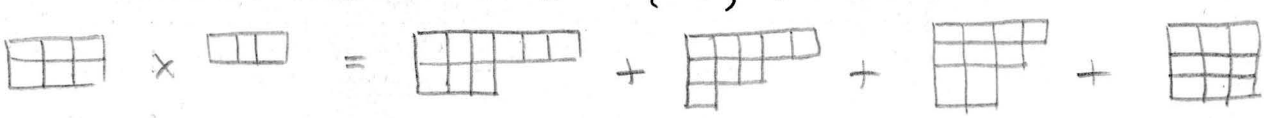
$B\bar{B}$:- $8 \times 8 = 27_s + 10_a + \overline{10}a + 8_s + 8_a + 1_s$

$M\bar{M}$:- $8 \times 8 = 27_s + 10_a + \overline{10}_a + 8_s + 8_a + 1_s$

where the symmetry and antisymmetry are got the same way as before.

So normally we have 8 amplitudes, according to the arrows marked, but under time reversal invariance $(8_a \rightarrow 8_s) = (8_s \rightarrow 8_a)$ so there are 7 independent. From equation (10), the number of independent amplitudes with the P.Th$^{\underline{m}}$ is 4, i.e. A_{27}, $A_{8_{ss}}$, $A_{8_{sa}}$ and A_1.

When the baryons belong to $\{10\}$, then we have for $B\bar{B}$



$\overline{10} \times 10 = 64 + 27 + 8 + 1$

So for MB scattering, there are 4 amplitudes $(27 \rightarrow 27)$, $(8_s \rightarrow 8)$ $(8a \rightarrow 8)$, $(1 \rightarrow 1)$ and this is reduced to 3 with the Pomeranchuk theorem.

e.g. 3 SU(6)

Mesons now belong to $\{35\}$, and we proceed exactly as before, writing (see appendix):-

$$M\bar{M} :- \ 35 \times 35 = 405_s + 280_a + 280_a + 35_s + 35_a + 189_s + 1_s$$

$$B\bar{B} :- \quad 2695 + 405 + 35 + 1$$

so we have 4 independent amplitudes, reduced to 3 by the Pomeranchuk theorem.(The symmetry and antisymmetry above comes, straightforwardly from the corresponding Young's diagrams, drawn in the appendix).

Note that this is a great reduction on SU(3), since there we need 7 + 7 + 1 + 4 + 4 + 1 = 24 amplitudes to describe all (0^- and 1^-)($\frac{1}{2}^+$ and $\frac{3}{2}^+$) reactions, which are here described by 4 amplitudes. So in principle we get many new relations, one of which is the famous Johnson-Treiman relation (1965).

A word about the conjecture. This states that only the singlet is exchanged in the crossed channel. This is clearly consistent with the theorem, but is only implied by it for the case of isospin. Since only the singlet is exchanged, in the direct (s) channel this means that all scattering amplitudes are equal, and so the scattering is independent of the isospin, or unitary spin, or "SU(6) spin". This is a general property of crossing matrices.*

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* I am grateful to Professors B.M. Udgaonkar and A.O. Barut for a discussion of this point, whilst I was at Trieste, at the Seminar on High Energy Physics, 1965.

(d) A, $\bar{A}$, B and $\bar{B}$ all belong to different multiplets.

In this case we may not apply the above reasoning. To see why, let us consider again equations (4) - (7). Altogether, we have, in the case of NK scattering, 4 independent amplitudes. The conjecture gives 2 relations between them, and the theorem also 2, but a different 2. The conjecture and the theorem together give the result that all 4 amplitudes are equal. In general neither the conjecture implies the theorem nor vice versa. (However, conjecture for (NK) + theorem $\Rightarrow$ conjecture for ($N\bar{K}$)).

Let us consider the statement of the theorem

$$(pK^+ \mid pK^+) \quad = \quad (pK^- \mid pK^-)$$

which lead to equation (6).

In the t channel, this reads

$$(p\bar{p} \mid K^- K^+) \quad = \quad (p\bar{p} \mid K^+ K^-) \tag{11}$$

and it may appear that this equation just states that only amplitudes which are symmetric with respect to $K^- \leftrightarrow K^+$ interchange contribute to the scattering. This is true, of course, but does not give any new information. This is what we expect, since equation (6) relates different amplitudes, and also the left and right hand sides of (11) refer to different physical processes - the scattering angle one observes is between different particles.

The essential reason that equation (11) gives no information is that K^+ and K^- belong to different isospin multiplets.

Using the basis vectors

$$\phi_{\frac{1}{2}\frac{1}{2}} = |K^+\rangle \qquad\qquad \phi_{\frac{1}{2}\frac{1}{2}} = |\bar{K}^0\rangle$$

$$\phi_{\frac{1}{2}-\frac{1}{2}} = |K^0\rangle \qquad\qquad \phi_{\frac{1}{2}-\frac{1}{2}} = -|K^-\rangle$$

Let us construct the isospin 1 and 0 basis vectors, using the normal Clebsch-Gordan coefficients. We get

$$\left.\begin{aligned}
\chi_1^1 &= |K^+\rangle|\bar{K}^0\rangle \\
\chi_1^0 &= \frac{1}{\sqrt{2}}\left(-|K^+\rangle|K^-\rangle + |K^0\rangle|\bar{K}^0\rangle\right) \\
\chi_1^{-1} &= -|K^0\rangle|K^-\rangle \\
\chi_0^0 &= \frac{1}{\sqrt{2}}\left(|K^+\rangle|K^-\rangle + |K^0\rangle|\bar{K}^0\rangle\right)
\end{aligned}\right\} \qquad (12)$$

Compare this with the particle-particle (as distinct from the above particle-antiparticle) states

$$\left.\begin{aligned}
\chi_1^1 &= |K^+\rangle_1|K^+\rangle_2 \\
\chi_1^0 &= \frac{1}{\sqrt{2}}\left(|K^+\rangle_1|K^0\rangle_2 + |K^0\rangle_1|K^+\rangle_2\right) \\
\chi_1^{-1} &= |K^0\rangle_1|K^0\rangle_2 \\
\chi_0^0 &= \frac{1}{\sqrt{2}}\left(-|K^+\rangle_1|K^0\rangle_2 + |K^0\rangle_1|K^+\rangle_2\right)
\end{aligned}\right\} \qquad (13)$$

Under interchange $|K^+\rangle_1 \leftrightarrow |K^+\rangle_2$, $|K^0\rangle_1 \longleftrightarrow |K^0\rangle_2$, χ_1^0 is symmetric and χ_0^0 is antisymmetric. This is a well-known result, and can of course be seen straight away from Young's diagrams, where the unit (singlet) is constructed by putting one box, or set of boxes, under another, and so is antisymmetric with respect to their interchange.

In (13), though, there is no particular symmetry with respect to $K^+ \leftrightarrow K^-$ interchange, which is the interchange we are concerned with, so we may draw no conclusions, for example, of the sort that certain (symmetric or antisymmetric) amplitudes vanish.

So if particles and antiparticles belong to different representations, the conjecture and the theorem are independent. Let us note in conclusion that the Pomeranchuk conjecture in SU(6) implies vanishing of spin flip processes asymptotically, and also implies that spin and orbital angular momentum are conserved separately, i.e. that SU(6) is exact. This has already been conjectured by Serber (private communication with Gürsey) as being true in the high energy limit.

CHAPTER VII

THE COVARIANT MAGNETIC MOMENT OPERATOR AND SU(6).

1. Introduction

Very soon after SU(6) symmetry was proposed, it was realized that it predicted, in the static limit,

$$\mu(n)/\mu(p) = -2/3 \tag{1}$$

for the total magnetic moments of proton and neutron. (Bég, Lee and Pais (1964) and Sakita (1964b)). This is in remarkable agreement with the experimental value of ≈ -0.684. In the following we shall give a short review of magnetic moment predictions for SU(3), and outline the derivation of equation (1) for static SU(6). We shall then show that it is possible to find a covariant magnetic moment operator such that equation (1) holds at all momentum transfers. In fact, it has already been shown by Barnes et al. (1965) how to generalise equation (1) to arbitrary momentum transfer, but their derivation was not manifestly covariant.

One's first reaction to the SU(6) prediction is to jump for joy, but one is arrested in mid-air due to two difficulties, both of which are non-trivial. First, if we assume <u>minimal</u> electromagnetic couplings, which is the usual convention, then $\mu(p) = \frac{eh}{2mc}$ and $\mu(n) = 0$, so $\mu(n)/\mu(p) = 0$. We would then expect the ratio to be zero. SU(6) is somehow taking into account for us the Pauli (derivative) term corresponding to the electromagnetic interaction with other particles (mesons in this case). But in any case we cannot assume both that SU(6) is valid and that local field theory with minimal electromagnetic interactions applies to nucleons.

The second difficulty relates to quarks. One could say, in partial solution to the above difficulty that the quarks do have minimal electromagnetic interactions, and that the nucleons are composed of physical quarks. But to account for the large anomalous term in the nucleon magnetic moment, the quark would have to have a small mass. As an order of magnitude calculation, let us note that the physical proton (say) is composed of two "proton" quarks and one "neutron" quark. If each of these has only a Dirac magnetic moment, then we would have

$$2\,\frac{e\hbar}{2m_Q c} = 2.79\,\frac{e\hbar}{2m_p c}$$

$$m_Q = \frac{2}{2.79}\,m_p \sim 0.7\,m_p$$

which predicts a quark mass in such a region that it would almost certainly have been observed. If $m_Q \gtrsim 3$ Gev, then the quark itself would have to have a large anomalous magnetic moment. This problem is discussed by Freund (1965) and a relativistic model of composite quark states, with the quarks having minimal interaction, is discussed by Bogolubov et al. (1965) and by Tavkhelidze (1965 where they obtain equation (1).

2. Magnetic Moments in SU(3).

Electromagnetism violates SU(3) invariance; this is clear, since it even violates SU(2)! But if we know exactly how it violates the symmetry, then all is far from lost, and we can in fact get some results. And in fact we do know how the electromagnetic interaction transforms. In the limit in which SU(3) is exact the form factor

$$< B \mid J_\mu^Q \mid B >$$

of the baryons transforms like J_μ^Q, but this is just the transformation law of Q itself. A mathematically identical way of saying this is to say that the electromagnetic interaction coupled to a charge (Q) "spurion", is SU(3) invariant. SU(3) invariants are traces and there are two independent traces of these matrices, so, for instance, the magnetic form factor is given by the combination

$$\mu = f_1 \operatorname{tr} \overline{\psi} \psi Q + f_2 \operatorname{tr} \overline{\psi} Q \psi \qquad (2)$$

where

$$\psi = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}$$

$$\overline{\psi} = \begin{pmatrix} \frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Sigma^-} & \overline{\Xi^-} \\ \overline{\Sigma^+} & -\frac{\overline{\Sigma^0}}{\sqrt{2}} + \frac{\overline{\Lambda}}{\sqrt{6}} & \overline{\Xi^0} \\ \overline{p} & \overline{n} & -\frac{2\overline{\Lambda}}{\sqrt{6}} \end{pmatrix}$$

$$Q = \frac{1}{3}\begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix} \qquad (3)$$

In terms of f_1 and f_2, we now have

$$\mu(\Lambda^{q}) = -\tfrac{1}{2}(f_1 + f_2)$$

$$\mu(\Sigma^{\circ}) = \tfrac{1}{2}(f_1 + f_2)$$

$$\mu(p) = -f_1 + 2f_2$$

$$\mu(n) = -f_1 - f_2 \qquad \text{etc.}$$

So we have the predictions

$$\mu(\Sigma^{+}) = \mu(p)$$

$$\mu(\Lambda) = \tfrac{1}{2}\,\mu(n)$$

$$\mu(\Xi^{\circ}) = \mu(n)$$

$$\mu(\Xi^{-}) = \mu(\Sigma^{-}) = -\left[\mu(p) + \mu(n)\right]$$

$$\mu(\Sigma^{\circ}) = -\tfrac{1}{2}\,\mu(n) \qquad (4)$$

The prediction $\mu(\Lambda) = \frac{1}{2}\mu(n)$ is to be compared with the Sakata model which gives $\mu(\Lambda) = \mu(n)$. In (4) the magnetic moments (which are all anomalous) of proton and neutron remain unrelated. The two parameters f_1 and f_2 are of course related to f/d ratio, and it is known that SU(6) fixes this ratio, so we expect now to determine numerically the ratio $\mu(n)/\mu(p)$.

3. Magnetic Moments in Static SU(6).

We rederive the result we require to illustrate that it is not necessary to use Clebsch-Gordan coefficients as is done in the literature. All we need to know is our angular momentum algebra.

It is easiest to consider the 3 quarks out of which the baryons are made; it does not matter of course, here, whether quarks exist or not. Let us call the three basic quarks p, n and λ, in small letters, and the physical nucleons P and N, in big letters. The basic 6-dimensional representation of SU(6) has as components

$$\psi = \begin{pmatrix} p_u \\ n_u \\ \lambda_u \\ p_v \\ n_v \\ \lambda_v \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \\ \psi_5 \\ \psi_6 \end{pmatrix} \qquad (5)$$

where the suffix u = spin up, v = spin down. We consider the rest frame of the particle where the spin matrices are just .

The physical baryons belong to the 56 dimensional representation. The highest state (highest weight, in mathematical language) of this representation clearly has spin $^3/2$, isospin $^3/2$ and $I_3 = {}^3/2$, and so is

$$N^{*++}_{3/2} = p_u\, p_u\, p_u \qquad (6)$$

Remembering that

$$J_- \,|\, j, m \rangle = \sqrt{j(j+1) - m(m+1)} \,|\, j, m-1 \rangle$$

we reduce (6) in charge space using τ_-, and get

$$\sqrt{3}\ N^{*++}_{3/2} = \{n_u\, p_u\, p_u\}$$

$$\therefore \qquad N^{*+}_{3/2} = \frac{1}{\sqrt{3}} \{n_u\, p_u\, p_u\} \tag{7}$$

where the curly brackets denote a completely symmetrical state, so that the state in brackets above is short for

$$(n_u\, p_u\, p_u + p_u\, n_u\, p_u + p_u\, p_u\, n_u)\,.$$

It is because there are 3 terms here that there is a $1/\sqrt{3}$ outside in equation (7) -- that is where Clebsch-Gordan coefficients come from.

Now do a spin reduction using σ_- on equation (7) and get

$$N^{*+}_{1/2} = \frac{1}{\sqrt{3}}\{n_v p_u p_u\} + \sqrt{\frac{2}{3}}\{n_u p_v p_u\}$$

$$= \frac{1}{\sqrt{3}}\Big[\{n_v p_u p_u\} + \sqrt{2}\,\{n_u p_v p_u\}\Big] \tag{8}$$

where there are now 3 terms inside the first curly bracket and 6 in the second. Now $N^{*+}_{1/2}$ of equation (8) has exactly the same I_3, Y and S_3 as the proton; they are degenerate on the weight diagram. So they are orthogonal, and

$$P = \frac{1}{\sqrt{3}}\Big[\sqrt{2}\ \{n_v\, p_u\, p_u\} - \{n_u\, p_v\, p_u\}\Big] \tag{9}$$

Similarly by taking the orthogonal state to that obtained by applying τ_- in equation (8), we obtain

$$N = \frac{1}{\sqrt{3}}\left[- \left\{ n_v\, n_u\, p_u \right\} + 2 \left\{ n_u\, n_u\, p_v \right\} \right] \tag{10}$$

for the neutron.

Converting into the notation of equation (5), and remembering that indices appearing together, above or below, are to be symmetrised, we write (9) and (10) as

$$\begin{aligned} P &= \frac{1}{\sqrt{3}}\left(\sqrt{2}\;\psi_{511} - \psi_{241}\right) \\ N &= \frac{1}{\sqrt{3}}\left(-\psi_{521} + \sqrt{2}\,\psi_{224}\right) \end{aligned} \tag{11}$$

Now we want the current which couples to $\bar{P}P$ and $\bar{N}N$, so we write

$$\begin{aligned} \bar{P}P &= \tfrac{1}{3}\left(\sqrt{2}\;\bar{\psi}^{511} - \bar{\psi}^{241}\right)\left(\sqrt{2}\;\psi_{511} - \psi_{241}\right) \\ &= \tfrac{1}{3}\left(2\,\bar{\psi}^{511}\psi_{511} - \sqrt{2}\;\bar{\psi}^{511}\psi_{241} - \sqrt{2}\;\bar{\psi}^{241}\psi_{511} \right. \\ &\qquad \left. + \bar{\psi}^{241}\psi_{241} \right) \end{aligned} \tag{12}$$

As it stands, equation (12) corresponds to the highest representation in the reduction of $56 \times \overline{56}$, i.e. to 2695, with tensor representation T^{abc}_{def}. The electromagnetic current transforms as 35, the spin zero octet corresponding to the charge form factor, and the spin one octet corresponding to the magnetic moment form factor. So, to couple to 35 invariantly, we have to reduce, e.g. (12) to a tensor of the form T^{a}_{b}, which belongs to 35. This we do by applying all possible combinations of delta functions two at a time to equation (12). Clearly, then, the only terms to give a non-zero contribution are those with 2 or more indices equal, and so we get

$$
\begin{aligned}
(\bar{P}P)_{35} &= 2(T^5_5 + 2T^1_1) + T^2_2 + T^4_4 + T^1_1 \\
&= 5T^1_1 + T^2_2 + T^4_4 + 2T^5_5 \qquad (13)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(\bar{N}N)_{35} &= T^5_5 + T^2_2 + T^1_1 + 2(2T^2_2 + T^4_4) \\
&= T^1_1 + 5T^2_2 + 2T^4_4 + T^5_5 \qquad (14)
\end{aligned}
$$

From the way in which the spinor (5) was constructed, we see that the "magnetic current" (or "charge-spin" current) matrix is

$$
M = \sigma_3 \otimes Q = \frac{1}{6}\begin{pmatrix} 2 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& -2 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} \qquad (15)
$$

where Q is given by equation (6). So, exactly as for SU(3), we see that the magnetic moments of proton and neutron are given by

$$
\text{proton} \qquad \operatorname{tr} \begin{pmatrix} 5 &&&&& \\ & 1 &&&& \\ && 0 &&& \\ &&& 1 && \\ &&&& 2 & \\ &&&&& 0 \end{pmatrix} \times \begin{pmatrix} 2 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& -2 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} = 9
$$

$$
\text{Neutron} \qquad \operatorname{tr} \begin{pmatrix} 1 &&&&& \\ & 5 &&&& \\ && 0 &&& \\ &&& 2 && \\ &&&& 1 & \\ &&&&& 0 \end{pmatrix} \times \begin{pmatrix} 2 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& -2 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix} = -6
$$

giving

$$
\mu(p)/\mu(n) = -\,{}^{3}/{}_{2}\,.
$$

4. The Covariant Magnetic Moment Operator.

Let us begin by stating our notation:-

γ_μ, γ_5 all hermitian

$x_4 = ix_o, \; x_o$ real

$p_\mu = -i\partial_\mu$

The hermitian electromagnetic current is then written

$$j_\mu = \left\langle \bar{\psi}(p_2) \middle| F_1 \, i\gamma_\mu - \frac{F_2}{2m} \, i \, \sigma_{\mu\nu} \, q_\nu \middle| \psi(p_1) \right\rangle \tag{16}$$

between states of momentum p_1 and p_2. F_1 and F_2 are real functions of q^2, and $q_\nu = (p_2 - p_1)_\nu$.

The Dirac equations for the spinor and conjugate spinor are

$$\gamma_\mu \, \partial_\mu \, \psi = - m\psi$$

$$\therefore -\gamma_\mu \, \gamma_\nu \, \partial_\nu \, \psi = m\gamma_\mu \, \psi$$

$$\gamma_\mu \, \psi = -\frac{1}{m}\left[\partial_\mu + i \, \sigma_{\mu\nu} \, \partial_\nu \right] \psi \tag{17}$$

and $\bar{\psi} \, \gamma_\mu \, \partial_\mu = m\bar{\psi}$

$$\bar{\psi} \, \gamma_\nu \, \gamma_\mu \, \partial_\nu = m\bar{\psi} \, \gamma_\mu$$

$$\bar{\psi}(\partial_\mu - i \, \sigma_{\mu\nu} \, \partial_\nu) = m\bar{\psi} \, \gamma_\mu \tag{18}$$

where in both cases we have used the relation

$$\gamma_\mu \, \gamma_\nu = \delta_{\mu\nu} + i \, \sigma_{\mu\nu} \tag{19}$$

(17) and (18) are

$$\gamma_\mu \psi = -\frac{1}{m}\left[\partial_\mu + i\sigma_{\mu\nu}\partial_\nu\right]\psi$$

$$\bar\psi \gamma_\mu = -\frac{\bar\psi}{m}\left[-\partial_\mu + i\sigma_{\mu\nu}\partial_\nu\right]$$

$$\therefore\ \bar\psi \gamma_\mu \psi = \frac{\bar\psi(\gamma_\mu\psi) + (\bar\psi\gamma_\mu)\psi}{2}$$

$$= \frac{-1}{2m}\left\langle \bar\psi(p_2) \middle| i(p_1 + p_2)_\mu - \sigma_{\mu\nu}(p_1 - p_2)_\nu \middle| \psi(p_1)\right\rangle$$

$$= \frac{1}{2m}\left\langle \bar\psi \middle| -iP_\mu - \sigma_{\mu\nu} q_\nu \middle| \psi\right\rangle \qquad (20)$$

where $P_\mu = (p_1 + p_2)_\mu$ is the c.of.m. momentum. So between spinor states, we may write

$$i\gamma_\mu = \frac{P_\mu}{2m} - \frac{i\sigma_{\mu\nu}q_\nu}{2m}$$

and substituting this in equation (16) gives

$$j_\mu = \left\langle \bar\psi(p_2) \middle| F_1 \frac{P_\mu}{2m} - (F_1 + F_2)\frac{i}{2m}\sigma_{\mu\nu} q_\nu \middle| \psi(p_1)\right\rangle \qquad (21)$$

This is in a covariant form now, since under the little group, P_μ is invariant, and the coefficient of the second term, $F_1 + F_2$, is the <u>total</u> (Dirac + anomalous) magnetic moment, when F_1 and F_2 are taken at $q^2 = 0$.

We now cast $\sigma_{\mu\nu} q_\nu$ into a form where its behaviour under the little group is manifest. We must, at this stage, refer back to equations in Chapter V. Substituting $J_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu}$ in the definition V(13A) of W_μ (since the orbital part of $J_{\mu\nu}$ does not contribute), we have

$$W_\kappa = \frac{1}{2i} \varepsilon_{\kappa\lambda\mu\nu} J_{\lambda\mu} P_\nu \qquad \text{V (13A)}$$

$$= \frac{1}{4i} \varepsilon_{\kappa\lambda\mu\nu} \sigma_{\lambda\mu} P_\nu$$

$$= \frac{1}{4i} \varepsilon_{\kappa\nu\lambda\mu} \sigma_{\lambda\mu} P_\nu \qquad (22)$$

Now let us use the relation

$$2\gamma_5 \sigma_{\mu\nu} = - \varepsilon_{\mu\nu\rho\sigma} \sigma_{\rho\sigma} \qquad (23)$$

(23) substituted into (22) gives

$$W_\kappa = \frac{-1}{2i} \gamma_5 \sigma_{\kappa\nu} P_\nu$$

$$\therefore \quad -2i\, \gamma_5 W_\kappa = \sigma_{\kappa\nu} P_\nu \qquad (24)$$

Now let us turn to the functions X defined towards the end of Chapter V. From equations V(60) and V(17) we have

$$X_{\mu\nu} = \frac{i}{M^2}\left\{ [W_\mu, W_\nu] + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} [W_\rho , W_\sigma] \right\}$$

$$= \frac{i}{M^2}\left\{ \varepsilon_{\mu\nu\kappa\lambda} W_\kappa P_\lambda + \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \varepsilon_{\rho\sigma\alpha\beta} W_\alpha P_\beta \right\}$$

$$= \frac{i}{M^2}\left\{ \varepsilon_{\mu\nu\kappa\lambda} W_\kappa P_\lambda + (\delta_{\mu\alpha} \delta_{\nu\beta} - \delta_{\mu\beta} \delta_{\nu\alpha}) W_\alpha P_\beta \right\}$$

$$= \frac{i}{M^2}\left\{ \varepsilon_{\mu\nu\kappa\lambda} W_\kappa P_\lambda + (W_\mu P_\nu - W_\nu P_\mu) \right\}$$

$$\therefore \quad X_{\mu\nu} P_\nu = \frac{i}{M^2} (-M^2) W_\mu \qquad (25)$$

since $W_\nu P_\nu = 0$

$$\therefore \quad W_\mu = i\, X_{\mu\nu} P_\nu \qquad (26)$$

So from (24) and (26)

$$\sigma_{\kappa\nu} P_\nu = 2\gamma_5 X_{\kappa\nu} P_\nu \tag{27}$$

and, from (21)

$$j_\mu = \left\langle \overline{\psi}(p_2) \left| F_1 \frac{P_\mu}{2m} - (F_1 + F_2) \frac{i\gamma_5}{m} X_{\mu\nu} q_\nu \right| \psi(p_1) \right\rangle \tag{28}$$

The space part of this current is

$$j_i = \left\langle \overline{\psi}(p_2) \left| F_1 \frac{p_i}{2m} - (F_1 + F_2) i\gamma_5 (X_{i\nu} q_\nu) \right| \psi(p_1) \right\rangle$$

and
$$X_{i\nu} q_\nu = X_{ij} q_j + X_{i4} q_4$$

$$= X_{ij} q_j + \varepsilon_{ijk} X_{jk} q_4$$

$$= \varepsilon_{ijk} X_k q_j + 2Xi\, q_4$$

$$= -(\vec{X} \times \vec{q})_i + 2i\, X_i\, q_o \tag{29}$$

So

$$\vec{j} = \left\langle \overline{\psi} \left| F_1 \frac{\vec{P}}{2m} + (F_1 + F_2) i\gamma_5 \left[\vec{X} \times \vec{q} - 2iq_o \vec{X} \right] \right| \psi \right\rangle \tag{30}$$

Under the little group, P_μ is a constant, being the c.of.m. momentum, and $\vec{q}$ transforms as a vector. $\vec{X}$ of course is a vector, so the coefficient of F_1 is constant, and of $(F_1 + F_2)$ transforms as a vector. $\vec{X}$ is the covariant spin operator, and the above is therefore manifestly covariant. We may then evaluate the ratios of the coefficients of F_1 and $(F_1 + F_2)$ in any frame, and so we choose the rest frame, where we obtain precisely the results of the previous section, since as we saw in Chapter V, in the rest frame

$$\vec{X} \to \vec{\sigma}$$

$$X_{\mu\nu} \to \sigma_{\mu\nu} \quad .$$

So the results of the last section are valid at all momentum transfers, and also, the charge form factor, of, say, the neutron ($F_1(q^2)$) is <u>always</u> zero. Notice that $F_1 + F_2$ gives the total magnetic form factor, not just the anomalous one.

CHAPTER VIII

SU(6) AND THE 3-POINT VERTEX

We shall now gather together our remarks on SU(6) to consider how to form the Lagrangian for the interaction of a meson with a baryon-antibaryon system in an SU(6) invariant way.

1. Choice of Lagrangians

For simplicity, let us consider the SU(4) subgroup of SU(6), corresponding to non-strange particles. Here SU(6) boils down to a symmetry between spin and isospin, or between $\vec{\sigma}$ and $\vec{\tau}$. We may use this as a criterion to decide which Lagrangian to choose. We will therefore take the vertex $\overline{B}BM$ where M is alternately a meson with spin and no isospin (ρ) and isospin and with no spin (π), and expect to see here the $\vec{\sigma} \leftrightarrow \vec{\tau}$ interchange symmetry. For each meson there is a basic freedom, since we may choose either direct or derivative interactions. This is a well-known phenomenon in field theory, and we can easily see how it comes about.

First, let us consider the operator W_μ between spin ½ particle states at arbitrary momentum. Starting from equation V (13A) we have

$$W_\kappa = \frac{1}{2i} \varepsilon_{\kappa\lambda\mu\nu} J_{\lambda\mu} P_\nu \qquad (1)$$

$$= \frac{1}{4i} \varepsilon_{\kappa\lambda\mu\nu} \sigma_{\lambda\mu} P_\nu$$

$$= \frac{1}{4i} \varepsilon_{\kappa\nu\lambda\mu} \sigma_{\lambda\mu} P_\nu \qquad (2)$$

Using the identity

$$2\gamma_5 \sigma_{\kappa\nu} = - \varepsilon_{\kappa\nu\lambda\mu} \sigma_{\lambda\mu} \qquad (3)$$

We have

$$W_\mu = -\frac{1}{2i}\gamma_5 \sigma_{\kappa\nu} P_\nu \tag{4}$$

Substituting

$$i\sigma_{\mu\nu} = \gamma_\mu\gamma_\nu - \delta_{\mu\nu} \tag{5}$$

Equation (4) gives

$$W_\mu = \tfrac{1}{2}\gamma_5(\gamma_\mu\gamma_\lambda + \delta_{\mu\lambda})P_\lambda$$

$$= \tfrac{1}{2}\gamma_5(\gamma_\mu\gamma_\lambda P_\lambda + P_\mu) \tag{6}$$

When this acts on a spinor state of momentum p, we may substitute the Dirac equation

$$-i\gamma_\lambda P_\lambda \psi(p) = m\psi(p) \tag{7}$$

so equation (6) gives

$$W_\mu \psi(p) = \tfrac{1}{2}\gamma_5(im\gamma_\mu + P_\mu)\psi(p)$$

$$= \tfrac{1}{2}\gamma_5(m\gamma_\mu - \partial_\mu)\psi(p) \tag{8}$$

We know already, but in any case it is clearly seen from equation (8), that W_μ transforms as a pseudovector under the Lorentz group. We therefore couple $\bar{\psi} W_\mu \psi$ to a pseudovector, which we call ϕ_μ, and the invariant interaction is

$$\bar{\psi} W_\mu \psi \phi_\mu = \frac{im}{2}\bar{\psi}\gamma_5\gamma_\mu\psi\phi_\mu - \frac{i}{2}\bar{\psi}\gamma\,\partial_\mu\psi\phi_\mu \tag{9}$$

which, by partial integration of the second term, gives

$$\bar{\psi} W_\mu \psi \phi_\mu = \frac{im}{2}\bar{\psi}\gamma_5\gamma_\mu\psi\phi_\mu - \frac{i}{2}\bar{\psi}\gamma_5\psi\,\partial_\mu\phi_\mu \tag{10}$$

In order that $\partial_\mu \phi_\mu$ is non-zero, ϕ_μ must transform as $\partial_\mu\pi$, where π is a pseudoscalar field. In this case

$$\partial_\mu \phi_\nu - \partial_\nu \phi_\mu = 0 \qquad (11)$$

We note that equation (11) is three conditions, which eliminate the spin 1 field from the field which transforms as (½, ½) under the homogeneous Lorentz group, and is known to contain both spin 0 and spin 1 parts. Here we are left only with the spin 0 part. So let us put

$$\phi_\mu = \frac{1}{\mu} \partial_\mu \pi \qquad (12)$$

where π is the pion field, and μ its mass. Substituting (12) in (10) gives

$$\bar{\psi} W_\mu \psi \phi_\mu = \frac{im}{2\mu} \bar{\psi} \gamma_5 \gamma_\mu \psi \partial_\mu \pi - \frac{i\mu}{2} \bar{\psi} \gamma_5 \psi \pi \qquad (13)$$

where we have used the fact that $\square^2 \pi = \mu^2 \pi$. Equation (13) gives us the usual couplings of direct and derivative type. There is nothing special about the relative coefficients of these two terms, since each term separately is an invariant. In fact, one term may be transformed into the other by the Foldy-Dyson transformation (see e.g. Schweber (1961), p. 301), and since this transformation is a unitary transformation on the Hamiltonian, the S-matrix will remain unchanged, and the direct and derivative couplings are equivalent to first order in the scattering. This is the statement of the equivalence theorem, and it is important that the coefficients of the two terms in equation (13) are <u>not</u> related, otherwise the equivalence theorem will not hold.

This has taken care of the pion field. To complete it, let us give the pion isospin, and write equation (13) in generalised form

$$\mathcal{L}_{B\pi} = ig_1 \bar{\psi} \gamma_5 \underline{\tau} \psi \cdot \underline{\phi} + \frac{ig_2}{\mu} \bar{\psi} \gamma_5 \gamma_\mu \underline{\tau} \psi \cdot \partial_\mu \underline{\phi} \tag{14}$$

To find the corresponding interaction Lagrangian for a spin 1 field, we consider, instead of W_μ, $W_{\mu\nu} = \frac{-i}{m^2}\left[W_\mu, W_\nu\right]$. From the commutation relations of the W_μ (eq. (17), page 74), we see that

$$W_{\kappa\lambda} = \frac{-i}{m^2} \varepsilon_{\kappa\lambda\mu\nu} W_\mu P_\nu \tag{15}$$

$$= -\frac{1}{4m^2} \varepsilon_{\kappa\lambda\mu\nu} \varepsilon_{\mu\rho\sigma\tau} \sigma_{\rho\sigma} P_\tau P_\nu$$

$$= -\frac{1}{4m^2} \varepsilon_{\mu\kappa\lambda\nu} \varepsilon_{\mu\rho\sigma\tau} \sigma_{\rho\sigma} P_\tau P_\nu$$

$$= -\frac{1}{2m^2} \left(-m^2 \sigma_{\kappa\lambda} - \sigma_{\kappa\nu} P_\lambda P_\nu - \sigma_{\nu\lambda} P_\kappa P_\nu\right) \tag{16}$$

and substituting $\sigma_{\kappa\nu}$ from equation (5) gives

$$W_{\kappa\lambda} \psi(p) = \left[\tfrac{1}{2} \sigma_{\kappa\lambda} - \frac{i}{2m} (\gamma_\kappa \partial_\lambda - \gamma_\lambda \partial_\kappa)\right] \psi(p) \tag{17}$$

The invariant is now formed by taking the product of this with the field $\Omega_{\kappa\lambda}$, an antisymmetric tensor of second rank. So the invariant interaction is

$$\bar{\psi} W_{\mu\nu} \psi \Omega_{\mu\nu} = \bar{\psi} \sigma_{\mu\nu} \psi \Omega_{\mu\nu} + \frac{2}{m} \bar{\psi} \gamma_\mu \partial_\nu \psi \Omega_{\mu\nu}$$

which, by partial integration of the second term, becomes

$$\bar{\psi} W_{\mu\nu} \psi \Omega_{\mu\nu} = \bar{\psi} \sigma_{\mu\nu} \psi \Omega_{\mu\nu} + \frac{2\mu}{m} \bar{\psi} \gamma_\mu \psi V_\mu \tag{18}$$

where $$V_\mu = \frac{1}{\mu} \partial_\nu \Omega_{\mu\nu} \tag{19}$$

We again have our two familiar types of coupling for a spin 1 field, but let us note that the field is $\Omega_{\mu\nu}$ and not V_μ, so the first term in (18) is the <u>direct</u> coupling and the second one is the <u>derivative</u> one. This is the other way round from what we are accustomed to in electrodynamics. In that case, $\Omega_{\mu\nu}$ becomes $F_{\mu\nu}$ and V_μ becomes A_μ, the vector potential. We may redefine the fields this way as long as we are dealing with a vector field of <u>non-vanishing mass</u> M. (For the freedom we have of defining spin 1 fields, see the papers by Kemmer (1938) and (1960).) In this case Proca's equations are

$$\left.\begin{aligned} \partial_\nu \Omega_{\mu\nu} &= M V_\mu \\ (\partial_\mu V_\nu - \partial_\nu V_\mu) &= M \Omega_{\mu\nu} \end{aligned}\right\} \qquad (20)$$

To bring out the analogy with electromagnetism, we may redefine fields

$$v_\mu = M^{-1/2} V_\mu \ ; \quad \omega_{\mu\nu} = M^{1/2} \Omega_{\mu\nu}$$

so as to give equation (20) the form

$$\left.\begin{aligned} \partial_\nu \omega_{\mu\nu} &= M^2 v_\mu \\ (\partial_\mu v_\nu - \partial_\nu v_\mu) &= \omega_{\mu\nu} \end{aligned}\right\} \qquad (20')$$

where now the second equation (20') is gauge invariant, but the first is not unless $M = 0$.

Hence the SU(6) interaction of spin 1 fields necessitates non-zero mass for the field. If there were to be no other grounds for ruling out electromagnetism, this would suffice, but in fact we know that electromagnetic interactions do not respect SU(3) invariance, and also the little group for massless particles is not SU(2) but E(2), as discussed before.

We may write equation (18) generally, for the case of mesons with spin but no isospin, as

$$\mathcal{L}_{B\omega} = g_3 \bar{\psi} \sigma_{\mu\nu} \psi \Omega_{\mu\nu} + \frac{ig_4}{\mu} \bar{\psi} \gamma_\mu \psi V_\mu \qquad (21)$$

where the i's in equations (14) and (21) are arranged to make the expression hermitian, using a real meson field.

2. Form of the Interaction

To write down the combined interaction Lagrangian corresponding to the desired symmetry between spin 0 mesons with isospin and spin 1 mesons without isospin ($\vec{\sigma} \leftrightarrow \underline{\tau}$ symmetry) we combine equations (14) and (21) term by term, taking the two direct and the two derivative terms together. Let us do this in detail for the derivative coupling

$$\mathcal{L}_1 = \frac{ig_2}{\mu} \bar{\psi} \gamma_5 \gamma_\mu \underline{\tau} \psi \cdot \partial_\mu \underline{\phi} + \frac{ig_4}{\mu} \bar{\psi} \gamma_\mu \psi V_\mu \qquad (22)$$

Taking now the representation in which γ_5 is diagonal, i.e. $\gamma_i = \rho_2 \sigma_i$; $\gamma_4 = \rho_1$, $\gamma_5 = \rho_3$, we may set

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

where

$$\psi_L = \left(\frac{E \gamma_4}{m}\right)^{1/2} (a e^{ipx} + \hat{b}\, e^{-ipx})$$

is a left- handed chiral spinor, a is the particle annihilation operator, and

$$\hat{b} = i \sigma_2 b^*$$

is the time-reversal antiparticle annihilation operator. ψ_L and ψ_R are converted into each other by the parity operator γ_4. Writing out explicitly the bilinear forms, and separating their space and time components gives

$$i\bar{\psi}\gamma_\mu\psi = i(\psi_L^+ \ \psi_R^+)\gamma_4\gamma_\mu\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$$

$$= \left\{i(\psi_L^+\psi_L + \psi_R^+\psi_R),\ -(\psi_L^+ \ \psi_R^+)\gamma_5\sigma i\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}\right\}$$

$$= \left\{i(\psi^+\psi_L + \psi_R^+\psi_R),\ -(\psi_L^+\vec{\sigma}\psi_L - \psi_R^+\vec{\sigma}\psi_R)\right\} \qquad (23)$$

and

$$i\bar{\psi}\gamma_5\gamma_\mu\psi = \left\{i(\psi_L^+\psi_L - \psi_R^+\psi_R),\ (\psi_L^+\vec{\sigma}\psi_L + \psi_R^+\vec{\sigma}\psi_R)\right\} \qquad (24)$$

So on putting $g_2 = g_4$, $\mathcal{L}_1$ of equation (22) becomes

$$\mathcal{L}_1 = \frac{ig}{\mu}\Big\{-(\psi_L^+\psi_L + \psi_R^+\psi_R)V_0 - (\psi_L^+\vec{\sigma}\psi_L - \psi_R^+\vec{\sigma}\psi_R)\cdot\vec{V}$$

$$+ (\psi_L^+\underline{\tau}\psi_L - \psi_R^+\underline{\tau}\psi_R)\partial_t\underline{\phi} + (\psi_L^+\vec{\sigma}\underline{\tau}\psi_L + \psi_R^+\vec{\sigma}\underline{\tau}\psi_R)\cdot\vec{\nabla}\underline{\phi}\Big\} \qquad (25)$$

where $V_4 = iV_o$ and $\partial_4 = -i\partial_t$. The expansion for the spinors ψ_L and ψ_R, as we may see from equation (41), page 84, is

$$\psi_L(p) = \left(\frac{\not{p}}{m}\gamma_4\right)^{1/2}\left[a_p e^{ipx} + \hat{b}_p e^{-ipx}\right] \qquad (26)$$

$$\psi_R(p) = \left(\frac{\not{p}}{m}\gamma_4\right)^{-1/2}\left[a_p e^{ipx} - \hat{b}_p e^{-ipx}\right] \qquad (27)$$

and so

$$\psi_L^+(p) = \left(\frac{\not{p}}{m}\gamma_4\right)^{1/2}\left[a_p^+ e^{-ipx} + \hat{b}_p^+ e^{ipx}\right] \qquad (28)$$

where a and b are annihilation operators for particle and antiparticle, and

$$\hat{b} = i\sigma_2 b^*$$

is the time reversed antiparticle annihilation operator.

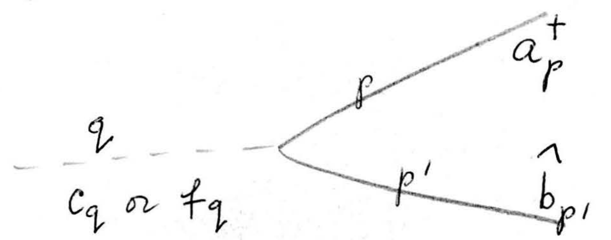
Similarly for the mesons, we may write

$$- i\partial_\mu \phi(q) = q_\mu \left(f_q\, e^{iq.x} + f_q^+\, e^{-iq.x}\right) \tag{29}$$

$$V_\mu(q) = \sum_{i=1}^{3} q_\mu^{(i)} \left(c_{q(i)}\, e^{iq.x} + c^+_{q(i)}\, e^{-iq.x}\right) \tag{30}$$

where the sum is over the three polarisation states of a massive spin one particle.

Now let us consider only the "creation" channel, i.e. let us pick out of equations (26) - (30) only those terms corresponding to the process



In the rest system of the meson, the only contributions come from $\partial_t \phi = q_o \phi$ and $\vec{V}$, since we can eliminate V_o by the Lorentz condition. The overall exponential is

$$e^{ipx - ip'x + iqx}$$

and so

$$p_\mu + p_\mu' = q_\mu$$

and in the rest frame, $q_o = \mu$ (meson mass). For the vector meson

$$q_4^{(i)} = 0 \qquad \left.\begin{array}{l} q_1^{(1)} = \mu \\ q_2^{(1)} = 0 \\ q_3^{(1)} = 0 \end{array}\right\} \qquad \left.\begin{array}{l} q_1^{(2)} = 0 \\ q_2^{(2)} = \mu \\ q_3^{(2)} = 0 \end{array}\right\}, \text{ etc.}$$

$$\therefore \quad q_n^{(i)} = \mu\, \delta_n^i \,.$$

Picking out the terms in (26) - (29) marked with a red cross, the expression (25) takes the form

$$\frac{ig}{\mu} \cdot \mu \quad a^{+}_{\vec{p}} \left(\frac{\not{p}}{m}\gamma_4\right)^{1/2} \left(\underline{\tau} \cdot \underline{f} + \vec{\sigma} \cdot \vec{C}\right) \left(\frac{\not{p}}{m}\gamma_4\right)^{-1/2} \hat{b}_{-\vec{p}} \tag{31}$$

since $\vec{p} = -\vec{p}'$, and where $\underline{f} = \underline{f}_{\vec{q}=0}$, $\vec{C} = \vec{C}_{\vec{q}=0}$. In the frame $\vec{p} = 0$, (31) is

$$g\mu a^{+} \left(\underline{\tau} \cdot \underline{f} + \vec{\sigma} \cdot \vec{C}\right) \hat{b} \tag{32}$$

and (25) takes the form

$$\mathcal{L}_1 = ig \left\{ a^{+}\left(\underline{\tau} \cdot \underline{f} + \vec{\sigma} \cdot \vec{C}\right) \hat{b} + \hat{b}^{+}\left(\underline{\tau} \cdot \underline{f}^{+} + \vec{\sigma} \cdot \vec{C}^{+}\right) a \right\} \tag{33}$$

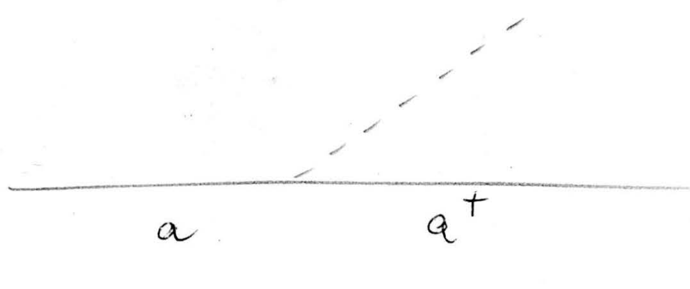
where the second term corresponds to the annihilation channel, and is the time reversed first term (Ryder, 1965).

Equation (33) exhibits the desired $\vec{\sigma} \leftrightarrow \underline{\tau}$ interchange symmetry, and under SU(2) transformations

$$\psi(x) \longrightarrow e^{i\vec{X} \cdot \vec{\omega}} \psi(x)$$

then $a \rightarrow e^{i\vec{\sigma} \cdot \vec{\omega}/2} a$, $b \longrightarrow e^{i\vec{\sigma} \cdot \vec{\omega}/2} b$

and so (33) is invariant. But clearly terms in the Lagrangian corresponding to processes



are not SU(6) invariant since, if the momenta of a and a^+ are time-like, that of the meson is space-like. This part of the Lagrangian is called $\mathcal{L}_2$, and violates SU(6). It so happens that in the static limit it gives zero contribution. (With scalar mesons $\mathcal{L}_2$ is invariant, and $\mathcal{L}_1$ breaks the symmetry). Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are related by crossing, SU(6) symmetry is not compatible with crossing, and therefore necessitates a non-local field theory, such as that of the Zachariasen model (1961), where $\mathcal{L}_1$ but not $\mathcal{L}_2$ is present. This is discussed by Gürsey (1965b) and Schroer (1964, 1965).

APPENDIX

THE UNITARY GROUPS AND YOUNG'S DIAGRAMS

The group SU(n) is the group of all unitary $n \times n$ matrices, with complex elements. Let us denote a vector in this complex n-dimensional space by x^i and its complex conjugate by x_i ; thus $x_i = (x^i)^*$. Under a transformation belonging to the group, the vectors x^i and x_i get transformed into the vectors $x^{i'}$ and x_i' according to

$$x^{i'} = \alpha_{ij} x^j \tag{1}$$

$$x_i' = \alpha^*_{ij} x_j = \alpha^{-1}_{ji} x_j \tag{2}$$

because unitarity of α implies

$$\alpha^+ = \alpha^{-1} \quad \text{or} \quad \alpha^*_{ij} = \alpha^{-1}_{ji} \tag{3}$$

(The symbol convention is $^+$ for hermitian conjugate and * for complex conjugate; we use the Einstein summation convention).

In the vector space we can define mixed tensors $A^{\alpha\beta\ldots\gamma}_{ij\ldots k}$ which transform according to

$$A^{\alpha\beta\ldots\gamma}_{ij\ldots k} = \alpha_{\alpha\lambda}\,\alpha_{\beta\mu}\cdots\cdots\alpha_{\gamma\nu}\,\alpha^{-1}_{li}\,\alpha^{-1}_{mj}\cdots\alpha^{-1}_{nk}\,A^{\lambda\mu\ldots\nu}_{lm\ldots n} \tag{4}$$

Very special tensors are δ^i_j , $\varepsilon^{ij\ldots k}$ and $\varepsilon_{ij\ldots k}$; they are unchanged under a transformation of the group. (The ε symbol has n components). We have

$$\delta^{i'}_j = \alpha_{ik}\alpha^{-1}_{lj}\delta^k_l = \alpha_{ik}\alpha^{-1}_{kj} = \delta^i_j$$

and

$$(\varepsilon^{ij\ldots k})' = \alpha_{i\ell}\,\alpha_{jm}\ldots\alpha_{kn}\,\varepsilon^{\ell m\ldots n}$$

$$= \det\alpha\;\varepsilon^{ij\ldots k}$$

$$= \varepsilon^{ij\ldots k}$$

because of the restriction to unimodular transformations.

A representation by a mixed tensor will in general be reducible because of the existence of the tensors δ^i_j, $\varepsilon^{ij\ldots k}$ and $\varepsilon_{ij\ldots k}$. For simplicity, let us deal first with SU(3), where these tensors are now δ^i_j, ε^{ijk} and ε_{ijk}.

With the help of these tensors, we can construct, from the general mixed tensor $A^{\alpha\beta\ldots\gamma}_{ij\ldots k}$ with p upper and q lower indices, the mixed tensors B, C and D, where

$$B^{\beta\ldots\gamma}_{j\ldots k} = \delta^i_\alpha\,A^{\alpha\beta\ldots\gamma}_{ij\ldots k}$$

has (p-1) upper and (q-1) lower indices

$$C^{\gamma\ldots\delta}_{\mu ij\ldots\ell} = \varepsilon_{\mu\alpha\beta}\,A^{\alpha\beta\gamma\ldots\delta}_{ij\ldots\ell}$$

has (p-2) upper and (q+1) lower indices, and

$$D^{m\alpha\beta\ldots\delta}_{k\ldots\ell} = \varepsilon^{mij}\,A^{\alpha\beta\ldots\delta}_{ijk\ldots\ell}$$

is a tensor with (p+1) upper and (q-2) lower indices. The tensors B, C and D are linear combinations of the elements of the tensor, A, with p upper and q lower indices. The transformation properties of B, C and D are however <u>different</u> from those of a tensor with p upper and q lower indices. The tensor A is therefore <u>reducible</u>, <u>unless</u> B, C and D are identically zero.

B = 0 when $A^{i\beta\ldots\gamma}_{ij\ldots k} = 0$, thus when the trace of A with respect to the indices α and i is zero. C = 0 when A is symmetric in the indices α and β; and D = 0 when A is symmetric in the indices i and j.

So, to construct bases for irreducible representations of SU(n), we take mixed tensors which are

(a) totally symmetric in all p upper indices,

(b) totally symmetric in all q lower indices,

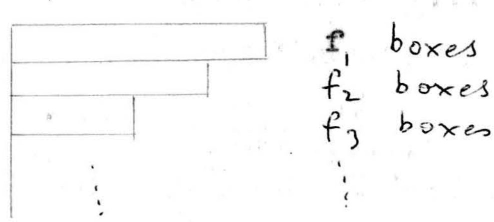
(c) traceless.

For SU(3) this gives the well-known dimensionality formula

$$N = \tfrac{1}{2}(p+1)(q+1)(p+q+2).$$

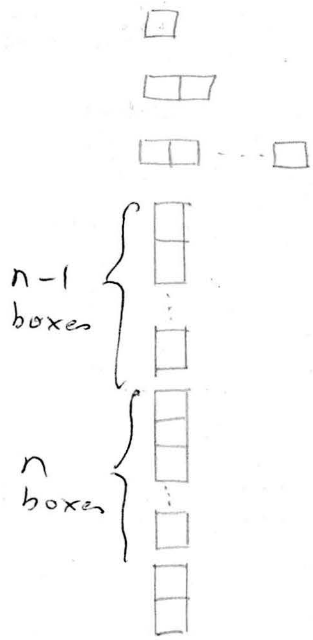
It is clear that there is an intimate connexion between the representations of SU(n) and the symmetric group, since irreducible representations of SU(n) are obtained by symmetrising indices in a space of n dimensions. The usual tool for dealing with the symmetric group is the use of Young's diagrams, and we shall now indicate, without proof, which can be found in the textbooks, how to use Young's diagrams with SU(n).

A Young's diagram is a collection of boxes, and the number of boxes is equal to the number of tensor indices. Corresponding to a general tensor, we have a general diagram, say



which is symmetric with respect to interchange of boxes in the same row, and antisymmetric with respect to interchange of boxes in the same column. With our tensors, we see we have to symmetrise in certain indices; but Hund's analysis (see, e.g., Hamermesh, Group Theory, p. 231) shows us that <u>symmetrising</u> on a certain combination of indices is <u>equivalent</u> to <u>antisymmetrising</u> on a "conjugate" combination of indices; this is where Young's diagrams become relevant.

We write, for introduction, a correspondence between simple diagrams and tensors:

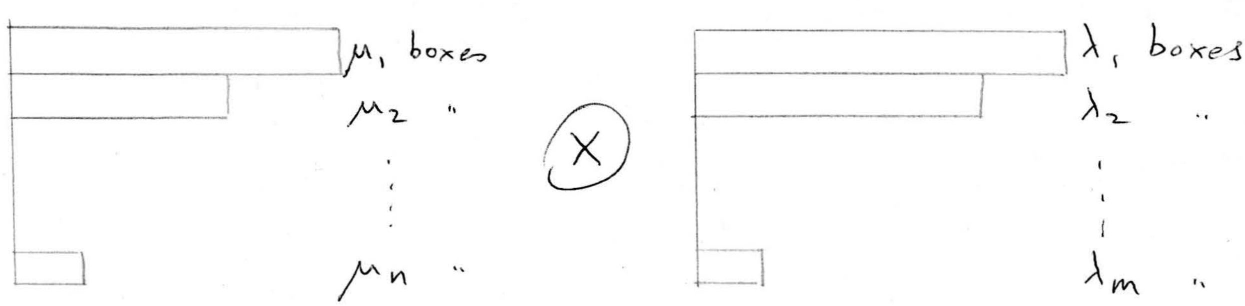


A^{α}	
$A^{\alpha\beta}$	α,β symmetric
$A^{\alpha\beta\ldots\gamma}$	$\alpha,\beta,\ldots\gamma$ "
A_{α}	conjugate to A^{α}
1 unit tensor	
$A^{[\alpha\beta]}$	antisymmetric in α,β.

We now state the rules for decomposing product diagrams into their irreducible parts, and give examples for SU(3) and SU(6), quoting the formulae for dimensionality. The behaviour of the general case is evident from that of SU(6).

Diagram Multiplication

Consider the decomposition of the product representation



To obtain the reduction of the product representation for U(n), we add to the $[\mu]$ Young diagram

λ_1 identical symbols α

λ_2 " " β

λ_3 " " γ

.

.

.

in this order, such that

(1) a regular scheme is formed with no two identical symbols in the same column,

(2) if all the added symbols are read from right to left in consecutive rows, starting at the top, we obtain a lattice permutation of

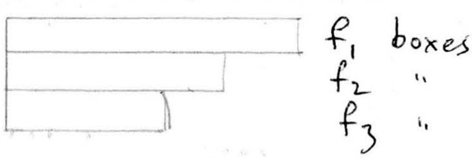
$$\alpha^{\lambda_1} \; \beta^{\lambda_2} \; \gamma^{\lambda_3} \; \ldots\ldots$$

i.e. such that among the first r terms, the number of times α occurs is <u>not less</u> than the number of times β occurs; similarly β does not occur fewer times than γ, etc., <u>for all r</u>.

In SU(n), a column of n squares has dimension 1. Also, we may never have a column of $>n$ boxes.

<u>SU(3)</u>

Dimensionality formula:-

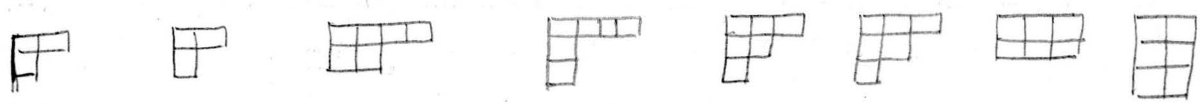


$$N = \frac{1}{2!}(f_1 - f_2 + 1)(f_1 - f_3 + 2)(f_2 - f_3 + 1) \quad .$$

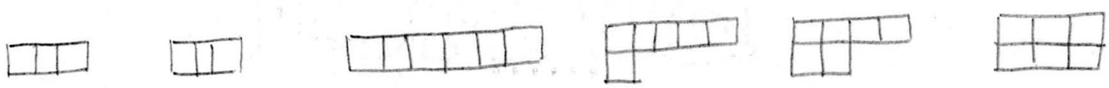
Decompositions

$3 \times 3 = 6 + \bar{3}$

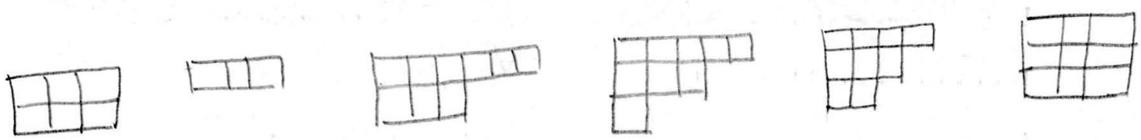
$\bar{3} \times 3 = 8 + 1$



$8 \times 8 = 27 + 10 + 8 + 8 + \overline{10} + 1$



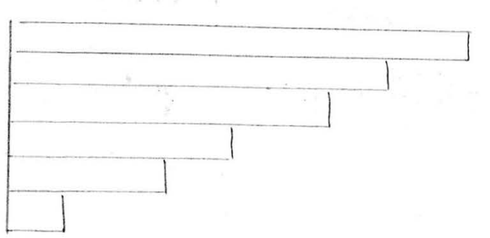
$10 \times 10 = 28 + 35 + 27 + \overline{10}$



$\overline{10} \times 10 = 64 + 27 + 8 + 1$

SU(6)

Dimensionality formula:-



f_1 boxes
f_2 "
f_3 "
f_4 "
f_5 "
f_6 "

$$N = \frac{1}{5!4!3!2!}(f_1-f_2+1)(f_1-f_3+2)(f_1-f_4+3)(f_1-f_5+4)(f_1-f_6+5)$$

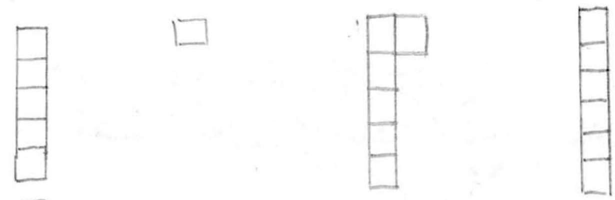
$$\times(f_2-f_3+1)(f_2-f_4+2)(f_2-f_5+3)(f_2-f_6+4)$$

$$\times(f_3-f_4+1)(f_3-f_5+2)(f_3-f_6+3)$$

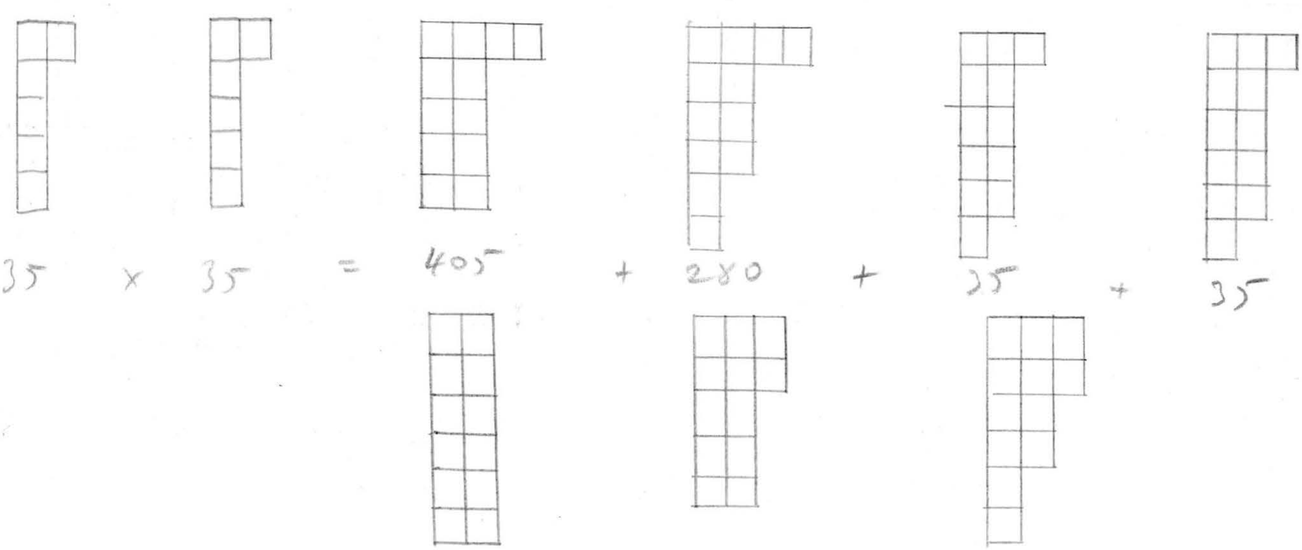
$$\times(f_4-f_5+1)(f_4-f_6+2)$$

$$\times(f_5-f_6+1).$$

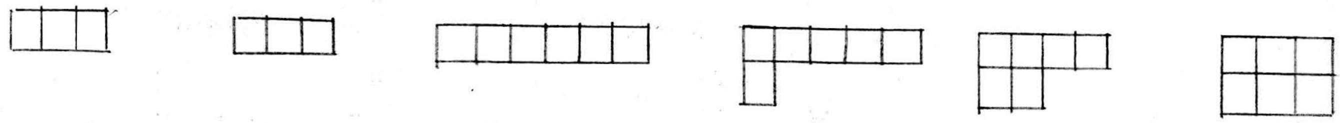
$6 \times 6 = 21 + 15$



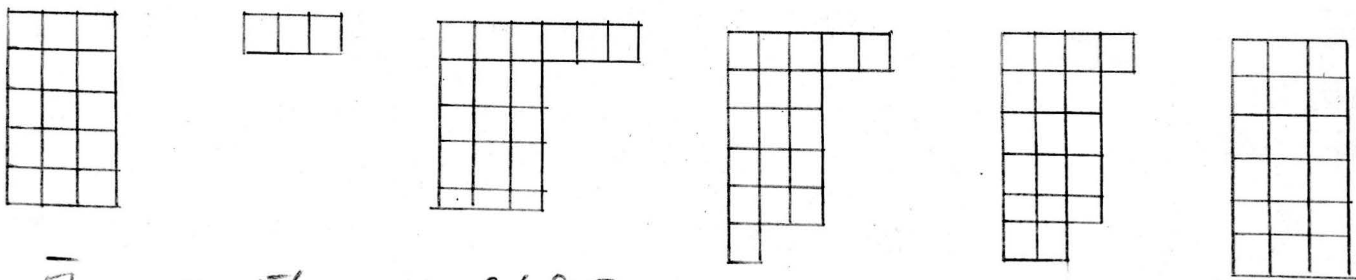
$\bar{6} \times 6 = 35 + 1$



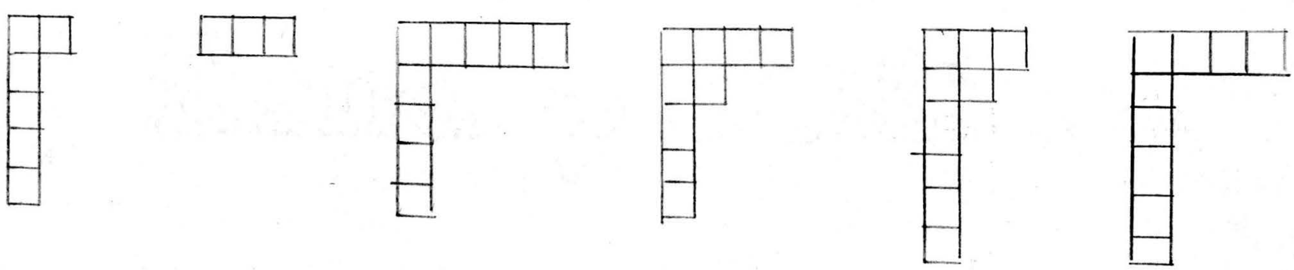
$35 \times 35 = 405 + 280 + 35 + 35 + 1 + \overline{280} + 189$



$$56 \times 56 = 462 + 1050 + 1134 + 490$$



$$\overline{56} \times 56 = 2695 + 405 + 35 + \underline{1}$$



$$35 \times 56 = 700 + 1134 + 70 + 56$$

REFERENCES

Amati, D., Prentki, J. & Stanghellini, A. (1962), Nuovo Cim. 26, 1003.

Babu, P. (1965), Phys. Rev. Letters 14, 166.

Bargmann, V. & Wigner, E.P. (1948), Proc. Nat. Acad. Sci. U.S. 34, 211.

Barnes, K.J., Carruthers, P. & von Hippel, E. (1965), Phys. Rev. Letters, 14, 82.

Barnes, V.E. et al. (1964), Phys. Rev. Letters 12, 204.

Bég, M.A.B., Lee, B.W. & Pais, A. (1964), Phys. Rev. Letters 13, 514.

Bég, M.A.B. & Singh, V. (1964a), Phys. Rev. Letters 13, 418.

Bég, M.A.B. & Singh, V. (1964b), Phys. Rev. Letters 13, 509.

Behrends, R. et al. (1962), Rev. Mod. Phys. 34, 1.

Bogolubov, N.N. et al. (1965), J.I.N.R. preprints, Dubna, D-1968, D-2075, P-2141.

Cabibbo, N. (1963), Phys. Rev. Letters 10, 531.

Cabibbo, N. (1964), Phys. Rev. Letters 12, 62.

Christenson, J.H. et al. (1964), Phys. Rev. Letters 13, 138.

Cutkosky, R.E. (1963), Ann. Phys. (N.Y.) 23, 415.

Cutkosky, R.E. (1964), Ann. Rev. Nucl. Sci., p. 175.

Dalitz, R.H. (1964), Lectures on Weak Interactions, Varenna.

d'Espagnat, B. (1962), C.E.R.N. Conference, p. 915.

de Swart, J.J. (1963), Rev. Mod. Phys. 35, 916 and erratum, do. 37, 326 (1965).

Dothan, Y., Gell-Mann, M. & Ne'eman, Y. (1965), Phys. Letters 17, 148.

Dyson, F.J. & Xuong N.-H. (1964), Phys. Rev. Letters 13, 815 and erratum do., 14, 339 (1965).

Feynman, R.P. & Gell-Mann, M. (1958), Phys. Rev. 109, 193.

Feynman, R.P., Gell-Mann, M. & Zweig, G., (1964), Phys. Rev. Letters 13, 678.

Foldy, L.L. & Wouthuysen, S.A. (1950), Phys. Rev. 78, 29.

Fradkin, D.M. and Good, R.H.Jr., (1961), Rev. Mod. Phys. 33, 343.

REFERENCES (Contd.)

Franzini, P. & Radicati, L.A. (1963), Phys. Letters 6, 322.

Freund, P.G.O., (1965), Phys. Letters 15, 352.

Fronsdal, C. (1962), Lectures on Group Theory, Brandeis Summer School.

Gasiorowitz, S. & Glashow, S.L. (1965), to be published.

Gell-Mann, M. (1961), Caltech. Report CTSL-20.

Gell-Mann, M. (1962), Phys. Rev. 125, 1067.

Gell-Mann, M. (1964a) in "Physics", Vol. 1.

Gell-Mann, M. (1964b) Phys. Letters 8, 214.

Gell-Mann, M. (1965a) Phys. Rev. Letters 14, 77.

Gell-Mann, M. (1965b) Lectures at Trieste Conference on High Energy Physics.

Gell-Mann, M. & Lévy, M. (1958), Nuovo Cim. 16, 705.

Gell-Mann, M. & Ne'eman, Y. (1965), The Eightfold Way. Published by W.A. Benjamin & Son Ltd., N.Y.

Gershtein, S.S. & Zeldovich, J.B. (1955), Zh. Eks. Teor. Fys. 29, 678; English trans., Society Physics, J.E.T.P. 2, 576.

Glashow, S.L. & Rosenfeld, A.H. (1963), Phys. Rev. Letters 10, 192.

Glashow, S.L. & Sakurai, J.J. (1962), Nuovo Cim. 25, 337 and 26, 622.

Gürsey, F. (1963) in "Relativity, Groups and Topology", Gordon & Breach, 1965.

Gürsey, F. (1964) in "Group Theoretical Concepts and Methods in Elementary Particle Physics," Gordon & Breach (1964).

Gürsey, F. (1965a), Phys. Letters 14, 330; Brookhaven National Laboratory Report No. 8767 (5th Jan.)

Gürsey, F. (1965b), Proceedings of Trieste Conference on High Energy Physics.

Gürsey, F. (1965c), Lectures on the Poincaré Group, Ankara (unpublished).

Gürsey, F. & Lee, T.D. (1963), Proc. Nat. Acad. Sci. U.S. 49, 179.

Gürsey, F., Lee, T.D. & Nauenberg, M., (1964), Phys. Rev. 135B, 467.

Gürsey, F. & Radicati, L.A. (1964), Phys. Rev. Letters 13, 173.

Gürsey, F., Pais, A. & Radicati, L.A. (1964), Phys. Rev. Letters 13, 29

Ikeda M., Ogawa, S. & Ohnuki, Y., (1959), Prog. Theor. Phys. 22, 715.

Jackson, J.D. (1962), Lectures on Weak Interactions, Brandeis Summer School.

Johnson, K. & Treiman, S.B. (1965), Phys. Rev. Letters 14, 189.

Joos, H. (1962), Fortschritte der Physik, 10, 65.

REFERENCES (Contd.)

Kemmer, N. (1938), Proc. Roy. Soc. A 166, 127.

Kemmer, N. (1960), Helv. Phys. Acta, 33, 829.

Kuo, T.K. & Yao, T. (1964), Phys. Rev. Letters 13, 415.

Lee, B.W. (1964), Phys. Rev. Letters 12, 83.

Lee, T.D. (1965), Nuovo Cim. 35, 933.

Levinson, C. Lipkin, H. & Meshkov, S. (1962), Nuovo Cim. 23, 236; Phys. Letters 1, 307.

Lipkin, H. (1965), Trieste Conference Proceedings, "Now we are SU(6)".

Lipkin, H. & Meshkov, S. (1965), Phys. Rev. Letters 14, 670.

Macfarlane, A.J. (1963), J. Math. Phys. 4, 490.

Marshak, R. & Sudarshan, E.C.G. (1958), Phys. Rev. 109, 1860.

Martin, André (1965), preprint.

Martin, A.W. & Wali, K.C. (1963), Phys. Rev. 130, 2455.

Martin, A.W. & Wali, K.C. (1964), Nuovo Cim. 31, 1324.

Matthews, P.T. & Salam, A. (1962), Proc. Phys. Soc, (London) 80, 28.

Matthews, P.T. & Salam, A. (1964), Phys. Letters 8, 357.

Ne'eman, Y. (1961), Nucl. Phys. 26, 222.

Nambu, Y. (1962)in Group Theoretical Concepts and Methods in Elementary Particle Physics, Gordon & Breach, 1964.

Oehme, R., Phys. Rev. Letters (1964), 12, 550, 604.

Oehme, R. & Segré, G. (1964), C.E.R.N. Report 8991/th 431.

Okubo, S. (1962), Prog. Theor. Phys. 27, 949; 28, 24.

Pais, A. (1964), Phys. Rev. Letters 13, 175.

Pais, A. & Treiman, S.N. (1964), Report at International Conference, (Dubna), to be published.

Pomeranchuk, I.Y. (1956), Zhur. Eks. Teor. Fys. 30, 423; English trans., Soviet Physics, J.E.T.P. 3, 306.

Pomeranchuk, I.Y. (1958), Zhur. Eks. Teor. Fys. 34, 725; English trans., Soviet Physics, J.E.T.P. 7, 499.

Pomeranchuk, I.Y. & Okun, L.B. (1956), Zhur. Eks. Teor. Fys. 30, 424; English trans. Soviet Physics, J.E.T.P. 3, 307.

Roos, M. (1963), Rev. Mod. Phys. 35, 314.

Rose, M.E. (1961), Relativistic Electron Theory. John Wiley, N.Y.

Ryder, L.H. (1965), Phys. Letters 17, 163.

Salam, A. et al. (1965a), Proc. Roy. Soc. A 284, 146.

Salam, A. (1965b), Lectures at Trieste Conference on High Energy Physics (to be published).

REFERENCES (Contd.)

Sakita, B. (1964a), Phys. Rev. 136B, 1756.

Sakita, B. (1964b), Phys. Rev. Letters 13, 643.

Sakurai, J.J. (1963), Lectures in Trieste Conference 1962. "Theoretical Physics", pub. I.A.E.A., Vienna.

Sakurai, J.J. (1964), Phys. Rev. Letters 12, 79.

Schroer, B. (1964) in Proc. Eastern Theoretical Physics Conference, Maryland.

Schroer, B. (1965) in Proc. Trieste Conference on High Energy Physics.

Schweber, S.S. (1961), "Relativistic Quantum Field Theory", Row, Peterson; reprinted Weatherhill, Inc., Tokyo, 1964.

Shaw, R. (1964), Nuovo Cim. 33, 1074.

Speiser, D. (1964) in "Group Theoretical Concepts and Methods in Elementary Particle Physics," Gordon & Breach, 1964.

Speiser, D. & Tarski, J. (1963). J. Math. Phys. 4, 588.

Sugawara, H. (1964), Prog. Theor. Phys. 31, 213.

Tarimer, N. (1965), Ankara preprint, to be published in Trieste Conference on High Energy Physics.

Tavkhelidze, A. (1965), in Proc. Trieste Conference on High Energy Physics.

Weinberg, S. (1964), Phys. Rev. 133B, 1318.

Wightman, A.S. (1960) in Relations de dispersicn et particules elementaires (Hermann, Paris & Wiley, N.Y.).

Wigner, E.P. (1937), Phys. Rev. 51, 105.

Wigner, E.P. (1939), Ann. Math. (U.S.) 40, 149.

Wigner, E.P. (1963) in Theoretical Physics (I.A.E.A., Vienna).

Wigner, E.P. (1964) in Group Theoretical Concepts and Methods in Elementary Particle Physics, Gordon & Breach.

Wigner, E.P. & Philips, T. (1962), Thesis of Philips, Princeton University (Unpublished).

Willis, W. et al. (1964), Phys. Rev. Letters 13, 291.

Wu, C.S. (1964), Rev. Mod. Phys. 36, 618.

Zachariasen, F. (1961), Phys. Rev. 121, 1851.

Zachariasen, F. & Zweig, G. (1965), Phys. Rev. Letters 14, 794.