Three Inadequate Models

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1 Introduction

The connection between operational and denotational semantics is of longstanding interest in the study of programming languages. One naturally seeks positive results. For example in [FP94, Sim99] adequacy results are given for models in a variety of categories. Again, the failure of full abstraction in the standard models constructed using complete partial orders and continuous functions [Plo77, Mil77] prompted the exploration of other categories (see, e.g., [BCL85, FJM96, AM98, AC98]) with varying degrees of success.

In this paper we interest ourselves in counterexamples in order to make a case that these natural avenues of research had a degree of necessity. To this end, we construct inadequate models and investigate whether one can do better than the standard model, but still stay in the category of complete partial orders. (In contrast, an inadequate standard model of PCF is given in [Sim99]—but in a specially constructed category.)

We consider just one example, an untyped call-by-name λ-calculus $\mathcal{L}$, whose *expressions* M are given by

$$\begin{array}{lll} M & ::= & x \mid \underline{0} \mid \mathrm{succ}(M) \mid \lambda x.M \mid \\ & & \partial(M,M,M) \mid \mathrm{pred}(M) \mid \textbf{if } M = 0 \textbf{ then } M \textbf{ else } M \mid MM \end{array}$$

where x runs over a fixed countably infinite set of *variables*. This is, essentially, the language considered by Pitts in [Pit93], but with the trivial variation of using natural numbers rather than integers (as will be seen from the operational semantics) and with the more significant addition of ∂ (to discriminate between functions and natural numbers).

There is a natural domain equation associated to this language

$$D \cong (D \Rightarrow D)_\perp + \boldsymbol{N}_\perp$$

and its standard solution is adequate. We give a non-standard solution to the domain equation which is not adequate, confirming that some restriction in the class of models is needed for adequacy. The idea of the construction dates back to Park [Par76], who gave a non-standard model of the pure untyped λ-calculus in which the paradoxical combinator does not denote the least fixed-point operator.

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One would hope next to show that the standard model, even though itself not fully abstract, gets as close as possible to that ideal among the models available or, if not, at least among the adequate models. The position here is not quite what one might expect. It turns out that if, in a suitable sense, one model is less abstract than another, then they make the same termination predictions for $\mathcal{L}^+$, an extension of $\mathcal{L}$ by a "parallel or" construct. It then follows that a model is comparably abstract with the standard one iff it is $\mathcal{L}^+$-adequate. This is somewhat surprising as one might, for example, have expected that the equational theory of any model adequate for $\mathcal{L}$ would be included in that of the standard model, rather than the prima facie stronger requirement of adequacy for $\mathcal{L}^+$. We are able to give a non-standard solution of the domain equation which is adequate for $\mathcal{L}$ but not for $\mathcal{L}^+$ and so consideration of the stronger adequacy condition really is required; this is our technically most involved result.

So far the standard model is the only example we have that is adequate for $\mathcal{L}^+$, but it is not difficult to find other solutions of the domain equation which are also adequate for $\mathcal{L}^+$, and in fact we can even find one in which the Y-combinator does not denote the least fixed-point operator. This model is inadequate in the sense that no analogue of the Hyland-Wadsworth approximation theorem [Bar84] can hold.

Section 2 below presents various technical preliminaries. The three inadequate models are presented in Sections 3, 5 and 6, with the "less abstract than" relation being considered in Section 4; Section 7 is a discussion section, placing our work in a broader context. The Appendix presents two results on full abstraction for the standard model; while perhaps not quite folklore, they are hardly unexpected: that the model is not fully abstract for $\mathcal{L}$, but is for $\mathcal{L}^+$. While we do define the particular flavour of domain used (cppos) the paper is not completely self-contained, and appropriate knowledge of domain theory is assumed.

2 Technical Preliminaries

2.1 Syntax

We have given the syntax of $\mathcal{L}$ above. Free and bound variables are defined as usual (λ is the only binding operator), as is simultaneous substitution $M[N_1/x_1, \ldots, N_n/x_n]$; we write $\mathcal{L}^o$ for the set of closed terms of $\mathcal{L}$. The operational semantics of $\mathcal{L}$ is given by the rules in Figure 2.1, giving an inductive definition of an *evaluation* relation $M \Rightarrow_{\mathcal{L}} V$ between closed terms M and (syntactic) values V; the latter are taken to be those closed terms which are either *abstractions* $\lambda x.N$ or *numerals* $\underline{n} =_{\text{def}} \text{succ}^n(0)$. We define the *termination* property $M \Downarrow_{\mathcal{L}}$ for $\mathcal{L}$-terms M to be that $M \Rightarrow_{\mathcal{L}} V$, for some V.

The language $\mathcal{L}^+$ is obtained from $\mathcal{L}$ by adding a "parallel or" construct $\mathbf{por}(M, N)$; its values are again closed abstractions and numerals. The evaluation rules are as before, together with:

$$\frac{M \Rightarrow \underline{0} \quad N \Rightarrow \underline{0}}{\mathbf{por}(M, N) \Rightarrow \underline{0}}$$

$$\frac{M \Rightarrow \underline{1}}{\mathbf{por}(M, N) \Rightarrow \underline{1}}$$

$$\underline{0} \Rightarrow \underline{0} \qquad \frac{M \Rightarrow \underline{n}}{\mathrm{succ}(M) \Rightarrow \underline{n+1}} \qquad \lambda x.M \Rightarrow \lambda x.M$$

$$\frac{L \Rightarrow \lambda x.L' \quad M \Rightarrow V}{\partial(L,M,N) \Rightarrow V} \qquad \frac{L \Rightarrow \underline{n} \quad N \Rightarrow V}{\partial(L,M,N) \Rightarrow V}$$

$$\frac{M \Rightarrow \underline{n+1}}{\mathrm{pred}(M) \Rightarrow \underline{n}}$$

$$\frac{L \Rightarrow \underline{0} \quad M \Rightarrow V}{\mathbf{if}\ L = 0\ \mathbf{then}\ M\ \mathbf{else}\ N \Rightarrow V} \qquad \frac{L \Rightarrow \underline{n+1} \quad N \Rightarrow V}{\mathbf{if}\ L = 0\ \mathbf{then}\ M\ \mathbf{else}\ N \Rightarrow V}$$

$$\frac{M \Rightarrow \lambda x.M' \quad M'[N/x] \Rightarrow V}{MN \Rightarrow V}$$

Figure 1: Operational Semantics of $\mathcal{L}$.

$$\frac{N \Rightarrow \underline{1}}{\mathbf{por}(M,N) \Rightarrow \underline{1}}$$

yielding an evaluation relation $M \Rightarrow_{\mathcal{L}^+} V$ for $\mathcal{L}^+$ (and a termination predicate $M \Downarrow_{\mathcal{L}^+}$)

There is a natural equational theory of "β-rules" for $\mathcal{L}$ with axioms:

1. $\partial((\lambda x.L), M, N) = M$
2. $\partial(\underline{n}, M, N) = N$
3. $\mathrm{pred}(\underline{n+1}) = \underline{n}$
4. $(\mathbf{if}\ \underline{0} = 0\ \mathbf{then}\ M\ \mathbf{else}\ N) = M$
5. $(\mathbf{if}\ \underline{n+1} = 0\ \mathbf{then}\ M\ \mathbf{else}\ N) = N$
6. $(\lambda x.M)N = M[N/x]$

and the usual rules for equality, including the ξ-rule for λ; we write $M =_\beta N$ if $M = N$ is provable in this theory. It is easily verified that $M =_\beta V$ holds when $M \Rightarrow_{\mathcal{L}} V$; the same is true for $\mathcal{L}^+$, extending 1–6 to the terms of that language and adding the evident three equations for parallel or.

2.2 Denotational Semantics

We take our models in the category **CPPO** of cppos and strict continuous functions. A *cpo (complete partial order)* is a partial order with lubs of increasing ω-sequences; a *continuous* function $f : P \to Q$ between such cpos is a monotone function which preserves the lubs; we write **CPO** for the cartesian closed category of cpos and continuous functions. A *cppo (complete pointed partial order)* is a cpo with a least element; a function between such cppos is *strict* if it preserves the least elements, and we write $f : D \to_\perp E$ for strict continuous functions. We write $D \Rightarrow E$ for the cppo of continuous functions from D to E, ordered pointwise; this yields a functor $\Rightarrow$: $\mathbf{CPPO}^{\mathrm{op}} \times \mathbf{CPPO} \to \mathbf{CPPO}$. We write $D + E$ for the sum of D and E

amalgamating the least elements; this is the categorical sum. For any set X, $X_\perp$ is the *flat* cppo whose elements are those of X plus a new least element; finally for any cppo D, $D_\perp$ is D with the addition of a new least element, yielding the *lifting* functor on **CPPO**.

A *model* of $\mathcal{L}$ is a structure $\mathcal{E} = <E, s, r>$ where

$$((E \Rightarrow E)_\perp + \boldsymbol{N}_\perp) \xrightarrow{s}_\perp E \xrightarrow{r}_\perp ((E \Rightarrow E)_\perp + \boldsymbol{N}_\perp)$$

is a retraction pair in **CPPO** (i.e., $r \circ s = \mathrm{id}$); elements x of E such that $r(x) \neq \perp$ are called *(semantic) values*; and we say that a model is *extensional* if s is an isomorphism. We set

$$\mathrm{num} =_{\mathrm{def}} \mathrm{inr} : \boldsymbol{N}_\perp \longrightarrow (E \Rightarrow E)_\perp + \boldsymbol{N}_\perp$$

and

$$\mathrm{fun} =_{\mathrm{def}} \mathrm{inl} \circ \mathrm{up} : (E \Rightarrow E) \longrightarrow (E \Rightarrow E)_\perp + \boldsymbol{N}_\perp$$

and it will often prove convenient to regard s as an inclusion and omit it, writing num (respectively fun) for $s \circ$num (respectively $s \circ$fun). We also define an "application" function by

$$e \cdot e' = \begin{cases} f(e') & (\text{if } r(e) = \mathrm{fun} f) \\ \perp & (\text{otherwise}) \end{cases}$$

Given any such model $\mathcal{E}$ we define the *interpretation* (or *denotation*) $\mathcal{E}[\![M]\!](\rho)$ of an $\mathcal{L}$-term as in Figure 2.2. Here ρ is an *environment*, i.e., a function from variables to E; note the use of "semantic substitution": $\rho[e_1/x_1, \ldots, e_n/x_n]$ is the environment that is equal to ρ except at an x_i where it is equal to e_i.

As usual, $\mathcal{E}[\![M]\!](\rho)$ is continuous in ρ, depends only on the values ρ assigns to the free variables of M, and the Substitution Lemma holds, viz.

$$\mathcal{E}[\![M[N/x]]\!](\rho) = \mathcal{E}[\![M]\!](\rho[\mathcal{E}[\![N]\!](\rho)/x])$$

We say an equation $M = N$ is *true* in $\mathcal{E}$, writing $\mathcal{E} \models_{\mathcal{L}} M = N$, if $\mathcal{E}[\![M]\!](\rho) = \mathcal{E}[\![N]\!](\rho)$ for all ρ (and we write $\mathrm{EqTh}_{\mathcal{L}}$ for $\{<M, N> \mid \mathcal{E} \models_{\mathcal{L}} M = N\}$). All the above axioms for $\mathcal{L}$ are true in this sense and the rules for equality preserve truth. Consequently we have that if $M \Rightarrow_{\mathcal{L}} V$ then $\mathcal{E} \models_{\mathcal{L}} M = V$. So, writing $\mathcal{E} \models_{\mathcal{L}} M\downarrow$ to mean that $\mathcal{E}[\![M]\!](\rho)$ is a semantic value for all ρ, we have that for all closed terms M

$$M \Downarrow_{\mathcal{L}} \text{ implies } \mathcal{E} \models_{\mathcal{L}} M\downarrow$$

as $\mathcal{E}[\![V]\!](\perp)$ is always a semantic value; $\mathcal{E}$ is said to be $\mathcal{L}$-*adequate* if the converse holds. We write $\mathrm{TerTh}_{\mathcal{L}}$ for $\{M \in \mathcal{L}^o \mid \mathcal{E} \models_{\mathcal{L}} M\downarrow\}$.

Operational equivalence is also defined as usual, taking $\mathcal{L}$-contexts $C[\ ]$ to be $\mathcal{L}$-terms with a "hole" and defining $M \simeq_{\mathcal{L}} N$ to mean that for all contexts $C[\,]$ such that $C[M]$ and $C[N]$ are both closed, $C[M]\Downarrow_{\mathcal{L}}$ iff $C[N]\Downarrow_{\mathcal{L}}$.

Adequacy is then equivalent to the *soundness* assertion that for all terms M, N

$$\mathcal{E} \models_{\mathcal{L}} M = N \text{ implies } M \simeq_{\mathcal{L}} N$$

That adequacy implies this implication follows as the congruence rules are sound for equations true in $\mathcal{E}$. For the converse one uses the fact that $\mathcal{E} \models_{\mathcal{L}} M\downarrow$ implies $\mathcal{E} \models_{\mathcal{L}} \partial(M, \underline{0}, \underline{0}) = \underline{0}$.

$$\mathcal{E}[\![x]\!](\rho) = \rho(x)$$

$$\mathcal{E}[\![\underline{0}]\!](\rho) = s(\mathrm{num}(0))$$

$$\mathcal{E}[\![\mathrm{succ}(M)]\!](\rho) = s(\mathrm{num}(\mathcal{E}[\![M]\!](\rho) + 1))$$

$$\mathcal{E}[\![\lambda x.M]\!](\rho) = s(\mathrm{fun}(\lambda e : E.\mathcal{E}[\![M]\!](\rho[e/x])))$$

$$\mathcal{E}[\![\partial(L, M, N)]\!](\rho) = \begin{cases} \mathcal{E}[\![M]\!](\rho) & (\text{if } r(\mathcal{E}[\![L]\!](\rho)) = \mathrm{fun}(f)) \\ \mathcal{E}[\![N]\!](\rho) & (\text{if } r(\mathcal{E}[\![L]\!](\rho)) = \mathrm{num}(n)) \\ \bot & (\text{otherwise}) \end{cases}$$

$$\mathcal{E}[\![\mathrm{pred}(M)]\!](\rho) = \begin{cases} s(\mathrm{num}(n)) & (\text{if } r(\mathcal{E}[\![M]\!](\rho)) = \mathrm{num}(n+1)) \\ \bot & (\text{otherwise}) \end{cases}$$

$$\mathcal{E}[\![\textbf{if } L = 0 \textbf{ then } M \textbf{ else } N]\!](\rho) = \begin{cases} \mathcal{E}[\![M]\!](\rho) & (\text{if } r(\mathcal{E}[\![L]\!](\rho)) = \mathrm{num}(0)) \\ \mathcal{E}[\![N]\!](\rho) & (\text{if } r(\mathcal{E}[\![L]\!](\rho)) = \mathrm{num}(n+1)) \\ \bot & (\text{otherwise}) \end{cases}$$

$$\mathcal{E}[\![MN]\!](\rho) = \mathcal{E}[\![M]\!](\rho) \cdot \mathcal{E}[\![N]\!](\rho)$$

Figure 2: Denotational Semantics of $\mathcal{L}$

We say that $\mathcal{E}$ is *(equationally) fully abstract for $\mathcal{L}$* iff it is adequate, and the converse holds, that for all terms M, N:

$$M \simeq_{\mathcal{L}} N \text{ implies } \mathcal{E} \models_{\mathcal{L}} M = N$$

Models of $\mathcal{L}$ can also be used as models of $\mathcal{L}^+$ if one sets

$$\mathcal{E}[\![\mathbf{por}(M, N)]\!](\rho) = \vee_p(\mathcal{E}[\![M]\!](\rho), \mathcal{E}[\![N]\!](\rho))$$

where the *parallel or* function is defined by

$$\vee_p(e, e') = \begin{cases} s(\text{num}(1)) & (\text{if one of } r(e),\, r(e') \text{ is num}(1)) \\ s(\text{num}(0)) & (\text{if } r(e) = r(e') = \text{num}(0)) \\ \bot & (\text{otherwise}) \end{cases}$$

Analogues of all the above remarks then hold and we can define $\mathcal{E} \models_{\mathcal{L}^+} M = N$, $\mathcal{E} \models_{\mathcal{L}^+} M\!\downarrow$, $M \simeq_{\mathcal{L}^+} N$, $\mathcal{L}^+$-adequacy and $\mathcal{L}^+$-full abstraction.

In any model $\mathcal{E}$ of $\mathcal{L}$ we identify the *least fixed-point operator* as the functional value fun(F) such that

$$F(x) = \begin{cases} \bigvee_{n \geq 0} f^n(\bot) & (\text{if } x = \text{fun}(f)) \\ \bot & (\text{otherwise}) \end{cases}$$

The *paradoxical combinator* $\mathrm{Y} =_{\text{def}} \lambda f.\Delta\Delta$, where $\Delta =_{\text{def}} \lambda x.f(xx)$, need not denote the least fixed-point operator—as will be seen below—although it does in $\mathcal{S}$ (proof as in, e.g., [Pit96]). However

Proposition 1 *There is an $\mathcal{L}$-term Y_μ that denotes the least fixed-point operator in any model.*

Proof The presence of $\boldsymbol{N}$ in all models allows us enough "standard" material to find a uniform definition, using a standard chain; the idea comes from the proof of the uniformity of the least fixed-point combinator in, e.g., [Fre91].

So consider the term

$$L =_{\text{def}} \lambda c.\lambda f.\mathrm{Y}(\lambda h.\lambda z.\mathbf{if}\ cz = 0\ \mathbf{then}\ f(h(z+1))\ \mathbf{else}\ \underline{0})$$

The following equation holds in any model

$$Lcfz = \mathbf{if}\ cz = 0\ \mathbf{then}\ f(Lcf(z+1))\ \mathbf{else}\ \underline{0}$$

Now fix a model $\mathcal{E}$ and let $\gamma_n : E \to E$ be num(0) on any num(m) with $m < n$ and $\bot$ elsewhere. Then γ_n is an increasing sequence with least upper bound the function $\lambda e : E.\mathcal{E}[\![\mathbf{if}\ z = 0\ \mathbf{then}\ \underline{0}\ \mathbf{else}\ \underline{0}]\!](\bot\ [e/z])$. Using the above equation it is straightforward to show that

$$\mathcal{E}[\![Lcfn]\!](\bot\ [\gamma_n/c, \text{fun}h/f, m/n]) = h^{n-m}(\bot)$$

for $m \leq n$ and $h : E \to E$, and so

$$\mathcal{E}[\![L(\lambda x.\mathbf{if}\ x = 0\ \mathbf{then}\ \underline{0}\ \mathbf{else}\ \underline{0})f\underline{0}]\!](\bot\ [\text{fun}h/f]) = \bigvee_{n \geq 0} h^n(\bot)$$

the least fixed-point of h, and we can take

$$\mathrm{Y}_\mu =_{\mathrm{def}} \lambda f.L(\lambda x.(\mathbf{if}\ x = 0\ \mathbf{then}\ \underline{0}\ \mathbf{else}\ \underline{0}))f\underline{0}$$

□

Some abbreviations will prove useful. We write **if** L **then** M **else** N as an abbreviation for the term **if** $L = 0$ **then** N **else** (**if** $\mathrm{pred}(L) = 0$ **then** M **else** Ω_μ) treating 1 and 0 as the truthvalues. Then *sequential conjunction* M **and** N is read as **if** M **then** N **else** $\underline{0}$ and *negation* $\neg M$ has an evident definition. Finally, we write **por** for $\lambda x.\lambda y.\mathbf{por}(x, y)$.

So far our considerations have been equational. However there are analogous *inequational* concepts. Define $\mathcal{E} \models_{\mathcal{L}} M \leq N$ to hold iff $\mathcal{E}[\![M]\!](\rho) \leq \mathcal{E}[\![N]\!](\rho)$ for all ρ and define the *operational inequivalence relation* $M \preceq_{\mathcal{L}} N$ to hold iff for all contexts $C[\ ]$ such that both $C[M]$ and $C[N]$ are closed, $C[N]\Downarrow_{\mathcal{L}}$ if $C[M]\Downarrow_{\mathcal{L}}$. Then we can say that say that $\mathcal{E}$ is *(inequationally) fully abstract for* $\mathcal{L}$ iff for all $\mathcal{L}$-terms M, N

$$M \preceq_{\mathcal{L}} N \text{ iff } \mathcal{E} \models_{\mathcal{L}} M \leq N$$

Similar definitions can be made for $\mathcal{L}^+$.

Fortunately this does not cause a duplication of material as the inequivalences can all be reduced to equivalences. Consider two closed $\mathcal{L}$-terms M and N, and define

$$A = \lambda z.\mathbf{if}\ zM\ \mathbf{then}\ \underline{1}\ \mathbf{else}\ \Omega_\mu$$

and

$$B = \lambda z.\mathbf{if}\ zM\ \mathbf{and}\ zN\ \mathbf{then}\ \underline{1}\ \mathbf{else}\ \Omega_\mu$$

Then one has that

$$\mathcal{E} \models_{\mathcal{L}} M \leq N \text{ iff } \mathcal{E} \models_{\mathcal{L}} A = B$$

and

$$M \preceq_{\mathcal{L}} N \text{ iff } A \simeq_{\mathcal{L}} B$$

This gives reductions for closed terms; for open terms note that the relations at hand are closed under λ-abstraction, for example, for any terms M' and N'

$$M' \preceq_{\mathcal{L}} N' \text{ iff } \lambda x.M' \preceq_{\mathcal{L}} \lambda x.N'$$

and so one can reduce to the case of closed terms. We just give some hints of the proofs of the reductions for closed terms. The first is not hard: one uses the fact that in any cppo D the ordering is *extensional* in the sense that

$$\forall x, y \in D.\, x \leq y \text{ iff } \forall f : D \to \boldsymbol{O}.\, f(x) \leq f(y)$$

where $\boldsymbol{O}$ is "Sierpinski space," the two-point cppo $\{\bot, \top\}$. For the second, the key fact needed is that for any $\mathcal{L}$-terms A' and B'

$$\lambda z.A' \preceq_{\mathcal{L}} \lambda z.B' \text{ iff } \forall C.\, A'[C/z] \preceq_{\mathcal{L}} B'[C/z]$$

where C ranges over closed $\mathcal{L}$-terms.

This is by no means easy to establish. One way is to use Abramsky's notion of applicative bisimilarity, introduced in the study of the lazy λ-calculus [Abr90].

Here one takes *simulation* to be the largest relation R between closed $\mathcal{L}$-terms such that if $R(A, B)$ holds then

(a) if $A \Rightarrow_{\mathcal{L}} \underline{n}$ then $B \Rightarrow_{\mathcal{L}} \underline{n}$

(b) if $A \Rightarrow_{\mathcal{L}} \lambda z.A'$ then for some B', $B \Rightarrow_{\mathcal{L}} \lambda z.B'$ and for all closed $\mathcal{L}$-terms C, $R(A'[C/z], B'[C/z])$ holds

The simulation relation is then extended to all terms by substitution. It turns out that it is *identical* to $\preceq_{\mathcal{L}}$; for a proof, one can, for example, follow [Pit97]. The key fact is then an immediate consequence.

The corresponding reductions hold for $\mathcal{L}^+$, with the same semantical reasoning and using, for the operational assertion, the inequational full abstraction of the standard model is established by an easy modification of the proof of equational full abstraction in the Appendix. It follows that the equational and inequational notions of full abstraction coincide.

2.3 Constructing Models

Let **K** be a **CPO**-enriched category with ω^{op}-limits, and let $F : \mathbf{K}^{op} \times \mathbf{K} \to \mathbf{K}$ be **CPO**-enriched. Then one can construct a functor $T : \mathbf{K}^e \to \mathbf{K}^e$ on the category of embeddings in **K** by putting:

$$T(x) = F(x, x)$$

and

$$T(\phi) = F(\phi^R, \phi)$$

(and then $T(\phi)^R = F(\phi, \phi^R)$). See [SP82, AC98] for details.

Now one can solve the domain equation

$$x \cong F(x, x)$$

provided one has an embedding

$$\phi_0 : x_0 \to T(x_0)$$

Define a chain $\Delta = < x_n, \phi_n >$, setting $x_{n+1} = T(x_n)$ and $\phi_{n+1} = T(\phi_n)$ for $n \geq 0$. Let $\rho : \Delta \to x_\infty$ be colimiting. Then the ρ_n are embeddings and there is an isomorphism $\eta : T(x_\infty) \to x_\infty$, the mediating morphism from $T(\rho_n)$ to ρ_{n+1}, i.e., the unique η such that $\rho_{n+1} = \eta \circ T(\rho_n)$ for $n \geq 0$. There are explicit formulae for it and its inverse, viz

$$\eta = \bigvee_{n \geq 0} \rho_{n+1} \circ T(\rho_n)^R$$

and

$$\eta^{-1} = \bigvee_{n \geq 0} T(\rho_n) \circ \rho_{n+1}^R$$

We say the solution $< x_\infty, \eta, \eta^{-1} >$ so generated is *standard* in the case that x_0 is initial in $\mathbf{K}^e$. Our primary application is to **CPPO** and the functor F given by

$$F(E, D) = (E \Rightarrow D)_\perp + \boldsymbol{N}_\perp$$

Note that such solutions are extensional models of $\mathcal{L}$. The category $\mathbf{CPPO}^e$ has an initial object, the one-point cppo $\mathbf{1}$ and we write $\mathcal{S} =< S, \eta, \eta^{-1} >$ for the standard model (i.e., solution). $\mathcal{L}$-adequacy can be established for $\mathcal{S}$ by using a recursively defined relation between $\mathcal{S}$ and closed terms, as in [Pit93]. In addition, although $\mathcal{S}$ is not fully abstract for $\mathcal{L}$, it is for $\mathcal{L}^+$; these latter facts do not seem to have been published elsewhere, but are hardly unexpected given the plethora of closely related material—see, for example, [AO93]. We give (fairly detailed) proofs in the Appendix.

2.4 Recursively Defined Predicates

We (more or less) follow the ideas of Pitts [Pit96], but rather than using Tarski's theorem, we give an abstract version of the treatment of predicates in [SP82]. We take a fibrational approach following Hermida [Her93], Hermida and Jacobs [HJ98] and what we do is essentially a special case of Fiore, Cattani and Winskel [FCW99]. So let us begin with a category $\mathbf{K}$ and a functor $\mathcal{P} : \mathbf{K}^{op} \to \mathbf{Pos}$ (the category of partial orders). We think of elements P of $\mathcal{P}(x)$ as predicates over x and of $\mathcal{P}(f)Q$ as $f^{-1}(Q)$; we write $f^*(Q)$ for $\mathcal{P}(f)Q$.

We can form the total category $\mathbf{K}_t$ with objects of the form $< x, P >$ (where $P \in \mathcal{P}(x)$) and morphisms of the form $f :< x, P > \to < y, Q >$, where $f : x \to y$ and $P \leq f^*Q$. This yields an evident fibration, $U : \mathbf{K}_t \to \mathbf{K}$.

We are going to obtain recursively defined predicates on solutions to domain equations in $\mathbf{K}$ by solving associated domain equations in $\mathbf{K}_t$. To this end, suppose we have a functor $F : \mathbf{K}^{op} \times \mathbf{K} \to \mathbf{K}$, and we wish to extend it to $\mathbf{K}_t$. For this we need an action of F on the fibres, $\mathcal{P}(x)$. Specifically we need for all pairs of objects y, x in $\mathbf{K}$ a monotone function:

$$F_{y,x} : \mathcal{P}(y)^{op} \times \mathcal{P}(x) \longrightarrow \mathcal{P}(F(y, x))$$

such that for all $y' \xrightarrow{g} y$, $x \xrightarrow{f} x'$, Q in $\mathcal{P}(y)$, and P' in $\mathcal{P}(x')$

$$F_{y,x}(Q, f^* P') \leq F(g, f)^* F_{y',x'}(g^* Q, P')$$

We may then define $F_t : {\mathbf{K}_t}^{op} \times \mathbf{K}_t \to \mathbf{K}_t$ by putting

$$F_t(< y, Q >, < x, P >) =< F(y, x), F_{y,x}(Q, P) >$$

and

$$F_t(g, f) = F(g, f)$$

(and any functor lifting F to $\mathbf{K}_t$ arises in this way). Now suppose we have a solution

$$\eta : F(< x, P >, < x, P >) \cong < x, P >$$

to the domain equation in $\mathbf{K}_t$. Then certainly we have a solution

$$\eta : F(x, x) \cong x$$

of the domain equation in $\mathbf{K}$ and also

$$P = (\eta^{-1})^* F_{x,x}(P, P)$$

since we also have that $F_{x,x}(p,P) \leq \eta^* P$ and $P \leq (\eta^{-1})^* F_{x,x}(P,P)$. One easily sees that this is a 1-1 correspondence between solutions in $\mathbf{K}_t$ and solutions in $\mathbf{K}$, together with such "recursively defined" predicates.

Now assume $\mathbf{K}$ has ω^{op}-limits and that $\mathcal{P}$ is in fact a functor $\mathcal{P} : \mathbf{K}^{op} \to \mathbf{mCPO}$, where $\mathbf{mCPO}$ is the category of posets with meets of decreasing sequences. Then $\mathbf{K}_t$ has ω^{op}-limits too, and U preserves them. To see this, let $\Delta = << x_n, P_n >, g_n >$ be an ω^{op}-chain in $\mathcal{K}_t$, and suppose that $\sigma : x_\infty \to < x_n, g_n >$ is limiting in $\mathcal{K}$. Then $(\sigma_n)^* P_n$ is decreasing as

$$\begin{aligned} (\sigma_n)^* P_n &= (\sigma_{n+1})^*((g_n)^* P_n) \\ &\geq (\sigma_{n+1})^* P_{n+1} \text{ (as } g_n : < x_{n+1}, P_{n+1} > \to < x_n, P_n >) \end{aligned}$$

and we may set $P_\infty = \bigwedge_n (\sigma_n)^* P$. One then easily verifies that $\sigma : < x_\infty, P_\infty > \to \Delta$ is limiting in $\mathcal{K}_t$.

Putting all this together, suppose now that $\mathbf{K}$ is $\mathbf{CPO}$-enriched, and that this yields a $\mathbf{CPO}$-enrichment of $\mathbf{K}_t$. Suppose we also have a functor $F : \mathbf{K}^{op} \times \mathbf{K} \to \mathbf{K}$, equipped with an action, as above, and now also assumed to be $\mathbf{CPO}$-enriched; then we have that $F_t : \mathbf{K}_t{}^{op} \times \mathbf{K}_t \to \mathbf{K}_t$ is also $\mathbf{CPO}$-enriched. Now, assuming $\mathbf{K}$ has ω^{op}-limits, suppose we have an embedding

$$\phi_0 : < x_0, P_0 > \to T_t(< x_0, P_0 >)$$

in $\mathbf{K}^e_t$. Then following the above remarks we can construct a solution

$$\eta : T_t(< x_\infty, P_\infty >) \cong < x_\infty, P_\infty >$$

to the domain equation in $\mathbf{K}_t$, following Section 2.3; and this solution can be taken so that

$$\eta : T(x_\infty) \cong x_\infty$$

is the solution to the domain equation in $\mathbf{K}$ constructed as in Section 2.3, starting now from $\phi_0 : x_0 \to T(x_0)$. Finally, we have that P_∞ satisfies the recursive predicate equation

$$P = (\eta^{-1})^* F(P,P)$$

One application is to predicates on $\mathbf{CPPO}$. A relation $R \subset D^n$ is *strict* iff $R(\bot, \ldots, \bot)$ holds; it is ω-*inductive* iff for any n increasing sequences $< x_{ij} >_{j \geq 0}$, $R(\bigvee_{j\geq 0} x_{1j}, \ldots, \bigvee_{j \geq 0} x_{nj})$ holds whenever $R(x_{1j}, \ldots, x_{nj})$ does for all $j \geq 0$; it is *admissible* if it is both strict and ω-inductive. Now set

$$\mathrm{APred}_n(D) = \{R \subset D^n \mid R \text{ is admissible}\}$$

Then we can take to be $\mathcal{P}(D)$ to be $\mathrm{APred}_n(D)$ or even $\mathrm{APred}_n(D)^X$, for some fixed set X. For example Pitts [Pit93] proof of adequacy for his language $\mathcal{L}$ uses $n = 1$ and $X = \mathcal{L}^o$. To use these predicates one has to prove a so-called *Logical Relations Lemma*, that the terms of the language at hand satisfy the predicate. We do not give any general formulation of such a lemma here; we rather content ourselves with stating the required version in each case, omitting the routine proof by structural induction.

3 The First Inadequate Model

As already remarked, in [Par76] Park showed that in non-standard models of the untyped λ-calculus the paradoxical combinator need not denote the least fixed-point operator. Now while Y may not be the least fixed-point operator semantically, it is syntactically, and we will show that inadequacy can result from this difference (though it need not—see Section 6).

The next lemma gives a general condition under which inadequacy occurs in this way. The statement of the lemma uses *step functions*: for x, y in cppos D, E, we take $x \Rightarrow y$ to be the function from D to E with value y at elements of D above (or equal to) x and $\perp$ elsewhere. Say that an element of a cppo is *ω-finite* iff whenever it is less than (or equal to) the lub of an increasing sequence it is less than (or equal to) a member of the sequence. Then $x \Rightarrow y$ is continuous iff x is ω-finite and it is then an ω-finite element of $D \Rightarrow E$ iff y is also ω-finite.

We should add that by the above operational assertion about Y we mean that the sentence

$$\forall f, x. fx \leq x \supset \mathrm{Y}f \leq x$$

is operationally true in the sense of [LP98], that is that for any closed $\mathcal{L}$-terms F, X, if $FX \preceq_{\mathcal{L}} X$ then $YF \preceq_{\mathcal{L}} X$. It is straightforward to prove using the properties of the standard model, particularly that Y denotes the least fixed-point operator there. The idea dates back to Morris (see [Bar84]), who proved a similar theorem in the context of the untyped λ-calculus.

Lemma 1 *Let $\mathcal{E}$ be a model of $\mathcal{L}$ and suppose that a, b are ω-finite elements of E such that $a = fun(a \Rightarrow b)$. Then $\mathcal{E}$ is not $\mathcal{L}$-adequate: specifically $Y(\lambda x.x)$ does not denote $\perp$, although it fails to terminate.*

Proof Since $b \leq \mathcal{E}[\![xx]\!](\perp [a/x])$ we have that $a = \mathrm{fun}(a \Rightarrow b) \leq \mathcal{E}[\![\lambda x.xx]\!](\perp)$ and so $a \leq \mathcal{E}[\![(\lambda x.xx)(\lambda x.xx)]\!](\perp) = \mathcal{E}[\![\mathrm{Y}(\lambda x.x)]\!](\perp)$. □

And now we may prove

Theorem 1 *$\mathcal{L}$ has an extensional model which is not $\mathcal{L}$-adequate.*

Proof We use the construction of section 2.3 with D_0 taken to be Sierpinski space and with ϕ_0 given by $\phi_0(\top) = \mathrm{inl}(\mathrm{up}(\top \Rightarrow \top))$. Note that ϕ_0 is an embedding, that $\phi_0^R(\mathrm{inr}(m)) = \perp$ and that $\phi_0^R(\mathrm{inl}(\mathrm{up}(g))) = g(\top)$.

Now set $a = \rho_0(\top)$; this is ω-finite as $\top$ is and embeddings preserve ω-finiteness. Next we have

$$\begin{aligned} a &= \rho_1(\phi_0(\top)) \\ &= \eta(T(\rho_0)(\mathrm{inl}(\mathrm{up}(\top \Rightarrow \top)))) \\ &= \eta(\mathrm{inl}(\mathrm{up}(\rho_0 \circ (\top \Rightarrow \top) \circ \rho_0^R))) \\ &= \mathrm{fun}(a \Rightarrow a) \end{aligned}$$

But now we may apply Lemma 1 to see that $< D_\infty, \eta, \eta^{-1} >$ is not $\mathcal{L}$-adequate. □

4 Comparing Models

How should we compare models? We would like to say that $\mathcal{D} \leq \mathcal{E}$ iff $\mathcal{E}$ is closer to being fully abstract than $\mathcal{D}$. It is natural to take this to mean that if $\mathcal{D}$ correctly predicts an observational equivalence then so does $\mathcal{E}$, leading to the following definition

Definition 1 *Let $\mathcal{D}$ and $\mathcal{E}$ be models of $\mathcal{L}$. Then $\mathcal{D} \leq \mathcal{E}$ iff for all expressions M, N such that $M \simeq_{\mathcal{L}} N$, if $\mathcal{D} \models_{\mathcal{L}} M = N$ then $\mathcal{E} \models_{\mathcal{L}} M = N$.*

Note that this definition is formulated so as to apply whether or not the models are $\mathcal{L}$-adequate. Also, by remarks in Section 2.2, the equational and inequational "less abstract than" relations coincide (with the evident definition of the inequational notion). The next proposition gives some apparently stronger but in fact equivalent conditions, one of which involves $\mathcal{L}^+$-notions; it also shows that only models which are "equally $\mathcal{L}^+$-adequate" can be compared.

Proposition 2 *Let $\mathcal{D}$ and $\mathcal{E}$ be models of $\mathcal{L}$. Then the following three conditions are equivalent and imply the fourth.*

1. $\mathcal{D} \leq \mathcal{E}$
2. $EqTh_{\mathcal{L}}(\mathcal{D}) \subset EqTh_{\mathcal{L}}(\mathcal{E})$
3. $EqTh_{\mathcal{L}^+}(\mathcal{D}) \subset EqTh_{\mathcal{L}^+}(\mathcal{E})$
4. $TerTh_{\mathcal{L}^+}(\mathcal{D}) = TerTh_{\mathcal{L}^+}(\mathcal{E})$

Proof Clearly 3 implies 2 implies 1, and so we show that 1 implies 3. To this end suppose $\mathcal{D} \models_{\mathcal{L}^+} M = N$. Then $M =_\beta M_0$**por** and $N =_\beta N_0$**por** for some M_0 and N_0 in $\mathcal{L}$. Now set

$$M_1 =_{\text{def}} \lambda z.\textbf{if } \lceil z \geq \vee_p \rceil \textbf{ then } M_0(Rz) \textbf{ else } \Omega_\mu$$

and

$$N_1 =_{\text{def}} \lambda z.\textbf{if } \lceil z \geq \vee_p \rceil \textbf{ then } N_0(Rz) \textbf{ else } \Omega_\mu$$

Here $\lceil z \geq \vee_p \rceil$ abbreviates $(z\,\underline{1}\,\bot)$ **and** $(z\,\bot\,\underline{1})$ **and** $\neg(z\,\underline{0}\,\underline{0})$ and R is the expression $\lambda z.\lambda x.\lambda y.(\textbf{if } x \leq 1 \textbf{ and } y \leq 1 \textbf{ then } zxy \textbf{ else } \Omega_\mu)$, where, in turn, $w \leq 1$ abbreviates **if** w **then** $\underline{1}$ **else** $\underline{1}$.

Then M_1 and N_1 are in $\mathcal{L}$ and $M_1 \simeq_{\mathcal{L}} \lambda x.\Omega_\mu \simeq_{\mathcal{L}} N_1$ follows from Lemma 7. Since $\mathcal{D} \models_{\mathcal{L}} M = N$ we also have that $\mathcal{D} \models_{\mathcal{L}} M_1 = N_1$, and so, as $\mathcal{D} \leq \mathcal{E}$, $\mathcal{E} \models_{\mathcal{L}} M_1 = N_1$. Now, applying both sides to **por**, we see that $\mathcal{E} \models_{\mathcal{L}^+} M = N$ as required.

That 3 implies 4 is immediate once we have noticed that for any model $\mathcal{D}$ and closed M in $\mathcal{L}^+$,

$$\mathcal{D} \models_{\mathcal{L}^+} M\!\downarrow \text{ iff } \mathcal{D} \models_{\mathcal{L}^+} \partial(M, \underline{0}, \underline{0}) = \underline{0}$$

and

$$\mathcal{D} \not\models_{\mathcal{L}^+} M\!\downarrow \text{ iff } \mathcal{D} \models_{\mathcal{L}^+} \partial(M, \underline{0}, \underline{0}) = \Omega_\mu$$

□

As a Corollary we can characterise comparability with the standard model:

Corollary 1 *Let $\mathcal{D}$ be a model of $\mathcal{L}$. The following three conditions are equivalent:*

1. *$\mathcal{D}$ and $\mathcal{S}$ are comparable*
2. *$\mathcal{D}$ is $\mathcal{L}^+$-adequate*
3. *$\mathcal{D} \leq \mathcal{S}$*

Proof That 1 implies 2 follows from the Proposition and the $\mathcal{L}^+$-adequacy of $\mathcal{S}$. That 2 implies 3 follows from the full abstraction of $\mathcal{S}$ for $\mathcal{L}^+$. □

This result is, perhaps, not quite what one might hope for. To show $\mathcal{S}$ is the "best" model one would like to show it is better than any $\mathcal{L}$-adequate models. However this is true only for $\mathcal{L}^+$-adequate models, and models which are not $\mathcal{L}^+$-adequate are, by the Proposition, incomparable with $\mathcal{S}$. It may bear emphasising that this implies the latter make true predictions of equalities that the standard model does not.

Corollary 1 implies that $\mathcal{S}$ is maximal in the abstraction ordering; this can be strengthened. The standard model is characterised by its minimality, where a model $\mathcal{E} =< E, s, r >$ is said to be *minimal* iff id_E is the least $g : E \to_\perp E$ such that

$$g = s{\circ}F(g, g){\circ}r$$

Then $\mathcal{E}$ is minimal iff it is isomorphic to $\mathcal{S}$. To see this, follow the discussion in [Pit96], but adapted to models rather than Pitts' invariants, which are just our extensional models. (One can also show that id_E is in fact the unique such g.)

The transformation $g \mapsto s{\circ}F(g, g){\circ}r$ is definable by the term

$$T =_{\text{def}} \lambda x.\partial(x, \lambda z.f(x(fz)), x)$$

in the sense that $\mathcal{E}[\![T]\!](\perp [\text{fun}g/f]) = s{\circ}F(g, g){\circ}r$. We therefore have that $\mathcal{E}$ is minimal iff the following equation holds

$$\mathrm{Y}_\mu(\lambda f.T) = \lambda x.x$$

So $\mathcal{S}$ is not only maximal in the abstraction ordering, but even strictly so in the sense that any model equivalent to it in that ordering is isomorphic to it.

5 The Second Inadequate Model

The last section left open the possibility that every $\mathcal{L}$-adequate model was actually $\mathcal{L}^+$-adequate, when the standard model would indeed be the closest to full abstraction among the $\mathcal{L}$-adequate models. We now show that this is, unfortunately, not the case, proving

Theorem 2 *There is an extensional model which is adequate for $\mathcal{L}$, but not for $\mathcal{L}^+$.*

The idea of the proof is to modify Lemma 1 by adding a "parallel or guard," inaccessible to $\mathcal{L}$-terms, but not to $\mathcal{L}^+$-terms. We find it convenient here, and below, to write $(a \Rightarrow b)$ rather than $\text{fun}(a \Rightarrow b)$, omitting fun.

Lemma 2 *Let $\mathcal{E}$ be a model containing an ω-finite element $a = (\vee_p \Rightarrow (a \Rightarrow a))$. Then $\mathcal{E}$ is not $\mathcal{L}^+$-adequate.*

Proof Let $T = \lambda z.\lambda x.xzx$. Then TMT terminates for no closed M. Since $a \cdot \vee_p \cdot a \geq a$ we get that $\mathcal{E}[\![T]\!](\perp) \geq (\vee_p \Rightarrow (a \Rightarrow a)) = a$. So $\mathcal{E}[\![T\,\mathbf{por}\,T]\!](\perp) \geq a \neq \perp$ and it follows that $\mathcal{E}$ is not $\mathcal{L}^+$-adequate. □

To apply this lemma we proceed analogously to the proof of Theorem 1, but with complications caused by the appearance of $\vee_p$ and the nested $\Rightarrow$. The idea is to find elements a, b, v in $\rho_0(D_0)$ such that $a = (v \Rightarrow b)$, $b = (a \Rightarrow a)$ and $v = \vee_p$; the latter requires further decompositions into elements corresponding to $(1 \Rightarrow (\perp \Rightarrow 1))$ etc. We achieve this by taking D_0 to be the set of configurations of E_0, an event structure in the sense of [NPW81].

Specifically, the events in E are: $\alpha, \beta, \nu, \nu_\perp, \nu_1, \nu_0, 1, 0$ with the ordering: $\nu_1 \leq \nu_\perp$ and with inconsistency relation the symmetric closure of

$$\{<i,e>\mid i \in \{0,1\}, e \neq i\} \cup$$
$$\{<\nu_0,\nu_\perp>, <\nu,\nu_1>, <\nu,\nu_0>, <\nu,\nu_\perp>, <\beta,\nu_\perp>, <\alpha,\nu_\perp>, <\alpha,\beta>\}$$

This determines D_0 as $\Gamma(E_0)$, the cppo of configurations of the event structure. Next we define $\phi : E_0 \to D_1$ by

$$\begin{array}{lcl}
\phi(\alpha) & = & \{\nu\} \Rightarrow \{\beta\} \\
\phi(\beta) & = & \{\alpha\} \Rightarrow \{\alpha\} \\
\phi(\nu) & = & (\{1\} \Rightarrow \{\nu_\perp, \nu_1\}) \vee (\perp \Rightarrow \{\nu_1\}) \vee (\{0\} \Rightarrow \{\nu_0\}) \\
\phi(\nu_\perp) & = & \perp \Rightarrow \{1\} \\
\phi(\nu_1) & = & \{1\} \Rightarrow \{1\} \\
\phi(\nu_0) & = & \{0\} \Rightarrow \{0\} \\
\phi(1) & = & 1 \\
\phi(0) & = & 0
\end{array}$$

which should make clear the intended rôles of the events. One can then verify that for any e, e' in E_0

$$e \leq e' \text{ iff } \phi(e) \leq \phi(e')$$

and

$$e \,\#\, e' \text{ iff } \phi(e) \,\#\, \phi(e')$$

(In fact, requiring these conditions forces the choices of $\leq$ and $\#$ for E_0 except for whether or not $\alpha \# \beta$.)

We can now define an embedding-projection pair $D_0 \overset{\phi_0}{\to} D_1 \overset{\psi_0}{\to} D_0$ by

$$\phi_0(d_0) = \bigvee\{\phi(e) \mid e \in d_0\}$$

and

$$\psi_0(d_1) = \{e \mid \phi_0(e) \leq d_1\}$$

Clearly ϕ_0 and ψ_0 are monotone: we now show they form an embedding-projection pair. We have $\psi_0 \circ \phi_0 \geq \mathrm{id}_{D_0}$ since:

$$\begin{array}{rcl}
\psi_0(\phi_0(d_0)) & = & \{e' \mid \phi_0(e') \leq \bigvee\{\phi(e) \mid e \in d_0\}\} \\
& \geq & \{e' \mid e' \in d_0\} \\
& = & d_0
\end{array}$$

and $\phi_0 \circ \psi_0 \leq \mathrm{id}_{D_1}$ as

$$\begin{array}{rcl} \phi_0(\psi_0(d_1)) & = & \bigvee\{\phi(e) \mid \phi(e) \leq d_1\} \\ & \leq & d_1 \end{array}$$

To show $\psi_0 \circ \phi_0 \leq \mathrm{id}_{D_0}$, consider a d_0 and an e' such that $\phi_0(e') \leq \bigvee\{\phi(e) \mid e \in d_0\}$. If $\phi_0(e')$ is prime, then $\phi(e') \leq \phi(e)$ for some e in d_0 and then $e' \leq e \in d$ and so $e' \in d_0$. The only non-prime $\phi(e')$ is $\phi(\nu)$. In that case we have $(1 \Rightarrow \nu_\perp) \leq \phi(\nu)$ and so, as $(1 \Rightarrow \nu_\perp)$ is prime, $(1 \Rightarrow \nu_\perp) \leq \phi(e)$ for some e in d_0; but then, as an inspection of the possibilities shows $e = \nu$. So, in all cases $e' \in d_0$ showing that $\psi_0 \circ \phi_0 \leq \mathrm{id}_{D_0}$, as required.

We may now form D_∞ in the usual way, yielding an extensional model $\mathcal{D} = < D_\infty, \eta, \eta^{-1} >$. Clearly $\rho_0(\{\nu\}) = \vee_p$; so, setting $a = \rho_0(\{\alpha\})$ and $b = \rho_0(\{\beta\})$, we get

$$\begin{array}{rcl} a & = & \rho_0(\{\alpha\}) \\ & = & \rho_1(\{\nu\} \Rightarrow \{\beta\}) \\ & = & \rho_0(\{\nu\}) \Rightarrow \rho_0(\{\beta\}) \\ & = & \vee_p \Rightarrow (\rho_1(\{\alpha\} \Rightarrow \{\alpha\})) \\ & = & \vee_p \Rightarrow (a \Rightarrow a) \end{array}$$

So by Lemma 2, $\mathcal{D}$ is not $\mathcal{L}^+$-adequate.

We now prove that is $\mathcal{L}$-adequate by combining the usual "$d \leq M$ method" with the ternary logical predicate used to show parallel or is not definable in [Sie92], defining the functor $\mathcal{P}$ as follows

$$\mathcal{P}(D) = \mathrm{APred}_3(D)^{(\mathcal{L}^o)^3}$$

and

$$\mathcal{P}(f) = \mathrm{APred}_3(f)^{(\mathcal{L}^o)^3}$$

and then define the action of F on predicates by taking $F_{E,D}(Q,P)_{L_1,L_2,L_3}(u_1,u_2,u_3)$ to hold iff one of the following is true

(a) $u_1 = \perp \vee u_2 = \perp \vee u_3 = \perp$

(b) for some m, $u_1 = u_2 = u_3 = \mathrm{inr}(m)$ and $L_1, L_2, L_3 \Rightarrow \underline{m}$

(c) for $i = 1, 3$, u_i is functional and $L_i \Rightarrow \lambda x.L'_i$, and whenever $Q_{M_1,M_2,M_3}(v_1,v_2,v_3)$ holds then so does $Q_{L'_1[M_1/x],L'_2[M_2/x],L'_3[M_3/x]}(u_1 \cdot v_1, u_2 \cdot v_2, u_3 \cdot v_3)$.

where u is said to be functional if it has the form inl(up(f)), and then $u \cdot v =_{\mathrm{def}} f(v)$. One can verify the required condition on the action, including those on enrichment.

We think of these predicates as order relations between semantics and syntax and write $u_1,u_2,u_3 \leq L_1,L_2,L_3$ rather than $P_{L_1,L_2,L_3}(u_1,u_2,u_3)$. We need to define such a $\leq_0$ on D_0 and begin with a corresponding predicate on E_0, $e_1,e_2,e_3 \in_0 L_1,L_2,L_3$, which we define by cases.

First consider the diagonal $e,e,e \in_0 L_1,L_2,L_3$; this holds iff one of the following cases hold.

(a) $e = 0$ and $L_i \Rightarrow \underline{0}$ (for all i)

(b) $e = 1$ and $L_i \Rightarrow \underline{1}$ (for all i)

(c) $e = \nu_0$ and $L_i \Rightarrow \lambda x.L'_i$ and $L'_i[M_i/x] \Rightarrow \underline{0}$ (for all i) whenever all the M_i evaluate to $\underline{0}$.

(d) $e = \nu_1$ and $L_i \Rightarrow \lambda x.L'_i$ and $L'_i[M_i/x] \Rightarrow \underline{1}$ (for all i) whenever all the M_i evaluate to $\underline{1}$.

(e) $e = \nu_\perp$ and $L_i \Rightarrow \lambda x.L'_i$ and $L'_i[M_i/x] \Rightarrow \underline{1}$ for all M_i.

(f) $e = \beta$ and $L_i \Rightarrow \lambda x.L'_i$ (for all i) and all $L'_i[M_i/x]$ evaluate to a functional value whenever all the M_i do.

(g) $e = \alpha$ and $L_i \Rightarrow \lambda x.L'_i$ (for all i).

Next, $e_1,e_2,e_3 \in_0 L_1,L_2,L_3$ holds off the diagonal iff all the L_i evaluate to functional values $\lambda x.L'_i$ and one of the following mutually exclusive conditions hold:

(a) $\nu_\perp$ appears among the e_1, e_2, e_3 and all the others are either $\nu_\perp$ or ν_1, and for all M_i, $L'_i[M_i/x] \Rightarrow \underline{1}$.

(b) Otherwise ν appears among the e_1, e_2, e_3 and all the others are either β or α and for all M_i, all the $L'_i[M_i/x]$ evaluate to functional values.

(c) Otherwise ν_0 or ν_1 appears among the e_1, e_2, e_3 but neither 0 nor 1 do.

(d) Otherwise β appears twice among the e_i, and α once, and all the $L'_i[M_i/x]$ evaluate to functional values for all M_i such that: (i) $M_i \Rightarrow \lambda x.M'_i$, and (ii) for any N_i, $M'_i[N_i/x]$ evaluates to a functional value.

(e) Otherwise the condition holds (here α appears twice among the e_i, and β once).

Note that $e_1,e_2,e_3 \in_0 L_1,L_2,L_3$ does not hold if any two of the e_i are equal to ν.

Now define $\leq_0$ by: $u_1,u_2,u_3 \leq_0 L_1,L_2,L_3$ iff $e_1,e_2,e_3 \in_0 L_1,L_2,L_3$ for all e_i in u_i.

Lemma 3 *If* $u_1,u_2,u_3 \leq_0 L_1,L_2,L_3$ *then* $\phi_0(u_1),\phi_0(u_2),\phi_0(u_3) \leq_1 L_1,L_2,L_3$.

Proof We proceed according to the three cases in the definition of the action of F on predicates (here $\leq_0$). If one of the u_i is $\perp$ then the conclusion follows by case (a) as ϕ_0 is strict.

Otherwise, suppose one of the u_i contains 0, then, since $e_1,e_2,e_3 \in_0 L_1,L_2,L_3$ for all e_i in u_i, we must have that all the u_i are equal to $\{0\}$. But then we have that $0,0,0 \in_0 L_1,L_2,L_3$ and so $L_i \Rightarrow 0$ and the conclusion follows by case (b) as $\phi_0(\{0\}) = 0$. The case where one of the u_i contains 1 is similar.

Otherwise we must proceed according to case (c). First, since here we have e_i in u_i such that $e_1,e_2,e_3 \in_0 L_1,L_2,L_3$ and none of the e_i are 0 or 1, it follows by the definition of ϕ_0 that the $\phi_0(u_i)$ are all functional and by the definition of $\in_0$ that the L_i evaluate to functional values $\lambda x.L'_i$.

Now we must take v_i in D_0 such that v_1,v_2,v_3 $\leq_0$ M_1,M_2,M_3, and show that $u_1 \cdot v_1$,$u_2 \cdot v_2$,$u_3 \cdot v_3$ $\leq_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$. The proof again splits, into mutually exclusive cases.

Case 1 Here $\nu_\perp$ is in some u_i. Then $u_i \subset \{\nu_\perp, \nu_1\}$, for all i, and we have e_i in u_i such that e_1,e_2,e_3 $\in_0$ L_1,L_2,L_3 one of the e_i is $\nu_\perp$, and the others are $\nu_\perp$ or ν_1. From the former it follows that each $u_i \cdot v_i$ is either 1 or $\perp$, and from the latter that $L'_i[M_i/x] \Rightarrow \underline{1}$; but then $u_1 \cdot v_1$,$u_2 \cdot v_2$,$u_3 \cdot v_3$ $\leq_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ holds, as required.

Case 2 Here ν is in some u_i, say u_1 (the situation is symmetrical). Here, by case (b) of the off-diagonal part of the definition of $\in_0$ we know that for all M_i, that $L'_i[M_i/x]$ evaluates to a functional value. Further, by consistency considerations, we have that $u_1 \subset \{\nu, \alpha, \beta\}$, and also that each of u_2,u_3 is $\{\alpha\}$ or $\{\beta\}$ (using case (b) of the off-diagonal part of the definition of $\in_0$ again too).

We may assume that no $u_i \cdot v_i$ is $\perp$. Then neither 0 or 1 is in v_1 as otherwise v_2 is $\{0\}$ or $\{1\}$ and then $u_2 \cdot v_2 =\perp$; it follows that $u_1 \cdot v_1 \subset \{\nu_1, \beta, \alpha\}$. Now, we cannot have β occurring in two of the $u_i \cdot v_i$ as otherwise ν occurs in two of the v_i, contradicting the definition of $\in_0$. As each of u_2, u_3 is $\{\alpha\}$ or $\{\beta\}$, the remaining possibilities are:

1. $\beta \in u_1 \cdot v_1 \subset \{\nu_1, \beta\}$ and $u_2 \cdot v_2 = u_3 \cdot v_3 = \{\alpha\}$
2. $\beta \not\in u_1 \cdot v_1 \subset \{\nu_1, \alpha\}$ and one of $u_2 \cdot v_2$, $u_3 \cdot v_3$ is $\{\alpha\}$ (the other is $\{\alpha\}$ or $\{\beta\}$)

In both cases we $u_1 \cdot v_1$,$u_2 \cdot v_2$,$u_3 \cdot v_3$ $\leq_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$, as required, as e_1,e_2,e_3 $\in_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ for any $e_1, e_2, e_3 \in u_1, u_2, u_3$, since all the $L'_i[M_i/x]$ all evaluate to functional values, and such a triple is either all α's, or else fall under one of the off-diagonal cases (c) or (e).

Case 3 Here neither $\nu_\perp$ nor ν is in any u_i. Here if some v_i is $\perp$ then so is the corresponding $u_i \cdot v_i$, and we are done. So we may assume that no v_i is $\perp$, and the proof splits into subcases.

Case 3.1 Here 0 is in some v_i. Then they are all equal to $\{0\}$, and $M_i \Rightarrow \underline{0}$. Now, if some ν_0 is not in some u_i then the corresponding $u_i \cdot v_i$ is $\perp$, and we are done. So we may assume that every u_i contains ν_0. Then we have that ν_0,ν_0,ν_0 $\in_0$ L_1,L_2,L_3 and so, as $M_i \Rightarrow \underline{0}$, we have that $L'[M_i/x] \Rightarrow \underline{0}$ and as we also have that $u_i \cdot v_i = \{0\}$ we are done.

Case 3.2 Here 1 is in some v_i; this case is similar to the last.

Case 3.3 Without 0 or 1 in any v_i, or $\nu_\perp$ or ν in any u_i we have

$$u_i \cdot v_i = (u_i \cap \{\alpha, \beta\}) \cdot (v_i \cap \{\alpha, \nu\})$$

If some $(u_i \cap \{\alpha, \beta\})$ is empty, we are done. Otherwise the proof again splits into subcases.

Case 3.3.1 Here β is in every u_i; we then have that β,β,β $\in_0$ L_1,L_2,L_3. We may also assume that α is in every v_i; we then have that α,α,α $\in_0$ M_1,M_2,M_3 and also that $u_i \cdot v_i = \{\alpha\}$. Now, since α,α,α $\in_0$ M_1,M_2,M_3, all the M_i evaluate to functional values. But then, since β,β,β $\in_0$ L_1,L_2,L_3, it follows that all the $L'_i[M_i/x]$ do too, and so $\{\alpha\}$,$\{\alpha\}$,$\{\alpha\}$ $\leq_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ as required.

Case 3.3.2 Here β is in exactly two u_i's—say u_1 and u_2 (and then $\alpha \in u_3$); we then have that β,β,$\alpha \in_0 L_1$,L_2,L_3. We may also then assume that α is in v_1 and v_2 and $\nu \in v_3$; we then have that α,α,$\nu \in_0 M_1$,M_2,M_3 and also that $u_1 \cdot v_1 = u_2 \cdot v_2 = \{\alpha\}$ and $u_3 \cdot v_3 = \{\beta\}$. But then, by similar reasoning to the previous case, from the facts that β,β,$\alpha \in_0 L_1$,L_2,L_3 and α,α,$\nu \in_0 M_1$,M_2,M_3 it follows that the $L'_i[M_i/x]$ evaluate to functional values and so that α,α,$\beta \in_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$.

Case 3.3.3 Here α is in at least two u_i's. Some $u_i \cdot v_i$ is then $\perp$, for we cannot have ν in two v_i's as there is no triple of events in $\in_0$ with two ν's. □

Lemma 4 *If* v_1,v_2,$v_3 \leq_1 L_1$,L_2,L_3 *then* $\psi_0(v_1)$,$\psi_0(v_2)$,$\psi_0(v_3) \leq_0 L_1$,L_2,L_3.

Proof

Suppose that v_1,v_2,$v_3 \leq_1 L_1$,L_2,L_3. The case where some v_i is $\perp$ is trivial (ψ_0 is strict) as is the case where $v_1 = v_2 = v_3 = \text{num}m$ for some m. So we may assume that all the v_i are functional values, that $L_i \Rightarrow \lambda x.L'_i$, and that whenever w_1,w_2,$w_3 \leq_0 M_1$,M_2,M_3 then $v_1 \cdot w_1$,$v_2 \cdot w_2$,$v_3 \cdot w_3 \leq_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ (where we write $v \cdot w$ for $f(w)$ if $v = \text{fun}f$).

Now, in order to show that $\psi_0(v_1)$,$\psi_0(v_2)$,$\psi_0(v_3) \leq_0 L_1$,L_2,L_3, we take e_i in $\psi_0(v_i)$ and show that e_1,e_2,$e_3 \in_0 L_1$,L_2,L_3. Note that none of the e_i are 0 or 1, and that $\phi(e_i) \subset v_i$. It is convenient to define $\overline{e}$ in D_0, for $e \neq 0, 1$ by

$$\begin{array}{lcl} \overline{\alpha} & = & \{\nu\} \\ \overline{\beta} & = & \{\alpha\} \\ \overline{\nu} & = & \perp \\ \overline{\nu_\perp} & = & \perp \\ \overline{\nu_1} & = & \{1\} \\ \overline{\nu_0} & = & \{0\} \end{array}$$

The proof now divides into mutually exclusive cases.

Case 1 Here some e_i is $\nu_\perp$—say e_1. Then we have $\perp$,$\overline{e_2}$,$\overline{e_3} \leq_0 M_1$,$M_2$,$M_3$ (for any M_i) and so $v_1 \cdot \perp$,$v_2 \cdot \overline{e_2}$,$v_3 \cdot \overline{e_3} \leq_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$. But $1 \in v_1 \cdot \perp$ and so $v_1 \cdot \perp = v_2 \cdot \overline{e_2} = v_3 \cdot \overline{e_3} = \{1\}$ and $L'_i[M_i/x] \Rightarrow \underline{0}$, for all i. As $v_2 \cdot \overline{e_2} = \{1\}$, e_2 is $\nu_\perp$ or ν_1, and similarly for e_3. Summarising, we have that $e_1 = \nu_\perp$, e_2, e_3 are each $\nu_\perp$ or ν_1 and for all M_i, $L'_i[M_i/x] \Rightarrow \underline{0}$. This shows that e_1,e_2,$e_3 \in_0 L_1$,L_2,L_3 as required.

Case 2 Here some e_i is ν—say e_1. Then neither e_2 nor e_3 can be ν_0 or ν_1. For example if $e_2 = \nu_0$, then as $\perp$,$\{0\}$,$\overline{e_3} \leq_0 M_1$,$M_2$,$M_3$ (for any M_i), by the definition of $\leq_1$, we get that ν_1,0,$e \in v_1 \cdot \perp$,$v_2 \cdot \{0\}$, $v_3 \cdot \overline{e_3} \leq_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$, for some e, which is impossible. So e_2, e_3 are each ν, α or β.

We now show that in fact neither can be ν. First, suppose that both are. Then, as $\perp$,$\{1\}$,$\{0\} \leq_0 M_1$,M_2,M_3 (for any M_i), by the definition of $\leq_1$, we get that ν_1,$\nu_\perp$,$\nu_0 \in_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$, which is impossible. Next, suppose that (say) only e_2 is. Then, as $\perp$,$\{1\}$,$\overline{e_2} \leq_0 M_1$,$M_2$,$M_3$ (for any M_i) we get that ν_1,$\nu_\perp$,$e \in_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$, for some e distinct from ν_1 and $\nu_\perp$, which is impossible.

So e_2, e_3 are each α or β. Now, as, for any M_i, $\perp$,$\overline{e_2}$,$\overline{e_3} \leq_0 M_1$,$M_2$,$M_3$, by the definition of $\leq_1$, we get that ν_1,e'_2,$e'_3 \in_0 L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ for some e'_2,

e'_3 which are α or β, showing that the $L'_i[M_i/x]$ all evaluate to functional values, as required.

Case 3 Here some e_i is ν_0—say e_1. If some other e_i is not ν_0 then we are done by case (c) of the off-diagonal part of the definition of $\in_0$. Otherwise suppose that they are all ν_0. Take M_i which all evaluate to $\underline{0}$. Then $\{0\}$,$\{0\}$,$\{0\}$ $\leq_0$ M_1,M_2,M_3, and so $v_1 \cdot \{0\}$,$v_2 \cdot \{0\}$,$v_3 \cdot \{0\}$ $\leq_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$, which implies that 0,0,0 $\in_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ and so all the $L'_i[M_i/x]$ evaluate to $\underline{0}$, showing that ν_0,ν_0,ν_0 $\in_0$ L_1,L_2,L_3, as required.

Case 4 Here some e_i is ν_1. This case is similar to the previous one.

Case 5 Here all the e_i's are α or β. The proof splits into subcases.

Case 5.1 Here all the e_i's are β. Now suppose that M_i are terms which all evaluate to functional values. Then $\{\alpha\}$,$\{\alpha\}$,$\{\alpha\}$ $\leq_0$ M_1,M_2,M_3, and we have that α,α,α $\in$ $v_1 \cdot \{\alpha\}$,$v_2 \cdot \{\alpha\}$,$v_3 \cdot \{\alpha\}$ $\leq_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ and so the $L'_i[M_i/x]$ evaluate to functional values, and it follows that β,β,β $\in_0$ L_1,L_2,L_3 as required.

Case 5.2 Here two of the e_i's are β—say e_1 and e_2, and the other is α. Now suppose that M_i are terms such that: (i) $M_i \Rightarrow \lambda x.M'_i$, and (ii) for any N_i, $M'_i[N_i/x]$ evaluates to a functional value. Then $\{\alpha\}$,$\{\alpha\}$,$\{\nu\}$ $\leq_0$ M_1,M_2,M_3 and, arguing as usual, we find that α,α,β $\in_0$ $L'_1[M_1/x]$,$L'_2[M_2/x]$,$L'_3[M_3/x]$ showing that all the $L'_i[M_i/x]$ evaluate to functional values, as required.

Case 5.3 Here at least two of the e_i's are α. This case is immediate as we already know that the L_i evaluate to functional values. □

We therefore obtain a relation

$$d_1,d_2,d_3 \leq_\infty L_1,L_2,L_3$$

between triples of elements of D_∞ and triples of closed terms of $\mathcal{L}$ which holds iff one of the following is true

(a) $d_1 =\perp \vee d_2 =\perp \vee d_3 =\perp$

(b) for some m, $d_1 = d_2 = d_3 = \mathrm{num}(m)$ and $L_1, L_2, L_3 \Rightarrow \underline{m}$

(c) for $i = 1, 3$, d_i is functional and $L_i \Rightarrow \lambda x.L'_i$, and whenever

$$d'_1,d'_2,d'_3 \leq_\infty M_1,M_2,M_3$$

holds, then so does

$$d_1 \cdot d'_1, d_2 \cdot d'_2, d_3 \cdot d'_3 \leq_\infty L'_1[M_1/x], L'_2[M_2/x], L'_3[M_3/x]$$

We then have the following Logical Relations Lemma.

Lemma 5 *Let M be an $\mathcal{L}$-term with free variables $x_1, \ldots, x_n$. Suppose for $i = 1, n$ that $d_{i1}, d_{i2}, d_{i3} \leq_\infty N_{i1}, N_{i2}, N_{i3}$. Then*

$$\mathcal{D}[\![M]\!](\rho_1), \mathcal{D}[\![M]\!](\rho_2), \mathcal{D}[\![M]\!](\rho_3) \leq_\infty \overline{M_1}, \overline{M_2}, \overline{M_3}$$

where $\rho_j =\perp [d_{1j}, \ldots, d_{nj}/x_1, \ldots, x_n]$ and $\overline{M_j} = M[N_{1j}, \ldots, N_{nj}/x_1, \ldots, x_n]$.

It follows that $\mathcal{D}$ is $\mathcal{L}$-adequate. For suppose that M is a closed term with non-$\perp$ denotation. Then by the lemma we have that

$$\mathcal{D}[\![M]\!](\perp), \mathcal{D}[\![M]\!](\perp), \mathcal{D}[\![M]\!](\perp) \leq_\infty M, M, M$$

and so either (b) or (c), as above, holds, and in either case, $M \Downarrow_{\mathcal{L}}$.

6 The Third Inadequate Model

In this section we concentrate on $\mathcal{L}^+$-adequate extensional models, beginning by displaying class many distinct from the standard model. We then give our third inadequate model. It is again $\mathcal{L}^+$-adequate; its "inadequacy" lies rather that that in it Y does not denote the least fixed-point operator; thus no Hyland-Wadsworth-type theorem [Bar84], that the meaning of a term is the lub of the meanings of its approximants, is going to hold. Finally we extract some information on the abstraction ordering below the standard model.

Consider the embedding

$$\phi_0 : D_0 \to T(D_0)$$

where $D_0 = \mathcal{P}(X)$ (where $\|X\| \geq 2$) and, for $x \neq \emptyset$, $\phi_0(x) = \bigvee\{\mathrm{inl}(\mathrm{up}(\bot \Rightarrow i)) \mid i \in x\}$ (and then $\phi_0^R(\mathrm{inl}(m)) = \bot$ and $\phi_0^R(\mathrm{inl}(\mathrm{up}(g))) = g(\bot)$. We write $\mathcal{D}_X$ for the resulting model $<D_\infty, \eta, \eta^{-1}>$.

Say that an *automorphism* of a model $\mathcal{E} = <E, s, r>$ is an automorphism of cppos $\theta : E \cong E$ such that the following diagram commutes

$$\begin{array}{ccc} T(E) & \xrightarrow{\;s\;} & E \\ {\scriptstyle T(\theta)}\downarrow & & \downarrow{\scriptstyle \theta} \\ T(E) & \xrightarrow[\;s\;]{} & E \end{array}$$

It turns out that $\mathcal{D}_X$ has at least as many automorphisms as there are bijections of X, yielding the required class many mutually non-isomorphic models, all of which are distinct from the standard model (which only has one automorphism).

Let θ_X be a bijection of X. Define $\theta_n : D_n \cong D_n$ by $\theta_0(x) = \{\theta_X(i) \mid i \in x\}$ and $\theta_{n+1} = T(\theta_n)$. This gives an automorphism $\theta : \Delta \cong \Delta$ of chains, i.e., the following diagram commutes for all n

$$\begin{array}{ccc} D_n & \xrightarrow{\;\phi_n\;} & D_{n+1} \\ {\scriptstyle \theta_n}\downarrow & & \downarrow{\scriptstyle \theta_{n+1}} \\ D_n & \xrightarrow[\;\phi_n\;]{} & D_{n+1} \end{array}$$

This is easily verified for $n = 0$, and the general case immediately follows by induction. Then the mediating morphism $\theta_\infty : D_\infty \cong D_\infty$ from ρ to $\rho\circ\theta$ is an automorphism of cppos (with inverse the mediating morphism from ρ to $\rho\circ\theta^{-1}$). Some straightforward diagram chasing shows that both $\eta\circ T(\theta_\infty)$ and $\theta_\infty\circ\eta$ are mediating morphisms from $T(\rho)$ to $\rho^+\circ T(\theta)$. They are therefore equal and so θ_∞ is indeed an automorphism of $\mathcal{D}_X$.

To show $\mathcal{L}^+$-adequacy we construct a predicate $\leq_\infty$ in $\mathrm{APred}_1(D_\infty)^{\mathcal{L}^o}$ such that $d \leq_\infty L$ iff one of the following is true

(a) $d = \perp$

(b) For some m, $d = \mathrm{num}(m)$ and $L \Rightarrow \underline{m}$

(c) d is functional, $L \Rightarrow \lambda x.L'$, and if $e \leq_\infty M$ holds so does $d \cdot e \leq_\infty L'[M/x]$.

This can be done as usual, using the evident action on predicates and starting from $d \leq_0 M$ iff $d = \perp$. We just check that ϕ_0 and ψ_0 preserve the predicate in the usual sense. This is trivial for ϕ_0 as it is strict. For ψ_0 suppose that $d_1 \leq_1 M$ in order to show that $\psi_0(d_1) \leq_0 M$. If d_1 is $\perp$ or num$m$ this is immediate, so only case (c) remains. Here we have that $d_1 = \mathrm{fun}(f)$, $L \Rightarrow \lambda x.L'$ and $f(x) \leq_0 L'[M/x]$ whenever $x \leq_0 M$. So as $\perp \leq_0 \lambda y.y$ we get that $f(\perp) \leq_0 L'[\lambda y.y/x]$, and therefore $f(\perp) = \perp$, showing that $\psi_0(d_1) \leq_0 M$, as desired, since $\psi_0(d_1) = f(\perp)$.

Finally, we outline a proof of a property needed below, that Y denotes the least fixed-point operator in $\mathcal{D}_X$ (see Section 2.2). It is not hard to show that $\mathcal{D}_X$ enjoys a modified form of minimality—as does any model constructed following the general pattern of Section 2.3—that the identity is the least $g : D_\infty \rightarrow_\perp D_\infty$ such that

$$p_0 \leq g = \eta \circ F(g, g) \circ \eta^{-1}$$

(where, in general, $p_n = \rho_n \circ \rho_n^R$). We now follow the proof in [Pit96]. It is enough to prove that $\mathcal{D}_X[\![\mathrm{Y}]\!](\perp) \leq \mathrm{fun}(F)$ as the opposite inclusion follows from the β-equality $\mathrm{Y}z = z(\mathrm{Y}z)$. Choose a functional value d in D_∞, and consider the set

$$P = \{p : D_\infty \rightarrow_\perp D_\infty \mid p \leq \mathrm{id} \wedge p(\delta_d) \cdot \delta_d \leq F(d)\}$$

where $\delta_d = \mathcal{D}_X[\![\Delta]\!](\perp\ [d/f])$. This set contains p_0—as can be seen using the facts that $p_0(\mathrm{fun}(h)) = p_0(h(\perp))$ and $p_0(u) \cdot v = p_0(u)$—and it is ω-inductive and closed under the mapping $g \mapsto \eta \circ F(g, g) \circ \eta^{-1}$. So, by the above modified notion of minimality it contains the identity, which concludes the proof as $\mathcal{D}_X[\![\mathrm{Y}]\!](\perp) \cdot d = \delta_d \cdot \delta_d$.

The construction of our third inadequate model is based on the following refinement of Lemma 1.

Lemma 6 *Let $\mathcal{E}$ be a model. Suppose it contains ω-finite elements a, b, c, with $b \neq \perp$ and such that $a \Rightarrow b \leq a \leq a \Rightarrow c$. Then $\mathcal{E} \not\models Y = Y_\mu$.*

Proof We have $(b \Rightarrow c) \cdot (a \cdot a) \geq (b \Rightarrow c) \cdot b \geq c$ from which it follows that $\mathcal{E}[\![\lambda x.f(xx)]\!][(b \Rightarrow c)/f] \geq (a \Rightarrow c) \geq a$ and hence

$$\mathcal{E}[\![\mathrm{Y}f]\!][(b \Rightarrow c)/f] \geq (a \Rightarrow c) \cdot a = c$$

But c is not the least fixed-point of $b \Rightarrow c$. □

Note that such a $\mathcal{E}$ cannot be isomorphic to $\mathcal{S}$, as there Y denotes the least fixed-point operator. To construct an $\mathcal{L}^+$-adequate such $\mathcal{E}$, let A be the cppo which is an increasing sequence of four objects: $\perp \leq \beta \leq \alpha \leq \gamma$ and take D_0' to be $A + \mathcal{P}(B)$, where $B = \{0, 1\}$ (we are following the usual construction, but using primes to distinguish this instance from that of the construction of $\mathcal{D}_X$).

Now define $\phi'_0 : D'_0 \to_\perp D'_1$ as the strict map such that:

$$\begin{array}{lcl} \phi'_0(\gamma) & = & \perp\Rightarrow \gamma \\ \phi'_0(\alpha) & = & (\perp\Rightarrow \beta) \vee (\gamma \Rightarrow \gamma) \\ \phi'_0(\beta) & = & \perp\Rightarrow \beta \\ \phi'_0(\{0\}) & = & \perp\Rightarrow \{0\} \\ \phi'_0(\{1\}) & = & \perp\Rightarrow \{1\} \\ \phi'_0(\{0,1\}) & = & \perp\Rightarrow \{0,1\} \end{array}$$

where A and $\mathcal{P}(B)$ are identified with the corresponding subsets of $A + \mathcal{P}(B)$. Then ϕ'_0 is an embedding, and for functional g in D'_1, $\psi'_0(g)$ is $g{\cdot}\perp$, unless $g \geq \phi'_0(\alpha)$ when it is $(g{\cdot}\perp) \vee \alpha$. We therefore obtain an extensional model $\mathcal{D}' = < D'_\infty, \eta', (\eta')^{-1} >$ as usual. (This construction is more complicated than is needed to get an inadequate model—one could instead just take $D'_0 = A$; the point of adding $\mathcal{P}(B)$ is, as will become clear, to be able to relate $\mathcal{D}'$ and $\mathcal{D}_B$.)

Theorem 3 *$\mathcal{D}'$ is an extensional $\mathcal{L}^+$-adequate model in which Y does not denote the least fixed-point operator.*

Proof In D'_∞ set $a = \rho_0(\alpha)$, $b = \rho_0(\beta)$ $(\neq\perp)$ and $c = \rho_0(\gamma)$. Then we have

$$\begin{array}{lcl} a & = & \rho_0(\alpha) \\ & = & \rho_1((\perp\Rightarrow \beta) \vee (\gamma \Rightarrow \gamma)) \\ & \geq & \rho_1(\perp\Rightarrow \beta) \\ & = & \perp\Rightarrow b \end{array}$$

Also,

$$\begin{array}{lcl} a & = & \rho_1((\perp\Rightarrow \beta) \vee (\gamma \Rightarrow \gamma)) \\ & = & \rho_1(\perp\Rightarrow \beta) \vee \rho_1(\gamma \Rightarrow \gamma) \\ & = & (\perp\Rightarrow b) \vee (c \Rightarrow c) \\ & \leq & \perp\Rightarrow c \end{array}$$

So, by Lemma 6, Y does not denote the least fixed-point operator in $\mathcal{D}'$. For $\mathcal{L}^+$-adequacy we use the same properties and action as before and again set $x \leq_0 M$ to hold iff $x =\perp$; we omit the details. □

We can exploit these results to learn a little about the abstraction relation $\leq$ below $\mathcal{S}$. First in any of the $\mathcal{D}_X$, Y is the least fixed-point operator and so the equation $\mathrm{Y} = \mathrm{Y}_\mu$ holds there but not in the model $\mathcal{D}'$ just constructed, showing that $\mathcal{D}_X \not\leq \mathcal{D}'$. We now outline a proof that, $\mathcal{D}' \leq \mathcal{D}_B$ and so that we have a proper chain

$$\mathcal{D}' < \mathcal{D}_B < \mathcal{S}$$

(It would also be interesting to construct two mutually incomparable models strictly below $\mathcal{S}$; perhaps this could be done by refining the current methods to construct models which do not satisfy some of the iteration theory axioms of Bloom and Ésik[BE93]—reading $\mu x.M$ as $\mathrm{Y}(\lambda x.M)$.)

The idea of the proof is to find a projection $\beta_\infty : D'_\infty \to D_\infty$ such that

$$\beta_\infty(\mathcal{D}'[\![M]\!](\perp)) = \mathcal{D}[\![M]\!](\perp) \qquad (1)$$

for all $\mathcal{L}^+$-terms M. For this implies that $\mathrm{EqTh}_{\mathcal{L}}(\mathcal{D}') \subset \mathrm{EqTh}_{\mathcal{L}}(\mathcal{D})$ and so $\mathcal{D}' \leq \mathcal{D}_B$. Equation 1 will be established by a logical relations argument; we package up the construction of β_∞ and the relation and its properties in a category $\mathbf{L}$ and apply the method of Section 2.3 to show existence.

The objects of $\mathbf{L}$ are structures

$$<D, D', \alpha, \beta, R>$$

where D and D' are cppos, $\alpha : D \to D'$ is an embedding with right adjoint β, $R \subset D \times D'$ is an admissible relation and the following hold:

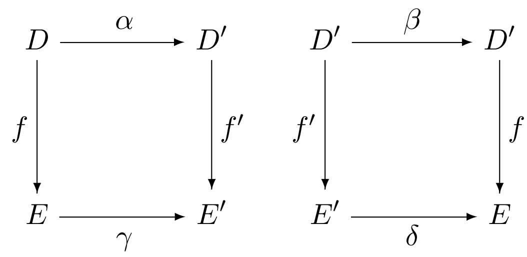
(i) $\forall d \in D.\ R(d, \alpha(d))$

(ii) $\forall d \in D, d' \in D'.\ R(d, d') \supset d = \beta(d')$

Morphisms are pairs

$$<f, f'>:<D, D', \alpha, \beta, R> \to <E, E', \gamma, \delta, S>$$

of strict continuous functions $f : D \to_\perp D'$, $E \to_\perp E'$ such that the following two diagrams commute



and such that

$$\forall d \in D, d' \in D'.\ R(d, d') \supset S(f(d), f'(d'))$$

This category is $\mathbf{CPO}$-enriched. It also has ω^{op}-limits, as we now sketch. Let

$$\Delta = << D_n, D'_n, \alpha_n, \beta_n, R_n >, < g_n, g'_n >>$$

be an ω^{op}-chain in $\mathbf{L}$. Let $\rho_n : D_\infty \to < D_n, g_n >$ and $\rho'_n : D'_\infty \to < D'_n, g'_n >$ be limiting cones. Set α_∞ to be the mediating morphism from $\alpha_n \circ \rho_n$ to ρ'_n, and set β_∞ to be the mediating morphism from $\beta_n \circ \rho'_n$ to ρ_n and define $R_\infty \subset D_\infty \times D'_\infty$ by: $R_\infty(d, d')$ iff $\forall n. R_n(d_n, d'_n)$. Then

$$< \rho_n, \rho'_n >:< D_\infty, D'_\infty, \alpha_\infty, \beta_\infty, R_\infty > \to \Delta$$

is limiting in $\mathbf{L}$.

We now define a $\mathbf{CPO}$-enriched functor $G : \mathbf{L}^{\mathrm{op}} \times \mathbf{L} \to \mathbf{L}$. For objects we set

$$G(<E, E', \gamma, \delta, S>, <D, D', \alpha, \beta, R>) = <F(E, D), F(E', D'), F(\delta, \alpha), F(\gamma, \beta), Q>$$

where $Q(u, u')$ holds iff one of the following is true

(a) $u = \perp\ \wedge u' = \perp$

(b) for some m, $u = \mathrm{inr}(m)$ and $u' = \mathrm{inr}(m)$

(c) for some h and h', $u = \mathrm{fun}(h)$, $u' = \mathrm{fun}(h')$ and whenever $S(v, v')$ holds so does $R(h(v), h'(v'))$

For morphisms we set

$$G(<g, g'>, <f, f'>) = <F(g, f), F(g', f')>$$

We now construct a solution to the domain equation $x \cong U(x)$ where $U : \mathbf{L}^e \to \mathbf{L}^e$ is the "diagonalisation" $U(x) = G(x, x)$ of G. For this, follow section 2.3, starting with the embedding

$$<\phi_0, \phi'_0>:<D_0, D'_0, \mathrm{inr}, \mathrm{inr}^R, R_0> \to U(<D_0, D'_0, \mathrm{inr}, \mathrm{inr}^R, R_0>)$$

where R_0 is the graph of inr. By the above construction of ω^{op}-limits the solution has the form

$$<\eta, \eta'>: U(<D_\infty, D'_\infty, \alpha_\infty, \beta_\infty, R_\infty>) \cong <D_\infty, D'_\infty, \alpha_\infty, \beta_\infty, R_\infty>$$

This yields that $R_\infty(u, u')$ holds iff one of the following is true

(a) $u = \perp \wedge u' = \perp$

(b) for some m, $u = \mathrm{num}(m)$ and $u' = \mathrm{num}(m)$

(c) for some h and h', $u = \mathrm{fun}(h)$, $u' = \mathrm{fun}(h')$ and whenever $R_\infty(v, v')$ holds so does $R_\infty(h(v), h'(v'))$

The usual argument by structural induction on terms then gives us that for every closed $\mathcal{L}^+$-term M, $R_\infty(\mathcal{D}[\![M]\!](\perp), \mathcal{D}'[\![M]\!](\perp))$ holds. But then equation 1 follows by the construction of $\mathbf{L}$ (specifically the second required property of the relation).

7 Discussion and Conclusions

The techniques used here originate, as we have already remarked, in work on the pure untyped λ-calculus. Much is now known about the construction of models of this calculus by means other than inverse limits—see, e.g., [Kri93, Plo93, Ber00]. It may be that these techniques can also be of use for the construction of inadequate models for such applied λ-calculi as $\mathcal{L}$. In particular it would be good to find a more conceptual account of our second inadequate model.

Having looked at one example—$\mathcal{L}$—it is natural to ask how typical it is. Suppose instead we took an applied typed λ-calculus such as PCF [Plo77, FJM96, AC98]. We could say that a model consists of a collection of cppos D_σ such that each $D_\sigma \Rightarrow D_\tau$ is a specified retract of $D_{\sigma \to \tau}$ and the flat cppos of the natural numbers and booleans are specified retracts of D_ι and, respectively, D_o; we could then interpret PCF and its extension PCF$^+$ in the D_σ, taking Y as the appropriate least fixed-point operator, modulo the retractions. One then easily shows that any such model is adequate for PCF$^+$ and so below the standard model in the evident abstraction ordering.

Thus, with no possibility of inadequacy, none of the phenomena of the type-free case arise and the picture is as perfect as it could be: the standard model, bad as it is, is the best we can do in the category of cppos. Note that this does not contradict the fact that stable models etc. are cppos as we have built the continuous function space into our notion of model.

We conjecture this pleasant situation should extend further, certainly to a fragment of, say, FPC [Fio96] with recursion but restricting recursive types $\mu X.\sigma$ so that σ contains no occurrence of lifting or function space (it may even be possible to allow lifting and function space, asking only that all occurrences of X are positive).

In contrast we would expect the same phenomena as arose for our type-free λ-calculus to arise for FPC itself; it would be interesting to see what kinds of counter-example one could construct. Presumably, too, much the same phenomena would arise for a call-by-value variant of our untyped λ-calculus. All in all, our language seems to be the simplest functional language for which inadequacy phenomena arise; it might be interesting to explore what happens with other language features.

It could also be interesting to look at other categories. Nothing much seems to hinge on the peculiarities of parallel or and we would expect similar inadequacy phenomena to occur in the various categories of stable functions that have been explored (e.g., see [AC98]). On the other hand, in categories of games there are strong uniqueness phenomena arising from intensionality, as in [DFH99, DF99]; it may be that no inadequate models of $\mathcal{L}$ exist there.

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Appendix

To show that the standard model $\mathcal{S} =< S, \eta, \eta^{-1} >$ is not fully abstract for $\mathcal{L}$ let us consider the following two terms

$$M = \lambda z.\mathbf{if}\ \lceil z \geq \vee_p \rceil\ \mathbf{then}\ \underline{1}\ \mathbf{else}\ \Omega_\mu$$

$$N = \lambda z.\Omega_\mu$$

Clearly $\mathcal{S}[\![M]\!] \neq \mathcal{S}[\![N]\!]$, since $\mathcal{S}[\![M]\!](\perp)(\vee_p) = \text{num}(1)$ and $\mathcal{S}[\![N]\!](\perp)(\vee_p) =\perp$ (we are identifying $\vee_p$ with the corresponding element in S, viz $\mathcal{S}[\![\mathbf{por}]\!](\perp)$). It therefore suffices to show M and N are operationally equivalent:

Lemma 7 $M \simeq_{\mathcal{L}} N$

We use a logical relations argument. In [Sie92] a ternary relation is given to show that parallel or is not definable in PCF. We adapt this to $\mathcal{L}$, "doubling it up" to enable the connection between M and N to be made. We take $\mathcal{P}(D)$ to be the 6-ary admissible relations on D and define the action of F by taking $F(Q,P)(u,v,w,u',v',w')$ to be true iff one of the following hold:

(a) $u = u' =\perp \vee v = v' =\perp \vee w = w' =\perp$

(b) $\exists m.\ u = v = w = u' = v' = w' = \text{num}(m)$

(c) u, v, w, u', v', w' are all functional values, and whenever $Q(x, y, z, x', y', z')$ holds, so does $P(u \cdot x, v \cdot y, w \cdot z, u' \cdot x', v' \cdot y', w' \cdot z')$.

One checks that this is indeed an action and then proves the appropriate Logical Relations Lemma for the resulting recursively defined relation $P_\infty \subset S^6$. This is that for any term L whose free variables are included in $x_1, \ldots, x_n$, if u_{ij} $(i = 1, m; j = 1, 6)$ are such that $P_\infty(u_{i1}, \ldots, u_{i6})$ hold (for $i = 1, m$) then $P_\infty(\mathcal{S}[\![L]\!](\rho_1), \ldots, \mathcal{S}[\![L]\!](\rho_6))$ holds where $\rho_j =\perp [u_{1j}/x_1] \ldots [u_{mj}/x_m]$.

We now show that $P_\infty(a, a, a, b, b, b)$ holds where $a = \mathcal{S}[\![M]\!](\perp)$ and $b = \mathcal{S}[\![N]\!](\perp)$. For this we must establish case (c), so we suppose $P_\infty(u, v, w, u', v', w')$ and have to show $P_\infty(a \cdot u, a \cdot v, a \cdot w, b \cdot u', b \cdot v', b \cdot w')$. We establish (a) by contradiction. So, assuming its negation, none of $a \cdot u$,$a \cdot v$,$a \cdot w$ are $\perp$ since $b \cdot u' = b \cdot v' = b \cdot w' =\perp$. But then u,v,w all extend $\vee_p$, and so as

$$P_\infty(u \cdot \perp \cdot 1, v \cdot 1 \cdot \perp, w \cdot 0 \cdot 0, u' \cdot \perp \cdot 1, v' \cdot 1 \cdot \perp, w' \cdot 0 \cdot 0)$$

holds, we obtain that $P_\infty(1, 1, 0, u' \cdot \perp \cdot 1, v' \cdot 1 \cdot \perp, w' \cdot 0 \cdot 0)$, providing the required contradiction. (We have allowed ourselves the freedom to omit writing num here.)

Now to show $M \simeq_{\mathcal{L}} N$, consider a context $C[\ ]$ with no free variables to show that $C[M]\Downarrow$ iff $C[N]\Downarrow$. By adequacy it is enough to show that $\mathcal{S}[\![C[M]]\!](\perp) =\perp$ holds iff $\mathcal{S}[\![C[N]]\!](\perp) =\perp$ does. Set $c = \mathcal{S}[\![\lambda x.C[x]]\!](\perp)$, where x does not appear in $C[\ ]$.

By the Logical Relations Lemma, $P_\infty(c, c, c, c, c, c)$ holds. Therefore we also have that $P_\infty(c \cdot a, c \cdot a, c \cdot a, c \cdot b, c \cdot b, c \cdot b)$ holds from which we clearly have $c \cdot a =\perp$ iff $c \cdot b =\perp$, concluding the proof as $\mathcal{S}[\![C[M]]\!](\perp) = c \cdot a$ and $\mathcal{S}[\![C[N]]\!](\perp) = c \cdot b$. $\Box$

The case for $\mathcal{L}^+$ is different, and we will show that the standard model is fully abstract for $\mathcal{L}^+$. Full abstraction is a consequence of the definability in $\mathcal{L}^+$ of all ω-finite elements of S; we build up to that by a sequence of definability results, beginning with a suitable "parallel conditional" for S. First we define one for the "truthvalues" 0,1 by the term:

$$\supset_T =_{\text{def}} \lambda b.\lambda x.\lambda y.(b \wedge_p x) \vee_p (\neg b \wedge_p y) \vee_p (x \wedge_p y)$$

where $M \wedge_p N$ abbreviates $\neg(\neg M \vee_p \neg N)$.

Using that we can define a parallel conditional for the natural numbers, by a technique of "determining one bit at a time" (and see [Sto91])

$$\supset_N =_{\text{def}} \mathrm{Y}_\mu(\lambda c.\lambda b.\lambda x.\lambda y.\textbf{if} \ \supset_T b(\text{zero}x)(\text{zero}y) \ \textbf{then} \ \underline{0} \ \textbf{else} \ \text{succ}(cb(\text{pred}x)(\text{pred}y)))$$

where zero$M$ abbreviates **if** $M = 0$ **then** $\underline{1}$ **else** $\underline{0}$. With that we can define a parallel conditional for all of S by

$$\supset_S =_{\text{def}} \mathrm{Y}_\mu(\lambda c.\lambda b.\lambda x.\lambda y.\textbf{if} \ \supset_T b(\text{fun}x)(\text{fun}y) \ \textbf{then} \ \lambda z.cb(xz)(yz) \ \textbf{else} \ \supset_N bxy)$$

Then one has for x, y in S that: $\supset_S \cdot \underline{1} \cdot x \cdot y = x$, $\supset_S \cdot \underline{0} \cdot x \cdot y = y$ and $\supset_S \cdot \perp \cdot x \cdot y = x \wedge y$.

In the proof of the following lemma we allow ourselves further notational freedom, identifying $\boldsymbol{N}$ and $S \Rightarrow S$ with subsets of S and confusing syntax and semantics.

Lemma 8 *Every ω-finite element of S is definable in $\mathcal{L}^+$.*

Proof For every ω-finite element a of S there is an ω-finite b in some S_n such that $a = \rho_n(b)$; let the *rank* of a be the smallest such n. Then every ω-finite element of S is either $\perp$, an integer or of the form

$$\bigvee_{i=1,n} b_i \Rightarrow c_i$$

where the b_i, c_i are ω-finite elements of strictly smaller rank than a. We show by induction on rank that for any ω-finite element a two elements $\#_a$ and $\vee_a$ are definable such that:

$$\#_a \cdot x = \begin{cases} 1 & \text{if } x \# a \\ 0 & \text{if } x \geq a \\ \perp & \text{otherwise} \end{cases}$$

and

$$\vee_a \cdot x = a \vee x \ (\text{if } x \uparrow a)$$

(where we write $x \# y$ to mean that x and y have no upper bound and $x \uparrow y$ to mean the opposite). The result will then follow as $\vee_a \cdot \perp = a$.

Now let a be an ω-finite element of S. If it is $\perp$ or an integer then it is straightforward to define $\#_a$ and a suitable $\vee_a$ in $\mathcal{L}$. Otherwise we have b_i and c_i $(i = 1, n)$ as above. In that case, $a \# x$ iff x is an integer or else $x \cdot b_i \# c_i$ for some i; also $x \geq a$ if $x \cdot b_i \geq c_i$ for all i. We therefore have

$$\#_a \cdot x = \partial(x, (\#_{c_1} \cdot (x \cdot b_1) \vee_p \ldots \vee_p \#_{c_n} \cdot (x \cdot b_n)), 1)$$

and so as the $\#_{c_i}$ are all definable in $\mathcal{L}^+$ by the induction hypothesis, $\#_a$ is too.

For $\vee_a$ we first establish the result in the case $n = 1$, setting $b = b_1$ and $c = c_1$. We claim that $\vee_{a \Rightarrow b}$ can be defined by

$$\vee_{a \Rightarrow b} \cdot x \cdot u \ = \supset_S \cdot (\#_a \cdot u) \cdot (x \cdot u) \cdot (\vee_b \cdot (x \cdot u))$$

So, assume $x \uparrow (a \Rightarrow b)$ in which case x is a function or $\perp$. There are three cases. In the first $u \geq a$. Here $x \cdot u \uparrow b$ and so the right hand side is $b \vee (x \cdot u)$ which is $(\vee_{a \Rightarrow b} \vee x) \cdot u$ as required. In the second case $u \# a$ and the right hand side is $x \cdot u$ which again is $(\vee_{a \Rightarrow b} \vee x) \cdot u$. In the third and last case $u \uparrow a$ but $u \not\geq a$. Here

$(x \cdot u) \uparrow b$ and the right hand side is therefore $(b \vee (x \cdot u)) \wedge (x \cdot u)$ which is $(x \cdot u)$, again as required.

Returning to the general case, we may define $\vee_a$ by

$$\vee_a \cdot x = \vee_{a_1 \Rightarrow b_1} \cdot (\vee_{a_2 \Rightarrow b_2} \cdot \ldots \cdot (\vee_{a_n \Rightarrow b_n} \cdot x) \ldots)$$

concluding the proof. □

Full abstraction now follows by a standard argument.

Theorem 4 *The standard model is fully abstract for $\mathcal{L}^+$.*

Proof It is enough to consider the case of two closed terms M and N such that $M \simeq_{\mathcal{L}^+} N$. Let $a \leq \mathcal{S}[\![M]\!](\bot)$ be ω-finite and let D define $a \Rightarrow 1$; it follows that $\mathcal{S}[\![DM]\!](\bot) \neq \bot$. We then have, successively, that $DM \Downarrow$ (by $\mathcal{L}^+$-adequacy), that $DN \Downarrow$ (as $M \simeq_{\mathcal{L}^+} N$), that $\mathcal{S}[\![DN]\!](\bot) \neq \bot$ (again by $\mathcal{L}^+$-adequacy) and so that $a \leq \mathcal{S}[\![N]\!](\bot)$. This shows that $\mathcal{S}[\![M]\!](\bot) \leq \mathcal{S}[\![N]\!](\bot)$ and the converse inequality follows by a similar argument. □

Note that, by the equational formulation of minimality, it follows that the standard model is the unique fully abstract model for $\mathcal{L}^+$.