# ORDER AND TRACE RESULTS FOR JORDAN ALGEBRAS

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ABSTRACT

The substance of this thesis falls into two parts. The first gives various results concerning the order structure of Jordan and von Neumann algebras and their pre-duals, relating these to such ideas as commutativity and factors.

The second part deals with the existance and uniqueness of a trace and a centre-valued trace on modular JW algebras - giving new proofs of these results, and shows that the closure of a subalgebra in the topology induced by the trace coincides with weak closure.

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## Declaration

I declare that this thesis has been composed by myself and that, except where otherwise stated, the results herein are my own

# CHAPTER I - INTRODUCTION

As is mentioned in the abstract, the first subject of this thesis, which is covered in Chapter II, is the relationship between properties of the natural ordering (i.e. that induced by the definition of elements of an algebra as positive ) of JC and von Neumann algebras, and their degree of commutativity.

Sections II.1 and II.4 give sufficient conditions on the order structure of a JC algebra for the ordinary operator product to commute on the algebra, and hence for the operator and jordan products to coincide. Section II.3 gives a converse of II.1, showing that minimum lattice structure goes with the minimum of commutativity.

Section II.2 gives a result for von Neumann algebras involving the concepts used in the rest of the chapter.

Chapter III begins with a brief resumé of definitions and known results to be used, and proceeds to demonstrate that modular JW algebras are characterised by possessing a (unique) faithful, normal centre-valued trace. The proof of this result, which is already known, is a new one based on von Neumann algebra work of F. J. Yeadon.

It is also shown that the JW subalgebras of a modular JW algebra are just those JC subalgebras that are closed in the toppology induced by the real-valued trace.

# CHAPTER II - ORDER RESULTS

## §1 - SHERMAN'S THEOREM FOR JORDAN ALGEBRAS

### 1.1 Definitions

A functional $\varphi$ on a partially ordered vector space V is <u>positive</u> if $\varphi(a) \geqslant 0$ for all $a \in V$ such that $a \geqslant 0$.

A partially ordered set S is a lattice if each pair of elements of S has a least upper bound and a greatest lower bound in S.

### 1.2 Lemma

Let V be a partially ordered normed real vector space such that for all $v \in V$ there exist $v_1, v_2 \in V^+$ such that $v = v_1 - v_2$, and $\|v\| = \max\{\|v_1\|, \|v_2\|\}$.

If V is a lattice, then so is its dual V* (the set of all bounded real-valued linear functions on V.)

The proof is that of Bratelli and Robinson [1] section 4.2.6.

### Proof

For any $\varphi \in V^*$, define $\varphi^{(+)}$ on $V^+$, the positive cone of V by

$$\varphi^{(+)}(a) = \sup\{\varphi(b) : b \in V,\ 0 \leqslant b \leqslant a\}.$$

Let $a_1, a_2 \in V^+$ and $0 \leqslant b \leqslant a_1 + a_2$.

Define $b_1 = b \wedge a_1$, $b_2 = b - b_1$ (this is possible as V is a lattice.)

So $b - a_2 \leqslant b$ as $a_2$ is positive, and $b - a_2 \leqslant a_1$ by the choice of b.

Therefore $b - a_2 \leqslant a_1 \wedge b = b_1$, and

therefore $b - b_1 \leqslant a_2$.

i.e. $b_2 \leqslant a_2$.

It follows immediately from the definitions of $b_1$ and $b_2$ that

$$0 \leqslant b_1 \leqslant a_1 \qquad \text{and}$$

$$b_1+b_2 \leqslant a_1+a_2.$$

Therefore $\varphi^{(+)}(a_1+a_2) = \sup \{\varphi(b_1+b_2) : 0 \leqslant b_1 \leqslant a_1,\ 0 \leqslant b_2 \leqslant a_2\}$

$$= \varphi^{(+)}(a_1) + \varphi^{(+)}(a_2).$$

i.e. $\varphi^{(+)}$ is additive on $V^+$, and hence linear on $V$.

Let $\omega_1, \omega_2 \in (V^*)+$.

Define $\omega_1 \vee \omega_2 = (\omega_1 - \omega_2)^{(+)} + \omega_2$.

For any $a \in V$ such that $a \geqslant 0$, and any $\varepsilon > 0$ there exists $b \in V$ such that $0 \leqslant b \leqslant a$ and

$$(\omega_1 - \omega_2)^{(+)}(a) \leqslant \omega_1(b) - \omega_2(b) + \varepsilon$$

Therefore $\omega_1 \vee \omega_2 \leqslant \omega$, $\forall \omega \geqslant \omega_1, \omega_2$.

I.e. $\omega_1 \vee \omega_2$ is the least upper bound of $\omega_1$ and $\omega_2$ in $V^*$.

Similarly, the greatest lower bound of $\omega_1$ and $\omega_2$ exists and is given by

$$\omega_1 \wedge \omega_2 = \omega_1 - (\omega_1 - \omega_2)^{(+)}$$

So $V^*$ is a lattice. □

<u>1.3 Lemma</u> (Kadison [5] Lemma 2 )

If $H$ is a Hilbert space and $e, f$ bounded projections from $H$ onto the manifolds $M$ and $N$ respectively, then $e \wedge f$, the projection onto $M \cap N$ is the greatest lower bound of $e$ and $f$ with respect to all positive bounded operators on H.

<u>Proof</u>

Let $0 \leqslant a \leqslant e,f$ be a linear operator on H.

We shall show that $a \leqslant e \wedge f$ :

For $\xi \in M^{\perp}$, $0 = (e\xi,\xi) \geqslant (a\xi,\xi) \geqslant 0$.

Therefore $a^{\frac{1}{2}}\xi = 0$.

Therefore $a\xi = 0$.

I.e. a annihilates $M^{\perp}$.

For all $\eta \in H$, $\xi \in M^{\perp}$, $(a\eta,\xi) = (\eta,a\xi) = 0$.

Therefore $a\eta \in M^{\perp\perp} = M$.

Therefore $aH \subset M$.

Similarly, $aH \subset N$.

Any $\zeta \in H$ can be expressed uniquely as $\xi + \eta$ with $\xi \in M \cap N$ and $\eta \in (M \cap N)^{\perp}$.

$$
\begin{aligned}
(e \wedge f\zeta,\zeta) &= (\xi,\xi) \\
&= (e\xi,\xi) \\
&\geqslant (a\xi,\xi) \\
&= (a\zeta,\zeta)
\end{aligned}
$$

I.e. $a \leqslant e \wedge f$ as required. □

<u>1.4 Lemma</u> ( Kadison [5] theorem 1 )

Let e,f be projections and S a real linear space of self-adjoint operators such that :

i) $e,\ f,\ e \wedge f,\ e \vee f \in S$

ii) e and f have an infimum in S ( in the order defined by the positive cone of S )

Then e and f commute.

Proof

$e \wedge f \leqslant e,f$, but by 1.3, $e \wedge f \geqslant \inf(e,f)$

So $e \wedge f = \inf(e,f)$.

Set $e' = e - \inf(e,f)$ and

$f' = f - \inf(e,f)$.

Then $(e')^2 = ( e - e \wedge f )^2$

$= e^2 - e(e \wedge f) - (e \wedge f)e + (e \wedge f)^2$

$= e - (e \wedge f) - (e \wedge f) + (e \wedge f)$

$= e'$.

I.e. e is a projection.

Similarly, f' is a projection.

$e \vee f - f' \geqslant e \vee f - f \geqslant 0$.

So $e' - ( e \vee f - f ) \leqslant e'$

and $e' - e \vee f \leqslant e - e \vee f \leqslant 0$.

So $e' - ( e \vee f - f' ) \leqslant f'$

Therefore $e' - ( e \vee f - f') \leqslant \inf (e',f') = 0$

So $e' \leqslant e \vee f - f'$

and $f'e'f' \leqslant f'( e \vee f - f')f'$

$= f' - f' = 0$.

But $f'e'f' = (e'f')^*(e'f')$

Therefore $e'f' = 0$.

$$ef = (e \wedge f + e')(e \wedge f + f')$$

$$= (e \wedge f)^2 + (e \wedge f)f' + e'(e \wedge f) + e'f'.$$

But it follows immediately from the definitions of $e'$ and $f'$ that

$$(e \wedge f)f' = e'(e \wedge f) = 0.$$

herefore $\quad ef = (e \wedge f)^2 + e'f'$

$$= e \wedge f.$$

Similarly $fe = e \wedge f$

I.e. e commutes with f. □

<u>1.5 Theorem</u>

Let J be a JC algebra.

If J is a lattice in the operator order, then J is commutative.

<u>Proof</u>

J is a lattice, so by 1.2 J* and J** are also.

J** is a JW algebra (Effros and Størmer [2]) and so contains the projection lattice meet (and hence join) of all pairs of projections.

Therefore any pair of projections in J** commutes (§1.4)

Therefore any pair of operators in J** commute (Finite linear combinations of projections are uniformly dence in J**.)

Therefore J, which is isomorphic to a subalgebra of J** is commutative.

## 2 - VON NEUMANN ALGEBRAS WHOSE PREDUALS ARE ANTILATTICES

The techniques of this theorem are those used by Green [3] in his result for the dual of C* algebras.

2.1 Theorem

Let M be a $\sigma$-finite von Neumann algebra ( i.e. let every family of non-zero orthogonal projections be at most countable. ), acting in its standard representation on a Hilbert space H, and let $\rho, \sigma \in (M_*)^+$. Also let $\xi, \eta \in H$ be such that $\rho = \omega_\xi$ and $\sigma = \omega_\eta$ ( for a proof that this is always possible, see e.g. Bratelli and Robinson [1] 2.5.31 ).

The following are then equivalent :

i) Inf $(\rho, \sigma) = 0$ in $(M_*)^+$

ii) There exists a projection e in the centre of M such that

$$\rho(e) = 1 \quad , \quad \sigma(e) = 0$$

iii) $(ab'\eta, \xi) = 0$ for all $a \in M$, $b' \in M'$ ( the commutant of M in this representation.

Corollary

For $\sigma$-finite algebras M, $M_*$ is an antilattice iff M is a factor.

( for proof of 'only if', see Green [3] ).

Note  Attempts to prove the analogous result for JW algebras ran into difficulties due to the need for the Double Commutant Theorem in iii) $\Rightarrow$ ii).

<u>Proof of Theorem</u>

iii) $\Rightarrow$ ii)

Let p be the projection onto $\{b'\xi : b' \in M'\}$

clearly $p\xi = \xi$

For any projection $q \in M'$

$$q(b'\xi) = (qb')\xi \in pH = M'\xi \quad \text{since } qb' \in M'.$$

Taking limits :

$$q\lambda \in [M'\xi] \quad \text{whenever } \lambda \in [M'\xi]$$

Therefore $qpH \subset pH$.

$$qp\xi \in pH \Rightarrow pqp\xi = qp\xi$$
$$\Rightarrow qp = pqp$$

Therefore $pq = (pqp)^*$

$$= pqp$$

$$= qp$$

I.e. p commutes with every projection in M'.

I.e. $p \in (M')' = M$.

Let e be the centralsupport of p in M.

Then e is the projection onto $\{ap\zeta : a \in M, \zeta \in H\}$ and $e\xi = \xi$.

$$(ab'\eta, \xi) = 0 \text{ for all } a \in M, \ b \in M'$$

$$\Rightarrow (ap\zeta, \eta) = 0 \quad \text{for all } a \in M, \ \zeta \in H$$

$$\Rightarrow (e\zeta, \eta) = 0 \quad \text{for all } \zeta \in H$$

$$\Rightarrow e\eta = 0.$$

So $\rho(e) = (\xi, e\xi) = (\xi, \xi) = 1$,

$$\sigma(e) = (\eta, e\eta) = (\eta, 0) = 0$$

i) $\Rightarrow$ iii)

Suppose $\inf(\rho,\sigma)=0$ and there exists $a \in M$, $b' \in M'$ such that $(ab'\eta, \xi) \neq 0$.

M is spanned by its positive elements, so we may assume that a and b' are positive.

There exists $\theta \in [0,2\pi[$ such that $(ab'\eta, e^{i\theta}\xi) < 0$.

$\rho = \omega_{e^{i\theta}\xi}$ , so we may replace $\xi$ by $e^{i\theta}\xi$

Thus we can assume $(ab'\eta, \xi) < 0$.

For all $\alpha \in \mathbb{R}$, define the functional on M :

$$\psi_\alpha(x) = \alpha^2(xb'\xi, b'\xi) + \alpha^2(xb'\eta, b'\eta) + \alpha(x\xi, b'\eta) + \alpha(xb'\eta, \xi)$$

Then $\psi_\alpha \in (M_*)_h$ ( $\psi_\alpha$ is defined by a finite sum of innerproducts and so is ultraweakly continuous.)

Let $x \in M^+$.

We then have :

$$\rho(x) + \psi_\alpha(x) = (x(\xi+\alpha b'\eta), \xi+\alpha b'\eta) + \alpha^2(xb'\xi, b'\xi) \geqslant 0$$

Since $b' \in M'$, $a, b' > 0$ we have :

$$(xb'\xi, \eta) = (x\xi, b'\eta)$$

and $\quad (x\eta, b'\xi) = (xb'\eta, \xi)$

and so we have :

$$\sigma(x) + \psi_\alpha(x) = (x(\eta+\alpha b'\xi), (\eta+\alpha b'\xi)) + \alpha^2(xb'\eta, b'\eta) \geqslant 0$$

So $-\psi_\alpha \leqslant \rho$, $\quad -\psi_\alpha \leqslant \sigma$

Therefore $-\psi_\alpha \leq \inf(\rho,\sigma) = 0$

I.e. $\psi_\alpha \geq 0$

Therefore $\psi_\alpha(a) \geq 0$ as a is positive.

But $\psi_\alpha(a) = \alpha^2(ab'\xi, b'\xi) + \alpha^2(ab'\eta, b'\eta) + 2\alpha(ab'\eta, \xi)$

$< 0$ for small $\alpha > 0$, which is a contradiction.

Hence $(ab'\eta, \xi) = 0$ for all $a \in M$, $b' \in M'$

ii) $\Rightarrow$ i)

Let $\psi \in (M_*)_h$, $\psi \leq \rho, \sigma$

Since e is in the centre of M, if $a \in M^+$ we have :

$$ea \geq 0 \;, \quad (1-e)a \geq 0$$

( $ea = e^2a = eae$. Therefore $(eae\xi, \xi) = (ae\xi, e\xi) \geq 0$ )

Therefore $\psi(ea) \leq \sigma(ea) = 0$

and $\psi((1-e)a) \leq \rho((1-e)a) = 0$

Thus $\psi(a) = \psi(ea) + \psi((1-e)a) \leq 0 \qquad a \in M^+$

Hence $\inf(\rho, \sigma) = 0$ □

## §3 - A JW Algebra is an Antilattice iff it is a Factor

### 3.1

Topping has shown ([9] proposition 22) that if a JW algebra forms an antilattice in the usual ordering then it is a factor. The purpose of this section is to show that the converse is also true. Lemma 3.2 is based on ideas of Green [3].

### 3.2 Lemma

Let J be a JW algebra and $a,b \in J$.

Then $\inf(a,b)=0 \Rightarrow \{axb\} = 0$ for all $x \in J$

**Proof**

We may assume that $\|a\|, \|b\| \leqslant 1$.

For $\alpha \in \mathbb{R}$ let :

$$x_\alpha = \alpha^2 ax^2 a + \alpha^2 bx^2 b + \alpha( axb + bxa )$$

Then $x_\alpha + b^2 = (\alpha ax + b)(\alpha xa + b) + \alpha^2 bx^2 b \geqslant 0$

$x_\alpha + a^2 = (\alpha bx + a)(\alpha xb + a) + \alpha^2 ax^2 a \geqslant 0$

So $\left.\begin{array}{l} -x_\alpha \leqslant a^2 \leqslant a \\ -x_\alpha \leqslant b^2 \leqslant b \end{array}\right\}$ since $\|a\|, \|b\| \leqslant 1$

So $-x_\alpha \leqslant \inf(a,b) = 0$

I.e. $x_\alpha \geqslant 0$

But if $\{axb\} \neq 0$ there exists $\xi \in H$ such that $(\{axb\}\xi, \xi) \neq 0$

and $(x_\alpha \xi, \xi) = \alpha^2(\|xa\xi\|^2 + \|xb\xi\|^2) + 2\alpha(\{axb\}\xi, \xi)$

$< 0$ for some small $\alpha \in R$

Therefore $\{axb\} = 0$ $\square$

3.3 Corollary

If A and B are the respective range projections of a and b then

$$\text{Inf}(a,b) = 0 \Rightarrow \{AJB\} = 0$$

Proof

Assume as before that $a,b \leqslant 1$

Then $0 \leqslant a^n \leqslant a$ and $0 \leqslant b^m \leqslant b$

and $0 \leqslant \inf(a^n,b^m) \leqslant \inf(a,b)$ for all $m,n \in \mathbb{N}$

I.e. $\inf(a^n,b^m) = 0$

So by §3.2, $\{a^n x b^m\} = 0$ for all $x \in J$

Therefore $\{p(a)xq(b)\} = 0$ for all polynomials p,q with zero constant term.

But A is the strong limit of a sequence of such polynomials ( see e.g. Topping [8] lemma 2 ) , and similarly so is B

Therefore $\{AxB\} = 0$ for all $x \in J$

Therefore $\{AJB\} = 0$ . $\square$

3.4 Lemma (Topping [9], corollary 18)

If e and f are any two projections in a JW algebra J they can be written as orthogonal sums :

$$e = e_1 + e_2 \qquad f = f_1 + f_2$$

where $e_1$ and $f_1$ are exchanged by a symmetry in J and $C(e_2) \perp C(f_2)$

<u>3.5 Lemma</u> ( Topping [9] lemma 24 )

Let e,f be non-zero projections in a JW algebra J.

Then $\{eJf\} = 0 \Rightarrow C(e) \perp C(f)$.

<u>Proof</u>

$$eof = e1f + f1e \in \{eJf\} = \{0\}.$$

Therefore $2efe = eo(eof) + (eof)oe - 2eof = 0$ .

I.e. $(ef)(ef)^* = 0$ .

Therefore $ef = 0$ .

Therefore $eJf = eJf^2 + fJ(ef)$

$$= \{eJf\}f$$

$$= 0.$$

Suppose $C(e) \not\perp C(f)$.

Then, in the notation of §3.4 either $e \neq e_2$ or $f \neq f_2$ : say the former, in which case $e_1 \neq 0$ and so $f_1 \neq 0$.

Let s be the symmetry exchanging $e_1$ and $f_1$.

Now $f_1 = f_1^2$

$= se_1sf_1$

$= se_1(esf)f_1$

$= 0$ , which is a contradiction.

So $C(e) \perp C(f)$ □

<u>3.6 Theorem</u>

Any JW factor is an antilattice.

<u>Proof</u>

If J is not an antilattice, then there exist $0 < a,b \leqslant 1$ such that

$$\text{Inf}(a,b) = 0.$$

By §3.3, (and using the notation of § 3.3) $\{AJB\} = 0$

Therefore by §3.5, $C(A) \perp C(B)$.

Since $C(A) \neq 0 \neq C(B)$ we have $0 \neq C(A) \neq 1$

I.e. J has a non-trivial centre and so is not a factor. □

## § 4 – A CONDITION IMPLYING COMMUTATIVITY OF A JORDAN ALGEBRA

Theorem

For any JC algebra J, if :

$$0 \leq x \leq y \Rightarrow x^2 \leq y^2 \qquad \text{for all } x,y \in J$$

then J is commutative.

Proof

Take $x, y \in J^+$ and $\varepsilon > 0$

Then $x \leq x + \varepsilon y$ whence

$$x^2 \leq (x + \varepsilon y)^2$$

$$= x^2 + 2\varepsilon x \circ y + \varepsilon^2 y^2$$

which gives $\quad 0 \leq xy + yx + \varepsilon^2 y^2 \qquad$ for all $\varepsilon > 0$

i.e. $\quad 0 \leq xy + yx \qquad (*)$

Set $\quad xy = a + ib$

where $\quad a = \frac{1}{2}(xy + yx) \in J \quad$ ( clearly $a \geq 0$ )

and $\quad b = \frac{1}{2}i(xy - yx) \in C^*(J)$

The positive elements of J are positive in $C^*(J)$, so the ordering of $C^*(J)$ extends that of J.

Also if $c, d \in J$ then $c \leq d$ in $C^*(J)$ iff $c \leq d$ in J

i.e. for such c, d , $\leq$ is unambiguous.

xyx and y are positive

and $\quad (xyx)y = (xy)^2$

$$= a^2 - b^2 + i(\, ab + ba \,) \qquad (**)$$

Therefore $a^2 - b^2 \geq 0$ by (*) with x replaced by xyx.

The set E of numbers $\alpha \geqslant 1$ such that $\alpha b^2 \leqslant a^2$ for all $x,y \in J^+$ with

$xy = a+ib$ ( $a \in J^+$ , $b \in C^*(J)$ ) is therefore nonempty.

E is also closed, so if it were bounded, it would have a largest

element, say $\lambda$

Thus if $x,y \in J^+$ and $xy=a+ib$, then $a^2 - \lambda b^2 \geqslant 0$ , and therefore by (*) :

$$0 \leqslant b^2( a^2 - \lambda b^2) + ( a^2 - \lambda b^2)b^2$$
$$= ( b^2a^2 + a^2b^2 ) - (2\lambda b^4) \qquad (***)$$

From (**) we have :

$$\lambda( ab+ba )^2 \leqslant ( a^2-b^2)^2$$

That is,

$$\lambda( ab^2a + ba^2b + a(bab) + (bab)a ) \leqslant a^4 + b^4 - a^2b^2 - b^2a^2 \quad (\dagger)$$

On LHS, $a(bab) + (bab)a \geqslant 0$ (in $C^*(J)$ ) by (*)

By assumption, $a^2 \geqslant \lambda b^2$, so $ba^2b \geqslant \lambda b^4$

And finally, $ab^2a \geqslant 0$ in J

Using this, and inserting (***) on RHS of ($\dagger$)

$$\lambda^2 b^4 \leqslant a^4 + (1-2\lambda)b^4$$

That is

$$( \lambda^2 + 2\lambda - 1)b^4 \leqslant a^4$$

By Pedersen [12] chapter 1.3.8 we have

$( \lambda^2 + 2\lambda - 1)^{\frac{1}{2}}b^2 \leqslant a^2$ contradicting the maximality of $\lambda$,

since $\lambda \geqslant 1$ .

Therefore E is unbounded,

therefore $\mu b^2 \leqslant a^2$ for all $\forall \mu > 0$

Therefore $b = 0$.

So $xy = a$

$$= \tfrac{1}{2}( xy + yx )$$

therefore $xy = yx$. $\square$

This result also follows from the work of Topping [17], which contains simplifications of Kadison's work quoted in this chapter.

# CHAPTER III - THE CENTRE VALUED TRACE

## § 1 Types of JW Algebras

The purpose of this section is to bring together definitions and structure results that will be used in the rest of the chapter. All of it, except the classification of type I JW algebras which can be found in [10] chapter 5.3 can be found in Topping [3].

**Definition**

A lattice L is called modular if

$$(e \cup f) \cap g = e \cup (f \cap g) \text{ whenever } e \leq f \quad (\forall\, e,f,g \in L)$$

A projection e in a JW algebra J is modular if the projection lattice of eJe is modular.

**Theorem**

For a JW algebra J the following are equivalent :

i) J is modular

ii) if $e,f \in J$ are projections such that $e \sim f$ and $e \leq f$, then $e = f$

iii) Every orthogonal family of equivalent projections in J is finite.

**Theorem**

If e and f are modular projections in a JW algebra J, then $e \cup f$ is modular.

**corollary**

Two equivalent modular projections in a JW algebra J can be exchanged by a symmetry in J.

Definitions

A projection e in a JC algebra J is minimal if there exists no projection $f \in J$ such that $0 < f < e$

A projection $e \in J$ is abelian if eJe is commutative.

All minimal projections are abelian, for if e is minimal and $f \in eJe$ is a projection, then $f \leqslant e$ and so $f = 0$ or e, in which case $eJe = \mathbb{R}e$ which is commutative.

If J is a factor, all abelian projections are also minimal.

A JW algebra J is :

type I if J contains a faithful abelian projection;

continuous if J contains no abelian projection;

type II if J is continuous and contains a faithful modular projection ( i.e. is locally modular );

type III or purely non-modular if J contains no non-zero, modular projection.

A JW algebra J is properly non-modular if J contains no central modular projection.

Theorem

Any JW algebra decomposes uniquely into five summands as follows:

i) type I modular

ii) type I properly non-modular, locally modular

iii) type II modular ( i.e. type $II_1$ )

iv) type II properly non-modular, locally modular ( i.e. type $II_\infty$ )

v) type III

A JW factor has one and only one of these types.

Definition

A JW algebra J is homogeneous if there exists an orthogonal family $(e_\alpha)_{\alpha \in A}$ of abelian projections such that $C(e_\alpha) = 1 = \sum_{\alpha \in A} e_\alpha$

If card( A ) = n ( n finite or infinite ) we say that J has type $I_n$

Theorem

Each JW algebra J of type I has a unique decomposition :

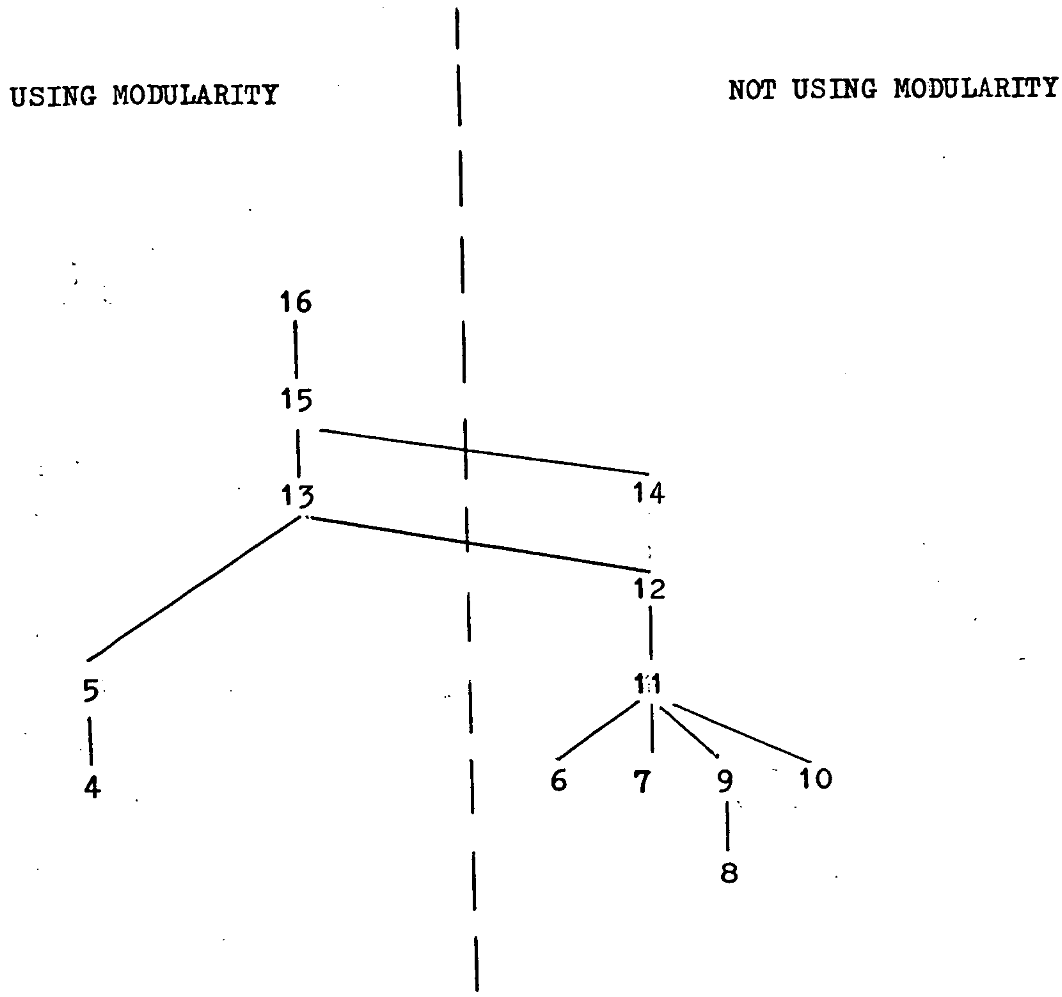
$$J = J_1 \oplus J_2 \oplus \cdots \oplus J_\infty$$

where each $J_n$ is either zero or a JW algebra of type $I_n$.

Definition

Let $J_*$ be the set of all real valued ultraweakly continuous linear functionals on J. Schultz has shown ( [14] Thm. 2.3 ) that $(J_*)^*$, the Banach space dual of $J_*$ is J. $J_*$ can therefore be called the predual of J.

THE DEPENDENCE OF RESULTS AND THE USE OF MODULARITY IN §2



## § 2 – Existence of a Centre-Valued Trace on a Modular JW Algebra

### 2.1 Definition

A centre - valued trace on a JW algebra $J$ is a map $T$ from $J$ to its centre $Z$ satisfying :

i) $T$ is linear

ii) $T(za) = zT(a)$ for all $a \in J$, $z \in Z$

iii) $T(a) \geq 0$ if $a \geq 0$, $a \in J$

iv) $T(sas) = T(a)$ for $a \in J$, $s$ a symmetry in $J$

v) $T(z) = z$ for all $z \in Z$

$T$ is faithful if $a \geq 0$, $T(a) = 0$ implies $a = 0$.

$T$ is normal if for every bounded increasing family $I \subset J^+$

$$\sup T(I) = T(\sup I).$$

Topping ([9]Cor 28 ) gives a proof that a JW algebra has at most one centre-valued trace. An alternative proof of this is given in §3 of this chapter.

### 2.2 Definitions

Two projections $e,f$ in a JC algebra $J$ are said to be perspective if they possess a common complement, i.e. there exists a projection $g \in J$ such that :

$$eg = 0 = fg \qquad e \cup g = 1 = f \cup g$$

Projections $e,f$ in a JC algebra $J$ are said to be equivalent if there exist symmetries $s_1, \dots, s_n \in J$ such that

$$e = s_1 \dots s_n f s_n \dots s_1$$

$e$ and $f$ bear the relation $e \lesssim f$ if there exists $f_1 \leq f$ such that $e$ is equivalent to $f_1$ (written $e \sim f_1$ ). If $e \lesssim f$ but $e \not\sim f$, we write $e \prec f$. Equivalently, $e \lesssim f$ if there exists $e_1 \geq e$ such that $e_1 \sim f$

Proof

If $e \sim f_1 \leqslant f$ let

$$e = s_1 \ldots s_n f_1 s_n \ldots s_1 \quad \text{then}$$

$$e \leqslant e_1 = s_1 \ldots s_n f\, s_n \ldots s_1 \sim f$$

Conversely, if $e \leqslant e_1 \sim f$ let

$$r_1 \ldots r_m e_1 r_m \ldots r_1 = f \text{ then}$$

$$r_1 \ldots r_m e r_m \ldots r_1 = f - r_1 \ldots r_m(e_1 - e)r_m \ldots r_1,$$ which is a projection and $\leqslant f$ .

An algebraic property of modular JW algebras is given by Topping ([9] corollary 12 )

Theorem

Let J be a modular JW algebra.

Then $e \sim f$ iff e and f are perspective in J,

Both $\sim$ and perspectivity are completely additive in J

2.3

If $\tau$ is a faithful, normal centre-valued trace on a JW algebra J then for projections $e,f \in J$ and any symmetry $s \in J$

i) $\tau(e) = 0 \Rightarrow e = 0$ (faithfulness )

ii) $\tau(ses) = \tau(e)$

iii) $e \perp f \Rightarrow \tau(e \cup f) = \tau(e) + \tau(f)$ (trivial case of normality )

i.e. the centre-valued trace satisfies the conditions of Topping ([9] corollary 9 ):

Corollary

Let J be a JW algebra on whose projection lattice L a mapping $e \to d(e)$ is defined into some abelian group such that :

i) $d(e \cup f) = d(e) + d(f)$ if $e \perp f$

ii) $d(ses) = d(e)$

iii) $d(e) = 0 \Rightarrow e=0$ Then L is modular i.e. the existence of a faithful, normal centre-valued trace on a JW algebra implies that the algebra is modular. The rest of this section is devoted to the proof of the reverse implication: that a modular JW algebra possesses a faithful, normal centre-valued trace (not necessarily positive ) and the following section to the proof that a modular JW algebra possesses at most one such trace, and that is positive. We then have :

Theorem

A JW algebra is modular iff it possesses a faithful, normal centre-valued trace. If it possesses such a trace it is unique

This result is already known, but a new proof is given here along the lines devised by Yeadon [10] . While writing this proof the author made use of an unpublished set of notes of lectures on von Neumann algebras given by Professor Ringrose at Newcastle.

In all the results of this and the following section, J is a JW algebra.

2.4 Lemma

Let J be modular and $(e_k)$ an increasing sequence of projections in J. If for all k, $e_k \lesssim f$ for some projection $f \in J$, then $\sup(e_k) \lesssim f$

Proof

The sequence $e_1$, $(e_2-e_1)$, $(e_3-e_2)$, ... is orthogonal and has supremum $e = \sup(e_k)$. If, therefore, we can construct a sequence $(f_k)$ of of orthogonal subprojections of f such that :

$$f_1 \sim e_1 \quad \text{and} \quad f_k \sim e_k - e_{k-1}$$

then, by the complete additivity of equivalence on the projection lattice of a modular JW algebra ( §2.2 above ),

$$e \sim \sum f_k \leq f.$$

$(f_n)$ is constructed inductively :

Since $e_1 \lesssim f$ there exists $f_1 \leq f$ such that $e_1 \sim f_1$

If $f_1, \ldots, f_r$ are given, it is sufficient to construct $f_{r+1}$ such that

$$f - (f_1 + \ldots + f_r) \geq f_{r+1} \sim e_{r+1} - e_r .$$

Since $e_{r+1} \lesssim f$, $1 - e_{r+1} \gtrsim 1 - f$ (if $e_{r+1} \sim z \leq f$, then $1 - e_{r+1} \sim 1 - z \geq 1 - f$ )

i.e. there exists g such that $1 - f \sim g \leq 1 - e_{r+1}$.

$g \perp e_r$.

$f_1 + \ldots + f_r \sim e_1 + (e_2 - e_1) + \ldots + (e_r - e_{r-1}) = e_r$ by additivity of $\sim$, and

$g \sim 1 - f$.

so $\quad 1 - f + f_1 + \ldots + f_r \sim g + e_r$ ,

and so $\quad f - (f_1 + \ldots + f_r) \sim 1 - g - e_r$

$$\geq 1 - (1 - e_{r+1}) - e_r = e_{r+1} - e_r .$$

So by the remark following the definition of equivalence, there exists

$f_{r+1} \leq f - (f_1 + \ldots + f_r)$ such that

$f_{r+1} \sim e_{r+1} - e_r$ as required.

### 2.5 Definition

Let J be a JC algebra and U the group:

$$\{s_1 \dots s_n : s_i \text{ is a symmetry in } J,\ n \in \mathbb{N}\}$$

Given $u \in U$, define an isometric isomorphism :

$L_u : J_* \to J_*$ by :

$$(L_u \omega)(a) = \omega(u^* a u) \qquad (\omega \in J_*,\ a \in A)$$

where $(s_1 \dots s_n)^* = s_n \dots s_1$

We denote by $Q_\omega$ the norm-closed convex hull of the set :

$$K_\omega = \{L_u \omega : u \in U\}$$

**Lemma**

If J is modular, $(e_n)$ an orthogonal sequence of projections in J and $\omega \in J_*$, then $\tau(e_n) \to 0$ as $n \to \infty$, uniformly for all $\tau \in Q_\omega$

**Proof**

It suffices to show that $\tau(e_n) \to 0$ uniformly for $\tau \in K_\omega$.

If not, there exists $\delta > 0$, a subsequence $(f_n)$ of $(e_n)$ and a sequence $(\tau_n)$ in $K_\omega$ such that :

$$|\tau_n(f_n)| \geq \delta \quad \text{for each } n$$

Then $\tau_n(.) = \omega(u_n^* .\ u_n)$ for some $u_n \in U$.

If $g_n = u_n^* f_n u_n$, then $g_n \sim f_n$ and $|\omega(g_n)| \geq \delta$

Define: $\quad p_{m,n} = \sup(g_m, g_{m+1}, \dots, g_n)$

$$p_m = \sup\{g_j : j \geq m\} = \sup\{p_{m,n} : n \geq m\}$$

$$p = \inf\{p_m : m \geqslant 1\}.$$

Then $p_{n,n} \leqslant p_{n,n+1} \leqslant \cdots$

We claim $p_{m,n} \precsim f_m + \dots + f_n \qquad (m \leqslant n) \qquad (*)$

Since $p_{n,n} = g_n \sim f_n$ (*) is true when m=n

If $n \geqslant m$ and (*) is known to be true for n then :

$$p_{m,n} \precsim f_m + f_{m+1} + \dots + f_n$$

and $$g_{n+1} \vee p_{m,n} - p_{m,n} = g_{n+1} - g_{n+1} \wedge p_{m,n}$$
$$\leqslant g_{n+1}$$
$$\sim f_{n+1} .$$

So (*) holds for n+1, and so for all $n \geqslant m$

Therefore $p_{m,n} \precsim \sum_{j=m}^{\infty} f_j$ .

By §2.4 $\quad p_m \precsim \sum_{j=m}^{\infty} f_j \quad$ hence

$$1 - \sum_{j=m}^{\infty} f_j \precsim 1 - p_m \leqslant 1 - p .$$

Again by § 2.4,

$$1 = \sup\left\{1 - \sum_{j=m}^{\infty} f_j : n \in \mathbb{N}\right\} \precsim 1-p$$

Since J is modular, p=0 (Topping [9] prop 14 ), and so since $p = \wedge p_n$

and $p_1 \geqslant p_2 \geqslant \dots$ we have :

$$0 \leqslant (g_n \xi, \xi) \leqslant (p_n \xi, \xi) \leqslant \|p_n \xi\| \|\xi\| \to 0 \quad \text{for all } \xi \in H .$$

Hence $g_n \to 0$ ultraweakly, and so $\lim \omega(g_n) = 0$. I.e. $\tau_n(f_n) \to 0$ , contradicting the choice of $(\tau_n)$ and $(f_n)$.

So $(e_n) \to 0$ uniformly for $\tau \in K_\omega$ as required. □

2.6 Lemma

Suppose $\omega \in J^*$ and $\eta \in \mathbb{R}$. Then :

i) If $\omega(a) > \eta$ for some positive $a \in J_1$, then $\omega(e) > \eta$ for some projection $e \in J$

ii) If $|\omega(f)| \leqslant \eta$ for every projection $f \in J$, then $\|\omega\| \leqslant 2\eta$

iii) If $\omega(f) \geqslant 0$ for every projection $f \in J$, then $\omega \in J^{*+}$.

See over for Proof

Proof of Lemma 2.6

i) By the spectral theorem, there exists an orthogonal family of projections $e_1, \dots, e_n \in J$ and $\lambda_1, \dots, \lambda_n \in [0,1]$ such that

$$\| a - \sum_1^n \lambda_j e_j \| \leqslant \|\omega\|^{-1} (\omega(a) - \eta)$$

Hence $\omega(a) - \sum \lambda_j \omega(e_j) < \omega(a) - \eta$

and $\sum \lambda_j \omega(e_j) > \eta$

We can assume that $\omega(e_j) > 0 \quad (1 \leqslant j \leqslant m)$

$\omega(e_j) \leqslant 0 \quad (m < j \leqslant n)$

Let $e = e_1 + \dots + e_m$

$$\omega(e) = \sum_{j=1}^{m} \omega(e_j)$$

$$\geqslant \sum_{j=1}^{m} \lambda_j \omega(e_j)$$

$$\geqslant \sum_{j=1}^{n} \lambda_j \omega(e_j)$$

$$> \eta$$

ii) If $|\omega(a)| > \eta$ for some $a \in J_1^+$, then :

either $-\omega(a) > \eta$

or $\omega(a) > \eta$

Therefore by i) there exists a projection f such that :

either $-\omega(f) > \eta$

or $\qquad \omega(f) > \eta$

i.e. $|\omega(f)| > \eta$ , contradicting hypothesis

Therefore $|\omega(a)| \leqslant \eta$ for all $a \in J_1^+$.

Each $b \in J_1$ can be expressed in the form :

$$b = b_1 - b_2$$

where $b_1, b_2 \in J_1^+$

whence $|\omega(b)| \leqslant |\omega(b_1)| + |\omega(b_2)|$

$\leqslant 2\eta$ .

iii) By hypothesis, $-\omega(f) \leqslant 0$ for all projections $f \in J$

Therefore by i) $-\omega(a) \leqslant 0$ for all $a \in J_1^+$

i.e. $\omega$ is positive. □

<u>2.7 Lemma</u>

If $\omega \in J^*$ is compl̶etely additive, and $e \in J$ is a projection, then there exists a subprojection f of e in J such that

$$\omega(f) \geqslant \omega(e)$$

and the restriction $\omega|fJf$ is a positive linear functional on the JW algebra fJf.

<u>Proof</u>

Let $(e_\alpha)$ be a maximal orthogonal family of projections in J such that $e_\alpha \leqslant e$ and $\omega(e_\alpha) < 0$.

With $f = e - \sum_\alpha e_\alpha$, the maximality of $(e_\alpha)$ implies that $\omega(g) \geqslant 0$ for every projection $g \in J$ such that $g \leqslant f$, that is for every projection $g \in fJf$.

Thus $\omega | fJf$ is positive.

By the complete additivity of $\omega$ :

$$\omega(f) = \omega(e) - \omega(\sum e_\alpha)$$
$$= \omega(e) - \sum \omega(e_\alpha)$$
$$\geqslant \omega(e) \quad \text{as } \omega(e_\alpha) \leqslant 0 \text{ for all } \alpha \qquad \square$$

2.8 Lemma

Let $\omega \in J^{*+}$ be completely additive, and $\omega'$ an extension of $\omega$ to the enveloping von Neumann algebra of J, with $\|\omega'\| = \|\omega\|$

If $e \in J$ is a non-zero projection, then there exists a non-zero subprojection f of e in J and a vector $\xi$ such that

$$|\omega'(a)| \leqslant \|a(\xi)\| \quad \text{for all } a \in Jf$$

Proof

Let $\eta$ be a vector such that $\|e\eta\|^2 > \omega(e)$

i.e. $(\omega_\eta - \omega)(e) = \|e\eta\|^2 - \omega(e) > 0$

By § 2.7 there exists a subprojection f of e in J such that

$$(\omega_\eta - \omega)(f) \geqslant (\omega_\eta - \omega)(e) > 0$$

and $(\omega_\eta - \omega) | fJf$ is positive.

If $a \in Jf$, then $a^*a \in fJf$

therefore $0 \leqslant (\omega_\eta - \omega)(a^*a) = \|a\eta\|^2 - \omega(a^*a)$.

By the Cauchy - Schwartz inequality :

$$|\omega(a)|^2 \leqslant |\omega'(I)| \, \omega(a^*a)$$
$$\leqslant |\omega'(I)| \, \|a\eta\|^2$$

So if $\xi = \sqrt{|\omega'(I)|}\,\eta$

then $|\omega'(a)| \leqslant \|a\xi\|$ as required □

2.9 Lemma

Let $\omega \in J^*$ be positive.

If $\omega$ is completely additive, then $\omega$ is ultraweakly continuous.

Proof

Let $(f_\alpha)_{\alpha \in A} \in J$ be a maximal orthogonal family of projections with

the property :

(P) There exists $\xi_\alpha$ such that $|\omega'(a)| \leqslant \|a\xi_\alpha\|$ whenever $a \in Jf_\alpha$
($\omega'$ defined as in § 2.8 )

If $\sum f_j \neq I$, it follows from §2.8 that there exists a vector $\xi$ and

a projection $f \in J$ such that :

$$0 < f \leqslant I - \sum f_j$$

and $|\omega'(a)| \leqslant \|a\xi\|$ ( for $a \in Jf$ ) , contradicting

the maximality of $(f_\alpha)$.

Therefore $\sum f_\alpha = I$.

Take $\varepsilon > 0$.

By the complete additivity of $\omega$ :

$$\sum_{\alpha \in A} \omega(f_\alpha) = \omega(I) ,$$

so there exists a finite subset B of A (if A is finite take B=A)

such that :

$$\omega\Big(\sum_{\alpha \in A \setminus B} f_\alpha\Big) = \sum_{A \setminus B} \omega(f_\alpha)$$

$$\leqslant \varepsilon^2 \|\omega\|^{-1}$$

Let $e = \sum_{\alpha \in A \setminus B} f_\alpha$

Then $\omega(e) \leqslant \varepsilon^2 \|\omega\|^{-1}$ and $e + \sum_{\alpha \in B} f_\alpha = I$

Thus $\omega' = \omega_1' + \omega_2'$ where

$$\omega_1'(a) = \sum_{\alpha \in B} \omega'(af_\alpha) \qquad a \in J$$

$$\omega'_2(a) = \omega'(ae).$$

$|\omega'_1(a)| \leqslant \sum_{\alpha \in B} \|af_\alpha \xi_\alpha\|$ $(a \in J)$, so $\omega'_1$ is strongly continuous on J.

I.e. $\omega'_1 | J \in J_*$

Also $\omega'_2(a)^2 = \omega'(ae)^2$

$$\leqslant \omega(a^2)\omega(e)$$

$$\leqslant \|\omega\| \|a\|^2 \|\omega\|^{-1} \varepsilon^2$$

$$= \varepsilon^2 \|a\|^2$$

$J_*$ is a Banach space embedded in its second dual $J^*$. □

Therefore $J_*$ is norm closed in $J^*$ and so $\omega \in J_*$.

2.10 Lemma

Suppose x,y,z are real numbers such that $x, y, xy-z^2 \geqslant 0$ and that $e, a \in B(H)$ are such that e is a projection and $\|a\| \leqslant 1$.

Then $xe + y(I-e) + z( ea(I-e) + (I-e)a^*e ) \geqslant 0$

Proof

$x(e\xi,\xi) + y((I-e)\xi,\xi) + z((ea(I-e) + (I-e)a^*e)\xi,\xi) =$

$$x\|e\xi\|^2 + y\|(I-e)\xi\|^2 + 2z\mathrm{Re}(ea(I-e)\xi,\xi) \geqslant$$

$$x\|e\xi\|^2 + y\|(I-e)\xi\|^2 - 2|z|\|(I-e)\xi\|\|e\xi\| \geqslant 0 \text{ , for all } \xi \in H$$

from which the result follows.

2.11 Theorem

$\omega \in J^*$ is ultraweakly continuous iff $\omega$ is completely additive.

Proof

If $(e_\alpha)_{\alpha\, A}$ is an orthogonal family of projection ( of norm $\leqslant 1$ ) then $e = \sum e_\alpha$ is the strong and hence weak limit of the net of finite subsums of $(e_\alpha)$ and $\|e\| \leqslant 1$. On the unit ball of J the weak and ultraweak topologies coincide, and so if $\omega$ is ultra-weakly continuous it is weakly continuous there. Hence $\omega(\sum e_\alpha) = \sum \omega(e_\alpha)$, from which it follows immediately that $\omega$ is completely additive on J.

The proof of the reverse implication is divided into two parts :

Part I

We can assume that $\|\omega\| \leqslant 1$

Let $\mu = \sup \{\omega(a) : 0 \leqslant a \leqslant I\}$

So $0 \leqslant \mu \leqslant \|\omega\| \leqslant 1$ .

Given $\varepsilon$ satisfying $0 < \varepsilon < \frac{3}{4}$ , there exists a positive element $e_1$ of the unit ball of J satisfying :

$$\omega(e_1) > \mu - \varepsilon$$

By §2.6(i) we can assume that $e_1$ is a projection.

By §2.7 we may also assume that $\omega | e_1Je_1$ is a positive, linear functional on $e_1Je_1$, and by § 2.9 we have that $\omega | e_1Je_1$ is ultra-weakly continuous.

If we put $e_2 = I-e_1$ we have :

$$\omega(a) = \omega(\{e_1ae_2\}) + \omega(e_1ae_1) + \omega(e_2ae_2) \text{ for all } a \in J$$

$$= \omega_{12}(a) + \omega_{11}(a) + \omega_{22}(a), \text{ say.}$$

By considering the maps :

$$a \mapsto e_1ae_1 \mapsto \omega(e_1ae_1) = \omega_{11}(a)$$

it is clear from the fact that $\omega$ is ultraweakly continuous on $e_1Je_1$ that $\omega_{11}$ is also ultraweakly continuous.

Any projection $f \in e_2Je_2$ is orthogonal to $I-e_2 = e_1$, so $e_1+f \leqslant I$ and hence:

$$\mu \geqslant \omega(e_1+f)$$

$$= \omega(e_1) + \omega(f)$$

$$> \mu - \varepsilon + \omega(f)$$

and so $\omega(f) < \varepsilon$ for all $f \in e_2Je_2$ (*)

Suppose $t \in J_1$, the unit ball of J, and let

$$s = (1-\varepsilon)e_1 + \varepsilon e_2 + \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}\{e_1te_2\}$$

Then,

$$1-s = \varepsilon e_1 + (1-\varepsilon)e_2 - \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}\{e_1te_2\}$$

By § 2.10, $s \geqslant 0$ and $I-s \geqslant 0$

i.e. s is positive and belongs to $J_1$

Therefore $\mu \geqslant \omega(s)$, by the definition of $\mu$

$$= (1-\varepsilon)\omega(e_1) + \varepsilon\omega(e_2) + \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}\omega(\{e_1te_2\})$$

$$\geqslant (1-\varepsilon)(\mu-\varepsilon) - \quad \varepsilon \quad + \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}\omega(\{e_1te_2\})$$

Therefore $(\mu+2)\varepsilon^{\frac{1}{2}}(1-\varepsilon)^{-\frac{1}{2}} \geqslant \omega(\{e_1te_2\})$.

$\mu \leqslant 1$, and $(1-\varepsilon)^{-\frac{1}{2}} \leqslant 2$ (since $\varepsilon < \frac{3}{4}$ )

therefore $\omega(\{e_1te_2\}) \leqslant 6\sqrt{\varepsilon}$.

I.e. $\omega_{12}(t) \leqslant 6\sqrt{\varepsilon}$ for all $t \in J_1$

I.e. $\|\omega_{12}\| \leqslant 6\sqrt{\varepsilon}$ and so:

$$\|\omega - \omega_{11} - \omega_{22}\| = \|\omega_{12}\| \leqslant 6\sqrt{\varepsilon}$$

Part II

We next show that there exists $\omega_0 \in J_*$ such that

$$\|\omega_0 + \omega_{22}\| \leqslant 2\varepsilon + 6\sqrt{\varepsilon}.$$

This completes the proof of the theorem : for then

$$\begin{aligned}\|\omega - \omega_{11} + \omega_0\| &\leqslant \|\omega - \omega_{11} - \omega_{22}\| + \|\omega_0 + \omega_{22}\| \\ &\leqslant 6\sqrt{\varepsilon} + 2\varepsilon + 6\sqrt{\varepsilon} \\ &= 2\varepsilon + 12\sqrt{\varepsilon}.\end{aligned}$$

Since this can be done for all $\varepsilon$, since $\omega_0 - \omega_{11} \in J_*$, and since $J_*$ is norm closed, it follows that $\omega \in J_*$ as required.

Let $\nu = -\omega|e_2Je_2$.

Then $\nu$ is a completely additive linear functional on $e_2Je_2$ and $\|\nu\| \leqslant 1$.

By (*) $\nu(f) > -\varepsilon$ for every projection $f \leqslant e_2Je_2$

Reasoning as in Part I, there exist projections $f_1, f_2 \in e_2Je_2$ with $f_1+f_2 = e_2$, satisfying the following conditions:

i) $\nu_{11}$ is ultraweakly continuous ( $\nu_{11}(a) = \nu(f_1af_1)$ )

ii) $\|\nu_{12}\| \leqslant 6\sqrt{\varepsilon}$ ( $\nu_{12} = \nu(\{f_1af_2\})$ )

iii) $\nu(f) < \varepsilon$ for every projection $f \in f_2Jf_2$.

But we already have that $\nu(f) > -\varepsilon$

so $|\nu(f)| < \varepsilon$ .

By §2.6 $\|\nu | f_2Jf_2\| \leqslant 2\varepsilon$

so we have :

$$|\nu_{22}(a)| = |\nu(f_2af_2)| \quad \text{for all } a \in f_2Jf_2$$

$$\leqslant 2\varepsilon\|f_2af_2\|$$

$$\leqslant 2\varepsilon\|a\|$$

So $\|\nu_{22}\| \leqslant 2\varepsilon$

$$\|\nu - \nu_{11}\| = \|\nu_{12} - \nu_{22}\|$$

$$\leqslant 2\varepsilon + 6\sqrt{\varepsilon}$$

Define $\omega_0 \in J_*$ by :

$$\omega_0(a) = \nu_{11}(e_2ae_2)$$

$$|(\omega_0 + \omega_{22})(a)| = |\nu_{11}(e_2ae_2) + \omega(e_2ae_2)|$$

$$= (\nu_{11} - \nu)(e_2ae_2)$$

$$\leqslant \|\nu_1 - \nu\| \, \|e_2ae_2$$

$$\leqslant (2\varepsilon + 6\sqrt{\varepsilon})\|a\|$$

So $\|\omega_0 + \omega_{22}\| \leqslant 2\varepsilon + 6\sqrt{\varepsilon}$ as required □

2.12 Lemma

If $K \subset J_*$ is bounded, and for any sequence $(e_n)$ of orthogonal projections in J, $\omega(e_n) \to 0$ uniformly for $\omega \in K$,

then K is relatively compact in the topology $\sigma(J_*, J)$ of $J_*$.

Proof

Let $K_1$ be the $\sigma(J^*, J)$ closure of K in $J^*$

Then $K_1$ is $\sigma(J^*, J)$ compact in $J^*$, and the theorem follows if it is shown that $K_1 \subset J_*$.

Let $(e_j)$ be a net of orthogonal projections in J and $e = \sum e_j$.

For all $\omega \in J_*$, $\sum_{j=1}^{N} \omega(e_j) \to \omega(e)$

Therefore if the convergence is not uniform on K then there exists $\delta > 0$, a sequence $(\omega_n) \in K$ and an increasing sequence $(N_i) \in \mathbb{N}$ such that

$$\left| \sum_{j=1}^{N_i} \omega_i(e_j) - \omega_i(e) \right| < \delta \qquad \forall i \tag{1}$$

$$\left| \sum_{j=1}^{N_i} \omega_{i+1}(e_j) - \omega_{i+1}(e) \right| > 2\delta \tag{2}$$

$$\left| \sum_{j=1}^{N_{i+1}} \omega_{i+1}(e_j) - \omega_{i+1}(e) \right| < \delta \tag{3}$$

(2) and (3) imply :

$$\left| \sum_{j=1}^{N_i} \omega_{i+1}(e_j) - \omega_{i+1}(e) \right| - \left| \sum_{j=1}^{N_{i+1}} \omega_{i+1}(e_j) - \omega_{i+1}(e) \right| > \delta$$

Therefore

$$\left| \sum_{j=1}^{N_i} \omega_{i+1}(e_j) - \left( \sum_{j=1}^{N_{i+1}} \omega_{i+1}(e_j) - \omega_{i+1}(e) \right) - \omega_{i+1}(e) \right| > \delta$$

i.e. $$\left| \sum_{j=N_i+1}^{N_{i+1}} \omega_{i+1}(e_j) \right| > \delta$$

Let $$f_i = \sum_{j=N_i}^{N_{i+1}} e_j$$

Then $\omega(f_i) \not\to 0$ uniformly on K, contradicting the hypothesis of the lemma.

So $$\sum_{j=1}^{N} \omega(e_j) \to \omega(e) \text{ uniformly on K.}$$

Let $\sigma \in K_1$

Then there exists $(\omega_\alpha) \in K$ such that $\omega_\alpha \to \sigma$ in $\sigma(J^*, J)$

$$\omega_\alpha(e_j) \to \sigma(e_j) \quad \text{and} \quad \omega_\alpha(e) \to \omega(e)$$

$$\begin{aligned} \sigma(e) &= \lim_\alpha \omega_\alpha(e) \\ &= \lim_\alpha \sum_i \omega_\alpha(f_i) \\ &= \sum_i \lim_\alpha \omega_\alpha(f_i) \\ &= \sum_i \sigma(e_j). \end{aligned}$$

Thus $\sigma$ is completely additive, and so by §2.12 $\sigma \in J_*$

Hence $K_1 \subseteq J_*$ as required. □

2.13 Lemma

Let J be modular.

If $\omega \in J_*$, then $Q_\omega$ is $\sigma(J_*,J)$-compact ( see §2.5 for notation )

Proof

Since $Q_\omega$ is norm closed and convex, it is also closed in the weak topology $\sigma(J_*,J)$ on $J_*$.

$Q_\omega$ is $\sigma(J_*,J)$-compact in $J_*$ by §2.5 and 2.12.

2.14

The following is the Ryll-Nardzewski fixed point theorem, a short proof of which is given by Namioka and Asplund [6].

Theorem

If Q is a non-empty weakly compact convex subset of a Banach space X and U is a semi-group of weakly continuous affine maps, $X \to X$, then there exists $x \in Q$ such that $ux = x$ for all $u \in U$.

2.15 Theorem

Let J be modular, and Z the centre of J.

Then each $\omega \in Z_*$ extends to an element $\tau \in J_*$ such that :

$\tau(u^*au) = \tau(a)$ for all $a \in J$, where u is any product of symmetries in J.

Moreover, $\|\omega\| = \|\tau\|$

Proof

Z is ultra-weakly closed, closed under the Jordan product, and so is a real von Neumann algebra.

Therefore $\omega \in Z_*$ can be expressed as a countable sum of vector states

$$\omega = \sum_1^\infty \omega_{\xi_j \eta_j}|Z$$

where $\sum_j \|\xi_j\|^2 \leqslant \|\omega\|$ and $\sum_j \|\eta_j\|^2 \leqslant \|\omega\|$

Let $$\rho = \sum_1^\infty \omega_{\xi_j \eta_j}|J$$

Then $\rho \in J_*$ and $\rho | Z = \omega$

$$\|\rho\| \leqslant \sum \|\xi_j\| \|\eta_j\| \leqslant \|\omega\|$$

$$( \|\omega\| \geqslant \tfrac{1}{2}\sum( \|\xi_j\|^2 + \|\eta_j\|^2) \geqslant \sum \|\xi_j\| \|\eta_j\| )$$

For each product of symmetries, u

$$\|L_u \rho\| = \|\rho\| \leqslant \|\omega\|$$

and $(L_u \rho)(z) = \rho(u^*zu) = \rho(z) = \omega(z)$ for all $z \in Z$

Thus $\sigma | Z = \omega$ (whence $\|\omega\| \leqslant \|\sigma\|$ ) for all $\sigma \in Q_\rho$

so we have :

$$\|\sigma\| = \|\omega\| \text{ and } \sigma | Z = \omega \text{ for all } \sigma \in Q_\rho) \qquad (*)$$

The isometric linear maps $L_u : J_* \rightarrow J_*$ form a group, are $\sigma(J_*,J)$ continuous ( i.e. pointwise continuous ), and leave invariant the $\sigma(J_*,J)$ compact convex set $Q_\rho$ in $J_*$.

The Ryll-Nardzewski theorem therefore asserts that there exists $\tau \in Q_\rho$ such that $L_u \tau = \tau$ for all $u \in U$

Hence $\tau \in J_*$, $\tau(u^*au) = L_u(\tau)(a) = \tau(a)$ for all $a \in J$

By (*) $\|\tau\| = \|\omega\|$ and $\tau | z = \omega$ □

2.16 Corollory

There exists a weakly continuous linear map $T : J \to Z$ such that

$$T(a) = T(u^*au) \quad \text{and } T(z) = z$$

whenever $a \in J$, $u \in U$ and $z \in Z$.

Moreover, T is norm decreasing and $T(za) = zT(a)$ whenever $a \in J$ and $z \in Z$.

$T(a) > 0$ if $a > 0$.

Proof

By §3.15 (and using the notation of §3.15 ) it is possible to define an isometric linear map

$$S : Z_* \to J_*$$

by $S(\omega) = \tau$

Identifying J with the Banach space dual of $J_*$, and Z with the dual of $Z_*$, the adjoint of S is a norm-decreasing linear operator $T : J \to Z$ and is continuous with respect to the topologies $\sigma(J,J_*)$ and $\sigma(Z,Z_*)$.

$$\begin{aligned}(T(u^*au)) &= (S\,\omega)(u^*au)\\ &= (S\,\omega)(a)\\ &= \omega(Ta) \quad \text{for all } \omega \in Z_*,\ a \in J,\ u \in U\end{aligned}$$

Also, if $a \in J$, $\omega \in J_*$ and $a \geq 0$, $\omega \geq 0$

$$\begin{aligned}\omega(Ta) &= (S\omega)(a)\\ &\geq 0 \quad \text{since } S\omega \text{ is positive (see § 4.6)}\end{aligned}$$

$Z_*$ is separating for Z (see footnote over )

Therefore $T(u^*au) = Ta$ as required and $a > 0 \Rightarrow Ta > 0$

Similarly $\omega(Tz) = (S\omega)z = \omega(z)$ for all $\omega \in Z_*$, $z \in Z$

So $T(z) = z$. □

Note

If $a, b \in Z$ and $a \neq b$, then there exists $\xi \in H$ such that $a\xi \neq b\xi$
Therefore there exists $\eta \in H$ such that $(a\xi, \eta) \neq (b\xi, \eta)$
$x \mapsto (x\xi, \eta)$ is an ultraweakly continuous function.

## § 3 - UNIQUENESS OF CENTRE-VALUED TRACE

### 3.1 Lemma

For a non-zero projection e in a JC algebra J to be abelian it is sufficient that :

Each subprojection f of e in J has the form f=qe, where q is a projection in the centre of J.

**Proof**

If the condition holds, and a,b are any two projections in eJe then :

$$a = q_1 e, \qquad b = q_2 e \qquad \text{where } 0 \leqslant q_i$$

But that the $q_i$ are central implies that

$$\begin{aligned} ab &= q_1 e q_2 e \\ &= q_2 e q_1 e \\ &= ba \end{aligned}$$

so since any two projections in eJe commute, eJe is abelian, and so e is an abelian projection. □

### 3.2 Theorem

Suppose e is a projection in a continuous modular JW algebra J and that $r \in \mathbb{N}$.

Then there exist projections $e_1, \ldots, e_r \in J$ such that

$$e_1 \sim e_2 \quad \ldots \sim e_r \quad \text{and}$$

$$e_1 + e_2 + \ldots + e_r = e.$$

**Proof**

The proof is in three stages. Stage I does not require modularity.

**Stage 1**

We assert that each non-zero projection f in J contains two non-zero equivalent orthogonal projections :

Since J is continuous, f is not abelian and so not minimal.

Therefore f has a proper non-zero subprojection $g$ , and by §3.1

$$g \neq C_g f$$

Let $f_1 = C_g f - g \leqslant 1-g$

$f_2 = gC_{f_1} \leqslant g$

So $f_1$ and $f_2$ are orthogonal subprojection of f and

$$C_{f_2} = C_{f_1} C_g = C_{f_1}$$

So by Topping [9] , lemma 18, $f_1$ and $f_2$ can be written as orthogonal sums :

$$f_1 = f_1^{(1)} + f_1^{(2)}$$

$$f_2 = f_2^{(1)} + f_2^{(2)}$$

where $f_1^{(1)}$ and $f_2^{(1)}$ are exchanged by symmetry, and $C(f_1^{(2)}) \perp C(f_2^{(2)})$

If $f_1^{(1)}$ were zero, then $f_2^{(1)}$ would be zero also, as they are exchanged by symmetry, in which case :

$$C(f_1^{(2)}) = C(f_1) = C(f_2) = C(f_2^{(2)}) \neq 0$$

I.e. $C(f_1^{(2)}) \not\perp C(f_2^{(2)})$

Therefore $f_1$ and $f_2$ have nonzero subprojections exchanged by symmetry, and f has two non-zero orthogonal equivalent subprojections

<u>Stage II</u>

We assert that each nonzero projection f in J contains r nonzero equivalent, orthogonal subprojections in J.

The proof is by induction on r :

For r=1 the result is obvious, and r=2 is the case above.

Assume the result for r=p.

Choose nonzero equivalent, orthogonal projections $g_1, \ldots, g_r$ with each $g_j \leqslant f$ and $g_1 \sim g_2 \sim \cdots \sim g_p$.

By stage I, there exist orthogonal nonzero subprojections $h_0$ and $h_1$ of $g_1$ with $h_0 \sim h_1$.

If $g_j = u_j^* g_1 u_j$, where $u_j$ is a product of symmetries, then

$$g_j \geq u_j^* h_1 u_j = h_j, \text{ say.}$$

Now $h_0, \ldots, h_p$ have the required properties for the result when $r=p+1$.

Stage III

Let $(f_\alpha)$ be a maximal family of orthogonal nonzero projections in J such that $f_\alpha \leq e$ and

$$f_\alpha = f_\alpha^{(1)} + f_\alpha^{(2)} + \ldots + f_\alpha^{(r)}$$

where the $f_\alpha^{(j)}$ are projections in J and $f_\alpha^{(1)} \sim f_\alpha^{(2)} \sim \ldots \sim f_\alpha^{(r)}$

If $f_\alpha \neq e$, we can by stage II find a subprojection of $e - \sum f_\alpha$ with the properties of an $f_\alpha$, contradicting maximality.

So $e = \sum f_\alpha$

The projections $e_j = \sum_\alpha f_\alpha^{(j)}$ are equivalent, since equivalence is completely additive on a modular JW algebra, and are orthogonal.

3.3 Lemma

Let J be modular, and $e, f \in J$ projections such that $f \not\leq e$.

Then there exists a projection p in the centre of J such that

$$0 < p \leq C_f \qquad pe \prec pf$$

Proof (This result can also be deduced from Topping [9] thm 12. )

Let $(e_\alpha, f_\alpha)_{\alpha \in A}$ be a family of pairs of projections, maximal subject to the conditions :

i) $(e_\alpha)$ is an orthogonal family and $0 < e_\alpha \leq e$

ii) $(f_\alpha)$ is an orthogonal family and $0 < f_\alpha \leq f$

iii) $e_\alpha \sim f_\alpha$

Let $\quad e_0 = \sum e_\alpha \qquad\qquad f_o = \sum f_\alpha$

$\qquad e_1 = e - e_0 \qquad\qquad f_1 = f - f_0$

Then $f_0 \sim e_0 \leqslant e$ by the complete additivity of equivalence

I.e. $f_0 \precsim e$ .

But $f \not\precsim e$, so $f_0 \neq f$ and $f_1 \neq 0$.

$C_{e_1} C_{f_1} = 0$, for if it did not, then there would exist $e_2 \leqslant e_1$ and

$f_2 \leqslant f_1$ such that $e_2 \sim f_2$, which would contradict the maximality of

$(e_\alpha, f_\alpha)_{\alpha \in A}$.

Take $p = C_{f_1}$.

Then $0 < p \leqslant C_f$ since $0 < f_1 \leqslant f$

$$pe - pe_0 = pe_1 = C_{f_1} C_{e_1} = 0.$$

Therefore $pe = pe_0 \sim pf_0$

So $pe \prec pf$. □

<u>3.4 Definition</u>

A nonzero projection e in a modular JW algebra J is <u>rational</u> if there exists a finite orthogonal family $e_1, \ldots, e_n$ of projections in J and an integer $k \leqslant n$ such that

$$e_1 \sim e_2 \quad \ldots \sim e_n$$

$$e_1 + e_2 + \ldots + e_k = e$$

$e_1 + e_2 + \ldots + e_n$ is a central projection in J, necessarily $C_e$.

If e and f are equivalent projections in J and e is rational, then so is f.

Proof

Let $e_1, \ldots, e_n$ and k be as above.

If $f \sim e$ then there exists a product u of symmetries in J such that

$$f_i = \sum_{i=1}^{k} u^* e_i u$$

Put $f_i = u^* e_i u \quad ( i \leqslant k )$

$C_e = C_f$ and $e \sim f$ imply $C_e - e \sim C_f - f$

i.e. $$C_f - f = \sum_{k+1}^{m} v^* e_i v$$ where v is again a product of unitaries in J.

Let $f_i = v^* e_i v \qquad ( k < i \leqslant n)$

Then $f = f_1 + f_2 + \ldots + f_k$

$$C_f = f_1 + f_2 + \ldots + f_n$$

$$f_1 \sim f_2 \quad \ldots \sim f_n.$$

I.e. f is rational. □

35 Theorem

Each projection e in a modular JW algebra J can be expressed as the sum of an orthogonal family of rational projections.

Proof

Let $(e_\alpha)$ be a maximal orthogonal family of rational projections in J with each $e_\alpha \leqslant e$, and let $f = e - \sum e_\alpha$

It suffices to prove f=0.

Since J is modular, there exist central projections $(p_{I_n})$ and $p_{II_1}$ such that :

$$p_{I_n} + p_{II_1} = 1$$

$$Jp_{I_n} \text{ is type } I_n$$

$$\text{and } Jp_{II_1} \text{ is type } II_1$$

If $f \neq 0$ then either $fp_{I_n} \neq 0$ for some n, or $fp_{II_1} \neq 0$

<u>Case I</u> $g = fp_{I_n} \neq 0$

Since $0 < C_g \leq p_{I_n}$, $JC_g$ is type $I_n$

i.e. there exist abelian projections $g_1, \ldots g_n \in JC_g$ such that

$$C_{g_m} = C_g \quad \text{for all } m \quad n$$

$$g_1 + \ldots + g_n = C_g$$

A lemma will now be proved before the proof of the theorem proceeds :

<u>Lemma</u>

Let K be a modular JC algebra, and e, f projections in K with e abelian and $C_e \leq C_f$.

Then $e \precsim f$.

If e and f are both abelian and $C_e = C_f$, then $e \sim f$.

<u>Proof</u>

The second statement follows from the first.

Suppose e is abelian and $C_e \leq C_f$.

If $e \not\precsim f$ , then by § 3.3 there exists a projection p in the centre of K such that $0 < p \leq C_f$ ,

$$pf \prec pe .$$

Replacing p by $pC_e$ , we can suppose :

$$0 < p \leq C_e .$$

Since $pf \prec pe$ there exists a projection $f_1$ such that :

$$pf \sim f_1 < pe \leq e .$$

Since e is abelian, eKe coincides with its centre Ze.

So $$f_1 = ef_1e \in eKe = Ze.$$

I.e. $f_1 = qe$ for some projection $q \in Z$.

Since $f_1 = pf_1$ , we can replace q by pq, and assume

$$0 < q \leq p \leq C_e \leq C_f .$$

Now, $p = pC_f = C_{pf}$ (by Topping [9] prop 18.)

$= C_{f_1}$ (by Topping [9] cor 14.)

$= C_{qe}$

$= qC_e = q.$

But $f_1 = qe = pe$, contradicting $f_1 < pe$.

Hence $e \precsim f$.

Hence the lemma follows. □

By the lemma, $$g_1 \sim \cdots \sim g_n \precsim g.$$

Therefore $g_1$ is rational and there exists a projection $g_1' \sim g_1$ which will also be rational, such that

$g_1' \leq g < f$ contradicting the definition of f.

<u>Case II</u> $g = fp_{II_1} \neq 0$

Since 0: $C_g$ $p_{II_1}$ , $JC_g$ is type $II_1$.

Let $g_1, \dots, g_n$ be a maximal orthogonal family of projections in $JC_g$ with $g_j \sim g$ ( $j=1, \dots, n$ )

( Such a family is necessarily finite since $g$ is modular : see e.g. Topping [9] thm 11 ).

Let $x = C_g - ( g_1 + \dots + g_n)$

$g \not\lesssim x$, by the maximality of the family $\{g_i\}$ (*)

Since $JC_g$ is type $II_1$ and hence continuous, Theorem 3.2 applies, and hence there exist projections $y_0, \dots, y_n \leqslant JC_g$ such that

$$y_0 \sim y_1 \sim \dots \sim y_n \quad \text{and}$$

$$y_0 + y_1 + \dots + y_n = C_g.$$

<u>If $g \lesssim y_0$</u> then $g_j \lesssim y_j$ ( $j = 1, \dots, n$ ) and

$$\begin{aligned} C_g - x &= g_1 + \dots + g_n \\ &\lesssim y_1 + \dots + y_n \\ &= C_g - y_0 \end{aligned}$$

I.e. $C_g - x \sim h \leqslant C_g - y_0$

Since $C_g$ is the identity of $JC_g$ this would imply

$$\begin{aligned} x &= C_g - ( C_g - x ) \\ &\sim C_g - h \\ &\geqslant y_0 \gtrsim g \quad \text{contradicting (*)} \end{aligned}$$

<u>Hence $g \not\lesssim y_0$</u>, and so by lemma 3.3 there exists a central projection $q$ with $0 < q \leqslant C_g$ and $qy_0 \prec qg$.

Thus $f$ ($\geq g \geq qg$ ) contains an projection equivalent to $qy_0$, and this is rational since

$$qy_0 \sim qy_1 \sim \dots \sim qy_n \quad \text{and}$$

$$qy_0 + qy_1 + \dots + qy_n = q$$

which contradicts the maximality of $(e_\alpha)_{\alpha \in A}$ □

3.6 Theorem

If J is a JW algebra with centre Z, the extension of any $\omega \in Z_*$ to a trace $\tau \in J_*$ is unique , and $\tau$ is positive if $\omega$ is.

The ultraweakly continuous centre-valued trace $T : J \to Z$ is also unique.

Proof

If e is a rational projection in J, choose equivalent projections $e_1, \dots, e_n \in J$ such that :

$$\sum_1^n e_i = c_e \qquad \sum_1^k e_i = e \quad \text{for some } k \leq n$$

there exists a product of symmetries $u_j$ such that

$$e_j = u_j^* e_1 u_j$$

So $\qquad \tau(e_j) = \tau(e_1) \qquad$ for all $j \leq n$

and $\qquad \tau(e) = \frac{k}{n}\tau(c_e) = \frac{k}{n}\omega(c_e).$

Thus $\tau(e)$ is determined by $\omega$ and is positive if $\omega$ is.

But any projection is J is the orthogonal sum of rational projections in J (§ 3.5) and $\tau$ is completely additive since it is ultraweakly continuous (§2.11).

Thus $\tau$ is determined by $\omega$ for all projections in J, and therefore is determined by $\omega$ on J, the closed linear span of its projections. If $\omega$ is positive, then $\tau(e) \geq 0$ for each projection e.

Hence $\tau$ is positive by lemma 2.6 (iii).

The same arguement proves the uniqueness of T. $\square$

## §4 - WEAK AND TRACE CLOSURE OF JC SUBALGEBRAS

### 4.1 Definition

Let J be a modular JW factor, and $\tau$ its trace. Then an inner product can be defined on J by :

$$(a,b) = \tau(a \circ b) \qquad \text{for all } a,b \in J$$

The associated norm is denoted by $| \ |_2$ .

### 4.2 Lemma

If $\psi$, $\phi$ are normal linear functionals on a JW algebra J such that $0 \leqslant \psi \leqslant \varphi$ , then there exists $h \in J_1^+$ such that :

$$\psi(a) = \phi(h \circ a)$$

If $\phi$ is faithful, then h is unique.

### Proof

The proof follows that for the analogous von Neumann algebra result given by Pedersen [12] chapter 5.3.2 .

Let $\quad X = \{\varphi(h \circ .) : h \in J_1^+\}$

X is convex, since $J_1^+$ is convex, and is compact in $J_*$ since $J_1^+$ is ultra-weakly compact ( $J_*$ is the set of ultraweakly continuous functionals on J ).
If $\psi \notin X$ then,
y the Hahn-Banach theorem, there exists $a \in (J_*)^* = J$ and $t \in \mathbb{R}$ such that : $\qquad \psi(a) > t \qquad$ and $\qquad X(a) \leqslant t$

Write a as $a_+ - a_-$ , the difference of positive elements.

Let $h = [a_+]$, the range projection of $a_+$

$h \in J_1^+$

$$a_- a_+ = 0$$

therefore $a_- [a_+] = 0$

therefore $ah = a[a_+]$

$= a_+[a_+] = a_+$

Also $ha = (ah)^*$

$= a_+^*$

$= a_+$

Therefore $\varphi(a_+) = \varphi(h \circ a)$.

$$\psi(a_+) \geq \psi(a_+ - a_-)$$

$> t$

$\geq \varphi(h \circ a)$

$= \varphi(a_+)$ contradicting $\psi \leq \varphi$

Let k also have the property required of h.

Then $(h-k)^2 = h \circ (h-k) - k \circ (h-k)$

therefore $\varphi((h-k)^2) = \varphi(h \circ (h-k)) - \varphi(k \circ (h-k))$

$= \psi(h-k) - \psi(h-k)$

$= 0$

Therefore if $\varphi$ is faithful then h=k : i.e. h is unique. □

4.3 Lemma

Let J be a modular JW factor and K a JW subalgebra of J containing the unit of J.

Then there exists a unital positive projection p: J→K such that

$$\tau(a \circ b) = \tau(p(a) \circ b) \qquad \text{for all } a \in J,\ b \in K,$$

where $\tau$ is the faithful normal tracial state of J.

Proof.

The proof follows that of Sakai [13] chapter 4.4.23

For $h \in J$, $h \geq 0$, $\|h\| = 1$. define $\varphi_h$ on K by :

$$\varphi_h(y) = \tau(h \circ y) \qquad \text{for all } y \in K$$

$$\tau(y) - \varphi_h(y) \geq 0 \qquad \text{( Pedersen \& Størmer [16] Theorem. )}$$

Therefore, $\tau \geq \varphi_h$

Therefore by § 4.2 there exists a unique $k \in K_1^+$ such that

$$\tau(k \circ y) = \varphi_h(y) .$$

Define $p(h) = k$.

p extends by linearity to the whole of J. Clearly $p^2 = p$

p is positive and so bounded ( Russo & Dye [15] ).

Also $p(1)=1$ so $\|p\| = 1$ .

The uniqueness follows from the faithfullness of $\tau$ □

4.4 Theorem

Let J be a modular JW factor and K a Jordan subalgebra of J.

Then K is closed in the trace-norm topology iff K is weakly closed.

Proof

$\Leftarrow$

By § 4.3, $(a-p(a),b) = 0 \qquad$ for all $a \in J$, $b \in K$

Therefore $p(a)$ is the best $|\ |_2$ approximation to a from K.

If $a \notin K$, $\quad \operatorname{dist}(a,K) = |a-p(a)|_2$

$> 0$ since $p(a) \neq a$, and $|\cdot|_2$ is a norm.

Therefore K is $|\ |_2$ closed.

$\Rightarrow$

Let $(a_\alpha)$ be a net in K and $a_\alpha \to a$ ultrastrongly.

Since $\tau$ is ultraweakly continuous and positive it can be expressed in the form

$$\tau = \sum_1^\infty \omega_{\xi_i}$$

Therefore

$$\begin{aligned} |a_\alpha - a|_2 &= \tau((a_\alpha - a)^*(a_\alpha - a)) \\ &= \sum_1^\infty ((a_\alpha - a)^*(a_\alpha - a)\xi_i, \xi_i) \\ &= \sum_1^\infty |(a_\alpha - a)\xi_i|^2 \\ &\to 0 \qquad \text{since } a_\alpha \to a \text{ ultrastrongly.} \end{aligned}$$

Therefore $a \in K$

Therefore K is ultrastrongly closed.

But in that case K is weakly closed ( Størmer[11] Lemma 4.2) □

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