Potential Based Prediction Markets

A Machine Learning Perspective

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Abstract

A prediction market is a special type of market which offers trades for securities associated with future states that are observable at a certain time in the future. Recently, prediction markets have shown the promise of being an abstract framework for designing distributed, scalable and self-incentivized machine learning systems which could then apply to large scale problems. However, existing designs of prediction markets are far from achieving such machine learning goal, due to (1) the limited belief modelling power and also (2) an inadequate understanding of the market dynamics. This work is thus motivated by improving and extending current prediction market design in both aspects.

This research is focused on potential based prediction markets, that is, prediction markets that are administered by potential (or cost function) based market makers (PMM). To improve the market's modelling power, we first propose the partially-observable potential based market maker (PoPMM), which generalizes the standard PMM such that it allows securities to be defined and evaluated on future states that are only partially-observable, while also maintaining the key properties of the standard PMM. Next, we complete and extend the theory of generalized exponential families (GEFs), and use GEFs to free the belief models encoded in the PMM/PoPMM from always being in exponential families.

To have a better understanding of the market dynamics and its link to model learning, we discuss the market equilibrium and convergence in two main settings: convergence driven by traders, and convergence driven by the market maker. In the former case, we show that a market-wise objective will emerge from the traders' personal objectives and will be optimized through traders' selfish behaviours in trading. We then draw intimate links between the convergence result to popular algorithms in convex optimization and machine learning. In the latter case, we augment the PMM with an extra belief model and a bid-ask spread, and model the market dynamics as an optimal control problem. This convergence result requires no specific models on traders, and is suitable for understanding the markets involving less controllable traders.

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My deepest gratitude goes to my parents, for their greatest love and for their support in all aspects of my life; and to my wife, for her appearance in my life at the perfect time. I love you.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

(*Jinli Hu*)

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Chapter 1

Introduction

> "It is far better to foresee even without certainty than not to foresee at all."
>
> — Henri Poincaré in *The Foundations of Science*

Respecting the stochastic nature of the world, probabilistic predictions have become, especially in machine learning, a more appropriate way for describing the future than deterministic approaches. During the construction of probabilistic predictors, aggregating information from different algorithms or agents is often preferred or even becomes necessary. This is guided by the intuition that the collective prediction should act better than any individuals and also by many practical evidences (e.g. Netflix challenge, PASCAL challenge, Kaggle). Today, in the era of "big data", this aggregation structure emerges not merely from the requirement of boosting prediction performance, but more importantly, from the interests of building distributed, scalable systems for solving large scaled data-driven or crowdsourcing tasks.

Prediction markets are marketplaces for trading securities whose values depend on some future states, which remain uncertain before being realized at a certain time in the future (Wolfers and Zitzewitz, 2004; Arrow et al., 2008). Recently, prediction markets have shown the promise of being an abstract framework for designing probabilistic belief aggregation systems that are favoured by the big data setting. On the one hand, the association between securities and future states allows beliefs and predictions to be encoded by the market quantities (e.g. prices, traded shares); on the other hand, the market structure

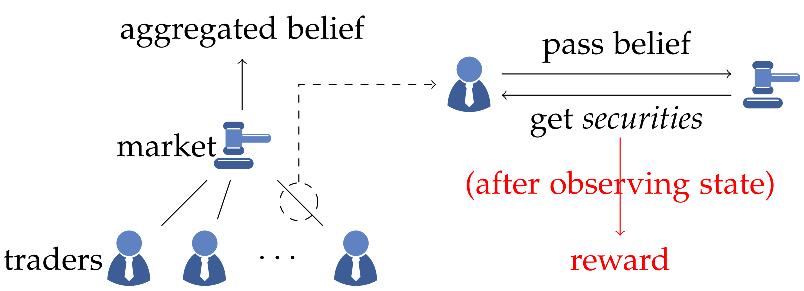

Figure 1.1: The structure of prediction market. The market structure is a natural distributed, large-scale environment where each trader learn their own probabilistic beliefs about the future state and then aggregate their beliefs to market by trading securities. With values dependent on the future state, securities can encode beliefs into trades, and also rewarding traders according to the quality of their beliefs assessed based on the actual observed state.

defines a natural distributed, large-scale environment, and also implements an information aggregation flow through transactions. In addition, the match between the trading reward and the performance of prediction can incentivize agents to submit results of better qualities (Figure 1.1). This environment also provide a way of measuring their skills, which makes prediction markets very suitable for crowdsourcing tasks (Abernethy and Frongillo, 2011).

In fact, prediction markets have been widely applied to public prediction tasks in many fields, such as politics (Iowa Electronic Market, Intrade), sports (TradeSports, STOCCER) and entertainment (Hollywood Stock Exchange). They are also used by companies to help improve the internal information aggression and decision-making process (Cowgill et al., 2009). For example, to predict the winner of the 2012 US presidential election between Obama and Romney, the prediction market defines two Arrow-Debreu (or indicator, winner-take-all) securities, one for each candidate, which will pay the holder $1 if the associated candidate wins and nothing otherwise. Trades in the market finally lead to two market prices that together give a probabilistic prediction for the winner of the election (Figure 1.2).

In most applications, prediction markets are set up with simple (e.g. indicator) securities. These have limited power for more general probabilistic belief aggregation problems. In addition, the instability of the behaviour and performance of prediction markets observed in practice (e.g. Berg et al. (1997);

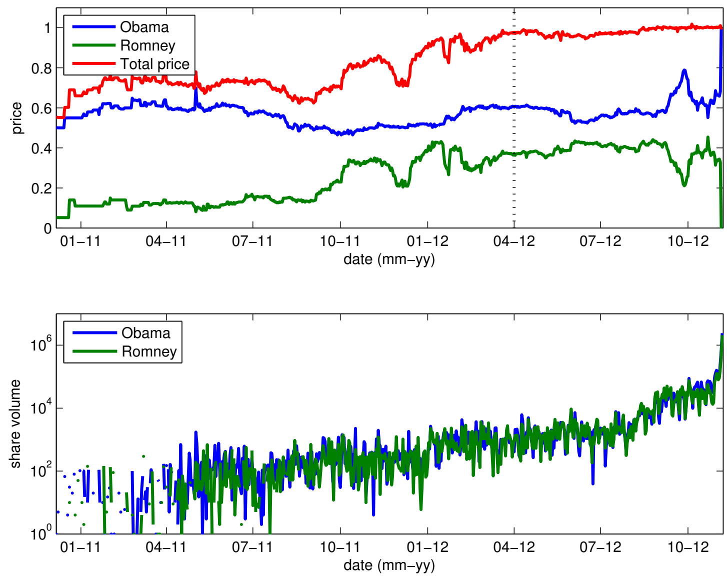

Figure 1.2: The prediction market for 2012 US presidential election, run by Intrade using continuous double auction. Two securities are defined, each pays $1 to the holder if Obama/Romney wins and nothing otherwise. After adequate amount of trades, the market prices sum up to one and start to interpret a Bernoulli distribution after April 2012 (black dotted line).

Luckner et al. (2008); Rieg and Schoder (2010)) is not well explained due to the lack of understanding of the market dynamics, which will prevent the further improvement and application of prediction markets as a machine learning system. These two concerns provide the motivation for our research: to develop prediction markets towards a general probabilistic belief aggregation system, it is necessary to improve their modelling powers and to better understand their dynamics.

1.1 Research scope and objectives

The framework of prediction markets still leaves open many design decisions for basic market structures. The focus in this thesis is on the potential based prediction markets, that is, prediction markets that are run by potential based market makers (also referred to as cost function based market makers) (Chen and Pennock, 2007; Abernethy et al., 2011, 2013). Such choice is based on the following considerations.

- For technical considerations, a market maker will usually introduce a simpler dynamics for trading, making the prediction market analysable, especially when the market is outside of its equilibrium.
- Potential based market makers are of special interest to recent prediction market research, due to their tight relevance to market scoring rules (one of the most popular prediction market making mechanism) and the direct link to machine learning method.

The research goal of the work described in this thesis is to design better potential based prediction markets encoding more powerful probabilistic belief models. These markets should also provide more accurate aggregation processes, and help pave the way for leveraging potential based prediction markets in building distributed, scalable, self-incentivized machine learning systems. To achieve this research goal, two research objectives have been identified.

1. Improve the modelling power of the potential based prediction market; and
2. Obtain deeper and more accurate understanding of prediction market dynamics.

The first objective consists of two aspects. First, the market belief models need to be powerful enough to be able to capture correlations between states in complicated spaces (such as combinatorial or continuous state spaces). In addition, the form of the model should also be flexible enough to express various types of distributions given the same space. One way to explain such complex correlations is through the latent variables, which then asks markets to be able to encode latent variable belief models and to run efficiently on spaces involving latent variables; while the flexibility of the model forms can be increased by characterizing more families of distributions that are encodable to markets. To achieve the second objective, a more complete discussion about the market convergence and equilibrium, as well as a revelation of the related optimization and machine learning topics will be required.

1.2 Thesis structure

Chapter 2 prepares this research by providing the background information and reviews the key related work. Chapter 3 and 4 focuses on increasing the modelling power of the potential based market maker. In particular, in Chapter 3, the partially-observable potential based market maker is proposed, which runs a market on a space of partially-observable states, and expresses the market belief through a probabilistic model containing latent variables. In Chapter 4, the theory of generalized exponential families is developed and applied in the mechanism design, providing more models for representing market beliefs. Chapter 5 analyses the market dynamics in the potential based prediction markets, and present market convergence and equilibrium results in different settings. Links between the market dynamics and many optimization/machine leaning methods are also given. Finally, conclusion is drawn in Chapter 6.

Chapter 2

Background and Literature Review

2.1 Market Setup

Before our work, a prediction market was associated with a future state ω whose value remains uncertain in Ω until it is fully revealed at a certain time in the future. In the prediction market, a security is defined to be a function of the future state, $\phi : \Omega \to \mathbb{R}$. More specifically, a security is characterized by a payment function ϕ such that one unit share of the security will pay $\phi(\omega)$ to the traders who hold it, when the future state turns out to be ω. This security is referred to as *the complex security* (Abernethy et al., 2011, 2013) and generalizes the Arrow-Debreu or the indicator security. An indicator security has a payment function $\phi_i(\omega) = \mathbf{1}_{A_i}(\omega)$, where $\{A_i\}$ are mutually exclusive and collectively exhaustive subsets of Ω. Multiple securities can be defined, which we index by $i = 1, 2, \ldots, K$ and collect into a vector $\boldsymbol{\phi}(\omega) := (\phi_1(\omega), \phi_2(\omega), \ldots, \phi_K(\omega))^\top$. These predefined securities are the objects traders aim to buy and sell.

Each trader i is characterized by the total amount of shares she holds $\boldsymbol{\vartheta}_i \in \mathbb{R}^K$, her budget $w_i \in \mathbb{R}$, and her personal trading preference $f_i(\boldsymbol{\vartheta}_i, w_i)$, a function of her holdings and her budget. The preference function measures the value of holding the share bundle before the realization of the future state.[1] Similarly, each trade made at time t is characterized by the amount of

[1]Here the "value" of the holdings is not necessarily measured by a monetary value. In

transactions δ_t and the cost $c_t(\delta_t)$. For a rational trader, her decision about a trade is obtained by maximizing her preference:

$$\delta_t = \arg\max_{\delta_t} \; f_i(\boldsymbol{\vartheta}_i + \delta_t, w_i - c_t(\delta_t)), \; s.t. \; w_i - c_t(\delta_t) \text{ meets budget constraints.} \tag{2.1}$$

In the thesis, we consider a number of different trader models in different contexts. The simplest model for traders is one of *risk-neutral myopic* traders. Risk-neutral myopic traders trade in each round to maximize the expected profit of the holdings w.r.t. her private belief over the future state p_i

$$\delta_t = \arg\max_{\delta_t} \; \delta_t^\top \mathbb{E}_{p_i}[\boldsymbol{\phi}] - c_t(\delta_t), \; s.t. \text{ budget constraints} \tag{2.2}$$

Note that all terms that are constants w.r.t. the t-th trade have been ignored. Risk-neutral myopic traders are the standard trader model for analysing the potential based market mechanisms (Hanson, 2007; Chen and Pennock, 2007; Abernethy et al., 2011). They are also involved in building up the convergence results driven by the market makers. In addition, we will also use risk-averse traders and niche traders when studying other types of convergence, in Chapter 5. These types of traders are introduced in (Storkey, 2011).

2.2 Prediction markets with market makers

A market maker is a special trader who is always willing to accept a trade offer from any trader as long as the trader agrees to pay according to the market maker's pricing. The motivation for using market maker was given in Chapter 1. One rationale is that, with a market maker, the prediction market will have a simpler dynamics outside of its equilibrium. Here, I will explain from intuition why a market maker can potentially simplify the trading dynamics.

Suppose a market maker set the price for security k to p_k. Let b_k be the highest price per share for buying k that traders can offer, and a_t the lowest price per share for selling k. Then it must hold that $b_k \leq p_k$, otherwise the trader who wants to buy at b_k will end up with spending less money if she trades

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fact, if the preference function is introduced based on the expected utility theory (EUT), the value simply measures the degree of satisfaction of holding the corresponding shares. For detailed discussions see Chapter 5.

directly with the market maker. Similarly, $p_k \leq a_k$. As the result, the market maker's unit price is always lower than lowest selling price among traders and is also always higher than the highest price, making traders more willing to interact with the market maker before the other traders. Since every trader prefers to trade with the market maker, the position of the market maker then indicates the state of the whole market, and the change of positions via trades characterizes the market dynamics. The market maker hence provides an easy interface for describing a prediction market out side of its equilibrium.

Note that the above intuitively discussion holds for transactions with infinitesimal shares that do not change the price before and after the trade. For discussion about trading finite shares of securities we refer to Chakraborty et al. (2015).

2.2.1 Potential based market maker (PMM)

A potential based market maker (PMM), first introduced by Chen and Pennock (2007), is a market maker who has a potential function F, which is a convex differentiable function whose effective domain $\boldsymbol{\Theta} \subseteq \mathbb{R}^K$ consists of all possible amount of shares that can be *sold* to the traders, such that a trade of $\boldsymbol{\delta}_t := \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ shares, $\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1} \in \boldsymbol{\Theta}$ is priced by the difference of F

$$c_t(\boldsymbol{\delta}_t) := F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1}). \tag{2.3}$$

The cost function is not an inner product of traded shares and the unit prices of securities, implying that the prices of security is varying during the trade. However, one can define the (instantaneous unit) price of each security at each position $\boldsymbol{\theta}$ based on an infinitesimal trade:

$$p_k(\boldsymbol{\theta}) := \lim_{\delta_k \to 0} \frac{F(\boldsymbol{\theta} + \delta_k \mathbf{e}_k) - F(\boldsymbol{\theta})}{\delta_k} = \frac{\partial}{\partial \theta_k} F(\boldsymbol{\theta}). \tag{2.4}$$

Therefore, the prices of all securities give the gradient of the potential function. They form a conservative potential field, which guarantees that any trades that begin with the same pre-trade positions and end with the same post-trade positions will have the same cost (path-independence). The term *potential* results from the fact that F is a potential field on $\boldsymbol{\Theta}$. All prices are collected into the price vector $\mathbf{p} := \{p_1, \ldots, p_K\}^\top$.

The PMM emerges naturally from an axiomatic approach. In fact, Abernethy et al. (2011, 2013) show that a PMM is the market maker that meets (1) path-independence; (2) existence of instantaneous prices (implying differentiable); (3) information incorporation, that for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and δ such that $\boldsymbol{\theta} - \delta, \boldsymbol{\theta} + \delta \in \boldsymbol{\Theta}$, $F(\boldsymbol{\theta} + \delta) - F(\boldsymbol{\theta}) \geq F(\boldsymbol{\theta}) - F(\boldsymbol{\theta} - \delta)$; (4) no arbitrage, that for any $\boldsymbol{\theta}, \boldsymbol{\theta} + \delta \in \boldsymbol{\Theta}$ and $\omega \in \Omega$, $F(\boldsymbol{\theta} + \delta) - F(\boldsymbol{\theta}) \geq \delta^\top \boldsymbol{\phi}(\omega)$; and (5) expressiveness, that for any probability over ω, there exists a $\boldsymbol{\theta}$ such that $\nabla F(\boldsymbol{\theta}) = \mathbb{E}_p[\boldsymbol{\phi}]$. All of these axioms are proposed based on the intuitive requirements for making a good market maker.

Typically, a PMM will subsidize the market and end up with losing money (in return for better market prices gathered from traders). Fortunately, the total amount of money that PMM may lose is proved to be bounded from above (Chen and Pennock, 2007; Abernethy et al., 2011). The idea that stays at the centre of the analysis is the so-called *Bregman divergence*. The Bregman divergence from $\mathbf{u}$ to $\mathbf{v}$ w.r.t. a convex differentiable function f,$D_f(\mathbf{u}, \mathbf{v})$, is defined by the value difference at $\mathbf{u}$ between f and the tangent line of f at $\mathbf{v}$ (Bregman, 1967; Banerjee et al., 2005)

$$D_f(\mathbf{u}, \mathbf{v}) := f(\mathbf{u}) - f(\mathbf{v}) - (\mathbf{u} - \mathbf{v})^\top \nabla f(\mathbf{v}). \tag{2.5}$$

The geometric interpretation of the Bregman divergence is shown in Figure 2.1. For strictly convex f, the induced Bregman divergence $D_f(\mathbf{u}, \mathbf{v}) \geq 0$ and the equality holds if and only if $\mathbf{u} = \mathbf{v}$. Therefore, a Bregman divergence can be thought of as some distance measure between its two input arguments, despite the fact that it is asymmetric $D_f(\mathbf{u}, \mathbf{v}) \neq D_f(\mathbf{v}, \mathbf{u})$ in general.

The Bregman divergence can be written in a primal-dual form. Denote f^* the convex conjugate of f

$$f^*(\mathbf{x}^*) := \sup_{\mathbf{x} \in \mathcal{X}} \langle \mathbf{x}^*, \mathbf{x} \rangle - f(\mathbf{x}). \tag{2.6}$$

Since here f is assumed to be differentiable at $\mathbf{x}$, the supremum of the above equation is achieved at $\mathbf{x}^* = \nabla f(\mathbf{x})$, and so $f^*(\mathbf{x}^*) = \mathbf{x}^\top \nabla f(\mathbf{x}) - f(\mathbf{x})$. Substituting them back to (2.5), one has

$$\begin{aligned} D_f(\mathbf{u}, \mathbf{v}) &= f(\mathbf{u}) - f(\mathbf{v}) - (\mathbf{u} - \mathbf{v})^\top \nabla f(\mathbf{v}) \\ &= f(\mathbf{u}) + (\mathbf{v}^\top \nabla f(\mathbf{v}) - f(\mathbf{v})) - \mathbf{u}^\top \nabla f(\mathbf{v}) \\ &= f(\mathbf{u}) + f^*(\mathbf{v}^*) - \langle \mathbf{u}, \mathbf{v}^* \rangle. \end{aligned} \tag{2.7}$$

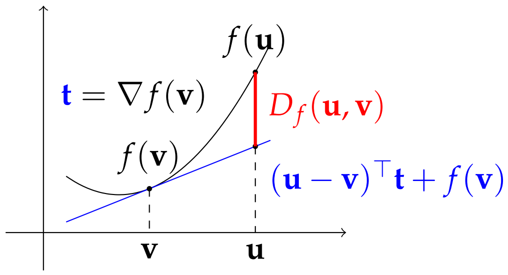


Figure 2.1: Bergman divergence from $\mathbf{u}$ to $\mathbf{v}$ w.r.t. f.

From this new form one can easily observe that duality exists between $D_f(\cdot, \cdot)$ and $D_{f^*}(\cdot, \cdot)$. More specifically, applying $f^{**} = f$ and $\mathbf{x}^{**} = \mathbf{x}$

$$\begin{aligned} D_f(\mathbf{u}, \mathbf{v}) &= f(\mathbf{u}) + f^*(\mathbf{v}^*) - \langle \mathbf{u}, \mathbf{v}^* \rangle \\ &= f^*(\mathbf{v}^*) + f^{**}(\mathbf{u}^{**}) - \langle \mathbf{v}^*, \mathbf{u}^{**} \rangle \\ &= D_{f^*}(\mathbf{v}^*, \mathbf{u}^*) \end{aligned} \tag{2.8}$$

It turns out that the total loss of the PMM can be written in Bregman divergences. By definition, given observed state ω, the total loss of the PMM is equal to the total profit obtained by traders, that is

$$\begin{aligned} M_T(\omega) &= \sum_{t=1}^{T} \boldsymbol{\delta}_t^\top \boldsymbol{\phi}(\omega) - c_t(\boldsymbol{\delta}_t) = (\boldsymbol{\theta}_T - \boldsymbol{\theta}_0)^\top \boldsymbol{\phi}(\omega) - (F(\boldsymbol{\theta}_T) - F(\boldsymbol{\theta}_0)) \\ &= \Big(F^*(\boldsymbol{\phi}(\omega)) + F(\boldsymbol{\theta}_0) - \boldsymbol{\theta}_0^\top \boldsymbol{\phi}(\omega) \Big) - \Big(F^*(\boldsymbol{\phi}(\omega)) + F(\boldsymbol{\theta}_T) - \boldsymbol{\theta}_T^\top \boldsymbol{\phi}(\omega) \Big) \\ &= D_{F^*}(\boldsymbol{\phi}(\omega), \mathbf{p}(\boldsymbol{\theta}_0)) - D_{F^*}(\boldsymbol{\phi}(\omega), \mathbf{p}(\boldsymbol{\theta}_T)) \leq D_{F^*}(\boldsymbol{\phi}(\omega), \mathbf{p}(\boldsymbol{\theta}_0)). \end{aligned} \tag{2.9}$$

Here F^* is the convex conjugate of the potential F, $\mathbf{p}(\boldsymbol{\theta}) = \nabla F(\boldsymbol{\theta}) = \boldsymbol{\theta}^*$ denotes the dual of $\boldsymbol{\theta}$, and in the third equation $F^*(\boldsymbol{\phi}(\omega))$ is added and subtracted to construct the two Bregman divergences. From (2.9), if the distance between the market initial prices $\mathbf{p}(\boldsymbol{\theta}_0)$ and the actual values of the securities $\boldsymbol{\phi}(\omega)$ is uniformly bounded, i.e. there exists a $M > 0$ such that

$$D_{F^*}(\boldsymbol{\phi}(\omega), \mathbf{p}(\boldsymbol{\theta}_0)) \leq M^2, \ \forall \omega \in \Omega, \tag{2.10}$$

then the market maker's loss will always be bounded from above. It is worth noting that this bound will not depend on any intermediate market states or the particular sequence of trade.

In addition, a PMM is incentive-compatible w.r.t. a risk-neutral myopic trader, in the sense that the post-trade market prices reflect precisely the expected values of the securities w.r.t. the trader's belief. Given a risk-neutral myopic trader with belief p_i sufficient budget, by (2.2) her maximum expected profit is

$$\begin{aligned}\max_{\delta_t} \mathbb{E}_{p_i}[\delta_t^\top \boldsymbol{\phi}(\omega) - c_t(\delta_t)] &= \max_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbb{E}_{p_i}[\boldsymbol{\phi}] - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \\ &= \max_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} \left(F^*(\mathbf{p}_i) + F(\boldsymbol{\theta}_{t-1}) - \boldsymbol{\theta}_{t-1}^\top \mathbf{p}_i\right) - \left(F^*(\mathbf{p}_i) + F(\boldsymbol{\theta}_t) - \boldsymbol{\theta}_t^\top \mathbf{p}_i\right) \\ &= \max_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} D_{F^*}(\mathbf{p}_i, \mathbf{p}(\boldsymbol{\theta}_{t-1})) - D_{F^*}(\mathbf{p}_i, \mathbf{p}(\boldsymbol{\theta}_t)) \\ &= D_{F^*}(\mathbf{p}_i, \mathbf{p}(\boldsymbol{\theta}_{t-1})) - \min_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} D_{F^*}(\mathbf{p}_i, \mathbf{p}(\boldsymbol{\theta}_t)), \end{aligned} \tag{2.11}$$

where $\mathbf{p}_i := \mathbb{E}_{p_i}[\boldsymbol{\phi}]$. Similar to how we derive (2.9), the term $F^*(\mathbf{p}_i)$ is added and then subtracted to construct two Bregman divergences. The last equation is given by the fact that the minimum is taken w.r.t. $\boldsymbol{\theta}_t$, and thus not affected by the first Bregman term which is only involved with $\boldsymbol{\theta}_t$. Therefore, the expected profit is maximized at $\mathbb{E}_{p_i}[\boldsymbol{\phi}] = \nabla F(\boldsymbol{\theta}_t)$.

2.2.2 Market scoring rules

A scoring rule assesses the quality of probabilistic forecasts by assigning a numerical score based on the submitted belief and the realized future state. It was first introduced by Brier (1950) for verifying the quality of probabilistic weather forecasts. Hanson (2007) uses scoring rules to develop a class of prediction market maker named the *market scoring rules* (MSRs).

Consider the space of future states Ω and a set of probabilities $\mathcal{P}$ on it. A scoring rule is a function $S : \Omega \times \mathcal{P} \to \mathbb{R}$, such that if a probabilistic belief $p \in \mathcal{P}$ is submitted, $S(\omega, p)$ is the score of p that evaluates the quality of the belief p. Note that $S(\omega, p)$ is uncertain before observing the state. A scoring rule is said *proper* if the expected score w.r.t. any distribution of the future state, $p^* \in \mathcal{P}$, is maximized when the submitted belief $p = p^*$. In other words,

$$S(p^*, p) := \int S(\omega, p) p^*(\omega) \mathrm{d}\omega \leq S(p^*, p^*), \ \forall p, p^* \in \mathcal{P}. \tag{2.12}$$

Popular scoring scoring rules/functions include: logarithmic score $S(\omega) = \log p(\omega)$; quadratic (Brier) score $S(\omega, p) = 2p(\omega) - \|p(\omega)\|_2^2$; and spherical score $S(\omega, p) = p(\omega)/\|p\|_2$. Among them, the log scoring rule is most commonly used due to its simplicity. First, it is the only *local* scoring rule whose score will depend only on the probability of the realized state (Bernardo, 1979). As a comparison, both quadratic and spherical scores involve the norm of probabilities. In addition, when extending to the continuous space, the log scoring rule does not suffer from complicated measure issues and still remains the same form as in the finite discrete space (Gneiting and Raftery, 2007).

Given a proper scoring rule S, and denote p_t the *market belief* over the target future state, maintained by the market maker. Then a market scoring rule (MSR) S_t is defined such that the agent who modifies the market belief from p_{t-1} to p_t will gain the profit

$$S_t(\omega, p_t) := S(\omega, p_t) - S(\omega, p_{t-1}). \tag{2.13}$$

That is, the profit is given by the improvement in the score from the previous submission. Same as a PMM, a MSR will typically subsidize a market but its loss will be bounded, and a MSR is incentive-compatible w.r.t. risk-neutral myopic traders. The former follows from the fact that the MSR pays to each agent based on the relative improvement of the market belief rather than the absolute quality. As a consequence, the total loss depends only on the initial and final market beliefs

$$\sum_{t=1}^{T} S_t(\omega, p_t) = \sum_{t=1}^{T} S(\omega, p_t) - S(\omega, p_{t-1}) = S(\omega, p_T) - S(\omega, p_0). \tag{2.14}$$

The latter results from the definition of a proper scoring rule (2.12).

The MSR introduces a market belief, and links the profit directly to the quality of the market belief measured by the scores. At the first glance, the MSR seems set up a different environment from a prediction market, since no securities are defined and no trades are involved. However, it turns out that a MSR, in particular a logarithmic MSR (LMSR) can be implemented by a potential based market maker through the exponential family (Chen and Pennock, 2007; Abernethy et al., 2014). Consider an exponential family $\mathcal{P}_{\boldsymbol{\Theta}}$ where $\boldsymbol{\Theta}$ is the parameter domain. Then the density of the distribution $p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}$ has the form

$$p_{\boldsymbol{\theta}}(\omega) = \frac{1}{Z(\boldsymbol{\theta})} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega)\right) \nu(\omega). \tag{2.15}$$

Then the corresponding LMSR is

$$S_t(\omega, p_t) = \log p_{\boldsymbol{\theta}_t}(\omega) - \log p_{\boldsymbol{\theta}_{t-1}}(\omega) = \boldsymbol{\delta}_t^\top \boldsymbol{\phi}(\omega) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})). \quad (2.16)$$

That is, if we define securities to be the sufficient statistics $\boldsymbol{\phi}$ involved in the exponential family, and set up a PMM with the potential function $F(\boldsymbol{\theta}) = \log Z(\boldsymbol{\theta})$, then the profit of the t-th trade will be precisely to the LMSR profit at time t, and so this PMM is also a LMSR. This connection also explains how a PMM encodes the market belief into the securities and its potential function. However, the market belief is restricted to exponential family.

2.2.3 Other designs of market makers

It is worth noting that PMMs and MSRs are not the only designs of prediction market makers. Other methods include dynamic parimutuel mechanisms (DPMs) of Pennock (2004) and Bayesian market makers (BMMs) of Das and Magdon-Ismail (2009). A DPM has a continuous cost function like a PMM and also prices each trade by the difference of the cost function. Different from a PMM, the final payout of securities is determined by the parimutuel rule which redistributes the money invested by traders among the winning shares, rather than the predefined payment functions. Therefore, no subsidy is required for running a DPM.[2] The drawbacks of DPMs are (1) that its the cost function is more expensive to compute, and (2) it is difficult to extend a DPM to large state spaces. The BMM is a market maker who mantains a belief about the *true prices* of the securities via Bayesian updates, and prices each security by utilizing this belief. In Chapter 5, we show that BMM can be thought of as an augmented version of the PMM, although they were viewed as two different models (Brahma et al., 2012).

2.3 Analysing prediction markets

Apart from market maker design, another central task in prediction markets is to analyse market dynamics.

[2]In fact, the dynamic parimutuel mechanism needs extra money to start the market, but this amount of money can be arbitrarily small.

Interestingly, compared to work on prediction market maker design which has started booming just recently, the equilibrium analysis for prediction markets can be dated back to Eisenberg and Gale (1959), and different discussions have been done from various aspects by Pennock and Wellman (1996, 1997, 1998); Storkey (2011); Barbu and Lay (2012); Storkey et al. (2012, 2015).[3] These discussions are all based on the generic market setup and do not involve any specific market mechanisms. The results show that the prediction market at the equilibrium state can be viewed as a belief aggregation model with different aggregation structures depending on the traders' behaviours. These structures include mixture (or average, or linear pooling) and product (or log pooling). Storkey (2011); Storkey et al. (2012, 2015) finally provide a unified view of these models. The authors formalize the traders' behaviours by modelling traders as expected utility maximizers, and derive a spectrum of belief aggregation model by varying continuously the form of the utility function. Note that, however, these results focus only on the market equilibrium, and do not involve the dynamics towards the equilibrium.

The first trial to explicitly link the PMM driven dynamics to machine learning is given by Chen and Wortman Vaughan (2010). In this work, the authors equate Arrow-Debreu securities to the experts in the online learning setting, prices of securities to the weights on the experts, and total sold shares of each security to the losses in each learning step. They show that the PMM is equivalent to the *Follow the Regularized Leader* (FtRL) algorithm. It is also worth mentioning that the analysis does not make any assumption about the trader's behaviour. The result is a bit less intuitive as securities instead of traders are matched to the learning experts.

Frongillo et al. (2012) shows that adding a PMM will not affect the equilibrium result based on risk-averse traders. More importantly, the interaction between a PMM and a set of stochastic risk-averse traders will approximately implement *stochastic mirror descent* (SMD). Premachandra and Reid (2013) introduce a modified PMM which fixed the prices to the pre-trade market prices for each trade, allowing the match between PMM and SMD to become exact. However, this modification exposes the PMM to arbitrage opportunities, since for any

[3]Admittedly, the formal concept of prediction markets does not exist in early work, but is replaced a similar concept such as a parimutuel betting market.

sequence of "closed" trades that move position $\boldsymbol{\theta}_0 \to \boldsymbol{\theta}_1 \to \cdots \to \boldsymbol{\theta}_T = \boldsymbol{\theta}_0$, the cost

$$\sum_{t=1}^{T} (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbf{p}(\boldsymbol{\theta}_t) \leq \oint F(\boldsymbol{\theta}) = 0. \tag{2.17}$$

Hence such PMM will not be realistic. A question remains if the PMM dynamics with risk-averse traders can match any learning process in an *exact* sense.

Another concern for the existing trader driven convergence results is that, they suggest using expected utility to model traders' preferences, but due to high computational complexity, behaviours under expected utilities cannot be solved or analysed except for simple cases (Sethi and Vaughan, 2013; Chakraborty and Das, 2015), and will prohibit deriving results in general circumstances.

2.4 Summary

In this chapter, necessary background was provided and related work has been reviewed. In particular, we reviewed the PMM defined on states that are fully observable, showing its properties and how it represents a market belief model by viewing it as an implementation of the LMSR. We found that the belief represented is currently restricted to the fully-observable state space and also to the exponential family, which motivates our work in Chapter 3 and 4. The existing market analysis results all imply a connection between market equilibria and belief aggregation models in probabilistic modelling, and a connection between market dynamics towards the equilibrium and the model learning process. However, these discussions are either incomplete or based on inappropriate approximations of the true potential based mechanism, thus motivating our analysis in Chapter 5.

Chapter 3

Partially-observable State Spaces

In the conventional design, a prediction market works on a space of possible future states Ω where the true state $\omega \in \Omega$ can eventually be determined. The observability of the true future state is fundamental to the conventional market design, since it guarantees that the values of all securities will finally become deterministic, and further, that all payouts promised by the purchased shares can really be executed. This condition provides the basic incentive for traders: certainly no one is willing to trade securities giving empty promises.

However, such state spaces (i.e. the spaces requiring the observability of the true future state) are restrictive in representing the unknown future states. Instead, what we commonly see is that the true state can only be determined up to a subset A in the space, that is, $\omega \in A \subseteq \Omega$. This inexact determination of the true state can be caused by various reasons that often emerge in the daily life, such as the limit budget of perform the accurate observation or the stochastic nature of the future state itself. We name a state space that allows the inexact determination of the true state a *partially-observable state space*, since we do get a better idea of the true state than before after we make the observation. Comparatively, we name a state space that requires the exact determination of the true state a *fully-observable state space*. As will be shown more formally later, a fully-observable state space can be characterized by a realizable random variable v, while a partially-observable state space can be characterized by a pair of random variables (v, h) with only v being realizable.[1] In machine learning language, v is called the visible/observable variable and h is the hidden/latent variable.

To have a concrete example of the partially-observable state space, let's consider a revised version of the *Spot-the-Ball* game (Figure 3.1). In a standard Spot-the-Ball game, people are presented with an image, and asked to place bet on the position of the football; the true position of the ball is then realized by the true image at the end of the game and each one is rewarded according to the accuracy of her bet. The standard Spot-the-Ball game defines a fully-observable state space: the true state of the ball is guaranteed to be exactly determined by the rule of the game. In the revised version of the game, called *Who-Touched-the-Ball* game, people are presented with the same image, but instead they are asked to bet on who last touched the ball and where the ball is given the player who touched it. The Who-Touched-the-Ball defines a partially-observable state space: each state consists of the player who last touched the ball and the ball position, and it cannot be determined exactly as we cannot read the true player directly from the true image. In particular, the true image only reveals the true position of the ball $\mathbf{x}$. For a Who-Touched-the-Ball game this could only determine the true state up to the set $\{$(player Gerrard with ball at $\mathbf{x}$, player Van Persie with ball at $\mathbf{x}$)$\}$.

The partially-observable state spaces are beyond the conventional setting of the prediction market design. In this chapter, we extend the conventional market design to allow prediction markets to work on partially-observable state spaces. Our solution is based on the potential-based market maker (PMM) but it is a non-trivial extension. We name our mechanism the *partially-observable potential-based market maker* (PoPMM).

3.1 Motivation

Although partially-observable state spaces are more commonly seen than the fully-observable state spaces and they provide better representations for the unknown future events, the doubt often comes out that whether the partially-observable state space is really necessary for prediction market design. Here

[1] Here the term "random variable" can be a variable or a vector in the normal sense.

Figure 3.1: A *Spot-the-Ball/Who-Touched-the-Ball* game (left) and the true answer (right). In the standard Spot-the-Ball game, only the position of the ball is asked; in the Who-Touched-the-Ball game, both the player who last touched the ball and the position of the ball are asked. From the true image we can only read off the position of the ball. The state space of a Spot-the-Ball game is a fully-observable state space. Comparatively, the state space of a Who-Touched-the-Ball game is a partially-observable state space: the true image reveals only the true position of the ball $\mathbf{x}$, which determines the true state up to the set $\{$(player Gerrard with ball at $\mathbf{x}$, player Van Persie with ball at $\mathbf{x}$)$\}$. (Figure credits: *Spot the ball Round 1* at `nytimes.com`)

the work on partially-observable state spaces is motivated and justified.

Compared to the fully-observable state space, the partially-observable state space provides a more interesting setting for running prediction markets. Benefit comes from the following aspects:

More straightforward and flexible in market design One criticism about the partially-observable state space is that, given a partially-observable space one can always introduce a fully-observable space with a redefined meaning of the future states. For example, consider a partially-observable space Ω in which we can only tell if the true state is either in the subset A_0 or its complement $A_1 := A^c$ after the observation. Then the space $I = \{0, 1\}$ with the state redefined to be the index i is a fully-observable state space. The conventional design of the prediction market can thus be applied on I, which avoids involving the partially-observable state space.

However, such workaround does not set up prediction markets on the state space we care about, and cannot necessarily extract the desired information from the market. In fact, all beliefs extracted from these markets are for the space I, not for Ω. Comparatively, prediction markets built directly on partially-observable spaces lead to a more straightforward design, and also allow us to obtain the information about the space we care.

In the Who-Touched-the-Ball game example (Figure 3.1), the true state could only be determined up to the set $\{($player Gerrard with ball at $\mathbf{x}$, player Van Persie with ball at $\mathbf{x})\}$. Each set is indexed by the ball position $i = \mathbf{x}$, thus the index space I, which contains all possible ball positions, is a fully-observable state space. This space I coincides with the state space of the standard Spot-the-Ball game. Therefore, when running a prediction market on I we are in essence running a Spot-the-Ball game, which is only about betting on the ball position. On the other hand, by running markets on the joint space we could extract the belief about who touched the ball, in addition to the ball position.

Using a partially-observable space also improves the design flexibility. To infer beliefs for any variables whose observations are expensive or impossible to obtain, we can construct a partially-observable space from them and their relevant variables with cheap observations, and run prediction markets on that space.

More effective in belief modelling The partially-observable space contains more correlations between states than the fully-observable space derived from it. This allows a prediction market to define more interesting belief models with fewer parameters/securities. Also, complicated correlations in the marginal fully-observable space I can potentially be replaced by simple correlations in Ω. For example, a prediction market can define a belief model that is an exponential family on the partially-observable space Ω but has a non-trivial marginal (such as multi-modal) on the derived fully-observable space I.

Again consider the Who-Touched-the-Ball game. The ball may end up with rather different positions conditioned on which player last touched the ball. It implies that the ball position follows a multi-modal distribution, which is difficult for the market's belief model to capture if the market is built on the derived fully-observable space I. One way to capture the belief is to divide the whole canvas into many small patches and associated with each patch an indicator security. Here numerous securities will be needed, but even with this setup the ball position could only be modelled down to a tiny area instead

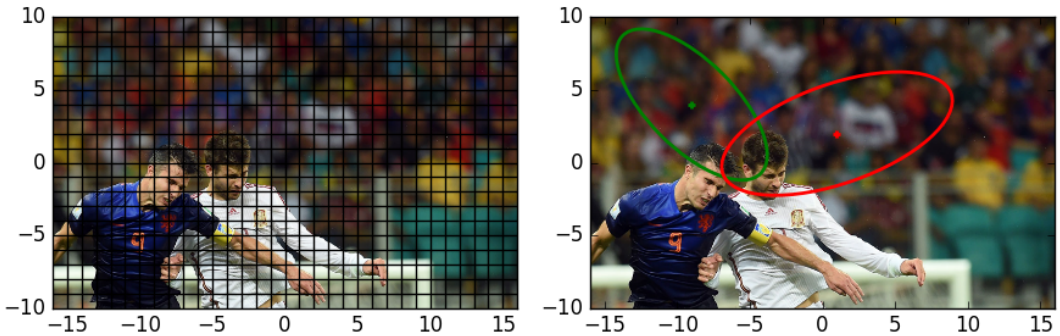


Figure 3.2: Effective belief modelling for the *Who-Touched-the-Ball* game. Left: belief model built on the derived fully-observable space. In order to capture the potential multi-modality, the market has to split the canvas into various tiny pieces and associate each one with an indicator security. Despite the large number of securities, ball positions are always described by areas instead of exact positions on the canvas. Right: belief model built directly on the partially-observable state space that expresses a mixture of two Gaussians. The model needs only 16 securities and can output exact ball positions.

of an exact position. If on the other hand the prediction market is built on the state space Ω, the market can define its belief model using a mixture of two Gaussians, with no more than fourteen securities: six securities for a Gaussian component, and two securities for the binary mixing (Figure 3.2).

Obtaining unified framework for free A fully-observable state space can be understood as a special case of a partially-observable state space whose observed subsets always contain a single state, i.e. $A = \{\omega\}$. As a consequence, prediction markets that are originally designed for partially-observable state spaces can also work on fully-observable spaces. Therefore, by building prediction markets on partially-observable state spaces we will obtain a unified market framework for free.

Bridging the gap The problem of eliciting beliefs on partially-observable state spaces sits in between the problem of eliciting beliefs on fully-observable state spaces, and the problem of eliciting beliefs on spaces with no observations. The former problem can be achieved by the conventional design of prediction markets while the latter can be achieved by more game-theoretic based methods such as peer prediction (Miller et al., 2005) and Bayesian truth serum (Prelec, 2004). Previously, prediction markets are thought very different from either peer prediction or Bayesian truth serum. However, discussing the market design in this new setting with partially-observable state space may shed light on the connections between these methods.

3.2 The setting of partially-observable state space

Consider a state space Ω which contains all possible states of the future we care about. These states are mutually exclusive and collectively exhaustive, that is, each $\omega \in \Omega$ represents a unique state of the future and the true state must fall in Ω. At a definite future time, an observation will be made that tries to determine the true state. Then Ω is a *partially-observable state space*, if the observation can only determine that the state ω is in one of the mutually exclusive and collectively exhaustive *subsets* of Ω: $A(\omega) \in \{A_1, A_2, \ldots, A_I\}$ where $A_i \cap A_j = \emptyset, \forall i, j \in I, i \neq j$ and $\bigcup_{i \in I} A_i = \Omega$ with I the index set of these subsets.

Requiring subsets $\{A_i\}_{i \in I}$ to be collectively exhaustive is to ensure that every possible state can be observed, while the mutual exclusiveness guarantees that the true state is determinable up to these subsets. If otherwise two subsets $A_i \cap A_j \neq \emptyset$ which allows the true state ω to be found in these two subsets at the same time, we can always split them into three subsets, $A_{ij} := A_i \cap A_j, A'_i = A_i - A_{ij}, A'_j = A_j - A_{ij}$ that are mutually exclusive. These are the actual subsets up to which the true state can be determined by the observation.

The property of subsets $\{A_i\}_{i \in I}$ allow themselves to be represented by a random variable/vector. This random variable, denote by v, is a function $\mathsf{v} : \Omega \to \mathcal{V}$ such that states in the same subset will be mapped to the same value, while those in different subsets will be mapped to different values. More accurately, this random variable v is defined on the probability space $(\Omega, \mathcal{A}, \mathsf{P})$ where $\mathcal{A}$ is the σ-algebra generated from the subsets $\{A_i\}_{i \in I}$, and P is the unknown distribution of the state in nature. Given a realization v of v, the true state is then determined up to the set $\{\omega \in \Omega \mid \mathsf{v}(\omega) = v\}$.

Denote B_l the set formed by the l-th elements in each $A_i, i \in I$ (if A_i has less than m elements then restart from the first). The subsets $\{B_l\}_{l \in L}$ have several properties. First, given an observation that the true state is in A_i, it still remains *uncertain* which B_l the true state belongs to since by construction each B_l contains one element from each A_i. Second, if we know the true state is in both A_i and B_l, then we can determine the state exactly by $A_i \cap B_l$. Finally, similar to $\{A_i\}_{i \in I}$ subsets $\{B_l\}_{l \in L}$ are mutual exclusive and collectively exhaustive. Therefore, they can also be represented by a random variable $\mathsf{h} : \Omega \rightarrow \mathcal{H}$.

The properties together with the definitions of v and h imply that a partially-observable state space can be represented by a pair of random variables (v, h). Each state of the space is uniquely represented by a pair of realizations (v, h) of (v, h), but only v is realizable by the observation. In machine learning language, v is called the visible/observable variable and h is called the hidden/latent variable. This gives an alternative interpretation of the term "partially-observable". Compared to subsets $\{A_i\}_{i \in I}$, the random variable pair (v, h) gives a simpler representation of the partially-observable state space. It is also easier for people to understand a partially-observable space, and constructing an interesting space can also be achieved by simply concatenating random variables without working out the underlying subsets (σ-algebras).

Reduction to the fully-observable state space When every set in $\{A_i\}_{i \in I}$ contains a single state, the partially-observable state space reduces to a fully-observable space. It follows that the subsets $\{B_l\}_{l \in L}$ has only one set B_1 that is equal to Ω. Then, the random variable h that represents $\{B_l\}_{l \in L}$ is actually a constant function, as all possible states belong to B_1 and will be mapped to the same value. Therefore, a fully-observable state space is represented by (v, h) where h is a constant. Since the constant function provides no information, we can also ignore h and represent the space using only v.

3.3 Partially-observable potential-based market maker (PoPMM)

The partially-observable potential-based market maker (PoPMM) extends the conventional potential-based market (PMM) to partially-observable spaces. In the rest of this section, we first review the key structure of the PMM, especially the PMM that defines its belief model via an exponential family, and then point out its deficiency on partially-observable spaces. Next, we discuss the equivalence between the market prices and expected security values, and introduce the *conditional market price*. Finally, we propose our mechanism, PoPMM, which is constructed from the conventional PMM and the conditional market prices.

3.3.1 PMM with exponential family belief model

Let (v, h) be a state space. Define K securities. Each security is a function $\phi_k : \mathcal{V} \times \mathcal{H} \to \mathbb{R}$ and they are collected into a vector $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_K)^\top$. Here we assumed that the K securities are defined such that the partition function

$$Z(\boldsymbol{\theta}) = \int_{v \in \mathcal{V}} \int_{h \times \mathcal{H}} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(v,h)\right) \nu(v,h) \mathrm{d}v \mathrm{d}h < +\infty \tag{3.1}$$

for every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, an open set in $\mathbb{R}^K$, where $\nu(\cdot)$ represents a certain base measure. Then $\boldsymbol{\phi}$ and $\boldsymbol{\Theta}$ together define an exponential family $\mathcal{P}_{\boldsymbol{\Theta}}$ in which each distribution $p_{\boldsymbol{\theta}}$ has density function

$$p_{\boldsymbol{\theta}}(v,h) = \frac{1}{Z(\boldsymbol{\theta})} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(v,h)\right) \nu(v,h), \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}. \tag{3.2}$$

Meanwhile, the log partition function $F(\boldsymbol{\theta}) := \log Z(\boldsymbol{\theta})$ is convex. Therefore $\boldsymbol{\phi}, \boldsymbol{\theta}$ and F together define a PMM who offers securities $\boldsymbol{\phi}$, sets prices based on the potential F, and has all possible positions (i.e. sold shares) given by $\boldsymbol{\Theta}$.

To show that the exponential family $\mathcal{P}_{\boldsymbol{\Theta}}$ is the belief model the PMM implicitly defines, we rewrite the PMM as a logarithmic market scoring rule (LMSR). An LMSR is a market maker who asks traders to *directly* submit a probability distribution and, after the true state is realized, pays each submission a profit equal to the increase in the log score of the distribution w.r.t. the previous one. If all traders' beliefs are restricted to $\mathcal{P}_{\boldsymbol{\Theta}}$, then the profit is given by

$$m_t(v,h) = \log p_{\boldsymbol{\theta}_t}(v,h) - \log p_{\boldsymbol{\theta}_{t-1}}(v,h) \tag{3.3}$$

$$= (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \boldsymbol{\phi}(v,h) - (\log Z(\boldsymbol{\theta}_t) - \log Z(\boldsymbol{\theta}_{t-1})), \tag{3.4}$$

which matches precisely to the profit of the t-th trade in the PMM defined by $F, \boldsymbol{\phi}$ and $\boldsymbol{\Theta}$. Therefore, $\mathcal{P}_{\boldsymbol{\Theta}}$ is the belief model for the PMM, and each $p_{\boldsymbol{\theta}}$ represents the belief of the market maker when it has the position $\boldsymbol{\theta}$.

3.3.2 Deficiency of the standard PMM

Full observability of the state space is a necessary condition for a PMM to work. First, it guarantees that some state will be drawn from the true distribution of nature, which is *not* manipulated by any trading strategies. Then each trade with the PMM can reflect the trader's true belief about how the state is distributed in nature, rather than a belief of strategy such as how the trader thinks the others will do. In addition, fully observability also guarantees that the true values of securities is determinable by the observation and the payouts of all purchased shares can always be cleared. If otherwise the space is partially-observable, securities will remain uncertain and a PMM cannot generate monetary payouts for purchased shares, ending up with buying and selling empty promises. Therefore, when the state space becomes partially-observable, the PMM will lose both guarantees, and any direct implementation of the PMM will not work in the desired way.

3.3.3 Conditional market prices

In order to let a PMM work on partially-observable state spaces, the first key issue we have to handle is to make the security values determinable by the observation. In other words, we need to find some deterministic quantities that can summarize the uncertain security values, taking into account the observed information. These quantities should be closely related to market prices. This is because market prices can be thought of as such quantities which summarize the uncertain security values into deterministic values, before the market observes the future state.

In fact, in a PMM with an exponential family belief model, such quantities can be constructed by using an alternative interpretation of the market prices. We name them *conditional market prices* or *conditional expected security values*, and the reason will be seen shortly.

Equivalence between markets prices and security values Originally the market prices at market position $\boldsymbol{\theta}$ are defined by the gradient of the market potential F at $\boldsymbol{\theta}$: $\mathbf{p}(\boldsymbol{\theta}) = \nabla F(\boldsymbol{\theta})$. Under the belief model, the market prices can also be expressed by the expectation of securities w.r.t. the current market belief $p_{\boldsymbol{\theta}}$, that is, $\mathbf{p}(\boldsymbol{\theta}) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}]$. This new interpretation of market prices results from an important property of the exponential family: the gradient of the log partition function of $p_{\boldsymbol{\theta}}$ always matches to the expectation of sufficient statistics w.r.t. $p_{\boldsymbol{\theta}}$.

The alternative interpretation of market prices also gives us a unified view of market prices and the true security values. When (v, h) is fully-observable such that h is ignored, the security values $\boldsymbol{\phi}(v)$ determined by the realization $\mathsf{v} = v$ can also be represented as expectations w.r.t. the current market belief, just like the market prices before observation, but conditioned on the deterministic observation $\mathsf{v} = v$.

$$\boldsymbol{\phi}(v) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v} = v]. \tag{3.5}$$

For fully-observable markets, although being written in the above form, the security values $\boldsymbol{\phi}(v)$ relies only on the observation v and is independent of the market position θ. This is because the conditional of the market belief $p_{\boldsymbol{\theta}}$ given $\mathsf{v} = v$ is a degenerate distribution at v with a Dirac Delta density. Nevertheless, with this new interpretation we can immediately generalize market prices to situations beyond no and full observation.

Conditional market prices Following the unified view and (3.5), if the PMM is defined on a partially-observable space (h, v) such that h cannot be ignored, we construct a deterministic quantity by taking the expectation of $\boldsymbol{\phi}$ w.r.t. the market belief $p_{\boldsymbol{\theta}}$ conditioned on the observed $\mathsf{v} = v$

$$\mathbf{p}(\boldsymbol{\theta}, v) := \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v} = v] = \int_{h \in \mathcal{H}} \boldsymbol{\phi}(v, h) p_{\boldsymbol{\theta}}(h \mid v) \mathrm{d}h. \tag{3.6}$$

We name $\mathbf{p}(\boldsymbol{\theta}, v)$ the *conditional market prices* or the *conditional expected security values* given $\mathsf{v} = v$, as $\mathbf{p}(\boldsymbol{\theta}, v)$ is computed the same way as $\mathbf{p}(\boldsymbol{\theta})$ but is derived using the conditional form of the market belief $p_{\boldsymbol{\theta}}$. Unlike for $\boldsymbol{\phi}(v)$ in (3.5), $\mathbf{p}(\boldsymbol{\theta}, v)$ will depend on $\boldsymbol{\theta}$ due to the partial observability, despite that they are defined via the same conditional expectation. This extra dependency on $\boldsymbol{\theta}$ also leaves the question of which $\boldsymbol{\theta}$ should be used for the computation.

The conditional market prices can be calculated in two ways. We can explicitly write down the market belief $p_{\boldsymbol{\theta}}$ and compute (exactly or approximately if necessary) the conditional expectation. Alternatively, the conditional of $p_{\boldsymbol{\theta}}$ given $\mathsf{v} = v$ is also an exponential family distribution with sufficient statistics $\boldsymbol{\phi}(v, \mathsf{h})$. Therefore, we can write down the log partition function of the conditional

$$F(\boldsymbol{\theta}, v) = \log \int_{h \in \mathcal{H}} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(v, h)\right) \nu(v, h) \mathrm{d}h. \tag{3.7}$$

Its gradient w.r.t. $\boldsymbol{\theta}$ will immediately give the conditional market prices, that is, $\mathbf{p}(\boldsymbol{\theta}, v) = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, v)$ when $\mathsf{v} = v$. We name $F(\cdot, \mathsf{v})$ the *conditional potential* given v. Note that $F(\boldsymbol{\theta}, \mathsf{v})$ is a function of v. Therefore, it is a random variable before v is realized.

By definition, given realization $\mathsf{v} = v$ and the conditional market price computed at market position $\boldsymbol{\theta}'$, $\mathbf{p}(\boldsymbol{\theta}', v)$, the profit of a trade $\boldsymbol{\delta}_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ is

$$m_t(v) = \boldsymbol{\delta}_t^\top \mathbf{p}(\boldsymbol{\theta}', v) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})). \tag{3.8}$$

3.3.4 PoPMM: the mechanism

A PoPMM is a potential-based market maker who evaluates security payouts using the conditional market prices at the *penultimate* market position. In practice, the penultimate position can be easily tracked by maintaining a record of the pre-trade market position with a K-dimensional vector. When computing the conditional market prices, the choice of penultimate position instead of the other positions is critical: it guarantees that the PoPMM will generate desired incentives for trading. The detailed analysis will be given later in Section 3.6.

A practical implementation of the PoPMM mechanism is given by Figure 3.3b. The resulting mechanism is simple. In fact, it is almost the same as a PMM

Input: $\boldsymbol{\phi}$, F, initial position $\boldsymbol{\theta}_0$;
for $t = 1$ **to** T **do**
  δ_t trade costs $F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})$;
  update position $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_{t-1} + \delta_t$;
end
at T observe $\mathsf{v} = v$, close the market;
evaluate securities $\mathbf{r} = \boldsymbol{\phi}(v)$;
a trader holding δ gets payout $\delta^\top \mathbf{r}$;

(a) PMM for fully-observable space

Input: $\boldsymbol{\phi}$, F, $\mathcal{P}_{\boldsymbol{\Theta}}$ (or $F(\cdot, \mathsf{v})$) and $\boldsymbol{\theta}_0$;
for $t = 1$ **to** T **do**
  record pre-position $\boldsymbol{\theta}^{\text{pre}} \leftarrow \boldsymbol{\theta}_{t-1}$;
  δ_t trade costs $F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})$;
  update position $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_{t-1} + \delta_t$;
end
at T observe $\mathsf{v} = v$, close the market;
evaluate securities $\mathbf{r} = \mathbf{p}(\boldsymbol{\theta}^{\text{pre}}, v) = \mathbb{E}_{p_{\boldsymbol{\theta}^{\text{pre}}}}[\boldsymbol{\phi} \mid v] = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}^{\text{pre}}, v)$;
a trader holding δ gets payout $\delta^\top \mathbf{r}$;

(b) PoPMM for partially-observable space

Figure 3.3: PoPMM and its comparison to PMM. Running on a partially-obsrvable space, a PoPMM is almost the same as a PMM except for two differences (marked in red): 1. an extra vector $\boldsymbol{\theta}^{\text{pre}}$ is defined, and is then used to record the pre-trade market position whenever a trade happens; 2. after observing $\mathsf{v} = v$, securities are evaluated by the conditional market prices at $\boldsymbol{\theta}^{\text{pre}}$.

running on a fully-observable state space (Figure 3.3a) but differs in two places, which we highlight:

1. an extra vector $\boldsymbol{\theta}^{\text{pre}}$ is defined, and is then used to record the pre-trade market position whenever a trade happens;
2. after observing $\mathsf{v} = v$, securities are evaluated by the conditional market prices at $\boldsymbol{\theta}^{\text{pre}}$.

3.4 Examples

In this section we give two examples of PoPMM. The first one defines its belief model by a mixture of exponential family distributions. The second one offers securities that are actually only dependent on the latent variable; this PoPMM has a belief model with adjustable marginal $p(\mathsf{h})$ on the latent variable but a fixed conditional $p(\mathsf{v} \mid \mathsf{h})$.

3.4.1 Defining belief model by a mixture

Consider a partially-observable state space (v,h) where $\mathsf{v} := \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^D$ and $\mathsf{h} := \mathsf{z} \in \mathcal{M} = \{1,2,\ldots,M\}$. Define M indicator functions on $\mathcal{H}$, $\boldsymbol{\varphi} : \mathcal{H} \to \mathbb{R}^M$, such that $\varphi_m(\mathsf{z}) := \mathbf{1}_{\{m\}}(\mathsf{z})$, $m \in \mathcal{M}$; also define M sets of statistics on $\mathcal{V}$, such that for the m-th set of statistics $\boldsymbol{\psi}_m : \mathcal{V} \to \mathbb{R}^{K_m}$, an exponential family distribution is defined with the partition function

$$Z_m(\boldsymbol{\theta}_m) := \int_{\mathbf{x}\in\mathcal{X}} \exp\left(\boldsymbol{\theta}_m^\top \boldsymbol{\psi}_m(\mathbf{x})\right) \nu(\mathbf{x})\mathrm{d}\mathbf{x} < +\infty \tag{3.9}$$

for every $\boldsymbol{\theta}_m \in \boldsymbol{\Theta}_m \subseteq \mathbb{R}^{K_m}$, where $\nu(\cdot)$ represents the base measure of the exponential family.

Then a PoPMM can be defined on this partially-observable space $(\mathbf{x},\mathsf{z})$. In particular, the securities $\boldsymbol{\phi}$ are given by

$$\boldsymbol{\phi}(\mathbf{x},\mathsf{z}) := \Bigg(\underbrace{\boldsymbol{\varphi}(\mathsf{z})}_{\boldsymbol{\phi}_0}{}^\top, \underbrace{\varphi_1(\mathsf{z})\boldsymbol{\psi}_1(\mathbf{x})}_{\boldsymbol{\phi}_1}{}^\top, \ldots, \underbrace{\varphi_M(\mathsf{z})\boldsymbol{\psi}_M(\mathbf{x})}_{\boldsymbol{\phi}_M}{}^\top\Bigg)^\top, \tag{3.10}$$

where for simplicity each sub-vector in $\boldsymbol{\phi}(\mathbf{x},\mathsf{z})$ is indexed by the new notation underneath. The PoPMM's potential and the set of all possible positions are

$$\boldsymbol{\Theta} := \mathbb{R}^M \times \prod_{m\in\mathcal{M}} \boldsymbol{\Theta}_m, \tag{3.11}$$

$$F(\boldsymbol{\theta}) := \log \sum_{m\in\mathcal{M}} \exp\left(\theta_{0m} + \log Z_m(\boldsymbol{\theta}_m)\right). \tag{3.12}$$

Here $\boldsymbol{\theta}$ is the vector of sold shares and is indexed the same way as $\boldsymbol{\phi}$, such that $\boldsymbol{\theta}_i$ is the shares associated with securities $\boldsymbol{\phi}_i$. θ_{0m} is the m-th element of $\boldsymbol{\theta}_0$ corresponding to security φ_m. Note that F is a convex function.

Now we show that the belief model this PoPMM defines is a mixture of M exponential family components. Following the argument in Section 3.3.1, we assume the space is fully-observable, and write down the profit $m_t(\mathbf{x},z)$ of a trade $\delta_t := \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$, given an observed $(\mathbf{x},z)$

$$m_t(\mathbf{x},z) = \delta_t^\top \boldsymbol{\phi}(\mathbf{x},z) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.13}$$

$$= (\boldsymbol{\theta}_t^\top \boldsymbol{\phi}(\mathbf{x},z) - F(\boldsymbol{\theta}_t)) - (\boldsymbol{\theta}_{t-1}^\top \boldsymbol{\phi}(\mathbf{x},z) - F(\boldsymbol{\theta})). \tag{3.14}$$

It turns out that

$$\boldsymbol{\theta}^\top \boldsymbol{\phi} - F(\boldsymbol{\theta}) = \boldsymbol{\theta}_0^\top \boldsymbol{\phi}_0 + \sum_m \boldsymbol{\theta}_m^\top \boldsymbol{\phi}_m - \log \sum_m \exp\left(\theta_{0m} + \log Z_m(\boldsymbol{\theta}_m)\right) \tag{3.15}$$

$$(\text{let } \xi_m := \theta_{0m} + \log Z_m(\boldsymbol{\theta}_m)) \tag{3.16}$$

$$= \underbrace{\sum_m \xi_m^\top \varphi_m - \log \sum_m e^{\xi_m}}_{\text{a categorical dist. over z}} + \sum_m \underbrace{\left(\boldsymbol{\theta}_m^\top \boldsymbol{\psi}_m - \log Z_m(\boldsymbol{\theta}_m)\right)}_{\text{an exponential family dist. over } \mathbf{x}}^\top \varphi_m \tag{3.17}$$

$$= \log(p(\mathrm{z})p(\mathbf{x} \mid z)). \tag{3.18}$$

Therefore, $m_t(\mathbf{x}, z)$ matches to an increase in the log density of a mixture model $p(\mathrm{z})p(\mathbf{x} \mid \mathrm{z})$ defined over $(\mathbf{x}, \mathrm{z})$ with parameter $\boldsymbol{\theta}$. In addition, this model is parametrized by $\boldsymbol{\theta}$, such that $p(\mathrm{z})$ is a categorical distribution with natural parameters $\boldsymbol{\xi} = \{\theta_{0m} + \log Z_m(\boldsymbol{\theta}_m)\}_{m \in \mathcal{M}}$, and $p(\mathbf{x} \mid \mathrm{z} = m)$ is an exponential family distribution with statistics $\boldsymbol{\psi}_m$ and natural parameters $\boldsymbol{\theta}_m$. Therefore, the proposed PoPMM defines a belief model $p_{\boldsymbol{\theta}}(\mathbf{x}, \mathrm{z}) = p(\mathrm{z})p(\mathbf{x} \mid \mathrm{z})$ with a mixture of M exponential family components.

The price vector is given by the gradient of the corresponding potential, or equivalently the expectation w.r.t. current market belief. Before observing $\mathbf{x}$, the prices are $\mathbf{p}(\boldsymbol{\theta}) = (\mathbf{p}_0(\boldsymbol{\theta}_0), \mathbf{p}_1(\boldsymbol{\theta}_1), \ldots, \mathbf{p}_M(\boldsymbol{\theta}_M))^\top$. For each $m \in \mathcal{M}$

$$p_{0m}(\theta_{0m}) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\phi_{0m}] = \frac{\exp(\theta_{0m} + \log Z_m(\boldsymbol{\theta}_m))}{\sum_m \exp(\theta_{0m} + \log Z_m(\boldsymbol{\theta}_m))} \tag{3.19}$$

$$\mathbf{p}_m(\boldsymbol{\theta}_m) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}_m] = p_{0m}(\theta_{0m}) \nabla \log Z_m(\boldsymbol{\theta}_m). \tag{3.20}$$

The conditional market prices are computed by taking the conditional expectation given $\mathbf{x} = \mathbf{x}$. Also, we can first derive the conditional potential and obtain the conditional market prices from its gradient. Here, the conditional potential is analytical:

$$F(\boldsymbol{\theta}, \mathbf{x}) = \log \sum_m \exp\left(\theta_{0m} + \boldsymbol{\theta}_m^\top \boldsymbol{\psi}_m(\mathbf{x})\right). \tag{3.21}$$

After observing $\mathbf{x}$, the prices change to

$$p_{0m}(\theta_{0m}, \mathbf{x}) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\phi_{0m} \mid \mathbf{x} = \mathbf{x}] = \frac{\exp(\theta_{0m} + \boldsymbol{\theta}_m^\top \boldsymbol{\psi}_m(\mathbf{x}))}{\sum_m \exp(\theta_{0m} + \boldsymbol{\theta}_m^\top \boldsymbol{\psi}_m(\mathbf{x}))} \tag{3.22}$$

$$\mathbf{p}_m(\boldsymbol{\theta}_m, \mathbf{x}) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}_m \mid \mathbf{x} = \mathbf{x}] = p_{0m}(\theta_{0m}, \mathbf{x}) \boldsymbol{\psi}_m(\mathbf{x}). \tag{3.23}$$

After the observation, prices for $\boldsymbol{\phi}_m$ clamps to the realization of $\mathbf{x}$. Since $\boldsymbol{\phi}_0$ are indicators, each price in (3.22) is in essence the conditional probability $p(\mathrm{z} = m \mid \mathbf{x} = \mathbf{x})$.

A PoPMM representing a mixture model is suitable for solving the Who-Touched-the-Ball game (Figure 3.1). The latent component indicator naturally encodes the player, while each component represents the possible ball position given that corresponding player last touched the ball. To complete the design, what remains is to specify the exponential family for each component. We will return to this market design later in Section 3.7.

3.4.2 Offering securities only for the latent variables

Consider a PoPMM with a mixture belief defined in the previous example, but now it only offers $\boldsymbol{\phi}_0$ for trading. Then only $\boldsymbol{\theta}_0$ can be adjusted via trades, and all the other parameters $\{\boldsymbol{\theta}_m\}_{m\in\mathcal{M}}$ will be fixed at their original values. Notice that $\boldsymbol{\theta}_m$ is the parameters of the m-th component of the belief mixture. Therefore, the belief model of this PoPMM has an adjustable marginal $p(\mathsf{z})$ over the latent indicator z but a fixed conditional $p(\mathbf{x} \mid \mathsf{z})$. Since $\log Z_m(\boldsymbol{\theta}_m^{\text{fix}})$ is fixed, by changing the variable $\xi_m := \boldsymbol{\theta}_{0m} + \log Z_m(\boldsymbol{\theta}_m^{\text{fix}})$ and $F'(\boldsymbol{\xi}) := F(\boldsymbol{\theta}_0)$, the market prices will be invariant $\nabla F'(\boldsymbol{\xi}) = \nabla F(\boldsymbol{\theta}_0)$, so will be a trade $\boldsymbol{\delta}_t = \boldsymbol{\theta}_{t,0} - \boldsymbol{\theta}_{t-1,0} = \boldsymbol{\xi}_t - \boldsymbol{\xi}_{t-1}$. In addition, the cost of a trade will also be simplified

$$F'(\boldsymbol{\xi}_t) - F'(\boldsymbol{\xi}_{t-1}) = \log \sum_m e^{\xi_{t,m}} - \log \sum_m e^{\xi_{t-1,m}}. \tag{3.24}$$

In general, simpler structures can be found in a PoPMM whose belief is only adjustable on the latent variable. Given a *fixed* conditional $p(\mathsf{v} \mid \mathsf{h})$, which determines the meaning of a set of latent variable and its correlation to the visible variables, a PoPMM can be set up to model $p(\mathsf{h})$ by an exponential family who has the partition function

$$Z(\boldsymbol{\theta}) := \int_{h\in\mathcal{H}} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(h)\right) \nu(h) \mathrm{d}h, \tag{3.25}$$

by first defining securities $\boldsymbol{\phi}$ matching to the sufficient statistics of the exponential family, and then defining the potential $F(\boldsymbol{\theta}) := \log Z(\boldsymbol{\theta})$, and finally setting the parameter set $\boldsymbol{\Theta} := \{\boldsymbol{\theta} \in \mathbb{R}^K \mid F(\boldsymbol{\theta}) < +\infty\}$. The profit of a trade

$\delta_t := \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$, given an observed (v, h), is

$$m_t(v,h) = \delta_t^\top \boldsymbol{\phi}(v,h) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.26}$$

$$= (\boldsymbol{\theta}_t^\top \boldsymbol{\phi}(h) - F(\boldsymbol{\theta}_t)) - (\boldsymbol{\theta}_{t-1}^\top \boldsymbol{\phi}(h) - F(\boldsymbol{\theta}_{t-1})) \tag{3.27}$$

$$= \log p_{\boldsymbol{\theta}_t}(h) - \log p_{\boldsymbol{\theta}_{t-1}}(h) \tag{3.28}$$

$$= \log(p_{\boldsymbol{\theta}_t}(h)p(v \mid h)) - \log(p_{\boldsymbol{\theta}_{t-1}}(h)p(v \mid h)), \tag{3.29}$$

where the last equality is obtained by introducing the fixed term $\log p(\mathsf{v} \mid \mathsf{h})$. Hence $p_{\boldsymbol{\theta}}(\mathsf{v},\mathsf{h}) := p_{\boldsymbol{\theta}}(\mathsf{h})p(\mathsf{v} \mid \mathsf{h})$ is the joint belief of this PoPMM.

From the partition function of $p(h \mid v)$ the conditional potential $\mathsf{v} = v$ can be derived, which has the form

$$F(\boldsymbol{\theta}, v) = \log \int_{h \in \mathcal{H}} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(h)\right) \nu(h) p(v \mid h) \mathrm{d}h. \tag{3.30}$$

The conditional market prices can be computed by either the conditional expectation w.r.t. $p_{\boldsymbol{\theta}}$ or by the gradient of the conditional potential.

$$\mathbf{p}(\boldsymbol{\theta}, v) = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, v) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v} = v]. \tag{3.31}$$

3.5 Bounding monetary losses

To be able to do prediction market making under real world circumstances, the market maker has to possess two fundamental properties: (1) always having bounded monetary losses that do not scale with the number of trades; and (2) providing incentives for traders such that they are willing to hand in their true beliefs. The first property allows a market maker to run large markets with many traders using a limited subsidy. The second property ensures that the market maker can learn from trades some useful information about the future state, which the market maker cares about.

In this section, we prove that PoPMM provides bounded monetary losses. The market incentives will be discussed later in the next section.

Consider a state space (v, h) and a PoPMM with securities $\boldsymbol{\phi}$, potential F and the set of positions $\boldsymbol{\Theta}$. Denote $\mathcal{P}_{\boldsymbol{\Theta}}$ the PoPMM's belief model and $p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}$ the specific distribution at market position $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Assume that the market lifetime is T and one trade is made at each time point. Then the t-trade with $\delta_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ costs $F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})$. Given the realization $\mathbf{v} = v$, securities are evaluated by the conditional market prices at the penultimate market position $\boldsymbol{\theta}_{T-1}$, $\mathbf{p}(\boldsymbol{\theta}_{T-1}, v)$, and the payout of the t-th trade is $\delta^\top \mathbf{p}(\boldsymbol{\theta}_{T-1}, v)$. Therefore, the profit of the t-trade is given by

$$m_t(v) = \delta_t^\top \mathbf{p}(\boldsymbol{\theta}_{T-1}, v) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.32}$$

Since the profit of the t-trade is the loss of the market maker at time t, then the market maker's accumulated monetary loss up to time T is

$$M_T(v) = \sum_{t=1}^{T} m_t(v) = \left(\sum_{t=1}^{T} \delta_t\right)^\top \mathbf{p}(\boldsymbol{\theta}_{T-1}, v) - \sum_{t=1}^{T}(F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.33}$$

$$= (\boldsymbol{\theta}_T - \boldsymbol{\theta}_0)^\top \mathbf{p}(\boldsymbol{\theta}_{T-1}, v) - (F(\boldsymbol{\theta}_T) - F(\boldsymbol{\theta}_0)), \tag{3.34}$$

Now we show that under certain conditions, $M_T(v)$ is bounded from above.

Lemma 3.1. *If* $\mathbf{p}(\boldsymbol{\theta}_{T-1}, v) \in \mathrm{dom}(F^*)$, *then*

$$M_T(x) \leq D_{F^*}(\mathbf{p}(\boldsymbol{\theta}_{T-1}, v), \mathbf{p}_0). \tag{3.35}$$

Here $\mathbf{p}_0 := \mathbf{p}(\boldsymbol{\theta}_0)$ *is the initial market price,* F^* *is the convex conjugate of* F *with effective domain* $\mathrm{dom}(F^*) := \{\mathbf{p} \in \mathbb{R}^K \mid F^*(\mathbf{p}) < +\infty\}$, *and* $D_f(\cdot, \cdot)$ *is the Bregman divergence generated by a convex function* f.

Proof. Denote $\mathbf{p} := \mathbf{p}(\boldsymbol{\theta}_{T-1}, v)$. Since $\mathbf{p}$ sits in the effective domain of F^*, $F^*(\mathbf{p})$ is defined. Adding and subtracting $F^*(\mathbf{p})$ we have

$$M_T(v) = (F^*(\mathbf{p}) + F(\boldsymbol{\theta}_0) - \boldsymbol{\theta}_0^\top \mathbf{p}) - (F^*(\mathbf{p}) + F(\boldsymbol{\theta}_T) - \boldsymbol{\theta}_T^\top \mathbf{p}) \tag{3.36}$$

By definition of convex conjugate and Bregman divergence (recall (2.7))

$$D_{F^*}(\mathbf{p}, \mathbf{p}_0) = F^*(\mathbf{p}) + F(\boldsymbol{\theta}_0) - \mathbf{p}^\top \boldsymbol{\theta}_0, \tag{3.37}$$

$$D_{F^*}(\mathbf{p}, \mathbf{p}_T) = F^*(\mathbf{p}) + F(\boldsymbol{\theta}_T) - \mathbf{p}^\top \boldsymbol{\theta}_T. \tag{3.38}$$

Therefore, $M_T(v)$ can be written as a difference between two Bregman divergence

$$M_T(x) = D_{F^*}(\mathbf{p}, \mathbf{p}(\boldsymbol{\theta}_0)) - D_{F^*}(\mathbf{p}, \mathbf{p}(\boldsymbol{\theta}_T)). \tag{3.39}$$

Apply non-negativity of Bregman divergence to complete the proof. □

Similar to a PMM, the monetary loss of a PoPMM is bounded from above by a Bregman divergence which depends only on the initial market positions, the penultimate market positions, and the realization of v.

On the one hand, we would desire a uniform bound on $M_T(v)$ regardless of what $\mathsf{v} = v$ is observed. On the other hand, we want to make sure the condition in Lemma 3.1 is mild enough to hold in general situations. It turns out that a simple and mild condition is sufficient to help achieve both goals. This condition is essentially the same as the one that guarantees the boundedness of PMMs in fully-observable settings: it simply requires that a PMM's loss is *uniformly* bounded for any possible complete realization of the state (cf. (2.10)).

Theorem 3.2. *If* $\forall(v,h) \in \mathcal{V} \times \mathcal{H}$ *we have* $\boldsymbol{\phi} \in \mathrm{dom}(F^*)$ *and* $D_{F^*}(\boldsymbol{\phi}, \mathbf{p}_0) \leq M^2$ *for some* $M > 0$*, then*

$$M_T(v) \leq M^2, \ \forall v \in \mathcal{V}. \tag{3.40}$$

Proof. When $\boldsymbol{\phi} \in \mathrm{dom}(F^*)$, any conditional expectation of $\boldsymbol{\phi}$ is simply a convex combination of $\boldsymbol{\phi}$ and must belong to $\mathrm{dom}(F^*)$, thus we get the condition $\mathbf{p}(\boldsymbol{\theta}_{T-1}, v) \in \mathrm{dom}(F^*)$ for free.

The Bregman divergence is convex in its first argument, as for any $\mathbf{u} := \lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2$ with $\mathbf{u}_1, \mathbf{u}_2$ in the domain and $\lambda \in [0,1]$

$$\begin{aligned} D_f(\mathbf{u}, \mathbf{v}) &= f(\lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2) + f^*(\mathbf{v}^*) - (\lambda\mathbf{u}_1 + (1-\lambda)\mathbf{u}_2)^\top \mathbf{v}^* \\ &\leq \lambda(f(\mathbf{u}_1) + f^*(\mathbf{v}^*) - \mathbf{u}_1^\top \mathbf{v}^*) + (1-\lambda)(f(\mathbf{u}_2) + f^*(\mathbf{v}^*) - \mathbf{u}_2^\top \mathbf{v}^*) \\ &= \lambda D_f(\mathbf{u}_1, \mathbf{v}) + (1-\lambda) D_f(\mathbf{u}_2, \mathbf{v}). \end{aligned} \tag{3.41}$$

Therefore, by applying Jensen's inequality we have

$$D_{F^*}(\mathbf{p}(\boldsymbol{\theta}_{T-1}, v), \mathbf{p}_0) \leq \mathbb{E}_{\boldsymbol{\theta}_{T-1}}[D_{F^*}(\boldsymbol{\phi}, \mathbf{p}_0) \mid \mathsf{v} = v] \leq M^2. \tag{3.42}$$

This completes the proof. □

Remark. Although in Lemmar 3.1 and Theorem 3.2 we specify conditional market prices to be $\mathbf{p}(\boldsymbol{\theta}_{T-1}, v)$, both results will hold for all choices of conditional market prices $\mathbf{p}(\boldsymbol{\theta}', v)$ as the proofs does not dependent on particular conditional market prices.

3.6 Market incentives

Similar to a PMM, a PoPMM encourages traders to move the market belief $p_{\boldsymbol{\theta}}$ towards their true beliefs. However, different from a PMM which allows traders' beliefs to be elicited precisely in a single trade, a PoPMM only guarantees that each trade moves the market position in the correct direction, in the sense that the post-trade market beliefs will always be closer to the traders' true beliefs than the pre-trade ones. The essential reason is that: a PMM implements an exact logarithmic market scoring rule (LMSR), while, as we will see, a PoPMM implements an approximated version.

3.6.1 LMSR and its double-potential form

Let (v, h) be a partially-observable state space and $\mathcal{P}_{\boldsymbol{\Theta}}$ be an exponential family defined on it. We consider a LMSR which pays the t-th trade the following profit

$$S_t(v, p_{\boldsymbol{\theta}_t}) := \log p_{\boldsymbol{\theta}_t}(v) - \log p_{\boldsymbol{\theta}_{t-1}}(v) \tag{3.43}$$

$$= \log \int_{h \in \mathcal{H}} p_{\boldsymbol{\theta}_t}(v, h) \mathrm{d}h - \log \int_{h \in \mathcal{H}} p_{\boldsymbol{\theta}_{t-1}}(v, h) \mathrm{d}h. \tag{3.44}$$

In other words, the LMSR asks marginal distributions over v to be submitted, with the set of belief distributions restricted to $\mathcal{P}_{\boldsymbol{\Theta}}$.

Substituting in the canonical form of the exponential family, we can rewrite (3.43) in a different form. Also recall that that the potential of a PoPMM with a belief model $\mathcal{P}_{\boldsymbol{\Theta}}$ has its potential defined by $F := \log Z$, and the conditional potential defined by (3.7).

$$S_t(v, p_{\boldsymbol{\theta}_t}) = \log \int_{h \in \mathcal{H}} \frac{e^{\boldsymbol{\theta}_t^\top \boldsymbol{\phi}(v,h)}}{Z(\boldsymbol{\theta}_t)} \nu(h) \mathrm{d}h - \log \int_{h \in \mathcal{H}} \frac{e^{\boldsymbol{\theta}_{t-1}^\top \boldsymbol{\phi}(v,h)}}{Z(\boldsymbol{\theta}_t)} \nu(h) \mathrm{d}h \tag{3.45}$$

$$= (F(\boldsymbol{\theta}_t, v) - F(\boldsymbol{\theta}_t)) - (F(\boldsymbol{\theta}_{t-1}, v) - F(\boldsymbol{\theta}_{t-1})) \tag{3.46}$$

$$= (F(\boldsymbol{\theta}_t, v) - F(\boldsymbol{\theta}_{t-1}, v)) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})). \tag{3.47}$$

The profit of a trade under the LMSR involves simultaneous changes of both the potential F and the conditional potential $F(\cdot, \mathsf{v})$, w.r.t. the same move in $\boldsymbol{\theta}$. Therefore, intuitively this LMSR can be understood as a hedge mechanism implemented by hedging between two PMMs which offer the same securities but price them in different ways. Thus a LMSR can be thought of as a hedge mechanism between two PMMs which offer the same security but have different potentials.

To be specific, consider two PMMs on (v, h) (Figure 3.4). The first PMM is a standard PMM with securities $\boldsymbol{\phi}$ and potential F. The second PMM offers the same securities, but its potential is set to be the conditional potential $F(\cdot, \mathsf{v})$. Traders can only interact with these two PMMs in a restrictive manner: each trade must trade a share bundle with both PMMs at the same time, such that whenever a trade purchases $\delta_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ shares of securities from one PMM, an inverse purchase $-\delta_t = \boldsymbol{\theta}_{t-1} - \boldsymbol{\theta}_t$ from the other is always automatically attached.

However, such a hedge mechanism is purposed vacuously only for the purpose of analysis, and cannot really be implemented. The problem is that the second PMM is not a standard PMM. In fact, the potential of the second PMM $F(\cdot, \mathsf{v})$ is a random variable, thus the cost of a trade with the second PMM is indeterminable until we observe v.

3.6.2 Weak incentive-compatibility

A well-known result given by Hanson (2007) is that the market scoring rule (MSR), which includes LMSR as a special case, is an incentive-compatible mechanism w.r.t. risk-neutral myopic traders. More precisely, given a risk-neutral myopic trader with her true belief $p_{\boldsymbol{\theta}^*} \in \mathcal{P}_{\boldsymbol{\Theta}}$, the highest MSR profit she expect will be obtained when the belief she submits $p_{\boldsymbol{\theta}_t}$ coincides with her true belief $p_{\boldsymbol{\theta}^*}$

$$S_t(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}^*}) - S_t(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}_t}) \geq 0, \ \forall p_{\boldsymbol{\theta}_t} \in \mathcal{P}_{\boldsymbol{\Theta}}. \tag{3.48}$$

Here $S_t(p, \cdot) := \mathbb{E}_p[S_t(v, \cdot)]$ denotes the expected MSR profit w.r.t. p.

A MSR interprets the incentive-compatibility of a mechanism as the ability to maximize the expected MSR profit. It allows us to define the conventional concept of incentive-compatibility through an optimization problem, and further introduce a weaker concept of incentive-compatibility.

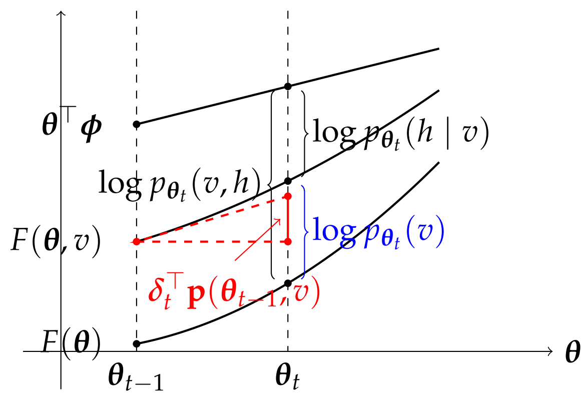


Figure 3.4: The double-potential form of LMSR. Consider a LMSR that asks traders to submit a marginal distribution over v. When the belief distributions are restricted to an exponential family $\mathcal{P}_\Theta$, the profit given by the LMSR can be rewritten as simultaneous changes of both F and the $F(\cdot, \mathrm{v})$ w.r.t. the same move in θ. Therefore, the LMSR can be thought of as a hedge mechanism: whenever a trade purchases $\delta_t = \theta_t - \theta_{t-1}$ from one PMM, an inverse purchase $-\delta_t = \theta_{t-1} - \theta_t$ from the other is always attached. From this double-potential point of view, the PoPMM approximates a LMSR by linearizing the change of the conditional potential $F(\theta_t, v) - F(\theta_{t-1}, v)$ to the payout $\theta_t^\top \mathbf{p}(\theta_{t-1}, v)$ (red), which always bounds the change form below.

Definition 3.3. A mechanism is *(S_t) incentive-compatible* (w.r.t. risk-neutral myopic traders), if there exists a MSR S_t such that for any risk-neutral myopic trader with belief $p_{\theta^*} \in \mathcal{P}_\Theta$, $S_t(p_{\theta^*}, \cdot)$ is always maximized at the post-trade market belief p_{θ_t} resulting from her optimal trade

$$S_t(p_{\theta^*}, p_{\theta_t}) - S_t(p_{\theta^*}, p_\theta) \geq 0, \ \forall p_\theta \in \mathcal{P}_\Theta. \tag{3.49}$$

Definition 3.4. A mechanism is said *weakly (S_t) incentive-compatible* (w.r.t. risk-neutral myopic traders), if there exists a MSR S_t such that for any risk-neutral myopic trader with belief $p_{\theta^*} \in \mathcal{P}_\Theta$, $S_t(p_{\theta^*}, \cdot)$ is always positive at the post-trade market belief p_{θ_t}, where p_{θ_t} results from her maximum expected profit trade. That is, for all t, $S_t(p_{\theta^*}, p_{\theta_t}) > 0$.

A weakly incentive-compatible mechanism can not elicit the traders' true beliefs exactly via its post-trade market beliefs, since there is no guarantee that the post-trade market beliefs will maximize the expected LMSR profits. However, the mechanism can guarantee that the market position moves in a correct direction, such that each post-trade belief is always closer to the current trader's true belief than the pre-trade one. Here, the closeness is measured by the KL-divergence. In fact, from the definition of weak incentive-compatibility

$$0 < S_t(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}_t}) = \mathbb{E}_{p_{\boldsymbol{\theta}^*}}[\log p_{\boldsymbol{\theta}_t}(v)] - \mathbb{E}_{p_{\boldsymbol{\theta}^*}}[\log p_{\boldsymbol{\theta}_{t-1}}(v)] \tag{3.50}$$

$$= KL(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}_{t-1}}) - KL(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}_t}). \tag{3.51}$$

The link between a PoPMM and the LMSR defined in (3.43) can be established through the double-potential form in (3.47). Given $\mathsf{v} = v$, the conditional potential $F(\cdot, v)$ is a convex differentiable function over $\boldsymbol{\Theta}$, as its Hessian

$$\nabla^2_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, v) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[(\boldsymbol{\phi} - \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid v])^2 \mid v] \geq 0, \ \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}. \tag{3.52}$$

Applying the convexity to (3.47), we have

$$S_t(v, p_{\boldsymbol{\theta}_t}) = (F(\boldsymbol{\theta}_t, v) - F(\boldsymbol{\theta}_{t-1}, v)) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.53}$$

$$\geq (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}_{t-1}, v) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.54}$$

Recall that the gradient of $F(\cdot, v)$ at $\boldsymbol{\theta}_{t-1}$ gives the conditional market prices $\mathbf{p}(\boldsymbol{\theta}_{t-1}, v)$. Then (3.54) matches precisely to the profit of the *most recent* trade made with a PoPMM. In other words, the profit of a trade made with a PoPMM always bounds the LMSR profit, or equivalently the increase in the marginal log probability at $\mathsf{v} = v$, from below (Figure 3.4).

Applying the above inequality, we can prove the weak incentive-compatibility of the PoPMM.

Theorem 3.5. *A PoPMM is weakly incentive-compatible.*

Proof. Consider a PoPMM which, after $t-1$ trades, has a market position $\boldsymbol{\theta}_{t-1}$. Since the t-th trader is risk-neutral and myopic, her trading objective is

$$\max \mathbb{E}_{p_{\boldsymbol{\theta}^*}}[m_t(\mathsf{v})] = \mathbb{E}_{p_{\boldsymbol{\theta}^*}}[(\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbf{p}(\boldsymbol{\theta}_{t-1}, \mathsf{v}) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1}))] \tag{3.55}$$

$$\leq \mathbb{E}_{p_{\boldsymbol{\theta}^*}}[S_t(\mathsf{v}, p_{\boldsymbol{\theta}_t})] = S_t(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}_t}) \tag{3.56}$$

On the other hand, notice that if no trade happens $\delta_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1} = 0$, the profit is zero. It follows that the maximum expected profit given by the optimal trade must be positive. Therefore,

$$0 < \max \mathbb{E}_{p_{\boldsymbol{\theta}^*}}[m_t(\mathsf{v})] \leq S_t(p_{\boldsymbol{\theta}^*}, p_{\boldsymbol{\theta}_t}). \tag{3.57}$$

Apply Definition 3.4 to complete the proof. □

Finally, we end up this section by introducing a special type of risk-neutral myopic trades, with respect to which the PoPMM (or in general any weakly incentive-compatible mechanism) becomes incentive-compatible. We say a trader is risk-neutral and *myopically greedy*, if she repeatedly trades with the market maker until there is no more myopically profitable trade.

Corollary 3.6. *A PoPMM is incentive-compatible w.r.t. a risk-neutral myopically greedy traders.*

This incentive-compatibility results from the fact that each trade in a PoPMM is an optimization step towards maximizing the expected LMSR profit. When the trader is myopically greedy, she will keep trading until the optimal market belief that maximizes the expected LMSR profit is reached. By the property of the LMSR, this optimal market belief reveals the trader's true belief.

3.7 Experiments

We first complete the PoPMM design for the toy example of the *Who-Touched-the-Ball* game (Figure 3.1). We simulate the market using synthetic traders. In the second example, we present a PoPMM for a more realistic problem of betting the players' skills, and simulate it using real data.

The "Who-Touched-the-Ball" game For the "Who-Touched-the-Ball" game in Figure 3.1, we build a PoPMM which represents a mixture of two Gaussians (2-GMM) (Section 3.4.1). Each Gaussian is associated with a single player in the image and describes the possible position of the ball given that the player it represents last touched the ball. Trades can adjust the mean and covariance matrix of each Gaussian as well as the mixing weights.

The market will offer securities

$$\boldsymbol{\phi}(\mathbf{x}, z) = \text{vect}\left(\underbrace{\mathbf{1}_{\{1\}}(z), \mathbf{1}_{\{2\}}(z)}_{\boldsymbol{\phi}_0}, \underbrace{\mathbf{1}_{\{1\}}(z)\mathbf{x}, \mathbf{1}_{\{1\}}(z)\mathbf{x}\mathbf{x}^\top}_{\boldsymbol{\phi}_1}, \underbrace{\mathbf{1}_{\{2\}}(z)\mathbf{x}, \mathbf{1}_{\{2\}}(z)\mathbf{x}\mathbf{x}^\top}_{\boldsymbol{\phi}_2}\right)^\top. \tag{3.58}$$

Note that $\mathbf{x}\mathbf{x}^\top$ is a two-by-two matrix, and $\text{vect}(\cdot)$ denotes the vectorization of

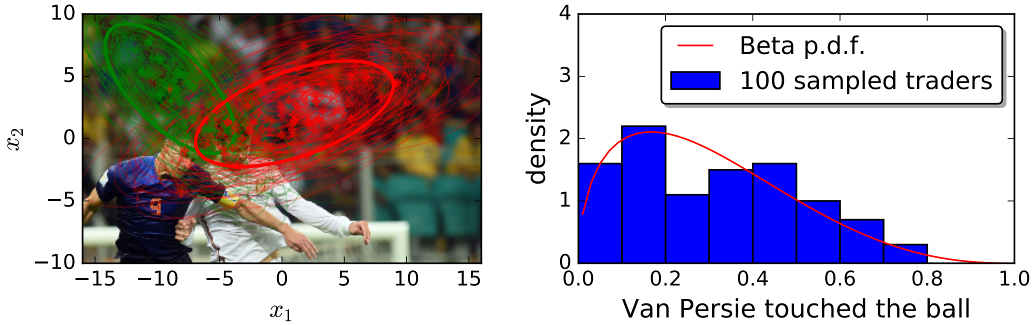


Figure 3.5: Traders drawn from the ensemble. Left: plot of first 100 sampled Gaussian components. For each component the mean and the 3σ contour (transparent thin) are shown. The thick opaque contours in red and green are the components of the ensemble mean while the transparent ones are samples. Right: plot of first 100 samples of the component mixing weights. The ensemble mean is an even mixture of two components, while traders deviate from it through a Beta distribution.

the securities. The corresponding share positions are denoted by

$$\boldsymbol{\theta} = \operatorname{vect}\left(\underbrace{\theta_{01}, \theta_{02}}_{\boldsymbol{\theta}_0}, \underbrace{\boldsymbol{\theta}_{11}, \boldsymbol{\theta}_{12}}_{\boldsymbol{\theta}_1}, \underbrace{\boldsymbol{\theta}_{21}, \boldsymbol{\theta}_{22}}_{\boldsymbol{\theta}_2}\right)^{\top}, \tag{3.59}$$

where $\boldsymbol{\theta}_{11}, \boldsymbol{\theta}_{21}$ are vectors of length two, and $\boldsymbol{\theta}_{12}, \boldsymbol{\theta}_{22}$ are two-by-two matrices. The potential and the conditional potential are

$$F(\boldsymbol{\theta}) = \log \sum_{m\in\{1,2\}} \exp(\theta_{0m} + \log Z_m(\boldsymbol{\theta}_m)), \tag{3.60}$$

$$F(\boldsymbol{\theta}, \mathbf{x}) = \log \sum_{m\in\{1,2\}} \exp\left(\theta_{0m} + \mathbf{x}^{\top}\boldsymbol{\theta}_{m1} + \operatorname{tr}(\mathbf{x}\mathbf{x}^{\top}\boldsymbol{\theta}_{m2})\right), \tag{3.61}$$

with the log partition function

$$\log Z_m(\boldsymbol{\theta}_m) = -\frac{1}{4}\boldsymbol{\theta}_{m1}^{\top}\boldsymbol{\theta}_{m2}^{-1}\boldsymbol{\theta}_{m1} - \frac{1}{2}\log|-2\boldsymbol{\theta}_{m2}|. \tag{3.62}$$

for each Gaussian component $\forall\theta \in \{1,2\}$.

To simulate the market, let's assume that there exists an ensemble of traders, and each trader's belief is a noise sample of the ensemble mean. In particular, we generate each trader's 2-GMM belief by drawing the mixing weights from a Beta distribution $\text{Beta}(3,2)$ and drawing each Gaussian component from a Normal-Inverse-Wishart distribution, $\text{NIW}(\lambda = 50, \nu = 0.6)$. The mean belief of the generated traders has mixing weights $\mathbf{w} = (0.6, 0.4)$, means of Gaussians $\boldsymbol{\mu}_1 = (-9, 4)^\top$, $\boldsymbol{\mu}_2 = (1, 2)^\top$ and covariance matrices $\boldsymbol{\Sigma}_1 = (3, -2; -2, 3)$, $\boldsymbol{\Sigma}_2 = (7, 2; 2, 2)$ (Figure 3.5). We expect that by interacting with these traders the market will somehow recover the mean belief of the ensemble.

Each trader will interact with the PoPMM once in a sequential order. The expected profit involves an integration w.r.t. the trader's belief distribution $p_t(\mathbf{x})$, which is usually expensive to compute. To avoid the integral, we represent the t-th trader's marginal belief by N_t sampled points $\{\mathbf{s}_t^n\}_{t=1}^{N_t}$ of the marginal. Then the trader's belief is approximated by $p_t(\mathbf{x}) \approx 1/N_t \sum_n \mathbf{1}(\mathbf{x} - \mathbf{s}_t^n)$, and the objective of a risk-neutral myopic trader is

$$\max_{\boldsymbol{\theta}_t \in \Theta} \frac{1}{N_t} \sum_{n=1}^{N_t} \delta^\top \mathbf{p}(\boldsymbol{\theta}_{t-1}, \mathbf{s}_t^n) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) \tag{3.63}$$

Now let's view the trading dynamics as an implementation of the Expectation-Maximisation (EM) algorithm (Dempster et al., 1977) for learning the market belief model $p_{\boldsymbol{\theta}}(\mathbf{x})$. Recall that the market belief in the joint space of $(\mathbf{x}, \mathrm{z})$ is an exponential family distribution. The key of the view is to think of each trader as a *dataset* $\mathcal{D}_t$, consisting of N_t data points distributed under the trader's individual belief p_t. Note that this trader dataset can contain infinite data points depending on how many we want to draw from the distribution. Then the E-step is to construct Q-objective using the old conditional $p_{\boldsymbol{\theta}_{t-1}}(z \mid \mathbf{x})$

$$Q(\boldsymbol{\theta}_t; \boldsymbol{\theta}_{t-1}, \mathcal{D}_t) = \sum_{\mathbf{s}^n \sim p_t} \int p_{\boldsymbol{\theta}_{t-1}}(z^n \mid \mathbf{s}^n) \log p_{\boldsymbol{\theta}_t}(\mathbf{s}^n, z^n) \mathrm{d}z^n. \tag{3.64}$$

and the M-step is to find the $\boldsymbol{\theta}_t$ that maximizes the Q-objective. Since the joint $p_{\boldsymbol{\theta}_t}(\mathbf{x}, \mathrm{z})$ is an exponential family distribution, substituting the canonical form, we obtain

$$\begin{aligned} Q(\boldsymbol{\theta}_t; \boldsymbol{\theta}_{t-1}, \mathcal{D}_t) &= \sum_{\mathbf{s}^n \sim p_t} \int p_{\boldsymbol{\theta}_{t-1}}(z^n \mid \mathbf{s}^n)(\boldsymbol{\theta}_t^\top \boldsymbol{\phi}(\mathbf{s}^n, z^n) - F(\boldsymbol{\theta}_t)) \mathrm{d}z^n + const. \\ &= \sum_{\mathbf{s}^n \sim p_t} \boldsymbol{\theta}_t^\top \Big(\int p_{\boldsymbol{\theta}_{t-1}}(z^n \mid \mathbf{s}^n) \boldsymbol{\phi}(\mathbf{s}^n, z^n) \mathrm{d}z^n \Big) - F(\boldsymbol{\theta}_t) + const. \\ &= \sum_{\mathbf{s}^n \sim p_t} \boldsymbol{\theta}_t^\top \mathbf{p}(\boldsymbol{\theta}_{t-1}, \mathbf{s}^n) - F(\boldsymbol{\theta}_t) + const. \end{aligned} \tag{3.65}$$

The Q-objective obtained above matches exactly the objective in (3.63) up to a set of terms independent of $\boldsymbol{\theta}_t$. Hence the M-step will give the same $\boldsymbol{\theta}_t$ with

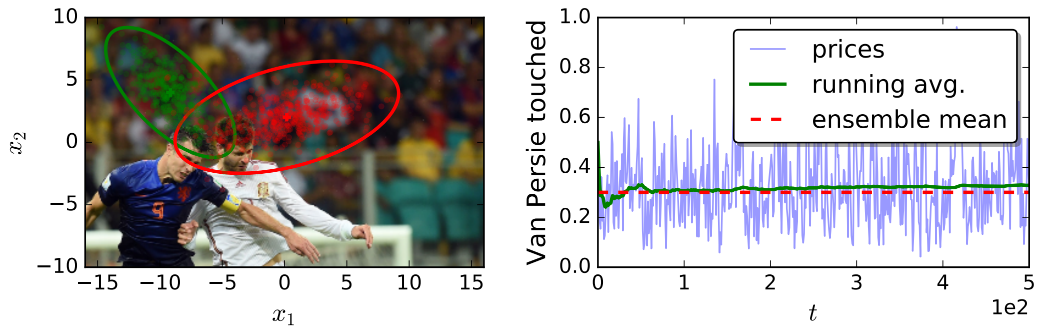


Figure 3.6: Convergence in running averaged prices. Left: The 3σ contours of the averaged Gaussian components, with transparent dots the Gaussian centres reported by each trade. Right: the running averaged prices of the security favouring Van Persie converge to the traders' mean belief.

the one resulting from the trading dynamics. From this point of view, the trading process is precisely one EM step performed on the dataset $\{s_t^n\}_{t=1}^{N_t} \sim p_t$. If we further assume that each trader may iterate multiple times in her own mind to determine the final trade, then this corresponds to running the EM steps a few more steps on the same trader's dataset. In the experiment we set this number of virtual iteration to $C = 5$, that is, each trader will actually perform five EM iterations to establish her final trade. We set $N_t = 500$ as an appropriate balance between the computational efficiency and the accuracy of representing the trader beliefs. We denote the optimal trade $\delta_t^{\text{neutral}}$ and the resulting post-trade position $\boldsymbol{\theta}_t^{\text{neutral}}$.

If the trader is risk-averse, her trade decision will be a compromise between her personal belief and the market belief, leading to a post-trade position sitting in between $\boldsymbol{\theta}_t^{\text{neutral}}$ and $\boldsymbol{\theta}_{t-1}$:

$$\boldsymbol{\theta}_t^{\text{averse}} = (1-\beta)\boldsymbol{\theta}_{t-1} + \beta\boldsymbol{\theta}_t^{\text{neutral}} = \boldsymbol{\theta}_{t-1} + \beta(\boldsymbol{\theta}_t^{\text{neutral}} - \boldsymbol{\theta}_{t-1}) \tag{3.66}$$

$$= \boldsymbol{\theta}_{t-1} + \beta\delta_t^{\text{neutral}} = \boldsymbol{\theta}_{t-1} + \delta_t^{\text{averse}}, \tag{3.67}$$

where $\delta_t^{\text{averse}} = \beta\delta_t^{\text{neutral}}$, $\beta \in [0, 1]$ denotes the actual transaction made by the risk-averse trader, and the parameter β measures the degree of risk-aversion. When $\beta = 1$, the trader becomes risk-neutral; when $\beta = 0$, the trader is too risk-averse to make any effective trade. In the experiment, we set $\beta = 0.8$.

The market starts from $t = 0$ and ends at T, during which a single trade occurs at each time. After time T, we close the market and reveal the true position

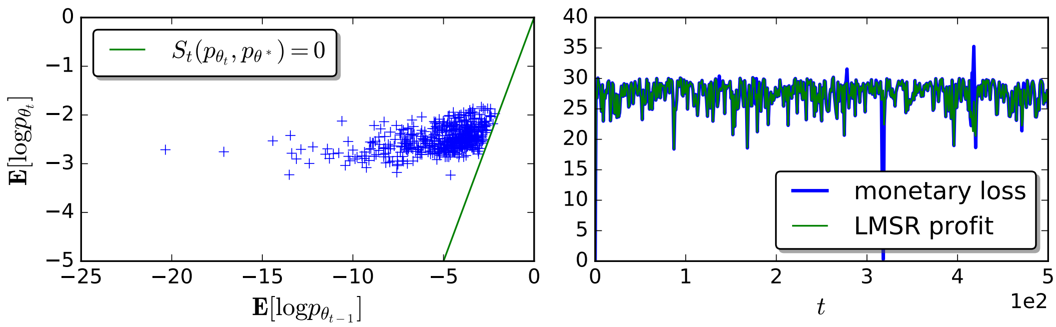


Figure 3.7: Experimental support for boundedness and incentive-compatibility. Left: comparison of the expected log probability of the market belief before and after each trade. All points lie above the green line, indicating that the expected LMSR profit w.r.t. each trader is always positive. Right: the money spent and LMSR profit gained by the market maker up to the t-th trade, given the observed $\mathbf{x}$. The monetary loss is bounded from above, but it is not always the same as the LMSR profit.

of the ball $\mathbf{x} = (-10.1, 6.5)^\top$ for this game. Finally, we evaluate securities and clearing payouts.

During trading, the market instantaneous prices show high volatility due to the interactions with stochastic natures. However, as the number of trades increases through time, the running averaged prices converge to the ensemble mean belief (Figure 3.6). Such convergence behaviour has been suggested by Frongillo et al. (2012). The authors analyse the convergence by approximating each trade to one step of the stochastic mirror descent. A more accurate discussion about the market convergence and equilibrium will be presented later in Chapter 5.

Figure 3.7 shows the results for boundedness and incentive-compatibility of the PoPMM. The left figure plots the expected log marginal market probability w.r.t. the trader's belief before and after each trade. All trades lie above the zero LMSR profit line, which means that all trades gain positive LMSR profits, thus implying the weak incentive-compatibility of the PoPMM. The right figure plots the total money spent and the total LMSR profit gained by the market maker up to the t-th trade, given the observed $\mathbf{x}$. For the PoPMM, the monetary loss is still bounded from above, though it does not always match the LMSR profit.

A skill betting system In this second example we use the PoPMM to implement a system that trades to bet on players' skills. This system can be viewed as a simple TrueSkill™ rating system with no teams (Herbrich et al., 2007), but driven by transactions between market and informative traders rather than performing Bayesian inference directly on historical data. Though based on TrueSkill™, this skill betting system is of interest on its own. In fact, compared to TrueSkill™, the system prior is enriched by various traders' beliefs. In addition, any new information learned by traders can be quickly captured by the system, such as an injury of a player.

The events we care about are the results of a pairwise comparison game without tie (e.g. go, tennis), repeatedly played by K players $\mathcal{K} = \{1, \ldots, K\}$. The m-th game is between players $i_m, j_m \in \mathcal{K}$. To model the game result, the actual performance of the two players are first drawn according their skill levels, $y_{i_m} \sim \mathcal{N}(h_{i_m}, \epsilon^2), y_{j_m} \sim \mathcal{N}(h_{j_m}, \epsilon^2)$, and the winner v_m is assigned to the player with the higher performance.

The belief that PoPMM encodes has the form $p(v, \mathbf{h}) = p(v \mid \mathbf{h})p(\mathbf{h})$. Here $p(\mathbf{h})$ is a multivariate Gaussian, which is a product of K univariate Gaussians and captures the skill levels $\mathbf{h}$ of players. The mean parameter of $p(\mathbf{h})$ is thus referred to as the players' *true skills*. The conditional $p(v \mid \mathbf{h})$ is the probability of v being the winner given players' skills $\mathbf{h}$. More specifically, given a game between k_m^1, k_m^2, the probability that $v_m = k_m^1$ is

$$p(v_m = i_m \mid \mathbf{h}) = \iint \mathbf{1}_{\{>0\}}(y_{i_m} - y_{j_m}) p(y_{i_m} \mid h_{i_m}) p(y_{j_m} \mid h_{j_m}) \mathrm{d}y_{i_m} \mathrm{d}y_{j_m} \tag{3.68}$$

$$= \Phi\left(\frac{h_{i_m} - h_{j_m}}{\sqrt{2}\epsilon}\right), \tag{3.69}$$

where Φ the c.d.f. of the standard Normal distribution. Therefore, $p(v \mid \mathbf{h})$ is a probit model.

A PoPMM with securities defined only on the latent variables is used to encode the above belief (Section 3.4.2). For simplicity, we will pre-set the variances of the skills, and only let traders bet on the players' true skills (mean of skills). In particular, we introduce K securities with $\phi_k(\mathbf{h}) = h_k$, and define potential function $F(\boldsymbol{\theta}) = \boldsymbol{\theta}^\top \boldsymbol{\Sigma} \boldsymbol{\theta}/2$ with fixed $\boldsymbol{\Sigma} = \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_K^2\}$. Then each position of the PoPMM is associated with the distribution $\mathcal{N}(\boldsymbol{\Sigma\theta}, \boldsymbol{\Sigma})$.

All tested game results are revealed simultaneously after market closes. The exact computation of the conditional market price is intractable, and we approximate it using (online) moment-matching (Minka, 2001). Given the game results $\{i_m, j_m, v_m\}_{m=1}^{M}$ in a specified order, starting with the market belief $p_{\boldsymbol{\theta}_{t-1}}(\mathbf{h})$, moment-matching sequentially approximates the posterior $p(\mathbf{h} \mid v_m)$ to a Gaussian by matching their means and covariance matrices, and uses the Gaussian as the prior to compute the next posterior $p(\mathbf{h} \mid v_{m+1})$. As described by Herbrich (2005), the approximated Gaussian has the form $\mathcal{N}(\boldsymbol{\mu}', \Sigma')$, with $\Sigma' = \text{diag}\{{\sigma'_1}^2, \ldots, {\sigma'_K}^2\}$

$$\mu'_k = \begin{cases} \mu_k + \sigma_k^2 g_k & k \in \{i_m, j_m\} \\ \mu_k & \text{otherwise} \end{cases}, {\sigma'_k}^2 = \begin{cases} \sigma_k^2 - \sigma_k^2 (g_k^2 - 2G_k)\sigma_k^2 & k \in \{i_m, j_m\} \\ \sigma_k^2 & \text{otherwise} \end{cases}, \tag{3.70}$$

where g_k and G_k are the derivatives of the log partition function of the posterior w.r.t. μ_k and σ_k^2, respectively. Denote w_m the winner of $\{i_m, j_m\}$ and l_m the loser, then for $k \in \{i_m, j_m\}$

$$g_k = \frac{\partial}{\partial \mu_k} \log \Phi \left(\frac{\mu_{w_m} - \mu_{l_m}}{\sqrt{2\epsilon^2 + \sigma_{w_m}^2 + \sigma_{l_m}^2}} \right) = (-1)^{(k=l_m)} \frac{1}{b} \varphi(a) \tag{3.71}$$

$$G_k = \frac{\partial}{\partial \sigma_k^2} \log \Phi \left(\frac{\mu_{w_m} - \mu_{l_m}}{\sqrt{2\epsilon^2 + \sigma_{w_m}^2 + \sigma_{l_m}^2}} \right) = -\frac{a}{2b^2} \varphi(a) \tag{3.72}$$

where $b = \sqrt{2\epsilon^2 + \sigma_{w_m}^2 + \sigma_{l_m}^2}, a = (\mu_{w_m} - \mu_{l_m})/b, \varphi(a) = \Phi'(a)/\Phi(a)$. Note that Φ' is simply the p.d.f. of the standard normal distribution. Finally, the mean of the final moment-matched Gaussian is then used as the approximation of the conditional market price.

We collect a total number of 51413 games for the top $K = 1049$ players from `Go4Go.net`, split 10% out (5141 games) for tested game results. The rest 90% games can be viewed as historical data available to traders to build their beliefs. We introduce two types of traders, both of which only care about betting the game between two specific players: a *simple trader* builds her belief up by simply counting the historical games between the two players; while a *smart trader* first learns a TrueSkill model on the historical data, and then uses only the marginal over the game she is interested in.

Two tests are run to evaluate our skill betting PoPMM. In the first test, we introduce $N = 20000$ traders, all of whom are smart traders, and let PoPMM

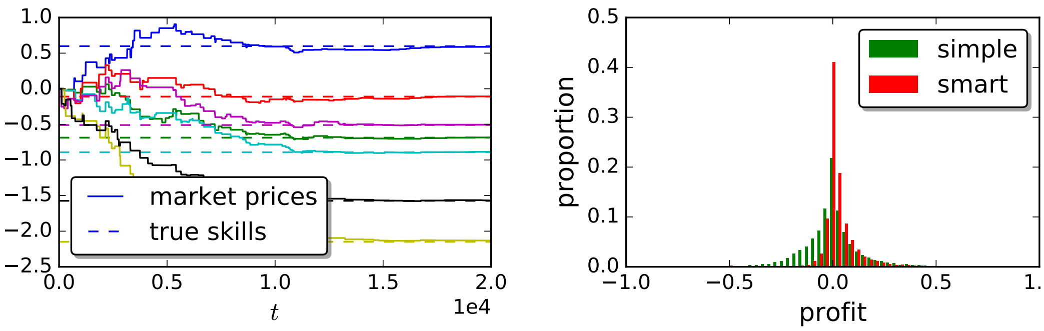


Figure 3.8: Betting Go players' skills using PoPMM. Left: prices of 7 sampled skill securities and the means of corresponding skills given by the TrueSkill model. The TrueSkill model is the underlying model for the belief construction of smart traders. The prices prices converge to TrueSkill model, implying that the PoPMM can recover the aggregated belief from only marginal beliefs over the game by leveraging the bets on the latent structure it defines. Right: the profit distributions for simple traders and smart traders when both of them interact with the PoPMM. The histograms are normalized in corresponding groups such that the heights of all bars in the same group sum up to one. The PoPMM gives higher profits collectively to smart traders, who by design have beliefs of higher quality than the simple traders.

interact with them sequentially. The left of Figure 3.8 shows that the market prices for skill securities all converge to the mean of the TrueSkill model, the underlying model from which each smart trader sets up her belief. We emphasize that traders have only marginal distribution on games. It is the latent structure the PoPMM defines that enables different marginal beliefs to be aggregated into a single belief via transactions.

In the second test, we again introduce $N = 20000$ traders, but this time we set half of the population of the traders to be simple traders and the rest to be smart traders. The PoPMM interacts with traders sequentially. The right of Figure 3.8 shows the final profits traders obtain after the results of the test games are revealed. Since by design the quality of the smart trader's belief is higher than that of the simple trader, collectively smart traders should be rewarded more than simple traders. The result shows that the smart traders' profit distribution shift positively from zero while the simple traders' profit distribution does not, implying a higher quality in smart traders' beliefs. In addition, the smart traders also have a smaller variance in their belief quality than simple traders.

3.8 Extensions

Our current PoPMM uses penultimate market position to compute conditional market prices. Although we have shown both theoretically and experimentally that the PoPMM inherits desired properties from the PMM, two important issues still remain.

Weak incentive-compatibility is enough to guarantee that the post-trade market belief will always have a higher quality than the pre-trade belief in terms of describing the traders' private information. However, under weak incentive-compatibility the post-trade market beliefs usually do not reflect traders' true beliefs, and so we cannot use current version of PoPMM for true belief revelation.

In addition, PoPMM is only weakly incentive-compatible in terms of risk-neutral, myopic sequential traders, that is, traders are expected profit maximizers and cares about their profits only in current trading round, and they interact with the PoPMM sequentially such that each trader will only trade once. Intuitively, suppose a trader can revisit the market at any time. Then this trader can manipulate the penultimate market price by following her trade with another infinitesimal trade that changes the penultimate market position to the post-trade position. Mathematically, if otherwise traders are risk-neutral and myopic but now allowed to trade multiple times, then the objective of trader i who trades at time t is to maximize

$$m_{i,t}(p_i) = \mathbb{E}_{p_i}[(\boldsymbol{\vartheta}_i + \delta_t)^\top \mathbf{p}(\boldsymbol{\theta}_{t-1}, \mathsf{v})] - (F(\boldsymbol{\theta}_{t-1} + \delta_t) - F(\boldsymbol{\theta}_{t-1})). \tag{3.73}$$

Here p_i is the trader's true belief, $\boldsymbol{\vartheta}_i$ is her holdings before the trade, $\boldsymbol{\theta}_{t-1}$ the market pre-trade position, and δ_t the shares traded in this round. Denote $\boldsymbol{\theta}_{t-1,\setminus i} = \boldsymbol{\theta}_{t-1} - \boldsymbol{\vartheta}_i$ and $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1} + \delta_t$. Then $m_{i,t}(p_i)$ can be rewritten into

$$\begin{aligned} m_{i,t}(p_i) = \mathbb{E}_{p_i}&[\log p_{\boldsymbol{\theta}_t}(\mathsf{v}) - \log p_{\boldsymbol{\theta}_{t-1,\setminus i}}(\mathsf{v})] \\ &- \mathbb{E}_{p_i}[D_{F(\cdot,\mathsf{v})}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1})] + \mathbb{E}_{p_i}[D_{F(\cdot,\mathsf{v})}(\boldsymbol{\theta}_{t-1,\setminus i}, \boldsymbol{\theta}_{t-1})] + C, \end{aligned} \tag{3.74}$$

where $C = F(\boldsymbol{\theta}_{t-1}) - F(\boldsymbol{\theta}_{t-1,\setminus i})$. It is a constant w.r.t. the trade δ_t. The first term can be thought of as the LMSR profit generated by all holdings of the trader, while he second and the third terms are two expected divergences generated by the conditional potential $F(\cdot, \mathsf{v})$. For traders who trade only once, $\boldsymbol{\vartheta}_i = 0$ and $\boldsymbol{\theta}_{t-1,\setminus i} = \boldsymbol{\theta}_{t-1}$. The second divergence is zero, then $m_{i,t}(p_i)$ bounds the LMSR profit from below, recovering weak incentive-compatibility in (3.55). However, for traders who have obtained non-zero $\boldsymbol{\vartheta}_i$ in previous trades and would like to trade once more, the second divergence remains positive, which may destroy the bound. In fact, each trader can always end her multiple trades with an infinitesimal trade such that $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1}$. Under this circumstance, the first divergence vanishes, and the profit of the trader now becomes an upper bound of the LMSR profit. This opposite bound is much less beneficial for the PoPMM, since PoPMM has to pay more money for better beliefs, and sometime may even end up with paying for worse beliefs.

Fortunately, we can solve both problems by choosing better market positions for computing conditional market prices. We first discuss how we can make our PoPMM incentive-compatible in the strict sense. Then the PoPMM is further improved to maintain (weak) incentive-compatibility under repeated trading.

3.8.1 Making PoPMM incentive-compatible

The key to recovering strict incentive-compatibility is to match the trading profit *precisely* to the corresponding LMSR profit. Given observation $\mathsf{v} = v$, for the t-th trade, the profits are given by (3.8) and (3.53), that is,

$$m_t(v) = (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbf{p}(\boldsymbol{\theta}', v) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})), \tag{3.75}$$

$$S_t(v, p_{\boldsymbol{\theta}_t}) = F(\boldsymbol{\theta}_t, \mathsf{v}) - F(\boldsymbol{\theta}_{t-1}, v) - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})). \tag{3.76}$$

Here $\boldsymbol{\theta}'$ is the market position used for computing the conditional market prices. In our current design, $\boldsymbol{\theta}'$ is the penultimate position $\boldsymbol{\theta}_{T-1}$. Recall that $\boldsymbol{\theta}' = \boldsymbol{\theta}_{T-1}$, the convexity of $F(\cdot, \mathsf{v})$ and $\mathbf{p}(\boldsymbol{\theta}, \mathsf{v}) = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, \mathsf{v})$ together result in $m_t(v) \leq S_t(v, p_{\boldsymbol{\theta}_t})$ when the t-th trade is the last trade.

Since the second terms on the RHS of both equations are the same, to match $m_t(v)$ and $S_t(v, p_{\boldsymbol{\theta}_t})$ we need to find a $\boldsymbol{\theta}'$ such that

$$(\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbf{p}(\boldsymbol{\theta}', v) = F(\boldsymbol{\theta}_t, v) - F(\boldsymbol{\theta}_{t-1}, v). \tag{3.77}$$

We must make sure the $\boldsymbol{\theta}'$ we find always exists. In addition, it should be easily constructed in order to keep the efficiency of potential market making.

It turns out that we can define $\boldsymbol{\theta}' = \lambda\boldsymbol{\theta}_t - (1-\lambda)\boldsymbol{\theta}_{t-1}$, where $\lambda \in [0,1]$ is such that (3.77) holds. The existence of $\boldsymbol{\theta}'$ is given by the *mean value theorem*. More specifically, define real-valued function

$$f(\lambda) := F(\lambda\boldsymbol{\theta}_t + (1-\lambda)\boldsymbol{\theta}_{t-1}, v). \tag{3.78}$$

It is continuous on $[0,1]$ and differentiable on $(0,1)$. Then, by the mean value theorem there exists a $\lambda' \in (0,1)$ such that $\mathrm{d}f(\lambda')/\mathrm{d}\lambda = f(1) - f(0)$, that is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(\lambda') = (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}', v) = F(\boldsymbol{\theta}_t, v) - F(\boldsymbol{\theta}_{t-1}, v). \tag{3.79}$$

Finally, the existence of λ' implies the existence of $\boldsymbol{\theta}'$.

The mean value theorem also implies an optimization based approach for finding $\boldsymbol{\theta}'$. If we define $g(\lambda) := f(\lambda) - \lambda(f(1) - f(0))$, then g is convex due to convexity of f. Taking the derivative of g at λ' and using $\mathrm{d}f(\lambda')/\mathrm{d}\lambda = f(1) - f(0)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda') = \frac{\mathrm{d}}{\mathrm{d}\lambda} f(\lambda') - (f(1) - f(0)) = 0. \tag{3.80}$$

That is, λ' happens to be the minimum of g. Finding λ' and corresponding $\boldsymbol{\theta}'$ now becomes a one dimensional minimization problem. If the conditional potential function $F(\cdot, v)$ is analytical, this minimization can be solved efficiently; otherwise, we can first compute the value of $F(\cdot, v)$ at $\boldsymbol{\theta}_t$ and $\boldsymbol{\theta}_{t-1}$ (i.e. $f(1)$ and $f(0)$), and then run root-finding algorithm for (3.80) to get λ'. Note that in the root-finding case, we only need to computing conditional market prices $\mathbf{p}(\cdot, v)$.

3.8.2 Repeated trades

Consider trader i who trades at time t, and assume her holdings $\boldsymbol{\vartheta}_i \neq 0$. Denote $\boldsymbol{\theta}_{t-1,\setminus i} := \boldsymbol{\theta}_{t-1} - \boldsymbol{\vartheta}_i$ the total shares the PoPMM sold to all traders except i. Intuitively, $\boldsymbol{\theta}_{t-1,\setminus i}$ can be thought of as a market position taking out the effect of the i-th trader. If we compute the conditional market prices using position $\boldsymbol{\theta}_{t-1,\setminus i}$, the trader's profit is

$$m_{i,t}(v) = (\log p_{\boldsymbol{\theta}_t}(v) - \log p_{\boldsymbol{\theta}_{t-1,\setminus i}}(v)) - D_{F(\cdot,v)}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_{t-1,\setminus i}) + C \quad (3.81)$$

$$\leq S_{i,t}(v, p_{\boldsymbol{\theta}_t}) + C. \quad (3.82)$$

where $S_{i,t}(v, p_{\boldsymbol{\theta}_t}) := \log p_{\boldsymbol{\theta}_t}(v) - \log p_{\boldsymbol{\theta}_{t-1,\setminus i}}(v)$ is the LMSR profit contributed by trader i, and $C = F(\boldsymbol{\theta}_{t-1}) - F(\boldsymbol{\theta}_{t-1,\setminus i})$ is the guaranteed/risk-free profit obtained by selling all holdings. Also, the equality holds when the trader sells all her holdings (resulting in $\boldsymbol{\theta}_t = \boldsymbol{\theta}_{t-1,\setminus i}$). Rearrange the inequality:

$$m_{i,t}(v) - C \leq S_{i,t}(v, p_{\boldsymbol{\theta}_t}). \quad (3.83)$$

That is, the part exceeding the guaranteed profit bounds the LMSR profit from below. Intuitively, it makes sense since trader i can contribute to the market belief only when she actually purchases securities.

Therefore, by computing conditional market prices at $\boldsymbol{\theta}_{t-1,\setminus i}$, the PoPMM is weak incentive-compatible for risk-neutral myopic traders, regardless of whether they trade sequentially or repeatedly. We refer this extended PoPMM as Re-PoPMM. Re-PopMM includes the original PoPMM as a special case, since in sequential setting $\boldsymbol{\theta}_{t-1,\setminus i}$ reduces to the penultimate position. However, as a price Re-PoPMM has to record all traders' holdings. The mechanism is described by Figure 3.9a.

We can further construct a strictly incentive-compatible Re-PoPMM for repeated trades by combining these two ideas. Similar to the PoPMM case, the strict incentive-compatible Re-PoPMM searches for a market position $\boldsymbol{\theta}'$ for computing the conditional market prices, such that the monetary profit of each trade $m_{i,t} - C$ matches exactly to the LMSR profit $S_{i,t}$ (i.e. (3.83) holds with equality). Such $\boldsymbol{\theta}'$ is found by

$$\boldsymbol{\theta}' = \operatorname*{arg\,min}_{\boldsymbol{\theta}_\lambda : \lambda \in [0,1]} F(\boldsymbol{\theta}_\lambda, v) - \lambda(F(\boldsymbol{\theta}_t, v) - F(\boldsymbol{\theta}_{t-1,\setminus i}, v)). \quad (3.84)$$

where $\boldsymbol{\theta}_\lambda = \lambda\boldsymbol{\theta}_t + (1-\lambda)\boldsymbol{\theta}_{t-1,\setminus i}$. The mechanism is given by Figure 3.9b.

3.9 Discussion and Summary

In this chapter we proposed a potential based market maker for handling partially-observable state spaces (PoPMM). The design of the PoPMM relies

Input: F, $F(\cdot, \mathsf{v})$, initial market position $\boldsymbol{\theta}_0$, and trader's id map γ;
for $t = 1$ **to** T **do**
  retrieve trader $i = \gamma(t)$ and holdings $\boldsymbol{\vartheta}_i$;
  record $\boldsymbol{\theta}_{\setminus i} = \boldsymbol{\theta}_{t-1} - \boldsymbol{\vartheta}_i$;
  δ_t trade costs $F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})$;
  update position $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_{t-1} + \delta_t$;
  update and store new holdings $\boldsymbol{\vartheta}_i$;
end
at T observe $\mathsf{v} = v$, close the market;
evaluate securities $\mathbf{r} = \mathbf{p}(\boldsymbol{\theta}_{\setminus i}, v)$;
a trader holding $\boldsymbol{\vartheta}$ gets payout $\boldsymbol{\vartheta}^\top \mathbf{r}$;

(a) Weakly incentive-compatible

Input: F, $F(\cdot, \mathsf{v})$, initial market position $\boldsymbol{\theta}_0$, and trader's id map γ;
for $t = 1$ **to** T **do**
  retrieve trader $i = \gamma(t)$ and holdings $\boldsymbol{\vartheta}_i$;
  record $\boldsymbol{\theta}_{\setminus i} = \boldsymbol{\theta}_{t-1} - \boldsymbol{\vartheta}_i$;
  δ_t trade costs $F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})$;
  update position $\boldsymbol{\theta}_t \leftarrow \boldsymbol{\theta}_{t-1} + \delta_t$;
  update and store new holdings $\boldsymbol{\vartheta}_i$;
end
at T observe $\mathsf{v} = v$, close the market;
find market position $\boldsymbol{\theta}'$ using (3.84);
evaluate securities $\mathbf{r} = \mathbf{p}(\boldsymbol{\theta}', v)$;
a trader holding $\boldsymbol{\vartheta}$ gets payout $\boldsymbol{\vartheta}^\top \mathbf{r}$;

(b) Incentive-compatible

Figure 3.9: Re-PoPMM: PoPMM allowing repeated trades. The market maker now has to keep track of the positions of all traders. For the (strictly) incentive-compatible PoPMM, it is more expensive to find the conditional market price as an optimization (3.84) is involved.

heavily on exponential families. In particular, the belief model that a PoPMM encodes is either an exponential family on the joint space (e.g. the mixture of Gaussians for Who-Touched-the-Ball game), or has an exponential family component for the latent variables (e.g. skill betting model for Go players), and the potential function of the PoPMM is defined to be the log partition function of the involved exponential family. We analysed theoretically and demonstrated by experiments how the PoPMM maintains bounded monetary loss and how it incentivizes traders to improve the quality of the market belief via transactions.

To the best of our knowledge, this is the first work on designing PMM for partially observable events. Prior to our work, Kutty (2014) has suggested using EM algorithm to model traders' activities when they temporarily have no access to the complete observation of the events. However, her market still falls into the fully-observable framework, as in the end the events must be completely observed to ensure the trading incentive. Another interesting work is given by Dudík et al. (2014). Although the setting is the fully-observable one, they design a PMM for sequentially observable events whose states will become less uncertain through time and are finally completely determined. The events can be treated as partially observable any time before the complete observation.

The first criticism would be the necessity of using exponential families. Admittedly, the PoPMM can encode flexible joint beliefs $p(\mathsf{v}, \mathsf{h})$ by having a fixed $p(\mathsf{v} \mid \mathsf{h})$ and only requiring $p(\mathsf{h})$ to vary in an exponential family (Section 3.4.2). Nevertheless, requiring exponential families in the design is restrictive. Do there exist other families of distributions that a PMM/PoPMM can represent efficiently? In Chapter 4, we complete the existing theory for the *generalized exponential families* (GEFs), established by Grünwald and Dawid (2004); Frongillo and Reid (2013), and show how GEFs can be applied to extend the design of potential based market maker.

In addition, it remains unclear how PoPMMs and in general potential based market makers aggregate traders' beliefs via transactions, despite that some convergence patterns in market prices have been discovered in the experiments. A thorough discussion is needed to characterize the equilibria of potential based markets and to understand the dynamics behind the price convergences. This will be done in Chapter 5.

Chapter 4

Representing Market Beliefs

A market belief is a distribution, expressed by current market status, over the uncertain future state upon which the prediction market is built. In a prediction market driven by a potential-based market maker (PMM), the market belief varies in a family of distributions parametrized by market maker's position, such that the expectation of security values w.r.t. the market belief coincides with the instantanous market prices. In previous chapters, we have seen how a conventional PMM can further defines a belief model that represents its belief via an exponential family, and how the exponential family belief model plays a central role in designing a *partially-observable PMM* (PoPMM). In both discussions, the exponential family is the only family of distributions used for modelling the market beliefs, due to its popularity in representing distributions and analytical simplicity. However, always representing market beliefs in an exponential family seems restrictive, especially when we consider the fact that set of all possible probability distributions over the future state is much larger than a particular exponential family.

In this chapter, we seek for a general parametrized family of distributions that be used to represent market beliefs beyond the exponential family, and further to improve the PoPMM design such that it will no longer depend on the exponential family. Our goal is achieved by allowing a PMM or a PoPMM to model its belief in a more general family of distributions, which is referred to as the *generalized exponential family* (GEF). On the one hand, as is implied by its name, a GEF maintains the key properties of an exponential family. On the other hand, a GEF can contain distributions very different from those in an exponential family. The former guarantees that such a PMM/PoPMM with its beliefs efficiently represented in a GEF can be constructed, while the latter increases flexibility of the representation.

This work is based on the GEF theory, first established by Grünwald and Dawid (2004) via game theory and later extended by Frongillo and Reid (2013) via convex analysis. The existing GEF theory can be applied to the design of the PMM directly, as is shown by Frongillo and Reid (2013). However, not all GEFs characterized by the existing theory is suitable for the PoPMM as some of them will suffer from high computational costs. We start by reviewing the existing theory, especially the key properties that makes a GEF representable by a PMM. We also contribute to the theory by reintroducing *regularity*, the central concept of the theory, more vigorously using convex analysis. Next, based on the existing theory, we characterize a special class of GEFs that possess certain desired conditioning structure. Finally, we incorporate GEFs into a PoPMM, to free it from the usage of the exponential family and make it efficient for practical application.

4.1 Motivation

Before we start, let's motivate this work a bit further. More specifically, we must justify (1) the necessity of including a model for market beliefs in the PMM design, (2) the non-necessity of representing market beliefs in an exponential family, and (3) the benefit of using GEF.

Necessity of modelling market beliefs At the first glance, it does not seem necessary to even introduce the concept of market beliefs in a PMM, let alone an explicit model of it. Indeed, Abernethy et al. (2011) show that the minimum structure of a PMM resulting from several desired axioms is a convex differential function with effective domain containing the convex hull of all the possible values of the securities. The concept of market beliefs is not explicitly defined in this minimum structure. Although market beliefs are introduced in later designs such as Abernethy et al. (2014), they are treated more like an add-on component and not essential to the functioning of the PMM.

However, despite being implicit, market beliefs are actually included in the minimum structure as a result of the *expressiveness* of the PMM. Expressiveness is a condition put on the price space of the PMM, such that for any risk-neutral trader there always exists a market price matched to the trader's expectation of the security values. That is, for any $p \in \mathcal{P}$, there exists $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ such that $\mathbb{E}_p[\boldsymbol{\phi}] = \nabla F(\boldsymbol{\theta})$. When designing the minimum structure, Abernethy et al. (2011) introduce the expressiveness condition as one axiom only for the purpose of characterizing the possible market positions of the PMM (or equivalently the effective domain of the potential function). But as we will show below, expressiveness can tell us more than the domain properties.

Expressiveness guarantees the existence of market beliefs, since it maps each market state to at least one distribution w.r.t. which the expectation of security values matches the market prices. Furthermore, in a *complete* market (i.e. a market where the number of linearly independent securities K is no smaller than $\dim(\Omega) - 1$), expressiveness can determine the model used for representing the market beliefs. For illustrative purpose, consider a complete prediction market on a fully-observable binary state space $\Omega = \{-1, 1\}$ offering a single security $\phi(\omega) := \omega$. For any trader with belief $p(\omega = 1) = p, p \in [0, 1]$, her expectation of ϕ is $\mathbb{E}[\phi] = 2p - 1$. Then by definition, a PMM with potential F and possible market positions Θ is expressive, if for every $p \in [0, 1]$ there always exists a position $\theta \in \Theta$ such that the market prices $\mathrm{d}F(\theta)/\mathrm{d}\theta = \mathbb{E}[\phi] = 2p - 1$. Therefore, this PMM models its market belief by a parametric family $\mathcal{P}_\Theta := \{(1 - p, p) \mid p = (1 + \mathrm{d}F(\theta)/\mathrm{d}\theta)/2\}_{\theta \in \Theta}$. In general, an expressive PMM in a complete market must have a belief model defined via its prices and parametrized by the market maker's position. If the market is incomplete (i.e. $K < \dim(\Omega) - 1$), then $\mathrm{d}F(\theta)/\mathrm{d}\theta = \mathbb{E}[\phi]$ together with $\sum_{\omega \in \Omega} p(\omega) = 1$ will not be enough to determine a unique distribution for each $\boldsymbol{\theta}$, hence the map from the market position to the market beliefs is one-to-many.

The necessity of modelling market beliefs is also supported by the need for building *arbitrage-free* markets. According to the *fundamental theorem of asset pricing* (see e.g. Schachermayer (2008); Haugh (2010)), a model of financial

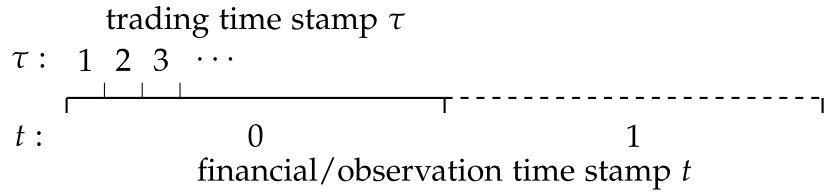


Figure 4.1: Two different types of time stamps in prediction markets. The trading time stamp labels the trade at each time τ, while the finiancial/observation time stamp tells whether the future state has been realized. Given fixed $\boldsymbol{\theta}$, the prediciton market can be viewed as a financial market with $(0, 1)$-time stamps.

market is free of arbitrage if and only if an equivalent measure p of the original measure P (i.e. the true probability measure of the future state) exists such that the prices for securities are martingales under p. A PMM with small enough trades (such that the market prices keep fixed at $\mathbf{p}(\boldsymbol{\theta})$) can be thought of as a special financial market model with only two financial time stamps: $t = 0$ for trading, and $t = 1$ for realizing state and clearing payouts of securities. Notice that the time stamp defined here is *not* for distinguishing different transactions. The financial market prices S_0 for securities $\boldsymbol{\phi}$ at $t = 0$ is equal to the market maker's prices $\mathbf{p}(\boldsymbol{\theta})$, and the prices S_1 at $t = 1$ is equal to the security values at the realized state $\boldsymbol{\phi}(\omega)$. Then, to build arbitrage-free market we must find a probability measure p under which the price $\{S_0, S_1\}$ is a martingale, that is,

$$\mathbf{p}(\boldsymbol{\theta}) = \underbrace{S_0 = \mathbb{E}_p[S_1]}_{p\text{-martingale}} = \mathbb{E}_p[\boldsymbol{\phi}]. \tag{4.1}$$

Based on this equality, we can then model the market belief at $\boldsymbol{\theta}$ by the martingale measure. Finally, to ensure the PMM is free of arbitrage at any position in $\boldsymbol{\Theta}$, we must associate with every position a market belief which is modelled by a martingale measure.

In summary, it is necessary to model the market belief to guarantee the expressiveness of the prediction market and the absence of arbitrage.

Non-necessity of using exponential family Representing market beliefs in an exponential family is restrictive. In particular, in a complete market built on a finite discrete state space, we can easily find a PMM representing beliefs

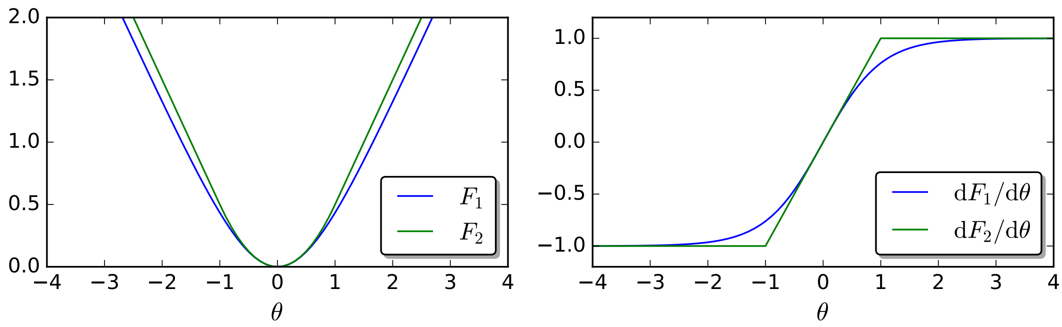


Figure 4.2: A binary complete market driven by two PMMs. Left: the potential functions of the PMMs. Right: the market prices for $\phi(\omega) = \omega \in \{-1, 1\}$. In this complete market, the market belief is represented in $p(\omega = 1) = p$ with $p \in \{(1 + \mathrm{d}F(\theta)/\mathrm{d}\theta)/2\}_{\theta \in \mathbb{R}}$. The PMM with potential F_1 models its belief by an exponential family $p(\omega) \propto \exp(\theta\phi(\omega))$, while the one with F_2 does not.

beyond exponential families. For example, again consider a binary state space $\Omega = \{-1, 1\}$, and two PMMs offering the same single security $\phi(\omega) = \omega$ and having the same possible positions $\Theta = \mathbb{R}$, but running under different potentials

$$F_1(\theta) = \log(\cosh(\theta)),\ F_2(\theta) = \begin{cases} -\theta - 1/2 & \theta \in [-\infty, -1] \\ \theta^2/2 & \theta \in (-1, 1) \\ \theta - 1/2 & \theta \in [1, +\infty] \end{cases} \tag{4.2}$$

The resulting market prices are

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F_1(\theta) = \tanh(\theta),\ \frac{\mathrm{d}}{\mathrm{d}\theta}F_2(\theta) = \begin{cases} -1 & \theta \in [-\infty, -1] \\ \theta & \theta \in (-1, 1) \\ 1 & \theta \in [1, +\infty] \end{cases}. \tag{4.3}$$

Figure 4.2 presents the plots of the potentials as well as the prices. By expressiveness, The market belief is computed by $p(\omega = 1) = p$ where $p = (1 + \mathrm{d}F(\theta)/\mathrm{d}\theta)/2$, that is

$$p_1 = \frac{\exp(\theta\phi(\omega = 1))}{\sum_{\omega \in \Omega} \exp(\theta\phi(\omega))},\ p_2 = \begin{cases} 0 & \theta \in [-\infty, -1] \\ (1 + \theta)/2 & \theta \in (-1, 1) \\ 1 & \theta \in [1, +\infty] \end{cases}. \tag{4.4}$$

The first PMM models its belief by an exponential family, while the second one does not. This is an example for a complete market with binary state space. For general incomplete markets, it is also possible to design a PMM with a belief model that is not an exponential family.

Benefit of using the GEF The exponential family provides an efficient way of encoding market beliefs into market prices and positions. The GEF provides a reasonable balance between generalizing the exponential family for flexible market design, and inheriting the key properties of the exponential family for efficient market belief representation. In fact, one important goal of the GEF theory is to characterize GEFs by directly using those key properties found in the exponential family.

4.2 Key properties of the exponential family

The market belief serves the PMM in two aspects. It makes the market expressive, by *encoding* each trader's true belief to the closest exponential family distribution under the KL-divergence. It also ensures the absence of arbitrage, by *decoding* the market prices as martingales. The exponential family can represent market beliefs in an efficient way due to the following two properties:

1. the gradient of the cumulant function $F := \log Z$ at $\boldsymbol{\theta}$ is matched to the mean statistics w.r.t. the exponential family distribution with parameter $\boldsymbol{\theta}$, that is, $\mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}] = \nabla F(\boldsymbol{\theta})$; and

2. each exponential family distribution $p_{\boldsymbol{\theta}}$ maximizes the Shannon entropy among all distributions that give the same mean statistics as $p_{\boldsymbol{\theta}}$, that is, $p_{\boldsymbol{\theta}} = \arg\inf_{p \in \Gamma_{\boldsymbol{\mu}}} -H(p)$, where $\Gamma_{\boldsymbol{\mu}} := \{p \in \mathcal{P} \mid \mathbb{E}_p[\boldsymbol{\phi}] = \boldsymbol{\mu}\}$ and $\boldsymbol{\mu} = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}]$.

The first property directly implements the price decoding, while the second property implements the belief encoding by matching the expected profit of a trade to the KL-divergence through a loss function defined below

$$L(p, p_{\boldsymbol{\theta}}) := F(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \mathbb{E}_p[\boldsymbol{\phi}] = -\mathbb{E}_p[\log p_{\boldsymbol{\theta}}], \tag{4.5}$$

$$= KL(p, p_{\boldsymbol{\theta}}) + H(p). \tag{4.6}$$

It is worth noting that if p is in the same exponential family as $p_{\boldsymbol{\theta}}$, then $L(p, p_{\boldsymbol{\theta}})$ is the negated expected score of the log scoring rule $S(\omega, p_{\boldsymbol{\theta}}) := \boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega) -$

$F(\boldsymbol{\theta}) = \log p_{\boldsymbol{\theta}}(\omega)$.

Then for a risk-neutral myopic trader, the highest profit she expects by trading $\delta_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ is

$$\max \mathbb{E}_p[m_t(\omega)] = L(p, p_{\boldsymbol{\theta}_{t-1}}) - \min_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} L(p, p_{\boldsymbol{\theta}_t}) \tag{4.7}$$

$$= L(p, p_{\boldsymbol{\theta}_{t-1}}) - H(p) - \min_{p_{\boldsymbol{\theta}_t} \in \mathcal{P}_{\boldsymbol{\Theta}}} KL(p, p_{\boldsymbol{\theta}_t}). \tag{4.8}$$

Therefore, her true belief p will be encoded into $p_{\boldsymbol{\theta}_t}$, which minimizes the KL-divergence in the exponential family. The optimal $p_{\boldsymbol{\theta}_t}$ is such that its mean statistics are matched to that of p, $\mathbb{E}_p[\boldsymbol{\phi}] = \mathbb{E}_{p_{\boldsymbol{\theta}_t}}[\boldsymbol{\phi}]$,

However, these properties are *not* exclusive to the exponential family. The first property simply requires that a cumulant function exists for the whole family of distributions. The second property in essence only asks each distribution in the family maximizes a certain objective, which only needs to be convex for defining a Bregman divergence, and does not have to be the Shannon entropy. In principle, any family with these two properties can potentially be used in the PMM design, serving the same role as the exponential family, and more importantly, generalizing the processes of belief encoding and price decoding.

4.3 Generalized exponential family (GEF)

A generalized exponential family (GEF) is a family of distributions derived from the maximum entropy (MaxEnt) principle with a generalized entropy function instead of the Shannon entropy. A GEF maintains both key properties of the exponential family.

In this section, we review the existing theory for the GEF. The content of the review will mainly follow the work of Frongillo and Reid (2013) based on convex analysis, but will differ in some details. The concepts and results are organized in a way to parallel the formulation of Frongillo and Reid (2013) to the original formulation of Grünwald and Dawid (2004) based on game theory. We also make our own contribution by reintroducing *regularity*, and then unifying it with similar concepts that have been previously defined in both formulations.

4.3.1 Convex analysis in the space of measures

To define generalized entropies and GEFs we need to analyse convex functions defined on the space of (probability) measures. In general, the space of measures is no longer $\mathbb{R}^n$, the domain space for deriving classic results of convex analysis (Rockafellar, 1970). Fortunately, the space of measures has a rich enough structure that maintains most of the results in $\mathbb{R}^n$. In mathematical jargon, the space that probability measures sit in carries a Hausdorff locally convex topology, a structure that introduces the sense of closeness between points in the space as well as the continuity of functions defined on the space, and is dually paired with another Hausdorff locally convex topological vector space through a bilinear functional.

In the rest of this section, the details of the convex structure of the space of measures are explained without mathematical rigour. For readers who are less interested in the details, please jump to the next section, and keep in mind that most the results in the space of measures coincide with the classic ones due to the convex structure. For a thorough introduction to the convex analysis on general topological spaces we refer to e.g. Cheridito (2013); Barbu and Precupanu (2012); Zalinescu (2002).

Consider a measurable space $(\Omega, \mathcal{F})$, where Ω is the sample space of the uncertain events equipped with some σ-algebra $\mathcal{F}$. Let $\mathcal{M}$ be the set of all finite *signed* measures (i.e. measures that allow using negative values) on $(\Omega, \mathcal{F})$. The space $\mathcal{M}$ is a vector space. Meanwhile, the set of all real-valued measurable functions on $(\Omega, \mathcal{F})$ forms another vector space, denoted by $\mathcal{X}$. The integral of a function in $\mathcal{X}$ w.r.t. a measure in $\mathcal{M}$

$$\langle m, \mathsf{x} \rangle := \int_{\omega \in \Omega} \mathsf{x}(\omega) \mathrm{d}m(\omega), \ \forall m \in \mathcal{M}, \forall \mathsf{x} \in \mathcal{X}, \tag{4.9}$$

then forms a bilinear functional on these two spaces $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{X} \to \mathbb{R}$. In addition, this bilinear functional separates points in both $\mathcal{M}$ and $\mathcal{X}$, i.e. for every $m \in \mathcal{M}$, there exists a $\mathsf{x} \in \mathcal{X}$ such that $\langle m, \mathsf{x} \rangle \neq 0$ and similar for every $\mathsf{x} \in \mathcal{X}$. Through the bilinear functional, a family of linear functionals on $\mathcal{M}$ with index set $\mathcal{X}$, $\{f_{\mathsf{x}} := \langle \cdot, \mathsf{x} \rangle\}_{\mathsf{x} \in \mathcal{X}}$, is also introduced, which then induces a Hausdorff locally convex topology on $\mathcal{M}$ such that every linear functional f_{x} is continuous on $\mathcal{M}$[1]. Symmetrically, a Hausdorff locally convex topology

[1]In topological sense, a real-valued function is *continuous* if the pre-image of every open

is also added to $\mathcal{X}$. The two spaces together with the bilinear functional $(\mathcal{M}, \mathcal{X}, \langle\cdot,\cdot\rangle)$ define a *dual system*, which carries a rich enough structure to support convex analysis.

The set of all probability measures $\mathcal{M}_1^+$ is then characterized by the intersection of the convex cone of positive measures $\mathcal{M}^+ := \{m \in \mathcal{M} \mid m \geq 0\}$ and the affine subspace containing all measures that assign a measure of one to the whole sample space $\mathcal{M}_1 := \{m \in \mathcal{M} \mid m(\Omega) = 1\}$, that is, $\mathcal{M}_1^+ = \mathcal{M}^+ \cap \mathcal{M}_1$. Let $\mathcal{P}$ be a convex subset of $\mathcal{M}_1^+$ with a non-empty interior[2] and contains all probability measures that we are interested in. Denote $\text{int}(\mathcal{P})$ the interior of $\mathcal{P}$, then it follows that $\text{int}(\mathcal{P}) \neq \emptyset$. For example, in the prediction market setting $\mathcal{P}$ will contain all possible beliefs a trader can have. Our discussions will be conducted on $\mathcal{P}$.

4.3.2 Generalized negated entropy and GEF

Entropies are (strictly) concave functions of the probability distribution. To directly apply convex analysis, in the following discussion we will always refer to the *negation of the entropy*. Also for simplicity, the negated entropy will be just called "entropy" when the context is clear. For readers who prefer the original concave definition of entropies, an extra negation sign should always be attached to the entropy functions.

First we extend the concept of entropy beyond Shannon. This generalized negated entropy is mainly characterized by convexity.

Definition 4.1. A *generalized (negated) entropy* on $\mathcal{P}$ is a function $G : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ that is *lower semi-continuous (l.s.c.)* and *strictly convex* with effective domain $\text{dom}(G) = \mathcal{P}$.

The properties of a generalized entropy G include that: (1) $G = G^{**}$ which results directly from the Fenchel-Moreau theorem (the convex conjugate G^* of G will be introduced shortly); (2) since G is strictly convex, $(\partial G)^{-1} = \partial G^*$ is single-valued on $\text{int}(\mathcal{P})$ and contains the gradient as its unique element,

interval in $\mathbb{R}$ through the function is an open set (and so falls into the topology).

[2]Here the interior of $\mathcal{P}$ is w.r.t. the affine subspace $\mathcal{M}_1$ with the *subspace topology* induced from $\mathcal{M}$. When w.r.t. $\mathcal{M}$, it then becomes the relative interior of $\mathcal{P}$.

$\partial G^* = \{\nabla G^*\}$; and (3) G is continuous on $\text{int}(\mathcal{P})$, by Theorem 3.4.1 of Cheridito (2013). According to the definition, the Shannon entropy is a generalized entropy.

Lemma 4.2. *The convex conjugate of a generalized entropy G,*

$$G^*(\mathrm{x}) := \sup_{p \in \mathcal{P}} \langle p, \mathrm{x} \rangle - G(p), \; \mathrm{x} \in \mathcal{X} \tag{4.10}$$

is a l.s.c. and proper convex function on $\mathcal{X}$, and is continuous on $\text{int}(\text{dom}(G^*))$.

Proof. The lower semi-continuity and proper convexity follows directly from the Fenchel-Moreau theorem.

Given a $p_0 \in \text{int}(\mathcal{P})$, the image of the sub-differential map at p_0 is characterized by (Barbu and Precupanu, 2012, Proposition 2.33)

$$\partial G(p_0) = \{\mathrm{x} \in \mathcal{X} \mid G(p_0) + G^*(\mathrm{x}) \leq \langle p_0, \mathrm{x} \rangle\}. \tag{4.11}$$

Convexity and lower semi-continuity of $G^*(\mathrm{x}) - \langle p_0, \mathrm{x} \rangle$ implies that $\partial G(p_0)$ is a closed convex subset of $\mathcal{X}$. Since $\text{int}(\mathcal{P}) \neq \emptyset$, there exists a neighbourhood U of p_0 such that the sub-differential exists for all points in U. Let $\partial G(U) := \bigcup_{p \in U} \partial G(p)$ be the image of U through the sub-differential map. Since the reverse sub-differential map is single-valued, different points in U will map to different subsets of $\partial G(U)$, which implies that $\text{int}(\partial G(U)) \neq \emptyset$. Therefore, G^* is finite on some neighbourhood $\text{int}(\partial G(U))$ of $\mathrm{x}_0 \in \partial G(p_0)$. By Theorem 3.4.1 of Cheridito (2013), G^* is continuous on $\text{int}(\text{dom}(G^*))$. □

Definition 4.3. A *statistic* ϕ is a function in $\mathcal{X}$.

Thus a statistic is a real-valued measurable function on $(\Omega, \mathcal{F})$. It evaluates each state by a real number. Usually a set of K statistics will be considered. We collect them into a vector, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_K)^\top$ with ϕ_k indexing the k-th statistic.

Based on a generalized entropy and statistics we can introduce two functions. Both of them will play key roles in characterizing the GEF.

Definition 4.4. Given a generalized entropy G and statistics $\boldsymbol{\phi}$, the *MaxEnt* *function* and the *dual MaxEnt function* are defined to be

$$g(\boldsymbol{\mu}) := \inf_{p \in \Gamma_{\boldsymbol{\mu}}} G(p) \text{ where } \Gamma_{\boldsymbol{\mu}} := \{p \in \mathcal{P} \mid \mathbb{E}_p[\boldsymbol{\phi}] = \boldsymbol{\mu}\}, \tag{4.12}$$

$$F(\boldsymbol{\theta}) := \sup_{p \in \mathcal{P}} \boldsymbol{\theta}^\top \mathbb{E}_p[\boldsymbol{\phi}] - G(p) = G^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}), \tag{4.13}$$

The effective domains of these two functions are given by $\mathrm{dom}(g) := \{\boldsymbol{\mu} \in \mathbb{R}^K \mid g(\boldsymbol{\mu}) < +\infty\}$ and $\mathrm{dom}(F) := \{\boldsymbol{\theta} \in \mathbb{R}^K \mid F(\boldsymbol{\theta}) < +\infty\}$.

Again remind that our generalized entropy is negated, thus the MaxEnt formula here involves infimum instead of supremum. We name g the MaxEnt function as it directly results from the generalized MaxEnt principle with the Shannon entropy replaced by the generalized entropy G. We name F the dual MaxEnt function since it is precisely the convex conjugate of g.

Lemma 4.5 (Lemma 2 of Frongillo and Reid (2013))**.** *Let g and F be the MaxEnt and dual MaxEnt functions generated by the generalized entropy G and statistics $\boldsymbol{\phi}$. Then g is convex and $F = g^*$.*

Now we are ready to define the GEF.

Definition 4.6 (Condition 7.3 of Grünwald and Dawid (2004))**.** Given a generalized entropy G and statistics $\boldsymbol{\phi}$, define the dual MaxEnt function F in (4.13). Then the *generalized exponential family* (GEF) generated by G and $\boldsymbol{\phi}$ is a set of distributions $\mathcal{P}_{\boldsymbol{\Theta}} := \{p_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta} \in \boldsymbol{\Theta}}$ with parameter set $\boldsymbol{\Theta} \subseteq \mathrm{dom}(F)$, such that the supremum in F is *attained* at $p_{\boldsymbol{\theta}} \in \mathcal{P}$ for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. The dual MaxEnt function F is also called the *cumulant function* of the GEF.

By definition a GEF $\mathcal{P}_{\boldsymbol{\Theta}}$ is a subset of $\mathcal{P}$. $\mathcal{P}_{\boldsymbol{\Theta}}$ might trivially be an empty set, which means that no GEF exists for current configurations of $(G, \boldsymbol{\phi})$. Without loss of generality we assume $\mathcal{P}_{\boldsymbol{\Theta}} \neq \emptyset$.

Now we show that Definition 4.6 of the GEF is *almost* identical to the one given by Frongillo and Reid (2013), and so will be the properties.

Theorem 4.7. *Given a generalized entropy G and statistics $\boldsymbol{\phi}$, let g, F and $\mathcal{P}_{\boldsymbol{\Theta}}$ be the MaxEnt, dual MaxEnt and the GEF generated by G and $\boldsymbol{\phi}$. Then*

1. *the GEF $\mathcal{P}_{\boldsymbol{\Theta}}$ is characterized by the sub-differential of G^* at each $\boldsymbol{\theta}^\top \boldsymbol{\phi}$, that is, $\mathcal{P}_{\boldsymbol{\Theta}} = \{p_{\boldsymbol{\theta}} = \nabla G^*(\boldsymbol{\theta}^\top \boldsymbol{\phi})\}_{\boldsymbol{\theta} \in \boldsymbol{\Theta}}$;*
2. *$\boldsymbol{\Theta} \subseteq \mathrm{dom}(\partial F)$ and $\mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}] = \nabla F(\boldsymbol{\theta}), \forall \boldsymbol{\theta} \in \boldsymbol{\Theta}$;*

3. $\inf_{p \in \Gamma_\mu} G(p) = G(p_\theta) = g(\mu)$ *with* $\mu = \mathbb{E}_{p_\theta}[\phi], \forall \theta \in \Theta$.

Proof. For every $\theta \in \Theta$ the supremum in F is attained at $p_\theta \in \mathcal{P}$, implying that $\theta^\top \phi \in \partial G(p_\theta)$. Since G is l.s.c. and proper convex, we have $G = G^{**}$. Then for any dual pair $p \in \mathcal{P}$ and $\mathrm{x} \in \mathcal{X}$, $\mathrm{x} \in \partial G(p)$ if and only if $p \in \partial G^*(\mathrm{x})$. This implies $p_\theta \in \partial G^*(\theta^\top \phi)$. The strict convexity of G gives that $\partial G^* = \{\nabla G^*\}$ and so $p_\theta = \nabla G^*(\theta^\top \phi)$.

For every $\theta_0 \in \Theta$, it holds that $G^*(\mathrm{x}) \geq G^*(\theta_0^\top \phi) + \langle p_{\theta_0}, \mathrm{x} - \theta_0^\top \phi \rangle, \forall \mathrm{x} \in \mathcal{X}$. In particular, $\forall \theta \in \mathbb{R}^K$, $\theta^\top \phi \in \mathrm{span}(\phi) \subseteq \mathcal{X}$ and we have $G^*(\theta^\top \phi) \geq G^*(\theta_0^\top \phi) + \langle p_{\theta_0}, \theta^\top \phi - \theta_0^\top \phi \rangle = G^*(\theta_0^\top \phi) + (\theta - \theta_0)^\top \mathbb{E}_{p_\theta}[\phi]$. It together with the single-valued property of ∂G^* gives $\mathbb{E}_{p_\theta}[\phi] = \nabla F(\theta)$ and $\Theta \subseteq \mathrm{dom}(\partial F)$.

Finally, for every $p_\theta \in \mathcal{P}_\Theta$, choose $\mu := \mathbb{E}_{p_\theta}[\phi]$. Then $\mu \in \mathbb{R}^K$ since $\theta \in \mathbb{R}^K$ and $\theta^\top \mu = \langle p_\theta, \theta^\top \phi \rangle \in \mathbb{R}^K$. Thus Γ_μ is well defined and we have

$$p_\theta = \arg\sup_{p \in \Gamma_\mu} \; \theta^\top \mathbb{E}_p[\phi] - G(p) = \arg\inf_{p \in \Gamma_\mu} \; G(p). \tag{4.14}$$

It follows that $g(\mu) = G(p_\theta)$. □

However, there exists some differences between Theorem 4.7 and the corresponding results in Frongillo and Reid (2013). First, in the original definition of Frongillo and Reid (2013) the GEF has domain $\Theta = \mathrm{dom}(F)$, which may cause trouble as the sub-differential may not exist for some $\theta \in \mathrm{dom}(F)$, leaving the GEF distribution undefined at this point. Comparatively, in our definition, the domain of a GEF is $\Theta \subseteq \mathrm{dom}(\partial F) \subseteq \mathrm{dom}(F)$ which guarantees the existence of sub-differential and thus the existence of a GEF distribution for every θ in Θ. Second, the strict convexity we require for a generalized entropy leads to a single-valued map (through gradient) at each θ, resulting in a unique GEF distribution for each θ. This eliminates the potential ambiguity in the original definition, in which each GEF distribution is only said to belong to the sub-differential. Finally, we refine our proof of Property 2 such that it no longer requires the GEF to be regular in the sense of Definition 3 of Frongillo and Reid (2013). Hence now Property 2 should hold for all GEFs.

Property 2 of Theorem 4.7 enables the GEF to maintain the first key property of the exponential family. It shows that the expectation of ϕ w.r.t. every GEF distribution has been encoded into the gradient of the cumulant F (and this is the reason why F is called the cumulant function). However, by having just the cumulant F we cannot completely characterize a GEF since its parameter set $\boldsymbol{\Theta}$ is only known to be a subset of $\mathrm{dom}(\partial F)$. If further the parameter set $\boldsymbol{\Theta}$ coincides with $\mathrm{dom}(\partial F)$, then $\boldsymbol{\Theta}$ will become an intrinsic feature of F, and the GEF will be characterized by the cumulant. Such a GEF can then be applied to the market design efficiently through its cumulant.

It turns out that there exists a simple sufficient condition to make $\boldsymbol{\Theta}$ determinable by $\mathrm{dom}(\partial F)$.

Theorem 4.8. *Given a GEF $\mathcal{P}_{\boldsymbol{\Theta}}$ with its cumulant function F, if the effective domain of ∂F is non-empty and open, then $\boldsymbol{\Theta} = \mathrm{dom}(\partial F)$.*

Proof. Since $\mathrm{int}(\mathrm{dom}(F)) \subseteq \mathrm{dom}(\partial F) \subseteq \mathrm{dom}(F)$, $\mathrm{dom}(\partial F)$ being non-empty and open implies that $\mathrm{dom}(\partial F) = \mathrm{int}(\mathrm{dom}(F))$ and $\mathrm{int}(\mathrm{dom}(F)) \neq \emptyset$. Hence $\{\boldsymbol{\theta}^\top \boldsymbol{\phi} \mid \boldsymbol{\theta} \in \mathrm{dom}(\partial F)\} \subseteq \mathrm{int}(\mathrm{dom}(G^*))$. By Lemma 4.2 G^* is continuous on $\mathrm{int}(\mathrm{dom}(G^*))$, then in particular G^* is continuous at $\boldsymbol{\theta}^\top \boldsymbol{\phi}$ for every $\boldsymbol{\theta} \in \mathrm{dom}(F)$. It follows that, given a fixed $\boldsymbol{\theta}_0^\top \boldsymbol{\phi}$, the directional derivative

$$(G^*)'(\boldsymbol{\theta}_0^\top \boldsymbol{\phi}, \mathrm{x}) := \lim_{\lambda \to 0^+} \frac{G^*(\boldsymbol{\theta}_0^\top \boldsymbol{\phi} + \lambda \mathrm{x}) - G^*(\boldsymbol{\theta}_0^\top \boldsymbol{\phi})}{\lambda} \tag{4.15}$$

is a real-valued continuous sub-linear function of $\mathrm{x} \in \mathcal{X}$. In addition, when being restricted to the linear subspace $\mathrm{span}(\boldsymbol{\phi})$ of $\mathcal{X}$, $(G^*)'(\boldsymbol{\theta}_0^\top \boldsymbol{\phi}, \mathrm{x})$ dominates the linear functional $f(\boldsymbol{\theta}) := \boldsymbol{\theta}^\top \boldsymbol{\mu}_0$ where $\boldsymbol{\mu}_0 = \nabla F(\boldsymbol{\theta}_0)$, since by convexity $F(\boldsymbol{\theta}_0 + \lambda \boldsymbol{\theta}) - F(\boldsymbol{\theta}_0) \geq \lambda \boldsymbol{\theta}^\top \nabla F(\boldsymbol{\theta}_0)$ and thus we have

$$(G^*)'(\boldsymbol{\theta}_0^\top \boldsymbol{\phi}, \mathrm{x}) = \lim_{\lambda \to 0^+} \frac{F(\boldsymbol{\theta}_0 + \lambda \boldsymbol{\theta}) - F(\boldsymbol{\theta}_0)}{\lambda} \geq \boldsymbol{\theta}^\top \boldsymbol{\mu}_0. \tag{4.16}$$

By the Hahn-Banach theorem (Rudin, 1991, Theorem 3.3) there exists a continuous linear functional $\langle p_0, \cdot \rangle$ with $p_0 \in \mathcal{P}$, such that $\langle p_0, \boldsymbol{\theta}^\top \boldsymbol{\phi} \rangle = \boldsymbol{\theta}^\top \boldsymbol{\mu}_0$ on the subspace $\mathrm{span}(\boldsymbol{\phi})$ and that $\langle p_0, \mathrm{x} \rangle \leq (G^*)'(\boldsymbol{\theta}_0^\top \boldsymbol{\phi}; \mathrm{x})$ for all $\mathrm{x} \in \mathcal{X}$. Therefore, for every $\boldsymbol{\mu}_0 \in \partial F(\boldsymbol{\theta}_0)$ there exists a sub-gradient $p_0 \in \partial G^*(\boldsymbol{\theta}_0^\top \boldsymbol{\phi})$, implying $\mathrm{dom}(\partial F) \subseteq \boldsymbol{\Theta}$. Combine it with Property 2 of Theorem 4.7 to complete the proof. □

4.3.3 Regular GEF

Motivated by Theorem 4.8, we can introduce a special GEFs by adding the sufficient condition as regularity.

Definition 4.9. A GEF $\mathcal{P}_{\boldsymbol{\Theta}}$ with its cumulant function F is *regular* if $\operatorname{dom}(\partial F)$ is non-empty and open.

According to Theorem 4.8, a regular GEF has its parameter set determined by ∂F, that is, $\boldsymbol{\Theta} = \operatorname{dom}(\partial F)$.

It is worth mentioning that our definition of the regular GEF directly generalizes the regular exponential family, which is characterized by having an open parameter set (Barndorff-Nielsen, 1978). In fact, every exponential family is by definition also a GEF derived from the generalized entropy $G = -H$.

The regular GEF has been introduced in different ways by Frongillo and Reid (2013) and Grünwald and Dawid (2004). In Frongillo and Reid (2013) a GEF is regular if its cumulant is l.s.c. and proper convex, while in Grünwald and Dawid (2004) a regular GEF is the set of all GEF distributions whose mean statistics are *regular points* (Definition 7.3). Now Definition 4.9 gives a new definition of the regular GEF that differs from the previous two. This leaves us a question that how the three different concepts of regularity are connected and whether it is even necessary to introduce a new definition of regularity as it seems only makes the theory more complicated. Fortunately, with our new definition we are now able to unify these concepts of regularity in different contexts. In particular, it can be shown that the regular GEF defined in our way is also regular in the other two senses. Note that, to the best of our knowledge, no work has been done to explicitly link the regular GEF of Frongillo and Reid (2013) and that of Grünwald and Dawid (2004).

We first show that the regular GEF given by Definition 4.9 also interprets regularity in the sense of Frongillo and Reid (2013). We also refine the important property of the regular GEF: there exists a bijection between the regular GEFs and certain class of Bregman divergences. Then we match the regular GEF to the one defined by Grünwald and Dawid (2004) via a game-theoretic approach. Unless otherwise stated, in the following discussion when mentioning the term regularity alone we mean regularity given by our Definition.

Regularity of Frongillo and Reid (2013) Under their definition, a GEF is regular if its cumulant is l.s.c. and proper convex. Therefore, the following theorem shows that every regular GEF is also regular in the sense of Frongillo and Reid (2013).

Theorem 4.10. *The cumulant function F of a regular GEF $\mathcal{P}_\Theta$ is proper convex, and is l.s.c. on* $\mathrm{dom}(\partial F)$.

Proof. By Definition 4.9 the sub-differential exists on $\mathrm{dom}(\partial F)$. It then follows directly from Lemma 3.5.7 of Cheridito (2013). □

Notice that a regular GEF in the sense of Frongillo and Reid (2013) may have a parameter set $\mathrm{dom}(F)$ that is not open, hence not regular in our sense of Definition 4.9. However, the distributions on the boundary of the parameter set, $\mathrm{dom}(F) \setminus \mathrm{int}(\mathrm{dom}(F))$ are very few compared to those in $\mathrm{int}(\mathrm{dom}(F))$, and the properties of the GEF will still be mainly contributed by the distribtuions in $\mathrm{int}(\mathrm{dom}(F))$.

Bijection between regular GEFs and Bregman divergences Let $(U, V, \langle\cdot,\cdot\rangle)$ be a dual pair of topological vector spaces, and $f : U \to \mathbb{R} \cup \{+\infty\}$ be a l.s.c. and proper convex function on U, the Bregman divergence generated by f is defined by $D_{f,\mathrm{d}f} : U \times U \to \mathbb{R} \cup \{+\infty\}$

$$D_{f,\mathrm{d}f}(u, u_0) := f(u) - f(u_0) - \langle u - u_0, v_0\rangle \tag{4.17}$$

$$= f(u) + f^*(v_0) - \langle u, v_0\rangle, \tag{4.18}$$

where $\mathrm{d}f$ is a *sub-gradient map* (not the sub-differential map) which is *single-valued* and it maps each $u_0 \in \mathrm{dom}(\partial f)$ to a sub-gradient $v_0 \in \partial f(u_0)$, and $v_0 = \mathrm{d}f(u_0)$. If either $f(u) = +\infty$ or $\partial f(u_0) = \emptyset$, $D(u, u_0) = +\infty$. Geometrically, the Bregman divergence $D_{f,\mathrm{d}f}(\cdot, u_0)$ is the vertical distance between f and the hyperplane $h_0 := \langle\cdot, v_0\rangle - f^*(v_0)$ that supports f at u_0. The lower semi-continuity and proper convexity of f guarantee the existence of such supporting hyperplanes (Barbu and Precupanu, 2012, Proposition 2.20).

The bijection is established between regular GEFs and Bregman divergences generated by a special class of MaxEnt functions. The original statement of the bijection is given by Theorem 2 of Frongillo and Reid (2013). Here we improve it a bit to fit our new definition of regularity.

Theorem 4.11. *Given the generalized entropy G, the set of all regular GEFs is in bijection with the set of Bregman divergences generated by the MaxEnt functions* *whose sub-differential map has an* open image.

Proof. By Lemma 4.5, g is convex and $F = g^*$. Furthermore, if there exists x_0 such that $\partial g(x_0) \neq \emptyset$, then g is proper convex on its effective domain and is l.s.c. at x_0 (Cheridito, 2013, Lemma 3.5.7). This property of g will be used for the proof.

Proof of sufficiency. For every configuration of statistics $\boldsymbol{\phi}$ such that the GEF $\mathcal{P}_{\boldsymbol{\Theta}}$ generated by G and $\boldsymbol{\phi}$ is regular, $\boldsymbol{\Theta} = \text{dom}(\partial F)$ and it is open. By Property 3 of Theorem 4.7, ∂g is non-empty at every $\boldsymbol{\mu} = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}]$ with $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, since $\boldsymbol{\theta} \in \partial g(\boldsymbol{\mu})$. Therefore, g is proper convex and l.s.c. on $\text{dom}(\partial g)$. In addition, by Corollary 23.5.1 of Rockafellar (1970) $\text{ran}(\partial g) = \boldsymbol{\Theta}$ is open.

Proof of necessity. For every MaxEnt function whose sub-differential map has an open image, it is proper convex and l.s.c. on $\text{dom}(\partial g)$. Then $\text{dom}(\partial F) = \text{ran}(\partial g)$ is open. The proof is completed by the definition of regularity. □

Regularity of Grünwald and Dawid (2004) This regularity is based on a loss function $L : \Omega \times \mathcal{P} \to \mathbb{R}$ defined based on the generalized entropy as follows (see Section 3.5.4)

$$L(\omega, p_0) := -G(p_0) - (\mathsf{x}(\omega) - \langle p_0, \mathsf{x} \rangle) = G^*(\mathsf{x}) - \mathsf{x}(\omega), \tag{4.19}$$

where $\mathsf{x} \in \partial G(p_0)$ is a sub-gradient at p_0 (thus being the dual of p_0). Given belief $p \in \mathcal{P}$, the minimum expected loss

$$\inf_{p_0 \in \mathcal{Q}} L(p, p_0) := \inf_{p_0 \in \mathcal{Q}} \mathbb{E}_p[L(\omega, p_0)] = \inf_{p_0 \in \mathcal{Q}} D_{G,\mathrm{d}G}(p, p_0) - G(p) \tag{4.20}$$

If $\mathcal{Q} = \mathcal{P}$, then the minimum is attained at $p_0 = p$ and has a minimum value equal to $-G(p)$, the (negated) generalized entropy of p.

Given the generalized entropy G and statistics $\boldsymbol{\phi}$, define the MaxEnt function g, loss function L and GEF $\mathcal{P}_{\boldsymbol{\Theta}}$. Then $\boldsymbol{\mu}_0 \in \text{dom}(g)$ is a *regular point*, if there exists a $p_0 \in \mathcal{P}_{\boldsymbol{\Theta}}$ such that $L(\cdot, p_0)$ can hold the following linear form p_0-almost surely

$$L(\omega, p_0) = \beta_0 + \boldsymbol{\beta}^\top \boldsymbol{\phi}(\omega), \tag{4.21}$$

where $\boldsymbol{\beta} = (\beta_1 \ldots \beta_K)^\top$ and $\beta_0, \ldots, \beta_K \in \mathbb{R}$. GEF is regular in the sense of Grünwald and Dawid (2004) if $\boldsymbol{\mu} = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}]$ is a regular point for every $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Theorem 4.12. *For every distribution in a regular GEF $\mathcal{P}_{\boldsymbol{\Theta}}$, its mean statistics are a regular point.*

Proof. For every $p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}$, we set $p_0 = p_{\boldsymbol{\theta}}$ and choose the sub-gradient map such that $\mathrm{d}G : p_{\boldsymbol{\theta}} \mapsto \boldsymbol{\theta}^\top \boldsymbol{\phi}$. Then the loss function L in (4.19) becomes

$$L(\omega, p_{\boldsymbol{\theta}}) = F(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega). \tag{4.22}$$

Therefore, the loss function has a linear form, with $\beta_0 = F(\boldsymbol{\theta}), \boldsymbol{\beta} = -\boldsymbol{\theta}$. □

We end this section by computing the minimum expected loss when the report distribution p_0 is constrained in the regular GEF. Given belief $p \in \mathcal{P}$, we have

$$\begin{aligned}\inf_{p_0 \in \mathcal{P}_{\boldsymbol{\Theta}}} L(p, p_0) &= \inf_{p_0 \in \mathcal{P}_{\boldsymbol{\Theta}}} D_{G,\mathrm{d}G}(p, p_0) - G(p) = (G(p) - g(\mathbb{E}_p[\boldsymbol{\phi}])) - G(p) \\ &= -g(\mathbb{E}_p[\boldsymbol{\phi}]).\end{aligned} \tag{4.23}$$

Namely, the minimum loss is the (negated) entropy of the GEF distribution which sits in the set $\Gamma_{\mathbb{E}_p[\boldsymbol{\phi}]}$ the belief p belongs to.

4.3.4 Examples of (regular) GEFs

Here we present two examples of GEFs. The first one defines distributions on finite discrete space while the second one defines distributions on an interval in $\mathbb{R}$.

A discrete GEF Recall that, in the motivation section we present a very simple non-exponential family on a binary state $\omega \in \{-1, 1\}$. It has one sufficient statistic $\phi(\omega) = \omega$ and its potential F and probability $p = p(\omega = 1)$ are

$$F(\theta) = \begin{cases} -\theta - 1/2 & \theta \in [-\infty, -1] \\ \theta^2/2 & \theta \in (-1, 1) \\ \theta - 1/2 & \theta \in [1, +\infty] \end{cases}, p = \begin{cases} 0 & \theta \in [-\infty, -1] \\ (1+\theta)/2 & \theta \in (-1, 1) \\ 1 & \theta \in [1, +\infty] \end{cases}. \tag{4.24}$$

It turns out that this family of distributions is a GEF generated by a *quadratic generalized entropy* G and statistic ϕ, where this quadratic generalized entropy has the following form

$$G(p) := \sum_{\omega \in \Omega} p(\omega)^2 - \frac{1}{2}. \tag{4.25}$$

To obtain the potential and the probability, we need to solve the optimization problem contained in the convex conjugate $G^*(\mathsf{x})$ (4.10). Assuming that $p(\omega) > 0, \forall\omega \in \Omega$, using calculus of variations with constraint $\mathbb{E}_p[1] = 1$, we have $p(\omega) = (\mathsf{x}(\omega) + \lambda)/2$ where λ is the Lagrange multiplier for the constraint. Applying the constraint, $\lambda = 1 - \sum_{\omega\in\Omega} \mathsf{x}(\omega)/2$, and the convex conjugate as a function of x is

$$G^*(\mathsf{x}) = \frac{1}{4}\sum_{\omega\in\Omega} \mathsf{x}(\omega)^2 - \frac{1}{4}\sum_{\omega\in\Omega} \lambda^2 + \frac{1}{2}. \tag{4.26}$$

By Theorem 4.7, set $\mathsf{x} = \theta\phi = \theta\omega$ to obtain the potential of the GEF. Since $\sum_{\omega\in\Omega}\omega = 0$ and $\sum_{\omega\in\Omega}\omega^2 = 2$, we have $\lambda = 1, F(\theta) = G^*(\theta\phi) = \theta^2/2$ and $p_\theta(\omega) = (1 + \theta\omega)/2$.

We can check if the derivative of F gives the desired expectation of ϕ w.r.t. the corresponding GEF distribution p_θ

$$\mathbb{E}_{p_\theta}[\phi] = \sum_{\omega\in\Omega} \frac{1+\theta\omega}{2}\omega = \theta = \frac{\mathrm{d}}{\mathrm{d}\theta}F(\theta). \tag{4.27}$$

Finally, since $\Theta = \mathbb{R}$, this GEF is by definition a regular GEF.

Calculus of variations adds an arbitrary perturbation $\delta p(\omega)$ around $p(\omega)$ for any $\omega \in \Omega$. When the support of p changes such that $p(\omega) = 0$ for some ω, the permutation $\delta p(\omega)$ will cause negative probability masses and thus lead to invalid result. In fact, this is the reason we assume $p(\omega) > 0, \forall\omega \in \Omega$ in the above derivation. To keep using this method, every time the support changes we need to re-apply calculus of variations only on the support of the distribution. In current example, this means we need to consider two more cases, with support $\{\omega = -1\}$ and support $\{\omega = 1\}$, respectively. However, both cases will uniquely determine one distribution $p(\omega = 1) = 0$ and $p(\omega = 1) = 1$. It is then straightforward to work out the potential functions for these cases, which is precisely the potential function in (4.24).

For a bit more complicated case, let's consider the space $\Omega = \{-1, 0, 1\}$. Then the GEF generated by the quadratic entropy G and statistic $\phi(\omega) = \omega$ has the form: for $\theta \in (\infty, -2]$,

$$F(\theta) = -\theta,\ p_\theta(\omega) = \begin{cases} 1 & \omega = -1 \\ 0 & \omega \in \{0,1\} \end{cases}; \tag{4.28}$$

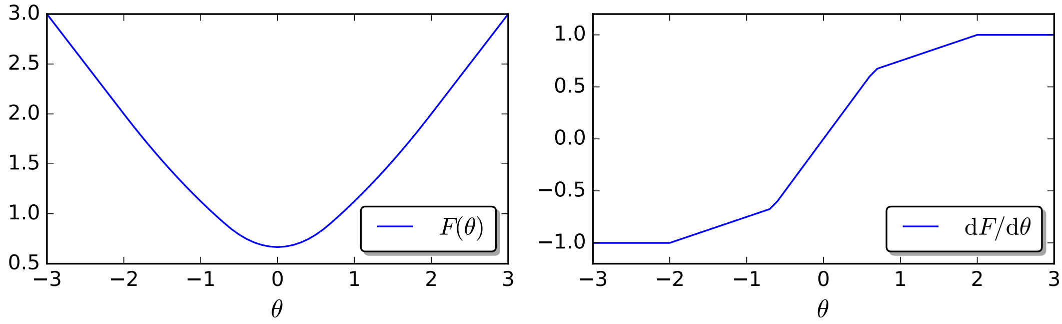


Figure 4.3: The potential/cumulant function of the GEF generated by the quadratic entropy G and statistic $\phi(\omega) = \omega$ with $\omega \in \{-1, 0, 1\}$ (left), and its gradient (right). The gradient of the potential matches the expectation of ϕ w.r.t. the GEF.

for $\theta \in (-2, -2/3]$,

$$F(\theta) = \frac{1}{8}(\theta - 2)^2,\ p_\theta(\omega) = \begin{cases} \frac{1}{2}\theta\phi(\omega) + \frac{1}{4}\theta + \frac{1}{2} & \omega \in \{-1, 0\} \\ 0 & \omega = 1 \end{cases}; \tag{4.29}$$

for $\theta \in (-2/3, 2/3)$,

$$F(\theta) = \frac{1}{2}\theta^2 + \frac{2}{3},\ p_\theta(\omega) = \frac{1}{2}\theta\phi(\omega) + \frac{1}{3}; \tag{4.30}$$

for $\theta \in [2/3, 2)$,

$$F(\theta) = \frac{1}{8}(\theta + 2)^2,\ p_\theta(\omega) = \begin{cases} 0 & \omega = -1 \\ \frac{1}{2}\theta\phi(\omega) - \frac{1}{4}\theta + \frac{1}{2} & \omega \in \{0, 1\} \end{cases}; \tag{4.31}$$

for $\theta \in [2, +\infty)$,

$$F(\theta) = \theta,\ p_\theta(\omega) = \begin{cases} 0 & \omega \in \{-1, 0\} \\ 1 & \omega = 1 \end{cases}. \tag{4.32}$$

Note that the potential function F, though written in parts, is a smooth function in $\mathbb{R}$. In addition, the support of the GEF varies as θ changes, and the change points are $\theta = -2, -2/3, 2/3$ and 2. With the probability and the potential functions, it can be verified that $\mathrm{d}F(\theta)/\mathrm{d}\theta = \mathbb{E}_{p_\theta}[\phi] = \mu$ always holds, and that the resulting $\mu \in [-1, 1]$. The change points of support in terms of μ are $\mu = -1, -2/3, 2/3$ and 1 (Figure 4.3).

Figure 4.4 provides a geometric view of this GEF (left) and compares it to the exponential family (EF) generated by the Shannon entropy $-H$ and ϕ

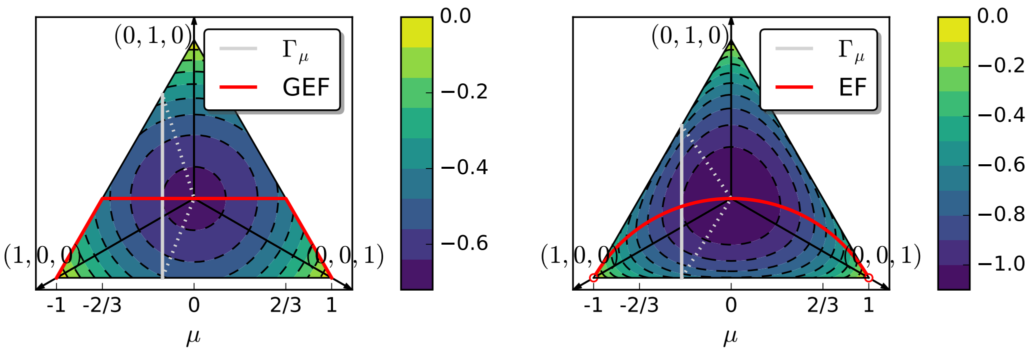


Figure 4.4: The GEF and EF on $\Omega = \{-1, 0, 1\}$. Left: the GEF generated by the quadratic entropy G and statistic $\phi(\omega) = \omega$. Right: the EF generated by the Shannon entropy $-H$ and the same ϕ. All distributions on Ω form a Δ^2-simplex, and each point is the distribution $(p(-1), p(0), p(1))$. Contours on the simplex show the entropy levels. The set of distributions Γ_μ happens to be a vertical line (solid grey) under this equilateral view. GEF/EF are plotted in red curves. Each GEF/EF distribution reaches the lowest entropy level along Γ_μ. Different from the EF, the support of GEF is dependent on θ. Figures are recreated from (Grünwald and Dawid, 2004).

(right). All distributions on Ω form a Δ^2-simplex, which is shown under the *equilateral* angle. Each point on this simplex represents a distribution p with probabilities matching the coordinate $(p(-1), p(0), p(1))$. The contour plot on the simplex shows the level of G ($-H$ for EF). Under this equilateral view, each Γ_μ, i.e. the set of distributions sharing the same mean statistic, happens to be a vertical line (solid grey). The corresponding value of μ is ticked on the bottom. The GEF and EF are represented by red curves on the simplex. Each of them is a one-dimensional sub-manifold as it is parametrized by a single parameter θ. Each point on the red curve also reaches the lowest level of the contour along Γ_μ, which demonstrates Property 3 of Theorem 4.7, that for each GEF/EF p_θ, $G(p_\theta) = \arg\inf_{p \in \Gamma_\mu} G(p)$. The GEF changes its support as θ varies, while the EF has a support independent of θ.

A continuous GEF Given an interval $\Omega = (-a, a) \in \mathbb{R}$, the GEF $\mathcal{P}_\Theta$ generated by the quadratic generalized entropy and statistic $\phi(\omega) = \omega$ has the following form: for $\theta \in (-1/a^2, 1/a^2)$,

$$F(\theta) = \frac{a^3}{6}\theta^2 - \frac{1}{2a}, \; p_\theta(\omega) = \frac{1}{2}\theta\omega + \frac{1}{2a}; \tag{4.33}$$

for $\theta \in (-\infty, -1/a^2]$

$$F(\theta) = \frac{a_+^3 + a^3}{12}\theta^2 - \frac{\left(4 - \theta(a_+^2 - a^2)\right)^2}{16(a_+ + a)},\ p_\theta(\omega) = \frac{1}{2}\theta\omega + \frac{4 - \theta(a_+^2 - a^2)}{4(a_+ + a)}, \tag{4.34}$$

with $a_+ = -a + 2/\sqrt{-\theta} \leq a$, and $(-a, a_+)$ the support of p_θ; similarly, for $\theta \in [1/a^2, \infty)$,

$$F(\theta) = \frac{a^3 - a_-^3}{12}\theta^2 - \frac{\left(4 - \theta(a^2 - a_-^2)\right)^2}{16(a - a_-)},\ p_\theta(\omega) = \frac{1}{2}\theta\omega + \frac{4 - \theta(a^2 - a_-^2)}{4(a - a_-)}, \tag{4.35}$$

with $a_- = a - 2/\sqrt{\theta} \geq -a$, and (a_-, a) the support of the distribution.

4.3.5 Discussion

One main characteristics of the GEF is that its support depends on the parameters. The varying support divides GEF derivations into many sub-cases, and deduction has to be done for each of these cases. Therefore, it is generally computationally expensive to derive a GEF with varying support for large spaces.

One way to mitigate this issue is to only consider the subset of the full GEF that contains only distributions with full support. More specifically, given a GEF $\mathcal{P}_{\boldsymbol{\Theta}}$, we consider its subset $\mathcal{P}_{\boldsymbol{\Theta}_s} := \{p_{\boldsymbol{\theta}} \mid \boldsymbol{\theta} \in \boldsymbol{\Theta}, \mathrm{supp}(p_{\boldsymbol{\theta}}) = \Omega\}$. We name this GEF the *fully-supported* GEF. The potential of this fully-supported GEF is related to the potential of the full set by

$$F_s(\boldsymbol{\theta}) = \begin{cases} F(\boldsymbol{\theta}) & \mathrm{supp}(p_{\boldsymbol{\theta}}) = \Omega \\ +\infty & \text{otherwise} \end{cases}. \tag{4.36}$$

Regularity can also be reintroduced for $\mathcal{P}_{\boldsymbol{\Theta}_s}$, which again requires $\mathrm{dom}(\partial F_s)$ is non-empty and open. We *do not* need $\mathcal{P}_{\boldsymbol{\Theta}}$ in the first place to derive $\mathcal{P}_{\boldsymbol{\Theta}_s}$. In stead, we can derive $\mathcal{P}_{\boldsymbol{\theta}_s}$ directly via calculus of variations, which now will only be applied once to distributions with full support on Ω.

We again consider previous examples. For the discrete case with binary space $\Omega = \{-1, 1\}$, the fully-supported GEF is the branch $\Theta_s = (-1, 1)$; for the discreate case with $\Omega = \{-1, 0, 1\}$, the fully-support GEF is the branch $\Theta_s = (-2/3, 2/3)$; for the continuous case, the fully-supported GEF is the branch $\Theta_s = (-1/a^2, 1/a^2)$. All of them are regular.

4.4 GEF with conditioning structure

In this section we will add an additional conditioning structure to the GEF. It allows us to efficiently express its conditional distributions, as we could do for the exponential family. The GEF with the conditioning structure will play a central role in making our PoPMM more general while keeping its market making efficiency. However, this important structure has not been discussed by either version of the existing GEF theory.

Consider a measurable space $(\Omega, \mathcal{F})$ and an exponential family distribution $p_{\boldsymbol{\theta}} \propto \exp(\boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega))\nu(\omega)$ defined on it. The cumulant function of $p_{\boldsymbol{\theta}}$ has the form $F := \log Z$, that is

$$F(\boldsymbol{\theta}) := \log \int_{\omega \in \Omega} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega)\right) \nu(\omega) \mathrm{d}\omega. \tag{4.37}$$

When being conditioned on a random variable $\mathsf{v}(\omega)$, each conditional of $p_{\boldsymbol{\theta}}$ given $\mathsf{v} = v$, which we denote by $p_{\boldsymbol{\theta},|v}$, remains in an exponential family, and therefore it will inherit the properties of the exponential family and in particular those in Section 4.2. More specifically, first, the conditional distribution has a cumulant

$$F(\boldsymbol{\theta}, v) := \log \int_{\omega \in \mathsf{v}^{-1}(v)} \exp\left(\boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega)\right) \nu(\omega) \mathrm{d}\omega. \tag{4.38}$$

which gives the *conditional expectation* of $\boldsymbol{\phi}$ efficiently via it gradient w.r.t. $\boldsymbol{\theta}$,

$$\mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v} = v] = \nabla_{\boldsymbol{\theta}} F(\boldsymbol{\theta}, v). \tag{4.39}$$

In addition, each conditional $p_{\boldsymbol{\theta},|v}$ maximizes the Shannon entropy among all conditional distributions on $\mathsf{v} = v$ sharing the same mean statistics with $p_{\boldsymbol{\theta},|v}$, that is, $\mathbb{E}_{p_{\boldsymbol{\theta},|v}}[\boldsymbol{\phi}] = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid v]$. Note that if v has not been realized, then both $F(\boldsymbol{\theta}, \mathsf{v})$ and the MaxEnt objective (which takes a conditional distribution as the input) mentioned above are functions of v and are also random variables.

For the GEF, it does not hold universally that the conditional distribution of a GEF distribution is still in some GEF. As a consequence, there will be no more efficient way of expressing the conditional expectation of the GEF. To obtain the conditional expectation we have compute integral explicitly w.r.t. the conditional distribution, but it is usually too expensive or even intractable and cannot be applied to the PoPMM design.

4.4.1 Conditional and nested GEF

To characterize the GEF with desired conditionals, we should first characterize those distributions which themselves may *not* be in any GEF but have conditionals to be GEF distributions. This characterization will be more complicated than that in the conditional exponential family: the latter can be directly defined by its density function, while the former is defined indirectly by the generalized entropy and the cumulant function via convex analysis.

Definition 4.13. Given a generalized entropy G and a random variable v, the *conditional generalized entropy* (induced by G) is a function $G(\cdot,\cdot) : \mathcal{P} \times \mathcal{V} \to \mathbb{R}$, such that $\forall p \in \mathcal{P}$, $G(\cdot, v) = G(p_{|v})$ almost surely.

A conditional generalized entropy is a *random variable* due to its dependency on v. On the other hand, given $\mathsf{v} = v$, the conditional generalized entropy is simply the generalized entropy of the conditional $p_{|v}$. For Shannon entropy, the induced conditional entropy has the form $-H(p, v) = -H(p_{|v}) = \mathbb{E}_{p_{|v}}[\log p_{|v}]$.

Both of the conditional MaxEnt and conditional dual MaxEnt functions are defined in the similar way as before, expect that the generalized entropy is now replaced by the conditional generalized entropy.

Definition 4.14. Given a generalized entropy G, statistics $\boldsymbol{\phi}$, and a random variable v, define the conditional generalized entropy $G(\cdot,\cdot)$. Then the *conditional MaxEnt function* and the *conditional dual MaxEnt function* are defined to be

$$g(\boldsymbol{\mu}) := \operatorname*{ess\,inf}_{p\in\Gamma_{\boldsymbol{\mu}}} G(p, \mathsf{v}) \text{ where } \Gamma_{\boldsymbol{\mu}} := \{p \in \mathcal{P} \mid \mathbb{E}_p[\boldsymbol{\phi} \mid \mathsf{v}] = \boldsymbol{\mu}\} \tag{4.40}$$

$$F(\boldsymbol{\theta}, \mathsf{v}) := \operatorname*{ess\,sup}_{p\in\mathcal{P}} \boldsymbol{\theta}^\top \mathbb{E}_p[\boldsymbol{\phi} \mid \mathsf{v}] - G(p, \mathsf{v}) = G^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}, \mathsf{v}). \tag{4.41}$$

Here the convex conjugate $G^*(\mathsf{x}, \mathsf{v})$ of $G(p, \mathsf{v})$ is w.r.t. the first argument p.

Since g and F are random variables, they in general give infimum and supremum almost surely instead of point-wisely. Therefore, the definition uses *essential infimum/supremum* instead of infimum/supremum.

Definition 4.15 (Conditional GEF)**.** Given a generalized entropy G, statistics $\boldsymbol{\phi}$, and a random variable v, define the conditional generalized entropy

$G(\cdot,\cdot)$ and the dual MaxEnt function $F(\cdot,\cdot)$ in (4.41). Let $\boldsymbol{\Theta}$ be the set of all $\boldsymbol{\theta} \in \mathrm{dom}(F)$ such that for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ the essential supremum of (4.41) is *achievable* at $p'_{\boldsymbol{\theta}}$. Then the *conditional GEF* generated by $G, \boldsymbol{\phi}$ and $\mathsf{v} = v$ is a set of distributions $\mathcal{P}_{\boldsymbol{\theta},|v}$ with parameter set $\boldsymbol{\Theta}$, such that $p_{\boldsymbol{\theta},|v} \in \mathcal{P}_{\boldsymbol{\theta},|v}$ is the conditional of $p'_{\boldsymbol{\theta}}$.

Here the generalized entropy G is introduced not for defining a GEF as before, but instead for defining the conditional entropy based on which the conditional GEF is introduced. As a result, the distribution p_0, which generates a conditional GEF distribution $p_{\boldsymbol{\theta},|\mathsf{v}}$, itself is not necessarily a GEF distribution.

To characterize the GEF distributions whose conditionals also belong to some GEF, we need to introduce the generalized entropy that can effectively generate the GEF with the desired conditioning structure. Recall that for the Shannon entropy, the entropy $H(p)$ and the conditional entropy $H(p, \mathsf{v})$ are related in the following way

$$H(p) := H(p_{\mathsf{v}}) + \mathbb{E}_{p_{\mathsf{v}}}[H(p, \mathsf{v})]. \tag{4.42}$$

Here p_{v} is the marginal distribution of p and $H(p_{\mathsf{v}})$ is the Shannon entropy defined on the marginal. If similar relation can be found for generalized entropies, the generated GEF should have GEF distributions as conditionals as the exponential family. The problem is that (4.42) is usually too strong to be held for a generalized entropy since it requires both the marginal entropy $G(p_{\mathsf{v}})$ and the conditional entropy $G(p, \mathsf{v})$ to be induced from the same entropy G. Fortunately, to achieve our goal we only need to maintain a much weaker relation than (4.42), by simply ensuring the term on the left to be some generalized entropy, which is allowed to be different from the generalized entropies that induce the marginal or the conditional entropies.

Definition 4.16. A generalized entropy G_0 is a *nested generalized entropy* w.r.t. the random variable v if there *exists* two generalized entropies $\bar{G}$ and G_1 such that for any $p \in \mathcal{P}$

$$G_0(p) = \bar{G}(p_{\mathsf{v}}) + \mathbb{E}_{p_{\mathsf{v}}}[G_1(p, \mathsf{v})] \tag{4.43}$$

where p_{v} is the marginal of p and $G_1(\cdot,\cdot)$ the conditional generalized entropy induced by G_1.

Remark. This definition considers only the existence of the decomposition of a nested generalized entropy, and does not require such decomposition to be unique. In practice, a nested generalized entropy is usually found by a direct construction from two given entropies via (4.43). The uniqueness will be a fundamental and also valuable problem, but discussion of it is beyond the scope of this thesis.

Definition 4.17 (Nested GEF)**.** Given generalized entropy G_0, statistics $\boldsymbol{\phi}$ and random variable v, then the resulting GEF $\mathcal{P}_{\boldsymbol{\Theta}}$ is a *nested GEF* generated by $G_0, \boldsymbol{\phi}$ and v if G_0 is a nested generalized entropy w.r.t. v.

A nested GEF inherits all the properties of a GEF (recall Theorem 4.7). Additional properties also emerges because of the richer conditioning structure. The following theorem summarizes the properties of a nested GEF.

Theorem 4.18. *Let $\mathcal{P}_{\boldsymbol{\Theta}}$ be a nested GEF generated by nested generalized entropies G_0, statistics $\boldsymbol{\phi}$ and a random variable v. Then*

1. *the cumulant function of $\mathcal{P}_{\boldsymbol{\Theta}}$ is*

$$F_0(\boldsymbol{\theta}) = G_0^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}) = \bar{G}^*(G_1^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}, \mathsf{v})) = \bar{G}^*(F_1(\boldsymbol{\theta}, \mathsf{v})); \tag{4.44}$$

2. *for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, $p_{\boldsymbol{\theta}} = p_{\boldsymbol{\theta},\mathsf{v}} \cdot p_{\boldsymbol{\theta},|\mathsf{v}}$ where*

$$p_{\boldsymbol{\theta}} = \nabla G_0^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}),\; p_{\boldsymbol{\theta},|\mathsf{v}} = \nabla G_1^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}, \mathsf{v}),\; p_{\boldsymbol{\theta},\mathsf{v}} = \nabla \bar{G}(\mathsf{x}) \mid_{\mathsf{x}=F_1(\boldsymbol{\theta},\mathsf{v})}; \tag{4.45}$$

3. *for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$*

$$\mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}] = \nabla F_0(\boldsymbol{\theta}),\; \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid v] = \nabla_{\boldsymbol{\theta}} F_1(\boldsymbol{\theta}, v); \tag{4.46}$$

   and

4. *for each $\boldsymbol{\theta} \in \boldsymbol{\Theta}$,*

$$\inf_{p \in \Gamma_{\boldsymbol{\mu}}} G_0(p) = G_0(p_{\boldsymbol{\theta}}) = g_0(\boldsymbol{\mu}) \qquad \text{with } \boldsymbol{\mu} = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}] \tag{4.47}$$

$$\operatorname*{ess\,inf}_{p \in \Gamma_{\boldsymbol{\mu}}} G_1(p, \mathsf{v}) = G_1(p_{\boldsymbol{\theta}}, \mathsf{v}) = g_1(\boldsymbol{\mu}) \qquad \text{with } \boldsymbol{\mu} = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v}]. \tag{4.48}$$

Property 1 gives the exact form of the cumulant function for the nested GEF. It is derived from the convex duality between MaxEnt and dual MaxEnt functions $F = g^*$. The cumulant of the nested GEF wraps the cumulant of a conditional GEF, implying the name "nested GEF". Property 2, 3, 4 extend the Property 1, 2, 3 of Theorem 4.7 respectively. These results are based on

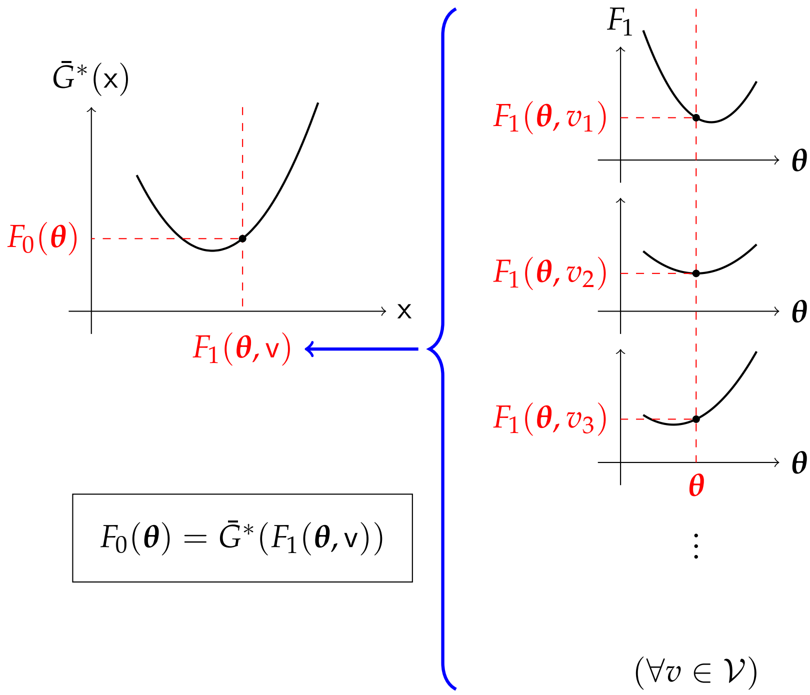


Figure 4.5: The structure of the potential of a nested GEF. Given each $\mathsf{v} = v$, $F_1(\boldsymbol{\theta}, v)$ is a potential of a GEF distribution. $F_1(\boldsymbol{\theta}, \mathsf{v})$ characterizes a family of conditional distributions, which we name the conditional GEF. $F_1(\boldsymbol{\theta}, \mathsf{v})$ is then feeded to $\bar{G}^*$, the convex conjugate of the marginal entropy $\bar{G}$. Such nesting reflects precisely how the joint probability is constructed from the marginal and conditional: we first obtain the coarse probability $p(v)$, and then obtain the finer one by further working out $p(h \mid v)$.

Property 1 and can be derived by using the fact that both the distribution and its conditionals are GEF distributions. More specifically, Property 2 characterizes each nest GEF distribution together with its conditional and marginal by the sub-differential of G_0, G_1 and $\bar{G}$. Property 3 shows that the expectation and conditional expectation of $\boldsymbol{\phi}$ can be represented by the gradients of the cumulants. Property 4 states the MaxEnt argument for the nested GEF and its conditional.

Proof. We will mainly prove Property 1. Denote $\mathcal{P}_{\mathsf{v}}$ the marginals of the distributions in $\mathcal{P}$ over v. By definition

$$F_0(\boldsymbol{\theta}) = \sup_{p \in \mathcal{P}} \boldsymbol{\theta}^\top \mathbb{E}_p[\boldsymbol{\phi}] - G_0(p) \tag{4.49}$$

$$= \sup_{p_{\mathsf{v}} \in \mathcal{P}_{\mathsf{v}}} \mathbb{E}_{p_{\mathsf{v}}} \left[\operatorname*{ess\,sup}_{p \in \mathcal{P}} \; \boldsymbol{\theta}^\top \mathbb{E}_p[\boldsymbol{\phi} \mid \mathsf{v}] - G_1(p, \mathsf{v}) \right] - \bar{G}(p_{\mathsf{v}}) \tag{4.50}$$

$$= \sup_{p_{\mathsf{v}} \in \mathcal{P}_{\mathsf{v}}} \langle p_{\mathsf{v}}, F_1(\boldsymbol{\theta}, \mathsf{v}) \rangle - \bar{G}(p_{\mathsf{v}}). \tag{4.51}$$

$$= \bar{G}^*(F_1(\boldsymbol{\theta}, \mathsf{v})). \tag{4.52}$$

The second is equality is obtained by substituting (4.43) in and regrouping, the third equality uses (4.41) and the last equality uses (4.10). □

4.4.2 Regular nested GEF, Bregman divergences and losses

Similar to the regular GEF, a regular nested GEF is determinable by the sub-differential domain of its cumulant functions. When only the mean statistics is involved in the application, we can avoid explicitly writing down the regular nested GEF but simply represent it by its cumulant functions.

Definition 4.19. A nested GEF with its cumulant function $F_0(\cdot) = \bar{G}^*(F_1(\cdot, \mathsf{v}))$ is *regular* if $\operatorname{dom}(\partial F_1)$ is non-empty and open, and $\operatorname{ran}(F_1) \subseteq \operatorname{dom}(\partial \bar{G}^*)$.

The second condition $\operatorname{ran}(F_1) \subseteq \operatorname{dom}(\partial \bar{G}^*)$ ensures that the conditional GEF is fully wrapped into the nested GEF, such that each conditional GEF distribution is always associated with a nested GEF.

Bregman divergences A nested GEF is involved with three generalized entropies G_0, G_1 and $\bar{G}$. Each of them can generate a Bregman divergence (recall

(4.17)). Similar to a regular GEF, the Bregman divergences between two regular nested GEF distributions or two conditional GEF distributions will be transformed to the Bregman divergences between their parameters. To be more specific, let's consider a nested GEF $\mathcal{P}_{\boldsymbol{\Theta}}$. For $\boldsymbol{\theta}, \boldsymbol{\theta}_0 \in \boldsymbol{\Theta}$, using G_0 and choosing sub-gradient maps $\mathrm{d}G_0 : p_{\boldsymbol{\theta}} \mapsto \boldsymbol{\theta}^\top \boldsymbol{\phi}, \mathrm{d}g_0 : \boldsymbol{\mu} \mapsto \boldsymbol{\theta}$ we have

$$D_{G_0,\mathrm{d}G_0}(p_{\boldsymbol{\theta}}, p_{\boldsymbol{\theta}_0}) = G_0(p_{\boldsymbol{\theta}}) - G_0(p_{\boldsymbol{\theta}_0}) - \langle p_{\boldsymbol{\theta}} - p_{\boldsymbol{\theta}_0}, \mathrm{d}G_0(p_{\boldsymbol{\theta}_0})\rangle \tag{4.53}$$

$$= g_0(\boldsymbol{\mu}) + F_0(\boldsymbol{\theta}_0) - \langle \boldsymbol{\mu}, \boldsymbol{\theta}_0\rangle \tag{4.54}$$

$$= D_{g_0,\mathrm{d}g_0}(\boldsymbol{\mu}, \boldsymbol{\mu}_0) = D_{F_0,\nabla F_0}(\boldsymbol{\theta}_0, \boldsymbol{\theta}) \tag{4.55}$$

where $\boldsymbol{\mu} := \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}]$ represents the mean statistics of the distribution $p_{\boldsymbol{\theta}}$; for the conditionals, using $G_1(\cdot,\cdot)$ and the sub-gradient maps $\mathrm{d}G_1(\cdot, \mathsf{v}) : p_{\boldsymbol{\theta},|\mathsf{v}} \mapsto \boldsymbol{\theta}^\top \boldsymbol{\phi}, \mathrm{d}g_1 : \boldsymbol{\mu} = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v}] \mapsto \boldsymbol{\theta}$ we have

$$D_{G_1(\cdot,\mathsf{v}),\mathrm{d}G_1(\cdot,\mathsf{v})}(p_{\boldsymbol{\theta},\mathsf{v}}, p_{\boldsymbol{\theta}_0,\mathsf{v}}) = D_{g_1,\mathrm{d}g_1}(\boldsymbol{\mu}, \boldsymbol{\mu}_0) = D_{F_1(\cdot,\mathsf{v}),\nabla F_1(\cdot,\mathsf{v})}(\boldsymbol{\theta}_0, \boldsymbol{\theta}). \tag{4.56}$$

Comparatively, the Bregman divergence between the marginals of two regular nested GEF distributions $D_{\bar{G},\mathrm{d}\bar{G}}(\cdot,\cdot)$ cannot be represented in divergences between the parameters of the distribution. This is due to the fact that the marginals may not belong to any GEF.

Losses For a regular nest GEF and its conditionals, the loss functions introduced based on the generalized entropy (4.19) have a linar form in (4.22)

$$L_0(\omega, p_{\boldsymbol{\theta}}) = F_0(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega) \tag{4.57}$$

$$L_1(\omega, p_{\boldsymbol{\theta},|\mathsf{v}}) = F_1(\boldsymbol{\theta}, \mathsf{v}(\omega)) - \boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega). \tag{4.58}$$

L_0 defines some measure of the loss of $p_{\boldsymbol{\theta}}$ w.r.t. the true state ω, while L_1 measures the loss of $p_{\boldsymbol{\theta},|\mathsf{v}}$ w.r.t. ω, given that v has been realized. The difference of L_0 and L_1 defines the loss of the marginal $p_{\boldsymbol{\theta},\mathsf{v}}$ naturally

$$\bar{L}(\omega, p_{\boldsymbol{\theta},\mathsf{v}}) = L_0(\omega, p_{\boldsymbol{\theta}}) - L_1(\omega, p_{\boldsymbol{\theta},|\mathsf{v}}) = F_0(\boldsymbol{\theta}) - F_1(\boldsymbol{\theta}, \mathsf{v}(\omega)). \tag{4.59}$$

Although $\bar{L}$ is written as a function of ω, its value is determined on the level of the random variable v and does not need the full observation of the state.

Given a belief $p \in \mathcal{P}$, the expected losses for L_0 and L_1 are

$$L_0(p, p_{\boldsymbol{\theta}}) = D_{g_0,\mathrm{d}g_0}(\mathbb{E}_p[\boldsymbol{\phi}], \boldsymbol{\mu}) - g_0(p_{\boldsymbol{\theta}}) \tag{4.60}$$

$$L_1(p, p_{\boldsymbol{\theta},|\mathsf{v}}) = \mathbb{E}_{p_{\mathsf{v}}}[D_{g_1,\mathrm{d}g_1}(\mathbb{E}_p[\boldsymbol{\phi} \mid \mathsf{v}], \boldsymbol{\mu}) - g_1(p_{\boldsymbol{\theta}}, \mathsf{v})]. \tag{4.61}$$

Therefore, L_0 is minimized at the nested GEF distribution that shares the same mean statistics with p, and L_1 is minimized at the nested GEF distribution that shares the same conditional mean statistics with p. If p is a distribution picked from the same regular nested GEF, then the losses are minimized when $p_{\boldsymbol{\theta}} = p$.

More interestingly, the expected loss for $\bar{L}$ also has a divergence form. Recall that for a nested GEF, $F_0(\boldsymbol{\theta}) = \langle p_{\boldsymbol{\theta},\mathsf{v}}, F_1(\boldsymbol{\theta}, \mathsf{v})\rangle - \bar{G}(p_{\boldsymbol{\theta},\mathsf{v}})$

$$\begin{aligned}\bar{L}(p, p_{\boldsymbol{\theta},\mathsf{v}}) &= F_0(\boldsymbol{\theta}) - \mathbb{E}_{p_\mathsf{v}}[F_1(\boldsymbol{\theta}, \mathsf{v})] = \langle p_{\boldsymbol{\theta},\mathsf{v}}, F_1(\boldsymbol{\theta}, \mathsf{v})\rangle - \bar{G}(p_{\boldsymbol{\theta},\mathsf{v}}) - \mathbb{E}_{p_\mathsf{v}}[F_1(\boldsymbol{\theta}, \mathsf{v})] \\ &= D_{\bar{G},\mathrm{d}\bar{G}}(p_\mathsf{v}, p_{\boldsymbol{\theta},\mathsf{v}}) - \bar{G}(p_\mathsf{v}). \end{aligned} \tag{4.62}$$

Similar to the previous losses, $\bar{L}$ is minimized by the $p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}$ whose marginal is closest to the marginal of p under divergence $D_{\bar{G},\mathrm{d}\bar{G}}(\cdot,\cdot)$. Since the marginal may not belong to any GEF, the minimum divergence will in general not be characterized by matching the mean statistics as in L_0 and L_1. If p is in the same GEF as $p_{\boldsymbol{\theta}}$, then the minimum loss is obtained at $p_{\boldsymbol{\theta}} = p$.

4.4.3 Examples

By definition an exponential family is a nested GEF, with entropies $G_1(\cdot, \mathsf{v}) = -H(p_{|\mathsf{v}}), \bar{G} = -H(p_\mathsf{v})$, and $G_0 = \bar{G}(p_\mathsf{v}) + \mathbb{E}_{p_\mathsf{v}}[G_1(\cdot, \mathsf{v})] = -H(p)$. The convex conjugates of G_1 and G_0 are (c.f. Frongillo and Reid (2013))

$$G_1^*(\mathsf{x}) = \log \int_{\omega \in \mathsf{v}^{-1}(v)} \exp(\mathsf{x}(\omega))\nu(\omega)\mathrm{d}\omega, \tag{4.63}$$

$$G_0^*(\mathsf{x}) = \log \int_{\omega \in \Omega} \exp(\mathsf{x}(\omega))\nu(\omega)\mathrm{d}\omega = \log \int_{v \in \mathcal{V}} \exp(G_1^*(\mathsf{x}))\mathrm{d}vi = \bar{G}^*(G_1^*(\mathsf{x})), \tag{4.64}$$

which verifies Property 1 of Theorem 4.18. The density functions of the GEF distribution and the conditional GEF distribution are

$$p_{\boldsymbol{\theta}}(\omega) = e^{\boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega) - F(\boldsymbol{\theta})}\nu(\omega), \; p_{\boldsymbol{\theta},|v}(\omega) = e^{\boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega) - F(\boldsymbol{\theta},v)}\nu(\omega) \tag{4.65}$$

where $F(\boldsymbol{\theta})$, and $F(\boldsymbol{\theta}, v)$ are the log partition functions in (4.37) and (4.38). It follows that $p_{\boldsymbol{\theta}} = G_0^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}), p_{\boldsymbol{\theta},|v} = G_1^*(\boldsymbol{\theta}^\top \boldsymbol{\phi}), \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}] = \nabla F_0(\boldsymbol{\theta}), \mathbb{E}_{p_{\boldsymbol{\theta},|v}}[\boldsymbol{\phi}] = \nabla_{\boldsymbol{\theta}} F_1(\boldsymbol{\theta}, v)$, which verify Property 2 and 3 of Theorem 4.18.

The marginal distribution is

$$p_{\boldsymbol{\theta},v}(\omega) = \frac{p_{\boldsymbol{\theta}}(\omega)}{p_{\boldsymbol{\theta},|v}(\omega)} = \frac{e^{F(\boldsymbol{\theta},v)}}{e^{F(\boldsymbol{\theta})}} = e^{F(\boldsymbol{\theta},v)-F(\boldsymbol{\theta})}. \tag{4.66}$$

Therefore, $p_{\boldsymbol{\theta},v} = \nabla \bar{G}^*(F_1(\boldsymbol{\theta},v))$.

In the second example, we construct a nested GEF on partially-observable state space (v,h) with $\mathsf{v} \in (-a,a)$ where $a \leq 6$ and $\mathsf{h} \in \{-1,1\}$, using the quadratic generalized entropy. It can be understood as a mixture model in which the latent variable h is the component indicator.

First, we induce both the conditional entropy $G_1(\cdot,\mathsf{v})$ and the marginal entropy $\bar{G}$ from the quadratic generalized entropy

$$G_1(p,v) = \sum_{h\in\{-1,1\}} p(h \mid v)^2,\ \bar{G}(p_v) = \int_{-a}^{a} p(v)^2 \mathrm{d}v \tag{4.67}$$

Define a single statistic $\phi(v,h) = vh$. Then by applying the first example in Section 4.3.4, the conditional GEF given $\mathsf{v} = v$ has potential and probability[3]

$$F_1(\theta,v) = \frac{\theta^2 v^2}{2},\ p_\theta(h \mid v) = \frac{1}{2} + \frac{\theta v h}{2},\ \theta \in (-\frac{1}{a},\frac{1}{a}). \tag{4.68}$$

The convex conjugate of $\bar{G}$ is (similar to (4.26))

$$\bar{G}^*(\mathsf{x}) = \frac{1}{4}\int_{-a}^{a} \mathsf{x}(v)^2 \mathrm{d}v - \frac{1}{4}\int_{-a}^{a} \lambda^2,\ \text{with } \lambda = \frac{1}{a} - \frac{1}{2a}\int_{-a}^{a} \mathsf{x}(v)\mathrm{d}v. \tag{4.69}$$

The marginal distribution is $p(v) = \nabla G^*(\mathsf{x}) = (\mathsf{x}(v)+\lambda)/2$. The potential of the nested GEF is obtained by feeding $\mathsf{x}(\mathsf{v}) = F_1(\theta,\mathsf{v})$ to $\bar{G}^*$

$$F_0(\theta) = \bar{G}^*(F_1(\theta,\mathsf{v})) = \frac{a^5}{90}\theta^4 + \frac{a^2}{6}\theta^2 - \frac{1}{2a}. \tag{4.70}$$

and the resulting marginal is

$$p_\theta(v) = \frac{1}{2a} + \frac{\theta^2}{4}\left(v^2 - \frac{a^2}{3}\right). \tag{4.71}$$

For $a \leq 6$, $p_\theta \geq 0$ thus it is a valid distribution. Finally, the joint distribution over (v,h) is obtained by $p_{\boldsymbol{\theta}}(v,h) = p_{\boldsymbol{\theta}}(v)p(h \mid v)$.

We can check if the mean statistics match the gradient of the corresponding potentials. For the conditional expectation, $\forall v \in (-a,a)$

$$\begin{aligned}\mathbb{E}_{p_\theta}[\phi \mid v] &= (-v)p(h=-1 \mid v) + vp(h=1 \mid v) \\ &= (-v)\left(\frac{1}{2} + \frac{-\theta v}{2}\right) + v\left(\frac{1}{2} + \frac{\theta v}{2}\right) = \theta v^2 = \frac{\partial}{\partial\theta}F_1(\theta,v).\end{aligned} \tag{4.72}$$

[3]For simplicity, here we will consider the fully-support GEF.

For the full expectation

$$\mathbb{E}_{p_\theta}[\phi] = \int_{-a}^{a} \theta v^2 p_\theta(v)\mathrm{d}v = \frac{2a^5}{45}\theta^3 + \frac{a^2}{3}\theta. \tag{4.73}$$

Since $\theta \in (-1/a, 1/a)$, this nested GEF is regular, which allows Bregman divergences and losses to be defined.

4.5 Generalizing PMM and PoPMM design

In previous two sections, we reviewed the existing theory for GEF, and also established an extended theory for characterizing the special GEF with certain desired conditioning structure. With these results, we are ready to tackle our main task: designing potential-based mechanisms that can represent more general market beliefs.

4.5.1 Summary of the inherited properties

The two properties of the exponential family in Section 4.2 are inherited by the regular GEF. In particular, given a regular GEF,

1. the gradient of its cumulant F at $\boldsymbol{\theta}$ matches the mean statistics w.r.t. $p_{\boldsymbol{\theta}}$ for every $\boldsymbol{\theta} \in \boldsymbol{\Theta} = \mathrm{dom}(\partial F)$, which is guaranteed by Property 2 of Theorem 4.7; and
2. each $p_{\boldsymbol{\theta}}$ minimizes the generalized entropy G among the distributions $\Gamma_{\boldsymbol{\mu}} := \{p \in \mathcal{P} \mid \mathbb{E}_p[\boldsymbol{\phi}] = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}]\}$, which is guaranteed by Property 3 of Theorem 4.7.

For a regular nested GEF, an additional property of the exponential family is inherited

3. given random variable v, the conditional distributions also form a GEF with cumulant $F(\cdot, \mathsf{v})$.

4.5.2 Design with regular GEF

In this section we will extend the design of both the PMM and the PoPMM using only regular GEFs. The design of the PMM was first given by Frongillo and Reid (2013). Here, we make more analysis to show how the GEF fits into the design, encoding traders' beliefs and decoding prices naturally. It is possible to design the PoPMM using just the regular GEF, but the mechanism may suffer from some deficiency.

Designing PMM To represent market beliefs in a regular GEF, we simply set up the PMM as follows:

- the PMM will offer securities that coincide with the statistics $\boldsymbol{\phi}$ used for defining the GEF;
- the PMM will use the cumulant function F of the GEF as its potential function; and
- the PMM will have possible market positions in $\boldsymbol{\Theta} = \mathrm{dom}(\partial F)$.

Based on this setup, each instantaneous market price at market position $\boldsymbol{\theta}$ is decoded as a martingale under the corresponding GEF distribution $p_{\boldsymbol{\theta}}$, since $\mathbf{p}(\boldsymbol{\theta}) := \nabla F(\boldsymbol{\theta}) = \mathbb{E}_p[\boldsymbol{\phi}]$. The resulting PMM also effectively encodes each trader's true belief to the closest GEF distribution under the Bregman divergence generated by the generalized entropy. In fact, this encoding process is implemented through the loss function $L(\omega, p_0)$ defined in (4.19). Recall that, when p_0 is in a regular GEF, the loss becomes a linear with the form in (4.22), which we rewrite here for reference

$$L(\omega, p_0) = F(\boldsymbol{\theta}) - \boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega). \tag{4.74}$$

Then, given a risk-neutral myopic trader, the highest profit she expects for a trade $\delta_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ is

$$\max \mathbb{E}_p[m_t(\omega)] = \max_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbb{E}_p[\boldsymbol{\phi}] - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_t)) \tag{4.75}$$

$$= L(p, p_{\boldsymbol{\theta}_{t-1}}) - \min_{p_{\boldsymbol{\theta}_t} \in \mathcal{P}_{\boldsymbol{\Theta}}} L(p, p_{\boldsymbol{\theta}_t}). \tag{4.76}$$

Thus the optimal trade will minimize the loss. By further using the relation between the loss function and the Bregman divergence (4.20),we obtain

$$\max \mathbb{E}_p[m_t(\omega)] = L(p, p_{\boldsymbol{\theta}_{t-1}}) + G(p) - \min_{p_{\boldsymbol{\theta}_t} \in \mathcal{P}_{\boldsymbol{\Theta}}} D_{G,\mathrm{d}G}(p, p_{\boldsymbol{\theta}_t}). \tag{4.77}$$

With the optimal trade, the p is encoded to $p_{\boldsymbol{\theta}_t}$ that minimizes the Bregman divergence. For the GEF, the optimal $p_{\boldsymbol{\theta}_t}$ is such that its mean statistics are matched to that of p, $\mathbb{E}_p[\boldsymbol{\phi}] = \mathbb{E}_{p_{\boldsymbol{\theta}_t}}[\boldsymbol{\phi}]$.

Note that the PMM involves only the cumulant function of the regular GEF. Once the cumulant function is given, there is no need to ask for the explicit p.d.f. or the c.d.f. of the family, which is usually expensive to obtain.

This benefit will go even further in practical PMM design. In practice, we will assume that every potential function we encounter in the design is matched to the cumulant of certain regular GEF. Using this assumption and the fact that the PMM requires no explicit form of the family, all we need to do is to construct a convex function and set it as the potential, without worrying about exactly which GEF is behind the potential function, just like Abernethy et al. (2011).

Designing PoPMM Let (v, h) be the partially-observable state space upon which the PoPMM is built. Setting up the PoPMM that represent a GEF over this space is the same as setting up a PMM by assuming that (v, h) is fully-observable. The conditional market prices at market position $\boldsymbol{\theta}$, given $\mathsf{v} = v$, are defined by the conditional expectation w.r.t. the GEF distribution $p_{\boldsymbol{\theta}}$

$$\mathbf{p}(\boldsymbol{\theta}, v) = \mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi} \mid \mathsf{v} = v] = \int_{h \in \mathcal{H}} \boldsymbol{\phi}(v, h) p_{\boldsymbol{\theta}}(h \mid v) \mathrm{d}h. \tag{4.78}$$

Since the regular GEF lacks a conditioning structure, there are no alternative ways but only to compute the conditional expectation via its definition, that is, explicitly working out the density, deriving the conditional $p(h \mid v)$ and computing the integral. Therefore, computing the conditional prices could be expensive or even intractable, making this generalized PoPMM design less interesting for practical application.

4.5.3 Efficient PoPMM design with regular nested GEF

When the regular GEF used for the PoPMM design is a regular nested GEF with cumulant $F_0 = \bar{G}^*(F_1(\cdot, \mathsf{v}))$, the conditional market prices at $\boldsymbol{\theta}$ is given by the gradient of $F_1(\cdot, \mathsf{v})$ at $\boldsymbol{\theta}$. Once F_1 is given, it will be cheaper than computing the conditional prices via explicit integrals.

With the regular GEF as the underlying belief model, the incentive property of the PoPMM is also well maintained and can be illustrated in a similar way as before (Section 3.6). Recall that a mechanism is *weakly incentive-compatible* if there exists a Market Scoring Rule (MSR) S_t such that given trader with belief $p_{\boldsymbol{\theta}^*} \in \mathcal{P}_{\boldsymbol{\Theta}}$, $S_t(p_{\boldsymbol{\theta}^*}, \cdot)$ is always positive w.r.t. the post-trade market belief resulting from an optimal trade of this trader (Definition 3.4). In particular, the expected profit of each trade under this PoPMM are related to the following MSR

$$S_t(\mathsf{v}(\omega), p_{\boldsymbol{\theta}_t}) := \bar{L}(\omega, p_{\boldsymbol{\theta}_{t-1}, \mathsf{v}}) - \bar{L}(\omega, p_{\boldsymbol{\theta}_t, \mathsf{v}}). \tag{4.79}$$

It is a MSR since $S_t(p, \cdot)$ for any $p \in \mathcal{P}_{\boldsymbol{\Theta}}$.

$$S_t(p, p_{\boldsymbol{\theta}_t}) := \bar{L}(p_{\mathsf{v}}, p_{\boldsymbol{\theta}_{t-1}, \mathsf{v}}) - \bar{L}(p_{\mathsf{v}}, p_{\boldsymbol{\theta}_t, \mathsf{v}}) = D_{\bar{G}, \mathrm{d}\bar{G}}(p_{\mathsf{v}}, p_{\boldsymbol{\theta}_{t-1}, \mathsf{v}}) - D_{\bar{G}, \mathrm{d}\bar{G}}(p_{\mathsf{v}}, p_{\boldsymbol{\theta}_t, \mathsf{v}}) \tag{4.80}$$

and is maximized at $p_{\boldsymbol{\theta}_t} = p$. Then for a risk-neutral myopic trader with belief $p_{\boldsymbol{\theta}^*} \in \mathcal{P}_{\boldsymbol{\Theta}}$, applying convexity of F_1 we have

$$S_t(v, p_{\boldsymbol{\theta}}) = (F_1(\boldsymbol{\theta}_t, \mathsf{v}) - F_1(\boldsymbol{\theta}_{t-1}, \mathsf{v})) - (F_0(\boldsymbol{\theta}_t) - F_0(\boldsymbol{\theta}_{t-1})) \tag{4.81}$$

$$\geq (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \nabla_{\boldsymbol{\theta}} F_1(\boldsymbol{\theta}_{t-1}, v) - (F_0(\boldsymbol{\theta}_t) - F_0(\boldsymbol{\theta}_{t-1})). \tag{4.82}$$

That is, the trade profit bounds the MSR profit from below exactly the same way as in the exponential family case. Therefore, all the following results will hold.

Since the explicit form of the GEF is not required in price computation, it is also simpler to design a PoPMM in practice with a regular nested GEF as the underlying belief model. By assuming that every conditional potential function F_1 we encounter in the design is matched to the cumulant of certain conditional GEF, then all we need to do is to (1) design a potential function $F_1(\cdot, \mathsf{v})$, (2) design a generalized entropy $\bar{G}$, and (3) compute $F_0 = \bar{G}^*(F_1(\cdot, \mathsf{v}))$, without worrying about which regular nested GEF is encoded into the PoPMM.

4.6 Summary

In this chapter, we discussed a general method for interpreting market beliefs in prediction markets, which was achieved by decoding market prices in generalized exponential families (GEFs). GEFs are not only more flexible than exponential families in terms of representing beliefs, but also the same efficient as the exponential families in terms of market making (price calculation). For partially-observable PoPMMs (Chapter 3), GEFs free the design and make exponential family unnecessary to define the market maker.

For PMMs, the generalization is supported by the existing GEF theory. We complete the theory (mainly the concept of regularity) to trigger a more natural design in PMM potentials. The existing theory can also be used to generalize PoPMMs, but it cannot guarantee the efficiency of computing the conditional market prices. We introduced a special class of GEFs, which we named nested GEFs, and showed that nested GEFs have all desired properties that will make the resulting PoPMMs efficient.

Chapter 5

Convergence and Equilibrium

Up to now our discussion of prediction market has focused on the market mechanism design. To be specific, we propose the partially-observable potential based market maker (PoPMM), which generalizes the standard PMM and can build prediction markets on events containing latent variables. As a market maker, the PMM/PoPMM guarantees bounded loss and (weak) incentive-compatibility. As a probabilistic model, the PMM/PoPMM is able to encode market beliefs in a variety of distribution families.

Another important topic on markets is the equilibria of markets as well as the dynamics of achieving them. It is a classic problem, and a vast amount of work can be found in both the economics and the game theory literature, among which famous work includes Walras (1877); Nash (1950); Shapley and Shubik (1969). However, these results on market convergence and equilibrium do not fit the prediction market setting very well, since a prediction market is not only a market in the standard economic sense, but also interpreted as a probabilistic model, especially an information aggregation model. Therefore, when analysing a prediction market, we need to characterize its equilibrium and convergence dynamics further in the machine learning context, by considering the extra modelling role it plays. In particular, we are not only interested in *what* the equilibrium and convergence dynamics are, but more importantly, *how* these concepts are linked to the learning objective and learning process of the underlying model. This gives the main motivation of this work.

5.1 Overview

In contrast to the market mechanism design which focuses on the market maker while involving almost no traders (expect that risk-neutral myopic traders are called to help show incentive-compatibility), the convergence and equilibrium analysis is based on the whole market. We present convergence results in two different settings: trader driven convergence and market maker driven convergence.

We start our discussion by presenting the convergence and equilibrium results driven by *risk-averse traders*. A risk-averse trader will not only consider how to reach high expected profits, but also how to avoid the risks of possibly obtaining low or even negative profits. As a result, the post-trade market prices will no longer encode the true belief of the trader, but instead her *effective belief*, which is a compromise between the trader's true belief and the market belief before the trade. Intuitively speaking, the more trades have been accomplished, the better the market prices will reflect the market-wise consensus, and the more risky for a risk-averse trader to move the market prices/belief away from the current position via transactions. Eventually, the risk-averse traders will agree on certain market prices, which none of them is willing to unilaterally change.

An alternative way of modelling the behaviour of a trader is to think that the trader is always risk-neutral but adjusts her true belief depending on the market positions. In fact, this alternative approach transforms each risk-averse trader to an equivalent risk-neutral trader, such that the risk-neutral trader's true belief matches the original risk-averse trader's effective belief, the latter of which depends on the market and varies through time. Under this model, the equilibrium is established simply when all trader's believed prices (i.e. expected security values) coincide with each other and also with the market ones. We refer to traders modelled by this alternative approached as *risk-neutral niche traders*. The model and convergence results for niche traders are discussed in Section 5.3.

The challenge of the trader driven convergence setting is that, distinct from most machine learning models that explicitly define and optimize a global learning objective, a prediction market is a distributed environment, where individual traders behave independently under their own local objectives. To interpret a market as a machine learning model, we must find, if there is any, the global objective the market aims to optimize, and also how each trader's selfish behaviour contributes to the optimization.

Trader driven convergence relies on the model of traders that is assumed to be able to capture the traders' trading behaviours. However, in situations where no good models for traders are available, such as due to the lack of the knowledge about traders or due to information privacy issues, the trader driven convergence may not be well established. To aggregate belief information from trades, an alternative approach is developed. Unlike the trader driven convergence, this approach avoids building the convergence results on particular trader models. Instead, the convergence is purely driven by the market maker. More specifically, we make the assumption that traders are i.i.d. sampled from an ensemble with fixed ensemble mean price (also referred to the true mean price in this context), and the goal of the market dynamics is to discover this mean price from trades. To discover the true mean price, the PMM is augmented by another prior belief over the mean price. This prior belief helps the PMM set the price for each trade, and is updated via the accomplished trades. As more trades occur, the posterior becomes more certain about the true mean price, and will gradually stabilize the market price at or around the true mean price. The details of market maker driven convergence is presented in Section 5.4.

5.2 Risk-averse traders

The quantitative analysis for the convergence and equilibrium of prediction markets will depend on the models of the risk-averse traders. A preference model is a function that assigns values to each possible position a trader can take, so as to reflect the trader's preference orders on these positions. The assigned values are only required to be ordinal, and may or may not reflect a real quantity such as a risk-free asset like money, although having an additional relation to the real quantity will make the preference model more quantitative and expressive.

Before this work, the dominant model for the analysis of the trader behaviour in prediction markets is the *expected utility theory (EUT)* introduced firstly by Daniel Bernoulli in the eighteenth century. EUT is well supported by axiomatic approaches and has become the most popular theory for modelling preferences in various areas including economics and game theory (Von Neumann and Morgenstern, 2007). In prediction market literature, analyses and discussions have been made by work such as Barbu and Lay (2012); Frongillo et al. (2012); Premachandra and Reid (2013); Storkey (2011); Storkey et al. (2012, 2015); Chakraborty and Das (2015); Sethi and Vaughan (2013).

This work will model the behaviours of risk-averse traders differently, by using a new approach based on *risk measures*. Risk measures were developed in the finance literature and have been widely applied to risk management tasks (Artzner et al., 1999; Linsmeier and Pearson, 2000; Föllmer and Schied, 2004). Since a prediction market can be thought of as a special financial market (which has only two financial time stamps $t = 0, 1$) risk measures fit naturally into the prediction market setting. In addition, there are more important reasons to choose risk measures instead of EUT: with risk measures we will provide a simple market model in which analysis will become tractable, and the link between the prediction markets and machine learning will be drawn more explicitly.

5.2.1 Modelling preferences in share space

We first build a preference model in the space of shares $\boldsymbol{\Theta}$, or equivalently $\operatorname{span}(\boldsymbol{\phi})$, by using risk measures. Remember that $\operatorname{span}(\boldsymbol{\phi})$ is a subspace of $\mathcal{X}$, the *dual space* in the dual system $(\mathcal{P}, \mathcal{X}, \langle\cdot,\cdot\rangle)$ (Section 4.3.1). Consider a prediction market that is built on the state space Ω and offers K securities $\boldsymbol{\phi}$. Then the position of a trader is characterized by a vector of $K+1$ dimensions $\hat{\boldsymbol{\theta}}$ referred to as the *portfolio*, which records the amount of money w and the shares of securities $\boldsymbol{\theta}$ she holds

$$\hat{\boldsymbol{\theta}} := (w, \boldsymbol{\theta}^\top)^\top = (w, \theta_1, \ldots, \theta_K)^\top. \tag{5.1}$$

The payout of the portfolio is computed by

$$\hat{\mathsf{x}}(\omega) = \hat{\boldsymbol{\theta}}^\top \hat{\boldsymbol{\phi}}(\omega) = w + \mathsf{x}(\omega) = w + \boldsymbol{\theta}^\top \boldsymbol{\phi}(\omega), \tag{5.2}$$

where $\hat{\boldsymbol{\phi}} := (1, \boldsymbol{\phi}^\top)^\top$ denotes the unit payout vector of the portfolio, and $\mathsf{x} := \boldsymbol{\theta}^\top \boldsymbol{\phi}$ denotes the payout of the securities. The payout consists of two parts: the *risk-free* part w which always guarantees a deterministic payout, and the *risky* part x with an uncertain payout dependent on the realization of the true state. Thus, we name x the *risky asset* of the trader, w the *risk-free asset*, and $\hat{\mathsf{x}}$ the *gross asset* or simply *asset*.

In this setting, the preference model is a function of the portfolio, $f : \hat{\boldsymbol{\Theta}} \to \mathbb{R}$, such that the trader prefers one portfolio $\hat{\boldsymbol{\theta}}_1$ than the other asset $\hat{\boldsymbol{\theta}}_2$ if and only if $f(\hat{\boldsymbol{\theta}}_1) > f(\hat{\boldsymbol{\theta}}_2)$, and that the agent is indifferent between asset $\hat{\boldsymbol{\theta}}_1$ and asset $\hat{\boldsymbol{\theta}}_2$ if and only if $f(\hat{\boldsymbol{\theta}}_1) = f(\hat{\boldsymbol{\theta}}_2)$.

There are plenty of theories on selecting and analysing a specific form of f, such as EUT of Von Neumann and Morgenstern (2007), dual utility theory of Yaari (1987) and risk measures of Artzner et al. (1999); Föllmer and Schied (2002). Unlike other work, this work uses risk measures to model traders' behaviours. We first introduce risk measures, then make a detailed justification of using risk measures and its relation to EUT.

Risk measures Risk measures assign higher scores to assets that are more *risky*. They can also be understood as measures of the potential loss of choosing a certain asset. A *(monetary) risk measure* is defined as a function $\rho : \mathcal{X} \to \mathbb{R}$ such that $\rho(0)$ is finite and ρ satisfies the following conditions (Artzner et al., 1999):

Translation invariance If $\mathsf{x} \in \mathcal{X}$ and $m \in \mathbb{R}$, then

$$\rho(\mathsf{x} + m) = \rho(\mathsf{x}) - m. \tag{5.3}$$

Monotonicity If $\mathsf{x}, \mathsf{y} \in \mathcal{X}$ and $\mathsf{x} \leq \mathsf{y}$, then

$$\rho(\mathsf{x}) \geq \rho(\mathsf{y}). \tag{5.4}$$

Here $\mathsf{x} \leq \mathsf{y}$ should be understood as $p(\mathsf{x} \leq \mathsf{y}) = 1$, that is, with the probability of one that x will generate a lower return than Y. Thus monotonicity indicates that an asset with a better return deserves a lower risk. Due to translation invariance, a risk measure maps any risk-free asset to itself, and is additive w.r.t. any risk-free asset. Therefore, the output of a risk measure has the same unit as a risk-free asset, and can be effectively treated as an asset.

Risk measures are very generic. In our discussion we will use both risk measures and a specific class of them, the *convex* risk measures. Föllmer and Schied (2002) defines a risk measure to be convex if $\forall \mathsf{x}_1, \mathsf{x}_2 \in \mathcal{X}$ and $\lambda \in [0,1]$

$$\rho(\lambda \mathsf{x}_1 + (1-\lambda)\mathsf{x}_2) \leq \lambda\rho(\mathsf{x}_1) + (1-\lambda)\rho(\mathsf{x}_2). \tag{5.5}$$

It says that the risk of a combination of two assets should not be higher than holding them separately. In other words, convex risk measures encourage diversification, which is a natural condition for managing risky-assets. A convex risk measure can always be written as a convex conjugate of a function f defined on the dual space:

$$\rho(\mathsf{x}) = \sup_{p \in \mathcal{P}} \langle p, \mathsf{x} \rangle - f(p). \tag{5.6}$$

Examples of risk measures A famous non-convex risk measure is the *Value at Risk* (V@R) (Linsmeier and Pearson, 2000), which outputs a threshold loss l such that the probability of $-\mathsf{x}$ exceeding l is smaller than a predefined level

$$\mathrm{VaR}_\alpha(\mathsf{x}) \equiv \inf\{l \in \mathbb{R} \mid P(-\mathsf{x} > l) \leq 1-\alpha\}. \tag{5.7}$$

A famous convex risk measure is the *Entropic risk measure* (Föllmer and Schied, 2004)

$$\frac{1}{\eta}\rho_E(\eta\mathsf{x}, p_0) = \frac{1}{\eta}\log M_{\mathsf{x},p_0}(-\eta) = \sup_{p \in \mathcal{P}} \mathbb{E}_p[-\mathsf{x}] - \frac{1}{\eta}KL(p, p_0). \tag{5.8}$$

Here p_0 is the trader's belief distribution, $M_{\mathsf{x},p_0}(t) := \mathbb{E}_{p_0}[e^{t\mathsf{x}}]$ is the moment-generating function, $KL(\cdot,\cdot)$ is the KL-divergence (and this is the reason the term *entropic* is used), and $\eta > 0$ controls the degree of risk aversion. The larger η is, the more risk averse the trader will be. In general, we can introduce the degree of risk aversion for any risk measures $\rho(\mathsf{x})$ by considering another risk measure $\rho_\eta(\mathsf{x}) := \eta^{-1}\rho(\eta\mathsf{x})$. From the second equality, we see the entropic risk measure is represented as a convex conjugate of the function $f := KL(\cdot, p_0)$.

As the final example, the convex conjugate of a generalized entropy is in essence a convex risk measure. Let G be a generalized entropy, and define

$\rho(\mathsf{x}) := G^*(-\mathsf{x})$. Then according to Theorem 4.2, $\rho(\mathsf{x})$ meets all conditions of being a convex risk measure.

It immediately follows that the potential function of a PMM/PoPMM defined using a GEF (cf. Section 4.5.2 and 4.5.3) is a convex risk measure. More specifically, let $\boldsymbol{\theta}$ be the market maker's inventory or holdings, which is the negation of the total number of sold shares, and define $\rho(\boldsymbol{\theta}^\top \boldsymbol{\phi}) := F(-\boldsymbol{\theta}) = G^*(-\boldsymbol{\theta}^\top \boldsymbol{\theta})$. Then ρ is a convex risk measure on span$(\boldsymbol{\phi})$.

Risk measures v.s. EUT To justify using risk measures for modelling preferences, several reasons why risk measures are preferred to EUT are given. First, the output value of a risk measure can be treated as a risk-free asset and standard linear operations are well defined for it. In comparison, an expected utility outputs a number that only has abstract meaning, that is, to measure the degree of agent's satisfaction. In addition, risk measures force translation invariance by definition, while expected utilities do not have this property in general. With the help of translation invariance, the wealth w can always be separated from the risky asset x, which implies that the optimal portfolio does not depend on w. This saves us from the trouble of associating w with the aggregation weights, as the relationship between them is highly inconsistent and varies dramatically under different utilities (Storkey et al., 2012). Finally, according to Föllmer and Schied (2004), we could always derive a convex risk measure ρ_u from any expected utility

$$\rho_u(\mathsf{x}, p_0) \equiv \inf\{m \in \mathbb{R} \mid \mathbb{E}_{p_0}[u(\mathsf{x}+m)] \geq u_0\}, \tag{5.9}$$

where p_0 is the true belief of the trader. In fact, the output of this risk measure is the *risk premium*, the least amount of money that one would like to borrow in order to accept this risky asset. Then a sensible decision rule should be to find an asset that minimizes the premium, which leads to risk minimization. For example, consider the isoelastic utility

$$u_I(x) = \begin{cases} -\frac{1}{\eta}x^\eta & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.10}$$

where $\eta > 1$. The resulting convex risk measure is the convex conjugate of the following function

$$\alpha(p) = (-\eta u_0)^{1/\eta} \mathbb{E}_{p_0}\left[\left(\frac{\mathrm{d}p}{\mathrm{d}p_0}\right)^{\gamma}\right]^{1/\gamma}, \tag{5.11}$$

where $1/\eta + 1/\gamma = 1$. The risk measure itself can be computed from (5.6). Another example is the entropic risk measure in (5.8). It can be derived from the exponential utility $u_E(x) = -\exp(-ax)$, with $\eta = a$. In fact, the entropic risk measure coincides with the log expected exponential utility.

Deriving preference models from risk measures A risk measure ranks assets naturally by their risks, which enables us to define a preference function f based on it. More specifically, let $\hat{\mathrm{x}}$ be the asset resulting from the portfolio $\hat{\boldsymbol{\theta}} = (w, \boldsymbol{\theta}^\top)^\top$ through (5.2). Then our preference model is defined by

$$f(\hat{\boldsymbol{\theta}}) := -\rho(\hat{\mathrm{x}}) = -\rho(w + \mathrm{x}) = w - \rho(\boldsymbol{\theta}^\top \boldsymbol{\phi}) = w - \varrho(\boldsymbol{\theta}). \tag{5.12}$$

where for simplicity we define $\varrho := \rho \circ \boldsymbol{\phi}^\top$, which is a function of $\boldsymbol{\theta}$. The third equality is obtained by applying translation invariance in (5.3). This model is going to be our model for describing the behaviours of risk-averse traders.

We say a trader is *myopic-rational* or *myopic* if she always chooses the portfolio that maximizes her preference values by only considering current round of trading and neglecting the influence of the future trades. Consider a myopic trader whose preference is modelled by our preference function in eq. (5.12). Then when she interacts with a PMM at time t, her behaviour will be described by the following optimization problem

$$\max_{\hat{\boldsymbol{\vartheta}}_t \in \hat{\Theta}} f(\hat{\boldsymbol{\vartheta}}_t) = \max_{\hat{\boldsymbol{\vartheta}}_t \in \hat{\Theta}} \underbrace{(w_{t-1} - (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})))}_{w_t\text{: money after trade}} - \varrho(\boldsymbol{\vartheta}_t) \tag{5.13}$$

$$= w_{t-1} - \min_{\boldsymbol{\vartheta}_t \in \Theta} (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) + \varrho(\boldsymbol{\vartheta}_t) \tag{5.14}$$

$$= f(\hat{\boldsymbol{\vartheta}}_{t-1}) - \min_{\boldsymbol{\vartheta}_t \in \Theta} (F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1})) + (\varrho(\boldsymbol{\vartheta}_t) - \varrho(\boldsymbol{\vartheta}_{t-1})). \tag{5.15}$$

where $\boldsymbol{\theta}_{t-1}, \boldsymbol{\theta}_t, \boldsymbol{\vartheta}_{t-1}, \boldsymbol{\vartheta}_t$ are positions of the PMM and the trader before and after the trade, respectively. $f(\boldsymbol{\vartheta}_{t-1}) = -\rho(\boldsymbol{\vartheta}_{t-1})$ is added and subtracted in the last equality, and (5.12) is applied. The four positions are linked via the trade $\boldsymbol{\delta}_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1} = \boldsymbol{\vartheta}_t - \boldsymbol{\vartheta}_{t-1}$, that is, the amount of shares the PMM sells matches to the amount of shares the trader buys. Note that w_{t-1} is a constant w.r.t. time t and hence can be moved out of the maximization. (5.15) interprets the trader's behaviour as searching for the minimum risk of updating her holdings from $\boldsymbol{\theta}_{t-1}$ to a new position $\boldsymbol{\theta}_t$.

Since the potential function of a PMM/PoPMM is a convex risk measure w.r.t. its inventory, a PMM can be treated simply as a risk-averse trader. Under this view, (5.13) becomes symmetric, thus there is no actual difference in the trading behaviour between a market maker and a normal trader. To identify the market maker from traders, we need an extra model such as a trade network (Frongillo and Reid, 2014), which characterizes a market maker as a trader who is linked to (is able to trade with) all the other traders.

5.2.2 Modelling preferences in price space

Dually, a preference model can also be established in the space of possible prices, which is derived from the *primal space* $\mathcal{P}$ in the dual system. If we review the behaviours of traders in the price space, then at each trade a risk-neutral myopic trader will always adjust the market prices to her expected securities values, regardless of the market status. A risk-averse instead myopic trader will adjust the market prices to a position that compromises between her expectations and the market prices before trading.

Building a model in the price space is less straightforward than in the share space. We will begin with the model established in the previous section, discussing its counterpart in the price space, and then proposing the modelling framework based on what we have learned from this specific example. To model risk-averse traders in the price space, let's write down the equivalent dual problem of the minimization problem in (5.15), which is

$$\min_{\mathbf{p}_t \in \mathbf{p}(\boldsymbol{\Theta})} D_{F^*}(\mathbf{p}_t, \mathbf{p}_{t-1}) + D_{\varrho^*}(\mathbf{p}_t, \mathbf{p}'), \tag{5.16}$$

where $\mathbf{p}_{t-1}, \mathbf{p}_t$ are the market prices before and after the trade; $\mathbf{p}'$ is the market prices the trader beliefs before this trade happens; $D_{F^*}(\cdot,\cdot)$ and $D_{\varrho^*}(\cdot,\cdot)$ are Bregman divergences generated by the convex conjugates of F and ϱ, respectively. In addition, $\mathbf{p}(\boldsymbol{\Theta})$ is the image of the set of all possible positions $\boldsymbol{\Theta}$ under the price (duality) map $\mathbf{p}(\cdot)$. It contains all possible prices the market can have. This dual form relies heavily on the fact, that for a potential function F (and similarly for a convex risk measure) the convex conjugate of its difference $C(\delta_t) := \eta_t^{-1}(F(\eta_t\boldsymbol{\theta}_t) - F(\eta_t\boldsymbol{\theta}_{t-1}))$ is a Bregman divergence

$$\begin{aligned} C^*(\mathbf{p}_t) := \sup_{\boldsymbol{\theta}_t\in\boldsymbol{\Theta}} \delta_t^\top \mathbf{p}_t - C(\delta_t) &= \sup_{\boldsymbol{\theta}_t\in\boldsymbol{\Theta}} \boldsymbol{\theta}_t^\top \mathbf{p}_t - \boldsymbol{\theta}_{t-1}^\top \mathbf{p}_t - \frac{1}{\eta_t}(F(\eta_t\boldsymbol{\theta}_t) - F(\eta_t\boldsymbol{\theta}_{t-1})) \\ &= \frac{1}{\eta_t}(F^*(\mathbf{p}_t) + F(\eta_t\boldsymbol{\theta}_{t-1}) - \eta_t\boldsymbol{\theta}_{t-1}^\top\mathbf{p}_t) = \frac{1}{\eta_t}D_{F^*}(\mathbf{p}_t, \mathbf{p}_{t-1}) \end{aligned} \tag{5.17}$$

Here $\eta_t > 0$ is an augmented parameter for F that controls the liquidity of the market (Chen and Wortman Vaughan, 2010). As we will see shortly, the liquidity parameter controls the step-size of the market optimization towards the market equilibrium.

In convex analysis, such augmentation of $f(x) \to \eta^{-1}f(\eta x)$ is called the *perspective transform*. The perspective transform will maintain the convexity of the original function f. However, the gradient will be scaled in the sense that the gradient map is maintained, but the input of gradient map is scaled multiplicatively by η, $\nabla f(x) \to \nabla f(\eta x)$.

The liquidity parameter η is fixed during each trade, and may only be updated by the market maker after each trade. Therefore, with the liquidity augmented potential function, the trading dynamics will almost be the same as before except for an extra update of η at the end of each trade t and before the next trade $t+1$. However, if the market maker only changes $\eta_t \to \eta_{t+1}$ but keeps $\boldsymbol{\theta}_t$ fixed, an inconsistency in price $\mathbf{p}_t$ will occur as the gradient $\nabla F(\eta_t\boldsymbol{\theta}_t) \neq \nabla F(\eta_{t+1}\boldsymbol{\theta}_t)$. To maintain the same price, the market maker will also update $\boldsymbol{\theta}_t' \leftarrow \eta_t\boldsymbol{\theta}_t/\eta_{t+1}$, such that $\eta_t\boldsymbol{\theta}_t = \eta_{t+1}\boldsymbol{\theta}_t'$ with this updated $\boldsymbol{\theta}_t'$. Then under this update, the market price $\mathbf{p}_t$ is maintained while varying the liquidity.

Notice that by updating $\boldsymbol{\theta}_t$ in accordance with liquidity η_t, The conventional meaning of $\boldsymbol{\theta}_t$, that it represents the market inventory which records the total amount of shares sold to traders, is no longer valid. Instead, $\boldsymbol{\theta}_t$ now simply describes an abstract market position of the market maker. If the liquidity is fixed throughout the market lifetime, then $\boldsymbol{\theta}_t$ can again represent the market inventory.

The convex conjugate relation (5.17) enables us to write the cost of purchasing securities as well as the full objective into the Bregman divergences as in (5.16). The term $D_{F^*}(\mathbf{p}_t, \mathbf{p}_{t-1})$ reflects the price change under the PMM, while the term $D_{\varrho^*}(\mathbf{p}_t, \mathbf{p}')$ describes the trader's behaviour. In fact, when not interacting with the PMM, then $\mathbf{p}_t = \mathbf{p}'$ minimizes the divergence, implying that the trader will prefer market prices that match her belief. When trading with a PMM, the final prices shift to a position between her belief $\mathbf{p}'$ and the pre-trade market prices $\mathbf{p}_{t-1}$ to minimize the risks.

In general, we can model a risk-averse myopic trader by a *convex* risk function defined on the price space $f_t : \mathbf{p}(\boldsymbol{\Theta}) \to \mathbb{R}$, such that when not interacting with the market maker the trader's minimum risk is achieved at her expectations. Then similar to (5.16) a trade with the market maker will be characterized by

$$\mathbf{p}_t^* = \underset{\mathbf{p}_t \in \mathbf{p}(\boldsymbol{\Theta})}{\arg\min}\ f_t(\mathbf{p}_t) + \frac{1}{\eta_t} D_{F^*}(\mathbf{p}_t, \mathbf{p}_{t-1}) \tag{5.18}$$

whose dual problem in the share space is

$$\delta_t^* = \underset{\delta_t : \delta_t + \boldsymbol{\theta}_{t-1} \in \boldsymbol{\Theta}}{\arg\max}\ -f_t^*(-\delta_t) - \frac{1}{\eta_t}\left(F(\eta_t(\boldsymbol{\theta}_{t-1} + \delta_t)) - F(\eta_t \boldsymbol{\theta}_{t-1})\right). \tag{5.19}$$

Here the second term inside the maximization is just the cost of trading δ_t shares, while the first term $-f_t^*(-\delta_t)$ is understood as the corresponding preference model in the dual space, which estimates the values of δ_t shares of securities that takes the trader's risk preference into consideration. When f_t is set to a characteristic function such that $f(\mathbf{v}_t) = 0$ and $f(\mathbf{p}_t) = +\infty$ for all $\mathbf{p}_t \neq \mathbf{v}_t$, then $-f_t^*(-\delta_t) = \delta_t^T \mathbf{v}_t$ which reduces to the risk-neutral case.

5.2.3 Convergence w.r.t. sequential traders

The first market setting for convergence analysis is a prediction market involving a set of sequential traders, such that each trader interacts with the PMM only once. From the PMM point of view, each time it will always trade with a new trader. In this setting, the total number of traders will increase through time. Hence the cash flow and the total amount of information carried by the traders also increase. Typically, the increasing number of traders will prevent the market prices from converging to a fixed point, and will not result in any equilibrium in the conventional fixed point sense. Instead, we show that an alternative concept of market equilibrium does exist in the stochastic sense, and this equilibrium is reached by having the PMM sequentially trade with risk-averse trader, who are assumed to be drawn an ensemble of agents with its mean preference invariant through time.

5.2.3.1 Trading for no regret

Recall that the interaction between a PMM and our risk-averse trader is represented by an optimization problem in (5.18), which we rewrite here for reference

$$\mathbf{p}_t = \mathop{\arg\min}_{\mathbf{p}\in\mathbf{p}(\Theta)} \; f_t(\mathbf{p}) + \frac{1}{\eta_t} D(\mathbf{p}, \mathbf{p}_{t-1}). \tag{5.20}$$

For simplicity, here and later we reload $\mathbf{p}_t$ for the optimal solution, and we shorten $D_{F^*}(\cdot,\cdot)$ to $D(\cdot,\cdot)$ when context is clear. (5.20) is referred to as *proximal iteration* in optimization literature (Parikh and Boyd, 2013, Section 4.1).

On the other hand, from the machine learning point of view, a prediction market with sequential traders is an online learning system, with the sequence of traders being the streaming data, and the trader's (primal) risk function f_t being the loss function associated with the data point. The performance of an online learning system is usually measured by the *(averaged) regret* (Shalev-Shwartz, 2011). For this particular market it has the form

$$R_T(\mathbf{p}^*) := \frac{1}{T}\sum_{t=1}^{T} f_t(\mathbf{p}_{t-1}) - \frac{1}{T}\sum_{t=1}^{T} f_t(\mathbf{p}^*) \tag{5.21}$$

with $\mathbf{p}^* = \arg\min_{\mathbf{p}\in\mathbf{p}(\Theta)} 1/T \sum_{t=1}^{T} f_t(\mathbf{p})$ the optimal *hindsight* model obtained given all data points and the losses associated with them. Note that $\mathbf{p}_{t-1}$ is used instead of $\mathbf{p}_t$ in order to capture the predictability of the market prices for the next trade. In other words, regret measures the performance of the system *not* by the absolute (averaged) loss, but by its difference to that of the optimal hindsight model. Compared to the absolute loss, regret provides a consistent measure of performance in the online setting, as the data stream is assumed to be arbitrary or even adversarial, w.r.t. which the absolute performance of the system cannot be guaranteed.

Since each trade is equivalent to a proximal iteration, the convergence of the market can be analysed by using standard techniques. In particular, our results is obtained with the help of Chen and Teboulle (1993), who analyse the convergence for this type of algorithm without sequential setting, and Duchi et al. (2010), who analyse the convergence for *online mirror descent*, a close neighbour of the proximal iteration algorithm.

We assume from now on that the potential function of a PMM is always augmented by the liquidity parameter, whose value η_t at time t is adjusted by the market maker right before the t-th trade. In addition, we assume

1. that all preference functions are L-Lipschitz continuous, and
2. that the PMM's potential function F is α-strongly smooth, which implies the α-strong convexity of its convex conjugate F^* (Kakade et al., 2009).

The key to the analysis The proximal iteration (5.20) tells us that at each time t, the following inequality holds for any $\mathbf{p}' \in \mathbf{p}(\boldsymbol{\Theta})$

$$\langle \mathbf{p}' - \mathbf{p}_t, \partial D(\mathbf{p}, \mathbf{p}_{t-1})|_{\mathbf{p}=\mathbf{p}_t} + \eta_t \partial f_t(\mathbf{p}_t) \rangle \geq 0, \tag{5.22}$$

and in particular for $\mathbf{p}' = \mathbf{p}^* = \arg\min_{\mathbf{p}\in\mathbf{p}(\boldsymbol{\Theta})} 1/T \sum_{t=1}^{T} f_t(\mathbf{p})$. From (5.22) an important lemma could be obtained.

Lemma 5.1. *Given a PMM with potential function F and a set of sequential risk-averse traders with preference models $\{f_t(\mathbf{p})\}_{t=1}^{T}$, the market regret in (5.21), after time T, is bounded from above by*

$$R_T(\mathbf{p}^*) \leq \frac{1}{T} \sum_{t=1}^{T} \left(\frac{1}{\eta_t} D(\mathbf{p}^*, \mathbf{p}_{t-1}) - \frac{1}{\eta_t} D(\mathbf{p}^*, \mathbf{p}_t) + \frac{\eta_t}{2\alpha} L^2 \right). \tag{5.23}$$

Proof. Rearrange (5.22)

$$\eta_t \langle \mathbf{p}_t - \mathbf{p}^*, \partial f_t(\mathbf{p}_t) \rangle \leq \langle \mathbf{p}^* - \mathbf{p}_t, \partial D(\mathbf{p}, \mathbf{p}_{t-1})|_{\mathbf{p}=\mathbf{p}_t} \rangle. \tag{5.24}$$

The LHS bounds $\eta_t(f_t(\mathbf{p}_t) - f_t(\mathbf{p}^*))$ from above by convexity of f_t. By definition of Bregmain divergences, $D(\mathbf{p}, \mathbf{p}_0) = F^*(\mathbf{p}) - F^*(\mathbf{p}_0) - \langle \partial F^*(\mathbf{p}_0), \mathbf{p} - \mathbf{p}_0 \rangle$, the RHS can be rewritten into

$$\begin{aligned}
&\langle \mathbf{p}^* - \mathbf{p}_t, \partial D(\mathbf{p}, \mathbf{p}_{t-1})|_{\mathbf{p}=\mathbf{p}_t} \rangle \\
=& -\langle \mathbf{p}^* - \mathbf{p}_{t-1}, \partial F^*(\mathbf{p}_{t-1}) \rangle + \langle \mathbf{p}^* - \mathbf{p}_t, \partial F^*(\mathbf{p}_t) \rangle + \langle \mathbf{p}_t - \mathbf{p}_{t-1}, \partial F^*(\mathbf{p}_{t-1}) \rangle \\
=& D(\mathbf{p}^*, \mathbf{p}_{t-1}) - D(\mathbf{p}^*, \mathbf{p}_t) - D(\mathbf{p}_t, \mathbf{p}_{t-1}).
\end{aligned} \tag{5.25}$$

These together lead to

$$\eta_t(f_t(\mathbf{p}_t) - f_t(\mathbf{p}^*)) \leq D(\mathbf{p}^*, \mathbf{p}_{t-1}) - D(\mathbf{p}^*, \mathbf{p}_t) - D(\mathbf{p}_t, \mathbf{p}_{t-1}). \tag{5.26}$$

The LHS gives $f_t(\mathbf{p}_t)$ while what we want is $f_t(\mathbf{p}_{t-1})$. Adding $\eta_t f_t(\mathbf{p}_{t-1})$ to both sides and moving $\eta_t f_t(\mathbf{p}_t)$ to the RHS in the we obtain

$$\eta_t(f_t(\mathbf{p}_{t-1}) - f_t(\mathbf{p}^*)) \leq D(\mathbf{p}^*, \mathbf{p}_{t-1}) - D(\mathbf{p}^*, \mathbf{p}_t) \\ - D(\mathbf{p}_t, \mathbf{p}_{t-1}) + \eta_t(f_t(\mathbf{p}_{t-1}) - f_t(\mathbf{p}_t)). \quad (5.27)$$

First by convexity we have $f_t(\mathbf{p}_{t-1}) - f_t(\mathbf{p}_t) \leq \langle \mathbf{p}_{t-1} - \mathbf{p}_t, \partial f_t(\mathbf{p}_{t-1})\rangle$. Second, by Fenchel-Young inequality we obtain

$$\langle \mathbf{p}_{t-1} - \mathbf{p}_t, \partial f_t(\mathbf{p}_{t-1})\rangle \leq \frac{\alpha}{2\eta_t}\|\mathbf{p}_{t-1} - \mathbf{p}_t\|_*^2 + \frac{\eta_t}{2\alpha}\|\partial f_t(\mathbf{p}_{t-1})\|^2, \quad (5.28)$$

where $\|\cdot\|$ is the norm w.r.t. which the α-strong smoothness of F is defined, and $\|\cdot\|_*$ is the corresponding dual norm. Then, by the α-strong convexity of F^*, the Bregman divergence generated by F^* is bounded from below

$$D(\mathbf{p}_{t-1}, \mathbf{p}_t) \geq \frac{\alpha}{2}\|\mathbf{p}_{t-1} - \mathbf{p}_t\|_*^2. \quad (5.29)$$

Finally by the L-Lipschitz continuity of f_t, $\|\partial f_t(\cdot)\| \leq L$. These three relations simplify the last three terms on the RHS of (5.27) to

$$- D(\mathbf{p}_t, \mathbf{p}_{t-1}) + \eta_t(f_t(\mathbf{p}_{t-1}) - f_t(\mathbf{p}_t)) \leq \frac{\eta_t^2}{2\alpha}\|\partial f_t(\mathbf{p}_{t-1})\|^2 \leq \frac{\eta_t^2}{2\alpha}L^2. \quad (5.30)$$

Now substitute (5.30) back to (5.27). We see that each step in the regret is bounded from above

$$f_t(\mathbf{p}_{t-1}) - f_t(\mathbf{p}^*) \leq \frac{1}{\eta_t}D(\mathbf{p}^*, \mathbf{p}_{t-1}) - \frac{1}{\eta_t}D(\mathbf{p}^*, \mathbf{p}_t) + \frac{\eta_t}{2\alpha}L^2. \quad (5.31)$$

Summing over t gives the target bound. □

Minimum regret bound with fixed liquidity Assume a fixed $\eta_t = \eta$. Then (5.23) is simplified to

$$R_T(\mathbf{p}^*) \leq \frac{1}{T\eta}D(\mathbf{p}^*, \mathbf{p}_0) - \frac{1}{T\eta}D(\mathbf{p}^*, \mathbf{p}_T) + \frac{\eta}{2\alpha}L^2 \leq \frac{1}{T\eta}D(\mathbf{p}^*, \mathbf{p}_0) + \frac{\eta}{2\alpha}L^2. \quad (5.32)$$

Recall that the bounded loss property of the market maker gives $D(\mathbf{p}, \mathbf{p}_0) \leq M^2$ for any $\mathbf{p} \in \mathbf{p}(\boldsymbol{\Theta})$ and thus in particular for $\mathbf{p}^*$ (Theorem 3.2). Then the minimum bound for the regret is

$$R_t(\mathbf{p}^*) \leq \frac{\sqrt{2}LM}{\sqrt{\alpha T}} = \mathcal{O}(\frac{1}{\sqrt{T}}). \quad (5.33)$$

which is obtained at $\eta = M/L\sqrt{2\alpha/T}$.

Minimum regret bound with adaptive liquidity Assume that η_t is adaptive through time. Rewrite (5.23) into

$$R_T(\mathbf{p}^*) \leq \frac{1}{T\eta_1} D(\mathbf{p}^*, \mathbf{p}_0) - \frac{1}{T\eta_T} D(\mathbf{p}^*, \mathbf{p}_T) + \frac{L^2}{2\alpha T} \sum_{t=1}^{T} \eta_t + \frac{1}{T} \sum_{t=1}^{T-1} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) D(\mathbf{p}^*, \mathbf{p}_t). \quad (5.34)$$

Choosing $\eta_t = M/L\sqrt{\alpha/t}$ leads to the minimum bound

$$R_T(\mathbf{p}^*) \leq \frac{M^2}{T} \left(\frac{1}{\eta_1} + \sum_{t=1}^{T-1} \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) \right) + \frac{L^2}{2\alpha T} \int_0^T \eta_t \mathrm{d}t \leq \frac{2LM}{\sqrt{\alpha T}}. \quad (5.35)$$

Similar to the fixed liquidity case, the regret bounded is $\mathcal{O}(1/\sqrt{T})$.

In both cases the regret objective is bounded by $\mathcal{O}(1/\sqrt{T})$ that shrinks through time. Thus the sequential trading scheme in a prediction market implements a *no regret* algorithm, such that with a sufficiently large time scale T, the market prices $\mathbf{p}_{T-1}$ will finally coincide with the prices $\mathbf{p}^*$ that achieves the minimum averaged loss. It is also worth noting that the liquidity η_t, in either fixed or adaptive setting, plays a role as the step-size parameter in the underlying optimization.

5.2.3.2 Stochastic traders

Now we further assume that the sequence of traders are in fact i.i.d. samples from the same distribution $p(z)$. More specifically, for each trader t, her risk function f_t is sampled from a collection $\{f(\cdot, z)\}_{z \in \mathcal{Z}}$, such that $f_t := f(\cdot, z_t)$ with $z_t \sim p(z)$. We also denote $f := \mathbb{E}[f(\cdot, z)]$ the mean risk function of the collection.

We refer to the sequence of traders under this extra assumption as the *stochastic traders*, as used by Frongillo et al. (2012). Stochastic traders are special cases of sequential traders, and so they will maintain the boundedness on the market regret. In addition to that, it will be shown that the market prices can be directly linked to the mean risk function f.

Lemma 5.2. *Consider a PMM with potential function F and a set of stochastic traders with preference models $\{f_t = f(\cdot, z_t) \mid z_t \sim p(z)\}$. Denote $f := \mathbb{E}[f(\cdot, z)]$*

the mean risk function and $\bar{\mathbf{p}}_T = 1/T \sum_{t=0}^{T} \mathbf{p}_t$ *the running averaged prices. Then after time T*

$$f(\bar{\mathbf{p}}_{T-1}) - f(\mathbf{p}^*) \leq \frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\eta_t}D(\mathbf{p}^*,\mathbf{p}_{t-1}) - \frac{1}{\eta_t}D(\mathbf{p}^*,\mathbf{p}_t) + \frac{\eta_t}{2\alpha}L^2\right) + 4LM\sqrt{\frac{1}{\alpha T}\log\frac{1}{\delta}}, \quad (5.36)$$

with probability at least $1-\delta$*.*

Proof. First rewrite eq. (5.31) to involve f instead of f_t in each single trade

$$f(\mathbf{p}_{t-1}) - f(\mathbf{p}^*) \leq \frac{1}{\eta_t}D(\mathbf{p}^*,\mathbf{p}_{t-1}) - \frac{1}{\eta_t}D(\mathbf{p}^*,\mathbf{p}_t) + \frac{\eta_t}{2\alpha}L^2 + (f(\mathbf{p}_{t-1}) - f_t(\mathbf{p}_{t-1})) - (f(\mathbf{p}^*) - f_t(\mathbf{p}^*)), \quad (5.37)$$

Denote the last two terms by $Y_t := (f(\mathbf{p}_{t-1}) - f_t(\mathbf{p}_{t-1})) - (f(\mathbf{p}^*) - f_t(\mathbf{p}^*))$. Then by definition of Y_t and f, $\mathbb{E}[Y_t \mid z_1,\ldots,z_{t-1}] = 0$. On the other hand, applying convexity of f, the lower and higher bounds on $D(\mathbf{p}^*,\mathbf{p}_{t-1})$: $\alpha/2\|\mathbf{p}^* - \mathbf{p}_{t-1}\|_*^2 \leq D(\mathbf{p}^*,\mathbf{p}_{t-1}) \leq M^2$, and Cauchy-Schwarz, we obtain

$$Y_t \leq \langle \mathbf{p}^* - \mathbf{p}_{t-1}, \partial f(\mathbf{p}_{t-1}) - \partial f(\mathbf{p}^*)\rangle \leq \|\mathbf{p}^* - \mathbf{p}_{t-1}\|_* \|\partial f(\mathbf{p}_{t-1}) - \partial f(\mathbf{p}^*)\| \leq 2\sqrt{2/\alpha}LM. \quad (5.38)$$

Thus $\{Y_t\}_{t=1}^T$ is a *bounded martingale difference sequence*, to which the Azuma's inequality can apply. Denote $\gamma_T := \sum_{t=1}^T Y_t$, then Azuma's inequality tells us that

$$p(\gamma_T \geq \epsilon) \leq \exp\left(\frac{-\alpha\epsilon^2}{16TL^2M^2}\right). \quad (5.39)$$

Summing (5.37) over t and applying the Jensen's inequality $\sum_{t=1}^T f(\mathbf{p}_{t-1}) \geq Tf(\bar{\mathbf{p}}_{T-1})$, we obtain

$$T(f(\bar{\mathbf{p}}_{T-1}) - f(\mathbf{p}^*)) - \sum_{t=1}^{T}\left(\frac{1}{\eta_t}D(\mathbf{p}^*,\mathbf{p}_{t-1}) - \frac{1}{\eta_t}D(\mathbf{p}^*,\mathbf{p}_t) + \frac{\eta_t}{2\alpha}L^2\right) \leq \gamma_T. \quad (5.40)$$

Since LHS is no larger than γ_T, it implies that the probability the LHS being equal or greater than ϵ is no more than the probability $p(\gamma_T \geq \epsilon)$. Finally, let

$$\delta = \exp\left(\frac{-\alpha\epsilon^2}{16TL^2M^2}\right), \quad (5.41)$$

and so we have $\epsilon = 4LM\sqrt{T/\alpha\log(1/\delta)}$. Replacing ϵ by δ leads to (5.36). □

As $T \to \infty$, $f(\mathbf{p}_{T-1})$ will finally convergence to $f(\mathbf{p}^*)$. Given a finite time, the variance of $f(\mathbf{p}_{T-1})$ can be calulated using the fact that

$$\begin{aligned}\text{var}[f(\mathbf{p}_{T-1})] &= \mathbb{E}[(f(\bar{\mathbf{p}}_{T-1}) - f(\mathbf{p}^*))^2] - (\mathbb{E}[f(\bar{\mathbf{p}}_{T-1})] - f(\mathbf{p}^*))^2 \\ &\leq \mathbb{E}[(f(\bar{\mathbf{p}}_{T-1}) - f(\mathbf{p}^*))^2]. \end{aligned} \tag{5.42}$$

and then by applying (5.36). Since $f(\bar{\mathbf{p}}_{T-1}) - f(\mathbf{p}^*)$ vanishes as $T \to \infty$, so does the variance of $f(\mathbf{p}_{T-1})$.

Minimum regret bound with fixed liquidity Similar to the regret analysis, if η_t is fixed to η, we have

$$\frac{1}{T}\sum_{t=1}^{T}\left(\frac{1}{\eta_t}D(\mathbf{p}^*, \mathbf{p}_{t-1}) - \frac{1}{\eta_t}D(\mathbf{p}^*, \mathbf{p}_t) + \frac{\eta_t}{2\alpha}L^2\right) \leq \frac{1}{T\eta}M^2 + \frac{\eta}{2\alpha}L^2. \tag{5.43}$$

The upper bound is tight when $\eta = M/L\sqrt{2\alpha/T}$. Therefore, in this case with probability at least $1-\delta$

$$f(\bar{\mathbf{p}}_{T-1}) \leq f(\mathbf{p}^*) + \frac{\sqrt{2}LM}{\sqrt{\alpha T}}\left(1 + 2\sqrt{2\log\frac{1}{\delta}}\right). \tag{5.44}$$

That is, the running averaged market prices $\bar{\mathbf{p}}_{T-1}$ converges to the optimum $\mathbf{p}^*$ in rate $\mathcal{O}(1/\sqrt{T})$.

Minimum regret bound with adaptive liquidity If η_t is adaptive, the minimum bound in probability is given by $\eta_t = M/L\sqrt{\alpha/t}$. A similar result will be obtained, that with probability at least $1-\delta$

$$f(\bar{\mathbf{p}}_{T-1}) \leq f(\mathbf{p}^*) + \frac{2LM}{\sqrt{\alpha T}}\left(1 + 2\sqrt{\log\frac{1}{\delta}}\right). \tag{5.45}$$

5.2.3.3 Example

Without loss of generality, consider a prediction market on a binary future state $\omega \in \{0, 1\}$. Each trader is assumed to have a beta prior $\pi = \text{Beta}(\alpha, \beta)$ about the true distribution $p(\omega)$ of ω, and she builds her belief over ω after observing several private i.i.d. samples $\mathbf{s} \sim p(\omega)$ using Bayesian update:

$$p_t(\omega = 0) = \int p'(\omega = 0)\pi(p' \mid \mathbf{s})\mathrm{d}p' = \frac{n_{t,0} + \alpha}{n_t + (\alpha + \beta)}. \tag{5.46}$$

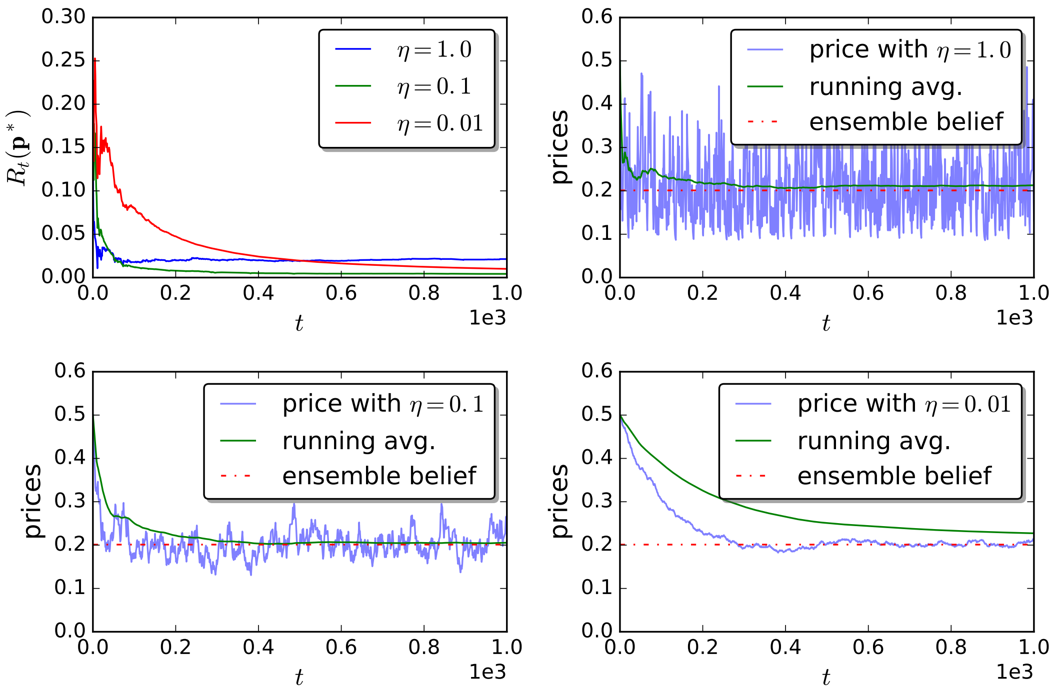


Figure 5.1: Sequential trading between market maker and traders under different liquidity parameters $\eta = 1, 0.1, 0.01$. In sequential trading, each trader acts only once and leaves the market immediately after the transaction. Given a fixed trading period $T = 1000$, the minimum market regret (5.21) as well as the convergence of the market price depend on η. Reducing η will decrease the price volatility, but will also slow down the convergence. According to the analysis, the optimal η is at the scale of $1/\sqrt{T} \approx 0.03$, which is supported by the result.

Here n_t is the number of private signals the t-th trader observes, and $n_{t,0}$ is the count of $\omega = 0$. Each trader models her preference through an entropic risk measure (5.8). The primal preference function for the t-th trader is thus $f_t(\cdot) = KL(\cdot, p_t)$. Note that the private signal is not necessary the historical observations of the same event we want to predict. In fact, many interesting events we want to predict are unique, such as each US presidential election. Instead, the private signal should be understood as the virtual result generated by each trader using her own prediction method. Here, we assume that each trader's prediction method is powerful enough to (almost) catch the true process of generating the future outcome, but is too expensive to run. Hence each trader can only get very few sample points.

We use a market maker with standard log-sum-exp potential, augmented by the liquidity parameter: $1/\eta F(\eta\boldsymbol{\theta}) = 1/\eta \log \sum_{i=0}^{1} \exp(\eta\theta_i)$. Trading period is $T = 1000$, and each trader observes $n_t = 5$ private signals for belief construction. The simulation result is shown in Figure 5.1.

Given a fixed trading period (in this case $T = 1000$), the minimum market regret (5.21) as well as the convergence of the market price to the ensemble mean depend on the choice of the liquidity parameters η. Recall that the liquidity of the market η also plays a role as the step-size parameter in the corresponding optimization process. Under a smaller η, the price is more stable but will converge more slowly. According to the previous analysis, the optimal η should be at the scale of $1/\sqrt{T} \approx 0.03$. This is supported by the simulations.

5.2.4 Convergence under repeated trading

In the second setting we consider a finite number of traders repeatedly trading in the market. Since the number of traders is finite, both the traders' budgets and the total amount of the information carried by the traders are bounded through time. We can thus show that a real convergence in the market prices rather than in the running averages will be established. To get the convergence result, we will first work in the share space, giving the condition (in the dual form) under which the market reaches its equilibrium and showing how the trades drive the market to this equilibrium; then we transform the result to its primal form described in the space of market prices. The significance of the convergence result in the primal form is beyond the pure market sense. In fact, in addition to giving the price condition for the market equilibrium, it will also link explicitly the prediction markets to a generic class of machine learning problems.

5.2.4.1 Market global objective

In the repeated trading market, despite that each trade is made to achieve the trader's personal goal, it will effectively act as an optimization step towards the minimum of a market-wise objective. In particular, let $\mathcal{A} = \{1, 2, \ldots, A\}$ be the index set of involved traders, $\boldsymbol{\vartheta}_a$ be the trader a's holdings and $\boldsymbol{\vartheta}_{a,0}$ be the trader's initial holdings. We will show that the market's global objective has the following form

$$L = \frac{1}{\eta}(F(\eta\boldsymbol{\theta}) - F(\eta\boldsymbol{\theta}_0)) + \sum_{a\in\mathcal{A}} \frac{1}{\eta_a}(\varrho_a(\eta_a\boldsymbol{\vartheta}_a) - \varrho_a(\eta_a\boldsymbol{\vartheta}_{a,0})), \tag{5.47}$$

where $\boldsymbol{\theta} - \boldsymbol{\theta}_0 = \sum_{a\in\mathcal{A}} \boldsymbol{\vartheta}_a - \boldsymbol{\vartheta}_{a,0}$, η the liquidity parameter, and η_a the degree of risk-aversion for trader a. The constraint on the shares interprets the conservation in shares: the total amount of shares sold by the market maker should match to the total amount of shares purchased by traders.

To derive the global objective, we recall from (5.15) that each trade at time t the optimal trade between trader a and the market maker minimizes the following function

$$l_t(\boldsymbol{\delta}_t) := \frac{1}{\eta}(F(\eta\boldsymbol{\theta}_t) - F(\eta\boldsymbol{\theta}_{t-1})) + \frac{1}{\eta_a}(\varrho_a(\eta_a\boldsymbol{\vartheta}_{a,t}) - \varrho_a(\eta_a\boldsymbol{\vartheta}_{a,t-1})). \tag{5.48}$$

where $\boldsymbol{\delta}_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1} = \boldsymbol{\vartheta}_{a,t} - \boldsymbol{\vartheta}_{a,t-1}$. Since one trade happens each time, for each trader $a' \neq a$ her holdings keep invariant $\boldsymbol{\vartheta}_{a',t} = \boldsymbol{\vartheta}_{a',t-1}$ during the t-th trade. Thus we can rewrite l_t in

$$l_t(\boldsymbol{\delta}_t) := \frac{1}{\eta}(F(\eta\boldsymbol{\theta}_t) - F(\eta\boldsymbol{\theta}_{t-1})) + \sum_{a\in\mathcal{A}} \frac{1}{\eta_a}(\varrho_a(\eta_a\boldsymbol{\vartheta}_{a,t}) - \varrho_a(\eta_a\boldsymbol{\vartheta}_{a,t-1})). \tag{5.49}$$

Summing up the minimum objective $\min_{\boldsymbol{\delta}_t} l_t$

$$\begin{aligned}
&\sum_{t=1}^{T} \min_{\boldsymbol{\delta}_t} l_t(\boldsymbol{\delta}_t) \\
&\quad\geq \min_{\{\boldsymbol{\delta}_t\}_{t=1}^T} \sum_{t=1}^{T} \frac{1}{\eta}(F(\eta\boldsymbol{\theta}_t) - F(\eta\boldsymbol{\theta}_{t-1})) + \sum_{a\in\mathcal{A}} \frac{1}{\eta_a}(\varrho_a(\eta_a\boldsymbol{\vartheta}_{a,t}) - \varrho_a(\eta_a\boldsymbol{\vartheta}_{a,t-1})) \\
&\quad= \min_{\{\boldsymbol{\delta}_t\}_{t=1}^T} \frac{1}{\eta}(F(\eta\boldsymbol{\theta}_T) - F(\eta\boldsymbol{\theta}_0)) + \sum_{a\in\mathcal{A}} \frac{1}{\eta_a}(\varrho_a(\eta_a\boldsymbol{\vartheta}_{a,T}) - \varrho_a(\eta_a\boldsymbol{\vartheta}_{a,0})).
\end{aligned} \tag{5.50}$$

Here $\boldsymbol{\delta} = \sum_{t=1}^{T} \boldsymbol{\delta}_t = \boldsymbol{\theta}_T - \boldsymbol{\theta}_0 = \sum_{a\in\mathcal{A}} \boldsymbol{\vartheta}_{a,T} - \boldsymbol{\vartheta}_{a,0}$. (5.50) is a sequential minimization scheme for minimizing the market objective L in (5.47). Finally, if the market has converged at time T, then for $t \geq T$, $\boldsymbol{\theta}_t = \boldsymbol{\theta}_T$ and $\boldsymbol{\vartheta}_{a,t} = \boldsymbol{\vartheta}_{a,T}$. We denote $\boldsymbol{\theta} = \boldsymbol{\theta}_T$ and $\boldsymbol{\vartheta}_a = \boldsymbol{\vartheta}_{a,T}$. Then $\{\boldsymbol{\theta}, \{\boldsymbol{\vartheta}_a\}_{a\in\mathcal{A}}\}$ gives a (potentially local) minimum of L.

Convergence The convergence analysis of our repeated trading market (and its generalization) is done by Frongillo and Reid (2014, 2015). They prove that the market prices converge to the minimum of L in $\mathcal{O}(1/T)$. We also mention that the repeated trading market effectively implements a generalized version of *stochastic dual coordinate accent (SDCA)* of Shalev-Shwartz and Zhang (2013), in which the PMM potential F instead of a standard quadratic function is used as the proximal function. For SDCA, the authors also show a convergence rate $\mathcal{O}(1/T)$.

5.2.4.2 Market objective in primal form

The minimization of the above market global objective in eq. (5.47) is known as *the sharing problem* in convex optimization (Boyd et al., 2011, Chapter 7). It is the Lagrange dual problem of *the consensus problem* which has the following general form

$$\min_{x\in\mathcal{X}} f(x) = \min_{x\in\mathcal{X}} \sum_{n=1}^{N} f_n(x), \tag{5.51}$$

where each f_i is convex. For our problem, the primal form of the market objective is obtained by directly applying the *generalized Fenchel's duality* of Shalev-Shwartz and Singer (2007) and (5.17). As a result, the primal objective is

$$\min_{\mathbf{p}\in\mathbf{p}(\boldsymbol{\Theta})} \frac{1}{\eta} D_{F^*}(\mathbf{p},\mathbf{p}_0) + \sum_{a\in\mathcal{A}} \frac{1}{\eta_a} D_{\varrho_a^*}(\mathbf{p},\mathbf{p}_{a,0}), \tag{5.52}$$

with $\mathbf{p}_0 = \nabla F(\boldsymbol{\theta}_0)$ the market maker's initial price, and $\mathbf{p}_{a,0} = \nabla \varrho_a(\boldsymbol{\theta}_{a,0})$. If we further assume $\boldsymbol{\theta}_0 = 0, \boldsymbol{\theta}_{a,0} = 0$, then the primal objective is simplified to

$$\min_{\mathbf{p}\in\mathbf{p}(\boldsymbol{\Theta})} \frac{1}{\eta} F^*(\mathbf{p}) + \sum_{a\in\mathcal{A}} \frac{1}{\eta_a} \varrho_a^*(\mathbf{p}) + \text{constant}. \tag{5.53}$$

The trading dynamics also has an interesting primal representation, which is illustrated by Figure 5.2. Under this representation, the trading dynamics can be understood as a *magnetic ball* algorithm: the market maker as well as the traders are treated as balls located at their prices; at each time t, the trade magnetizes the market maker (with prices $\mathbf{p}_{t-1}$) and the corresponding trader (with belief $\mathbf{p}_{a,t-1}$); the market maker and the trader are attracted towards each other until they collide at a new position (i.e. $\mathbf{p}_t$), after which they are demagnetized. The new position is characterized by

$$\mathbf{p}_t = \arg\min_{\mathbf{p}\in\mathbf{p}(\boldsymbol{\Theta})} \frac{1}{\eta} D_{F^*}(\mathbf{p},\mathbf{p}_{t-1}) + \frac{1}{\eta_a} D_{\varrho^*}(\mathbf{p},\mathbf{p}_{a,t-1}). \tag{5.54}$$

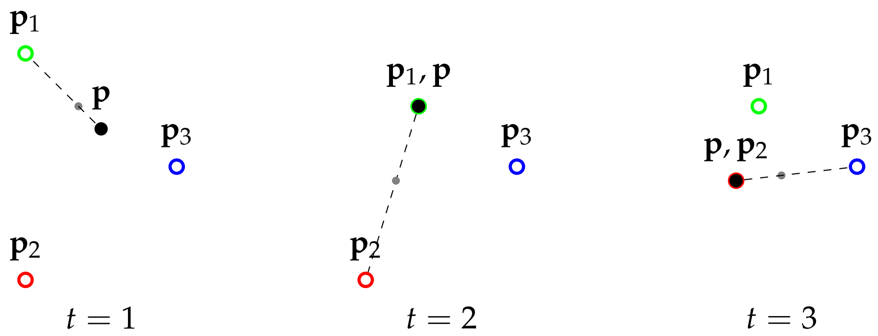


Figure 5.2: The repeated trading dynamics in the price space. The black dot represents the market prices $\mathbf{p}$ while the coloured circles represent traders' beliefs $\{\mathbf{p}_a\}$. The dashed line indicates the trade that happens at current time. Each trade could be understood as follows: at each time t, the black dot is linked with one of the circles, and the dot and the linked circle are attracted towards each other until they collide at a new position, marked by a small grey dot.

It is also interesting to note that, while each trader a's belief about the future state upon which the market is built is *fixed* through time, the prices $\mathbf{p}_a$ that the trader believes in, however, *keeps varying* after each trade. Due to risk aversion, the trader will prefer prices not only based on her belief about the future state, but also based on the amount of shares she has held (inventory). Thus the prices $\mathbf{p}_a$ can be thought of as the trader's *effective belief* which considers the impact of risk aversion. As a comparison, for a risk-neutral trader, the prices she believes in will always match her expected values of the securities and are thus invariant.

Finally, using the primal form we can compare the convergence results between the repeated trading setting and the sequential trading setting, especially the setting with stochastic traders. The sequential trading has an objective that excludes the market maker and scales with trading times t, while the repeated trading has an objective that consists of the market maker and a fixed number of repeated traders. In terms of the convergence rate, we proved that the sequential trading converges in $\mathcal{O}(1/\sqrt{T})$, while Shalev-Shwartz and Zhang (2013); Frongillo and Reid (2014, 2015) show that the repeated trading converges in $\mathcal{O}(1/T)$.

5.2.4.3 Link to machine learning

Many machine learning tasks can be interpreted under the following generic framework: given a set of data sampled from a space Ω, and a hypothesis space $\mathcal{P}_{\boldsymbol{\Theta}}$ parametrized by $\boldsymbol{\Theta}$ which contains a class of accessible probabilistic models on Ω, we would like to find a probability distribution/model in $\mathcal{P}_{\boldsymbol{\Theta}}$ that can best describe the data. Usually we use a function $f : \mathcal{P}_{\boldsymbol{\Theta}} \to \mathbb{R}$ to measure the performance, such that the best model is the one that minimize f, that is,

$$p^*_{\boldsymbol{\theta}} = \arg\min_{p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}} f(p_{\boldsymbol{\theta}}). \tag{5.55}$$

For specific problems in which the information comes from different parts of the data or the models, f can often be written as the sum of a set of functions which share the same domain $\mathcal{P}_{\boldsymbol{\Theta}}$, that is, the form in (5.51). Notice that the data is necessary for constructing the learning objective f, although it is not explicitly presented in the above learning objective. For example, if $f_n(p_{\boldsymbol{\theta}})$ is the negative log likelihood of the model $p_{\boldsymbol{\theta}}$ on the n-th data point, then $f(p_{\boldsymbol{\theta}}) = \sum_n f_n(p_{\boldsymbol{\theta}})$ is the negative log likelihood of the model on the full dataset, and the minimization problem will simply implement maximum likelihood estimation. Here we do not explicitly write the data as we would like to emphasize the argument we are trying to optimize, and also to draw the link to prediction market more straightforwardly.

A market global objective (5.52) can match exactly to the machine learning objective. In fact, if we assume that F and $\{\varrho_a\}$ are cumulant functions of GEFs $\mathcal{P}_{\boldsymbol{\Theta}}$ and $\{\mathcal{P}_a\}$ respectively (see the previous chapter), then the Bregman divergences in (5.52) can all be rewritten into the Bregman divergences between the corresponding GEF distributions. More specifically, note that $\mathcal{P}_{\Theta}$ and $\mathcal{P}_a$ may be different families. Then for any $p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}$, we denote $\text{proj}_a(p_{\boldsymbol{\theta}}) := \min_{p_a \in \mathcal{P}_a} D(p_{\boldsymbol{\theta}}, p_a)$ its projection onto $\mathcal{P}_a$. With the projection defined we can rewrite the objective in the following form

$$\min_{p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}} \frac{1}{\eta} D_G(p_{\boldsymbol{\theta}}, p_0) + \sum_{a \in \mathcal{A}} \frac{1}{\eta_a} D_{\rho^*_a}(\text{proj}_a(p_{\boldsymbol{\theta}}), p_{a,0}). \tag{5.56}$$

Here $p_{a,0}$ is the true belief of the trader a, and p_0 is the initial market belief. When F and ϱ_a coincides, we have $\mathcal{P}_{\boldsymbol{\Theta}} = \mathcal{P}_a$ and thus $\text{proj}_a(p_{\boldsymbol{\theta}}) = p_{\boldsymbol{\theta}}$.

We use two examples to illustrate how a repeated trading market can implement certain machine learning algorithms.

Opinion Pooling The opinion pooling problem is a common setting for prediction market models (Barbu and Lay, 2012; Storkey et al., 2012). Garg et al. (2004) show that the objective of an opinion pool is to minimise a weighted sum of a set of divergences. Particularly, for *logarithmic opinion pooling* (LogOP) the objective is to

$$\min_{p \in \mathcal{P}} \sum_{n} w_n KL(p, p_n). \tag{5.57}$$

where $\{w_n\}$ are weight parameters.

A repeated trading prediction market implements LogOP. Consider a LogOP of the Δ^{K-1}-simplex of probabilities on a finite discrete state space Ω with K future states. Define a market on the same space Ω and introduce K indicator securities. We introduce a set $\mathcal{A}$ of traders, and assign a unique probability $p_a \in \Delta^{K-1}$ to trader $a \in \mathcal{A}$ as its personal belief. Model each trader's preference by the entropic risk measure in (5.8)

$$\varrho_a(\boldsymbol{\vartheta}_a) = \frac{1}{\eta_a} \log \sum_{k=1}^{K} p_{a,k} e^{-\eta_a \vartheta_{a,k}}, \tag{5.58}$$

where we let η_a match the weight w_a by $w_a = 1/\eta_a$. Let the traders interact with the PMM who has the following potential function

$$F(\boldsymbol{\theta}) = \frac{1}{\eta} \log \sum_{k=1}^{K} e^{\eta \theta_k}. \tag{5.59}$$

Two typical simulation results are shown in Figure 5.3 and 5.4. As the number of trades increases, the market objective gradually reaches the optimum, and the market prices converge to a fixed level. When the market scales up, the converge process slows down but the convergence is guaranteed to happen.

In this case, we can analytically work out the primal objective of this market using (5.56). Since all distributions lie in the Δ^{K-1}-simplex, we have $\text{proj}_a(p) = p$ and so

$$\min_{p \in \Delta^{K-1}} \frac{1}{\eta} KL(p, p_0) + \sum_{a \in \mathcal{A}} \frac{1}{\eta_a} D(p, p_a), \tag{5.60}$$

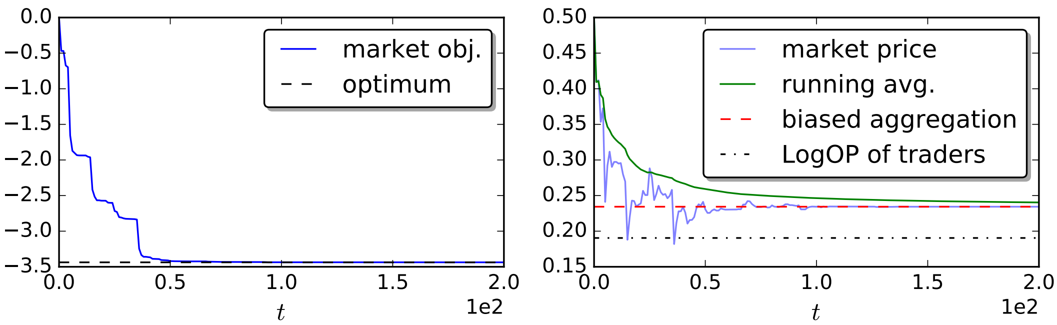


Figure 5.3: A market with indicator securities defined on a binary future state ω (i.e. $K = 2$). $N = 10$ traders are involved. All traders start with a uniform prior on ω and each one builds its own posterior belief after observing 5 private samples of ω. The market price (for one of the securities) converges to a position which is close to the unbiased agent aggregation but with a bias towards 0.5. This bias is introduced by the market maker (cf. (5.61)).

where $p_0 = \text{uniform}(K)$ is the discrete uniform distribution in Δ^{K-1}. In this case the optimal p can be analytically solved. Recall that $w_a = 1/\eta_a$ and we have

$$p \propto \prod_{a \in \mathcal{A}} p_a^{w_a/(1/\eta_0 + \sum_{a \in \mathcal{A}} w_a)}. \tag{5.61}$$

The aggregated belief p is not a pure weighted product of trader beliefs but has a bias towards p_0, due to the presence of the PMM. However, when the population is sufficiently large such that the market maker's weight is negligible compared to the traders' collective weight, $\sum_a 1/\eta_a \gg 1/\eta$, then the market maker's bias could be ignored and we will end up with a pure LogOP of traders' beliefs.

Logistic Regression Consider a logistic regression model with l_2 regularization learnt on the dataset $\mathcal{D} = \{\mathbf{x}_m, \mathbf{y}_m\}_{m=1}^M, \mathbf{x}_m \in \mathbb{R}^K, \mathbf{y}_m \in \{+1, -1\}$. It has the following objective

$$L = \min_{\mathbf{w} \in \mathbb{R}^K} \frac{1}{M} \sum_{m=1}^{M} \log\left(1 + e^{y_m(\mathbf{w} \cdot \mathbf{x}_m)}\right) + \frac{\lambda}{2} \|\mathbf{w}\|^2, \tag{5.62}$$

where $\|\cdot\|$ denotes the l_2 norm.

We build a repeated trading market that has a dual objective equal to (5.62). Let the sample space be the space that generates the data $\Omega := \mathbb{R}^K \cup \{+1, -1\}$

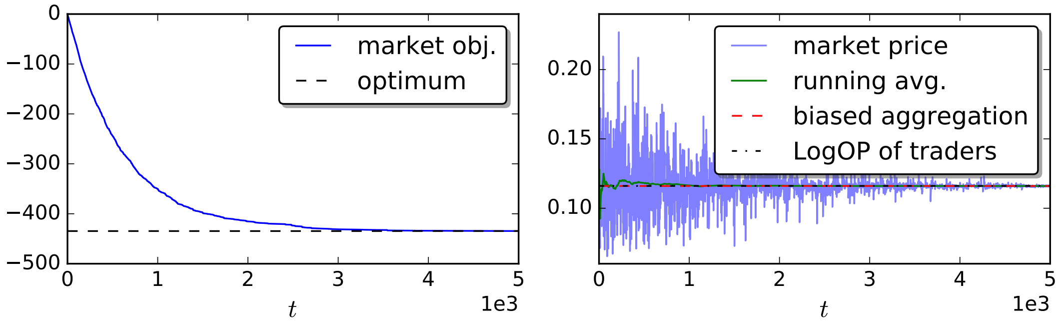


Figure 5.4: A similar setting to Figure 5.3 but this time the market has indicator securities defined on a space of $K = 10$ possible future states. $N = 500$ traders are involved. After increasing the population, the convergence of price slows down. The impact of the market maker is also negligible compared to the population of traders, and thus the biased aggregated belief matches the LogOP of traders.

and each future state is associated with a data in Ω, $\omega = \{\mathbf{x}, y\}$. Define K securities, each of which is $\phi_k(\mathbf{x}, y) = yx_k$. We introduce a set $\mathcal{A}$ of K traders, such that the trader $a = k$ is only interested in trading the k-th security ϕ_k. Thus the shares of the k-th security held by trader a is $\vartheta_{a,k} = \mathbf{1}(n = k)w_k$, and the asset is $\mathrm{x}_a = \boldsymbol{\vartheta}_a^\top \boldsymbol{\phi} = w_a\phi_a$. The market inventory is $\boldsymbol{\theta} = \sum_a \boldsymbol{\vartheta}_a = \mathbf{w}$. Let $F(\mathbf{w})$ be the first term on the RHS of (5.62) and define the risk measure of trader a to be $\varrho_n(\boldsymbol{\vartheta}_n) = \lambda\boldsymbol{\vartheta}_n^2/2$. We end up with

$$L = \min_{\mathbf{w}} F(\mathbf{w}) + \sum_{k=1}^{K} \frac{\lambda}{2} w_k^2 = \min_{\{\boldsymbol{\vartheta}_n\}} c(\boldsymbol{\vartheta}_0) + \sum_{a\in\mathcal{A}} \varrho_a(\boldsymbol{\vartheta}_n), \tag{5.63}$$

which matches the market global objective in (5.47). Therefore, by running this market we will effectively solve a logistic regression problem.

In order to show a slightly deeper connection to a specific learning method, notice that the objective of trader a at each round is $\min_{\Delta w_{k,t}} F(\mathbf{w}_{t-1} + \Delta w_{k,t}) + (w_{k,t-1} + \Delta w_{k,t})^2/2$. The minimum is not analytic, and it is costly to solve for the exactly minimum of this objective at each time. To get rid of this problem, we relax the condition on the trader's behaviour, such that instead of being strictly profit maximizer, a trader will accept a portfolio as long as it is better than her current position, $\varrho(\hat{\boldsymbol{\vartheta}}_{a,t}) < \varrho_a(\hat{\boldsymbol{\vartheta}}_{a,t-1})$. To find a better position, a trader can simply update her portfolio along the (reversed) gradient direction, leading to the following portfolio updating rule

$$\Delta w_{k,t} = -\alpha \frac{\mathrm{d}}{\mathrm{d}w_k} \left(c(\mathbf{w}) + \frac{\lambda}{2} w_k^2 \right) \Big|_{\mathbf{w}=\mathbf{w}_{t-1}}, \tag{5.64}$$

where $\alpha > 0$ is a step size adjusted such that $\varrho(\hat{\boldsymbol{\vartheta}}_{a,t}) < \varrho(\hat{\mathbf{s}}_{a,t-1})$. In practice α could be chosen by backtracking line search (Boyd and Vandenberghe, 2004). This updating rule implements a *coordinate descent* algorithm.

Instead of introducing K agents, we can match the logistic regression problem by using only one agent and allowing it to trade all securities. This will result in a standard gradient descent method.

5.3 Risk-neutral niche traders

A niche trader is a trader who believes that the true distribution of the future state differs from the current market belief in some limited way, by learning and representing her belief relative to the market. When a niche trader interact with a PMM, she exploits this belief difference for possible profits, and at the same time, updates her belief according to current market position.

Storkey (2011) proposes a model for niche traders and uses it for equilibrium analysis. In this model, a niche trader represents her belief as a factor on the equilibrium belief. The author shows that when all niche traders behave risk-aversely under the same entropic risk measure (or equivalently the exponential utility), the equilibrium belief is the product of all traders' factors.

5.3.1 Modelling niche traders

Our model for niche traders is similar to the original model of Storkey (2011). We improve the design of the original model in order to better capture the trading dynamics, which is not addressed in the equilibrium analysis. In particular, consider a market with fixed set $\mathcal{A}$ of repeated traders. Denote $\boldsymbol{\theta}_t, \boldsymbol{\vartheta}_{a,t}$ the total shares of securities sold by the market maker and the trader a's holdings after the t-th trade, respectively, and $f_a(\cdot)$ the belief factor of the trader. Then the niche trader's true belief at time t is

$$p_{a,t} := f_a \cdot f_{\setminus a} = f_a \cdot p_{\boldsymbol{\theta}_{t-1} - \boldsymbol{\vartheta}_{a,t-1}}, \tag{5.65}$$

where $f_{\backslash a} := p_{\boldsymbol{\theta}_{t-1}-\boldsymbol{\vartheta}_{a,t-1}}$ is the corresponding market belief for the market position $\boldsymbol{\theta}_{t-1} - \boldsymbol{\vartheta}_{a,t-1}$. It can be understood as the pre-trade market collective belief excluding trader a, and it encodes the information that is not represented by f_a. In fact, if trader a sells all her current holdings, which is $\boldsymbol{\theta}_{t-1}$ before the t-th trade happens, and leaves the market, the market belief will just be $p_{\boldsymbol{\theta}_{t-1}-\boldsymbol{\vartheta}_{a,t-1}}$.

Assuming that the belief model behind the PMM is a GEF $\mathcal{P}_{\boldsymbol{\Theta}}$, and that the trader is risk-neutral and myopic, then the optimal trade at time t between the market maker and trader a is such that

$$\max_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} m_t = \max_{\boldsymbol{\theta}_t \in \boldsymbol{\Theta}} (\boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1})^\top \mathbb{E}_{p_{a,t}}[\boldsymbol{\phi}_t] - F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1}), \tag{5.66}$$

whose optimal is obtained at $\mathbb{E}_{p_{a,t}}[\boldsymbol{\phi}] = \nabla F(\boldsymbol{\theta}_t)$. Since the market belief is a GEF distribution, the optimal post-trade market position $\boldsymbol{\theta}_t^*$ is obtained by matching the moments of the market belief $p_{\boldsymbol{\theta}_t^*}$ to that of the trader's belief $p_{a,t}$, that is, $p_{\boldsymbol{\theta}_t^*}[\boldsymbol{\phi}] = \mathbb{E}_{p_{a,t}}[\boldsymbol{\phi}]$.

Implementing expectation propagation (EP) Similar to the risk-averse setting, the prediction market with risk-neutral niche traders can also be linked to specific machine learning algorithms. In fact, this market in essence implements *expectation propagation (EP)* of Minka (2001). In particular, the target factor model we aim to approximate here is the product of trader's factors $\{f_a\}$ and the PMM's initial belief $p_{\boldsymbol{\theta}_0}$

$$p \propto p_{\boldsymbol{\theta}_0} \cdot \prod_{a \in \mathcal{A}} f_a, \tag{5.67}$$

While the approximate model $\tilde{p}$ is given by the market belief at the equilibrium $p_{\boldsymbol{\theta}}$. When the market beliefs are represented by an exponential family, the approximate distribution can be rewritten as a product of trader's approximate factors $\tilde{f}_a$ and the PMM's initial belief

$$p_{\boldsymbol{\theta}} \propto p_{\boldsymbol{\theta}_0} \cdot \prod_{a \in \mathcal{A}} \tilde{f}_a, \tag{5.68}$$

where $\tilde{f}_a$ is proportional to the exponential family distribution parametrized by the trader a's final holdings $p_{\boldsymbol{\theta}_a}$. The EP algorithm starts with an initial guess of $p_{\boldsymbol{\theta}}$, and in each step it replaces one approximate factor $\tilde{f}_a$ by the true factor f_a, and updates the approximate distribution $\tilde{p}$ via moment-matching:

$$\tilde{p} \leftarrow \arg_{p_{\boldsymbol{\theta}} \in \mathcal{P}_{\boldsymbol{\Theta}}} \left(\mathbb{E}_{p_{\boldsymbol{\theta}}}[\boldsymbol{\phi}] = \mathbb{E}_{\tilde{p} f_a / \tilde{f}_a}[\boldsymbol{\phi}] \right). \tag{5.69}$$

Since $\tilde{p} f_a / \tilde{f}_a = f_a p_{\boldsymbol{\theta} - \boldsymbol{\vartheta}_a}$, each trade is exactly one EP step.

When the market beliefs are GEF distributions, the market equilibrium belief $p_{\boldsymbol{\theta}}$ will still serve as the EP approximate distribution $\tilde{p}$, although it may no longer have a factorized form as (5.68). The algorithm also remains the same. Now the true factor model will be approximated to a GEF distribution rather than an exponential family distribution.

By using risk-neutral niche traders, we can avoid the issue of choosing specific preference functions for traders, at the cost of a more complicated model for their beliefs. The complete match between this market model and EP has both advantages and drawbacks. On the one hand, it makes the market more machine learning oriented, and one can understand and analyse the market though EP; on the other hand, the convergence result of the market is weaker than the risk-averse setting, since it will rely on the convergence of the corresponding EP algorithm, which is case-dependent.

5.3.2 Example

Consider a market betting players' skills. As distinct from the setting in Chapter 3, here we assume that the players' true skills are observable, and define a fully-observable PMM to offer traders directly on the players' skills. More specifically, given a set $\mathcal{K}$ of K players, denote the skill of player k by x_k. We define securities $\{\phi_{k1} = x_k, \phi_{k2} = x_k^2\}_{k=1}^K$, and a fully-observable PMM with the following potential function

$$F(\boldsymbol{\theta}) = \sum_{k=1}^{K} -\frac{\theta_{k1}^2}{4\theta_{k2}} - \frac{1}{2}\log(-2\theta_{k2}). \tag{5.70}$$

That is, the PMM expresses a multivariate Gaussian with diagonal covariance matrix: $\mathcal{N}(\boldsymbol{\mu}, \Sigma) = \prod_{k=1}^{K} \mathcal{N}(\mu_k, \sigma_k^2)$, where $\mu_k = -\theta_{k1}/2\theta_{k2}, \sigma_k^2 = -1/2\theta_{k2}, \Sigma = \mathrm{diag}(\sigma_1^2, \ldots, \sigma_K^2)$.

Each trader observes one game result between two players and keeps it as her private information. The trader has a niche belief, that is, she only has a belief about a player's actual game performance $y \sim \mathcal{N}(x_k, \epsilon^2)$, but does not have the belief over the skills $\{x_k\}$. Then the trader's niche belief about the game result between two player $k_1, k_2 \in \mathcal{K}$, conditioned on the players' skills, is

$$p(y_1 > y_2 \mid \mathbf{x}, k_1, k_2) = \Phi\left(\frac{x_{k_1} - x_{k_2}}{\sqrt{2}\epsilon}\right) \tag{5.71}$$

With the private game result, the niche factor is obtained by

$$f_a(\mathbf{x}) := \Phi\left(\frac{x_{k_1} - x_{k_2}}{\sqrt{2}\epsilon}\right)^{(y_1>y_2)} \cdot \Phi\left(\frac{x_{k_2} - x_{k_1}}{\sqrt{2}\epsilon}\right)^{(y_1<y_2)}. \tag{5.72}$$

The niche trader combines her niche factor with the market belief over skills to form her complete belief $p_a(\mathbf{x}) \propto f_a(\mathbf{x}) p_{\boldsymbol{\theta}_{\setminus a}}(\mathbf{x})$, where $p_{\boldsymbol{\theta}_{\setminus a}}(\mathbf{x})$ is the market belief at expressed at position $\boldsymbol{\theta}_{\setminus a}$ which excludes the trader a's traded shares.

Since trader is risk-neutral and myopic, the post-trade market position is determined by $\nabla F(\boldsymbol{\theta}) = \mathbb{E}_{p_a}[\boldsymbol{\phi}]$, or by moment-matching between p_a and the market belief $p_{\boldsymbol{\theta}}$. The trader a's holding $\boldsymbol{\vartheta}_a$ determines the approximate factor $\tilde{f}$, which is a Gaussian with natural parameter $\boldsymbol{\vartheta}_a$.

Since each trade reflects a moment-matching step, the repeated trading among traders implements the EP algorithm for inference in the true skill system.

5.4 Market maker driven convergence

Trader driven convergence relies on the model of traders that is assumed to be able to capture the traders' trading behaviours. Thus the trader driven convergence analysis asks traders to actually behave, at least approximately, in the way as the model describes. However, in situations where no good models for traders are available, such as due to the lack of the knowledge about traders or due to information privacy issues, the trader driven convergence may not be well established. In this section, we discuss an alternative convergence result that is directly driven by the market maker.

This alternative approach does not build any specific risk-aversion models for traders. Instead, it treats traders as i.i.d. samples from a trader ensemble which has a fixed ensemble mean belief about the future state, or equivalently, fixed ensemble mean prices for the securities. The ensemble mean prices are also referred to as the true mean prices in this context, and the goal of the market maker is to discover the true mean prices from the trades with traders.

This convergence solution is heavily inspired by the Bayesian market maker of Das and Magdon-Ismail (2009). Viewed from the high level, the PMM is augmented by a probabilistic model representing the market maker's belief about the ensemble mean prices. This model is updated via accomplished trades using Bayes rule. As more trades occur, the belief will gradually converge to the ensemble mean prices.

For simplicity, convergence discussion in this section will be restricted to a PMM that encodes her belief in an exponential family and offers just one security. Settings that involve multiple securities and GEFs can be discussed in similar ways.

5.4.1 Potential function revisited

In a PMM, an equivalence between the market maker's inventory and maker maker's belief could be found. The key idea is that, the market maker can be treated as a special trader who always has *an inventory fixed to zero* but keeps updating her belief after every trade.

Consider a PMM with initial belief p_0. If we treat the market maker as a trader, that is, we record how many shares the market maker holds instead of how many shares she sells, then the potential function is defined to be

$$F_{p_0}(\theta) := \log \mathbb{E}_{p_0}[e^{-\theta\phi(\omega)}], \tag{5.73}$$

where an extra negation appears due to this change of view. The inventory of the market maker θ will decrease when the market maker sells and increases when she buys. The instantaneous price is also redefined to be the *negation* of the gradient of F_{p_0}

$$p(\theta) = -\frac{\mathrm{d}}{\mathrm{d}\theta} F_{p_0}(\theta) \tag{5.74}$$

and when $\theta = 0$, the price matches the expectation of ϕ w.r.t. p_0, as expected. At time t, a trade $\delta_t := \theta_t - \theta_{t-1}$, with $\delta_t > 0$ indicating that the market maker buys securities from the trader, is priced by the difference in potential $F_{p_0}(\theta_t) - F_{p_0}(\theta_{t-1})$.

We assume that the market maker's belief p_0 also belongs to the encoded exponential family, and has parameter θ_0. Then (5.73) becomes

$$F_{p_0}(\theta) = \log \int \frac{1}{Z(\theta_0)} e^{(\theta_0-\theta)\phi(\omega)} \nu(\omega)\mathrm{d}\omega = \log Z(\theta_0 - \theta) - \log Z(\theta_0), \quad (5.75)$$

and so the cost of the trade δ_t can be rewritten as

$$F_{p_0}(\theta_t) - F_{p_0}(\theta_{t-1}) = \log Z(\theta_0 - \theta_{t-1} - \delta_t) - \log Z(\theta_0 - \theta_{t-1}) \quad (5.76)$$

$$= F_{p_{t-1}}(\delta_t) - F_{p_{t-1}}(0), \quad (5.77)$$

where p_{t-1} is the market belief before the t-th trade, and has natural parameter $\theta_0 - \theta_{t-1}$. Now, we end up with two equivalent ways of interpreting the market maker's behaviour.

1. The market maker has a fixed initial belief throughout the market lifetime, and maintains a record of the inventory. To determine prices she needs to compute the change of potential using the pre-trade and post-trade inventories.

2. The market maker varies her initial belief between trades. After each trade she updates her initial belief to the post-trade market belief, and resets the inventory to zero. As a consequence, the market maker's pre-trade inventory will always be zero. Using $F_{p_{t-1}}(\delta_t)$ is enough to compute the price of a trade δ_t, as the pre-trade state is implicitly adjusted through p_{t-1}.

The second view is of special importance to building the belief augmented PMM. Since the PMM under the second view always has an empty inventory, then each price update is caused by a pure update in the market belief. Therefore, if the market belief could be updated in a better way such that it can finally converge to ensemble mean, a market convergence result will also be obtained based on this belief update.

To simplify our notation, the later discussion will denote the market maker's potential function F_{p_t} by F_t.

5.4.2 Belief augmented PMM

Under the basic potential based mechanism, the market prices are updated to match the most recent trader's expectation of the security values. Hence if the traders are stochastically drawn i.i.d. from a trader ensemble, the market prices will also form i.i.d. samples of ensemble. In other words, the basic potential based mechanism can be thought of as a *model-free sampler* of the trader ensemble. Since market prices are samples, their running averages will converge to the ensemble mean price v, but the market prices themselves will not converge to any fixed price.

To obtain converged market prices, more trades than just the most recent one must be used for setting up the market instantaneous prices. This goal is achieved by adding an extra belief model on the top of the existing PMM. This belief model treats the ensemble mean price v as a random variable and maintains a distribution $p_t(v)$ over it. Note that the price belief $p_t(v)$ is different from $p_t(\omega)$ or its abbreviation p_t, which is market belief over the future state represented by the PMM (recall Chapter 4). To distinguish them we will always write the belief over the price in $p_t(v)$ with random variable shown explicitly, while p_t (without the argument) always refers to the market belief at time t.

The mean of v under $p_t(v)$, denoted by μ_t, is used for setting up the initial market price for the $(t+1)$-th trade. On the other hand, after the $(t+1)$-th trade occurs, the belief $p_t(v)$ is updated to $p_{t+1}(v)$ based on the trade. More specifically, the belief model assumes that the $(t+1)$-th trader's true price q_{t+1} is a noise around v: $q_{t+1} = v + \epsilon_{t+1}$, $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma_n^2)$. Since traders are risk-neutral and myopic, q_{t+1} can be recovered from the traded shares δ_t through $q_{t+1} = \nabla F_t(\delta_{t+1})$. Then by Bayes rule, the belief about the ensemble mean price is updated to

$$p_{t+1}(v) := p(v \mid q_{t+1}) \propto p(q_{t+1} \mid v) p_t(v). \tag{5.78}$$

Assume that $p_t(v)$ is a Normal distribution $\mathcal{N}(\mu_t, \sigma_t^2)$. Then the updated belief $p_{t+1}(v)$ is also a Normal distribution $\mathcal{N}(\mu_{t+1}, \sigma_{t+1}^2)$ with parameters

$$\mu_{t+1} = \sigma_{t+1}^2 \left(\frac{\mu_t}{\sigma_t^2} + \frac{q_{t+1}}{\sigma_n^2} \right), \sigma_{t+1}^2 = \left(\frac{1}{\sigma_t^2} + \frac{1}{\sigma_n^2} \right)^{-1}. \tag{5.79}$$

After the $(t+1)$-th trade, the market maker will reset her inventory to zero and sets a new initial price for the next trade. However, distinct from the standard PMM who matches the new initial price $-\mathrm{d}F_{t+1}(0)/\mathrm{d}\delta$ to q_{t+1}, the

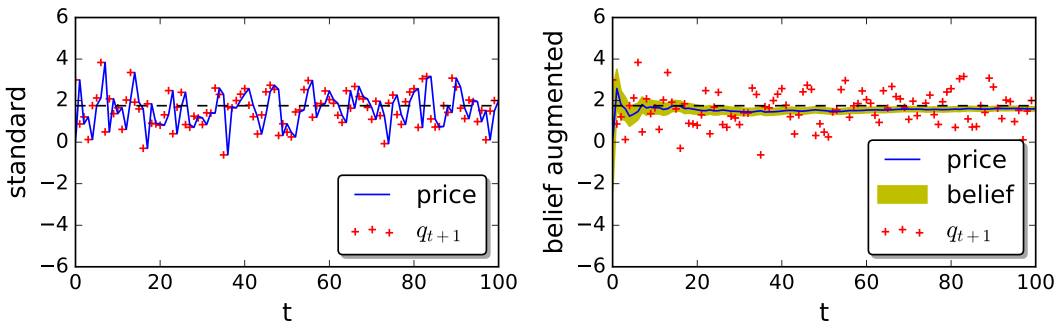


Figure 5.5: The price dynamics of the standard PMM (left) and the belief augmented PMM (right). While the standard PMM follows precisely the most recent trader's belief, the belief augmented PMM discovers the true price using trades and drive the market price to converge.

belief augmented PMM uses the updated belief in (5.79) and matches the new initial price to μ_{t+1}. In other words, the market maker is adjusted to the new position such that the market price is equal to the posterior mean μ_{t+1}, rather than the post-trade price q_{t+1} in the standard PMM case.

Evaluation A simulated environment is used to test the performance of the belief augmented PMM in comparison with a standard PMM. For each simulation, we generate a true ensemble mean price from a Normal distribution $v^* \sim \mathcal{N}(0, 2.5^2)$, and generate a set of risk-neutral myopic traders with true beliefs $\{q_{t+1}\}_{t=0}^{T-1}, T = 100, q_{t+1} \sim \mathcal{N}(v^*, 1^2)$. Traders interact with the market maker sequentially. The simulation is repeated for $N = 10,000$ times.

Figure 5.5 shows the price dynamics in a simulation. The standard PMM's price follows precisely the traders' true beliefs, showing completely a stochastic pattern, while the prices of the belief augmented PMM do not follow the traders but converge to the ensemble mean. On the right figure the belief interval $(\mu_t - \sigma_t, \mu_t + \sigma_t)$ of $p_t(v)$ is also plotted. As more trades occur, the belief becomes more and more certain around the true ensemble mean.

The averaged reward per trade of the both market makers are given by Figure 5.6. The left figure plots the profit of each trade given the true observation of the event being the ensemble mean. The right figure treats the market maker as a risk-averse trader whose risk/preference function is given by its

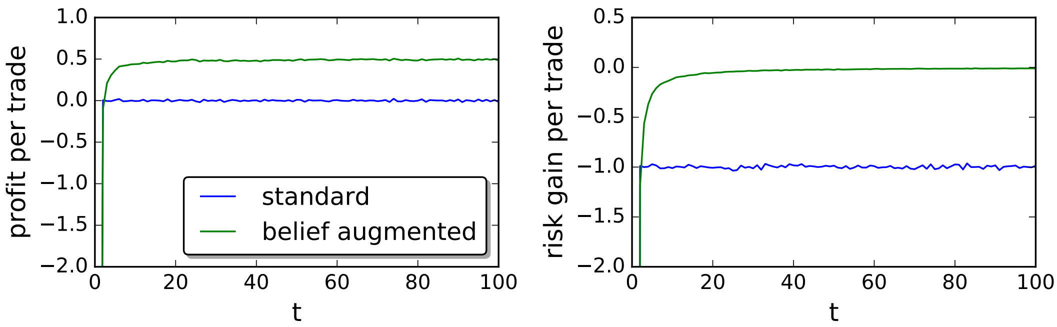


Figure 5.6: The profits (left) and risk gains (right) of standard PMM and belief augmented PMM in each trade. Results are averaged over $N = 10,000$ simulations. The belief augmented PMM outperforms the standard PMM under both measures.

potential (recall Section 5.2.1), and plots the maker maker's risk gain in each trade. Under both measures the belief augmented PMM outperforms the standard PMM. At the beginning of the trades, both mechanisms lose some money in order to move their positions close to v^*. The standard PMM follows precisely the stochastic traders, such on average gains zero profit in each round as traders are i.i.d. sampled around v^*. It also obtains negative risks per trade since the price does not converge and has a constant volatility. The belief augmented PMM gradually discovers the true price v^*, and sets the true belief as the initial price for each trade. Then each trader who drives the price away from v^* will lose money to the market maker, and the converged price also reduces the risks to zero.

One drawback of the belief augmented PMM is that it does not provide a theoretical bound on its monetary loss. Intuitively, if the price that the market price converged to turns out to be incorrect, then those traders with the true mean price v^* can trade to earn money from the market maker before the price is readjusted to v^*. In a Bayesian update, the more uncertain that $p_t(v)$ is, the slower for $p_t(v)$ to readjust the market price to the true one. Thus the belief augmented PMM may end up with losing arbitrary amount of money. Fortunately, this issue can be avoided by a mild condition on the traders: if the mean of stochastic traders' beliefs forms a consistent estimator of the ensemble mean v^*, then the belief augmented PMM will finally discover the true price, since

$$\lim_{t\to+\infty}\mu_{t+1}=\lim_{t\to+\infty}\left(\frac{1}{\sigma_0^2}+\frac{t}{\sigma_n^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2}+\frac{\sum_{t'=0}^{t-1}q_{t'+1}}{\sigma_n^2}\right)=\lim_{t\to+\infty}\frac{1}{t}\sum_{t'=1}^{t}q_{t'}=v^*. \tag{5.80}$$

Another drawback of the belief augmented PMM is that it inevitably loses money at the beginning of the trade and can never obtain positive risk gains. Can we exploit the belief augmented mechanism more efficiently, to avoid the money loss in the early market and to earn positive risk gains? It turns out that a belief augmented PMM with bid-ask spread can kill two birds with one stone.

5.4.3 Adding bid-ask spread

To further leverage the belief augmented PMM for ensemble price discovery while reducing the costs and risks of running such market makers, a bid-ask spread is introduced. A bid-ask spread is defined by a bid price b_t and a ask price a_t. They bound the uni-price μ_t of the belief augmented PMM from below and above, respectively: $b_t \le \mu_t \le a_t$. The bid price b_t is the price at which the market maker would like to pay for buying securities, while the ask price a_t is the price at which the market maker would like to sell. Both bid-ask prices are derived from the potentials, hence the prices will change during a trade. More specifically, at the end of each time t (also the beginning of time $t+1$), the bid-ask potentials (F_t^b, F_t^a) are published by the PMM, with bid (ask) price initialized at $b_t = -\mathrm{d}F_t^b(0)/\mathrm{d}\theta$ and $a_t = -\mathrm{d}F_t^a(0)/\mathrm{d}\theta$. At time $t+1$, one trade will occur. If the trader wants to buy $\delta_1 > 0$ shares, then the ask potential F_t^a will be used for pricing and the purchase costs $F_t^a(-\delta_1) - F_t^a(0)$. Otherwise if a trader wants to sell $\delta_2 > 0$, the other potential F_t^b will be used and the trader's gain is computed by $F_t^b(0) - F_t^b(\delta_2)$. Notice that each trade will *always* increase the bid-ask spread. A market maker with such bid-ask spread in essence implicitly maintains a *limit order book* (F_t^b, F_t^a), waiting

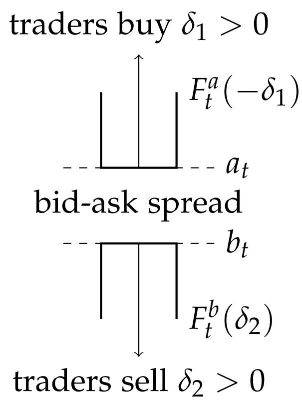


Figure 5.7: Market maker's limit order book (F_t^b, F_t^a).

for a trader to neutralize some limit orders via transactions (Figure 5.7).

Intuitively, the bid-ask spread is a better way to encode and express the market belief than the uni-price. The uni-price gives a point estimate of the mean value of the security w.r.t. current market belief, while the bid-ask prices give an interval that the mean value falls in. In other words, the bid-ask prices can capture the uncertainly of the security value under the market belief. Another benefit of using the bid-ask spread is that, with a bid-ask spread the market maker can avoid trading with less informative traders, thus achieving more efficient market making with less monetary loss and risks.

Since each trade always increases the bid-ask spread, a maintenance for the spread (or equivalently, the limit order book) is needed after the trade in order to correctly reflect the market belief and also to encourage future trading. Similar to the uni-priced belief augmented PMM, the bid-ask prices/potentials should somehow update based on the latest belief model for the ensemble mean price. That is, after the $(t+1)$-th trade, the market maker first updates her belief over the ensemble mean price $p_t(v) \rightarrow p_{t+1}(v)$, and then sets the new potentials (F^b_{t+1}, F^b_{t+1}) for the next trade. However, unlike the uni-priced PMM who simply matches the new price to the latest mean of $p_t(v)$, there is no trivial way to set the bid-ask prices. In fact, the following two issues are the key to designing a belief augmented PMM with a bid-ask spread.

1. How to update the belief $p_t(v) \rightarrow p_{t+1}(v)$ from the latest trade; and
2. How to set new potentials (F^b_{t+1}, F^b_{t+1}) using the new market belief.

The belief update is addressed by Section 5.4.3.1 using a standard Bayesian approach. Section 5.4.3.2 then builds the new potentials as the solution to an optimal control problem defined on the updated belief. Finally, in the last section this bid-ask system is evaluated and the results are discussed.

5.4.3.1 Belief update in bid-ask systems

Now the market maker controls two prices: bid for buying (from the trader), and ask for selling (to the trader). The price discovery process remains almost the same except that there will be an explicit "null-trade" interval (b_t, a_t). Given a trader with expected security values q_{t+1}, if $q_{t+1} \in (b_t, a_t)$, then the trader will not see any profitable offer from the market maker. Outside the null-trade interval the model is as previous but with a modified partition function to ensure the probabilities sum up to one. Recall that v is the ensemble mean price; μ_t is the market price at end of time t (and also the beginning of time $t+1$). The market maker's noise model for the trader is changed to

$$p(q_{t+1} \mid v, a_t, b_t) = \begin{cases} \frac{1}{a_t - b_t}\Phi\left(\frac{a_t - v}{\sigma_n}\right) - \Phi\left(\frac{b_t - v}{\sigma_n}\right) & q_{t+1} \in (b_t, a_t) \\ \frac{1}{\sigma_n}\phi\left(\frac{q_{t+1} - v}{\sigma_n}\right) & \text{otherwise.} \end{cases} \tag{5.81}$$

The probability mass on $q_{t+1} \in \mathbb{R} \setminus (b_t, a_t)$ is

$$\int_{q_{t+1} \in \mathbb{R}\setminus(b_t, a_t)} \frac{1}{\sigma_n}\phi\left(\frac{q_{t+1} - v}{\sigma_n}\right) \mathrm{d}q_{t+1} = \Phi\left(\frac{b_t - v}{\sigma_n}\right) + \Phi\left(\frac{v - a_t}{\sigma_n}\right) \tag{5.82}$$

which by definition is equal to one minus the probability of a null trade (with the corresponding trader's belief in (b_t, a_t)) in (5.81).

The true ensemble mean price v could be integrated out by $p_t(v)$. The exact marginal distribution over the trader's belief q_{t+1} is

$$p(q_{t+1} \mid a_t, b_t) = \int p(q_{t+1} \mid v, a_t, b_t) p_t(v) \mathrm{d}v = \begin{cases} \frac{1}{a_t - b_t} Z_{t+1} \\ \frac{1}{\sqrt{\sigma_t^2 + \sigma_n^2}}\phi\left(\frac{q_{t+1} - \mu_t}{\sqrt{\sigma_t^2 + \sigma_n^2}}\right) \end{cases} \tag{5.83}$$

where

$$Z_{t+1} = \Phi\left(\frac{a_t - \mu_t}{\sqrt{\sigma_t^2 + \sigma_n^2}}\right) - \Phi\left(\frac{b_t - \mu_t}{\sqrt{\sigma_t^2 + \sigma_n^2}}\right). \tag{5.84}$$

The price update after trade δ_t, which adjusts the market price from μ_t to q_{t+1}, is given by the Bayes rule $p_{t+1}(v) \propto p_t(v) p(q_{t+1} \mid v, a_t, b_t)$

$$p_{t+1}(v) = \begin{cases} \frac{1}{\sigma_t Z_{t+1}}\phi\left(\frac{v - \mu_t}{\sigma_t}\right)\left(\Phi\left(\frac{a_t - v}{\sigma_n}\right) - \Phi\left(\frac{b_t - v}{\sigma_n}\right)\right) \\ \frac{1}{\sigma_{t+1}}\phi\left(\frac{v - \mu_{t+1}}{\sigma_{t+1}}\right) \end{cases} \tag{5.85}$$

Here μ_{t+1} and σ_{t+1}^2 are computed from (5.79). It is worth pointing out that a trader who does not make a trade still provides information. However, the update rule for null trade is different.

After a null-trade update, $p_{t+1}(v)$ is no longer a Normal but it can be well approximated by a Normal via moment-matching. The approximated posterior is what is actually used as the prior for the next trade. The moment-matched Normal is with

$$\mu_{t+1} = \mu_t + \sigma_t^2 g,\ \sigma_{t+1}^2 = \sigma_t^2 - \sigma_t^2 (gg^\top - 2G)\sigma_t^2, \tag{5.86}$$

where

$$g = \frac{\partial}{\partial \mu_t} \log Z_{t+1} = -\frac{\phi(u_t^+) - \phi(u_t^-)}{\sigma_n \eta_t Z_{t+1}} \tag{5.87}$$

$$G = \frac{\partial}{\partial \sigma_t^2} \log Z_{t+1} = -\frac{u_t^+ \phi(u_t^+) + u_t^- \phi(u_t^-)}{2\sigma_n^2 \eta_t^2 Z_{t+1}} \tag{5.88}$$

with dimensionless quantities

$$\rho_t = \frac{\sigma_t}{\sigma_n}, \eta_t = \sqrt{1+\rho_t^2}, z_t^+ = \frac{a_t - \mu_t}{\sigma_n}, z_t^- = \frac{\mu_t - b_t}{\sigma_n}, u_t^+ = \frac{z_t^+}{\eta_t}, u_t^- = \frac{z_t^-}{\eta_t}. \tag{5.89}$$

By applying this approximation, we will always have a Normal distribution $\mathcal{N}(\mu_{t+1}, \sigma_{t+1}^2)$ no matter whether the trade is a null trader or not. This Normal distribution is then used as the new belief over the ensemble mean price for the next round pricing.

To sum up

$$Z_{t+1} = \Phi(u_t^+) - \Phi(-u_t^-), \tag{5.90}$$

$$\mu_{t+1} = \begin{cases} \mu_t - \sigma_t \dfrac{\rho_t}{\eta_t} \dfrac{\phi(u_t^+) - \phi(u_t^-)}{Z_{t+1}} & \text{null trade} \\ \mu_t - \dfrac{\rho_t^2}{\eta_t^2}(\mu_t - q_{t+1}) & \text{otherwise} \end{cases}, \tag{5.91}$$

$$\frac{\sigma_{t+1}^2}{\sigma_t^2} = \begin{cases} 1 - \dfrac{\rho_t^2}{\eta_t^2} \dfrac{(\phi(u_t^+) - \phi(u_t^-))^2 + (u_t^+ \phi(u_t^+) + u_t^- \phi(u_t^-)) Z_{t+1}}{Z_{t+1}^2} \\ \dfrac{1}{\eta_t^2} \end{cases}. \tag{5.92}$$

5.4.3.2 Bid-ask potentials for the next trade

Previous sections discussed how the belief updates, given the bid-ask prices and the trade. The problem remains is that how this updated belief can help set the bid and ask prices for the next trade. In the uni-price system, the price is simply set to match the mean μ_{t+1}. However, there is no trivial way to set the bid-ask prices.

As pointed out by Das and Magdon-Ismail (2009), the price update process can be modelled as an optimal control problem. Intuitively, with the latest belief over the ensemble mean price, the belief augmented PMM can estimate the behaviours of the following traders and also the accumulated reward generated from the later trades. Maximizing the accumulated reward involves the interactive process between the trading and the bid-ask pricing. On the one hand, the bid-ask prices will affect the behaviour of each trade; on the other hand, each trade will bring in new information that helps update the PMM's belief over the ensemble mean price, which in turn affects the bid-ask pricing. An optimal control problem thus emerges from this trading-pricing interaction.

In the rest of this section, the model for the price update process will be introduced. The model is mainly based on the Glosten-Milgrom model (O'hara, 1995, Chap. 3.3), which is originally used by Das and Magdon-Ismail (2009). The model will cover several settings that result from the different choices on the reward function and on the decision-making preference of the market maker. For each setting, the optimal pricing strategy will be solved.

Deriving the model Denote $r(q_{t+1}, v, a_t, b_t)$ the gain that the market maker obtains in the $(t+1)$-th trade, given that the ensemble mean v is revealed. Then the expected gain for the $(t+1)$-th trade is

$$\bar{r}_{t+1}(a_t, b_t) = \int\int r(q_{t+1}, v, a_t, b_t) p(q_{t+1} \mid v, a_t, b_t) p_t(v) \mathrm{d}v \mathrm{d}q_{t+1}. \tag{5.93}$$

If no trade happens, $r(q_{t+1}, v, a_t, b_t) = 0$. Thus the expected gain decomposes to two parts: gain of bidding and gain of asking

$$\bar{r}^{\text{bid}}_{t+1}(a_t, b_t) = \int\int_{q_{t+1}<b_t} r(q_{t+1}, v, a_t, b_t) p(q_{t+1} \mid v, a_t, b_t) p_t(v) \mathrm{d}v_t \mathrm{d}q_{t+1} \tag{5.94}$$

$$\bar{r}^{\text{ask}}_{t+1}(a_t, b_t) = \int\int_{q_{t+1}>a_t} r(q_{t+1}, v, a_t, b_t) p(q_{t+1} \mid v, a_t, b_t) p_t(v) \mathrm{d}v_t \mathrm{d}q_{t+1} \tag{5.95}$$

and no gain is generated for $q_{t+1} \in (b_t, a_t)$.

The vanilla Glosten-Milgrom model assumes that the market maker stays in a competitive environment. That is, there exists many other market makers who provide the same services, and in order to survive the market maker has to set her bid-ask returns to zero, $\bar{r}_t^{\text{bid}}(a_t, b_t) = \bar{r}_t^{\text{ask}}(a_t, b_t) = 0$. If we drop the competitive market assumption and let the market maker be a monopoly, then the price update will be characterized by a *Markov Decision Process* (MDP), governed by the following Bellman equation

$$V^\pi(p_t(v)) = \max_\pi \bar{r}_{t+1}(a_t^\pi, b_t^\pi) + \gamma \mathbb{E}[V^\pi(p_{t+1}(v)) \mid a_t^\pi, b_t^\pi]. \tag{5.96}$$

Here each state of the MDP is a belief over the ensemble mean price $p_t(v)$, and the value function V is defined as the discounted total expected reward through time $V(p_t(v)) := \sum_{t=0}^\infty \gamma^t r_{t+1}(a_t, b_t)$. Map $\pi : p_t(v) \mapsto (a_t, b_t)$ is the *policy* that tells how the market maker will set her bid-ask prices when she has belief $p_t(v)$. The bid-ask prices as well as the value function resulting from π are denoted by a_t^π, b_t^π, V^π, respectively. Note that at the beginning of time t right before the $(t+1)$-th trade $V^\pi(p_{t+1}(v))$ is random as q_{t+1} is not observed, and the expectation of $V^\pi(p_{t+1}(v))$ is taken w.r.t. $p(q_{t+1} \mid v, a_t, b_t)$.

Measuring the gain Intuitively, the gain $r(q_{t+1}, v, a_t, b_t)$ can be measured by the market maker's profit obtained for the $(t+1)$-th trade. Consider a trader with belief $q_{t+1} < b_t$. She will sell securities to the market maker until the bid price matches q_{t+1}. According to potential based mechanism, this trader will keep trading until

$$q_{t+1} = -\partial F_t^b(\delta_{t+1}), \tag{5.97}$$

which costs her $F_t^b(\delta_{t+1}) - F_t^b(0) = F_t^b(\delta_{t+1})$. Similarly, for a trader with belief $q_t > a_t$ the trade stops at $q_{t+1} = -\partial F_t^a(\delta_{t+1})$ and the agent pays $F_t^a(\delta_{t+1})$ to (gets $-F_t^a(\delta_{t+1})$ from) the market maker. If the market market know the true ensemble mean price for the security v, the profit is

$$r(q_{t+1}, v, a_t, b_t) = v\delta_{t+1} - c_{t+1}, \tag{5.98}$$

where c_{t+1} is the money the market maker pays to the trader, and δ_{t+1} is the shares she bought

$$c_{t+1} = \begin{cases} -F_t^b(\delta_{t+1}) & q_{t+1} < b_t \\ -F_t^a(\delta_{t+1}) & q_{t+1} > a_t \end{cases}, \; \delta_{t+1} = \begin{cases} (-\partial F_t^b)^{-1}(q_{t+1}) & q_{t+1} < b_t \\ (-\partial F_t^a)^{-1}(q_{t+1}) & q_{t+1} > a_t \end{cases}. \tag{5.99}$$

Note that $\delta_{t+1} > 0$ when $q_{t+1} < b_t$ and $\delta_{t+1} < 0$ when $q_{t+1} > a_t$.

In general, we can think the market maker, as a trader, is risk-averse and include the risks when measuring its gain. Let ρ be the market maker's risk measure and $\varrho := \rho \circ \phi$, then the gain for the trade δ_{t+1} is

$$r(q_{t+1}, v, a_t, b_t) = -\rho(\delta_{t+1}\phi - c_{t+1}) = -\varrho(\delta_{t+1}) - c_{t+1}. \tag{5.100}$$

Solving optimal pricing strategies For simplicity we will use the *quadratic potential* for F_t^a and F_t^b

$$F_t^a(\delta) = -a_t\delta + \frac{1}{2}\sigma_p^2\delta^2, \tag{5.101}$$

$$F_t^b(\delta) = -b_t\delta + \frac{1}{2}\sigma_p^2\delta^2, \tag{5.102}$$

and choose the risk measure

$$\varrho_t(\delta) = -v\delta + \frac{1}{2}\sigma_r^2\delta^2. \tag{5.103}$$

Note that when $\sigma_r = 0$, the market maker becomes risk neutral and measures its gain via profit. We say the risk measure is *consistent* with the potential if $\sigma_r = \sigma_p$, and *inconsistent* otherwise.

After the trade δ_t, the price is adjusted to the trader's belief q_t. From (5.99), we obtain the amount of traded shares $\delta_{t+1} = (b_t - q_{t+1})/\sigma^2$ if $q_{t+1} < b_t$ (i.e. market maker buys), and $\delta_{t+1} = (a_t - q_{t+1})/\sigma^2$ if $q_{t+1} > a_t$ (i.e. market maker sells).

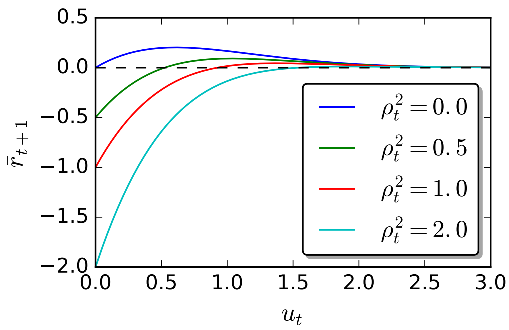
The gain is

$$r(q_{t+1}, v, a_t, b_t) = \begin{cases} \frac{1}{\sigma_p^2}(v - b_t)(b_t - q_{t+1}) - \frac{\sigma_r^2 - \sigma_p^2}{2\sigma_p^4}(b_t - q_{t+1})^2 & q_{t+1} < b_t \\ \frac{1}{\sigma_p^2}(v - a_t)(a_t - q_{t+1}) - \frac{\sigma_r^2 - \sigma_p^2}{2\sigma_p^4}(a_t - q_{t+1})^2 & q_{t+1} > a_t \end{cases}. \tag{5.104}$$

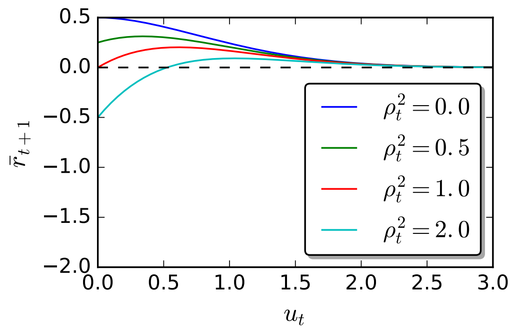
Denote constants $\alpha^+ = (\sigma_r^2 + \sigma_p^2)/2\sigma_p^2$, $\alpha^- = (\sigma_r^2 - \sigma_p^2)/2\sigma_p^2$, and their quotient $\alpha = \alpha^-/\alpha^+$. According to (5.94) and (5.95), we compute the expected gains

$$\bar{r}_{t+1}^{\text{bid}}(b_t) = \alpha^+\eta_t^2\frac{\sigma_n^2}{\sigma_p^2}\left(u_t^-\phi(u_t^-) - ((u_t^-)^2 + \frac{\rho_t^2}{\eta_t^2})\Phi(-u_t^-)\right) - \alpha^-\frac{\sigma_n^2}{\sigma_p^2}\Phi(-u_t^-), \tag{5.105}$$

$$\bar{r}_{t+1}^{\text{ask}}(a_t) = \alpha^+\eta_t^2\frac{\sigma_n^2}{\sigma_p^2}\left(u_t^+\phi(u_t^+) - ((u_t^+)^2 + \frac{\rho_t^2}{\eta_t^2})\Phi(-u_t^+)\right) - \alpha^-\frac{\sigma_n^2}{\sigma_p^2}\Phi(-u_t^+). \tag{5.106}$$



(a) $\alpha = 0$ ($\alpha^+ = 1, \alpha^- = 0$)



(b) $\alpha = -1$ ($\alpha^+ = 1/2, \alpha^- = -1/2$)

Figure 5.8: The reward function $\bar{r}_{t+1}$ under different circumstances. For $\alpha = -1$, $\bar{r}_{t+1}$ is positive everywhere when $\rho^2 < 1$. In general, the reward $\bar{r}_{t+1}(u)$ will always be positive if $\rho^2 + \alpha < 0$.

Here we use the dimensionless quantities introduced in (5.89).

A symmetric pattern emerges in between $\bar{r}_t^{\text{bid}}(b_t)$ and $\bar{r}_t^{\text{ask}}(a_t)$. In particular, if b_t^* is a solution to $\bar{r}_t^{\text{bid}}(b_t) = r_0$, then the solution to $\bar{r}_t^{\text{ask}}(a_t) = r_0$ is such that $\mu_t - b_t^* = a_t^* - \mu_t$. Due to this symmetric pattern, the optimal solution to the Bellman equation (a_t, b_t) will meet the condition $u_t^+ = u_t^- = u_t$. Therefore, the $\bar{r}_{t+1} = \bar{r}_{t+1}^{\text{bid}} + \bar{r}_{t+1}^{\text{ask}}$ is simplified to

$$\bar{r}_{t+1}(u_t) = 2\alpha^+ \eta_t^2 \frac{\sigma_n^2}{\sigma_p^2} \left(u_t \phi(u_t) - (u_t^2 + \frac{\rho_t^2 + \alpha}{\eta_t^2}) \Phi(-u_t) \right). \tag{5.107}$$

When $\rho_t^2 + \alpha < 0$, the reward $\bar{r}_{t+1}(u_t)$ will always be greater than zero.

Also, under the symmetric condition the belief update in (5.90–5.92) is simplified to

$$Z_{t+1} = 1 - 2\Phi(-u_t), \tag{5.108}$$

$$\mu_{t+1} = \begin{cases} \mu_t & \text{null trade} \\ \mu_t - \dfrac{\rho_t^2}{\eta_t^2}(\mu_t - q_{t+1}) & \text{otherwise} \end{cases}, \tag{5.109}$$

$$\frac{\sigma_{t+1}^2}{\sigma_t^2} = \begin{cases} A_t := 1 - \dfrac{2}{Z_{t+1}} \dfrac{\rho_t^2}{\eta_t^2} u_t \phi(u_t) & \text{null trade} \\ \dfrac{1}{\eta_t^2} & \text{otherwise} \end{cases}. \tag{5.110}$$

There are three solution concepts (1) a market maker in a competitive environment; (2) a myopic monopoly; and (3) a Bellman optimal monopoly. A market maker in a competitive environment maintains zero expected gain $\bar{r}_{t+1} = 0$, while a myopic monopoly maximizes $\bar{r}_{t+1}$ at each time t. The optimal solution for these two cases can be easily obtained by using Newton's method. When the market maker is a Bellman optimal monopoly, the Bellman equation (5.96) will have the following specific form, which is derived by applying (5.83) and (5.110), and the symmetry of the bid-ask prices

$$V(\rho_t^2) = \max_{u_t} 2\alpha^+ \eta_t^2 \frac{\sigma_n^2}{\sigma_p^2} \left(u_t \phi(u_t) - (u_t^2 + \frac{\rho_t^2 + \alpha}{\eta_t^2}) \Phi(-u_t) \right) + \gamma \left(2\Phi(-u_t) V(\rho_t^2/\eta_t^2) + (1 - 2\Phi(-u_t)) V(A_t \rho_t^2) \right). \quad (5.111)$$

Notice that for both coefficient of ρ_t^2 we have $0 < A_t, \eta_t < 1$. Thus the state ρ^2 is bounded in the interval $[0, \rho_0^2]$. To solve the Bellman equation, we first find the boundary solution at $\rho_t^2 = 0$, and then build up $V(\rho^2)$ for other $\rho^2 > 0$ by sweeping from $\rho^2 = 0$ to the initial state $\rho^2 = \rho_0^2$.

At $\rho^2 = 0$, (5.111) reduces to

$$V(0) = \max_{u_t} 2\alpha^+ \frac{\sigma_n^2}{\sigma_p^2} \left(u_t \phi(u_t) - (u_t^2 + \alpha) \Phi(-u_t) \right) + \gamma V(0). \quad (5.112)$$

Hence the optimal u^* is the root of $(1 + \alpha)\phi(u) - 2u\Phi(-u) = 0$. In particular, when $\alpha = 0$, that is, the market maker is risk-averse about the gain and the risk measure is consistent with the bid-ask potentials

$$u^* = 0.6120, \;\; V(0) = \frac{0.2025}{1 - \gamma} \frac{\sigma_n^2}{\sigma_p^2}; \quad (5.113)$$

when $\alpha = -1$, that is, the market maker uses profit to measure the gain

$$u^* = 0, \;\; V(0) = \frac{0.5}{1 - \gamma} \frac{\sigma_n^2}{\sigma_p^2}. \quad (5.114)$$

To compute the value function at other states, we build a discrete grid on the state space $0 = \hat{\rho}_0^2 < \hat{\rho}_1^2 < \cdots < \hat{\rho}_{N+1}^2 = \rho_0^2$, and approximate $V(\rho_t^2)$ using a piecewise linear function $\hat{V}$ defined on this grid. Specifically

$$\hat{V}(\rho^2) = \hat{V}_n + \frac{\rho^2 - \hat{\rho}_n^2}{\hat{\rho}_{n+1}^2 - \hat{\rho}_n^2} (\hat{V}_{n+1} - \hat{V}_n), \; \hat{\rho}_n^2 \le \rho^2 \le \hat{\rho}_{n+1}^2. \quad (5.115)$$

Figure 5.9 and 5.10 shows the optimal pricing strategy under different solution concepts. Notice that the belief augmented PMM with single price (Section 5.4.2) is a special bid-ask market maker who always has a zero bid-ask

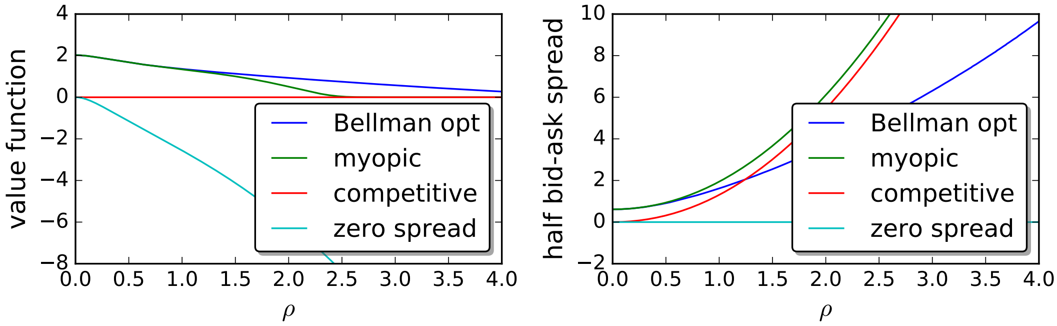


Figure 5.9: The value function (left) and the half bid-ask spread (right) for $\alpha = 0$. The Bellman optimal market maker has the highest value function, as expected. The minimum spread of the competitive market maker is zero, while the myopic and the Bellman optimal market makers have the same non-zero minimum spread. The belief augmented PMM with single price can be treated as a special bid-ask market maker with bid-ask spread always being zero.

spread. The optimal strategy depends on how the market maker measures her gain of each trade. When the market maker is risk-averse and her risk measure is consistent with the potential function she uses for pricing, both the myopic and the Bellman optimal solutions have the same non-zero bid-ask spread when her belief $\mathcal{N}(\mu_t, \sigma_t^2)$ converges (i.e. $\rho_t^2 \to 0$), while the competitive market maker shrinks the spread to zero. However, when the market maker is aiming to maximize the expected profits, all optimal strategies will finally lead to zero bid-ask spread. Another interesting observation is that, when the expected profit is used to measure the gain, the value function of the competitive market maker is actually not zero. The reason is that for $\alpha < 0$, if ρ_t^2 is small enough such that $\rho_t^2 + \alpha < 0$, then the expected profit $\bar{r}_{t+1}(u_t)$ is positive for any $u_t \geq 0$. In this situation, the market maker still gains a positive reward even if she sets a zero bid-ask spread.

5.4.3.3 Evaluation

This section evaluates the optimal pricing strategies in a simulated market environment. More specifically, we draw the *true* ensemble mean v^* from a Normal distribution with zero mean and standard deviation $\sigma = 5$. Traders are risk-neutral and myopic, with beliefs i.i.d. drawn $q_{t+1} \sim \mathcal{N}(v^*, 1)$. They visit the market in a sequential manner, such that each trader interacts with

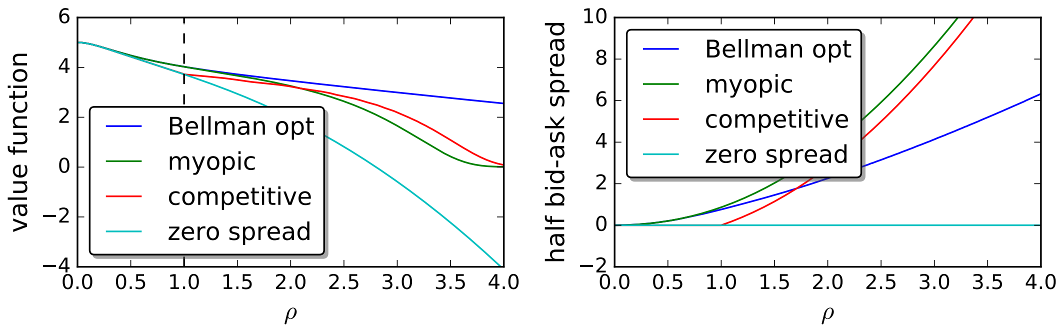


Figure 5.10: The value function (left) and the half bid-ask spread (right) for $\alpha = -1$. The minimum spread is zero for all three types of market makers. When $\rho^2 + \alpha < 0$, $\bar{r}_{t+1} > 0$ and so the market maker in the competitive environment actually receives a positive reward when $\rho < 1$.

the market maker only once, and then leaves the market. All market makers have the same initial prior belief over v, which is a Normal distribution $\mathcal{N}(0, \sigma_0^2)$. The optimal strategies of the Bellman optimal maker maker are derived under the discount factor $\gamma = 0.9$ (i.e. same as the results plotted in Figure 5.9 and 5.10). At the end of the market lifetime, we reveal v^* to evaluate the market makers' actual profits and risk gains. Finally, the simulation is repeated for $N = 10,000$ and we report the averaged result.

Figure 5.11 shows the pricing processes of market makers in one of the N simulations. The optimal strategies of the market makers are derived from the risk consistent measure of gains ($\alpha = 0$). The Bellman optimal market maker has a much smaller initial bid-ask spread than the myopic and the competitive ones, allowing her to accept more non-zero transactions (red cross outside of the bid-ask interval) in the early stage. Since the coefficients of the variance update in (5.110) has the relation $A_t \geq 1/\eta_t^2$, these early non-zero transactions will lead to a faster convergence of the market maker's belief. The large bid-ask spread is the main drawback of the competitive and the myopic optimal maker makers, which can even make them impractical: if the market maker has a broad initial belief over v, $p_0(v)$, the bid-ask spread will be too large to allow any non-zero trade to happen in the early market, and the belief convergence will be very slow (Figure 5.12).

All market makers' beliefs converge to the true ensemble mean (black dashed line) as time increases. The pricing process is consistent with the optimal

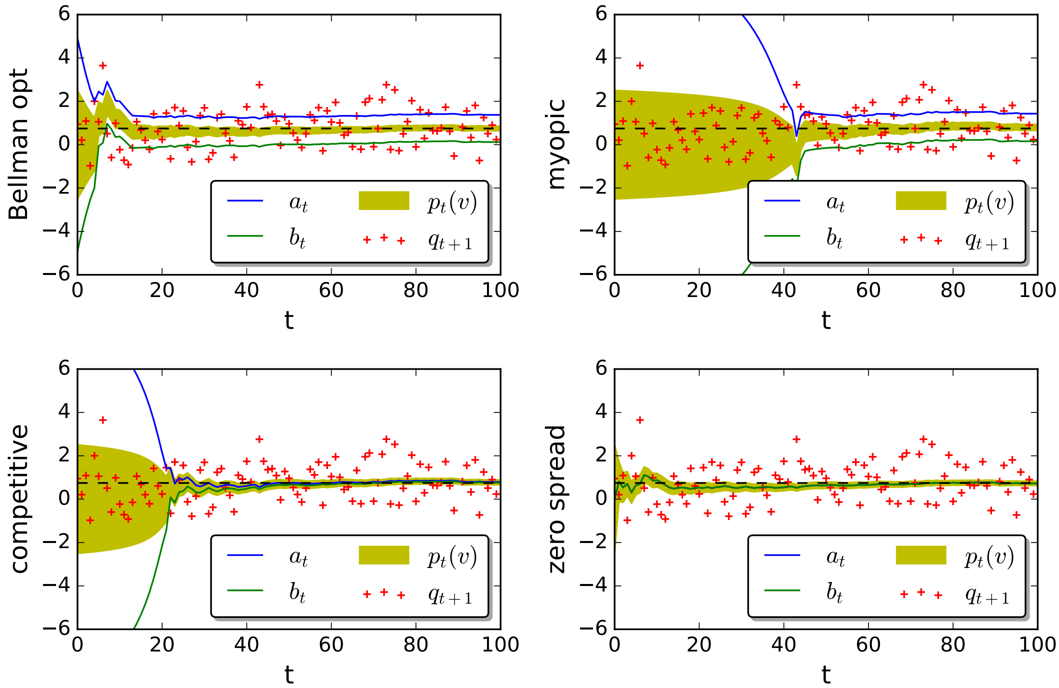


Figure 5.11: The pricing process of the market makers with $\sigma_0 = 2.5$ and strategies derived from the risk consistent measure of gains ($\alpha = 0$). A non-zero trade happens when the trader' belief (red cross) is outside of the bid-ask spread. All market makers' beliefs converge to the ensemble mean price v^*. The Bellman optimal and the myopic optimal market makers maintain a non-zero bid-ask spread.

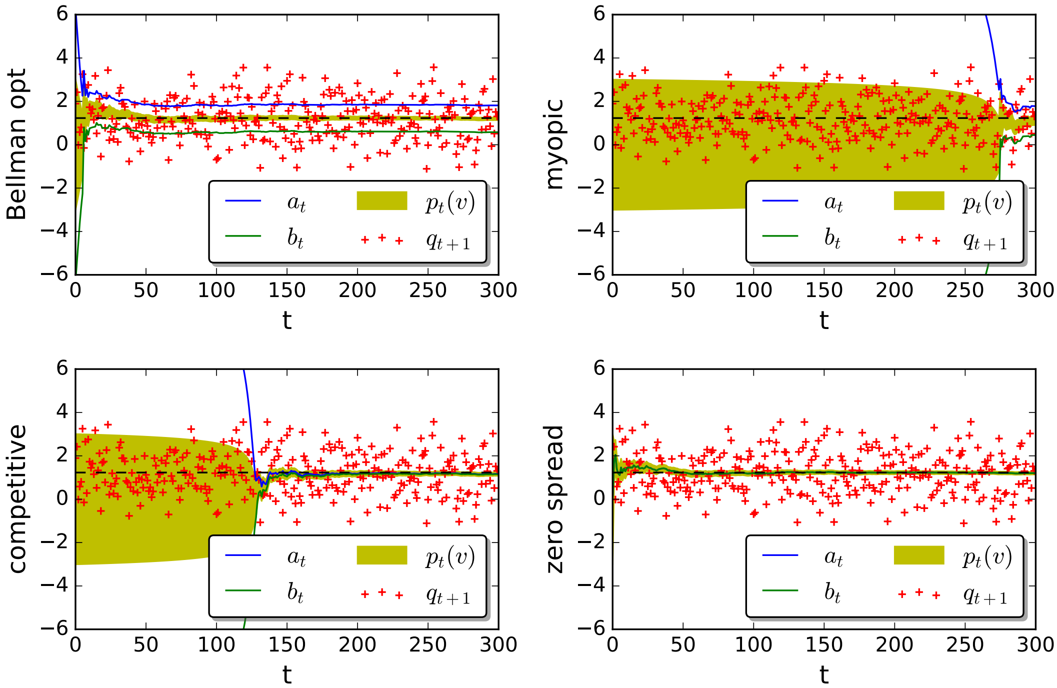


Figure 5.12: The pricing process of the market makers with $\sigma_0 = 3$ and strategies derived from the risk consistent measure of gains ($\alpha = 0$). For the competitive and the myopic market makers, a broader initial belief $p_0(v)$ leads to a much larger bid-ask spread, and a much slower convergence. Note that the σ_0 only increases by 0.5.

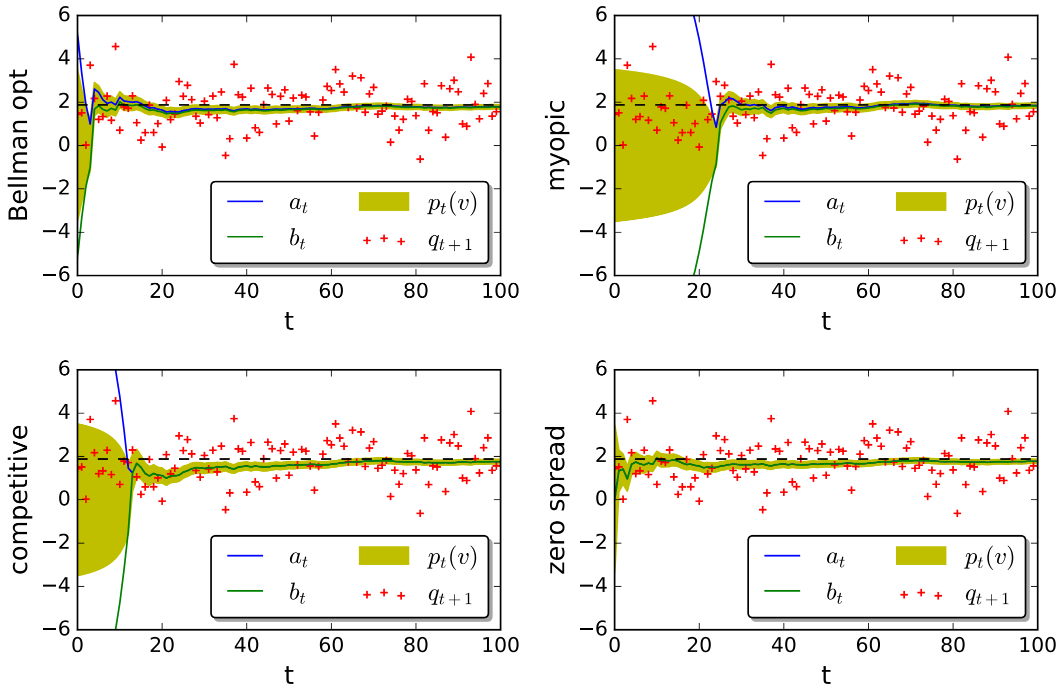


Figure 5.13: The pricing process of the market makers with $\sigma_0 = 3.5$ and strategies derived form the profit measure of gains ($\alpha = -1$). A non-zero trade happens when the trader' belief (red cross) is outside of the bid-ask spread. All market makers' beliefs and bid-ask spread converge to the ensemble mean price v^*.

strategies in Figure 5.9: the zero spread market maker has a zero bid-ask spread by definition; the competitive market maker starts with a non-zero but ends up with a zero bid-ask spread; the Bellman optimal market maker and the myopic optimal market maker maintains a non-zero bid-ask spread through out the market lifetime. The pricing processes of the market makers derived from $\alpha = 0$ is also consistent with their optimal strategies (Figure 5.13).

Figures 5.14, 5.15 and 5.16 show the actual profits, or the actual risk gains of the market makers, averaged over simulations. Note that when $\alpha = -1$, the variance of the market maker's risk measure $\sigma_r^2 = 0$, and so the risk gains coincide with the profits. For the market makers derived from $\alpha = 0$, we see a trade-off between profits and risk gains (Figure 5.14 and 5.15). In particular, the Bellman optimal market maker has the highest risk gains among all market makers. However, to guarantee low risks the market maker has to give up those strategies that are more profitable but more risky, which in av-

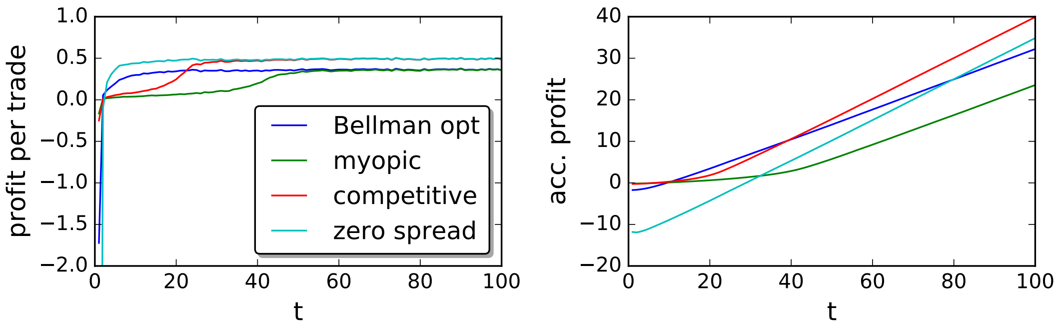


Figure 5.14: The actual profits of the market makers with $\sigma_0 = 2.5$ and pricing strategies derived from the risk consistent measure of gains ($\alpha = 0$). Results are averaged over $N = 10,000$ simulations.

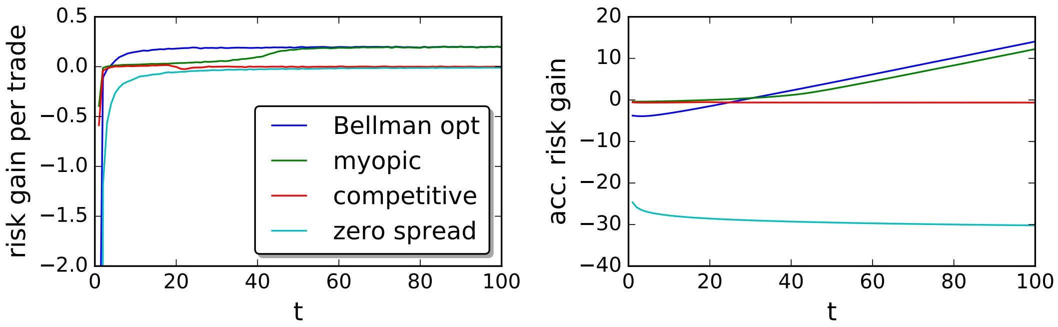


Figure 5.15: The actual risk gains of the market makers with $\sigma_0 = 2.5$ and pricing strategies derived from the risk consistent measure of gains ($\alpha = 0$). Results are averaged over $N = 10,000$ simulations.

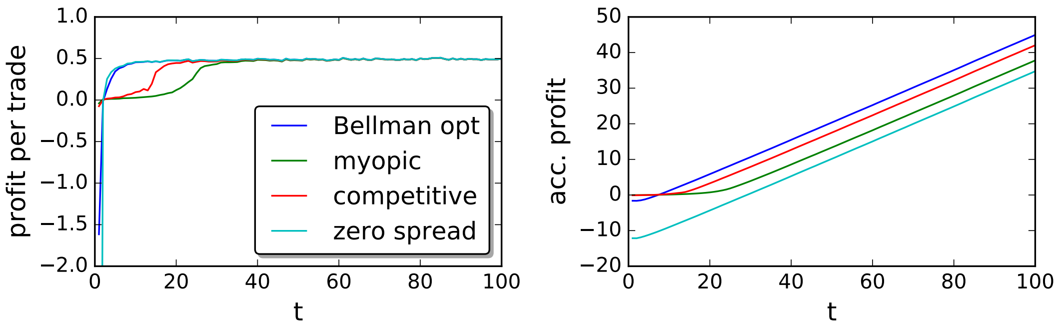


Figure 5.16: The actual profits/risk gains of the market makers with $\sigma_0 = 3.5$ and pricing strategies derived from the profit measure of gains ($\alpha = -1$). Here the actual profits and the risk gains coincide since $\sigma_r^2 = 0$. Results are averaged over $N = 10,000$ simulations.

erage sense will lead to worse actual profits than the competitive and the zero spread market makers. When $\alpha = -1$, the Bellman optimal market maker aims to maximize the monetary profit, and it does achieve the highest actual profits among all market makers (Figure 5.16). With a smaller initial bid-ask spread the Bellman optimal market maker may lose some money and risk-averse gain in the early stage. However, the smaller bid-ask spread speeds up the price discovery and, with a better belief, the Bellman optimal market maker gains more in the long run. Therefore, by using bid-ask market maker we can avoid investing money for price discovery, and usually we even end up with earning profits.

5.4.4 Discussion

The market maker driven convergence is heavily inspired by the *Bayesian market maker* (BMM) of Das and Magdon-Ismail (2009), but differs in fundamental design. While the original BMM was proposed as a mechanism distinct from the standard PMM, here we actually construct a similar mechanism by directly augmenting the standard PMM, thus providing a unified view for both mechanisms. In addition, our mechanism can naturally model an arbitrary amount of shares in each transaction using the embedded PMM, while the BMM relies on the assumption that either a unit share or nothing is traded in each round, and can only model up to integer shares by trading multiple times (Brahma et al., 2012).

Alternatively, we can view our bid-ask mechanism as a BMM but with a finite limit order book. The original BMM has infinite many limit orders at bid and ask prices, which makes the bid-ask prices unmovable by during a trade. Comparatively, our bid-ask model has finite limit orders at each price. When the orders at the initial bid (ask) price is not enough to meet the demand of the trade, orders at lower (higher) prices will start to supply, implying a decrease (increase) in the bid (ask) price. The finite limit order book is a more realistic model for limit order books, and also works better with risk-neutral myopic traders who, with sufficient budgets, will keep trading until the market prices are moved to their true beliefs.

One underlying assumption of the market maker driven convergence is that all traders are drawn consistently from an invariant trader ensemble. This assumption enables the price belief $p_t(v)$ to be updated under Bayes rule, which guarantees the shrinkage of $p_t(v)$ conditioned on new trades (recall (5.79) and (5.110)). However, if the trader ensemble is dynamically changing though time, the convergence result will not apply. For example, when the market price has almost converged but the true ensemble mean suddenly jumps to a different value, it will take a considerably long time for the market price to adapt to this jump, due to the small variance that $p_t(v)$ has already acquired. One solution to this problem is to model the traders with a linear dynamical system, such that we can use the Markov chain on the latent state to capture the dynamics of the ensemble mean belief while still viewing traders as noises around the ensemble mean. Another solution is to drop the assumption of stochastic traders completely, and search for a convergence result based on online learning. We will study these ideas in our future work.

5.5 Summary

In this chapter, we discussed in details the convergence and equilibrium of potential based prediction markets in various settings. These settings and results were classified into two categories: (1) the convergence driven by traders, and (2) the convergence driven by the market maker.

In the trader driven convergence, we considered more realistic models for trader behaviours, then analysed based on them how a market-wise objective emerges from the trading preference of each individual, and how this global objective is optimized via each selfish trade. Our analysis showed that the trader driven convergence is closely linked, or even equivalent to popular convex optimization techniques (e.g. Chen and Teboulle (1993); Duchi et al. (2010)) and machine learning methods (e.g. Minka (2001); Garg et al. (2004)). The trader driven convergence result is suitable for situations where traders are controllable, or are designed artificially.

In the market maker driven convergence, we simply left traders being risk-neutral myopic, instead augmented PMM with an extra belief model that allows the market maker to learn the equilibrium price from the past trades, and with an extra bid-ask spread that allows the market maker to set better initial prices for the upcoming trades. We modelled this convergence process as an optimal control problem, governed by a Bellman equation which we derive by combining the potential based mechanism and the classic Glosten-Milgrom model. In addition to achieving price convergence, the belief augmented PMMs can usually avoid subsidizing the market and even make profits. However, the market maker driven convergence requires that the traders are stochastic and consistently drawn from an invariant ensemble.

The results in this chapter can help us understand and even solve the instability that has been observed in practical prediction markets. By comparing the real-world settings to that of the ideal convergence models, one could know how a prediction market violates the necessary assumptions and hence deviates from the expected equilibria. Furthermore, one can also increase the stability of a prediction market by adding appropriate risk-aversion regularities to traders, or by augmenting the PMM with a proper price belief.

Chapter 6

Conclusion

After having spread our discussion of potential prediction markets over three chapters, let's now review the research problem and contributions of the work in this thesis, point out some future directions, and draw the conclusion.

6.1 Review of research problems and achievements

The goal of our research is to improve the potential based prediction market designs which currently suffer from limited belief modelling powers and inadequate understandings of market dynamics. Improvement in both aspects will help pave the way for using potential based prediction markets as distributed, scalable and self-incentivized machine learning systems, which models beliefs over the target problems using the probabilistic models encoded in the market maker, and learns the beliefs by aggregating local information from potentially large population of traders via transactions. The main results and contributions are summarized below.

- In Chapter 3, we introduce the partially-observable state space as a new target space for running prediction markets, and argue how this space could potentially improve the flexibility and effectiveness of the market design and the belief modelling. Then, based on the latent variable models involving exponential families, we propose the partially-observable potential based market maker (PoPMM) that allows prediction markets to be run on partially-observable spaces, and expressing beliefs on both observable and latent variables. We prove that the PoPMM maintains the key properties of the standard PMM including bounded monetary loss and weak incentive-compatibility. We show in details with two practical examples how the PoPMM is designed from the latent variable models involving exponential families, and demonstrate the PoPMM properties by running experiments on them. Finally, we discuss how we can further improve the PoPMM design, such that it can be strictly incentive-compatible and can provide correct incentives for traders who may revisit markets.

- To represent the market belief models beyond exponential families, in Chapter 4, we focus on the development of the theory of generalized exponential families (GEFs). We first complete the existing GEF theory, and then develop the theory to characterize GEFs with conditioning structures. For the first part, our key contribution is Theorem 4.8, which gives the condition for when the whole GEF can be precisely captured by the internal domain of its cumulant function. Based on this theorem we redefine the concept of regularity and generalized Bregman divergences, and further unify the existing concepts of regularity in different formulations of GEF theory. We also give the first example of GEF on the continuous sample space. For the second part, our key contribution is Theorem 4.18, which characterizes those GEFs with additional conditioning structures by a cumulant function with a simple nested form. We then show all concepts and properties of the nested GEFs parallel the GEFs.

  We then replace exponential families with GEFs in the PMM design, and replace exponential family involved latent variable models with nested GEFs in the PoPMM design. We show both design generalization will result in a much broader range of distributions for representing market beliefs, while maintaining the market making efficiency.

- Chapter 5 covers market convergence and equilibrium analyses under various circumstances, which are classified in two categories: convergence driven by the traders, and the convergence that are led by the market maker itself.

In the trader driven convergence analysis, we first introduce a realistic model for risk-averse traders using risk measures, providing two equivalent formulations in share (dual) and price (primal) spaces. Then we consider settings in which traders are trading sequentially or repeatedly, and for each setting we derive the market-wise objective from each individual trading goal, and match precisely the trading rule (without any approximation) to the single optimization step towards the global objective. We also present an alternative niche model for risk-averse traders. Along with our analysis, the connections between the market dynamics and several optimization as well as machine learning problems are drawn explicitly.

Then we discuss the convergence driven by the market maker, aiming to obtain the convergence result when trader models are inaccurate or traders are less controllable. Instead of modelling traders, we augment market makers with an extra belief model for the equilibrium prices and bid-ask spreads, allowing the market maker to learn the equilibrium prices from the trades and to set market prices efficiently. We convert the market dynamics to an optimal control problem and solve its Bellman equation. We demonstrate the convergence result using simulations. The convergence result is suitable for understanding the market involving stochastic traders from a fixed trader ensemble.

6.2 Future directions

This section points out several potential directions that are worth investigating in the future research. For some topics, preliminary ideas are also presented.

Efficient computation of trading costs The cost of a trade is defined as the difference between the values of the potential function at the pre-trade and post-trade market positions. In this thesis the potential function is assumed to be tractable. Hence the trading costs are cheap to obtain from the potential function. However, for PMMs with intractable potential functions, we have to seek for another way of computing the trading costs. It is worth noting that even for PMMs derived from exponential families, their potential functions could be expensive or intractable to evaluate. For example, for a PMM encoding a d-dimensional multivariate Gaussian, its potential function involves the log determinant and the inverse of the covariance matrix, both of which has a time complexity of $\mathcal{O}(d^3)$. As d increases, the computation quickly becomes expensive. Another example is a PMM which encodes a Restricted Boltzmann Machine (RBM), which belongs to the exponential family but has an intractable log partition function.

For a PMM that encodes its market belief in the exponential family, the cost of a trade $\boldsymbol{\delta}_t = \boldsymbol{\theta}_t - \boldsymbol{\theta}_{t-1}$ is bounded from above

$$F(\boldsymbol{\theta}_t) - F(\boldsymbol{\theta}_{t-1}) \leq \boldsymbol{\delta}_t^\top \nabla F(\boldsymbol{\theta}_t) = \boldsymbol{\delta}_t^\top \mathbb{E}_{p_{\boldsymbol{\theta}_t}}[\boldsymbol{\phi}]. \tag{6.1}$$

We can redefine the cost of a trade as the upper bound, which is simply the unit prices of securities multiplied by the amount of shares. Since the new cost bounds the original from above, one can interpret the excess part as the *commission*. Notice the subtle difference between the redefined cost and the one used by Premachandra and Reid (2013): here the unit price is computed at the post-trade position instead of the pre-trade position. For intractable potentials, computing the exact unit price is also expensive and we will estimate it using the samples $\{\mathbf{s}_t^n\}$ drawn from $p_{\boldsymbol{\theta}_t}$. However, if we estimate each new price by drawing a new set of samples, the estimation will suffer from high variances, making the mechanism unstable. One way of mitigating the problem is to run a *persistent Markov chain* for sampling $p_{\boldsymbol{\theta}}$, which is similar to the idea of Tieleman (2008). To make persistent chain work, each trade has to be restricted to a small amount of shares, such that the post-trade market position is close enough to the pre-trade position to ensure fast mixing of the samples.

Further work needs to be done to exploit the above ideas. Topics include but not limited to: (1) how the new cost and its estimation will affect the existing market results; (2) to what extent the persistent chain can reduce the estimation variance; (3) how many shares can be traded without breaking the persistent chain. It will be also interesting to explore other ways of efficient computing the trading costs.

PoPMM with sequentially observed information In our discussion all observable variables are revealed simultaneously, after which the market will close. If the variables are sequentially observed, the market will keep running before all variables are revealed but its potential function will somehow change in order to adapt the new revealed information. In the fully-observed setting, Dudík et al. (2014) design a PMM for sequentially observable information whose states will become less uncertain through time and are finally completely determined.

From the probabilistic modelling point of view, the market running on sequentially observed variables effectively represents its belief as a state-space model (e.g. hidden Markov model, linear dynamical system), and computation of the conditional market prices becomes the filtering problem. The simplest implementation for the sequential setting is to concatenate multiple independent PoPMMs, with the initial state of each PoPMM being the final state of the previous one conditioned on the observed information.

Advanced analysis of potential based prediction markets A link between the prediction markets and the financial markets has been revealed in Chapter 4. This link may enable us to apply numerous tools developed in the financial literature to prediction markets. On the other hand, since the potential based prediction markets have close dependencies on the exponential family which is the central object in information geometry (Amari and Nagaoka, 2007), advanced analysis can also be done from the geometry point of view.

In fact, consider a PoPMM encoding an exponential family $\mathcal{P}_{\boldsymbol{\Theta}}$. The potential of the PoPMM matches to the log partition of $\mathcal{P}_{\boldsymbol{\Theta}}$, $F(\boldsymbol{\theta}) = \log Z(\boldsymbol{\theta})$. Then $\boldsymbol{\Theta}$ is a Riemannian manifold with metric given by $\nabla^2 F(\boldsymbol{\theta})$. The tangent space $T_{\boldsymbol{\theta}}$ at each $\boldsymbol{\theta}$ is a vector space spanned by the market instantaneous prices at $\boldsymbol{\theta}$ since $\mathbf{p}(\boldsymbol{\theta}) = \nabla F(\boldsymbol{\theta})$. $T_{\boldsymbol{\theta}}$ can thus be understood as a market with *fixed prices*, and each vector $\mathbf{a}$ in $T_{\boldsymbol{\theta}}$ is a trade which costs $\mathbf{a}^\top \mathbf{p}(\boldsymbol{\theta})$. When new information (i.e. the value of the observable variable $\mathbf{v}$) is observed, the prices in the fixed-price market will be adjusted to the gradient of the conditional potential $F(\cdot, v)$. It can be verified that the prices of the fixed-price market at the same $\boldsymbol{\theta}$ across different information sets form a $p_{\boldsymbol{\theta}}$-martingale, implying

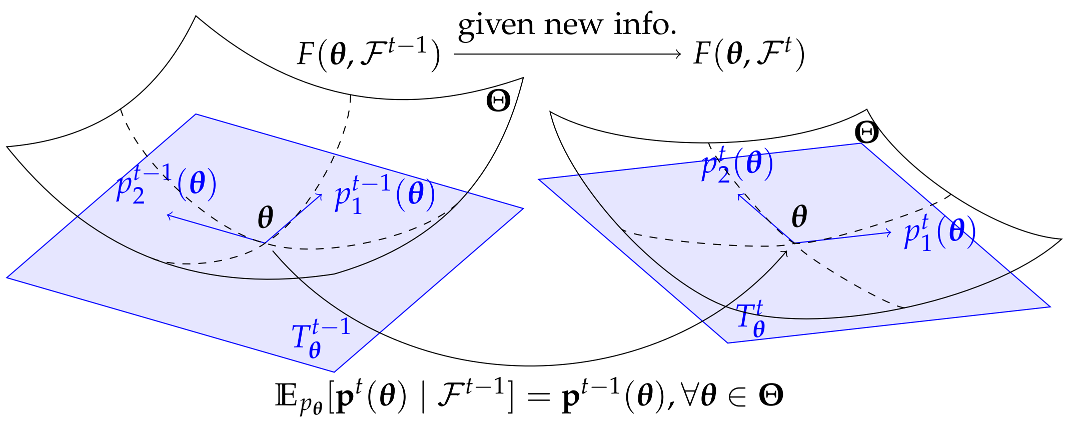


Figure 6.1: Geometric view of potential based prediction markets. A potential based prediction market could be viewed as a Riemannian manifold with metric $\nabla^2 F(\boldsymbol{\theta})$. The tangent space at each $\boldsymbol{\theta}$, $T_{\boldsymbol{\theta}}$, can be viewed as a market with fixed market prices $\mathbf{p}(\boldsymbol{\theta}) = \nabla F(\boldsymbol{\theta})$. When new information is observed (here a filtration of σ-algebras $\{\mathcal{F}^t\}$ is used to denote the information available at each time t), the potential is adjusted to its conditional potential, and the market prices are recalculated. In the same fixed-price market $T_{\boldsymbol{\theta}}$, its prices across different information sets form a $p_{\boldsymbol{\theta}}$-martingale, where $p_{\boldsymbol{\theta}}$ is the exponential family distribution encoded by the market position $\boldsymbol{\theta}$. Therefore, each $T_{\boldsymbol{\theta}}$ is a financial market model.

that each fixed-price market, or each tangent space $T_{\boldsymbol{\theta}}$, is a financial market model (Figure 6.1).

The above view sheds light on the intimate connections between prediction markets, financial markets, and information/differential geometry. It would be of interest to work further along this direction. It is also worth noting that researchers have started to use the geometry for solving economics problems in recent years (Marriott and Salmon, 2000).

Generalized exponential families in practice Generalized exponential families (GEFs) greatly expand the range of the probabilistic models that a potential based prediction market can encode. However, computation of the cumulants of the nested GEFs may not be efficient, if the underlying generalized entropy is complex. In practice, one would like to further characterize a special class of generalized entropy which can generate GEFs with cheap cumulants.

Robust market maker driven convergence As is pointed out, the convergence result is efficient if the true price the market aims to discover is varying through time. To obtain a robust result that can quickly adapt to the change of the underlying true price, we can improve the price belief model (i.e. the model used to augment the PMM) such that it now considers the dynamics of the true belief. Switching linear dynamical system (Shumway and Stoffer, 1991) and Bayesian change-point model (Western and Kleykamp, 2004) are promising candidates. Alternatively, similar to part of the trader driven convergence results, we can also seek for a model-free result by applying methods in online learning, which will hold for those true prices that change arbitrarily or even adversarially.

6.3 Concluding remarks

This work studied the potential based prediction markets from the machine learning perspective. Leveraging the links between the potential based prediction markets and machine learning, this work enriched the probabilistic models that a prediction market can encode, and matched the market dynamics to a variety of important optimization methods and machine learning algorithms. It is anticipated that this work will help inspire further developments of prediction markets, and will also trigger more applications of prediction markets in machine learning.

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