MACDOWELL SYMMETRY AND FERMION REGGE TRAJECTORIES

Thesis

Submitted by

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CONTENTS

INTRODUCTION

Phase shift analyses of pion-nucleon scattering have led to the discovery of a large number of excited baryonic states having positive and negative parity. A fascinating challenge is presented by the classification of these states, and the search for the fundamental laws of Nature which determine their spectrum.

Previous study of this problem has taken place along two different lines. In one, the use of symmetry groups is made, and the pion-nucleon resonances are allocated to different representations of these groups. The other has been the study of the underlying forces involved, including dynamical models such as bootstrap theory. Both these approaches have been adequately discussed in the report of the Trieste Conference (1965), and are not treated further in this work.

Recently, the importance of complex angular momentum and Regge theory in this problem was demonstrated by Barger and Cline[20], who showed that the known pion-nucleon resonances could be fitted on families of Regge trajectories.

An important theoretical concept in baryonic systems is MacDowell symmetry[1], which is a relationship between parity conserving partial wave amplitudes for one parity at positive energy, to the wave having opposite parity and negative energy. The application of MacDowell symmetry and Regge theory to the pion nucleon system shows that two Regge trajectories $\alpha(\pm w)$ may be defined. The physical Regge recurrences on $\alpha(+w)$ have one parity, and the trajectory $\alpha(-w)$ corresponds to a trajectory in which the Regge recurrences have opposite parity.

The work of Barger and Cline is discussed in detail in Chapter II, and from the experimental fits it is shown that the two trajectories $\alpha(w)$, $\alpha(-w)$ are approximately the same, so parity degeneracy occurs. There are some notable exceptions to this result. Several states predicted by this parity degeneracy are missing, such as the lowest member of the highest ranking N_β trajectory (the S_{11}), the lowest member of the N_∂ (P_{13}), and the lowest members of the Δ_γ (D_{33}, G_{37}). The usual spectroscopic notation $L_{2I,2J}$ is used in the classification of the baryons.

This thesis is concerned with a study of the pion-nucleon resonances in the framework of Regge theory and MacDowell symmetry. Attempts are made to explain the form of the Regge trajectories for the system, and special attention is paid to the missing mass states. The scope has been restricted to the nucleon N_α and N_β trajectories, but the theory may be generalised to other trajectories using SU(3) symmetry(41).

In Chapter I the concept of MacDowell symmetry is stated and proved for the parity conserving partial wave amplitudes of Gell-Mann, Goldberger, Low and Zachariasen(2). A discussion of generalised MacDowell symmetry which depends on field theoretic arguments has been given by Hara(5). The approach to MacDowell symmetry used in this thesis depends on crossing symmetry, and to the author's knowledge this has not been done before.

Chapter II is an introduction to pion-nucleon scattering and the application of MacDowell symmetry and Regge poles. The original work in this thesis starts at section 2.5, in which a discussion of Riemann sheets and their application to missing mass states, is given.

In Chapter III a potential scattering model is described, and its possible applications to the pion-nucleon system and missing mass states is discussed.

Chapter IV is concerned with parametrisations of Regge trajectories, and a critical discussion is given of models which produce distortions of the Regge trajectory near the missing mass states.

Finally, in Chapter V possible dynamical models for fermion Regge trajectories are discussed, and a review is given of their applications to the higher pion-nucleon resonances.

The proof of this takes place as follows.

1.2 Definition of Parity Conserving Helicity Amplitudes

In this section, the notation of GGLZ[2] is followed.

Consider scattering processes of the type $a + b \longrightarrow c + d$, where a, b, c, d describe both the particle and its helicity state. If a ket $|JM, cd\rangle$ is defined for the final state, the parity (P) operator has the effect:-

$$P\, |JM, cd\rangle = \eta_c \eta_d \, (-1)^{J - S_c - S_d} \, |JM, -c-d\rangle$$

(see Jacob & Wick[3]), where $\eta_c \eta_d$ are the intrinsic parities of c and d, and S_c, S_d the spins.

From this result, parity conserving helicity states are defined by GGLZ as follows

$$|JM, cd\rangle_{\pm} = \frac{1}{\sqrt{2}} \left[|JM, cd\rangle \pm \eta_c \eta_d (-1)^{S_c + S_d - v} |JM, -c-d\rangle \right]$$

where $P\, |JM, cd\rangle_{\pm} = \pm (-1)^{J-v} \, |JM, cd\rangle_{\pm}$

$v = \tfrac{1}{2}$ for J half integral

$v = 0$ for J integral .

Partial wave helicity amplitudes are defined by:-

$$F_{cd,ab} = \frac{1}{(k_{ab}\, k_{cd})^{1/2}} \left[S_{cd,ab} - \delta_{cd,ab} \right] / (2i)$$

where k_{ab}, k_{cd} are the c.m. momenta for ab, cd respectively, and $S_{cd,ab}$ is the S matrix element for the process.

Since F has no matrix elements between + and − states if parity is conserved, parity conserving partial wave amplitudes are defined by:-

$$_{\pm}\left\langle JM, cd \right| F \left| JM, ab \right\rangle_{\pm} = \left\langle cd \right| F^J \left| ab \right\rangle \pm \eta_c \eta_d (-1)^{S_c + S_d - v} \times \left\langle -c-d \right| F^J \left| ab \right\rangle$$

(omitting the JM terms in the kets).

Jacob and Wick[3] define helicity amplitudes by

$$f_{cd,ab}(\theta) = \sqrt{\frac{k_{cd}}{k_{ab}}} \sum_J (2J+1) \left\langle cd \right| F^J \left| ab \right\rangle d^J_{\lambda\mu}(\theta_S)$$

$\lambda = a-b$, $\mu = c-d$; $d^J_{\lambda\mu}(\theta_S)$ are the d functions of the rotation matrix (see Edmonds[4], p. 53).

θ_S is the direct channel scattering angle. In the c.m. system

$$\cos \theta_S = \left[2st + s^2 - s \sum_i m_i^2 + (m_a^2 - m_b^2)(m_c^2 - m_d^2) \right] / (S_{ab} S_{cd})$$

$$S^2_{ab} = \left[s - (m_a - m_b)^2 \right] \left[s - (m_a + m_b)^2 \right] = 4s\, k^2_{ab}$$

$$S^2_{cd} = \left[s - (m_c - m_d)^2 \right] \left[s - (m_c + m_d)^2 \right] = 4s\, k^2_{cd}$$

s, t, u are the usual Mandelstam variables for the reaction.

The $f_{cd,ab}$ helicity amplitudes are related to the differential cross-section by:-

$$\frac{d\sigma}{d\Omega} = \left| f_{cd,ab} \right|^2 .$$

If new amplitudes $T_{cd,ab} = 8\pi \left(\frac{s\, k_{ab}}{k_{cd}}\right)^{1/2} f_{cd,ab}$ are defined, then

$$\frac{d\sigma}{d\Omega} = \sqrt{\frac{k_{cd}}{k_{ab}}}\ \frac{T_{cd,ab}}{8\pi W}\ ; \qquad s = W^2, \quad \text{and}$$

$$S_{cd,ab} - \partial_{cd,ab} = (2\pi)^4 \frac{i\partial^4(p_c + p_d - p_a - p_b)}{(2p_a^o\ 2p_b^o\ 2p_i^o\ 2p_d^o)^{1/2}} T_{cd,ab}\ .$$

From field theoretic arguments, Hara[5] has shown that $T_{cd,ab}$ may be expressed as:

$$T_{cd,ab} = \sum_j B_j(s,t,u) \times \Big[\text{Polynomial in } p_i^o,\ p'^2,\ p'^2, pp', \sin\frac{\theta_s}{2},\ \cos\frac{\theta_s}{2}\Big] \times \Big[p^2 \text{ or } p'^2\Big]^{\eta}$$

$$\times \prod_{i:\text{fermion}} \Big[p_i^o + m_i\Big]^{-1/2} \qquad \text{(A)}$$

where $p_i = (p_i^o, \underline{p}_i)$ is the four momentum of the i-th particle.

$$p^2 = k_{ab}^2,\ p'^2 = k_{cd}^2$$

$$\eta = 0 \text{ if } \eta_a\,\eta_b\,\eta_c\,\eta_d = 1$$

$$\eta = 1 \text{ if } \eta_a\,\eta_b\,\eta_c\,\eta_d = -1.$$

The partial wave expansion of $T_{cd,ab}$ is

$$T_{cd,ab} = 8\pi W \sum_J (2J+1) \langle cd \,|\, F^J \,|\, ab\rangle\, d^J_{\lambda\mu}(\theta_s)$$

where
$$d^J_{\lambda\mu}(\theta_s) = \left[\frac{(J+\lambda)!(J-\lambda)!}{(J+\mu)!(J-\mu)!}\right]^{1/2} \Big[\cos\theta_s/2\Big]^{|\lambda+\mu|} \Big[\sin\theta_s/2\Big]^{|\lambda-\mu|} \cdot P_{J-\lambda}^{(|\lambda-\mu|,\,|\lambda+\mu|)}(\cos\theta_s)$$

(see Edmonds[4], p. 58).

From Hara's result (A), the function

$$T'_{cd,ab} = \left[\sqrt{2}\cos\theta_{s/2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\sin\theta_{s/2}\right]^{-|\lambda-\mu|} T_{cd,ab} \qquad \text{(B)}$$

has no singularities in $\cos\theta_s$ from the d function which arises due to spin. Since $t \sim \cos\theta_s$, $T'_{cd,ab}$ has only dynamical t singularities and satisfies a fixed s dispersion relation in t.

Finally, the parity conserving helicity amplitudes of GGLZ are defined by:

$$T^{\pm}_{cd,ab} = \left[\sqrt{2}\cos\theta_{s/2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\sin\theta_{s/2}\right]^{-|\lambda-\mu|} T_{cd,ab}$$

$$\pm\ (-1)^{\lambda+\lambda_m}\,\eta_c\eta_d\,(-1)^{s_c+s_d-v}\left[\sqrt{2}\sin\theta_{s/2}\right]^{-|\lambda+\mu|}\left[\sqrt{2}\cos\theta_{s/2}\right]^{-|\lambda-\mu|} T_{-c-d,ab}$$

where $\lambda_m = \max(|\lambda|, |\mu|)$.

The partial wave amplitudes are related to the T's by

$$T^{\pm}_{cd,ab} = 8\pi W \sum_J (2J+1)\left[e^{J+}_{\lambda\mu}(z)F^{J\pm}_{cd,ab} + e^{J-}(z)\,F^{J\mp}_{cd,ab}\right]$$

$$F^{J\pm}_{cd,ab} = \frac{1}{16\pi W}\int_{-1}^{+1} dz\left[C^{J+}_{\lambda\mu}(z)T^{\pm}_{cd,ab} + C^{J-}_{\lambda\mu}(z)\,T^{\mp}_{cd,ab}(z)\right].$$

The '+' sign signifies parity $(-1)^{J-\frac{1}{2}}$, and the '-' sign parity $-(-1)^{J-\frac{1}{2}}$, and the 'e' and 'c' functions are as defined in ref. 2 (see Appendix I).

From equation (B) above, $T^{\pm}_{cd,ab}$ has no kinematic singularities in t, and as before, satisfies a fixed s dispersion relation in t.

1.3 Analyticity of $T_{cd,ab}$ in s.

Let the direct (or s) channel denote the reaction $a+b \rightarrow c+d$. The 't' channel reaction will be :

$$D + b \longrightarrow c + A \; ,$$

where D, A denote the antiparticles of d, a, and their helicities.

The s and t channel amplitudes are related by crossing symmetry (Trueman & Wick[6]) :

$$T^{s}_{cd,ab} = \sum_{c'A',D'b'} \left[d^{J_a}_{A'a}(\chi_a)\, d^{J_b}_{b'b}(\chi_b)\, d^{J_c}_{c'c}(\chi_c)\, d^{J_d}_{D'd}(\chi_d) \right] \cdot T^{t}_{c'A,D'b}$$

where χ_a, χ_b, χ_c, χ_d are defined in the above reference (see Appendix I), and T^s, T^t denote s and t channel amplitudes respectively.

The crossing relations for the T''s (defined in equation (B), sec. 1.2) are :

$$T'^{s}_{cd,ab} = \left(\sqrt{2}\sin \tfrac{1}{2}\theta_s\right)^{-|\lambda-\mu|}\left(\sqrt{2}\cos \tfrac{1}{2}\theta_s\right)^{-|\lambda+\mu|} \times \sum_{A'b'c'D'}$$

$$d^{J_a}_{A'a}(\chi_a)\, d^{J_b}_{b'b}(\chi_b)\, d^{J_c}_{c'c}(\chi_c)\, d^{J_d}_{D'd}(\chi_d) \left(\sqrt{2}\sin\tfrac{1}{2}\theta_t\right)^{|\lambda'-\mu'|}\left(\sqrt{2}\cos\tfrac{1}{2}\theta_t\right)^{|\lambda'+\mu'|} \cdot T'^{t}_{c'A'D'b}$$

where $\lambda' = D - b$, $\mu' = C - A$, and $\sin\theta_s$, $\sin\theta_t$ are given in Appendix I.

In section (1.2) T'^s was shown to have only dynamical singularities in t, so T'^{t} has only dynamical singularities in s. Hence the kinematic singularities of T'^{s} in s are entirely contained in the crossing matrix elements which are known functions of s. Mixed s t singularities have been shown to cancel (see L.L. Wang[6]).

The negative helicity states in the above summation may be eliminated by the parity symmetry relation (see Jacob & Wick[3]).

$$T_{c'A',D'b'} = \eta_t \, T_{-c'-A',-D'-b'}$$

where

$$\eta_t = \frac{\eta_A \eta_C}{\eta_D \eta_b} (-1)^{s_C + s_A - s_b - s_D} = \frac{\eta_A \eta_c}{\eta_D \eta_b} (-1)^{J_c + J_A - J_D - J_b} (-1)^{\lambda' - \mu'} .$$

Let the crossing matrix relating T'^{s} and T'^{t} be M.

Then

$$T'^{s}_{cd,ab} = \sum_{A'b'c'D'} \mathcal{M}_{c'A',D'b'} \; T'^{t}_{c'A',D'b'}$$

$$= \mathcal{M}^{\not\Sigma}_{c'A',D'b'} \; T'^{t}_{c'A',D'b'}$$

where $\mathcal{M}_{c'A',D'b'} = \mathcal{M}_{c'A',D'b'} + \eta_t \, \mathcal{M}_{-c'-A',-D'-b'}$.

The parity conserving helicity amplitudes of GGLZ are defined in <u>1.2</u> as :

$$T^{\pm}_{cd,ab} = T'_{cd,ab} \pm T'_{-c-d,ab}$$

where for simplicity $(-1)^{\lambda+\lambda_m} \eta_c \eta_d \, (-1)^{s_c + s_d - v}$ is assumed equal to +1. Let $M^{\pm}$ be the crossing matrix between $T^{\pm s}$ and T'^{t}.

Then $T^{\pm s}_{cd,ab} = M^{\pm}_{c'A',D'b'} \; T^{t}_{c'A',D'b'}$

where, as before, all kinematic singularities in s are contained in the crossing matrix element $M^{\pm}_{c'A',D'b'}$.

Elementary manipulation using the results for the $d^{J}_{\lambda\mu}$ function given in Edmonds[(4)] (see Appendix I) enables $M^{\pm}$ to be found. This calculation was first done by L.L. Wang[(6)]. The results below differ slightly from those of Wang, because Edmonds'[(4)] notation has been followed, whereas Wang has used Rose's[(7)] notation. This difference does not affect the final result.

Since $\cos\chi_a \sim \frac{1}{S_{ab}}$, $\cos\chi_b \sim \frac{1}{S_{ab}}$, a form of $M^{\pm}$ which explicitly demonstrates the singularities as $S_{ab} = 0$ is:

$$M^{\pm}_{c'A',D'b'} \sim \left(\sin\tfrac{1}{2}\theta_t\right)^{|\lambda'-\mu'|}\left(\cos\tfrac{1}{2}\theta_t\right)^{|\lambda'+\mu'|}\left(\sin\theta_s\right)^{-|\lambda-\mu|}$$

$$\times\Big[\Big\{\left(\frac{1-\cos\chi_a}{\sin\chi_a}\right)^{|A'-a|}\left(\frac{1-\cos\chi_b}{\sin\chi_b}\right)^{|b'-b|} P^{(J_a-v_a/2)}(\cos\chi_a)$$

$$P^{(J_b-v_b/2)}(\cos\chi_b).\left(\cos\tfrac{1}{2}\theta_s\right)^{|\lambda-\mu|-|\lambda+\mu|}\left(\cos\tfrac{1}{2}\chi_a\right)^{v_a}\left(\cos\tfrac{1}{2}\chi_b\right)^{v_b}$$

$$\pm\,\eta_{ab}\left(\frac{1+\cos\chi_a}{-\sin\chi_a}\right)^{|A'-a|}\left(\frac{1+\cos\chi_b}{-\sin\chi_b}\right)^{|b'-b|} P^{(J_a-v_a/2)}(-\cos\chi_a)$$

$$P^{(J_b-v_b/2)}(-\cos\chi_b)\left(\sin\tfrac{1}{2}\theta_s\right)^{|\lambda-\mu|-|\lambda+\mu|}\left(\sin\tfrac{1}{2}\chi_a\right)^{v_a}\left(\sin\tfrac{1}{2}\chi_b\right)^{v_b}\Big\}$$

$$\times\, d^{J_c}_{c'c}(\chi_c)\, d^{J_d}_{D'd}(\chi_d)$$

$$\pm\,\eta_{cd}\Big\{(-1)^{2(c+d)}\left(\frac{1-\cos\chi_a}{-\sin\chi_a}\right)^{|A'-a|}\left(\frac{1-\cos\chi_b}{-\sin\chi_b}\right)^{|b'-b|}$$

$$P^{(J_a-v_a/2)}(\cos\chi_a)\,P^{(J_b-v_b/2)}(\cos\chi_b)\left(\sin\tfrac{1}{2}\theta_s\right)^{|\lambda-\mu|-|\lambda+\mu|}$$

$$\left(\cos\tfrac{1}{2}\chi_a\right)^{v_a}\left(\cos\tfrac{1}{2}\chi_b\right)^{v_b}\pm\,\eta_{ab}\left(\frac{1+\cos\chi_a}{\sin\chi_a}\right)^{|A'-a|}\left(\frac{1+\cos\chi_b}{\sin\chi}\right)^{|b'-b|}$$

$$P^{(J_a-v_a/2)}(-\cos\chi_a)\,P^{(J_b-v_b/2)}(-\cos\chi_b).\left(\cos\tfrac{1}{2}\theta_s\right)^{|\lambda-\mu|-|\lambda+\mu|}$$

$$\left.\left(\sin\tfrac{1}{2}\chi_a\right)^{v_a}\left(\sin\tfrac{1}{2}\chi_b\right)^{v_b}\right\} d^{J_c}_{c'c}(\pi-\chi_c)\, d^{J_d}_{D'd}(\pi-\chi_d)\Big] \quad (A)$$

$$\eta_{ab} = \frac{\eta_A\,\eta_c}{\eta_b\,\eta_D}(-1)^{J_c+J_d+c+d}$$

$$\eta_{cd} = (-1)^{J_c+J_d+c+d}$$

v_i = 1 if i-th particle is a fermion
= 0 otherwise.

$P^{(J-v/2)}(\cos\chi)$ is a polynomial in $\cos\chi$ of order $(J - v/2)$. Similarly, a form of $M^{\pm}$ which demonstrates the singularities at $S_{cd} = 0$, may be written down as described by Wang[6].

1.4 Analyticity in W and MacDowell Symmetry.

From equation (A), section (1.3), the singularity in $M^{\pm}$ at $s = 0$ and hence $T^{\pm}$ may be investigated. For $m_a \neq m_b$; $m_c \neq m_d$, since $\sin\theta_s \sim s^{1/2}$, the entire $s^{1/2}$ singularity is introduced by the terms $\sin\theta_s$, and $\sin\theta_{s/2}$ occurring in (A).

For reactions of the type boson + fermion $\longrightarrow$ boson + fermion, the term $|\lambda-\mu| - |\lambda+\mu|$ is always odd. Hence the term $(\sin\theta_{s/2})^{|\lambda-\mu|-|\lambda+\mu|}$ gives rise to a branch point at $s = 0$ which cannot be removed by multiplying by $s^{\frac{1}{2}\max[|\lambda-\mu|,|\lambda+\mu|]}$. This will remove the possible pole at $s = 0$, but not the square root branch point.

Although they are not analytic in s, the parity conserving helicity amplitudes of GGLZ are analytic in $w = \sqrt{s}$.

The proof of MacDowell symmetry for the above amplitudes is as follows.

$$T^{\pm}_{cd,ab}(w) = M^{\pm}_{c'A',D'b'}(w) \; T'^{t}_{c'A',D'b'} \text{ , since } T^{\pm}, M^{\pm}$$

are analytic in w, and T'^{t} is analytic in s.

$$\therefore \quad T^{\pm}_{cd,ab}(-w) = M^{\pm}_{c'A',D'b'}(-w) \; T'^{t}_{c'A',D'b'}(s) \; .$$

From equation (A), section <u>1.3</u> :

$$M^{\pm}_{c'A',D'b'}(-w) = (-1)^{\lambda-\mu} M^{\mp}_{c'A',D'b'}(w)$$

$$\therefore \quad T^{\pm}_{cd,ab}(-w) = (-1)^{\lambda-\mu} \; T^{\mp}_{cd,ab}(w) \qquad \text{(B)}$$

The parity conserving partial wave amplitudes are related to the $T^{\pm}$'s by :

$$F^{J^{\pm}}_{cd,ab}(w) = \frac{1}{16\pi w}\int_{-1}^{+1} dz \left[c^{J^{+}}_{\lambda\mu}(z) \; T^{\pm}_{cd,ab}(w,z) + c^{J^{-}}_{\lambda\mu}(z) \; T^{\mp}_{cd,ab}(w,z) \right]$$

Hence, from equation (B) above:

$$F^{J^{\pm}}_{cd,ab}(-w) = -(-1)^{\lambda-\mu} \; F^{J^{\mp}}_{cd,ab}(w) \qquad \text{(C)}$$

This is the general MacDowell symmetry relation for the above parity conserving amplitudes.

The above relation (C) has been proved only for $m_a \neq m_b$, $m_c \neq m_d$. When this relation does not hold, MacDowell symmetry is seen to break down.

Suppose $m_a = m_b = m_c = m_d = m$

$$\sin\theta_s = \frac{2\sqrt{ut}}{s-4m^2} \; ; \quad \sin\frac{\theta_s}{2} = \sqrt{\left[\frac{-t}{s-4m^2}\right]}$$

$$\cos\chi_a = -\cos\chi_b = -\cos\chi_c = \cos\chi_d = -\sqrt{\left(\frac{st}{(s-4m^2)t-4m^2}\right)}$$

In this case $\sin\theta_s$, $\sin\theta_{s/2}$ are analytic at $s = 0$, and the singularities are contained only in the $\cos\chi_i$ terms. These may be factored out from the crossing matrix, and the amplitudes made analytic in s by multiplication with appropriate powers of $s^{\frac{1}{2}}$. The interchange $w \rightarrow -w$ will not affect the parity of the state, so there is no relation between different parity states in this case, However there is no known case in Nature of a boson and a fermion having the same mass, so for all known reactions, $m_a \neq m_b$, $m_c \neq m_d$, and the MacDowell symmetry relation (C) is assumed to hold.

1.5 Application to Pion-nucleon Scattering

The following results may be obtained from any standard work on pion-nucleon scattering (see Jacob & Chew[8], Chapter II).

The nucleon has spin $\frac{1}{2}$, and helicity state $\pm\frac{1}{2}$. Invariance under parity implies

$$\left\langle \tfrac{1}{2} \middle| F^J \middle| \tfrac{1}{2} \right\rangle = \left\langle -\tfrac{1}{2} \middle| F^J \middle| -\tfrac{1}{2} \right\rangle$$

$$\left\langle \tfrac{1}{2} \middle| F^J \middle| -\tfrac{1}{2} \right\rangle = \left\langle -\tfrac{1}{2} \middle| F^J \middle| \tfrac{1}{2} \right\rangle .$$

Denote positive nucleon helicity by $+$, and negative nucleon helicity by $-$.

The partial wave expansion of Jacob & Wick[3] in terms of the $f_{\lambda\mu}$, and $f^J_{\lambda\mu}$ amplitudes is:-

$$f_{++}(\theta) = \sum_J (2J+1)\ f^J_{++}\ d^J_{1/2\,1/2}(\theta)$$

$$f_{+-}(\theta) = \sum_J (2J+1)\ f^J_{+-}\ d^J_{1/2\,1/2}(\theta)$$

where f^J_{++}, $f^J_{+-} \equiv \langle cd \,|\, F^J \,|\, ab \rangle$ amplitudes of GGLZ .

Parity conserving helicity amplitudes are defined by

$f_{\ell^+} = f^J_{++} + f^J_{+-}$ having parity $-(-1)^{\ell} = -(-1)^{J-\frac{1}{2}}$. This is equivalent to the amplitude $F^{J^-}_{cd,ab}$ used by GGLZ.

Similarly, $f_{(\ell+1)^-} = f^J_{++} - f^J_{+-}$ has parity $+(-1)^{J-\frac{1}{2}}$, and corresponds to $F^{J^+}_{cd,ab}$.

Comparison of the $f_{\ell^{\pm}}$ with the generalised parity conserving amplitudes

$$\langle cd \,|\, F^J \,|\, ab \rangle \pm (-1)^{\lambda+\lambda_m}\ \eta_c \eta_d (-1)^{s_c+s_d-v} \langle -c-d \,|\, F^J \,|\, ab \rangle$$

shows that $\lambda = a - b = \frac{1}{2}$, $\mu = c - d = \frac{1}{2}$, so $\lambda - \mu = 0$.

The relation

$$F^{J^+}_{cd,ab}(-w) = -(-1)^{\lambda-\mu}\, F^{J^-}_{cd,ab}(w)$$

reduces to $f_{\ell^+}(w) = -f_{(\ell+1)^-}(-w)$ for pion-nucleon scattering. This is identical to the result found by MacDowell[1].

The one important assumption made in proving the above MacDowell symmetry relation, is that invariant amplitudes analytic in s, t, u may be defined for any scattering process. This is a field theoretic result and is stated by Hara[5]. Other important results used in the proof, are analyticity of $T^{\pm}_{cd,ab}$ in w, and the crossing relations of Trueman & Wick.

CHAPTER II

MACDOWELL SYMMETRY AND COMPLEX ANGULAR MOMENTUM IN THE PION-NUCLEON SYSTEM

2.1 Introduction

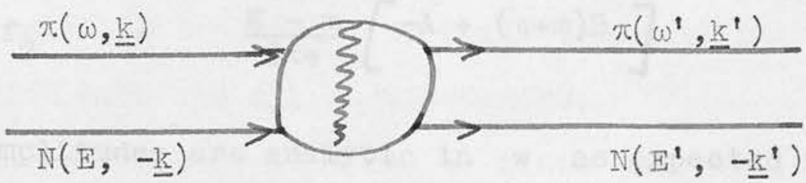


Fig. 1

The problem of complex angular momentum in the pion-nucleon system was first discussed by Singh[9], and his notation is used throughout this chapter.

Invariance of the pion-nucleon scattering amplitude under parity and time reversal leads to a T matrix of the form:

$$T = -A + \gamma.QB$$

where $A = A(s,t,u)$, $B = B(s,t,u)$ are invariant amplitudes which satisfy the Mandelstam Representation[10]. s,t,u are the Mandelstam variables, $Q = \frac{1}{2}(K_1 + K_2)$, where K_1, K_2 are the 4 momenta of the incoming and outgoing pion respectively.

The convention used for the γ matrices is defined in Appendix II.

Let k, E, w represent the magnitude of the pion three momentum in the centre of mass frame, the nucleon energy, and the total energy respectively. Let m = nucleon mass, μ = pion mass.

Then $E^2 = k^2 + m^2$; $E = \dfrac{w^2+m^2-\mu^2}{2w}$; $4k^2 = w^2 - 2(m^2+\mu^2) + \dfrac{(m^2-\mu^2)^2}{w^2}$.

From any standard treatise of pion-nucleon scattering (see Jacob & Chew[8]) amplitudes f_1, f_2 are defined by

$$f_1 = \frac{E+m}{8\pi w}\Big[A + (w-m)B\Big]$$

$$f_2 = \frac{E-m}{8\pi w}\Big[-A + (w+m)B\Big]$$

These amplitudes are analytic in w as expected from the results of Chapter I.

The partial wave expansion of the f_i's is

$$f_1 = \sum_\ell f_{\ell^+} P'_{\ell+1}(z) - f_{(\ell+1)^-} P'_\ell(z) \;; \quad z = \cos\theta,$$

and the prime denotes differentiation with respect to z.

$$f_2 = \sum_\ell f_{(\ell+1)^-} P'_{\ell+1}(z) - f_{\ell^+} P'_\ell(z)$$

where $$f_{\ell^+}(w) = \tfrac{1}{2}\int_{-1}^{+1} dz \Big[f_1 P_\ell(z) + f_2 P_{\ell+1}(z)\Big]$$

$$f_{(\ell+1)^-}(w) = \tfrac{1}{2}\int_{-1}^{+1} dz \Big[f_1 P_{\ell+1}(z) + f_2 P_\ell(z)\Big]$$

are partial wave amplitudes having parity $\mp(-1)^\ell = \mp(-1)^{J-\frac{1}{2}}$.

Consider forward ($\theta < 90^\circ$) pion nucleon scattering. According to the Regge pole model, forward (s channel) scattering is dominated by the exchange of Regge poles in the crossed (t and u) channels. t channel Regge poles have the same quantum numbers as the system $\pi\pi \rightarrow \bar{N}N$, i.e. J = integer, I = isospin = 0,1.

These correspond to boson Regge poles, and are not discussed further. The u channel contributes to s channel processes for which $\theta \sim 180^\circ$ (backward scattering) and is also represented by the process $\pi N \to \pi N$. The quantum numbers of a u channel Regge pole are J = half integer, $I = \frac{1}{2}, \frac{3}{2}$, and these correspond to fermion Regge poles. Regge poles having $I = \frac{1}{2}$ belong to the N trajectories, while those having $I = \frac{3}{2}$ are classified with the Δ trajectories.

The previous results of w plane analyticity and MacDowell symmetry may be applied to Regge poles in the u channel where $w = \sqrt{u}$, and f_{ℓ^+}, $f_{(\ell+1)^-}$ are functions of $w = \sqrt{u}$.

Suppose partial wave f_{ℓ^+} is dominated by a Regge pole at $\ell = \alpha(w)$.

$$\text{Then} \quad f_{\ell^+}(w) = \frac{\beta(w)}{\ell-\alpha(w)} = \frac{\beta_+(w)}{\ell-\alpha_+(w)} \; ; \quad \left.\begin{aligned} \beta_+(w) &\equiv \beta(w) \\ \alpha_+(w) &\equiv \alpha(w) \end{aligned}\right\} w \text{ +ve}$$

Continuing this relation to -w gives

$$f_{\ell^+}(-w) = \frac{\beta(-w)}{\ell-\alpha(-w)} = \frac{-\beta_-(w)}{\ell-\alpha_-(w)} \; ; \quad \left.\begin{aligned} \beta(-w) &\equiv -\beta_-(w) \\ \alpha(-w) &\equiv \alpha_-(w) \end{aligned}\right\} w \text{ +ve}$$

From MacDowell symmetry,

$$f_{(\ell+1)^-}(w) = \frac{\beta_-(w)}{\ell-\alpha_-(w)} \quad .$$

Hence if a Regge trajectory $\alpha_+(w)$ contributes to the partial wave f_{ℓ^+}, the Regge trajectory $\alpha_-(w)$ contribute to $f_{(\ell+1)^-}$, where

$$\alpha_+(w) \equiv \alpha(w) \equiv \alpha_-(-w) \quad .$$

Physical bound states and resonances which occur on the trajectory

$\alpha_+(w)$ have parity $-(-1)^{\ell} = (-1)^{J+\frac{1}{2}}$, and similar states on $\alpha_-(w)$ have parity $-(-1)^{J+\frac{1}{2}}$. Due to signature effects, physical resonances on Regge trajectories are separated by a multiple of 2 units of angular momentum J, so the parity will remain constant along a trajectory.

MacDowell symmetry and w plane analyticity thus enable a meaning to be given to $\alpha(-w)$. If $\alpha(w) = \alpha_+(w)$ describes resonances of positive parity, the function $\alpha(-w) = \alpha_-(w)$ will describe resonances of negative parity, and vice versa.

In this chapter, the properties of the above defined functions $f_{\ell\pm}(w)$, $\alpha_{\mp}(w)$ for the pion-nucleon system, are discussed. First, the results of an introductory survey of complex angular momentum in pion-nucleon scattering, are applied to u channel scattering. Next, the analyticity of $\alpha(w)$ is discussed, and a dispersion relation for $\alpha(w)$ given. This is followed by an examination of the continuation process $w \rightarrow -w$ in the MacDowell symmetry relation. Special attention is paid to the particular sheets on which the above functions are defined. Finally, a survey is given of the known backward pion nucleon resonances, and the problems discussed in this thesis are illustrated.

2.2. Complex Angular Momentum in the Pion Nucleon System

2.2.(1) Concept of signature, or J parity

The invariant amplitudes $A(s,t,u)$, $B(s,t,u)$ defined in (2.1) are assumed to satisfy a fixed s dispersion relation of the form(9)

$$A(s,t,u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \frac{A_t(s,t')}{t'-t} + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{A_u(s,u')}{u'-U} . du'$$

$$= \frac{1}{\pi}\int_{4\mu^2}^{\infty} dx \frac{A_t(s,x)}{x+2k^2(1-\cos\theta)} + \frac{1}{\pi}\int_{(m+\mu)^2-\frac{(m^2-\mu^2)^2}{s}}^{\infty} dx . \frac{A_u\left(s,x+\frac{(m^2-\mu^2)^2}{s}\right)}{x+2k^2(1+\cos\theta)} .$$

Define $A_\ell = \int_{-1}^{+1} d(\cos\theta) A(s,t,u) P_\ell(\cos\theta)$

$$= \frac{1}{\pi k^2}\int dx \quad A_t(s,x) + (-1)^\ell A_u\left(s,x+\frac{(m^2-\mu^2)^2}{s}\right) Q_\ell\left(1+\frac{x}{2k^2}\right) \quad (A)$$

All q uantities in the above expression (A) are suitable for continuation to complex ℓ except the term $(-1)^\ell = \exp(i\pi\ell)$, which diverges as $\ell \rightarrow i\infty$. Put $\ell = \lambda$ (complex).

To avoid this difficulty, define two separate continuations for values of ℓ which are even and odd respectively.

$$A^{\ell,0}(s,\lambda) = \frac{1}{\pi k^2}\int dx \quad Q_\lambda\left(1+\frac{x}{2k^2}\right) A^{\ell,0}_{tu}(s,x)$$

where $A^{\ell}_{tu}(s,x) = A_t + A_u$, and is defined for ℓ = even systems

$A^{o}_{tu}(s,x) = A_t - A_u$, and is defined for ℓ = odd systems.

The functions A^{ℓ}, A^{o} interpolate between partial wave amplitudes for even and odd values of ℓ respectively. A physical system having ℓ even is said to have positive "signature", or "J parity", while one having ℓ = odd has negative signature. The continuations A^{ℓ}, A^{o} are physically meaningful only for even and odd values of ℓ respectively, and bear no relation to each other. The signature is a well defined quantum number for a Regge trajectory, and its value is $(-1)^{\ell}$, where ℓ is the orbital angular momentum of physical bound states on the trajectory.

The concept of signature is entirely due to the presence of the third double spectral function A_u, so it is an essentially relativistic phenomenon. In potential scattering and other ~~more~~ non relativistic problems, $A_u = 0$, and the concept of signature does not arise.

2.2.(2) The Sommerfeld Watson transform for the pion nucleon system.

The partial wave expansion for amplitudes f_1, f_2 defined in (2.1) is

$$\left.\begin{aligned} f_1 &= \sum_{\ell} f_{\ell^+} P'_{\ell+1}(z) - f_{(\ell+1)^-} P'_{\ell}(z) \\ f_2 &= \sum_{\ell} f_{(\ell+1)^-} P'_{\ell+1}(z) - f_{\ell^+} P'_{\ell}(z) \end{aligned}\right\} \quad (A)$$

Now continue equation (A) above into the complex ℓ plane. f_{ℓ^+}, $f_{(\ell+1)^-}$ have two continuations $f^{e,o}_{\ell^+}$, $f^{e,o}_{(\ell+1)^-}$ according to whether ℓ is even or odd.

Let $f^{e,o}_{\ell^+} = f^{e,o}_{(J-\frac{1}{2})^+}$; $f^{e,o}_{(\ell+1)^-} = f^{e,o}_{(J+\frac{1}{2})^-}$

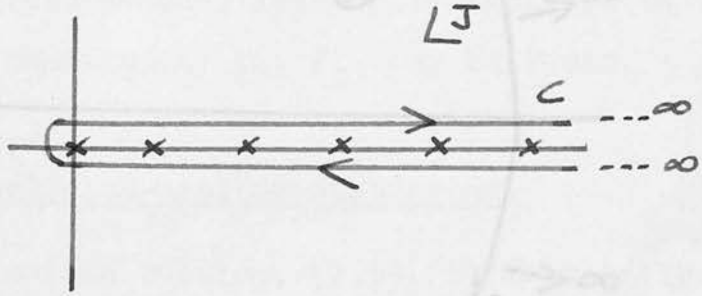


Fig. 2.

The above partial wave expansion (A) may be written as a contour integral around C shown in Figure 2. C is described clockwise, and is assumed closed at infinity.

$$
\begin{aligned}
f_1 = {} & \frac{i}{4}\int_C \frac{dJ}{\cos\pi J}\, f^{e}_{(J-\frac{1}{2})^+} \left[P'_{J+\frac{1}{2}}(-z) + P'_{J+\frac{1}{2}}(z) \right] \\
& + \frac{i}{4}\int_C \frac{dJ}{\cos\pi J}\, f^{o}_{(J-\frac{1}{2})^+} \left[P'_{J+\frac{1}{2}}(-z) - P'_{J+\frac{1}{2}}(z) \right] \\
& - \frac{i}{4}\int_C \frac{dJ}{\cos\pi J}\, f^{\ell}_{(J+\frac{1}{2})^-} \left[P'_{J-\frac{1}{2}}(-z) - P'_{J-\frac{1}{2}}(z) \right] \\
& - \frac{i}{4}\int_C \frac{dJ}{\cos\pi J}\, f^{o}_{(J+\frac{1}{2})^-} \left[P'_{J-\frac{1}{2}}(-z) + P'_{J-\frac{1}{2}}(z) \right]
\end{aligned}
\qquad (B)
$$

The Sommerfeld Watson transform consists of opening out the above contour as shown. The new contour consists of an infinite semi-circle, bounded by a line at Re $J = -\frac{1}{2}$.

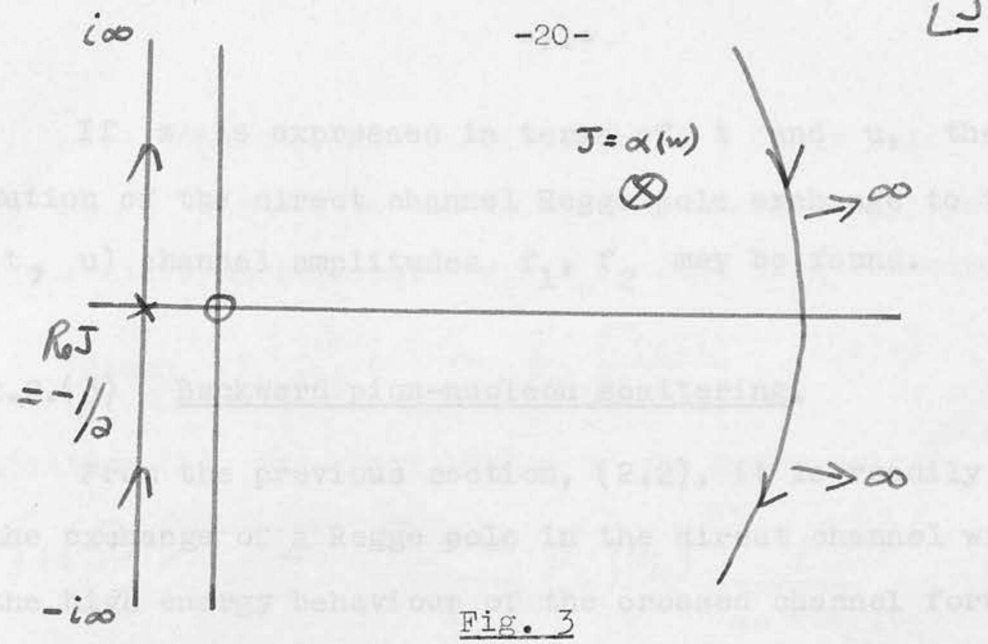


Fig. 3

The Sommerfeld-Watson Transform

The contribution of the Regge pole terms at $J = \alpha(w)$ are now included in the expression (B) above for f_1. Standard analysis(11) shows that for large z (i.e. large t or u) the integral along the contour tends to vanish, leaving only the Regge pole terms.

Suppose $f^{e,o}_{(J\mp\frac{1}{2})\pm} = \sum_{\text{Regge poles}} \dfrac{R^{\pm e,o}}{\alpha^{e,o}(w) - J}$ + regular terms

where $R^{\pm e,o} = \lim\limits_{J \to \alpha^{e,o}(w)} f^{e,o}_{(J\mp\frac{1}{2})\pm} \left(\alpha^{e,o}(w) - J\right)$.

Considering only the most divergent Regge pole contributions in the contour integral (B) gives

$$f_1 = \frac{\pi}{2} \sum_{\text{Regge poles}} \pm\left[\frac{1}{\cos\pi\alpha_i(w)} R_i^{\pm e,o}(w) \; . \; (1 + \eta_i \exp -i\pi\alpha_i) P'_{\alpha_i+\frac{1}{2}}(z)\right] \tag{D}$$

where η_i is the signature of the i-th Regge pole. A similar expression holds for f_2.

If z is expressed in terms of t and u, the contribution of the direct channel Regge pole exchange to the crossed (t, u) channel amplitudes f_1, f_2 may be found.

2.2.(3) <u>Backward pion-nucleon scattering.</u>

From the previous section, (2.2), it is readily seen that the exchange of a Regge pole in the direct channel will affect the high energy behaviour of the crossed channel forward scattering amplitude. Conversely, the asymptotic behaviour of the forward s channel amplitude is governed by Regge pole exchange in the t channel, and the asymptotic behaviour of backward $(\theta \sim \pi)$ s channel scattering depends on Regge pole exchange in the u channel. The quantum numbers of the u channel have been shown in the Introduction (2.1) to correspond to fermions.

The kinematics of the u channel are exactly similar to the s channel, except that u replaces s are the square of the centre of mass energy (see Singh[9]).

$$w_u = \sqrt{u} = \text{total c.m. energy}$$

$$4k_u^2 = u - 2m^2 - 2\mu^2 + \frac{(m^2-\mu^2)^2}{u} = \text{square of c.m. momentum.}$$

$$E_u = \left(\frac{w_u^2 + m^2 - \mu^2}{2w_u}\right) = \text{nucleon energy.}$$

$$z_c = \cos\theta_u = -\left[s - m^2 - \mu^2 + 2E_u(w_u - E_u)\right]/2k_u^2 .$$

Backward scattering is described by $\theta = \pi$, $\theta_u = 0$. This gives the condition

$$u - (m^2 - \mu^2)/s = 0 \qquad \text{(A)}$$

Now consider the contribution of a Regge pole $\alpha_i(w_u)$ in the u channel to the forward scattering amplitude. The MacDowell symmetric pole $\alpha_i(-w_u)$ will also contribute.

Since $P'_{\alpha(w)+\frac{1}{2}}(z) \sim \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \cdot z^{\alpha(w)-\frac{1}{2}}$, from equation (D), the contribution of the Regge pole to f_1 is

$$
\begin{aligned}
f_1(\sqrt{s},u) = \sum_i & \frac{E_u+m}{W_u} \beta_i(W_u) \cdot \frac{1}{\cos \pi \alpha_i(W_u)} \left(\frac{s}{s_0}\right)^{\alpha_i(W_u)-1/2} \\
& \times \left(1+\eta_i \exp\left[-i\pi\left(\alpha[W_u]-1/2\right)\right]\right) \\
& + \frac{E_u-m}{W_u} \beta_i(-W_u) \frac{1}{\cos \pi \alpha_i(-W_u)} \left(\frac{s}{s_0}\right)^{\alpha_i(-W_u)-1/2} \\
& \times \left(1+\eta_i \exp\left[-i\pi\left(\alpha[-W_u]-1/2\right)\right]\right)
\end{aligned}
\tag{B}
$$

where

$$\beta_i(W_u) = \sqrt{\pi} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)} \gamma(W_u)$$

$$\gamma(W_u) = \mathop{\mathrm{lt}}_{J \to \alpha(W_u)} \left[(\alpha[W_u]-J)(2k^2)^{\alpha(W_u)-1/2} h^+_{\alpha(W_u)-1/2}(W_u)\right]$$

$$h^+_{J-1/2}(W_u) = \frac{16\pi W_u}{E_u+m} \cdot f_{(J-1/2)+}(W_u) / (2k^2)^{J-1/2}$$

From the backward scattering condition, $E_u+m = \sqrt{u} + \sqrt{s} + 2m$, the result of Chiu and Stack[12] for the contribution of the N and Δ Regge poles to $f_1(\sqrt{s},u)$ may be proved in a similar manner, except

$$\gamma(W_u) = \mathop{\mathrm{lt}}_{J \to \alpha(W_u)} \left[(\alpha(W_u)-J)(2k^2)^{\alpha(W_u)-1/2} \frac{W}{E+m} h^+_{\alpha(W_u)-1/2}(W_u)\right]$$

Comparison of result (B) above with the observed s channel backward scattering amplitudes enables parametrisations of $\alpha(W_u)$ to be made.[12,13] The result of this work is discussed more fully in Chapter IV.

2.4 Analytic structure of $\alpha_{\pm}(w)$

In this section the analyticity of $\alpha_{\pm}(w)$ is examined. The functions $\alpha_{\pm}(w)$ are shown to be real analytic, having cuts from $-\infty \leq w \leq -(m+\mu)$, and $(m+\mu) \leq w \leq \infty$.

2.4.(1) Kinematic singularity free amplitudes

A Mandelstam representation for the invariant amplitudes A, B of pion-nucleon scattering, yields, as in (2.2)

$$A(s,t,u) = \frac{1}{\pi}\int_{t_o}^{\infty} \frac{A_t(s,x)dx}{x+2k^2(1-\cos\theta)} + \frac{1}{\pi}\int_{u_o-\frac{(m^2-\mu^2)^2}{s}}^{\infty} . \, dx \, \frac{A_u(s,x+\frac{(m^2-\mu^2)^2}{s})}{x+2k^2(1+\cos\theta)}$$

where

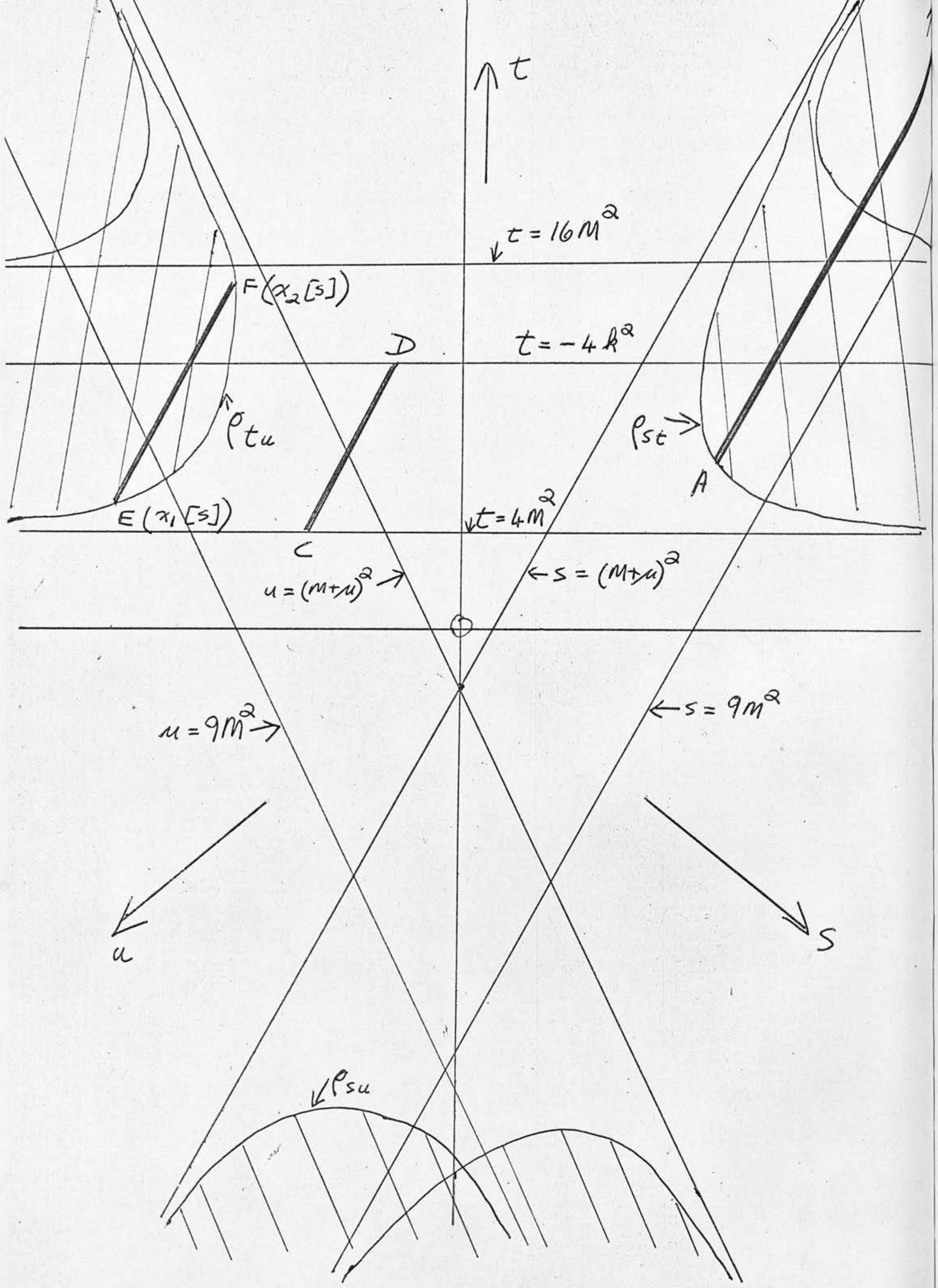
$$A_t(s,x) = \frac{1}{\pi}\int_{s_o}^{\infty} \frac{\rho_{st}(s',x)ds'}{s'-s} + \frac{1}{\pi}\int_{u_o}^{\infty} \frac{\rho_{tu}(x,u')}{u'-u}\, du'$$

Similarly for $A_u(s,x)$.

From the above dispersion relation, the analytic structure of $A(s,t,u)$ is a right-hand cut starting at $s = s_o$, and a left-hand cut given by $x + 2k^2(1 - \cos\theta) = 0$, which starts at $k^2 = - t_o/4$.

The function $A_\ell(s)$ is defined as in (2.2), and has the same analytic structure as $A(s,t,u)$. For continuation to complex $\ell = \lambda$, the functions $A^{e,o}(\lambda,s)$

$$= \frac{1}{\pi k^2}\int_{t_o}^{\infty} dx\, Q_\lambda(1+\frac{x}{2k^2})\, A^{e,o}_{tu}$$ are defined.


t
$t = 16M^2$
$F(\alpha_2[s])$
D
$t = -4k^2$
ρ_{tu}
ρ_{st}
A
$E(x_1[s])$
$t = 4M^2$
C
$u = (M+\mu)^2$
$s = (M+\mu)^2$
O
$u = 9M^2$
$s = 9M^2$
u
S
ρ_{su}

Since $Q_\lambda(z)$ has cuts from $z = -\infty$ to $z = -1$, and from $z = -1$ to $z = +1$, $A^{e,o}(\lambda, s)$ has an extra cut from $k^2 = 0$ to $k^2 = -t_o/4$.

However, it is a standard result(11,14) that the function $A^{e,o}(\lambda, s)/(2k^2)^\lambda$ has analytic properties exactly similar to $A_\ell(s)$ for real ℓ.

Define $C^{e,o}(\lambda, s) = A^{e,o}(\lambda, s)/(2k^2)^\lambda$.

Then

$$\frac{1}{2i}\,\text{disc}\left[C^{e,o}(\lambda,s)\right]_{R.H.} = \frac{(k^2)^{-\lambda}}{\pi k^2}\int_{t_o}^{\infty} dx\, Q_\lambda\left(1+\frac{x}{2k^2}\right) \cdot \left[\rho_{st}(s,x) \pm \rho_{su}(s,x)\right]$$

$$\frac{1}{2i}\,\text{disc}\left[C^{e,o}(\lambda,s)\right]_{L.H.} = \frac{(k^2+i\varepsilon)^{-\lambda}}{\pi k^2}\int_{x_1(s)}^{x_2(s)} dx.\, Q_\lambda\left(1+\frac{x}{2k^2+i\varepsilon}\right) \cdot \rho_{tu}(s,x)\left[1 \mp e^{-i\pi\lambda}\right]$$

$$- \frac{i(-k^2)^\lambda}{k^2}\int_{t_o}^{-4k^2} dx.\, P_\lambda\left(-1-\frac{x}{2k^2}\right) A^{e,o}_{tu}\,(s - i\varepsilon,\ t) \qquad \text{(A)}$$

where disc $[\ ]_{L.H.}$ is the discontinuity of $C^{e,o}$ across the left-hand cut. The regions of integration are shown on the Mandelstam diagram for pion-nucleon scattering(10).

The discontinuity across the right-hand cut $(s > 0)$ comes from the double spectral functions ρ_{st}, ρ_{su}, and the region of integration is the infinite line AB. CD, EF represent the regions of integration for the left-hand cuts.

The region of integration for the left-hand discontinuity is finite, and since P_λ, Q_λ are analytic functions, the left-hand discontinuity is an analytic function of λ. Exceptions occur at $\lambda = -n$, for Q_λ has simple poles at these points.

In contrast the right-hand discontinuity is given by an infinite integral, and is thus defined only for $\mathrm{Re}\lambda > N$, where N is the number of subtractions necessary to make the Mandelstam representation converge.

From the above results, the functions:-

$$h^{\pm}_{(J\mp\frac{1}{2})}(w) = \frac{16\pi w}{E \pm m} \; \frac{f^{\pm}_{(J\mp\frac{1}{2})}(w)}{(2k^2)^{J-\frac{1}{2}}} \qquad \text{(B)}$$

are free of kinematic singularities in the w plane, and contain only the right-hand unitarity cut for $w^2 > (m+\mu)^2$, and the left-hand cut starting at $k^2 = -\, t_o/4$.

2.4.(2) Analyticity of the trajectory function $\alpha(w)$

The analyticity of $\alpha(W)$ may be readily found from the implicit function theorem. This is stated by J.R. Taylor as follows[15]:

If $F(\lambda, w)$ is holomorphic in some domain $\mathcal{D}$ and $(\lambda_o, w_o) \varepsilon\, \mathcal{D}$ is such that $F(\lambda_o, w_o) = 0$, $\left.\frac{\partial F}{\partial \lambda}\right|_{\lambda=\lambda_o, w=w_o} \neq 0$, then there exists a neighbourhood $N_{\lambda_o} \times N_{w_o}$ of (λ_o, w_o) such that for each $w \,\varepsilon\, N_{w_o}$ there is a unique and holomorphic solution $\lambda = \alpha(w) \,\varepsilon\, N_{\lambda_o}$ of the equation $F(\lambda, w) = 0$.

From the previous section $h^{\pm}_{(J\mp\frac{1}{2})}(w)$ is a real analytic function. For simplicity, define $h^{\pm}(\lambda,w) = h_{(J\mp\frac{1}{2})^{\pm}}(w)$, where $\lambda = J$ = complex.

A Regge pole is given by the condition:

$$\left[h^{\pm}(\lambda,w)\right]^{-1} = 0$$

(The function $\left[h^{\pm}(\lambda,w)\right]^{-1}$ is holomorphic in the domain between the cuts of $h^{\pm}(\lambda,w)$, and in general $\frac{\partial}{\partial\lambda}\left[h^{\pm}(\lambda,w)\right]^{-1} \neq 0$ unless two trajectories intersect. The conditions of the implicit function theorem are satisfied, so $\lambda = \alpha(w)$ is holomorphic in the neighbourhood $N = N_{\lambda_o} \times N_{w_o}$.

Consider first the effect of the left-hand cut. From the previous section (2.4(1)), provided $\lambda \neq -n$, the discontinuity across the left-hand cut is always finite for all values of w. Consequently, no poles can pass from the unphysical to the physical sheet through the left hand cut, and the domain $N_{\lambda_o} \times N_{w_o} = (\lambda_o, w_o)$ will exclude this cut.

Next consider the right-hand cut. Unitarity gives

$$\frac{1}{2i}\left[h^{\pm}(\lambda,w+i\varepsilon) - h^{\pm}(\lambda,w-i\varepsilon)\right] = \rho^{\pm}\, h^{\pm}(\lambda,\ w+i\varepsilon)\, h^{\pm}(\lambda,\ w-i\varepsilon)$$

where $\rho^{\pm} = \dfrac{k(E\pm m)(2k^2)^{\lambda}}{16\pi W}$.

Thus $h^{\pm}(\lambda,w+i\varepsilon) = h^{\pm}(\lambda,w-i\varepsilon)/(1-2i\,\rho^{\pm}\,h^{\pm}\ \lambda,w-i\varepsilon\ )$

and poles are given by the condition $\rho^{\pm}\,h^{\pm}(\lambda,w) = \frac{1}{2i}$ (A) .

Poles pass through this cut provided condition (A) is satisfied, so the domain (λ_o, w_o) will include this cut.

For all w, λ can exist everywhere except on the left-hand cut. From the implicit function theorem, $\lambda = \alpha(w)$ inherits only the unitarity cuts in w, and is an analytic function of w.

Since $h^{\pm}(\lambda,w)^{-1}$ is real between the unitarity cuts, $\alpha(w)$ is also real between the cuts.

Consequently $\alpha(w)$ is a real analytic function having cuts from $-\infty \leq w \leq -(m+\mu)$, and $(m+\mu) \leq w \leq \infty$.

Define $\alpha_+(w) = \alpha(w)$; $\alpha_-(w) = \alpha(-w)$. The properties of $\alpha_{\pm}(w)$ are as follows.

1. α_+, α_- have the unitarity cut for $|w| \geq m+\mu$, so $\alpha_+(w)$ $\alpha_-(w)$ are complex and unrelated for $|w| \geq m+\mu$.
2. For $0 < |w| < m+\mu$, $\alpha_+(w)$, $\alpha_-(w)$ are both real and unrelated.
3. Since $\alpha(w)$ is a real analytic function, $\alpha_+(0) = \alpha_-(0) = \alpha(0)$. An illustration of this result is given by Gribov[16].
4. For w pure imaginary, $w = i\,|w|$, $\alpha_+(w) = \alpha_+^*(-i\,|w|) = \alpha_-^*(w)$. In this case, the trajectories $\alpha_+(w)$, $\alpha_-(w)$ are complex conjugate. This follows since $\alpha(w)$ is a real analytic function.

In Figure 4, a plot of $\operatorname{Im}\alpha/\operatorname{Re}\alpha$ is shown for a typical trajectory function. (Assume $\alpha_{\pm}(\infty)$ = a finite constant.

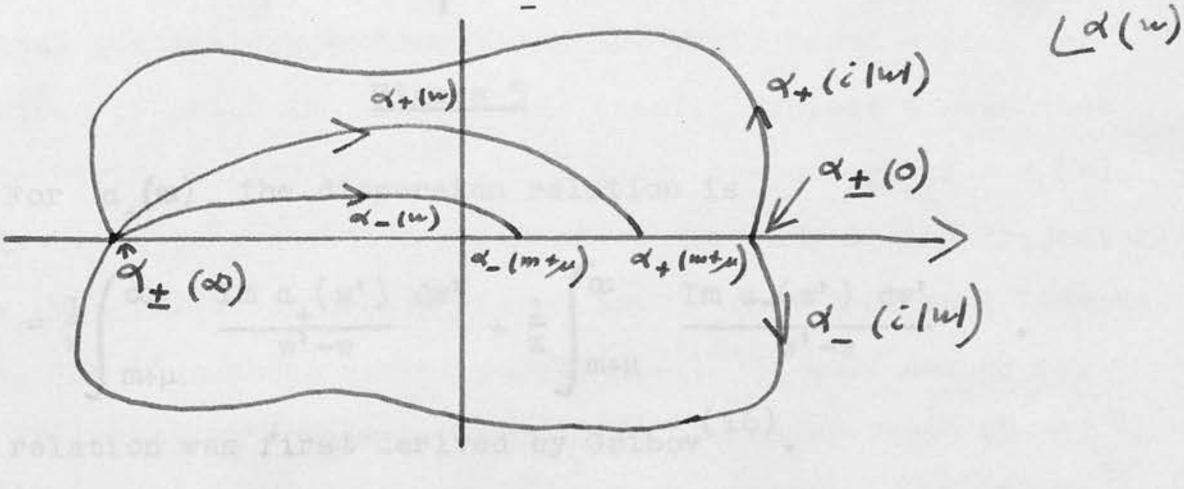


Figure 4

2.4(3) Dispersion relations for $\alpha_{\pm}(w)$

In the complex w plane the analytic structure of $\alpha(w)$ is a cut for w $m+\mu$. Dispersion relations satisfied by $\alpha_{\pm}(w)$ are obtained by integrating the function

$$\frac{1}{\pi}\int_C \frac{\alpha(w')}{w'-w}\,dw' = \alpha(w)$$

round the contour shown in Figure 5.

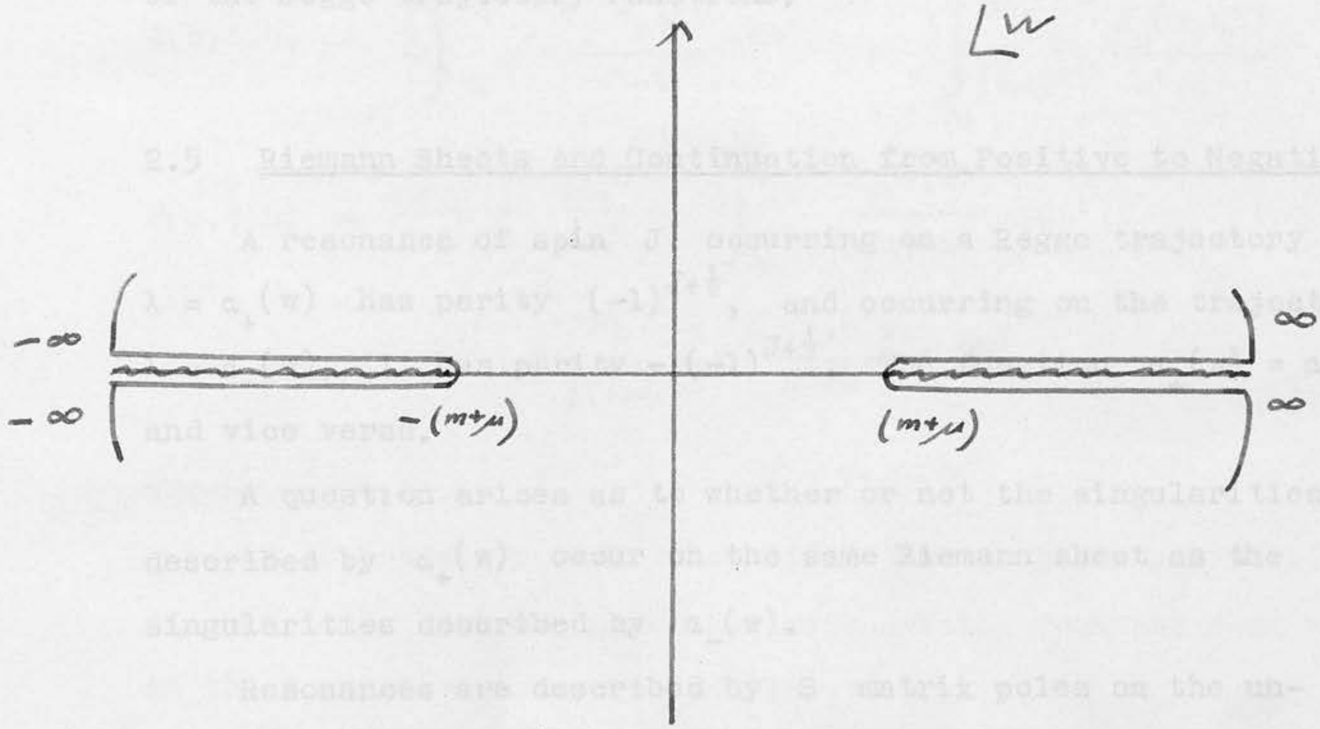


Figure 5

For $\alpha_+(w)$ the dispersion relation is

$$\alpha_+(w) = \frac{1}{\pi}\int_{m+\mu}^{\infty} \frac{\operatorname{Im}\alpha_+(w')\,dw'}{w'-w} + \frac{1}{\pi}\int_{m+\mu}^{\infty} \frac{\operatorname{Im}\alpha_-(w')\,dw'}{w'-w} .$$

This relation was first derived by Gribov[16].

The once subtracted dispersion relations are

$$\alpha_+(w) = \alpha(0) + \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{\mathrm{Im}\,\alpha_+(w')}{w'(w'-w)}\,dw' - \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{\mathrm{Im}\,\alpha_-(w')}{w'(w'+w)}\,dw'$$

$$\alpha_-(w) = \alpha(0) + \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{\mathrm{Im}\,\alpha_-(w')}{w'(w'-w)}\,dw' - \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{\mathrm{Im}\,\alpha_+(w')}{w'(w'+w)}\,dw' \qquad (A)$$

The applications and limitations of the above dispersion relations (A) are discussed in Chapter IV on the parameterisation of the Regge trajectory functions.

2.5 Riemann Sheets and Continuation from Positive to Negative w

A resonance of spin J occurring on a Regge trajectory $\lambda = \alpha_+(w)$ has parity $(-1)^{J+\frac{1}{2}}$, and occurring on the trajectory $\lambda = \alpha_-(w)$, it has parity $-(-1)^{J+\frac{1}{2}}$. The function $\alpha_+(w) = \alpha_-(-w)$, and vice versa.

A question arises as to whether or not the singularities described by $\alpha_+(w)$ occur on the same Riemann sheet as the singularities described by $\alpha_-(w)$.

Resonances are described by S matrix poles on the unphysical sheet close to the real axis, while bound states are described by poles in the physical sheet. Suppose a resonance occurs having parity $(-1)^{J+\frac{1}{2}}$ on a Regge trajectory $J = \alpha_+(w)$. Continuation to a negative value of w results in the trajectory $J = \alpha_-(w)$. If these singularities appear on a different Riemann sheet, the resonances having parity $-(-1)^{J+\frac{1}{2}}$ will not occur.

To investigate this problem it is necessary first of all to examine the analyticity of the invariant amplitudes A, B in terms of s. Then the analyticity in terms of w, and the possibility of continuation from $+w$ to $-w$ is examined.

2.5(1) Analyticity in s

The s plane analyticity of A & B is well known, and is given in the review article by Hamilton[17], also Hamilton and Spearman[18].

The dispersion relations for A, B are

$$A(s,t,u) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{A_t(s,t')}{t'-t}\,.dt' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{A_u(s,u')}{u'-u}\,.\,du'$$

$$B(s,t,u) = \frac{-4\pi G^2}{s-m^2} + \frac{4\pi G^2}{u-m^2} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{B_t(s,t')}{t'-t}\,.dt' + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{B_u(s,u')}{u'-u}\,.\,du'$$

where $G^2 \sim 15$.

It is necessary only to consider the s plane analyticity of $A(s,t,u)$, and the poles and cuts arising from the Born terms in the B amplitudes may be ignored.

The kinematics of the problem are

$$s = \left[(m^2+k^2)^{\frac{1}{2}} + (\mu^2+k^2)^{\frac{1}{2}}\right]^2$$

$$t = -2k^2(1-\cos\theta)$$

$$k^2 = \frac{1}{4}\left[s - 2(m^2+\mu^2) + \frac{(m^2-\mu^2)^2}{s}\right]$$

The complex s plane consists of a two sheeted Riemann surface joined across the line $-m^2 \leq k^2 \leq -\mu^2$. This enables the sign of the square root $\sqrt{(m^2+k^2)(\mu^2+k^2)}$ which appears in s to be uniquely determined.

Let the two sheets be denoted by I and II. When s is on sheet I, $s = \left[(m^2 + k^2)^{\frac{1}{2}} + (\mu^2 + k^2)^{\frac{1}{2}}\right]^2$, and for s on sheet II, $s = \left[(m^2 + k^2)^{\frac{1}{2}} - (\mu^2 + k^2)^{\frac{1}{2}}\right]^2$. The expression $\sqrt{(m^2 + k^2)(\mu^2 + k^2)}$ is defined to be positive in both cases.

The line $-m^2 \leq k^2 \leq -\mu^2$ corresponds to the circle cut $|s| = m^2 - \mu^2$ in the s plane. For k^2 on sheet I, s lies outside the circle, and for k^2 on sheet II, s lies inside the circle.

From section (2.4), further cuts arise from the double spectral function representation of $A(s,t,u)$. These consist of the unitarity cut for $(m+\mu)^2 \leq s \leq \infty$, and the left-hand cut for $-\infty \leq k^2 \leq -t_o/4$, where $t_o = 4\mu^2$. In terms of s the left hand cut, or "crossed" pion-nucleon cut, is given by $-\infty \leq s \leq (m-\mu)^2$.

The s plane analyticity of $A(s,t,u)$ is portrayed in Figure 6.

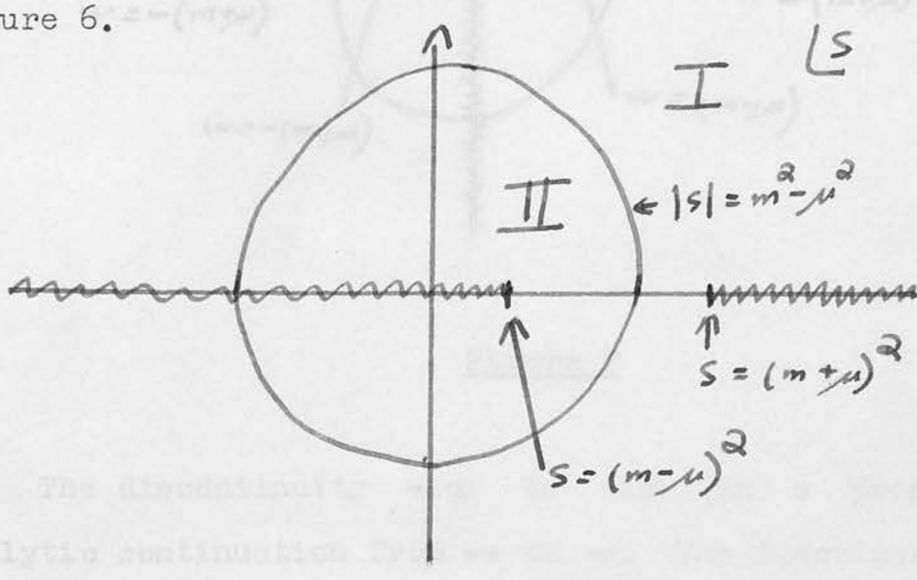


Figure 6

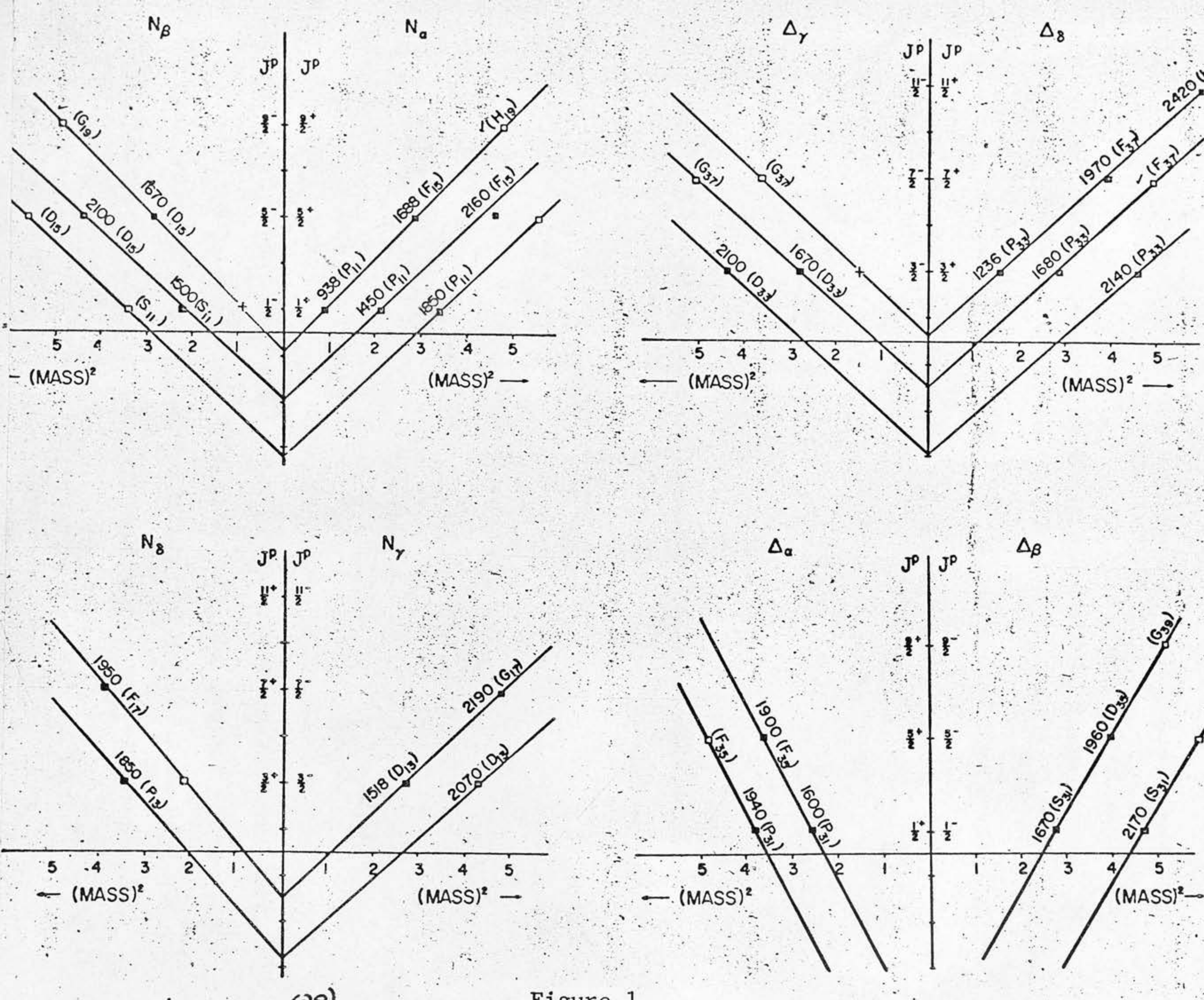


Figure 1

Barger and Cline's[20] proposed classification of πN resonances on parity degenerate Regge trajectories. Solid boxes denote known resonances, open boxes conjectured states.

for the above functions for $\mathrm{Re}\, w > 0$. A completely new set of surfaces and sheets must be defined for $\mathrm{Re}\, w < 0$. Hence if a singularity occurs in $h^{\pm}(\lambda, w)$ on a given Riemann sheet for positive w, no statement can be made about the sheet on which it occurs for $w \rightarrow -w$.

Since the w plane is effectively cut in two by the cut from $-i\infty$ to $+i\infty$, the definition of functions on the left of the cut is arbitrary. The restriction of the MacDowell symmetry relation is used to define functions for this region $\mathrm{Re}\, w < 0$.

$$h^{\pm}(\lambda, -w) = h^{\mp}(\lambda, w)$$

$$\alpha_{\pm}(-w) = \alpha_{\mp}(w) \quad .$$

2.6 <u>Classification of the $Y = 1$ Baryonic Resonances</u>

The $Y = 1$ baryonic resonances from the latest phase-shift analysis of Donnachie, Kirsopp and Lovelace[19], have been classified on Regge trajectories as shown opposite by Barger and Cline[20]. The Regge trajectories N_α, N_β give physical resonances $T = \frac{1}{2}$, $J = \frac{1}{2} + 2N (N=0,1,2..)$ and parity ± 1 respectively. N_γ, N_∂ are negative signature trajectories having $T = \frac{1}{2}$, $J = \frac{3}{2} + 2N$, and parity ∓ 1 respectively. Similarly for the Δ_α, Δ_β, Δ_γ, and Δ_∂ trajectories which have $T = \frac{3}{2}$.

Several interesting points arise from this analysis.

1. The trajectories are approximately linear functions of $(\mathrm{mass})^2$.

2. Trajectories having the same T, τ (signature), and opposite parity are approximately degenerate (e.g. N_α, N_β). In the current literature this property is frequently referred to as the

MacDowell symmetric property of the trajectories.

The consequence of this parity degeneracy is

$\alpha_+(w) \quad \alpha_-(w) = \alpha_+(-w)$, and the trajectories are approximately symmetric functions of w.

3. Several states predicted by the above mentioned parity degeneracy are missing, e.g. the lowest member of the highest ranking N_β trajectory (the S_{11}), and also the lowest members of the N_∂ (P_{13}) and Δ_γ (D_{33}, G_{37}). The possibility that these resonances are not seen because the S-matrix poles lie on the wrong Riemann sheet after continuation from positive to negative w is made, is ruled out by the discussion in Section 2.5. Two possibilities remain. Either the Regge trajectory has a sharp dip in the vicinity of the missing resonances, or else the residue function $\beta(w)$ vanishes when w is equal to the mass of a missing state. These two possibilities are investigated in this thesis.

4. Lower ranking trajectories are separated from each other by spin values $J = 1$. They correspond to the "daughter" trajectories of Freedman and Wang[21].

CHAPTER III

MACDOWELL SYMMETRY IN POTENTIAL SCATTERING

3.1 Introduction

In this chapter, the radial Dirac equation describing a spin $\frac{1}{2}$ particle in a central field is discussed, and the existence of Regge trajectories and MacDowell symmetry is demonstrated for this system. Finally, evidence is given for the possible absence of the MacDowell symmetric partner of the nucleon, and other states.

3.2 Central Field Equation and Parity States

The radial Dirac equation for a spin $\frac{1}{2}$ particle of mass m in a central field of potential $\phi(r)$ is[22]

$$\frac{dg}{dr} = \frac{Kg}{r} + \left[\frac{mc}{h} + \frac{E}{hc} - \frac{\phi(r)}{hc}\right] f$$
$$\frac{df}{dr} = \frac{-Kf}{r} + \left[\frac{mc}{h} - \frac{E}{hc} + \frac{\phi(r)}{hc}\right] g \qquad \text{(A)}$$

E is the total energy of the system.

K is a scalar which commutes with the Hamiltonian of the system, and is called by Weyl[22] the Auxiliary Quantum Number. When the total spin J is half integral, K has two values

$$K = \pm (J + \tfrac{1}{2})$$

The parity of the system is $(-1)^K$ when K is negative, and when K is positive, the parity is $-(-1)^K = -(-1)^{J+\frac{1}{2}}$.

For the pion-nucleon interaction, the parity $(-1)^K = (-1)^{J+\frac{1}{2}}$

is the parity of the amplitude f_{ℓ^+} and the parity $-(-1)^K = -(-1)^{J+\frac{1}{2}}$ is the parity of $f_{(\ell+1)^-}$. Hence if the Dirac equation is applied to the pion-nucleon interaction, the quantum number K determines the parity of the system.

Define $\nu + \frac{1}{2} = K = \pm(J + \frac{1}{2})$. The value $\nu = +J$ describes a parity state $(-1)^{J+\frac{1}{2}}$, and the value $\nu = -J-1$ describes a parity state $-(-1)^{J+\frac{1}{2}}$.

If the above units are redefined such that $h = c = 1$, then the Dirac equation (A) becomes

$$\frac{d}{dx}\psi(\nu,x) = \left[\underline{A} + \frac{U(x)}{k}\underline{\Gamma}\right]\psi(\nu,x) \qquad \text{(B)}$$

where $\psi(\nu,x) = \begin{bmatrix} g(\nu,x) \\ f(\nu,x) \end{bmatrix}$; $\underline{\Gamma} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$;

$$\underline{A} = \begin{bmatrix} \left(\frac{\nu+\frac{1}{2}}{x}\right) & -t \\ \frac{1}{t} & -\left(\frac{\nu+\frac{1}{2}}{x}\right) \end{bmatrix}$$

$$t = \sqrt{\left(\frac{E+m}{E-m}\right)} \; ; \quad U(x) = \phi(x) \; ; \quad k = \sqrt{(E^2-m^2)}; \quad x = kr$$

(See Favella and Reineri[23]).

3.3 Solution for Large x

At large x, $U(x) \longrightarrow 0$ and the Dirac equation (B) tends to the unperturbed form

$$\begin{aligned} \frac{dg}{dx} &= \frac{(\nu+\frac{1}{2})}{x} - ft \\ \frac{df}{dx} &= -\frac{(\nu+\frac{1}{2})}{x} + g/t \end{aligned} \qquad \text{(A)}$$

The general solution of the above free field equations is

$$\begin{bmatrix} g \\ f \end{bmatrix} = Cx^{\frac{1}{2}} \begin{bmatrix} H^{(1)}(x)\ \exp(i\partial) + H^{(2)}(x)\ \exp(-i\partial) \\ \frac{1}{t}\left(H^{(1)}_{\nu+1}(x)\ \exp(i\partial) + H^{(2)}_{\nu+1}(x)\ \exp(-i\partial)\right) \end{bmatrix}$$

where $H^{(1)}_{\nu}(x)$, $H^{(2)}_{\nu}(x)$ are Hankel functions of the first and second kind respectively. C and ∂ are arbitrary constants.

The functions $y = H^{(1)}_{\nu}(x)$, $H^{(2)}_{\nu}(x)$ are solutions of Bessel's equation

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0 .$$

From the results

$$H^{(1)}(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}}\ \exp\left[i\left(x - \frac{\pi}{2}\left\{\nu + \tfrac{1}{2}\right\}\right)\right]$$

$$H^{(2)}(x) \xrightarrow{x \to \infty} \sqrt{\frac{2}{\pi x}}\ \exp\left[-i\left(x - \frac{\pi}{2}\left\{\nu + \tfrac{1}{2}\right\}\right)\right]$$

the solutions of the free field equations (3.3A) at large x have the form

$$\begin{bmatrix} g \\ f \end{bmatrix} = \exp\left[i\left(\partial - \left\{\nu + \tfrac{1}{2}\right\}\frac{\pi}{2}\right)\right] \begin{bmatrix} 1 \\ \frac{-i}{t} \end{bmatrix} e^{ix} + \exp\left[-i\left(\partial - \left\{\nu + \tfrac{1}{2}\right\}\frac{\pi}{2}\right)\right] \begin{bmatrix} 1 \\ i/t \end{bmatrix} e^{-ix} \qquad \text{(B)}$$

The constant ∂ is the phase shift of the regular solution.

Since the central field Dirac equation (3.2B) tends to the free field form at large x, the above solutions (3.3B) also describe the solutions of (3.2B) as $x \to \infty$.

3.4 Conditions for Poles in the S-matrix

In the following, the potential $U(x)$ is a ssumed real, so the scattering is purely elastic. A complex potential implies some degree of inelasticity, and consideration of $U(x)$ complex is deferred to Section 3.7 on missing mass states.

From equation (3.2B) the Dirac equation for a real potential $U(x)$ is

$$\frac{d}{dx}\psi(\upsilon,x) = \left[\underline{A} + \frac{u(x)}{k} \underline{\Gamma} \right] \psi(\upsilon,x) \ .$$

Direct substitution yields

$$\frac{d}{dx}\left[\psi^{+}\Gamma\psi\right] = -2i\,\frac{\mathrm{Im}(\upsilon+\frac{1}{2})}{x}\ .\ \psi^{+}\sigma_1\psi \qquad \text{(A)}$$

where Γ is defined as before and σ_1, σ_2, σ_3 are the Pauli matrices. From the results of Section 3.3, the value of $\psi^{+}\Gamma\psi$ at large x is

$$\left[\psi^{+}\Gamma\psi\right]_{x\to\infty} = \frac{4i}{t}\sinh\left[2\,\mathrm{Im}(\partial - \{\upsilon+\tfrac{1}{2}\}\tfrac{\pi}{2})\right] \qquad \text{(B)}$$

Integrate equation (3.4A) above between $x = 0$ and $x = \infty$. At $x = \infty$ the value of the integrand is (3.4B). At $x = 0$, Favella & Reineri[23] have shown that $\psi^{+}, \psi \to 0$, so the value of the integrand is zero. The result is

$$-\frac{4}{t}\sinh\left[2\,\mathrm{Im}(\partial - \{\upsilon+\tfrac{1}{2}\}\tfrac{\pi}{2})\right] = 2\,\mathrm{Im}\upsilon\int_0^\infty \frac{\psi^{+}\sigma_1\psi.dx}{x}$$

$$= 2\mathrm{Im}\upsilon\int_0^\infty k\,|g|^2\left[\frac{d}{dx}\left(\frac{1}{x\,[E+m-u]}\right) + \frac{\mathrm{Re}[2\upsilon+1]}{x^2\,[E+m-u]}\right]dx \qquad \text{(C)}$$

The integrand of the above equation (3.4C) is certainly positive when

$$\frac{dU/dx}{(E+m-U)^2} + \frac{2\mathrm{Re}\,\nu}{x(E+m-U)} > 0 \qquad \text{(D)}$$

The conditions on dU/dx, $\mathrm{Re}\,\nu$, and $(E+m-u)$ are discussed in Section 3.6.

Suppose the above inequality (3.4D) is satisfied. Then

$$\sinh\left[2\mathrm{Im}\left(\partial - \left[\nu+\tfrac{1}{2}\right]\frac{\pi}{2}\right)\right] = -\frac{t}{2}\,\mathrm{Im}\,\nu\,.\,I, \quad \text{where} \quad t, I > 0\,.$$

If $\mathrm{Im}\,\nu < 0$, R.H.S. > 0, and a solution is possible only for

$$\mathrm{Im}\partial > {}^{\pi}/2\ \mathrm{Im}\,\nu \qquad \text{(E)}$$

If $\mathrm{Im}\,\nu > 0$, R.H.S. < 0, and a solution is possible only for

$$\mathrm{Im}\partial < {}^{\pi}/2\ \mathrm{Im}\,\nu \qquad \text{(F)}$$

From the inequalities (3.4E) and (3.4F) it is seen that the S matrix $S = \exp(2i\partial)$ can have poles only when $\mathrm{Im}\,\nu > 0$.

The above results are reversed when the inequality (3.4D) is not satisfied, and for this case, the S matrix $S = \exp(2i\partial)$ has poles only for $\mathrm{Im}\,\nu < 0$.

3.5 Resonance and Regge Trajectories

Poles in the S-matrix $S = e^{2i\partial}$ occur for $\mathrm{Im}\partial = -i\infty$. Since these phase shifts ∂ are, in general, functions of the total energy E, the position of these poles are functions of energy, and thus are Regge poles.

Further, the interchange $E \rightarrow -E$ is equivalent to $t \rightarrow {}^{1}/t$, and since this transformation does not alter the form of equation

(3.2B), Regge poles are expected for positive and negative values of E.

A Regge trajectory is described by $\alpha(\pm E)$, and a resonance of spin J which lies on the trajectory is given by $J = \mathrm{Re}\,\alpha(E_J)$

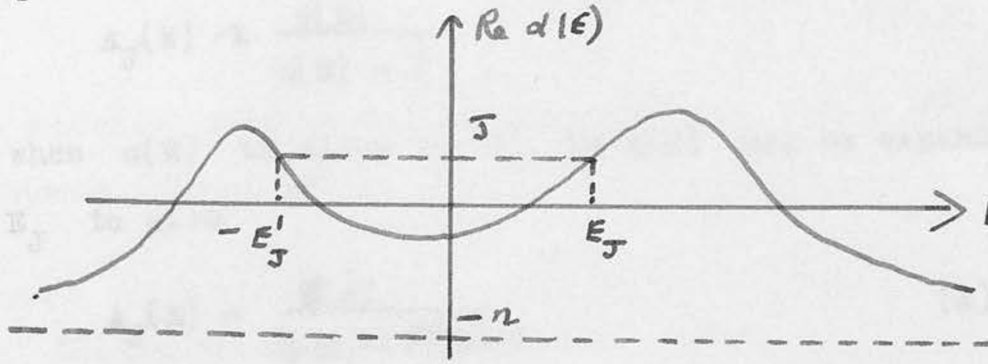


Figure 1

A typical Regge trajectory is shown in Figure 1. Experience with the Schrödinger equation[24] has shown that in general the trajectories turn over and tend towards a negative integer as $E \rightarrow \infty$. However, if the Dirac equation has any relevance at all to the pion-nucleon system, the trajectories must rise through some of the pion-nucleon resonances. Since the energy of a resonance increases with spin, the gradient of the trajectory at a resonance will, in general, be positive $(E > 0)$. When $E < 0$, $\left.\dfrac{d\alpha(-E')}{d(-E')}\right|_{E'=E_J'}$ will similarly be positive.

There are two values of $\nu(E)$ for a given Regge trajectory α:

(1) $\nu(E) = \alpha(E)$; for this value of $\nu(E)$, a resonance of spin J has parity $(-1)^{J+\frac{1}{2}}$.

(2) $\nu(E) = -\alpha(E)-1$; for this value of $\nu(E)$, a resonance of spin J has parity $-(-1)^{J+\frac{1}{2}}$.

3.5(1) Resonances for positive E

Let the partial wave amplitude for some value of J be $A_J(E)$. Near a Regge pole at $J = \mathrm{Re}\,\alpha(E_J)$, the amplitude has the form

$$A_J(E) \sim \frac{\beta(E)}{\alpha(E) - J}$$

and when $\alpha(E)$ is close to J, $\mathrm{Re}\,\alpha(E)$ may be expanded about $E = E_J$ to give

$$A_J(E) = \frac{\gamma(E)}{E-E_J+i\Gamma(E)/2} \qquad \text{(A)}$$

The above expression for $A_J(E)$ has a Breit-Wigner form, and provided $\gamma(E)$, $\Gamma(E) > 0$, a resonance is described of half width

$$\Gamma(E)/2 = \left[\frac{\mathrm{Im}\,\alpha(E)}{d(\mathrm{Re}\,\alpha(E))/dE}\right]_{E=E_J} \qquad \text{(B)}$$

The conditions $\gamma(E)$, $\Gamma(E) > 0$ are necessary, otherwise the Breit-Wigner equation describes the unphysical case of a resonance whose amplitude increases with time.

From the discussion following Figure 1, $\frac{d(\mathrm{Re}\,\alpha(E))}{dE} > 0$, so the condition $\Gamma(E) > 0$ implies $\mathrm{Im}\,\alpha(E) > 0$, E positive. Thi relation is necessary if physical resonances are described by the Regge trajectory.

3.5(2) Resonances for negative E

Suppose a resonance, spin J, occurs for $E < 0$, say $E = -E$

If the above results for E positive are reworked for $E = -E'$, th

$$A_J(-E') = \frac{\gamma(-E')}{-E'+E'_J + i\Gamma(-E')/2};$$

$$\frac{\Gamma(-E')}{2} = \left[\frac{\mathrm{Im}\,\alpha(-E')}{\frac{d}{d(-E')}\,\mathrm{Re}\,\alpha(-E')}\right]_{E'=E'_J}$$

Physical resonances are described for $\gamma(E')$, $\Gamma(-E')$, and hence $\mathrm{Im}\,\alpha(-E') < 0$.

3.6 Parity and MacDowell Symmetry

Suppose a resonance occurs having spin J. Let this correspond to an S-matrix pole at energies E, -E' respectively. For each value of $\mathrm{Re}\,\alpha(E) = \mathrm{Re}\,\alpha(-E') = J$, define $\upsilon(E)$, $\upsilon(-E')$ such that $\mathrm{Re}\,\upsilon(E) = \mathrm{Re}\,\upsilon(-E') = J$ describes a state having parity $(-1)^{J+\frac{1}{2}}$, and $\mathrm{Re}\,\upsilon(E) = \mathrm{Re}\,\upsilon(-E') = -J-1$ describes a state with parity $-(-1)^{J+\frac{1}{2}}$. Further, the restrictions $\mathrm{Im}\,\alpha(E) > 0$, and $\mathrm{Im}\,\alpha(-E') < 0$ are necessary for physical resonances to be described.

Favella and Reineri(23) have shown that solutions of the Dirac equation exist in the region $\mathrm{Re}\,\upsilon > 0$, $x > 0$. The following theory depends on the assumption that restrictions on the potential $U(x)$ are such that the known solutions of the Dirac equation can be analytically continued into the region $\mathrm{Re}\,\upsilon < 0$.

3.6(1) The parity of positive energy resonances

Consider the resonances in $A_J(E)$ which occur for $\mathrm{Im}\,\upsilon(E) > 0$.

(i) $\mathrm{Re}\,\nu(E) = J$; $\nu(E) = \alpha(E)$, so $\mathrm{Im}\,\alpha(E) > 0$; parity $(-1)^{J+\frac{1}{2}}$.

Resonance state poles can occur in the S-matrix, provided

(3.4D)

$$\frac{dU/dx}{(E+m-U)^2} + \frac{2\mathrm{Re}\,\nu(E)}{x(E+m-U)} > 0 \ .$$

A typical potential $U(x)$ which gives rise to a resonance is shown in Figure 2 below. This consists of an attractive well with an outer repulsive region. This is necessary to hold in the resonance.

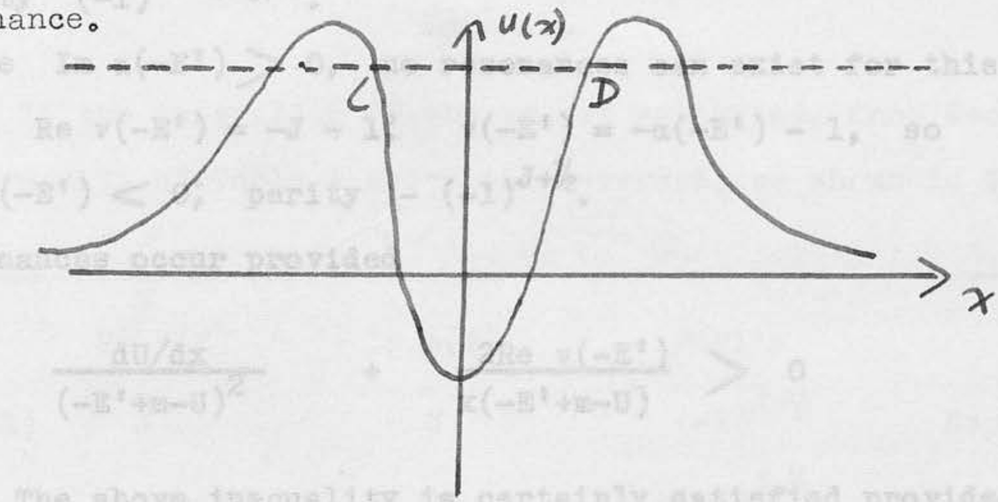


Figure 2

The resonance occurs in the region below the line CD. From the form of the potential, it is readily seen that in the resonance region $dU/dx > 0$ $(x > 0)$. Further, the expression $E + m - U(x) \rightarrow E + m$ = positive as $x \rightarrow \infty$. Since $E + m - U \neq 0$, otherwise the Dirac equation would have singularities between $x = 0$ and $x = \infty$, the expression $E + m - U$ remains positive.

• In this case, $dU/dx > 0$, $E+m-U > 0$, and $\mathrm{Re}\,\nu(E) = J > 0$, so the inequality is well satisfied, and a resonance occurs.

(ii) $\mathrm{Re}\,\nu(E) = J - 1$; $\nu(E) = -\alpha(E) - 1$, so $\mathrm{Im}\,\alpha(E) < 0$; parity $-(-1)^{J+1/2}$.

Since $\mathrm{Im}\,\alpha(E) < 0$, no resonance can exist for this parity.

3.6(2) <u>The parity of negative energy states</u>

Next, consider the resonances in $A_J(-E')$ for $\mathrm{Im}\,\nu(-E') > 0$.

(i) $\mathrm{Re}\,\nu(-E') = J$; $\nu(-E') = \alpha(-E')$, so $\mathrm{Im}\,\alpha(-E') > 0$; parity $(-1)^{J+1/2}$.

Since $\mathrm{Im}\,\alpha(-E') > 0$, no resonances can exist for this case.

(ii) $\mathrm{Re}\,\nu(-E') = -J - 1$; $\nu(-E') = -\alpha(-E') - 1$, so $\mathrm{Im}\,\alpha(-E') < 0$; parity $-(-1)^{J+1/2}$.

Resonances occur provided

$$\frac{dU/dx}{(-E'+m-U)^2} + \frac{2\mathrm{Re}\,\nu(-E')}{x(-E'+m-U)} > 0$$

The above inequality is certainly satisfied provided $(-E'+m-U) < 0$. If, however, $E' < m$, at $x = \infty$ the expressions $(-E'+m) > 0$, and the expression $-E'+m - U(x)$ remains positive. Thus the above inequality cannot be expected to hold for values of $E' < m$ (nucleon mass). This result is discussed more fully in section 3.7 on missing mass states.

If the inequality (3.4D) is satisfied, the above results may be summarised in Table 1.

	Re $\nu(E)$	Im $\nu(E)$	Parity	
$A_J(E)$	J	0	$(-1)^{J+\frac{1}{2}}$	Resonance
	$-J-1$	0	$-(-1)^{J+\frac{1}{2}}$	No Resonance
$A_J(-E')$	J	0	$(-1)^{J+\frac{1}{2}}$	No Resonance
	$-J-1$	0	$-(-1)^{J+\frac{1}{2}}$	Resonance

Table 1

If the inequality (3.4D) is not satisfied, from Section 3.4 the results of Table 1 above are reversed, as shown in Table 2.

	Re $\nu(E)$	Im $\nu(E)$	Parity	
$A_J(E)$	J	0	$(-1)^{J+\frac{1}{2}}$	No Resonance
	$-J-1$	0	$-(-1)^{J+\frac{1}{2}}$	Resonance
$A_J(-E')$	J	0	$(-1)^{J+\frac{1}{2}}$	Resonance
	$-J-1$	0	$-(-1)^{J+\frac{1}{2}}$	No Resonance

Table 2

Provided inequality (3.4D) holds, a resonance of spin J in $A_J(E)$, $E > 0$, occurs with parity $(-1)^{J+\frac{1}{2}}$ and a resonance of spin J in $A_J(-E')$ occurs with parity $-(-1)^{J+\frac{1}{2}}$.

This is a statement of MacDowell symmetry for the central field Dirac equation.

3.7 Missing Mass States

Evidence is given in this section for the possible non-occurrence of MacDowell symmetric states having negative E. If the above model has any relevance to the pion-nucleon system, this may explain why the MacDowell symmetric partner for the nucleon (the S_{11}) is missing on the N_β trajectory. Finally, the theory is generalised to include a complex potential, and the relevance to higher mass missing MacDowell symmetric states on the N_∂ and Δ_γ trajectories is discussed.

Suppose $E' < m$ so the inequality (3.4D) is not satisfied. If a resonance of spin J occurs in $A_J(E)$ having parity $(-1)^{J+\frac{1}{2}}$, from the results of Table 2, the MacDowell symmetric resonance in $A_J(-E')$ having parity $-(-1)^{J+\frac{1}{2}}$ will be absent. The resonance of parity $(-1)^{J+\frac{1}{2}}$ in $A_J(-E')$ will correspond physically to the resonance in $A_J(E)$.

The above results can hold only for $E' < m$. As E' increases, $(-E' + m - U) < 0$ and the inequality (3.4D) is re-established. Consequently, MacDowell symmetric states will start appearing for higher values of E', but the lowest order state will be missing. If potential scattering theory has any relevance to the pion-nucleon system, this may explain why the MacDowell symmetric partner for the nucleon (the S_{11}) on the N_β trajectory is not seen. The mass of the S_{11} has been estimated at 850 Mev., which is less than m (938 Mev.).

The above theory applies when the scattering is elastic and the potential $U(x)$ is real. If the Dirac equation is used to describe the higher pion-nucleon resonances, some degree of inelasticity must be allowed and the potential $U(x)$ becomes complex. The equations of section (3.4) may be reworked as follows.

$$-\frac{d}{dx}\left[\psi^+\Gamma\psi\right] = -2i\left[\frac{\mathrm{Im}(\nu+\frac{1}{2})}{x}\,\psi^+\sigma_1\psi + \frac{\mathrm{Im}\,U}{k}\,.\,\psi^+\psi\right] \quad \text{(A)}$$

$$-\frac{4}{t}\sinh\left[2\,\mathrm{Im}\left(\delta - \left[\nu+\tfrac{1}{2}\right]\frac{\pi}{2}\right)\right] =$$

$$2\left[\mathrm{Im}\,\nu\int_0^\infty dx\,k|g|^2\left\{\frac{d}{dx}\left(\frac{1}{x[E+m-U]}\right) + \frac{\mathrm{Re}[2\nu+1]}{x^2(E+m-U)}\right\}dx\right.$$

$$\left. + \frac{\mathrm{Im}\,U(x)}{k}\left\{|f|^2 + |g|^2\right\}\right] \quad \text{(B)}$$

Comparison of the above equation (3.7B) with equation (3.4C) shows that the presence of a complex potential makes the theory very complicated. The simple conditions (see (3.4B), (3.4E), and (3.4F)) which give rise to poles in the S matrix, are now lost.

If potential has relevance to the pion-nucleon system, an analysis similar to the above is necessary if the higher mass missing MacDowell symmetric states on the N_γ and Δ_γ trajectories are investigated.

3.8 Conclusion

MacDowell symmetry has been demonstrated in this chapter for the central field Dirac equation, and the possibility of missing mass states has been hinted. No calculations have been done for specific potentials. The calculation of Regge trajectories for real and complex potentials, and the demonstration of missing mass states is an interesting problem, but beyond the scope of this work.

It was also hoped that a possible form for $\mathrm{Im}\,\alpha(E)$ might be obtained suitable for use in a dispersion relation for $\alpha(E)$.

The expression, however,

$$-\frac{4}{t}\sinh\left[2\,\mathrm{Im}\left(\delta - \left[\nu + \tfrac{1}{2}\right]\frac{\pi}{2}\right)\right] = 2\,\mathrm{Im}\,\nu(E)\int_{0}^{\infty}\frac{\psi^{+}\sigma_{1}\psi}{x}\,dx$$

is complicated, and quite unsuitable for any parametrisations of $\mathrm{Im}\,\alpha(E)$.

CHAPTER IV

PARAMETRISATION OF $\alpha(w)$ AND CONFORMAL MAPPING TECHNIQUES

4.1 Introduction

In this chapter, parametrisations of the trajectory function $\alpha(w)$ are investigated. First, a conformal mapping technique is described which simplifies the calculations. Then simple forms of $\mathrm{Im}\,\alpha(w)$ are considered, and evenness or oddness of the resulting Regge trajectories is demonstrated. Next a parametrisation of $\mathrm{Im}\,\alpha(w)$ using the threshold condition is investigated, and a critical discussion of this method is given. The threshold form for $\mathrm{Im}\,\alpha(w)$ was used by Lyth[25] and Jones[26], who demonstrated the existence of threshold **cusps** in Regge trajectories. They considered models in which a threshold **cusp** caused a displacement of the N_α trajectory near the missing S_{11} state. Finally, a discussion is given of the difficulties of the above model.

4.2 The Dispersion Relation for $\alpha(w)$

Previous work (see section 2.2) has shown that the Regge trajectory function $\alpha(w)$ is analytic in w except for a cut for $-\infty \leq w \leq -(m+\mu)$, and $(m+\mu) \leq w \leq \infty$. The once subtracted dispersion relation satisfied by $\alpha(w)$ is

$$\alpha_+(w) = \alpha(0) + \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{\mathrm{Im}\,\alpha_+(w')\,dw'}{w'(w'-w)} - \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{\mathrm{Im}\alpha_-(w')\,.\,dw'}{w'(w'+w)}$$

where $\alpha_+(w) = \alpha(w)$

$$\alpha_-(w) = \alpha(-w)$$

$$\alpha_+(0) = \alpha_-(0) = \alpha(0)$$

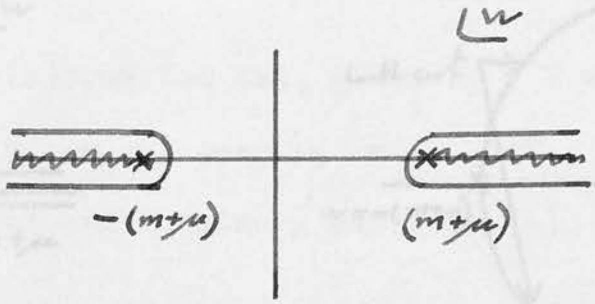


Fig. 1.

$\alpha(w)$ is a real analytic function of w, so for w real and $(m+\mu)$

$$\text{Im}\,\alpha_+(w) = \frac{1}{2i}\Big[\alpha(w+i\varepsilon) - \alpha(w-i\varepsilon)\Big]$$

$$= \frac{1}{2i}\Big[\alpha_+(w+i\varepsilon) - \alpha_+(w-i\varepsilon)\Big]$$

For w real and $> m+\mu$, $(w+i\varepsilon)$ defines the top part of the cut $m+\mu \leq w \leq \infty$. Similarly, $\text{Im}\,\alpha_-(w) = \text{Im}\alpha(-w)$

$$= \frac{1}{2i}\Big[\alpha(-w-i\varepsilon) - \alpha(-w+i\varepsilon)\Big]$$

$$= \frac{1}{2i}\Big[\alpha_-(w+i\varepsilon) - \alpha_-(w-i\varepsilon)\Big] \quad .$$

Further, $(-w-i\varepsilon)$ defines the lower part of the cut $-\infty \leq w \leq -(m+\mu)$. From the results of section 3.5, if an S matrix pole corresponds to a physical resonance:-

$\text{Im}\,\alpha(w) > 0$ along the top of the cut $(m+\mu) \leq w \leq \infty$

$\text{Im}\,\alpha(-w) < 0$ along the lower part of the cut $-\infty \leq w \leq -(m+\mu)$.

The above results imply $\text{Im}\,\alpha_+(w) > 0$, and $\text{Im}\,\alpha_-(w) < 0$ both along the top part of the cut $m+\mu \leq w \leq \infty$

4.3 Analyticity and Conformal Mapping

4.3(1) The conformal transformation

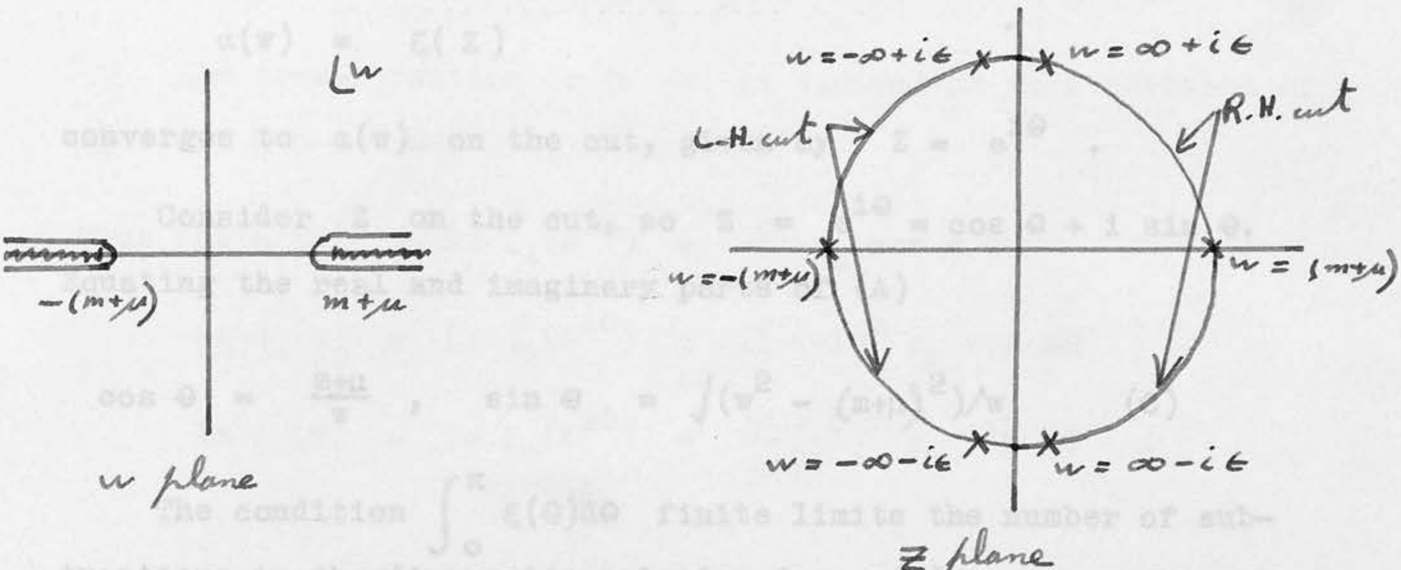


Figure 2.

Consider the w plane cut from $-\infty \leqslant w \leqslant -(m+\mu)$ and $(m+\mu) \leqslant w \leqslant \infty$. Then the transformation

$$Z = \frac{1 + i\sqrt{\left(\frac{w-m-\mu}{w+m+\mu}\right)}}{1 - i\sqrt{\left(\frac{w-m-\mu}{w+m+\mu}\right)}} \qquad \text{(A)}$$

transforms the cut w plane on to the interior of the unit circle $Z = 1$. The cuts map on to the boundary of the unit circle as shown in Figure 2. Critical points occur at $Z = \pm i$, for the mapping is no longer single valued at these points.

Atkinson[27] and Islam[28] have shown that provided

$$\int_0^\pi \xi(e^{i\theta})d\theta \qquad \text{(B)}$$

converges absolutely, then in a region free from branch points of $\alpha(w)$, the relation:-

$$\alpha(w) = \xi(Z)$$

converges to $\alpha(w)$ on the cut, given by $Z = e^{i\theta}$.

Consider Z on the cut, so $Z = e^{i\theta} = \cos\theta + i\sin\theta$. Equating the real and imaginary parts of (A)

$$\cos\theta = \frac{m+\mu}{w}, \quad \sin\theta = \sqrt{(w^2 - (m+\mu)^2)}/w \qquad (C)$$

The condition $\int_0^{\pi} \xi(\theta)d\theta$ finite limits the number of subtractions in the dispersion relation for $\alpha(w)$ to one. When the number of subtractions is more than one, linear, quadratic, and higher terms in w appear in the expression for $\mathrm{Re}\,\alpha(w)$.

$w = (m+\mu)/\cos\theta$ is divergent at $\theta = \pi/2$, and the above integral (B) is no longer convergent in the range $0 \leqslant \theta \leqslant \pi$.

Recently, Mandelstam[29] has used dispersion relations with two subtractions to generate infinitely rising Regge trajectories. The conformal mapping techniques described in this section cannot be applied to these dispersion relations.

The justification for the above conformal transformation is that $\alpha(w) = \xi(z)$ can be expanded as a power series in z inside the unit circle, with great mathematical simplifications.

$$\alpha(w) = \xi(z) = \sum_n a_n z^n \, .$$

On the cut, $z = e^{i\theta}$ so

$$\begin{aligned} \operatorname{Re} \alpha(w) &= \operatorname{Re} \xi(e^{i\theta}) = \sum_n a_n \cos n\theta \\ \operatorname{Im} \alpha_-(w) &= \operatorname{Im} \xi(e^{i\theta}) = \sum_n a_n \sin n\theta \end{aligned} \qquad \text{(D)}$$

The transformation $w \to -w$ is equivalent to a rotation of π, i.e. $\theta \to \pi + \theta$.

Thus
$$\begin{aligned} \operatorname{Re} \alpha_+(w) &= \operatorname{Re} \xi_+(e^{i\theta}) = \sum_n a_n \cos n\,\theta \\ \operatorname{Re} \alpha_-(w) &= \operatorname{Re} \xi_-(e^{i\theta}) = \sum_n (-1)^n a_n \cos n\theta \\ \operatorname{Im} \alpha_-(w) &= \operatorname{Im} \xi_-(e^{i\theta}) = \sum_n (-1)^n a_n \sin n\theta \quad . \end{aligned}$$

4.3(2) A possible self-consistency calculation

The above equations (D) suggest that a self-consistency calculation might be carried out in the following manner.

(i) Assume a form for $\operatorname{Im} \alpha_+(w)$ and calculate the expansion coefficients a_n from (D).

(ii) Substitute this form for $\operatorname{Im} \alpha_+(w)$ into the dispersion relation for $\operatorname{Re} \alpha_+(w)$. Once again calculate the coefficients a_n from the relations (D) for $\operatorname{Re} \alpha_+(w)$.

(iii) Compare the two solutions for α_n and equate them. The resulting equation does not involve α_n and may be solved for the unknown constants in $\operatorname{Im} \alpha_+(w)$ assumed initially.

The fallacy in this method is that the two values for the coefficient of α_n obtained in (i) and (ii) are exactly the same, and (iii) reduces to the trivial relation $0 = 0$. This is demonstrated below for the simple case $\operatorname{Im} \alpha(w) = (m+\mu)/w$.

Further, the calculations (i), (ii), and (iii) contain no dynamics other than unitarity and analyticity in the derivation of the dispersion relation for $\alpha_+(w)$. Thus no extra equations, and no new points on the trajectory can arise.

As an example, take $\mathrm{Im}\,\alpha_+(w) = (m+\mu)/w = \cos\theta$.

A Fourier sine series expansion of $\cos\theta$ in the range $0 \leqslant \theta \leqslant \pi$ gives for the coefficient a_n

$$\frac{\pi}{2}\,a_n = \frac{2n}{n^2-1} - \frac{2}{n^2-1}(-1)^p \; ; \quad \text{for } n = 2p+1 = \text{odd}$$

$$= 0 \quad (n \text{ even}).$$

Substitution of this value for $\mathrm{Im}\,\alpha_+(w)$ into the dispersion relation gives

$$\mathrm{Re}\,\alpha_+(w) = \alpha(0) + \frac{1}{\pi} - \frac{m+\mu}{w}\,\ell n\,\frac{(m+\mu)^2}{w^2 - {m+\mu}^2}$$

and a Fourier cosine expansion of this gives

$$\frac{\pi}{2}\,a_n = \frac{2n}{n^2-1} - \frac{2}{n^2-1}(-1)^p \quad ; \quad n = 2p+1 = \text{Odd}$$

$$= 0 \quad (n \text{ even}).$$

These two results for a_n are exactly the same as expected.

4.4 Parametrisation of the Trajectory Function

Simple parametrisations of $\text{Im}\,\alpha_{\pm}(w)$ which give rise to even and odd trajectories are discussed first in this section. Later, a form for $\text{Im}\,\alpha_{\pm}(w)$ which satisfies the threshold conditions is derived, and the resulting trajectory constrained to fit the nucleon trajectory. Two cases are considered:

(i) $\text{Re}\,\alpha_{+}(w_{\text{threshold}}) = \text{Re}\,\alpha_{-}(w_{\text{threshold}})$ (Exact symmetry)

(ii) $\text{Re}\,\alpha_{+}(w_{\text{threshold}}) = \alpha_{I}$; $\text{Re}\,\alpha_{-}(w_{\text{threshold}}) = \alpha_{II}$,

and α_{II} is adjusted until the Regge parameters fit the N (1688) width.

Simple parametrisations of $\text{Re}\,\alpha_{\pm}(w)$ may be obtained by using equation (2.3B), giving $f_1(\sqrt{s}, u)$ for backward pion-nucleon scattering. Comparison with experiment enables the unknown parameters in $\text{Re}\,\alpha_{\pm}(w)$ to be found, and details of this work are given in references 12 and 13. The latest results for the nucleon trajectory are[31]

$$\text{Re}\,\alpha(\sqrt{u}) = -0.38 + 0.88u \quad ; \quad \sqrt{u} \text{ in Mev.}$$

in the range u 1.5 Gev. Evidence is given for levelling off in the high u region.

These techniques do not give detailed information about the form of $\alpha_{\pm}(\sqrt{u})$, and to date only linear and parabolic forms for $\alpha_{\pm}(\sqrt{u})$ have been tried. Further study along these lines is not continued in this work.

4.4(1) Even and odd trajectories

The power series expansion of $\mathrm{Re}\,\alpha_{\pm}(w)$ given in section (4.2) is

$$\mathrm{Re}\,\alpha_{+}(w) = \mathrm{Re}\,\mathcal{E}_{+}(\theta) = a_o + a_1 \cos\theta + a_2 \cos 2\theta + \ldots.$$

$$\mathrm{Re}\,\alpha_{-}(w) = \mathrm{Re}\,\mathcal{E}_{-}(\theta) = \mathrm{Re}\,\mathcal{E}_{+}(\pi + \theta) = a_o - a_1 \cos\theta + a_2 \cos 2\theta.$$

In the presence of a subtraction at $w = 0$, the results are modified to

$$\mathrm{Re}\,\alpha_{+}(w) - \alpha(0) = \mathrm{Re}\,\mathcal{E}_{+}(\theta) - \alpha(0) = a_o + a_1 \cos\theta + a_2 \cos 2\theta + .$$

Two cases are considered:

(i) <u>Symmetrical trajectories, $\mathrm{Re}\,\alpha_{+}(w) = \mathrm{Re}\,\alpha_{-}(w)$</u>

The results of condition (i) give $a_1 = a_3 = a_5 = a_{2n+1} = 0$,

so $\mathrm{Im}\,\alpha_{+}(w) = \mathrm{Im}\,\mathcal{E}_{+}(\theta) = a_2 \sin 2\theta + a_4 \sin 4\theta \ldots.$

which is even under the interchange $w \rightarrow -w$. Thus the condition

$\mathrm{Re}\,\alpha_{+}(w) = \mathrm{Re}\,\alpha_{-}(w)$ (even trajectories)

requires the constraint (A)

$\mathrm{Im}\,\alpha_{+}(w) = \mathrm{Im}\,\alpha_{-}(w)$.

(ii) <u>Asymmetrical trajectories, $\mathrm{Re}\,\alpha_{+}(w) - \alpha(0) = -\mathrm{Re}\,\alpha_{+}(-w) - \alpha(0)$</u>

The trajectories are completely asymmetric about the point $w = 0$, $\mathrm{Re}\,\alpha_{+}(w) = \alpha(0)$.

The above condition (ii) requires the constraint (B)

$\mathrm{Im}\,\alpha_{+}(w) = -\mathrm{Im}\,\alpha_{-}(w)$.

The above results (A) and (B) were tested for simple forms of $\text{Im}\ \alpha_{\pm}(w)$.

(1) $\text{Im}\ \alpha_{+}(w) = \text{Im}\ \alpha_{-}(w) = c_o$.

The dispersion relation (4.1A) gives for $\text{Re}\ \alpha_{+}(w)$

$$\text{Re}\ \alpha_{+}(w) = \alpha(0) + \frac{c_o}{\pi} \ \ell n \left(\frac{(m+\mu)^2}{w^2 - m+\mu)^2}\right) \quad ; \quad w > (m+\mu)$$

which is symmetrical in w as required. The form of the coefficient of c_o is shown in Fig. 5.

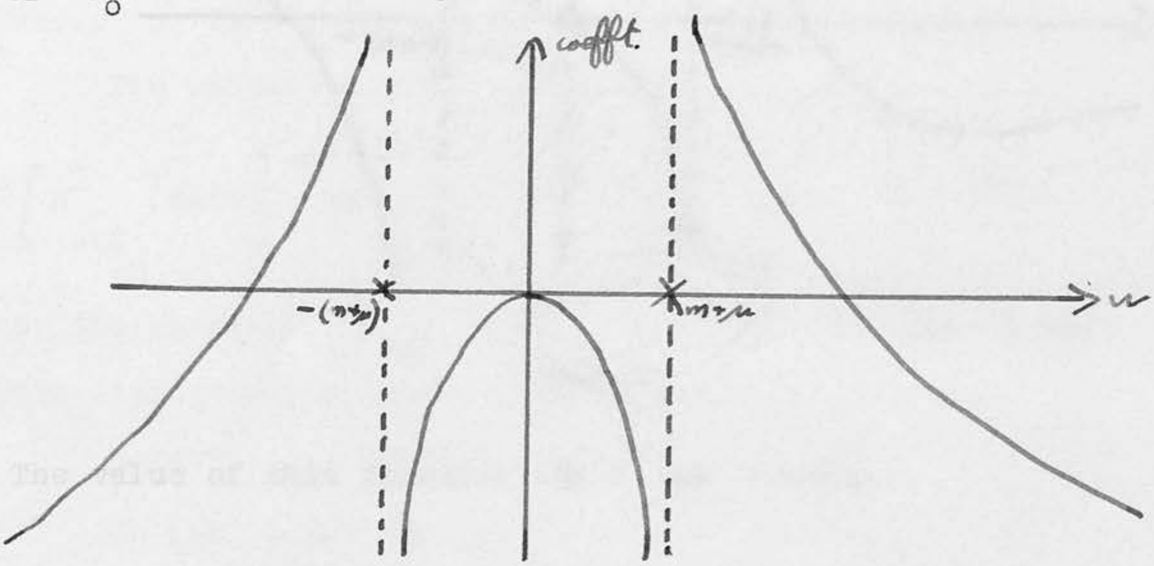


Fig. 5.

$\text{Re}\ \alpha_{\pm}(w) \xrightarrow{w \to \infty} \pm\infty$ logarithmically depending on whether $c_o \gtreqless 0$, so the once subtracted dispersion relation used in this work is capable of generating an infinitely rising Regge trajectory. This result is interesting in view of the recent speculation on infinitely rising trajectories[29, 30].

(2) $\quad \mathrm{Im}\, \alpha_+(w) = -\mathrm{Im}\, \alpha_-(w) = c_1/w.$

$$\mathrm{Re}\, \alpha_+(w) = \alpha_o + \frac{c_1}{\pi} \frac{1}{w} \ell n \left(\frac{(m+\mu)^2}{w^2 - (m+\mu)^2} \right) \quad ; \quad w > m+\mu$$

which is an odd trajectory as required. The form of the coefficient of c_1 is shown in Figure 6.

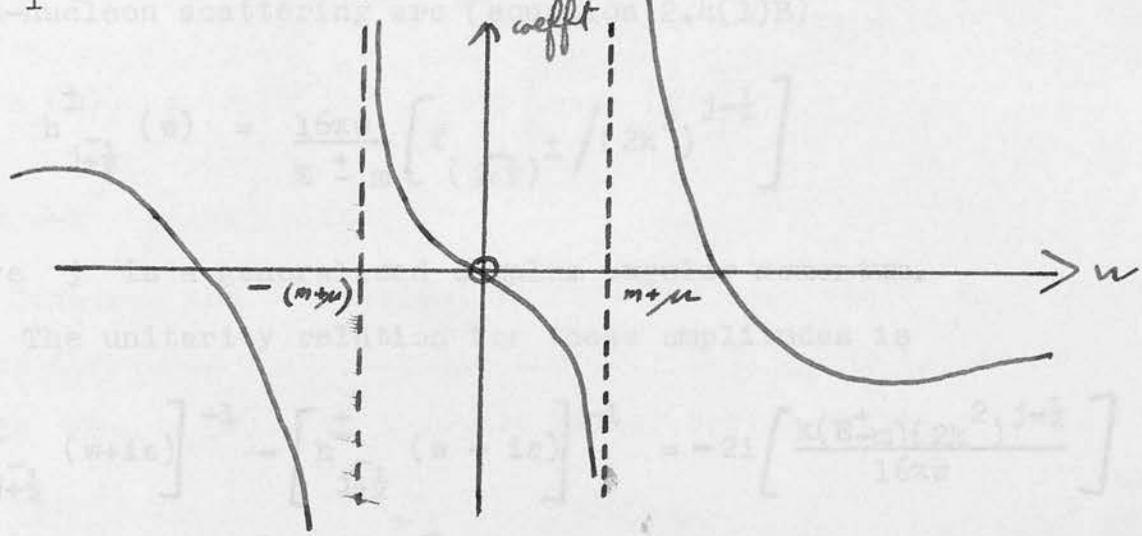


Fig. 6

The value of this function $\rightarrow 0$ as $w \rightarrow \infty$.

(3) $\quad \mathrm{Im}\, \alpha_+(w) = \mathrm{Im}\, \alpha_-(w) = c_2/w^2$.

Then $\mathrm{Re}\, \alpha_+(w) = \alpha(0) + \frac{c_2}{\pi} \frac{1}{w^2} \ell n \left(\frac{(m+\mu)^2}{w^2 - (m+\mu)^2} \right) - \frac{c_2}{\pi(m+\mu)^2}$

which is even in w as required.

Similar results follow for higher powers of $\frac{1}{w}$.

The divergence of these functions at threshold makes them unsuitable for parametrisation of $\alpha_{\pm}(w)$ in the lower energy region ($w \sim 1.5$ Gev). Other forms of $\mathrm{Im}\, \alpha(w)$ are now investigated notably the form obtained from the known threshold behaviour .

4.4(2) <u>The threshold approximation for $\operatorname{Im} \alpha_{\pm}(w)$</u>

The calculations described below follow Barut and Zwanziger[32], also Squires[11] (Chapters 2 and 3) for spinless particles.

The kinematic singularity free amplitudes of Singh[9] for pion-nucleon scattering are (equation 2.4(1)B)

$$h^{\pm}_{j\mp\frac{1}{2}}(w) = \frac{16\pi w}{E \pm m}\left[f^{\pm}_{(j\mp\frac{1}{2})} \Big/ (2k^2)^{j-\frac{1}{2}} \right]$$

where j is a generalised complex angular momentum.

The unitarity relation for these amplitudes is

$$\left[h^{\pm}_{j\mp\frac{1}{2}}(w+i\varepsilon) \right]^{-1} - \left[h^{\pm}_{j\mp\frac{1}{2}}(w-i\varepsilon) \right]^{-1} = -2i \left[\frac{k(E\pm m)(2k^2)^{j-\frac{1}{2}}}{16\pi w} \right] \qquad (A)$$

so the function $\left[h^{\pm}_{j\mp\frac{1}{2}}(w) \right]^{-1}$ has a cut for $|w| \geqslant m+\mu$ such that the discontinuity is given by equation (A) above.

However, use of the identities

$$\ell n(-k^2 - i\varepsilon) = \ell n(k^2) - i\pi$$

$$\ell n(-k^2 + i\varepsilon) = \ell n(k^2) + i\pi$$

shows that the function

$$\frac{1}{16\pi w}\left[\frac{(E\pm m)}{\cos\pi(j-\frac{1}{2})} \right] 2^{j-\frac{1}{2}} \exp\left[j\,\ell n(-k^2) \right] \qquad (B)$$

has the same discontinuity as $\left[h^{\pm}_{j\mp\frac{1}{2}}(w) \right]^{-1}$ for $(m+\mu) \leqslant w \leqslant w_{inel}$ and is regular for $w < (m+\mu)$.

Thus $\left[h^{\pm}_{j\mp\frac{1}{2}}(w) \right]^{-1}$ may be written

$$\left[h^{\pm}_{j\mp 1/2}(w)\right]^{-1} = \frac{1}{16\pi w}\left[\frac{(E\pm m)}{\cos\pi(j-\frac{1}{2})}\right] 2^{j-\frac{1}{2}}\left[Y(w,j) + \exp(j\,\ell n\left[-k^2\right])\right] \qquad \text{(C)}$$

where $Y(w,j)$ is a function which has no elastic cut and is analytic at $w = m+\mu$.

On a Regge trajectory $j = \alpha_{\pm}(w)$, and $\left[h^{\pm}_{j\mp 1/2}(w)\right]^{-1} = 0$, so

$$Y(w,\ \alpha_{\pm}(w)) = -(k^2)^{\alpha_{\pm}(w)} \exp\left[-i\pi\alpha_{\pm}(w)\right], \qquad \text{(D)}$$

since $-k^2 = -k^2 - i\varepsilon$.

Consider the "+" trajectory for $w \sim w_T$, where $w_T = (m+\mu)$ is the threshold value for the energy. Let $\alpha_+ \Rightarrow \alpha_I$ as w w_T. Expand $Y(w,\ \alpha_+(w))$ in a double Taylor series about $w = w_T$, $\alpha_+ = \alpha_I$.

$$Y(w,\ \alpha_+(w)) = Y(w_T,\ \alpha_I) + Y_2^+\left[\alpha_+(w) - \alpha_I\right] + Y_1^+\left[w - w_T\right] + \ldots$$

where $Y_1^+ = \left[\frac{\partial Y}{\partial w}(w,\ \alpha_+(w))\right]_{\substack{w=w_T\\ \alpha_+=\alpha_I}}$; $Y_2^+ = \left[\frac{\partial Y}{\partial j}(w,\alpha_+(w))\right]_{\substack{w=w_T\\ j=\alpha_I}}$.

Since $Y(w,j)$ is real and analytic at threshold, Y_1^+, Y_2^+ are real constants. Further, since $k^2 = 0$ at threshold, from equation (D) above

$$Y(w_T,\ \alpha_I) = 0 \qquad \text{(F)}$$

provided $\alpha_I > 0$.

The expansion (E) above gives for $\alpha_+(w)$

4.4(3) Parametrisation of the nucleon trajectory assuming exact symmetry

Exact symmetry between the trajectories $\alpha_+(w)$, $\alpha_-(w)$ gives that $\alpha_+(w_T) = \alpha_-(w_T) = \alpha_I$, and the threshold form for $\mathrm{Im}\,\alpha_+(w)$ from equation (4.3(2)K) above is

$$\mathrm{Im}\,\alpha_+(w) \sim \left[1 - \left(\frac{m+\mu}{w}\right)^2\right]^{\alpha_I}$$

$$\text{or} \quad \mathrm{Im}\,\alpha_+(w) = \mathrm{Im}\,\xi_+(\theta) = (\sin^2\theta)^{\alpha_I} .$$

The parametrisation

$$\mathrm{Im}\,\alpha_+(w) = \mathrm{Im}\,\xi_+(\theta) = (\sin^2\theta)^{\alpha_I}\left[c_1 + c_2 \cos\theta\right] \qquad \text{(L)}$$

c_1, c_2 constant, satisfies the threshold condition and allows for asymmetry between $\alpha_+(w)$ and $\alpha_-(w)$. A similar parametrisation was used by Islam[28] for the ρ meson trajectory, and is investigated below for the nucleon trajectory.

The form (L) for $\mathrm{Im}\,\alpha_+(w)$ is substituted into the dispersion relation 4.1 A, and a value of $\mathrm{Re}\,\alpha_\pm(w)$ is obtained subject to the constraints

$$\begin{aligned} \mathrm{Re}\,\alpha_+(938) &= \tfrac{1}{2} \\ \mathrm{Re}\,\alpha_+(1688) &= 5/2 \\ \mathrm{Re}\,\alpha_-(1670) &= 5/2 \end{aligned} \qquad \text{(M)}$$

If a linear trajectory in $s = w^2$ passes through the nucleon (938) and the $N^+(1688)$, the value of α_I is

$$\alpha_I = 0.7865 \qquad \text{(N)}$$

The expressions (L) for $\mathrm{Im}\,\alpha_+(w) = \mathrm{Im}\,\xi_+(\theta)$ represent truncations of the series $\mathrm{Im}\,\xi_+(\theta) = \sum_{n=1}^{\infty} a_n \sin n\theta$. Hence the value of α_I used in the parameterisation of $\mathrm{Im}\,\alpha_{\pm}(w)$ will not in general agree with the value $\alpha_I = \mathrm{Re}\,\alpha(w_T)$ obtained from the dispersion relation.

Substitution of the forms (L) above into the dispersion relation for $\alpha_{\pm}(w)$ gives

$$\alpha_{+}(w) = \alpha(0) + X(w)\left[c_1 + c_2\left(\frac{m+\mu}{w}\right)\right]$$

$$\text{where}\quad X(w) = \frac{w^2}{\pi}\int_{m+\mu}^{\infty} dw'\left(1 - \frac{m+\mu}{w'}^{2}\right)^{\alpha_I} \frac{1}{w'(w'^2-w^2)}$$

$$\text{Put}\quad \frac{m+\mu}{w'} = \cos\theta\,,$$

$$\therefore\ X(w) = Z\,J(w) \qquad \text{where}$$

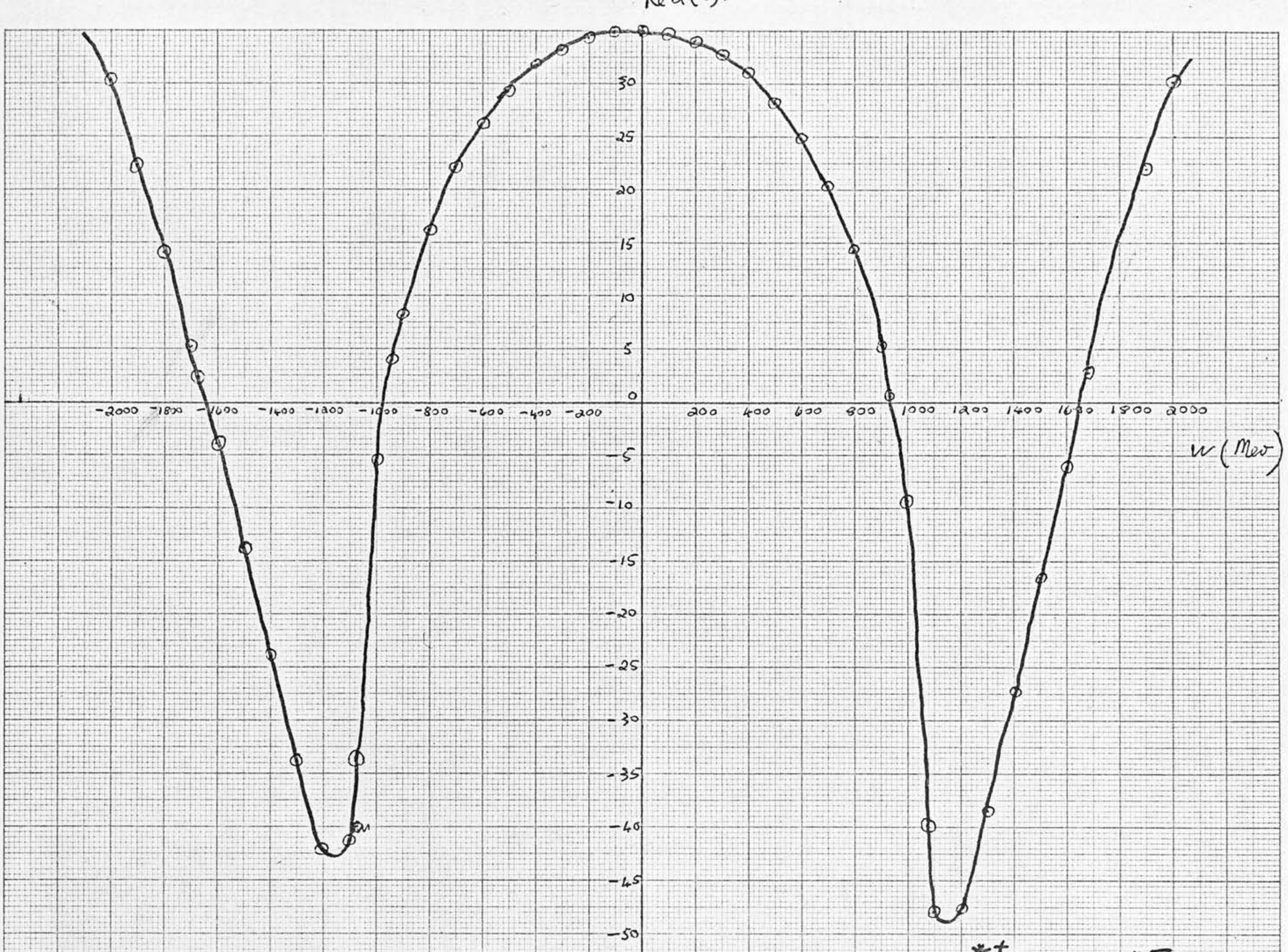
$$Z = \frac{w^2}{(m+\mu)^2-w^2}\,, \quad \text{and}$$

$$J(w) = \int_0^{\pi/2} d\theta\,\frac{(\sin\theta)^{2\alpha_I+1}\cos\theta}{1 - Z\sin^2\theta}$$

$$= \frac{\Gamma(\alpha_I+1)}{2\Gamma(\alpha_I+2)}\,F(1,\ \alpha_I+1,\ \alpha_I+2,\ Z)$$

(see reference 33, Section 2.12, p. 115, equation 7), where F is the hypergeometric function.

The above integral defined by $J(w)$ is uniformly convergent in any closed domain of the Z plane cut along the real axis from 1 to ∞. This implies cuts in w for $-\infty \leq w \leq -(m+\mu)$, and $(m+\mu) \leq w \leq \infty$, so the analyticity structure of $X(w)$, hence $\alpha_{+}(w)$ is explicitly demonstrated.



Regge trajectory for the threshold approximation passing through $N, N^{*+}(1688), N^{*-}(1670)$ and for which exact parity degeneracy occurs at threshold.

The integral defined by $X(w)$ becomes a principal value integral when $\mathrm{Re}\ \alpha_+(w)$ is calculated.

Behaviour of the trajectory as $w \rightarrow \infty$

The most divergent term in $X(w)$ is

$$X(w) \sim \frac{P}{\pi}\int_{m+\mu}^{\infty} dw' \left[\frac{1}{w'-w} + \frac{1}{w'+w} - \frac{2}{w'}\right]$$

$$= \frac{1}{\pi}\left[-\ell n\left(\frac{w^2-(m+\mu)^2}{(m+\mu)^2}\right)\right]$$

so if $c_1 > 0$ the trajectory diverges logarithmically to $-\infty$ and if $c_1 < 0$, it diverges logarithmically to $+\infty$. Hence the condition $c_1 < 0$ is necessary for an infinitely rising Regge trajectory.

Discussion of the results

The equations (M) may be fitted to the above parametrisation, giving the results

$$\begin{aligned} \alpha_I &= 0.7865 \\ c_1 &= -208.9671 \\ c_2 &= -8.60188332 \\ \alpha(0) &= 34.9202044 \quad . \end{aligned}$$

The principal value integrals were evaluated using a method described in Appendix III. All calculations were done on the English Electric KDF 9 computer at the Regional Centre in Edinburgh.

The trajectory for this parametrisation is shown opposite, and the following points of interest may be noted.

(i) The trajectory becomes large and negative, and has cusps at $w = \pm (m+\mu)$ respectively. This is readily seen from the threshold form of $\mathrm{Re}\, \alpha_{\pm}(w)$ (equation 4.3(2)G).

$$\mathrm{Re}\, \alpha_+(w) = \alpha(0) - \left[w - w_T \right] \frac{Y_1^+}{Y_2^+} - (k^2)^{\alpha_I} \cos \pi \alpha_I \frac{1}{Y_2^+} .$$

$\alpha_I = 0.7865 < 1$ for the nucleon trajectory, so the above equation has a cusp at threshold. For $\alpha_I < 0$, the above value of $\mathrm{Re}\, \alpha_+(w) \to -\infty$ as $w \to$ threshold. Similar results hold for the trajectory $\mathrm{Re}\, \alpha_-(w)$.

(ii) Since $c_1 < 0$, the trajectory is infinitely rising, and diverges logarithmically to $+\infty$.

(iii) The trajectory predicts a MacDowell symmetric partner for the nucleon at $w \sim 970$ Mev.

(iv) The width of the N (1688) is predicted to be

$$2 \mathrm{Im}\, \alpha_+(1688) / \left[\frac{d\alpha_+}{dw}(w) \right]_{w=1688} \sim -1000 \text{ Mev.}$$

The value quoted by Donnachie, Kirsopp and Lovelace[19] is $= 177$ Mev, so the predicted value has the wrong sign and is six times too large.

(v) The value of $d\alpha_+/dw$, $d\alpha_-/dw$ is of the order ten times too large to give satisfactory predictions of possible higher spin resonances for $w \gtrsim 1700$ Mev.

(vi) The value of $\mathrm{Im}\, \alpha_+(w) < 0$ for all values of $w > 0$. However, from the work of Chapter 3, $\mathrm{Im}\, \alpha_+(w) > 0$, $w > 0$, for the system to describe physical resonances.

(vii) The value of $\alpha(0)$ is ten times larger than the latest value given by Barger and Cline[31], and also suggested from the Chew Frautschi plot.

The only satisfactory prediction for this parametrisation is (ii), but in view of the other unsatisfactory features, such a parametrisation appears to have little relevance to physics.

4.3(4) Parametrisation of the nucleon trajectory which fits the N^* width

The previous section has shown that the assumption of exact symmetry for the trajectories appears to have little physical importance. Another possible parametrisation tried was the MacDowell symmetric form of $\mathrm{Im}\,\alpha_{\pm}(w)$ which has different threshold values $\mathrm{Im}\,\alpha_+(w_T) = \alpha_I$, and $\mathrm{Im}\,\alpha_-(w_T) = \alpha_{II}$.

$$\mathrm{Im}\,\alpha_{\pm}(w) = \left[1 \mp \left(\frac{m+\mu}{w}\right)\right]^{\alpha_I} \left[1 \pm \left(\frac{m+\mu}{w}\right)\right]^{\alpha_{II}} \left[c_1 \pm c_2 \left(\frac{m+\mu}{w}\right)\right] \qquad (P)$$

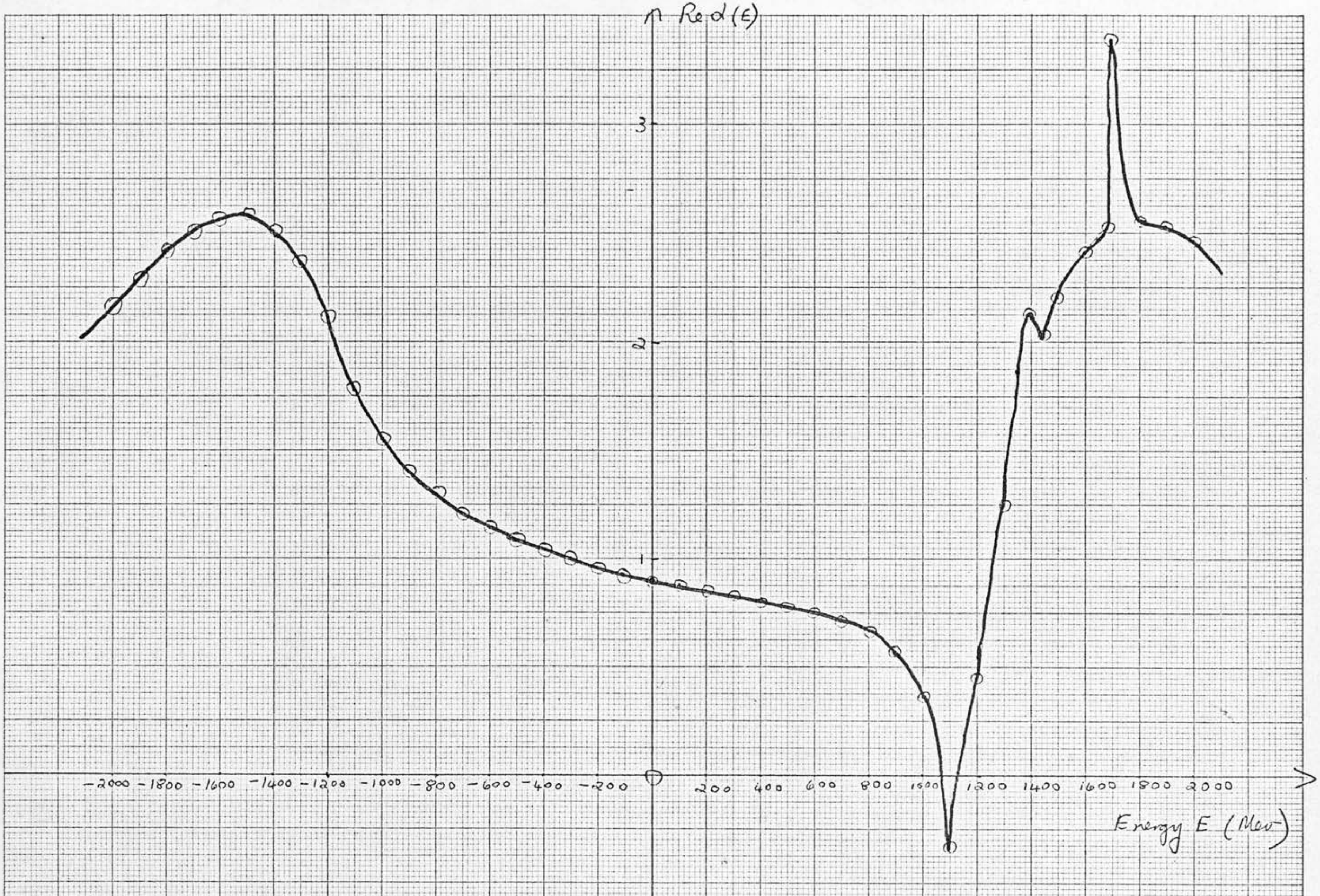
The value of α_{II} is varied until the correct value for the $N^*(1688)$ width is obtained (Γ = 177 Mev). α_I is assumed fixed at $\alpha_I = .7865$.

The substitution of equation (P) for $\mathrm{Im}\,\alpha_{\pm}(w)$ into the dispersion relation for $\mathrm{Re}\,\alpha_{\pm}(w)$ gives

$$\mathrm{Re}\,\alpha_+(w) = \alpha(0) + c_1\, D_1(w) + c_2\, D_2(w) \qquad (Q)$$

$$D_1(w) = \frac{w}{\pi}\int_{m+\mu}^{\infty} \frac{dw'}{w'}\left[A^{\alpha_I}\, B^{\alpha_{II}}\, \frac{1}{w'-w} - A^{\alpha_{II}}\, B^{\alpha_I}\, \frac{1}{w'+w}\right]$$

$$D_2(w) = \frac{w(m+\mu)}{\pi}\int_{m+\mu}^{\infty} \frac{dw'}{w'^2}\left[A^{\alpha_I}\, B^{\alpha_{II}}\, \frac{1}{w'-w} + A^{\alpha_{II}}\, B^{\alpha_I}\, \frac{1}{w'+w}\right]$$



Regge trajectory for threshold approximation which passes through N, $N^{*+}(1688)$, $N^{*-}(1670)$, and fits the $N^{*+}(1688)$ width

$$A = (1 - \frac{m+\mu}{w}) \; ; \quad B = (1 + \frac{m+\mu}{w}) \,.$$

The four unknowns c_1, c_2, $\alpha(0)$, α_{II} occurring in equation (Q) are obtained from the four constraints.

$$\begin{aligned}
\tfrac{1}{2} &= \alpha(0) + c_1 D_1(938) + c_2 D_2(938) \\
\tfrac{5}{2} &= \alpha(0) + c_1 D_1(1688) + c_2 D_2(1688) \\
\tfrac{5}{2} &= \alpha(0) + c_1 D_1(-1670) + c_2 D_2(-1670) \\
177 &= 2\mathrm{Im}\,\alpha_+(1688) \Big/ \left[\frac{d\alpha_+(w)}{dw}\right]_{w=1688}
\end{aligned} \qquad (R)$$

The equations (Q) for $\mathrm{Re}\,\alpha_\pm(w)$ are fitted to the constraints (R), and the following values obtained for the parameters.

$$\begin{aligned}
\alpha(0) &= 0.885488571 \\
c_1 &= 6.41091143 \\
c_2 &= -9.70803843 \\
\alpha_{II} &= 2.11748
\end{aligned}$$

The integrals $D_1(w)$, $D_2(w)$ are

(i) divergent if either α_I or $\alpha_{II} \leqslant 1$

(ii) divergent at $w = (m+\mu)$ if $-1 < \alpha_I < 0$

(iii) have threshold cusps at $w = (m+\mu)$ if $0 \leqslant \alpha_I < 1$.

Similar results hold at $w = -(m+\mu)$ for α_{II}.

As before, the trajectory is plotted out as shown and the following results are noted.

(i) A threshold cusp occurs at $w = +(m+\mu)$, and not at $w = -(m+\mu)$.

(ii) Since $c_1 > 0$, the trajectories will turn over and diverge logarithmically to $-\infty$. The highest spin value reached by the trajectory is $j = 3.3$ for $w = 1700$ Mev ($w > 0$) and $j = 2.6$ for $w = -1500$ Mev ($w < 0$). Hence the Regge trajectory cannot reach any higher spin resonance.

(iii) There is no nucleon MacDowell symmetric partner. Rather, there is a particle predicted of spin $5/2$ around $w = 950$ Mev.

(iv) The value of $\mathrm{Im}\,\alpha_+(w) > 0$ in the range $(m+\mu) < w < 1.5385\,(m+\mu)$. Otherwise it is < 0. Similarly $\mathrm{Im}\,\alpha_-(w) < 0$ for all $w > (m+\mu)$. Thus physical resonances on $\alpha_+(w)$ can be described in the range $\lesssim 2$ Gev, and resonances on $\alpha_-(w)$ occur for all $w > (m+\mu)$.

(v) $\alpha(0) = 0.885488571$ which is the same order of magnitude as the value obtained from Barger and Cline[31], but has the wrong sign.

The above results show that the above parametrisation gives reasonable physical predictions in the range $w \lesssim 2$ Gev, but fails hopelessly in the higher energy region. The trajectory turns over and does not reach any of the higher resonances.

4.3(5) Discussion of the above parametrisation

The above results are in general unsatisfactory, and cast grave doubts on the possibility of using the threshold condition to obtain physically meaningful Regge trajectories.

The threshold approximation has been discussed in potential scattering by Warburton[34], who investigated the case of a repulsive Yukawa potential having a short range attractive core. For spinless particle scattering

$$\mathrm{Re}\,\alpha(k^2) = \alpha(0) + A\,k^2 - B\,(k^2)^{\alpha(k^2)+\frac{1}{2}}\cos\pi(\alpha[k^2]+\tfrac{1}{2})$$

at threshold.

He found that the above term which gives the leading order behaviour at threshold, is small compared with other terms in the trajectory when the phase shifts are still of order only 10^{-4}.

Potential scattering shows that the expected threshold behaviour of Regge trajectories has no practical significance at points even slightly away from threshold. If potential scattering has any relevance to relativistic S matrix theory, the use of parametrisations 4.3(3)L and 4.3(4)R at points even slightly away from threshold must be viewed with extreme caution. The work in sections (4.3(3) and 4.3(4) shows that the threshold parametrisation has little relevance to the nucleon trajectory.

Other forms of $\text{Im}\ \alpha_{\pm}(w)$ may be considered as a means of parametrising Regge trajectories. Ahmadzadeh and Sakmar[35] considered the form

$$\text{Im}\ \alpha(x) \quad = \quad \frac{cx^{\lambda}}{c_1+(x-c_2)^2}$$

which was obtained from the Schrodinger equation so describes meson trajectories. c, c_1, c_2, λ are parameters which are obtained from experimental information about the trajectories. An open question is whether or not a similar form for $\text{Im}\ \alpha$ for fermion trajectories can be obtained from the Dirac equation.

4.4 Status of the missing MacDowell symmetric partner S_{11} of the nucleon

Recently, attempts have been made to parametrise the nucleon trajectory close to threshold using the threshold behaviour of $\text{Im}\ \alpha_{\pm}(w)$ as described above. (See Lyth[36], Jones[37])

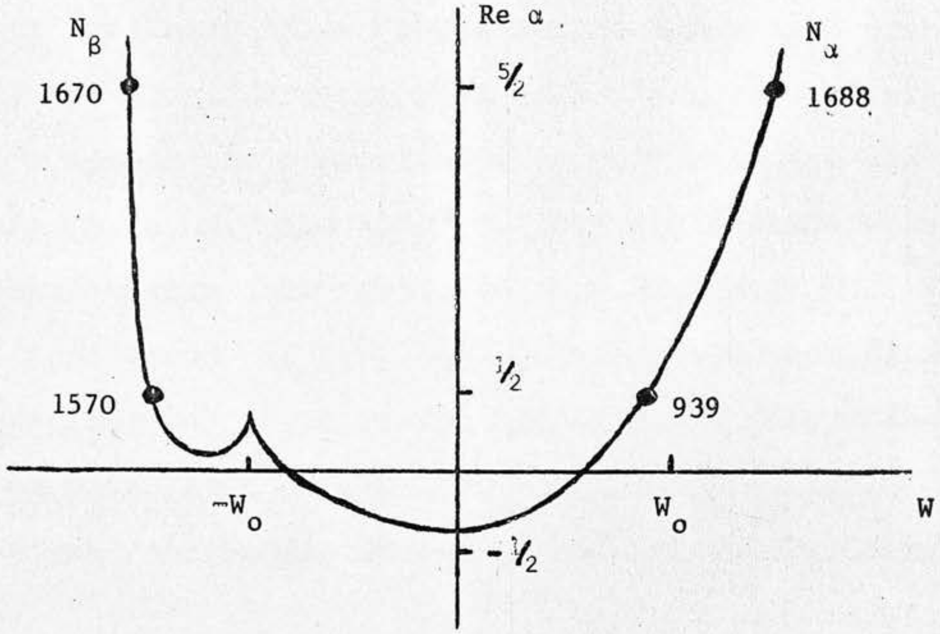
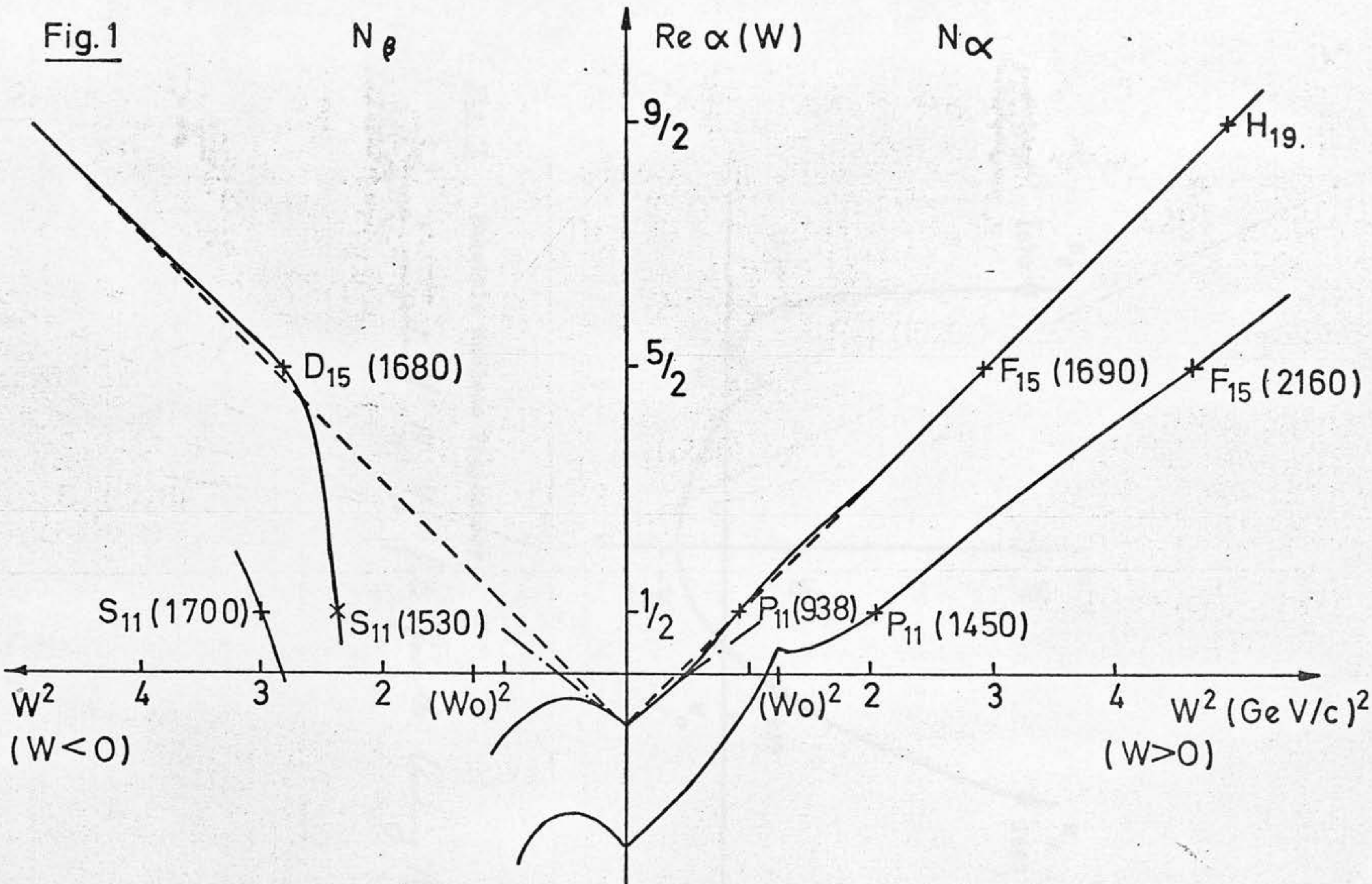


Fig. 1 - Possible Nucleon Trajectory

Jones' conjectured N_α, N_β fits for the leading N_α, N_β trajectories.

Fig. 1



Conjectured N_α, N_β trajectory of Lyth. The line ---- is the fit of Barger and Cline. The line —·—·— is the experimental slope and intercept at w = 0. The full line is Lyth's conjectured leading and daughter trajectories

In Lyth's preprint $\alpha_{II} < 0$ so that $\mathrm{Re}\,\alpha_{-}(m+\mu) \rightarrow -\infty$. In Jones' preprint a cusp is assumed in $\alpha_{-}(w)$ at $w = (m+\mu)$. This effect produces a distortion of the highest ranking N_{α} trajectory, so the highest S_{11} state (1591 Mev) is assigned to the leading trajectory, rather than the first daughter, as in the fits of Barger and Cline(20). Their results are shown opposite.

4.4(1) <u>Difficulties of the theory</u>(38)

The main difficulty of a theory of threshold cusps lies in explaining the relatively large width of the S_{11}. Donnachie, Kirsopp, and Lovelace(19) give a value $\Gamma(S_{11}) = 268$ Mev. This is nearly twice the width of the F_{15} ($\Gamma = 177$ Mev) and the D_{15} ($\Gamma = 173$ Mev).

However

$$\Gamma/_2 = \mathrm{Im}\,\alpha(w) \Big/ \left[\frac{d\alpha(w)}{dw}\right]_{w=w_R}$$

so if $(d\alpha(w)/dw)_{w=w_R}$ is large, then the width will tend to be small. In Lyth's preprint the Regge trajectory is diverging to $-\infty$ through the S_{11} (1591), and the value of $d\alpha(w)/dw\big|_{w=w_R}$ is very large. In Jones' preprint, the S_{11} passes through a point on the trajectory above the highest point of the cusp, and the values of

$$d\alpha(w)/dw\Big|_{S_{11}}, \quad d\alpha(w)/dw\Big|_{F_{15}}, \quad d\alpha(w)/dw\Big|_{D_{15}}$$

are approximately the same.

The above parametrisations have difficulty in explaining the large width of the S_{11}, unless $\mathrm{Im}\,\alpha_{-}(1591)$ is abnormally large.

An argument against this possibility may be seen from the work of Chiu and Stack[12] who considered the form of backward π^+p scattering at fixed large s and variable u. Two Regge trajectories contribute to this, the N and the Δ. The contribution of the Δ is small and may be ignored, leaving N as the dominant trajectory.

The equation for the amplitude $f(\sqrt{s}, u)$ has been given previously by equation 2.3B. The combination $\gamma(w)\Gamma(\alpha+1)$ is a smooth function without poles. Consider the combination

$$\left(1 + \eta \exp(-i\pi\left[\alpha-\tfrac{1}{2}\right]\right) \quad x \quad \frac{1}{\cos\pi\alpha\,\Gamma(\alpha+\frac{1}{2})} \qquad (S)$$

This contributes either a pole or a finite quantity for α a positive half integer according to whether the trajectory has "right" or "wrong" signature. When α is a negative half integer equation (S) contributes either a finite quantity or zero, depending on the signature's being right or wrong respectively. Zeros in the amplitude resulting from negative half integer values of α occurring at wrong signature points, are called "wrong signature nonsense points".

The point $\alpha(w) = -\frac{1}{2}$ is a wrong signature point for the nucleon trajectory. If there exists a value of w such that $\alpha_N(w)$ and $\alpha_N(-w)$ are both near $j = -\frac{1}{2}$, the contribution of the nucleon trajectory to the scattering will vanish, and a large dip occurs in the cross-section at this point.

Since a large dip in π^+p backward scattering occurs at $u \sim -0.2\ (\text{Gev})^2$, then

$$\mathrm{Re}\,\alpha_N(\sqrt{-0.2}) = \mathrm{Re}\,\alpha_N(\sqrt{0.2}) \sim -\tfrac{1}{2}$$

$$\mathrm{Im}\,\alpha_N(\sqrt{-0.2}) = \mathrm{Im}\,\alpha_N(\sqrt{0.2}) \sim 0\,.$$

As w increases from $\sqrt{(0.2)}$ to 1591, 1670, 1688 Mev the value of $\mathrm{Im}\,\alpha_N(w)$ is also expected to increase from zero, since the width of $F_{15}(1688)$ is slightly larger than the width of the $D_{15}(1670)$. The width of the $S_{11}(1591)$ would be expected smaller than the width of the D_{15}. Just the opposite is observed, since $\Gamma(S_{11}) = 268$ Mev, $\Gamma(D_{15}) = 173$ Mev, and $\Gamma(F_{15}) = 177$ Mev.

4.5 Conclusion

The possibility of parametrisations of $\alpha_{\pm}(w)$ has been investigated in this chapter, and special attention has been paid to the threshold parametrisation. The results show that extreme caution must be observed if any conclusion is drawn from this at points away from threshold. Further, the method of threshold cusps to explain the missing S_{11} on the leading N_β trajectory has been discussed critically. The explanation of the large width of the $S_{11}(1591)$ presents grave difficulties to this theory.

Another possible explanation of the missing S_{11} is the vanishing of the Regge residue function at exactly this point on the leading trajectory. This effect is considered in the next chapter on dynamical models.

CHAPTER V

DYNAMICAL MODELS FOR REGGE TRAJECTORIES

5.1 Introduction

Field theoretic models for fermion Regge trajectories are discussed in this chapter. The basis of the theory is a series of papers by Gell-Mann et al.[2], where it is shown that under certain conditions, Regge behaviour can be obtained by the iteration of Feynman diagrams. First of all, a brief survey is given of the above theory, and the concepts of "Reggeisation", "sense", and "nonsense" are discussed. Later, dynamical models describing the pion-nucleon interaction are considered, and the residue functions and the trajectories calculated. The resulting trajectories are examined for dips and cusps around the $S_{11}(1591)$ resonance, and the Regge residue functions are examined for zeros at this point.

5.2 The Theory of Reggeisation

Work done by Gell-Mann et al.[2] has shown that, under certain conditions, a Born approximation pole in a Feynman graph corresponding to elementary particle exchange, may lie on a Regge trajectory. The conditions required are that the system contains a "nonsense" channel, and the Born approximation residue factorises. The method used is to show that non continuable terms in the complex j plane are cancelled by similar terms from higher order Feynman graphs, and the resulting amplitude is shown to exhibit simple Regge behaviour.

The above process in which a fixed pole corresponding to the exchange of an elementary particle is made to lie on a Regge trajectory, is called "Reggeisation", and the conditions for Reggeisation are given in the previous paragraph.

5.2(1) Sense and Nonsense Amplitudes

The notation is that of GGLZ[2], and has been given in Chapter 1.

Consider the reaction $a + b \rightarrow c + d$, where a, b, c, d denote the helicities of particles a, b, c, d. Let $\lambda = a-b$, $\mu = c-d$, and J be the total (real) angular momentum for the system. λ, μ represent the helicity of the composite system, and for physical ("sense") amplitudes, $\lambda, \mu \leq J$. Amplitudes for which $\lambda, \mu > J$ cannot exist physically and are known as "nonsense" amplitudes. Thus "sense" states have $\lambda, \mu \leq J$ and can occur physically, while "nonsense" states have $\lambda, \mu > J$ and cannot occur in physical processes.

Gell-Mann et al.[2] considered the scattering of a neutral vector meson (a, c) from a nucleon (b, d) and defined $S_a = S_c = 1$, $\ell = J - \frac{1}{2}$, $b = d = \frac{1}{2}$, $F_{a\frac{1}{2}c\frac{1}{2}} = F_{ac}$, and $f_{a\frac{1}{2}c\frac{1}{2}} = f_{ac}$.

At $\ell = 0$ the channels having $a, c = 0, 1$ are sense, and those having $a, c = -1$ are nonsense. Greek letters κ, ν, describe the sense states, so a sense-sense transition is described by $f_{\kappa\nu}$, and a nonsense-nonsense transition is f_{-1-1}. Similarly, the parity conserving partial wave amplitudes corresponding to these transitions are $F_{\kappa\nu}$, $F_{-1\nu}$, and F_{-1-1} respectively.

Let a Regge pole occur at a value $\ell = \alpha(w)$. Previous work by Gell-Mann[39] has shown that the sense and nonsense amplitudes become decoupled at $\ell = 0$, so a Regge pole occurs entirely in the sense amplitude or in the nonsense amplitude.

The trajectory is said to choose sense if the Regge pole occurs in the sense amplitude, and Gell-Mann et al.[2] have shown that near a Regge pole

$$\begin{aligned} F_{K0} &\sim \eta_K \eta_0 / [\ell - \alpha(w)] \\ F_{-1} / [\ell(\ell+2)]^{\frac{1}{2}} &\sim \zeta_{-1} \eta_0 / [\ell - \alpha(w)] \\ F_{-1-1} &\sim (\zeta_{-1})^2 \, \alpha(w) [\alpha(w) + 2] / [\ell - \alpha(w)] \end{aligned} \qquad \text{(A)}$$

The residues η_K corresponding to the sense amplitudes approach finite constants as $\alpha(w) \rightarrow 0$, while the nonsense residues $\zeta_{-1} [\alpha(w)(\alpha(w) + 2)]^{\frac{1}{2}} \rightarrow 0$ as $[\alpha(w)]^{\frac{1}{2}}$ as $\alpha(w) \rightarrow 0$.

Similarly, if the trajectory chooses nonsense at $\ell = 0$

$$\begin{aligned} F_{K0} &\sim (\zeta_{-1})^2 \, \alpha(w) [\alpha(w) + 2] / [\ell - \alpha(w)] \\ F_{-10} / [\ell(\ell+2)]^{\frac{1}{2}} &\sim \zeta_{-1} \eta_0 / [\ell - \alpha(w)] \\ F_{-1-1} &\sim \eta_K \eta_0 / [\ell - \alpha(w)] \quad . \end{aligned} \qquad \text{(B)}$$

When the Regge trajectory chooses sense, near $\alpha(w) = 0$ the above amplitudes become

$$\begin{aligned} F_{K0} &\sim -(\eta_K \eta_0 \alpha^{-1}) \, \delta_{0\ell} \\ F_{-10} / [\ell(\ell+2)]^{\frac{1}{2}} &\sim \zeta_{-1} \eta_0 \, \ell^{-1} \\ F_{-1-1} &\sim 2(\zeta_{-1}^2 \, \alpha) \ell^{-1} \quad . \end{aligned} \qquad \text{(C)}$$

5.2(2) Regge behaviour and Feynman graphs

Gell-Mann et al.[2] illustrated the above concepts of Reggeisation, sense, and nonsense by considering the Born graph

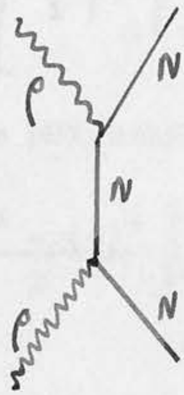


Fig. 1

It may be shown that the partial wave amplitudes for the above graph at large $Z = \cos\theta$ are given by formulae similar to equation 5.2(1)C. Thus the Born approximation at large Z corresponds exactly to the contribution of a sense choosing Regge pole with $\alpha \rightarrow 0$. Further study shows that this Regge pole chooses sense at $\ell = 0$, $w = m_0$ (nucleon mass) and thus corresponds to the physical nucleon.

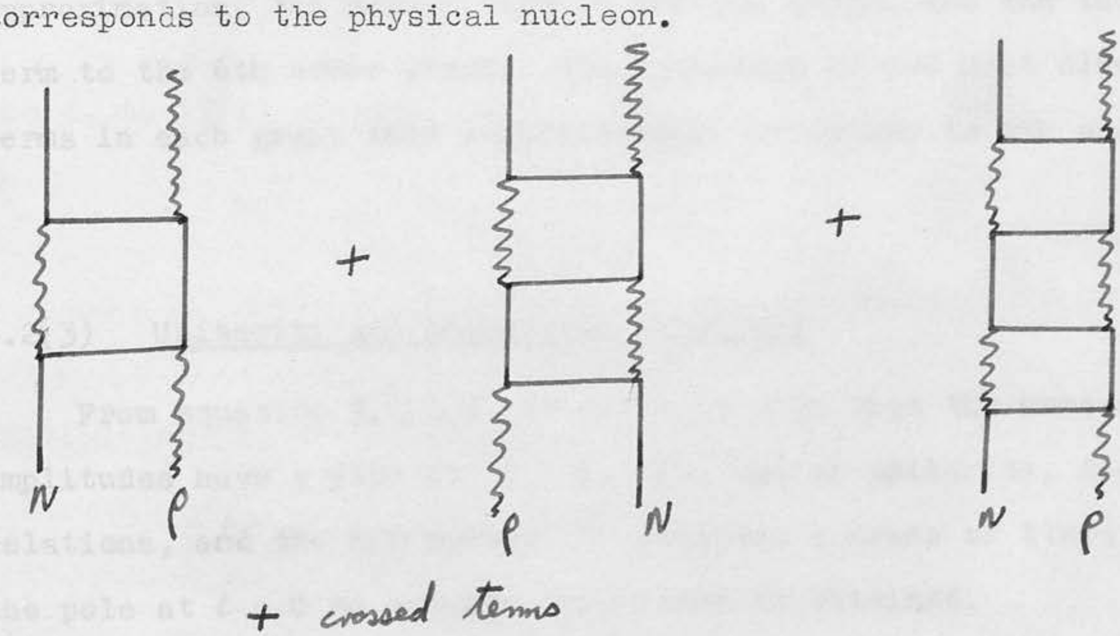


Fig. 2

If the Regge behaviour persists for all orders of Feynman graphs shown in Figure 2, the matrix element for the scattering is

$$M_{\mu\nu} = \gamma^2 \left[i \, \Gamma_\mu \left\{ \frac{iZ}{\gamma . p - m_o} \right\} i \, \Gamma_\nu \right]$$

where γ is the ρNN coupling constant, $\Gamma_{\mu,\nu}$ is a gauge, and

$$Z = \left[\frac{(-z)^\alpha + (z)^\alpha}{2} \right] - \left[\frac{(-z)^{-\alpha} - (z)^{-\alpha}}{2} \right]$$

$$= 1 + \alpha \, \ell n(-z) + \frac{\alpha^2}{2!} (\ell n \, z)^2 + \ldots.. \qquad \text{(D)}$$

The presence of terms having positive and negative z in equation (D) is due to the fact that the variables t and u are alternately the leading asymptotic variables in the graphs shown in Figure 2.

Gell-Mann et al.[2] verified directly that the first term in the series corresponds to the most divergent term in the Born approximation, the second term to the box graph, and the third term to the 6th order graph. The summation of the most divergent terms in each graph thus verifies Regge behaviour to 6th order.

5.2(3) <u>Unitarity and dispersion relations</u>

From equation 5.2(1)C above it is seen that the nonsense amplitudes have a pole at $\ell = 0$. The use of unitarity, dispersion relations, and the N/D method[40] provides a means of iterating the pole at $\ell = 0$ so a Regge trajectory is obtained.

$$\frac{1}{\ell} \longrightarrow \frac{1}{\ell} + \frac{a}{\ell^2} + \frac{a^2}{\ell^3} + \ldots\ldots = \frac{1}{\ell - a} .$$

If a Regge trajectory is generated by this method, the effect of higher order iterations on the non-continuable term $\partial_{\ell o}$ in the sense amplitude is

$$\partial_{\ell o} \longrightarrow - \frac{a}{\ell - a}$$

An example of this technique is as follows. Suppose the nonsense amplitude for the Born graph is

$$B_{-1-1}(w) = \frac{1}{\ell} h(w) \ ; \quad w = \sqrt{s}$$

and the unitarity relation for the Regge pole exchange amplitude is

$$\mathrm{Im}\, F_{-1-1}(w) = k \sum_{\lambda} \left| F_{-1\lambda} \right|^2 ; \quad k \text{ is the c.m. momentum.}$$

The amplitude $F_{-1-1}(w)$ may be written as

$$F_{-1-1}(w) = \frac{N(w)}{D(w)}$$

where, to first order

$$N(w) = B_{-1-1}(w)$$

$$D(w) = 1 - \int_{\substack{\text{unitarity}\\ \text{cuts}}} dw' \frac{k(w') B_{-1-1}(w')}{w' - w} \qquad \text{(E)}$$

Multiplying (E) by ℓ gives

$$N(w) = h(w)$$

$$D(w) = \ell - \int_{\substack{\text{unitarity}\\ \text{cuts}}} dw' \frac{k(w') h(w')}{w' - w} \qquad \text{(F)}$$

The above method of iterating the Born approximation pole to obtain a Regge trajectory has been discussed in detail by GGLZ[2]. Provided the conditions

(i) There exists a nonsense channel.

(ii) The Born approximation residue factorises into residues corresponding to the different sense and nonsense channels respectively, the effect of unitarity and dispersion relations is to transform the fixed Born approximation pole into a Regge pole. Furthermore the Regge trajectory obtained by this method is exactly similar to the trajectory obtained by summing the most divergent terms of the Feynman graphs described in the previous section.

The N/D method described above is a practical means of calculation, and has been used by Freedman[41] to construct models for the nucleon and baryon Regge trajectories.

5.2(4) Nonsense states and the S_{11} resonance[42]

In paragraphs 5.2(1) - (3) a model of Gell-Mann et al.[2] has been described in which the physical nucleon lies on a Regge trajectory. The Regge pole occurs in the sense amplitude at $\ell = 0$, so the nucleon is physical. Suppose the Regge pole chooses nonsense at $\ell = 0$, so it occurs in the nonsense amplitude. Since sense-nonsense decoupling occurs at $\ell = 0$[39] and the nonsense state is never reached physically, no bound state or resonance will be seen at this point. Should the N_β trajectory choose nonsense at $\ell = 0$, no S_{11} resonance will exist, and there will be no nucleon MacDowell symmetric partner.

The question as to whether or not the N_β trajectory chooses nonsense at $\ell = 0$ depends entirely on the dynamics of the system. If the Regge residue function $\longrightarrow$ zero as $(w - m')$ at $\ell = 0$, where m' is the S_{11} mass, the trajectory will indeed choose nonsense. Otherwise, the trajectory chooses sense.

5.2(5) <u>Constraints on the residue function</u>

The w plane is defined as before, so the N_α trajectory (positive parity trajectory) corresponds to $w > 0$, and the N_β trajectory (negative parity) corresponds to $w < 0$.

If the residue function $\beta(w)$ is to describe physical resonances, then $\beta(w) > 0$ for $w > w_T$ (threshold), and $\beta(w) < 0$ for $w < -w_T$. Thus the residue function changes sign in the region $-w_T \leqslant w \leqslant w_T$ (See Desai[43]).

Suppose a Regge pole occurs in a partial wave amplitude $F_J(w)$ at $w = w_R$. Near $w = w_R$

$$F_J(w) \sim \frac{\beta(w)}{\alpha(w) - J}$$

and expansion of $\alpha(w)$ about $w = w_R$ leads to a Breit-Wigner form for $F_J(w)$,

$$F_J(w) \sim \frac{\beta(w)}{\left[\frac{d}{dw}\ \mathrm{Re}\ \alpha(w)\right]_{w=w_R}} \Bigg/ \left[w - w_R + \frac{i\mathrm{Im}\ \alpha(w)}{\left[\frac{d}{dw}\ \mathrm{Re}\ \alpha(w)\right]_{w=w_R}} \right].$$

For $w > 0$, the work in Chapter III shows that

$$\mathrm{Im}\ \alpha(w) > 0, \quad \left[\frac{d}{dw}\mathrm{Re}\ \alpha(w)\right]_{w=w_R} > 0, \text{ so } \beta(w) > 0 \text{ if}$$

physical resonances are to be described.

Continuation of this equation to $w = -w'$ shows that if physical resonances are described in this region, $\beta(-w') < 0$. For physical resonances

$$\begin{aligned} \beta(w) &> 0 \\ \beta(-w') &< 0 \end{aligned} \qquad (G)$$

The Regge residue function is required to change sign in the range $-w_T \leq w \leq w_T$, and consequently has a zero (or infinity) in this range.

Dynamical models of Regge trajectories are constructed in the next section. The Regge residue functions are checked first for smooth positive behaviour in the range $w_T \leq w \leq \infty$, and for smooth negative behaviour in the range $-\infty \leq w \leq -w_T$. Next, the residue function is examined for zeros in the range $-w_T \leq w \leq w_T$, especially for zeros near the mass of the missing S_{11} on the highest ranking N_β trajectory.

5.3 Dynamical Models

In this section, dynamical models for the nucleon trajectory are considered. The first model was used by Freedman[41] to evaluate the slope of the N_α trajectory at the nucleon. This work has been extended, and the residue and trajectory functions calculated for this model. Other models consist of Born diagrams with σ meson (S wave $\pi\pi$ state) and ρ meson exchange, and residues and trajectories are calculated as before.

From Barger and Cline's classification of the baryon resonances[20], it is readily seen that the odd signature trajectory passing through $P_{11}(938)$ and $F_{15}(1688)$, lies very close to the even signature trajectory which passes through $D_{13}(1518)$, and $G_{17}(2190)$. This suggests that the exchange (u channel) forces responsible for even and odd signature effects are unimportant in this case. Only models which do not give rise to u channel cuts are considered in this work. Iteration of the Feynman graph used in the model of GGLZ[2], produces alternate t channel and u channel cuts, and gives rise to ~~stray~~ exchange forces. Such a model is expected to be not so successful in describing the physical nucleon trajectory, and is not considered further.

5.3(1) Freedman's model

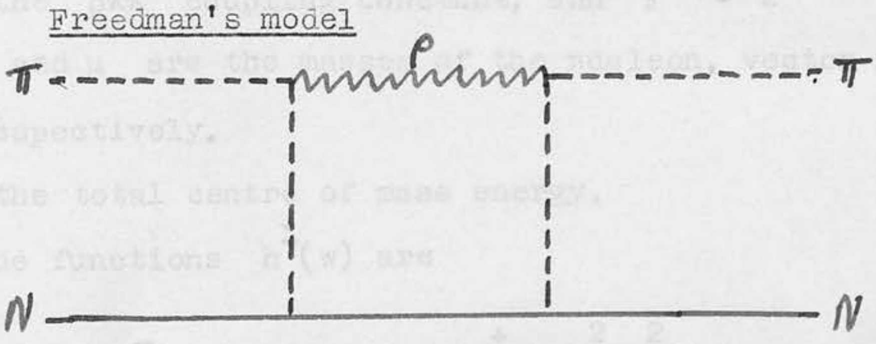


Fig. 3

This model consists of iteration of the Feynman graph shown in Figure 3 by unitarity and dispersion relations. This graph is seen to give rise only to t channel cuts, and there are no exchange forces as required. The effect of the vector meson acting transversely is to displace the fixed pole which occurs in the spinless particle partial wave amplitude at $\ell = -1$, by one unit. Thus the above

Feynman graph (Figure 3) has a fixed pole at $\ell = 0$, $J = \frac{1}{2}$, and iteration of this pole gives rise to the nucleon trajectory.

A weakness of the above model is that there is no nonsense channel for the Reggeisation theory of GGLZ[2], but this was ignored in the calculations of the Regge parameters.

After correction of an error has been made in Freedman's paper, the partial wave amplitude near $\ell = 1$, $J = \ell - \frac{1}{2} = \frac{1}{2}$, parity $-(-1) = +1$ is (see Appendix IV)

$$\begin{bmatrix} B^{\frac{1}{2}}_{\ell^-}(w) \\ B^{3/2}_{\ell^-}(w) \end{bmatrix} \sim \frac{(w-m_o)^2 - \mu^2}{w^2} \begin{bmatrix} 4 \\ 1 \end{bmatrix} \frac{G^2F^2}{8\pi}\; h^-(w)\; \frac{1}{\ell-1}$$

The conventions used in the calculation are given in Appendix II.
G is the pion-nucleon coupling constant, and $G^2 \sim 15$.
F is the $\rho\pi\pi$ coupling constant, and $F^2 \sim 2$
m_o, m, and μ are the masses of the nucleon, vector meson, and pion respectively.
W is the total centre of mass energy.

The functions $h^{\pm}(w)$ are

$$h^{\pm}(w) = \frac{1}{w}\left[\ell n\left(\frac{m}{m_o}\right) + \left(\frac{\overline{w \pm m_o}^{\,2} - m^2}{\left|(w^2 - [m+m_o]^2)(w^2 - [m-m_o]^2)\right|^{\frac{1}{2}}}\right) I(w)\right]$$

where (i)

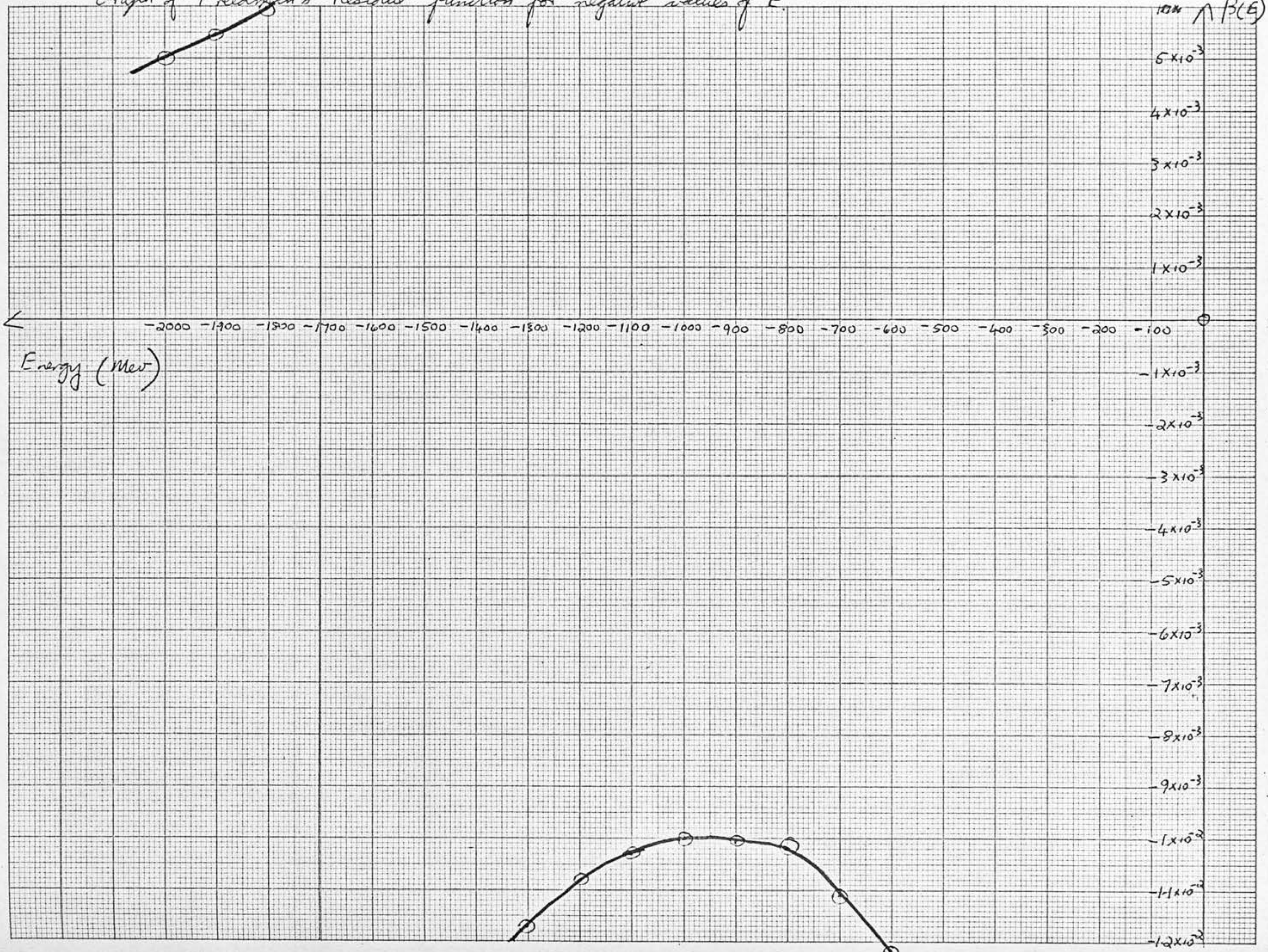
$$I(w) = \frac{\pi}{2} + \tan^{-1} \frac{w^2 - (m^2 + m_o^2)}{\left|(w^2 - [m+m_o]^2)(w^2 - [m-m_o]^2)\right|^{\frac{1}{2}}}$$

for $(m_o - m)^2 < w^2 < (m_o + m)^2$

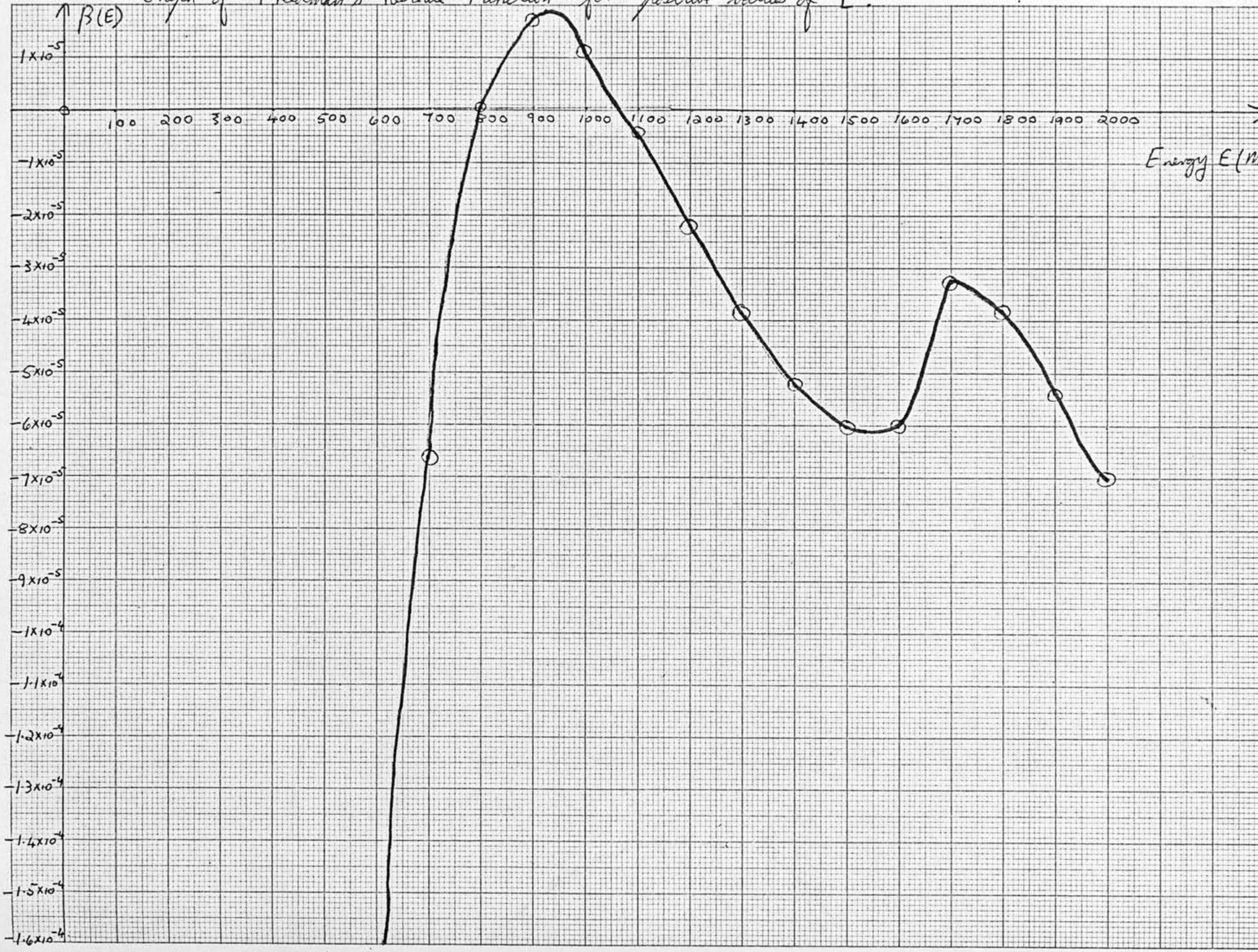
$$\text{(ii)}\quad I(w) = \ell n\left[\frac{w^2 - (m^2 + m_o^2) - \left|(w^2 - [m+m_o]^2)(w^2 - [m-m_o]^2)\right|^{\frac{1}{2}}}{2mm_o}\right]$$

$w^2 > (m_o + m)^2$, or $w^2 < (m_o - m)^2$

Graph of Freedman's Residue function for negative values of E.



Graph of Freedman's Residue Function for positive values of E.



Application of the N/D method as described in Freedman's paper gives the trajectory function

$$\alpha^{\frac{1}{2}}(w) = 1 + \frac{w-m_o}{2\pi} G^2 F^2$$

$$x \int_{m_o+\mu}^{\infty} dw' \frac{q(w')}{w'^2} \left[\frac{(w'-m_o)^2-\mu^2}{(w'-m_o)(w'-w)} h^-(w') - \frac{(w'+m_o)^2-\mu^2}{(w'+m_o)(w'+w)} h^+(w') \right]$$

The expression for the partial wave amplitude is

$$F^{\frac{1}{2}}_{\ell^-} = \frac{\beta(w)}{\ell-\alpha(w)} , \quad \text{where} \quad \beta(w) = \frac{(w-m_o)^2-\mu^2}{w^2} \frac{G^2F^2}{2\pi} h^-(w)$$

The Regge residue functions $\beta(w)$ and the trajectory $\alpha(w)$ were evaluated and the results shown on the graphs opposite.

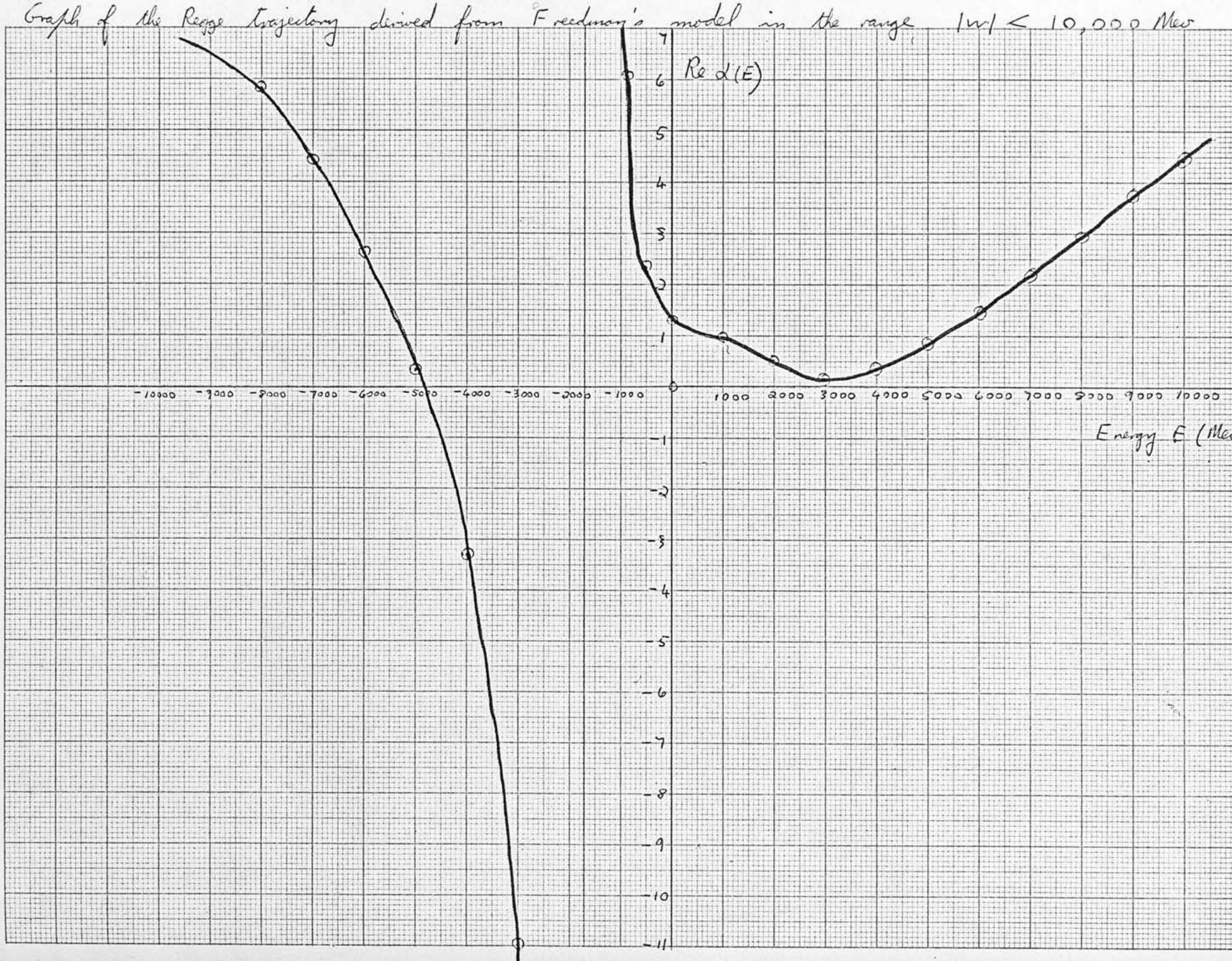
Conclusions and Discussion

For $w > 0$ the residue function $\beta(w)$ is negative except in the region $800 \leq w \leq 1065$ Mev. Zeros occur in $\beta(w)$ at $w = 800$ Mev and $w = 1065$ Mev, but these appear to have no physical significance. Since $\beta(w)$ is positive only in the region $800 \leq w \leq 1065$, physical resonances can be described only in this region. Further, $\beta(w) \to 0$ as $w \to \infty$. A cusp occurs at the ρ meson threshold, $w = 1703$ Mev. $\beta(w)$ diverges at $w = 0$.

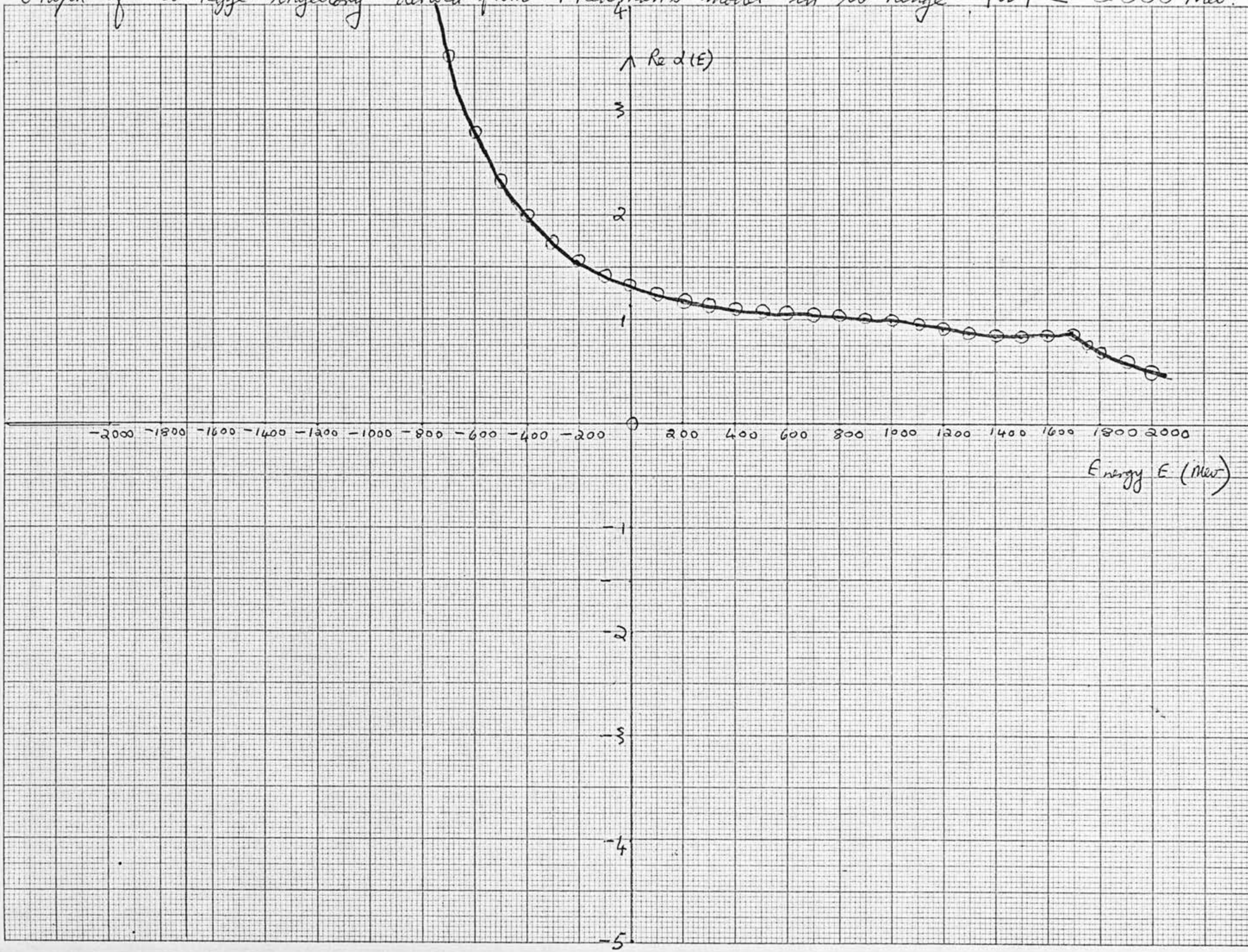
When $w < 0$, $\beta(w)$ diverges at $w = 0$ and $w = -(m+m_o) = 1703$ Mev, and is negative in between. For $|w| > 1703$ Mev, the residue is positive, and decreases to zero as $w \to -\infty$. $\beta(w)$ has no zeros except at $w = \pm \infty$.

The residue function changes sign for w both positive and negative, and in general has a very unsatisfactory behaviour. This

Graph of the Regge trajectory derived from Freedman's model in the range $|w| < 10,000$ Mev



Graph of the Regge trajectory derived from Freedman's model in the range $|w| < 2000$ Mev.



may not be due entirely to weaknesses in Freedman's model, since the first order N/D method used is a crude approximation, and mutilates the correct form of the partial wave amplitude $F^{\frac{1}{2}}_{\ell-}$ obtained from the iteration.

For $w > 0$, the trajectory function $\alpha(w)$ decreases slowly in the range $0 \leq w \leq 2000$ Mev, and has a small cusp at the ρ meson threshold m_o+m. The trajectory varies little from $\alpha(w) = 1$, and has little physical interest in this range.

For $w < 0$, $\alpha(w)$ is positive, increasing, and divergent at the ρ meson threshold. The divergence is too rapid for any physical significance in this region. For $-4800 \leq w \leq -1703$ the trajectory $\alpha(w)$ is negative.

When $|w| > 5000$ Mev, trajectories for both positive and negative w start diverging. This is confirmed by a simple calculation which shows that both trajectories diverge as $(\ell n w)^2$ as $w \to \infty$. Thus Freedman's model gives rise to infinitely rising Regge trajectories.

In Freedman's model, there is no MacDowell symmetric partner to the nucleon, since a state of spin ~ 10 is predicted at $w = -850$ Mev. However, the Regge trajectory is unrealistic in this range. The residue function $\beta(w)$ shows no evidence of a zero at $w = -850$ Mev, so the non appearance of the S_{11} is not a nonsense effect according to this model.

In general the forms of both the trajectory and the residue function have little relevance to the higher pion-nucleon resonances.

5.3(2) <u>σ-exchange model</u>

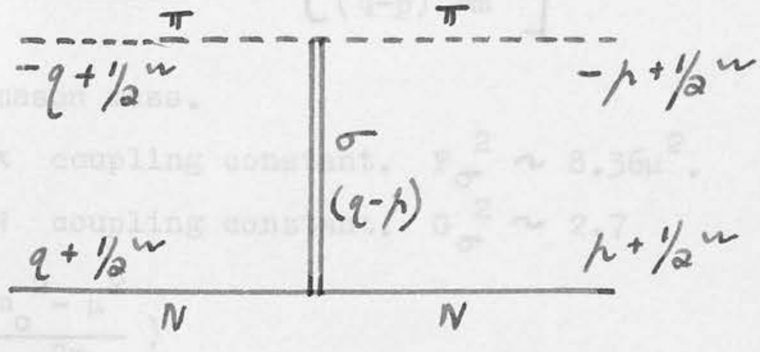


Fig. 4.

Single particle exchange models are considered in this and the following section. First the exchange $I = 0$, $J = 0$ is considered. This corresponds to an S-wave dipion state or sigma meson.

If coupling takes place between this state and the two pions in Figure 4, the pions are in an S state (ℓ=0).

The parity of the pions is $(-1)^2(-1)^\ell = +1$

The G parity of the pions is $(-1)^2 = +1$

Hence the quantum numbers of the exchanged sigma meson are 0^{++}. The question as to whether or not the sigma meson exists is unimportant, since the important effect is the exchange of an S wave dipion state having quantum numbers 0^{++}.

There are two weaknesses in the above model. The first is that there is no nonsense channel, and the second is that there is no vector meson to displace the fixed pole occurring at the unphysical value $\ell = -1$, to the physical value $\ell = 0$. In spite of this, however, calculations were done to see if the residue and trajectory function have any physical significance.

The Feynman amplitude is

$$M_{fi} = \frac{i}{\pi} F_o G_o \bar{u} (\hat{q}_f) \left[\frac{1}{(q-p)^2 - m^2} \right] u(\hat{q}_i) \quad .$$

m is the σ meson mass.

F_σ is the $\sigma\pi\pi$ coupling constant. $F_\sigma^2 \sim 8.36\mu^2$.

G_σ is the σNN coupling constant. $G_\sigma^2 \sim 2.7$

$$p = \left(\hat{q}_i , \frac{m_o^2 - \mu^2}{2w} \right)$$

$$q = \left(\hat{q}_f , \frac{m_o^2 - \mu^2}{2w} \right) .$$

Projection of the negative partial wave (near $\ell = -1$ this gives a trajectory of positive parity) takes place as before, and near the fixed pole at $\ell = -1$ the partial wave amplitude is

$$B_{\ell^-}(w) = \frac{F_\sigma G_\sigma}{w(E+m_o)} \left[1 + \frac{m^2}{2q^2(w)} \right] \frac{1}{\ell+1}$$

where $q(w)$ is the centre of mass momentum and E the nucleon energy. Put $\Upsilon(w) = \left[1 + \frac{m^2}{2q^2(w)} \right]$,

$\therefore$ $B_{\ell^-}(w) = \frac{F_\sigma G_\sigma}{w(E+m_o)} \frac{\Upsilon(w)}{\ell+1}$, and the first order N/D method gives

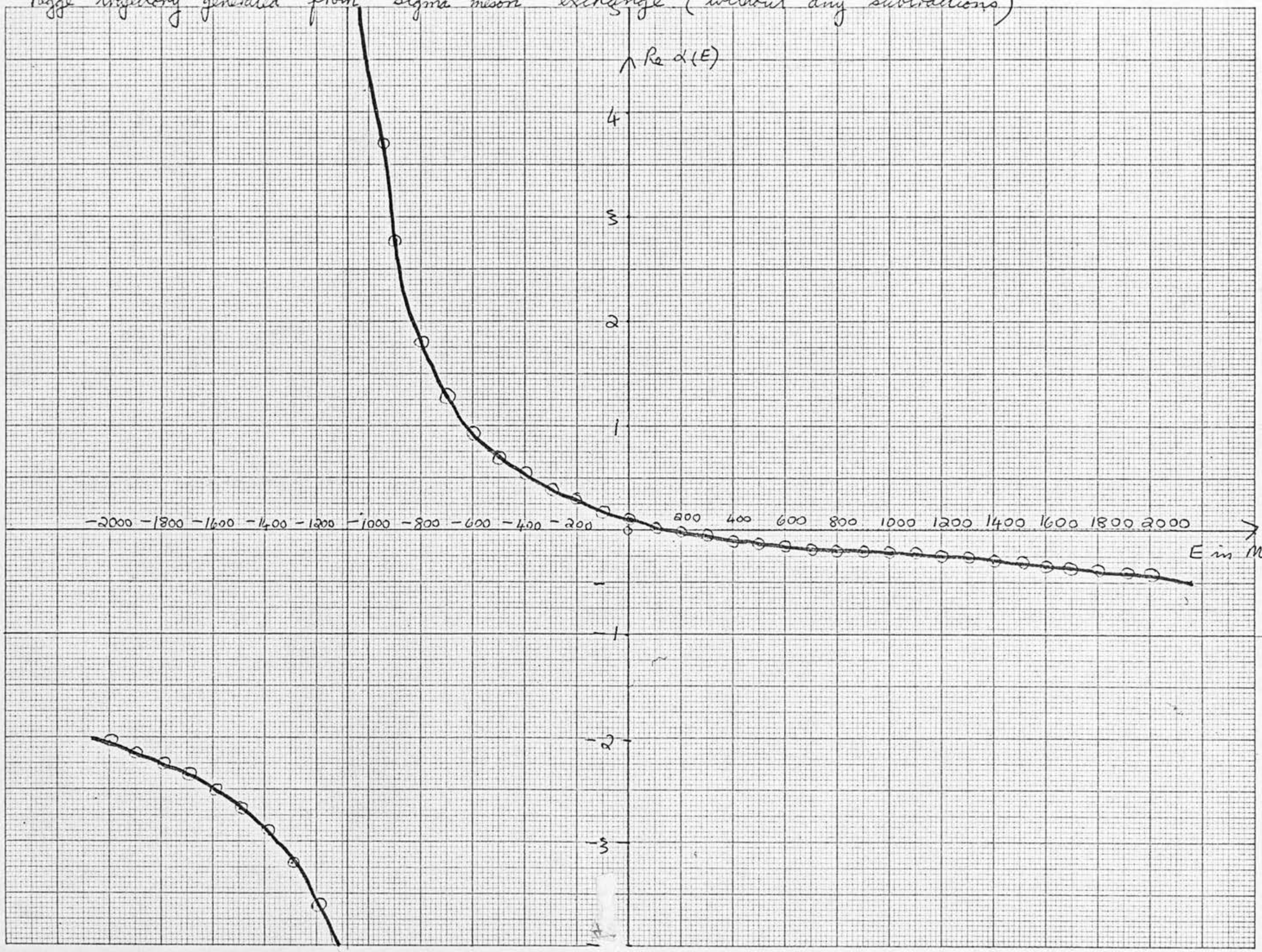
$$F_{\ell^-}(w) = \frac{N(w)}{D(w)}$$

$$N(w) = B_{\ell^-}(w)$$

$$D(w) = 1 - \frac{1}{\pi} \int_{m_o+\mu}^{\infty} dw' \, B_{\ell^-}(w') \frac{q(w')}{w'-w} + \int_{-\infty}^{-(m_o+\mu)} dw' B_{\ell^-}(w) \frac{q(w')}{w'-w}$$

Considering only the most divergent term of $B_{\ell^-}(w)$ gives

Regge trajectory generated from sigma meson exchange (without any subtractions)



$$D(w) = 1 - \frac{F_\sigma G_\sigma}{\pi(\ell+1)}\left\{\int_{m_o+\mu}^{\infty} dw'\left[\left(\sqrt{\frac{E-m_o}{E+m_o}}\right)\frac{1}{w'(w'-w)}\right.\right.$$

$$\left.\left. + \sqrt{\left(\frac{E+m_o}{E-m_o}\right)}\,\frac{1}{w'(w'+w)}\right]\right\} .$$

This gives Regge behaviour

$$F_{\ell^-}(w) = \frac{\beta(w)}{\ell - \alpha(w)}, \text{ where}$$

$$\beta(w) = \frac{F_\sigma G_\sigma}{w(E + m_o)} = \frac{F_\sigma G_\sigma}{(w + m_o - \mu)(w - m_o + \mu)}$$

$$\alpha(w) = -1 + \frac{F_\sigma G_\sigma}{\pi}\left(\int_{m_o+\mu}^{\infty} dw'\ \left(\sqrt{\left(\frac{E-m_o}{E+m_o}\right)}\ \frac{1}{w'(w'-w)}\right.\right.$$

$$\left.\left. + \sqrt{\left(\frac{E+m_o}{E-m_o}\right)}\ \frac{1}{w'(w'-w)}\right)\right) .$$

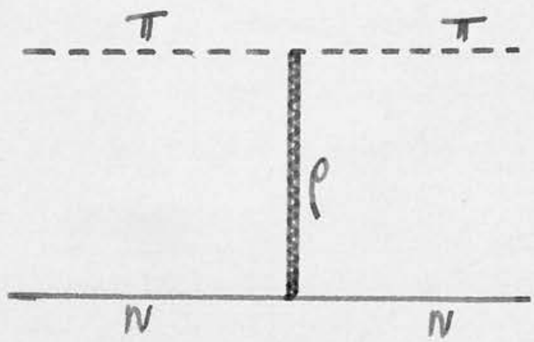
Discussion

The residue function is positive for both $w > m_o + \mu$ and $w < -(m_o + \mu)$ and is never zero. Divergence occurs at $w = \pm(m_o - \mu)$. There is no evidence of a zero near the S_{11} mass, $w \sim 850$ Mev. Such a residue function can give rise to resonances in the region $w > (m_o + \mu)$, but not in the region $w < -(m_o + \mu)$.

The trajectory function is shown opposite. For $w > 0$ the trajectory fails to rise to any physical value of J, and approaches $\mathrm{Re}\,\alpha(w) = -1$ as $w \to \infty$. For $w < 0$, a MacDowell symmetric partner for the nucleon is predicted at $w = 615$ Mev, and the trajectory diverges to $+\infty$ at threshold $m_o + \mu$. For $w < -(m_o+\mu)$ the trajectory is negative and slowly tends to $\mathrm{Re}\,\alpha(w) = -1$ as $w \Rightarrow -\infty$.

Such a model has no application at all to the higher pion-nucleon resonances.

5.3(3) <u>ρ meson exchange</u>



<u>Fig. 5.</u>

If vector meson exchange takes place in Figure 5, the quantum numbers of the exchanged state are $J^{PG} = 1^{-+}$ (ρ meson).

A similar calculation to that in section 5.3(2) shows that iteration of the above graph produces Regge behaviour

$$\frac{\beta^{\frac{1}{2}}(w)}{\ell - \alpha^{\frac{1}{2}}(w)} \quad .$$

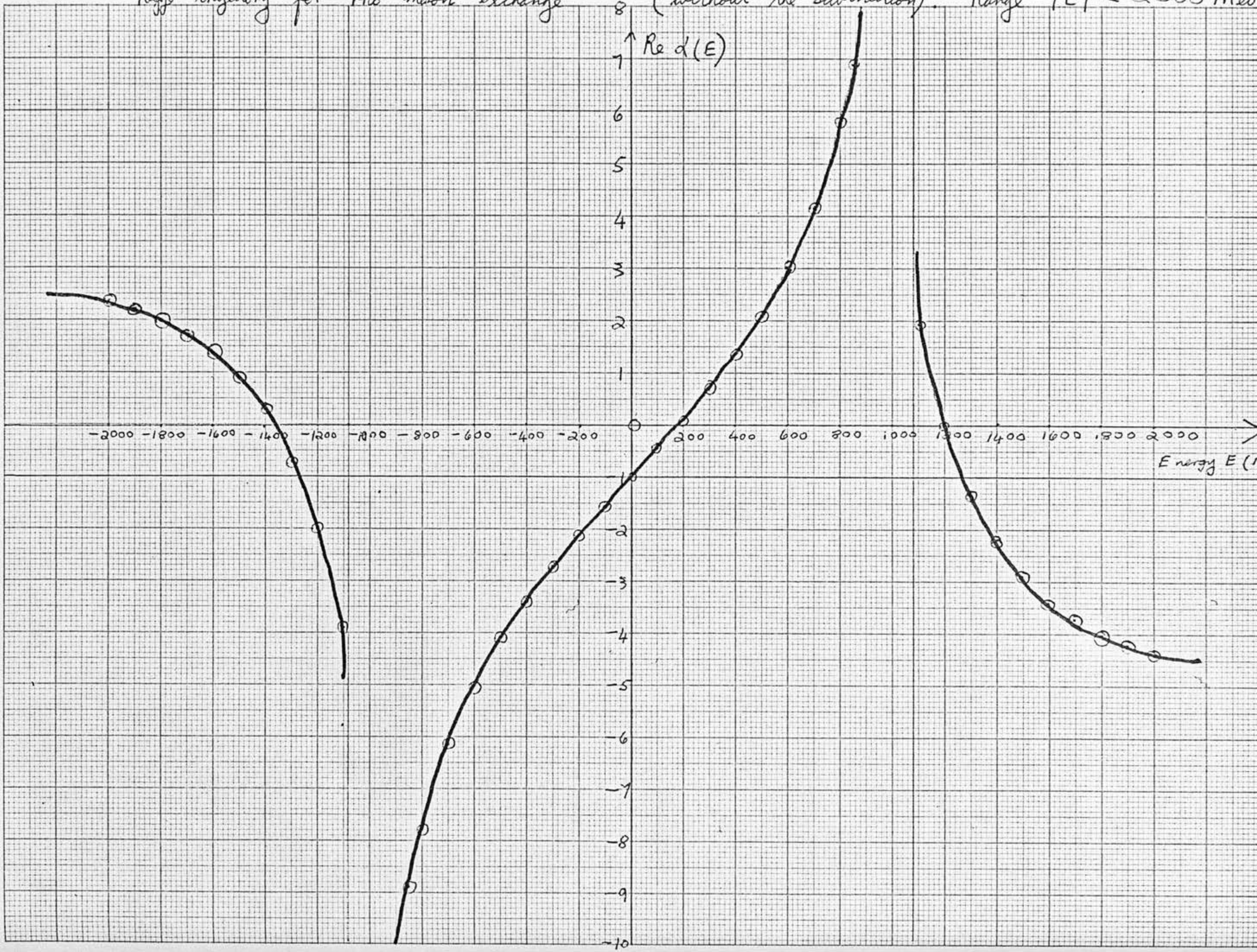
The superscript refers to the isospin state.

$$\beta^{\frac{1}{2}}(w) = \sqrt{\frac{2}{3}}\, 4 \;\; G_\rho \, F_\rho \, m_o \left[\frac{w - E(w)}{wq^2(w)} \right]$$

$$\alpha^{\frac{1}{2}}(w) = -1 + \sqrt{\frac{2}{3}} \;\; \frac{8m_o}{\pi} \;\; G_\rho \, F_\rho \;\; w \int_{m_o+\mu}^{\infty} dw' \left(\frac{w' - E(w')}{w' q(w')} \right) \frac{1}{w'^2 - w^2}$$

The factor $\sqrt{\frac{2}{3}}$ results after projection of the state $I = \frac{1}{2}$.

Regge trajectory for rho meson exchange (without the subtraction). Range $|E| < 2000$ Mev



G_ρ is the $N\rho N$ coupling constant, and F_ρ is the $\rho\pi\pi$ coupling constant (see Appendix II).

Discussion

Once again the residue function remains positive for $w < -(m_o + \mu)$, so physical resonances cannot be described in this region. Zeros occur at $w = \pm\sqrt{(m_o^2 - \mu^2)}$, i.e. $w = \pm 928$ Mev. The negative value is near the assumed mass of the S_{11} (850 Mev), so this model gives a possible explanation of the missing S_{11} in terms of the vanishing of the residue function.

The trajectory function is shown opposite. Divergence to $\pm\infty$ occurs at the two thresholds $w = \pm(m_o + \mu)$ respectively, and as $w \rightarrow \pm\infty$, $\mathrm{Re}\,\alpha(\pm w) \rightarrow -1$. This was confirmed by computer studies for w up to 10^{12} Mev, and is also evident from a simple calculation.

Since positive divergence occurs at $w = (m_o + \mu)$, a subtraction was made at $w = w_o$ in the dispersion relation for the D function so that $\alpha^{\frac{1}{2}}(w)$ passed through the nucleon ($\ell = 1$). The expression for the trajectory is now

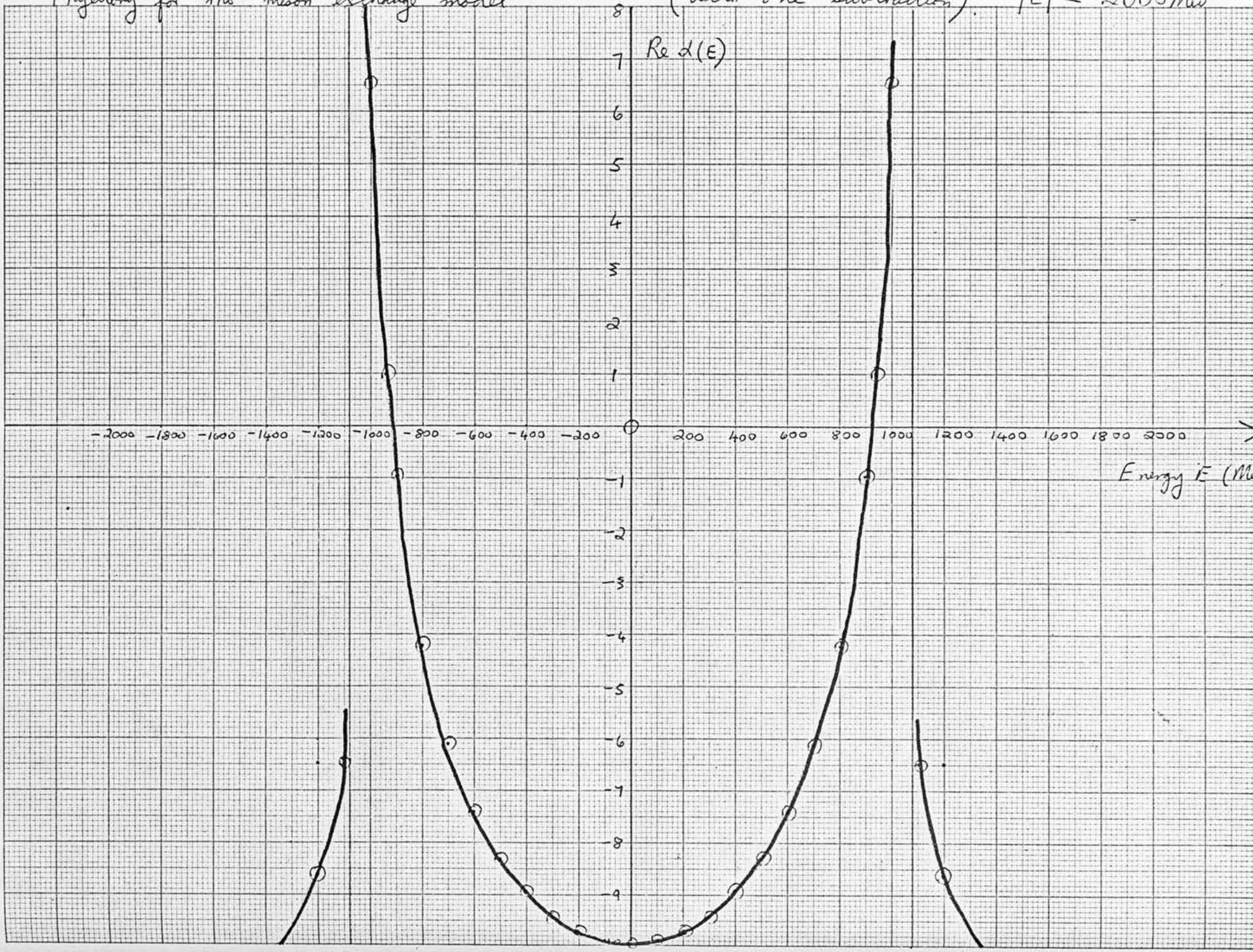
$$\alpha^{\frac{1}{2}}(w) = -1 + \sqrt{\frac{2}{3}}\ \frac{8m_o}{\pi}\ G_\rho F_\rho (w^2 - w_o^2) \int_{m_o+\mu}^{\infty} dw' \ \frac{w'-E(w')}{w'q(w')} \cdot$$

$$\cdot \frac{1}{(w'^2 - w^2)(w'^2 - w_o^2)}$$

where $w_o = 899.1408$ Mev and $\alpha^{\frac{1}{2}}(m_o) = 1$.

The trajectory again shows divergence to $+\infty$ at the thresholds $\pm(m_o + \mu)$. Above threshold the trajectory diverges logarithmically to $-\infty$ and does not reach any of the higher pion-nucleon resonances.

Trajectory for rho meson exchange model (With one subtraction) $|E| < 2000$ Mev



The unsubtracted form of the trajectory for ρ meson exchange has some satisfactory explanations for the missing S_{11} state, notably vanishing of the residue function in the vicinity of its mass. Furthermore, the trajectory function has a large dip (at $-\left[m_o + \mu\right]$) in its vicinity.

5.4 Conclusion

Both the residue functions and the trajectory functions for the three models discussed have very unsatisfactory behaviour and little relevance to the higher pion-nucleon resonances. This may be partly due to the models (in all three models there is no nonsense channel and also no transverse vector meson in the single particle exchange models), and also to the approximations made in the calculation (the first order N/D method).

The single particle exchange trajectories tend to -1 as $w \rightarrow \pm \infty$. There is an analogy with potential scattering, because Regge trajectories obtained from the Schrödinger equation[24] end on negative integers as $w \rightarrow \infty$.

The effects of the absence of a vector meson acting transversely in the single particle exchange models have been discussed by Abers and Zachariasen[44].

Due to the -1 term in the expressions for the trajectory, dynamical effects are necessary to bring the trajectory into the physical ℓ region. Since first order perturbation theory has been used for strong coupling, calculations involving these models are expected to be very unreliable.

For the box graph used by Freedman, due to the translation of the fixed pole by the vector meson, the fixed pole occurs at $J = \frac{1}{2}$. The trajectory obtained by iteration of this graph starts at a physical value of angular momentum $J = \frac{1}{2}$, and no dynamical effects are necessary to produce a bound state lying on this trajectory. For models of this type, the nucleon can lie on a Regge trajectory even if the approximation of weak coupling and first order perturbation theory are used.

The above considerations show that the box graph considered by Freedman is more likely to produce a physically meaningful Regge trajectory than single particle exchange models. A possible future research project is to consider the effect of a Feynman graph with both a vector meson acting transversely and a nonsense channel, e.g.

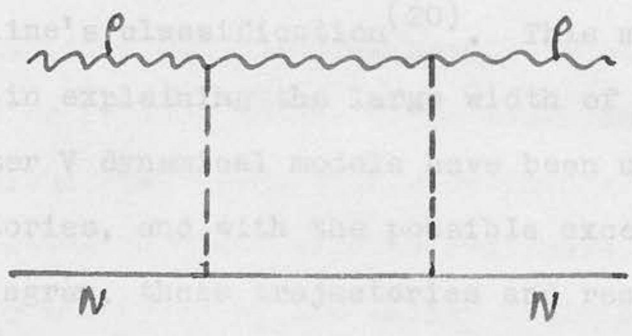


Fig. 6.

The dotted line in Figure 6 can represent any meson having positive G parity except the ρ meson, since a 3ρ vertex is forbidden by Furry's theorem.

CONCLUSION

In this thesis, attempts have been made, on the basis of the Regge model, to explain the missing nucleon MacDowell symmetric partner, (S_{11}) and these attempts have been highly unsuccessful.

In the potential scattering model of Chapter III, the possibility of missing mass states has been demonstrated for negative w, but no calculations have been done.

Parametrisations of the trajectory function are discussed in Chapter IV, especially those giving rise to dips and cusps near the missing S_{11}. A model of Lyth[25] and Jones[26] is considered, in which the leading N_β trajectory is distorted to include the $S_{11}(1591)$, which is assigned to a lower ranking trajectory in Barger and Cline's classification[20]. This model has grave difficulties in explaining the large width of the $S_{11}(1591)$.

In Chapter V dynamical models have been used to generate Regge trajectories, and with the possible exception of the single ρ exchange diagram, these trajectories and residue functions are totally at variance with the observed Regge trajectories for the pion-nucleon resonances. It is not clear whether or not the poor results are due to defects in the models used, such as the absence of a nonsense channel, or are due to the approximations made in the calculation. This would be partly answered by consideration of the graph in Figure 6, Chapter V, which has a fixed pole at $j = \frac{1}{2}$ and a nonsense channel. This is a possible future research project.

Recently, models other than the Regge model have been considered, notably the quark model. Squires[45] has obtained a form for a meson Regge trajectory by considering the exchange of a heavy quark in the t channel. Another possible future research project is the extension of this work to baryon Regge trajectories.

A model for the higher pion-nucleon resonances which does not involve complex angular momentum is the orbital excitation model of Dalitz[46], also Faiman and Hendry[47]. The higher baryon states are given by excitations of the internal orbital motions of the quarks composing the state. In this model, the approximate degeneracy between positive and negative parity states is accidental, so there is no trouble with missing MacDowell symmetric partners. However, this model has more difficulty than the Regge model in placing some of the higher mass resonances.

The present experimental status of the degeneracy between states of positive and negative parity is still rather shaky. The success or failure of the Regge model and MacDowell symmetry in pion-nucleon scattering will come when accurate data on the resonances above 2000 Mev becomes available.

In 1905 the study of the energy distribution in the spectrum of black body radiation led to a revolution in physical thought. As progressively higher energies are investigated, perhaps the study of the pion-nucleon resonances may lead to a similar revolution.

APPENDIX I

Definition of angles and projection operators occurring in the reaction $a + b \longrightarrow c + d$.

(1) Angles[6]

Consider the reaction $a + b \longrightarrow c + d$ where $m_a \neq m_b \neq m_c \neq m_d$. Define the Mandelstam variables s, t, u as before. Let θ_s, θ_t be the centre of mass scattering angles occurring in s channel and t channel scattering, respectively.

Define

$$S_{ab}^2 = \left[s - (m_a - m_b)^2\right]\left[s - (m_a + m_b)^2\right] = 4s\,P_{ab}^2$$

$$S_{cd}^2 = \left[s - (m_c - m_d)^2\right]\left[s - (m_c + m_d)^2\right] = 4s\,P_{cd}^2$$

$$T_{ac}^2 = \left[t - (m_a - m_c)^2\right]\left[t - (m_a + m_c)^2\right] = 4t\,p_{ac}^2$$

$$T_{bd}^2 = \left[t - (m_b - m_d)^2\right]\left[t - (m_b + m_d)^2\right] = 4t\,p_{bd}^2$$

$$\begin{aligned}\phi(s,t) = {} & st(\Sigma m_i^2 - s - t) - s(m_b^2 - m_d^2)(m_a^2 - m_c^2) \\ & - t(m_a^2 - m_b^2)(m_c^2 - m_d^2) - (m_a^2 m_d^2 - m_c^2 m_b^2) \\ & .(m_a^2 + m_d^2 - m_c^2 - m_b^2) .\end{aligned}$$

Then

$$\cos\theta_s = \left[2st + s^2 - s(\Sigma m_i^2) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)\right] / S_{ab}\,S_{cd}$$

$$\sin\theta_s = 2\left[s\phi(s,t)\right]^{\frac{1}{2}} / S_{ab}\,S_{cd}$$

$$\cos \theta_t \;=\; 2st + t^2 - t(\;\; m_i^{\,2}) + (m_a^{\,2} - m_b^{\,2})(m_c^{\,2} - m_d^{\,2}) \;/T_{ac}\, T_{bd}$$

$$\sin \theta_t \;=\; 2\;\; t\phi(s,t)^{\frac{1}{2}}/T_{ac}\, T_{bd} \,.$$

The angles $\;_a$, $\;_b$, $\;_c$, $\;_d$ occurring in the crossing formula (1.3A) are

$$\cos\;_a \;=\; -(s+m_a^{\,2} - m_b^{\,2})(t + m_a^{\,2} - m_c^{\,2}) - 2m_a^{\,2}(m_c^2 - m_a^2 + m_b^2 - m_d^2) \;/\; S_{ab}\, T_{ac}$$

$$\cos\;_b \;=\; (s+m_b^{\,2} - m_a^{\,2})(t + m_b^{\,2} - m_d^{\,2}) - 2m_b^{\,2}(m_c^2 - m_a^2 + m_b^2 - m_d^2) \;/\; S_{ab}\, T_{bd}$$

$$\cos\;_c \;=\; (s+m_c^{\,2} - m_d^{\,2})(t + m_c^{\,2} - m_a^{\,2}) - 2m_c^{\,2}(m_c^2 - m_a^2 + m_b^2 - m_d^2) \;/\; S_{cd}\, T_{ac}$$

$$\cos\;_d \;=\; -(s+m_d^{\,2} - m_c^{\,2})(t + m_d^{\,2} - m_b^{\,2}) - 2m_d^{\,2}(m_c^2 - m_a^2 + m_b^2 - m_d^2) \;/\; S_{cd}\, T_{bd}$$

$$\sin\;_a \;=\; 2m_a\;\; \phi(s,t)^{\frac{1}{2}}/S_{ab}\, T_{ac}$$

$$\sin\;_b \;=\; 2m_b\;\; \phi(s,t)^{\frac{1}{2}}\; S_{ab}\, T_{bd}$$

$$\sin\;_c \;=\; 2m_c\;\; \phi(s,t)^{\frac{1}{2}}\; S_{cd}\, T_{ac}$$

$$\sin\;_d \;=\; 2m_d\;\; \phi(s,t)^{\frac{1}{2}}\; S_{cd}\, T_{bd}$$

(ii) Define functions $e^{J}_{\lambda\mu}$, $e^{J^+}_{\lambda\mu}$, $e^{J^-}_{\lambda\mu}$, $c^{J}_{\lambda\mu}$, $c^{J^+}_{\lambda\mu}$, $c^{J^-}_{\lambda\mu}$ such that

$$e^{J}_{\lambda\mu} = e^{J^+}_{\lambda\mu} + e^{J^-}_{\lambda\mu} = \left[(\sqrt{2}\cos\theta/2)\right]^{-|\lambda+\mu|}\left[(\sqrt{2}\sin\theta/2)\right]^{-|\lambda-\mu|} d^{J}_{\lambda\mu}(\theta)$$

$$c^{J}_{\lambda\mu} = c^{J^+}_{\lambda\mu} + c^{J^-}_{\lambda\mu} = (\sqrt{2}\cos\theta/2)^{|\lambda+\mu|}(\sqrt{2}\sin\theta/2)^{|\lambda-\mu|} d^{J}_{\lambda\mu}(\theta)$$

The e^{J^+} and c^{J^+} functions occur in formula (1.2C). The $d^{J}_{\lambda\mu}(\theta)$ functions are the d functions of the rotation matrix[4]. Values of e^{J} and c^{J} are tabulated in Reference (2) for $\lambda, \mu \leq 3/2$

APPENDIX II

Conventions and Notation used in this thesis

The notation of Bjorken and Drell[48], p. 285 is used in this work.

(1) A four vector $k = (k_o, \underline{k})$; $k^2 = k_o^2 - \underline{k}^2$.

(2) The Dirac matrices are

$$\underline{\gamma} = \begin{bmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{bmatrix} ; \qquad \gamma_o = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} ;$$

σ_i are the Pauli matrices.

(3) The Dirac spinors are defined by

$$u_r(p) = \sqrt{\left(\frac{E+m}{2m}\right)} \begin{bmatrix} \chi_r \\ \frac{\underline{\sigma}.\underline{p}}{E+m} \chi_r \end{bmatrix} ,$$

where χ_r is a two component spinor describing the spin state. The spinors are normalised by $\bar{u}_r u_r = I$, where $\bar{u}_r = u_r^+ \gamma_o$.

(4) A factor $\frac{1}{(2\pi)^4} \frac{i}{\gamma p - m_o}$ for each internal nucleon line of 4 momentum p.

(5) A factor $\frac{1}{(2\pi)^4} \frac{i}{k^2 - \mu^2}$ for each internal meson line of 4 momentum k, and mass μ .

(6) A factor $- \frac{1}{(2\pi)^4} \frac{i}{k^2} g_{\mu\nu}$ for each internal photon line.

(7) A factor $- \frac{1}{(2\pi)^4} \frac{1}{k^2 - m^2}$ for each vector meson line of mass m

(8) A factor $\varepsilon_\mu^\lambda(\underline{k}) \frac{1}{(2\pi)^{3/2}} \frac{1}{2 k_o}^{1/2}$ for each external photon.

(9) A factor $\frac{1}{(2\pi)^{3/2}} \sqrt{\left(\frac{M}{E}\right)} \left\{ \begin{matrix} \bar{\mu}_r(p) \\ \mu_r(p) \end{matrix} \right\}$ for each external nucleon line $\begin{pmatrix}\text{entering} \\ \text{leaving}\end{pmatrix}$ the vertex.

(10) A factor $(2\pi)^4 \,\delta(p - p' \pm k)$ for each vertex, corresponding to 4 momentum conservation.

(11) A ps interaction occurs between pions and nucleons. The interaction Lagrangian is

$$-\mathcal{L}_{int} = i(4\pi)^{1/2}\, G\, \bar{\psi}\, \underline{\tau}\, \gamma_5\, \psi \cdot \phi \;.$$

G is the conventional πN coupling constant, where $G^2 \sim 15$. The above Lagrangian gives a factor $-i(4\pi)^{1/2}\, G\, \gamma_5\, \tau_\alpha$ at each pion-nucleon vertex, where τ_α is the isospin matrix. The relative coupling strength is $\pm\sqrt{2}$ for charged pions, and ± 1 for neutral pions.

(12) The ρNN interaction. Since the ρ meson is a vector and also an isovector, the interaction Lagrangian is

$$-\mathcal{L}_{int} = (4\pi)^{1/2}\, F\, \hat{\rho}_\lambda \cdot (\hat{\phi} \times \partial_\lambda \hat{\phi})$$

where $F^2 \sim 20$.
The ρNN vertex contributes a factor $-i(4\pi)^{1/2}\, F(p+p')_\mu$ where p, p' are the four momenta of the ingoing and outgoing pions.

(13) The $\sigma\pi\pi$ interaction. σ is a scalar meson, and thus

$$-\mathcal{L}_{int} = (4\pi)^{1/2}\, F_\sigma\, \sigma(\phi \cdot \phi)$$

$F_\sigma^2 = 8.36\mu^2$ (μ is pion mass).

(14) The σNN interaction. As before

$$- \quad _{int} = (4\pi)^{1/2} \, G_o \, \bar{\psi} \, \_ \, \psi \, \sigma$$

where $G_o^2 \quad 3$[49].

(15) The σNN interaction. Assuming vector and tensor interactions are possible, the interaction Lagrangian is

$$- \quad _{int} = i(4\pi)^{1/2} \, G_V \, \bar{\psi} \, \gamma_\lambda \, \_ \, \psi \cdot \sigma_\lambda$$

$$+ (4\pi)^{1/2} \, \frac{G_T}{2m_o} \, \bar{\psi} \, \sigma_{\lambda\mu} \, \_ \, \psi \cdot \partial_\lambda \, \sigma_\mu$$

where $\sigma_{\lambda\mu} = \frac{1}{2i} \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu$, $\frac{1}{2}$ m_o is the nucleon mass. This gives the vertex factor[50]

$$-(4\pi)^{1/2} \, i \quad (G_V + G_T)\gamma_\mu + \frac{iG_T}{m_o} p_\mu$$, where p is the 4 momentum of the incoming baryon.

In practical calculations the second term is ignored, and the vertex factor taken as

$$-(4\pi)^{1/2} \, i \, G_o \, \gamma_\mu, \quad \text{where} \quad G_o = G_V + G_T.$$

Values of G_V, G_T obtained by different methods have been summarised by Signell and Durso[51]. The value of $G_o \quad 3$.

(16) Since angular momentum is conserved, q_i, q_f (the unit directions of the incoming and outgoing momentum respectively) are coplanar vectors.

If $q_i = (x, 0, z)$, $q_f = (x', 0, z')$,

$q_i \times q_f = (0, \sin\theta, 0)$ where $\theta = (q_i, q_f)$.

(17) Partial wave projection operators may be defined for pion-nucleon scattering. (See Cuilli and Fischer[52]).

The operators

$$y_\ell^+ = \frac{2\ell+1}{4\pi\sqrt{(\ell+1)}}\left[(\ell+1)P_\ell(\hat{q}_i,\hat{q}_f) - i\underline{\sigma}.\hat{q}_i\times\hat{q}_f P'_\ell(\hat{q}_i,\hat{q}_f)\right]$$

$$y_\ell^- = \frac{2\ell+1}{4\pi\sqrt{\ell}}\ \ell\ P_\ell(\hat{q}_i,\hat{q}_f) + i\underline{\sigma}.\hat{q}_i\times\hat{q}_f\ P'_\ell(\hat{q}_i,\hat{q}_f)$$

project out partial waves having $J = \ell+\frac{1}{2}$, parity $(-1)^{J+\frac{1}{2}}$, and $J = \ell-\frac{1}{2}$, parity $-(-1)^{J+\frac{1}{2}}$ respectively.

The above angular operators are normalised to

$$\underset{\text{av.}}{\text{spin}}\ \mathrm{Tr}\int y^{\pm +}_{\ell'}(\hat{q}_f,\hat{q}_i)\, Y^{\pm}_{\ell}(\hat{q}_f,\hat{q}_i)\, d\hat{q}_i\, d\hat{q}_f = (2\ell+1)\ \partial_{\ell\ell'} \quad .$$

The projection of a partial wave $F^{\pm}_{\ell}$ from an amplitude $f(\hat{q}_i, \hat{q}_f)$ is

$$\underset{\text{av.}}{\text{spin}}\ \mathrm{Tr}\int_{\text{angles}} dq_i\, dq_f\, (Y^{\pm}_{\ell})^{+}\, f(q_i, q_f) = F^{\pm}_{\ell} \quad .$$

(18) In a pion-nucleon elastic scattering process

$$\pi N \longrightarrow \pi N$$

the Feynman matrix element is $M_{fi} = \bar{u}\,(\hat{q}_f)\left[-A + \frac{iB}{2}\,\gamma(q_i+q_f)\right]u(\hat{q}_i)$

This is related to the amplitude

$f_{fi} = \chi_f^{+}\left[f_1 + f_2(\underline{\sigma}.q_f)(\underline{\sigma}.q_i)\right]\chi_i$ by

$f_{fi} = \frac{m_o}{4\pi W}\cdot M_{fi}$, where m_o is the nucleon mass, and W is the total centre of mass energy.

(19) The elastic partial wave amplitudes B^+_ℓ, B^-_ℓ having definite parity have the form

$$B^{\pm}_{\ell} = \frac{e^{i\partial_{\ell^{\pm}}} \sin \partial_{\ell^{\pm}}}{k}$$

where $\partial_{\ell^{\pm}}$ is the phase shift, and k is the centre of mass momentum.

They obey the unitarity relation

$$\operatorname{Im} B^{\pm}_{\ell} = k \left| B^{\pm}_{\ell} \right|^2 .$$

(20) The partial waves $B^{\pm}$ are projected out from the Feynman matrix element by the relations

$$B^{\pm}_{\ell} = \frac{m_o}{W} \underset{\text{av.}}{\text{spin}} \operatorname{Tr} \int Y^{\pm}_{\ell} \, M_{fi} \, d\hat{q}_f \, d\hat{q}_i \quad .$$

APPENDIX III

The Evaluation of a Principal Value Integral

In this appendix the numerical method used to evaluate the principal value integral

$$I = P\int_{K}^{\infty} \frac{f(x)}{x - x_o} . dx$$

which occurs in Chapters IV and V is considered. This method has been described by Carruthers & Nieto[52], Appendix (A).

Consider the transformation

$$y = \frac{x + K}{x + V - 2K} \quad ; \qquad V - K > 0 \ .$$

Set $f(x) = g(y)$

$$\text{Then} \quad I = P\int_{o}^{1} \frac{\left(\frac{1-y_o}{1-y}\right) g(y).dy}{y-y_o} \quad ; \quad y_o = \frac{x_o + K}{x_o + V - 2K}$$

Adding and subtracting one gives

$$I = \int_{o}^{1} \frac{\left(\frac{1-y_o}{1-y}\right) g(y), - g(y_o)}{y - y_o} .dy + g(y_o)\ln\left(\frac{V - 2K}{y_o - K}\right) \ .$$

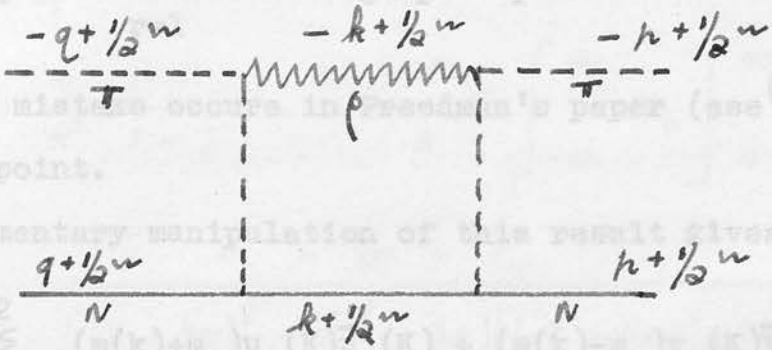
The integral above is no longer a principal value one, but it reduces to the meaningless form 0/0 when $y = y_o$.

This difficulty may be avoided if V is irrational, so y_o is irrational. The limits of y are 0 to 1, and as the interval (0, 1) is divided up for numerical integration, $y \neq y_o$ for any finite number of iterations.

This method is superior to the normal method of evaluating a principal value integral, in which the computer approaches the pole from both sides and subtracts the differencesuntil these become small compared with the total integral, since temporary cancellations may occur. The method described in this appendix avoids this difficulty by converting the principal value integrations into an ordinary integration.

APPENDIX IV

The Solution of the Square Graph in Freedman's Model



Application of the results of Appendix II gives for the matrix element

$$M_{fi} = \frac{1}{(2\pi)^6}\,\bar{u}\,(\underline{q}_f)\left[\int d^4k\left(\frac{\gamma\left[k+½w\right]+m_o}{\left[k+½w\right]^2-m_o^2+i\varepsilon}\right)\left(\frac{1}{\left[q-k\right]^2-\mu^2+i\varepsilon}\right)\right.$$

$$\left.\times\left(\frac{1}{\left[k-p\right]^2-\mu^2+i\varepsilon}\right)\left(\frac{g_{\mu\nu}}{-k+½w^2-m^2+i\varepsilon}\right)\right]\quad u(\underline{q}_i)\quad .$$

Next consider the expression

$$\bar{u}\,(\underline{q}_f)\left[\gamma.K+m_o\right]u(\underline{q}_i)\;;\qquad K = k+½w \qquad \text{(A)}$$

Use the projection operators

$$\Lambda_+(K) = \frac{\gamma.K+m(k)}{2m(k)} = \sum_{r=1}^{2} u_r(K)\,u_r(K)$$

$$\text{(B)}$$

$$\Lambda_-(K) = \frac{m(k)-\gamma.K}{2m(k)} = \sum_{r=1}^{2} v_r(K)\,v_r(K)$$

where u_r, v_r are the Dirac spinors corresponding to the particles and antiparticles respectively of mass $m(k)$.

Solving for $\gamma.K$ from equation (B) above gives

$$(\gamma.K + m_o) = \sum_{r=1}^{2} (m(k)+m_o)u_r(K)\bar{u}_r(K) + (m(k)-m_o)v_r(K)\bar{v}_r(K) .$$

The mistake occurs in Freedman's paper (see[41], Appendix II) at this point.

Elementary manipulation of this result gives

$$\bar{u}_f(q_f) \sum_{r=1}^{2} (m(k)+m_o)u_r(K)\bar{u}_r(K) + (m(k)-m_o)v_r(K)\bar{v}_r(K)$$

$$= \chi_f^+ \left(\left[\frac{E(q)+m_o}{2m_o}\right]\left[E(k)+m_o\right] - \frac{1}{2m_o}\left[(\underline{\sigma}.\underline{q}_f)(\underline{\sigma}.\underline{k}) + \underline{\sigma}.\underline{k})(\underline{\sigma}.q_i)\right]\right.$$

$$\left. + \left[\frac{E(q)+m_o}{2m_o}\right]\left[E(k) - m_o\right] \frac{\underline{\sigma}.q_f \ \ \underline{\sigma}.q_i}{\left[E(q) + m_o\right]^2} \right) \chi_i$$

where χ_f, χ_i are two component spinors.

This result was verified by direct calculation of $\bar{u}(q_f)(\gamma.K+m_o)$. $u(q_i)$ using the form for the Dirac spinors given in Appendix II.

The projection of the positive parity partial wave amplitude follows from Appendix II.

$$B_{\ell^-}(w) = \frac{2\ell+1}{\pi^2\sqrt{\ell}} G^2F^2 \frac{E(q)+m_o}{\left[q^2 w\right]} \int_{-\infty}^{\infty} dk_o \int_{o}^{\infty} dk$$

$$\frac{A_1 + A_2 + A_3}{\left[(k+\tfrac{1}{2}w)^2 - m_o^2 + i\varepsilon\right]\left[(-k+\tfrac{1}{2}w)^2 - m^2 + i\varepsilon\right]}$$

where A_1, A_2, A_3 are given by Freedman.

The most divergent term is A_3, and near $\ell = 1$

$$A_3 = \frac{q^4}{\ell - 1}\left[\frac{m_o - E(k)}{(E(q) + m_o)^2}\right] .$$

Near $\ell = 1$,

$$B_{\ell^-}(w) = \frac{3}{\pi^2}\,\frac{G^2F^2}{\ell - 1}\cdot\frac{E(q) - m_o}{w}\int_{-\infty}^{\infty} dk_o \int_o^{\infty} dk$$

$$\frac{m_o - \tfrac{1}{2}w - k_o}{\left[(k + \tfrac{1}{2}w)^2 - m_o{}^2 + i\varepsilon\right]\left[(-k + \tfrac{1}{2}w)^2 - m^2 + i\varepsilon\right]} .$$

The integral may be evaluated by Feynman technique[53] to yield

$$B_{\ell^-}(w) = \frac{3}{4\pi}\,\frac{(w - m_o)^2 - \mu^2}{w^2}\,\frac{G^2F^2}{\ell - 1}\; h^-(w)$$

where $h^{\mp}(w) = \displaystyle\int_o^1 dx \frac{(wx \mp m_o)}{w^2x^2 - x(w^2 + m_o{}^2 - m^2) + m_o{}^2 + i\varepsilon}$ as in Freedman.

The isotopic factors are

$$\begin{bmatrix} B_{\ell^-}^{1/2}(w) \\ B_{\ell^-}^{3/2}(w) \end{bmatrix} = \frac{3}{4\pi}\begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}\frac{(w - m_o)^2 - \mu^2}{w^2}\cdot\frac{h^-(w)}{\ell - 1}\cdot G^2F^2$$

$$= \frac{(w - m_o)^2 - \mu^2}{w^2}\begin{bmatrix} 4 \\ 1 \end{bmatrix}\frac{G^2F^2}{8\pi}\,\frac{h^-(w)}{\ell - 1} .$$

Hence the Regge trajectory is given in error by equation (8) of Freedman. The correct expression is

$$\begin{bmatrix} \alpha^{1/2}(w) \\ \alpha^{3/2}(w) \end{bmatrix} = 1 + \frac{(w - m_o)}{8\pi}\begin{bmatrix} 4 \\ 1 \end{bmatrix} G^2F^2 \int_{m_o+\mu}^{\infty} dw' \,\frac{q(w')}{w'^2}$$

$$\times\left[\frac{(w' - m_o)^2 - \mu^2}{(w' - m_o)(w' - w)}\,h^-(w') - \frac{(w' + m_o)^2 - \mu^2}{(w' + m_o)(w' + w)}\right] .$$

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