RESEARCHES ON THE RELATIVITY WAVE EQUATION OF THE ELECTRON

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P R E F A C E.

The first chapter contains an account of the extension of Dirac's equation to general relativity while the second one gives a summary of the generalised two-component spinor theory and its application to the wave equation. Spinors are used extensively in Chapter III to deal with the invariant theory of Dirac's equation. Here certain results of Prof. E.T. Whittaker are directly extended to general relativity and the complete scheme of the simpler tensorial quantities including all those with physical interpretations is developed, all the expressions and the relations they obey being derived in a perfectly general manner. A number of these relations are already known but now all of them are proved without the necessity of referring to a special coordinate system or of utilising a special set of mat:rices. The vector form of the wave equation valid in all space-times is derived from the spinor theory, agreeing in form with the vector obtained by Prof. Whittaker from the special relativity equation. In this formulation the wave equation is expressible in terms of four null world-vectors which can re:place the ψ-functions, and all the tensorial quantities are restated in terms of these vectors alone. The tensors and vec:tor wave equation are written out in detail in the case of a Galilean system and these are expressed in matrix notation by means of a special set of α-matrices. It is shown that the matrix with imaginary elements is distinguished from the ones with real elements in this form of the wave equation and the effect/

effect of similarity transformations is considered.

In Chapter IV it is shown that the criticism directed by T. Levi-Civita against the Dirac system in that it depended for its generalisation on specially distinguished directions in space time, does not hold. In the first place his considera:tions were really applied to an equation where the ψ- function was a world vector and so was not the usual wave equation and secondly, the argument does not hold when one deals with the actual Dirac equation which, because of the possibility of spin transformations is shown to distinguish no special directions.

The eigen functions for the hydrogen electron in momentum space are found in Chapter IV, these are a finite series of hypergeometric functions which do not reduce to elementary functions. A form of the wave equation in momentum space is used to derive the fine structure formula.

CHAPTER I.

THE GENERAL RELATIVITY WAVE EQUATION.

The relativity wave-equation of the electron as discovered by Dirac has the usual form:

$$\left(\frac{W}{c} + \alpha_1 p^1 + \alpha_2 p^2 + \alpha_3 p^3 + \alpha_4 mc \right) \psi = 0 \qquad (1.1)$$

where W, p^1, p^2, p^3 are operators representing the energy and the three components of momentum expressed by

$$-\frac{h}{2\pi i} \frac{\partial}{\partial t}, \quad + \frac{h}{2\pi i} \frac{\partial}{\partial x^i} \quad (i = 1, 2, 3) \text{ respectively.}$$

The four α_i are matrices which obey the conditions

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij}\, 1 \quad (i, j = 1, 2, 3, 4) \qquad (1.2)$$

and these are assumed to be hermitian, that is

$$\alpha_i^* = \tilde{\alpha}_i \quad \text{or} \quad \alpha_i = \alpha^+$$

The star * will be used to denote the conjugate complex while ~ will denote the transposed matrix, and + the hermitian adjoint.

From the wave equation and its conjugate complex which are respectively

$$\left. \begin{aligned} &\left(-\frac{\hbar}{i} \frac{\partial}{c\,\partial t} + \frac{\hbar}{i} \sum_1^3 \alpha_i \cdot \frac{\partial}{\partial x^i} + \alpha_4 mc \right) \psi = 0 \\ \text{and} \quad &\left(\frac{\hbar}{i} \frac{\partial}{c\,\partial t} - \frac{\hbar}{i} \sum_1^3 \alpha_i^* \frac{\partial}{\partial x^i} + \alpha_4 mc \right) \psi^* = 0 \end{aligned} \right\} \qquad (1.3)$$

we obtain a more symmetrical form. The first when multiplied by α_4 gives us the equation.

(P/

$$(\rho^{\nu}\frac{\partial}{\partial x^{\nu}} + \mu)\,\psi = 0$$

where summation is to be taken over the repeated index between 0 and 3. Here $\rho^{0} = -\alpha_4$ which is hermitian

and $\rho^{i} = \alpha_4\alpha_i$ $(i = 1,2,3)$ which are skew-hermitian. (1.5)

The imaginary constant $\mu = \frac{2\pi i}{h}mc$

As $x^{0} = ct$, we are dealing with real co-ordinates.

If we write $\phi = \rho^{0}\psi$

then $\psi^{*} = -\alpha_4^{*}\phi^{*} = -\tilde{\alpha}_4\phi^{*}$ (1.6)

and from the conjugate complex equation we at once obtain

$$(\rho^{\nu}\frac{\partial}{\partial x^{\nu}} - \mu)\,\phi^{*} = 0 \qquad (1.7)$$

From the definition of the ρ^{ν} (1.5) and the properties of the α_{μ} we have the relations

$$\rho^{\nu}\rho^{\sigma} + \rho^{\sigma}\rho^{\nu} = 2g^{\nu\sigma}, \qquad (1.8)$$

where in the space of special relativity we have

$$g^{\nu\sigma} = e_{\nu}\,\delta_{\nu\sigma}$$

with $e_0 = 1$ $e_1 = e_2 = e_3 = -1$

When there is an external electro-magnetic field specified by the four potential A_{μ} we add to the operator

$\frac{\partial}{\partial x^{\mu}}$ the quantity $-\frac{2\pi i}{h}\cdot\frac{e}{c}\cdot A_{\mu}$ when it operates on a ψ-function,

and $+\frac{2\pi i}{h}\cdot\frac{e}{c}\cdot A_{\mu}$ when it operates in the conjugate-complex of a ψ-function.

In/

In this notation the well known current vector is

$$J^{\mu} = e\,\phi^{+} \rho^{\mu} \psi \quad = e\,\psi^{+} \rho^{0} \rho^{\mu} \psi . \tag{1.9}$$

Its time component is $J^0 = -e\,\psi^{+}\psi$ which is the density of distribution of electrical charges according to quantum mechanics. That J^{μ} is a non-divergent vector is at once seen if we pre-:multiply (1.4) by ϕ^{+} and (1.7) by $\tilde{\psi}$ and add,

$$\phi^{+} \rho^{\nu} \left(\frac{\partial}{\partial x^{\nu}} \psi \right) + \tilde{\psi}\, \tilde{\rho}^{\nu} \left(\frac{\partial}{\partial x^{\nu}} \phi^{*} \right) + \mu \left(\phi^{+} \psi - \tilde{\psi} \phi^{*} \right) = 0 .$$

That is $\quad \phi^{+} \rho^{\nu} \left(\frac{\partial}{\partial x^{\nu}} \psi \right) + \left(\frac{\partial \phi^{+}}{\partial x^{\nu}} \right) \rho^{\nu} \psi = 0 \quad$ by transposition,

or $\quad \frac{\partial}{\partial x^{\nu}} \left(\phi^{+} \rho^{\nu} \psi \right) = \frac{\partial}{\partial x^{\nu}} \left(J^{\nu} \right) = \operatorname{div} J^{\nu} = 0 .$

Now all this relates to an electron referred to pseudo-orthogonal axes in the space of special relativity. The extension of the wave equation to general relativity has been effected by a number of investigators, who using various methods have ultimately reached similar results. The wave equation differed from other physical equations in that it was not completely tensorial in form so that an immediate generalisation was not possible. There were the α- matrices and a four component wave-function which transformed as a so-called semi-vector.

Fock (1) used an orthogonal ennuple as a system of refer-:ence in general space-time so was able to retain the α-matrices in the formulation. Tensors then have two different type of components, those referred to the co-ordinate axes and those referred to the orthogonal ennuple. From the idea of parallel transfer/

transfer of a semivector which introduces four coefficients C_i, he defined the covariant derivative of the spinor to be $\left(\frac{\partial}{\partial x^i} - C_i\right)\psi$ which is directly comparable with the terms $\left(\frac{\partial}{\partial x^i} - \frac{2\pi i}{h}\cdot\frac{e}{c}\cdot A_i\right)\psi$ occurring in the original wave equation. This held when one differentiated along the directions of the orthogonal ennuple, and the wave equation was expressed in terms of these covariant derivatives of ψ and the Dirac α-matrices.

Finally transitions from ennuple components to general co-ordinates components were accomplished by frequent use of the Ricci co-efficients of rotation. In this way the four compon-:ents of the potential were absorbed into the geometrical scheme, the ennuple component C_i being regarded as equivalent to them. The work is rather complicated on account of the retention of the two types of reference systems.

Other investigators started from a slightly modified form of Dirac's equation such as (1.4). In this way Tetrode (2) used matrices like the ρ^ν and treated them as vectors in space-time in so far as their index was concerned. Finally Schrödinger (3) in a similar way formalised the theory and pre-:sented a full development while Bargmann (4) added some modi-:fications and simplifications to the former's method of approach. In their treatment undue reference to orthogonal ennuples is eliminated/

eliminated, in fact they appear only in the initial stages of the development. In essence, however, their generalisation is equivalent to Fock's, the presentation is different, but the same results evolve. The account of the general relativity wave equation which we now give, follows the line of Bargmann and Schrödinger.

Still keeping to special relativity, letus examine the transformation of the wave equation in its symmetrical form. Make a linear substitution to introduce new co-ordinates

$$y^\mu = a^\mu{}_\nu \; x^\nu$$

or reciprocally $x^\mu = b_\nu{}^\mu y^\nu$

where $a^\mu{}_\nu \; b_\sigma{}^\nu = \delta^\mu_\sigma$.

The wave equation $(\rho^\nu \frac{\partial}{\partial x^\nu} + \mu)\psi = 0$

then becomes in this new system

$$(\rho^\nu a^\mu{}_\nu \frac{\partial}{\partial y^\mu} + \mu)\psi = 0$$

which is $(\gamma^\mu \frac{\partial}{\partial y^\mu} + \mu)\psi = 0$

if we write $\gamma^\mu = a^\mu{}_\nu \, \rho^\nu$

or $\gamma_\mu = b_\mu{}^\nu \rho_\nu$

and leave ψ unaltered. That is, the form of the wave-equation is preserved if we treat the ρ^ν matrices as an ordinary vector in space time when we are dealing with linear substitutions and at the same time leave the wave function unchanged.

In the new system we have

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^{\mu\nu}$$

and if we treat μ and ν as ordinary tensor indices we

can/

can lower them by means of the fundamental $g^{\mu\nu}$ tensor and obtain

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} .$$

At the same time the new current vector is

$$\acute{J}^\mu = a^\mu{}_\nu J^\nu = e a^\mu{}_\nu \phi^+ \rho^\nu \psi$$
$$= e \phi^+ \gamma^\mu \psi .$$

We observe that ϕ is still equal to $\rho^\circ \psi$, which we shall write hereafter as $\rho\psi$, but now ρ is not one of the four γ-matrices. The hermiticity of the matrix $\rho\gamma^\mu$ is preserved for this is $a^\mu{}_\nu \rho \rho^\nu$

and in this linear aggregate, we have real coefficients $a^\mu{}_\nu$ because we are dealing with real coordinates and real trans:formations and the matrix $\rho\rho^\nu$ in each term is hermitian so that the sum is also hermitian. Thereby is a real current vector obtained. It will be noted that the γ^μ are expressed as the sum of three skew and one hermitian matrix so that in:dividually these matrices are neither skew nor hermitian. The employment of the special ρ matrix simplifies the reality con:siderations.

The conditions (1.8) are not sufficient to define ρ^ν uniquely and we have now the possibility of applying "spin" - transformations to these quantities for each component ρ^ν is not a simple number but a matrix. A spin or similarity trans:formation such as

$$\acute{\gamma}^\mu = S^{-1} \gamma^\mu S \qquad (1.12)$$

where/

where S is a square matrix of the same type as γ^μ will produce new matrices which still satisfy the condition, and the wave equation will remain invariant in form provided that we simultaneously vary ψ according to the rule:-

$$\psi' = S^{-1}\psi . \tag{1.13}$$

Transformation of this second type must not affect quantities which have direct physical interpretation; we see that the current vector may be kept constant if we admit the law of varia:tion of the matrix ρ to be

$$\rho' = S^{+}\rho S \tag{1.14}$$

whence it follows that $\phi' = \rho'\psi' = S^{+}\phi$ (1.15)

Moreover we observe that

$$\rho'^{+} = (S^{+}\rho S)^{+} = S^{+}\rho^{+}S = S^{+}\rho S = \rho' \tag{1.16}$$

that is, a similarity transformation does not change the pro:perty of hermiticity of ρ. Likewise the matrix $\rho\gamma^\nu$ has its hermiticity preserved for, in the new system

$$\begin{aligned}(\rho'\gamma'^\nu)^{+} &= (\gamma'^\nu)^{+}\rho'^{+} = (S^{-1}\gamma^\nu S)^{+}(S^{+}\rho S)\\ &= S^{+}\gamma^{\nu+}(S^{-1})^{+}.S^{+}\rho S\\ &= S^{+}(\rho\gamma^\nu)^{+}S\\ &= \rho'\gamma'^\nu .\end{aligned}$$

Another simple relation

$$\rho\gamma^\nu = \gamma^{\nu+}\rho \tag{1.17}$$

which follows from the previous result as ρ is always hermitian, is also invariant under S - transformation. Similarly

$$\rho\gamma_\nu = \gamma_\nu^{+}\rho$$

is/

is an invariant relation.

Now define the commutator, introduced by Schrödinger

$$S^{\mu\nu} = \tfrac{1}{2}\left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right) \tag{1.18}$$

Then as a consequence of (1.17) we have invariantly

$$\rho\, S^{\mu\nu} + S^{\mu\nu +}\rho = 0 . \tag{1.19}$$

From the commutation rules for the γ -matrices follows the relation

$$\gamma_\alpha S^{\mu\nu} - S^{\mu\nu}\gamma_\alpha = 2\left(\delta^\mu_\alpha \gamma^\nu - \delta^\nu_\alpha \gamma^\mu\right) \tag{1.20}$$

which is very useful as it expresses a single γ -matrix as a commutator.

GENERAL RELATIVITY.

At this stage it becomes possible to consider the generalisation of this work to any space-time. The natural generalisation of (1.10) is of course

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, g^{\mu\nu}(x) \tag{1.21}$$

where $g^{\mu\nu}(x)$ is the metric tensor of the general space-time as is a function of the co-ordinates. To find a solution of this equation we choose an orthogonal ennuple as reference system and use Latin indices for components referred to it, while we continue to use Greek indices for components with respect to the co-ordinates x^μ. Then, if V_α is any vector, the relation between the co-ordinate components V_α and the ennuple components V_i is

$$V_\alpha = \sum_{i=0}^{3} e_i V_i \lambda_{i|\alpha} ,$$

with/

with $e_0 = 1 \quad e_1 = e_2 = e_3 = -1,$

and $\lambda_{i|\alpha}$ as the compnnents of the ennuple.

Conversely $\quad V_i = V_\alpha \lambda_{i|}{}^{\alpha},$

the summation convention still applying to repeated Greek indices.

Here $$\lambda_{i|}{}^{\alpha} \lambda_{j|\alpha} = \sum_i e_i \delta_{ij}$$

$$\sum e_i \lambda_{i|\alpha} \lambda_{i|\beta} = g_{\alpha\beta}$$

$$\lambda_{i|}{}^{\alpha} = g^{\alpha\beta} \lambda_{i|\beta}.$$

If we take $$\gamma_\alpha = \sum e_i \lambda_{i|\alpha} \rho_i$$

where the ρ_i are the same as those defined in (1.8), then we find that

$$\begin{aligned}\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha &= \sum_{ij} (e_i \lambda_{i|\alpha} \rho_i e_j \lambda_{j|\beta} \rho_\beta + e_j \lambda_{j|\beta} e_j e_i \lambda_{i|\alpha} \rho_i) \\ &= \sum_{ij} (e_i e_j \lambda_{i|\alpha} \lambda_{j|\beta} [\rho_i \rho_j + \rho_j \rho_i]) \\ &= \sum_{ij} (e_i e_j \lambda_{i|\alpha} \lambda_{j|\beta} \cdot 2 e_j \delta_{ij}) \\ &= 2 g_{\alpha\beta}.\end{aligned}$$

In this way we have obtained one possible solution for the at a definite point $P (x^\mu)$. We must see how it would hold if we proceeded to a neighbouring point $P' = (x^\mu + \delta x^\mu)$.

By differentiating the relation (1.21) we have at once

$$\begin{aligned}(\delta\gamma_\mu)\gamma_\nu + \gamma_\mu(\delta\gamma_\nu) + (\delta\gamma_\nu)\gamma_\mu + \gamma_\nu(\delta\gamma_\mu) &= 2\,\delta(g_{\mu\nu}) \\ &= 2 \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \delta x^\sigma. \qquad (1.22)\end{aligned}$$

But/

But the covariant derivation of the fundamental tensor is zero, that is

$$(g_{\mu\nu})_{\sigma} \equiv \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} - \Gamma^{\alpha}_{\sigma\mu}\, g_{\alpha\nu} - \Gamma^{\alpha}_{\sigma\nu}\, g_{\mu\alpha} = 0 \qquad (1.23)$$

Also

$$\delta \gamma_{\mu} = \sum e_i \; \delta(\lambda_{i|\mu})\, e_i$$

$$= e_i \; \frac{\partial \lambda_{i|\mu}}{\partial x^{\sigma}} (\delta x^{\sigma})\, \lambda_{i|}{}^{\alpha}\, \gamma_{\alpha}$$

which is of the form

$$\delta \gamma_{\mu} = C^{\alpha}{}_{\mu\sigma}\, \gamma_{\alpha}\, \delta x^{\sigma}, \qquad (1.24)$$

the coefficients $C^{\alpha}{}_{\mu\sigma}$ not necessarily being tensorial.

When the results of (1.23) and (1.24) are substituted in (1.22) we find that

$$\left\{ (C^{\alpha}{}_{\mu\sigma} - \Gamma^{\alpha}_{\mu\sigma})\, g_{\nu\alpha} + (C^{\alpha}{}_{\nu\sigma} - \Gamma^{\alpha}_{\nu\sigma})\, g_{\mu\alpha} \right\} \delta x^{\sigma} = 0 .$$

Now our results must be independent of the directions of the displacement, all directions at P must be treated equally. Therefore we demand that the coefficients of δx^{σ} should vanish separately.

Thus $\quad (C^{\alpha}{}_{\mu\sigma} - \Gamma^{\alpha}_{\mu\sigma})\, g_{\nu\alpha} + (C^{\alpha}{}_{\nu\sigma} - \Gamma^{\alpha}_{\nu\sigma})\, g_{\mu\alpha} = 0.$

Let $\quad A^{\alpha}{}_{\mu\sigma} = C^{\alpha}{}_{\mu\sigma} - \Gamma^{\alpha}_{\mu\sigma} \qquad (1.25)$

and $\quad A_{\alpha\mu\sigma} = g_{\alpha\beta}\, A^{\beta}{}_{\mu\sigma}.$

Then the condition is expressible as

$$A_{\nu\mu\sigma} + A_{\mu\nu\sigma} = 0 \qquad (1.26)$$

where the $A_{\nu\mu\sigma}$ are now tensor components.

Expressed in terms of these, the result (1.24) is

$$\frac{\partial \gamma_{\mu}}{\partial x^{\sigma}} - \Gamma^{\alpha}_{\mu\sigma}\, \gamma_{\alpha} = A^{\alpha}{}_{\mu\sigma}\, \gamma_{\alpha} = A_{\alpha\mu\sigma}\, \gamma^{\alpha}. \qquad (1.27)$$

Multiply/

Multiply (1.20) by $A_{\mu\nu\sigma}$ and sum over μ and ν :

$$A_{\mu\nu\sigma}\, 2(\delta^\mu_\alpha \gamma^\nu - \delta^\nu_\alpha \gamma^\mu) = A_{\mu\nu\sigma}(\gamma_\alpha s^{\mu\nu} - s^{\mu\nu}\gamma_\alpha)$$

which by (1.26) leads to

$$A_{\nu\alpha\sigma}\gamma^\nu = -\tfrac{1}{4} A_{\mu\nu\sigma}(\gamma_\alpha s^{\mu\nu} - s^{\mu\nu}\gamma_\alpha)$$

$$= +\left(\tfrac{1}{4} A_{\mu\nu\sigma} s^{\mu\nu} + a_\sigma 1\right)\gamma_\alpha - \gamma_\alpha\left(\tfrac{1}{4} A_{\mu\nu\sigma} s^{\mu\nu} + a_\sigma 1\right)$$

where 1 is the unit matrix and a_σ commutes with all γ_μ.
It is introduced arbitrarily. If we write

$$\tfrac{1}{4} A_{\mu\nu\sigma} s^{\mu\nu} + a_\sigma 1 = \Gamma_\sigma \tag{1.28}$$

equation (1.27) takes the form

$$\frac{\partial\gamma_\mu}{\partial x^\sigma} = \Gamma^\alpha_{\mu\sigma}\gamma_\alpha + \Gamma_\sigma\gamma_\mu - \gamma_\mu\Gamma_\sigma . \tag{1.29}$$

It is now a question of finding the conditions under which this equation is integrable. We require, therefore, that

$$\frac{\partial}{\partial x^\tau}\left(\frac{\partial\gamma_\mu}{\partial x^\sigma}\right) - \frac{\partial}{\partial x^\sigma}\left(\frac{\partial\gamma_\mu}{\partial x^\tau}\right) = 0.$$

This very quickly reduces to the condition that

$$R^{\alpha}_{\cdot\,\mu\tau\sigma}\gamma_\alpha + \phi_{\tau\sigma}\gamma_\mu - \gamma_\mu\phi_{\tau\sigma} = 0 \tag{1.30}$$

where as usual the Riemannian curvature-tensor is defined as

$$R^{\alpha}_{\cdot\,\mu\tau\sigma} = \frac{\partial}{\partial x^\tau}\Gamma^\alpha_{\mu\sigma} - \frac{\partial}{\partial x^\sigma}\Gamma^\alpha_{\mu\tau} + \Gamma^\beta_{\mu\sigma}\Gamma^\alpha_{\beta\tau} - \Gamma^\beta_{\mu\tau}\Gamma^\alpha_{\sigma\beta} ; \tag{1.31}$$

and where for brevity we write

$$\phi_{\tau\sigma} = \frac{\partial\Gamma_\sigma}{\partial x^\tau} - \frac{\partial\Gamma_\tau}{\partial x^\sigma} + \Gamma_\sigma\Gamma_\tau - \Gamma_\tau\Gamma_\sigma . \tag{1.32}$$

Another application of (1.20) to the equation (1.30) gives us the following explicit expression for $\phi_{\tau\sigma}$

$$\phi_{\tau\sigma} = -\tfrac{1}{4} R_{\alpha\beta\tau\sigma} s^{\alpha\beta} + f_{\alpha\beta} . 1 . \tag{1.33}$$

where/

where $f_{\tau\sigma}$ is, as yet, an arbitrary tensor.
We can connect up these arbitrary quantities by considering the spurs of some of the matrices. The spur of Γ_σ because $s^{\mu\nu}$ is a commutator of the γ matrices and therefore possesses zero spur, is from its definition (1.28) equal to $4\ a_\sigma$. From (1.32) we find the spur of $\phi_{\tau\sigma}$

$$\text{spur}\ \ \phi_{\tau\sigma} = 4\left(\frac{\partial a_\sigma}{\partial x^\tau} - \frac{\partial a_\tau}{\partial x^\sigma}\right),$$

while from (1.33) we have an alternative impression

$$\text{spur}\ \ \phi_{\tau\sigma} = 4\ f_{\tau\sigma}.$$

Hence we see that the a_σ and $f_{\tau\sigma}$ are not completely arbi:trary but are connected by the relation,

$$f_{\tau\sigma} = \frac{\partial a_\sigma}{\partial x^\tau} - \frac{\partial a_\tau}{\partial x^\sigma}. \qquad (1.34)$$

Therefore the a_σ completely determine the $f_{\tau\sigma}$ tensor com:ponents and the a_σ themselves are quite independent of the γ - matrices.
The a_σ are identified with the components of the four-potential (apart from a constant factor) so that the six-vector $f_{\tau\sigma}$ gives us the electro magnetic field strengths. We just summarise the previous line of work: a solution γ_μ of the matrix equation $\quad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}$
was found for a certain point. The conditions that this de:finition of γ_μ would hold at a near-by point independent of its direction, led to a differential condition which in its most general form contained an arbitrary quantity a_σ.
The/

The integrability conditions arising from the differential equation brought in the arbitrary six-vector $\phi_{\sigma\tau}$ of which the spur however was derivable as the curl of our former quantity a_{σ}.

Vectors and pure physical quantities will not be influenced by S-transformations. Quantities such as γ_{μ} and Γ_{σ} have a dual nature in that their tensor character, in so far as we are dealing with transformations of the co-ordinates, is correctly indicated by their indices, and that moreover they are affected by a spin-transformation. Let us now admit S-transformation

$$\acute{\gamma}_{\mu} = S^{-1} \gamma_{\mu} S$$

and find how equation (1.29) will be affected. It is at once seen that this equation will remain valid also in the accented system if at the same time the Γ_{σ} undergo the transformation,

$$\acute{\Gamma}_{\sigma} = S^{-1} \Gamma_{\sigma} S - S^{-1} \frac{\partial S}{\partial x^{\sigma}} \qquad (1.35)$$

As $S^{-1} S = S S^{-1} = 1$, differentiation gives

$$\frac{\partial S^{-1}}{\partial x^{\mu}} . S + S^{-1} \frac{\partial S}{\partial x^{\mu}} = 0$$

and spur $\left(\frac{\partial S^{-1}}{\partial x^{\mu}} . S \right) = \frac{\partial}{\partial x^{\mu}}$ (log det $|S|$).

Therefore, on taking the spurs in equation (1.35) we have

$$\acute{a}_{\sigma} = \frac{1}{4} \text{ spur } \acute{\Gamma}_{\sigma} = a_{\sigma} - \frac{1}{4} \frac{\partial}{\partial x^{\sigma}} (\log \det |S|). \qquad (1.36)$$

As a_{σ}, therefore, is not invariant under an S-transformation but is altered to the extent of an additional term which is the gradient of a scalar quantity, it is not a simple vector. But this/

this is precisely the type of variation permitted the four-potential by the principle of gauge invariance. On the other hand the quantities $\phi_{\tau\sigma}$ behave like the γ_μ as we see from (1.32) and (1.35),

$$\phi'_{\tau\sigma} = S^{-1}\,\phi_{\tau\sigma}\,S\ ,$$

and its spur, which is proportional to the field strength re-:mains unaltered as is necessary for a definite physical quantity. Thus we have further justification for our interpretation of a_σ and $f_{\tau\sigma}$.

As the $A_{\mu\nu\sigma}$ are all real, it follows from (1.19) and (1.28) that

$$\rho\,\Gamma_\sigma + \Gamma_\sigma^{+}\rho = (a_\sigma + a_\sigma^{*})\,\rho\ .$$

As a_σ is proportional to the potentials it would first appear very satisfactory to take

$$a_\sigma + a_\sigma^{*} = 0$$

that is, take the factor of proportionality to be pure imaginary and thereby obtain real values of the potential. However such an equation is obviously not invariant under S-transformations except when det /S/ is independent of the x^μ. For general coordinate systems the equation

$$\rho_{(\sigma)} = \frac{\partial\rho}{\partial x^\sigma} + \rho\,\Gamma_\sigma + \Gamma_\sigma^{+}\rho = 0 \qquad (1.37)$$

is found to be invariant. Under coordinate transforma-:tions $\rho_{(\sigma)}$ behaves as a covariant vector, while after a similarity transformation we have

$$\rho'_{(\sigma)} = \frac{\partial}{\partial x^\sigma}(S^{+}\rho S) + (S^{+}\rho S)\left(S^{-1}\Gamma_\sigma S - S^{-1}\frac{\partial S}{\partial x^\sigma}\right) + \left\{S^{+}\Gamma_\sigma^{+}(S^{-1})^{+} - \frac{\partial S^{+}}{\partial x^\sigma}(S^{-1})^{+}\right\}S^{+}\rho S$$

$$= S^{+}\left(\frac{\partial\rho}{\partial x^\sigma} + \rho\,\Gamma_\sigma + \Gamma_\sigma^{+}\rho\right)S = S^{+}\rho_{(\sigma)}S = 0\ .$$

Thus/

Thus equation possesses the required invariance property. Two more solutions of a similar type are obtainable: from (1.32) and (1.37) it follows that

$$\rho \, \phi_{\sigma\tau} + \phi^{+}_{\sigma\tau} \, \rho = \frac{\partial}{\partial x^{\sigma}} \frac{\partial \rho}{\partial x^{\tau}} - \frac{\partial}{\partial x^{\tau}} \frac{\partial \rho}{\partial x^{\sigma}} = 0 . \qquad (1.38)$$

using this result in (1.33) we have, on noting (1.19)

$$\rho \, f_{\sigma\tau} + f^{+}_{\sigma\tau} \, \rho = 0$$

so that the $f_{\sigma\tau}$ are always imaginary yielding real field strengths.

So far attention has been concentrated on the γ_{μ} -matrices and our considerations have brought in the quantities associated with the field. Now we must consider the ψ -functions and similar quantities and investigate their covariant derivatives. It has been seen that γ_{μ} behaves like a tensor, but because each component is not a simple number but a matrix, spin-trans:formations can be applied to it. A system of matrix operators denoted by $T^{\mu\nu\cdots}_{\sigma\tau\cdots}$ is said to constitute a tensor-operator or Schrödinger tensor of degree m + n when T, for point sub:stitutions behaves like an ordinary tensor with m contravariant and n covariant indices, and transforms into S^{-1} T S as as result of a similarity transformation upon the γ^{μ}. Ob:viously γ^{μ} itself is such a tensor-operator, while the work above shows that $s^{\mu\nu}$ and $\phi_{\sigma\tau}$ supply further examples. Also $\Psi^{\mu\nu\cdots}_{\sigma\tau\cdots}$, $\Phi^{\mu\nu\cdots}_{\sigma\tau\cdots}$ and $R^{\mu\nu\cdots}_{\sigma\tau\cdots}$ are called ψ- , ϕ- , or ρ - tensors, if Ψ , Φ , R transform as/

as tensors for point substitutions and into

$$S^{-1}\,\Psi$$
$$S^{+}\,\Phi$$
$$S^{+}\,R\,S$$

respectively for the similarity transformation,

ψ, ρ and ϕ are clearly ψ - , ρ - and ϕ - tensors of zero rank, and ψ and ϕ are often described as spin-tensors or spinors.

Products of these various types of tensors behave as ordinary tensors for coordinate transformations but the type may be altered.

For example:	$T\,\Psi$	is	a	ψ - tensor
	$T\,\Phi$	is	a	ϕ - tensor
$M =$	$\rho\,T$	is	a	ρ - tensor
$M^{+} =$	$T^{+}\rho$	is	a	ρ - tensor .

A tensor equation is invariant under transformation of coordinates and further, if all the terms are tensors of the same type, the invariance will hold for spin-transformations. For example

$$M \pm M^{+} = 0,$$

is a simple ρ - tensor equation, invariant for both kinds of transformation. Cases of such equations are afforded by (1.17), (1,19) and 1.38).

The various combinations of all these different tensors together with their reciprocals and hermitian adjoints lead to quantities/

quantities G which, under S-transformations, change according to

$$G \to P G Q$$

where P may be one of the set of operators $1,\ S^{-1},\ S^{+}$
and Q $1,\ S,\ (S^{+})^{-1}$.

COVARIANT DERIVATIVES.

The covariant derivative of G is symbolised by $\nabla_{\sigma} G$ or $(G)_{\sigma}$
Before a suitable meaning is sought, the task is simplified by admitting the two following postulates.

1. If G is of ordinary tensorial rank $m + n$, then $(G)_{\sigma}$ is of rank $m + \overline{n - 1}$
2. Under an S-transformation $(G)_{\sigma}$ should become $P.(G)_{\sigma}.Q$, that is we demand that the original tensor and its covariant de-:rivative should be of the same type with regard to S-trans-:formations.

The operator ∇_{σ} thus depends on the nature (i.e. its tensor rank and type) of its operand, but it will be of the form:

$$\nabla_{\sigma} = \mathring{\nabla}_{\sigma} + F\,(G, \Gamma_{\sigma})$$

where $\mathring{\nabla}_{\sigma}$ denotes the ordinary covariant derivation of Riemannian geometry and F is a linear function of G, Γ_{σ} and Γ^{+}_{σ}. The presence of this additional term is necessitated by the second postulate. Finally we wish to preserve the product rule so that if G is a product of two other tensors i.e. $G = G_1\, G_2$ which is possible if $Q_1\, P_2 = 1$, then it is required that

$$\nabla_{\sigma} G = (\nabla_{\sigma} G_1)\, G_2 + G_1 (\nabla_{\sigma} G_2).$$

These/

These requirements are suitably satisfied by the following definitions, -

(a) Tensor - operators.

$$\nabla_\sigma T = \mathring{\nabla}_\sigma T + T\Gamma_\sigma - \Gamma_\sigma T \tag{1.39}$$

in particular.

$$\nabla_\sigma \gamma_\mu = \frac{\partial \gamma_\mu}{\partial x^\sigma} - \Gamma^\alpha_{\mu\sigma}\gamma_\alpha + \gamma_\mu \Gamma_\sigma - \Gamma_\sigma \gamma_\mu \tag{1.40}$$

$$= 0 \quad \text{by equation (1.29).}$$

Hence, as $\nabla_\sigma \equiv \mathring{\nabla}_\sigma$ for ordinary (c) tensors

$$\nabla_\sigma \gamma^\nu = \nabla_\sigma(\gamma_\mu g^{\mu\nu}) = 0 \quad \text{by the product rule.}$$

(b) Ψ - tensors.

$$\nabla_\sigma \Psi = \mathring{\nabla}_\sigma \Psi - \Gamma_\sigma \Psi. \tag{1.41}$$

Thus for the ψ - spinor

$$\nabla_\sigma \psi = \frac{\partial \psi}{\partial x^\sigma} - \Gamma_\sigma \psi . \tag{1.42}$$

($\nabla_\sigma \psi$) is another ψ- tensor, and so differentiation may be repeated following the same rule. So we find that

$$\nabla_\tau(\nabla_\sigma \psi) - \nabla_\sigma(\nabla_\tau \psi) = \phi_{\sigma\tau}\psi \quad \text{by (1.32).} \tag{1.43}$$

(c) R-tensors.

$$\nabla_\sigma R = \mathring{\nabla}_\sigma R + R\Gamma_\sigma + \Gamma^+_\sigma R, \tag{1.44}$$

and $\nabla_\sigma \rho = 0$ by (1.37) (1.45)

By the product rule $\nabla_\sigma (\rho\gamma^\mu) = 0$.

(d) Φ - tensors.

$$\nabla_\sigma \Phi = \mathring{\nabla}_\sigma \Phi + \Gamma^+_\sigma \Phi , \tag{1.46}$$

so that $\nabla_\sigma \varphi = \dfrac{\partial \phi}{\partial x^\sigma} + \Gamma^+_\sigma \phi$ (1.47)

and $\nabla_\sigma \phi^+ = \dfrac{\partial \phi^+}{\partial x^\sigma} + \phi^+ \Gamma_\sigma$.

The/

The consistency of these definitions is easily verified. As an example, consider the current vector which should be an ordinary c - tensor,

$$\nabla_\sigma J^\mu = \nabla_\sigma (\phi^+ \gamma^\mu \psi) \qquad \text{by the product rule}$$

$$= (\nabla_\sigma \phi^+) \gamma^\mu \psi + \phi^+ \gamma^\mu (\nabla_\sigma \psi) \qquad \text{noting that} \quad \nabla_\sigma \gamma^\mu = 0.$$

$$= \frac{\partial \phi^+}{\partial x^\sigma} \gamma^\mu \psi + \phi^+ \gamma^\mu \frac{\partial \psi}{\partial x^\sigma} + \phi^+ \Gamma_\sigma \gamma^\mu \psi - \phi^+ \gamma^\mu \Gamma_\sigma \psi$$

$$= \frac{\partial \phi^+}{\partial x^\sigma} \gamma^\mu \psi + \phi^+ \gamma^\mu \frac{\partial \psi}{\partial x^\sigma} + \phi^+ \left(\frac{\partial \gamma^\mu}{\partial x^\sigma} + \Gamma^\mu_{\sigma\alpha} \gamma^\alpha \right) \psi \qquad \text{by (1.29)}$$

$$= \frac{\partial}{\partial x^\sigma} (\phi^+ \gamma^\mu \psi) + \Gamma^\mu_{\sigma\alpha} (\phi^+ \gamma^\alpha \psi)$$

$$= \mathring{\nabla}_\sigma J^\mu \quad \text{as required.}$$

In fact, it is clearly seen that any quantity of the form

$$\phi^+ T \psi$$

where T is a tensor operator of any rank, is a ordinary c-tensor of the same rank as T. This will be real if ρT is an hermitian matrix, as is the case when $T = \gamma^\mu$ or $= i s^{\mu\nu}$ for example. The advantage of Bargmann's treatment lies largely in the use of this special matrix ρ by means of which the hermiticity of certain other matrices leading to real physical quantities is the more easily assured. This direct method obviates the long and tedious investigations applied to each type of nature separately as in Schrödinger's paper.

THE GENERAL RELATIVITY EQUATIONS.

The Dirac equation in its symmetrical form is

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + \mu \right) \psi = 0$$

This/

This is now generalised to

$$(\gamma^{\mu} \nabla_{\mu} + \mu)\, \psi = 0 \qquad (1.48)$$

an equation which is a ψ-tensor one (each term is a ψ-scalar) and so it is invariant both for point substitutions and for spin-transformations. Moreover the general definition of the operator ∇_{μ} introduces quantities which have been identified with the components of the electro-magnetic potential so that this equation can be applied to an electron in any external electro-magnetic field. In special relativity, the effect of the field was accounted for when the operator

$\frac{\partial}{\partial x^{\sigma}}$ was replaced by $\frac{\partial}{\partial x^{\sigma}} - \frac{2\pi i}{h} \frac{e}{c} A_{\sigma}$.

Now we have appearing the operator

$$\nabla_{\sigma}(\psi) = \left\{ \frac{\partial}{\partial x^{\sigma}} - \Gamma_{\sigma} \right\} (\psi)$$

which reduces to the former when the general space-time reduces to that of special relativity if we take

$$a_{\sigma} = \frac{2\pi i}{h} \frac{e}{c} A_{\sigma} \qquad (1.49)$$

This comparison has revealed the factor of proportionality that exists between a_{σ} and the four-potential A_{σ}, and between $f_{\sigma\tau}$ and electromagnetic six-vector $F_{\sigma\tau}$.

This has been the chief purpose of the theory, to obtain this generalisation of the operator which occurs in the original Dirac equation and to describe and treat it geometrically, as an operator of covariant differentiation when it is applied to a ψ-function.

The/

The adjoint of the equation can be deduced from the new form, $\mu \phi^{+} = \mu \psi^{+} \rho = (\gamma^{\mu} \nabla_{\mu} \psi)^{+} \rho$ by the equation (1.48) as μ is an imaginary constant,

$$= (\nabla_{\mu} \psi^{+})(\gamma^{\mu +} \rho)$$

$$= (\nabla_{\mu} \psi^{+})(\rho \gamma^{\mu})$$

$$= \nabla_{\mu} \phi^{+} \gamma^{\mu}$$

Hence $\phi^{+} (\overleftarrow{\nabla}_{\mu} \gamma^{\mu} - \mu) = 0$ (1.50)

is the generalised form of the hermitian adjoint of Dirac's equation. (The arrow denotes that the covariant differential operator acts upon ϕ^{+} on the left).

This generalisation is perfectly satisfactory, in special relativity it reduces to the required form, and it preserves the non-divergence of the current vector. The latter fact is readily proved for

$$\nabla_{\mu} J^{\mu} = \nabla_{\mu} (\phi^{+} \gamma^{\mu} \psi)$$

$$= (\nabla_{\mu} \phi^{+}) \gamma^{\mu} \psi + \phi^{+} \gamma^{\mu} (\nabla_{\mu} \psi)$$

$$= (\mu \phi^{+}) \gamma^{\mu} \psi + \phi^{+} \gamma^{\mu} (-\mu \psi)$$

$$= 0 .$$

GAUGE/

GAUGE INVARIANCE.

This is demonstrated by means of the special spin-trans-:formation obtained by taking

$$S = e^{-i\lambda}\, 1$$

whence $\log.\det.|S| = -4\, i\lambda.$

Then by (1.36) $\acute{a}_\mu = a_\mu + i\,\dfrac{\partial\lambda}{\partial x^\mu}$

so that $\acute{A}_\mu = A_\mu + \dfrac{hc}{2\pi e}\,\dfrac{\partial\lambda}{\partial x^\mu}$

while $\acute{\psi} = S^{-1}\psi = e^{i\lambda}\,\psi$ (1.51)

Thus under this particular transformation the four-potential is changed to the extent of an added gradient (of a scalar function), while the phase of the wave function ψ is altered, the form of the wave equation remaining the same. This is Weyl's (5) principle of gauge-invariance.

THE SECOND-ORDER WAVE EQUATION.

From the first order equation

$$\gamma^\mu \nabla_\mu \psi = -\mu\psi$$

by operating on both sides with $\gamma^\nu \nabla_\nu$ we obtain

$$\gamma^\nu \nabla_\nu\, \gamma^\mu \nabla_\mu \psi = -\mu\, \gamma^\nu \nabla_\nu \psi$$
$$= \mu^2 \psi .$$

Since $\nabla_\mu \gamma^\mu = 0$ this becomes

$$\gamma^\nu \gamma^\mu \nabla_\nu \nabla_\mu \psi = \mu^2 \psi$$

Now $\gamma^\nu \gamma^\mu = g^{\mu\nu}\, 1 + S^{\mu\nu}$ so that we have

$$\left(g^{\nu\mu} \nabla_\nu \nabla_\mu + s^{\nu\mu} \nabla_\nu \nabla_\mu\right)\psi = \mu^2 \psi$$

and/

and $\quad (g^{\nu\mu} \nabla_\nu \nabla_\mu + s^{\mu\nu} \nabla_\mu \nabla_\nu)\psi = \mu^2\psi$ by interchange of dummy indices. Average the two equations, noting that $S^{\mu\nu}$ is skew and that

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\psi = \phi_{\mu\nu}\psi$$

Hence

$$\left(g^{\nu\mu} \nabla_\nu \nabla_\mu - \frac{s^{\mu\nu}}{2} \phi_{\mu\nu}\right)\psi = \mu^2\psi.$$

After introducing the implicit expression for $\phi_{\mu\nu}$ (1.33), one obtains the equation

$$\left(g^{\nu\mu} \nabla_\nu \nabla_\mu - \tfrac{1}{2} s^{\mu\nu} f_{\mu\nu} + \tfrac{1}{8} s^{\mu\nu} s^{\alpha\beta} R_{\alpha\beta\mu\nu} - \mu^2\right)\psi = 0. \tag{1.52}$$

This fourfold sum $\frac{1}{8} s^{\mu\nu} s^{\alpha\beta} R_{\alpha\beta\mu\nu}$ can be evaluated leading to a simple result. $R_{\alpha\beta\mu\nu}$ is skew in (α,β) and in (μ,ν), so that form (1.18) term considered is equivalent to

$\frac{1}{8} \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta R_{\mu\nu\alpha\beta}$

$= \frac{1}{8} \gamma^\mu (\gamma^\nu \gamma^\alpha + \gamma^\alpha \gamma^\nu - \gamma^\alpha \gamma^\nu) \gamma^\beta R_{\alpha\beta\mu\nu}$ which by (1.11)

$= \frac{1}{8} \gamma^\mu \gamma^\beta (2 g^{\nu\alpha}) R_{\alpha\beta\mu\nu} - \frac{1}{8} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta R_{\alpha\beta\mu\nu}$

$= \frac{1}{4} (g^{\mu\beta} + s^{\mu\beta}) R_{\beta\mu} - \frac{1}{24} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta (R_{\alpha\beta\mu\nu} + R_{\alpha\beta\mu\nu} + R_{\alpha\mu\mu\nu})$

$R_{\beta\mu}$ is the contracted Riemann or Ricci tensor; it is symmetric and when it is contracted it gives the curvature scalar R

where $R = g^{\mu\beta} R_{\mu\beta} = g^{\mu\beta} (R^{\nu}{}_{\cdot\beta\mu\nu})$

Therefore the term is now

$R/4$

$$\frac{R}{4} - \frac{1}{24}\Big\{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta R_{\alpha\beta\mu\nu} + \gamma^\mu\left(2g^{\nu\alpha} - \gamma^\nu\gamma^\alpha\right)\gamma^\beta R_{\alpha\beta\mu\nu} + \gamma^\mu\gamma^\alpha\left(2g^{\nu\beta} - \gamma^\beta\gamma^\nu\right) R_{\alpha\beta\mu\nu}\Big\}$$

$$= \frac{R}{4} - \frac{1}{24}\Big\{\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta R_{\alpha\beta\mu\nu} + 2\gamma^\mu\gamma^\beta R_{\beta\mu} - \gamma^\mu\gamma^\nu\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right) R_{\alpha\beta\mu\nu} - 2\gamma^\mu\gamma^\alpha R_{\alpha\mu} - \gamma^\mu\left(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha\right)\gamma^\nu R_{\alpha\beta\mu\nu}\Big\}$$

$$= \frac{R}{4} - \frac{1}{24}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\left\{R_{\alpha\beta\mu\nu} + R_{\beta\nu\mu\alpha} + R_{\nu\alpha\mu\beta}\right\}$$

when the dummy suffixes are changed.

The last bracket by the symmetry properties is

$$R_{\alpha\beta\mu\nu} + R_{\alpha\mu\nu\beta} + R_{\alpha\nu\beta\mu}$$

which by a well known identity is zero.
Hence finally the second order Dirac equation is

$$\left(g^{\mu\nu}\nabla_\mu\nabla_\nu - \tfrac{1}{2}s^{\mu\nu}f_{\mu\nu} + \tfrac{1}{4}R - \mu^2\right)\psi = 0. \qquad (1.53)$$

The first and last terms are but a generalised form of the Klein-Gordon wave-equation while the second term represents an interaction of the external field and the electronic spin, and the other term introduces the curvature scalar which vanishes in special relativity.

CHAPTER II.

TWO COMPONENT SPINORS in GENERAL RELATIVITY.

Van der Waerden (6) introduced a theory of spinor analysis applicable to the space-time of special relativity, the Dirac ψ - functions forming two pairs of two-component spinors. This theory has been extended to general relativity by Infeld and van der Waerden (7, 8) and also, in a more geometrical form by Veblen (9). Essentially equivalent to this is the theory of semi-vectors of Einstein and Mayer (10), the connection between the two theories having been expounded by Bargmann (11). The spinor theory will give results similar to those obtained from the principles of Schrödinger's generalisation of Dirac's equa:tion, but for many purposes its notation is extremely convenient. As we shall be making full use of its notation and results in considering the general universal theory of Dirac's equation, the properties of spinors will now be given here.

Instead of using matrix notation, we express all row and column indices in full and these will be treated as tensor in:dices in the spin-space, so that the new process will be a more formal one of tensor analysis with two types of tensors: world-tensors and spin-tensors (or spinors). Fundamentally however this theory is closely comparable with Schrödinger's extension.

At each point of the Riemannian space, V, of special re:lativity for which we have the usual metrical tensor g_{kl}, with/

with g_{11}, g_{22}, $g_{33} < o$ and $g_{44} > o$, we associate a complex two dimensional spin-space S, the vectors and tensors of which being referred to as spinors. The components of world tensors are always denoted by Latin indices (generally we write such tensors as Latin capitals) ranging from 1 to 4 while those of spinors are indicated by Greek indices, range 1 and 2. For both types of indices the summation convention for repeated indices is to be adopted.

Transformations in the spaces V and S are to be considered as being completely independent. If α^{λ} ($\lambda = 1,2$) are the components of a contrariant spinor, then under a general co-or-:dinate transformation in the spin-space they transform according to

$$\alpha'^{\lambda} = \Lambda^{\lambda}_{\rho}\,\alpha^{\rho} \qquad (2.1)$$

Both the Λ^{λ}_{ρ} and α^{ρ} are in general complex functions of the world point with which the spin space is associated. It is assumed that the Λ^{λ}_{ρ} are differentiable and that their deter-:minant is different from zero. The spinor which is the complex conjugate to α^{λ} undergoing the conjugate transformation to (2.1), is denoted by a dotted index

$$\alpha^{\dot{\lambda}} = \overline{\Lambda}^{\lambda}_{\rho}\,\alpha^{\dot{\rho}} \qquad (2.2)$$

Bars are used to denote the complex conjugate; and so $\alpha^{\dot{\lambda}} = \overline{\alpha^{\lambda}}$ Spinors of order greater than one transform like a product of appropriate spin-vectors, e.g.

$$\beta^{\dot{\lambda}\mu\dot{\nu}} \text{ transforms like } \alpha^{\dot{\lambda}}\beta^{\mu}\gamma^{\dot{\nu}}.$$

In/

In the theory of van der Waerden, one dealt only with unimodular transformations in the spin-space. There, two spinors α^{μ} and β^{μ} had the invariant,

$$\varepsilon_{\lambda\mu}\,\alpha^{\lambda}\beta^{\mu} \quad \text{where} \quad \varepsilon_{12} = -\varepsilon_{21} = 1\,, \quad \varepsilon_{11} = \varepsilon_{22} = 0\,; \tag{2.3}$$

and covariants were formed by the rule $\alpha_{\mu} = \varepsilon_{\mu\lambda}\,\alpha^{\lambda}$.

Here in the general theory of van der Waerden and Infeld transformations are not restricted to be unimodular and for raising and lowering indices a skew spinor $\gamma_{\lambda\mu} = -\gamma_{\mu\lambda}$ replaces $\varepsilon_{\lambda\mu}$ to play the part of the fundamental spinor of the S-space. The only non zero components are $\gamma_{12} = -\gamma_{21}$ which is an arbitrary function of the world point. $\gamma_{\dot\lambda\dot\mu}$ is its conjugate complex and $\gamma^{\mu\lambda}$ is its inverse where $\gamma^{12} = -\gamma^{21} = \dfrac{1}{\gamma^{12}}$.

We can write

$$\left.\begin{aligned} \gamma_{12} &= \sqrt{\gamma}\, e^{i\theta} \\ \gamma_{\dot1\dot2} &= \sqrt{\gamma}\, e^{-i\theta} \qquad \text{where} \quad \gamma = \gamma_{12}\,\gamma_{\dot1\dot2} \end{aligned}\right\} \tag{2.4}$$

Transitions from contravariant to covariant forms and vice-versa can be effected as follows:-

$$\left.\begin{aligned} \alpha_{\mu} &= \alpha^{\rho}\gamma_{\rho\mu} \qquad & \alpha^{\mu} &= \gamma^{\mu\rho}\alpha_{\rho} \\ \alpha_{\dot\mu} &= \alpha^{\dot\rho}\gamma_{\dot\rho\dot\mu} \qquad & \alpha^{\dot\mu} &= \gamma^{\dot\mu\dot\rho}\alpha_{\dot\rho}\,. \end{aligned}\right\} \tag{2.5}$$

The scalar product $-\alpha_{\rho}\beta^{\rho} = \alpha^{\rho}\beta_{\rho} = \gamma^{\rho\sigma}\alpha_{\sigma}\beta_{\rho} = \gamma_{\rho\sigma}\alpha^{\sigma}\beta^{\rho}$ is invariant, while in particular for $\beta = \alpha$

$$\alpha_{\lambda}\,\alpha^{\lambda} \equiv 0\,.$$

Spin/

Spin-space transformationsand co-ordinate transformations for world tensors are distinct and do not affect each other. There are however "mixed" quantities with both Latin and Greek indices such as $\sigma^{k}{}_{\dot\lambda\mu}$ which for world transformations behaves like a contravariant vector and for spin transformations like a spin tensor of the form $\alpha_{\dot\lambda\mu}$.

By means of the $\sigma^{k}{}_{\dot\lambda\mu}$ a relation is set up between world vectors and Hermitian spin-tensors (i,e. $\alpha_{\dot\lambda\mu} = \alpha_{\mu\dot\lambda}$). We obtain a real world vector a^k as a linear function of the $\alpha_{\dot\lambda\mu}$ by means of the relation

$$a^k = \sigma^{k\dot\lambda\mu}\alpha_{\dot\lambda\mu} = \sigma^{k}{}_{\dot\lambda\mu}\alpha^{\dot\lambda\mu} \qquad (2.6)$$

when $\sigma^{k\dot\lambda\mu}$ is Hermitian, i.e. $\sigma^{k\mu\dot\lambda} = \sigma^{k\dot\lambda\mu}$. (2.7)

In special relativity, there was a correspondence between Lorentz transformations and the unimodular transformations of the spinors. Both $\sigma^{k\dot\lambda\mu}$ and $g_{k\ell}$ were constants. Now in general relativity both of those are functions of position.

The invariant from the vector a^k is $g_{k\ell}a^k a^\ell$ while that from the spinor $\alpha_{\dot\lambda\mu}$ is $\gamma^{\dot\lambda\dot\varrho}\gamma^{\mu\sigma}\alpha_{\dot\lambda\mu}\alpha_{\dot\varrho\sigma}$

$$= \gamma\,(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}). \qquad (2.8)$$

As we are dealing with an Hermitian $\alpha_{\dot\lambda\mu}$, that is α_{11} α_{22} real and $\alpha_{12} = \overline{\alpha_{21}}$, we can express these components as a + b, a - b, c + id, c - id, respectively where a, b, c and d are real. Thus in terms of these quantities, the invariant .8 is $\gamma\,(a^2 - b^2 - c^2 - d^2)$ which has the same signature as the metricE metric of our space-time. Therefore a direct correspondence between the two invariants can be set up.

In fact we write

$$g_{kl}\, a^k a^l = \gamma^{\dot\lambda\dot\varrho}\gamma^{\mu\sigma}\alpha_{\dot\lambda\mu}\alpha_{\dot\varrho\sigma} \tag{2.9}$$

identically for all $\alpha_{\dot\lambda\mu}$.

By (2.6) : $g_{kl}\,\sigma^{k\dot\lambda\mu}\sigma^{l\dot\varrho\sigma}\alpha_{\dot\lambda\mu}\alpha_{\dot\varrho\sigma} = \gamma^{\dot\lambda\dot\varrho}\gamma^{\mu\sigma}\alpha_{\dot\lambda\mu}\alpha_{\dot\varrho\sigma}$.

As this is to be independent of $\alpha_{\dot\lambda\mu}$, this means that

$$\left.\begin{aligned} g_{kl}\,\sigma^{k\dot\lambda\mu}\sigma^{l\dot\varrho\sigma} &= \gamma^{\dot\lambda\dot\varrho}\gamma^{\mu\sigma} \\ \text{or}\qquad \sigma^{k\dot\lambda\mu}\sigma_{k\dot\sigma\rho} &= \delta^{\dot\lambda}_{\dot\sigma}\delta^{\mu}_{\rho}. \end{aligned}\right\} \tag{2.10}$$

From this result we have

$$\alpha_{\dot\lambda\mu} = \sigma_{k\dot\lambda\mu}\, a^k \tag{2.11}$$

Multiply by $\sigma^{l\dot\lambda\mu}$ and then from (2.6) for all a^l

$$a^l = \sigma^{l\dot\lambda\mu}\sigma_{k\dot\lambda\mu}\, a^k$$

so that

$$\sigma^{l\dot\lambda\mu}\sigma_{k\dot\lambda\mu} = \delta^l_k \tag{2.12}$$

The two relations also soon obtained:

$$\left.\begin{aligned} \sigma^{l\dot\lambda\mu}\sigma^{k}{}_{\dot\lambda\sigma} + \sigma^{k\dot\lambda\mu}\sigma^{l}{}_{\dot\lambda\sigma} &= g^{kl}\delta^{\mu}_{\sigma} \\ \sigma^{l\dot\varrho\mu}\sigma^{k}{}_{\dot\lambda\mu} + \sigma^{k\dot\varrho\mu}\sigma^{l}{}_{\dot\lambda\mu} &= g^{kl}\delta^{\dot\varrho}_{\dot\lambda}. \end{aligned}\right\} \tag{2.13}$$

The formulae (2.10) (2.12) and (2.13) find frequent application throughout the work.

COVARIANT DIFFERENTIATION.

As usual we define the covariant derivative of a spin vector by the following forms:-

$$\psi_{\alpha|k} = \partial_k\psi_\alpha - \Gamma^{\rho}_{\alpha k}\psi_\rho \tag{2.14}$$

$$\psi^{\alpha}{}_{|k} = \partial_k\psi^\alpha + \Gamma^{\alpha}_{\rho k}\psi^\rho$$

where/

where $\partial_k = \frac{\partial}{\partial x^k}$. The difference in sign is necessary so that the covariant derivation of the scalar ($\psi^\alpha \chi_\alpha$) as determined by the product rule reduces to the ordinary derivative.

$\psi^\alpha{}_{|k}$ is to transform, under the transformation (2.1), as a spinor with respect to the index α , so that we must have $\Gamma^\alpha_{\rho k}$ transforming according to the law

$$\Lambda^\alpha_\rho \Gamma^\rho_{\beta k} = \Lambda^\rho_\beta \Gamma'^\alpha_{\rho k} + \partial_k \Lambda^\alpha_\beta . \tag{2.15}$$

The relation between a spinor ψ^α and its conjugate $\psi^{\dot\alpha}$ is preserved after covariant differentiation if

$$\psi^{\dot\alpha}{}_{|k} = \partial_k \psi^{\dot\alpha} + \Gamma^{\dot\alpha}_{\dot\rho k} \psi^{\dot\rho} \tag{2.16}$$

$$\psi_{\dot\alpha|k} = \partial_k \psi_{\dot\alpha} - \Gamma^{\dot\rho}_{\dot\alpha k} \psi_{\dot\rho}$$

with $$\Gamma^{\dot\rho}_{\dot\alpha k} = \overline{\Gamma^\rho_{\alpha k}} .$$

Thus any spin-tensor can be differentiated as we have the rules for each undotted and dotted, covariant or contravariant index.

If μ_1, μ_2 are the complex spin variables in terms of ~~from~~ four real parameters p, q, r, s the volume element of the spin space is

$$\gamma_{12}\, \gamma_{\dot1\dot2} \frac{\partial(\mu_1, \mu_2, \bar\mu_1, \bar\mu_2)}{\partial(p, q, r, s)}\, dp\, dq\, dr\, ds .$$

For this element to remain unaltered we require $\gamma \equiv \gamma_{12}\gamma_{\dot1\dot2}$ to have zero covariant derivative.

$$\begin{aligned}(\gamma)_{|k} &= \partial_k \gamma - \Gamma^\alpha_{1k} \gamma_{\alpha 2} \gamma_{\dot1\dot2} - \Gamma^\alpha_{2k} \gamma_{1\alpha} \gamma_{\dot1\dot2} - \Gamma^{\dot\alpha}_{\dot1 k} \gamma_{12} \gamma_{\dot\alpha\dot2} - \Gamma^{\dot\alpha}_{\dot2 k} \gamma_{12} \gamma_{\dot1\dot\alpha} \\ &= \partial_k \gamma - \left(\Gamma^\alpha_{\alpha k} + \Gamma^{\dot\alpha}_{\dot\alpha k}\right)\gamma \\ &= 0\end{aligned}$$

$$\therefore \quad \Gamma^\alpha_{\alpha k} + \Gamma^{\dot\alpha}_{\dot\alpha k} = \partial_k (\log \gamma). \tag{2.17}$$

Finally/

Finally the covariant derivative of $\sigma^{k\dot{\lambda}\mu}$ is made zero. Then

$$a^{k}{}_{|l} = \left(\sigma^{k\dot{\lambda}\mu}\alpha_{\dot{\lambda}\mu}\right)_{l} = \sigma^{k\dot{\lambda}\mu}\alpha_{\dot{\lambda}\mu|l} \tag{2.18}$$

By (2.12)

$$g^{kl} = \sigma^{k\dot{\lambda}\mu}\sigma^{l}{}_{\dot{\lambda}\mu} = \sigma^{k\dot{\lambda}\mu}\sigma^{l\dot{\sigma}\nu}\gamma_{\dot{\lambda}\dot{\sigma}}\gamma_{\mu\nu}.$$

Therefore by the rules we have made

$$g^{kl}{}_{|s} = 0 \tag{2.19}$$

Thus as far as the world space is concerned, the connections are the Chrstoffel symbols $\{{}^{\,i}_{jk}\}$. So the covariant deriva:tive of $\sigma^{k\dot{\lambda}\mu}$ is

$$0 = \left(\sigma^{k\dot{\lambda}\mu}\right)_{s} = \partial_{s}\sigma^{k\dot{\lambda}\mu} + \Gamma^{k}_{rs}\sigma^{r\dot{\lambda}\mu} + \Gamma^{\dot{\lambda}}_{\dot{\rho}s}\sigma^{k\dot{\rho}\mu} + \Gamma^{\mu}_{\sigma s}\sigma^{k\dot{\lambda}\sigma}. \tag{2.20}$$

There are included in this statement 64 equations, which on account of the symmetry of Γ^{k}_{rs} supply 24 linear conditions for the components of $\Gamma^{\alpha}_{\beta s}$. There are actually 32 real parameters in the $\Gamma^{\alpha}_{\beta s}$, and these 24 conditions together with the 4 supplied by (2.17), leave four parameters undetermined. Indeed, if in (2.17) we replace

$\Gamma^{\mu}_{\rho s}$ and $\Gamma^{\dot{\mu}}_{\dot{\rho}s}$ by $\Gamma^{\mu}_{\rho s} + \frac{i}{2}\Phi_{s}\delta^{\mu}_{\rho}$ and $\Gamma^{\dot{\mu}}_{\dot{\rho}s} - \frac{i}{2}\Phi_{s}\delta^{\dot{\mu}}_{\dot{\rho}}$ respectively, the relation still is satisfied. Thus there are four arbitrary real parameters ϕ_{s} defined by

$$\Gamma^{\alpha}_{\alpha s} - \Gamma^{\dot{\alpha}}_{\dot{\alpha}s} = 2i\,\phi_{s} \tag{2.21}$$

For world transformations ϕ_s behaves as a covariant world vector. Under the spin-transformation (2.1), the new $\Gamma^{\alpha}_{\beta k}$ is given by (2.15). If this equation is multiplied by the cofactor of Λ^{α}_{β} in $|\Lambda^{\alpha}_{\beta}| = \Delta$ and summing for α and β we/

we immediately see that

$$\Delta\, \Gamma^{\alpha}_{\alpha k} = \Delta\, \acute{\Gamma}^{\alpha}_{\alpha k} + \partial_k \Delta$$

or
$$\acute{\Gamma}^{\alpha}_{\alpha k} = \Gamma^{\alpha}_{\alpha k} - \partial_k \log \Delta . \qquad (2.22)$$

Similarly
$$\acute{\Gamma}^{\dot\alpha}_{\dot\alpha k} = \Gamma^{\dot\alpha}_{\dot\alpha k} - \partial_k \log \bar{\Delta} .$$

Therefore
$$\acute{\phi}_k = \phi_k + \frac{i}{2}\, \partial_k \left(\log \frac{\Delta}{\bar{\Delta}} \right)$$

If
$$\Delta = |\Delta|\, e^{i\varphi}$$

then
$$\phi'_k = \phi_k - \partial_k \varphi \qquad (2.23)$$

In the special case $\Lambda = 1\, e^{i\varphi/2}$ $\quad \Delta = e^{i\varphi}$

and
$$\alpha'^{\lambda} = e^{i\varphi/2}\, \alpha^{\lambda}$$
$$\acute{\phi}_k = \phi_k - \partial_k \varphi$$

corresponding to the principle of gauge invariance with the ϕ as the electro-magnetic potential. Thus ϕ_k is not a pure vector as it is affected by the spin transformation.

Let us examine the effect of a spin transformation on $\quad \gamma_{12} = \sqrt{\gamma}\, e^{i\theta}$

$$\sqrt{\gamma'}\, e^{+i\theta'} = \gamma'_{12} = \gamma_{12}\, \Delta^{-1} = \gamma_{12}\, |\Delta|^{-1} e^{-i\varphi}$$

$$\sqrt{\gamma'}\, e^{-i\theta'} = \gamma'_{\dot1\dot2} = \gamma_{\dot1\dot2}\, \bar{\Delta}^{-1} = \gamma_{\dot1\dot2}\, |\bar{\Delta}|^{-1} e^{+i\varphi} .$$

Divide: $\quad e^{2i\theta'} = e^{2i\theta}\, e^{-2i\varphi}$

$$\theta' = \theta - \varphi .$$

Therefore $\partial_k \theta$ transforms like ϕ_k and

$$\phi^*_k = \phi_k - \partial_k \theta \qquad (2.24)$$

is an actual vector upon which spin transformations have no influence.

Now/

Now from (2.17) and (2.21)

$$\left.\begin{aligned} \Gamma^{\alpha}_{\alpha k} &= i\phi_k + \partial_k \log\sqrt{\gamma} &&= i\phi^*_k + \partial_k \log\sqrt{\gamma}\, e^{i\theta} \\ \Gamma^{\dot\alpha}_{\dot\alpha k} &= -i\phi_k + \partial_k \log\sqrt{\gamma} &&= -i\phi^*_k + \partial_k \log\sqrt{\gamma}\, e^{-i\theta} \end{aligned}\right\} \qquad (2.25)$$

whence we obtain the covariant derivatives of the $\gamma_{\lambda\mu}$ spinors as

$$\left.\begin{aligned} \gamma^{\lambda\mu}{}_{|k} &= i\gamma^{\lambda\mu}\phi^*_k &\qquad \gamma^{\dot\lambda\dot\mu}{}_{|k} &= -i\gamma^{\dot\lambda\dot\mu}\phi^*_k \\ \gamma_{\lambda\mu|k} &= -i\gamma_{\lambda\mu}\phi^*_k &\qquad \gamma_{\dot\lambda\dot\mu|k} &= i\gamma_{\dot\lambda\dot\mu}\phi^*_k \end{aligned}\right\} \qquad (2.26)$$

GEODESIC COORDINATES.

If the space considered is the pseudo-euclidean one of special relativity, then it is immediately seen that all our equations would be satisfied if we chose

(1) $g_{kl} = \overset{\circ}{g}_{kl}$. $\quad \overset{\circ}{g}_{kl} = \delta_{kl} e_l \qquad e_l = (- - - +)1.$

(2) $\gamma^{\lambda\mu} = \varepsilon^{\lambda\mu}$.

(3) $\sigma^{k\dot\lambda\mu} = \overset{\circ}{\sigma}{}^{k\dot\lambda\mu}$. $\quad \overset{\circ}{\sigma}{}^{1\dot\lambda\mu} = \frac{1}{\sqrt2}\begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix} \qquad \overset{\circ}{\sigma}{}^{2\dot\lambda\mu} = \frac{1}{\sqrt2}\begin{pmatrix} \cdot & -i \\ i & \cdot \end{pmatrix}$ (2.27)

$\qquad \overset{\circ}{\sigma}{}^{3\dot\lambda\mu} = \frac{1}{\sqrt2}\begin{pmatrix} 1 & \cdot \\ \cdot & -1 \end{pmatrix} \qquad \overset{\circ}{\sigma}{}^{4\dot\lambda\mu} = \frac{1}{\sqrt2}\begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$

(4) $\Gamma^{l}_{sk} = 0$.

(5) $\Gamma^{\beta}_{\alpha k} = \frac{i}{2}\phi_k \delta^{\beta}_{\alpha}$.

These values would hold after an arbitrary Lorentz-transformation followed by a suitable spin-transformation with determinent unity. For $g_{kl} = \overset{\circ}{g}_{kl}$ and $\gamma^{\lambda\mu} = \varepsilon^{\lambda\mu}$ the values of $\overset{\circ}{\sigma}{}^{k\dot\lambda\mu}$ are not unique, an arbitrary Lorentz-transformation could be applied to the index k and an arbitrary spin trans-:formation to $\dot\lambda\mu$.

In/

In general Riemannian spaces we can take a point P_0 and choose coordinate for which (1) is true. In the associated spin space we can make $\gamma^{\lambda\mu} = \varepsilon^{\lambda\mu}$ by a suitable choice of spin coordinates. This allows us then to take $\sigma^{k\dot{\lambda}\mu}$ as above. In this perfectly geodesic coordinate system $\Gamma^{l}_{sk} = 0$ and we have

$$\begin{aligned} (g_{kl})_{P_0} &= \overset{\circ}{g}_{kl} & (\partial_s g_{kl})_{P_0} &= 0 \\ (\gamma^{\lambda\mu})_{P_0} &= \varepsilon^{\lambda\mu} & (\partial_s \gamma^{\lambda\mu})_{P_0} &= 0 \\ (\sigma^{k\dot{\lambda}\mu})_{P_0} &= \overset{\circ}{\sigma}^{k\dot{\lambda}\mu} & (\partial_s \sigma^{k\dot{\lambda}\mu})_{P_0} &= 0 \end{aligned} \qquad (2.28)$$

and the only solution for $\Gamma^{\alpha}_{\beta k}$ is

$$\left(\Gamma^{\alpha}_{\beta k}\right)_{P_0} = \tfrac{1}{2} i \, \delta^{\alpha}_{\beta} \, \phi_k .$$

THE CURVATIVE TENSORS.

In the Riemannian space we have the usual curvative tensor

$$R^{l}{}_{kps} = -\partial_s \Gamma^{l}_{kp} + \partial_p \Gamma^{l}_{ks} - \Gamma^{n}_{kp}\Gamma^{l}_{ns} + \Gamma^{n}_{ks}\Gamma^{l}_{np}$$

Similarly for the spin space a mixed curvative can be formed

$$P^{\mu}{}_{\lambda ps} = -\partial_s \Gamma^{\mu}_{\lambda p} + \partial_p \Gamma^{\mu}_{\lambda s} - \Gamma^{\rho}_{\lambda p}\Gamma^{\mu}_{\rho s} + \Gamma^{\mu}_{\rho s}\Gamma^{\rho}_{\lambda p}. \qquad (2.29)$$

and similarly for $P^{\dot{\mu}}{}_{\dot{\lambda} ps}$ by dotting all Greek indices.

By contraction and application of (2.25) we have

$$\begin{aligned} P^{\mu}{}_{\mu ps} &= i\,(\partial_p \phi_s - \partial_s \phi_p) = i\, F_{ps} \\ P^{\dot{\mu}}{}_{\dot{\mu} ps} &= -i\,(\partial_p \phi_s - \partial_s \phi_p) = -i\, F_{ps} . \end{aligned} \qquad (2.30)$$

$F_{ps} = -F_{sp}$ being (apart from a real constant) the electro-magnetic six vector. The usual results making use of the curvative tensor, give/

give at once

$$\psi^{\rho}{}_{|kl} - \psi^{\rho}{}_{|lk} = \psi^{\sigma} P^{\rho}{}_{\sigma lk}. \tag{2.31}$$

$$\psi_{\dot{\rho}|kl} - \psi_{\dot{\rho}|lk} = \psi_{\dot{\sigma}} P^{\dot{\sigma}}{}_{\dot{\rho} lk}.$$

Also $$\sigma^{k\dot{\lambda}\mu}{}_{|\mu s} - \sigma^{k\dot{\lambda}\mu}{}_{|s\mu} = \sigma^{k\dot{\rho}\mu} P^{\dot{\lambda}}{}_{\dot{\rho}s\mu} + \sigma^{k\dot{\lambda}\rho} P^{\mu}{}_{\rho s\mu} + \sigma^{r\dot{\lambda}\mu} R^{k}{}_{rs\mu}.$$

$$= 0 \tag{2.32}$$

because $\sigma^{k\dot{\lambda}\mu}{}_{|\mu} \equiv 0$

Solving (2.30) and (2.32) we obtain the unique results

$$\left.\begin{aligned} P^{\lambda}{}_{\rho s\mu} &= \tfrac{1}{2} R_{krs\mu}\, \sigma^{k\lambda\dot{\nu}} \sigma^{r}{}_{\dot{\nu}\rho} + \tfrac{i}{2} F_{s\mu}\, \delta^{\lambda}_{\rho} \\ P^{\dot{\lambda}}{}_{\dot{\rho}s\mu} &= \tfrac{1}{2} R_{krs\mu}\, \sigma^{k\dot{\lambda}\nu} \sigma^{r}{}_{\nu\dot{\rho}} - \tfrac{i}{2} F_{s\mu}\, \delta^{\dot{\lambda}}_{\dot{\rho}}. \end{aligned}\right) \tag{2.33}$$

By thus fully exploiting the application of geometrical ideas to the spin-space one obtains the mixed curvature tensor ex-:pression equivalent to the $\Phi_{s\mu}$ matrix which appears in Schrödinger's treatment. The contracting of its pair of spinor indices is the same as forming the spur of the matrix.

THE DIRAC EQUATIONS.

We first assume that there is a current vector J^{k} which has zero divergence.

$$J^{k}{}_{|k} = 0.$$

Assume also that J^{k} correspondends to some spinor $\omega_{\dot{\lambda}\mu}$

where $J^{k} = \sigma^{k\dot{\lambda}\mu}\, \omega_{\dot{\lambda}\mu} = \sigma^{k}{}_{\dot{\lambda}\mu}\, \omega^{\dot{\lambda}\mu}$

Take the divergence : $J^{k}{}_{|k} = \sigma^{k\dot{\lambda}\mu}\, \omega_{\dot{\lambda}\mu|k} = 0$

For J^{k} to be real $\omega_{\dot{\lambda}\mu}$ must be hermitian; this condition and the condition that the time component of J^{k} is positive are satisfied/

satisfied by giving $\omega^{\dot{\lambda}\mu}$ the form

$$\omega^{\dot{\lambda}\mu} = \chi^{\dot{\lambda}}\chi^{\mu} + \psi^{\dot{\lambda}}\psi^{\mu}$$

where χ and ψ are two spin-vectors. Substituting in the expression ( ) for div J we have

$$\left(\sigma^{k}{}_{\dot{\lambda}\mu}\,\psi^{\dot{\lambda}}\psi^{\mu} + \sigma^{k\dot{\lambda}\mu}\,\chi_{\dot{\lambda}}\chi_{\mu}\right)_{k} = 0$$

which when expanded is

$$\psi^{\mu}\sigma^{k}{}_{\dot{\lambda}\mu}\psi^{\dot{\lambda}}{}_{|k} + \chi_{\mu}\sigma^{k\dot{\lambda}\mu}\chi_{\dot{\lambda}|k} + \psi^{\dot{\lambda}}\sigma^{k}{}_{\dot{\lambda}\mu}\psi^{\mu}{}_{|k} + \chi_{\dot{\lambda}}\sigma^{k\dot{\lambda}\mu}\chi_{\mu|k}\,.$$

This is satisfied if we put

$$\left.\begin{aligned} \sigma^{k}{}_{\dot{\lambda}\mu}\,\psi^{\dot{\lambda}}{}_{|k} &= \alpha\,\chi_{\mu} \\ \sigma^{k\lambda\dot{\mu}}\,\chi_{\lambda|k} &= -\bar{\alpha}\,\psi^{\dot{\mu}} \end{aligned}\right) \qquad (2.34)$$

where α is some constant.

In the geodesic system, from equations and their conjugates we derive the equation

$$\left\{-\frac{h}{2\pi i}\frac{\partial}{c\,\partial t} + \frac{e}{c}A_4 + \sum_{i=1}^{3}\alpha^{i}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x^{i}} - \frac{e}{c}A_i\right) + \alpha_4 mc\right\}\Psi = 0.$$

where

$$-\alpha_1 = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot \end{bmatrix} \qquad -\alpha_2 = \begin{bmatrix} \cdot & -i & \cdot & \cdot \\ i & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & i \\ \cdot & \cdot & -i & \cdot \end{bmatrix} \qquad -\alpha_3 = \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

$$-\alpha_4 = \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \end{bmatrix} \qquad \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \psi^{\dot{1}} \\ \psi^{\dot{2}} \end{bmatrix} \qquad (2.35)$$

The vector $\quad \phi_i = \dfrac{4\pi}{h}\dfrac{e}{c}A_i$

and $\quad \alpha = -\dfrac{2\pi i}{h}\dfrac{mc}{\sqrt{2}}\,,\quad$ a pure imaginary constant.

The/

The α_i here are set of hermitian anticommuting matrices and so the spinor equations have reduced to the well known form to Dirac's equation. The actual set of spinor equations used here is

$$\Lambda^{\dot\mu} \equiv \sigma^{k\lambda\dot\mu} \chi_{\lambda|k} - \alpha \psi^{\dot\mu} = 0 \qquad (2.36)$$

$$\Pi_{\mu} \equiv \sigma^{k}{}_{\dot\lambda\mu} \psi^{\dot\lambda}{}_{|k} - \alpha \chi_{\mu} = 0 .$$

and these are regarded as wave equation in general relatively expressed in spinor notation.

THE SECOND ORDER WAVE EQUATION.

From 2.34 we have

$$-\alpha(\sigma^{k\dot\lambda\mu} \chi_{\dot\lambda|k}) = \sigma^{k\dot\lambda\mu} (\sigma^{l}{}_{\dot\lambda\rho} \psi^{\rho}{}_{|l})_{k} = \alpha^2 \psi^{\mu} .$$

Therefore $\alpha^2 \psi^{\mu} = \frac{1}{2} (\sigma^{k\dot\lambda\mu} \sigma^{l}{}_{\dot\lambda\rho} \psi^{\rho}{}_{|lk} + \sigma^{l\dot\lambda\mu} \sigma^{k}{}_{\dot\lambda\rho} \psi^{\rho}{}_{|kl})$

after interchange of dummy suffixes.

By (2.31),(2.13) and (2.33) we have in succession

$$\alpha^2 \psi^{\mu} = \tfrac{1}{2} \left(\sigma^{k\dot\lambda\mu} \sigma^{l}{}_{\dot\lambda\rho} + \sigma^{l\dot\lambda\mu} \sigma^{k}{}_{\dot\lambda\rho} \left[\psi^{\rho}{}_{|lk} + \psi^{\sigma} P^{\rho}{}_{\sigma lk} \right] \right)$$

$$= \tfrac{1}{2} g^{lk} \psi^{\mu}{}_{|lk} + \tfrac{1}{2} \sigma^{l\dot\lambda\mu} \sigma^{k}{}_{\dot\lambda\rho} \psi^{\sigma} P^{\rho}{}_{\sigma lk}$$

$$= \tfrac{1}{2} g^{lk} \psi^{\mu}{}_{|lk} + \tfrac{1}{4} \sigma^{l\dot\lambda\mu} \sigma^{k}{}_{\dot\lambda\rho} R_{psl k} \sigma^{p\rho\dot\nu} \sigma^{s}{}_{\dot\nu\sigma} + \tfrac{i}{4} \sigma^{l\dot\nu\mu} \sigma^{k}{}_{\dot\nu\rho} F_{lk} .$$

By a reduction very similar to that for (1.52), the middle term here simplifies to $\frac{1}{8} R \psi^{\mu}$ so that the final form of the equa- :tion is

$$g^{lk} \psi^{\mu}{}_{|lk} + \left(\tfrac{1}{4} R - 2\alpha^2\right) \psi^{\mu} + \tfrac{i}{2} F_{lk} \sigma^{l\dot\lambda\mu} \sigma^{k}{}_{\dot\lambda\sigma} \psi^{\sigma} = 0 \qquad (2.37)$$

Similarly for $\chi_{\dot\mu}$ we have the equation

$$g^{lk} \chi_{\dot\mu|lk} + \left(\tfrac{1}{4} R - 2\alpha^2\right) \chi_{\dot\mu} + \tfrac{i}{2} F_{lk} \sigma^{l}{}_{\lambda\dot\mu} \sigma^{k\lambda\dot\sigma} \chi_{\dot\sigma} = 0. \qquad (2.38)$$

CHAPTER III.

THE INVARIANT THEORY of DIRAC'S EQUATIONS.

In the spinor formulation of Dirac's equation we have two spinors ψ and χ both having two dotted and two undotted components besides the spinor expressions Λ and Π the vanishing of which being the spinor form of the wave equation. We now wish to consider all the invariants and tensors derivable from these quantities. For this, the mathematical theory, we obtain forms which are not equivalent to the bilinear products of a ψ-function with its complex conjugate, so that some of the tensors found have no direct physical interpretation, although these have a mathematical importance.

A spinor has zero length $\phi_\alpha \phi^\alpha \equiv 0$ From two different spinors the inner product by contraction can be formed to produce a scalar. In this way we find the two fundamental scalars in the theory, these being

$$\left.\begin{aligned} K &= \psi^\alpha \chi_\alpha \\ \overline{K} &= \psi^{\dot\alpha} \chi_{\dot\alpha} \end{aligned}\right\} \qquad (3.1)$$

where K and $\overline{K}$ are complex conjugate quantities.

The vector formed from any two spinors whatever, say ϕ and ω by means of the relation

$$\underline{X}^{\kappa} = \sigma^{\kappa\dot\lambda\mu} \phi_{\dot\lambda} \omega_\mu$$

is always a null vector, for its length is

$$\begin{aligned} X^\kappa X_\kappa &= \sigma^{\kappa\dot\lambda\mu} \sigma_{\kappa\dot\nu\rho} \phi_{\dot\lambda} \omega_\mu \phi^{\dot\nu} \omega^\rho \\ &= \delta^{\dot\lambda}_{\dot\nu} \delta^\mu_\rho \phi_{\dot\lambda} \phi^{\dot\nu} \omega_\mu \omega^\rho \qquad \text{by (2.12)} \\ &= 0 \qquad \text{as the contracted spinors are always zero.} \end{aligned}$$

From/

From all possible combinations of ψ and χ to replace ϕ and ω we obtain four distinct vectors of this type, the first two are real since $\sigma^{k\dot{\lambda}\mu}$ is hermitian in its spinor indices while the other pair are complex conjugates. These null vectors are

$$\left.\begin{aligned} A^{k} &= \sigma^{k\dot{\lambda}\mu}\ \chi_{\dot{\lambda}}\,\chi_{\mu} \\ B^{k} &= \sigma^{k}{}_{\dot{\lambda}\mu}\ \psi^{\dot{\lambda}}\,\psi^{\mu} \\ C^{k} &= \sigma^{k}{}_{\dot{\lambda}\mu}\ \psi^{\dot{\lambda}}\,\chi^{\mu} \\ \overline{C}^{k} &= \sigma^{k}{}_{\dot{\lambda}\mu}\ \chi^{\dot{\lambda}}\,\psi^{\mu} \end{aligned}\right\} \qquad (3.2)$$

The inner products of these vectors lead to scalars, all of which are expressible in terms of K and $\overline{K}$ as a consequence of the properties of the correspondence between world vectors and spinors.

For example, again using (2.12) and from (3.1) we have

$$\begin{aligned} A^{k}\ B_{k} &= \sigma^{k\dot{\lambda}\mu}\,\chi_{\dot{\lambda}}\chi_{\mu}\ \sigma_{k\dot{\alpha}\beta}\,\psi^{\dot{\alpha}}\psi^{\beta} \\ &= \delta^{\dot{\lambda}}_{\dot{\alpha}}\ \chi_{\dot{\lambda}}\psi^{\dot{\alpha}}\ \ \delta^{\mu}_{\beta}\ \chi_{\mu}\psi^{\beta} \ = K\,\overline{K}. \end{aligned}$$

The complete scheme is quickly found to be

$$\left.\begin{array}{llll} A^{k}B_{k} = 0 \quad A^{k}B_{k} = K\overline{K} & A^{k}C_{k} = 0 & A^{k}\overline{C}_{k} = 0 \\ \qquad\qquad\quad B^{k}B_{k} = 0 & B^{k}C_{k} = 0 & B^{k}\overline{C}_{k} = 0 \\ & C^{k}C_{k} = 0 & C^{k}\overline{C}_{k} = -K\,\overline{K} \\ & & \overline{C}^{k}\overline{C}_{k} = 0 \end{array}\right\} \qquad (3.3)$$

Each of these four vectors is perpendicular to itself (i.e. it is null) and to two others but not to the fourth.

SECOND ORDER TENSORS.

There are three distinct tensors, together with their com:plex conjugate, quadratic in the wave functions and not involving/

involving covariant derivatives. Under these conditions, the two tensor indices must be introduced by means of the tensor in-:dices in the $\sigma^{k\lambda\mu}$ quantities and as there are no free spinor indices, it is seen that the following are the only possibilities for this class:-

$$\left.\begin{aligned} P^{lk} &= \chi_\nu\, \sigma^{l\mu\nu}\, \sigma^{k}{}_{\mu\sigma}\, \psi^\sigma \\ M^{lk} &= \chi_\nu\, \sigma^{l\mu\nu}\, \sigma^{k}{}_{\mu\sigma}\, \chi^\sigma \\ N^{lk} &= \psi^\nu\, \sigma^{l}{}_{\mu\nu}\, \sigma^{k\mu\sigma}\, \psi_\sigma . \end{aligned}\right\} \qquad (3.4)$$

Now $\quad P^{kl} = \chi_\nu\, \sigma^{k\mu\nu}\, \sigma^{l}{}_{\mu\sigma}\, \psi^\sigma$

$$= \chi_\nu \left(g^{kl} \delta^\nu_\sigma - \sigma^{l\mu\nu} \sigma^{k}{}_{\mu\sigma} \right) \psi^\sigma \qquad \text{by } (2.13)$$
$$= K g^{kl} - P^{lk} .$$

Therefore we see that the symmetrical part of the tensor P^{lk} is proportional to the metric tensor g^{kl} and is of no new in-:terest so that it will be quite sufficient to study the skew part which we now denote by

$$Q^{kl} = -Q^{lk} = P^{lk} - \tfrac{1}{2} g^{lk} K$$
$$= \tfrac{1}{2} \chi_\nu \left(\sigma^{l\mu\nu} \sigma^{k}{}_{\mu\sigma} - \sigma^{k\mu\nu} \sigma^{l}{}_{\mu\sigma} \right) \psi^\sigma . \qquad (3.5)$$

Also $\quad M^{kl} = \chi_\nu \left(g^{kl} \delta^\nu_\sigma - \sigma^{l}{}_{\mu\sigma}\, \sigma^{k\mu\nu} \right) \chi^\sigma$

$$= g^{kl} \chi^\sigma \chi_\sigma - M^{lk} \qquad = - M^{lk} .$$

In this way we find that both M^{lk} and N^{lk} are skew tensors. These are very simply related to the vectors A B and C for we find/

find that

$$A^l C^k - A^k C^l = (\sigma^{l\dot{\lambda}\mu} \chi_{\dot{\lambda}} \chi_\mu)(\sigma^k{}_{\dot{\alpha}\beta} \psi^{\dot{\alpha}} \chi^\beta) - (\sigma^k{}_{\dot{\alpha}\beta} \chi^{\dot{\alpha}} \chi^\beta)(\sigma^{l\dot{\lambda}\mu} \psi_{\dot{\lambda}} \chi_\mu)$$

$$= \chi_\mu \sigma^{l\dot{\lambda}\mu} \sigma^k{}_{\dot{\alpha}\beta} \chi^\beta \{ \psi^{\dot{\alpha}} \chi_{\dot{\lambda}} - \chi^{\dot{\alpha}} \psi_{\dot{\lambda}} \}$$

But $\psi^{\dot{\alpha}} \chi_{\dot{\lambda}} - \chi^{\dot{\alpha}} \psi_{\dot{\lambda}} = \delta^{\dot{\alpha}}_{\dot{\lambda}} \bar{K}$

because when $\dot{\alpha} = \dot{\lambda}$ each term is $\bar{K}$, while when $\dot{\alpha} \neq \dot{\lambda}$ the lowering of the index $\dot{\alpha}$ makes it $\dot{\lambda}$ so that both terms are identical except for the sign and so the expression is zero.

$$\left.\begin{aligned} &\text{Therefore} \quad A^l C^k - A^k C^l = \bar{K} M^{lk} \\ &\text{Similarly} \quad B^l \bar{C}^k - B^k \bar{C}^l = \bar{K} N^{lk} \end{aligned}\right\} \quad (3.6)$$

These null vectors and skew tensors were discovered in their special relativity form by E.T. Whittaker.

Whereas M and N are each expressed in terms of two of the small vectors, Q requires all four when it is written in a compar:able form.

Let us form

$$C^l \bar{C}^k - C^k \bar{C}^l = \sigma^l{}_{\dot{\mu}\lambda} \psi^{\dot{\mu}} \chi^\lambda \sigma^{k\dot{\alpha}\beta} \chi_{\dot{\alpha}} \psi_\beta - \sigma^{k\dot{\alpha}\beta} \psi_{\dot{\alpha}} \chi_\beta \sigma^l{}_{\dot{\mu}\lambda} \psi^\lambda \chi^{\dot{\mu}}$$

$$= \sigma^l{}_{\dot{\mu}\lambda} \sigma^{k\dot{\alpha}\beta} (\psi^{\dot{\mu}} \chi_{\dot{\alpha}} \psi_\beta \chi^\lambda - \chi^{\dot{\mu}} \psi_{\dot{\alpha}} \chi_\beta \psi^\lambda)$$

$$= \sigma^l{}_{\dot{\mu}\lambda} \sigma^{k\dot{\alpha}\beta} \left[(\psi^{\dot{\mu}} \chi_{\dot{\alpha}} - \chi^{\dot{\mu}} \psi_{\dot{\alpha}})(\psi_\beta \chi^\lambda + \chi_\beta \psi^\lambda) - \psi^{\dot{\mu}} \psi^\lambda \chi_{\dot{\alpha}} \chi_\beta + \chi^{\dot{\mu}} \chi^\lambda \psi_{\dot{\alpha}} \psi_\beta \right]$$

$$= \sigma^l{}_{\dot{\mu}\lambda} \sigma^{k\dot{\alpha}\beta} \left[\delta^{\dot{\mu}}_{\dot{\alpha}} \bar{K} (\psi_\beta \chi^\lambda + \chi_\beta \psi^\lambda) \right] - B^l A^k + A^l B^k$$

$$= 2\bar{K} Q^{kl} + A^l B^k - A^k B^l.$$

Hence we have Q^{kl} in the desired form:

$$2\bar{K} Q^{kl} = (C^l \bar{C}^k - C^k \bar{C}^l) - (A^l B^k - B^l A^k). \quad (3.7)$$

Various contractions can be made and we quickly have the

length/

length $M^{kl}M_{kl} = \frac{1}{\bar{K}^2}(A^k C^l - A^l C^k)(A_k C_l - A_l C_k)$

$= 0$

$$\left.\begin{array}{l} \text{also } N^{lk}N_{lk} = 0 \\ \text{while } M^{lk}N_{lk} = -2K^2 \quad \text{and } M^{lk}\bar{N}_{lk} = 0 \end{array}\right) \qquad (3.8)$$

It follows at once that

$$\left.\begin{array}{llll} M^{lk}A_k = 0\ , & M^{lk}B_k = -KC^l\ , & M^{lk}C_k = 0\ , & M^{lk}\bar{C}_k = -KA^l \\ N^{lk}A_k = -K\bar{C}^l, & N^{lk}B_k = 0 & N^{lk}C_k = -KB^l, & N^{lk}\bar{C}_k = 0 \end{array}\right) \qquad (3.9)$$

Either from the definition of Q^{lk} in terms of spinors or its expression in terms of A, B, C and $\bar{C}$ we have the following relations

$$\left.\begin{array}{ll} A_l\ Q^{lk} = -\frac{KA^k}{2} & B_l\ Q^{lk} = \frac{KB^k}{2} \\ C_l\ Q^{lk} = -\frac{KC^k}{2} & \bar{C}_l\ Q^{lk} = \frac{K}{2}\bar{C}^k \end{array}\right) \qquad (3.10)$$

the length of Q is then found to be

$$Q^{lk}Q_{lk} = -K^2 \qquad (3.11a)$$

Also the inner product with its conjugate complex is zero

$$Q^{lk}\bar{Q}_{lk} = 0 \qquad (3.11b)$$

The inner products of Q with the tensors M and N lead to the results

$$\left.\begin{array}{ll} M^{lk}Q_{lk} = 0 & N^{lk}Q_{lk} = 0 \\ \bar{M}^{lk}Q_{lk} = 0 & \bar{N}^{lk}Q_{lk} = 0 \end{array}\right) \qquad (3.12)$$

By contracting one pair of indices only, we obtain the following set

$$\left.\begin{array}{ll} M^{lk}Q_{km} = -\frac{K}{2}M^l_{\cdot m} & N^{lk}Q_{km} = \frac{K}{2}N^l_{\cdot m} \\ M^{lk}\bar{Q}_{km} = \frac{A^l C_m + A_m C^l}{2} & N^{lk}\bar{Q}_{km} = -\frac{B^l\bar{C}_m + B_m\bar{C}^l}{2} \end{array}\right) \qquad (3.13)$$

The/

The remainder of the results of these types are obtained by taking the complex conjugates of the relations given here.

All the results so far are consequences of the general theory of spinors. For these relations to be true ψ and χ can be any spinors whatever, for we have not, as yet, made use of the wave equation. When we deal with covariant derivative then we can use the connection between ψ and χ as expressed by the wave equation.

GEOMETRICAL INTERPRETATIONS.

As these results have simple geometrical interpretations, the geometrical aspect of this work will now be considered before we continue our list of tensors. The geometrical method of the following is very similar to Ruse's (12) treatment of the geometry of the electro-magnetic field.

We have the four dimensional space-time with coordinates x^1 to x^4, to which we shall refer as the underlying V4. At each point of V4 the metric tensor components g_{ab} have definite values. The totality of contravariant vectors at any point P in V4 constitute the tangent space T4 for which the quadratic form is $ds^2 = [g_{ab}]\, dx^a dx^b$ where $[g_{ab}]$, the value of g_{ab} at P, is fixed by the position of P and so is constant throughout T_4. Let X^a denote the coordinate system in T_4, it being understood that the origin $X^a = (o, o, o, o,)$ is the point P whose coordin-:ates are $\bar{x}^a$ with respect to the system in V4.

Thus any contravariant X^a in space-time (4) can be geometri-:cally interpreted as the coordinates of a point in T4.

The equation $g_{ab} X^a X^b = o$ (3.14)

is that of a quadratic cone in T4 with its vertex at the origin.

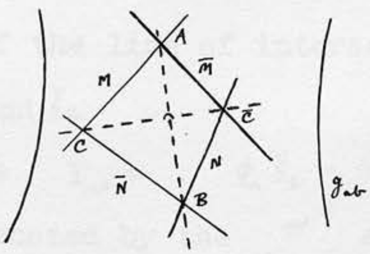
If we introduce a fifth variable X^5 and make the coordinates homogeneous by replacing X^a by $\frac{X^a}{X^5}$ then (3.14) is still the equation of the null cone and $X^5 = o$ is the hyperplane at infinity, S_3 (say). Now the cone intersects S_3 in a quadric,

so/

so that if we confine our attention to the hyperplane S_3 we may consider equation (3.14) as the homogeneous equation of an ordinary quadric. The cone is generated by lines through the origin and each of these lines determines a point on the quadric in S_3, and a plane in the cone determines a line (a generator) on the quadric. Similarly lines and planes in T4 which do not lie in the cone meet S_3 in points and lines not on the quadric. Instead of considering configurations in T_4, we can study their representations in S_3.

Spinor theory is essentially a parameterisation of the null cone or equally of this quadric. All null vectors of the underlying space at P lie on the null cone and they each determine a point on the quadric g_{ab}. We have four null vectors A^k, B^k, C^k and $\bar{C}^k$ which we now consider as the homogeneous coordinates of points on the quadric.

Call these points A B C and $\bar{C}$



Now the coordinates of the polar place of the point X^a with respect to the quadric are given by

$$X_a = X^b g_{ab}$$

and in fact, raising and lowering indices by means of g_{ab} corresponds geometrically to reciprocation with respect to the quadric g_{ab}

Thus/

Thus the polar plane of A^k has coordinates A_k and hence it has the point equation,

$$A_k X^k = 0$$

But $A_k C^k = 0 \quad A_k \bar{C}^k = 0$ while $A_k B^k = K\bar{K} \neq 0$ so that both C^k and $\bar{C}^k$ lie on the polar plane of A^k, while B^k does not. As C^k and $\bar{C}^k$ both lie on the quadric this means that $AC\bar{C}$ is the tangent plane at A with AC and $A\bar{C}$ as generators. Similarly it is clear that $B\bar{C}C$ is the polar plane of B

$\bar{C}AB$ C

CBA $\bar{C}$

$A_k B^k = -C_k \bar{C}^k \neq 0$ so that B does not lie on $AC\bar{C}$ and we obtain a skew quadrilateral $ABC\bar{C}$ on the quadric with the diagonals $C\bar{C}$ and AB non intersecting.

The Plücker coordinate of the line joining two points, X^a and Y^a are given by

$$\tilde{X}^{ab} = X^a Y^b - X^a Y^b \tag{3.15}$$

and those of the line of intersection of two planes with coordinates ϕ_a and ξ_b

$$\text{are} \quad Y_{ab} = \phi_a \xi_b - \phi_b \xi_a \tag{3.16}$$

Duals are denoted by the $\sim$ sign where

$$\left.\begin{aligned} \tilde{X}^{ab} &= \tfrac{1}{2}\, \varepsilon^{abcd} X_{cd} \\ \tilde{X}_{ab} &= -\tfrac{1}{2}\, \varepsilon_{abcd} X^{cd} \end{aligned}\right\} \tag{3.17}$$

with ε^{abcd} skew in all pairs of indices and

$$\varepsilon^{1234} = \frac{1}{\sqrt{-g}} \,.$$

$$\varepsilon_{1234} = \sqrt{-g} \,.$$

The/

The lines $\tilde{X}^{ab}$, Y_{ab} are the same if

$$\tilde{X}^{ab} = \rho\,\tilde{Y}^{ab} = \rho\tfrac{1}{2}\varepsilon^{abcd}\,Y_{cd}$$

where ρ is a factor of proportionality. The coordinates of a line satisfy identically the equation.

$$\tilde{X}^{ab}\;X_{ab} = 0 . \qquad (3.18)$$

A skew tensor not satisfying this identity determines a linear complex - if Z_{ab} is such a tensor, the complex consists of all lines whose Plücker coordinates satisfy the relation

$$Z_{ab}\,\tilde{X}^{ab} = 0$$

or dually $\tilde{Z}^{ab}\,X_{ab} = 0$.

Through any point X^a, the lines which belong to the complex all lie in a plane which is called the "polar plane" of X^a with respect to the complex. Its coordinates are given by $\phi_a = Z_{ab}X^b$.

A point lies in its own polar plane with respect to a linear complex.

The tensor $M^{ab} = (A^a\,C^b - A^b\,C^a)$ is represented in S, by the line A C, its six components being the (dual) Plucker coordinates of this line.

In this way we have the four skew tensors

$$M,\ \bar{M},\ N \text{ and } \bar{N}$$

represented by the lines, AC, A$\bar{C}$, B$\bar{C}$ and BC respectively

The meaning of some of the relations involving these tensors becomes evident. For example (3.9) $M^{lk}\,A_\kappa = 0$ expresses the fact that the plane A_κ namely AC$\bar{C}$ contains the line M.

$M^{lm}\,B_m = -\,K\,C^l$ means that the plane B_m, that is B$C\bar{C}$ intersects/

intersects the line M at the point C.

Now $M_{lk} = g_{ka} M^{ab} g_{bl}$ are the coordinates of the polar line of M with respect to the quadric g_{ab} and the equa:tion

$$M^{lk} M_{lk} = 0$$

means that these two lines M and its polar intersect. However as M actually lies on the quadric it and its polar not merely intersect but coincide.

$M^{lk} N_{lk} = -2 K^2$ shows us that M and the polar line of N with respect to the null quadric, i.e. the line N again, do not intersect.

As $\tilde{Q}^{lk} Q_{lk} \neq 0$ we cannot represent Q as a line but as a linear complex. The polar plane of the point A^a with respect to this linear complex is

$$\phi_b = A^a Q_{ab}$$

but $A^a Q_{ab} = -\frac{K A_b}{2}$ by (3.10) so that the polar plane of A^a with respect to both the complex and the quadric is the same plane $AC\bar{C}$. Similarly the two polar planes at each of the four points A B C and $\bar{C}$ are the same so that the skew quadrilateral is common to the quadric g_{ab} and the linear complex Q_{ab}.

From a repetition of the relation (3.12)

$$M^{lk} Q_{km} = -\frac{K M^l{}_m}{2}$$

we have
$$Q_{bc} M^{cd} Q_{da} = \frac{K^2 M_{ab}}{4} .$$

The/

The left hand side gives the coordinate of the polar of the line M with respect to the complex Q_{ab} and as we obviously expect it is the line M_{ab} itself.

Clearly the linear complex, conjugate to Q namely $\overline{Q}$ intersects the quadric g_{ab} again in the same skew-quadrilateral. The six-vector of the electric and megnetic moments we shall later find to be

$\mathcal{M}^{kl} \propto Q^{kl} - \overline{Q}^{kl}$ and it too determines a linear complex also containing the skew-quadrilateral A B C $\overline{C}$.

We may note that the line $C\overline{C}$ is the join of $C\overline{C}$ and also the intersections of the planes A_a and B_a. Therefore, for its Plucker coordinates we have either

$$X^{ab} = C^a \overline{C}^b - C^b \overline{C}^a ,$$

$$\text{or} \quad Y_{ab} = A_a B_b - A_b B_a .$$

As these are the coordinates and the same line, X^{ab} must be the dual of Y_{ab}, apart from a factor of proportionality. Hence we set

$$X^{ab} = \rho \, \tilde{Y}^{ab}.$$

Now from the relations between the null vectors we immediately find that

$$X^{ab} X_{ab} = -2(K \overline{K})^2$$

and again $Y^{ab} Y_{ab} = -2 (K \overline{K})^2 .$

Also, as the inner product of a pair of skew-tensors is minus the inner product of the dual pair we are now enabled to determine the factor ρ.

$$X^{ab} X_{ab} = \rho^2 \tilde{Y}^{ab} \tilde{Y}_{ab} = -\rho^2 Y^{ab} Y_{ab} .$$

But we have just noted that $X^{ab} X_{ab} = Y^{ab} Y_{ab}$.

Therefore/

Therefore

$$\rho^2 = -1$$

$$\rho = \pm i .$$

We take the + sign for reasons that will be revealed later.
Thus we have

$$C^a \bar{C}^b - C^b \bar{C}^a = \frac{i}{2} \varepsilon^{abcd} (A_c B_d - A_d B_c) = i \tilde{Y}^{ab} \quad (3.19)$$

As

$$2 \bar{K} \tilde{Q}^{ab} = -(C^a \bar{C}^b - C^b \bar{C}^a) + (A^a B^b - B^b A^a)$$

$$= i \tilde{Y}^{ba} - Y^{ba}$$

we can take the duals of both sides and obtain the relation

$$2 \bar{K} \tilde{Q}^{ab} = -i Y^{ba} - \tilde{Y}^{ba}$$

$$= 2 \bar{K} i Q^{ab}.$$

Therefore we have the simple result that

$$\tilde{Q}^{ab} = i Q^{ab}. \quad (3.20)$$

This relations means that the polars of the lines of the linear complex Q_{ab} with respect to the quadric form the same complex. For the lines X_{ab} belonging to Q_{ab} satisfy the linear equation $\tilde{Q}^{ab} X_{ab} = 0$ or $Q_{ab} \tilde{X}^{ab} = 0$

The polar of any line X_{ab} of the linear complex has coordinates

$$X'_{ab} = g_{ac} \tilde{X}^{cd} g_{db} = \tilde{X}_{ab}$$

But as $Q^{ab} \tilde{X}_{ab} = 0$

$$Q^{ab} X'_{ab} = 0 \qquad \text{also} .$$

Thus the polars constitute a linear complex which is the same as the original one if $\tilde{Q}^{ab} = \rho Q^{ab}$
and as we have seen such a relation is possible only if $\rho^2 = -1$.

From/

From the form of the tensor Q^{ab} namely

$$2\,\overline{K}\,Q_{ab} = (\,C_b\overline{C}_a - C_a\overline{C}_b\,) - (\,A_bB_a - A_aB_a\,)$$

we at once see that the linear complex Q_{ab} contains the congruence having the lines AB and $C\overline{C}$ (which are the intersec- :tions of the polar plane of C^a, C^b and of A^a and B^b respec- :tively) as directrices. The directrices are polar lines with respect to the quadric and from this fact we have just shown that the polar complex is the same as the original and it again contains the same congruences.

The tensors M^{ab} and N^{ab} are proportional to their duals. These represent the coordinates of self-polar lines. Let X_{ab} be the Plücker coordinate of a line and $\tilde{X}_{ab}$ be their duals, so that the polar of this line is the line with coordinates

$$\tilde{Y}^{ab} = g^{ac}\,X_{cd}\,g^{db} = X^{ab}.$$

The conditions for the coincidence of the line and its polar, that is, the conditions that the line should be a generator of the quadric is $\tilde{Y}^{ab} \equiv X^{ab} = k\,\tilde{X}^{ab}$.

The value of the constant again follows by considering the length of the six-vectors

$$X^{ab}X_{ab} = k^2\,\tilde{X}^{ab}\tilde{X}_{ab} = -k^2\,X^{ab}X_{ab}.$$
$$\therefore\quad k^2 = -1$$

Hence the line X_{ab} is self polar if

$$X^{ab} = \pm i\,\tilde{X}^{ab}.$$

In/

In this way we obtain simple relations connecting the tensors M^{ab} and N^{ab} with their duals. These are

$$\left.\begin{aligned} M^{ab} &= -i\,\tilde{M}^{ab} \\ N^{ab} &= -i\,\tilde{N}^{ab} \\ \text{and previously we had}\quad Q^{ab} &= -i\,\tilde{Q}^{ab} \end{aligned}\right\} \tag{3.21}$$

Here as before we have shown the sign of i which is immed- :iately obtained when the vectors and tensors are expressed in the special coordinate system.

Thus, given two different spinors, one can form three skew tensors which are quadratic in the spinor components each being (-i) times its own dual. The only symmetric tensor which can be obtained is the symmetric part of P^{ab} which however proves to be merely an invariant times the fundamental tensor namely $g^{ab}K$. These tensors together with their complex conjugates are the only types of second order tensors quadratic in the spinors which we can derive from the given pair of spinors.

Finally it should be noted that the skew quadrilateral on the null quadric is in general non-degenerate. To produce the coincidence of any two vertices one had to have $\psi_\alpha = \chi_\alpha$ and this makes all four vertices collapse into a single point. This case, $\psi \equiv \chi$ is the only degeneration that might possibly occur, but we shall see later that from the physical point of view this state is never reached by an ordinary electron.

THE/

THE DIVERGENCES of the NULL VECTORS.

Now we leave the geometry and consider the consequences of the Dirac equations which have not been used as yet. By appealing to the wave equation (2.35) we can evoluate the divergences of the vectors A^k, B^k, C^k and $\bar{C}^k$

Thus $\quad div\ A^k \equiv (A^k)_k = \sigma^{k\dot{\alpha}\beta}(\chi_{\dot{\alpha}|k}\chi_\beta + \chi_{\dot{\alpha}}\chi_{\beta|k})$

$$
\left.\begin{aligned}
&= (-\alpha\psi^\beta)\chi_\beta + (\alpha\psi^{\dot{\alpha}}\chi_{\dot{\alpha}}) \\
&= -\alpha\ (K - \bar{K}) \\
\text{while} \quad (B^k)_k &= +\alpha\ (K - \bar{K})
\end{aligned}\right\} \quad (3.22)
$$

Again $(C^k)_k = \sigma^{k\dot{\alpha}\beta}(\chi_{\beta|k}\psi_{\dot{\alpha}} + \psi_{\dot{\alpha}|k}\chi_\beta)$

$$= (\alpha\psi^{\dot{\alpha}})\psi_{\dot{\alpha}} + \sigma^{k\dot{\alpha}\beta}(\psi^{\dot{\sigma}}\gamma_{\dot{\sigma}\dot{\alpha}})_k\chi_\beta$$

(remembering that raising and lowering of a spinor index is not in general commutative with covariant differentiation) so that from (2.26) we have

$$(C^k)_k = i\,C^k\,\Phi_k \qquad (3.23a)$$

Henceforth we shall omit the star on the vector Φ_k^* which is proportional to the sum of the electro-magnetic four-potential and the gradient of a scalar (2.24). For the complex conjugate we have

$$(\bar{C}^k)_k = -i\ \bar{C}^k\,\Phi_k \qquad (3.23b)$$

We may note here that $(A^k + B^k)$ is a non divergent vector which is the current vector of Infeld and van der Waerden.

The/

The other vector $(A^k - B^k)$ has for its divergence the non vanishing scalar $-2\alpha(K - \bar{K})$.

VECTORS INVOLVING the FIRST COVARIANT DERIVATIVE of ONE SPINOR.

Vectors of this type are

$$\left.\begin{aligned} G_k &= \psi^\alpha \chi_{\alpha|k} \\ H_k &= \psi^\alpha{}_{|k} \chi_\alpha \\ E_k &= \psi^\alpha{}_{|k} \psi_\alpha \\ F_k &= \chi^\alpha \chi_{\alpha|k} \end{aligned}\right\} \tag{3.24}$$

together with their complex conjugates.

From the identity $\psi^\alpha \psi_\alpha = 0$, by differentiation we have

$$\psi^\alpha{}_{|k} \psi_\alpha + \psi^\alpha \psi_{\alpha|k} = 0$$

so that $\psi^\alpha \psi_{\alpha|k} = -E_k$.

Similarly $\chi^\alpha{}_{|k} \chi_\alpha = -F_k$.

These complete the list of the type under consideration.

Differentiation of the relation $\psi^\alpha \chi_\alpha = K$ leads to the relations

$$G_k + H_k = K_{,k} . \tag{3.25}$$

The divergences of the second order tensors M N and Q can be expressed in terms of these new vectors. Thus for Q^{lk} we have the divergence $(Q^{lk})_k = -(Q^{kl})_k$ which from (1.5) and the wave equations (2.36)

$$\begin{aligned} &= \chi_{\nu|k} \sigma^{l\dot\mu\nu} \sigma^k{}_{\dot\mu\sigma} \psi^\sigma + \chi_\nu \sigma^{l\dot\mu\nu} \sigma^k{}_{\dot\mu\sigma} \psi^\sigma{}_{|k} - \tfrac{1}{2} g^{lk} K_{,k} \\ &= \chi_{\nu|k} (g^{lk} \delta^\nu_\sigma - \sigma^{k\dot\mu\nu} \sigma^l{}_{\dot\mu\sigma}) \psi^\sigma + \chi_\nu \sigma^{l\dot\mu\nu}(-\alpha \chi_{\dot\mu}) - \tfrac{1}{2}(G^l + H^l) \\ &= G^l - \alpha A^l - \alpha B^l - \tfrac{1}{2} G^l - \tfrac{1}{2} H^l \\ &= \tfrac{1}{2}(G^l - H^l) - \alpha(A^l + B^l) . \end{aligned} \tag{3.26}$$

For/

For the vector M^{lk} we have

$$
\begin{aligned}
\operatorname{div} M^{lk} &= (M^{lk})_{k} \\
&= \chi_{\nu|k}\,\sigma^{l\dot{\mu}\nu}\,\sigma^{k}{}_{\dot{\mu}\sigma}\,\chi^{\sigma} + \chi_{\nu}\,\sigma^{l\dot{\mu}\nu}\,\sigma^{k}{}_{\dot{\mu}\sigma}\left(\gamma^{\sigma\rho}\chi_{\rho|k} + i\chi^{\sigma}\phi_{k}\right) \\
&= \chi_{\nu|k}\left(g^{lk}\,\delta^{\nu}_{\sigma} - \sigma^{k\dot{\mu}\nu}\sigma^{l}{}_{\dot{\mu}\sigma}\right)\chi^{\sigma} + \chi_{\nu}\,\sigma^{l\dot{\mu}\nu}\left(-\alpha\psi_{\dot{\mu}} + i\phi_{k}\,\sigma^{k}{}_{\dot{\mu}\sigma}\chi^{\sigma}\right) \\
&= F^{l} - 2\alpha C^{l} + i\phi_{k} M^{lk}. && (3.27)
\end{aligned}
$$

Likewise for N^{lk}

$$
\begin{aligned}
\operatorname{div} N^{lk} &= (N^{lk})_{k} = \psi^{\alpha}{}_{|k}\,\sigma^{l}{}_{\dot{\mu}\alpha}\,\sigma^{k\dot{\mu}\beta}\,\psi_{\beta} + \psi^{\alpha}\,\sigma^{l}{}_{\dot{\mu}\alpha}\,\sigma^{k\dot{\mu}\beta}\,\psi_{\beta|k}. \\
&= g^{lk}\,\psi^{\alpha}{}_{|k}\,\psi_{\alpha} - \psi^{\alpha}{}_{|k}\,\sigma^{k}{}_{\dot{\mu}\alpha}\,\sigma^{l\dot{\mu}\beta}\,\psi_{\beta} \\
&\quad + \psi^{\alpha}\,\sigma^{l}{}_{\dot{\mu}\alpha}\,\sigma^{k\dot{\mu}\beta}\left(-i\psi_{\beta}\phi_{k} + \psi^{\rho}{}_{|k}\,\gamma_{\rho\beta}\right) \\
&= E^{l} + 2\alpha\,\overline{C}^{l} - i\phi_{k} N^{lk}. && (3.28)
\end{aligned}
$$

As M N and Q have been expressed in terms of A B and C we see that E F G and H must also be simply connected with these vectors. In fact we have

$$
\begin{aligned}
\overline{C}_{n}\,B^{n}{}_{|k} &= \sigma_{n}{}^{\dot{\alpha}\beta}\,\psi_{\beta}\chi_{\dot{\alpha}}\,\sigma^{n}{}_{\dot{\mu}\nu}\left(\psi^{\dot{\mu}}\psi^{\nu}{}_{|k} + \psi^{\dot{\mu}}{}_{|k}\psi^{\nu}\right) \\
&= \psi_{\nu}\chi_{\dot{\mu}}\left(\psi^{\dot{\mu}}\psi^{\nu}{}_{|k} + \psi^{\dot{\mu}}{}_{|k}\psi^{\nu}\right) \\
&= \overline{K}\,E_{k}. && (3.29)
\end{aligned}
$$

while
$$
\begin{aligned}
C_{n}\,A^{n}{}_{|k} &= \sigma_{n}{}^{\dot{\alpha}\beta}\,\psi_{\dot{\alpha}}\chi_{\beta}\,\sigma^{n}{}_{\dot{\mu}\nu}\left(\chi^{\dot{\mu}}\chi^{\nu}{}_{|k} + \chi^{\dot{\mu}}\chi^{\nu}{}_{|k}\right) \\
&= \overline{K}\,F_{k}. && (3.30)
\end{aligned}
$$

Also
$$
\left.
\begin{aligned}
A_{n}\,B^{n}{}_{|k} &= \sigma_{n}{}^{\dot{\alpha}\beta}\,\chi_{\dot{\alpha}}\chi_{\beta}\,\sigma^{n}{}_{\dot{\mu}\nu}\left(\psi^{\dot{\mu}}\psi^{\nu}{}_{|k} + \psi^{\dot{\mu}}{}_{|k}\psi^{\nu}\right) \\
&= \chi_{\dot{\alpha}}\chi_{\beta}\left(\psi^{\dot{\alpha}}\psi^{\beta}{}_{|k} + \psi^{\dot{\alpha}}{}_{|k}\psi^{\beta}\right) \\
&= \overline{K}\,H_{k} + K\,\overline{H}_{k},
\end{aligned}
\right\} \qquad (3.31)
$$

and similarly $B_{n}\,A^{n}{}_{|k} = \overline{K}\,G_{k} + K\,\overline{G}_{k}.$

By/

By additions we have $\quad A_n B^n{}_{|k} + B_n A^n{}_{|k} = (A_n B^n)_{,k}$

$$= \bar{K}(H_k + G_k) + K(\bar{H}_k + \bar{G}_k)$$

$$= \bar{K} K_{,k} + \bar{K}_{,k} K$$

$$= (K\bar{K})_{,k} \qquad \text{as we should expect.}$$

Again $\quad \bar{C}_n C^n{}_{|k} = \sigma_n{}^{\dot{\alpha}\beta} \psi_\beta \chi_{\dot{\alpha}} \; \sigma^n{}_{\dot{\mu}\nu}(\psi^{\dot{\mu}}{}_{|k} \chi^\nu + \psi^{\dot{\mu}} \chi^\nu{}_{|k})$

$$= \psi_\beta \chi_{\dot{\alpha}} (\psi^{\dot{\alpha}}{}_{|k} \chi^\beta + \psi^{\dot{\alpha}} \chi^\beta{}_{|k})$$

$$= -K\bar{H}_k + \bar{K}\psi_\beta (\gamma^{\rho\beta} \chi_{\rho|k} + i\chi^\beta \phi_k)$$

$$= -K\bar{H}_k - \bar{K} G_k - i\phi_k K\bar{K}. \qquad (3.32)$$

From (3.25), (3.31) and (3.32) we can find H_k and G_k in terms of the null vectors and the scalars. For we have

$$2\bar{K} H_k = A_n B^n{}_{|k} + \bar{C}_n C^n{}_{|k} + \bar{K}(K_{,k} + i\phi_k K) \qquad (3.33)$$

and $\quad 2\bar{K} G_k = 2\bar{K}(K_{|k} - H_k)$

$$= -A_n B^n{}_{|k} - \bar{C}_n C^n{}_{|k} + \bar{K}(K_{,k} - i\phi_k K)$$

which, as $\quad + A_n B^n{}_{|k} = -A_{n|k} B^n + (K\bar{K})_{,k}$

and $\quad \bar{C}_n C^n{}_{|k} = -\bar{C}_{n|k} C^n - (K\bar{K})_{,k}$

can also be written,

$$2\bar{K} G_k = A_{n|k} B^n + \bar{C}_{n|k} C^n + \bar{K}(K_{,k} - i\phi_k K). \qquad (3.34)$$

The divergences of these vectors will now be found

$$\mathrm{Div}\, E^l = E^l{}_{|l} = g^{kl}(\psi^\alpha{}_{|kl} \psi_\alpha + \psi^\alpha{}_{|k} \psi_{\alpha|l}).$$

Now from the second order wave equation (2.37) we have

$$\psi_\alpha(g^{lk}\psi^\alpha{}_{|lk}) = \psi_\alpha\left[(2\alpha^2 - \tfrac{R}{4})\psi^\alpha - \tfrac{i}{2} F_{lk} \sigma^{l\dot{\mu}\alpha} \sigma^k{}_{\dot{\mu}\beta} \psi^\beta\right]$$

$$= \tfrac{i}{2} F_{lk} N^{lk}.$$

The/

The other term $g^{kl}\psi^{\alpha}{}_{|k}\psi_{\alpha|l}$

$$= \tfrac{1}{2} g^{lk}(\psi^{\alpha}{}_{|k}\psi_{\alpha|l} + \psi^{\alpha}{}_{|l}\psi_{\alpha|k})$$

$$= \tfrac{1}{2} g^{lk}\left[(i\psi^{\alpha}\phi_k + \psi_{\rho|k}\gamma^{\alpha\rho})(-i\psi_\alpha\phi_k + \psi^{\sigma}{}_{|l}\gamma_{\sigma\alpha}) + \psi^{\alpha}{}_{|l}\psi_{\alpha|k}\right]$$

$$= \tfrac{1}{2} g^{lk}\left[\psi^{\alpha}\psi_\alpha\phi_k\phi_l - i\phi_k\psi_\sigma\psi^{\sigma}{}_{|l} + i\phi_l\psi^{\rho}\psi_{\rho|k}\right]$$

$$= -iE^k\phi_k .$$

Therefore combining the two terms we have

$$\left.\begin{aligned} (E^l)_l &= -i\phi_l E^l + \tfrac{i}{2} F_{lk} N^{lk} \\ \text{Similarly}\quad (F^l)_l &= i\phi_l F^l - \tfrac{i}{2} F_{lk} M^{lk} \end{aligned}\right\} \quad (3.35)$$

For the divergence of G^l we find after using (2.38)

$$(G^l)_l = g^{kl}(\psi^{\alpha}\chi_{\alpha|k})_l = g^{kl}\psi^{\alpha}{}_{|l}\chi_{\alpha|k} + \psi^{\alpha} g^{kl}\chi_{\alpha|kl}$$

$$= g^{kl}\psi^{\alpha}{}_{|l}\chi_{\alpha|k} + \psi^{\alpha}\left[(2\alpha^2 - \tfrac{R}{4})\chi_\alpha + \tfrac{i}{2} F_{lk}\sigma^{l}{}_{\lambda\alpha}\sigma^{k\lambda\sigma}\chi_\sigma\right]$$

$$= g^{kl}\psi^{\alpha}{}_{|l}\chi_{\alpha|k} + (2\alpha^2 - \tfrac{R}{4})K + \tfrac{i}{2} F_{lk} Q^{kl} \qquad \text{(by 3.5)} \quad (3.36)$$

Also $(H^l)_l = g^{kl}(\psi^{\alpha}{}_{|l}\chi_{\alpha|k}) + \chi_\alpha g^{lk}\psi^{\alpha}{}_{|lk}$

$$= g^{kl}\psi^{\alpha}{}_{|l}\chi_{\alpha|k} + \chi_\alpha\left[(2\alpha^2 - \tfrac{R}{4})\psi^{\alpha} - \tfrac{i}{2} F_{lk}\sigma^{l\lambda\alpha}\sigma^{k}{}_{\lambda\sigma}\psi^{\sigma}\right]$$

$$= g^{kl}\psi^{\alpha}{}_{|l}\chi_{\alpha|k} + (2\alpha^2 - \tfrac{R}{4})K - \tfrac{i}{2} F_{lk} Q^{lk}$$

$$= (G^l)_l$$

As $\quad G_l + H_l = K_{,l} \qquad$ by (3.25)

$$G_{l,k} + H_{l,k} = K_{,lk}$$

and hence $\quad (G^l)_l = \tfrac{1}{2} g^{lk}(K)_{lk} = (H^l)_l \qquad (3.37)$

SECOND/

SECOND ORDER TENSORS WITH ONE COVARIANT DERIVATIVE.

In this class we have four distinct tensors defined as

$$\left.\begin{aligned} T^{l\cdot}_{\cdot k} &= \psi^{\lambda}\,\sigma^{l}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|k} \\ U^{l\cdot}_{\cdot k} &= \chi_{\lambda}\,\sigma^{l\lambda\dot\mu}\,\chi_{\dot\mu|k} \\ V^{l\cdot}_{\cdot k} &= \chi^{\lambda}\,\sigma^{l}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|k} \\ W^{l\cdot}_{\cdot k} &= \psi_{\lambda}\,\sigma^{l\lambda\dot\mu}\,\chi_{\dot\mu|k} \end{aligned}\right\} \qquad (3.38)$$

We require the divergences of these taken with respect to both of the tensor indices.

First consider $\left(T^{l\cdot}_{\cdot k}\right)_{|l} = \psi^{\lambda}{}_{|l}\,\sigma^{l}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|k} + \psi^{\lambda}\,\sigma^{l}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|kl}$

Now
$$\begin{aligned} \psi^{\dot\mu}{}_{|kl} &= \psi^{\dot\mu}{}_{|lk} + \psi^{\dot\sigma}\,P^{\dot\mu}{}_{\dot\sigma lk} \\ &= \psi^{\dot\mu}{}_{|lk} + \psi^{\dot\sigma}\left(\tfrac{1}{2}R_{prlk}\,\sigma^{p\dot\mu\nu}\,\sigma^{r}{}_{\nu\dot\sigma} - \tfrac{i}{2}F_{lk}\,\delta^{\dot\mu}_{\dot\sigma}\right) \end{aligned}$$

Therefore
$$\begin{aligned} \left(T^{l\cdot}_{\cdot k}\right)_{|l} &= \left(-\alpha\chi_{\dot\mu}\right)\psi^{\dot\mu}{}_{|k} + \psi^{\lambda}\left(\alpha\chi_{\lambda}\right)_{k} - \tfrac{i}{2}F_{lk}\,\psi^{\dot\mu}\psi^{\lambda}\sigma^{l}{}_{\lambda\dot\mu} \\ &\quad + \tfrac{1}{2}R_{prlk}\,\sigma^{l}{}_{\lambda\dot\mu}\,\sigma^{p\dot\mu\nu}\,\sigma^{r}{}_{\nu\dot\sigma}\,\psi^{\lambda}\psi^{\dot\sigma} \end{aligned}$$

$$= -\alpha\bar{H}_{k} + \alpha G_{k} - \tfrac{i}{2}F_{lk}\,B^{l} - \tfrac{1}{2}R_{kl}\,B^{l} \qquad (3.39)$$

where $R_{kl} = g^{pr}R_{lprk}$ and use has been made of the fact that $R_{prlk} + R_{rlpk} + R_{lprk} = 0$

A similar result holds for $U^{l\cdot}_{\cdot k}$:

$$\left(U^{l\cdot}_{\cdot k}\right)_{|l} = \alpha\bar{G}_{k} - \alpha H_{k} - \frac{R_{kl} + iF_{kl}}{2}A^{l}. \qquad (3.40)$$

If/

If the divergence is taken with respect to the other index we have

$$(T^{kl})_{|l} = (T^{k\cdot}{}_{\cdot m}\, g^{ml})_{|l}$$

$$= g^{ml}\left(\psi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|m} + \psi^{\lambda}\,\sigma^{k}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|ml}\right)$$

$$= g^{ml}\,\psi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|m} + \sigma^{k}{}_{\lambda\dot{\mu}}\,\psi^{\lambda}\left[\left(2\alpha^2 - \frac{R}{4}\right)\psi^{\dot{\mu}} + \frac{i}{2}F_{lp}\,\sigma^{l\sigma\dot{\mu}}\,\sigma^{p}{}_{\sigma\dot{\nu}}\,\psi^{\dot{\nu}}\right]$$

$$= g^{ml}\,\psi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|m} + \left(2\alpha^2 - \frac{R}{4}\right)B^{k} + \frac{i}{2}F_{lp}\,\psi^{\lambda}\,\sigma^{k}{}_{\lambda\dot{\mu}}\,\sigma^{l\dot{\mu}\sigma}\,\sigma^{p}{}_{\sigma\dot{\nu}}\,\psi^{\dot{\nu}}$$

The imaginary part of this is

$$\frac{i}{4}F_{lp}\left[\psi^{\lambda}\,\sigma^{k}{}_{\lambda\dot{\mu}}\,\sigma^{l\dot{\mu}\sigma}\,\sigma^{p}{}_{\dot{\nu}\sigma}\,\psi^{\dot{\nu}} + \psi^{\dot{\lambda}}\,\sigma^{k}{}_{\dot{\lambda}\mu}\,\sigma^{l\dot{\sigma}\mu}\,\sigma^{p}{}_{\nu\dot{\sigma}}\,\psi^{\nu}\right]$$

which by interchange of dummy suffixes in the second term

$$= \frac{i}{4}F_{lp}\,\psi^{\lambda}\psi^{\dot{\nu}}\left[\sigma^{k}{}_{\lambda\dot{\mu}}\,\sigma^{l\dot{\mu}\sigma}\,\sigma^{p}{}_{\dot{\nu}\sigma} + \sigma^{p}{}_{\lambda\dot{\mu}}\,\sigma^{l\sigma\dot{\mu}}\,\sigma^{k}{}_{\dot{\nu}\sigma}\right]$$

$$= \frac{i}{4}F_{lp}\,\psi^{\lambda}\psi^{\dot{\nu}}\left[-g^{kp}\,\delta^{\sigma}_{\lambda}\,\sigma^{l}{}_{\dot{\nu}\sigma} + \sigma^{p}{}_{\lambda\dot{\mu}}\,\sigma^{k\dot{\mu}\sigma}\,\sigma^{l}{}_{\dot{\nu}\sigma} + \sigma^{p}{}_{\lambda\dot{\mu}}\,\sigma^{l\dot{\mu}\sigma}\,\sigma^{k}{}_{\dot{\nu}\sigma}\right]$$

$$= \frac{i}{4}F_{lp}\,\psi^{\lambda}\psi^{\dot{\nu}}\left[-g^{kp}\,\sigma^{l}{}_{\dot{\nu}\lambda} + \sigma^{p}{}_{\lambda\dot{\mu}}\,g^{kl}\,\delta^{\dot{\mu}}_{\dot{\nu}}\right]$$

$$= \frac{i}{2}F_{lp}\,g^{lk}\,B^{p}. \tag{3.41}$$

A similar calculation for the imaginary part of $(U^{kl})_{|l}$ gives

$$(U^{kl})_{|l} - (\bar{U}^{kl})_{|l} = -i\,F_{lp}\,g^{lk}A^{p}. \tag{3.42}$$

The divergences of V and W are as follows

$$(V^{l}{}_{\cdot k})_{|l} = \chi^{\lambda}{}_{|l}\,\sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|k} + \chi^{\lambda}\,\sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|kl}$$

$$= \left(i\phi_l\,\chi^{\lambda} + \chi_{\rho|l}\,\gamma^{\lambda\rho}\right)\sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|k}$$

$$+ \chi^{\lambda}\,\sigma^{l}{}_{\lambda\dot{\mu}}\left(\psi^{\dot{\mu}}{}_{|lk} + \psi^{\dot{\rho}}\left[\frac{1}{2}R_{p\nu lk}\,\sigma^{p\dot{\mu}\nu}\,\sigma^{r}{}_{\dot{\rho}\nu} - \frac{i}{2}F_{lk}\,\delta^{\dot{\mu}}_{\dot{\rho}}\right]\right)$$

$$= i\phi_l\,V^{l}{}_{\cdot k} - \alpha\,\psi_{\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|k} + \chi^{\lambda}(\alpha\,\chi_{\lambda})_{|k} - \frac{iF_{lk} + R_{lk}}{2}C^{l}$$

$$= i\phi_l\,V^{l}{}_{\cdot k} - \alpha\,\bar{E}_{k} + \alpha\,F_{k} - \frac{C^{l}}{2}(iF_{lk} + R_{lk}). \tag{3.43}$$

Similarly/

Similarly

$$(W^{l}{}_{\cdot k})_{l} = -i\phi_l W^{l}{}_{\cdot k} + \alpha \bar{F}_k - \alpha E_k - \frac{\bar{c}^l}{2}\left(iF_{kl} + R_{lk}\right). \quad (3.44)$$

From (3.43) and (3.44) it follows that

$$(V^{l}{}_{\cdot k} + \bar{W}^{l}{}_{\cdot k})_{l} = i\phi_l \left(V^{l}{}_{\cdot k} + \bar{W}^{l}{}_{\cdot k}\right) - i c^l F_{lk} - c^l R_{lk}.$$

The divergences with respect to the other indices may also be determined

$$\begin{aligned}(V^{kl})_{l} &= g^{pl}(V^{k}{}_{\cdot p})_{l} = g^{pl}\left(\chi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|p} + \chi^{\lambda}\sigma^{l}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|pl}\right)\\ &= g^{pl}\chi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|p} + \chi^{\lambda}\sigma^{k}{}_{\lambda\dot\mu}\left[\left(2\alpha^2 - \frac{R}{4}\right)\psi^{\dot\mu} + \frac{i}{2}F_{pl}\,\sigma^{p\dot\mu\rho}\,\sigma^{l}{}_{\dot\sigma\rho}\,\psi^{\dot\sigma}\right]\\ &= g^{pl}\chi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|p} + \left(2\alpha^2 - \frac{R}{4}\right)c^{k} + \frac{i}{2}F_{pl}\left(\chi^{\lambda}\sigma^{k}{}_{\lambda\dot\mu}\,\sigma^{p\dot\mu\rho}\,\sigma^{l}{}_{\dot\sigma\rho}\,\psi^{\dot\sigma}\right) \quad (3.45)\end{aligned}$$

$$\begin{aligned}(W^{kl})_{l} &= g^{pl}(W^{k}{}_{p})_{l} = g^{pl}\left(\psi_{\lambda|l}\,\sigma^{k\lambda\dot\mu}\chi_{\dot\mu|p} + \psi_{\lambda}\sigma^{k\lambda\dot\mu}\chi_{\dot\mu|pl}\right)\\ &= g^{pl}\psi_{\lambda|l}\,\sigma^{k\lambda\dot\mu}\chi_{\dot\mu|p} + \psi_{\lambda}\sigma^{k\lambda\dot\mu}\left[\left(2\alpha^2 - \frac{R}{4}\right)\chi_{\dot\mu} - \frac{i}{2}F_{pl}\,\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{l\rho\dot\sigma}\chi_{\dot\sigma}\right]\\ &= g^{pl}\psi_{\lambda|l}\,\sigma^{k\lambda\dot\mu}\chi_{\dot\mu|p} + \left(2\alpha^2 - \frac{R}{4}\right)\bar{c}^{k} - \frac{i}{2}F_{pl}\left(\psi_{\lambda}\sigma^{k\lambda\dot\mu}\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{l\rho\dot\sigma}\right)\chi_{\dot\sigma}. \quad (3.46)\end{aligned}$$

These expressions (3.45) and 3.46) introduces new tensors which do not reduce to very simple relations

$$\begin{aligned}(V^{kl} + \bar{W}^{kl})_{l} &= 2\left(2\alpha^2 - \frac{R}{4}\right)c^{k} + g^{pl}\left[\left(\chi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|p}\right) + \left(\chi_{\dot\mu|p}\,\sigma^{k\lambda\dot\mu}\,\psi_{\lambda|l}\right)\right]\\ &\quad + \frac{i}{2}F_{pl}\left[\psi_{\lambda}\,\sigma^{k\lambda\dot\mu}\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{l\rho\sigma}\chi_{\sigma} + \psi_{\lambda}\,\sigma^{l\lambda\dot\mu}\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{k\rho\sigma}\chi_{\sigma}\right],\end{aligned}$$

after raising, lowering and changing dummy suffixes.

Now $\quad \left(\sigma^{k\lambda\dot\mu}\,\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{l\rho\sigma} + \sigma^{l\lambda\dot\mu}\,\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{k\rho\sigma}\right)F_{pl}$

$$= \left(\left[\sigma^{k\lambda\dot\mu}g^{pl}\delta^{\sigma}_{\dot\mu} - g^{kl}\sigma^{p\rho\sigma}\delta^{\lambda}_{\rho} + \sigma^{l\lambda\dot\mu}g^{pk}\delta^{\sigma}_{\dot\mu} - \sigma^{l\lambda\dot\mu}\sigma^{p}{}_{\rho\dot\mu}\sigma^{k\rho\sigma}\right] + \sigma^{l\lambda\dot\mu}\sigma^{p}{}_{\rho\dot\mu}\,\sigma^{k\rho\sigma}\right)F_{pl}$$

$$= 2\,\sigma^{p\lambda\sigma}F^{k}{}_{\cdot l} \qquad \text{since } F_{pl} \text{ is skew.}$$

Hence for the third term we have

$$\begin{aligned}&i\,F^{k}{}_{\cdot l}\,\sigma^{l\lambda\sigma}\psi_{\lambda}\chi^{\sigma}\\ = \;&i\,c^{l}F^{k}{}_{\cdot l}.\end{aligned}$$

The/

The first term which is $g^{pl}[\chi^{\lambda}{}_{|l}\,\sigma^{k}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|p} + \chi_{\mu|p}\,\sigma^{k\lambda\dot\mu}\,\psi_{\dot\lambda|l}]$

$$= g^{pl}\left[\left(\tau^{\lambda\rho}\chi_{\rho|l} + i\phi_l\chi^{\lambda}\right)\sigma^{k}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|p} + \chi_{\mu|p}\,\sigma^{k\lambda\dot\mu}\left(\psi^{\dot\rho}{}_{|l}\,\tau_{\dot\rho\dot\lambda} + i\phi_l\,\psi_{\dot\lambda}\right)\right]$$

$$= i\phi_l\, g^{pl}\left(V^{k}{}_{\cdot p} + \overline{W}{}^{k}{}_{\cdot p}\right) - 2g^{pl}\left(\chi_{\rho|l}\,\sigma^{k\rho}{}_{\dot\mu}\,\psi^{\dot\mu}{}_{|p}\right).$$

We shall just note the result of reducing the last term here, it is

$$-\frac{2}{K\overline{K}}\left\{A^{k}\overline{E}_{p}G^{p} + B^{k}\overline{H}_{p}F^{p} - C^{k}\overline{H}_{p}G^{p} - \overline{C}^{k}\overline{E}_{p}F^{p}\right\}$$

or from formulas (3.50) and (3.52) below, this is

$$\frac{2}{\overline{K}}\left(\overline{H}_{p}\,\overline{W}^{kp} - \overline{E}_{p}\,\overline{U}^{kp}\right).$$

Hence finally we have

$$\left(V^{kl} + \overline{W}^{kl}\right)_{l} = i\phi_l\left(V^{kl} + \overline{W}^{kl}\right) + iC^{l}F^{k}{}_{\cdot l}$$
$$+\left(4\alpha^2 - \frac{R}{2}\right)C^{k} - \frac{2}{\overline{K}}\left(\overline{E}_{p}\,\overline{U}^{kp} - \overline{H}_{p}\,\overline{W}^{kp}\right).$$

The expressions for these divergences are not so simple as those for T and U.

The contracted products of these tensors are easily found and from these one can express the tensors in terms of the null vectors. We have

$$\begin{aligned}
T^{l}{}_{\cdot k}A_{l} &= \psi^{\lambda}\sigma^{l}{}_{\lambda\dot\mu}\,\psi^{\dot\mu}{}_{|k}\;\sigma_{l}{}^{\dot\alpha\rho}\chi_{\dot\alpha}\chi_{\rho}\\
&= \psi^{\lambda}\psi^{\dot\mu}{}_{|k}\,\delta^{\rho}_{\lambda}\,\delta^{\dot\alpha}_{\dot\mu}\,\chi_{\dot\alpha}\chi_{\rho}\\
&= K\,\overline{H}_{k}\\
T^{l}{}_{k}B_{l} &= \psi^{\lambda}\psi^{\dot\mu}{}_{|k}\,\psi_{\dot\mu}\psi_{\lambda} = 0\\
T^{l}{}_{k}C_{l} &= \psi^{\lambda}\psi^{\dot\mu}{}_{|k}\,\psi_{\dot\mu}\chi_{\lambda} = K E_{k}\\
T^{l}{}_{k}\overline{C}_{l} &= \psi^{\lambda}\psi^{\dot\mu}{}_{|k}\,\chi_{\dot\mu}\psi_{\lambda} = 0
\end{aligned} \qquad (3.47)$$

This/

This corresponds to a resolution along the directions of a set of orthogonal vectors only here the orthogonality is different from the usual type for the vectors are self perpendicular. These results suggest then that

$$B^{l}\,\overline{H}_{k} \;-\; \overline{C}^{l}\,\overline{E}_{k} \;=\; \overline{K}\;T^{l}_{\cdot k}\,. \tag{3.48}$$

This is easily verified, for by (3.2) and 3.24)

$$\begin{aligned}(B^{l}\overline{H}_{k} - \overline{C}^{l}\overline{E}_{k}) &= \sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\lambda}\psi^{\dot{\mu}}\,\psi^{\dot{\sigma}}{}_{|k}\,\chi_{\dot{\sigma}} - \sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\lambda}\chi^{\dot{\mu}}\,\psi^{\dot{\sigma}}{}_{|k}\,\psi_{\dot{\sigma}}\\ &= \psi^{\lambda}\,\sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\sigma}}{}_{|k}\,(\psi^{\dot{\mu}}\chi_{\dot{\sigma}} - \chi^{\dot{\mu}}\psi_{\dot{\sigma}})\\ &= \psi^{\lambda}\,\sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\sigma}}{}_{|k}\;\;\delta^{\dot{\mu}}_{\dot{\sigma}}\,\overline{K}\\ &= \overline{K}\,T^{l}_{\cdot k}\end{aligned}$$

Likewise for the tensor $U^{l}_{\cdot k}$ we have the contracted products

$$\left.\begin{aligned} U^{l}_{\cdot k}\;A_{l} &= 0\\ U^{l}_{\cdot k}\;B_{l} &= K\,\overline{G}_{k}\\ U^{l}_{\cdot k}\;C_{l} &= 0\\ U^{l}_{\cdot k}\;\overline{C}_{l} &= K\,F_{k}\end{aligned}\right\} \tag{3.49}$$

whence it follows directly that

$$\overline{K}\,U^{l}_{\cdot k} = A^{l}\,\overline{G}_{k} \;-\; C^{l}\,\overline{F}_{k}\,. \tag{3.50}$$

In the same way

$$\left.\begin{aligned} V^{l}_{\cdot k}\,A_{l} &= 0 & W^{l}_{\cdot k}\;A_{l} &= -\,K\,\overline{F}_{k}\\ V^{l}_{\cdot k}\,B_{l} &= -\,K\,\overline{E}_{k} & W^{l}_{\cdot k}\;B_{l} &= 0\\ V^{l}_{\cdot k}\,C_{l} &= 0 & W^{l}_{\cdot k}\;C_{l} &= -\,K\,\overline{G}_{k}\\ V^{l}_{\cdot k}\,\overline{C}_{l} &= -\,K\,\overline{H}_{k} & W^{l}_{\cdot k}\;\overline{C}_{l} &= 0\end{aligned}\right\} \tag{3.51}$$

with/

$$\text{with } \bar{K}\, V^{l}_{\cdot k} = -\, A^{l}\bar{E}_{k} + C^{l}\bar{H}_{k} \qquad \bar{K}\, W^{l}_{\cdot k} = -\, B^{l}\bar{F}_{k} + \bar{C}^{l}\bar{G}_{k} \tag{3.52}$$

For the contracted tensors or spurs, by appealing to the wave equation we find

$$T^{l\cdot}_{\cdot l} = T^{\cdot l}_{l\cdot} = \psi^{\lambda}\,\sigma^{l}{}_{\lambda\dot{\mu}}\,\psi^{\dot{\mu}}{}_{|l} = \psi^{\lambda}\,(\alpha\,\chi_{\lambda})$$

$$\begin{aligned} \text{Therefore} \quad T^{l}_{l} &= \alpha K \\ \text{Similarly} \quad U^{l}_{l} &= -\alpha K \\ V^{l}_{l} &= 0 \\ W^{l}_{l} &= 0 \end{aligned} \tag{3.53}$$

Although there is no limit to the number of types of tensors we could form, this list contains all the important fundamental types including all those which combine to form tensorial quan:tities for which there are special physical interpretations. We have not concerned ourselves with tensors above the second order and of those of that order we have studied two classes. There are others of this order such as $\psi^{\mu}{}_{|lk}\,\chi_{\mu}$ and $\psi^{\mu}{}_{|l}\,\chi_{\mu|k}$ involving covariant derivatives twice over, but these are of no special interest and are not studied further. However the contracted scalar derived from the latter appeared in (3.36) and by equation (3.37) it can be linked up with the given tensorial quantities.

At this stage we may conveniently reproduce the list of quantities considered in the special relativity case. The component of the vectors and tensors will be written out in terms of the original/

original Dirac notation, that is by the four component function and the α - matrices. As we have already noted, for the transition to pseudo orthogonal axes in special relativity we have

$$\Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \psi^{\dot{1}} \\ \psi^{\dot{2}} \end{bmatrix}$$

with

$$\Psi_{,k} = \left(\frac{\partial}{\partial x^k} - \frac{2\pi i}{h}\frac{e}{c}A_k\right)\Psi = \left(\frac{\partial}{\partial x^k} - \frac{i}{2}\phi_k\right)\Psi \qquad (3.54)$$

and $$\overline{\Psi}_{,k} = \left(\frac{\partial}{\partial x^k} + \frac{2\pi i}{h}\frac{e}{c}A_k\right)\overline{\Psi} = \left(\frac{\partial}{\partial x^k} + \frac{i}{2}\phi_k\right)\overline{\Psi}.$$

The Dirac equations when written in full are

$$\left.\begin{aligned} \Pi_1 &\equiv -\psi_{4|1} + i\,\psi_{4|2} - \psi_{3|3} + \psi_{3|4} - \alpha\psi_1 = 0 \\ \Pi_2 &\equiv -\psi_{3|1} - i\,\psi_{3|2} + \psi_{4|3} + \psi_{4|4} - \alpha\psi_2 = 0 \\ \Lambda^{\dot{1}} &\equiv +\psi_{2|1} - i\,\psi_{2|2} + \psi_{1|3} + \psi_{1|4} - \alpha\psi_3 = 0 \\ \Lambda^{\dot{2}} &\equiv \psi_{1|1} + i\,\psi_{1|2} - \psi_{2|3} + \psi_{2|4} - \alpha\psi_4 = 0. \end{aligned}\right\} \qquad (3.55)$$

The various quantities will now be expressed in detail and also in matrix form where

Ψ represents the column vector

$\overline{\Psi}$ is its complex conjugate

$\widetilde{\Psi}$ is the transposed vector (i.e. a row vector)

$\Psi^{\dagger}$ is the adjoint or row-vector with complex conjugate components.

Then

Then by using the values of $\overset{\circ}{\sigma}{}^{k\dot{\lambda}\mu}$ (2.27) and noting the rules of lowering spinor indices we can write out the quantities in terms of spinor components and then rewrite these as Ψ-components. These bilinear forms are expressed in a matrix product by means of the α-matrices (which are written out in (2.35).

$$\left\{\begin{aligned}
\sqrt{2}A^1 &= \bar\psi_1\psi_2 + \bar\psi_2\psi_1 &&= \tfrac{1}{2}\,\psi^+(-\alpha_1)(1 + i\,\alpha_1\alpha_2\alpha_3)\,\psi \\
\sqrt{2}A^2 &= -i\,\bar\psi_1\psi_2 + i\,\bar\psi_2\psi_1 &&= \tfrac{1}{2}\,\psi^+(-\alpha_2)(1 + i\,\alpha_1\alpha_2\alpha_3)\,\psi \\
\sqrt{2}A^3 &= \bar\psi_1\psi_1 - \bar\psi_2\psi_2 &&= \tfrac{1}{2}\,\psi^+(-\alpha_3)(1 + i\,\alpha_1\alpha_2\alpha_3)\,\psi \\
\sqrt{2}A^4 &= \bar\psi_1\psi_1 + \bar\psi_2\psi_2 &&= \tfrac{1}{2}\,\psi^+\;1.(1 + i\,\alpha_1\alpha_2\alpha_3)\,\psi.
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\sqrt{2}B^1 &= -\bar\psi_3\psi_4 - \bar\psi_4\psi_3 &&= \tfrac{1}{2}\,\psi^+(-\alpha_1)(1 - i\,\alpha_1\alpha_2\alpha_3)\,\psi \\
\sqrt{2}B^2 &= i\,\bar\psi_3\psi_4 - i\,\bar\psi_4\psi_3 &&= \tfrac{1}{2}\,\psi^+(-\alpha_2)(1 - i\,\alpha_1\alpha_2\alpha_3)\,\psi \\
\sqrt{2}B^3 &= -\bar\psi_3\psi_3 + \bar\psi_4\psi_4 &&= \tfrac{1}{2}\,\psi^+(-\alpha_3)(1 - i\,\alpha_1\alpha_2\alpha_3)\,\psi \\
\sqrt{2}B^4 &= \bar\psi_3\psi_3 + \bar\psi_4\psi_4 &&= \tfrac{1}{2}\,\psi^+\;1\;(1 - i\,\alpha_1\alpha_2\alpha_3)\,\psi.
\end{aligned}\right.$$

$$\left\{\begin{aligned}
\sqrt{2}C &= \psi_1\psi_3 - \psi_2\psi_4 &&= \tfrac{1}{2}\,\tilde\psi\,(i\,\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)(-\alpha_1)\,\psi \\
\sqrt{2}C &= i\,\psi_1\psi_3 + i\,\psi_2\psi_4 &&= \tfrac{1}{2}\,\tilde\psi\,(i\,\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)(-\alpha_2)\,\psi \\
\sqrt{2}C &= -\psi_1\psi_4 - \psi_2\psi_3 &&= \tfrac{1}{2}\,\tilde\psi\,(i\,\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)(-\alpha_3)\,\psi \\
\sqrt{2}C &= -\psi_1\psi_4 + \psi_2\psi_3 &&= \tfrac{1}{2}\,\tilde\psi\,(i\,\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)(1)\,\psi.
\end{aligned}\right.$$

$$K = \psi_1\bar\psi_3 + \psi_2\bar\psi_4 = \tfrac{1}{2}\,\psi^+(-\alpha_4)(1 + i\,\alpha_1\alpha_2\alpha_3)\,\psi$$

$$E_k = \bar\psi_3\bar\psi_{4|k} - \bar\psi_4\bar\psi_{3|k} = \tfrac{1}{2}\,\psi^+\,(\alpha_3\alpha_1 + i\,\alpha_2)\,\bar\Psi_{1k}$$

F /

$$F_k = -\psi_2 \psi_{1|k} + \psi_1 \psi_{2|k} = \frac{1}{2} \tilde{\psi} (\alpha_3\alpha_1 - i\alpha_2) \psi_{1k}$$

$$G_k = \bar{\psi}_3 \psi_{1|k} + \bar{\psi}_4 \psi_{2|k} = \frac{1}{2} \psi^+ (-\alpha_4)(1 + i\alpha_1\alpha_2\alpha_3) \psi_{1k}$$

$$H_k = \bar{\psi}_{3|k} \psi_1 + \bar{\psi}_{4|k} \psi_2 = \frac{1}{2} \psi^+_{1k} (-\alpha_4)(1 + i\alpha_1\alpha_2\alpha_3) \psi$$

$$2Q^{23} = i\bar{\psi}_3 \psi_2 + i\bar{\psi}_4 \psi_1 = \frac{1}{2} \psi^+ (i\alpha_4\alpha_1 - \alpha_4\alpha_2\alpha_3) \psi$$

$$2Q^{31} = \bar{\psi}_3 \psi_2 - \bar{\psi}_4 \psi_1 = \frac{1}{2} \psi^+ (i\alpha_4\alpha_2 - \alpha_4\alpha_3\alpha_1) \psi$$

$$2Q^{12} = i\bar{\psi}_3 \psi_1 - i\bar{\psi}_4 \psi_2 = \frac{1}{2} \psi^+ (i\alpha_4\alpha_3 - \alpha_4\alpha_1\alpha_2) \psi$$

$$2Q^{14} = \bar{\psi}_3 \psi_2 + \bar{\psi}_4 \psi_1 = \frac{1}{2} \psi^+ (-i\alpha_4\alpha_2\alpha_3 + \alpha_4\alpha_1) \psi$$

$$2Q^{24} = -i\bar{\psi}_3 \psi_2 + i\bar{\psi}_4 \psi_1 = \frac{1}{2} \psi^+ (-i\alpha_4\alpha_3\alpha_1 + \alpha_4\alpha_2) \psi$$

$$2Q^{34} = \bar{\psi}_3 \psi_1 - \bar{\psi}_4 \psi_2 = \frac{1}{2} \psi^+ (-i\alpha_4\alpha_1\alpha_2 + \alpha_4\alpha_3) \psi$$

$$2M^{23} = 2\,i\,M^{14} = -i\psi_1^2 + i\psi_2^2 = \frac{1}{2} \tilde{\psi} (i\alpha_3\alpha_1 + \alpha_2) \alpha_1 \psi$$

$$2M^{31} = 2\,i\,M^{24} = \psi_1^2 + \psi_2^2 = \frac{1}{2} \tilde{\psi} (i\alpha_3\alpha_1 + \alpha_2) \alpha_2 \psi$$

$$2M^{12} = 2\,i\,M^{34} = 2i\psi_1\psi_2 = \frac{1}{2} \tilde{\psi} (i\alpha_3\alpha_1 + \alpha_2) \alpha_3 \psi$$

$$2\bar{N}^{23} = 2\,i\,\bar{N}^{41} = i\psi_3^2 - i\psi_4^2 = -\frac{1}{2} \tilde{\psi} (i\alpha_1\alpha_3 + \alpha_2) \alpha_1 \psi$$

$$2\bar{N}^{31} = 2\,i\,\bar{N}^{42} = -\psi_3^2 - \psi_4^2 = -\frac{1}{2} \tilde{\psi} (i\alpha_1\alpha_3 + \alpha_2) \alpha_2 \psi$$

$$2\bar{N}^{12} = 2\,i\,\bar{N}^{43} = -2i\psi_3\psi_4 = -\frac{1}{2} \tilde{\psi} (i\alpha_1\alpha_3 + \alpha_2) \alpha_3 \psi$$

$$\sqrt{2}T^1_{\cdot k} = -\bar{\psi}_3 \psi_{4|k} - \bar{\psi}_4 \psi_{3|k} = \frac{1}{2} \psi^+ (-\alpha_1)(1 - i\alpha_1\alpha_2\alpha_3) \psi_{1k}$$

$$\sqrt{2}T^2_{\cdot k} = i\bar{\psi}_3 \psi_{4|k} - i\bar{\psi}_4 \psi_{3|k} = \frac{1}{2} \psi^+ (-\alpha_2)(1 - i\alpha_1\alpha_2\alpha_3) \psi_{1k}$$

$$\sqrt{2}T^3_{\cdot k} = -\bar{\psi}_3 \psi_{3|k} + \bar{\psi}_4 \psi_{4|k} = \frac{1}{2} \psi^+ (-\alpha_3)(1 - i\alpha_1\alpha_2\alpha_3) \psi_{1k}$$

$$\sqrt{2}T^4_{\cdot k} = \psi_3 \psi_{3|k} + \psi_4 \psi_{4|k} = \frac{1}{2} \psi^+ (1)(1 - i\alpha_1\alpha_2\alpha_3) \psi_{1k}.$$

$$\sqrt{2}U^1_{\cdot k} = \psi_1 \bar{\psi}_{2|k} + \psi_2 \bar{\psi}_{1|k} = \frac{1}{2} \psi^+_{1k} (-\alpha_1)(1 + i\alpha_1\alpha_2\alpha_3) \psi$$

$$\sqrt{2}U^2_{\cdot k} = i\psi_1 \bar{\psi}_{2|k} - i\psi_2 \bar{\psi}_{1|k} = \frac{1}{2} \psi^+_{1k} (-\alpha_2)(1 + i\alpha_1\alpha_2\alpha_3) \psi$$

$$\sqrt{2}U^3_{\cdot k} = \psi_1 \bar{\psi}_{1|k} - \psi_2 \bar{\psi}_{2|k} = \frac{1}{2} \psi^+_{1k} (-\alpha_3)(1 + i\alpha_1\alpha_2\alpha_3) \psi$$

$$\sqrt{2}U^4_{\cdot k} = \psi_1 \bar{\psi}_{1|k} + \psi_2 \bar{\psi}_{2|k} = \frac{1}{2} \psi^+_{1k}\, 1\, (1 + i\alpha_1\alpha_2\alpha_3) \psi\,.$$

$$2U =$$

$$\begin{aligned}
\sqrt{2}V^{1}_{\cdot k} &= \psi_1 \psi_{31k} - \psi_2 \psi_{41k} &&= \tfrac{1}{2}\tilde{\psi}\,(i\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)\,(-\alpha_1)\,\psi_{1k} \\
\sqrt{2}V^{2}_{\cdot k} &= +i\psi_1 \psi_{31k} + i\psi_2 \psi_{41k} &&= \tfrac{1}{2}\tilde{\psi}\,(i\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)\,(-\alpha_2)\,\psi_{1k} \\
\sqrt{2}V^{3}_{\cdot k} &= -\psi_1 \psi_{41k} - \psi_2 \psi_{31k} &&= \tfrac{1}{2}\tilde{\psi}\,(i\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)\,(-\alpha_3)\,\psi_{1k} \\
\sqrt{2}V^{4}_{\cdot k} &= -\psi_1 \psi_{41k} + \psi_2 \psi_{31k} &&= \tfrac{1}{2}\tilde{\psi}\,(i\alpha_4\alpha_2 + \alpha_4\alpha_3\alpha_1)\,(1)\,\psi_{1k} \\
\sqrt{2}W^{1}_{\cdot k} &= \bar{\psi}_3 \bar{\psi}_{11k} - \bar{\psi}_4 \bar{\psi}_{21k} &&= \tfrac{1}{2}\psi^{+}(-\alpha_1)\,(i\alpha_4\alpha_2 - \alpha_4\alpha_3\alpha_1)\,\bar{\psi}_{1k} \\
\sqrt{2}W^{2}_{\cdot k} &= -i\bar{\psi}_3 \bar{\psi}_{11k} - i\bar{\psi}_4 \bar{\psi}_{21k} &&= \tfrac{1}{2}\psi^{+}(-\alpha_2)\,(i\alpha_4\alpha_2 - \alpha_4\alpha_3\alpha_1)\,\bar{\psi}_{1k} \\
\sqrt{2}W^{3}_{\cdot k} &= -\bar{\psi}_4 \bar{\psi}_{11k} - \bar{\psi}_3 \bar{\psi}_{21k} &&= \tfrac{1}{2}\psi^{+}(-\alpha_3)\,(i\alpha_4\alpha_2 - \alpha_4\alpha_3\alpha_1)\,\bar{\psi}_{1k} \\
\sqrt{2}W^{4}_{\cdot k} &= -\bar{\psi}_4 \bar{\psi}_{11k} + \bar{\psi}_3 \bar{\psi}_{21k} &&= \tfrac{1}{2}\psi^{+}(1)\,(i\alpha_4\alpha_2 - \alpha_4\alpha_3\alpha_1)\,\bar{\psi}_{1k}
\end{aligned} \qquad (3.56)$$

From geometrical considerations we found the relation (3.19)

$$C^{a}\,\bar{C}^{b} - C^{b}\,\bar{C}^{a} = \rho\,\frac{\varepsilon^{abcd}}{2}\,(A_c\,B_d - A_d\,B_c)$$

By substituting the above values in one of these equations we can determine the value of ρ. For example

$$C^{1}\,\bar{C}^{2} - C^{2}\,\bar{C}^{1} = \rho\,(A_3\,B_4 - A_4\,B_3)$$

This gives : $-i(\psi_1\psi_3 - \psi_2\psi_4)(\bar{\psi}_1\bar{\psi}_3 + \bar{\psi}_2\bar{\psi}_4) - i(\psi_1\psi_3 + \psi_2\psi_4)(\bar{\psi}_1\bar{\psi}_3 - \bar{\psi}_2\bar{\psi}_4)$

$$= -\rho\,(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2)(\bar{\psi}_3\psi_3 + \bar{\psi}_4\psi_4) + \rho\,(\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2)(-\bar{\psi}_3\psi_3 + \bar{\psi}_4\psi_4)$$

This/

This equation is true when $\rho = +i$; this enables us to make a suitable choice of the sign of $\sqrt{-1}$ to be taken so that all our results will be consistent.

The list of quantities shows us immediately that there are two types, those which involve the ψ -function and its complex conjugate and those which involve two ψ -functions (or two $\bar{\psi}$ -functions). For the former some physical interpretations may be possible, but, as far as the theory of quantum mechanics is concerned at least, no interpretation can be given to the latter. To the former class belong the vectors, A^k B^k G^k and H^k

the scalar K

and the tensors Q^{ik} T^{ik} and U^{ik}

all with their complex conjugates.

The latter class contains the remainder, namely

the vectors C^k E^k and F^k

the tensors M^{ik} N^{ik} V^{ik} and W^{ik}

together with the complex conjugates.

In the relations among the quantities in the first class the potential vector ϕ_k does not occur, but in the other class, where ever a covariant derivative appears there is always a term with $\pm i\,\phi_k$ replacing the covariant operator.

PHYSICAL/

PHYSICAL INTERPRETATIONS.

The vector $J^k = -ec\sqrt{2}\,(A^k + B^k)$ (3.57)

reduces in special relativity to $-ec\,\psi^+ \beta^k \psi$

when $\beta^k = (\beta^1 \beta^2 \beta^3 \beta^4) = (-\alpha_1, -\alpha_2, -\alpha_3, 1)$

which is the well known form of the vector which gives the density distribution of electric current and it was introduced by Dirac (13). As already noted, the vector is non-divergent in the most general case.

The vector $S^k = \frac{h}{4\pi}\sqrt{2}\,(A^k - B^k)$ (3.58)

reduces to $\frac{h}{4\pi}\,\psi^+ \sigma^k \psi$

when $(\sigma^1, \sigma^2, \sigma^3, \sigma^4) = (-i\alpha_2\alpha_3, -i\alpha_3\alpha_1, -i\alpha_1\alpha_2, i\alpha_1\alpha_2\alpha_3)$

This is interpreted as the density distribution of the angular (spin) momentum of the electron. This vector was first mention:ed by van Neumann (14)

The scalars:

$$\left.\begin{array}{l} -(K + \bar{K}) = V_1 \text{ reduces to } \psi^+ \alpha_4 \psi \\ -i(K - \bar{K}) = V_2 \text{ reduces to } \psi^+ \alpha_1\alpha_2\alpha_3\alpha_4 \psi \end{array}\right\} \quad (3.59)$$

which are the well known pair of scalars in the Dirac theory.

From (3.22) we have the divergence of the spin vector

$$\operatorname{div} S^k = -mc\,V_2 .$$

There is the real six-vector $\mathcal{M}^{lk} = \frac{hc}{2\pi mci}(Q^{lk} - \bar{Q}^{lk})$ (3.60)

which in the coordinate system considered becomes

M /

$$\mathcal{M}^{12} = \frac{he}{4\pi mc}\,\psi^{+} i\alpha_4\,\alpha_1\alpha_2\,\psi \qquad \text{etc}$$

$$\mathcal{M}^{14} = \frac{he}{4\pi mc}\,\psi^{+}\, i\,\alpha_1\alpha_4\,\psi \qquad \text{etc.}$$

$(\mathcal{M}^{23}, \mathcal{M}^{31}, \mathcal{M}^{12})$ are interpreted as the density of the compnnent of the magnetic moment of the electron which $(\mathcal{M}^{14}, \mathcal{M}^{24}, \mathcal{M}^{34})$ give the electric moment density. This six-vector and the two vectors V_1 and V_2 were discovered by Darwin (15).

This transition to special relativity has enabled us to recognise immediately the fundamental interpretation of our quantities and to supply the appropriate numerical factors. As defined here the new vectors J^k and S^k the scalars V_1 and V_2 and the six vector $\mathcal{M}^{lk}$ are perfectly general and the relationships we shall derive will be worked out using only the general relativity results we have already given.

The resolution of the electric current into its polarisation and convection components according to Gordon's (16) method can be effected $c(\mathcal{M}^{lk})_k = \frac{ec}{\alpha\sqrt{2}}(Q^{lk} - \overline{Q}^{lk})_k$ which from (3.26) becomes

$$= \frac{ec}{\alpha\sqrt{2}}\left(\frac{G^l}{2} - \frac{H^l}{2} - \alpha A^l - \alpha B^l - \frac{\overline{G}^l}{2} + \frac{\overline{H}^l}{2} - \alpha A^l - \alpha B^l\right)$$

$$= \frac{ec}{\alpha\sqrt{2}}\left(G^l - \overline{G}^l - H^l + \overline{H}^l\right) - J^l$$

or $\quad J^l = c(\mathcal{M}^{lk})_k + \mathcal{J}^l$

= "polarisation current" + "convection current".

The/

The convection current vector is

$$\mathcal{J}^{l} = - \frac{ec}{\sqrt{2}\,2\alpha} \left(G^{l} - H^{l} - \overline{G}^{l} + \overline{H}^{l} \right) \tag{3.61}$$

$c(\mathcal{M}^{lk})_k$ from its form has zero divergence and as J^{l} is also non-divergent $\mathcal{J}^{l}$ must be so as well. This fact also follows from the result (3.37) that $(G^{l})_l = (H^{l})_l$ which immediately shows that $(\mathcal{J}^{l})_l = 0$.

Certain relations between the vectors and tensors have been demonstrated at various times by using a special set of matrices. These relations are perfectly general and they follow directly from our previous results.

For example the length of the current vector is

$$\begin{aligned} J^{l} J_{l} &= 2(ec)^2 \,(A^{l} + B^{l})(A_{l} + B_{l}) \\ &= 4(ec)^2 \; K \overline{K} \\ &= (ec)^2 \,(V_1^2 + V_2^2). \end{aligned} \tag{3.62}$$

a relation given by Darwin (15).

The length of the spin vector S^{l} is

$$\begin{aligned} S^{l} S_{l} &= 2 \left(\frac{h}{4\pi} \right)^2 \cdot (A^{l} - B^{l})(A_{l} - B_{l}) \\ &= - \left(\frac{h}{4\pi} \right)^2 \cdot (V_1^{\,2} + V_2^{\,2}). \end{aligned} \tag{3.63}$$

The contracted product of these two vectors is

$$\begin{aligned} J^{l} S_{l} &= 2\, ec \,\frac{h}{4\pi} \,(A^{l} + B^{l})(A_{l} - B_{l}) \\ &= 0 . \end{aligned} \tag{3.64}$$

so/

so that the current and spin momentum vectors are perpendicular. These two relations (3.63) and (3.64) are given by Uhlenbeck and Laporte (17).

In his quaternion notation Lanczos (18) found four perpendi- :cular vectors all of equal length. These are essentially the same as the four vectors

$$(A^l + B^l)$$
$$(A^l - B^l)$$
$$i(C^l + \overline{C}^l)$$
$$(C^l - \overline{C}^l)$$

where the last pair are imaginary, having no direct physical meaning. These vectors are perpendicular to each other while each is of length $2K\overline{K}$. The first has zero divergence and the values of the divergences for the last two are, by equations (3.23) respectively:

$$-i\phi_k(C^k - \overline{C}^k)$$
$$i\phi_k(C^k + \overline{C}^k)\ .$$

These vanish only when the four potential is zero which was the case considered by Lanczos.

For the length of the six-vector of the electric and magnetic moments we have

M/

$$\mathcal{M}^{ik}\mathcal{M}_{ik} = \frac{e^2}{2\alpha^2}\ (Q^{ik} - \overline{Q}^{ik})\ (Q_{ik} - \overline{Q}_{ik})$$
$$= \frac{-e^2}{2\alpha^2}\ (K^2 + \overline{K}^2) \text{ by equations} \qquad (3.11)$$

$$= \frac{h^2 e^2}{8\pi^2 m^2 c^2}\ .\ (V_1^{\ 2} - V_2^{\ 2}) \qquad (3.65)$$

cf. Darwin (15).

The other invariant derived from this six vector is its inner product with its dual. This is

$$\tilde{\mathcal{M}}^{ik}\mathcal{M}_{ik} = \frac{e^2}{2\alpha^2}\ (\tilde{Q}^{ik} - \tilde{\overline{Q}}^{ik})(Q_{ik} - \overline{Q}_{ik})$$

But $\tilde{Q}^{ik} = i\, Q^{ik}$ equation (3.20)

$$Q^{ik}\, Q_{ik} = -K^2 \qquad (3.11)$$

so that
$$\tilde{\mathcal{M}}^{ik}\mathcal{M}_{ik} = \frac{e^2 i}{2\alpha^2}\ (Q^{ik} + \bar{Q}^{ik})(Q_{ik} - \overline{Q}_{ik})$$
$$= \frac{e^2 i}{2\alpha^2}\ (-K^2 + \overline{K}^2)$$
$$= \frac{h^2 e^2}{4\pi^2 m^2 c^2}\ .\ V_1\ V_2\ . \qquad \text{(Proca (19))} \qquad (3.66)$$

This six-vector can be contracted with the current and spin vectors leading to the following vectors

$$J_l\, \mathcal{M}^{lk} = \frac{e^2\, c}{\alpha}\ (A_l + B_l)\ (Q^{lk} - \overline{Q}^{l})$$
$$= \frac{e^2\, c}{\alpha}\ \left(\frac{K}{2}\ (A^k - B^k) - \frac{\overline{K}}{2}\ (A^k - B^k)\right)$$
$$= -\frac{e^2}{m}\ V_2\ S^k\ . \qquad (3.67)$$

Also/

Also $S_l \mathcal{M}^{lk} = \frac{e h^2}{8\pi^2 mc} \sqrt{2} \, (A_l - B_l)(Q^{lk} - \bar{Q}^{lk})$

$$= \frac{- e h^2}{8\pi^2 mc} \sqrt{2} \, \frac{K}{2} \, (A^k + B^k) - \frac{\bar{K}}{2} \, (A^k + B^k)$$

$$= \frac{- h^2}{16\pi^2 mc^2} \quad V_2 \quad J^k . \qquad (3.68)$$

Cf. Uhlenbeck and Laporte (17)

Similarly for the dual six vector we have

$$J_l \tilde{\mathcal{M}}^{lk} = \frac{e^2}{m} \; V_1 \; S^k$$

$$S_l \tilde{\mathcal{M}}^{lk} = \frac{h^2}{16\pi^2 mc^2} \quad V_1 \quad J^k$$

From U and T one can build up the energy tensor possessing the correct reality and symmetrical properties. This was found in generalised spinor form by Infeld and van der Waerden (7). In terms of U and T the required tensor is

$$\mathcal{J}^l_k = \frac{chi}{8\pi} \; \sqrt{2} \; (T^l_{\cdot k} - \bar{T}^l_{\cdot k} + T_k^{\cdot l} - \bar{T}_k^{\cdot l} - U^l_{\cdot k} + \bar{U}^l_{\cdot k} - U_k^{\cdot l} + \bar{U}_k^{\cdot l}) \qquad (3.69)$$

because, if we form its divergence using the results of (3.35), (3.40), (3.41) and (3.44) we have

$$(\mathcal{J}^l_k)_l = \frac{chi}{8\pi} \sqrt{2} \quad . \quad 2i \; F_{kl} (A^l + B^l)$$

$$= \frac{1}{c} \, . \; A_{kl} \, J^l \qquad (3.70)$$

after/

after the constants of proportionality between F_{kl} and A_{kl}, $A^l + B^l$ and J^l have been introduced. Thus $\mathcal{J}^l_k$ satisfies the usual requirements for the energy tensor. Its spur leads to the scalar

$$\mathcal{J} = \mathcal{J}^l_l = \frac{ch i}{8\pi} \cdot \sqrt{2} \cdot 4\alpha \ (K + \overline{K}) \qquad \text{by (3.53)}$$

$$= - mc^2 \, V_1 \qquad \text{by (3.59)} \qquad (3.71)$$

which in ordinary dynamics is c^2 times the invariant mass density. We may note that in special relativity the energy tensor takes the matrix form

$$\mathcal{J}^l_k = \frac{chi}{8\pi} \left\{ \psi^+ \beta^l \psi_{1k} - \psi^+_{1k} \beta^l \psi + g^{lq} g_{kr} \left(\psi^+ \beta^r \psi_{1q} - \psi^+_{1q} \beta^r \psi \right) \right\}$$

with $\beta^l = -\alpha_1, -\alpha_2, -\alpha_3, 1$

The energy tensor in this form was first derived by Tetrode (20) Also the scalar $\mathcal{J}$ reduces to $-mc^2 \psi^+ \alpha_4 \psi$.
This lends support to the suggestion of de Broglie (21) that the proper mass of classical theory should be represented in quantum mechanics by the operator

$$-m\alpha_4$$

of which the eigen values are $\pm m$. Then, assuming this idea is correct, we have the mass density distribution

$$\psi^+ (-m\alpha_4) \psi$$

in agreement with the usual interpretation of the spur of the energy/

energy tensor for a particle. Therefore, in general, the scalar $-m V_1$ may possibly be interpreted as the invariant mass density of the electron.

We may have to make one or two remarks upon the origin of this speculation of de Broglie. By comparing the second order wave equation in his theory with the Klein-Gordon equation, Dirac showed that two additional terms appeared which corresponded to the interaction of the electro-magnetic field upon an electron with magnetic and electric moments represented by the operators

$$\frac{he}{4\pi mc}\vec{\sigma} \qquad \frac{he}{4\pi mc} i\vec{\alpha}$$

Although the comparison was admittedly artificial it was never:theless unsatisfactory to obtain a skew-hermitian operator (for which the eigen values are pure imaginary) to describe a physical quantity such as the electric moment of an electron. This operator for the electric moment arises from the commutator

$$\left[P^{\circ} , \sum_{i=1}^{3} \alpha_i P^i \right]$$

where P°, P^i are the general energy, momentum operators. Through neglect of the fact that P° and $\sum \alpha_i P^i$ do not commute, Lees (22) by directly squaring each side of an equation containing these operators, one on each side, naturally found no electric moment.

De Broglie made the suggestion that the proper mass should be replaced by $-m\alpha_4$ in quantum mechanics, so that when the two wave equations are compared an operator $-\alpha_4$ is finally/

finally attached to the representatives of the electric and magnetic moments. This then gives real densities for both these moments which are

$-\psi^{+}\alpha_4\vec{\sigma}\psi$ and $-\psi^{+}\alpha_4 i\vec{\alpha}\psi$ respectively

and three together form a six-vector. From classical relativity considerations Frenkel (23) showed that the magnetic moment must necessarily be accompanied by an electric moment and that the two constitute a six-vector. In this way we obtain the same six vector $\mathcal{M}^{lk}$ in agreement with Darwin's work and with Gordon's resolution of the current so that $\mathcal{M}^{lk}$ seems to be quite correctly interpreted. These two successful applications of $-m\alpha_4$ as the operator representing proper mass give con:siderable value to de Broglie's suggestion.

There is also the dual to $\mathcal{M}^{lk}$ to be considered. This tensor is

$$\tilde{\mathcal{M}}^{lk} = \frac{e}{\alpha\sqrt{2}}\left(\tilde{Q}^{lk} - \tilde{\bar{Q}}^{lk}\right) = \frac{ei}{\alpha\sqrt{2}}\left(Q^{lk} + \bar{Q}^{lk}\right)$$

If we form its divergence we find

$$c\left(\mathcal{M}^{lk}\right)_{k} = \frac{eci}{\alpha\sqrt{2}}\left(Q^{lk}{}_{,k} + \bar{Q}^{lk}{}_{,k}\right) \quad \text{which by (3.26)}$$

$$= \frac{eci}{\alpha\sqrt{2}}\left(\frac{G^{l}}{2} - \frac{H^{l}}{2} - \alpha A^{l} - \alpha B^{l} + \frac{\bar{G}^{l}}{2} - \frac{\bar{H}^{l}}{2} + \alpha A^{l} + \alpha B^{l}\right)$$

$$= \frac{eci}{\alpha\sqrt{2}}\left(G^{l} + \bar{G}^{l} - H^{l} - \bar{H}^{l}\right)$$

or

$$c\left(\tilde{\mathcal{M}}^{lk}\right)_{k} + \frac{he}{4\pi m}\left(H^{l} - G^{l} + \bar{H}^{l} - \bar{G}^{l}\right) = 0.$$

Here/

Here the first term which is the divergence of the six vector dual to the tensor $\mathcal{M}_{ik}$ of the electric and magnetic moment can be interpreted by the principle of duality as the polarisa:tion magnetic current, while the second term, or an analogy with the electric convection current J^l may be taken as re:presenting a magnetic convection current. That is, the total magnetic current vanishes, the two neutralising components being

the magnetic polarisation current $K^l = c(\tilde{\mathcal{M}}^{lk})_k$ (3.72)

and the magnetic convection current

$$g^l = \frac{he}{4\pi m}\left(H^l - G^l + \bar{H}^l - \bar{G}^l \right) \tag{3.73}$$

This interpretation is due to Zaïcoff (24), our g^l in special relativity reducing to his vector

$$g^l = \frac{hei}{4\pi m}\left(\psi^{+}_{1k}\, \alpha_1\alpha_2\alpha_3\alpha_4 \psi - \psi^{+}\alpha_1\alpha_2\alpha_3\alpha_4 \psi_{1k} \right) g^{lk}.$$

The electric convection current has a corresponding form

$$J^l = \frac{-hei}{4\pi m}\left(\psi^{+}_{1k}\, \alpha_4 \psi - \psi^{+} \alpha_4 \psi_{1k} \right) g^{lk}.$$

which shows how J^l and g^l are similarly derived from the two invariants $V_1 = \psi^{+}\alpha_4\psi$ and $V_2 = \psi^{+}\alpha_1\alpha_2\alpha_3\alpha_4\psi$ respectively.

<u>Jehle's Equation</u>.

From the second order wave equation, Jehle (25) has deduced a linear equation involving only two-rowed matrices and a two-com:ponent ψ-function, his equation being valid in general re:lativity and for all spin transformations. In the wave equation/

equation appears not only ψ but also its conjugate $\overline{\psi}$.
His equation runs

$$\gamma^{\nu} \nabla_{\nu} \psi = \sigma \overline{\psi} \qquad \left(\sigma = \frac{\hbar m}{c}\right). \qquad (3.74)$$

The theory of covariant differentiation and of spin transforma-:tions besides the properties of the γ-matrices (2×2) are developed along the lines of Bargmann's treatment of the exten-:sion of the four component wave equations to general relativity. There is also a matrix α which makes $\alpha \gamma^{\nu}$ hermitian and leads to a real vector:

$$s^{\nu} = \psi^{+} \alpha \gamma^{\nu} \psi$$

which, as it is non-divergent, was taken as the current vector.

As this equation is based upon relativity principles and is invariant for spin-transformations it should be contained in the spinor theory. In fact we can straightforwardly express Jehle's equation in spinor form, for if it is multiplied by α we have

$$(\alpha \gamma^{\nu}) \nabla_{\nu} \psi = \sigma (\alpha \overline{\psi})$$

This is equivalent to the spinor equation

$$\sigma^{k}{}_{\dot{\alpha}\beta}\, \psi^{\beta}{}_{|k} = -\alpha\, \psi_{\dot{\alpha}} \qquad (3.75)$$

(this $\underline{\alpha}$ is now the old imaginary constant). The agreement of these two forms is easily tested by taking the values of the α-, γ^{k}- matrices and of $\sigma^{k}{}_{\dot{\iota}\rho}$ in the special relativity cases. So in Jehle's theory instead of two spinor equations/

equations to represent the wave equation we have only one, and only the one spinor appears in the invariant theory. For the second order equation we differentiate the first order equation with respect to l multiply by $\sigma^{l\dot{\alpha}\rho}$ and sum.

$$\begin{aligned}
\sigma^{l\dot{\alpha}\rho}\,\sigma^{k}{}_{\dot{\alpha}\beta}\,\psi^{\beta}{}_{|kl} &= -\alpha\,\sigma^{l\dot{\alpha}\rho}\,\psi_{\dot{\alpha}|l} = -\alpha\,\sigma^{l\dot{\alpha}\rho}(i\phi_l\,\gamma_{\dot{\nu}\dot{\alpha}}\,\psi^{\dot{\nu}} + \gamma_{\dot{\nu}\dot{\alpha}}\,\psi^{\dot{\nu}}{}_{|l}) \\
&= -\alpha\,\sigma^{l\dot{\alpha}\rho}\,\psi_{\dot{\alpha}}\,\phi_l + \alpha\,\sigma^{l}{}_{\dot{\alpha}}{}^{\rho}\,\psi^{\dot{\alpha}}{}_{|l} \\
&= -\alpha\,\sigma^{l\dot{\alpha}\rho}\,\psi_{\dot{\alpha}}\,\phi_l + \alpha^2\psi^{\rho}
\end{aligned}$$

By changing the dummy suffixes k, l on the left hand side and then adding we obtain the second order equation

$$g^{lk}\,\psi^{\rho}{}_{|lk} - 2\alpha^2\psi^{\rho} + \alpha\,\phi_l\,\sigma^{l\dot{\alpha}\rho}\,\psi_{\dot{\alpha}} = 0. \tag{3.76}$$

This is equivalent to the Klein-Gordon equation in the first two terms but the third term expresses some additional reaction between the electro-magnetic field and the particle. We note that it is the potential four-vector which occurs here instead of the field six-vector.

As we have only one spinor, the only scalar which is its length, is identically zero. The sole vector we can form is

$$s^{k} = \sigma^{k}{}_{\dot{\alpha}\beta}\,\psi^{\dot{\alpha}}\,\psi^{\beta} \tag{3.77}$$

which is null. From the wave equation and its conjugate we deduce

$$\begin{aligned}
\operatorname{div} s = s^{k}{}_{,k} &= (\sigma^{k\dot{\alpha}\beta}\,\psi_{\dot{\alpha}|k})\,\psi_{\beta} + (\sigma^{k\dot{\alpha}\beta}\,\psi_{\beta|k})\,\psi_{\dot{\alpha}} \\
&= (\alpha\,\psi^{\beta})\,\psi_{\beta} + (-\alpha\,\psi^{\dot{\alpha}})\,\psi_{\dot{\alpha}} \\
&= 0
\end{aligned}$$

This/

This null, non-divergent vector is in fact the same as Jehle's current vector which can easily be shewn to be of zero length by using the special values of the matrices.

There is one quadratic second order tensor

$$N^{lk} = \psi^{\nu} \sigma^{l}{}_{\mu\nu} \sigma^{k\mu\sigma} \psi_{\sigma}$$

but as this contains two ψ -functions we cannot, in accordance with accepted quantum mechanical principles form an electro-magnetic moment six-vector, although, of course, both wave equations introducing the conjugate function $\bar{\psi}$ as well as ψ do not conform to the accepted principles. There is the tensor $T^{l}{}_{\cdot k} = \psi^{\lambda} \sigma^{l}{}_{\lambda\mu} \psi^{\mu}{}_{,k}$ as before from which we might form an energy tensor. In fact we soon find that $(T^{l}{}_{\cdot k} - \bar{T}^{l}{}_{\cdot k})_{,l} = i F_{lk} s^{k}$

But the divergence with respect to the other index does not lead to a similar expression because the field strengths do not appear in the second order equation. Instead we have the result

$$(T_{k}{}^{\cdot l} - T^{l}{}_{k})_{,l} = \alpha \phi_{l} (N^{l}{}_{\cdot k} + \bar{N}^{l}{}_{k})$$

For real and symmetric tensor $\mathcal{T}^{l}_{k}$

$$\mathcal{T}^{l}_{k} = \frac{chi}{2\sqrt{2}\pi} (T^{l}{}_{\cdot k} - \bar{T}^{l}{}_{\cdot k} + T_{k}{}^{\cdot l} - \bar{T}_{k}{}^{\cdot l})$$

has its divergence

$$(\mathcal{T}^{l}_{k})_{,l} = \frac{1}{c} A_{kl} J^{l} + \frac{1}{2} mc^{2} A_{l} (N^{l}{}_{\cdot k} + \bar{N}^{l}{}_{\cdot k})$$

Although $\mathcal{T}^{l}_{k}$ has the correct symmetry, the extra term in this divergence relation makes it impossible to interprete $\mathcal{T}^{l}_{k}$ is energy tensor. Also the scalar $\mathcal{T}$ would be zero.

This/

This wave equation does not therefore, appear to contain much of physical significance. We can examine its connection with the Dirac theory. Normally Jehle's equation is to be con-:sidered as perfectly distinct from Dirac's equations, this one equation is supposed to represent the state of affairs complete-:ly. However in one special case both may be considered to-:gether, this being the case when there is degeneracy in the four component theory. It may happen that the two Dirac sys-:tems are equivalent, that is

$$\sigma^{k}{}_{\dot\alpha\beta}\,\psi^{\dot\alpha}{}_{|k} = \alpha\,\chi_{\beta}$$

may imply that $$\sigma^{k\dot\alpha\beta}\,\chi_{\beta|k} = \alpha\,\psi^{\dot\alpha}$$

The conjugate of the second equation when remodelled is

$$\sigma^{k}{}_{\dot\alpha\beta}\,\chi^{\dot\alpha}{}_{|k} + i\,\phi_k\,\sigma^{k}{}_{\dot\alpha\beta}\,\chi^{\dot\alpha} = \alpha\,\psi_{\alpha}$$

This is the same as the first equation when we have both

(a) $\psi_\alpha = \chi_\alpha$

and (b) $\phi_k = 0$.

The condition (a) supplies us with the case where the skew quadrilateral is degenerate, the four vectors A, B, C and $\bar{C}$ being all coincident, but such a case is permitted by the Dirac equations only when the four-potential is everywhere zero. It means that the last two components of the Dirac ψ-functions are the conjugate of the first pair. Hence if there is no external electro-magnetic field, the degenerate Dirac equations coincide with Jehle's equation for that case. We at once see that we have a null current vector,

$$J^{k} = -ec\,2\sqrt{2}\,A^{k}$$

while/

while the spin vector vanishes.

The tensors Q^{lk} M^{lk} and $-N^{lk}$ are identical while

$E^{k} = H^{k} = -G^{k} = -F^{k}$.

For the electro-magnetic moments we have the null six-vector

$$\mathfrak{M}^{lk} = \frac{ec}{\alpha\sqrt{2}}\left(Q^{lk} - \bar{Q}^{lk}\right)$$

with $$(\mathfrak{M}^{lk})_{k} = \frac{ec}{\alpha\sqrt{2}}\left(-E^{l} - 2\alpha A^{l} + \bar{E}^{l} - 2\alpha A^{l}\right)$$

so that the convection current is

$$g^{l} = \frac{ec}{\sqrt{2}\alpha}\left(E^{l} - \bar{E}^{l}\right)$$ which is also a null vector.

As $T^{l}_{\cdot k} = U^{l}_{\cdot k}$ the energy tensor J^{l}_{k} becomes identi-:cally zero. Hence we see that this degeneracy is a highly specialised case and it seems of little physical importance. Certainly it will not apply to any ordinary electron phenomenon.

THE/

THE TENSORISED DIRAC EQUATION.

The world vectors and tensors we have considered are derived from wave equations which are spinor equations, namely

$$\left.\begin{aligned} \Pi_{\dot{\mu}} &= 0 \\ \Lambda_{\dot{\mu}} &= 0 . \end{aligned}\right\} \qquad (3.78)$$

These spinors can be used together with the spinors ψ and χ to form world vectors. The vector

$$\Omega^{k} \equiv \sigma^{k\dot{\mu}\nu}\left(\Pi_{\dot{\mu}}\psi_{\nu} + \Lambda_{\dot{\mu}}\chi_{\nu}\right) \qquad (3.79)$$

is the expression in general relativity of the vector due to E. T. Whittaker who showed that the vanishing of the vector was equivalent to the four (spinor) Dirac equations.

$$\Omega^{k} = 0 \qquad (3.80)$$

supplies four homogeneous linear equations in $\Pi_{\dot{1}}\ \Pi_{\dot{2}}\ \Lambda_{\dot{1}}\ \Lambda_{\dot{2}}$ and the determinant of the system is

$$\Delta = \left| \sigma^{k\dot{1}\nu}\psi_{\nu} \quad \sigma^{k\dot{2}\nu}\psi_{\nu} \quad \sigma^{k\dot{1}\nu}\chi_{\nu} \quad \sigma^{k\dot{2}\nu}\chi_{\nu} \right|$$

where $k = 1, 2, 3, 4$ supplies the four rows. With the summations expressed this is

$$\Delta = \left| \sigma^{k\dot{1}1}\psi_1 + \sigma^{k\dot{1}2}\psi_2 \quad \sigma^{k\dot{2}1}\psi_1 + \sigma^{k\dot{2}2}\psi_2 \quad \sigma^{k\dot{1}1}\chi_1 + \sigma^{k\dot{1}2}\chi_2 \quad \sigma^{k\dot{2}1}\chi_1 + \sigma^{k\dot{2}2}\chi_2 \right|$$

To evaluate this, first

multiply column 1 by χ_2 and add to it $(-\psi_2)$ times column 3

2 χ_2 $(-\psi_2)$ 4

so/

so that

$$\Delta = \frac{1}{(\chi_2)^2}\left| \sigma^{k11}(\psi_1\chi_2 - \psi_2\chi_1) \quad \sigma^{k21}(\psi_1\chi_2 - \psi_2\chi_1) \quad \sigma^{k11}\chi_1 + \sigma^{k12}\chi_2 \quad \sigma^{k21}\chi_1 + \sigma^{k22}\chi_2 \right|$$

$$= \frac{(\gamma_{12}\psi^\nu\chi_\nu)^2}{(\chi_2)^2}\left| \sigma^{k11} \quad \sigma^{k21} \quad \sigma^{k11}\chi_1 + \sigma^{k12}\chi_2 \quad \sigma^{k21}\chi_1 + \sigma^{k22}\chi_2 \right|$$

From column 3 take χ_1 column 1

" " 4 " χ_1 " 2

and after the central columns are interchanged we have this simple expression

$$\Delta = -(\gamma_{12}\psi^\nu\chi_\nu)^2 \left| \sigma^{k11} \quad \sigma^{k12} \quad \sigma^{k21} \quad \sigma^{k22} \right|.$$

Thus we find that Δ is never zero, for γ_{12} is the now zero component of the fundamental spinor of the spin space, - $\psi^\nu\chi_\nu \equiv K = \frac{V_1 - iV_2}{2}$ is a complex scalar which never vanishes, (except in the degenerate case which is physically unattainable) and the determinant

$$\left| \sigma^{k11} \quad \sigma^{k12} \quad \sigma^{k21} \quad \sigma^{k22} \right|$$

is the one which occurs in the correspondence between world vectors X^k and spin tensors $\omega_{\lambda\mu}$ and is assumed throughout the whole theory to be non singular.

Therefore the vector equation

$$\Omega^k = 0 \qquad (k = 1 \text{ to } 4)$$

necessarily implies

$$\Pi_\mu = 0 \qquad (\mu = 1, 2)$$

$$\Lambda_\mu = 0$$

These give us the four Dirac equations, and to obtain them in a form/

form which reduces to the usual one where the space-time becomes that of special relativity, we form the conjugate complex of the first of these, and the contravariant spinor form of the second.

Let us now consider the vector Ω^k and evaluate it in terms of the tensors and vectors we have collected. In doing so, we wish to use spinor relations which do not involve the application of Dirac's equations in their proof. Thus we shall write

$$\sigma^{k\lambda\mu}\chi_{\lambda|k} = \Lambda^{\mu} + \alpha\psi^{\mu}$$

$$\sigma^{k}{}_{\lambda\mu}\psi^{\lambda}{}_{|k} = \Pi_{\mu} - \alpha\chi_{\mu}$$

and we shall not put Λ or Π equal to zero. If we do this in the expression for the divergences $(M^{lk})_k$ and $(N^{lk})_k$ we find that

$$(M^{lk})_k = \chi_{\nu|k}\chi^{\nu}g^{lk} - \sigma^{k\mu\nu}\chi_{\nu|k}\sigma^{l}{}_{\mu\sigma}\chi^{\sigma} + \chi_{\nu}\sigma^{l\mu\nu}\sigma^{k}{}_{\mu\sigma}\tau^{\sigma\rho}\chi_{\rho|k}$$
$$+ i\phi_k M^{lk}$$
$$= \chi_{\nu|k}\chi^{\nu}g^{lk} - 2(\Lambda^{\mu} + \alpha\psi^{\mu})\sigma^{l}{}_{\mu\sigma}\chi^{\sigma} + i\phi_k M^{lk}.$$

Therefore $\quad 2\Lambda^{\mu}\sigma^{l}{}_{\mu\sigma}\chi^{\sigma} = -(M^{lk})_k + F^{l} - 2\alpha C^{l} + i\phi_k M^{lk}.$

At the same time

$$(N^{lk})_k = g^{lk}\psi^{\alpha}{}_{|k}\psi_{\alpha} - 2\psi^{\alpha}{}_{|k}\sigma^{k}{}_{\mu\alpha}\sigma^{l\mu\beta}\psi_{\beta} - i\phi_k N^{lk}$$
$$= E^{l} - 2\Pi_{\mu}\psi_{\beta}\sigma^{l\mu\beta} + 2\alpha\bar{C}^{l} - i\phi_k N^{lk}.$$

Therefore/

Therefore

$$2\pi_{\dot\mu}\psi_\beta\sigma^{l\dot\mu\beta} = -(N^{lk})_{,k} + E^l + 2\alpha\,\bar{C}^l - i\phi_k N^{lk}.$$

So finally

$$\begin{aligned}\Omega^l &= \sigma^{l\dot\mu\beta}\left(\pi_{\dot\mu}\psi_\beta + \Lambda_{\dot\mu}\chi_\beta\right)\\ &= \tfrac{1}{2}\left[(M^{kl})_{,k} + (N^{kl})_{,k} + i\phi_k\left(N^{kl} - M^{kl}\right) + E^l + F^l + 2\alpha(\bar{C}^l - C^l)\right]. \end{aligned} \qquad (3.81)$$

But as we have already expressed all the quantities appearing here by means of the four null vectors and the invariant K, we can write this vector in the form

$$\begin{aligned}\Omega^l = {} & \tfrac{1}{2}\left[\tfrac{1}{\bar K}\left\{(A^k C^l - A^l C^k) + (B^k \bar{C}^l - B^l \bar{C}^k)\right\}\right]_{,k}\\ & + \frac{i\phi_k}{2\bar K}\left\{(-A^k C^l + A^k C^k) + (B^k \bar{C}^l - B^l \bar{C}^k)\right\}\\ & + \frac{1}{2\bar K} g^{lk}\left[C_n A^n{}_{,k} + \bar{C}_n B^n{}_{,k}\right] + \alpha\left[\bar{C}^l - C^l\right]. \end{aligned} \qquad (3.82)$$

when we use (3.6), (3.29) and (3.30).

Thus we have found an expression for Ω^l without reference to spinors, only the vectors A B C and $\bar{C}$ together with scalars K and $\bar{K}$ are present.

The quantity $\phi_k = \frac{4\pi}{h}\frac{e}{c}$ x four-potential (gauge invariant)

$$\alpha = -\frac{2\pi i}{h}\frac{mc}{\sqrt{2}}.$$

The whole of the invariant theory can be derived from the funda-:mental statement that $\Omega^l = 0$ when we use the null vectors instead of ψ - functions. We must, however, assume the perpendicularity relations and the reality properties of these vectors - namely that A^k and B^k are real while C^k and $\bar{C}^k$ are/

are complex conjugates and that each vector is perpendicular to itself and to two others with $A^k B_k = -C^k \bar{C}_k = +K\bar{K}$

These relations supply us with ten conditions. The equation $\Omega^l = 0$ together with its conjugate $\bar{\Omega}^l = 0$ yield eight simultaneous differential equations involving the vectors and scalars. As A and B are real we have essentially eighteen real quantities (sixteen from the vector components and two from the complex K) which is just the number of conditions and equations.

If we form the expression $(C_l + \bar{C}_l)(\Omega^l + \bar{\Omega}^l)$ which is zero as $\Omega^l = 0$, then the supplementary conditions reduce this to the result that

$$(A^k + B^k)_k = 0 ,$$

while the equation $(C_k - \bar{C}_k)(\Omega^l - \bar{\Omega}^l) = 0$ yields

$$(A^k - B^k)_k = 2(\bar{K} - K)\alpha$$

so that we have

$$\text{div } A = \alpha(\bar{K} - K)$$

$$\text{div } B = -\alpha(\bar{K} - K).$$

Again from the statement $A_l \bar{\Omega}^l - B_l \Omega^l = 0$ we derive

$$\text{div } C = i\phi_k C^k$$

of which the conjugate is $\text{div } \bar{C} = -i\phi_k \bar{C}^k$.

Similarly all the relations we have found are deductible from the null-vector expression of the wave equations.

The vectors and tensors appearing in the physical theory are now/

now collected and restated in terms of the null vectors.

As we have already seen the density of the four-current is expressed by

$$J^k = -ec\sqrt{2}\,(A^k + B^k).$$

The density of the open angular momentum is

$$S^k = \frac{h}{4\pi}\sqrt{2}\,(A^k - B^k).$$

For the density of the electric and magnetic moments of the electron we have the six-vector which from (3.7) and (3.60) is

$$\mathfrak{M}^{lk} = \frac{he}{4\pi m c i}\left[\left(\frac{1}{\kappa} + \frac{1}{\bar{\kappa}}\right)(C^k\bar{C}^l - C^l\bar{C}^k) + \left(\frac{1}{\kappa} - \frac{1}{\bar{\kappa}}\right)(A^k B^l - A^l B^k)\right].$$

An alternative form in which the C^k do not appear by (3.19) is

$$\frac{he}{4\pi m c i}\left[\left(\frac{1}{\kappa} + \frac{1}{\bar{\kappa}}\right)\frac{i}{2}\,\varepsilon^{klmn}(A_m B_n - A_n B_m) + \left(\frac{1}{\kappa} - \frac{1}{\bar{\kappa}}\right)(A^k B^l - A^l B^k)\right]$$

The convection current J^k in (3.61) is given as

$$J_k = \frac{hei}{4\pi m}\left[i(\kappa + \bar{\kappa})\phi_k + \left(\frac{1}{\bar{\kappa}} - \frac{1}{\kappa}\right)A_n B^n{}_{,k} + \frac{\bar{C}_n C^n{}_{,k}}{\bar{\kappa}} - \frac{C_n \bar{C}^n{}_{,k}}{\kappa}\right]$$

to which is clearly related the magnetic convection current $\mathfrak{J}^k$

$$\mathfrak{J}_k = \frac{he}{4\pi m}\left[i(\kappa - \bar{\kappa})\phi_k + \left(\frac{1}{\bar{\kappa}} - \frac{1}{\kappa}\right)A_n B^n{}_{,k} + \frac{\bar{C}_n C^n{}_{,k}}{\bar{\kappa}} + \frac{C_n \bar{C}^n{}_{,k}}{\kappa}\right]. \tag{3.73}$$

The last important tensor of physical interest is the energy tensor T_{lk}. Its expression in terms of the small vectors follows from (3.69), (3.48) and (3.50), is

$$T_{lk} = \frac{chi}{8\pi}\sqrt{2}\left[\frac{1}{\bar{\kappa}}(B_l\bar{H}_k - \bar{C}_l\bar{E}_k - A_l\bar{G}_k + C_l\bar{F}_k + B_k\bar{H}_l - \bar{C}_k\bar{E}_l - A_k\bar{G}_l + C_k\bar{F}_l)\right.$$
$$\left. - \frac{1}{\kappa}(B_l H_k - C_l E_k - A_l G_k + \bar{C}_l F_k + B_k H_l - C_k E_l - A_k G_l + \bar{C}_k F_l)\right]$$

which/

which by (3.29), (3.30), (3.33) and (3.34), take the form

$$\begin{aligned} J_{lk} = \frac{chi\sqrt{2}}{8\pi}\,\frac{1}{\kappa\bar{\kappa}}\Big[\, & A_l\,(c_n\,\bar{c}^n{}_{|k} + \bar{\kappa}\,\kappa_{,k}) + B_l\,(c_n\,\bar{c}^n{}_{|k} + \kappa\,\bar{\kappa}_{,k}) \\ & + (c_l\,\bar{c}_n - c_n\,\bar{c}_l)(A^n + B^n)_k \\ & + A_k\,(c_n\,\bar{c}^n{}_{|l} + \bar{\kappa}\,\kappa_{,l}) + B_k\,(c_n\,\bar{c}^n{}_{|l} + \kappa\,\bar{\kappa}_{,l}) \\ & + (c_k\,\bar{c}_n - c_n\,\bar{c}_k)(A^n + B^n)_l \Big]. \end{aligned} \tag{3.83}$$

AN ALTERNATIVE METHOD of GENERALISING DIRAC'S EQUATION.

The vector Ω^k has been derived on the basis of general relativity with a suitably generalised theory of spinors. From the theory of the Dirac equation in restricted relativity as expressed by van der Waerden's spinors, Whittaker found this vector which when expressed in terms of the null-vector gives exactly the same form as we have here, save for the potential terms. However the potential is easily introduced in the special relativity case at the end of the process for we replace

∂_k by $\partial_k \mp \frac{i}{2}\phi_k$ according as the operand is a ψ-function or a complex conjugate of one. The tensor M^{kl} is a bilinear form of the ψ-functions

$$\tilde{\psi}\,A\,\psi$$

so that instead of $\partial_k\,(M^{kl})_l$

we now have $\left[(\partial_k - \frac{i}{2}\phi_k)\,\tilde{\psi}\right]A\psi + \tilde{\psi}\,\partial_k A\psi + \tilde{\psi}\,A(\partial_k - \frac{i}{2}\phi_k)\psi$

$$= \partial_k(\tilde{\psi}A\psi) - i\,\phi_k(\tilde{\psi}A\psi)$$

$$= \partial_k M^{kl} - i\,\phi_k M^{kl}.$$

Similarly/

Similarly N^{kl} is a tensor which is a bilinear form arising from conjugate wave functions so that

$$\partial_k (N^{kl})$$

is replaced by

$$\partial_k (N^{kl}) + i\phi_k N^{kl}.$$

Hence to allow for the electro-magnetic potential we add to the divergences of the tensors N^{kl} and M^{kl} an extra term which has a different sign for each (3.81). Thus in this case in special relativity one obtains the vector expression for Dirac's equation which is identical with (3.81). In this way we have an interesting alternative method of generalising the wave equation. Firstly working with the simpler spinor theory in special relativity we can derive the vectorial form of the wave equation, inserting the additional terms involving the potential at the end. This potential vector must be regarded as a simple vector, that is, it is the vector denoted originally by ϕ^*_k which is the ordinary potential vector to which a gradient has been added so that the whole is purely tensorial in character and unaffected by any spin transformation, the gauge transformation being a special case. This equation is based on a Lorentz-in-:variant theory so that it holds in all co-ordinate systems in special relativity, but as it is entirely tensorial in form we can immediately assume that this equation can be taken directly over into the wider scheme of general relativity. Thereby we avoid the necessity of developing the fairly elaborate theory of parallel transfer of spinors, the mixed curvative tensors etc.

in/

in the general case. All reference to these is entirely omitted, and as we have already noted, both methods lead to identical results. There is one minor point which for the sake of com:pleteness must not be overlooked. The simpler spinor theory is applied to the Dirac equation with a definite set (the van der Waerden set) of matrices in which case the ψ -components are also spinor components. Provision must be made for spin-transformations, but this is a simple matter and we soon find that the expression for Ω^{*} in terms of world tensors is invariant for such transformations. All this of course is contained in the generalised theory which in the particular case of special relativity gives the simple two-component spinor theory together with a theory of spin-transformations.

In this way a simple and direct generalisation of Dirac's equation is made possible. All the calculations and formation of tensors are performed in the pseudo-euclidean space of special relativity leading to a pure vector equation, the generalisation of which is automatically performed. Then from this new vector equation the invariant properties can all be derived.

However, if one desires an equation in spinor form compar:able with the original form of Dirac's equation then the general spinor theory in all its detail would have to be established. When this was done the null vectors could be related to two spin-vectors. The supplementary conditions among the null vectors lead at once to our earlier geometrical configuration. When/

When the null quadric is parameterised in terms of the spinors ψ_α and χ_α we have the necessary correspondence between world tensors and spinors. Then the reverse process is carried out wherein the tensors and vectors appearing in Ω^κ are split up and expressed by means of spinors leading at last to the spinor wave equation which formed the commencement of this study on invariant properties.

THE MATRIX FORM OF Ω^k.

In the geodesic coordinate system and in special relativity let us write the Dirac equations as

$$L \equiv \begin{bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{bmatrix} \equiv \begin{bmatrix} -\frac{\partial}{c\,\partial t} + \frac{i}{2}\phi_4 - \sum_1^3 \alpha_i\left(\frac{\partial}{\partial x^i} - \frac{i}{2}\phi_i\right) + \alpha_4 \cdot \alpha \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = 0$$

Then as L in this case is

$$\begin{bmatrix} \Lambda^i \\ \Lambda^{\dot{i}} \\ \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} \Lambda_i \\ -\Lambda_{\dot{i}} \\ \Pi_1 \\ \Pi_2 \end{bmatrix} \qquad \Psi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \psi_{\dot{i}} \\ -\psi_i \end{bmatrix} \tag{3.84}$$

the vector $\Omega^k = \mathring{\sigma}^{k\mu\rho}(\Pi_\mu \psi_\rho + \Lambda_\mu \chi_\rho)$

has components

$$\sqrt{2}\,\Omega^1 = \Pi_i \psi_2 + \Pi_{\dot{i}} \psi_1 + \Lambda_i \chi_2 + \Lambda_{\dot{i}} \chi_1 = \bar{L}_3 \bar{\psi}_3 - \bar{L}_4 \bar{\psi}_4 + L_1 \psi_1 - L_2 \psi_2$$

$$\sqrt{2}\,\Omega^2 = -i\,\Pi_i \psi_2 + i\,\Pi_{\dot{i}} \psi_1 - i\,\Lambda_i \chi_2 + i\,\Lambda_{\dot{i}} \chi_1 = -i\,\bar{L}_3 \bar{\psi}_3 - i\,\bar{L}_3 \bar{\psi}_3 + i\,L_1 \psi_1 + i\,L_2 \psi_2$$

$$\sqrt{2}\,\Omega^3 = \Pi_i \psi_1 - \Pi_{\dot{i}} \psi_2 + \Lambda_i \chi_1 - \Lambda_2 \chi_2 = -\bar{L}_3 \bar{\psi}_4 - \bar{L}_3 \bar{\psi}_3 - L_1 \psi_2 - L_2 \psi_1$$

$$\sqrt{2}\,\Omega^4 = \Pi_i \psi_1 + \Pi_{\dot{i}} \psi_2 + \Lambda_i \chi_1 - \Lambda_2 \chi_2 = -\bar{L}_3 \bar{\psi}_4 + \bar{L}_4 \bar{\psi}_3 + L_1 \psi_2 - L_2 \psi_1$$

In matrix notation these components are expressible as

$$\begin{aligned}
2\sqrt{2}\,\Omega^1 &= \tilde{L}\,(i\alpha_2 + \alpha_1\alpha_3)(\alpha_1)\,\psi + L^{+}(\alpha_1)(-i\alpha_2 + \alpha_1\alpha_3)\,\bar{\psi} \\
2\sqrt{2}\,\Omega^2 &= \tilde{L}\,(i\alpha_2 + \alpha_1\alpha_3)(\alpha_2)\,\psi + L^{+}(\alpha_2)(-i\alpha_2 + \alpha_1\alpha_3)\,\bar{\psi} \\
2\sqrt{2}\,\Omega^3 &= \tilde{L}\,(i\alpha_2 + \alpha_1\alpha_3)(\alpha_3)\,\psi + L^{+}(\alpha_3)(-i\alpha_2 + \alpha_1\alpha_3)\,\bar{\psi} \\
2\sqrt{2}\,\Omega^4 &= \tilde{L}\,(i\alpha_2 + \alpha_1\alpha_3)(-1)\,\psi + L^{+}(1)(-i\alpha_2 + \alpha_1\alpha_3)\,\bar{\psi}
\end{aligned} \tag{3.85}$$

The α_i-matrices have the values stated in (2.35)

Here α_2 is distinguished from α_1 and α_3 ; it will be noted that is/

is the only matrix of the set $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ that has imagin-:ary elements. When the Dirac set of matrices is used, the ψ-functions are not themselves spinors, but linear sums of pairs of the ψ-components form spinors. The $\mathcal{R}$ vector that is finally obtained is ėxpressible in the same way as the results above with the α_2 distinguished from α_1 and α_3, but with the Dirac matrices as with the van der Waerden set (we are using them with reversed sign), α_2 has imaginary elements. We shall seek the reason for this distinction.

Results of Temple (26) and Eddington (27) relating to the group and other properties of the α-matrices are here assumed. In a complete perpendicular set, that is five hermitian anti-:commuting matrices with unit square such as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\alpha_5 \equiv \alpha_1\alpha_2\alpha_3\alpha_4$ it will be recalled that three of the mat-:rices must have real elements while the other two have imaginary ones so long as we are restricted solely to those matrices which have their elements all real or all imaginary.

Still referring to the ordinary relativity equation, we now briefly consider the effect of changing from one set of anti-commuting hermitian matrices α_i to another set α'_i. This corresponds to a similarity transformation

$$\alpha'_i = S \alpha_i S^{-1}$$

where S is a unitary matrix if both α_i and α'_i are hermitian. At the same time the wave formations would be transformed and likewise/

likewise the left hand sides of the equation by the law,

$$\acute{\psi} = S\psi$$
$$\acute{L} = SL$$

Then the sixteen quantities

$$\psi^{+} \alpha_j \psi$$

where α_j is any member of the fundamental set of operaters,

can be written $\quad \psi^{+}(S^{+}S)\ \alpha_j\,(S^{-1}S)\,\psi,$

because $\quad S^{+}S = 1 = S^{-1}S$

But this is $\quad \acute{\psi}^{+}\ \acute{\alpha}_j\ \acute{\psi}$

so that these quantities which give us the two scalars V_1, V_2, the two vectors J^{k} and S^{k} and the six-vector $\mathcal{M}^{lk}$ are in-:variant in form as far as a special choice of α-matrix is concerned.

Moreover as S is independent of the coordinates quantities of the type $\quad \psi^{+}_{1k}\ \alpha_j\ \psi \quad$ and $\quad \psi^{+}\ \alpha_j\ \psi_{1k}$

are also invariant.

However the vector Ω^{k} is not necessarily in invariant form for we have

$$\Omega^{k} = \tilde{L}\,(i\alpha_2 + \alpha_1\alpha_3)\,(\alpha^{k}_{A})\,\psi \;+\; L^{+}\,(\alpha^{k}_{B})(-i\alpha_2 + \alpha_1\alpha_3)\,\overline{\psi}$$

(where for brevity $\alpha^{k}_{A} = (\alpha_1, \alpha_2, \alpha_3, -1)$, $\alpha^{k}_{B} = (\alpha_1, \alpha_2, \alpha_3, +1)$)

$$= \tilde{L}\,S^{-1}S\,(i\alpha_2 + \alpha_1\alpha_3)(\alpha^{k}_{A})\,S^{-1}S\,\psi \;+\; L^{+}(S^{+}S)(\alpha^{k}_{B})(-i\alpha_2 + \alpha_1\alpha_3)\,S^{-1}S\,\overline{\psi}$$

$$= \tilde{\acute{L}}\,\tilde{S}^{-1}S^{-1}(i\acute{\alpha}_2 + \acute{\alpha}_1\acute{\alpha}_3)(\acute{\alpha}^{k}_{A})\,\acute{\psi} \;+\; \acute{L}^{+}\,(\acute{\alpha}^{k}_{B})(-i\acute{\alpha}_2 + \acute{\alpha}_1\acute{\alpha}_3)\,S\overline{S}^{-1}\,\overline{\acute{\psi}}\,.$$

This will have the original form if $\tilde{S}^{-1}S^{-1} = 1$ and $S\overline{S}^{-1} = 1$.

As/

As S is unitary both conditions are the same, namely

$$S = \bar{S} \tag{3.86}$$

that is, S has real elements.

Now when the two perpendicular sets α_i and α'_i (i = 1 to 5) are so arranged that matrices with the same numerical suffix have either all real or all imaginary elements, the same will hold throughout the sixteen matrices. The S- transformation connect-:ing such sets will be either purely real or imaginary. This follows quickly, for

$$\alpha'_i = S\alpha_i S^{-1}$$

Therefore $\qquad \bar{\alpha}'_i = \bar{S}\bar{\alpha}_i\bar{S}^{-1}$

But $\bar{\alpha}'_i = \pm \alpha'_i$ according as $\bar{\alpha}_i = \pm \alpha_i$

so that $\qquad \alpha'_i = \bar{S}\alpha_i\bar{S}^{-1} = S\alpha_i S^{-1}$

or $\qquad S^{-1}\bar{S}\alpha_i = \alpha_i S^{-1}\bar{S}$

As the matrix $S^{-1}\bar{S}$ commutes with all sixteen α_i it is proportional to the unit matrix, whence we have

$\bar{S} = \rho S = e^{i\theta} S$ where ρ is a number with unit modulus.

In the general case when $S = k\sum \alpha'^{+}_i \alpha_i$ (that is, when this sum is non vanishing) S is real, for each term of this sum is a product of two real or two imaginary matrices and hence is real, so that $\rho = 1$.

From Dirac's set and van der Waerden's (we have used his matrices with α_1 α_2 α_3 α_4 with reversed sign) α_2 and α_5 have imaginary element in both, while α_1, α_3 and α_4 are real.

The/

The S-matrix connecting them is real

$$S = S^{-1} = S^{+} = \tilde{S} = \bar{S} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \\ -1 & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & -1 \end{bmatrix}$$

In terms of either system this is

$$S = \frac{1}{\sqrt{2}} \left(i\,\alpha_1 \alpha_2 \alpha_3 + \alpha_4 \right).$$

Thus the matrix expressions of Ω^k and similarly of all the other quantities such as C^k are the same in both schemse.

It is possible to have the following cases, with the matrices with real and imaginary elements disposed as follows

	$\alpha_1\ \alpha_2\ \alpha_3$	α_4	α_5
(1)	3R	I	I
(2)	2R + I	R	I
(3)	2R + I	I	R
(4)	1R + 2I	R	R

We have just had example of case (2) when α_2 is the imaginary matrix. If these conditions are preserved in a new set $\bar{S} S^{-1}$ will be equal ρ^1 so that the same preference given to will continue.

If we reshuffle the matrices to make the first the imaginary one e.g. $\beta_1 = \alpha_2$ (I) $\quad \beta_2 = \alpha_1 \quad \beta_3 = \alpha_3 \quad \beta_4 = \alpha_4 \quad \beta_5 = -\alpha_5$ (I)

then

$$S = \frac{i}{\sqrt{2}} \left(\alpha_5 \alpha_1 - \alpha_5 \alpha_2 \right) = S^{-1}$$

$$= \frac{i}{\sqrt{2}} \left(\beta_5 \beta_1 - \beta_5 \beta_2 \right)$$

$$\bar{S} = -\frac{i}{\sqrt{2}} \left(\beta_5 \beta_1 + \beta_5 \beta_2 \right).$$

Then/

Then $\bar{S}\, S^{-1} = \beta_1 \beta_2$

so that in place of $(-i\alpha_2 + \alpha_1\alpha_3)$

we now have $(i\beta_1 + \beta_3\beta_2)$ in the matrix form of

β_i now being the imaginary matrix of the first trio occupies the special place. Similarly for the vectors such as C^k.

Consider case (1) by taking for an example

$$\beta_1 = \alpha_1 \quad \beta_2 = \alpha_4 \quad \beta_3 = \alpha_3 \quad \beta_4 = \alpha_2\ (I) \quad \beta_5 = -\alpha_5\ (I).$$

From it is easily seen that $S = \frac{i}{\sqrt{2}}(\alpha_5\alpha_4 - \alpha_5\alpha_2) = S^{-1}$

or $= \frac{i}{\sqrt{2}}(\beta_5\beta_4 - \beta_5\beta_2)$

$$\bar{S} = \frac{-i}{\sqrt{2}}(\beta_5\beta_4 + \beta_5\beta_2)$$

and $\bar{S}\, S^{-1} = \beta_4\beta_2$

Hence in place of $(i\alpha_2 + \alpha_1\alpha_3)$

now appears $(i\beta_4 + \beta_1\beta_2\beta_3\ \beta_4)$

so that the vector

$$\Omega^k = \tilde{L}\,(i\beta_4 + \beta_5)(\beta^k_A)\,\psi + L^+(\beta^k_B)(-i\beta_4 - \beta_5)\,\bar{\psi}$$

This illustrates how by making the three matrices associated with the space-like coordinates all real, we find no distinction among them. For similar dispositions the form of this vector holds apart from a possible introduction of a numerical factor of unit modulus

For case (3) by taking

$$\beta_1 = \alpha_1 \quad \beta_2 = \alpha_2\ (I) \quad \beta_3 = \alpha_3 \quad \beta_4 = \alpha_5\ (I) \quad \beta_5 = -\alpha_4$$

we/

we soon find that

$$(i\alpha_2 + \alpha_1\alpha_3)$$

is to be replaced by $(\beta_2 - i\beta_1\beta_3)$

As an example of case (4)

$$\beta_1 = \alpha_1 \qquad \beta_2 = \alpha_2\ (I) \qquad \beta_3 = \alpha_5\ (I) \qquad \beta_4 = \alpha_4 \qquad \beta_5 = -\alpha_3$$

$$(i\beta_1 + \beta_2\beta_3)\beta_4$$

appears in the formula.

Thus it is now evident that the imaginary nature of our original α_2 was the sole reason for its being distinguished from α_1 and α_3 . If a set of matrices with the first trio real had been used, no distinction in the vector form of Ω^k would have occurred. Of course the special value of the set we used is that the resulting wave function has its components as they stand also components of spinors. We note that no more than three real matrices are possible, a fact to which Eddington points as a reason for the difference of the fourth (time) coordinate from the other three (space-like) coordinates.

CHAPTER IV.

AN EXAMINATION OF A PAPER by T. Levi-Civita.

In this paper Levi-Civita (28) states that in order to generalise Dirac's equation to any ds^2 it is necessary to introduce an orthogonal ennuple into space-time. In the completed generalisation additive terms would appear, depending in an essential way upon the choice of this ennuple. As, he says, no directions nor examples of special importance exist there must be something at fault with the equation. He proposed to remove the ennuple from the work right from the beginnings and in so doing he suggests a modification of Dirac's equation.

It appears that there is some confusion about the role played by certain indices. As is well known the four components of the wave function ψ do not form a world tensor and its indices have no tensorial character. In the usual special relativity formulation, the Dirac α-matrices operate upon these complex ψ quantities,

$$\psi' = \alpha_\mu \psi \qquad (4.1)$$

which in detail means that

$$\psi'_i = \sum_{j=1}^{4} (\alpha_\mu)_{ij} \, \psi_j \qquad i, j = 1,2,3,4. \qquad (4.1a)$$

where $(\alpha_\mu)_{ij}$ is the component of the matrix α_μ in the i^{th} row and j^{th} column. The nature of i and j has not been directly discussed in the extension to general relativity, but at no time have these been given a tensor meaning as far as space-time has been concerned. In fact most of the proposed generalisations of the Dirac equation have treated i and j quite differently from tensor/

tensor indices but rather as "spinor" indices as in Infelds and van der Waerden's theory (7). Indeed when one discusses the Lorentz-invariance of Dirac's equations one can arrange that the ψ-function be left unaffected while the α_μ are treated as world vector components under the transformation or alternatively, the α_μ can be retained unaltered while the ψ_i are varied, their law of transformation being quite different from that of a world vector.

Levi-Civita, however, commences his considerations with the transformation

$$v' = \alpha v$$

which is interpreted as $$v'_\mu = \sum_0^3 \alpha^\nu_\mu v_\nu$$

where v is a world vector and ν, μ are tensor indices while α is a Dirac matrix.

We shall follow out his calculation, only in certain places corrections have been made in the use of the e_i which are $e_0 = 1 \quad e_1 = e_2 = e_3 = -1$.

Define the matrices $\mathring{\alpha}_h$ by the relations

$$\mathring{\alpha}_h \mathring{\alpha}_k + \mathring{\alpha}_k \mathring{\alpha}_h = 2 e_h \delta_{hk} 1 \qquad h\ k = 0\ 1\ 2\ 3 \tag{4.2}$$

that is $$\mathring{\alpha}_{h|\mu}^{\ \ \rho} \mathring{\alpha}_{k|\rho}^{\ \ \nu} + \mathring{\alpha}_{k|\mu}^{\ \ \rho} \mathring{\alpha}_{h|\rho}^{\ \ \nu} = 2 e_h \delta_{hk} \delta^\nu_\mu .$$

Regard h and k and all Latin indices as referring to an ortho-:gonal lattice and μ, ν, ρ and all Greek indices as tensor ones. In the general space time manifold with signature $(+ - - -)$ we have chosen quite arbitrarily at each point an orthogonal lattice/

order linear partial differential equation,

$$\alpha^{\tau\nu}{}_{\rho} D_\tau \psi_\nu + mc\,\psi_\rho = 0 . \tag{4.10}$$

In the special case when the metric is pseudo-euclidian, we take cartesian coordinates with $g_{\mu\nu} = g^{\mu\nu} = e_\mu \delta_{\mu\nu}$ $\quad \overset{\circ}{\nabla}_\mu = \frac{\partial}{\partial x^\mu}$
and we can choose $\lambda^\mu_h = \delta_{h\mu}$.
Then the equation (4.10) becomes

$$\sum e_h \overset{\circ}{\alpha}_{h1}{}^{\nu}_{\rho} \,.\, \delta_{h\tau} D_\tau \psi_\nu + mc\,\psi_\rho = 0$$

that is, $\left\{ \overset{\circ}{\alpha}_{01}{}^{\nu}_{\rho} D_0 - \sum_{j=1}^{3} \overset{\circ}{\alpha}_{j1}{}^{\nu}_{\rho} D_j + mc\,\delta^\nu_\rho \right\} \psi_\nu = 0 .$

If we operate upon this with
$\left(\overset{\circ}{\alpha}_{01}{}^{\rho}_{\mu} D_0 - \sum_1^3 \overset{\circ}{\alpha}_{i1}{}^{\rho}_{\mu} D_i - mc\,\delta^\rho_\mu \right)$ the result by (4.2)
is $\left\{ D_0^2 - D_1^2 - D_2^2 - D_3^2 - m^2c^2 + \sum\sum e_i e_j \overset{\circ}{\alpha}_{i1}{}^{\rho}_{\mu} \left(\overset{\circ}{\nabla}_h \overset{\circ}{\alpha}_{i1}{}^{\nu}_{\rho} \right) \right\} \psi_\nu = 0$

To obtain the Schrodinger equation when there is no field we require the condition

$$\sum\sum e_h e_i \overset{\circ}{\alpha}_{h1}{}^{\rho}_{\mu} \left(\overset{\circ}{\alpha}_{i1}{}^{\nu}_{\rho} \right)_h = 0 \tag{4.11}$$

which in tensor notation is

$$\alpha^{\sigma\rho}{}_{1\mu} \left(\alpha^{\tau\nu}{}_{1\rho} \right)_\sigma = 0 . \tag{4.12}$$

Thus if we expect (4.10) to be a generalisation of Dirac's equation we must have the tensor condition (4.12) fulfilled. This point seems to have been neglected by Levi-Civita, but appears in a work by Temple (29) in which the Dirac equation is given a tensor form.

Neglecting this condition for a moment, we have the second degree equation from (4.10) as

$$(\alpha^{\sigma}{}_{1}{}^{\rho}_{\mu} D_\sigma - \delta^\rho_\mu mc)(\alpha^{\tau\nu}{}_{\rho} D_\tau + \delta^\nu_\rho mc)\,\psi_\nu = 0 ,$$

that/

lattice whose components are given by the Ricci coefficients λ^{μ}_{h} or $\lambda_{h|\mu}$.

$$\lambda^{\mu}_{k}\,\lambda_{h|\mu} = \lambda_{k|\mu}\,\lambda^{\mu}_{h} = \delta_{hk}e_{h} = \delta_{hk}e_{k} \qquad (4.3)$$

$$\sum_{h} e_{h}\,\lambda_{h|\mu}\,\lambda_{h|\nu} = g_{\mu\nu}\,. \qquad (4.4)$$

As usual, summation is automatically extended over the range o-3 for all repeated Greek indices while the sign of summation will be always inserted for the Latin indices.

According to the usual rule, the third order tensor corres:ponding to the quantity $\overset{\circ}{\alpha}_{h|}{}^{\rho}_{\mu}$ is

$$\alpha_{\sigma|}{}^{\rho}_{\mu} = \sum_{h} e_{h}\,\overset{\circ}{\alpha}_{h|}{}^{\rho}_{\mu}\,\lambda_{h|\sigma} \qquad (4.5)$$

Hence $\quad \alpha_{\sigma|}{}^{\rho}_{\mu}\,\alpha_{\tau|}{}^{\nu}_{\rho} = \sum_{h}\sum_{k} e_{h}e_{k}\,\overset{\circ}{\alpha}_{h|}{}^{\rho}_{\mu}\,\overset{\circ}{\alpha}_{k|}{}^{\nu}_{\rho}\,\lambda_{h|\sigma}\,\lambda_{k|\tau}$

so that by interchange of σ and ρ and adding we get

$$\alpha_{\sigma|}{}^{\rho}_{\mu}\,\alpha_{\tau|}{}^{\nu}_{\rho} + \alpha_{\tau|}{}^{\rho}_{\mu}\,\alpha_{\sigma|}{}^{\nu}_{\rho} = \sum_{h}\sum_{k} e_{h}e_{k}\left(2\delta_{hk}\,e_{h}\,\delta^{\nu}_{\mu}\right)\lambda_{h|\sigma}\,\lambda_{k|\tau}$$

$$= 2\,g_{\sigma\tau}\,\delta^{\nu}_{\mu} \qquad (4.6)$$

The first index is raised with the fundamental tensor and written

$$\alpha^{\sigma\rho}{}_{|\mu} = g^{\sigma\tau}\,\alpha_{\tau|}{}^{\rho}_{\mu}\,. \text{ by (4.2) and (4.4)} \qquad (4.7)$$

Therefore (4.6) is equivalent to

$$\alpha^{\sigma\rho}{}_{|\mu}\,\alpha^{\tau\nu}{}_{|\rho} + \alpha^{\tau\rho}{}_{|\mu}\,\alpha^{\sigma\nu}{}_{|\rho} = 2\,g^{\sigma\tau}\,\delta^{\nu}_{\mu}. \qquad (4.8)$$

THE DIRAC EQUATION.

Let A^{μ} or A_{μ} represent the vector potential at each point, -e be the electronic charge. The operator D is introduced

$$D_{\mu} = \frac{h}{2\pi i}\,\overset{\circ}{\nabla}_{\mu} - \frac{e}{c}\,A_{\mu}. \qquad (4.9)$$

when $\overset{\circ}{\nabla}_{\mu}$ denotes ordinary covariant differentiation.

Finally let ψ^{μ} and ψ_{μ} be the components of a <u>world</u> <u>vector</u> ψ which are defined by the solutions of the following first order/

that is $(\alpha^{\sigma\rho}{}_{\mu} D_\sigma \alpha^{\tau\nu}{}_{\rho} D_\tau - m^2c^2\delta^{\nu}_{\mu})\psi_\nu = 0$

By (4.9) $\quad D_\sigma \alpha^{\tau\nu}{}_{\rho} = \frac{h}{2\pi i}(\alpha^{\tau\nu}{}_{\rho})_\sigma + (\alpha^{\tau\nu}{}_{\rho})D_\sigma$

and $\quad D_\sigma D_\tau - D_\tau D_\sigma = \frac{h}{2\pi i}\frac{e}{c}(A_{\sigma,\tau} - A_{\tau,\sigma})$

$$= \frac{h}{2\pi i}\frac{e}{c} F_{\tau\sigma}$$

where $F_{\sigma\tau} = E_\gamma$ the electric force

$(F_{32}\ \ F_{13}\ \ F_{21}) = (H_1,\ H_2,\ H_3)$ the magnetic force.

Therefore $\quad \alpha^{\sigma\rho}{}_{\mu} D_\sigma \alpha^{\tau\nu}{}_{\rho} D_\tau$

$$= \alpha^{\sigma\rho}{}_{\mu}\alpha^{\tau\nu}{}_{\rho} D_\sigma D_\tau + (\alpha^{\sigma\rho}{}_{\mu})(\alpha^{\tau\nu}{}_{\rho})_\sigma D_\tau \frac{h}{2\pi i}.$$

$$= \tfrac{1}{2}(\alpha^{\sigma\rho}{}_{\mu}\alpha^{\tau\nu}{}_{\rho} D_\sigma D_\tau + \alpha^{\tau\rho}{}_{,\mu}\alpha^{\sigma\nu}{}_{\rho} D_\tau D_\sigma) + \alpha^{\sigma\rho}{}_{\mu}(\alpha^{\tau\nu}{}_{\rho})_\sigma D_\tau \frac{h}{2\pi i}.$$

$$= g^{\sigma\tau} D_\sigma D_\tau + \tfrac{1}{2}\alpha^{\sigma\rho}{}_{\mu}\alpha^{\tau\nu}{}_{\rho}\frac{h}{2\pi i}\frac{e}{c}F_{\tau\sigma} + \alpha^{\sigma\rho}{}_{\mu}(\alpha^{\tau\nu}{}_{\rho})_\sigma D_\tau \frac{h}{2\pi i}.$$

Thus the second order equation is expressible as

$$S\psi_\mu + R^{\nu}_{\mu}\psi_\nu + N^{\tau\nu}{}_{\mu} D_\tau \psi_\nu = 0$$

where $S = g^{\sigma\tau}(\nabla_\sigma - \frac{2\pi i}{h}\frac{e}{c}A_\sigma)(\nabla_\tau - \frac{2\pi i}{h}\frac{e}{c}A_\tau) + \frac{4\pi^2 m^2c^2}{h^2}.$

$$R^{\nu}_{\mu} = \frac{hi}{2\pi}\frac{e}{2c}\alpha^{\sigma\rho}{}_{\mu}\alpha^{\tau\nu}{}_{\rho}F_{\sigma\tau} \tag{4.13}$$

$$N^{\tau\nu}{}_{\mu} = \frac{2\pi i}{h}\alpha^{\sigma\rho}{}_{,\mu}(\alpha^{\tau\nu}{}_{\rho})_\sigma$$

This is the equation derived in this paper, but the term in $N^{\tau\nu}{}_{\mu}$ must be zero by (4.12)

Furthermore it is necessary that a non-divergent current vector should be obtainable from our equation, and this vector must be real. It is evident from (4.2) that $\mathring{\alpha}_0$ can be taken as hermitian while $\mathring{\alpha}_1\ \mathring{\alpha}_2\ \mathring{\alpha}_3$ will be skew. Then from (4.5) we see that $\alpha_{\sigma|\mu}{}^{\rho}$ is not hermitian nor skew but that $\mathring{\alpha}_{0|\mu}{}^{\rho}\,\alpha^{\sigma}{}_{|\rho}{}^{\nu}$ is hermitian. This suggests that the

vector/

vector $\qquad J^{\sigma} = \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{1\rho} \psi_{\nu},$

might serve as current vector. Let us find its divergence

$$\operatorname{div} J = (J^{\sigma})_{\sigma} = \psi^{*\mu}{}_{1\sigma} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma}{}_{1\rho}{}^{\nu} \psi_{\nu} + \psi^{*\mu}(\mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho})_{\sigma} \psi_{\nu}$$
$$+ \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho} (\psi_{\nu})_{\sigma} .$$

Now from (4.9) and (4.10)

$$\alpha^{\sigma\nu}{}_{\rho} (\psi_{\nu})_{\sigma} = \alpha^{\sigma\nu}{}_{\rho} \left(\frac{2\pi i}{h} D_{\sigma} + \frac{2\pi i e}{c} A_{\sigma}\right) \psi_{\nu}$$
$$= \frac{2\pi i}{h} \left(- mc\, \psi_{\rho} + \frac{e}{c} \alpha^{\sigma\nu}{}_{\rho} A_{\sigma} \psi_{\nu} \right).$$

Multiply (4.10) by $\mathring{\alpha}_{01\mu}{}^{\rho}$ and take the hermitian adjoint

$$(D_{\tau} \psi_{\nu})^{+} (\mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho})^{+} + mc\, \psi^{*\rho} \mathring{\alpha}_{01}{}^{\mu}{}_{\rho} = 0$$

because

$$\psi_{\rho}^{+} = \psi^{\rho *},$$

that is

$$\left(-\frac{h}{2\pi i} \nabla_{\tau} - \frac{e}{c} A_{\tau}\right) \psi^{*\nu} \mathring{\alpha}_{01\nu}{}^{\rho} \alpha^{\sigma\mu}{}_{\rho} + mc\, \psi^{*\rho} \mathring{\alpha}_{01}{}^{\mu}{}_{\rho} = 0$$

$$\therefore (\psi^{*\mu})_{\sigma} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho} = \frac{2\pi i}{h} mc\, \psi^{*\rho} \mathring{\alpha}_{01}{}^{\nu}{}_{\rho} - \frac{2\pi i}{h}\frac{e}{c} A_{\sigma} \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho}.$$

Substitute this result in the expression for $(J^{\sigma})_{\sigma}$:

$$(J^{\sigma})_{\sigma} = \psi^{*\rho} \mathring{\alpha}_{01}{}^{\nu}{}_{\rho} \psi_{\nu} \frac{2\pi i mc}{h} - \frac{2\pi i e}{hc} A_{\sigma} \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho} \psi_{\nu} + \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho}$$
$$- \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \psi_{\rho} \frac{2\pi i mc}{h} + \frac{2\pi i e}{hc} A_{\sigma} \psi^{*\mu} \mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho} \psi_{\nu}$$
$$= \psi^{*\mu} (\mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho})_{\sigma} \psi_{\nu}$$

which is zero only if

$$\nabla_{\sigma} (\mathring{\alpha}_{01\mu}{}^{\rho} \alpha^{\sigma\nu}{}_{\rho}) = 0. \tag{4.14}$$

Thus for a satisfactory theory based upon ψ-quantities which are world vectors we require

$$\alpha^{\sigma} \alpha^{\tau} + \alpha^{\tau} \alpha^{\sigma} = 2 g^{\tau\sigma} \qquad \text{(a)}$$
$$\alpha^{\sigma} (\alpha^{\tau})_{\sigma} = 0 \qquad \text{(b)}$$
$$(\mathring{\alpha}_{0} \alpha^{\sigma})_{\sigma} = 0 , \qquad \text{(c)}$$

omitting/

omitting the row and column tensor indices.

In general it would be impossible to satisfy all these con-:ditions. A solution satisfying (a) would satisfy (b) and (c) only along special directions unless the space was flat. The conditions of integrability in a general ds^2 impost limita-:tions upon the α's and thereby certain directions would be singled out. This would make physical laws dependant in some ways on special directions in space-time, and as there are no such directions of physical importance this theory is quite unsatis-:factory.

The conclusion is not the Dirac's equation is at fault as Levi-Civita suspects - but that it is impossible to formulate a wave equation in general relativity in a form similar to Dirac's but involving no quantities other than world tensors, the four component ψ- function being a vector.

Levi-Civita proposed a new wave equation

$$S\psi_\mu + \chi^{\nu}_{\mu}\psi_\nu = 0$$

with S as before

and $\chi^{\nu}_{\mu} = C\, g_{\mu\rho}\, \varepsilon^{\rho\nu\sigma\tau} F_{\sigma\tau}$.

An equation of this form with a coupling between the different ψ - compnnents effected by the χ^{ν}_{μ} term would be satisfactory for explaining the Zeeman effect in a way similar to the Dirac theory. This equation, however, has this grave objection, it is more or less empirical and has no theoretical foundation in quantum theory. On the other hand Dirac's equation evolves from general/

general principles of present quantum theory and the spin effects, which provide the coupling just mentioned, are automatically ob-:tained when the demands of relativity are met.
While Levi-Civita's criticism is valid for a wave equation in Dirac form, but with a world-vector as wave-function, it fails for the usual Dirac equation where the ψ -function is a spinor. In the former, the α -matrices had three tensor indices, which in the latter they possess only one such index for the other two are to be considered as spinor indices. Therefore in the latter can we have two kinds of transformations: point substitutions and spin or similarity transformations which are independant of each other. We shall now illustrate how the results of Chapter I where on account of the generalisation of the Dirac equation to general re-:lativity was given, do not distinguish any special directions.

To begin with, we took a set of matrices ρ_i related to an orthogonal ennuple $\overset{\circ}{g}_{ij} = e_i \delta_{ij}$; e_i is the signature of the metric. Then

$$\rho_i \rho_j + \rho_j \rho_i = 2 e_i \delta_{ij} 1 \qquad (i,j = 0-3) \qquad (4.15)$$

The indices of the ρ_i matrices referring to rows and columns are suppressed - these spinor indices are not concerned in trans-:formations of coordinates. If ρ'_i is another set of matrices obeying (4.15) then the ρ_i and ρ'_i are related by a similarity transformation. We shall assume that the ρ_i are hermitian when $e_i = +1$ and skew-hermitian when $e_i = -1$. We can take, for example, the α_i chosen by Dirac and obtain the ρ_i by the/

the relation $\rho_i = \sqrt{e_i}\,\alpha_i$. The set ρ_i' need not however be limited by such hermiticity restrictions. Then if

$$S = \sum \rho_k' \rho_k^+ \tag{4.16}$$

where the summation is extended over the whole 32-termed group generated by the ρ_i and the cross again denotes the hermitian adjoint,

$$\rho_i' \, S \, \rho_i^+ = \sum_k \rho_i' \rho_k' \rho_k^+ \rho_i^+$$
$$= \sum_k \rho_i' \rho_k' (\rho_i \rho_k)^+ .$$

$\rho_i \rho_k$ forms a term ρ_j of the group and as k assumes all 32 values, ρ_j repeats the group in a new order. If the group ρ_j' is numbered in the same way,

$$\rho_i' \, S \, \rho_i^+ = \sum_j \rho_j' \rho_j^+ = S$$

Postmultiply by ρ_i^+ and note that $\rho_i^+ = e_i \rho_i$

and also that $(\rho_i^+)^2 = (\rho_i)^2 = e_i$

$$\rho_i' \, S \, e_i = S \rho_i^+ = e_i S \rho_i$$

Therefore
$$\rho_i' = S \rho_i S^{-1} \tag{4.17}$$

From the ρ_i we obtained the vector matrices γ_μ by the usual rule for forming vector components from the components re-:ferred to an orthogonal example namely

$$\gamma_\mu = \sum_i e_i \lambda_{i|\mu} \rho_i \tag{4.18}$$

and as we are using real coordinates the coefficients $\lambda_{i|\mu}$ are real.

From (4.4) and (4.15) we find

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu} \tag{4.19}$$

Thus (4.18) gives us a solution of (4.19), the ρ_i being in their/

their turn solutions of (4.15). Thus as the ρ_i are not unique neither are the r_μ. However if r_μ and $\acute{r}_\mu$ are two sets of solutions

$$\begin{aligned} r_\mu &= \sum_i e_i \lambda_{i|\mu} \rho_i \\ \acute{r}_\mu &= \sum e_i \lambda_{i|\mu} \acute{\rho}_i \\ &= \sum e_i \lambda_{i|\mu} S \rho_i S^{-1} \\ &= S\left(\sum e_i \lambda_{i|\mu} \rho_i\right) S^{-1} \\ &= S\, r_\mu S^{-1}. \end{aligned}$$

Therefore different solutions of the relations (4.19) are con:nected by means of a similarity transformation.

COORDINATE TRANSFORMATIONS.

The transformation of coordinates

$$\acute{x}^\mu = l^\mu{}_\nu x^\nu \tag{4.20}$$

which leaves the $g_{\mu\nu}$ unaltered in the invariant expression

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

will now be considered. The conditions imposed upon the coefficients $l^\mu{}_\nu$ are that

$$g_{\sigma\tau} l^\sigma{}_\mu l^\tau{}_\nu = g_{\mu\nu}$$

or

$$l^\mu{}_\sigma l^\nu{}_\tau g^{\sigma\tau} = g^{\mu\nu} \tag{4.21}$$

Under the transformation (4.20)

$$\acute{r}^\mu = l^\mu{}_\nu r^\nu$$

$$l^\mu{}_\nu \acute{r}_\mu = r_\nu$$

Then/

Then $\acute{\gamma}^{\mu}\acute{\gamma}^{\nu}+\acute{\gamma}^{\nu}\acute{\gamma}^{\mu} = l^{\mu}{}_{\sigma}\, l^{\nu}{}_{\tau}\,(\gamma^{\sigma}\gamma^{\tau}+\gamma^{\tau}\gamma^{\sigma})$

$= l^{\mu}{}_{\sigma}\, l^{\nu}{}_{\tau}\, 2g^{\sigma\tau}$ by (4.19)

$= 2g^{\mu\nu}$ by (4.21)

Similarly $\acute{\gamma}_{\mu}\acute{\gamma}_{\nu}+\acute{\gamma}_{\nu}\acute{\gamma}_{\mu} = 2g_{\mu\nu}$.

γ_{μ} and $\acute{\gamma}_{\mu}$ both satisfy the relations (4.19). But we have seen that two such sets of matrices are connected by a similarity transformation,

$$\left.\begin{array}{ll} \text{i.e.} & \acute{\gamma}^{\mu} = S\gamma^{\mu}S^{-1} = l^{\mu}{}_{\nu}\,\gamma^{\nu} \\ \text{or} & l^{\mu}{}_{\nu}\,\acute{\gamma}_{\mu} = l^{\mu}{}_{\nu}\, S\gamma_{\mu}S^{-1} = \gamma_{\nu} \end{array}\right\} \qquad (4.22)$$

This means that a coordinate transformation produces a change in the γ^{μ} but this change could also be effected by a similarity transformation. The S matrix is uniquely deter-:mined apart from an arbitrary numerical factor.

Again in the general theory there were given then rules for the general spin-transformations which leave the formulation invariant. Now suppose $g_{\mu\nu}$ given, and that we have found a set of suitable γ^{μ} matrices. Does this infer that we have distinguished special directions in space-time? Clearly not, for we can transform from our coordinate system to any other one with the result that

$$\acute{\gamma}^{\mu} = l^{\mu}{}_{\nu}\,\gamma^{\nu} = S\gamma^{\mu}S^{-1}$$

Let us at the same time, however, admit a similarity transforma-:tion so chosen that we restore our original γ^{μ}. This of

course/

course is the reciprocal of the S - transformation which pro-:duces the same change in γ^{ν} as the coordinate transforma-:tion. Thus with the same matrix S we have

$$\tilde{\gamma}^{\mu} = S^{-1}(\acute{\gamma}^{\mu})\,S = S^{-1}S\,\gamma^{\mu}\,S^{-1}S = \gamma^{\mu}$$

Therefore to be given $g_{\mu\nu}$ and γ^{μ} does not indicate any pre-:ferred coordinate system, all are equivalent, and no special directions in space time are selected thereby. In this way, the argument raised by Levi-Civita proves to be without founda-:tion when it is applied to the usual form of Dirac's equation and its generalisation by the method of Schrodinger and others. This necessary result could not be obtained from Levi-Civita's form of the wave equation as his α - matrices, being thrid rank tensors, left no provision for spin or similarity transformations. Or the difference may be expressed in this way: it was possible to displace the γ^{μ} matrices along any direction because the quantities Γ_{σ} were introduced into the law of covariant differ-:entiation and these were given the necessary properties to make the equation integrable, but the third order $\alpha^{\sigma\rho}_{\mu}$-tensors are purely tensorial and no such quantities can be introduced so that parallel displacement in general is not possible along all directions.

LEVI/

LEVI-CIVITA'S EQUATION.

We shall briefly examine the wave equation which Levi-Civita has proposed as a substitute for Dirac's relativity wave equa- :tion. The new equation is

$$\left\{g^{\sigma\tau}\left(\nabla_\sigma - \frac{ie}{c\hbar}A_\sigma\right)\left(\nabla_\tau - \frac{ie}{c\hbar}A_\tau\right)\right\}\psi_\mu + \frac{e}{2c\hbar}\, g_{\mu\rho}\,\varepsilon^{\rho\nu\sigma\tau}F_{\sigma\tau}\,\psi_\nu = 0$$

Write $\quad \frac{1}{2}\,\varepsilon^{\rho\nu\sigma\tau}F_{\sigma\tau} = \tilde{F}^{\rho\nu}$

and $\quad \frac{e}{c\hbar}A_\sigma = \phi_\sigma$

It will be sufficient to keep to special relativity so that we may write

$\partial_\sigma \equiv \frac{\partial_\sigma}{\partial x}$ for ∇_σ. Then the equation is now

$$\left[\left\{g^{\sigma\tau}\partial_\sigma\partial_\tau - (\phi^\sigma\phi_\sigma) - 2i\,\phi^\sigma\partial_\sigma + \frac{m^2c^2}{\hbar^2}\right\}\delta^\nu_\mu + \frac{e}{c\hbar}\, g_{\mu\rho}\tilde{F}^{\rho\nu}\right]\psi_\nu = 0 \tag{4.23}$$

The conjugate complex is

$$\left[\left\{g^{\sigma\tau}\partial_\sigma\partial_\tau - (\phi^\sigma\phi_\sigma) + 2i\,\phi^\sigma\partial_\sigma + \frac{m^2c^2}{\hbar^2}\right\}\delta^\nu_\mu + \frac{e}{c\hbar}\, g_{\mu\rho}\tilde{F}^{\rho\nu}\right]\overline{\psi}_\nu = 0$$

where the $^{-}$ denotes the complex conjugate. (4.24)

Premultiply these equations by $\overline{\psi}^\mu$ and ψ^μ respectively, summing with respect to μ :

$$\overline{\psi}^\mu\left[\partial^\sigma\partial_\sigma - \phi^\sigma\phi_\sigma - 2i\,\phi^\sigma\partial_\sigma + \frac{m^2c^2}{\hbar^2}\right]\psi_\mu + \frac{e}{c\hbar}\,\overline{\psi}_\rho\psi_\nu\tilde{F}^{\rho\nu} = 0$$

$$\psi^\mu\left[\partial^\sigma\partial_\sigma - \phi^\sigma\phi_\sigma + 2i\,\phi^\sigma\partial_\sigma + \frac{m^2c^2}{\hbar^2}\right]\overline{\psi}_\mu + \frac{e}{c\hbar}\,\psi_\rho\overline{\psi}_\nu\tilde{F}^{\rho\nu} = 0$$

Subtract and then we have

$$(\overline{\psi}^\mu\partial^\sigma\partial_\sigma\psi_\mu - \psi^\mu\partial^\sigma\partial_\sigma\overline{\psi}_\mu) - 2i\,\phi^\sigma(\overline{\psi}^\mu\psi_{\mu|\sigma} + \psi^\mu\overline{\psi}_{\mu|\sigma})$$
$$+ \frac{2e}{c\hbar}\,\overline{\psi}_\rho\psi_\nu\tilde{F}^{\rho\nu} = 0,$$

or/

or

$$\partial^\sigma \left[(\bar{\psi}^\mu \psi_{\mu|\sigma} - \psi^\mu \bar{\psi}_{\mu|\sigma}) - 2i\, \phi_\sigma \bar{\psi}^\mu \psi_\mu \right] + \frac{2e}{c\hbar}\, \bar{\psi}_\rho \psi_\nu \tilde{F}^{\rho\nu} = 0$$

Hence the vector

$$S_\sigma = (\bar{\psi}^\mu \psi_{\mu|\sigma} - \psi^\mu \bar{\psi}_{\mu|\sigma}) - 2i\, \phi_\sigma \bar{\psi}^\mu \psi_\mu \qquad (4.25)$$

has its divergence equal to the scalar

$$-\frac{2e}{c\hbar}\, \bar{\psi}_\rho \psi_\nu \tilde{F}^{\rho\nu}.$$

Now S^σ appears to be the only reasonable vector that one can use to form the charge current density - corresponding to the expression in the Klein-Gordon relativity equation where the function ψ is a scalar. As, however, S^σ is not of zero diver:gence it follows that there is no satisfactory current vector derivable from the present equation which has all the disadvan:tages inherent in second degree wave equations, one of the chief of these being that $\bar{\psi}^\mu \psi_\mu$ can no longer represent the electric charge density.

However it is at once noticed that if the imaginary factor i is introduced into the term in $\tilde{F}^{\rho\nu}$, then after the above cal:culation is carried out, the contributions from this term to $\operatorname{div} S$ cancel each other leaving us with

$$\partial_\sigma S^\sigma = 0$$

so that it would be all right to interprete S^σ as the current vector. In the case of this modification we find that the equation is derivable from the Lagrangian

$$L(\psi_\mu, \psi_{\mu|\sigma}, \bar{\psi}_\mu, \bar{\psi}_{\mu|\sigma}, \phi_\mu, \phi_{\mu|\sigma})$$

$$= (\psi^{\mu|\sigma} - i\phi^\sigma \psi^\mu)(\bar{\psi}_{\mu|\sigma} + i\phi_\sigma \bar{\psi}_\mu) + \frac{m^2c^2}{\hbar^2}\psi^\mu \bar{\psi}_\mu - \frac{ei}{c\hbar}\bar{\psi}_\rho \psi_\nu \tilde{F}^{\rho\nu}, \qquad (4.26)$$

the/

the wave equation in fact being

$$\frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial \overline{\psi}_{\mu|\sigma}}\right) - \frac{\partial L}{\partial \overline{\psi}_\mu} = 0 \tag{4.27}$$

This is closely parallel to Schrödinger's (30) derivation of the Klein-Gordon equation from a Lagrangian only now the ψ functions are vectors to which one tensor index is attached, and there is the additional term in the field strength. The current vector is expressible in the form

$$S^\sigma = i\,\frac{\partial L}{\partial \phi_\sigma} \tag{4.28}$$

If there is an energy tensor, we should have

$$\partial_\sigma\left(T^\sigma_\rho\right) = F_{\rho\sigma}\, S^\sigma \tag{4.29}$$

Neglecting all the numerical constants, we have for the right hand side

$$\left(\frac{\partial \phi_\sigma}{\partial x^\rho} - \frac{\partial \phi_\rho}{\partial x^\sigma}\right)\frac{\partial L}{\partial \phi_\sigma}$$

$$= \left(\frac{\partial L}{\partial \phi_\sigma}\right)\phi_{\sigma|\rho} - \frac{\partial}{\partial x^\sigma}\left(\phi_\rho\,\frac{\partial L}{\partial \phi_\sigma}\right)$$

since $\quad \frac{\partial}{\partial x^\sigma}\left(\frac{\partial L}{\partial \phi_\sigma}\right) = \operatorname{div} S = 0 .$

Now $\frac{\partial L}{\partial x^\rho}$

$$= \frac{\partial L}{\partial \phi_\sigma}\,\phi_{\sigma|\rho} + \frac{\partial L}{\partial \phi_{\mu|\sigma}}\,\phi_{\mu|\sigma\rho} + \frac{\partial L}{\partial \psi^\mu}\,\psi^\mu{}_{|\rho} + \frac{\partial L}{\partial \psi^\mu{}_{|\sigma}}\,\psi^\mu{}_{|\sigma\rho}$$

$$+ \frac{\partial L}{\partial \overline{\psi}_\mu}\,\overline{\psi}_{\mu|\rho} + \frac{\partial L}{\partial \overline{\psi}_{\mu|\sigma}}\,\overline{\psi}_{\mu|\sigma\rho}$$

$$= \frac{\partial L}{\partial \phi_\sigma}\,\phi_{\sigma|\rho} + \frac{\partial L}{\partial \phi_{\mu|\sigma}}\,\phi_{\mu|\sigma\rho} + \frac{\partial}{\partial x^\sigma}\left(\psi^\mu{}_{|\rho}\,\frac{\partial L}{\partial \psi^\mu{}_{|\sigma}} + \overline{\psi}_{\mu|\rho}\,\frac{\partial L}{\partial \overline{\psi}_{\mu|\sigma}}\right)$$

after using the wave equation and the fact that the order of differentiation is immaterial. On subtracting on two results

we/

we have $\lambda F_{\rho\sigma} S^{\sigma}$

$$= \frac{\partial}{\partial x^{\sigma}}\left(\psi^{\mu}{}_{|\rho} \frac{\partial L}{\partial \psi^{\mu}{}_{|\sigma}} + \bar{\psi}^{\mu}{}_{|\rho} \frac{\partial L}{\partial \bar{\psi}_{\mu|\sigma}} + \phi_{\rho} \frac{\partial L}{\partial \phi_{\sigma}} \right) - \frac{\partial L}{\partial x^{\rho}} + \frac{\partial L}{\partial \phi_{\mu|\sigma}} \phi_{\mu|\sigma\rho} \qquad (4.30)$$

But

$$\frac{\partial L}{\partial \phi_{\mu|\sigma}} = \frac{\partial}{\partial \phi_{\mu|\sigma}} \left(i \bar{\psi}_{\rho} \psi_{\nu} \, \varepsilon^{\rho\nu\alpha\beta} \phi_{\alpha|\beta} \right)$$

$$= i \bar{\psi}_{\rho} \psi_{\nu} \, \varepsilon^{\rho\nu\mu\sigma},$$

so that the final term of (4.30)

$$\phi_{\mu|\sigma\rho} \frac{\partial L}{\partial \phi_{\mu|\sigma}} = i \bar{\psi}_{\mu} \psi_{\nu} \left(\tilde{F}^{\mu\nu} \right)_{\rho}$$

The presence of this term prevents us from taking the tensor

$$T_{\rho\sigma} = \left(\psi^{\mu}{}_{|\rho} \frac{\partial L}{\partial \psi^{\mu}{}_{|\sigma}} + \bar{\psi}_{\mu|\rho} \frac{\partial L}{\partial \bar{\psi}_{\mu|\sigma}} + \phi_{\rho} \frac{\partial L}{\partial \phi_{\sigma}} - \delta_{\rho\sigma} L \right) \qquad (4.31)$$

as the material energy tensor except in the special case where the electro-magnetic field strength is constant. The difficulty lies in the fact that the Lagrangian contains a term depending on the derivatives of the electro-magnetic potential. As we have no satisfactory energy tensor, this wave equation will not give us the usual Lorentzian equations of motion for the electron, so that even with our modification it is hardly acceptable. Although it has no definite physical foundation it has emperically added to it extra terms to give the linkage between the wave function components to explain the Zeeman effect.

If one is willing to forego the theoretical advantage of a first order equation then a second order equation such as the one recently proposed by Proca (31) is much more satisfactory than the one of Levi-Civita. In Proca's work the Lagrangian depends on/

on the potentials but not on the field strength with the result a proper energy tensor is readily obtainable and the equation is such that a definite electromagnetic moment six-vector evolves from the theory. This equation, in the notation we have been using is

$$\left\{ g^{\sigma\tau} (\partial_\sigma - i\phi_\sigma)(\partial_\tau - i\phi_\tau) + \frac{m^2c^2}{\hbar^2} \right\} \psi_\mu - (\partial_\mu - i\phi_\mu)(\partial_\sigma - i\phi_\sigma)\psi^\sigma + \frac{ie}{c\hbar} F_{\mu\sigma}\psi^\sigma = 0,$$

where the last two terms which provide the linkage between the different ψ-functions are added to the Klein-Gordon expression. When ϕ_σ and $F_{\mu\sigma}$ are zero the equation which reduces to

$$\left\{ g^{\sigma\tau} \partial_\sigma \partial_\tau + \frac{m^2c^2}{\hbar^2} \right\} \psi_\mu - \partial_\mu \partial_\sigma \psi^\sigma = 0$$

implies

$$\text{(a)} \quad \left\{ g^{\sigma\tau} \partial_\sigma \partial_\tau + \frac{m^2c^2}{\hbar^2} \right\} \psi_\mu = 0 \qquad \text{(b)} \quad \partial_\sigma \psi^\sigma = 0$$

so that in the case of no field each ψ-component obeys the Schrödinger equation (a) and together they satisfy the relation (b).

CHAPTER V.

THE ATOMIC WAVE FUNCTIONS OF HYDROGEN IN MOMENTUM SPACE.

The wave equation of the orbital electron in a hydrogen atom is

$$\left(\frac{W}{c} + \frac{e^2}{r} + \alpha_1 p^1 + \alpha_2 p^2 + \alpha_3 p^3 + \alpha_4 mc \right) \psi = 0 \qquad (5.1)$$

In ordinary coordinate space (x^1, x^2, x^3) the ψ-quantities are functions of the x^i, while the momenta p^i are represented by differential operators $\frac{h}{2\pi i}\frac{\partial}{\partial x^i}$. Alternatively, we may interprete the equation as one in momentum space $(p_1\, p_2\, p_3)$ with ψ a function of the p_i while the ordinary coordinates x^i are now represented by the operators $-\frac{h}{2\pi i}\frac{\partial}{\partial p_i}$

The moment of momentum which is defined classically as the vector product

$$L = [x \times p]$$

is interpreted in coordinate space as the operator,

$$\left[x \times \frac{h}{2\pi i}\frac{\partial}{\partial x}\right]$$

or in momentum space as

$$\left[-\frac{h}{2\pi i}\frac{\partial}{\partial p} \times p\right] = \left[p \times \frac{h}{2\pi i}\frac{\partial}{\partial p}\right]$$

since $p_1\ p_2\ p_3$ are independent variables.

Therefore, the L - operators are exactly the same functions of the coordinates whether these are those of coordinate - or of momentum space. From the properties of the L - operator the eigen function dependence upon the angular coordinates $(\theta\ \phi)$ have/

have been determined and are well known. The same analysis can be used to determine the $(\theta' \phi')$ part of the eigen-function in momentum space with polar coordinates $(p\, \theta'\, \phi')$ leading to exactly the same function as is obtained for coordinate space.

Let the state represented by $\psi(x_i)$ in coordinate space be represented by $\phi(p_i)$ in momentum spaces. Then these two functions are related by a Fourier integral

$$\phi(p) = \frac{1}{h^{3/2}} \iiint_{-\infty}^{\infty} e^{-\frac{2\pi i}{h}(p_1 x_1 + p_2 x_2 + p_3 x_3)} \psi(x)\, dx_1\, dx_2\, dx_3 \qquad (5.2)$$

and a differential operator $F\left(x_i\, ,\, \frac{h}{2\pi i}\frac{\partial}{\partial x_i}\right)$ becomes $F\left(-\frac{h}{2\pi i}\frac{\partial}{\partial p_i}\, ,\, p_i\right)$.

The wave equation as it stands contains the term $r^{-1}\phi(p)$ which is represented as

$$r^{-1}\phi = \frac{1}{2\pi^2 h} \iiint_{-\infty}^{\infty} \frac{\phi(\bar{p}_1 \bar{p}_2 \bar{p}_3)\, d\bar{p}_1\, d\bar{p}_2\, d\bar{p}_3}{(\bar{p}_1 - p_1)^2 + (\bar{p}_2 - p_2)^2 + (\bar{p}_3 - p_3)^2}$$

so that the wave equation is an integral one. However after preliminary modifications it is possible to obtain an equation in which r^2 and no negative nor odd powers of r occur. As

$$r^2 \rightarrow -\left(\frac{h}{2\pi}\right)^2 \left(\frac{\partial^2}{\partial p_1^2} + \frac{\partial^2}{\partial p_2^2} + \frac{\partial^2}{\partial p_3^2}\right)$$

we obtain a rational differential equation (of the second degree) When it is transformed to polar coordinates we can introduce the angular solutions and so have four simultaneous differential equations for the four $\phi_i(p)$.

But, as the ψ_i - functions in x-space have already been de:termined those in the p - space can be found from the Fourier transform/

transform the same way as Elsasser (32) has used to find the corresponding functions for the Schrödinger wave equation. If we take spherical coordinates in both spaces $(r\theta\phi)$ and $(p\theta'\varphi')$ and denote the angle between the two radius vectors by ω:

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi'). \tag{5.3}$$

The integral transform in terms of these coordinates is

$$\phi(r'\theta'\phi') = h^{-3/2}\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{-\frac{2\pi i}{h}rr'\cos\omega}\,\psi(r\theta\phi)\,r^2\sin\theta\,dr\,d\theta\,d\varphi. \tag{5.4}$$

Now we must recall the full solution $\psi(r\theta\varphi)$ of the eigen value problem (5.1) for the hydrogen atom. (Cf Darwin (33) Gordon (34) Pauli (35)). The four components, when the Dirac set of matrices are used, are

$$\begin{aligned}
\psi_1 &= F(r) & (-k-m) & \quad P^m_{k-1}(\cos\theta) & e^{im\phi} \\
\psi_2 &= F(r) & 1 & \quad P^{m+1}_{k-1}(\cos\theta) & e^{i(m+1)\varphi} \\
\psi_3 &= G(r) & (k-m) & \quad P^m_k(\cos\theta) & e^{im\phi} \\
\psi_4 &= G(r) & 1 & \quad P^{m+1}_k(\cos\theta) & e^{i(m+1)\phi},
\end{aligned} \tag{5.5}$$

where the radial parts are

$$\begin{aligned}
F(r) &= -i\,.\,e^{-r/a}\left(\tfrac{2r}{a}\right)^{s-1}\left\{c_1F(1-n', 2s+1, 2r/a) + c_2F(-n', 2s+1, 2r/a)\right\} \\
G(r) &= \frac{\sqrt{p^2+a^2}+p}{a}\,.\,e^{-r/a}\left(\tfrac{2r}{a}\right)^{s-1}\left\{c_1F(1-n', 2s+1, 2r/a) - c_2F(-n', 2s+1, 2r/a)\right\}
\end{aligned} \tag{5.6}$$

Here $F(\alpha, \beta, x)$ denotes the Laguerre function which at the origin has the expansion

$$F(\alpha,\beta,x) = 1 + \frac{\alpha}{\beta}\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{x^3}{3!} + \cdots$$

The/

The constants c_1 and c_2 are connected by the relation

$$\frac{c_1}{c_2} = \frac{k + \frac{\alpha}{2}\left(b + \frac{1}{b}\right)}{s - \frac{\alpha}{2}\left(b - \frac{1}{b}\right)}.$$

k is the quantum number assuming one of the values

$\pm 1, \pm 2, \ldots \pm(n-1), -n,$ where n is the total quantum number

m is one of the series $0, \pm 1, \pm 2, \ldots \pm|k|-1, -|k|$

and $S = +\sqrt{k^2 - \alpha^2}$ where $\alpha = \frac{2\pi e^2}{h}$ is the fine structure constant

$$a = \frac{h}{2\pi}\left(m^2c^2 - \frac{W^2}{c^2}\right)^{-1/2}, \qquad (5.7)$$

$$b = \left(\frac{mc^2 - W}{mc^2 + W}\right)^{1/2}$$

$s + \frac{\alpha}{2}\left(b - \frac{1}{b}\right) = -n'$ and when $W < mc^2$ as in the case of the energy levels considered in line spectra n' must be a positive integer.

$p = n' + s$ and as s is not an integer neither is p, but it differs only slightly from the total quantum number n. Finally we may note that the energy levels are given by

$$W = \frac{mc^2}{(1 + \alpha^2/p^2)^{1/2}}$$

Now we can return to our integral and proceed to integrate with respect to θ and φ. The angular part of ψ will be denoted by $P_l^u(\cos\theta)\, e^{iu\phi}$ which is the form of the functions for the 4 ψ_i.

Then $h^{-3/2} \int_0^{\pi}\int_0^{2\pi} e^{-\frac{2\pi i r r'}{h}} P_l^u(\cos\theta)\, e^{iu\phi} \sin\theta \, d\theta \, d\phi$

is the required integral.

The/

The expansion of $e^{-\frac{2\pi i r r'}{h}\cos\omega}$ in terms of $P_n(\cos\omega)$ is

$$e^{-\frac{2\pi i}{h} r r' \cos\omega} = \sum_{n=0}^{\infty} (-i)^n (2n+1)\left(\frac{\pi}{2}\cdot\frac{h}{2\pi r r'}\right)^{1/2} J_{n+1/2}\left(\frac{2\pi r r'}{h}\right) P_n(\cos\omega).$$

(Frank-Mises Vol.1, IX § 2:3,

and the addition theorem is

$$P_n(\cos\omega) = P_n(\cos\theta) P_n(\cos\theta') + 2\sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\varphi-\varphi')$$

$$= \sum_{m=-n}^{n} P_n^m(\cos\theta) P_n^{-m}(\cos\theta')\, e^{im\phi} e^{-im\phi'}.$$

The non zero term in the integration with respect to ϕ is that for which $m = -u$. On account of the orthogonality properties of the P_n^m the sole surviving term after the integration is

$$h^{-3/2}(-i)^l \sqrt{\frac{h}{r r'}}\; J_{l+1/2}\left(\frac{2\pi r r'}{h}\right) .\ 2\pi\ .\ P_l^u(\cos\theta')\, e^{iu\phi'}$$

This means that the angular part of the solution is the same spherical harmonic as in coordinate space - which has already been proved.

The radial part is now

$$\phi(r') = \int_0^\infty (-i)^l \frac{2\pi}{h}\left(\frac{1}{r r'}\right)^{1/2} J_{l+1/2}\left(\frac{2\pi r r'}{h}\right) r^2 \left\{ e^{-r/a}\left(\frac{2r}{a}\right)^{s-1} \times \text{ a power series in ascending powers of } r \right\} dr.$$

Consider the integral arising from the term r^n in the series:

$$(-i)^l \frac{2\pi}{h r'^{1/2}} \int_0^\infty e^{-r/a} J_{l+1/2}\left(\frac{2\pi r r'}{h}\right)\left(\frac{2}{a}\right)^{s-1} r^{s+n+1/2}\, dr.$$

Write/

Write $q = \frac{2\pi r'}{h}$ and expand the Bessel function $J_{l+1/2}$:

$$(-i)^l \left(\frac{2\pi}{h}\right)^{1/2} \left(\frac{2}{a}\right)^{s-1} q^{-1/2} \int_0^\infty e^{-r/a} \left(\frac{qr}{2}\right)^{l+1/2} \sum_{t=0}^{\infty} \frac{\left(\frac{qr}{2}\right)^{2t} (-)^t \, r^{s+n+1/2}}{\Gamma(l+3/2+t) \quad t!} \, dr$$

$$= \left(-\frac{iq}{2}\right)^l a^{n+l+3} \left(\frac{2\pi}{h}\right)^{1/2} 2^{s-3/2} \sum_{t=0}^{\infty} \left\{ \int_0^\infty e^{-r/a} \left(\frac{r}{a}\right)^{l+s+n+2t+1} \frac{dr}{a} \right\} \frac{(-)^t}{t!} \left(\frac{qa}{2}\right)^{2t} \frac{1}{\Gamma(l+3/2+t)}$$

$$= \left(-\frac{iq}{2}\right)^l a^{n+l+3} \left(\frac{2\pi}{h}\right)^{1/2} 2^{s-3/2} \sum_{t=0}^{\infty} \frac{\Gamma(l+s+n+2+2t)}{\Gamma(l+3/2+t) \quad t!} \left(\frac{-q^2a^2}{4}\right)^t .$$

As s is not an integer, this series does not reduce to a set of simple binomial expansions. The Gamma-functions in the numerator may be expressed as follows

$$\Gamma(l+s+n+2+2t) = (l+s+n+1+2t)(l+s+n+2t) \ldots (l+s+n+3)(l+s+n+2)\Gamma(l+s+n+2)$$

Divide each of the $2t$ factors by 2 and group even and odd terms so that we have

$$2^{2t} \left(\frac{l+s+n+1}{2} + t\right)\left(\frac{l+s+n+1}{2} + t - 1\right) \cdots \left(\frac{l+s+n+1}{2} + 1\right)$$

$$\times \left(\frac{l+s+n}{2} + t\right)\left(\frac{l+s+n}{2} + t - 1\right) \cdots \left(\frac{l+s+n}{2} + 1\right) \Gamma(l+s+n+2)$$

$$= 2^t \frac{\Gamma\left(\frac{l+s+n+3}{2} + t\right) \Gamma\left(\frac{l+s+n+2}{2} + t\right)}{\Gamma\left(\frac{l+s+n+3}{2}\right) \quad \Gamma\left(\frac{l+s+n+2}{2}\right)} \Gamma(l+s+n+2).$$

Substitute/

Substitute this in our expression and obtain

$$\left(-\frac{iq}{2}\right)^{l} a^{n+l+3}\left(\frac{2\pi}{h}\right)^{1/2} 2^{s-3/2}\,\frac{\Gamma(l+s+n+2)}{\Gamma(l+3/2)}\sum_{t=0}^{\infty}\left\{\frac{\Gamma\left(\frac{l+s+n+3}{2}+t\right)\Gamma\left(\frac{l+s+n+2}{2}+t\right)\Gamma\left(l+3/2\right)\cdot\left(-q^2a^2\right)^t}{\Gamma\left(\frac{l+s+n+3}{2}\right)\Gamma\left(\frac{l+s+n+2}{2}\right)\Gamma(l+3/2+t)\;t!}\right\}$$

The sum in the bracket is the hypergeometric function

$$F\left(\frac{l+s+n+3}{2},\ \frac{l+s+n+2}{2};\ l+\frac{3}{2};\ -q^2a^2\right)$$

For ψ_1 and ψ_2 the coefficient of r^n in the expansion of the two Laguerre functions (5.6) is

$$-i\left\{c_1\ \frac{\Gamma(2s+1)\ \ \Gamma(1-n'+n)}{\Gamma(2s+1+n)\ \ \Gamma(1-n')}\left(\frac{2}{a}\right)^n\frac{1}{n!}+c_2\ \frac{\Gamma(2s+1)\ \ \Gamma(-n'+n)}{\Gamma(2s+1+n)\ \ \Gamma(-n')}\left(\frac{2}{a}\right)^n\frac{1}{n!}\right\}$$

$$=\ -i\ \frac{\Gamma(2s+1)}{\Gamma(2s+1+n)}\ \left(-\frac{2}{a}\right)^n\ \frac{1}{n!}\left\{c_1\ \frac{(n'-1)!}{(n'-n-1)!}\ +\ c_2\ \frac{n'!}{(n'-n)!}\right\}$$

These are terms which are non vanishing for $n=0,1,\dots n'$ only for the last value the term in c_1 is zero.

Similarly for ψ_3 and ψ_4 the coefficient is

$$\frac{\sqrt{p^2+\alpha^2}+p}{\alpha}\cdot\frac{\Gamma(2s+1)}{\Gamma(2s+1+n)}\cdot\left(-\frac{2}{a}\right)^n\frac{1}{n!}\left\{c_1\ \frac{(n'-1)!}{(n'-n-1)!}\ -\ c_2\ \frac{n'!}{(n'-n)!}\right\}$$

For the first pair of functions l is $(k-1)$, while for the second l is k.

Denoting the factor

$$(-ia)^{k+2}\ 2^{s-k-1/2}\left(\frac{2\pi}{h}\right)^{1/2}\ \frac{\Gamma(2s+1)}{\Gamma\ k+1/2}\qquad\text{by}\qquad\rho$$

we obtain the following values for the eigensolutions of the orbital electron:-

$$\phi_1(q\,\theta'\phi')=\rho\, q^k\sum_{n=0}^{n'}\frac{\Gamma(k+s+n+1)}{\Gamma(2s+n+1)}\frac{(-2)^n}{n!}\left\{\frac{c_1(n'-1)!}{(n'-n-1)!}+c_2\frac{n'!}{(n'-n)!}\right\}F\left(\frac{k+s+n+2}{2},\frac{k+s+n+1}{2};k+\frac{1}{2};-q^2a^2\right)$$
$$\times(-k-m)\cdot P_{k-1}^{m}(\cos\theta')\cdot e^{im\phi'}.$$

For/

For $\phi_2(q\,\theta'\phi')$ replace the last line by

$$\times\ 1 .\ P_{k-1}^{m+1}(\cos\theta')\ e^{i(m+1)\phi'}$$

$$\phi_3(q\,\theta'\phi') = \rho\, q^{k}\,\frac{a.\left(\sqrt{p^2+\alpha^2}+p\right)}{2\alpha\quad(k+1/2)}\cdot\sum_{n=0}^{n'}\frac{\Gamma(k+s+n+2)\ (-2)^{n}}{\Gamma(2s+n+1)\quad n!}\left\{\frac{c_1\,(n'-1)!}{(n'-n-1)!}-\frac{c_2\,n'!}{(n'-n)!}\right\}$$

$$\times\ F\left(\frac{k+s+n+3}{2},\ \frac{k+s+n+2}{2}\ ;\ k+\frac{3}{2}\ ;-q^2a^2\right)\times\ (k-m).\ P_{k}^{m}(\cos\theta')\ e^{im\phi'}.$$

For $\phi_4(q\,\theta'\phi')$ the last factor is

$$\times\ 1 .\ P_{k}^{m+1}(\cos\theta')\ e^{i(m+1)\phi'} \qquad (5.8)$$

$\left(q=\frac{2\pi}{h}p\right)$

As normalisation is preserved after the Fourier transformation is carried out, the functions possess the same normalising factor as the ψ- functions.

POLAR/

POLAR WAVE EQUATION and FINE STRUCTURE.

The fine structure formula can be derived by considering the wave equation in momentum space. This time we use the wave equation in its polar form which is

$$\left(\frac{W}{c} + \frac{e^2}{r} + \omega\left(p_r + i\alpha_4 \frac{hk}{2\pi r} \right) + \alpha_4 mc \right)\psi = 0 \qquad (5.9)$$

Where p_r is the radial component of momentum, i.e. the com- :ponent acting in the direction of r, and p_r and r are conjugate quantities. Therefore there is a representation in which these are

p_r and $\frac{h}{2\pi i}\frac{\partial}{\partial p_r}$ respectively

After a certain spin transformation the operator ω can be made equal to

$$\alpha_5 = \begin{pmatrix} \cdot & \cdot & -i & \cdot \\ \cdot & \cdot & \cdot & -i \\ i & \cdot & \cdot & \cdot \\ \cdot & i & \cdot & \cdot \end{pmatrix}$$

but the ψ- functions are changed so that their components are not identical with those in the original Dirac equation but are combinations of those depending on the directions of r.

After operating on the above equation by r we obtain the differential equations for the components of the wave func- :tion. These, in detail after the matrices have been intro- :duced, are

$$\left.\begin{aligned} \left(\frac{W}{c} + mc\right)\frac{d}{dp_r}\psi_1 + i\alpha\psi_1 - i\left(p_r\frac{d}{dp_r} + 1 + k\right)\psi_3 &= 0 \\ \left(\frac{W}{c} - mc\right)\frac{d}{dp_r}\psi_3 + i\alpha\psi_3 + i\left(p_r\frac{d}{dp_r} + 1 - k\right)\psi_1 &= 0 \end{aligned}\right\}$$

α being the fine structure constant.

With p_r written as $\frac{h}{2\pi}\frac{q}{a}$ and using the constants a and b as before (5.7)

with $A = \sqrt{ab}$ $B = \sqrt{\frac{a}{b}}$ we now obtain

$$\left.\begin{aligned}\left(\frac{a}{A}\frac{d}{dq} + i\alpha\right)\psi_1 - i\left(q\frac{d}{dq} + 1 + k\right)\psi_3 &= 0\\ \left(-\frac{a}{B}\frac{d}{dq} + i\alpha\right)\psi_3 + i\left(q\frac{d}{dq} + 1 - k\right)\psi_1 &= 0\,.\end{aligned}\right\}$$

Make the substitution
$$\psi_1 = (\phi + \chi)/\sqrt{B}\,,$$
$$\psi_3 = (\phi - \chi)/\sqrt{A}\,,$$

so that the equations become

$$\left.\begin{aligned}\left(i\frac{d}{dq} + b\alpha - q\frac{d}{dq} - k - 1\right)\phi + \left(-i\frac{d}{dq} + b\alpha + q\frac{d}{dq} + k + 1\right)\chi &= 0\\ \left(i\frac{d}{dq} + \frac{\alpha}{b} + q\frac{d}{dq} - k + 1\right)\phi + \left(-i\frac{d}{dq} - \frac{\alpha}{b} + q\frac{d}{dq} - k + 1\right)\chi &= 0\end{aligned}\right\}$$

After adding the two equations, and again after subtraction we derive the new equations

$$\left.\begin{aligned}\left[\frac{\alpha}{2}\left(b + \frac{1}{b}\right) - k\right]\phi + \left[(q-i)\frac{d}{dq} + \frac{\alpha}{2}\left(b - \frac{1}{b}\right) + 1\right]\chi &= 0\\ \left[\frac{\alpha}{2}\left(b + \frac{1}{b}\right) + k\right]\chi + \left[-(q+i)\frac{d}{dq} + \frac{\alpha}{2}\left(b - \frac{1}{b}\right) - 1\right]\phi &= 0\end{aligned}\right\}$$

Let us write :

$$-\frac{\alpha}{2}\left(b - \frac{1}{b}\right) = P$$
$$\frac{\alpha}{2}\left(b + \frac{1}{b}\right) = R\,,$$

so/

so that the simultaneous equations become

$$\left.\begin{aligned} (R-k)\,\phi + \left[(q-i)\frac{d}{dq} + 1 - P\right]\chi &= 0 \\ (R+k)\,\chi - \left[(q+i)\frac{d}{dq} + 1 + P\right]\phi &= 0 \end{aligned}\right\}$$

Solving for ϕ and χ by substitution we find the second order differential equation for these functions to be

$$\left.\begin{aligned} \left[(q^2+1)\frac{d^2}{dq^2} + (3q - 2iP - i)\frac{d}{dq} + 1 - s^2\right]\phi &= 0 \\ \left[(q^2+1)\frac{d^2}{dq^2} + (3q - 2iP + i)\frac{d}{dq} + 1 - s^2\right]\chi &= 0 \end{aligned}\right\}$$

where $\quad s = +\sqrt{k^2 - \alpha^2}$

If one puts $\quad x = -iq \quad$ one soon finds that the solution is expressible as a Riemann P-function.

$$\phi = P\left\{\begin{matrix} 1 & -1 & \infty & \\ 0 & 0 & 1+s & x \\ P & -P-1 & 1-s & \end{matrix}\right\}$$

or

$$= P\left\{\begin{matrix} 1 & -1 & 0 & \\ 0 & 0 & 1+s & \frac{1}{x} \\ P & -P-1 & 1-s & \end{matrix}\right\}$$

This is now expressible in terms of a hypergeometric function

$$\phi = \left(\frac{1}{1+x}\right)^{1+s} {}_2F_1\left(1+s,\ s-P;\ 1-P;\ \frac{x-1}{x+1}\right)$$

or $\quad \phi = \left(\frac{1}{1-iq}\right)^{1+s} {}_2F_1\left(1+s \quad s-P;\ 1-P;\ \frac{q+1}{q-1}\right).$

Similarly/

Similarly

$$\chi = C\left(\frac{1}{1-iq}\right)^{1+s} {}_2F_1\left(1+s,\ 1+s-P;\ 2-P;\ \frac{iq+1}{iq-1}\right)$$

When q is real, the argument of the hypergeometric function tends to unity as q increases to infinity. As $(\gamma-\alpha-\beta)$ which here is $-2S$ is negative the series will not be convergent at infinity nor will $\phi^*\phi$. Hence the series must terminate for the probability amplitude to remain finite, and this can happen only if α or β is a negative integer. As s by definition is non integral, our only choice is to make

$$s - P = -n' \qquad \text{where } n' \text{ is an integer}$$

or $$P = s + n' = p,$$

so that one finds directly the fine structure formula as before on solving the equation $\frac{\alpha}{2}\left(b - \frac{1}{b}\right) = -p$ where b is given by equation (5.7)

When $\underline{a}$ is imaginary, that is $W > mc^2$, q will have only imaginary values so that $x = -iq$ is always real. We see that the argument $\frac{x-1}{x+1}$ lies between -1 and $+1$ as x ranges from 0 to ∞. In this case the hypergeometric function is convergent and no restriction is required.

As an alternative form of the solution of the P-function is

$$\left(\frac{1}{1-x}\right)^{1+s} F\left(1+s,\ 1+s+P;\ 2+P;\ \frac{x+1}{x-1}\right)$$

we can apply this in the range x to 0 to $-\infty$ in which case/

case the argument is again within the limits of -1 and +1. Hence for all values of q we obtain a convergent function for $\phi^* \phi$ so that all values of $W > mc^2$ are possible giving a continuous spectrum.

The factor C is found by substitution in the first order equation and it is

$$C = \frac{\sqrt{P^2 + \alpha^2} - k}{P - 1}$$

For the Schrödinger mechanics, Rumer (36) found the eigenfunc-:tions in momentum space. His solution in atomic units for a state with total quantum number n is

$$\psi_n(p) = \frac{e^{i 2n \tan^{-1} np}}{n^2 p^2 + 1}$$

$$= \frac{(1 + inp)^{n-1}}{(1 - inp)^{n+1}}$$

In our notation $p = q \frac{k}{a} = q \frac{\alpha mc}{\sqrt{p^2 + \alpha^2}}$

which when we neglect fine structure gives us

$$p = \frac{q}{n}$$

in atomic units where $m = 1 = \alpha c$

Therefore, the Schrödinger equation gives us the functions

$$\frac{(1+iq)^{n-1}}{(1-iq)^{n+1}}$$

Similarly/

Similarly for the Dirac equation, when we neglect α compared with unity, taking $s = k$ an integer and $P = n$, the functions ϕ and χ can be expressed as

$$\frac{f(q)}{(1-iq)^{n+1}}$$

$$\frac{g(q)}{(1-iq)^{n}}$$ respectively

where f and g are polynominals of degree n-s and n-s-1 respectively. These functions depend not only on the total quantum number n but also on another quantum number $s = k$.

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