TENSOR PRODUCTS OF BANACH SPACES

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ABSTRACT

Chapter one consists of a general discussion of tensor products.

Chapter two is concerned with the relationship between tensor products and the approximation property. In Theorem 2.1 we give an equivalent condition to the approximation property which is due to Grothendieck. In Theorem 2.5 we prove that every complex Banach space is isometrically isomorphic to a complemented subspace of a uniform algebra. From this, we prove in Theorem 2.6 that there exists a uniform algebra not having the approximation property. Tomiyama has shown that if A and B are semi-simple commutative Banach algebras, and either A or B has the approximation property, then $A \hat{\otimes} B$ is semi-simple. In Theorem 2.8 we establish a converse to this result, namely that if A is a commutative Banach algebra not having the approximation property, then there is a uniform algebra B such that $A \hat{\otimes} B$ is not semi-simple. We next discuss the ε-product and the slice product, and their relationships with the injective tensor product and with

the approximation property. Then, in Theorem 2.11, we prove that a uniform algebra A has the approximation property if and only if $A \check{\otimes} B = A \# B$ for all uniform algebras B.

In chapter three we consider injective algebras. Using techniques similar to those used in the proof of Theorem 2.5, we give a proof in Theorem 3.2 of Varopoulos's characterisation of injective commutative Banach-algebras. This states that a commutative Banach-algebra A is injective if and only if there exists a uniform algebra B, a bounded algebra homomorphism h of B onto A, and a bounded linear operator j of A into B such that $h_o j = I_A$. In Theorem 3.4 we prove a sharpening of Varopoulos's result that a normed-algebra is injective if and only if its injective tensor product with any normed-algebra is a normed-algebra.

Chapter four is concerned with the question, also considered in chapter three, of whether the injective tensor product of two normed-algebras is a normed-algebra. We show that this is the case for the tensor product $l_p \check{\otimes} l_q$ (where p or q $\leqslant$ 2), and for the tensor product of two Banach-algebras which are $\mathcal{L}_1$ spaces.

In chapter five we consider measures orthogonal to injective tensor products of uniform algebras, and we obtain an analogue of Cole's decomposition theorem for orthogonal measures to the bidisc algebra. Through a general study of bands, we set up the decomposition in Lemma 5.4, and prove that this decomposition is of the form we want in Theorem 5.7. This then gives us our main result in Theorem 5.8.

CHAPTER ONE

In this chapter we discuss some of the basic properties of tensor products of normed spaces and Banach spaces. All the results of this chapter are well-known.

Definition: If X and Y are vector spaces over k (where k is the real or complex field), let Sp(X×Y) denote the vector space over k which has the elements of X×Y as a basis. Let J be the subspace of Sp(X×Y) generated by the elements:

$$(\alpha x + \beta x', y) - \alpha(x,y) - \beta(x',y),$$

and $(x, \alpha y + \beta y') - \alpha(x,y) - \beta(x,y')$,

where $x, x' \varepsilon X$, $y, y' \varepsilon Y$, and $\alpha, \beta \varepsilon k$. We define the tensor product $X \otimes Y$ of X and Y to be the vector space Sp(X×Y)/J. We write $x \otimes y$ for $(x,y) + J$.

We have: $(\alpha x + \beta x') \otimes y = \alpha(x \otimes y) + \beta(x' \otimes y)$

and $x \otimes (\alpha y + \beta y') = \alpha(x \otimes y) + \beta(x \otimes y')$

where $x, x' \varepsilon X$, $y, y' \varepsilon Y$, and $\alpha, \beta \varepsilon k$. Also $x \otimes y = 0$ if and only if $x = 0$ or $y = 0$.

If X and Y are algebras over k, then $X \otimes Y$ becomes an algebra under the multiplication:

$$\left(\sum_{i=1}^{n} x_i \otimes y_i\right).\left(\sum_{j=1}^{m} x_j' \otimes y_j'\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i x_j' \otimes y_i y_j'.$$

$X \otimes Y$ is then commutative if X and Y are, and it has a 1 if both X and Y have a 1.

Crossnorms on $X \otimes Y$

We now take X and Y to be normed spaces. We shall denote the closed unit ball of X by Ball X, and the topological dual of X by X*. If α is a norm on $X \otimes Y$, we shall denote the space $(X \otimes Y, \alpha)$ by $X \otimes_\alpha Y$.

Definition: If α is a norm on $X \otimes Y$, then α is said to be a crossnorm if $\| x \otimes y \|_\alpha = \| x \| \| y \|$ for all $x \in X$, $y \in Y$.

Definition: If α is a norm on $X \otimes Y$ such that for each $\sum_{i=1}^{n} f_i \otimes g_i \in X^* \otimes Y^*$, $\| \sum_{i=1}^{n} f_i \otimes g_i \|_{\alpha'} = m \sup_{\| \sum_{j=1} x_j \otimes y_j \|_\alpha \leq 1} \left| \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(x_j) g_i(y_j) \right| < \infty$,

then α' is a norm on $X^* \otimes Y^*$, called the associate of α.

There are two important crossnorms on $X \otimes Y$ which we shall be concerned with, and which we now define.

Definition : The projective norm γ on $X \otimes Y$ is given by :

$$\| z\|_{\gamma} = \inf \{\sum_{i=1}^{n} \| x_i\|\|y_i\|: z = \sum_{i=1}^{n} x_i\otimes y_i, x_i \varepsilon X, y_i \varepsilon Y\}.$$

The injective norm λ on $X \otimes Y$ is given by :

$$\| z \|_{\lambda} = \sup_{\substack{f\varepsilon \text{ Ball } X^* \\ g\varepsilon \text{ Ball } Y^*}} |\sum_{i=1}^{n} f(x_i)g(y_i)| \text{ if } z = \sum_{i=1}^{n} x_i\otimes y_i.$$

γ and λ are both crossnorms on $X \otimes Y$, and $\lambda \leqslant \gamma$. γ is the greatest crossnorm on $X \otimes Y$, for if α is any crossnorm, and $z \varepsilon X \otimes Y$, then if $z = \sum_{i=1}^{n} x_i\otimes y_i$, $\| z\|_{\alpha} \leqslant \sum_{i=1}^{n} \| x_i\otimes y_i\|_{\alpha}$

$= \sum_{i=1}^{n} \| x_i \| \| y_i \|$. Hence we have $\|z\|_{\alpha} \leqslant \| z \|_{\gamma}$.

Also λ is the least crossnorm whose associate is a crossnorm, for if α is a crossnorm, then its associate α' is a crossnorm if and only if $\alpha \geqslant \lambda$. To see this, suppose that $\alpha \geqslant \lambda$. Then for $f \varepsilon X^*$, $g \varepsilon Y^*$,

$$\begin{aligned}
\| f\| \| g\| &= \sup_{\|\sum_{i=1}^{n} x_i\otimes y_i\|_{\lambda}\leqslant 1} |\sum_{i=1}^{n} f(x_i)g(y_i)| \\
&\geqslant \sup_{\|\sum_{i=1}^{n} x_i\otimes y_i\|_{\alpha}\leqslant 1} |\sum_{i=1}^{n} f(x_i)g(y_i)| \\
&= \| f \otimes g \|_{\alpha'} \\
&\geqslant \sup_{\| x \otimes y \|_{\alpha}\leqslant 1} |f(x)g(y)| \\
&= \| f\| \| g\| .
\end{aligned}$$

So α' is a crossnorm. Now if α' is given to be a crossnorm, then $\|f\|\,\|g\| = \|f \otimes g\|_{\alpha'} \geqslant \dfrac{\left|\sum_{i=1}^{n} f(x_i)g(y_i)\right|}{\left\|\sum_{i=1}^{n} x_i \otimes y_i\right\|_{\alpha}}$,

so $\left\|\sum_{i=1}^{n} x_i \otimes y_i\right\|_{\alpha} \geqslant \dfrac{\left|\sum_{i=1}^{n} f(x_i)g(y_i)\right|}{\|f\|\,\|g\|}$ for all f and g.

Therefore $\alpha \geqslant \lambda$.

Also the injective norm on $X^* \otimes Y^*$ is the associate of the projective norm on $X \otimes Y$. For if $\sum_{i=1}^{n} f_i \otimes g_i \in X^* \otimes Y^*$, then

$$
\begin{aligned}
\left\|\sum_{i=1}^{n} f_i \otimes g_i\right\|_{\gamma'} &\geqslant \sup_{\|x \otimes y\|_{\gamma} \leqslant 1} \left|\sum_{i=1}^{n} f_i(x)g_i(y)\right| \\
&= \sup_{\|x\| \leqslant 1} \left\|\sum_{i=1}^{n} f_i(x)g_i\right\| \\
&= \sup_{\substack{\|x\| \leqslant 1 \\ \psi \in \text{Ball } Y^{**}}} \left|\sum_{i=1}^{n} f_i(x)\psi(g_i)\right| \\
&= \sup_{\substack{\phi \in \text{Ball } X^{**} \\ \psi \in \text{Ball } Y^{**}}} \left|\sum_{i=1}^{n} \phi(f_i)\psi(g_i)\right| \\
&= \left\|\sum_{i=1}^{n} f_i \otimes g_i\right\|_{\lambda}.
\end{aligned}
$$

And
$$
\begin{aligned}
\left|\sum_{i=1}^{n}\sum_{j=1}^{m} f_i(x_j)g_i(y_j)\right| &\leqslant \sum_{j=1}^{m} \left|\sum_{i=1}^{n} f_i(x_j)g_i(y_j)\right| \\
&\leqslant \sum_{j=1}^{m} \|x_j\|\,\|y_j\| \cdot \left\|\sum_{i=1}^{n} f_i \otimes g_i\right\|_{\lambda}.
\end{aligned}
$$

Therefore $\left\|\sum_{i=1}^{n} f_i \otimes g_i\right\|_{\gamma'} \leqslant \left\|\sum_{i=1}^{n} f_i \otimes g_i\right\|_{\lambda}$, and we have $\gamma' = \lambda$.

If X and Y are normed algebras, then $X \otimes_\gamma Y$ is a normed algebra, for if $z_1 = \sum_{i=1}^{n} x_i \otimes y_i$ and $z_2 = \sum_{j=1}^{m} x'_j \otimes y'_j$,

$$\| z_1 z_2 \|_\gamma \leq \sum_{i=1}^{n} \sum_{j=1}^{m} \| x_i x'_j \| \, \| y_i y'_j \|$$

$$\leq \sum_{i=1}^{n} \| x_i \| \, \| y_i \| \cdot \sum_{j=1}^{m} \| x'_j \| \, \| y'_j \| .$$

Hence $\| z_1 z_2 \|_\gamma \leq \| z_1 \|_\gamma \| z_2 \|_\gamma$. (For the purposes of this chapter and chapter 2, we shall take all Banach algebras and normed algebras to satisfy $\| z_1 z_2 \| \leq \| z_1 \| \, \| z_2 \|$ for all z_1 and z_2 in the algebra). In general, multiplication in $X \otimes Y$ is not bounded with respect to the injective norm. This will be discussed in chapters 3 and 4.

We now take X and Y to be Banach spaces. We denote the Cantor-Meray completion of $X \otimes_\gamma Y$ by $X \hat{\otimes} Y$. Any element z of $X \hat{\otimes} Y$ may be represented as $z = \sum_{i=1}^{\infty} x_i \otimes y_i$ where $x_i \in X$, $y_i \in Y$, and $\sum_{i=1}^{\infty} \| x_i \| \, \| y_i \| < \infty$. Also

$$\| z \| = \inf\{ \sum_{i=1}^{\infty} \| x_i \| \, \| y_i \| : z = \sum_{i=1}^{\infty} x_i \otimes y_i \}.$$

The dual of $X \hat{\otimes} Y$ is isometrically isomorphic to $B(X,Y^*)$ under

$$\langle \phi, \sum_{i=1}^{\infty} x_i \otimes y_i \rangle = \sum_{i=1}^{\infty} \phi(x_i)(y_i), \text{ where } \phi \in B(X,Y^*),$$

$x_i \in X$, $y_i \in Y$ and $\sum_{i=1}^{\infty} \| x_i \| \, \| y_i \| < \infty$.

If X and Y are Banach algebras, then $X \hat{\otimes} Y$ becomes a Banach algebra when the multiplication on $X \otimes_\gamma Y$ is extended by continuity.

We denote the completion of $X \otimes_\lambda Y$ by $X \check{\otimes} Y$.

There are a number of useful operators between different tensor products. Since the associate of the projective norm is the injective norm, we see that the linear mapping

$$\Psi : X^* \check{\otimes} Y^* \to (X \hat{\otimes} Y)^* \text{ given by}$$

$$\Psi\Big(\sum_{i=1}^{n} f_i \otimes g_i\Big)\Big(\sum_{j=1}^{\infty} x_j \otimes y_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{\infty} f_i(x_j) g_i(y_j) ,$$

is isometric.

Similarly , there is a bounded linear mapping of $X^* \hat{\otimes} Y^*$ into $(X \check{\otimes} Y)^*$.

If X_1, X_2, Y_1 and Y_2 are Banach spaces, there is a bounded linear mapping $\xi : B(X_1,X_2) \hat{\otimes} B(Y_1,Y_2) \to$

$B(X_1 \hat{\otimes} Y_1, X_2 \hat{\otimes} Y_2)$ such that

$$\xi\Big(\sum_{i=1}^{\infty} S_i \otimes T_i\Big)\Big(\sum_{j=1}^{\infty} x_j \otimes y_j\Big) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} S_i(x_j) \otimes T_i(y_j)$$

and $\|\xi(F)(z)\|_\gamma \leqslant \|F\|_\gamma \|z\|_\gamma$.

At this point we give a lemma which will be required in the next chapter.

<u>Lemma 1.1</u> Let X be a Banach space, and let E be a complemented subspace of X such that there exists a projection P of X onto

E of norm 1. Let Y be any Banach space, and let $I : E \to X$ be the inclusion mapping. Then the mapping $f = \xi(I \otimes I_Y)$: $E \hat{\otimes} Y \to X \hat{\otimes} Y$ is an isometry.

Proof: Let $g = \xi(P \otimes I_Y) : X \hat{\otimes} Y \to E \hat{\otimes} Y$. Then $\|f\| \leqslant \|I \otimes I_Y\| = 1$.

Similarly $\|g\| \leqslant 1$. But if $z \in E \hat{\otimes} Y$, $\|z\|_\gamma = \|g_0 f(z)\|_\gamma \leqslant \|f(z)\|_\gamma$. Hence f is isometric.

CHAPTER TWO

In this chapter we shall give some results concerning the relationship between tensor products and the approximation property.

Definition: If X is a Banach space, then X has the approximation property if the identity function on X, I_X, belongs to the closure of the finite rank operators in the topology of uniform convergence on compact sets.

So X has the approximation property if and only if there exists a net $\{P_\alpha\}$ of finite rank operators in B(X) such that $P_\alpha \to I_X$ uniformly on compact subsets of X.

Most of the standard separable Banach spaces such as the disc algebra, L_p, H^p; C(K) (where K is compact Hausdorff), and spaces of continuously differentiable functions are known to have the approximation property. It is not known whether B(H) or H^∞ have the approximation property. P.Enflo has recently constructed a Banach space which fails to have the approximation property (see [7] and [8]).

We shall give first a rather technical result due to Grothendieck ([1]) which we shall require later in the chapter. If E and F are Banach spaces, then there is a natural bounded linear mapping $\Theta : E \hat{\otimes} F \to B(E^*,F)$

such that $\Theta(\sum_{i=1}^{\infty} e_i \otimes f_i)(e^*) = \sum_{i=1}^{\infty} e^*(e_i)f_i$.

<u>Theorem 2.1</u> Let E be a Banach space. Then the following statements are equivalent :

(1) E has the approximation property.

(2) The mapping $E \hat{\otimes} E^* \to B(E^*,E^*)$ is one to one.

(3) The mapping $E \hat{\otimes} F \to B(E^*,F)$ is one to one for all Banach spaces F.

(4) The mapping $F \hat{\otimes} E \to B(F^*,E)$ is one to one for all Banach spaces F.

Proof : It is easy to see that (3) is equivalent to (4), and that (3) implies (2). We show first that (1) implies (3). Let $z = \sum_{i=1}^{\infty} e_i \otimes f_i \in E \hat{\otimes} F$, with $\sum_{i=1}^{\infty} \| e_i \| \| f_i \| < \infty$. Suppose that $\sum_{i=1}^{\infty} e^*(e_i)f_i = 0$ for all $e^* \in E^*$. There is a net $\{P_\alpha\}$ of finite rank operators in B(E) such that $P_\alpha \xrightarrow{\tau} I_E$ (where τ is the topology of uniform convergence on compact sets).

Now for each α, $\sum_{i=1}^{\infty} e^* \circ P_\alpha(e_i)f_i = 0$, so $\sum_{i=1}^{\infty} P_\alpha(e_i) \otimes f_i = 0$, for if the range of P_α has a basis $\{x_1, x_2, \ldots x_n\}$, and $P_\alpha(e_i) = \sum_{j=1}^{n} \beta_{ij} x_j$, there exists $e_j^* \in E^*$ such that $e_j^*(x_{j'}) = \delta_{j,j'}$, where δ is the Kronecker δ. So $\sum_{i=1}^{\infty} \beta_{ij} f_i = 0$. Therefore

$$\sum_{i=1}^{\infty} P_\alpha(e_i) \otimes f_i = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \beta_{ij} x_j \otimes f_i$$

$$= \sum_{j=1}^{n} x_j \otimes \sum_{i=1}^{\infty} \beta_{ij} f_i = 0 .$$

We have that $\sum_{i=1}^{\infty} \| e_i \| \| f_i \| < \infty$. So there is a sequence $\{\lambda_i\}$ such that $\lambda_i > 0$, $\lambda_i \to 0$, and $c = \sum_{i=1}^{\infty} \| e_i \| \| f_i \| / \lambda_i < \infty$. Let $K = \{ c\lambda_i e_i / \| e_i \| : i = 1, 2, \ldots \} \cup \{0\}$. K is compact, so if $\varepsilon > 0$ there exists an α_0 such that if $\alpha \geq \alpha_0$, then $\| P_\alpha(x) - x \| \leq \varepsilon$ for all x in K.

So $$\| \sum_{i=1}^{\infty} P_\alpha(e_i) \otimes f_i - \sum_{i=1}^{\infty} e_i \otimes f_i \| \leq \sum_{i=1}^{\infty} \| P_\alpha(e_i) - e_i \| \| f_i \|$$

$$\leq \sum_{i=1}^{\infty} \| e_i \| \| f_i \| \varepsilon / c\lambda_i$$

$$= \varepsilon. \quad \text{Therefore } z = 0.$$

In order to show that (2) implies (1) we shall require two lemmas.

<u>Lemma 2.2</u> Let K be a compact subset of a Banach space E. Then K is contained in the closed convex cover of a sequence in E which converges to zero.

Proof: For $x \in E$ and $\varepsilon > 0$, let $B(x, \varepsilon)$ denote the open ε-ball with centre x. Since K is compact, there exists a finite set S_0 in E such that $K \subset \bigcup_{s_0 \varepsilon S_0} B(s_0, 3^{-1})$. Again, since K is compact there exists a finite set S_1 in E such that

$$K \subset \bigcup_{s_0, s_1} B(s_0+s_1, 3^{-2}) \quad \text{and } S_1 \subset B(0, 3^{-1}).$$

Similarly there exists a finite set S_2 such that

$$K \subset \bigcup_{s_0, s_1, s_2} B(s_0+s_1+s_2, 3^{-3}) \text{ and } \quad S_2 \subset B(0, 3^{-2}).$$

Continuing in this fashion, we construct finite sets $S_3, S_4, \ldots$ such that $S_n \subset B(0, 3^{-n})$, and each point of K has distance less than 3^{-n-1} from $S_0+S_1+\ldots\ldots+S_n$.

Let $S = \{0\} \cup 2S_0 \cup 4S_1 \cup 8S_2 \cup \ldots\ldots$ Then S is a sequence converging to zero. If $s_i \varepsilon S_i$ for $i = 0,1,2,\ldots\ldots n$, then

$$s_0+s_1+\ldots\ldots+s_n = 2^{-n-1}.0 + \tfrac{1}{2}.2s_0 + \tfrac{1}{4}.4s_1+\ldots\ldots+ 2^{-n-1}.2^{n+1}.s_n$$
$$\varepsilon \; co(S) .$$

Hence $S_0+S_1+\ldots\ldots+S_n \subset co(S)$, and $K \subset \overline{co}(S)$.

<u>Lemma 2.3</u> Let E be a Banach space. Then there is a natural linear mapping $\zeta : E \hat{\otimes} E^* \xrightarrow{onto} (B(E), \tau)^*$ such that

$$\zeta\left(\sum_{i=1}^{\infty} e_i \otimes \psi_i\right)(T) = \sum_{i=1}^{\infty} \psi_i(T(e_i))$$

where $\sum_{i=1}^{\infty} \| e_i \| \, \| \psi_i \| < \infty$ and $T \varepsilon B(E)$.

Proof : We may certainly define $\zeta : E \hat{\otimes} E^* \to (B(E), \| . \|)^*$ by $\zeta\left(\sum_{i=1}^{\infty} e_i \otimes \psi_i\right)(T) = \sum_{i=1}^{\infty} \psi_i(T(e_i))$.

We show that the range of ζ is $(B(E), \tau)^*$. Let $z = \sum_{i=1}^{\infty} e_i \otimes \psi_i$ belong to $E \hat{\otimes} E^*$ with $\sum_{i=1}^{\infty} \| e_i \| \, \| \psi_i \| < \infty$. As before, let

$\{\lambda_i\}$ be a positive sequence such that $\lambda_i \to 0$ and

$c = \sum_{i=1}^{\infty} \| e_i \| \, \| \psi_i \| / \lambda_i < \infty$. Then

$K = \{ c e_i \lambda_i / \| e_i \| : i = 1,2 \ldots \} \cup \{0\}$ is compact, and

$\| T(x) \| \leqslant 1$ for all $x \varepsilon K$ implies

$$|\zeta(z)(T)| = \left| \sum_{i=1}^{\infty} \psi_i(T(e_i)) \right|$$

$$\leqslant \sum_{i=1}^{\infty} \| \psi_i \| \, \| T(e_i) \|$$

$$\leqslant \sum_{i=1}^{\infty} \| \psi_i \| \, \| e_i \| / \lambda_i c$$

$$= 1 \text{ . So } \zeta(z) \varepsilon (B(E), \tau)^*.$$

Now let $\phi \varepsilon (B(E),\tau)^*$. There exists a compact K such that if $\|T(x)\| \leqslant 1$ for all $x \varepsilon K$ then $|\phi(T)| \leqslant 1$. There exists a sequence $\{x_n\}$ in E such that $x_n \to 0$ and such that K is contained in the closed convex cover of $\{x_n\}$. So for $T \varepsilon B(E)$,

$$|\phi(T)| \leqslant \sup_{x \,\varepsilon\, K} \| T(x) \| \leq \sup_n \| T(x_n) \| .$$

In particular, $T(x_n) = 0$ for all n implies that $\phi(T) = 0$. So we may define a continuous linear functional ψ on the subspace $\{\{T(x_n)\} : T \varepsilon B(E)\}$ of $c_o(E)$ by $\psi(\{T(x_n)\}) = \phi(T)$. (If L is one of the usual sequence spaces and F is any Banach space, we define $L(F) = \{\{f_n\}_{n=1}^{\infty} \subset F : \| \{f_n\} \| = \| \{ \| f_n \| \} \|_L < \infty \}$).

By the Hahn-Banach Theorem we may extend ψ to $\psi' \varepsilon (c_o(E))^*$, which is isometrically isomorphic to $l_1(E^*)$. Therefore there exists $\{\psi_n\} \varepsilon l_1(E^*)$ with $\phi(T) = \psi'(\{ Tx_n\}) =$

$\sum_{i=1}^{\infty} \psi_i(T(x_i)) \forall\, T \in B(E)$. Let $z = \sum_{i=1}^{\infty} x_i \otimes \psi_i \in E \hat{\otimes} E^*$.

Then $\phi = \zeta(z)$, and the lemma is proved.

We now show (2) implies (1).

Suppose $\phi \in (B(E),\tau)^*$ and $\phi(T) = 0$ for all finite rank T in B(E). We show that $\phi(I_E) = 0$, and the result follows by a corollary to the Hahn-Banach Theorem. By the lemma above, there exists $z \in E \hat{\otimes} E^*$ such that $\zeta(z) = \phi$. Suppose that $z = \sum_{i=1}^{\infty} e_i \otimes \psi_i$, then $\phi(T) = \sum_{i=1}^{\infty} \psi_i(T(e_i))$. For $e^* \in E^*$, $y \in E$, let $T(x) = e^*(x)y$. T is finite rank, and

$$0 = \phi(T) = \sum_{i=1}^{\infty} e^*(e_i)\psi_i(y) \quad ,$$

so $\quad 0 = \sum_{i=1}^{\infty} e^*(e_i)\psi_i$.

By (2) we have $z = 0$, therefore $\phi(I_E) = \sum_{i=1}^{\infty} \psi_i(e_i) = 0$.

<u>Corollary</u> If E* has the approximation property, then the mapping $E \hat{\otimes} E^* \to B(E^*,E^*)$ is one to one by (4), so E has the approximation property.

<u>Theorem 2.4</u> Let E be a Banach space. Then E has the approximation property if and only if for all Banach spaces

X and Y and $T \in B(X,Y)$ with T one to one, $\xi(I_E \otimes T) : E \hat{\otimes} X \to E \hat{\otimes} Y$ is one to one.

Proof: The forward implication follows from an argument similar to that in the first part of the proof of Theorem 2.1.

For the backwards implication, suppose that E is a Banach space satisfying the conditions of the theorem, and let F be any Banach space. Let T be the natural embedding of F in C(K), where K = Ball F*. So $T(f)(k) = k(f)$ where $f \in F$ and $k \in K$. C(K) has the approximation property, therefore the mapping $E \hat{\otimes} C(K) \to B(E^*, C(K))$ is one to one. We have that the mapping $E \hat{\otimes} F \to E \hat{\otimes} C(K)$ is one to one, and so the mapping $E \hat{\otimes} F \to B(E^*, F)$ is one to one, and so E has the approximation property.

Throughout this thesis we shall be very concerned with uniform algebras, which we now define.

Definition: Let X be a compact Hausdorff space. A uniform algebra on X is a norm - closed subalgebra of C(X) which separates the points of X and contains the constants.

If we take $k = \mathbb{R}$, then by the Stone-Weierstrass Theorem, the only real uniform algebra on a compact Hausdorff space

X is C(X) itself. In any results or discussion involving uniform algebras, therefore, we shall normally assume that the underlying field k is the complexes.

<u>Theorem 2.5</u> Let Y be a Banach space, and let X be the closed unit ball of Y*, with the weak* topology. Then there is a uniform algebra A on X which has a complemented subspace isometrically isomorphic to Y and such that the projection has norm 1.

Proof: Let R be the subalgebra of C(X) generated by the function 1 and the functions $G_y (y \in Y)$ given by $G_y(x) = x(y) \quad (x \in X)$. Let A be the closure of R in C(X). Define $S : Y \to A$ by $S(y) = G_y$. Then

$$\| S(y) \| = \sup_{x \in X} |x(y)| = \| y \| .$$ So S is an isometric isomorphism of Y onto S(Y). Define $P : R \to S(Y)$ by

$$P(1) = 0$$

$$P(G_y) = G_y \qquad (y \in Y)$$

$$P(G_{y_1} . G_{y_2} \ldots\ldots G_{y_n}) = 0 \text{ if } n \geq 2 \ (y_i \in Y),$$

extending by linearity. We show that P is bounded. Let $g \in R$ with $\| g \| \leq 1$. There exist $y_1, y_2, \ldots . y_n \in Y$ such that $g = Q(G_{y_1}, G_{y_2}, \ldots . G_{y_n})$ where Q is a polynomial in n variables. Suppose

$$Q(\omega) = a_o + \sum_{i=1}^{n} a_i\omega_i + \text{higher order terms } (\omega = (\omega_1, \ldots \omega_n)).$$

Fix $x \in X$, and for $\zeta \in C$ let $\phi(\zeta) = g(\zeta x)$. Then $|\phi(\zeta)| \leqslant 1$ for $|\zeta| \leqslant 1$. We have

$$\phi(\zeta) = a_o + \left(\sum_{i=1}^{n} a_i x(y_i)\right)\zeta + (\quad)\zeta^2 + \ldots\ldots$$

Now $\phi'(0) = (1/2\pi i)\int_{|\zeta|=1} \phi(\zeta)/\zeta^2 \, d\zeta$.

Therefore $|\phi'(0)| \leqslant 1$. But $\phi'(0) = \sum_{i=1}^{n} a_i x(y_i)$. Thus we have $\left\|\sum_{i=1}^{n} a_i G_{y_i}\right\| \leqslant 1$, that is $\|P(g)\| \leqslant 1$. Therefore P is bounded with norm 1, and so P has a bounded extension $\bar{P}$ from A onto S(Y). $\bar{P}$ is clearly a projection.

<u>Corollary</u> If Y is a real Banach space, then Y is isometrically real-isomorphic to a real-complemented linear subspace of a complex Banach space, such that the projection has norm 1. Hence Y is isometrically real-isomorphic to a real-complemented subspace of a complex uniform algebra, such that the projection has norm 1.

<u>Theorem 2.6</u> There is a uniform algebra not having the approximation property.

Proof: We have, by [7]or [8], that there exists a Banach space not having the approximation property. So the result follows from Theorem 2.5 above, and the fact that if Y is a Banach space which has the approximation property then any complemented subspace of Y has the approximation property. To see this fact, suppose that E is the complemented subspace of Y, with projection P. Let $\{P_\alpha\}$ be a net of finite rank operators in B(Y) such that $P_\alpha \to I_Y$ uniformly on compact subsets of Y. Then $P_o P_\alpha|_E \to I_E$ uniformly on compact subsets of E.

We shall now consider projective tensor products of commutative Banach algebras (complex). We shall use the following result due to Tomiyama ([3]) (If A is a commutative Banach algebra, we denote the carrier space of A by Φ_A).

<u>Theorem 2.7</u> If A and B are commutative Banach algebras, then there is a homeomorphism $\chi : \Phi_A \times \Phi_B \to \Phi_{A\hat{\otimes}B}$ such that

$$\chi(\phi,\psi)\left(\sum_{i=1}^{\infty} f_i \otimes g_i\right) = \sum_{i=1}^{\infty} \phi(f_i)\psi(g_i), \quad \sum_{i=1}^{\infty} \|f_i\| \, \|g_i\| < \infty .$$

Proof: If $\phi \in \Phi_A$, $\psi \in \Phi_B$, then certainly $\chi(\phi,\psi) \in \Phi_{A\hat{\otimes}B}$. Suppose now $\theta \in \Phi_{A\hat{\otimes}B}$. $\theta \neq 0$, therefore there exist $a \in A$ and $b \in B$ such that $\theta(a \otimes b) \neq 0$. Define $\phi(f)=\theta(af \otimes b)/$

$\theta(a \otimes b)$ $(f \varepsilon A)$. If a', b' satisfy $\theta(a' \otimes b') \neq 0$, then we get

$$\theta(a'f \otimes b')/ \theta(a' \otimes b') = \theta(af \otimes b)/ \theta(a \otimes b)$$ by cross

multiplication, so ϕ is independent of the choice of a and b.

Now $\phi(f)\ \phi(f') = \theta(af \otimes b)\ \theta(af' \otimes b)/\theta(a \otimes b)\ \theta(a \otimes b)$

$$= \theta(a^2ff' \otimes b^2)/\theta(a^2 \otimes b^2) = \phi(ff').$$

Similarly, we may define a multiplicative linear functional ψ on B by $\psi(g) = \theta(a \otimes bg)/\theta(a \otimes b)$. Then

$$\phi(f)\psi(g) = \theta(af \otimes b)\theta(a \otimes bg)/\theta(a \otimes b)\theta(a \otimes b)$$

$$= \theta(f \otimes g)\theta(a^2 \otimes b^2)/\theta(a^2 \otimes b^2) = \theta(f \otimes g).$$

Therefore ϕ and ψ are non-zero and so belong to Φ_A and Φ_B respectively, and $\theta = \chi(\phi,\psi)$. So χ is onto. Also if $\phi, \phi' \varepsilon \Phi_A$, and $\psi,\psi' \varepsilon \Phi_B$, and $\chi(\phi',\psi') = \chi(\phi,\psi)$, then $\phi'(f)\psi'(g) = \phi(f)\psi(g)$ for all f and g. So if we take $g_o \varepsilon B$ such that $\psi'(g_o) \neq 0$ we get $\phi'(f) = \alpha\phi(f)$ $(f \varepsilon A)$ where α is a constant. Since ϕ and ϕ' are multiplicative, $\alpha = 1$, and so $\phi' = \phi$ and $\psi' = \psi$. Hence χ is one to one.

To show that χ is continuous, let $(\phi_\alpha,\psi_\alpha)$ be a net in $\Phi_A \times \Phi_B$ converging to (ϕ,ψ). Therefore $\phi_\alpha(f) \to \phi(f)$ and $\psi_\alpha(g) \to \psi(g)$ for all f and g and so $\chi(\phi_\alpha,\psi_\alpha)(f \otimes g)$ $= \phi_\alpha(f)\psi_\alpha(g) \to \phi(f)\ \psi(g) = \chi(\phi,\psi)(f \otimes g)$. Hence $\chi(\phi_\alpha,\psi_\alpha)(z)$ $\to \chi(\phi,\psi)(z)$ $(z \varepsilon A \hat{\otimes} B)$, so χ is continuous. To show that χ is bicontinuous it is enough now to show that each point (ϕ,ψ) of $\Phi_A \times \Phi_B$ is contained in a compact set whose χ-image is a neighbourhood of $\chi(\phi,\psi)$. Suppose then that $(\phi,\psi) \varepsilon \Phi_A \times \Phi_B$.

Take $f \varepsilon A$, $g \varepsilon B$ such that $|\phi(f)\psi(g)| > 1$. Then the set $\{\theta \varepsilon \Phi_{A\hat{\otimes}B} : |\theta(f \otimes g)| > 1\}$ is a neighbourhood of $\chi(\phi,\psi)$ and is contained in $\chi(K \times L)$, where $K = \{\phi' \varepsilon \Phi_A : |\phi'(f)| \geqslant 1/\|g\|\}$ and $L = \{\psi' \varepsilon \Phi_B : |\psi'(g)| \geqslant 1/\|f\|\}$ are compact. So χ is a homeomorphism.

We now have the following result, the first part of which is due to Tomiyama ([3]). We observe that if $A \hat{\otimes} B$ is semi-simple, then A and B are semi-simple.

<u>Theorem 2.8</u> (1). If A and B are semi-simple commutative Banach algebras, and either A or B has the approximation property, then $A \hat{\otimes} B$ is semi-simple.

(2). If A is a commutative Banach algebra not having the approximation property, then there is a uniform algebra B such that $A \hat{\otimes} B$ is not semi-simple.

Proof: (1). Suppose that A and B are semi-simple, and either A or B has the approximation property. Let $F = \sum_{i=1}^{\infty} f_i \otimes g_i$, $\sum_{i=1}^{\infty} \|f_i\| \|g_i\| < \infty$, $f_i \varepsilon A$, $g_i \varepsilon B$. Suppose that $\theta(F) = 0 \ \forall \theta \varepsilon \Phi_{A \hat{\otimes} B}$. Therefore

$$\sum_{i=1}^{\infty} \gamma(f_i)\delta(g_i) = 0 \ \forall \gamma \varepsilon \Phi_A, \delta \varepsilon \Phi_B.$$

For $\delta \in \Phi_B$, let $h_\delta = \sum_{i=1}^{\infty} \delta(g_i) f_i \in A$. Then for all $\gamma \in \Phi_A$,

$$\gamma(h_\delta) = \sum_{i=1}^{\infty} \delta(g_i)\gamma(f_i) = 0.$$

Since A is semi-simple, $h_\delta = 0$. Now fix $\phi \in A^*$, and define $G = \sum_{i=1}^{\infty} \phi(f_i) g_i \in B$. For each $\delta \in \Phi_B$,

$$\delta(G) = \sum_{i=1}^{\infty} \phi(f_i)\delta(g_i) = \phi(h_\delta) = 0.$$

Since B is semi-simple, G = 0, therefore

$$\sum_{i=1}^{\infty} \phi(f_i) g_i = 0 \ \forall \ \phi \in A^*.$$

The mapping $A \hat{\otimes} B \to B(A^*,B)$ is one to one, so F must equal 0. Thus $A \hat{\otimes} B$ is semi-simple.

(2) Let A be a commutative Banach algebra which fails to have the approximation property. Then for some Banach space E, the mapping $A \hat{\otimes} E \to B(A^*,E)$ is not one to one. There exist a uniform algebra B, a projection P on B of norm 1, and a linear isometry S of E onto P(B). By Lemma 1.1, the mapping $\eta : A \hat{\otimes} E \to A \hat{\otimes} B$ is isometric. There exists $z = \sum_{i=1}^{\infty} a_i \otimes e_i$ in $A \hat{\otimes} E$ with $z \neq 0$, and such that $\sum_{i=1}^{\infty} \phi(a_i) e_i = 0 \ \forall \ \phi \in A^*$. Let $F = \eta(z) = \sum_{i=1}^{\infty} a_i \otimes S(e_i)$. Then $F \neq 0$, but we have

$$\sum_{i=1}^{\infty} \phi(a_i)\psi(S(e_i)) = \psi \circ S\left(\sum_{i=1}^{\infty} \phi(a_i) e_i\right) = 0 \ \forall \ \phi \in \Phi_A, \psi \in \Phi_B.$$

Hence $\theta(F) = 0 \ \forall \ \theta \in \Phi_{A \hat{\otimes} B}$, therefore $A \hat{\otimes} B$ is not semi-simple.

We shall now study the notions of ε-products and slice products, and their relationship with the injective tensor product.

<u>The ε-Product</u>

The definition and results of this section are due to Waelbroeck in [6]. Before giving the definition of the ε-product, we state the following theorem.

<u>Theorem</u> Let X and Y be Banach spaces. Then the following Banach spaces are isometrically isomorphic:

(1) The space of linear functions from X* into Y whose restrictions to Ball X* are weak* continuous.

(2) The space of linear functions from Y* into X whose restrictions to Ball Y* are weak* continuous.

(3) The space of bilinear functionals on X* × Y* whose restrictions to Ball X*× Ball Y* are weak* continuous.

The norm in (1), (2) and (3) is the supremum on Ball X*, Ball Y* and Ball X* × Ball Y* respectively.

Definition: Let X and Y be Banach spaces. Then we define the ε-product XεY to be the Banach space (1) above.

We imbed the injective tensor product of two Banach spaces in the ε-product. Define $\xi: X \otimes_\lambda Y \to X\varepsilon Y$ by

$$\xi\left(\sum_{i=1}^{n} x_i \otimes y_i\right)(f) = \sum_{i=1}^{n} f(x_i)y_i \quad , x_i \in X, y_i \in Y, f \in X^*.$$

Then $\left\| \xi\left(\sum_{i=1}^{n} x_i \otimes y_i\right) \right\| = \sup_{f \in \text{Ball } X^*} \left\|\sum_{i=1}^{n} f(x_i)y_i\right\| = \left\|\sum_{i=1}^{n} x_i \otimes y_i\right\|_\lambda$

Thus ξ is a linear isometry. We identify $X \otimes_\lambda Y$ with its image in $X \varepsilon Y$, and identify $X \check{\otimes} Y$ with the closure of $X \otimes Y$ in $X \varepsilon Y$. In fact $\xi(X \otimes Y)$ is the set of finite rank elements in $X \varepsilon Y$.

<u>Theorem 2.9</u> Let X be a Banach space. Then X has the approximation property if and only if $X \check{\otimes} Y = X \varepsilon Y$ for all Banach spaces Y.

Proof: Suppose first that X has the approximation property, and let Y be any Banach space. Let u belong to $X \varepsilon Y$, and suppose that $u : Y^* \to X$. Let $\varepsilon > 0$. Then u (Ball Y*) is compact in X, so there exists a finite rank P in B(X) such that

$$\sup_{g \in \text{Ball } Y^*} \| P(u(g)) - u(g) \| < \varepsilon .$$

Therefore $\|P_0 u - u\| < \varepsilon$, and $P_0 u$ is finite rank in $X \varepsilon Y$. Hence $X \otimes_\lambda Y$ is dense in $X \varepsilon Y$, and so $X \check{\otimes} Y = X \varepsilon Y$.

Now suppose that $X \check{\otimes} Y = X \varepsilon Y$ for all Banach spaces Y. Let K be compact in X. Define Y to be the norm closure in $C(K)$ of $X^*|_K$. Therefore $X \check{\otimes} Y = X \varepsilon Y$. Define u in $X \varepsilon Y$ by $u(f) = f|_K$ ($f \varepsilon X^*$). There exists $\sum_{i=1}^{n} x_i \otimes y_i$ in $X \otimes Y$ such that

$$\|\xi(\sum_{i=1}^{n} x_i \otimes y_i) - u\| \leqslant \tfrac{1}{2},$$ and we may suppose that

$\sum_{i=1}^{n} \|x_i\| = 1$. Also for each i there is an f_i in X^* such that

$\|f_i|_K - y_i\|_K \leqslant \frac{1}{2}$. Let $P(x) = \sum_{i=1}^{n} f_i(x) x_i$ for x in X. Then P is finite rank, and $\|P(x) - x\| \leqslant 1$ for x in K. Hence by one of the definitions of the topology of compact convergence, X has the approximation property.

The Slice Product

If X and Y are compact Hausdorff spaces, then we may identify $C(X) \check{\otimes} C(Y)$ with $C(X \times Y)$. For define $\Gamma : C(X) \otimes_\lambda C(Y) \to C(X \times Y)$ by $\Gamma(\sum_{i=1}^{n} f_i \otimes g_i)(x,y) = \sum_{i=1}^{n} f_i(x) g_i(y)$.

$$\|\Gamma(\sum_{i=1}^{n} f_i \otimes g_i)\| = \sup_{\substack{x \varepsilon X \\ y \varepsilon Y}} |\sum_{i=1}^{n} f_i(x) g_i(y)|$$

$$= \sup_{x \varepsilon X} \|\sum_{i=1}^{n} f_i(x) g_i\|$$

$$= \sup_{\substack{x \varepsilon X \\ \psi \varepsilon \text{Ball } (C(Y))^*}} |\sum_{i=1}^{n} f_i(x) \psi(g_i)|$$

$$= \sup_{\psi \in \text{Ball}(C(Y))^*} \left\| \sum_{i=1}^{n} \psi(g_i) f_i \right\|$$

$$= \left\| \sum_{i=1}^{n} f_i \otimes g_i \right\|_\lambda .$$

So Γ is isometric and is an algebra homomorphism. We extend Γ to $C(X) \check{\otimes} C(Y)$ and extend multiplication on $C(X) \otimes_\lambda C(Y)$ to $C(X) \check{\otimes} C(Y)$ so that Γ remains an isometric algebra homomorphism. Then $\Gamma(C(X) \check{\otimes} C(Y))$ is a closed subalgebra of $C(X \times Y)$ which contains the function 1, separates the points of $X \times Y$, and is closed under complex conjugation (if $k = C$). Hence by the Stone-Weierstrass Theorem, we have $\Gamma(C(X) \check{\otimes} C(Y)) = C(X \times Y)$

If A and B are uniform algebras on compact Hausdorff spaces X and Y respectively, then we may similarly imbed $A \check{\otimes} B$ homomorphically and isometrically in $C(X \times Y)$, and we thus get that $A \check{\otimes} B$ is a uniform algebra on $X \times Y$.

Definition: If X is a compact Hausdorff space, and B is a Banach space, $C(X,B)$ is the set of continuous functions from X into B.

$C(X,B)$ is then a Banach space under the norm $\|f\| = \sup_{x \in X} \|f(x)\|$. If B is a Banach algebra, then $C(X,B)$ is a Banach algebra under pointwise multiplication, and is commutative if B is commutative.

If A is a uniform algebra on X, and B is any Banach algebra, we may define $\Delta : A \otimes_\lambda B \to C(X,B)$ by

$\Delta(\sum_{i=1}^{n} f_i \otimes b_i)(x) = \sum_{i=1}^{n} f_i(x) b_i, f_i \in A, b_i \in B$ and $x \in X$.

As before, Δ is isometric and an algebra homomorphism, and $A \check{\otimes} B$ is therefore a Banach algebra.

Now take X and Y to be fixed compact Hausdorff spaces. For $h \in C(X \times Y)$ and $x \in X$, define $h_x \in C(Y)$ by $h_x(y) = h(x,y)$. Define $h^y \in C(X)$ by $h^y(x) = h(x,y)$. We may define an isometric (algebra) isomorphism of $C(X \times Y)$ onto $C(X, C(Y))$ by $\Lambda(h)(x) = h_x$. Then $\|\Lambda(h)\| = \sup_{x \in X} \|h_x\| = \sup_{\substack{x \in X \\ y \in Y}} |h(x,y)| = \|h\|$.

Similarly $C(X \times Y)$ is isometrically (algebra) isomorphic to $C(Y, C(X))$.

Definition: If A and B are uniform algebras on X and Y respectively, then the slice product $A \# B$ is the space $\{h \in C(X \times Y) : h_x \in B \ \forall x \in X$ and $h^y \in A \ \forall y \in Y\}$.

We may define $\Omega : A \varepsilon B \to A \# B$ by

$\Omega(u)(x,y) = u(\phi_x)(y)$ where $x \in X, y \in Y, u \in A \varepsilon B$, and ϕ_x is the evaluation functional at x, so $\phi_x(f) = f(x)$ $(f \in A)$.

We then have the following result, relating the slice product and the ε-product (Proposition 15, [4]).

Theorem 2.10 If A and B are uniform algebras, then Ω defined above is an isometric isomorphism of $A \varepsilon B$ with $A \# B$ (as Banach spaces).

Proof: For u in $A \varepsilon B$, the mapping $x \to u(\phi_x)$ belongs to $C(X,C(Y))$, so by the remarks above, the mapping $(x,y) \to u(\phi_x)(y)$ belongs to $C(X \times Y)$. If $h = \Omega(u)$, $h_x = u(\phi_x) \varepsilon B$ for each x in X. If y belongs to Y, the mapping $\phi \to u(\phi)(y)$ is a linear functional on A* with weak* continuous restriction to Ball A*. Therefore there exists an f in A such that $\phi(f) = u(\phi)(y)$ $(\phi \varepsilon A^*)$. Then

$f(x) = \phi_x(f) = u(\phi_x)(y) = h(x,y)$ $(x \varepsilon X)$.

Hence $h^y = f \varepsilon A$, and $h = \Omega(u)$ belongs to $A \# B$.

Now
$$\|\Omega(u)\| = \sup_{\substack{x\varepsilon X \\ y\varepsilon Y}} |u(\phi_x)(y)| = \sup_{x\varepsilon X} \|u(\phi_x)\|$$

$$= \sup_{\phi\ \varepsilon\ \text{Ball } A^*} \|u(\phi)\| \text{ by the bipolar theorem}$$

$$= \|u\|.$$

So Ω is isometric. Now let h belong to $A \# B$. Define $u \varepsilon B(A^*,C(Y))$ by $u(\phi)(y) = \phi(h^y)$ $(\phi \varepsilon A^*, y \varepsilon Y)$. Since the mapping $y \to h^y$ belongs to $C(Y,A)$, we get by compactness that the restriction of u to Ball A* is weak* continuous. We now show the range of u is contained in B. If $x \varepsilon X$, $u(\phi_x) = h_x \varepsilon B$. The set Ball $A^* \cap u^{-1}(B)$ is weak* closed in Ball A*, and therefore in A*. Ball $A^* \cap u^{-1}(B)$ contains

$\{\phi_x : x \in X\}$ and therefore contains the (weak*) closed convex circled cover of $\{\phi_x : x \in X\}$, which equals Ball A* by the bipolar theorem. Hence the range of $u \subset B$, and so u belongs to $A \varepsilon B$. We have $\Omega(u)(x,y) = u(\phi_x)(y) = \phi_x(h^y) = h(x,y)$, therefore Ω is onto.

(In fact the above result holds more generally, for we have not used the uniform algebra properties of A and B. We may define the slice product of any two closed subspaces of C(X) and C(Y), (with $k = \mathbb{C}$ or $\mathbb{R}$), and we still get the slice product equals the ε-product).

The equivalence of the slice product and the ε-product for uniform algebras now allows us to establish a relationship between the slice product of uniform algebras and the approximation property. If A and B are uniform algebras on X and Y respectively, then by our remarks at the beginning of this section, we may regard $A \check{\otimes} B$ as a subspace (in fact a subalgebra) of $C(X \times Y)$. Then we have:

<u>Theorem 2.11</u> Let A be a uniform algebra. Then A has the approximation property if and only if $A \check{\otimes} B = A \mathbin{\#} B$ for all uniform algebras B.

Proof: If A has the approximation property, and B is any uniform algebra, then $A \check{\otimes} B = A \varepsilon B = A \# B$. Now suppose A is a uniform algebra satisfying the given conditions. Let E be any Banach space. There exists a uniform algebra B with a projection P of norm 1, and an isometric isomorphism S of E onto P(B). Let u belong to $A \varepsilon E$, and let $\varepsilon > 0$. Define u_1 in $A \varepsilon B$ by $u_1 = S_o u$. Now $A \check{\otimes} B = A \# B = A \varepsilon B$, hence there exists

$$\sum_{i=1}^{n} f_i \otimes g_i \text{ in } A \otimes B \text{ such that}$$

$$\left\| \sum_{i=1}^{n} f_i \otimes g_i - u_1 \right\| < \varepsilon . \qquad \text{Therefore,}$$

$$\left\| \sum_{i=1}^{n} f_i \otimes S_o^{-1} P(g_i) - u \right\| < \varepsilon .$$

Thus $A \otimes_\lambda E$ is dense in $A \varepsilon E$, and so we have that A has the approximation property.

CHAPTER THREE

Injective Algebras

In this chapter and chapter 4 it will be convenient to generalise Banach algebras and normed algebras so that multiplication may be bounded by a constant other than 1.

Definition: If A is a normed space (a Banach space) and an algebra over k, then A is a (K)-normed algebra ((K)-Banach algebra) if

$$\| ab \| \leqslant K\| a \| \| b\| \quad (a,b \in A).$$ If such a K exists, we say A is a normed-algebra (a Banach-algebra).

With this definition, a (K)-normed algebra A may always be re-normed (by $\| . \|' = K\| . \|$) to become a (1)-normed algebra, so A is isomorphic to a (1)-normed algebra.

Definition: If A is a normed-algebra, then A is said to be K-injective ($K \geqslant 0$) if

$$\| \sum_{i=1}^{n} x_i y_i \| \leqslant K \| \sum_{i=1}^{n} x_i \otimes y_i \|_\lambda \quad (x_i, y_i \in A).$$

A is said to be injective if it is K-injective for some K.

Then A is K-injective if and only if the natural mapping of $A \otimes_\lambda A$ into A is bounded with norm $\leqslant K$.

Every uniform algebra is 1-injective, for if A is a uniform algebra on X, and $\sum_{i=1}^{n} f_i \otimes g_i \ \varepsilon\ A \otimes_\lambda A$,

$$\left\| \sum_{i=1}^{n} f_i . g_i \right\| = \sup_{x \varepsilon X} \left| \sum_{i=1}^{n} f_i(x) g_i(x) \right|$$

$$\leqslant \sup_{\phi,\psi\ \varepsilon \mathrm{Ball}\ A^*} \left| \sum_{i=1}^{n} \phi(f_i)\psi(g_i) \right|$$

$$= \left\| \sum_{i=1}^{n} f_i \otimes g_i \right\|_\lambda .$$

Also the space l_1 with pointwise multiplication is 1-injective, for if $\sum_{r=1}^{n} x^{(r)} \otimes y^{(r)}\ \varepsilon\ l_1 \otimes_\lambda l_1$,

let $a_{ij} = \sum_{r=1}^{n} x_i^{(r)} y_j^{(r)}$, and let γ_i be a scalar of modulus 1 such that $\gamma_i a_{ii} = |a_{ii}|$. Let $m\ \varepsilon\ \mathbb{P}$ and let $\Delta = \{-1,1\}^m$. Then

$$\sum_{\delta\varepsilon\Delta} \left| \sum_{i,j=1}^{m} \gamma_i \delta_i a_{ij} \delta_j \right| \geqslant \left| \sum_{\delta\varepsilon\Delta} \sum_{i,j=1}^{m} \gamma_i \delta_i a_{ij} \delta_j \right|$$

$$= \left| \sum_{i,j=1}^{m} \gamma_i a_{ij} \sum_{\delta\varepsilon\Delta} \delta_i \delta_j \right| = 2^m \sum_{i=1}^{m} |a_{ii}| .$$

Hence there exists $\delta\varepsilon\Delta$ such that $\sum_{i=1}^{m} |a_{ii}| \leqslant \left| \sum_{i,j=1}^{m} \gamma_i \delta_i a_{ij} \delta_j \right|$

$$\leqslant \sup_{\phi,\psi\ \varepsilon\ \mathrm{Ball}\ l_\infty} \left| \sum_{r=1}^{n} \phi(x^{(r)}) \psi(y^{(r)}) \right| .$$

Therefore $\left\| \sum_{r=1}^{n} x^{(r)} y^{(r)} \right\| = \sum_{i=1}^{\infty} |a_{ii}| \leqslant \left\| \sum_{r=1}^{n} x^{(r)} \otimes y^{(r)} \right\|_\lambda$, and so l_1 is injective.

The space l_1 with convolution multiplication is not injective. Also for $1 < p < \infty$, l_p with pointwise multiplication is not injective (l_∞ is a uniform algebra). Also the

Banach algebra $C^p[0,1]$ of all functions on $[0,1]$ with continuous derivatives of order p (normed by

$$\|f\| = \sum_{j=0}^{p} \sup_{t\varepsilon I} |f^{(j)}(t)|, \text{ where } I = [0,1]\ ) \text{ is injective.}$$

If A is a K-injective normed-algebra, then for $x_i, y_i, z_i \in A$, and $i = 1,2,\ldots n$,

$$\|\sum_{i=1}^{n} x_i(y_i z_i)\| \leqslant K \sup_{\phi_1 \varepsilon \text{ Ball } A^*} \|\sum_{i=1}^{n} \phi_1(x_i) y_i z_i\|$$

$$\leqslant K^2 \sup_{\phi_1,\phi_2,\phi_3 \ \varepsilon \text{ Ball } A^*} |\sum_{i=1}^{n} \phi_1(x_i)\phi_2(y_i)\phi_3(z_i)|.$$

In general,

$$\|\sum_{i=1}^{n} x_i^{(1)} \ldots\ldots x_i^{(r)}\| \leqslant K^{r-1} \sup_{\phi_1,\ldots\phi_r \varepsilon \text{Ball } A^*} |\sum_{i=1}^{n} \phi_1(x_i^{(1)})\ldots\phi_r(x_i^{(r)})|.$$

<u>Commutative Injective Algebras</u>

We shall now consider injective commutative Banach-algebras. We shall require a standard symmetrisation result.

<u>Theorem 3.1</u> Let X be a vector space, let $x_1, x_2 \ldots . x_n \varepsilon X$, and let $\phi_1, \phi_2, \ldots \phi_n \varepsilon X'$. Then if S_n is the group of permutations on n letters, and $K_n = \{1,2,\ldots.n\}$ and the cardinality of a set Ω is $|\Omega|$,

$$\sum_{\pi \ \varepsilon \ S_n} \phi_1(x_{\pi_1})\ldots\phi_n(x_{\pi_n}) = \sum_{\Omega \subset K_n} (-1)^{n-|\Omega|} \prod_{r=1}^{n} (\sum_{j\varepsilon\Omega} \phi_j)(x_r).$$

Proof : If Y is a non-empty finite set and Y_1 is a proper subset, then

$$\sum_{\substack{\Omega\subset Y\\ \Omega\supset Y_1}} (-1)^{|\Omega|} = 0.$$

To see this it is enough to assume Y_1 is empty and show $\sum_{\Omega\subset Y}(-1)^{|\Omega|} = 0$ (where the summation is over all subsets of Y, including the empty set). If $|Y| = 1$, this holds. If $|Y| > 1$, choose $y \in Y$ and let $Z = Y\setminus\{y\}$. Then $\sum_{\Omega\subset Y} (-1)^{|\Omega|} = \sum_{\Omega\subset Z}(-1)^{|\Omega|} + \sum_{\Lambda\subset Z}(-1)^{|\Lambda|+1}$. The result then follows by induction.

The right hand side of the equation in the statement of the theorem equals

$$\sum_{\Omega\subset K_n} (-1)^{n-|\Omega|} \sum_{j_1,j_2,..j_n \in\Omega} \phi_{j_1}(x_1)\ldots\phi_{j_n}(x_n)$$

$$= \sum_{j_1,j_2,..j_n \in K_n} \phi_{j_1}(x_1)\ldots\phi_{j_n}(x_n) \sum_{\substack{\Omega\subset K_n\\ \Omega\supset\{j_1,j_2,..j_n\}}} (-1)^{n-|\Omega|} .$$

Now

$$\sum_{\substack{\Omega\subset K_n\\ \Omega\supset\{j_1,j_2,..j_n\}}} (-1)^{n-|\Omega|} = 1 \quad \text{if } \{j_1,\ldots j_n\} \text{ is a permutation of } K_n$$

$$= 0 \quad \text{otherwise}$$

So the right hand side of the equation equals

$\sum_{\pi\,\in\, S_n} \phi_{\pi_1}(x_1)\ldots\phi_{\pi_n}(x_n)$, which equals the left hand side.

We now establish a characterisation of injective commutative Banach-algebras which was proved by Varopoulos in [9].

We use techniques akin to those used in the proof of Theorem 2.5. For the purposes of this result we do not require that a uniform algebra must have an identity, and we take the scalar field to be the complexes.

Theorem 3.2 Let A be a commutative Banach-algebra. Then A is injective if and only if there exists a uniform algebra B, a bounded algebra homomorphism h of B onto A, and a linear (bounded) operator $j : A \to B$ such that $h_o j = I_A$, the identity function from A onto itself.

Proof : Suppose A satisfies the conditions above, and let $\sum_{i=1}^{n} x_i \otimes y_i \in A \otimes_\lambda A$. Then

$$\left\| \sum_{i=1}^{n} x_i y_i \right\| = \left\| \sum_{i=1}^{n} h(j(x_i)j(y_i)) \right\|$$

$$\leq \|h\| \left\| \sum_{i=1}^{n} j(x_i)j(y_i) \right\|$$

$$\leq \|h\| \sup_{\phi,\phi' \in \text{Ball } B^*} \left| \sum_{i=1}^{n} \phi(j(x_i))\phi'(j(y_i)) \right|$$

(since B is a uniform algebra)

$$\leq \|h\| \, \|j\|^2 \sup_{\psi,\psi' \in \text{Ball } A^*} \left| \sum_{i=1}^{n} \psi(x_i)\psi'(y_i) \right|.$$

Hence A is injective.

Suppose now that A is K-injective. Let $m,k \in \mathbb{P}$, and let $x_{ir} \in A$ for $i = 1,2,\ldots k$, $r = 1,2,\ldots m$. Then

$$m!\left\|\sum_{i=1}^{k} x_{i_1}\cdots x_{i_m}\right\| = \left\|\sum_{i=1}^{k}\sum_{\pi\in S_m} x_{i_{\pi_1}}\cdots x_{i_{\pi_m}}\right\|$$

$$\leqslant K^{m-1}\sup_{\phi_1,\ldots\phi_m\in \text{ Ball } A^*}\left|\sum_{i=1}^{k}\sum_{\pi\in S_m}\phi_1(x_{i_{\pi_1}})\cdots\phi_m(x_{i_{\pi_m}})\right|$$

$$\leqslant K^{m-1}\sup_{\phi_j\in\text{Ball } A^*}\left|\sum_{i=1}^{k}\sum_{\Omega\subset K_m}(-1)^{m-|\Omega|}\prod_{r=1}^{m}\left(\sum_{j\in\Omega}\phi_j\right)(x_{ir})\right|$$

$$\leqslant m^m K^{m-1}\sum_{\Omega\subset K_m}\sup_{\phi_j\in\text{ Ball } A^*}\left|\sum_{i=1}^{k}\prod_{r=1}^{m}\left(\sum_{j\in\Omega}\phi_j/m\right)(x_{ir})\right|$$

$$\leqslant m^m K^{m-1}2^m\sup_{\phi\in\text{Ball } A^*}\left|\sum_{i=1}^{k}\phi(x_{i_1})\cdots\phi(x_{im})\right|.$$

Now $m^m/m! \leqslant e^m$, hence

$$\left\|\sum_{i=1}^{k} x_{i_1}\cdots x_{im}\right\| \leqslant (2e)^m K^{m-1}\sup_{\phi\in\text{Ball } A^*}\left|\sum_{i=1}^{k}\phi(x_{i_1})\cdots\phi(x_{im})\right|.$$

It follows that if P is a polynomial in n variables, and P is homogeneous of degree m, then if $x_1, x_2,\ldots x_n \in A$,

$$\|P(x_1,\ldots x_n)\| \leqslant (2e)^m K^{m-1}\sup_{\phi\in\text{Ball } A^*}|P(\phi(x_1),\ldots,\phi(x_n))|.$$

Now let $Y = \lambda$ Ball A^*, where $\lambda = 4eK$, and let Y have the weak* topology. Let B be the closed subalgebra of C(Y) generated by the functions $G_x (x \in A)$ given by $G_x(\theta) = \theta(x)\ (\theta \in Y)$. Let P be a polynomial in n variables with no constant term, and let $x_1,\ldots,x_n$ belong to A. Suppose $P = P_1 + P_2 + \ldots + P_r$ where P_i is homogeneous of degree i (r = degree of P). Then

$$\|P_i(G_{x_1},\ldots,G_{x_n})\| = \sup_{\theta\in Y}|P_i(\theta(x_1),\ldots,\theta(x_n))|$$

$$\leqslant \sup_{\theta\in Y}\sup_{|\alpha|\leqslant 1}\left|\sum_{j=1}^{r}\alpha^j P_j(\theta(x_1),\ldots,\theta(x_n))\right|$$

(by the Cauchy coefficient inequalities)

$$= \sup_{\theta \varepsilon Y} \sup_{|\alpha| \leqslant 1} |P(\alpha\theta(x_1),\ldots,\alpha\theta(x_n))|$$

$$= \| P(G_{x_1},\ldots,G_{x_n}) \| .$$

Also $\|P_i(x_1,\ldots,x_n)\| \leqslant ((2e)^i K^{i-1}/\lambda^i) \sup_{\phi \varepsilon \mathrm{Ball}\ A^*} |P_i(\lambda\phi(x_1),\ldots\lambda\phi(x_n))|$

$$= (1/K2^i) \| P_i(G_{x_1},\ldots,G_{x_n}) \| .$$

Thus $\| P(x_1,\ldots x_n) \| \leqslant \sum_{i=1}^{r} \|P_i(x_1,\ldots,x_n)\|$

$$\leqslant (1/K) \sum_{i=1}^{r} (1/2^i) \| P_i(G_{x_1},\ldots,G_{x_n}) \|$$

$$\leqslant (1/K) \| P(G_{x_1},\ldots,G_{x_n}) \| .$$

Hence we may define $h : B \rightarrow A$ by

$h(P(G_{x_1},\ldots,G_{x_n})) = P(x_1,\ldots,x_n)$, extending by

continuity. h is bounded and is clearly an algebra homomorphism. Define $j : A \rightarrow B$ by $j(x) = G_x (x \varepsilon A)$. Then

$\| j(x) \| = \sup_{\phi \varepsilon \mathrm{Ball}\ A^*} |\lambda\phi(x)| = \lambda \| x \| \quad (x \varepsilon A)$. So j is linear and bounded, and clearly $h_o j = I_A$.

<u>Corollary</u> If A is a K-injective commutative Banach-algebra with an identity 1_A, then we may assume that the uniform algebra B in the statement of the theorem has a one. For if in the above proof we take B' to be the closed subalgebra of C(Y) generated by the functions G_x and the function 1, and we take

Q to be any polynomial in n variables, with

$Q = Q_o + Q_1 + \dots + Q_r$ (Q_i homogeneous of degree i) then

$$\| Q(x_1,\dots x_n)\| \leqslant \sum_{i=0}^{r} \| Q_i(x_1,\dots\dots,x_n)\|$$

$$\leqslant \| Q_o 1_A\| + (1/K)\sum_{i=1}^{r}(1/2^i)\| Q_i(G_{x_1},\dots,G_{x_n})\|$$

$$\leqslant (\| 1_A\| + 1/K)\| Q(G_{x_1},\dots,G_{x_n})\|.$$

We may therefore define in an analogous fashion a bounded algebra homomorphism h' of B' onto A and a bounded linear $j' : A \to B'$ such that $h'_o j' = I_A$.

Q-algebras

Definition : A commutative Banach-algebra A is a Q-algebra if it is isomorphic to a quotient algebra B/I where B is a uniform algebra and I is a closed ideal in B. Equivalently, there is a bounded (algebra) homomorphism of B onto A. (Again we do not require that a uniform algebra must have a one).

For a study of Q-algebras, see [11].

From Theorem 3.2, we have that every injective commutative Banach-algebra is a Q-algebra. Not all Q-algebras are injective, for example, l_p ($1< p <\infty$) is a Q-algebra, but is not injective.

We now return to general normed-algebras (not necessarily complex, commutative or Banach). We shall show that a normed-algebra A is injective if and only if $A \otimes_\lambda B$ is a normed-algebra for every normed-algebra B. This was proved by Varopoulos in [10]. In this paper Varopoulos showed that if A is a 1-injective normed-algebra and B is a (1)-normed algebra, then $A \otimes_\lambda B$ is a (K)-normed algebra for some K (for a commutative Banach-algebra A over C this already follows from Theorem 3.2). In fact the following is true.

<u>Theorem 3.3</u> If A is a 1-injective normed-algebra, and B is a (1)-normed algebra, then $A \otimes_\lambda B$ is a (1)-normed algebra (and so $A \check{\otimes} B$ is a (1)-Banach algebra when multiplication is extended by continuity from $A \otimes_\lambda B$).

Proof : Let $z_1 = \sum_{i=1}^{n} x_i \otimes y_i, z_2 = \sum_{j=1}^{m} a_j \otimes b_j$ belong to $A \otimes_\lambda B$.

For $\psi \in \text{Ball } B^*$,

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{m} x_i a_j \psi(y_i b_j) \right\| \leq \sup_{\phi,\phi' \in \text{Ball}A^*} \left| \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(x_i)\phi'(a_j)\psi(y_i b_j) \right|$$

$$\leq \sup_{\phi,\phi' \in \text{Ball } A^*} \left\| \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(x_i)\phi'(a_j) y_i b_j \right\|$$

$$\leq \sup_{\phi,\phi' \in \text{Ball } A^*} \left\| \sum_{i=1}^{n} \phi(x_i) y_i \right\| \left\| \sum_{j=1}^{m} \phi'(a_j) b_j \right\|$$

$$= \|z_1\| \|z_2\|.$$

Hence $\|z_1 z_2\| \leq \|z_1\| \|z_2\|$.

Corollary If A is K-injective, and B is an (L)-normed algebra, then $A \otimes_\lambda B$ is a (KL)-normed algebra.

In order to establish the converse result, we wish to show that if A is a non-injective normed-algebra, then there exists a normed-algebra B such that $A \otimes_\lambda B$ is not a normed-algebra, i.e. for each $K > 0$ $A \otimes_\lambda B$ is not (K)-normed. It is sufficient to show that if A is not injective then for each $K > 0$ there exists a (1)-normed algebra B such that $A \otimes_\lambda B$ is not (K)-normed. For if $\{B_n\}$ is a sequence of (1)-normed algebras such that $A \otimes_\lambda B_n$ is not (n)-normed, let

$$l_\infty(\{B_n\}) = \{\{b_n\}_{n=1}^{\infty} : b_n \in B_n, n \in \mathbb{P}, \|\{b_n\}\| = \sup_n \|b_n\| < \infty\}.$$

$l_\infty(\{B_n\})$ is a (1)-normed algebra under pointwise operations. By Lemma 1.1, the natural imbedding of $A \otimes_\lambda B_m$ in $A \otimes_\lambda l_\infty(\{B_n\})$ given by $\sum_{i=1}^{r} a_i \otimes b_i \to \sum_{i=1}^{r} a_i \otimes (0,0,\ldots 0,b_i,.)$ $(a_i \in A, b_i \in B_m)$ is isometric. This imbedding is an algebra homomorphism, hence since $A \otimes_\lambda B_m$ is not (m)-normed, $A \otimes_\lambda l_\infty(\{B_n\})$ is not (m)-normed. So $A \otimes_\lambda l_\infty(\{B_n\})$ is not a normed-algebra.

Theorem 3.4 Let A be a normed-algebra and let $K \geq 0$. Then A is K-injective if and only if $A \otimes_\lambda B$ is a (K)-normed algebra for each (1)-normed algebra B.

Proof : We already have the forward implication. The following proof of the reverse implication was pointed out to me by Dr. A.M. Davie.

Suppose that A is not K-injective. Then there exists $\sum_{i=1}^{n} x_i \otimes y_i \in A \otimes_\lambda A$ such that $\|\sum_{i=1}^{n} x_i y_i\| > K \|\sum_{i=1}^{n} x_i \otimes y_i\|_\lambda$.

We may assume without loss of generality that the sets $\{x_i\}$ and $\{y_i\}$ are both linearly independent, and $\sum_{i=1}^{n} \|x_i\| \leq 1$ and $\sum_{i=1}^{n} \|y_i\| \leq 1$.

Choose $L > K$ such that $\|\sum_{i=1}^{n} x_i y_i\| > L \|\sum_{i=1}^{n} x_i \otimes y_i\|$.

Let B be the algebra over k of polynomials in 2n indeterminates $z_1, z_2, \ldots z_n, w_1, \ldots w_n$.

Let $H = \{1\} \cup \{\sum_{i=1}^{n} \phi(x_i) z_i : \phi \in \text{Ball } A^*\} \cup \{\sum_{i=1}^{n} \psi(y_i) w_i : \psi \in \text{Ball } A^*\}$.

Let N be the convex circled semigroup in B generated by H.So

$N = \{\sum_{k=1}^{m} \lambda_k h_1^{(k)} h_2^{(k)} \ldots h_{r_k}^{(k)} : h_j^{(k)} \in H, \sum_{k=1}^{m} |\lambda_k| \leq 1\}$.

For each i=1,2,..n, there exists $\phi \in A^*$ such that $\phi(x_j) = \delta_{ij}$ (j=1,2,...n). Hence $z_i/\|\phi\| \in H \subset N$. Similarly some positive multiple of $w_i \in N$. Hence N absorbs the monomials, and therefore N absorbs all polynomials, i.e. N is absorbent.

If, for $p \in B$, $\sigma(p)$ denotes the sum of the moduli of the coefficients of p, then $\sigma(h) \leqslant 1$ for $h \in H$, so $\sigma(h_1 h_2 \dots h_r) \leqslant 1$ for $h_j \in H$, and hence $\sigma(u) \leqslant 1$ for $u \in N$. Thus if $p \in B$ is non-zero, $\sigma(p) > 0$, so some positive multiple of p does not belong to N. Hence the Minkowski functional of N is a norm, given by

$$\|p\|_N = \inf\{\lambda > 0 : p/\lambda \in N\} \quad (p \in B).$$

Since N is closed under multiplication, $\|pq\|_N \leqslant \|p\|_N \|q\|_N$. So $(B, \|.\|_N)$ is a (1)-normed algebra. (Also B is commutative and $\|1\|_N \leqslant 1$ since $1 \in N$, therefore $\|1\|_N = 1$, i.e. B is unital).

Now if $A \otimes_\lambda B$ is a (K)-normed algebra, then

$$\Big\|\sum_{i=1}^n \sum_{j=1}^n x_i y_j \otimes z_i w_j\Big\| \leqslant K\Big\|\sum_{i=1}^n x_i \otimes z_i\Big\| \, \Big\|\sum_{j=1}^n y_j \otimes w_j\Big\|$$

$$= K \sup_{\phi \in \text{Ball } A^*} \Big\|\sum_{i=1}^n \phi(x_i) z_i\Big\| \sup_{\psi \in \text{Ball } A^*} \Big\|\sum_{j=1}^n \psi(y_j) w_j\Big\|$$

$$\leqslant K.$$

Hence if $\phi \in \text{Ball } A^*$, $\Big\|\sum_{i=1}^n \sum_{j=1}^n \phi(x_i y_j) z_i w_j\Big\| \leqslant K < L$. Therefore

$\sum_{i=1}^n \sum_{j=1}^n \phi(x_i y_j) z_i w_j \in L.N.$ Therefore there exist

$\phi_k, \psi_k \in \text{Ball } A^*$ $(k=1,2,..m)$ such that

$$\sum_{i=1}^n \sum_{j=1}^n \phi(x_i y_j) z_i w_j = L \sum_{k=1}^m \lambda_k \Big(\sum_{i=1}^n \phi_k(x_i) z_i\Big)\Big(\sum_{j=1}^n \psi_k(y_j) w_j\Big) \text{ and}$$

$\sum_{k=1}^m |\lambda_k| \leqslant 1$. Equating coefficients of $z_i w_i$ in this polynomial identity, we get $\phi(x_i y_i) = L \sum_{k=1}^{m} \lambda_k \phi_k(x_i)\psi_k(y_i)$.

So $\sum_{i=1}^{n} \phi(x_i y_i) = L \sum_{k=1}^{m} \lambda_k \sum_{i=1}^{n} \phi_k(x_i)\psi_k(y_i)$. Therefore

$$|\phi(\sum_{i=1}^{n} x_i y_i)| \leq L \sum_{k=1}^{m} |\lambda_k| \sup_{\substack{\phi'\varepsilon \text{Ball } A^* \\ \psi'\varepsilon \text{Ball } A^*}} |\sum_{i=1}^{n} \phi'(x_i)\psi'(y_i)|.$$

Hence $\|\sum_{i=1}^{n} x_i y_i\| \leq L \|\sum_{i=1}^{n} x_i \otimes y_i\|_\lambda$, and we have a contradiction, so $A \otimes_\lambda B$ is not a (K)-normed algebra.

<u>Corollary</u> Since the algebra B of the above proof is commutative and unital, and since the algebra $l_\infty(\{B_n\})$ is commutative and unital if each B_n is, we have that if $A \otimes_\lambda B$ is a normed algebra for each commutative unital B, then A is injective.

CHAPTER FOUR

In this chapter we shall again be concerned with the question of whether the injective tensor product of two normed-algebras is a normed-algebra. We prove that this is the case for the tensor product $l_p \otimes_\lambda l_q$ (where either p or q $\leqslant 2$), and for the injective tensor product of two Banach-algebras which are $\mathcal{L}_1$ spaces.

Tensor Products of l_p Spaces

In this section l_p will always have pointwise multiplication and p will be $\geqslant 1$. We already know that the Banach algebras l_1 and l_∞ are 1-injective, hence $l_1 \otimes_\lambda l_q$ and $l_\infty \otimes_\lambda l_q$ are (1)-normed algebras for every q.

Now let $1 \leqslant p,q < \infty$. Let p' and q' satisfy $1/_p + 1/_{p'} = 1 = 1/_q + 1/_{q'}$. We may represent an element of $l_p \otimes_\lambda l_q$ uniquely as an infinite scalar matrix as follows.

Let $z = \sum_{r=1}^{n} x^{(r)} \otimes y^{(r)} \varepsilon\ l_p \otimes_\lambda l_q$.

Define $(a_{ij})_{i,j=1}^{\infty}$ by $a_{ij} = \sum_{r=1}^{n} x_i^{(r)} y_j^{(r)}$.

Then we have $z = \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} a_{ij} e^{(i)} \otimes e^{(j)})$, and

$\|z\|_{\lambda} = \sup_{\substack{\phi \varepsilon \text{Ball } l_{p'} \\ \psi \varepsilon \text{Ball } l_{q'}}} |\sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} \phi_i a_{ij} \psi_j)|$. We write $z \sim (a_{ij})$.

If also $w = \sum_{k=1}^{m} u^{(k)} \otimes v^{(k)}$ and $w \sim (b_{ij})$, i.e. $b_{ij} = \sum_{k=1}^{m} u_i^{(k)} \cdot v_j^{(k)}$,

$$\sum_{r=1}^{n} \sum_{k=1}^{m} x_i^{(r)} u_i^{(k)} y_j^{(r)} v_j^{(k)} = (\sum_{r=1}^{n} x_i^{(r)} y_j^{(r)})(\sum_{k=1}^{m} u_i^{(k)} v_j^{(k)}) = a_{ij} b_{ij}.$$

Hence $z.w \sim (a_{ij} b_{ij})_{i,j}$.

We now require the theory of finite tensor algebras in order to establish our result for $l_p \otimes_{\lambda} l_q$.

<u>Finite Tensor Algebras (§2,[11])</u>

If m and n are positive integers, we denote by K_m the set $\{1,2,\ldots,m\}$, and by K_m^n its n-fold Cartesian product. Let C_m^n denote the m^n-dimensional vector space of all scalar valued functions on K_m^n. We write C_m for C_m^1.

If $a \varepsilon C_m^n$, then we have

$$a(\beta_1, \beta_2, \ldots, \beta_n) = \sum_{(\alpha_1, \ldots, \alpha_n) \varepsilon K_m^n} a(\alpha_1, \ldots, \alpha_n) \delta_{\alpha_1 \beta_1} \cdots \delta_{\alpha_n \beta_n}$$

where δ is the Kronecker δ. Thus we may define the tensor algebra norm on C_m^n by

$$\|a\|_V = \inf\{ \sum_{r=1}^{r_o} |\lambda_r| \; : a(\beta_1,..,\beta_n) = \sum_{r=1}^{r_o} \lambda_r f_1^{(r)}(\beta_1)\ldots f_n^{(r)}(\beta_n)$$

where $\lambda_r \varepsilon k$ and $f_i^{(r)} \varepsilon C_m$ with $|f_i^{(r)}(\alpha)| \leqslant 1$ $(\alpha \varepsilon K_m)$,
$1 \leqslant i \leqslant n, 1 \leqslant r \leqslant r_o\}$.

We may identify C_m^n with its own dual by defining

$$\langle a,b\rangle = \sum_{\beta \varepsilon K_m^n} a(\beta)b(\beta) \qquad (a,b\varepsilon\, C_m^n).$$

This gives us the dual norm on C_m^n

$$\|a\|_{V*} = \sup\{|\langle a,b\rangle| : b \;\varepsilon\; C_m^n, \|b\|_V \leqslant 1\}$$
$$= \sup\{|\sum_{\beta \varepsilon K_m^n} a(\beta) f_1(\beta_1).f_2(\beta_2)\ldots.f_n(\beta_n)| : f_i \varepsilon\, C_m \text{ with}$$
$$|f_i(\alpha)| \leqslant 1 \;(\alpha \;\varepsilon\, K_m) \text{ for } 1 \leqslant i \leqslant n\}.$$

In Theorem 1.1 of [12], Littlewood gave estimates for these norms in the case $n = 2$. He showed that if $a \;\varepsilon\; C_m^2$, then

$$3^{\frac{1}{2}}\|a\|_{V*} \geqslant \sum_{i=1}^{m} (\sum_{j=1}^{m} |a_{ij}|^2)^{\frac{1}{2}} \quad (\text{writing } a_{ij} \text{ for } a(i,j)).$$

Hence $\|a\|_V = \sup\{|\langle a,b\rangle| \; b\varepsilon\; C_m^2, \|b\|_{V*} \leqslant 1\}$

$$\leqslant 3^{\frac{1}{2}} \sup\{|\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}b_{ij}| \; : \sum_{i=1}^{m}(\sum_{j=1}^{m}|b_{ij}|^2)^{\frac{1}{2}} \leqslant 1\}$$
$$= 3^{\frac{1}{2}} \sup_{i\varepsilon K_m} (\sum_{j=1}^{m} |a_{ij}|^2)^{\frac{1}{2}} \text{ by the Cauchy-Schwartz}$$

inequality and the fact that for each i

$(\sum_{j=1}^{m} |a_{ij}|^2)^{\frac{1}{2}} = |\sum_{j=1}^{m} a_{ij}x_j|$ for some $\{x_j\} \varepsilon$ Ball l_2. The inequality

$\|a\|_V \leqslant 3^{\frac{1}{2}} \sup_{i\,\varepsilon K_m} (\sum_{j=1}^{m}|a_{ij}|^2)^{\frac{1}{2}}$ is called Littlewood's inequality.

Littlewood also showed that if $a \in C_m^2$, then $2^{\frac{3}{4}}3^{\frac{1}{2}} \|a\|_{V^*} \geq (\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^{\frac{4}{3}})^{\frac{3}{4}}$. Hence

$$\|a\|_V = \sup \{|\langle a, b\rangle| : b \in C_m^2, \|b\|_{V^*} \leq 1\}$$

$$\leq 2^{\frac{3}{4}}3^{\frac{1}{2}} \sup \{|\sum_{i,j=1}^{m} a_{ij}b_{ij}| : (\sum_{i,j=1}^{m} |b_{ij}|^{\frac{4}{3}})^{\frac{3}{4}} \leq 1\}$$

$$\leq 2^{\frac{3}{4}}3^{\frac{1}{2}} (\sum_{i,j=1}^{m} |a_{ij}|^4)^{\frac{1}{4}} \quad \text{by Hölder's inequality.}$$

These results may be extended to C_m^n for general n, and we get for $a \in C_m^n$,

$$\|a\|_V \leq 3^{(n-1)/2} . \sup_{\beta_1 \in K_m} (\sum_{\beta_2,..,\beta_n \in K_m} |a(\beta_1,..,\beta_n)|^2)^{\frac{1}{2}}$$

and $\|a\|_V \leq 3^{(n-1)/2} . n^{(n+1)/2n} (\sum_{\beta \in K_m^n} |a(\beta)|^{2n/(n-1)})^{(n-1)/2n}$.

We now apply these ideas to tensor products of l_p spaces.

<u>Theorem 4.1</u> Let $1 \leq p < \infty$, $1 \leq q \leq 2$. Then $l_p \otimes_\lambda l_q$ is a $(3^{\frac{1}{2}})$-normed algebra, and $l_p \check{\otimes} l_q$ is a $(3^{\frac{1}{2}})$-Banach algebra.

Proof: Let p' and q' satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$.
Let z_1 and z_2 belong to $l_p \otimes_\lambda l_q$ and let $z_1 \sim (a_{ij}), z_2 \sim (b_{ij})$.
Then
$$\|z_1 z_2\| = \sup_{\substack{\phi \in \text{Ball } l_{p'} \\ \psi \in \text{Ball } l_{q'}}} |\sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} \phi_i a_{ij} b_{ij} \psi_j)|.$$

So it is enough to show that if $m \in \mathbb{P}$, $\phi \in$ Ball $l_{p'}$ and $\psi \in$ Ball $l_{q'}$ then

$$\left|\sum_{i,j=1}^{m} \phi_i a_{ij} b_{ij} \psi_j\right| \leq 3^{\frac{1}{2}} \|z_1\| \|z_2\| \quad \text{, since then}$$

$$\left|\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} \phi_i a_{ij} b_{ij} \psi_j\right| \leq 3^{\frac{1}{2}} \|z_1\| \|z_2\| \quad \text{for all } m_1, m_2, \phi \text{ and } \psi .$$

Given $\varepsilon > 0$, there exist scalars $\lambda_r, f_i^{(r)}$ and $g_i^{(r)}$ $(1 \leq r \leq n,\ i \in K_m)$ with $|f_i^{(r)}| \leq 1$, $|g_i^{(r)}| \leq 1$, such that

$b_{ij} = \sum_{r=1}^{n} \lambda_r f_i^{(r)} g_j^{(r)}$ for $1 \leq i,j \leq m$ and

$$\begin{aligned}
\sum_{r=1}^{n} |\lambda_r| - \varepsilon &\leq \|(b_{ij})_{i,j=1}^{m}\|_V \\
&\leq 3^{\frac{1}{2}} \sup_{i \in K_m} \Big(\sum_{j=1}^{m} |b_{ij}|^2 \Big)^{\frac{1}{2}} \quad \text{by Littlewood's inequality} \\
&\leq 3^{\frac{1}{2}} \sup_{i \in K_m} \sup_{\delta \in \text{Ball } l_{q'}} \Big|\sum_{j=1}^{m} b_{ij} \delta_j\Big| \quad \text{since Ball } l_2 \subset \text{Ball } l_{q'} \\
&\leq 3^{\frac{1}{2}} \sup_{\substack{\gamma \in \text{Ball } l_{p'} \\ \delta \in \text{Ball } l_{q'}}} \Big|\sum_{i,j=1}^{m} \gamma_i b_{ij} \delta_j\Big| \\
&\leq 3^{\frac{1}{2}} \|z_2\| .
\end{aligned}$$

$$\begin{aligned}
\text{Now } \Big|\sum_{i,j=1}^{m} \phi_i a_{ij} b_{ij} \psi_j\Big| &= \Big|\sum_{r=1}^{n} \sum_{i,j=1}^{m} \phi_i a_{ij} \lambda_r f_i^{(r)} g_j^{(r)} \psi_j\Big| \\
&\leq \sum_{r=1}^{n} |\lambda_r| . \Big|\sum_{i,j=1}^{m} \phi_i f_i^{(r)} a_{ij} g_j^{(r)} \psi_j\Big| \\
&\leq \Big(\sum_{r=1}^{n} |\lambda_r|\Big) \|z_1\|
\end{aligned}$$

since each $|f_i^{(r)}| \leq 1$ & $|g_j^{(r)}| \leq 1$.

It therefore follows that

$$\left|\sum_{i,j=1}^{m} \phi_i a_{ij} b_{ij} \psi_j\right| \leqslant 3^{\frac{1}{2}} \|z_1\| \|z_2\|.$$

The constant $3^{\frac{1}{2}}$ of the above result need not be the best possible. In fact $l_2 \otimes_\lambda l_2$ can be shown to be a (1)-normed algebra.

$\mathcal{L}_p$ Spaces

The definitions and background results as given here are taken from [13].

Definition : For $p \geqslant 1$ and $n \in \mathbb{P}$ we shall denote by l_p^n the space of sequences $\{x_r\}$ in l_p such that $x_r = 0$ for $r \geqslant n+1$. If X and Y are Banach spaces, then $d(X,Y) =$

$$\inf\{\|T\| \|T^{-1}\| : T \in B(X,Y) \text{ with } T \text{ invertible}\}.$$

So if X and Y are not isomorphic, $d(X,Y) = \infty$.
A Banach space X is called an $L_{p,\alpha}$ space ($1 \leqslant p \leqslant \infty, 1 \leqslant \alpha < \infty$) if for every finite dimensional subspace B of X there is a finite dimensional subspace E of X containing B, such that $d(E, l_p^n) \leqslant \alpha$, where n = dimension of E. X is called an $\mathcal{L}_p$ space if it is an $L_{p,\alpha}$ space for some $\alpha < \infty$.

For every positive measure space (μ,Σ), $L_p(\mu,\Sigma)$ is an $L_{p,\alpha}$ space for each $\alpha > 1$ ($1 \leq p \leq \infty$). In particular l_p is an $L_{p,\alpha}$ space for each $\alpha > 1$, although if $p \neq 2$ l_p is not an $L_{p,1}$ space. Also if K is compact Hausdorff, C(K) is an $L_{\infty,\alpha}$ space for each $\alpha > 1$. Conversely, every infinite-dimensional $L_{p,\alpha}$ space (for $1 \leq p < \infty$) has a complemented subspace isomorphic to l_p. Also there are no infinite-dimensional $L_{p,1}$ spaces for $1 \leq p < \infty$ and $p \neq 2$.

Every Hilbert space is an $L_{2,1}$ space, and every $L_{2,\alpha}$ space is isomorphic to a Hilbert space. These and other basic properties of $L_{p,\alpha}$ spaces are to be found in [14].

Definition : Let X and Y be Banach spaces, let $T \in B(X,Y)$ and let $1 \leq p < \infty$. Put

$$a_p(T) = \inf\{C>0 : (\sum_{i=1}^{n} \| T(x_i) \|^p)^{1/p} \leq C \sup_{\phi \in \text{Ball } X^*} (\sum_{i=1}^{n} |\phi(x_i)|^p)^{1/p} \ \forall x_1, .. x_n \in X, n \in \mathbb{P}\}.$$

If $a_p(T) < \infty$, we say T is p-absolutely summing.

The main result which we shall require is Grothendieck's inequality, which was proved in [15]. Our statement of the result is as in Theorem 2.1 of [13].

Theorem 4.2 Let $(a_{ij})_{i,j=1}^{n}$ be a scalar matrix, and let $M > 0$ satisfy $|\sum_{i,j=1}^{n} a_{ij}s_i t_j| \leqslant M$ $\forall$ scalars s_i and t_j with $|s_i| \leqslant 1$ and $|t_j| \leqslant 1$.

Then if $x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n \in$ any inner product space H,

$$|\sum_{i,j=1}^{n} a_{ij}(x_i, y_j)| \leqslant K\, M \sup_i \|x_i\| \sup_j \|y_j\| .$$

Here K is Grothendieck's constant. If $k = \mathbb{R}$, then $K \leqslant \sinh \pi/2$ and if $k = \mathbb{C}$, $K \leqslant 2 \sinh \pi/2$

As a corollary to this result we have :

Theorem 4.3 Let $(a_{ij})_{i,j}$ be an infinite scalar matrix such that $|\sum_{i,j=1}^{n} a_{ij}s_i t_j| \leqslant M$ whenever $|s_i| \leqslant 1$ and $|t_j| \leqslant 1$ for $i,j = 1,2,\ldots n$ and $n \in \mathbb{P}$. Let $(x_{ki})_{k,i}$ be an infinite matrix such that $(\sum_{k=1}^{\infty} |x_{ki}|^2)^{\frac{1}{2}} \leqslant C$ for each $i \in \mathbb{P}$. Then

$$(\sum_{k=1}^{\infty} (\sum_{j=1}^{\infty} |\sum_{i=1}^{\infty} x_{ki} a_{ij}|)^2)^{\frac{1}{2}} \leqslant K\, C\, M.$$

From this theorem it immediately follows that if $z = \sum_{r=1}^{r_0} b^{(r)} \otimes c^{(r)}$ belongs to $l_1 \otimes_\lambda l_1$, and ζ is the isometric imbedding of $l_1 \otimes_\lambda l_1$ in $B(l_\infty, l_1)$, so that $\zeta(z)(x) = \sum_{r=1}^{r_0} < x, b^{(r)} > J c^{(r)}$ $(x \in l_\infty)$, then $T = \zeta(z)$ is 2-absolutely summing and $a_2(T) \leq K \|z\|_\lambda$. For let $x^{(1)}, x^{(2)}, \ldots x^{(m)} \in l_\infty$. If $a_{ij} = \sum_{r=1}^{r_o} b_i^{(r)} c_j^{(r)}$ then $|\sum_{i,j=1}^{n} a_{ij} s_i t_j| \leq \|z\|_\lambda$ whenever $|s_i| \leq 1$ and $|t_j| \leq 1$ and $n \in \mathbb{P}$. Now

$$
\begin{aligned}
\left(\sum_{k=1}^{m} \|T(x^{(k)})\|^2\right)^{\frac{1}{2}} &= \left(\sum_{k=1}^{m} \|\sum_{r=1}^{r_o} \langle x^{(k)}, b^{(r)}\rangle c^{(r)}\|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{m} \left(\sum_{j=1}^{\infty} |\sum_{r=1}^{r_o} \langle x^{(k)}, b^{(r)}\rangle c_j^{(r)}|\right)^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{m} \left(\sum_{j=1}^{\infty} |\sum_{i=1}^{\infty} x_i^{(k)} a_{ij}|\right)^2\right)^{\frac{1}{2}} \\
&\leq K \|z\|_\lambda \sup_i \left(\sum_{k=1}^{m} |x_i^{(k)}|^2\right)^{\frac{1}{2}} \\
&\leq K \|z\|_\lambda \sup_{\phi \in \mathrm{Ball}\, l_1} \left(\sum_{k=1}^{m} |\langle \phi, x^{(k)}\rangle|^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

(In fact by Theorem 4.3 of [13], if X is any $L_{\infty,\alpha}$ space and Y is any $L_{p,\beta}$ space with $1 \leq p \leq 2$, then every $T \in B(X,Y)$ is 2-absolutely summing).

The next result is due to Pietsch. Our proof is effectively that in [13] .(The underlying field may as usual be either $\mathbb{R}$ or C).

<u>Theorem 4.4</u> Let X and Y be Banach spaces, and let $T \in B(X,Y)$ be 2-absolutely summing. Let $L = \mathrm{Ball}\, X^*$. Then there is a probability measure μ on L and an operator $S: L_2(\mu) \to Y$ such that $\|S\| = a_2(T)$ and $T = S_o J_o I$, where $I: X \to C(L)$ is the canonical isometry $I(x)(\phi) = \phi(x)$ $(x \in X, \phi \in L)$ and

$J : C(L) \to L_2(\mu)$ is the formal identity mapping.

Proof : Let $W = \{a_2(T)^2 \sum_{i=1}^{n} | I(x_i)|^2 : \sum_{i=1}^{n} \| T(x_i)\|^2$

$= 1, x_i \varepsilon X, n \varepsilon \mathbb{P}\}$.

Then $W \subset C_{\mathbb{R}}(L)$ (the space of continuous real valued functions on L). Let $N = \{f \varepsilon C_{\mathbb{R}}(L) : \sup_{\phi \varepsilon L} f(\phi) < 1\}$. W and N are convex, and N is open. It follows by the separation theorem and the Riesz representation theorem that there exists a real regular Borel measure ν on L such that

$$\int_L f d\nu \leqslant 1 \quad (f \varepsilon N)$$

and $$\int_L f d\nu \geqslant 1 \quad (f \varepsilon W).$$

If $f \varepsilon C_{\mathbb{R}}(L)$ is non-negative, then for $\lambda > 0$ $-f/\lambda \varepsilon N$, hence

$\int_L f d\nu \geqslant -\lambda$, so $\int_L f d\nu \geqslant 0$. Thus ν is a positive measure.

If $f \varepsilon C_{\mathbb{R}}(L)$ and $\| f \| < 1$, $|\int_L f d\nu| \leqslant 1$. Hence $\| \nu \| \leqslant 1$, and there exists α with $0 < \alpha \leqslant 1$ and a probability measure μ on L such that $\nu = \alpha\mu$.

Now if $x \varepsilon X$ and $T(x) \neq 0$, let

$g = a_2(T)^2 \, |I(x)|^2 / \| T(x)\|^2 \varepsilon W$. Then $1 \leqslant \int_L g d\nu \leqslant \int_L g d\mu$.

Therefore $\| T(x)\|^2 \leqslant a_2(T)^2 \int_L |I(x)|^2 d\mu$ and so

$$\| T(x)\| \leqslant a_2(T) \| J_o I(x)\|_2 \quad (x \varepsilon X).$$

Thus there exists $Q \in B(\overline{JI(X)},Y)$ such that $Q(JI(x)) = T(x)$ $(x \in X)$ and $\| Q\| \leqslant a_2(T)$. Now

$$\sum_{i=1}^{n} \| T(x_i)\|^2 \leqslant \| Q\|^2 \sum_{i=1}^{n} \| JI(x_i)\|^2$$

$$= \| Q\|^2 \int_L \sum_{i=1}^{n} |I(x_i)|^2 d\mu$$

$$\leqslant \| Q\|^2 \, \| \sum_{i=1}^{n} |I(x_i)|^2 \|.$$

Hence $\| Q\| = a_2(T)$. In the Hilbert space $L_2(\mu)$ there is a projection P of norm one onto $\overline{JI(X)}$. Let $S = Q_0P$. Then $\| S\| = \| Q\| = a_2(T)$ and $S_0J_0I(x) = T(x)$ $(x \in X)$.

Theorem 4.5 Let X and Y be (1)-Banach algebras, such that X is an $L_{1,\alpha}$ space and Y is an $L_{1,\beta}$ space. Then $X \otimes_\lambda Y$ is a $(K^2\alpha^3\beta^3)$-normed algebra and $X \check{\otimes} Y$ is a $(K^2\alpha^3\beta^3)$-Banach algebra.

Proof : Let $\sum_{s=1}^{s_0} x^{(s)} \otimes y^{(s)}$, $\sum_{t=1}^{t_0} w^{(t)} \otimes z^{(t)} \in X \otimes_\lambda Y$, each with norm $\leqslant 1$. Let $\alpha' > \alpha, \beta' > \beta$. There exists a finite dimensional subspace X_0 of X containing $x^{(s)}$ and $w^{(t)}$ for each s and t, and an isomorphism U of $l_1^{(m_0)}$ with X_0 ($m_0 = \dim X_0$) such that $\| U\| = 1$ and $\| U^{-1}\| \leqslant \alpha'$. There exists a finite dimensional subspace X_1 of X containing $U(e^{(i)}).U(e^{(k)})$ for $i,k = 1,2,..m_0$ and an isomorphism U_1 of $l_1^{(m_1)}$ with X_1 such that $\| U_1\| = 1$ and $\| U_1^{-1}\| \leqslant \alpha'$. There exists a finite dimensional subspace Y_0 of Y containing $y^{(s)}$ and $z^{(t)}$ for each s and t, and an isomorphism V of $l_1^{(n_0)}$ with Y_0 ($n_0 = \dim Y_0$) such that $\| V\| = 1$ and $\| V^{-1}\| \leqslant \beta'$. There exists a finite dimensional subspace Y_1 of Y containing $V(e^{(j)}).V(e^{(r)})$ for $j,r = 1,2,..n_0$ and an isomorphism V_1 of $l_1^{(n_1)}$ with X_1

such that $\|V_1\| = 1$ and $\|V_1^{-1}\| \leqslant \beta'$.

Let $U^{-1}(x^{(s)}) = a^{(s)}$, $V^{-1}(y^{(s)}) = b^{(s)}$, $U^{-1}(w^{(t)}) = c^{(t)}$ and $V^{-1}(z^{(t)}) = d^{(t)}$ $(s=1,2,..s_o,\ t=1,2,..t_o)$. Define a_{ikm} by

$$\{a_{ikm}\}_{m=1}^{m_1} = U_1^{-1}(U(e^{(i)}).U(e^{(k)})) \quad (i,k = 1,2,...m_o).$$

Define b_{jrn} by

$$\{b_{jrn}\}_{n=1}^{n_1} = V_1^{-1}(V(e^{(j)}).\ V(e^{(r)})) \quad (j,r = 1,2,...n_o).$$

Now $\displaystyle \Big\| \sum_{s=1}^{s_o} \sum_{t=1}^{t_o} x^{(s)}.w^{(t)} \otimes y^{(s)}.z^{(t)} \Big\|$

$$= \sup_{\substack{\theta \in \text{Ball } X^* \\ \eta \in \text{Ball } Y^*}} \Big| \sum_{s=1}^{s_o} \sum_{t=1}^{t_o} \theta(x^{(s)}.w^{(t)})\eta(y^{(s)}.z^{(t)}) \Big|$$

$$= \sup_{\substack{\theta \in \text{Ball } X^* \\ \eta \in \text{Ball } Y^*}} \Big| \sum_{s=1}^{s_o} \sum_{t=1}^{t_o} \theta_o U_1(\sum_{i,k=1}^{m_o m_1} a_i^{(s)} c_k^{(t)} \{a_{ikm}\}_m) \eta_o V_1(\sum_{j,r=1}^{n_o n_1} b_j^{(s)} d_r^{(t)} \{b_{jrn}\}_n) \Big|$$

$$\leqslant \sup_{\phi,\psi \in \text{Ball } l_\infty} \Big| \sum_{s=1}^{s_o} \sum_{t=1}^{t_o} \sum_{i,k,m=1}^{m_o} \phi_m a_i^{(s)} c_k^{(t)} a_{ikm} \sum_{j,r,n=1}^{n_o} \psi_n b_j^{(s)} d_r^{(t)} b_{jrn} \Big|$$

$$= \sup_{\phi,\psi \in \text{Ball } l_\infty} \Big| \sum_{k=1}^{m_o} \sum_{r=1}^{n_o} \gamma_{kr} \langle T(f^{(k)}), g^{(r)} \rangle \Big|$$

where $\gamma_{kr} = \sum_{t=1}^{t_o} c_k^{(t)} d_r^{(t)}$, and $f^{(k)} \in l_\infty^{m_o}$, $g^{(r)} \in l_\infty^{n_o}$

are given by $\quad f_i^{(k)} = \sum_{m=1}^{m_1} \phi_m a_{ikm} \quad (i,k = 1,2,...m_o)$

$$g_j^{(r)} = \sum_{n=1}^{n_1} \psi_n b_{jrn} \quad (j,r = 1,2,...n_o)$$

and $T \in B(l_\infty, l_1)$ is given by $T(u) = \sum_{s=1}^{s_o} \langle u, a^{(s)} \rangle b^{(s)} \quad (u \in l_\infty)$.

For each k and i, $|f_i^{(k)}| = |\sum_{m=1}^{m_1} \phi_m a_{ikm}|$

$$\leqslant \sum_{m=1}^{m_1} |a_{ikm}| \quad \text{since } \phi \in \text{Ball } l_\infty$$

$$= \| U_1^{-1}(U(e^{(i)}).\ U(e^{(k)}))\|$$

$$\leqslant \| U_1^{-1}\| \, \| U(e^{(i)})\| \, \| U(e^{(k)})\|$$

$$\leqslant \alpha'.$$

Thus $\|f^{(k)}\|_\infty \leqslant \alpha'$ for each k, and similarly $\| g^{(r)}\|_\infty \leqslant \beta'$ for each r. By the remarks following Theorem 4.3, T is 2-absolutely summing with $a_2(T) \leqslant K\| \sum_{s=1}^{s_o} a^{(s)} \otimes b^{(s)} \|_\lambda$

$$\leqslant K\alpha'\beta'.$$

By Theorem 4.4 there exists a Hilbert space H, and operators $S \in B(H,l_1)$ with $\|S\| = a_2(T)$ and $R \in B(l_\infty, H)$ with $\|R\| \leqslant 1$, such that $T = S_oR$. We have

$$<T(f^{(k)}),g^{(r)}> \;=\; <S_oR(f^{(k)}),g^{(r)}>$$

$$= (R(f^{(k)}),S^*(g^{(r)}))$$ where S^* is the adjoint of S. Also if $|\sigma_k| \leqslant 1$ $(k = 1,2,..m_o)$ and $|\tau_r| \leqslant 1$ $(r = 1,2,...n_o)$

$$|\sum_{k=1}^{m_o} \sum_{r=1}^{n_o} \gamma_{kr}\sigma_k\tau_r| \leqslant \| \sum_{t=1}^{t_o} c^{(t)} \otimes d^{(t)} \|_\lambda \leqslant \alpha'\beta'.$$

Hence by Grothendieck's inequality,

$$|\sum_{k=1}^{m_o} \sum_{r=1}^{n_o} \gamma_{kr} <T(f^{(k)}),g^{(r)}>|$$

$$= |\sum_{k=1}^{m_o} \sum_{r=1}^{n_o} \gamma_{kr} (R(f^{(k)}),\ S^*(g^{(r)}))|$$

$$\leqslant K\alpha'\beta' \sup_k \| R(f^{(k)})\| \sup_r \| S^*(g^{(r)})\|$$

$$\leqslant \quad K\,\alpha'\beta'\alpha'\,a_2(T)\beta'$$

$$\leqslant \quad K^2\alpha'^3\,\beta'^3 \quad .$$

Hence $\left\| \sum_{s=1}^{s_0} \sum_{t=1}^{t_0} x^{(s)} w^{(t)} \otimes y^{(s)} z^{(t)} \right\| \leqslant K^2 \alpha^3 \beta^3$,

and the proof is complete.

Suppose now that $l_1 = \{\{x_n\}_{n=0}^{\infty} : x_n \in k, \sum_{n=0}^{\infty} |x_n| < \infty\}$

and let l_1 be equipped with any bounded multiplication. By our remarks at the beginning of this section, l_1 is an $L_{1,\alpha}$ space for each $\alpha > 1$, and it follows that $l_1 \otimes_\lambda l_1$ is a normed-algebra. In particular, if l_1 is equipped with convolution multiplication, then $l_1 \otimes_\lambda l_1$ is a (K^2)-normed algebra.

However $l_1 \otimes_\lambda l_1$ with convolution multiplication is not a (1)-normed algebra. For as in the first section of this chapter we may represent an element of $l_1 \otimes_\lambda l_1$ by an infinite matrix $(a_{ij})_{i,j=0}^{\infty}$. Then if $w \sim (a_{ij})$ and $z \sim (b_{ij})$, $w.z \sim (a_{ij})*(b_{ij})$ where $*$ represents matrix convolution.
First take $k = \mathbb{R}$,

let $w \sim \begin{bmatrix} 1 & 1 & 0 & . \\ 1 & -1 & 0 & . \\ 0 & 0 & & \\ . & . & & \end{bmatrix}$ and let $z \sim \begin{bmatrix} 1 & 0 & 1 & 0 & . \\ 0 & 0 & 0 & 0 & . \\ 1 & 0 & -1 & 0 & . \\ 0 & 0 & 0 & & \\ . & . & . & & \end{bmatrix}$.

Then $\|w\|_\lambda = 2 = \|z\|_\lambda$.

$$\text{Now } w.z \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & \cdot \\ 1 & -1 & 1 & -1 & 0 & \cdot \\ 1 & 1 & -1 & -1 & 0 & \cdot \\ 1 & -1 & -1 & 1 & 0 & \cdot \\ 0 & 0 & 0 & 0 & & \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}$$

Therefore $\|w.z\|_\lambda = 8$.

Now take k to be the complex field

$$\text{Let } z_1 \sim \begin{bmatrix} 1 & 1 & 1 & 0 & \cdot \\ 1 & w & w^2 & 0 & \cdot \\ 0 & 0 & 0 & & \\ \cdot & \cdot & \cdot & & \end{bmatrix}, \; w^3=1, w \neq 1. \text{ Then } \|z_1\|_\lambda = 4.$$

$$\text{Let } z_2 \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & w & 0 & 0 & w^2 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

Then $\|z_2\|_\lambda = \|z_1\|_\lambda = 4$ and

$$z_1.z_2 \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \cdot \\ 1 & w & w^2 & 1 & w & w^2 & 1 & w & w^2 & 0 & \cdot\cdot \\ 1 & 1 & 1 & w & w & w & w^2 & w^2 & w^2 & 0 & \cdot \\ 1 & w & w^2 & w & w^2 & 1 & w^2 & 1 & w & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \end{bmatrix}$$

By our remarks on finite tensor algebras, we have

$$\sqrt{3}\ \| z_1 z_2 \|_\lambda \geq 9\sqrt{4} = 18.$$

We may similarly construct $z_3, z_4, \ldots, z_n \in l_1 \otimes_\lambda l_1$ such that $\| z_i \|_\lambda = \| z_1 \|_\lambda = 4$ and

$$\sqrt{3} \| z_1 z_2 \ldots z_n \|_\lambda \geq 3^n 2^{n/2}.$$ Now if $l_1 \otimes_\lambda l_1$ is (1)-normed,

then $3^n 2^{n/2} \leq \sqrt{3}\ 4^n$, i.e. $18^n \leq 3.16^n$ for all n. This is a contradiction.

CHAPTER FIVE

In this chapter we shall discuss measures orthogonal to injective tensor products of uniform algebras. Throughout this chapter the scalar field will be C.

Brian Cole has shown that if A is the bidisc algebra (the space of continuous functions on $\overline{\Delta}^2$ which are analytic in Δ^2, where Δ is the open unit ball in C), then any measure μ which is orthogonal to A has a unique decomposition

$$\mu = \sigma + \tau + \nu$$

where σ is concentrated on E x T, with E σ-compact and m(E) = O, τ is concentrated on T x F, with F σ-compact and m(F) = O, and $\nu \ll$ a representing measure for some point of Δ^2, and σ, τ and ν are orthogonal to A. Here m represents Lebesgue measure on the unit circle T.

In [16], Otto Bekken obtained a Cole-type decomposition of orthogonal measures for the algebras A(U x V) and $R(K_1 \times K_2)$. By A(U x V) we mean the algebra of continuous functions on $\overline{U} \times \overline{V}$ which are analytic in U x V, where U and V are bounded open subsets of C. $R(K_1 \times K_2)$ is the uniform closure on $K_1 \times K_2$ of the rational functions with singularities off

$K_1 \times K_2$, where K_1 and K_2 are compact sets in C.
Bekken's results state that if μ is a measure orthogonal to $A(U \times V)$ then μ has a unique decomposition

$\mu = \sigma + \tau + \nu$ where σ, τ and ν are orthogonal to $A(U \times V)$ and σ is concentrated on $E \times \partial V$, with E a nullset for $A(U)^{\perp}$,

τ is concentrated on $\partial U \times F$, with F a nullset for $A(V)^{\perp}$, and ν belongs to the band of measures generated by the representing measures for points of $U \times V$.

If μ is a measure orthogonal to $R(K_1 \times K_2)$ then μ has a unique decomposition

$\mu = \sigma + \tau + \nu$ where σ, τ and ν are orthogonal to $R(K_1 \times K_2)$ and σ is concentrated on $E \times K_2$, with E a nullset for $R(K_1)^{\perp}$,

τ is concentrated on $K_1 \times F$, with F a nullset for $R(K_2)^{\perp}$, and ν belongs to the band of measures generated by the representing measures for $Q_1 \times Q_2$, where Q_i is the set of non-peak points for $R(K_i)$.

We shall obtain an analogous decomposition for the injective tensor product of a uniform algebra fulfilling certain conditions, with A(U), where U is bounded and open in C. (A(U) is the space of continuous functions on $\overline{U}$ which are analytic in U, regarded as a uniform algebra on ∂U).

If A is a uniform algebra on a compact Hausdorff space X, we denote the space of (regular Borel complex) measures on X by M(X). We denote the set of measures orthogonal to A by $A^{\perp}$. A Borel set E in X is a nullset for $A^{\perp}$ if for every $\mu \in A^{\perp}$ $\mu_E = 0$, where μ_E is the restriction of μ to E. If $\phi \in \Phi_A$, we write M_ϕ for the set of representing measures for ϕ. We say a measure μ is completely singular if it is singular to M_ϕ for every $\phi \in \Phi_A$.

We say a subset E of X is a peak set for A if there exists $f \in A$ such that $f(x) = 1$ $(x \in E)$ and $|f(x)| < 1 (x \in X \setminus E)$. A point x of X is called a peak point for A if $\{x\}$ is a peak set for A. A peak set E satisfying $A|_E = C(E)$ is called a peak interpolation set.

We now state two results from the theory of uniform algebras. The first result is a simplified form of Lemma 2.2.7 of [16].

<u>Lemma 5.1</u> Let A be a uniform algebra on a compact Hausdorff space X. Let $\phi_1, \phi_2, \ldots \phi_m \in \Phi_A$ have representing measures $\mu_1, \mu_2, \ldots \mu_m$. Let E be an F_σ set such that $M_{\phi_i}(E) = 0$ for $i = 1,2,\ldots m$. Then there is a sequence $\{f_n\}_{n=1}^{\infty}$ in A such that $\|f_n\| \leq 1$, $f_n \to 1$ pointwise on E, and $f_n \to 0$ weak* in $L_\infty(\mu_i)$ for each i.

<u>Theorem 5.2</u> (Theorem 2.12.7, [17]). Let A be a uniform algebra on a compact metric space X, and let E be a closed subset of X. E is a peak set for A if and only if $\mu_E \in A^{\perp}$ for each

$\mu \in A^{\perp}$. E is a peak interpolation set if and only if $\mu_E = 0$ for all $\mu \in A^{\perp}$.

We now discuss the theory of bands.

Definition: Let X be a compact Hausdorff space. A norm-closed linear subspace M of M(X) is a band if whenever $\mu \in M$ and $\lambda << |\mu|$, $\lambda \in M$.

For an arbitrary subset S of M(X), we write S' for the set of measures singular to every measure in S. We easily have that S' is a band.

The following result is well known.

<u>Theorem 5.3</u> Let S be a band. Then $M(X) = S \oplus S'$.

Proof: Let $\mu \in M(X)$. Let $K = \sup\{|\mu|(G) : \mu_G \in S\}$. Choose F_n such that $\mu_{F_n} \in S$ and $|\mu|(F_n) \to K$. Let $F = \bigcup_{n=1}^{\infty} F_n$. Then $\mu_F << \sum_{n=1}^{\infty} |\mu_{F_n}|/2^n \in S$. And $|\mu|(F) \geq |\mu|(F_n)$ for all n, therefore $|\mu|(F) = K$. Also if $\mu_G \in S$, $\mu_{F \cup G} << |\mu_F| + |\mu_G| \in S$. Hence $|\mu|(F \cup G) = |\mu|(F)$, so $|\mu|(G \setminus F) = 0$. Now let $\gamma \in S$. There exists a subset H of X such that $\mu_H << |\gamma|$ and $\mu_{X \setminus H} \perp |\gamma|$. So $\mu_H \in S$ & $\mu_{H \setminus F} = 0$. So $\mu_{X \setminus F} \perp \gamma$, and $\mu = \mu_F + \mu_{X \setminus F} \in S + S'$.

Corollary If S is a band, then $S'' = S$. For if $\mu \varepsilon S''$ we have $\mu = \nu + \eta$ where $\nu \varepsilon S$, $\eta \varepsilon S'$. Then $\eta = \mu - \nu \varepsilon S''$. Therefore $\eta = 0$ and $\mu \varepsilon S$.

Corollary If S is an arbitrary subset of M(X), S'' is the smallest band containing S.

Now let A and B be uniform algebras on compact metric spaces X and Y respectively. Let $C = A \check{\otimes} B$, regarded as a uniform algebra on X x Y. Let $S_1 = \{\lambda \varepsilon M(X \times Y): \lambda$ is concentrated on $E \times Y$, E a nullset for $A^{\perp}\}$. Then S_1 is a band, and

$S_1' = \{\lambda \varepsilon M(X \times Y): |\lambda|(E \times Y) = 0$ for all nullsets E for $A^{\perp}\}$.

We observe that if λ is concentrated on $E \times Y$, when E is a nullset for $A^{\perp}$, then we may suppose without loss of generality that E is σ-compact. For there is a σ-compact subset Ω of $E \times Y$ such that $|\lambda|((E \times Y) \setminus \Omega) = 0$. Then if p is the projection of X x Y onto X, $p(\Omega)$ is σ-compact and $\Omega \subset p(\Omega) \times Y \subset E \times Y$. So λ is concentrated on $p(\Omega) \times Y$.

We also define a band

$S_2 = \{\mu \varepsilon M(X \times Y) : \mu$ is concentrated on $X \times F$, F a nullset for $B^{\perp}\}$, then

$S_2' = \{\mu \varepsilon M(X \times Y) : |\mu|(X \times F) = 0$ for all nullsets F for $B^{\perp}\}$.

We now have $M(X) = S_1 \oplus S_1'$

and $M(X) = S_2 \oplus S_2'$

so $M(X) = S_1 + S_2 + (S_1' \cap S_2')$

and in fact

<u>Lemma 5.4</u> $C^{\perp} = (C^{\perp} \cap S_1) \oplus (C^{\perp} \cap S_2) \oplus (C^{\perp} \cap S_1' \cap S_2')$.

Proof: Suppose $\lambda \varepsilon C^{\perp} \cap S_1$, $\mu \varepsilon C^{\perp} \cap S_2$, $\nu \varepsilon C^{\perp} \cap S_1' \cap S_2'$ and $\lambda + \mu + \nu = 0$.

Suppose λ is concentrated on $E \times Y$, E σ-compact and a nullset for $A^{\perp}$ and μ is concentrated on $X \times F$, F σ-compact and a nullset for $B^{\perp}$. Then $|\nu|((E \times Y) \cup (X \times F)) = 0$, and so $\nu = 0$. Now we have $\lambda = -\mu$ is concentrated on $E \times F$. There exist closed sets E_n and F_n such that $E \times F = \bigcup_{n=1}^{\infty} E_n \times F_n$.

For each n, E_n is a peak interpolation set for A, and F_n is a peak interpolation set for B. Therefore $E_n \times F_n$ is a peak interpolation set for C, and so a nullset for $C^{\perp}$. Therefore $|\lambda|(E_n \times F_n) = 0$, and hence $|\lambda|(E \times F) = 0$. So $\lambda = \mu = 0$.

Now let $Q_1 = \Phi_A \setminus P_A$ be the non-peak points for A, and let $Q_2 = \Phi_B \setminus P_B$ be the non-peak points for B. Let M be the band generated by the representing measures for points of $Q_1 \times Q_2$. We wish to find conditions such that $C^{\perp} \cap S_1' \cap S_2'$ will equal $C^{\perp} \cap M$. We always have the following.

<u>Lemma 5.5</u> (Lemma 3.1.7, [16]). $M \subset S_1' \cap S_2'$.

Proof: We show that if ν is a representing measure for a point (ϕ,ψ) in $Q_1 \times Q_2$, then $\nu \in S_1' \cap S_2'$. Let E be a nullset for $A^{\perp}$. We may suppose E is σ-compact. Let $E = \bigcup_{i=1}^{\infty} E_i$, where E_i is compact. Each E_i is a peak interpolation set for A. Let $f \in A$ peak on E_i.

Then $\phi(f)^n = \int f^n \otimes 1 \, d\nu \to \nu(E_i \times Y)$. Since ϕ is a non-peak point, $\phi(f)^n \to 0$. Hence $\nu(E_i \times Y) = 0$ for each i, and so $\nu(E \times Y) = 0$ and $\nu \in S_1'$. Similarly $\nu \in S_2'$.

Definition: If A is a uniform algebra on a compact Hausdorff space, then ϕ and $\psi \in \Phi_A$ are in the same part if $\|\phi-\psi\| < 2$.

Definition: If A is a uniform algebra on a compact Hausdorff space X, then a band $\mathcal{M} \subset M(X)$ is a reducing band for A if whenever $\mu \in A^{\perp}$ decomposes $\mu = \mu_a + \mu_s$ relative to $\mathcal{M}$, μ_a and $\mu_s \in A^{\perp}$.

Now take A to be a uniform algebra on a compact Hausdorff space X, and let R be a Borel subset of $\Phi_A \setminus P_A$, the set of non-peak points for A. We denote by M_R the band of measures generated by the representing measures for points of R.

Lemma 5.6 (Proposition 2.1.12,[16]). With A,X and R as above, M_R is a reducing band for A.

Proof: Let $\mu \in A^{\perp}$ have Glicksberg-Wermer decomposition $\mu = \mu_o + \sum_{n=1}^{\infty} \mu_n$ where μ_o is completely singular and $\mu_n << \lambda_n$ where λ_n is a representing measure for some non-peak point $\phi_n \in \Phi_A$. The μ_n's are pairwise mutually singular and $\mu_n \in A^{\perp}$, n= 0,1,2,.... . Let D consist of those indices n for which ϕ_n belongs to the same part as some point in R. For each $n \in D$, there exists a representing measure ν_n for a point in R such that $\lambda_n << \nu_n$ (Corollary 6.1.2, [17]). Let $\mu_a = \sum_{n \in D} \mu_n \in M_R$. For each $n \notin D$, λ_n is singular to all representing measures for points in R (Theorem 6.2.2,[17]), so $\mu_n \in M_R'$. Let $\mu_s = \mu_o + \sum_{n \notin D} \mu_n \in M_R'$. The decomposition $\mu = \mu_a + \mu_s$ is the decomposition of μ relative to M_R and μ_a and $\mu_s \in A^{\perp}$, so M_R is reducing.

We can now obtain our decomposition in the desired form.

Theorem 5.7 Let A be a uniform algebra on a compact metric space X, such that A has no completely singular annihilating measures except zero, $A^{\perp}$ is (norm) separable, and A has countably many non peak point parts. Let Q be the set of non-peak points for A. Let U be a bounded open subset of the complex plane, and let $C = A \check{\otimes} A(U)$. Let $\nu \in C^{\perp}$ satisfy

$$|\nu|(E \times \partial U) = 0 \quad \text{if } E \text{ is a nullset for } A^{\perp},$$

$$|\nu|(X \times F) = 0 \quad \text{if } F \text{ is a nullset for } A(U)^{\perp}.$$

Then $\nu \in M = M_{Q \times U}$.

Proof : Since M is a reducing band for C, we may assume that $\nu \in M'$. We show first that if $g \in A$ and $h \in C(\partial U)$, then

$$\int g(x)h(z)d\nu(x,z) = 0 .$$

By Lemma 1.1 of [18], $C(\partial U)$ is the closed linear span of A(U) and the functions $1/(z-z_0)$ ($z_0 \in U$). It is therefore enough to show that

$$\int g(x)/(z-z_0)d\nu(x,z) = 0 \quad (g \in A,\ z_0 \in U) .$$

Define $\lambda \in M(X)$ by $\lambda(E) = \int_{E \times \partial U} 1/(z-z_0)d\nu(x,z)$.

Then if E is a nullset for $A^{\perp}$, $\lambda(E) = 0$. Since A has no non-zero completely singular annihilating measures and countably many non-trivial parts, it follows that $\lambda \in M_Q$. Hence there exist $\alpha_i \geq 0$, and $\phi_i \in Q$ with representing measures μ_i such that $\lambda << \mu = \sum_{i=1}^{\infty} \alpha_i \mu_i$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. There exists $k \in L_1(\mu, X)$ such that $d\lambda = kd\mu$.

Therefore $\sum_{i=1}^{\infty} \alpha_i \int |k(x)| d\mu_i(x) = \int |k(x)| d\mu(x) < \infty$.

Given $\varepsilon > 0$, there exists $n_o \in \mathbb{P}$ such that

$$\sum_{i=n_o+1}^{\infty} \alpha_i \int |k(x)| d\mu_i(x) < \varepsilon / \|g\| .$$

Now let $\tau \in M(\partial U)$ be a representing measure for z_o. Then $\mu_i \otimes \tau \in M(X \times \partial U)$ is a representing measure for (ϕ_i, z_o). ν is orthogonal to $M_{(\phi_i, z_o)}$ for $i = 1,2,\ldots n_o$. It follows from Lemma 2.7.4 of [17]that there exists an F_σ set E in $X \times \partial U$ such that ν is concentrated on E, and $M_{(\phi_i, z_o)}(E) = 0$ for $i = 1,2,\ldots . n_o$. By Lemma 5.1, there is a sequence $\{f_n\}$ in C such that $\|f_n\| \leq 1$, $f_n \to 1$ pointwise on E, and $f_n \to 0$ weak* in $L_\infty(\mu_i \otimes \tau)$ for $i = 1,2,\ldots,n_o$.

Now the function $g(x)(f_n(x,z) - f_n(x,z_o))/(z-z_o)$ is in C.

Therefore $\int_{X \times \partial U} g(x) \; f_n(x,z)/(z-z_o) d\nu(x,z)$

$$= \int_{X \times \partial U} g(x) \; f_n(x,z_o)/(z-z_o) d\nu(x,z)$$

$$= \int_X g(x) \; f_n(x,z_o) d\lambda(x).$$

Since ν is concentrated on E,

$$\int g(x) \; f_n(x,z)/(z-z_o) d\nu(x,z) \to \int g(x)/(z-z_o) d\nu(x,z).$$

But $\left| \int_X g(x) \; f_n(x,z_o) d\lambda(x) \right|$

$$= \left| \sum_{i=1}^{\infty} \alpha_i \int_X g(x) k(x) f_n(x,z_o) d\mu_i(x) \right|$$

$$\leq \sum_{i=1}^{n_o} \alpha_i \left| \int_{\partial U} \left(\int_X g(x) k(x) f_n(x,z) d\mu_i(x) \right) d\tau(z) \right| + \|g\| \varepsilon / \|g\|$$

$$\to 0 + \varepsilon .$$

It follows that $\int g(x)/(z-z_0)d\nu = 0$, and hence

$$\int g(x)h(z)d\nu = 0 \text{ for all } g \in A,\ h \in C(\partial U).$$

We now disintegrate ν, and we get

$$\int_{X\times\partial U} F(x,z)d\nu(x,z) = \int_{\partial U} \left(\int_X F(x,z)d\eta_z(x)\right)d\sigma(z).$$

The mapping $z \to \eta_z$ is weak* measurable, and σ is the compression of $|\nu|$ onto ∂U, i.e. $\sigma(F) = |\nu|(X\times F)$ for each Borel set F. Therefore $\sigma(F) = 0$ for all nullsets F for $A(U)^{\perp}$, and so $\sigma \in M_U$. We now have

$$\int_{\partial U} \left(\int_X g(x)d\eta_z(x)\right)h(z)d\sigma(z) = 0 \text{ for all } g \in A,\ h \in C(\partial U).$$

It follows that $\eta_z \in A^{\perp}$ for σ-almost all z, since A is separable.

Since $A^{\perp}$ is separable, we may choose a countable dense set $\{\mu_n : n \in \mathbb{P}\}$ in $A^{\perp}$. Let $\gamma = \sum_{n=1}^{\infty} (1/2^n\|\mu_n\|)|\mu_n| \in M_Q$. Then if $\mu \in A^{\perp}$, and E is a Borel set, $\gamma(E) = 0$ implies $|\mu_n|(E) = 0$ for all $n \in \mathbb{P}$, and so $\mu(E) = 0$. Therefore $\mu << \gamma$ for all $\mu \in A^{\perp}$.

We thus have that $\eta_z << \gamma$ for σ- almost all z. Now define H by $d\eta_z(x) = H(x,z)d\gamma(x)$. Also define h such that $h(x,z)H(x,z) = |H(x,z)|$ and $|h(x,z)| = 1$.

Then $\int_{\partial U} \left(\int_X |H(x,z)|d\gamma(x)\right)d\sigma(z)$

$$= \int_{\partial U} \left(\int_X h(x,z)d\eta_z(x)\right)d\sigma(z)$$

$$= \int_{X\times\partial U} h(x,z)d\nu(x,z) < \infty .$$

And $d\nu(x,z) = H(x,z)d\gamma(x)d\sigma(z)$,

so $\nu << \gamma \otimes \sigma \in M$.

Combining this result with Lemmas 5.4 and 5.5, we now have

<u>Theorem 5.8</u> Let A be a uniform algebra on a compact metric space X, such that A has no completely singular annihilating measures except zero, $A^{\perp}$ is (norm) separable, and A has countably many non-trivial parts. Let Q be the set of non-peak points for A. Let U be a bounded open set in the complex plane, and let $C = A \check{\otimes} A(U)$. Then any measure μ in $C^{\perp}$ has a unique decomposition

$$\mu = \sigma + \tau + \nu \text{ , where } \sigma, \tau \text{ and } \nu \in C^{\perp},$$

σ is concentrated on $E \times \partial U$, with E a nullset for $A^{\perp}$, τ is concentrated on $X \times F$, with F a nullset for $A(U)^{\perp}$, and ν belongs to the band of measures generated by the representing measures for points of $Q \times U$.

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