Practical Activities

in

Mathematics Learning

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στους γονείς μου,
Αθηνὰ και Αλὲξανδρο
και στον αδελφὸ μου,
Νὶκο

Declaration

This thesis is my work.

I have not submitted it, or part of it, in a previous application for a degree.

N.A. Triadafillidis

August 1993

Abstract

The effectiveness of activity-based learning has been discussed by many authors over the past 4,000 years. Despite the suggested strength of a 'hands-on' approach, learning in secondary school mathematics classes has become abstract and analytic. Students are taught out-of-context and are seldom given the opportunity to act upon their educational experiences.

To evaluate the effectiveness of practical activities in classroom situations, materials were developed by the author. These concerned areas from the mathematics syllabus of the first and second years of secondary school. Data were collected from urban and rural schools in both Greece and Scotland. The students' performance on the practical activities was investigated in terms of the cognitive difficulty of the introduced mathematical concepts. Culture was also investigated as a differentiating factor in the performance and attitudes of the students.

The results of the study indicated a differentiation in performance and attitudes between students of the two countries, in favour of the Greek students. In some tasks first grade students performed better than the second grade ones, in both countries. Cultural differences, as these are reflected in the educational systems, indicated the existence of a 'classroom culture'. This 'classroom culture' appears as the ethos of a school class, created and sustained by the teacher and the students. In this respect more similarities were found between Greece and Scotland rather than differences. These similarities address the formation of values in the mathematics classroom about the nature of mathematics, about

understanding mathematics, about the role of the teacher and about education in general.

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Chapter 1

Practical Activities in Mathematics Learning

1.1   A Historical Retrospective

Eastern mathematics started in countries with big river-valleys (Egypt, Babylonia, India, China) serving the economic and technological needs of the time (Struik 1982, Gheverghese 1992). The Ahmes and Moscow papyri (1650 BC and 1850 BC) give evidence of the problems that most concerned the Egyptians: distributions of loaves and wine, remuneration of temple personnel, feeding animals, land surveying, mensuration, volumes of granaries and pyramids, astronomical calculations. Some of these practical problems, though, presented theoretical interests that were pursued by the scholars of the time. The algebra was rhetorical (verbally expressed, detailed instructions) so that any theoretical motivation was hidden behind rules for computation (Gheverghese 1992). The Egyptian scholars, responsible for the teaching of mathematics, encouraged learning through play and with activities that corresponded to the practical character of the subject (Yannicopoulos 1989).

Babylonian mathematics represented a practical tool as well, rather than an intellectual pursuit. Problems were similar to those handled by the Egyptians (Struik 1982). There was a small elite class that developed a non-practical pursuit of mathematical science. The rhetorical algebra of the Egyptians became syncopated, with abbreviations for recurring quantities and operations (Gheverghese

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1992). The extent to which the teaching methods of the priests, responsible for the teaching of sciences, were practically based is doubtful (Freudenthal 1973). Struik (1982) suggests that the rationale of the Indian and Chinese mathematics teaching had a similar orientation to that of the Egyptians and the Babylonians.

The transcending of the utilitarian origins of mathematics appears most clearly in Ancient Greece. The question of 'why?' was added to that of 'how?' of the East (Struik 1982). A knowledge of mathematics became a prerequisite for the study of philosophy, as the Greeks were striving to identify fundamental principles that would bring some order in the 'chaos' of existence. All activities have an end, a completion (teleological dimension of activity, Danassis 1985a). In mathematics, being a theoretical science (Aristotle, in Apostle 1952), this end is found in the simple activity of knowing, of the acquisition of scientific truth (Burnet 1905). On the other hand, practical sciences are closely related to the actual construction of a product beyond the activity itself. Then knowledge is sought in so far as it is useful to that construction (opcit). Despite the ideological orientation given to mathematics, its practical applications were also appreciated. For Aristotle, theoretical philosophy acquired value and interest only if it was related to practical philosophy (Danassis 1985a).

In the two great schools of antiquity, Plato's Academy (387 BC - 529 AD) and Aristotle's Lyceum (335 BC - 86 BC), mathematics was taught beginning from physical objects (Anapolitanos 1985, Yannicopoulos 1989). Plato, despite the clear theoretical and philosophical orientation of his Academy, believed that the abstraction of real knowledge could be achieved either directly from the awakening of the senses, or indirectly through the stimulation of imagination using the Socratic dialectic method (Anapolitanos 1985). In Aristotle's Lyceum the prevailing belief was that the building of the intellect was impossible without the experience of objects (Yannicopoulos 1989). This belief was reflected in the empiriocratic and scientific character of his school. Moreover Plato advised the teachers of the time to use play as a method for teaching, as the Egyptians were doing (opcit). In the Republic he says that knowledge that is imposed never takes root in the soul of a child (Yannicopoulos 1983). A similar view was adopted by Aristotle in his Ethics

(opcit). The teaching practice though, despite these suggestions, relied heavily on memorisation (Yannicopoulos 1989).

In contrast, the Roman view was that the development of human nature should be guided by tradition rather than reason (Wilkins 1914). Education took place in the home, based essentially on apprenticeship. After the second half of the third century BC and during the Hellenistic period (until 313 AD), the Greek influence became apparent. The teaching of mathematics was largely performed by Greeks. Roman education was still based though in the teaching of the 'trivium' (grammar, rhetoric, logic) with the 'quadrivium' having a secondary role (arithmetic, geometry, music, astronomy). As Cicero reported (1st century BC), geometry and arithmetic had many practical applications in land surveying, in military applications, in navigation and astronomy (Bonner 1977). Quintilian (1st century AD) suggested that what enters through the ears stirs the mind less vividly than what is presented to the trusty eyes (in Horace's words, opcit). Teaching methods in mathematics then concerned practical demonstrations, counting using fingers and calculi (counters), abacuses (reckoning boards), group-tutoring, tutoring by older students, singing and play (Wilkins 1914, Bonner 1977).

During the Byzantine years and until the fall of the empire to the West in 1204 AD, the dominant pedagogical ideas were those of Basil the Great, Gregory Nazianzen and John Chrysostom (4th century AD). Greatly influenced by Christianity, they believed that the best teaching was achieved through experiences closely related to the learner's own interests and past experience (Yannicopoulos 1983). In Chrysostom's words *"You cannot teach as effectively by words as you can by objects"*, *"Nothing is less constructive than teaching by words only; for this is not suitable for a teacher but for a hypocrite"* (writer's translation, opcit: 190-191). Consolidation for each subject was achieved by frequent repetitions and by application of the learned knowledge in everyday life experiences.

During the 'Dark Ages' (4th to 13th century) human energy was absorbed in the struggle for survival (Ulich 1963). Intellectual activity revolved around learning Latin, liturgical observance and rudiments of the seven liberal arts (trivium and quadrivium) from few and second hand sources (Freeman 1985, Ulich

1963). Teaching took place in monastic, parish or cathedral schools with methods based on memorisation. Non-school learning involved apprenticeship to prepare the youth for their occupations. The spirit that dominated the thinking of this period is captured in the words of Hrabanus Maurus (priest, 776- 856 AD) and his view about arithmetic:

> *"... those eager to cultivate arithmetic are right because in large measure it turns the mind from the fleshly desires and furthermore awakens the wish to comprehend what with God's help we can merely receive with the heart."* (Ulich 1963: 178)

Geometry for Maurus was also a way to appreciate the well ordered arrangement of the world by the *"almighty creator"* (opcit).

The writings of the great Fathers of Christianity reached the West with the first emigration of Byzantine scholars during the ninth century and with the Crusades (Yannicopoulos 1983). The learning preserved by the Arabs also transmitted, slowly, to the Western Europe during the transition period from 540 to 1500 AD (Burton 1988). During the Renaissance years learning was based on memorisation, with understanding as a desirable but secondary aim (Freeman 1985).

Some influences from classical Greece and from the Byzantine scholars can be detected in the work of John Locke (1632-1704). He was an advocate of empirical knowledge (Aristotelian perspective). The materials for this knowledge are to be acquired through our senses (Danassis 1985b). Even more influences can be found in Comenius' ideas (1592-1670), who believed in learning through the senses with the use of teaching aids (Laurie 1899). The familiar should be combined with the pleasant and should be presented to as many of the senses as possible (Danassis 1985b, Freudenthal 1973).

> *"And in order that everything may be imprinted the more easily, let the senses be applied to the subject as often as possible - e.g. let hearing be joined with vision and the hand with speech.*

"For the beginning of knowledge is from pure sense, not from words;
truth and certitude are testified to by the evidence of the senses." (Di-
dactica Magna, Laurie 1899: 125, 146)

We find a continuation of these pedagogical ideas during the 18th century in
the works of Rousseau, Pestalozzi and Froëbel. They all suggested the training of
the senses and the transition from concrete to abstract in learning (Kramer 1976,
Danassis 1985b). Pestalozzi and Froëbel actually developed a series of toys or ap-
paratus ('gifts' as Froëbel called them), to heighten the awareness of relationships
between things and to stimulate learning through play (Kramer 1976, Exarchakos
1988).

At about the same time the pioneering (and successful) work of Itard and
Seguin with feeble-minded children had its foundations on similar principles. They
nourished a respect for individuality in learning and acknowledged that the build-
ing of the intellect can begin with physical movement, 'the education of action'
(Kramer 1976). The contribution of Froëbel in the development of pedagogical
thinking can be seen as the bridge between the work of Pestalozzi and Montessori.
Montessori's ideas on 'auto-education' marked the passage to the 20th century,
along with those of Dewey and the development of his 'theory of experience'.
Montessori strongly believed that the tendency to establish relations is innate to
all children and underlies sense perception (Hunt 1912). She spoke about the
'hunger of the senses' and advocated that pedagogical exercises should not leave
the child inactive, preventing him/her from controlling the material (Montessori
1912). The materials that she used were to some extent a development of those
used by Itard, Seguin, Froëbel and Pestalozzi (two and three dimensional shapes,
of differing sizes, colours, matched into holes, blindfolded activities, etc.). Decroly,
another educator of this time, approached the formation of knowledge in the same
way as Pestalozzi (Exarchakos 1988).

Dewey's theory of experience followed the belief that education was a devel-
opment within, by and for experience (Dewey 1963). Not all experiences, though,
are genuinely or equally educative. He identified experiences of educative value

according to the principles of continuity and interaction. Continuity refers to the suggestion that each activity takes up something from things that happened in the past and also modifies in some way the quality of things which come after. This suggestion presupposes the specification of a direction in which growth derived from experiences should progress. What is done by the educator during the experience - words used, tone of voice, equipment, books, apparatus, toys, games played, etc, constitute the objective conditions under which experiences are had. The internal conditions of an experience are what goes on 'within' the individuals having the experience - personal needs, desires, purposes, conditions of the experience. Every experience has an 'active' side which changes to some degree the objective conditions under which experiences are had. The interaction principle suggests that the objective conditions (interactions between the individual and objects and other persons) should be subordinated to the internal ones. Experience is truly experience only if these conditions are assigned equal 'rights'. With his theory of experience Dewey aimed at an ultimate freedom of the intelligence, that is to say *"freedom of observation and of judgement exercised in behalf of purposes that are intrinsically worthwhile"* (Dewey 1963: 61). To resume Dewey's suggestions:

> *"Anything that can be called a study, whether arithmetic, history, geography, or one of the natural sciences, must be derived from the materials which at the outset fall within the scope of ordinary life-experience.*
>
> *"Finding the materials for learning within experience is the first step. The next step is the progressive development of what is already experienced into a fuller and richer and also more organised form, a form that gradually approximates that in which subject-matter is presented to the skilled, mature person."* (opcit: 73-74)

The more recently developed theories of learning were built upon the ideas of Dewey and his predecessors. Piaget saw acting upon our experiences as the only way to learn about our world. Manipulating objects in a concrete and action-

oriented context is the first step of the internalisation of actions. By reducing the perceptual and motor supports we increase the level of internalization and the strength of abstraction (Flavell 1963). Abstraction for Piaget *"is only a kind of trickery and deflection of the mind if it does not constitute the crowning stage of a series of previously uninterrupted concrete actions"* (Labinowicz 1980: 181). Acting upon our experiences involves transforming them within the mind, so to fit the existing cognitive structure and adjusting the mind to the new experiences (Sutherland 1992). The former mechanism provides continuity and stability, while the latter is akin to novelty and change. Learning is seen as the process of balancing between these two mechanisms. It is through successive, essentially discontinuous equilibrations that organised systems of actions are formed, as the learner's intellect develops from the stage of sensory-motor operations to the stage of formal operations (Flavell 1963).

Ausubel (1968, 1985) and Skemp (1986) saw learning, like Piaget, as a process of interpreting unfamiliar incoming information and fitting it into a schema. In this way great emphasis is placed on the existing cognitive structure of the learner. The interaction between the incoming and potentially meaningful material and the established knowledge (in the form of schemata) causes changes to both. Therefore, what is actually stored in memory may not be exactly as it was when the process had started. Furthermore, the accommodating schema (or schemata) may not have the same composition as it had before the assimilation of the new material. Ausubel et al. suggested that sitting still and listening does not rule out thinking (Entwistle 1988). Therefore the materials used to introduce a new concept do not always have to be of concrete-empirical nature. They can take the form of primary ideas, as in the acquisition of 'secondary concepts' (Ausubel 1968, Skemp 1986). Returning to Piaget's work, it has to be stressed that he considered that the manipulation of objects was critical to the later development of logical thinking only up to the twelfth year of age, when the stage of formal operations begins (Labinowicz 1980).

The work of the American psychologist Bruner concerning the process of learning, corresponds in many ways to the ideas introduced by Piaget. He opposed,

however, Piaget's belief that learning is subordinate to biological development (Liebeck 1984). Bruner (1967) suggested that the process of learning and of intellectual growth is the translation of our experiences in the world into increasingly more elaborate and powerful modes of representation. Our actions within an experience (or experiences) are the means for the formation of the initial mode of representation, the 'enactive'. As soon as the existence of objects involved in our actions ceases to depend upon these actions, we begin to operate in an 'iconic' mode of representation. Objects become visual or sensory representations that summarise our actions on the objects. When the final mode of representation is reached, the 'symbolic', some kind of language is formed to represent, internalise and manipulate actions. This language is used as an instrument of thought rather than as language per se (Bruner 1967). Bruner (1965) emphasised the importance for the learner of grasping the structure of an idea in a way that permits many other things to be related to it meaningfully. Reaching this structure requires the provision of opportunities to operate in the modes described by the model. He believed in giving visible embodiments to ideas and in the use of models to lead the learner through the different modes of representing ideas. Falling back on initial experiences can provide help, when symbolic representations fail the learner in solving a problem. By-passing the first two modes of representation then, may deprive him/her of the ability to fall back on these formative experiences.

Liebeck (1984) suggested a similar theory of learning to that of Bruner. However, she emphasised the importance of spoken words to represent and communicate our actions, by introducing a fourth stage in the model between the enactive and symbolic stages (Experience - Language - Picture - Symbol). This approaches Skemp's definition of 'logical understanding', that is the difference between being convinced oneself and being able to convince others (Skemp 1976, 1979, Byers & Herscovics 1977). Bruner had indicated the importance of language *"Intellectual growth involves an increasing capacity to say to oneself and others, by means of words or symbols, what one has or what one will do"* (Bruner 1967: 5).

An approach to learning from a different perspective, specific to mathematics, was introduced by Dienes (1960, 1963, 1964, 1973). He saw the process of learning

through increasingly more structured play activities. By *"fiddling around"* with the provided material the child adapts to a certain environment ('free play' stage). From free play the child begins to realise some constraints or regularities (rules of the game) on which relevant mathematical structures will depend (second stage). The child is then introduced to perceptually different activities based on a common structure (principle of 'multiple embodiment'). The underlying structure is thus extracted from these various activities (third stage). In the fourth and fifth stages, representations (auditory or visual) and language are introduced, to reflect upon and discuss the abstracted structure. Finally (sixth stage) the child draws on what has been abstracted from the activities and by analytic thinking proceeds to further generalisations (Dienes 1973). To provide the maximum amount of experience of the learned knowledge, Dienes suggested further activities following the principles of 'mathematical variability' (all possible variables vary keeping the concept intact) and of 'contrast' (use of non-exemplars to ensure that situations not addressed by the concept will be identified as such) (Dienes 1960, 1964).

Contemporary views on mathematics learning suggest that knowledge is constructed by the learner through an active learning process. This corresponds to Ausubel's assimilation theory (constructivism, Jaworski 1991). Knowledge *"is not out there in the world waiting to be discovered"* (opcit: 11), contrary to what Plato and Gibson suggested by saying that knowledge is actually discovered through a process of learning to perceive what has always been there (Anapolitanos 1985, Miller 1989). Constructivism accounts for the individuality of knowledge and draws attention to the fact that teaching is more like developing shared knowledge rather than giving it to the students (Jaworski 1991).

Pirie (1992) and Mason (1992) proposed models of learning which are an amalgamation of Bruner's thoughts, based on constructivist philosophy. The process of concept building passes through levels of growing understanding about the concept. Manipulation of objects (physical or mental) gives an insight into a sense of structure, which further becomes more articulate and detached from the initial experiences with the objects. Even though at each level the learner can perform without reference to experiences from past levels, understanding may break down.

The learner then would have to draw from these lower level experiences and skills to find help. This 'folding back' feature of the model is essential if understanding is to continue its growth. If lower level references do not exist, or are not adequate to provide ideational scaffolding (Ausubel's term, 1968), understanding is bound to cease at the level that the difficulty emerged. It might be substituted for by the use of some rule or technique.

1.2   The Present Situation

Mathematics has been addressed for millennia as a theoretical science, partly because of its nature and partly because of the objectives of those who have pursued mathematics ('lovers' of wisdom, seeking to discover eternal truths, Apostle 1952). Arithmetic and geometry though, started as practical sciences serving the needs of everyday life. Sympathies for abstraction developed as soon as mathematics moved beyond the purposes for which it was initially invented. Despite that, practical applications have never been divorced from the mathematical disciplines (arithmetic, geometry, astronomy, musical theory). In addition, many contributors to pedagogical thinking in the past 4,000 years suggest that pedagogical exercises should not leave the child inactive, preventing the learner from controlling the material.

Instruction in mathematics, though, followed the pattern of teaching the young by telling 'out of context' rather than showing 'in context' (Bruner 1967). The great problem of mathematics education then seems to be, as Freudenthal (1973) suggests, the gap between the use of mathematics and the aim of learning mathematics. This gap seems to widen in secondary education. Is the aim of learning mathematics 'to think logically'? Is that aim valueless if logical thinking can only be applied in the mathematics classroom? Is it worthwhile, then, to teach only through 'sentences' disregarding Dewey's suggestions that learning should be a development within, by and for experience? How can we shorten the distance be-

tween a *"useless aim"* and an *"aimless use"* in the field of mathematics instruction (opcit)?

In life outside school, mental work is distributed over several individuals and the use of 'tools' expand people's mental power. In contrast, in schools learning and performing become individualised and the focus is on symbolic activities (Resnick 1987, 1989). Bruner suggested the use of devices for *"vicarious"* experience to substitute for experiences taking place in everyday life. Giving visible embodiments to ideas and making provision for sequential programs can promote students' understanding of basic ideas and structures in mathematics (opcit). This approach in teaching could meet the cognitive and affective needs of the students, if we are to accept that learning progresses from concrete to more abstract knowledge. It satisfies both children's need for movement (hunger of the senses) and innate disposition for learning (growth of intellect).

In contrast to the suggestions supporting this way of learning mathematics, practical activities are used only sporadically with older children. This decision seems to be supported by arguments that draw not particularly on their effectiveness in promoting learning but on functional difficulties caused by their use as a teaching aid. As Desforges said *"Teachers, of course, will not teach what they do not value and cannot teach what they do not know"* (1985: 93). We could go even further and suggest that sometimes teachers cannot teach what they already know. Based on these views, arguments against the use of practical activities will be discussed next.

To 'not value' practical activities can be translated as not valuing them as an approach to learning or as an approach in teaching. Either of the views of course may lead to the other. Some may advocate that practical activities are not essential in secondary education since learning becomes formal and analytic. This very argument implies that understanding comes at a higher level. We can suggest then that difficulties faced at these higher levels can only be dealt with by the use of tricks or techniques provided by the teacher instead of by folding back to past experience. The implicit knowledge acquired from practical work, at the initial levels of understanding, cannot be taught didactically at a later stage

(Giles 1981). Obtaining the correct answers then replaces understanding. Not to value practical activities is also closely related to beliefs about the nature of mathematics. If mathematics is to be addressed only as a theoretical and highly abstract subject, then such a view may have some validity. However, this still would not divorce these abstract tools from their practical origins.

For those who 'value' and therefore use practical activities in the classroom, there is the danger of perceiving similar but essentially different activities as practical. James (1985) distinguished activity-based learning from the *"rules with tools for ticks"* approach. In the latter students are using 'tools' while working but they are still tuned into the teacher's way of doing things. They have to follow a path that the teacher and the 'tools' have set for them. 'Tools' become techniques for reaching the correct answers (the 'ticks'). Discussion is not usually encouraged and students may fail to appreciate the mathematical and everyday life implications of those practical activities. In another, similar approach, practical activities may be used as a 'fun' activity. This definitely redirects students and teachers from the real nature of the approach, perceiving them as 'not mathematics' but as play. Learning from such activities is mainly incidental. It seems then as if teachers cannot teach what they already know, not always though because of their inadequacy.

We can suggest that there are also teachers who 'value' practical activities as a way of learning but are hesitant in using them for 'security' reasons. Leinhardt and Greeno described teaching as a cognitive skill, which actually requires *"... a complex knowledge structure composed of interrelated sets of organised actions"* [schemata] (1986: 75). Practically-introduced tasks may hide unexpected difficulties in their progress (cognitive and functional), which could impose a 'threat' to these 'teaching schemata'. The teacher then may decide to adopt a teaching style that would 'secure' the students' success (Freudenthal 1973) and his/her authority in the classroom (the 'informer' and the 'problem solver', non-pupil initiated activities, Khale 1987).

Closely related to this security feeling are the difficulties that may occur in the use of practical activities. Equipment needs to be prepared beforehand, which

may take a lot of the teacher's time. Equipment needs to be cheap so that the schools can afford it. Assistance might be needed during the sessions using the practical activities, to allocate the materials and make sure that each student has the appropriate support and challenge. There is pressure on teachers to have calm and quiet classes. Students should be lively and interested (not too lively and therefore disruptive), conscientious, little trouble in the class, etc (Walden & Walkerdine 1985). Practical activities encourage talking and active participation by students. This could be contrary to the commonly accepted organisation of a class. Other difficulties may concern the evaluation of the outcomes, since it is likely that they will be long- rather than short-term. All the discussed difficulties are heightened by the large number of students in a class. In science classes the numbers of students are limited to make practical activities easier for the teachers to manage. This is not the case in maths classes. The students themselves can play a role in overcoming such organisational difficulties by adopting a purpose for learning (Dewey 1963). To transcend over and above the requirements of a task is to pursue cognitive processes that have learning as a goal (intentional learning, Bereiter & Scardamalia 1989). The formation of such an attitude can be promoted by activity-based teaching. A further objection to practical activities is that they are time-consuming in an already full schedule for preparing the students for their final examinations.

To address the second part of Desforges' citation, teachers will not teach with practical activities if they do not know how to. Most secondary school mathematics teachers are trained within a period of one year. Their classroom training takes place in schools, where the situation may coincide with the one described. Moreover many teachers are not aware of already existing teaching materials and it is difficult for teachers to produce their own materials due to time limitations. If we do not use practical activities on the grounds of their being time-consuming (the burden of completing the syllabus), we have to reconsider the criteria for success in mathematics learning. Are we aiming to teach an isolated subject, with tasks that are neither unusually extensive nor profound? Should we prefer teaching that aims to promote life-long learning and portrays honestly the nature of the

subject? The earlier aim concerns short-term achievements and provides cheap success (Freudenthal 1973). Time in this case is important, since this is an aim that can be achieved only during the school-years. The latter aim though, concerns a will for life-long learning where time is irrelevant since learning continues after school.

1.2.1   Creating a Learning Environment

So what should we choose? The 'realistic' but possibly sterile approach, or the more 'unrealistic' and maybe metaphysical one? There is not always 'one' answer. There are though answers that serve the needs of situational and intrinsic factors. One of the disasters of education was described by Dewey (1963) as the 'either-or' philosophy. It is not then a matter of choosing between the two ends of the argument. There is rather a need for finding a balance between the situational factors of teaching mathematics and the intrinsic ones that account for the individuality of the learner. It is not a matter of teaching everything we possibly can within a prescribed period of time, in a way convenient to us. It is rather a matter of providing experiences (as already described) and support, so to promote understanding and build positive attitudes towards mathematics and learning in general.

Chapter 2

Focusing on Certain Mathematical Areas

2.1   Stranding the Concept of Similarity

The theme of the first part of this chapter will be the similarity of rectangles. Similarity will be discussed in general along with studies pinpointing the difficulty and importance of the concept. We shall identify its relations to other topics in mathematics and also distinguish rectangle-similarity from the broader concept of similarity. The complexity, difficulty and importance of all the related concepts will be established through a number of researches and surveys. We shall quote theories concerning the development of these concepts and we shall conclude with a summary of teaching approaches towards them.

In order to trace the origin of the idea of similarity we have to investigate the very early experiences in our childhood (Williams & Shuard 1991). Long before children can think in terms of similarity, they can make judgements on whether figures possess similar relationships. Van den Brink and Streefland (1979) suggest that 6 to 8 year-olds can deal with similarity as an operative equivalence while they are trying to order the visual perceptive reality. Researchers like Freudenthal and Dudwell (opcit) suggest that congruences and similarities are ways of processing our visual perceptions and they are built into our central nervous system. So in order to seek the similarity of figures we have to seek for the actual perception of the figures (Piaget & Inhelder 1956). Piaget called the perceptual activity of recognising two shapes as similar *"transposition"* and regarded it as one of the fundamental properties of perception (opcit). Perceptual transposition then is

the initial form of the concept of similarity. We have a long development though before we reach the point where figures, similar to a given one, can be constructed operationally, that is, to acquire a geometric sense of the similarity concept.

Similarity assists in perceiving, categorising and organising the world around us. How does it fit though in the mathematics curriculum? As Lappan and Even (1988) indicate, similarity is an important topic in geometry, since it is basic in understanding other topics like the geometry of indirect measurement, proportion and ratio, scale drawing, modelling and the nature of growing (shapes etc). Piaget and Inhelder (1956) suggest that similarity tasks are easier than tasks involving proportion. Therefore, as Friedlander (in Lappan & Even 1988) says, geometrical similarity is a concept that may lead to an understanding of proportionality. Similarity also relates closely to equivalent fractions (Hart 1984). Teachers in the Second IEA Study of Mathematics (Garden & Robitaille 1989) rated similarity of plane figures as an important topic for 13-year-olds (see also Hüsen 1967). Along with its importance in the mathematics curriculum, similarity is regarded as a difficult concept. Evidence from research studies and achievement surveys support this view (Robitaille 1989, Robitaille & Taylor 1989, Lappan & Even 1988, Cresswell & Cubb 1987, Hart 1978, 1981c, 1987, 1989, Hüsen 1967). The most popular items in these studies though involved applications to similar triangles and indirect measurement.

2.1.1 Similarity of Rectangles

The similarity of rectangles as a topic deserves to be examined separately, due to its peculiarity compared to the tasks discussed previously.

Transposition of similar shapes can be done according to overall shape, dimensional relations or angles. Intuitive understanding of similar figures involves such comments as 'having the same shape', while a more analytic and powerful understanding has to do with 'angle size is preserved', 'all lengths are multiplied by a constant', or 'ratios of corresponding sides are equal' (Lappan & Even 1988).

With rectangles there are no angles to compare. It may seem then that identi-
fying similar rectangles is a simpler task than that of triangles, since in rectangles
one need to focus only on two cues (size and sides' proportions). This may be
the case when a formal understanding of similarity has been acquired. For a child
though who is perceiving similarity intuitively, comparing rectangles for their sim-
ilarity is a complex task. The ratio of the sides has to be estimated and no help
can come from any correlative change in angle (Piaget & Inhelder 1956).

A distinction has to be made between tasks involving perceptual comparison
of rectangles and pictorial construction. In the first case intelligence is governed
by perception, therefore we speak of a perceptual estimation. In the second case
perception is governed by intelligence and we speak then of an intellectual con-
struction (opcit). In the research reports reviewed by the author, items on similar
rectangles involve intellectual constructions. The child has the length and breadth
of the original rectangle (either given or needing to be measured) and has to en-
large it by a ratio (either given or has to be calculated) (Hart 1978, 1981b, 1988,
Lappan & Even 1988, Clarkson 1989). These tasks were proved to be difficult for
12 to 15-year-olds when the enlargement ratio was other than 2:1.

2.1.2 Strategies on Rectangle-Similarity Tasks

Clarkson (1989) analysed students' responses (12 to 13 years of age) on enlarge-
ment tasks (not limited to rectangles). He suggested a number of strategies that
students use when confronted with such tasks. A summary of these strategies
adapted to the case of rectangles follows:

- Linear Scale Factor: the child multiplies the sides of the original rectangle
  by the scale factor to find the lengths of the sides of the wanted rectangle

- Addition Scale Factor: the child finds the increase in length for one side to
  determine the scale factor, eg. a side of 3 units becomes 9, which is two
  times bigger than 3, so scale factor is 2

- Area: the child has doubts as to whether an enlargement is concerned with a linear or an area scale factor, so s/he finds the scale factor by using the areas of the original and the enlarged rectangle

- Area/Addition: this is a combination of area and addition strategies, eg. areas of 5 and 20 units, 20-5 is 15, 5 into 15 is 3, so scale factor is 3

- Border: the child regards the original rectangle as being in the corner or the middle of the wanted one, and tries to complete the missing part

- Enlarge One Side Only: it involves *"centration"* to one dimension. Piaget and Inhelder (1965) suggested that the longer the rectangle is, relative to the height, the more rectangular it appears to a child. Therefore the general tendency is to make the wanted rectangle too long to be similar to the original.

We shall come upon some of these strategies later on, when similarity will be discussed in relation to the development of proportional reasoning.

2.1.3 Similarity and Measuring

Measuring is closely related to similarity. Students have to measure sides of shapes (especially in practical tasks) in order to identify and apply enlargement ratios. Findings coming from National and International surveys revealed that not all students of 9 to 15 years of age use a ruler competently (Hart/CSMS 1981a, AAP/SED 1983, 1989, Johnson/CSMS 1989, Dickson et al. 1991, EMU/SEAC 1991; NCES 1991, Lapointe et al./IAEP 1992, SOED/IAEP 1992, Semple 1992). Most common errors relate to the failure to count units correctly (i.e. placing properly the zero point, rounding lengths that do not align with an indication). Measuring mistakes can affect students' performance on similarity tasks and hinder also the strategies used (Clarkson 1989).

2.1.4   A Word About Terminology

Hart (1981a) draws attention to the terminology used in similar figures tasks. In interviews with students she found that the word 'similar' means little to many children. Due to its everyday life use it tends to mean 'approximately the same'. Piaget and Inhelder (1956) in their clinical interviews on similarities and proportions used the expressions 'which one is the daddy of the little one and looks most like it' with the youngest children, and 'looks like' or 'is the same shape but bigger' with older children. Teachers that took part in Clarkson's (1989) study used terms like 'times', 'bigger', 'much bigger', 'times as long'. Williams and Shuard (1991) are closer to an expression 'of the same shape but larger'. Hart (1981b) cautions that words like 'bigger', 'larger' do not automatically infer multiplication. They may infer addition and lead students to adopt erroneous strategies. She suggested the expression 'times larger' to avoid such misunderstandings.

2.1.5   Proportion, Ratio and Similarity

From a perspective of a network model of memory, learning involves both the acquisition of concepts and the construction of hierarchical relations among these concepts (J.L. McDonald 1989, Branca 1980). As Vergnaud (1988) suggests no single concept refers to only one type of situation and no single situation can be addressed and analysed with only one concept.

> *"Each concept should be seen as a triplet of sets: C:=(S,I,S), where C is the concept, S a set of situations that make this concept meaningful, I is a set of invariants (objects, properties and relationships) that can be recognised and used by the subjects to analyse and master these situations, and S is a set of symbolic representations that can be used to point and represent these invariants and therefore represent the situations and procedures to deal with them."* (Vergnaud 1988: 141)

The previous quotation suggests the complexity, and therefore difficulty, of studying a single concept. A way of dealing with this difficulty is by studying

conceptual fields (Vergnaud 1983, 1989). These are comprehensive systems and are defined as a set of situations, the mastering of which requires the mastery of several concepts of different nature. Therefore in order to study similarity we should examine the concept from the perspective of other associated concepts as well.

2.1.6 Ratio and Proportion

Ratio and proportion are widely studied concepts. The earliest research evidence comes from Winch around 80 years ago (Karplus et al. 1983a). Researchers agree both on their importance and difficulty (eg Lovell & Butterworth 1966, Wollman & Lawson 1978, Noelting 1980a, Turner 1982, Hart 1978, 1984, 1985, Tourniaire & Pulos 1985, Lesh et al. 1988). This belief is supported by international surveys of performance on mathematics (see studies in §2.1.2, §2.1.3).

Ratio (a/b) is the numeric relationship between two entities (Hart 1988) or in other words a numerical expression of how much there is of (a) refers to the first quantity and (b) to the second, while the value of a/b is the numerical expression of the comparison. Quantities can be extensive, intensive or scalar. Extensive quantities represent *"how much of a quantity is associated with a given object"* (Lesh et al. 1988: 109). For example 45 degrees for an angle, 45 degrees of temperature, 45 gallons of petrol. Fractions are a special kind of extensive quantities (internal ratio). *"Intensive are the quantities that are ordinarily not either counted or measured directly"* (Schwartz 1988: 42). They are the 'per' quantities and express how much of a quantity is related to a unit of another quantity (rates are such quantities - external ratio). Scalar quantities are a type of intensive quantities, where the two quantities are measured in the same unit (eg weight of sugar/weight of all recipe's ingredients) (Lesh et al. 1988). When ratio is seen as a correspondence between two sets (Skemp 1987), this correspondence can involve ordered pairs of any of the previous types of quantities.

The Investigation of British Secondary School Mathematics textbooks (in Hart 1978, 1981a) indicated the following aspects as being basic to ratio:

> *"– doubling or halving*
> *– multiplication by an integer*
> *– given a rate per unit apply this rate*
> *– find the rate per unit and then apply it*
> *– enlarge drawing in ratio 2:1, 3:2 , 5:3, etc.*
> *– find a ratio a:b using an intermediate quantity, c, i.e. given*
> * relationships a to c, b to c*
> *– using a fractional multiplier*
> *– simple percentages ...*
> *...fractions...*
> *...similar triangles."* (Hart 1978: 4)

Other manifestations of ratio can also be found in the following activities (adapted from Streefland 1984: 336):

- comparing magnitudes not only of the same kind but also of different kinds, such as length and number

- considering mixtures of intensive quantities

- recipes as a separate theme from mixtures

- distinguishing internal and external ratios

- stressing the ratio in the operator (5 from x to 5x, 1/5 from y to (1/5)×y).

Ratio then is a concept that 'participates' in many classroom activities. Its relationship to similarity and proportion is strong and development of proportional reasoning relies much on ratio activities, as we shall see.

Proportion involves the equivalence of two ratios (a/b = c/d) (Skemp 1987, Hart 1988). Therefore to make a proportional judgement two ratios have to be recognised as being equal or as belonging to the same equivalence class. For Lesh et al. proportional reasoning is *"reasoning about the holistic relationship between two rational expressions such as rates, ratios, quotients and fractions"* (1988: 43). They continue, saying:

> *" Proportional reasoning is a form of mathematical reasoning that involves a sense of co-variation and of multiple comparisons and the ability to mentally store and process several pieces of information. Proportional reasoning is very much concerned with inference and prediction and involves qualitative and quantitative methods of thought ... "*

Proportional reasoning also involves:

> *"... mental assimilation and synthesis of the various complements of those expressions and an ability to infer the equality or inequality of pairs or series of such expressions based on this analysis and synthesis. It also involves the ability to generate successfully missing components regardless of numerical aspects of the problem situation."* (opcit: 93)

This quotation from Lesh et al. encapsulates the meaning of the concept of proportion. As Tourniaire and Pulos (1985) suggest most people can use proportion in familiar contexts. Various difficulties arise though even in the definition of proportion. In mathematical discourse it refers to an equality of ratios as we stated earlier. In everyday life context though proportion is more closely related to other quotient terms than the mathematical. *"Proportion is a part considered in respect to the whole"* (Collins Dictionary). It is easy therefore to confuse proportions with fractions, since the subtle difference resides in the reference of the denominator (Ohlsson 1988). Proportions also may appear as percentages or even as the probability of an event.

Lesh et al. (1988) characterised proportion as a 'watershed' concept, which is justified by the previous discussion. It is expected then that students will face

difficulties with proportions, since misunderstandings from related concepts may result in adopting erroneous strategies when dealing with proportion tasks.

2.1.7 The Development of Proportional Reasoning

It is suggested that the development of proportional reasoning is slow and the concept is acquired late (Tourniaire & Pulos 1985, Hart 1984, Lovell & Butterworth 1966, Lunzer & Pumphrey 1966). Renner (in Hart 1984) investigated freshmen in four American universities and found that students had a basic deficiency in problems requiring ratio or proportion of any kind. Capon and Kuhn (1979) suggested that many adults do not exhibit mastery of the concept of proportion at all. It is essential then to examine proportional reasoning further in order to reveal the reasons for such a slow and late development.

Studies in proportional reasoning have employed different methodologies and tasks. Students were asked either to find from the given data an additional value for an extensive quantity, or to compare the two values of the intensive variable computed by the data (missing value and comparison problems correspondingly) (Karplus et al. 1983b). In some tasks students had to simply provide an answer, in others they were asked explanations on their strategy (Tourniaire & Pulos 1985). Finally physical tasks have been employed, especially by Piaget (Inhelder & Piaget 1958), like the beam and projection of shadows experiments. These latter tasks have been criticized for their potential in assessing proportional reasoning, since they require the understanding of some physical principle in addition to understanding proportions (Karplus et al. 1983a).

Piaget suggests (Piaget & Inhelder 1956, Inhelder & Piaget 1958) that proportional reasoning characterises formal thinking. Therefore, proportional reasoning is integrally linked to other reasoning patterns that may be used under different circumstances and at various levels. In the beginning only qualitative comparisons can be made (until the second year of age). At the intuitive level comparisons between terms are possible. The child has to reach the level of concrete operations in order to involve joint multiplication or division of terms and equivalence classes.

At the level of formal operations equivalencies are mentally reconstructed before comparison can take place (Noelting 1980a). Bryant and Lawrence though showed that young children can logically connect two discrete perceptual experiences by contrasting a common identity element, eg. colour, size or proportion (in Muller 1978). Errors occur only because children fail to make the correct initial analysis, e.g. on size instead of proportion.

Case (in Karplus et al. 1983a) suggested that the gradual growth in the effectiveness of working memory may account for the development of proportional reasoning. Karplus et al. (opcit) reject the Piagetian theory suggesting that proportional reasoning constitutes an independent entity and Noelting (1980b) deals with proportional reasoning in separation from other cognitive operations.

Suarez (in Karplus et al. 1983a) investigated proportions as linear functions, where the slope of the function is one value of the intensive variable $[y=(a/b) \times x]$. Vergnaud (1983) sees proportion as an isomorphism of measures, which is a structure that consists of a simple proportion between two measure-spaces M1 and M2 (values of the intensive variable) (see also Lamon 1990). These approaches have been criticized as dealing only with arithmetic relations between sets of numbers, ignoring the relations between the variables represented by the numbers (Karplus et al. 1983a).

From the previous evidence it is clear that the development of proportional reasoning is hierachical. Initially proportionality is mastered in small and restricted classes of tasks. As children become more competent, they restructure their strategies and the classes to which these strategies apply to are gradually extended (restructuring theory/ adaptive restructuring, Noelting 1980b) . The change that takes place from one stage to the next is both qualitative and quantitative. Children restructure their strategies in order to comply with new situations but within each stage strategies are extended to a variety of applications of the new situation.

Children's strategies in solving proportional problems $(a/b=c/d)$ fall into two large categories: the with-in and the between strategies. In the former, students

find the ratio of corresponding (extensive) quantities (a to c, b to d), while in the latter they find the rates representing the values of the intensive quantity (a/b to c/d) (Tourniaire & Pulos 1985). Correct strategies involve multiplicative and building-up strategies. In multiplicative strategies a relation is obtained between two terms of the proportion, either following the with-in or between approach, and then is extended to the remaining terms (opcit). The building-up strategies are more elementary (Hart 1981b) and involve finding parts of the answer which will be added together eventually.

Erroneous strategies may involve either using an inappropriate strategy or misusing a correct one (Tourniaire & Pulos 1985). One of the most commonly used inappropriate strategies is the additive strategy or strategy of constant difference (Hart 1981b). Students concentrate on the difference between the extensive quantities that form the intensive variable. Another commonly used error strategy is that of ignoring part of the data and concentrating on one term of the proportion only (centration) (Piaget and Inhelder 1956). Erroneous strategies may also be used as fall back strategies when children are working on difficult tasks. For example a child capable of using the multiplicative method with integer ratios may fail to do so with non-integer ratios. In such a case the child employs an 'elementary' strategy (eg additive method).

The strategies mentioned correspond to different levels of development of proportional reasoning. Starting from additive methods we progress to the stage of logical proportions which is characterised by understanding all four terms of a proportion (for a further discussion see Tourniaire & Pulos 1985).

2.1.8 Variables Affecting Performance on Proportion Tasks

Variables that affect performance on proportion tasks fall into two categories. The task-centred variables and the student-centred ones (Tourniaire & Pulos 1985). A summary of these variables follows.

Hart (1981b) and Noelting (1980a, b) suggest that the presence of integer ratios make the task easier. Rupley (in Karplus et al. 1983a) adds that the place of the

number to be found in the proportion and the inclusion of numerical values larger than 30 increase task difficulty. The presence of a unit makes a problem easier (Hart 1981b), while comparing unequal ratios is more difficult than comparing equal ones (Karplus et al. 1983a, b). Other variables, besides structural, concern the context of the task (see Tourniaire & Pulos 1985). The familiarity of the context, the presence of intensive or extensive quantities and the way a task is mediated, all affect students' performance.

Karplus et al. (1983a) suggested that the number of schemes one can 'attend to' at one time (M-capacity) is related to performance on proportion tasks. The extent to which a child can apply his/her understanding of proportions in different contexts, is related to success in proportion tasks (FDI, Field Dependence-Independence). Field independent students tend to perform better (Tourniaire & Pulos 1985). Other variables that seem to correlate positively with success in proportion tasks are intelligence (Hart 1981b), attitudes towards mathematics and students' metacognitive abilities (in Tourniaire & Pulos 1985, Karplus et al. 1983a).

2.1.9 Teaching Approaches to Ratio and Proportion

Since learning cannot be isolated from teaching, we now examine some teaching approaches which try to build proportional reasoning. Ratio and proportion are concepts that require a lengthy learning process before they are mastered. The usual approaches followed by school teachers are criticized as impoverished and overconcise (see Streefland 1985). The concepts are taught in isolation from other concepts, there is lack of visualisation (opcit) and there is no transfer of knowledge from math-classrooms to other disciplines and to everyday situations (Carraher et al. 1984).

The most common methods taught for solving problems in ratio and proportion are the 'unitary method' (find how much it is for one unit), the 'rule-of-three' or 'cross multiplication' (Hart 1984) to test for the equality of ratios, converting unequal ratios to fractions with a common denominator in order to compare them

(Karplus et al. 1983a). The teaching of these methods have been widely criticized, especially the cross multiplication method (Carraher et al. 1984, Lesh et al. 1988). The rule-of-three is a method poorly understood by the students, is seldom used as a solution method (Hart 1984, 1981b, Streefland 1985) and in cases where it is used it can destroy instead of facilitate proportional reasoning.

Training studies have employed group (eg. Lunzer & Pumphrey 1966), individual (eg Wollman & Lawson 1978, Nesher & Sukenik 1991) and classroom (eg Carraher et al. 1984, Hart 1981b, 1984) approaches. Lunzer and Pumphrey (1966) used an approach working with Cuisenaire rods. Wollman and Lawson (1976) compared an active method (Cuisenaire rods, series of geometric shapes, etc) to a verbal method (textbook, discussion with the experimenter) and found that active-method students outperform other students, in most tasks. Nesher and Sukenik (1991) taught students of 7, 8, and 9 years of age the concept of ratio in a formal way and had satisfactory results. Hart (1981b) suggests that through attempting practical problems students may abandon erroneous strategies (additive strategy). Lamon (1990) supports the view that concrete activities are important in abstracting the concept of ratio. Hart (1984) and Szetela (1980) introduced the use of calculators to remove from students the demand of having to perform computations as well as think through a solution. The calculator-based instruction groups achieved higher scores than the students that did not use a calculator but the difference was significant only in less familiar problems.

Hart (1981b) suggests that visual confrontation with erroneous responses may eradicate constant-difference strategies (a-b difference results in gross distortions from the correct answer). Streefland (1984) suggested an interdisciplinary approach and a spiral curriculum to account for a long term development through sound connections with other concepts. He also suggested (1985) the use of visual models (sector diagram, ratio table) to support the learning process and to broaden the applicability of the newly acquired knowledge. Lovell and Butterworth suggested that differences may exist in the development of proportional reasoning across cultures, since *"the form of thinking skills practised and valued*

by society seems to make a difference to the ease with which formal thought can be elaborated" (1966: 8).

The general notion though coming from the previous studies is that teaching approaches should take into account the students' initial, intuitive methods and introduce formal strategies and algorithms after having revealed the inadequacy of their own methods. Through conflict (confrontation with limitations of their methods) and reflection, the development of consciousness and therefore a more formal proportional reasoning can be achieved (Streefland 1984).

The importance of similarity in our everyday life and in the mathematics curriculum are well established. Ratio and proportion are among the concepts closely related to similarity. They require a long term and hierarchical development, based both on qualitative and quantitative changes. We find though that these concepts are taught in isolation, with no reference to everyday life situations. Therefore we would propose a teaching approach that takes into account everyday life applications of these concepts and appreciates the strength of a more active, practical perspective.

Rectangle-similarity has to be considered separately due to its peculiarity compared to similarity of other shapes. Difficulties can arise from the perception (features) of a rectangle (eg length-breadth confusion). Such aspects that may hinder performance on similarity tasks have to be 'disclosed', identified and remedied through this approach. For this approach to be applicable in many cases, it should also take into consideration students' individual differences in the cognitive and affective domains. It remains to be seen whether such a teaching approach can be achieved.

2.2 Haptic Exploration of Geometric Shapes

Denmark and Kepner (1980) reported that 74 per cent of the teachers that took part in their survey agreed on the importance of students being able to recall properties of simple geometric shapes. National and International surveys of mathematical performance revealed that secondary students cannot identify and name shapes like the kite, rhombus, trapezium, parallelogram and triangle (AAP 1983, Chicago Project in Hoffer 1983, APU 1980 in Dickson et al., 1984; NAEP 1980 in Dickson et al. 1984). Students' performance is even poorer when it comes to items involving the understanding of features and properties of shapes. For instance, only 14 per cent of the 13 year olds could select correctly the necessary conditions for a figure to be a rectangle (NAEP 1980 in Dickson et al. 1984). These findings suggest perhaps that the approach to the teaching of geometric shapes is faulty.

In the following section haptic exploration will be suggested as a different approach for the teaching of geometric shapes. By haptic exploration we mean the intentional and conscious movements of our hands about an object (Weber 1978). The importance of visualisation and action will be addressed along with the limitations of visual perceptions in the formation of geometric meaning in general. An account of touch modality and the development of haptic exploration strategies will follow, in relation to the development of geometric thought. Views on coding of perceptual information and on mental representations of concepts will be discussed in an effort to distinguish the perceptual, cognitive and other possible factors that characterise the haptic exploration of shapes.

2.2.1 Visual Limitations and Geometric Thought

Much learning of geometric meanings involves the use of diagrams (Dickson et al., 1984). Bishop observed *"one problem with geometry is that it is impossible to draw a generalised diagram"* (1983: 180). Whenever we draw a diagram of a

geometrical object, for instance, there is a loss of information. The restitution of the meaning of what the diagram represents comes due to a common (to some extent) geometrical culture (Parzysz 1988). Restitution comes only after geometric meaning has been acquired. This 'limitation' of geometry is accentuated by the way concepts are introduced. The formation of a geometric concept has to be based on a number of critical attributes of the concept (Hershkowitz 1989).

These critical attributes are the features and properties that characterise and differentiate the particular concept from all the others. When concepts are introduced by a few examples, some of them tend to be more popular than others (Roth 1986). Prototype examples constrain students' knowledge of a concept to these cases that are more often addressed in a book or by a teacher, neglecting particular cases.

Moreover, prototype examples may 'attach' additional, non-critical attributes to a concept (eg orientation effects), in the same way as they 'detach' other attributes (eg being able to draw an isosceles and a right-angled triangle but not a right-angled isosceles one). Being taught in this receptive way, students may attempt to idealise any task at hand by transforming it into a special case, or seeking non-critical attributes to support their reasoning (Hoffer 1983).

Fisher suggests that *"the additional non-critical attribute of a prototypical example draws our attention because it is visually strong and usually registers our minds spontaneously"*, in addition to the fact that prototypical examples are usually presented before any other example (in Hershkowitz 1989: 73-74).

This view proposes a visual limitation which facilitates the misleading potential of prototypical examples. Indeed, as Eysenk and Keane (1990) suggest, visual stimuli are often incomplete and ambiguous. It is suggested that such simple illusions as centration (the extent to which vision is centred on one point, side, relationship, rather than on another), diminishes very little in the course of development (Piaget & Inhelder 1956).

The effect of these visual-perceptual limitations cannot be ameliorated by the provision of information obtained in other ways. Hershkowitz (1989) provided her

subjects (students and teachers) with the verbal definition of isosceles and right-angled triangles. Despite these verbal cues the subjects' identification ability of the concepts did not change. Michotte (1991) suggests that such conflicts between perceptual evidence and information obtained from other sources can be resolved only by knowledge. Knowledge in our case is related to initial learning of a concept and the 'exposure' of the concept's critical attributes.

2.2.2   Circumventing Visual Limitations

Visualisation, or spatial ability, or spatial perception is, in part, an intuitive feel for one's surroundings and the objects in them (Del Grande 1990). It is this intuitive character of visualisation that creates difficulties when it comes to addressing the actual abilities from which is constituted. For the purposes of geometry we can define visualisation as the ability to interpret figural information (the figural language of geometry) and the ability to carry out visual processing (manipulating figural stimuli, associating them with previous experiences, translation of non-figural stimuli into visual terms) (Bishop 1983, Del Grande 1987).

The ability to carry out visual processing places emphasis on the process and not on the nature of the stimulus. The stimulus does not have to be figural to be processed visually (Bishop 1983, Dickson et al. 1984). This 'versatility' of visualisation increases its applicability and therefore its importance. Initially, visual processing can help to discard geometry's visual limitations by the recognition and discrimination of a concept's critical attributes and by not being deceived by the additive ones (eg mental rotation). Furthermore, it can facilitate performance in geometry by the visualisation of the elements of a concept or concepts, providing support for deductive reasoning (Hershkowitz 1989).

Not all people employ visual processing in their mathematical thinking (in general) to the same extent but visualisation is to some extent a trainable skill. Such a training should be approached in an 'active' way (Bishop 1983, Del Grande 1987, 1990). This implies that the individual perceives stimuli referring to the concept

with other senses apart from vision, eg by touching, manipulating, constructing, drawing etc.

Apart from its role in the development of visualisation, action plays an important role in the formation of geometrical concepts themselves. There is always a correlation between the concept and the aspects of the actual activity addressed by the concept (Lakoff & Johnson 1980). In other words, a concept fits an experience or a whole reference of experiences. These experiences define the concept in terms of interactional properties, which precede its inherent properties (Piaget & Inhelder 1956, Lakoff & Johnson 1980). These interactional properties may have to do with motor activity, purpose, function, size, etc. Taking for example the concept of a geometric shape, the 'affordances' are discovered initially by the individual by using his/her motor skills. The recognition of the shape just by its internal geometry follows, drawing on an abstraction of all the past experiences concerning this particular shape, in the same way that a natural number is the characteristic property of a certain collection of sets (Skemp 1986).

It follows that visualisation and action can ameliorate misunderstandings imposed on geometric thought by the limitations of vision and of inadequate teaching. Haptic exploration of geometric shapes incorporates both visualisation and action. It relates to the perception of these shapes using tactile-kinesthetic information without the assistance of any visual input (Pick 1980, Schiff 1980). It involves the translation of the haptically-obtained stimuli into a spatial image of a visual kind (Piaget & Inhelder 1956, Weber 1978) (but see later discussion on mental representations of concepts). For the shape to be identified (ie named), some kind of match is sought between this spatial image and the already existing representations (concepts) in the individual's mind. In the case of recognising an unfamiliar geometric shape, the visual image is usually compared to simultaneously or continuously perceived visual stimuli (the shape is recognised from a set of visually displayed shapes).

From the descriptions of identification and recognition processes, it is obvious that visual processing ability and action play an important role in the haptic exploration of geometric shapes. Action is what actually provides the individual with

the required information, that after being processed visually will lead to a mental representation of the shape. Moreover the evocation of the mental representations of already acquired concepts suggests that possible misunderstandings could be detected. Piaget and Inhelder (1956) considered haptic exploration as the border-line between perception and mental representation of concepts. The perceptual stimuli can act as remediation agents, assuming that haptic exploration strategies of the individual are sufficient to provide him/her with accurate information.

Thus there is evidence to support the view that haptic exploration of geometric shapes may provide opportunity for training of visual processing skills and to support an 'active' approach in teaching geometry. Combining the positive elements of these two aspects of geometric thought, haptic exploration can provide us with a comprehensive way of circumventing some visual limitations in the teaching of geometry.

2.2.3 Touch Modality and Haptic Perception

The sense of touch has an essential role in exploratory and manipulatory activities. These activities require the integration of motor and sensory activities. Active exploration of shapes involving the sense of touch is performed by moving our hands and fingers about the object. The glabrous skin (soles of our feet, palms of our hands and on the smooth surfaces of our toes and fingers) is 'equipped' with a number of mechanosensitive receptors (four types of sensory units). Two types of these units are responsible for spatial discrimination, allowing localisation of stimuli (local sign). These types of sensory units occur in high densities at the fingertips as compared to other parts of our palm. The other two types of units respond to indentations, protuberances, stretching of the skin and tangential forces in the skin created when manipulating an object (vibratory stimulation, intensity of stimulus). These types of receptors have a much lower density compared to the previous units and are allocated uniformly in all parts of the palm (Pick 1980, Vallbo 1987).

In addition to the previous information, the individual perceives also proprioceptive and efferent information (Pick 1980). Proprioceptive information concerns the position and movement of the body, particularly those parts involved in active exploration. This information is gathered by kinesthetic, vestibular[1] and visual receptors (Sugden & Keogh 1990). Efferent information is available because the individual signals consciously an action plan to perform the exploration. The action plan and the exploration while performed may be altered because of the gathered proprioceptive information (see §3.3).

2.2.4 The Development of Haptic Exploration Strategies

Abravanel (1981) suggests that when a shape is explored, a series of sequential steps are required for the shape to be identified or recognised. Initially the individual has to be aware that exploration should be attuned not in manipulating the object (mere performance) but to perceiving its shape. A generic identification of the principal form characteristics follows, which is supplemented by strategic exploration of features and relations, leading to an integrated percept. Thereafter, gross or fine comparisons can be made with the mental representation or the visual stimuli.

These sequential steps are regulated by developmental changes. Children of 3-4 years perceive exploration of shapes as mere performance. They tend to grope the object, pat it with their fingers in a more or less meaningful manner. Recognition of the shape usually arises as a matter of accident. By the 5th or 6th year of age children use both hands to manipulate and/or explore the object. They seem to attend to the major features of the object but their strategies are more systematic, starting to discover relationships among features (Williams 1983, Piaget & Inhelder 1956, Abravanel 1981). After the age of 6 exploration becomes methodical. Movements of the two hands are coordinated, they move in succession,

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[1]This is information about the position and movement of the head in relation to the body (Sugden & Keogh 1990)

having one or more reference points. Reference points involve systematic return to these points, so that the movements can be reversed and therefore repeated and more easily integrated (Piaget & Inhelder 1956, Pick 1980). By the age of 9 children's fingers become also organs of perception, increasing the efficiency of the strategies. Moreover, children start to use strategies that involve simultaneous information pickup, besides the non-mobile and sequential scanning strategies (Millar 1981, Abravanel 1981). It is suggested that blind people explore objects in ways that are likely to provide unified percepts, which in general lead to more accurate mental representation of the shape (Abravanel 1981).

The perception and memory of individual objects are heavily influenced by their relations to their surroundings. Visual and tactile experiences with objects can provide the individual with external frames of reference. The activation of an external frame of reference in haptic exploration of objects can assist the perceiver. To constitute, though, a step of an exploring strategy, an external frame of reference has to be evoked intentionally. This means that the individual has to be capable of deductive inference (Bryant 1974).

It is suggested that perceiving shape and length by the means of haptic exploration may enclose qualitative differences (Bryant 1978, Weber 1987, Abravanel 1981). The exploration strategies for length follow a development analogous to those for shape. The coding though of stimuli from haptic perception of length may be of more quantitative nature (absolute coding). On the contrary, shape perception involves relative coding, which makes it a more elaborate and tedious process. Absolute values assist intra- but not inter-tactile explorations. In the latter cases relative values are more successful. Thus internal geometry of a shape involves the relationships between the shape's characteristics as well as the characteristics themselves. One cannot give absolute values to such characteristics and relations (Bryant 1974).

2.2.5   The Development of Haptic Exploration Strategies and Geometric Thought

The development and refinement of haptic exploration strategies correspond to the development of the tactile-kinesthetic system in young children. Additionally, haptic exploration strategies reflect in some way the geometric thinking of children.

In the van-Hieles model of the development of geometric thought, the child moves from a global recognition of shapes (level 1) to a more analytic appreciation of the shapes' internal geometry (level 2). The emergence of interrelationships of characteristics both within shapes and amongst shapes follows (level 3). The ability for deductive reasoning (level 4) and the construction of theory in complete absence of concrete models (level 5) are the final stages in the model (Dickson et al. 1983).

Usiskin addresses the difficulty in classifying, reliably, a student according to the model. Students may move back and forth while they are in transition from one level to the next. Moreover, there may well be students performing at different levels for different concepts (Burger & Shaugnessy 1986). These findings suggest that assigning ages to each level of the model would be a difficult task and possibly of no practical value.

From the above description we find a correlation between the development of geometric thought and the strategies employed in haptic exploration of shapes. A more-or-less global exploration strategy would agree with the first and second levels of geometric thought. From the appreciation of the interrelationships between a shape's characteristics (level 3), more analytic and methodological strategies are developed. When the child reaches the level of deductive reasoning, simultaneous perception of information and external frames of reference are employed to enrich and improve exploring strategies.

2.2.6 Perception of Geometric Shapes and Mental Processes

We discussed earlier the ways in which the individual can gather the tactile-kinesthetic information that will lead to an identification or recognition of geometric shapes. How do these perceptions translate into a mental representation? How do we decide whether this mental representation matches the corresponding concept in our mind? Can we identify haptically a shape that has been abstracted initially only by visual stimuli? Roth (1986) observed that geometric figures seem to be well defined 'in themselves'. That is, a conceptual rule exists that defines the necessary and sufficient characteristics of the geometric shape. Most concepts in geometry though are conjunctive and their formation depends on the number of critical attributes of its concept (Hershkowitz 1989). Therefore:

> *"In mathematics a definition does not serve to explain to people*
> *what is meant by a certain word. In mathematics definitions are links*
> *in deductive chains but how can you forge such a link unless you know*
> *in which it should fit?"* (Freunenthal 1973: 416)

What is actually coded then in these representations? Bruner and his colleagues suggested that individual concepts are represented by lists of properties or features, which actually define the concept (Roth 1986). Rosch though argued that in some categories some of its members are more typical than others (opcit). This view is in accord with the earlier discussion on prototype phenomenon and suggests that a list of features simply typifies rather than defines a certain category. An alternative view is provided by the Gestalt approach, where the whole is more than the sum of its parts and shape identification depends on the overall shape of a perceptual stimulus rather than on its individual features (Eysenk & Keane 1990).

All the previous approaches neglect the fact that people know many things about the relationships between the features of a shape. These relationships should be coded in mental representations and can be identified as 'higher order' features

characterising a concept (Roth 1986). These higher order features, are the means of forging the links within a chain and between chains. There may well be, then, different representations of one concept. These may serve the needs of different tasks (drawing, verbal response), different contexts (maths-classes, everyday world) and depend heavily on the individual's knowledge (opcit).

The form in which such representations are stored mentally is unknown as yet. Piaget (1956) and Weber (1978) believed that physical tactual features, encountered when exploring shapes, are coded as images or in terms of spatial features in exactly the same way as in vision. It would be an oversimplification, though, to assume that all people, always, process perceptual stimuli as images. Mental representations can be symbolic or distributed (Eysenk & Keane 1990).

Symbolic representations can be either analogical (nondiscrete, implicit, modality specific), or propositional (language-like, discrete, explicit, amodal). As Millar (1981) suggested though, it is next to impossible to distinguish totally between these two forms of representation. Furthermore it was suggested that 'verbal' and 'non-verbal' representational systems are interconnected by referential links, or that a special, spatial medium exists where representations are constructed by information coming from image and propositional files. An alternative consideration of the issue came from Johnson-Laird. He introduces mental models as representation of concepts, which can be wholly analogical, or partly analogical and partly propositional, which also are distinct from but related to images (images are considered as mental models 'viewed' from a particular perspective/angle) (Eysenk & Keane 1990).

Distributed representations of concepts account for a computational model of mental processes, consisting of networks of neuron-like units (connectionist approach). According to this view, information about a shape is stored in modality-specific units that are all interrelated with multiple synaptic connections. Concepts are represented by a pattern (or many equivalent patterns) of activation of neuronal networks. Each unit has an activation level which distinguishes whether the information stored in it will be disclosed and used or not during a certain task. An activation pattern then 'excites' only those units that hold useful information

for the task at hand (Anderson 1990). As Hinton et al. suggest, it is possible that all the previous models complement each other. We can accept that symbolic representation may characterise higher levels of cognition, while lower levels may be represented in a distributed fashion (to account for the mechanical and eruptive fashion of mental processes) (Eysenk & Keane 1990).

Since there is no consensus, either on how perceptions are coded or on how these codes are represented mentally, it is not surprising that there are many theories about shape identification. 'Template' theories suggest that for a shape to be identified a best match has to be achieved between the perceptual information and the mental representations stored in the mind. Each template corresponds to one shape and it is obvious that a shape cannot be identified if no corresponding template exists in the mind from past experience (Anderson 1990). 'Feature' theories suggest that shapes are recognised after a feature analysis of the perceptual input. These features are combined and compared against information stored in memory (Eysenk & Keane 1990). Finally, 'prototype' theories seek for a match between the basic or most crucial elements of a set of stimuli and information from past experience (Roth 1986).

Recognition of shapes does not rely on stored mental representations to the same extent as identification (Anderson 1990). The perceptual information gathered from haptic exploration is compared directly to an existing visual image. This process is more a function of cross modal communication of information and is considered to be more successful than identification (recall of concept).

2.2.7 Visual and Tactile-Kinesthetic Systems Integration

'Unity vs. separateness' of sensory modalities has been a long lasting argument in the field of perception (Gregory 1974). Molyneux, more than two centuries ago, *"asked his celebrated question whether a blind man, made to see, would recognise by sight alone an object that he had hitherto perceived only through touch"* (Millar 1981: 281). Knowledge is perceptually based but can it be classified as being either visually based or tactually based?

Many researchers agreed to the 'unity' of sensory modalities. Information retained in the nervous system is thought to be represented without specific reference to one single modality (Abravanel 1981). In 'separateness' theories a translation medium is required to relate the otherwise different and separate forms of information. Neurophysiology findings tend to suggest that modalities are complementary and convergent. *"At higher levels of the nervous system space is represented by overlapping integrated inputs from number of different modalities rather than in independent visual, auditory and somesthetic spaces"* (Jones 1981: 110).

Therefore we should consider representations of a certain shape derived from visual and tactile-kinesthetic systems as equivalent. The organ differences between vision and haptics however preclude complete isomorphisms of the activities used for information gathering. Under normal conditions vision is dominant over haptics (Matlin 1988). Vision is considered to be more holistic[2] as compared to a sequential gathering of information by the haptic subsystem. A haptic exploration strategy that allows simultaneous information pickup can smooth this inherent difference between the two systems. Gregory (1974) reported the difficulty that a blind person had, after having his sight restored, in identifying relatively unfamiliar objects until he had explored them by hand. Moreover, blind and sighted individuals perform similarly in recognising haptically unfamiliar shapes (Millar 1981). This evidence suggest that the term 'cross-modal' should not be used in a 'blanket manner'. The development of intersensory integration relies heavily upon intrasensory development and vice versa. Until the age of 5-7 visual-visual recognition of shapes is the most advanced, with visual-tactile and tactile-visual following and tactile-tactile being the least advanced. By the age of 9-11, as tactual perception becomes more efficient, intravisual recognition is the most efficient with the other conditions being equally advanced (Williams 1983, Jones 1981).

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[2]Except for very small objects or those fixated at a distance great enough to produce a small retinal image, visual forms are not considered unified (Abravanel 1981).

Moreover, as Williams suggests, intersensory integration takes place at three levels. The first level *"involves simple, low level or automatic integration of basic sensory information"*. The next level *"involves the higher order integration of perceptual features of stimulus information"* and finally *"a cognitive-conceptual integration process that involves the transfer of ideas or concepts across modalities"* (1983: 142). It is reasonable to suggest that intersensory experiences can refine mental representations and remedy possible misunderstandings. Therefore, a fusion of the visual and tactile-kinesthetic systems can be obtained through intermodal and particularly intramodal experiences.

Evidence reveals students' discomfort with tasks relating to an understanding of features and properties of geometric shapes. Haptic exploration of geometric shapes may provide an approach suitable for identifying and remedying possible misunderstandings in the mental representations of these shapes. This approach can improve visualisation skills and assist in the development of geometric and deductive thinking. The mental process of exploring geometric shapes haptically is complex. This complexity is reflected in a lack of consensus about the ways in which physical features are coded mentally, the form that they take when represented mentally and the extent to which sense modalities are integrable.

There are still, other agents that can determine behaviour in haptic exploration tasks which have not been addressed in the previous text. For sensory stimuli to be perceived they first have to be noticed. During the haptic exploration of geometric shapes by the individual, surface and material variables are sensed as well as the geometric variables of the object. Moreover, sound and olfactory senses and other external distractions may interfere with concentration on the appropriate tactile stimuli (Pick 1980, Matlin 1988).

Attention on the appropriate stimuli is closely related to the intentions of the individual. Tactile shape perception necessarily involves conscious deliberation. Therefore there should be a relationship between the individual's goals and the information extracted from the haptic exploring task (Gibsonian theory, Miller 1989). Knowledge and memory are also important since they influence the ease with which mental representations are retrieved from the mind (Roth 1986, Schiff

1980). Personality factors may differentiate performance in haptic exploration tasks (Schiff 1980). These factors lead an individual to perform a wide variety of perceptual tasks in a certain fashion. If, then, haptic exploration of geometric shapes is to be adopted for enriching geometric experiences, a consideration of all the previous factors has to be undertaken.

Chapter 3

Practical Activities and Motor Skills

Working on practical activities demands a certain level of motor-control competency. Materials accompanying a practical activity have to be handled with dexterity. In the following text we shall describe a movement situation in general terms and motor skills development will be addressed. We shall investigate the interaction of a student's developmental stage of motor functioning with performance on a practical activity. Other factors affecting movement performance will be considered as well.

3.1 Describing a Movement Situation

By the term 'movement' we mean any body movement. A movement skill is an organised sequence of movements initiated to achieve a certain outcome. Movement skills are *"goal directed, organised, adaptive and involve input and direction from sensory, perceptual and cognitive processes"* (Sugden & Keogh 1990: 1). We are mostly interested in 'fine control' movements of the hand and especially of a functional asymmetry form (both hands make different movements in a coordinated and complementary manner) (Keogh & Sugden 1985).

A movement situation is characterised by the mover and the environment in which the movement takes place. The mover carries a repertoire of resources to

cope with the requirements of the task and the environmental conditions (physical and social) (opcit). The actual movement is produced by the neuromotor system, since every movement involves muscular contractions. The mover's resource repertoire functions under the constraints of the biological and psychological conditions existing when the neuromotor system produces the movement (Laszlo & Bairstow 1985). Biological conditions refer to the current state of the individual's physiological systems, neuromuscular system, nonmuscle tissue and bone structure. Psychological conditions refer to the perceptual, cognitive and emotional systems. We see then that movement performance is regulated by endogenous and heterogeneous conditions. There is a constant interplay between these conditions and the task's requirements. Finally each movement situation is characterised by a level of demand which characterises the challenge that faces the individual (Keogh & Sugden 1985).

3.2 The Development of Motor Skills

Performing in a movement situation requires processing of body and environment information. As individuals grow older they become more proficient in their processing abilities and their sensory modalities are further refined. The best performance is achieved when the task's requirements match or are congruent to the individual's processing abilities. Every movement task is assigned an M-demand which is the demand on the individual's mental space. Performers with the same mental space structure may still perform differently according to the actual mental space that are able to use. Individuals, then, should be classified not only by their chronological age but also by their processing abilities (Keogh & Sugden 1985). Attention is conceived of as being a very limited mental resource (Anderson 1990). It implies withdrawal from some things in order to deal effectively with others (Klein 1976). Therefore, when young children are confronted with a novel motor task, some kind of time sharing takes place to compensate for the attentional capacity limitation. Moreover, as children grow older, they develop more

efficient strategies for selecting and organising a task's relevant variables (Keogh & Sugden 1985).

Another information processing faculty is memory, which also improves with age during school years. Addressing memory not merely as remembering and forgetting ,it is related to knowing, knowing how to know and knowing about knowing (Sugden & Keogh 1990). Short term motor memory limitations require the use of control operations (eg rehearsal) to maintain the attended information. Very young children seem to have few and unsophisticated processing strategies, which become refined with age and are used more spontaneously (Keogh & Sugden 1990, Laszlo & Bairstow 1985). Contrary to the study of short term motor memory, studying long term movement retention is a difficult task. Little research has been undertaken with children. Research with adults has proved that continuous everyday tasks (like swimming) are particularly resistant to decay. The tasks that are affected by time are the discrete tasks (eg using an instrument) (Keogh & Sugden 1990). It is suggested that important components of a skill may be learned without actually performing the movement (Suzuki method for violin, Keele & Summers 1976). This stresses the importance of mental strategies in the learning of a movement.

Along with the development of the individual's processing abilities, sensory modalities develop with age as well. This development takes place both in an intrasensory and in an intersensory sense. Studies with 12 to 14 year olds suggest that children with more advanced development of tactile-kinesthetic abilities perform better on conceptual and intellectual functioning tasks (Williams 1983). There is little doubt also about the role that vision plays, especially in the development of fine motor skills. Full development of fine motor skills involves the regulation of movement patterns by visual information (opcit). The individual's ability to use or combine simultaneous information from different senses appears to be rather important. Even though some relationships between modalities are developed by the first year of life, development continues in childhood and even in adulthood. The level of intersensory development seems to be more impor-

tant in the acquisition or refinement of fine motor skills (Laszlo & Bairstow 1985, Williams 1983).

3.3 Learning and Movement

Fowler and Turvey (1978) suggest that learning a movement skill involves the discovery of an optimal self-organisation in the sense of organising the neuromotor system in coordinative structures. Any particular movement pattern is then assigned with a relative *"attractiveness"* for solving a particular movement problem (Whiting 1980). This attractiveness may be considered as a mere personal preference or may be determined by the actual production of success or of economy in effort.

Models of learning in movement tasks fall into two categories whether or not feedback information is used (Adams 1976). 'Open-loop' models have no feedback or mechanisms for error regulation and therefore no compensatory capability. 'Closed-loop' models have feedback, error detection and error correction as key elements. Classroom learning activities can be better addressed by closed-loop models because of the appreciation ascribed to feedback. Even in well learned tasks, performance cannot become independent of feedback information. On the other hand, there are studies suggesting that reliance on feedback increases as a skill becomes well learned (Laszlo & Bairstow 1985).

Every movement task has a goal which initiates the learning process. According to the closed-loop models sensory and kinesthetic input, along with instructions, are attended to and stored in short term motor memory (Keogh & Sugden 1985). Long term motor memory is then sought for relevant information from past motor experiences and a plan of action is created. The motor programming unit then selects and activates the relevant muscles required for the performance of the specific movement (Laszlo & Bairstow, 1985). It is suggested that as the efferent outflow reaches the muscles, an efferent copy the plan of action is also sent to a comparison centre (Keele & Summers 1976). There, kinesthetic and other sensory feedback is

compared to the efferent copy. In that way the success of the movement is being assessed while performed and corrections of the efferent outflow and efferent copy may follow.

Closed-loop models have been criticized on a number of issues. They face storage problems, since for every movement a reference of correctness must exist against which the movement must be compared. These models also seem to treat performance in novel motor situations inadequately. In relation to the storage problem, success in novel movements cannot be justified only by a library of limited action plans or references of correctness (Whiting 1980). Finally, a persistent problem of theorists in motor control is the detection and correction of errors by the mover. In closed-loop models corrections cannot be made before the commands for actions are generated (Schmidt 1976). Only then feedback from the efferent outflow can be compared against the efferent copy. Thus the only error that the performer can detect is the failure, for some reason, to execute the plan of action effectively. Also the model cannot explain how the mover deals with an error in which the environmental goal is not reached despite the fact that the plan of action has been followed successfully (Keogh & Sugden 1985).

Schmidt's (1976) schema theory provides a possible solution for these problems. Schema theory postulates two separate states of memory, one for recall and one for recognition. The recall schema is responsible for the generation of impulses to the neuromotor system and is built up from past experience, taking into account the actual outcome and the response specifications. The initial conditions of the movement situation direct the recall schema to this particular movement. The recognition schema makes possible the generation of error information about a movement and is built up similarly, based on sensory consequences and actual outcomes and regulated by the initial conditions of the task.

3.4 Factors Affecting Performance and Learning of Motor Skills

Apart from the developmental issues discussed earlier, there are a number of factors that may affect the learning of motor skills and performance in movement situations. The initial instructions and the feedback received after the completion of the movement are crucial to the performance in a movement situation. Initial instructions can be verbal or demonstrated, or both. They direct attention to certain aspects of the task at hand or even indicate strategies for dealing with the task. Designated strategies may increase the rate of initial skill acquisition but do not facilitate learning in transfer situations (Singer 1980).

Any event that follows a response is considered as a reinforcer increasing the response's probability of occurrence. In motor learning this event is called *"knowledge of results"* (KR). Knowledge of results gives information about the goal achievement and if given in well defined quantitative, demonstrated terms can lead to improved performance (Laszlo & Bairstow 1985). The optimal level of KR becomes more precise with age. Caution is needed though in its use, since extreme levels, very imprecise or very precise, can destroy performance on the task (Keogh & Sugden 1985). It is suggested that artificial feedback should be used only temporarily in cases where natural feedback resources are impoverished. Individuals should be encouraged to use their own mental qualities to evaluate movement performance (Keele & Summers 1976).

Personal and social influences are probably indirect and can either facilitate or interfere with the interplay of other factors. Differentiation in task proficiency may reflect different opportunities for practising a movement skill or different interests and motivation (Laszlo & Bairstow 1985, Keogh & Sugden 1985). Therefore concentrating only on what the mover can or cannot perform may be misleading at times and may not show what the individual is actually capable of doing.

The demand that a movement task may pose to an individual may be either on motor control or cognition, or both. During the acquisition of a new skill the learner attempts to understand the requirements of the task, with a consequent deterioration in motor performance (cognitive stage). As the situation becomes more familiar, motor control is refined by the use of KR (associative stage). The movement skill becomes autonomous when the demand for cognitive control is further reduced (autonomous stage) (Fitts' stages of learning, Wall 1986). It is obvious then that an interplay between motor and cognitive functioning does exist. As schema theory suggests, movement memory is cognitive since it is stored as rules and principles and can be applied in a range of movement situations. The efficiency of an action plan then depends on the amount and quality of information contained in the long term motor memory (Singer 1980). The link, though, between motor and cognitive functioning is far from being direct, especially after the seventh year of age (Williams 1983). It is worthwhile mentioning that fine motor behaviour is the most important contributor to whatever relationships exist between these modes of functioning, motor and cognitive (Williams 1983).

3.5   Motor Control Skills In Practical Activities

Learning situations in mathematics classes, and particularly practical activities, require fine movements of a functional asymmetry form. Practical activities can be considered as novel movement tasks, that at times require the use of other, supposedly well learned skills (eg the use of drawing instruments).

If we accept Bruner's view about motor behaviour, every movement comprises a number of movement units, which are part of a movement vocabulary with movement syntax (modularisation) (Sugden & Keogh 1990). Every movement unit can be seen as a vector, having as parameters direction, extent, velocity and force (Laszlo & Bairstow 1985). All these factors have to be considered before the activation of a movement unit. The production of the movement becomes even more complicated when the movement requires the release of objects (eg in

construction tasks) (Keogh & Sugden 1985). It is then obvious that applying even well learned movement units in a novel practical activity reflects a fair amount of difficulty.

The relationship between motor skills and performance in mathematics is an area, though, that considerably lacks research. Motor skill learning resembles that of cognitive skills. Fine motor movements require the production and execution of an action plan. This functioning 'appears' as motor only after the cognitive difficulties of mastering the certain skill have been superseded. Attention should be drawn to the fact that the quoted research findings attempt to explain how the development of motor functioning as a whole correlates with cognitive functioning as a whole. This perspective cannot address cases where manual dexterity is accompanied by underdeveloped cognitive functioning, or where sound cognitive development is impaired with movement skill deficiencies.

Motor skills are only the means to exploit an activity's potential in practically demonstrating mathematical experiences. The research findings on the development of motor functioning processes are important. When developing a practical activity special care should be given to the demand on motor control skills. There is the danger of going beyond the students' level of motor development and simply asking too much of them.

In practical activities there is also a cognitive demand, of a different nature, concerning the mathematical (or other) concepts involved in the problem situation. This cognitive demand may interact further with motor performance. Well mastered movement units may be proved inefficient if they cannot be collated to form an appropriate action plan (eg the use of drawing instruments in geometric construction). Moreover failure to resolve a novel problem situation can temporarily affect the actual motor skills (negative affective predisposition). Therefore the cognitive demand or difficulty of the task is of critical importance in the successful application of movement skills.

Chapter 4

Methodological Background

4.1 The Greek and the Scottish Educational Systems: An Outline

Formal education in Greece begins at the fifth year of age, with children spending one year in nursery school (Neipiagogeon). Primary education is completed in six years (Demoticon, 6th to 12th years of age), while secondary education covers six more years of schooling (12th to 18th years of age). Secondary education is further divided, in equal parts, between the Gymnasium (12th - 15th) and Lyceum (15th - 18th). Lyceums are mainly of three types: the 'General', the 'Technical Vocational' and the 'Integrated'. The Integrated Lyceum provides experience in a wide range of practical fields. Other types are available as well, emphasising the teaching of specific subjects (Classical, 'Ecclesiastical', Musical, Physical Education). The General Lyceum is usually attended by students intending to proceed to university, even though all Lyceums give access to Higher Education (universities, technological institutions). The national curriculum is common to all schools and the same textbooks are used nationwide (issued free of charge). There are no clear suggestions concerning individual pupil differences in ability and attainment. Assessment is informal in the first four years of Demoticon. It becomes formal but still internal in the last two years, when grades are awarded. In general, promotion to the next class is automatic. In secondary education assessment remains internal, except in the last year of Lyceum when students have to pass examinations for entrance into Higher Education. Grading is emphasised

in these years and students may have to re-sit examinations in September, or even repeat the class during the next academic year.

In Scotland schooling begins at the fifth year of age. Primary education covers seven years of schooling (5th to 12th years of age), followed by four to six years of secondary education (12th to 16th, 17th or 18th years of age). The curriculum up to age 14 is based on guidelines issued by the Scottish Office Education Department, with schools having the freedom to shape it according to their particular needs. Provisions are made for individual pupil differences in achievement and attainment. Assessment is internal until the fourth year of secondary school. At the end of the fourth year students sit external examinations that determine their later schooling. Some students may leave school after this stage, holding a Scottish Certificate of Education (Standard Grade). The higher achievers continue their education for one more year before sitting Higher Grade examinations for entrance to university. It is recommended though that they continue their studies for one more year (Certificate of Sixth Year Studies) before starting courses at the university. For the less academic pupils a wide range of vocational modules is available (SCOTVEC modules), which usually head on to Further Education.

Mathematics syllabuses in Greece are prescriptive, in the sense that the hours spent in each area are predetermined. This phenomenon is more apparent in Demoticon. There is a discontinuity between primary and secondary school syllabuses, with many areas being repeated. Recent recommendations on possible reforms addressed the need for continuity in the syllabuses for Demoticon and Gymnasium. In Lyceum the syllabus is broken into Algebra and Geometry, each being taught in separate classes. Overall, mathematics syllabuses give an impression of being rather ambitious, with emphasis on arithmetic in early years and geometry in later years. In Scotland, mathematics syllabuses are determined in each school following the national guidelines. Decisions taken at school level concern the textbooks and other materials to be used, as well as their time allocation over the school year. The syllabus is common for all pupils until the third year of secondary school, when students are set according to ability and attainment. Great emphasis is given, overall, in building problem solving and inquiry skills.

4.2 Culture: Making a Difference in the Results?

As indicated above the Greek educational system is centrally managed, compared to the decentralised management of the Scottish one. This difference is reflected in the organisation and functioning of the mathematics classrooms in the two countries. The rationale underlying the Greek system is 'equal opportunities for everyone' at all levels of education. Mathematics in the classroom is taught formally by talking to the students from the front. Instruction is restricted to the textbook provided by the state. Students sit in rows and activities (questioning, problem solving) usually engage the whole class. Everyday evaluation is public and visible to all students, with academic feedback being immediate in a verbal form. Types of academic and behavioural feedback can vary markedly, though, among teachers. Homework is assigned to the students for practice and consolidation. Private tuition is common from the early years of secondary education. The pervasive extent of this phenomenon ('parapaedia') has created a form of education that runs in spite of and at times may take the role of school in preparing students for their examinations.

On the other side the Scottish educational system is more meritocratic, with a tendency to move the able students faster (the 'learner' educational principle, J.I.H. McDonald 1989). Teaching mathematics in classrooms follows an individualised approach. Students do not often engage in tasks as a whole class and usually have fewer chances to receive instruction from their teacher. Evaluation is mostly in written form and personal. Educational feedback may be delayed in many cases, since students usually work their way through a set of tasks before correcting their answers (they are encouraged to correct them by themselves using answer-books). Private tuition exists in Scotland but not to the same extent as in Greece.

It is suggested that beliefs about children are the result of history, culture

and personal disposition and serve as a grounding rationale for the actions taken by parents and educational institutions (Gergen et al. 1990). Since schools are considered as institutions responsible for the transmission and construction of culture (Eisner 1977c), it is suggested that differences between the mathematics classrooms of the two countries are reflected in sociocultural differences.

Stigler and his colleagues (Stigler et al. 1990, 1987, 1986; Stevenson et al. 1986) observed Chinese, Japanese and American primary classes and suggested that differences in mathematics classrooms, like those existing between Greece and Scotland (classroom organisation, functioning), may well be related to differences in learning. They also found that parental beliefs about teaching and mothers' evaluations of their children's mathematics performance corresponded to the actions taken by teachers and educators. For example, American mothers attributed performance in mathematics to innate abilities of the child and therefore emphasise individualised teaching. In contrast, Asian mothers assigned more weight to effort and hard work and favoured uniform educational experiences for all students in a classroom (Stigler et al. 1990, Stevenson et al. 1986). Gergen et al. further suggested that parents' *"beliefs about child development may have their origins in and be sustained by a substantial array of conventionally related activities"* (1990: 122). They investigated German and American women's beliefs about competition in the work place, solidarity of the family and centrality of motherhood. Beliefs about these actions correlated with beliefs about social or independence needs of a child and about attention to autonomy, emotion and cognition. It is similar beliefs that may shape decisions for the setting of educational experiences in schools. We should consider, then, the possibility of culturally appropriate, or even acceptable teaching methods. These constitute a capitalisation on culturally well-practised routines to determine participant structures in the classroom (Brown & Palincsar 1989). Learning in school, then, may correspond to the 'situated' learning of personal life outside school, with this approach in learning forming the basis for possible curricula (Resnick 1989, Burton et al. 1984, Greenfield 1984).

When comparing mathematics classrooms across cultures though, surface features appear to be more similar than different. The research findings presented

above focus on subtle features that may affect decisions in education and therefore learning in the classroom. Surface features though may prove to have great influence on classroom learning as suggested below.

Commonalities in characterising classrooms across cultures concentrate on the taught subject itself and are reflected in the act of 'educating the young' in general. Goodnow suggests *"cognitive development is marked by the acquisition of values"* (1990: 259). Learning in the classroom is in a *"collateral"* fashion, where the formation of knowledge is accompanied by the formation of enduring attitudes, likes and dislikes (Dewey 1963). This 'other' knowledge has a sociocultural basis and has been described as tacit knowledge (Hundeide 1985), as frameworks of interpretation (Gergen et al. 1990) or as implicitly modelled messages (Goodnow 1990). This knowledge can determine behaviours, problem solving approaches and performances in particular contexts (different 'senses' of action). It can also attribute significance to certain kinds of problems and shape beliefs about areas of knowledge and skill (opcit) and therefore affect learning in a general fashion.

Bishop (1988) suggests there is no explicit attention paid to values in mathematics teaching. He distinguishes between mathematical education that contributes to development of values and mathematical training that treats mathematics as a body of knowledge. This distinction is reflected in the way mathematics is taught in the classroom. Skemp suggested *"that there are two effectively different subjects being taught under the same name 'mathematics' "* (1976: 27). The factors differentiating between the two subjects are located in and characterised by the difference between the instrumental and relational understanding in mathematics (Skemp 1979, 1976, Byers & Herscovics 1977). This is the difference between applying appropriate remembered rules and deducing specific rules or procedures from more general mathematical relationships. Teaching mathematics out of context and in ways that do not correspond to everyday life learning emphasises this latter difference (Resnick 1989, 1987). Mathematics, then, is seen as a body of knowledge, separated from real life, a subject in school taught through books or by experts (Goodnow 1990).

There is also a tendency in mathematics classrooms to perceive the ability to

cover a large number of problems in a single lesson as characteristic of expert teaching (Stigler & Perry 1990, Leinhardt 1986, Leinhardt & Greeno 1986). It is suggested that the coherence of text enables or allows the learner to infer relations between events and therefore promotes understanding (Beck & McKeown 1989). Stigler and Perry, speculating, suggested that *"mathematics lessons may be easier to comprehend and students likely to learn more, when the episodes that comprise the class are coherent"* (1990: 345). This speculation is supported by Leinhardt and Putnam (1987), who emphasised the recognition and anticipation of the components of a lesson as a factor that promotes learning in class ('lesson parsing'). To cover a large number of tasks then, in a single lesson, may endanger the coherence of the lesson, especially if the transition between activities is not clear and if the amount of time between these is too small (Stigler & Perry 1990). Moreover, students may come to believe that homework and test problems are impossible tasks if these cannot be solved in a few minutes (less than 12, Schoenfeld 1988, 1989). Along with the demand for formal records of procedures in the classrooms (Desforges 1985), learning activities become concretised and are interpreted as jobs to be done (Bereiter & Scardamalia 1989). Various strategies may be adopted by the students then in order to avoid work, please or challenge the teacher's authority, which do not necessarily lead to learning (Woods 1985). Related to the issue of coherence of mathematics lessons are the teachers' own beliefs about what comprises a coherent lesson. The 'cognitive mediational paradigm' for research on teaching (Winne & Marx 1982) suggests that there is a *"Noticeable lack of one-to-one correspondence between instructional stimuli that the teachers identified and the cognitive processing that these cued for the students"* (opcit: 513, Ben-Chaim et al. 1990). Therefore what teachers perceive as a coherent lesson may not coincide with the students' perceptions about lesson coherence. Messages mediated by the teacher should be as clear as possible so as to smooth the mismatch between instructional stimuli and students' cognitive responses.

Reflecting upon the arguments discussed above, we could suggest that a 'classroom culture' exists and heavily determines learning in the classrooms. This 'classroom culture' appears to have a uniform pattern across different cultures, mainly

due to their common characteristics. Schooling seems to be closely associated with modernisation, economic growth and national aspirations. There is a danger of education being reduced to training, with academic success being arbitrarily related to life success (Resnick 1987). This reflects the more widespread subsumption of culture by civilisation (Marcuse 1984). In reality though, the 'classroom culture' is the ethos of a group created and sustained by the participants in the educational experience. Dewey (1963) suggested that the teachers can readily alter only the objective conditions of an educational experience. This of course may require modifications of internal conditions and more specifically the teachers' possible predisposition towards learning and teaching. Teachers' initiatives can form the scaffold for students to build positive attitudes towards mathematics, if these are not sacrificed for immediate, short-term success. Teachers provide a model in the classroom whether they intend to or not. If, then, teachers are seen more as a model and less as an instructor, it is possible that students' beliefs about mathematics and learning will be altered or enriched.

4.3   Materials Used in the Study

The practical activities used in this research are the 'Feely Box' (FB), the 'Same Shape As' (SSA) and the 'Drawing and Geometric Constructions' (DGC) activities.

The FB activities are based on an idea introduced by Geoff Giles. The feely box is a cardboard cubic box with two holes cut on opposite sides. Students are asked to explore haptically objects placed in the feely box. Seeing the object while exploring it is considered 'cheating'. In the FB activity the objects are two and three dimensional and some composite shapes. The first three worksheets (1.1-1.3) involve identifying, sketching and discussing properties of these shapes (edges, vertices, faces, characteristic properties of each shape). Worksheet 1.4 involves recognising composite shapes, while worksheet 1.5 introduces some work on perimeter, area and proportion. The final worksheet of the activity (1.6) requires

students to draw two dimensional shapes on dotty and isometric paper following the instructions given (see Appendix A).

The SSA sequence of activities use rectangles to introduce the simplest ideas about similarity. Due to the colloquial meaning of the word 'similar' the term 'same shape' is used. The activity starts with an example sheet displaying pairs of rectangles that have or do not have the same shape. This sheet encourages the students to start thinking about the relationship which characterises same-shaped rectangles. Worksheets 5.1 to 5.3 provide the foundation for the succeeding activities by requiring the students to identify and construct same-shaped rectangles. Worksheet 5.4 provides students with a practical way of investigating the 'sameness' of rectangles, while 5.5 leads students to a more formal definition/explanation of 'sameness' of rectangles. The ratio test (worksheet 5.5 - ratio of length to breadth) has some potential for misleading the students, since it does not apply to other shapes. This has to be stressed at some stage during the SSA sequence of activities. The activities that follow (worksheets 5.6-5.10) reinforce the concepts and skills acquired during the preceding worksheets. They extend into aspects of problem solving and provide the opportunity for students to discover and appreciate the aesthetic and practical aspect of A (the standardised A-sizes of sheets of paper) and golden rectangles. The worksheets also provide practice for number, ratio work, enquiry and drawing skills (see Appendix B).

The DGC activities consists of two collections of activities ('Be a Geometric Constructor' and 'Balancing Polygons'). They are accompanied by a booklet ('Measuring and Drawing Library') with illustrated information and hints on the use of measuring and drawing instruments. The work included in worksheet 1.6 of the FB activity is also considered as an extension of the DGC activity. All these activities require accurate use of drawing and measuring instruments (compasses, set square, ruler, protractor) and the ability to follow instructions. Aspects of problem solving are reinforced by the worksheets on geometric constructions. Finally most worksheets have an outcome that is pleasing to the eye and can be intriguing mathematically due to the simplicity and generality of the construction (see Appendix C).

The material was developed over a period of six months (September 1991 to February 1992). These activities were designed to conform with the 5-14 Scottish Guidelines for Mathematics consulting also the Greek Mathematics Syllabus. More specifically, problem-solving and inquiry, multiply and divide, round numbers, ratio, measure and estimate, perimeter and formulae, range of shapes (Levels D and E). Drawing skills and the use of calculators (calculating, checking, investigation, problem solving) are also reinforced.

A group of teachers from Scottish secondary schools was set up by the Edinburgh Centre for Mathematical Education (ECME), under the name 'Practical Work in S1/S2'. Members of ECME participated in this group as well. The preliminary meeting took place in June 1991. The main objective at that time was to collect views and thoughts on practical work in S1 and S2 and identify areas of difficulty in mathematics which may benefit from some practical activities. These suggestions were taken into consideration in the development of the material. They concerned the acceptability of the learning approach, the appropriateness of the use of language (enough but not too many words, appropriate for the students' level of understanding) and the demand for simple and readily available apparatus accompanying the activities.

The 'Practical Work in S1/S2' group remained active for one year. Over this period the FB and the SSA activities were tried in schools in Edinburgh, Greece and Dumfries and Galloway. The author was present at these schools during the sessions with the practical activities. Other schools, apart from these mentioned, tried the material without providing any feedback. Inappropriate sequencing of the tasks for each activity was detected, along with mistakes in the layout of the worksheets. In May 1992 a workshop was offered to other teachers of Lothian Region. The FB, SSA and DGC activities were introduced during this workshop along with other material. Susan MacGillivray (Annan Academy) led the work on the FB activity following a group approach. Jim McGregor (Whitburn Academy) led the work for the SSA and DGC activities using a mixed approach (stations and individualised learning approaches). The three activities were also offered as

a part of a workshop at the Mathematics Teaching Conference 1992 in Edinburgh by Ann Aughwane (St. Georges School, Edinburgh).

4.4 Methodology

The evaluation of educational material necessitates decisions on possible methodological rationales. Such decisions cannot be taken before taking account of the complexity and subtlety of the phenomenon studied. Studying attitudes of students and teachers towards an innovatory learning approach implies studying classrooms as a fusion of intentional worlds that participants carry with them and live within the school.

Intentional worlds draw their existence from the people who live in them and who are, in their turn, influenced by intentional objects (feelings, beliefs, attitudes, concepts, percepts and so forth, Shweder 1990). In the learning environment of a classroom such worlds are those constituting the learning milieu, a nexus of cultural, social, institutional and psychological variables (Parlett & Hamilton 1972). It would be unrealistic to suggest that an educational study can analyse all these parameters. We can only hope to address and illuminate an array of questions (Stake 1977b, 1967, Parlett & Hamilton 1972, Eisner 1977b, Centre for New Schools 1977). These questions form the issues that the research is dealing with. Such a *"thick description"* of the study's ambitions implies continuing openness and responsiveness to the kaleidoscopic nature of a classroom setting. An explicit statement of aims and objectives made too early may hide an intention to discover respective outcomes, excluding from the expected outcomes-spectrum those that become apparent later in the study (Atkin 1977a, Eisner 1977a, Parlett & Hamilton 1972, Stenhouse 1977).

This suggests that methodological steps should follow an understanding of the framework within which participants interpret their thoughts, feelings and actions. Since all thinking is thinking about something (Chamberlin 1974), one direction for reaching people's perceptions and mental states about intentional objects would

be to study the consciousness which participants have of them (phenomenological approach). Such an intentional analysis, as Husserl (1969, in Chamberlin 1974) suggests:

> *"...has to place before its own eyes as instances certain pure conscious events, to bring these to complete clearness, and within this zone of clearness subject them to analysis and the apprehension of their essence, to follow up the essential connections that can be clearly understood, to grasp what is momentarily perceived in faithful conceptual expressions, of which the meaning is prescribed by the object perceived or in some way transparently understood."* (Chamberlin 1974: 128)

Intentional analysis is an intuitive analysis striving to give meaning to explicit and implicit data. It is not a construction of meaning. It is a description of reality drawing its determination from the situation that it describes and is further determined by keeping at a distance everything that does not account for this determination (Chamberlin 1974).

The philosophy of evaluation just described provides a direction, not a stance. Means of reaching a determination of reality, then, vary according to the particular conditions of the study. Any condition existing prior to teaching and learning which may relate to outcomes has to be identified. Observations and discussion with the participants could provide a profile of such information. This approach can also familiarise the researcher/observer with the environment and the participants and vice versa. Data from the many encounters of students and teachers with the materials, of students with students, of students with teachers, of students and teachers with the researcher/observer follow. Recordings of such encounters, interviewing the participants after the completion of the study or administering a questionnaire can assist in penetrating the immediate perceived situation (Stake 1977a, Parlett & Hamilton 1972). The teaching material is also judged by the students' actual performance on the materials. Numerical data, though, are not of chief importance here. It is the nature or an underlying structure of the students' responses and blunders that is important in evaluating this teaching

material (Scriven 1967). Moreover, the possibility of an educational study finding a universal truth may well be characterised as utopian. It would be better to speak in terms of analogies applicable to some other situations, an approach that is tangential to an intentional analysis. In this way we shift our attention from things and minds to the relations between the experiencing subject and the experienced objects (Moore, 1917 in Chamberlin 1977).

Subjectivity in intentional analysis/description is a feature as in all studies conducted by humans. *"...the thinker never thinks from any starting-point but from the one constituted by what he is."* (Merleau-Ponty, 1970 in Chamberlin 1977). One characteristic of intentional worlds is that they are both different and the same for different subjects. This addresses the fact that individuals may perceive certain features of these worlds as the same and others as different, according to their will, emotions, experiences or knowledge (Lauer, 1958 in Chamberlin 1977). Our aim with an honest description of such worlds is to establish 'commonness' of meaning by investigating, collectively, the intersubjectivities of the participants (Pramling 1983, Chamberlin 1977). Moreover, a 'thick description' of the research can demonstrate the evaluator's presuppositions, making him/her account for them and expose them for what they are (House 1977).

This report will strive to follow the ideas discussed above. A main objective is to evaluate the 'hands-on' method of learning mathematics, taking into account the participants' intentional worlds. This implies the need to identify possible cultural differences that might illuminate the differences which have occurred in the data collected from the two countries. Students' performance on the materials will also be seen in conjunction with the imposed cognitive demand of the concepts involved in the process.

Chapter 5

The Study and Its Results

5.1 Describing the Study

The research took place in two phases. The first phase concerns work done in Greece during March and April of 1992. The practical activities were used in four Gymnasiums in urban and rural geographic areas of Greece. These schools were 'chosen' by means of personal acquaintances with teachers or head teachers. A letter was sent to all schools explaining the objectives of the research and describing the practical activities.

Patras is a city of about 300,000 inhabitants, one of the bigger ports. Patras Experimental Gymnasium, the only High School in the city affiliated to the University of Patras, was the first school visited. Students at this school are randomly selected but an application is required beforehand. They usually come from middle-upper class families and their parents are mostly well educated. The three year groups at the Gymnasium had 180 students allocated in six classes, two for each grade. A rural school, Vlachokerasias Gymnasium, was next to be visited. Vlachokerasias is a small village of 1000 inhabitants. The school serves the needs of a wider area. It has Demoticon and Gymnasium classes. The number of students has decreased over the past years, resulting in the closure of the Demoticon classes as from the current school year. During the term of the study the Gymnasium had 40 students, one class at each grade. Most students came from families of farmers or other manual occupations mainly of low to average level of education.

Patras and Vlachokerasias are both in the southern part of the mainland. The remaining two schools were located in the middle area of the Greek mainland, on the eastern side. Volos is another big port with a strong cultural heritage, especially in education. The city's population is around 120,000. The Ninth Gymnasium of Volos was the third school visited, a school of 270 students and three classes in each grade. Due to its location at the centre of the city most students came from middle-upper class families with well educated backgrounds. The last school that took part in this phase of the research was the Second Gymnasium of Almyros, a school accommodating a total of 220 students in three classes for every grade. Almyros is a small town of around 10,000 inhabitants, 30 km S-SW of Volos. The school is close to the edge of the town, with students coming from a wide range of family backgrounds (from lower to middle class).

The second phase of the research concerns work done in Scotland during November and December 1992 and February and March 1993. Four schools were chosen in order to match the sizes of the Greek schools and also to provide a good mixture of urban and country settings. A letter was sent to all schools with details of the objectives of the research and the research materials. In three a preliminary visit preceded the actual sessions on the practical activities. This was not practicable at the fourth school, Tobermory High School. The purpose of this contact was to meet the teachers, organise the later formal visits and become familiar with the school's environment. Sanquhar Academy was the first school visited. The school is situated at the one end of the town, serving a population of about 10,000. There were 130 students in Secondary 1 (S1) and Secondary 2 (S2) allocated in three classes for each grade (only the first two Secondary years are quoted so to obtain a more objective comparison with the sizes of the Greek Gymnasiums). The economy is based on farming and other manual occupations with an evident unemployment problems. Tobermory High School was the other school visited before Christmas. Tobermory is a fishing village being at the north part of the Isle of Mull, west of the Scottish mainland. It was the only Secondary school on the island, with 60 students in S1 and S2 (two classes in each grade).

After Christmas of 1992 I visited two more schools. Selkirk High School,

in the small town of Selkirk near the Scottish Borders, serves the needs of a wider area with a population of 10,000 having 200 students in S1 and S2 (four Secondary 1 and three Secondary 2). The community is organised around farming and industry with some unemployment problem. Drummond Community High School was the last school taking part in the research. It is in Edinburgh, the capital of Scotland (620,000 inhabitants). The school has a strong multicultural identity, serving a community with many minorities (mainly Asian). It has also a role in Adult Education, providing the opportunity for adults to take up classes in various subjects. The first two grades are comprised of six classes, three in each grade, accommodating 130 students.

Apart from Vlachokerasias, the Greek Gymnasiums were accommodated in two-floored, fairly modern buildings. At Patras and Volos three more schools shared the same building (another Gymnasium and two Lyceums). This phenomenon is common for urban Greek schools due to the small number of schools as compared to the number of students. Gymnasiums and Lyceums have to rotate their timetable from morning to evening every week. The schools had a large playground, with courts for popular sports, where students spend their breaks. I followed each school's timetable for at least three days. During that time I lived in the community served by the school. During these days I had the chance to observe first and second year classes, studying other subjects as well as mathematics.

Greek school days start with a prayer, which all students have to while attend lined up in the playground. Announcements by the Head Teacher follow if necessary and then students enter their classrooms. The teacher follows after a few minutes, while one student usually waits for him/her at the door. All subjects are taught in the same classroom, the homeroom of each class. The number of students in each class varies from 13 at Vlachokerasias Gymnasium to 34 at Volos (the number of students in a class cannot exceed 35). Students sit in twos allocated in rows, with the teacher's desk usually in front of one of the side rows. In two of the schools (Volos and Almyros) the teacher's desk was placed on a podium. One of the students in each class is given the task of filling in the 'absences book' for every session. This student is the student that graduated the previous year with

the highest overall mark. Teachers have to sign the absences book at some stage during the session. All these books are stored in the head teacher's room after the end of the school day. There were no resource materials in the classrooms apart from blackboard-drawing instruments. Any materials were usually kept at the Head Teacher's room and teachers had to borrow them for each period. At Vlachokerasias Gymnasium a set of geometric shapes was also available that was locked inside a cupboard. Volos Gymnasium owned an overhead projector but teachers never used it. As a teacher explained they did not know how to operate it and also it was not convenient to carry it up and down stairs. No posters concerning mathematics were displayed on the walls. A portrait of Christ was hanging above the blackboard, a characteristic of all Greek classrooms.

Each day's schedule consists of (at most) six sessions of 45 minutes each. A five to ten minute break separates two consecutive sessions, during which students have to leave their classroom. Only two students remain in the classroom tidying up the desks, cleaning the blackboard and ventilating the room by opening the windows (these two students change every week). During the breaks students may engage in popular sports in the playground while teachers patrol the building in order to prevent misbehaviour. Each school had two mathematics teachers, except from Vlachokerasias which had one. Patras Gymnasium had a separate room as a library for the teachers. The other schools had reference books in the teachers' common room. At Volos students were running a borrowing library with a small number of books, none about mathematics. All the teachers were friendly, particularly at Volos and Almyros. They were willing to answer any of my queries and were interested to know about the objectives of my research.

In the mathematics sessions teaching usually followed three phases. Questioning on the previous day's lesson (oral and blackboard assessment), presenting the next lesson and finally working on applications from the school book and giving out homework. The style of teaching can be described as the standard, 'talking to the whole class' style. The teacher spends most of the time talking to the students, defining and initiating activity mainly by asking questions. Students' activity was limited to raising arms to answer a question or to performing a task

on the board. At Patras and at Vlachokerasias Gymnasiums and for one of the teachers at Volos, the questioning had in no sense the characteristics of a dialogue. It was too formal, merely a matter of waiting for the correct answer. Mistakes were not discussed. Correction came only through verbalising the correct answer, either by the teacher or by a student. Explaining was reduced to rephrasing the answer which had already been given. There was no exposition in groups nor to any individual. The sessions were not coherent since the transition from one topic to another was swift and not clear. In general teachers appeared to be remote from the students due to a lack of feedback (either cognitive or psychological). Their authority over the class was merely positional and sapiential. Comments to students were sarcastic at times. At Patras, the teacher's sapiential authority was challenged by the older students, with students showing a clear satisfaction. During the sessions students at Patras engaged in irrelevant activities, especially those sitting in the back desks of each row. At Vlachokerasias and Volos students spoke only when teacher permitted it. At some stage of the session the teacher inspected the students' jotters where they keep their homework. This inspection was not meticulous though. Homework was given in the end of each session, usually after the bell had gone. It consisted of exercises from the school book.

The other teacher at Volos Gymnasium and the teacher at Almyros favoured a discussion with the students. They did not give the impression of assessing them. They were not just seeking a correct response. On the contrary they seemed interested in exploring students' understanding (tolerant of blunders, giving adequate time for students to think after a question, rephrasing a question, giving hints). New concepts, definitions, rules arose naturally from the discussion, a fact that made their teaching smooth and coherent. Dictating to the class was used for reducing the chances for misconceptions and only after students had expressed their own opinions on the discussed concept. Exposition to individuals often occurred during the sessions and checking the students' jotters was an activity with an educational value placed upon it. Homework was given again after the bell had rung. The students' participation was strong, even amazing, during the sessions. At times students were 'begging' for the teacher's permission to answer a ques-

tion. The teacher always made an effort to provide equal opportunities for every student in participating in the session's progress. The climate of both classes was relaxed and understanding, teachers were polite and friendly and it seemed that their positional and sapiential authorities were also well established.

In Scotland, the only school that was accommodated in an old building was Drummond Community High School. It was also the only school that had a large playground attached to its premises. I followed each school's timetable for at least one week, living close to the school for all these days. I had the opportunity to attend first and second year classes in various subjects. At Tobermory and Selkirk I did not have the chance to attend any mathematics sessions.

The schools' timetables were very different. The number of sessions for a day varied from six at Tobermory and Selkirk to eight at Sanquhar and nine at Drummond. Sessions lasted from 40 to 55 minutes, with a long lunch break and one short break during the morning. Students had to change classroom in between sessions since rooms are assigned to teachers and subjects (departments). The first event of the day is registration. Each teacher is responsible for one class and during registration time s/he keeps a record of the absentees and makes any announcements. Teachers look for absentees every session thereafter and report them to the school's administration office. Students usually have to line up outside the classroom waiting for the teacher to allow them to take their seat. In all mathematics classes they were sitting in rows of two. The teacher's desk was at the front at the same level as the students. The classes were well equipped. A BBC computer and an overhead projector were in most rooms besides the separate facilities for computer studies.

Posters were displayed on the walls with some of the students' work as well. Drawing and writing instruments, calculators, even jotters are provided for the students. School books do not belong to the students but they are allowed to take them home if the circumstances demand it. All schools had libraries with a large selection of books, journals and periodicals for students to borrow or consult for their projects. Tobermory and Selkirk High Schools even had CD-ROM facilities in their library. Libraries could also be used as study rooms. Students are allowed

to leave the school during breaks. Schools provide proper lunches for students and teachers. The mathematics departments visited each had three teachers except for Tobermory which had two. These teachers were teaching all grades, from Secondary 1 to Secondary 6. All teachers were friendly and cooperative, contributing to a warm and relaxed school environment. Sanquhar and Selkirk schools were particularly welcoming.

In three of the schools the second year students were placed in classes by ability (top, middle and bottom). Setting was a characteristic of the mathematics departments only. Students are assessed towards the end of their first year. This setting is also determined by the overall behaviour of the students. Tobermory High School was the only school that did not follow this setting approach. Teaching in all schools was individualised. Every year's mathematics syllabus, for each grade, is organised by the staff of the school's mathematics department following the general guidelines of the Scottish 5-14 Curriculum. Students work at their own pace on material that their teacher has assigned to them. Interaction between students and the teacher was limited. There were times when teachers addressed the whole class to introduce a new aspect of a topic or clarify a difficult point. Teachers, in general, initiated activity by assigning work to the students or by directing them to the place where the material was stored in the room. Otherwise the interaction between students and the teacher was limited. Students were supposed to consult the answers booklet after they had completed their work. Usually they had to move to the teacher's desk to show them their work or to a ask a question. At classes with 'weak' students, a support teacher might be in the room helping them individually with their work. Interaction between students was limited as well. They were not allowed to speak during the lesson, even though they could move around the class if they wanted to find something concerning their work. Group work was also not favoured by the teachers (for this particular age), mainly for discipline reasons. Students never had homework assigned to them by the teacher. They only had to complete a revision sheet, at home, as soon as they had finished a topic of the syllabus. The teachers' authority in the class was positional and sapiential, with their behaviour becoming informal only towards

the end of each session. Talking to the students at times resembled giving definite orders (apart from Tobermory and one teacher at Selkirk). Work in each session stops five minutes before the bell to give students time to tidy up the material used and prepare for the next session.

5.2 The Results

The following results concern observations made during and after the sessions on the practical activities. These were drawn from participant observation, from recorded data (portable tape recorder), from information coming from taped interviews with groups of students (in Greece interviews took place only at Patras and Vlachokerasias), from student questionnaires (see Appendix D) and from discussions with the teachers. In addition an analysis of the completed worksheets on the activities is presented.

In total 203 Greek students participated (average age 13), 99 Gymnasium 1 (G1) and 104 Gymnasium 2 (G2) and 313 Scottish students (average age 12.5), 133 Secondary 1 (S1) and 180 Secondary 2 (S2). They worked in groups of two to four members. In Greek schools groups of four were more common due to the smaller number of copies of the materials. On the other hand, groups of four, or even three, were the exception in Scotland. In Greece the author was alone with the class, except in two cases. In Scotland there were classes where two other teachers were assisting. Students in each class were given, at random, one of the three activities to work on. There were cases of students starting on a different activity after completing the one initially assigned to them. In Sanquhar Academy the S1 students tried both the SSA and FB activity during different sessions. There were also isolated cases of students who demanded to change activity, from the SSA to the FB. For these reasons the number of completed worksheets for all activities exceed the number of students that participated in the research.

Not all students completed the activities that were assigned to them within the arranged time limit, especially the SSA activity. There were also cases where

students, for various reasons, did not complete some of the worksheets, or tasks on a worksheet, resulting in non-uniform data. Moreover, the students did not work on the activities for the same amount of time. In Greece, where in all schools sessions are of 45 minutes, most students worked during four sessions spread over two days (180 minutes). In three classes though they worked for three sessions (135 minutes). A similar difficulty appeared in Scotland due to variations in the duration of sessions from school to school. In Sanquhar and Drummond students worked for four sessions (140 minutes) and in the other two schools for three sessions (150 minutes). Therefore the only way to analyse the students' responses was to work with each worksheet separately and then try to give a profile for each activity. Each worksheet is further divided according to the tasks that comprise it. The success rates given for each task of every worksheet are calculated according to the number of students that actually worked on this task. The results for each activity and each worksheet follow. All the presented success rates are rounded.

5.2.1   The 'Same Shape As' Activities

After a very low success rate in the first worksheet (5.1) during the pilot study, an example sheet 5.0 was added to the activity. The purpose of that sheet was to provide students with examples of rectangles that do and do not have the 'same' shape. The abstraction of the concept of 'sameness' was assisted by a discussion in each group. During these discussions on 'rectangle-sameness' the most common response of the students concerned the area of the rectangles. Two rectangles were the same if the smaller one could fit a certain number of times in the larger one. This certain number of times had to be, preferably, an integer. In the second example of the second part of 5.0 students could not distinguish length from breadth of the rotated rectangle. In the third example they could not extend the definition for 'rectangle-sameness' (multiply both sides by the same number) to non-integer numbers. They would rather add to the sides of the small rectangle in order to reach the length and breadth of the large rectangle. In isolated cases

students thought that two rectangles are the 'same' if they have exactly the same size.

Worksheet 5.1: Overall 68 Greek (29 G1 and 39 G2) and 144 Scottish students (78 S1 and 66 S2) worked on 5.1. The success rate for the first and second parts of the worksheet for each country and each grade is given below.

Class	A-part	B-part
G1	41%	86%
G2	49%	72%
S1	21%	49%
S2	20%	42%

These success rates were higher than those in the pilot study. The success rate in the second part of the worksheet though was higher than the one in the first part in both studies. In 121 cases (21 in Greece and 100 in Scotland) students perceived rectangle no.7 (an 1.5 enlargement) as not having the 'same' shape as the given one. Most of the remaining incorrect responses concerned additive strategies, centration to one side, while the rest may be characterised as careless mistakes (possibly mis-measurement of the rectangles' sides).

Worksheet 5.2: 68 Greek (29 G1 and 39 G2) and 143 Scottish students (78 S1 and 65 S2) completed this worksheet. The success rates for the first, second and third parts of the worksheet follow:

Class	A-part	B-part	C-part
G1	100%	100%	93%
G2	92%	87%	87%
S1	96%	91%	91%
S2	100%	99%	94%

Mistakes concerned incorrect counting of the unit squares, especially for those who attempted a big enlargement, centration to one of the rectangles' sides and additive strategies.

Worksheet 5.3: This worksheet was completed by 65 Greek (26 G1 and 39 G2) and 143 Scottish students (78 S1 and 65 S2). The success rates for the three parts of the worksheet follow:

Class	A-part	B-part	C-part
G1	92%	96%	77%
G2	100%	87%	69%
S1	94%	96%	82%
S2	100%	92%	71%

In the third part of the worksheet most students reduced the given rectangle three times. From those who attempted a $\div 2$ reduction some failed giving a 3units x 4units or 3units x 5units rectangle. Other incorrect responses concerned additive strategies and centration to one of the sides.

Worksheet 5.4: 63 Greek (23 G1 and 40 G2) and 141 Scottish students (77 S1 and 64 S2) worked on this worksheet. The success rates for the use of the string test with the rectangles of sets 1 and 2 are shown below:

Class	Set 1	Set 2
G1	78%	100%
G2	95%	93%
S1	82%	72%
S2	70%	56%

In the first question on the worksheet the rate of success was 52%, 75% for the Greek students (G1, G2 respectively) and 22%, 27% for the Scottish (S1, S2 respectively). In this question students were advised to find a way to compare the rectangles of set 1 for their 'sameness'. The most common strategies were by measuring or superimposing the sides of the rectangles. From the students' responses to this first question, we can infer whether they perceived 'sameness' as an equivalence property of rectangles. About one third of the students had written the 'same' rectangles in pairs. For example '1-2, 1-3, 1-5' instead of '1, 2, 3, 5'.

Worksheet 5.5: This worksheet was completed by 53 Greek (16 G1 and 37 G2) and 142 Scottish students (78 S1 and 64 S2). Following are the percentages of students who measured the sides of the rectangles in sets 3 and 4 accurately and of those that identified the ratio test.

Class	Meas1	Test1	Meas2	Test2
G1	88%	63%	94%	75%
G2	87%	78%	62%	84%
S1	54%	51%	72%	73%
S2	63%	56%	67%	78%

It was clear from the students' responses to the first question that they used the string test of worksheet 5.4, to find the 'same' rectangles of set 3. The success rates in using this test were slightly better than those of 5.4, especially for the Scottish students. We can also infer that almost all students had appreciated by this stage 'sameness' of rectangles as an equivalence property, apart from some G1 Greek and S2 Scottish students. The percentage success in the fourth part of the worksheet is bigger than that of the third part (except for G1 students). This was caused by students who identified the ratio test but inaccurate measurements kept them from giving a complete answer.

Worksheet 5.6: 55 Greek (21 G1 and 34 G2) and 123 Scottish students (66 S1 and 57 S2) worked on this worksheet. Students had to use the string test and the ratio test for the A3, A4, A5 and A6 rectangles. On the back of the worksheet, they had to give the ratio of any A-rectangle and work out a problem and the Temple of Zeus tasks. The success rates on these follow:

Class	String	Ratio	A-ratio	Problem	Zeus
G1	81%	71%	87%	73%	53%
G2	82%	38%	91%	91%	47%
S1	65%	43%	79%	42%	16%
S2	61%	60%	75%	65%	22%

Worksheet 5.7: The number of students who worked on this worksheet was limited. Only 9 G1 and 19 G2 Greek students and 28 S1 and 24 S2 Scottish. Omitting this worksheet was a deliberate decision due to time constraints. The success rate reached 90% for the Greek students for both grades and 32% for the S1 and 58% for the S2 in Scotland.

Worksheet 5.8: 34 Greek students (6 G1 and 28 G2) and 66 Scottish (25 S1 41 S2) started this worksheet. This number decreased for the subsequent tasks on the sheet, as this worksheet was as far as some students reached. The success rates for the pentalpha measurements, the calculation of the golden ratio, the tasks on Parthenon and Epidaurus and the final calculations with the golden ratio are listed below:

Class	Pental.	G.R.	Parthenon	Epidaurus	Decimals
G1	100%	100%	100%	100%	100%
G2	64%	100%	88%	44%	68%
S1	60%	61%	78%	44%	38%
S2	61%	44%	85%	35%	33%

On the Epidaurus task, when students are asked to name the ratio that they found, 26% of the students failed to identify it as the golden ratio. There were also cases of students who calculated the golden ratio on the front page correctly but failed to do so on the task at the back of the sheet. They found for example ratios of 1.81... or 2.1... which did not raise any queries.

Worksheet 5.9: Only 12 G2 students completed this worksheet in Greece and 2 S1 and 10 S2 in Scotland. From all those students 20 found the correct rectangle but only 8 of them named the ratio and the rectangle they had found as the golden ratio and golden rectangle.

In the first steps of the activity students faced difficulties in abstracting the rule for 'sameness' of rectangles. The strategies adopted by students coincide with those discussed in §2.1.2. Area and additive strategies, centration to one of the rectangle's sides, fall back strategies when facing difficult tasks (1.5 enlargement).

Their persistence with these strategies is worth mentioning, even after these were shown to be defective by the examples used. The students' difficulty (especially in Scotland), in verbalising their thoughts was evident. Even at the stage when all the examples in 5.0 had been discussed, they could not state the rule for 'sameness' in their own words.

Students were expected to face difficulties with this activity because of the un-familiarity of the concept introduced. There were many cases, though, where diffi-culties arose from misconceptions concerning past knowledge ('length and breadth' confusion, working with decimal numbers). In worksheet 5.1 students were not confident in applying the rule for 'rectangle-sameness'. They needed support and immediate feedback on their answers. The choice of the order of the examples in 5.0 and in the tasks in 5.1 did not prove efficient (examples on lack of rectangle-sameness before examples of the concept). This did not interfere with the abstrac-tion of the target concept, since the initial discussion in each group first addressed the examples of the concept. No comments can be made though for the affect that this had on the students' later performance.

Students did not need much help with worksheets 5.2 and 5.3. The amount of time spent completing these worksheets was small compared to the time spent on sheets 5.0 and 5.1. The Greek students may have had difficulties in counting the unit squares on the grid due to the poor quality of the xerox copies. Also, in the second part of 5.3 there was a typing mistake, naming the side of the wanted rectangle as 'breadth' instead of 'length'. There was no indication though that it caused difficulties to the students. These two weaknesses of the materials were corrected in the copies used in Scotland. The students' responses in the third part of worksheet 5.3 were yet another indication of their unfamiliarity in working with decimal numbers. Most of those who attempted to decrease the given rectangle two times failed to do so, ending up with a length of 4 or 5 units.

In worksheet 5.4 the students' main difficulty was to accept the fact that the string should not, necessarily, have to pass from all the rectangles' vertices. They needed help in placing the rectangles in the lid and in drawing the diagonals for each rectangle of set 1. A common response was that of drawing a crooked line

connecting the vertices of all rectangles instead of drawing their diagonals. Many of them failed to accept 'sameness' as an equivalence property of rectangles. We observe though that in worksheet 5.5 almost all overcame this misunderstanding, probably because of the use of the string test. Even though students measurements in 5.5 were fairly accurate, not all of them observed that the ratios of the same shaped rectangles were equal or nearly equal. One reason for this may be that many of the students kept all the decimal places displayed on the calculator. Some responses were far from the wanted answer: *"they all are decimal numbers"*, *"they all start from 1"*, *"they have the same number of digits"*. These responses may reflect the students' unfamiliarity in working with decimal numbers and some lack of intuitive thought. Interestingly, some groups argued about whether dividing two measurements in millimetres will give the same result as dividing the same measurements in centimetres. Students also had difficulties in going from millimetres (markings on the rulers) to centimetres and in deciding how many decimal places to keep in their ratios. For those who decided to cut the decimal places, rounding to the nearest ten was not always their immediate choice. 'Length and breadth' confusion was apparent in this worksheet, despite the instructions given during earlier worksheets. The term 'ratio' was often mispronounced by Scottish students.

In the first task on worksheet 5.6 students had to improvise in order to use the string test for the A-shaped rectangles. The string from 5.4 was too short to be used with these rectangles, so they used a metre ruler or two ordinary rulers placed together. Students seemed to expect all ratios of the 'A-rectangles' to be close to 1.41. Some of these rectangles were not cut accurately resulting in incorrect ratios. Some students corrected their measurements accordingly in order to obtain a ratio close to 1.41! The request to measure in millimetres for better accuracy created difficulties once more as did the rounding of the ratios found. The 'length and breadth' confusion appeared on this worksheet as well but not often. In the problem of question 6, students could not easily use the fact that the rectangle was an A-rectangle. Moreover, choosing the correct operation caused confusion for most students, even though the answer was in a bubble. In

general, students failed to exploit the clues that were given in bubbles throughout the whole activity. *"Oh...is this all we had to do?"*, they might say. Possibly they were not familiar with that technique, or perhaps they could not believe that the task itself provided them with clues for solving it. Greater difficulties occurred on the Temple of Zeus task. Students failed to identify the marked rectangle as a clue, so they could not easily apply the ratio of 1.41 to find an A-shaped rectangle. In addition to that, identification of the correct operation combined with the need for accurate measuring resulted in a low success rate (especially in Scotland). Worksheet 5.6 was also a first indication that multi-stepped worksheets have a potential for exposing the students' inadequate grasp of the concepts. This conclusion is drawn from the failure of students to use and combine information given or found in the earlier tasks of the worksheet (identify the 'A-ratio' and use it in question 6 and Temple of Zeus task).

In worksheet 5.7 problems were caused by the extended instructions. Even Greek students, who eventually achieved a high success rate on this worksheet, needed support. This support mainly concerned the supervision of the reading of the instructions together with providing a summary of the text. Some students did not actually realise that they had to work on the rectangle at the bottom of the page and were placing the mapping pins on the drawn diagrams. Difficulties arose from the motor-coordination of the cross form as well, more evident with the Scottish students. In Greece the two arms of some of the cross forms did not cross at 90 deg due to faulty construction. Answers then close to the wanted response were considered as correct.

Worksheet 5.8 is another multi-stepped sheet. Students faced difficulties in measuring, in using results from earlier tasks on the sheet (Epidaurus task, calculations with golden ratio) and in working with decimal numbers (calculations with golden ratio). Students had difficulties in using the calculator, especially G1 and S1 students. They did not know how to find the square root of a number and the most common mistake was to key the calculation in the following order: $(1 + \sqrt{5} \div 2)$. We have to bear in mind here that Greek students are not familiar

with the use of the calculator and in Scotland not all teachers favour the use of calculators in S1.

Worksheet 5.9 is a rather complex one since it requires elements from most of the past worksheets (string and ratio tests, golden rectangles). Moreover the translation of the small rectangle from the right side of the large rectangle to its bottom left corner is a further conceptual step. Apart from these hurdles, students faced difficulty mainly in coordinating their motor skills to perform the required moves with the cardboard rectangles. Worksheet 5.10 was not completed by any of the students.

The success rates of worksheet 5.1 indicate that a significant number of first year students did not have a complete understanding of 'sameness'. Especially in the first part of the worksheet, the success rates are the lowest in the activity. The tasks of enlarging and reducing rectangles in worksheets 5.2 and 5.3 and the guessing task in 5.4, could be considered as foundation activities for the concept of 'rectangle-sameness'. The high success rates in these worksheets then contradict the low performance on 5.1. Is this because students had not abstracted the rule of 'sameness' or is it also due to the fact that the tasks in 5.1 were hard to tackle? We should consider here the interference from students' other difficulties and misconceptions and the effects of multi-tasked worksheets. Students seemed to have grasped the string and ratio tests, since they used them in subsequent tasks (worksheets 5.5, 5.6, 5.9).

The tasks involving measuring and ratio work proved demanding, especially for the Scottish students. There were students that used unorthodox methods of measuring (measuring the sides of the rectangles by finding the distance between two points of opposite sides that seemed to be at the same level; taking the measurements keeping the ruler and the rectangle up in the air). In problem-solving and 'forming-conclusions' tasks (set 1 of 5.4, set 3 of 5.5, question 6 and Zeus Temple in 5.6, calculations with the golden ratio in 5.8) students were not very successful, especially the Scottish. Motor difficulties were also evident in manipulating the accompanying material in worksheets 5.4, 5.7 and 5.9. There were indications that the Scottish students did not appreciate the illustrations from

Ancient Greek architecture. For example in one recording a student says: *"Here is a picture of something..."* referring to the picture of Parthenon in Athens. We also notice that the Zeus Temple and Epidaurus tasks received the lowest success rates of all the tasks in Scotland. This attitude may be justified by the unfamiliarity of the names and the context. In contrast, in cases where the author had the chance to give information about these illustrations, students seemed to enjoy it.

Students, then, were fairly successful on the tasks directly related to the concept of 'rectangle-sameness' (worksheets 5.2, 5.3 and the first question on 5.4). There was poorer performance on worksheets where secondary cognitive demands interfere with the concept of 'sameness' (past difficulties and misconceptions, multi-tasked worksheets), especially for the Scottish students. Keeping in mind that the number of Scottish students that participated in the research was double the number of Greek students, the results show a differentiation in performance between Greek and Scottish students in almost all of the tasks. Tasks involving measuring and ratio work proved particularly difficult for the Scottish students. In problem-solving and 'forming-conclusions' tasks (set 3 of 5.5, problem and Zeus Temple in 5.6, calculations with the golden ratio in 5.8) students were not very successful, especially in Scotland. We would normally expect second year students to do better than the first year students, as this is the philosophy of any curriculum. A differentiation exists, though, between the performance of first and second year students, in both countries, in favour of the former in several of the tasks. Second year students may simply have found the tasks not to their level of competency and therefore did not invest in them their best effort. Care has to be taken to give students tasks which are appropriate to their level of competence.

Only a small number of students addressed the difficulties discussed above in the recorded interviews and in the questionnaires. More specifically, about half of the students mentioned vaguely that they had some difficulty with the SSA activity. Only one sixth of them explicitly mentioned their difficulties (example sheet 5.0, worksheets 5.1, 5.4, 5.5, 5.6-Zeus Temple, 5.7). These observations, compared to the presented success rates, indicate that students were not fully aware of the choices they were making or of the level of their competency. When

students were asked to describe what they learned from doing the activity, almost half of them answered that they learned something new and consolidated past knowledge. The other half of the students simply referred to a certain task of the activity. This indicated that 'sameness' of rectangles was not what many of the students recalled as the main recurring concept of the activity. Their enjoyment of working on the SSA activity shows in the following table:

Response	Greece	Scotland
Did not Enjoy	0%	5%
Enjoyed A Little	2%	16%
So-So	2%	36%
Enjoyed Much	30%	31%
Enjoyed Very Much	66%	12%

Two thirds of the Greek students, then, enjoyed the activity very much, as compared to one fifth of the Scottish. The main reasons for liking the activity were the group work, the practical aspect of the activity, the fact that there was no marking, lost sessions from the day's schedule, it was easy to cope with and because they liked the new teacher. Often responses were of the kind: *"It was better than school maths"*. The main reason for not liking the activity was the fact that they became bored with all the measuring and calculations. They acknowledged that there was an interesting discovery at the end of each worksheet but it took too long to reach it. They offered as an alternative outdoor measuring instead of measuring cardboard rectangles. Other responses suggested that the activity was not challenging enough or that they could not see why they had to know about 'same' rectangles.

These responses have to be examined with caution, taking into consideration the 'Hawthorne effect' of any innovatory program, that is, the tendency for an innovatory program to produce results that overestimate its long term effectiveness (see Ausubel 1968). Greek students, being less experienced in group-work, in manipulating materials, in working with calculators, may be more 'vulnerable' to such an effect. Even the change from having their teacher and following the day's

schedule might lead to misleading conclusions about the program's impact on the students. Scottish students on the other hand, are more used to this style of learning, as described by the 5-14 Mathematics Document. This was not the case, though, in some schools, where practical and calculator (for the S1 students) work was limited and group work was not favoured (mainly for behaviour reasons).

5.2.2 The 'Feely Box' Activity

The data from this activity appear to be more uniform than those of the SSA activity. Almost all students that worked on the FB activity completed worksheets 1.1 to 1.5. A different kind of difficulty arose though, since not all students worked with the same number of shapes for each worksheet. Thus the data is not directly comparable even for students of the same class. For this reason, emphasis will be placed on the students' blunders and not on their success rates. The results follow (for 1.6 see the DGC activity). Percentages for each shape are calculated according to the number of students who worked with that particular shape.

Worksheet 1.1: This worksheet was completed by 86 Greek (45 G1 and 41 G2) students and 185 Scottish (86 S1 and 99 S2). On average, the Scottish students worked with seven shapes as compared to six for Greek students. The percentages of students who completed this worksheet with absolute success (sides, vertices and names) were very low. Only 4% and 3% for the G1 and S1 students respectively and 46% and 10% for the G2 and S2 ones. All these percentages rise above 50% if we consider only the responses to the 'sides' and 'vertices' columns. All together 45 students (19 Greek and 16 Scottish) perceived the inverted kite as having three sides and 158 students (49 and 109) as having three vertices. Other difficulties concerned naming the shapes. The following table shows these difficulties for each country and grade. Percentages correspond to the incorrect responses:

Shape	G1	G2	S1	S2
Square	–	–	–	5%
Rectangle	–	–	–	3%
Parallelogram	21%	21%	32%	26%
Trapezium	18%	–	91%	81%
Isosc. Trapez.	10%	–	97%	87%
Rhombus	–	3%	24%	14%
Kite	72%	18%	8%	11%
Inverted Kite	80%	47%	60%	38%

In Scotland a large part of the above error rates are due to students who did not attempt to name certain shapes. This was not the case in Greece. Most students did not know the name of the inverted kite. They were encouraged to describe it or name it after something familiar with the same shape. The most common responses were *"triangle with a bit missing"*, *"broken triangle"*, *"triangle with two triangles on the bottom"*, *"arrow"*, *"boomerang"*, *"rocket"*. All these were considered as correct responses and were given mainly in the further observations column. Incorrect responses referred to this shape as a triangle. Interesting are the large number of Scottish students who failed to name the two trapezia. Most of them tried to describe these shapes with the help of other familiar objects (roof of a house, plant pot, thing an elephant stands on, etc). This time such responses were not considered as correct, since these two shapes were known to them. The 'further observations' column caused some confusion to the students due to the lack of definite instructions. With familiar shapes students followed a descriptive approach based on one or two of the shape's characteristic properties. For shapes where they could not remember the name, they tried to describe the physical appearance. The Scottish students were more efficient at that. Less than ten students, from both countries, gave a complete description of the shapes. With unfamiliar shapes a 'fall back' strategy was often adopted, changing to an inappropriate descriptive approach based on the material the shapes were made of (eg one side smooth, one side rough).

Worksheet 1.2: In Greece 87 students worked on this sheet (46 G1 and 41 G2) and in Scotland 184 (89 S1 and 95 S2). On average the Greek students worked with five shapes and the Scottish with six shapes. Students were asked to produce a simple sketch of each shape they 'felt'. On that basis sketches were rated for their accuracy as poor, fair and good. Percentages for each category follow. These numbers are subject to the writer's interpretation of the students' drawings.

Class	Poor	Fair	Good
G1	41%	24%	35%
G2	22%	32%	46%
S1	22%	65%	15%
S2	10%	63%	27%

In many of the students' sketches (around half of them), it was evident that they had not discriminated between two of the triangles, the equilateral and right-angled isosceles. The regular hexagons and pentagons were poorly sketched. In general students tended to sketch small figures with not particularly straight lines.

Worksheet 1.3: This worksheet was completed by 83 Greek students (43 G1 and 40 G2) and 190 Scottish (90 S1 and 100 S2). Students worked with five solids on average. As in 1.1, the percentages of students with complete success on the sheet (edges, vertices, faces, names) was very lower for the Scottish students. Only 3% and 4% for the S1 and S2 students respectively and 21% and 50% for the G1 and G2 ones. Ignoring the responses to the 'names' column these success rates become 14%, 22%, 77% and 85% respectively, still considerably low for the Scottish students. In the 'edges' and 'faces' columns students faced difficulties with the cylinder. They could not decide on the number of faces and edges, with many students answering that a cylinder has no edges. The Scottish students were also not very successful in giving the correct number of faces for the cube and the prism (one fourth to one third of them). Most of the difficulties, once more, appeared in naming the shapes. These follow, with percentages corresponding to the incorrect responses:

Shape	G1	G2	S1	S2
Cube	5%	–	16%	22%
Cuboid	24%	11%	38%	36%
Cylinder	15%	3%	20%	10%
Triang. Prism	34%	20%	30%	26%
Sq. Based Pyram.	36%	16%	15%	11%
Tetrahedron	41%	24%	31%	33%

The error rates for the cube and the cuboid are surprisingly high. Students named them after their corresponding two dimensional shape. The shapes that attracted the most diverse responses were the cylinder and the prism. Some names given for these shapes follow:

- cylinder: circular based cylinder, sphere, cone, tube, circular prism, circular cube, circle.

- prism: rectangular based pyramid, tent, 'Toblerone', rectangle-triangle.

These names reveal that students based their responses on the shapes' characteristics, improvising on the names of other shapes with equivalent features to the one in question. The Euler column was completed by almost all students. Success depended on whether students had their previous numbers correct. Actually the success rates resemble those given before for the 'edges', 'vertices' and 'faces' columns.

Worksheet 1.4: In Greece 80 students worked on this worksheet (40 G1 and 40 G2) and in Scotland 191 (87 S1 and 104 S2). The success rates were very high as the following table shows:

Class	G1	G2	S1	S2
	70%	90%	94%	96%

From the 26 incorrect responses, 24 concerned the L-shaped solids. Due to faulty construction, one of the L shapes did not correspond exactly to its figure on

the worksheet. Many students asked about this while working on the sheet and were directed accordingly. It can be argued, though, that some of the blunders might have been due to this faulty construction.

Worksheet 1.5: This worksheet was completed by 80 Greek students (40 from each grade) and 176 Scottish ones (80 S1 and 96 S2). The success rates for the three parts of the worksheet follow:

Class	Seq.A	Seq.B	Seq.C
G1	95%	94%	20%
G2	100%	85%	80%
S1	86%	80%	20%
S2	89%	85%	28%

In finding the relationship between the two cuboids, the most common incorrect responses referred to the solids' common geometrical characteristics: *"they have the same number of sides, edges,...their edges are parallel"*, and so on.

Overall students faced difficulties in remembering the names of certain shapes. These were shapes that students do not meet very often (parallelogram, trapezia, rhombus, prism, tetrahedron). The students' unfamiliarity with these shapes was indicated by their bad spelling and pronunciation of their names. In Scotland only a few students spelled the names of all shapes correctly. Most students were confused by the inverted kite in worksheet 1.1. These students failed to identify the fourth vertex of the shape, possibly because it was not something that 'pricks' as vertices usually do. Moreover, many of their responses concerned a 3-sided shape with four vertices, or a 4-sided shape with three vertices. This was an indication of how easily students' logical thought failed when confronted with unfamiliar problems.

The percentages of the students, especially in Scotland, that had difficulty in naming the cuboids in worksheet 1.3 was surprisingly high. In the same worksheet students found it more difficult to count the faces of the solids than to count the edges or the vertices. The high success rates in worksheet 1.4 are explained by the

nature of the task. It involved recognition of shapes that were at present, drawn on the sheet. Discomfort was evident when students had to sketch the shapes they were exploring haptically (worksheet 1.2). Most of them kept their figures small in size, obviously to give the appearance of better accuracy. Students failed to discriminate between the equilateral and the right-angled isosceles triangles in this worksheet. Finally, the low success rates in the third part of worksheet 1.5 may indicate students' difficulty in problem-solving and forming-conclusions.

In exploring the shapes students used both their hands, either placing the shape in the palm of their hand or tracing it with their fingers. Their strategies were exploratory with fast movements, keeping their hands in the air. They used vertices as reference points but they used external frames of reference only in an accidental fashion. In worksheet 1.5 they usually superimposed the shapes of each sequence to compare them for their perimeter or area. Students did not have many queries while working on this activity. In Scotland students did not understand the terminology used on the worksheets, more specifically the term 'vertices' to name the corners of a shape. Also there was no consensus between schools about the terms used. For some schools faces meant sides and sides meant edges and for other schools the opposite. To overcome the delay caused by the big number of students in a group, the Greek students formed two groups within groups of four and explored half of the shapes keeping their hands under the desk. Then they would exchange their shapes with the other half. Students, generally, found the work pleasant and not very demanding. Most of them completed worksheets 1.1 to 1.5 within the two thirds of the time available (one and a half to two hours). When students were asked what they had learned from the activity, most of them described what they had done in the activity. A large number of Greek students replied that they recalled relevant past knowledge from working, though, on something different. Their enjoyment gained from working on the activity appears on the next table:

Response	Greece	Scotland
Did not Enjoy	0%	1%
Enjoyed A Little	4%	7%
So-So	4%	30%
Enjoyed Much	22%	30%
Enjoyed Very Much	70%	32%

The most popular reasons given for enjoying the activity were the practical aspect of the activity, working in a group, its easiness and the fact that it was better than the usual work in mathematics. There were students, though, who expressed clearly their discomfort at the cognitive difficulty of the tasks, suggesting that they were too easy. Some of the students who made these comments had blunders in their responses. This may indicate the students' frivolous approach to a new learning situation. Some students also felt bored after some time, because as they said they were just 'feeling' shapes. They proposed having various shapes, two-dimensional and three-dimensional accompanying every worksheet.

5.2.3 The 'Drawing and Geometric' Constructions Activities

In Greece 47 students started working on the 'Be a Geometric Constructor' and 'Balancing Polygons' sequences of activities and 16 in Scotland. The actual number of students who worked on these activities, though, is much bigger. That is because students from the FB activity continued on these activities after completing worksheet 1.6. The tasks of the 'Balancing Polygons' sequence that involve balancing were not tried in Scotland. The acetate shapes were not robust enough to use in the classroom. Also some students found it tricky passing the thread through the holes.

Only 23 Greek students tried worksheet 1.6 (they were directed to the DGC activities) and 141 Scottish. Most of them faced difficulties when they had to draw in directions other than the horizontal and vertical. The triangles, the rhombus,

the kite, the pentagon and the hexagon were the hardest to draw. Very few of the students used drawing instruments other than a ruler, most of them in Greece. At each step they concentrated on the length of the side to draw next but not on the angle determining the direction of the side. Whether they followed the given instructions then was a matter of chance. The following quotation is revealing: *"The ones with diagonal lines, they end up the wrong size".*

The students' responses on the ease or difficulty of drawing on the two kinds of papers were interesting. Some noticed that shapes like the square, the pentagon and the circles are hard to draw on isometric paper and shapes like the pentagon, the hexagon, the rhombus, the kite and the circles are hard to draw on squared paper. Others responded that all shapes were easy to draw on both papers. These answers contradicted their actual performance, since most of the students had faced difficulties. Possibly students ignored the fact that they did not only have to draw a number of shapes but they had to draw them following the given instructions. Otherwise we would have to suggest that the students unfamiliarity with some tasks is the reason for their inability to judge their own performance. From these answers we can also conclude that students failed to exploit the properties of the isometric paper. On the squared paper they also counted the distance of a diagonal of a unit square as one unit of length. We have to note here that the Greek students at least were not familiar with drawing on squared or on isometric paper.

The 'Measuring and Drawing Library' booklet preceded the allocation of the 'Be a Geometric Constructor' and 'Balancing Polygons' sequences of activities. Students simply riffled through the pages and did not use it afterwards. Some of the students' queries while working on the tasks had their answers in the booklet. Either the students had not studied the booklet thoroughly or they had not understood its instructions.

Students were mostly accurate in the triangles' constructions and the bisecting of a line. The construction that created most of the difficulties was the one for finding the centre of a circle. Some of the problems caused might have been by the distortion of the circle due to photocopying. Students' quality of drawing

was at high levels, especially that of the Greek students. But even though they were competent in the actual use of the drawing instruments, their measurements were not absolutely correct. In other words, they could cope with completing the construction but they could not follow the given instructions for the lengths in these constructions. Moreover, in cases where the students were not positively sure about the outcome, they preferred to reproduce the given illustrations in the instructions. This was common for the Pascal's line and less common for Pappus', where they chose the points in the way they were marked on the diagrams in the instructions. In Scotland, many students had difficulties in reading the instructions. This was less common in Greece. An interesting situation was created while students were working on the construction of a regular hexagon. In this task students are asked at some stage to measure the distance between certain vertices of the hexagon. Distance for most of them was a line of a certain length. So to measure the distance between two vertices they first drew the line to connect these two points. Students also had difficulties in going from millimetres to centimetres. Finally, the students' inability to use information from past constructions was evident in, for example, the construction of a perpendicular on the worksheet of constructing two parallel lines.

In general, students used the drawing instruments adeptly, particularly the Greek students. Lines were straight, arcs were competently drawn but the outcome did not always match the wanted one. Either students did not manage to follow the instructions, or they did not double-check their constructions. The former may indicate unfamiliarity in drawing and the latter an improper working approach. The students' difficulties in relating information from worksheets already completed to the one at hand is in accordance with the improper working approach. Students enjoyed working on the constructions, to judge from their comments and their teachers' remarks. It was a novel task, or maybe as they themselves said a task that they are not usually given the opportunity to work on. They felt challenged by the accuracy that the tasks required and by the large number of the worksheet. They also seemed to enjoy, to some extent, the narrative that accompanied each worksheet.

5.3 Further Observations

The organisation of the classes in Greece for the practical activities was more or less the same in all schools. Students formed groups by themselves and then they were assigned to one of the activities at random. The author was alone in all except two of the classes. The students were very demanding when asking for the next worksheet. Some of them preferred to ask the author for explanation before even reading the instructions, saying *"teacher knows better"*. For these reasons the task of providing the groups with each worksheet along with the accompanying material seemed too much for one person to tackle. On the other hand in Scotland much help was given by the teacher of each class and quite often by a third teacher. In every school in Scotland the organisation of the class during the sessions on the practical activities was affected by the suggestions of the class' teacher. In Sanquhar classes were divided into two large groups, according to the activity students were working on. These groups occupied separate rooms with a teacher responsible for each one of them. At Tobermory and Selkirk High Schools these two large groups occupied different areas of one room. The same arrangement was made at Drummond, with the difference that students working on the same activity were sitting around the same table. Also one teacher was responsible for each table and each activity. Students in Scotland were less demanding than Greek students. They still preferred to hear the instructions from the teacher but not to the same extent as in Greece.

Students tended to seek approval for their answers before proceeding with the next task. This was more evident at some schools in Scotland, despite the individualised, self-paced teaching approach. In general, group work was favoured by the vast majority of the students, as they indicated in the questionnaires. They acknowledged the social aspects in group work and its advantages in coping with difficult tasks. These remarks, though, should be interpreted with caution, because in many groups discussion was akin to chatting and cooperation was limited to simply reading the instructions together. There were cases in Scotland where

students in a group needed encouragement before they even started talking to each other. This reflected the discipline demanded by the mathematics teachers. It seems that the students were not accustomed to group work, either in Greece or in Scotland. Those students who opposed working in groups did so because their group members were cheating or because they did not like them. There were also some students who stated that they enjoyed group work but they would rather work individually.

Students seemed very worried at times about the confidentiality of their work, mainly because they were afraid that their teachers might see it. The author made it clear that the sessions on the practical activities was not an assessment and their teachers would not have access to any of the collected material. Despite this comment, many students, especially Greek, decided to use nicknames on their worksheets. Due to the demands of the sessions, unattended recordings were the only possible option for capturing students' discussions. The students though, were fascinated by the tape-recorder and ended up expending much of their energy in saying silly things to it. The following recurring behaviour was recorded by the tape-recorder. Students appeared to adopt a different behaviour when they were facing a teacher and when they were hiding in anonymity. When a teacher approached them, they always had some question to ask about the task on which they were working on. As soon as the teacher had moved away from their desk, they would start chatting again. This immature attitude was the students' response to the 'no-talk, just-work' discipline rationale of most of the mathematics classes. Pretend that you are working (asking questions is an indication of doing so) when the teacher looks at you and chat whenever you find the chance.

In Greece students remained at their seats, working, even during the breaks between the sessions. The situation at Vlachokerasias was amazing, where students worked through all four sessions in one day, some of them without having any breaks. In Greece, the classrooms with the practical activities was an attraction for the rest of the school's students. Students kept asking questions about the materials and were trying to persuade the author to try them in their classes as well. All students were curious about the materials. In Greece the calculators

attracted most attention. Some of the students even treated them as 'sacred' objects. Teachers, not only of mathematics, were very interested to know about the materials and the objectives of the project. Greek teachers wanted to know about teaching practice in Scotland and the other way round. No mathematics teacher in Greece, though, requested any details about the activities before trying them in their classes. Only one of them had some interaction with the materials, by helping to its allocation and clearing up queries whenever he could. On the contrary, Scottish teachers were actively involved during the sessions, usually taking charge of the 'feely box' activity. Their comments on the practical activities were encouraging. They said many of their students performed better than expected (judging by the amount of work they completed and their attitude during the sessions). Teachers at the Mathematics Teaching conference in Edinburgh also liked the activities. Their main worries concerned the organisation of such sessions and the cost and availability of the materials.

When students were asked on the questionnaires to propose possible areas in mathematics suitable for practical activities, their responses covered, more or less, the mathematics syllabus of the first two High School grades. These proposals might indicate the working area for many research projects but also the many difficulties that students face and the need for help. The students' comments in the interviews concerning mathematics and their school experiences with this subject were revealing. In Greece only 21 students were interviewed as compared to 87 in Scotland. The interviews were unstructured discussions. They took place in groups, as this was proved anxiety-relieving for the students.

The similarity of students responses concerning the description of a typical mathematics session is remarkable. These descriptions coincide with those given at an earlier stage by the author (see §5.1). Students' studying habits were fairly similar. Provided that they had homework (rare for the Scottish students), they read the school book and then tried to solve the problems set. The answer-books, in both countries, play a misleading role at this point. Students in Greece are tempted and usually copy the answers from the books without understanding being involved. Similarly in Scotland, students correct their work by themselves at the

end of each booklet. They tick the correct answers and change the incorrect ones. Understanding why some answers are correct and others are not is not essential at this stage. It is when students have to pass a revision sheet that teacher and students might be surprised by the performance of the latter.

In Greece all tasks come from the school book and in Scotland mostly from the booklets. Students expressed clearly that they would like a change in that process, with more practical work and more everyday applications. Moreover, they would like teachers to be less strict, allow them to do some group work and let them speak amongst themselves during mathematics classes. The relationship between students and their mathematics teachers is a matter of concern. The teacher reprents positional authority and in the best of cases sapiential authority as well. Most students do not see their teacher as a friend and this gradually leads to an avoidance of him/her. This situation may easily lead to cognitive as well as affective difficulties. As many students commented, consulting their teacher was not their first choice when they needed help. They would prefer to ask their parents about their homework, or ask a classmate 'who knows', instead of asking their teacher. The result are misunderstandings and, especially in Greece, this signals the phenomenon of parapaedia. In addition, students with well educated parents are likely to have better chances in schooling, since help is more readily available to them. Related to this issue may be the students' difficulty in seeing themselves as future mathematics teachers or even as teachers.

Of all the different types of assessment used, the blackboard has the most pitfalls for the students' ego (used only in Greece). It is during this type of assessment that students' performance is there to be judged by teacher and students. Even friends can be malicious at such times. Competitiveness, jealousy, etc, can produce unkind comments about a student's performance, abilities, even about his/her personality. The author observed similar attitudes from some teachers during the observation sessions. It seems then that this type of assessment has undesirable effects, mainly because of the way it is used in the classroom. In general, Greek students sounded more frustrated about their interactions with their teacher, while the Scottish were more frustrated about the teaching and

the tasks they were experiencing. This is possibly because the teaching style in Greece is more interactive between teachers and students, while the Scottish is individualised and more content oriented.

The students in general did not like mathematics very much and a reason for that was its difficulty. Students who could cope well generally liked it. All of them anticipated its essential role in finding a job later in their lives. They could not find, though, everyday examples of its use other than involving money. Such a utilitarian approach is revealing of the bad view of the subject students are receiving in class. Some students even said that 'grocery' mathematics (money give-and-take) is not mathematics, as it is too easy. Despite all the unfavourable comments about mathematics and about the way it is taught, it was surprising to hear from the Greek students the demand for more mathematics sessions, more time to cope with their difficulties, more time to interact with their teacher in and outside the classroom. They also suggested the establishment of mathematics libraries within each school, where students could study their homework, find books to read and get help with their queries. These remarks came from students at Patras High School, where students might have learned about foreign schools from their well educated parents.

The students' effort to please the author on the one side and not to expose their teacher on the other was evident. This was obvious at times from the contradictory responses to different but related questions.

Chapter 6

Discussion and Conclusions

The main intention of the study was to investigate the effectiveness of practical work in lower secondary school mathematics learning. This called for the consideration of students' performance on the developed materials and students' attitudes towards the suggested approach. The effectiveness of the practical activities was investigated in terms of the cognitive difficulty associated with the concepts that these introduce.

Caution should be exercised in judging the extent to which the collected data allows for a direct comparison between Greece and Scotland. This refers both to the quantitative and the qualitative results of the study. The sample from Greece was much smaller. The time that students worked on the practical activities was not equal. It appears that Greek students had the chance to work on the activities for more time than the Scottish students. The conditions, though, in the Greek classrooms during the sessions using practical activities caused considerable delays. Also the mathematics syllabuses and the students' educational experiences in the two countries do not correspond. The Greek students' experience of non-traditional teaching methods is very limited. This fact allows us to suggest a greater influence of the Hawthorne effect on the students' attitudes and performance. Other inadequacies may refer to the author, who played the roles of both the researcher and the evaluator. The language and the cultural barrier may also have restrained him from meeting the demands of the research that took place in

Scotland. All these issues should be kept in mind, along with the methodological rationale of the study (see §4.4). These issues then, may not weaken the study's potential for investigating its main objectives. On the contrary, they emphasize the complexity of establishing a commonness of meaning in the learning milieu.

The results of the study indicate a differentiation between the performance and the attitudes of the Greek and the Scottish students, in favour of the Greeks. Overall, students faced difficulties in multi-stepped worksheets. They failed to make use of information from earlier tasks on the worksheet or from past worksheets. Interference from secondary cognitive demands also considerably affected their performance. It seems then, that the practical activities played a part in 'exposing' students' past misunderstandings. Their discomfort in tasks that required intuitive thinking and problem solving skills was evident. They also seemed to lack what Dewey (1963) called the *"stop and think"* quality. In other words, they did not pay much attention to the task's context, as this was described by the instructions. They were satisfied by reaching 'an' answer for the task at hand, without being interested in checking their answer and comparing it to their earlier responses. Moreover, tasks with illustrations from Ancient Greek architecture failed to draw students attention to the aesthetic projections of the discussed concepts, especially the Scottish. This was possibly due to an unfamiliarity with the archaeological sites in the illustrations and possibly with the relationship that mathematics has with the arts.

Other students' difficulties were related to language and motor control skills. Inability to express their thoughts verbally characterised most of the students (especially in Scotland). Many students faced reading difficulties, especially in worksheets with extended instructions. This was accentuated by many students' preference (especially in Greece) for listening to the instructions from the teacher rather than trusting their own abilities. Also, most of the students in Scotland pronounced and spelled the names of shapes and concepts incorrectly. Shapes like the parallelogram, the trapezium, the isosceles triangle, the tetrahedron, are shapes (and names) that students do not come upon often in their school and everyday life. Possibly language worked to the advantage of the Greek students,

since most of these names have Greek origins. Everyday words and metaphors may be used by Scottish mathematics teachers (to a lesser extent in Greece), to compensate for this unfamiliarity. These expressions can vary among schools around the country. They may provide a *"vivid and memorable"* way of interpreting and describing knowledge (Lopez-Real 1989, 1990). Their meanings, though, are not always exploited in the classroom (opcit). It is suggested then, that they may have proved to be a disadvantage for the Scottish students, since there are the dangers of over-simplification and confusion between the different meanings that these metaphors have in everyday life and in the mathematics classroom (see Fielker 1988). Irrespective of these reasons, the percentages of Scottish students especially but also of Greek students, who failed to name solids like the cube, the cuboid and the cylinder were worrying. These are familiar solids, that appear early in the primary school syllabuses in both countries.

In tasks that required dexterous use of equipment, several students required adult help. Instead of the equipment helping them to overcome the cognitive demands of the tasks, the opposite occurred. Either the motor-skills demand was merely beyond them or they lacked the appropriate practice. The latter is supported by the haptic exploration strategies that students used in the 'feely box' activities, which in general were unsystematic. Motor skills difficulties may account for some students' poor drawing and sketching skills.

There was also a differentiation in performance, in several tasks, between first and second grade students, in favour of the former. Enjoyment was closely related to the cognitive difficulty imposed by the tasks in the activities. There were students who liked the activities because they were 'easy' and others who did not like them because they were 'too hard'. There were few who did not like the activities because they were not challenged by the tasks. No clear conclusion can, then, be drawn from the differentiation in the performance between the students of the two grades (see later discussion on students' strategies). In general all students rated their experience on the practical activities as better than the school mathematics, even those students who did not enjoy working on them very much. In the interviews they specified that they liked the 'hands-on' aspect of the activities.

Another indication of students' preference for practical work can be seen from the higher enjoyment that they received from the 'feely box' activities, as compared to the 'same shape as' ones. Practical work is more clear and vivid in the 'feely box' activities, since all worksheets are accompanied with equipment which is also pleasing to the eye. The possibility of Hawthorne effects exists then, even for the Scottish students. There is an obvious danger of students being influenced by characteristics, irrelevant to the the nature of the method: being allowed to discuss with their classmates, working in groups, avoiding the usual work, interacting with a new person (the researcher). These characteristics, important in activity-based learning, make it more difficult to judge the method's potential in learning mathematics.

The underlying rationale of the Greek and the Scottish educational system appears to be challenged by reality, as revealed by the results of the study. In Greece many students turn to private tuition (parapaedia) to compensate for inadequacies of the educational system. Being able to afford a private tutor in mathematics or not contradicts the alleged equality of opportunities for all students. The meritocratic rationale of the Scottish educational system seems to be challenged by the performance and attitudes of the students in lower ability classes. Furthermore, despite the differences in the educational systems of the two countries (mathematics syllabuses, functioning and organisation of mathematics classes), the differences in the students' performance were not significant (the actual success rates). In addition, the difficulties that the students faced during the practical activities were more similar than different. Bearing in mind the limitations of the study (see opening discussion), this prompts the demand for an appreciation of how fundamentally different educational systems may lead to similar performance outcomes. It raises the need for open-mindedness among those who work in the field of mathematics, irrespective of their role. By subjecting educational ideas and rationales to cross-cultural comparison and criticism, social and cultural generalisations may be discovered, which may in turn lead to more refined educational systems.

Students adopted strategies, in an effort to look as if they were working on and

interested in the assigned tasks. Other strategies also indicated avoidance of work (eg delaying asking for the next worksheet). We can suggest that the first year students may be more likely to adopt strategies for pleasing the teacher with their performance and attitude. In order to come to terms with a new environment, the more appropriate strategy is a cooperative rather an opposing one. On the other hand, second year students appeared more challenging towards authority figures. This may explain, to some extent, differences in performance between first and second year students. Students' evident worries, as expressed to the author, about the confidentiality of their work and behaviour indicate differences in behaviour during educational experiences. The persons that are involved in these experiences and their behaviour. The organisation and nature of these educational experiences seem to interact with the people that are involved in these experiences and greatly affect their behaviour.

The tasks that students are asked to work on in the mathematics classroom, are not perceived by the students to correspond to their personal experiences and interests. Despite these remarks they could not find everyday uses of mathematics other than in money transactions. Their beliefs about the nature of mathematics corresponded with the formal and analytic way in which the discipline is taught in high school. Also their aspirations about learning mathematics chiefly concerned its applications in possible future occupations. In most cases interactions with the teacher were limited to receiving instructions. Interactions with other students were also limited, almost non-existent in Greek classrooms. This may explain the observed difficulties in communicating their mathematical ideas verbally and also the unfamiliarity with functioning in a group. Understanding was hidden behind the effort of reaching an answer, preferably the correct one. The phenomenon of the wrong use of the answers-books was worrying. Many students (especially in Greece) said that they preferred to ask their parents, or a friend 'who knows', when facing difficulties with homework or other assignments. Teachers are seen as distant authority figures. Students said that they would prefer them to be less strict, more caring, allowing and encouraging discussion. They added that they would not like them to be very tolerant of bad behaviour and performance, since

this was also perceived as not caring enough for the students. All the pedagogical ideas that suggest the teacher should be a 'model' for the student in and out of the classroom (encouraging the formation of values and attitudes about the discipline, about learning and about life), seem to be overlooked by reducing the teachers' role to giving instructions.

The above discussion does not apply to all the observed mathematics classrooms, since a few of them were functioning in a relaxed and caring atmosphere (see §4.1). For this reason, even classrooms within a country were not directly comparable. It was part of the intention of the study to investigate differentiation in performance and attitudes between urban and rural high schools. Only in one first grade class in Greece (at Vlachokerasias Gymnasium) was students' performance significantly lower than in the other schools. This low performance was paired with students' low motivation for learning (not only in mathematics classes). It was also found that in classes with authoritarian teachers students received the practical activities very warmly. Moreover, despite the fact that some classes were classified as lower ability, students' achievement and behaviour surprised their teachers in a positive way.

Considering, then, the students' behaviour in the mathematics classrooms during the study, we observe common characteristics which give evidence for what was described earlier as 'classroom culture' (see §4.2). These common characteristics mirror the acquisition of values in the mathematics classroom about the nature of the discipline itself, about mathematics understanding, about work in the mathematics classroom, about the role of the teacher and about education in general. Schooling nowadays seems to be characterised by an 'industrial' ethic. Teaching used to be addressed as a 'vocation'. The compartmentalised structure of school, though, with a clear prescription of rules and regularities, based upon efficiency and effectiveness, has made the roles of teachers and students resemble those of contract-workers. If education is perceived merely as a job to be done, it is expected that 'workers' will try to bring this job to an end with the least possible effort. Learning then is likely to be an incidental outcome of schooling. The values acquired in the mathematics classrooms nowadays mark the widespread

subsumption of culture by civilisation. Civilisation, of course, is not a reprehensible characteristic of human race. It is reprehensible, though, to limit students' aims to short-term aspirations, even if these have to do with everyday life pursuits. To transcend above the utilitarian, everyday life activities (not to reject them), does not necessarily mean to indulge in a theoretical pursuit of mathematics. This may form the means but not the end. Short term values appear to be strong but not unalterable or lacking the need for enrichment. This of course may require re-evaluation of beliefs about the aims and use of mathematical education.

Should we stop thinking only in terms of 'producing' future mathematicians, or even in terms of teaching better mathematics to our students? Should our aim be to teach to students as much 'mathematics' as possible during their schooling career? It is deceitful to suggest that acquisition of skills and techniques is a preparation for the students' future life. Nor is sterile mathematical knowledge that confines the learner to the discipline of mathematics alone a preparation for future life, even if this knowledge is learned in a meaningful way. Would it be better to think in terms of teaching for 'better' people and cultivate the ability to make decisions in later life? This approach would also reflect an honesty about the nature and use of mathematics in the process of life. Learning, in general, should be an open-ended activity/experience and not dependent on instruction. The aim of mathematics education should be the 'end' of dependency on instruction. This corresponds to the meaning that Aristotle gave to the word 'end' of an action as the acquisition of an *"agatho"* (possession, quality), with all actions leading to the acquisition of the ultimate *agatho*, that of the *"politiki techni"* (the 'art' of forming social groups) as described in Plato's myth about Protagoras. It envisages an adult independent of pedagogic relations, being able to survive in and contribute to his/her culture, considering the existing societal and historical situation. It is wrong then to perceive education as schooling and the role of mathematics teachers as that of instructing the young about mathematics. Schooling should provide the students with opportunities that would bring a fusion to the Apollonian and the Dionysian modes of life. It should engage students in experiences that would 'unravel' and cultivate their potential qualities and prepare them for 'political'

life. Teachers should be seen as models for life (they are anyway) and not only as authority figures or as bearers of knowledge. This would require freeing the teachers from contract-like demands and the development of curricula that would appreciate teachers' initiative and personal qualities.

The following words by Nietzsche (1872 in Breazeale 1979: 16) sound so contemporary, summarising the above discussion:

> *"The Greeks as discoverers, voyagers and colonizers. They knew how to learn: an immense power of appropriation. Our age should not think that it stands so much higher in terms of its knowledge drive - except that in the case of the Greeks everything was life! With us it remains knowledge!"*

Aiming for a mathematical knowledge drive, then, may alienate its owner from life, if this drive is limited to the mere acquisition of knowledge.

It would be 'convenient' to suggest that activity-based learning can secure intentional learning of mathematics and also serve the former objective. The results of this study suggest though, that practical activities deserve serious consideration in the field of mathematics learning. The extent to which they can promote learning is something that cannot be evaluated by a single study. To pretend so would be to abide by the short-term aims that characterise mathematics learning and education overall nowadays. Their potential in disclosing past misunderstandings (even about familiar concepts), is an indication of their value as a teaching aid. As far as the formation of values is concerned, we have to accept that this process takes place in the classroom irrespective of the kind of values that are formed. It is an ethical issue then, that of determining what 'kind' of values these should be and what purposes they should serve for the learner and for the society. Possibly, it would be more 'honest' for the learners if they were aware both of the utilitarian role of education and the imaginary one. Practical activities, by encouraging deliberate action, interpersonal communication and functioning in a group and activity-based learning can play a role in educating 'political' human beings. By demanding interaction with the teacher they can promote the formation of values.

Why then are practical activities not used in secondary school mathematics classes? What can be the reasons for this? Do teachers not value them as a way of learning mathematics, or is it simply that they are not convinced about their use as an efficient teaching aid? How can the process of value-formation in the classroom be described in detail? Is there a need to re-evaluate the role of the teacher? Is there a need to re-evaluate the role of mathematics education and that of education in general? Is it 'egotistic' to direct our effort to mathematics education only, or would it be more honest, if we think in terms of a metaphysical appreciation of schooling, to strive to cultivate the whole individual? What is the role that parents can play in this process? These are some questions that future research may try to address.

Appendix A

The 'Feely Box' Activities

The worksheets of the 'feely box' activities follow , including worksheet 1.6.

FEELING SHAPES / PART A   WORKSHEET 1.1   YOUR NAME:
                                             YOUR SCHOOL:

1. Place box A in the feely box.
2. Put your hands in the feely box and pick one
    shape from the box.

For each shape of box A, follow steps 3-5.

3. Identify certain properties (sides, vertices),
    and name the shape that you are touching.
4. Copy your answers on your worksheet and write
    down any further observations.
5. Check your answers by looking at the shape
    afterwards.

	No. of SIDES	No. of VERTICES	NAME	OBSERVATIONS
FIRST SHAPE				
SECOND SHAPE				
THIRD SHAPE				
FOURTH SHAPE				
FIFTH SHAPE				
SIXTH SHAPE				
SEVENTH SHAPE				

USE THE BACK OF THE PAGE IF YOU NEED TO DRAW ANYTHING.

T.A.T.

FEELING SHAPES / PART B   WORKSHEET 1.2   YOUR NAME:

YOUR SCHOOL:
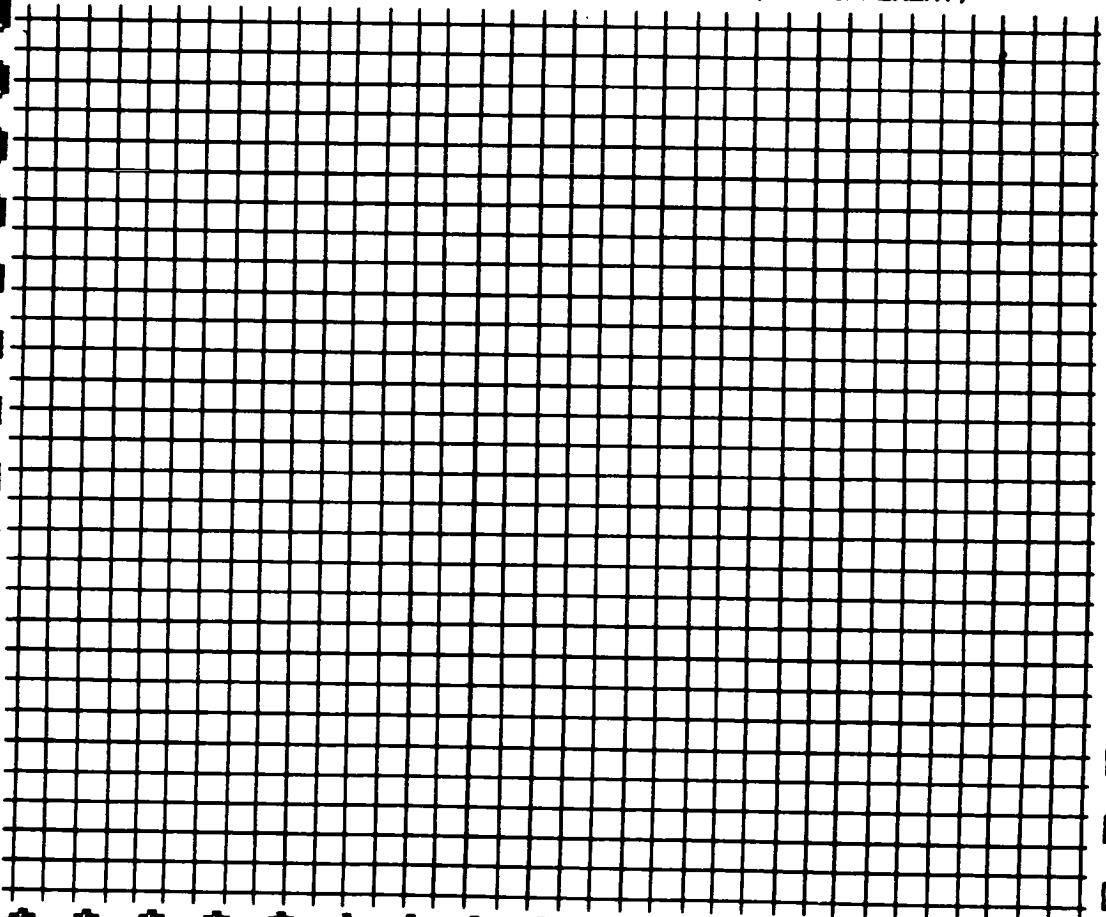
1. Place box A in the feely box.
2. Put your hands in the feely box and pick up one shape from the box.

For each shape of box B, follow steps 3-4.

3. Using squared paper, <u>sketch</u> the shape that you are touching.
4. Check your answers by looking at the shape afterwards. If that shape is different than yours, <u>sketch</u> the right one beside.

YOUR SKETCH   RIGHT ANSWER (IF DIFFERENT)

T.A.T.

FEELING SHAPES / PART C   WORKSHEET 1.3   YOUR NAME:

————————————   ————————   YOUR SCHOOL:

1. Place box C in the feely box.
2. Put your hands in the feely box and pick up one solid shape from the box.

For each solid shape of box C, follow steps 3-4.

3. Identify certain properties (faces, edges, vertices), and name the shape that you are touching.
4. Copy your answers on your worksheet, and check them afterwards by looking at the solid shape.

	No. of FACES (F)	No. of VERTICES (V)	No. of EDGES (E)	NAME	(F)+(V)−(E)=
1st SOLID					
2nd SOLID					
3rd SOLID					
4th SOLID					
5th SOLID					
6th SOLID					

What do you observe about the last column?

(No. of faces) + (No. of vertices) − (No. of edges) =

USE THE BACK OF YOUR PAGE IF YOU NEED MORE SPACE.

T.A.T.

FEELING SHAPES / PART D   WORKSHEET 1.4   YOUR NAME:

                                                      YOUR SCHOOL:

   1.   Place box C in the feely box.

For each solid shape of box D follow steps 2-4.

   2.   Put your hands in the feely box and pick up
          one solid shape.

   3.   Identify the solid shape that you are
          touching from a range of solid shapes drawn
          on your worksheet.

   4.   Check your answer by looking at the solid
          shape afterwards. Tick the circle if you had
          it right, or cross it if you had it wrong.

T.A.T.
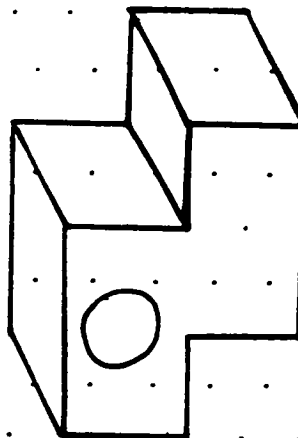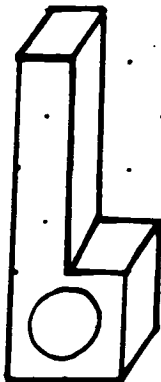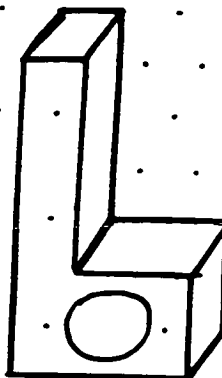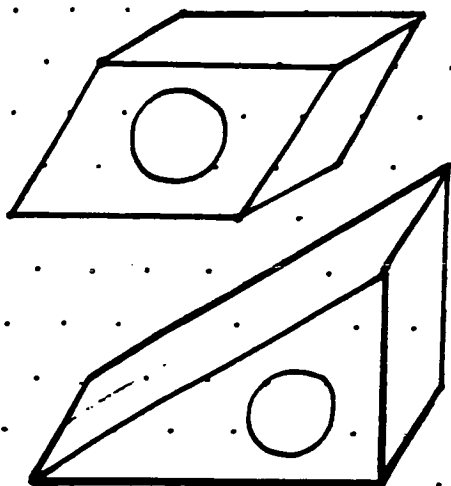
FEELING SHAPES / PART E   WORKSHEET 1.5   YOUR NAME:
_____   _____   YOUR SCHOOL:

1.   Place sequence A in the feely box (a square, a
     triangle and a trapezium). Ignore the thickness
     of the material.

2.   Try to order them, by marking first on the
     worksheet the one with the smaller perimeter.

3.   Check your answers by looking (or measuring) at the
     the shapes.

```
                 SHAPE 1        SHAPE 2        SHAPE 3
             ┌────────────┐  ┌────────────┐  ┌────────────┐
SEQUENCE A:  │            │  │            │  │            │
             └────────────┘  └────────────┘  └────────────┘
```

4.   Now use sequence B (two squares and a
     parallelogram). Try to order them by marking first
     on your worksheet the one with the smallest area.

5.   Chech your answers by looking (or measuring) at the
     shapes.

```
                 SHAPE 1        SHAPE 2        SHAPE 3
             ┌────────────┐  ┌────────────┐  ┌────────────┐
SEQUENCE B:  │            │  │            │  │            │
             └────────────┘  └────────────┘  └────────────┘
```

6.   Now put sequence C (two cuboids) in the feely box.
     Can you identify ay relation between the dimensions
     of the two solids?

7.   Copy your answer in any form you like (sketch,
     words, numbers), and check it by looking at the
     solids afterwards.

USE THE BACK OF YOUR PAGE IF YOU NEED MORE SPACE.

T.A.T.
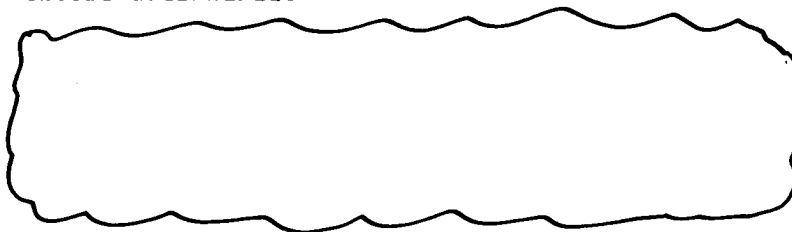
FEELING SHAPES / PART F   WORKSHEET 1.6   YOUR NAME:
                                                 YOUR SCHOOL:

Draw the following shapes on squared paper according to the
instructions. Use whatever drawing instrument you need.

        square: side of 4 units
        rectangle: side of 4 and 6 units
        rhombus: side of 4 units
        kite: sides of 3 and 5 units
        tarpezium: top side of 4 and bottom side of 7 units
        parallelogram: sides of 4 and 6 units
        pentagon: side of 3 units
        hexagon: side of 3 units
        equilateral triangle: side of 5 units
        circle, half-circle and quarter circle : radius of 3 units

Are there any shapes that are hard to draw? Can you describe
why?

Now try to draw the same shapes on isometric paper following
the same instructions. Are there any shapes that can be
drawn more easily on the isometric paper? Are there any
shapes that are hard to draw on both papers? (squared and
isometric).

T.A.T.

Appendix B

The 'Same Shape As' Activities

The worksheets of the 'same shape as' rectangles activities follow.

SAME SHAPED RECTANGLES   EXAMPLE SHEET

All rectangles are the same kind of shape, but not all rectangles have the same shape.

For instance, the following pairs of rectangles do not have the same shape:
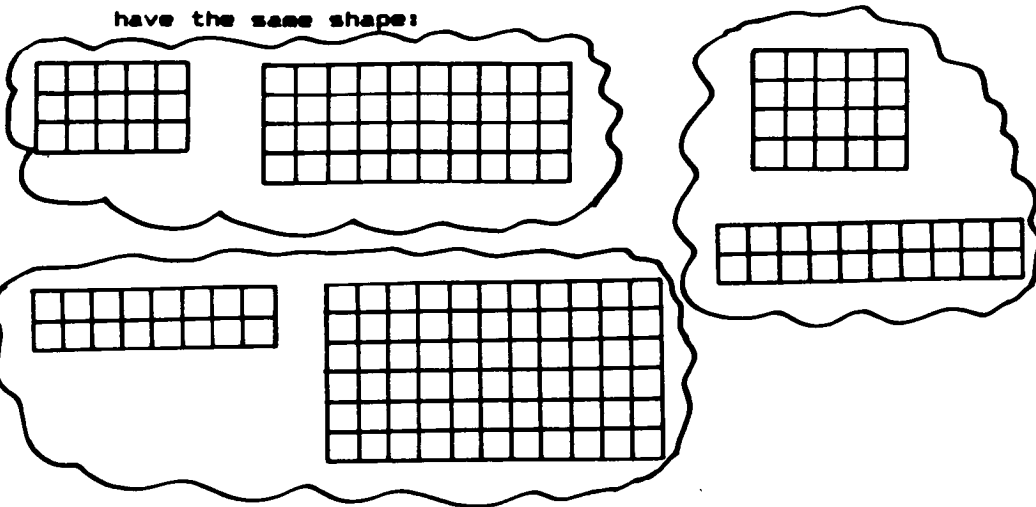
But the following pairs do have the same shape:

T.A.T
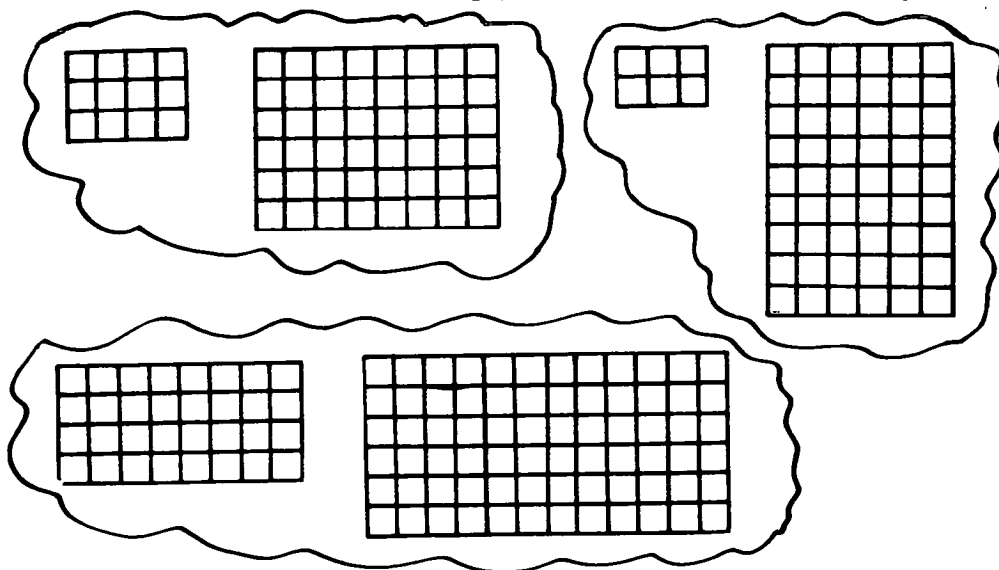
SAME SHAPED RECTANGLES   WORKSHEET 5.1

Your Name :
Your School :

1. Here are some rectangles. Tick with an X those rectangles which do NOT have the same shape as :
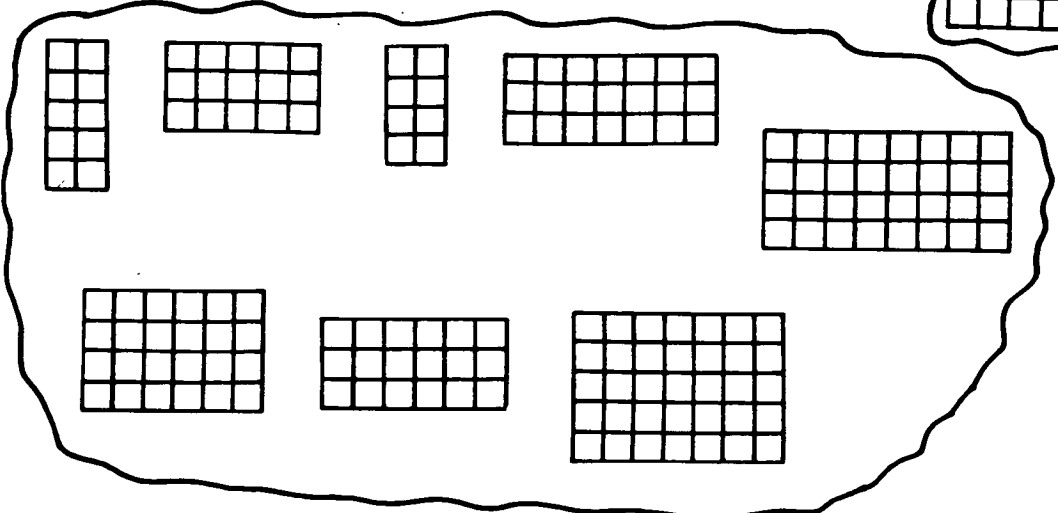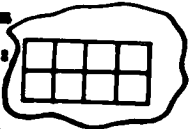
2. Here are some more rectangles. Mark with a √ those rectangles which have the same shape as :

T·A·T
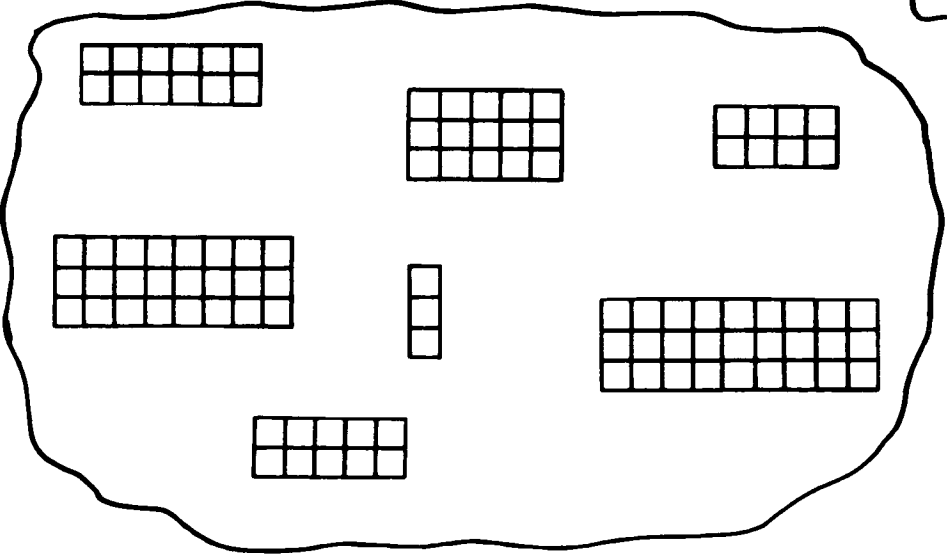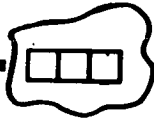
ENLARGING RECTANGLES WORKSHEET 5.2

Your Name :

Your School :

1. Complete the rectangle, so to have the same shape as the given one.
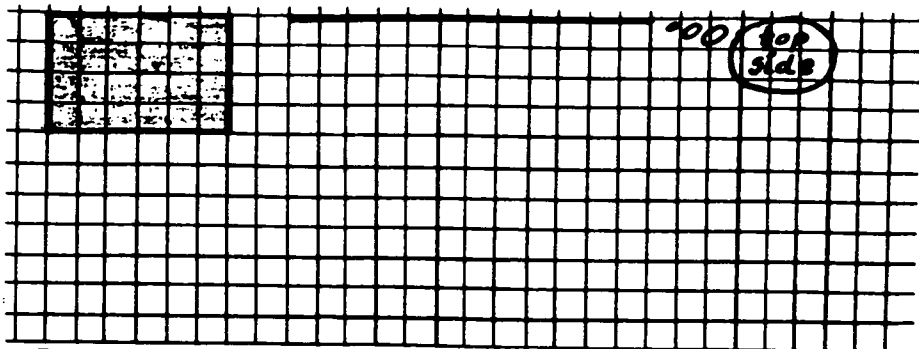
2. Complete the rectangle, so to have the same shape as the given one.
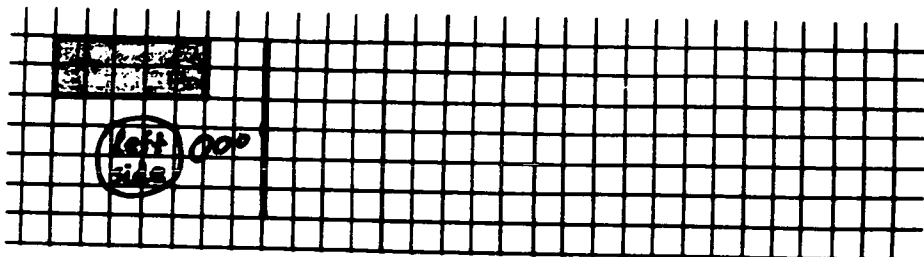
3. Enlarge the given rectangle as many times as you want.
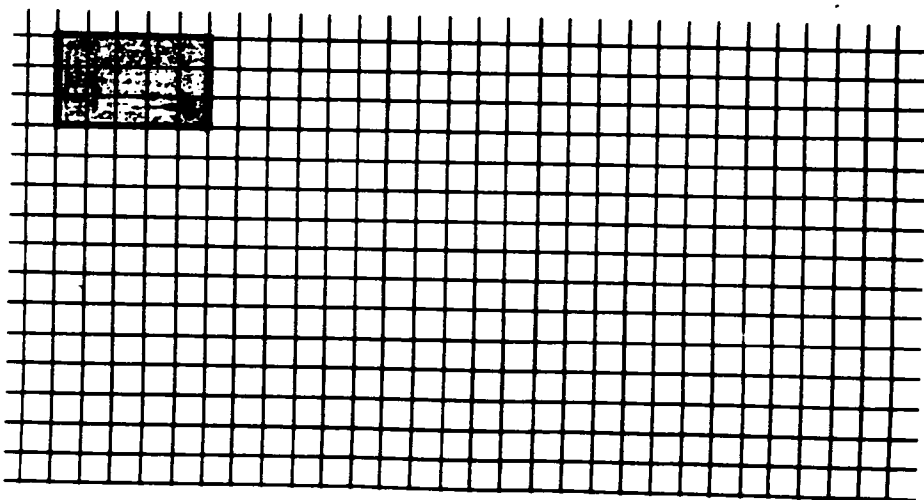
REDUCING RECTANGLES

WORKSHEET 5.3

Your Name :

Your School :

1. Complete the rectangle so to have the same shape as the given one.
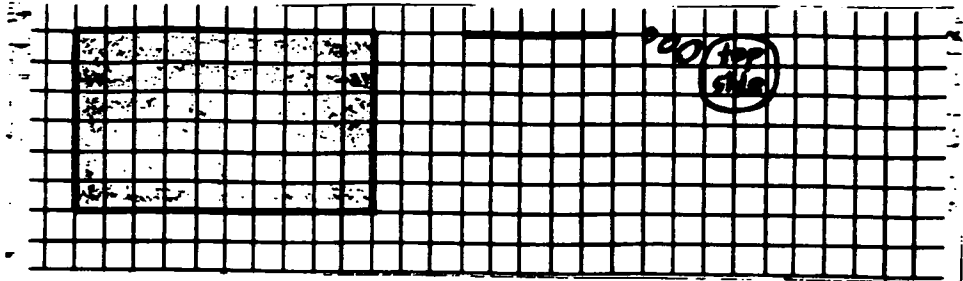
2. Complete the rectangle so to have the same shape as the given one.
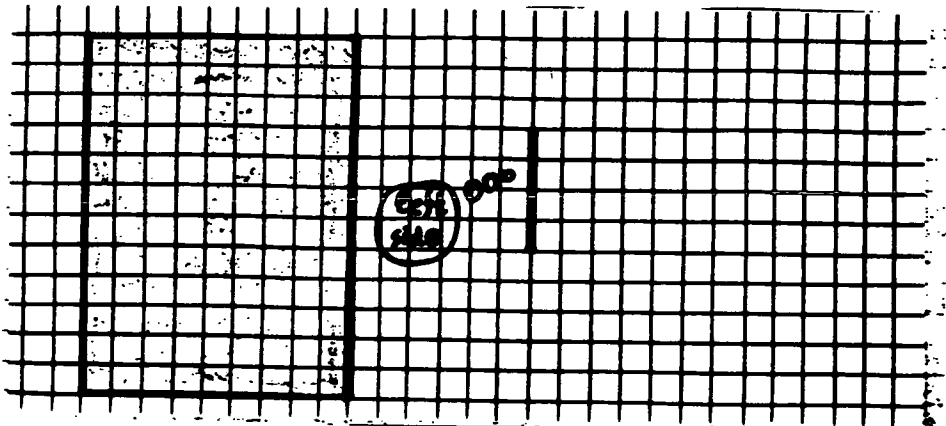
3. Reduce the given rectangle as many times as you want.
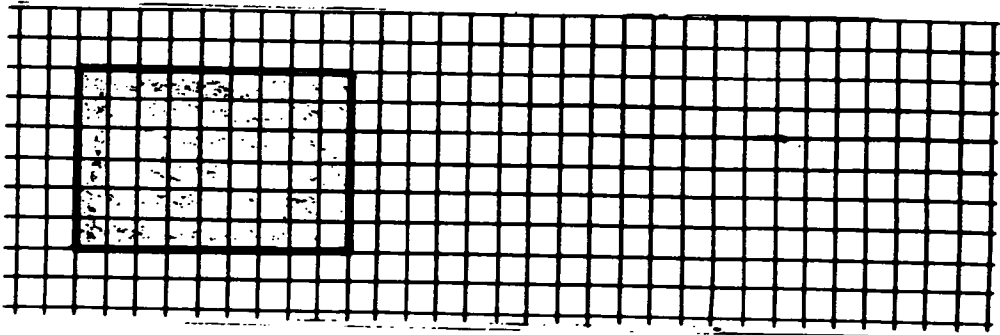
STRING TEST   WORKSHEET 5.4

                                     Your name :
                                     Your School :

1. Decide on which of the rectangles in set 1 have the
    same shape.
RECTANGLES OF THE SAME SHAPE :

2. Fit a piece of paper in the bottom left corner of
    the box.

3. Place the smallest rectangle of set 1 in
    the bottom left corner of the box (like in fig.1).
    Follow the sides of the rectangle with your pencil,
    to draw the same rectangle on the paper.
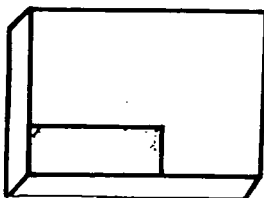
fig.1

4. Repeat step 3 for each rectangle in set 1.

5. Draw on the paper the same diagonal for all
    rectangles. What do you notice?

5. Place the rectangles in set 2 as in step 3.   Use a
    piece of string instead of drawing their diagonals.
    Which rectangles in set 2 have the same shape ?

        (see back cover for SET1 and SET2 rectangles)

T.A.T.

RATIO TEST   WORKSHEET 5.5

Your name :
Your School :

1. Decide on which of the rectangles in set 3 have the same shape.

RECTANGLES OF THE SAME SHAPE :

2. Find the length and the breadth for each rectangle in set 3, to complete the following table.

	LENGTH	BREADTH
RECTANGLE A		
RECTANGLE B		
RECTANGLE C		
RECTANGLE D		

3. Divide the length by the breadth for each rectangle, to find the ratio of length to breadth.

LENGTH ÷ BREADTH

RATIO A	
RATIO B	
RATIO C	
RATIO D	

4. What do you notice about the ratios of the rectangles in set 3?

T.A.T.

118A
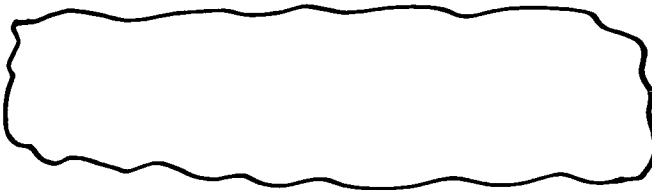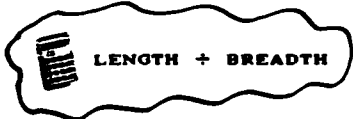
5. Find the length and the breadth for each rectangle
   in set 4, to complete the following table.

	LENGTH	BREADTH
RECTANGLE A		
RECTANGLE B		
RECTANGLE C		
RECTANGLE D		
RECTANGLE E		

6. Find the ratio of length over breadth for each
   rectangle in set 4, to complete the following
   table.

LENGTH ÷ BREADTH

RATIO A	
RATIO B	
RATIO C	
RATIO D	
RATIO E	

7. Can you decide which rectangles in set 4 have the
   same shape, just by comparing their ratios from
   step 6 ?

   RECTANGLES OF THE SAME SHAPE :

T.A.T.

SET 3 - RATIO TEST

A   B   C   D

SET 2 - RATIO TEST

A   B   C   D   E
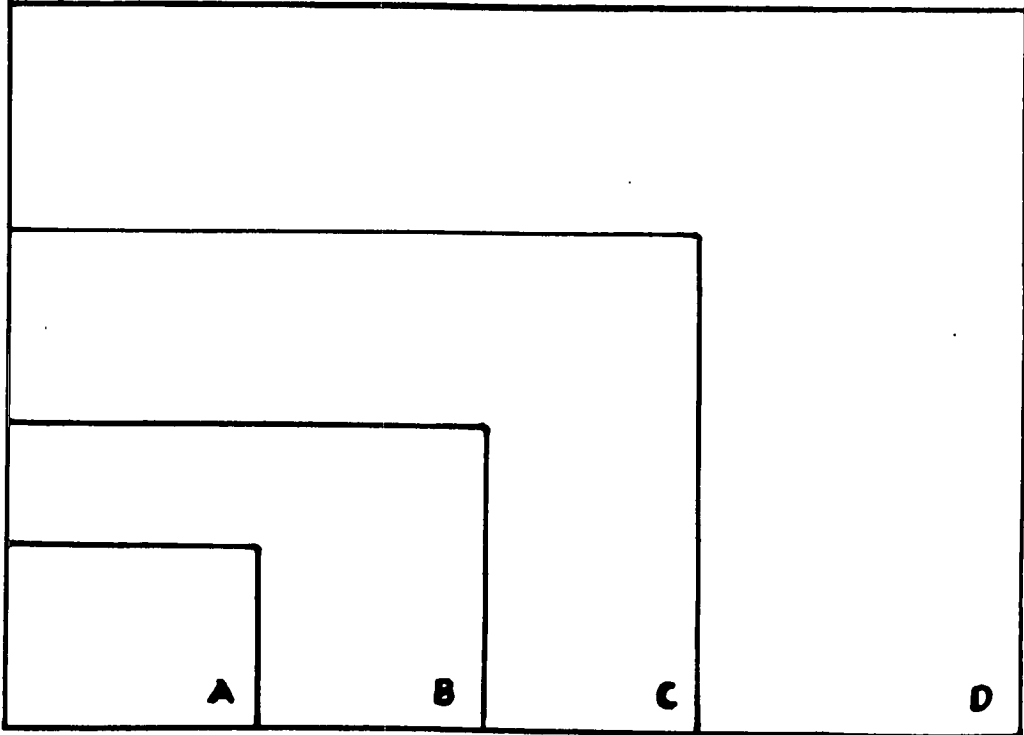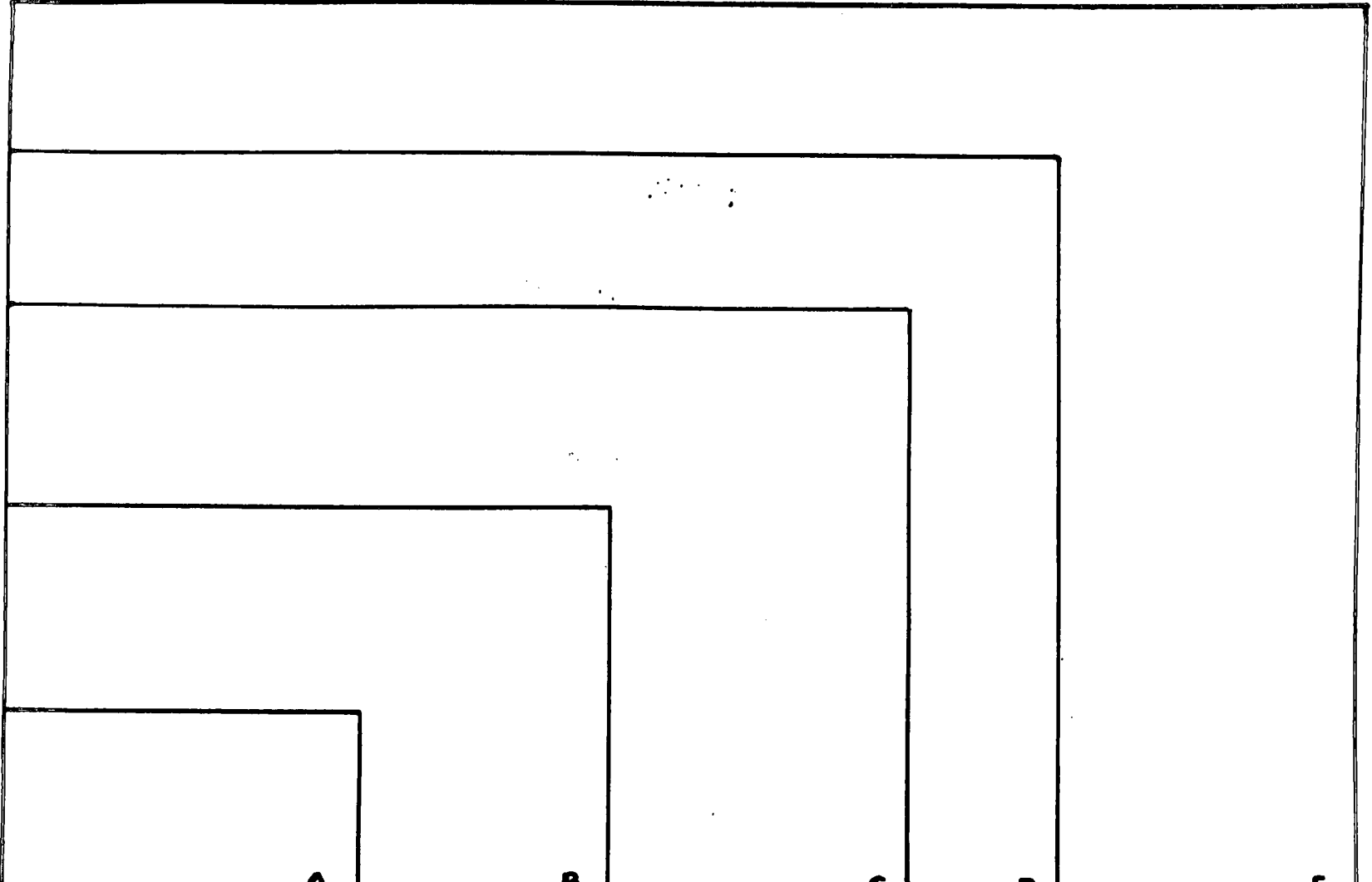
A - RECTANGLES   WORKSHEET 5.6   YOUR NAME:

YOUR SCHOOL:

Begin with two sheet each of A3, A4, A5, and A6 paper.

1. Check that :

    a. 2 pieces of A4 laid side by side is the same as one piece of A3

    b. 2 pieces of A5 laid side by side is the same as one piece of A4

    c. 2 pieces of A6 laid side by side is the same as one piece of A5

2. How many pieces of A5 are the same as one piece of A3?

    How many pieces of A6 are the same as one piece of A3?

3. Use the "string test". Do these sheets (A3, A4, A5, & A6), have the same shape?

    YES / NO

4. Now check your answer using the "ratio test". Measure the sides of the rectangles (in millimitres for better accuracy). Work out the ratio of the length to the breadth for each size.

    CHECK ON YOUR CALCULATOR

SIZE	LENGTH	BREADTH	LENGTH÷BREADTH
A3	420	297	1.41
A4			
A5			
A6			
	MILLIMITRES	MILLIMITRES	

Are your ratios equal (or nearly equal)?   YES / NO

T.A.T.

5. An A-shaped rectangle will have the ratio

LENGTH÷BREADTH =

6. An A-shaped rectangle has breadth 141mm. What is its length?

LENGTH=BREADTH x RATIO

7. Identify any A-shaped rectangle on the Temple of Zeus in Olympia, Greece (the marked rectangle is a clue).

breadth

length

T.A.T.
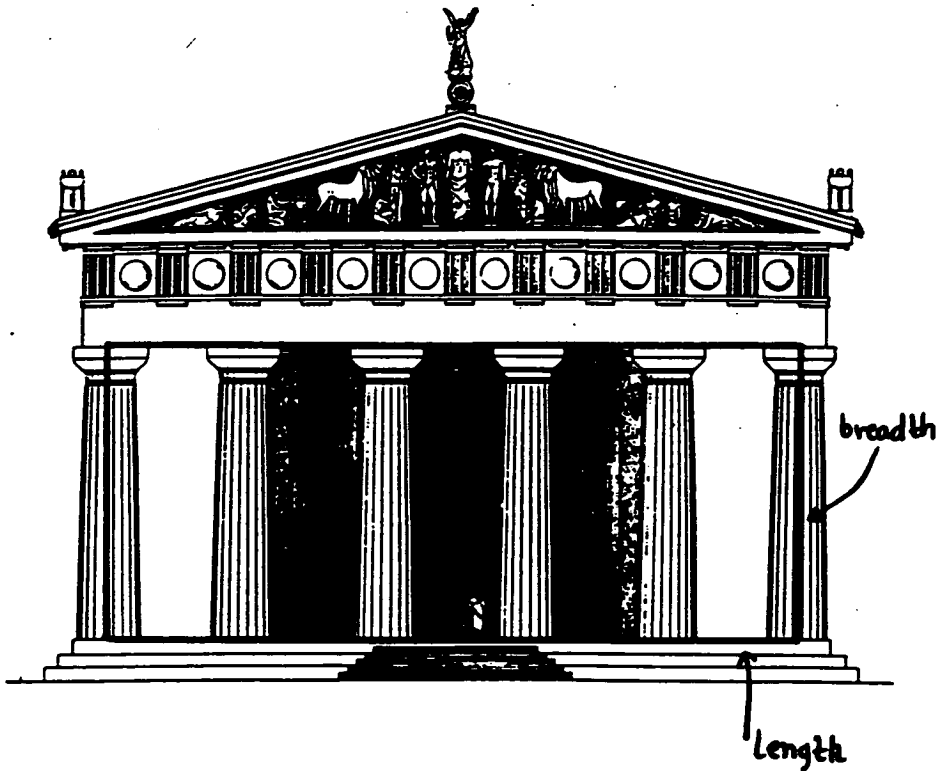
DRAWING A - RECTANGLES.   WORKSHEET 5.7   YOUR NAME:

YOUR SCHOOL:

1.   Fix this piece of paper to a drawing board.

2.   Check that the length AB equals the length BC.

3.   Put in mapping pins at A and B. Take
     the cardboard cross form and place it
     so that two adjacent arms rest on the
     two pins at A and B (see figure 1).

4.   Slide the cross form about, keeping it
     in contact with the pins at A and B.
     Points P and Q are the points where
     the other two arms of the cross cut the
     vertical lines through A and C.
     Notice how these points, P & Q, move
     when you slide the cross about.

5.   Now find the position of the cross, so
     that the points P and Q are at the
     same height (see figure 2).

6.   Mark these positions of P and Q. Remove the cross form
     and draw the line PQ.

7.   Is the rectangle ACQP A-shaped?   YES / NO
     (use the ratio test)

fig.1

fig.2

T.A.T.
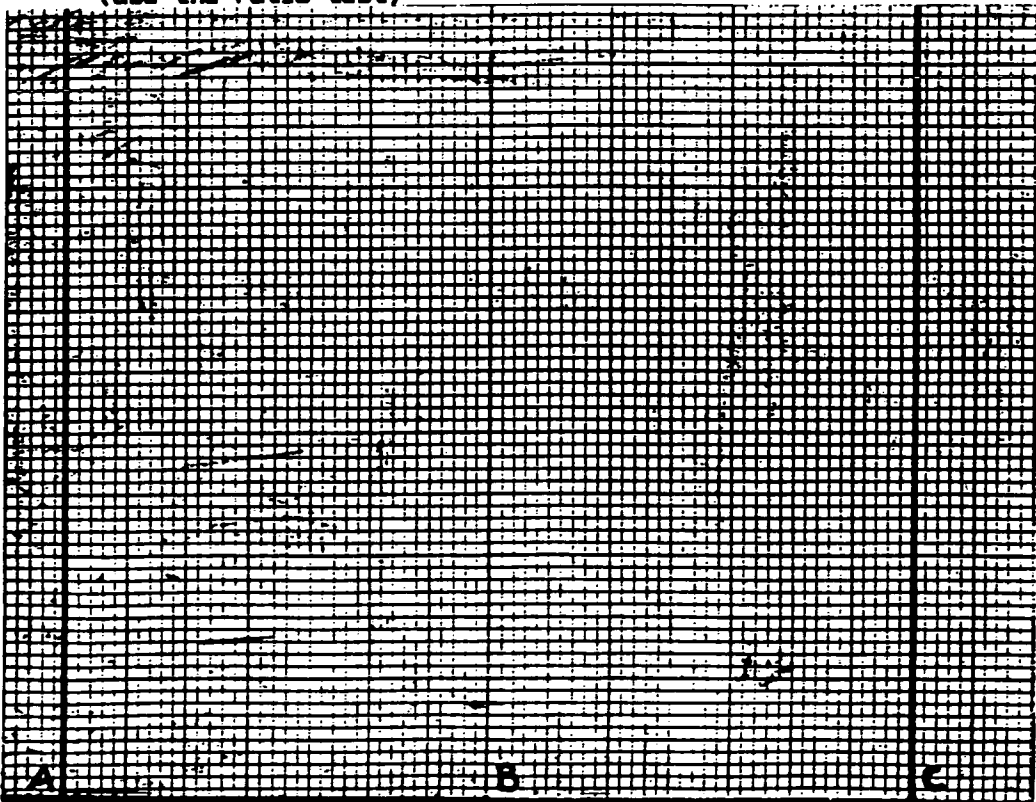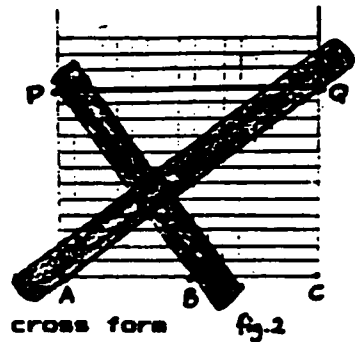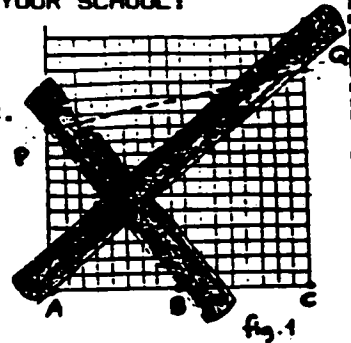
GOLDEN RATIO

WORKSHEET 5.8

YOUR NAME:

YOUR SCHOOL:

The pentagram (or *pentalpha*, 5 A's in the Greek), was thought to be a magic charm to ward off evil spirits. The ratio of the lengths of its parts is called the GOLDEN RATIO.
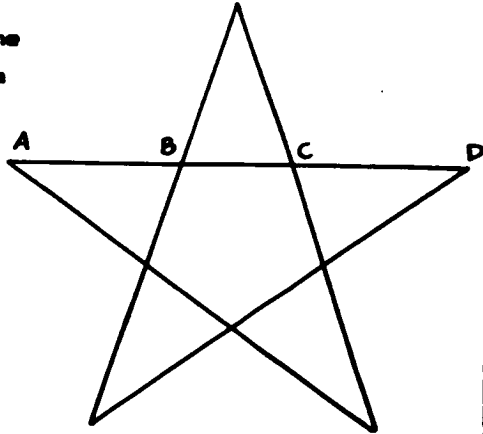
$$\frac{\text{LENGTH OF AD}}{\text{LENGTH OF AC}} = \text{———} = $$

$$\frac{\text{LENGTH OF AC}}{\text{LENGTH OF AB}} = \text{———} = $$

This special number was used in the design of sacred buildings. Its exact value is (1+√5)÷2. Work out this value on your calculator, and write down your answer with two decimal places.

Here is a picture of the Parthenon in Athens.
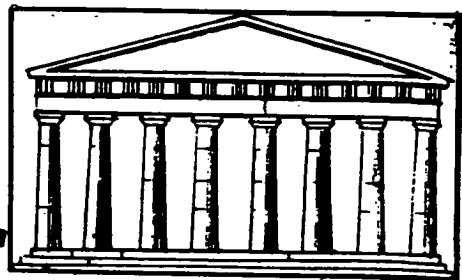
breadth

length

Measure the length and the breadth of the marked rectangle, and work out their ratio.

LENGTH =

BREADTH =

RATIO = LENGTH÷BREADTH

T.A.T.

Here is a view of the theatre at Epidaurous in Greece, and a cross-section.
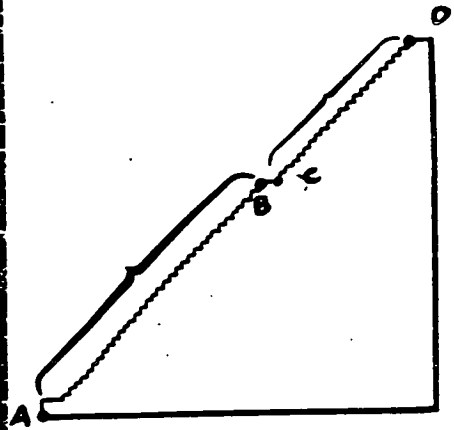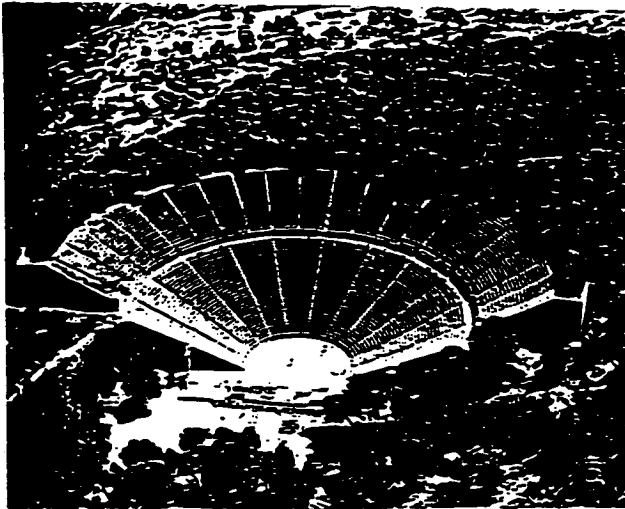
Measure the lengths AB and CD, and work out their ratio.

LENGTH AB = ⬡   LENGTH CD = ⬡   RATIO = ⬡

What does this ratio reminds you of?

Calculate for once more the value of $\frac{1+\sqrt{5}}{2}$. Write down your answer keeping all the decimal places on your calculator:

(GOLDEN RATIO)=

Now multiply the previous value with itself and write down your answer keeping all decimal places on your calculator:

(GOLDEN RATIO) X (GOLDEN RATIO)=

Now devide number 1 by the value of the golden ratio and write down your answer keeping all decimal places again:

1 ÷ (GOLDEN RATIO)=

What do you observe comparing the 3 values that you calculated?

INVESTIGATION   WORKSHEET 5.9   YOUR NAME:

YOUR SCHOOL:

We wish to find a rectangle which have the special property
that when wew add on a square we do not change its shape!

$$ \boxed{S} + \boxed{R_1} = \boxed{R_2 = S + R_1} $$

$\{$ and R_1, R_2 have
$\{$ the same shape
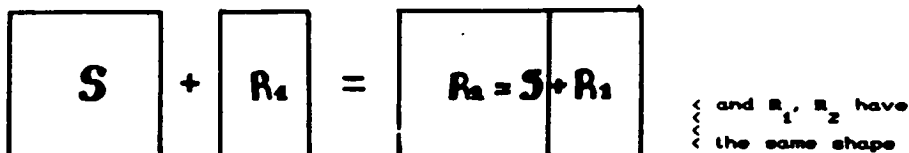
TRYING OUT (SEE NEXT SHEET) :

Slide the rectangular piece of cardboard under the square
one, in the direction that the arrows indicate (use the lid
from the string test).

a) Stop when the side of the square has reached the line CD
    drawn on the rectangle (see fig.1). For CDFE to be the
    wanted rectangle, it should have the same shape as ABEF
    rectangle. You can check that using the string test:

    1. Find a rectangle from group A that has exactly the
        same shape and size as CDFE.
    2. Fit this rectangle at the bottom left corner of
        rectangle ABEF.
        Now use the string test.

    Is then CDFE the wanted rectangle?   YES/NO

b) Slide the square until its side has reached the line GH drawn
    on the rectangle (see fig.2).

    Is then GHFE the wanted rectangle?   YES/NO

T.A.T j

SOLUTION :

Keep on trying for other positions of the square. When you
have found the wanted rectangle, measure its length and
breadth and work out their ratio.

RATIO = LENGTH ÷ BREADTH = ⬭ ÷ ⬭ = ⬭ CM

What is the name of that ratio? ⬭

How would you name then the rectangle that we were looking
for? ⬭

Now find out about "foolscap" paper!

TRYING OUT:
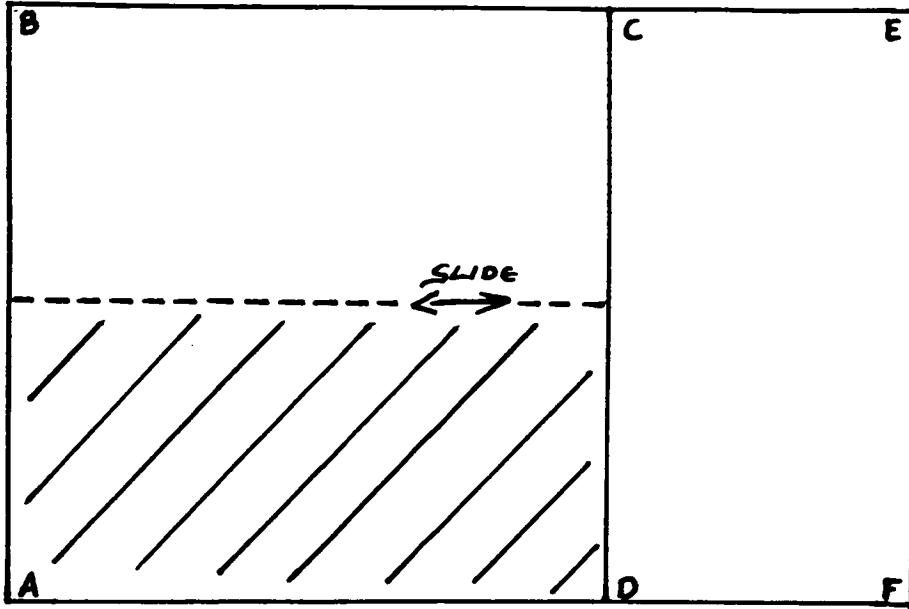
fig.1
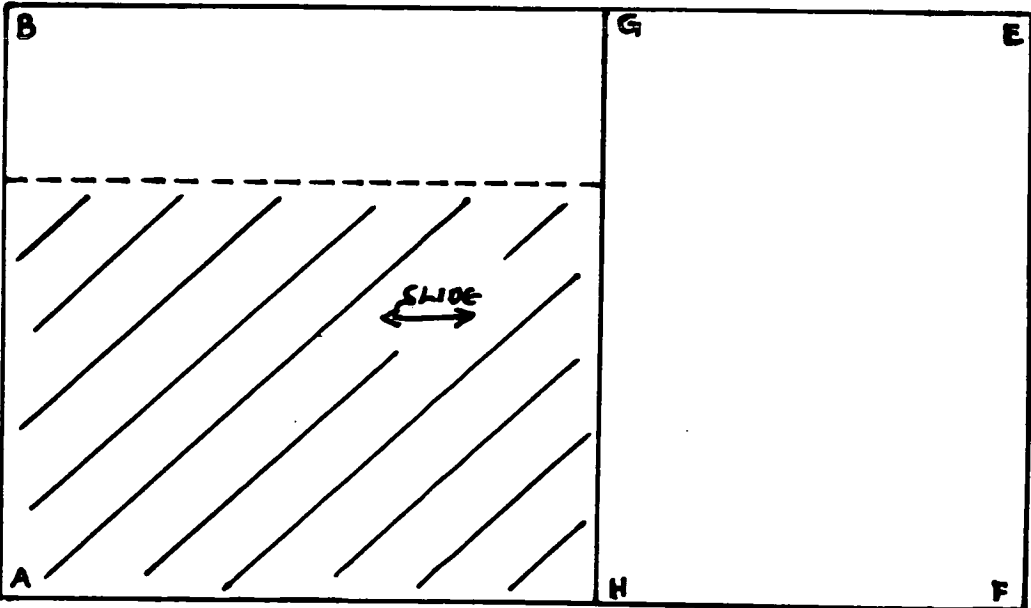
fig.2

DRAWING GOLDEN RECTANGLES   WORKSHEET 5.10   YOUR NAME:

YOUR SCHOOL:

1. Fix this piece of paper to a drawing board.
2. Put in mapping pins at B and D. Take the cardboard cross and place it so that two adjacent arms rest on two pins at B & D.
3. Slide the cross form about, keeping it in contact with the pins at B and D. Points P and Q are the points where the other two arms of the cross cut the extended sides AD & BC. Notice how these points, P & Q, move when you slide the cross form about.
4. Now find the position of the cross, so that the points P and Q are at the same vertical line (see figure 2).
5. Mark these positions of P and Q. Remove the cross form and draw the line PQ.
6. Is the rectangle ABPQ a golden rectangle?   YES / NO
7. Now use the back of this page, to draw any golden rectangle following the same technique.
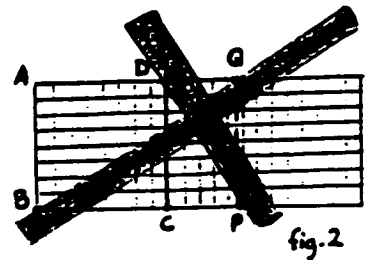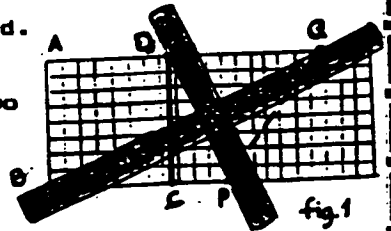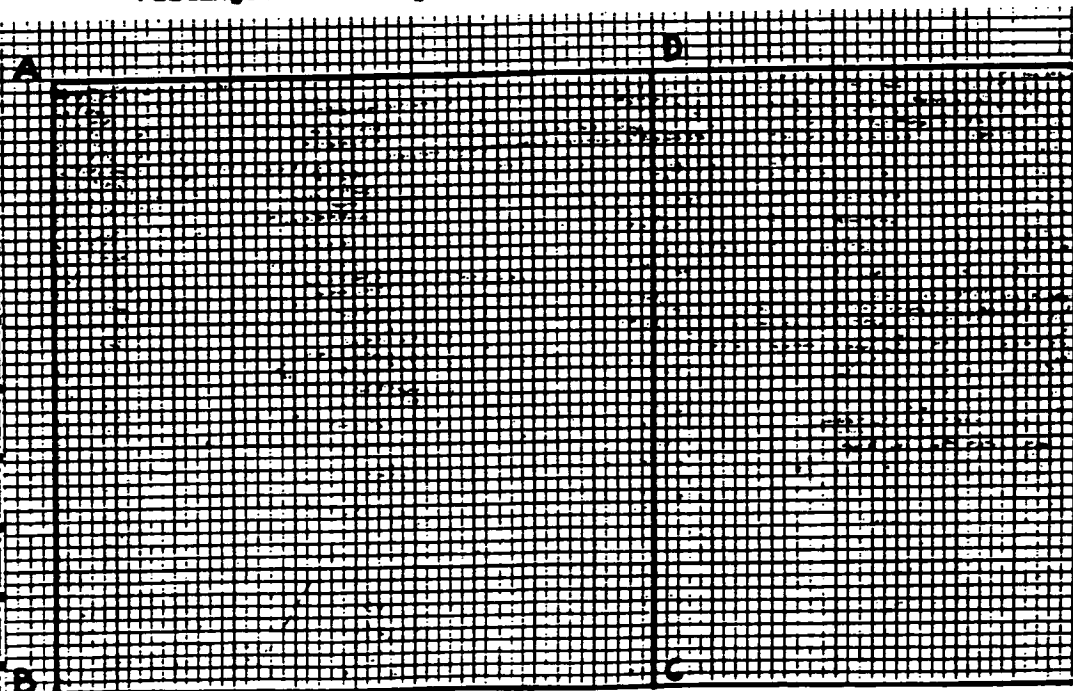
Appendix C

The 'Drawing and Geometric Constructions' Activities

The worksheets that comprise the 'drawing and geometric constructions' activities follow. See also back cover for the 'Measuring and Drawing Library' booklet.

BE A GEOMETRIC CONSTRUCTOR.

All you have to do to become a "geometric constructor",
is to be able to draw geometric shapes using only a
straightedge and a pair of compasses.

That is:
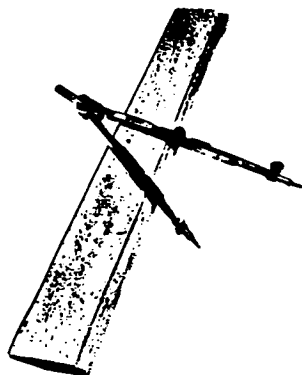
In some constructions though we shall allow some
cheating, by using a ruler instead of a straightedge!

T. A. T.
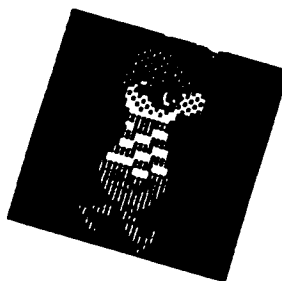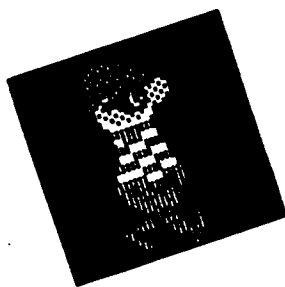
TO CONSTRUCT A PERPENDICULAR TO A CERTAIN POINT OF A LINE

Having the line drawn beside, I want to construct the perpendicular to that line at the point A.

1. I adjust the compasses to any radius, not a very long one, and I draw a circle with centre A.
2. I mark with B and C the points where the circle meets the line.
3. I adjust the compasses to any radius, longer than before, and I draw two circles with centres B and C.
4. I mark with D the point where these circles meet, and I draw a line to connect D with A.
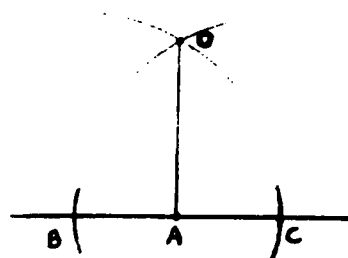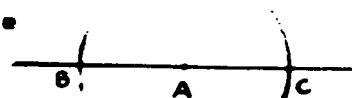
Then AD is perpendicular to the initial line (check with your protractor or a set square).

NOW IT IS YOUR TURN: construct a perpendicular to the line drawn below at the point A, following the previous steps, labelling the points B, C, and D used in your construction.

CHECK YOUR CONSTRUCTION USING A SET SQUARE OR A PROTRACTOR!

4

T.A.T.

To Construct a Perpendicular to a Line From an Outside Point

Having the line drawn beside, I
want to construct the perpendicular
to that line that passes from A.

A

1. With centre A I draw a circle to
    take C and B (any radius will do
    as long as the circle meets the
    line at two points).
2. Keeping the same radius I draw two
    circles with centres B and C.
    I mark D the point where they meet.
3. I connect A with D with a line.
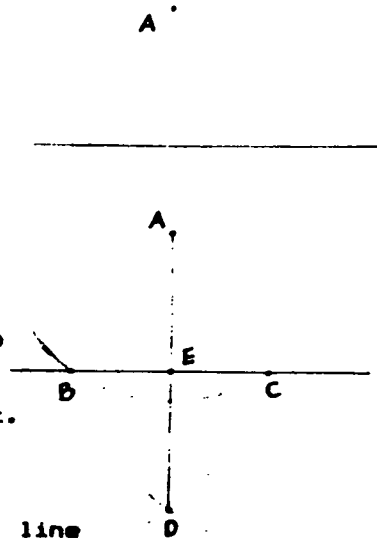
Then AD is perpendicular to the initial line
(check with a protractor or a set square).

NOW IT IS YOUR TURN: construct a perpendicular to the line
drawn below that passes from A, labelling the points B, C, D,
and E used in your construction.

. A

CHECK YOUR CONSTRUCTION USING A SET SQUARE OR A PROTRACTOR!

2

T.A.T.

TO BISECT A LINE

Given a line AB,

1. I adjust the compasses to a radius longer than half the of AB.

2. With centres A and B I draw two circles and I mark C and D the points that these meet.

3. I draw a line to connect C with D, and I mark M the point where this line meets line AB.
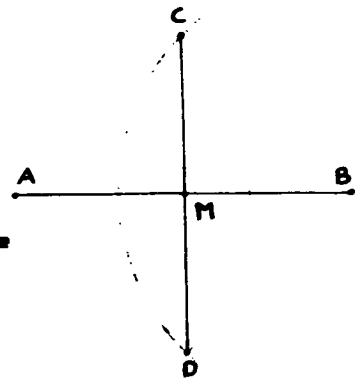
Then the lengths of AM and BM are equal (check with your ruler).

Line CD is called the bisector of AB.

NOW IT IS YOUR TURN: bisect line AB labelling the points C, D, and M used in your construction.

A ——————————————————————— B

CHECK YOUR CONSTRUCTION WITH A RULER!

3

T.A.T.
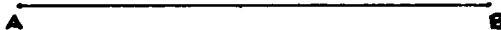
To Translate an Angle

I want to translate the <BCA shown.

1. I draw a line and I mark a point D on it. D will be the vertex of the new angle.

2. With centres A and D I draw two circles of the same radius (any length will do).
   I mark the points B, C, and E.

3. I adjust the compasses to the distance between B and C. With that as a radius and centre E I draw a circle.

4. I mark F the point where circles meet. I draw a line to connect D and F.
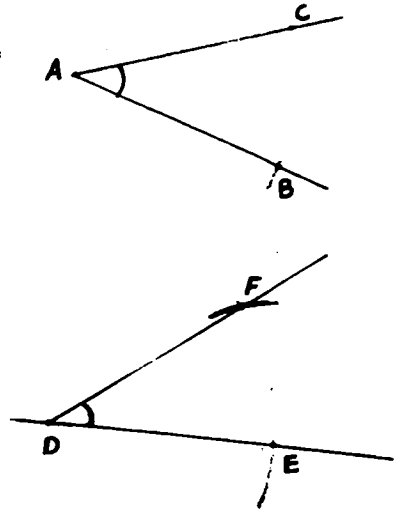
Then <EDF and <BAC are equal (check it with your protractor).

NOW IT IS YOUR TURN: translate the angle given below, labelling the points B, C, E, and F used in your construction.
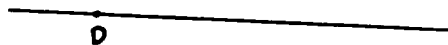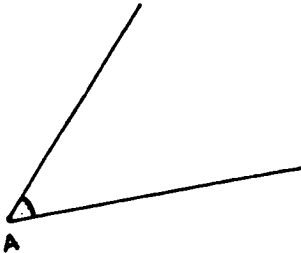
CHECK YOUR CONSTRUCTION USING A PROTRACTOR!

4

T.A.T.

TO CONSTRUCT AN EQUILATERAL TRIANGLE

1.   I draw a line AB which will be
     one side of the triangle. Then
     I adjust the compasses to a radius
     equal to the length of AB.
2.   With centres A and B I draw two
     circles with the previous radius.
3.   I mark C one point where the circles
     meet, and I draw lines to connect
     C with A and B.
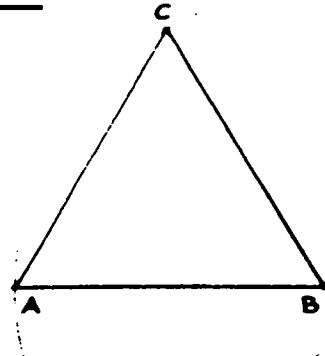
Then ABC is an equilateral triangle
(check the sides with your ruler and
the angles with your protractor).

NOW IT IS YOUR TURN: construct an equilateral triangle for
the given line AB.
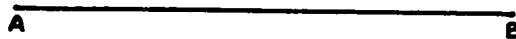
CHECK YOUR CONSTRUCTION WITH YOUR RULER AND PROTRACTOR!

5

T.A.T.

TO CONSTRUCT AN ISOSCELES TRIANGLE

1. I draw a line AB which will be the base of the triangle. I adjust the compasses to a radius longer than the length of AB.
2. With centres A and B I draw two circles with the previous radius.
3. I mark C one point where the circles meet, and I draw lines to connect it with A and B.
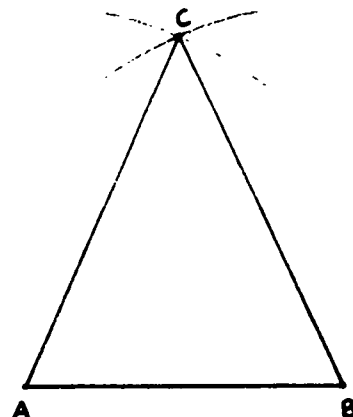
Then ABC is an isosceles triangle (check with a ruler and a protractor).

NOW IT IS YOUR TURN: construct an isosceles triangle for the given base AB.

A ─────────────────────── B
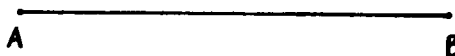
CHECK YOUR CONSTRUCTION WITH A RULER AND PROTRACTOR!

6

T.A.T.

To Construct a Right-Angled Triangle

1. I draw any line and I mark a point A on it.
2. I adjust the compasses to any radius, not very long, and I draw a circle with centre A.
3. I mark with B and D the points where the circle meets the line.
4. I adjust the compasses to any radius, longer than before, and I draw two circles with centres B and D.
5. I mark with C one point where the circles meet, and I draw lines to connect it with A and B.
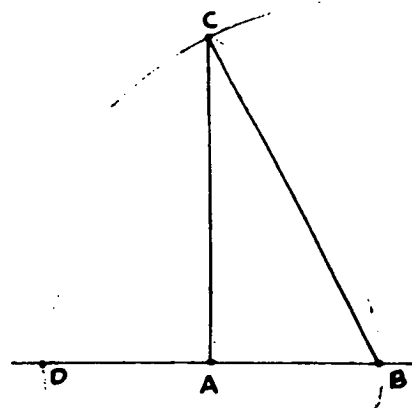
Then ABC is a right-angled triangle (check with your protractor and a set square).

NOW IT IS YOUR TURN: construct a right-angled triangle labelling the points B, D and C used in your construction.

A

CHECK YOUR CONSTRUCTION WITH A PROTRACTOR AND A SET SQUARE!

7

T.A.T.

To Construct a Scalene Triangle
(with sides 6,4, & 3cm)

1. I use the ruler to draw a line 6cm long. I name its ends A & B.

2. With the help of a ruler, I adjust the compasses to a radius of 4cm, and I draw a circle with centre A.

3. I adjust the compasses to a radius of 3cm, and I draw a circle with centre B.

4. I mark with a C one pint where the circles meet, and I draw lines to connect it with A & B.
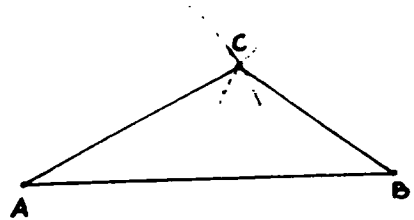
Then ABC is a scalene triangle of sides 6, 4, and 3cm (check the sides using your ruler).

NOW IT IS YOUR TURN: construct a scalene triangle of sides 8, 6, and 3cm.

CHECK YOUR CONSTRUCTION USING A RULER!

8

T.A.T.

To Construct a Regular Hexagon

1. I draw a circle of any radius and I mark a point A on it.
2. Leaving the compasses adjusted to the radius of the circle and starting from A, I mark points B, C, D, E, and F.
3. I connect A with B, B with C,..., and F with A.
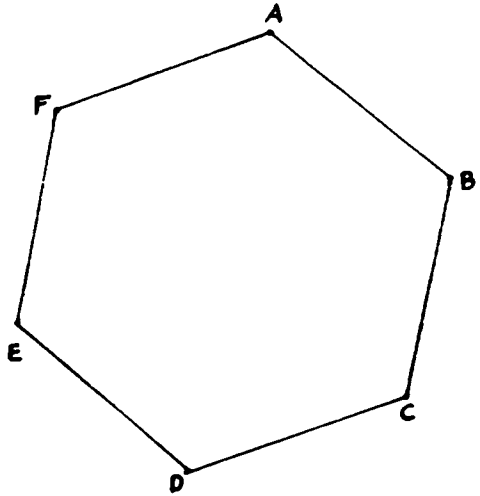
Then ABCDEF is a regular hexagon (check the sides and the angles).
Compare the diagonals AC and AD to the side of the hexagon.

LENGTH AB=⬭cm LENGTH AC=⬭cm LENGTH AD=⬭cm

NOW IT IS YOUR TURN: construct a regular hexagon ABCDEF with a side of your choice.

LENGTH AB=⬭cm LENGTH AC=⬭cm LENGTH AD=⬭cm

CHECK YOUR CONSTRUCTION USING A RULER AND PROTRACTOR!

9

T. A. T.

To Construct Two Parallel Lines

I want to construct a line parallel
to the line drawn beside.

1. I mark two points on the line,
    A and B.
2. I construct a perpendicular
    to the point A of the line.
3. I construct a perpendicular
    to the point B of the line,
    using exactly the same radii
    as for the point A.
4. I draw a line passing through
    C and D.
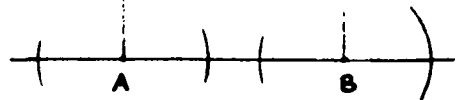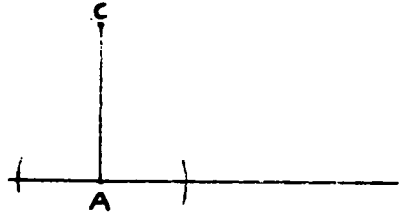
Then the line that passes through C
and D is parallel to that passing from
A and B (check with a set square and
a ruler).

NOW IT IS YOUR TURN: construct a parallel line to the line
drawn below labelling the points A, B, C, and D used in your
construction.

CHECK YOUR CONSTRUCTION USING SET SQUARE AND RULER!

10

T.A.T.

To Find the Centre of a Circle

I want to find the centre of the
circle drawn beside.

1.   I draw any line that meets the
        circle in two points, A and B.
2.   I bisect the line AB.
        I mark C and D the points
        where the bisector of AB meets
        the circle.
3.   I bisect the line CD.
        I mark E the point where the
        two bisectors, of AB and CD, meet.
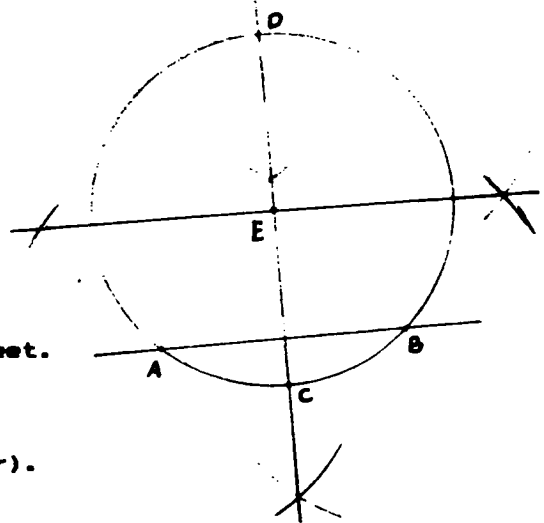
Then E is the centre of the circle
(check with your compasses and ruler).

NOW IT IS YOUR TURN: find the centre of the following circle
labelling the points A, B, C, D, and E used in your construction.

CHECK YOUR CONSTRUCTION USING A RULER AND COMPASSES!

11

T.A.T.

THALES OF MILETUS

Thales was a great Greek mathematician that lived between 624 and 547 B.C. He was renown for his interest in Astronomy. They say that once he fell into a well while star gazing!! He was so eager to know what goes on in the heavens that he could not see what was straight in front of his very feet.

Draw the lines on the sketch between E (eye) and S (star), between E (eye) and W (well), and between F (foot) and W (well).

Measure these lines ES, EW, and FW carefully, and mark their lengths on the the following lines appropriately.

ES = ( ) cm   EW = ( ) cm   FW = ( ) cm

E

E

F

CHECK: if your measurements are accurate, then you should
           have :

                (LENGTH ES) = (LENGTH EW) + (LENGTH FW)

12

T.A.T.

The Bermuda Triangle

The "Bermuda Triangle" is an infamous location in the Atlantic Ocean, west of the U.S.A.
Many tragic accidents have happened there, all unexplained.

To draw the "Bermuda Triangle" follow the next steps:

1. Draw a line AB 10cm long.
2. Knowing that the angle at B is 70, draw a line BC 14cm long.
3. Draw AC.

If your drawing is accurate, AC should be 14.2cm long.

13

T.A.T.

BILLIARDS

Billiards is a game similar to snooker. Two white balls and a red one are on the table. In order to score a point, a player has to make his ball hit the two other balls.

On the table shown, the player aims to hit ball B first, hoping to hit E eventually.

Using a ruler and protractor copy the path of the ball A.

14

T.A.T.
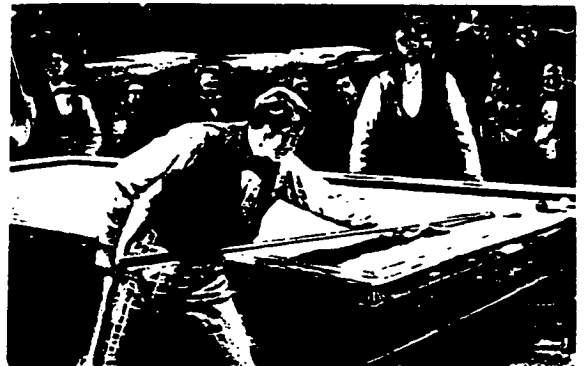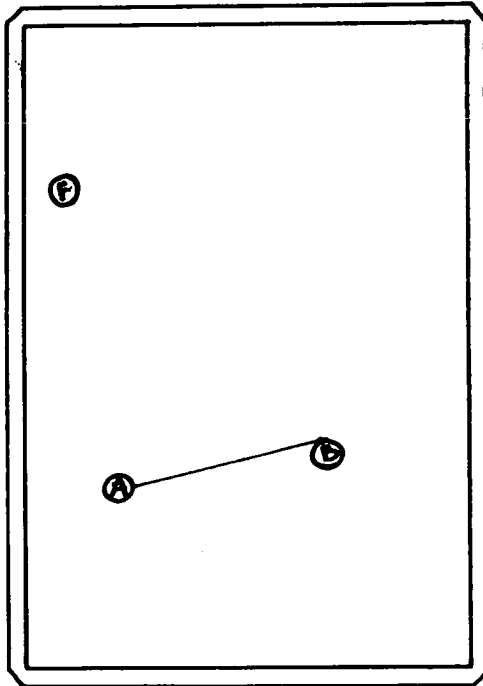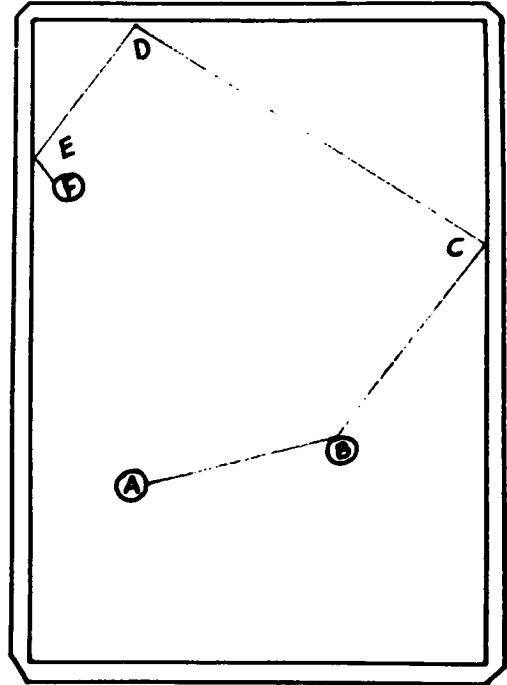
TREASURE HUNT

You are on a treasure-hunt trip. A
is the point where the boat left you.
The "hunting" starts from point B
beside the palm tree.
Follow the directions written on the
papyrus to reach the treasure.

! Note that 1cm represents 100 steps !

When you reach the palm
tree turn 70° left and walk
1360 steps. Turn 150° left
and walk 1150 steps.
Then turn 60° right and
walk 1100 steps. Turn 140°
to the right and walk
1500 steps. Turn 135° left
and walk 1520 steps.
Turn 155° left and walk
1640 steps.
Now you're rich - enjoy life!

been used only in Greece -
wrong instructions

B

A
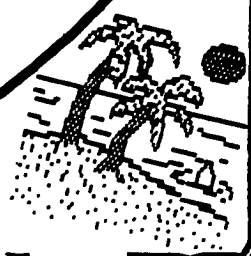
15

T.A.T

5-GON AND 6-GON MAGIC

By drawing all the diagonals from each vertex of the
following pentagon, you should be able to create a second
pentagon within the first one.
Repeat the process to create a third pentagon within the
second. Repeat the process to ......
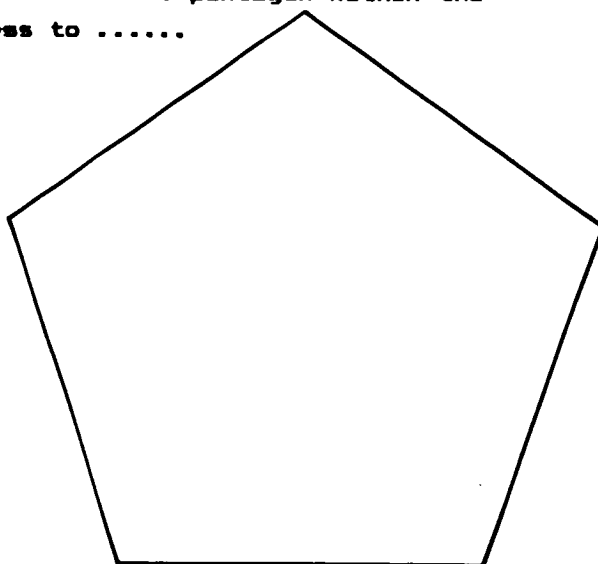
Follow the same process for the hexagon below:

16

T.A.T.
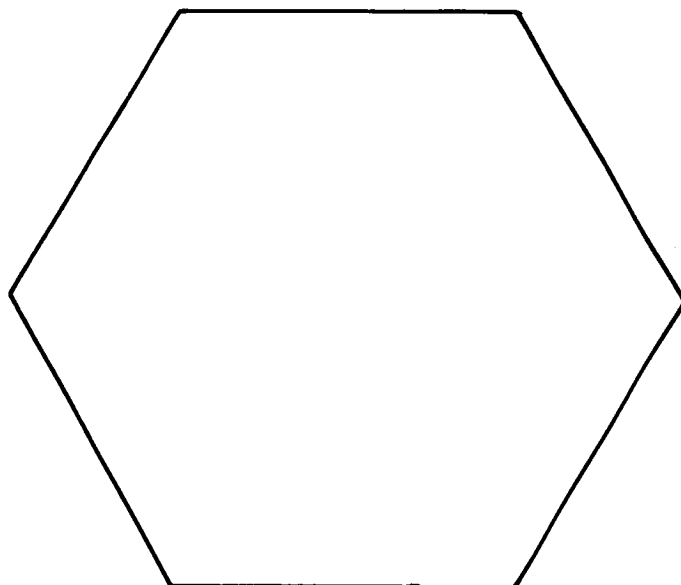
PENTAGRAM

The pentagram, or star-pentagon, was used as a symbol of recognition between the members of the school of Phythagoras in Ancient Greece.

Notice how the sides of the three triangles interweave to give the pentagram symbol.

Use the space below to draw a pentagram of your own.

17

T.A.T.
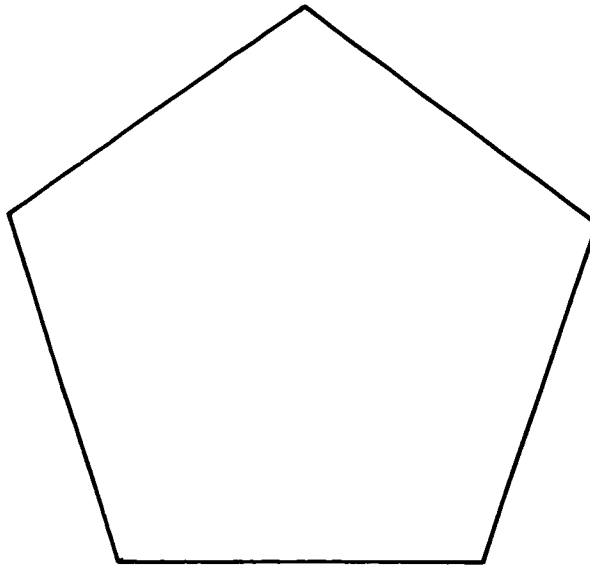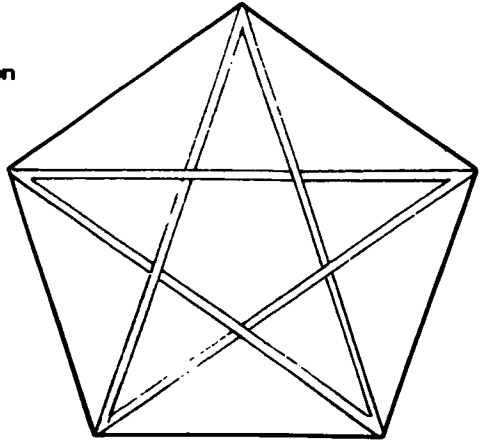
THE VERNIER RULER

The Vernier ruler is one that can help us measure more easily and with better accuracy the length of various lines.
Its scale is different than that of the ordinary rulers. Ten units on the Vernier scale represent a length of 9 units on the ordinary scale.

ORDINARY SCALE:

```
0   1   2   3   4   5   6   7   8   9   10
```

VERNIER SCALE:

```
0   1   2   3   4   5   6   7   8   9   10
```
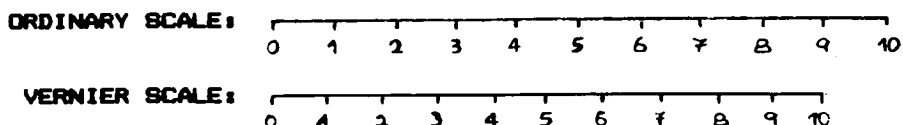
A Vernier ruler is used together with an ordinary ruler to find the length of a line. For the following line for instance:
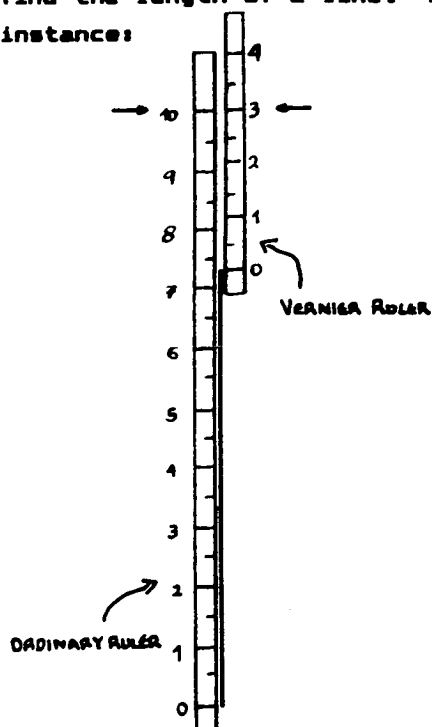
I place the ordinary ruler at the one side of the line. The length of the line is a little more than 7cm.
I place the Vernier ruler at the other side of the line, so that the zero (0) mark is exactly beside the end of the line.
I look along the Vernier ruler, and I stop where the mark on the Vernier scale is exactly beside a mark on the ordinary scale. Here it is 3. Then the length of the line is 7.3cm (check with an ordinary ruler).

VERNIER RULER

ORDINARY RULER

NOW IT IS YOUR TURN: use an ordinary ruler and a Vernier ruler, as before, to find the length of the lines on the separate sheet.

18

T.A.T.

VERNIER RULER

1.

2.

3.

4.

5.

19

NAUTILUS SHELL

Often we find a strong relationship between beauty and mathematics. That applies not only to artistic creations but to natural beauty as well.

An example of the relationship between mathematics and nature's beauty is the shell of the chambered nautilus, a creature that leaves in the S. Pacific.

The shape of the nautilus shell is a spiral. We can draw a spiral like the nautilus' shell as follows:

1. I start with a certain rectangle ABCD, and I mark off a square ABEF. With centre F and radius AF I draw a circle.
2. From rectangle ECDF I mark off a square ECGH. With centre H and radius HE I draw a circle.
3. From rectangle GHFD I mark off a square DGIJ. With centre I and radius IG I draw a circle.

I continue with rectangle IJFH in the same manner.

NOW IT IS YOUR TURN: start with rectangle ABCD (on a separate sheet) and draw a spiral, labelling the points E, F, G, H, I, and J used in your drawing.

20

T.A.T.

Nautilous Shell

T.A.T.
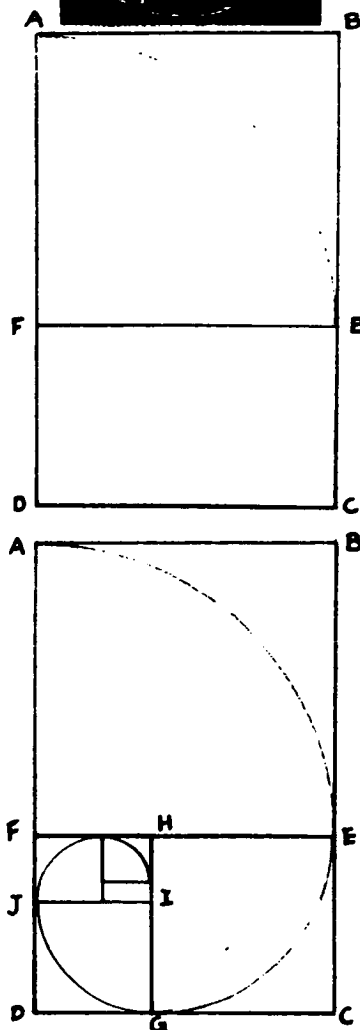
...MYSTIKAL GEOMETRY..

In many cases various geometric designs have been invested with magical or supernatural powers.

Religious, Magic, astrology, witchery, superstitions. The most favourable geometric shapes used are the regular polygons (equilateral triangle, square, pentagon, hexagon, ...).

Some of these magical geometric designs are on this page. Magical rings and brooches, horoscopic tables, Holy Tables that provide a means of communication with the Angels, Holy Signs with inscriptions to protect from Demons.

Use a separate sheet to design your own magical design that will protect you from the powers of Evil. You can even draw horoscopic tables, and try to predict the future....

11.			9.
12. Friends	10.		Religion
Enemies	Magistracy		8.
1.			Death
Life			
2.			Marriage
Riches	4.		6.
3.	Parents	5.	Health
Family		Children	

Propositi Genesis.

22   T A T
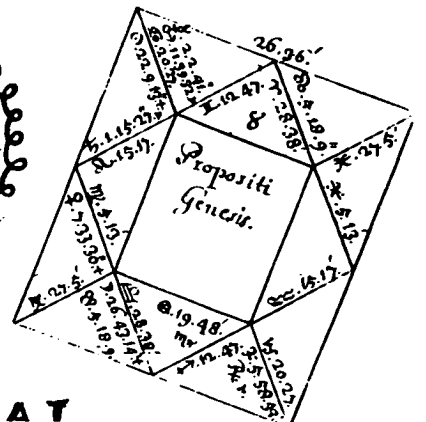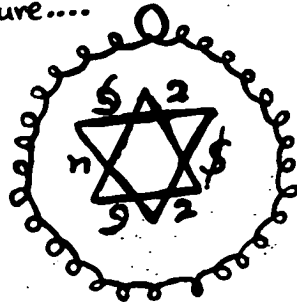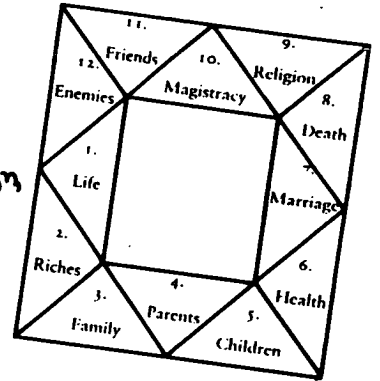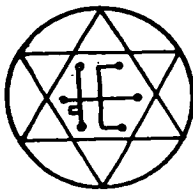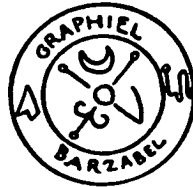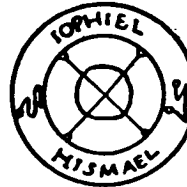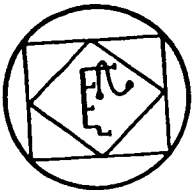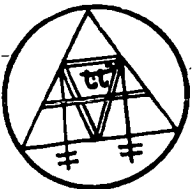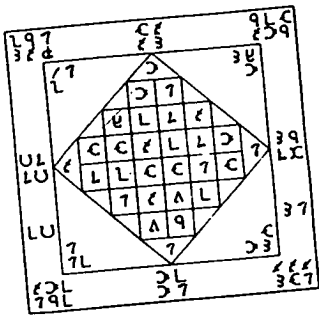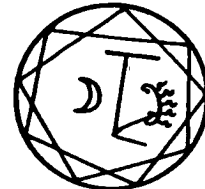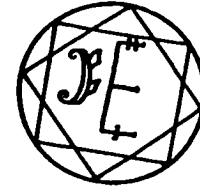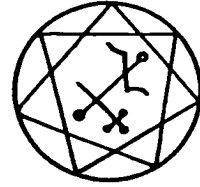
1/ B A L A N C I N G P O L Y G O N S

YOUR NAME:

YOUR SCHOOL:

In this activity you are going to
find the point on a plane figure
about which it will balance.
(see figure beside)
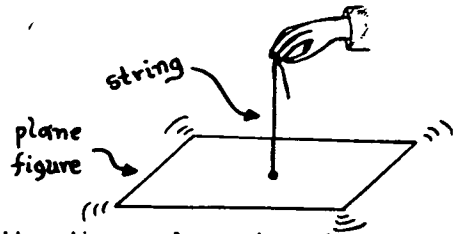
Start with the <u>square</u>, below. Draw the diagonals and mark
the point where they meet. This point is called the
<u>centroid</u> of the square.
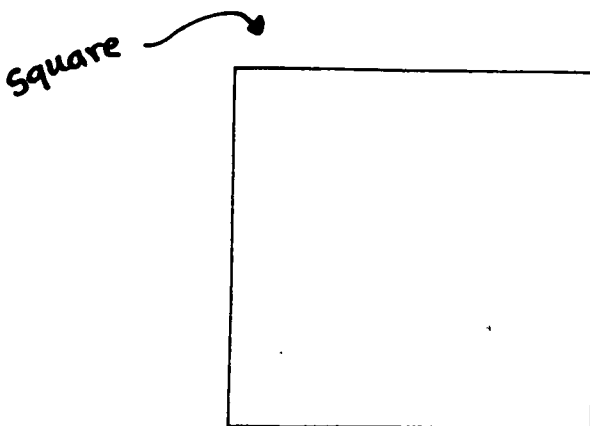
Place the acetate square carefully on top of the drawn one,
so that it matches exactly. Suspend the square by passing a
thread through the hole on the acetate that corresponds to
the centroid. Now suspend the square using the other
two holes in turn. Describe with a sketch what happened.

THREAD THROUGH CENTROID   **THREAD THROUGH OTHER HOLES**

In the same way you can balance a hexagon. Draw the
diagonals to connect opposite vertices only, to find the
centroid of the following hexagon. Check that it balances
with the acetate hexagon.
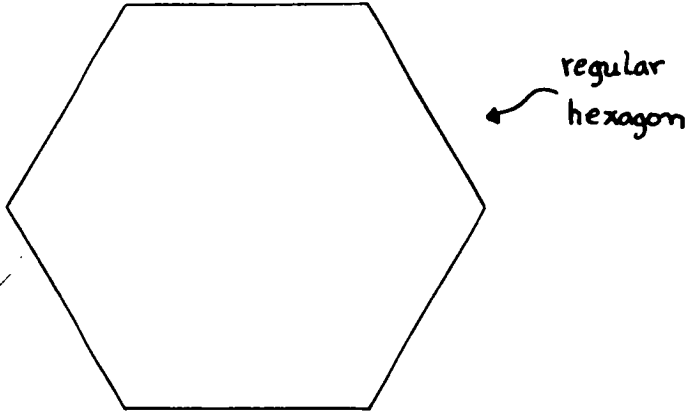
regular
hexagon

The situation becomes a bit more tricky when it comes in balancing a triangle or a regular pentagon.

For the first triangle below draw carefully the lines that connect each vertex with the mid-point of the opposite side. These lines are called <u>medians</u>. If you have drawn them carefully you will have found that the medians have met in a single point. This point is called the <u>centroid</u> of the triangle. Now check that it balances with the acetate triangle. Try it again with the second triangle.

2nd triangle

1st triangle
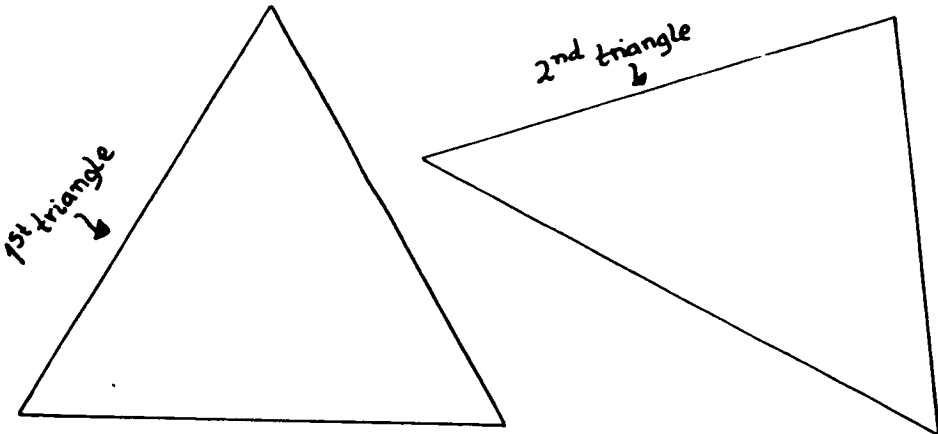
Now find in the same way the centroid of the following regular pentagon, and then try to balance the acetate one.

regular pentagon
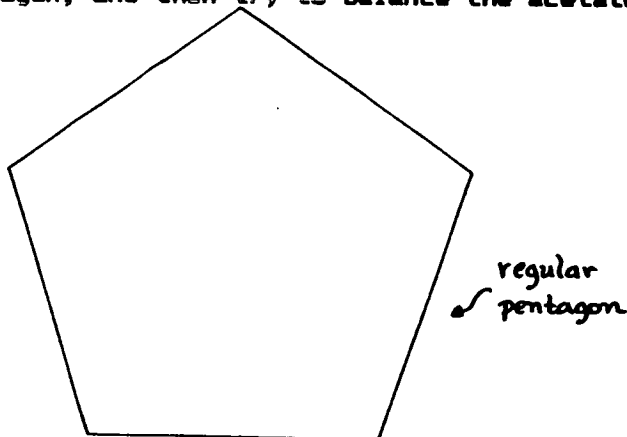
Another name for the centroid is <u>"centre of gravity"</u>, since we use this point to balance shapes.

T.A.T.

ANOTHER WAY OF BALANCING POLYGONS

There is a more general way of finding the
centre of gravity. Start with a pentagon A,
that has two holes marked close to its
vertices and four marked inside.

Hang the pentagon from one of the holes
close to its vertices, using the thread with
the paper clip tied at its one end (to keep
it straight). Mark the hole(s) inside the
pentagon which the thread passed across.
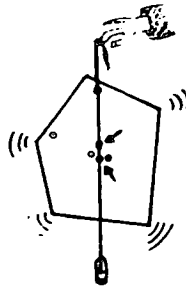
Hung the pentagon from a different hole
close to the pentagon's sides, and mark the
hole(s) inside the pentagon from which the
thread passed across.
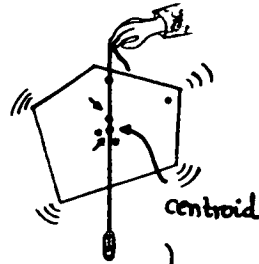
There is one hole from which the thread
passed across both times. This is the <u>centroid</u>.
Check it by balancing as before.

centroid
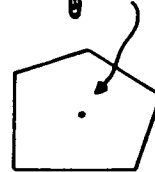
This process will work for other shapes as well. Try it with
the shapes B and C (a hexagon and an heptagon).

Did you manage to balance the hexagon with the first turn?

Did you manage to balance the heptagon with the first turn?

>>>   Discuss with your friends why this "hanging" process <<<
>>>   works. Write your ideas below.   <<<

2/ TRIANGLE'S MEETING POINTS

YOUR NAME:

YOUR SCHOOL:

Every triangle has a special group of "meeting points". In the "balancing polygons" activity we found one such "meeting point", the <u>centroid</u>. Here are some more:

A/ For the following triangle use a set square to draw through each vertex the line that is perpendicular to the opposite side. These lines are called the <u>altitudes</u> of the triangle. The point where the altitude meets the opposite side is called the <u>foot of the altitude</u>.

foot of the altitude
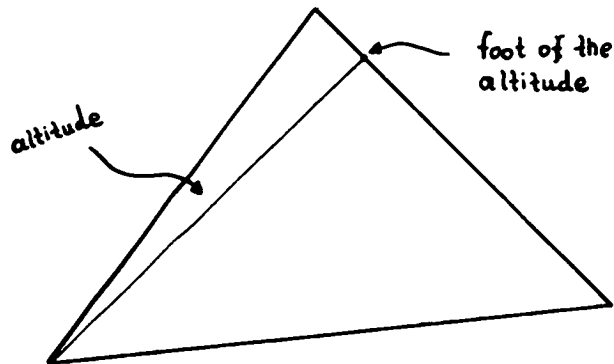
altitude

If your drawing is accurate, the three altitudes will meet in one point. This point is called the <u>orthocentre</u> of the triangle.

Try it again with another triangle:
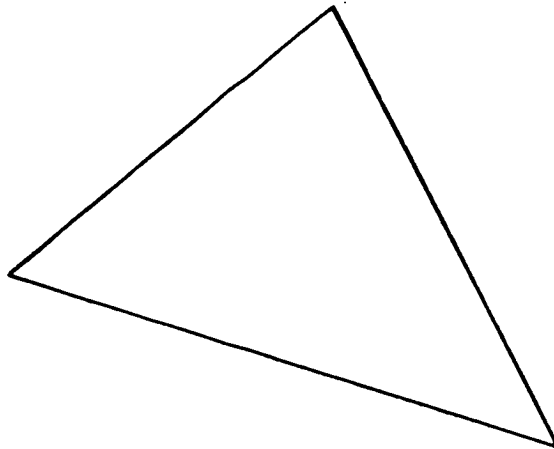
>>>   Discuss where is the orthocentre for a   <<<
>>>   right-angled triangle.   <<<
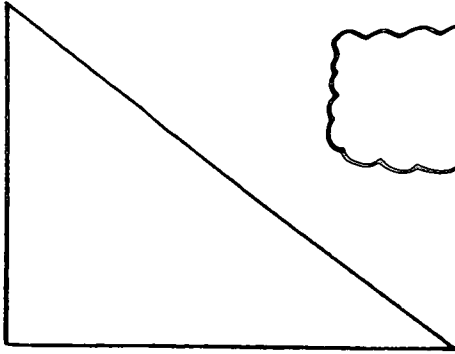
>>>   Discuss where is the orthocentre for an   <<<
>>>   equilateral triangle.   <<<
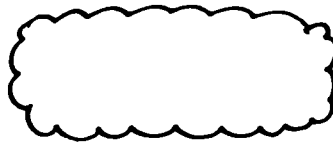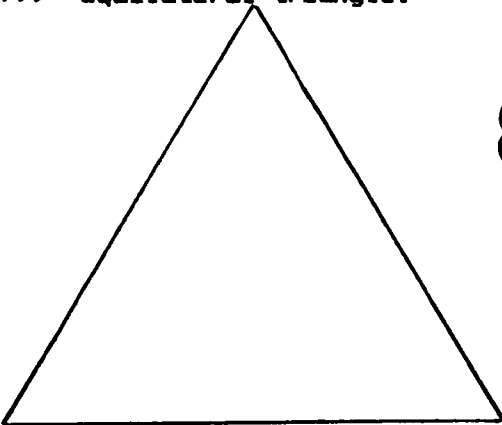
>>>   Discuss where is the orthocentre when the <<<
>>>   triangle has an obtuse angle.   <<<

I.A.T.
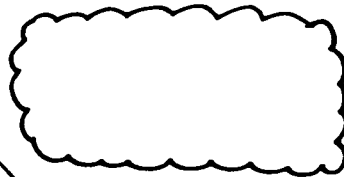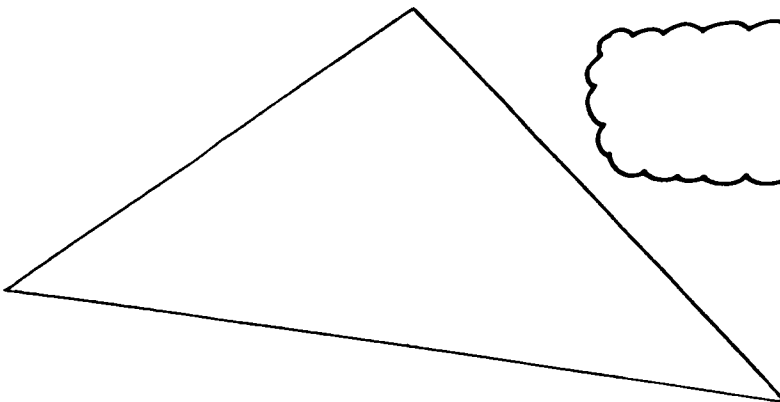
B/ For the following triangle draw the bisector of each angle.
That is the line that divides the angle into two smaller,
equal angles.
Use your protractor and a ruler to draw the angle bisectors.
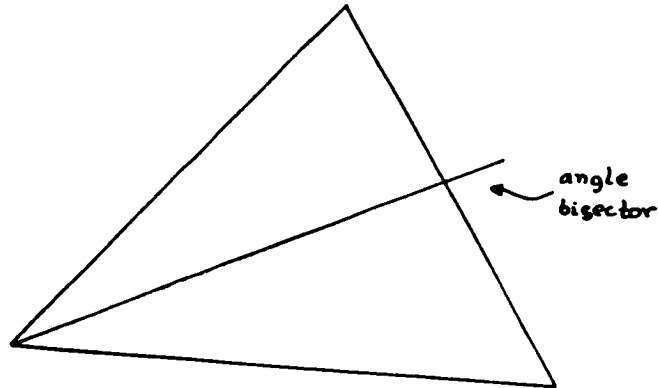
angle bisector

If your drawing is accurate, the three angle bisectors will
meet in one point called the incentre of the triangle. It
has a special property. With the incentre as centre we can
draw a circle that touches each side of the triangle at one
point. We call that circle the inscribed circle of the
triangle. Try it with the previous triangle.

For the triangle below find the incentre and then draw the
inscribed circle.
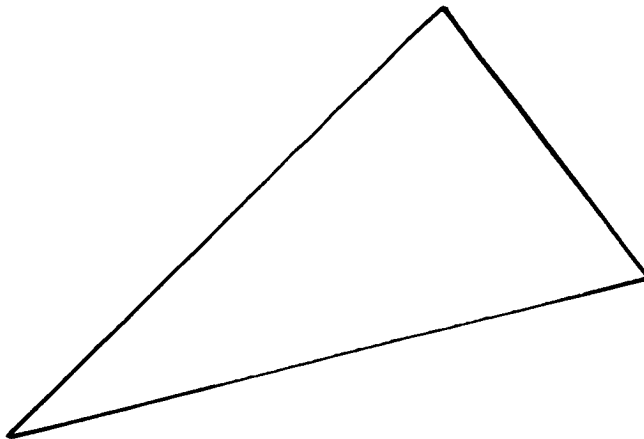
C/ For the following triangle use a set square to draw the
perpendicular bisector to each side. That is the
perpendicular that starts from the mid-point of the side.

perpendicular
bisector to
the side

mid-point
of the
side
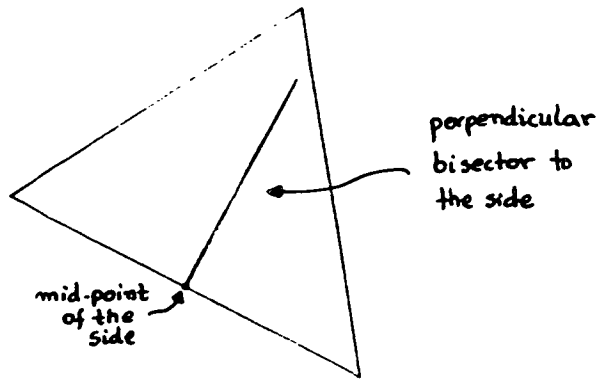
If your drawing is accurate, all three perpendicular
bisectors will meet in one point called the _circumcentre_ of
the triangle. With this point as centre we can draw a circle
that passes through the vertices of the triangle. We call
that circle the _circumscribed circle_. Try it with the
previous triangle.

For the triangle below find the circumcentre and then draw
the circumscribed circle.
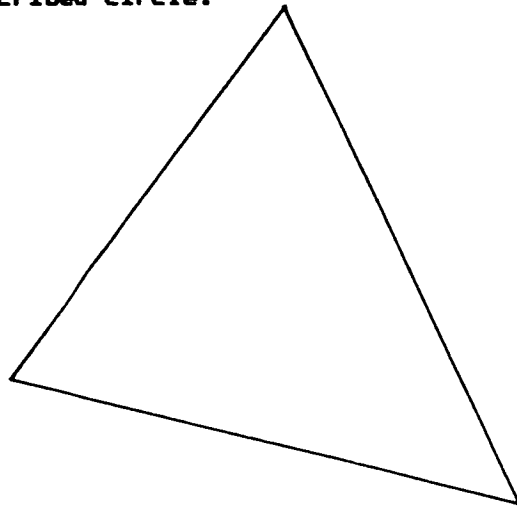
T.A.T.

>>> Discuss where is the centroid, the orthocentre,   <<<
>>> the incentre, and the circumcentre of an   <<<
>>> equilateral triangle.   <<<
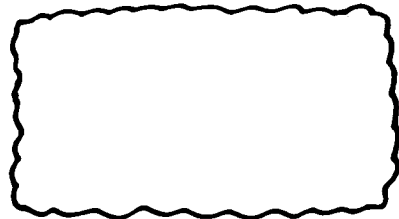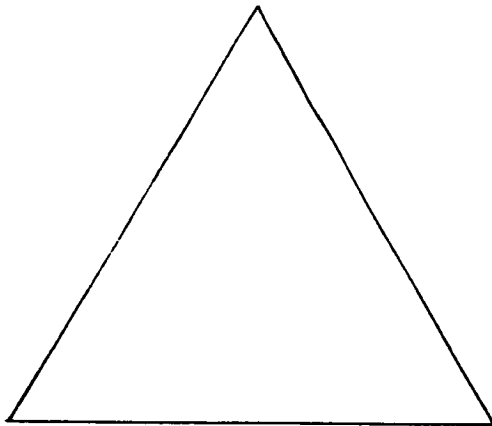
>>> Discuss where is the centroid, the orthocentre,   <<<
>>> the incentre, and the circumcentre of an   <<<
>>> isosceles triangle.   <<<
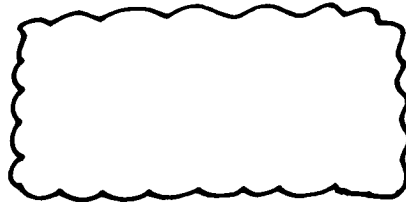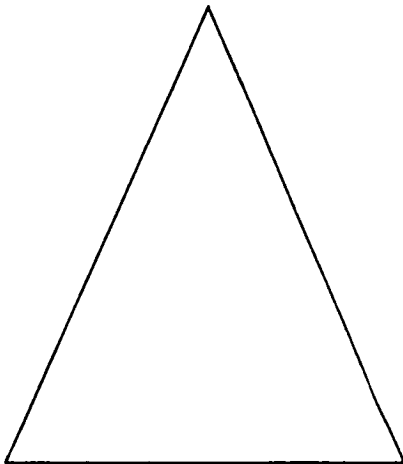
/3/ T R I A N G L E ' S   M E E T I N G   P O I N T S   YOUR NAME:

YOUR SCHOOL:

For the following triangle find the orthocentre, the centroid
and the circumcentre. Mark them with A, B, and C.
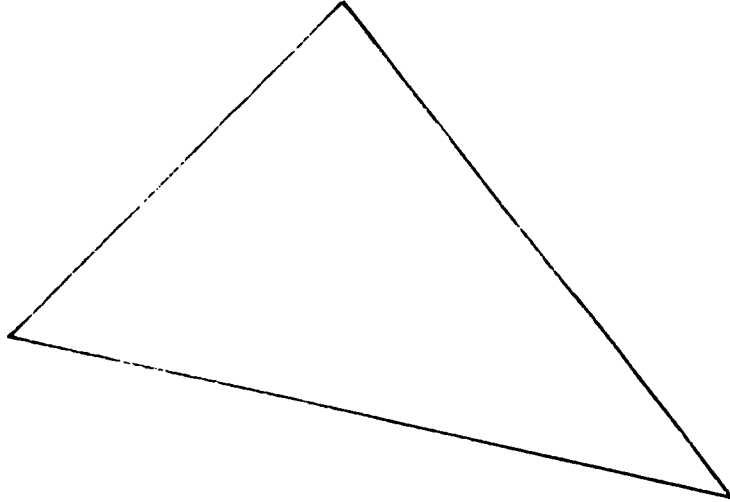
If your drawing is accurate, all three points A, B, and C
will lie on a straight line.

This line is called the Euler's line, because Euler, a great
mathematician of the 18th century, was the first to prove
that these three points lie on a straight line.

Now try again with the triangle at the back of the page.
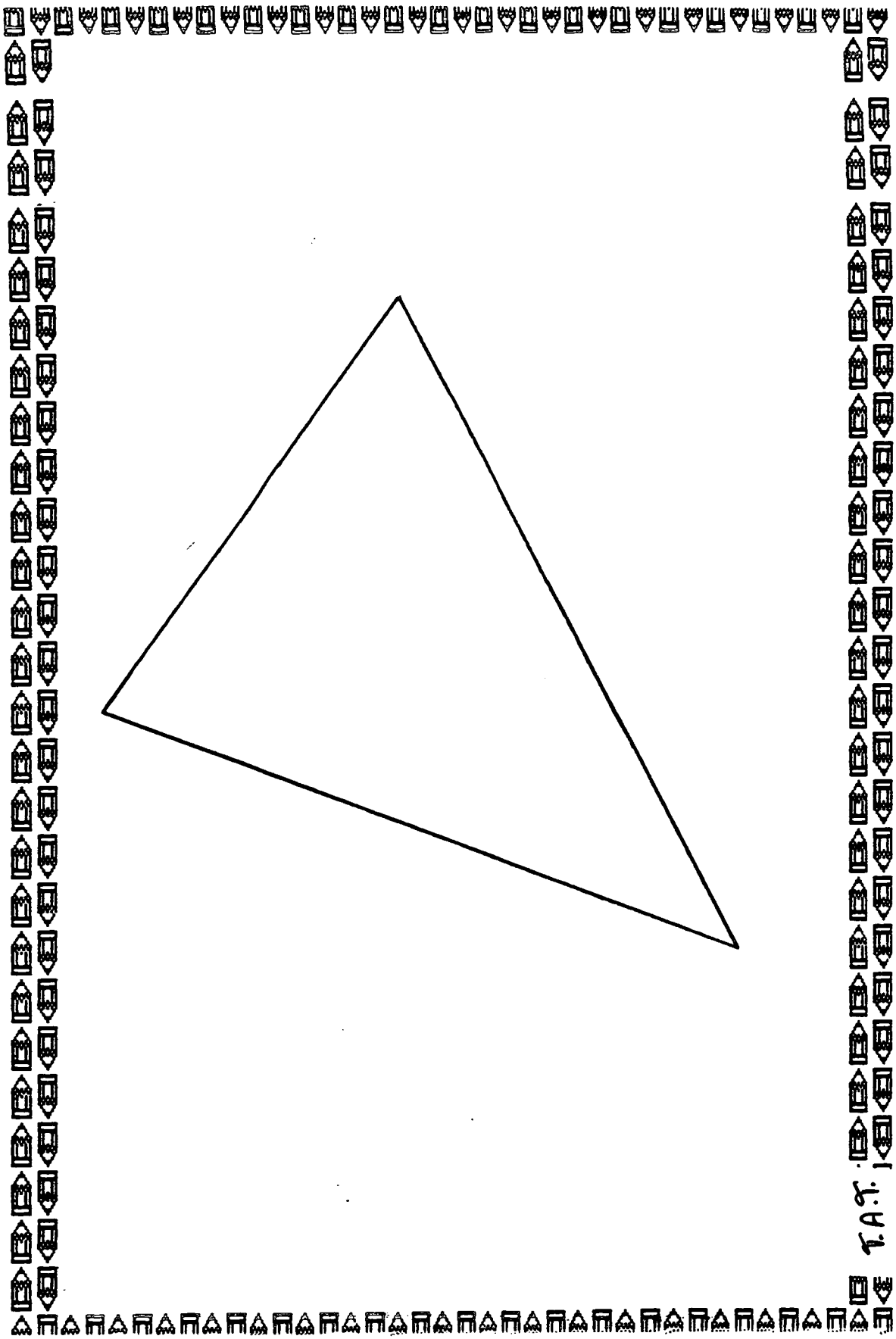
4/ P A P P U S ' s L I N E

YOUR NAME:

YOUR SCHOOL:

Pappus was a great Greek geometer who lived 23 centuries ago. A famous piece of his work is the Pappus' line.

Begin with two parallel lines, drawn with the help of a set square and a ruler. Mark three points A, B, C; D, E, F on each line, and draw the lines AE, AF, BD, BF, CD, and CE.

The lines: AE and BD meet at G

AF and CD meet at H

BF and CE meet at I

For accurate drawing, points G, H, and I lie on a straight line, the Pappus' line. (check with your ruler)
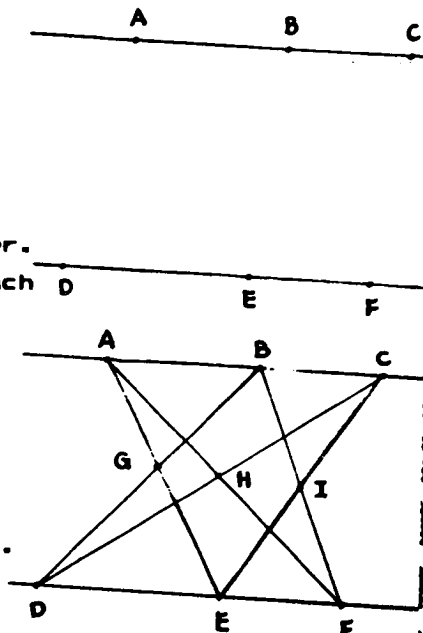
NOW IT IS YOUR TURN: draw two parallel lines using a ruler and a set square, and follow the previous steps labelling the points A, B, C, D, E, F, G, H, and I on your drawing.

ARE POINTS G, H, AND I ALL ON THE SAME LINE? CHECK WITH A RULER.

5/   P A S C A L ' S   L I N E

YOUR NAME:

YOUR SCHOOL:

Draw any circle, and mark 6 points'
on it A, B, C, D, E, and F.

Draw the following lines:
AE, EB, BD, DF, FC, and CA.

The lines: AE and FD meet at G

FC and BE meet at H

BD and AC meet at I

For accurate drawing the points G,
H, and I all lie on the same line
(check with your ruler).

That line is the Pascal's line, one
of the most famous in mathematics.

fig.1

fig.2

NOW IT IS YOUR TURN: draw a circle and follow the previous
steps labelling the points A, B, C, D, E, F, G, H, and I on
your drawing.
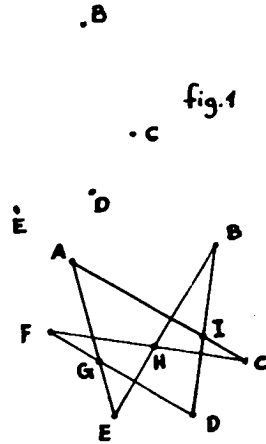
ARE POINTS G, H, AND I ALL ON THE SAME LINE?
CHECK WITH YOUR RULER.

6/   QUADRILATERAL MAGIC

YOUR NAME:

YOUR SCHOOL:

Start with any quadrilateral ABCD.
Mark the mid points of its sides
and I connect them with lines as in
the figure.

Mark the mid points of the sides
of the new quadrilateral and
connect them as in the figure.

Keep on repeating the same work.
Notice that after some steps the
quadrilateral looks more and more
like a square.

Actually after many steps you will
end up with a square!!

NOW IT IS YOUR TURN: start from a quadrilateral ABCD, and
by repeating the previous process try to reach to a square.
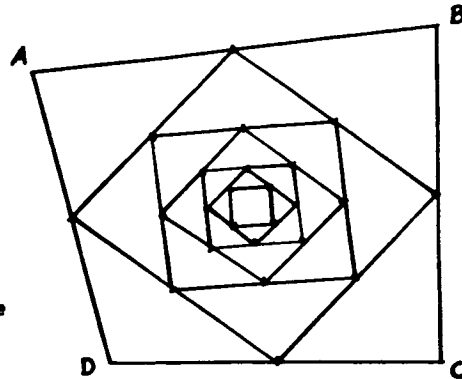
7/ BEAM TRIANGLES

YOUR NAME:

YOUR SCHOOL:

Draw three lines starting from the same point O, that look like a beam of light.
Draw then a triangle ABC with its vertices lying on each of the beam lines.
Then for any triangle DEF, having its vertices lying on each of the beam lines, find the points:

I: the point where sides AB and DE meet if I extend them.
J: the point where sides AC and DF meet if I extend them.
K: the point where sides BC and EF meet if I extend them.

For accurate drawing points I, J, and K lie on the same line, as Desargués showed in the 17th century.
(check with your ruler)

NOW IT IS YOUR TURN: follow the previous process for the beam lines drawn below, labelling the points A, B, C, D, E, F, I, J, and K on your drawing.
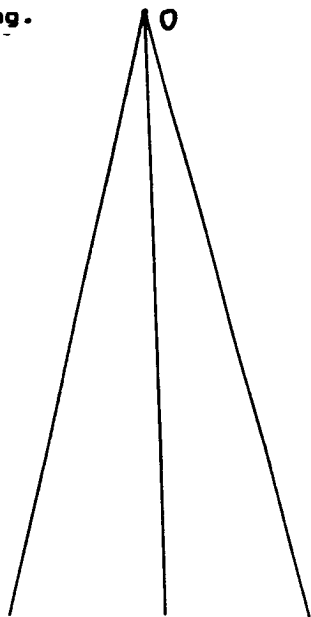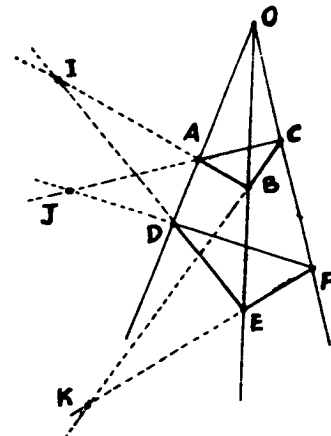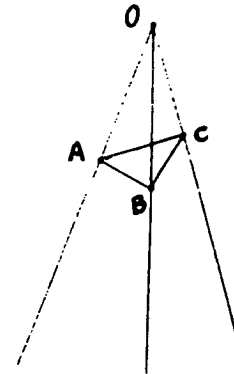
8/ D R A W I N G   A   H E A R T

YOUR NAME:

YOUR SCHOOL:

INSTRUCTIONS:

1. Start with a circle of radius about 3cm, and take a point O on it.

2. Take another point A on the circle and with centre A and radius AO draw a circle.

3. Take another point B on your initial circle (the shaded one), and draw a circle with centre B and radius BO.

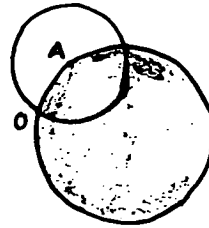4. Repeat for many points all around your initial circle.

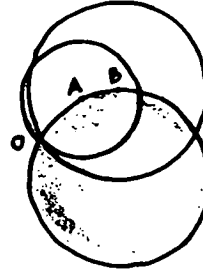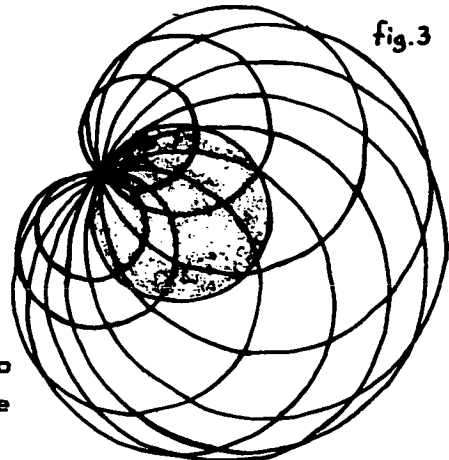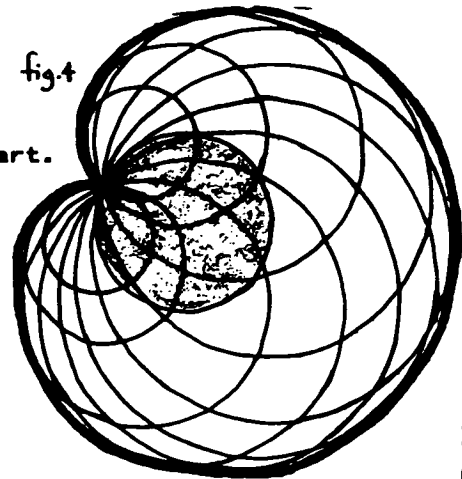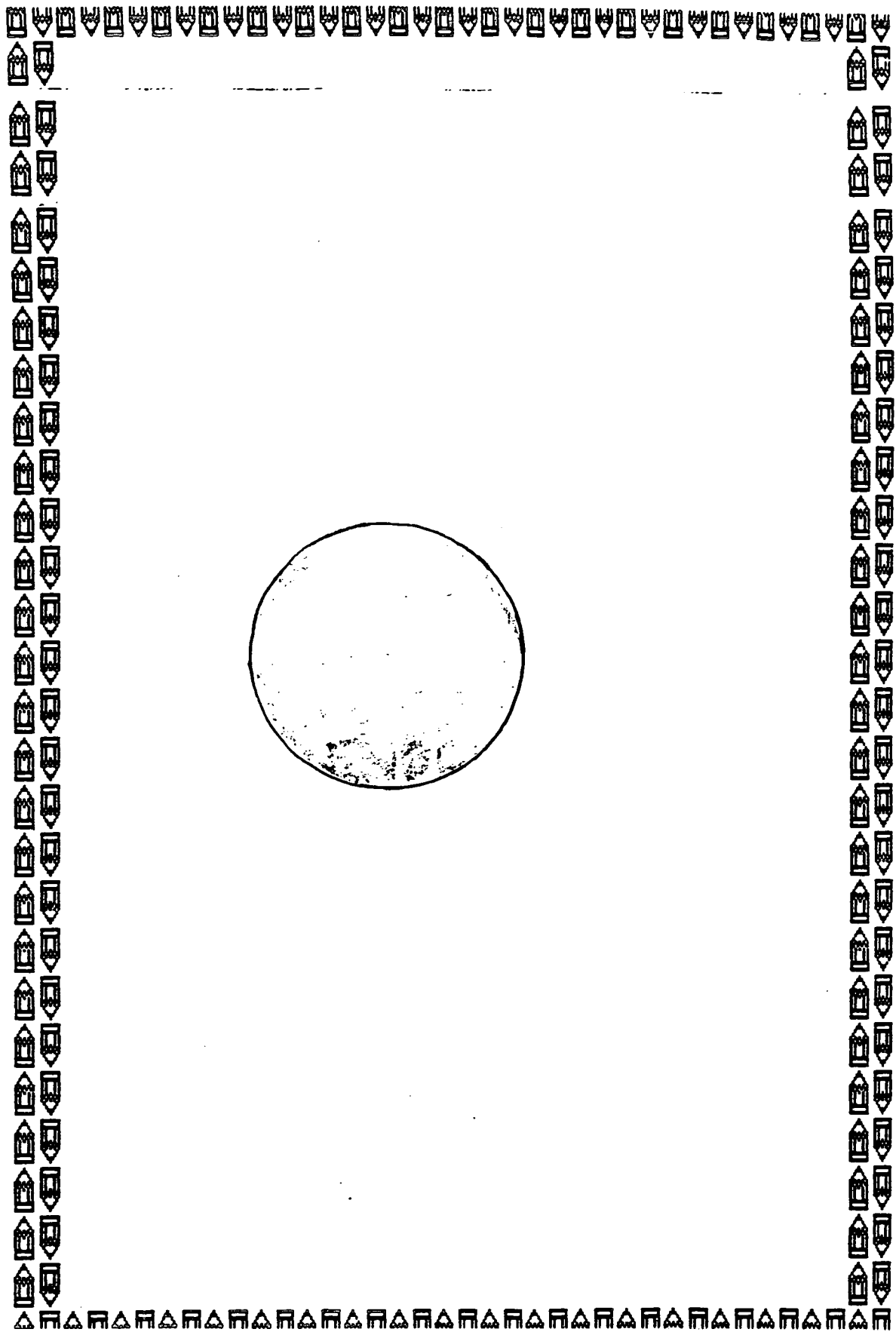By drawing carefully all the circles you will end up with a drawing like in figure 3.

To draw a heart then, all you have to is to enclose the drawing with a line like in figure 4.

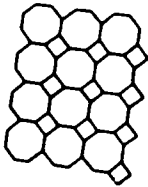Use the back of the page to draw a heart. The initial circle is drawn for you.

fig.1

fig.2

fig.3

fig.4

9/ MAKING A MOSAIC

YOUR NAME:

YOUR SCHOOL:
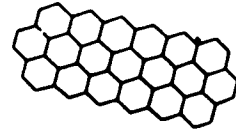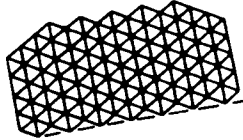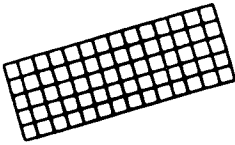
Using regular polygons only
(equilateral triangle, square,
pentagon, hexagon,...) we can
fill a plane creating various
patterns.

Some of these patterns shown
on this page have been used for
making mosaics and pavements.

On a separate sheet create your
own pattern and colour it.
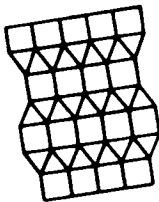
In the end you can make a poster
with all the different patterns
that your group created.
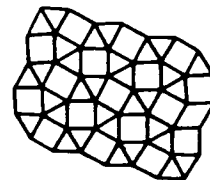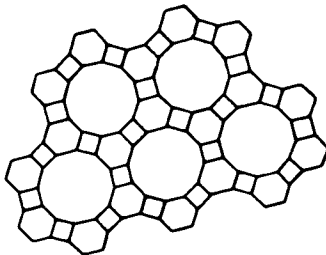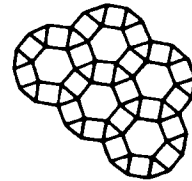
Appendix D

Students' Questionnaire

YOUR NAME:

AGE:

SCHOOL & GRADE:

ON WHICH ACTIVITY DID YOU WORK:

>>> Describe what do you think you learned from the activity you worked on.

>>> Were there any points that seemed easy or difficult, or that had mistakes?

>>> How much did you enjoy working on these activities?

NOT AT ALL:

A BIT:

SO SO:

A LOT:

VERY MUCH:

Can you explain your answer?

>>> Did you like working in groups? Please explain.

>>> Can you suggest other areas in mathematics that could be taught or assisted by some practical activity?

Bibliography

AAP (1983). *Achievement of Scottish School Children in Mathematics: Performance of Pupils at Primary Seven/Secondary Two.* SED.

AAP (1989). *Mathematics.* HMSO. Edinburgh. SED.

Abravanel, E. (1981). Integrating the Information from Eyes and Hands: A Developmental Approach. in R.D. Walk and H.L. Pick, Jr. (eds). *Intersensory Perception and Sensory Integration.* Plenum Press, New York: 71-108.

Adams, J.A. (1976). Issues of a Closed-Loop Theory of Motor Learning. in G.E. Stelmach (ed) *Motor Control: Issues and Trends.* Academic Press, New York: 87-107.

Anapolitanos, D.A. (1985). *Eisagogi stin Philosophia ton Mathematikon.* Nefeli, Athens, Greece.

Anderson, J.R. (1990). *Cognitive Psychology and Its Implications.* 3rd Edition. W.H. Freeman & Co, N.Y.

Apostle, H.G. (1952). *Aristotle's Philosophy of Mathematics.* The University of Chicago Press, Chicago.

Atkin, M. (1977). Evaluation for content improvement. in D. Hamilton, D. Jenkins, C. King, B. McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 75-77.

Ausubel, D.P. (1968). *Educational Psychology: A Cognitive View.* Holt, Rinehart and Winston, Inc., U.S.A.

Ausubel, D.P. (1985). Learning as Constructing Meaning. in N.J. Entwistle (ed). *New Directions in Educational Psychology.* Vol.I. Falmer Press.

Beck & McKeown (1986). Expository Text for Young Readers: The Issue of Coherence. in L.B. Resnick (ed). *Knowing, Learning and Instruction: Essays in Honor of Robert Glaser.* Lawrence Erlbaum Assoc., Publ., U.S.A.: 47-68.

Ben-Chaim, D., Fresko, B. & Carneli, M. (1990). Comparison of the teacher and pupil perceptions of the learning environment in mathematics classes. *Educational Studies in Mathematics*, 21(5): 415-429.

Bereiter, C. and Scardamalia, M. (1989). Intentional Learning As A Goal Of Instruction. in L.B. Resnick (ed). *Knowing, Learning and Instruction: Essays in Honor of Robert Glaser.* Lawrence Erlbaum Assoc., Publ., U.S.A.: 361-392.

Bishop, A.J. (1988). Mathematics Education in Its Cultural Context. *Educational Studies in Mathematics*, 19: 179-191.

Bishop, A.J. (1983). Space and Geometry. in R. Lesh and M. Landau (eds). *Acquisition of Mathematics Concepts and Processes.* Academic Press, Inc., USA: 175-203.

Bonner, S.F. (1977). *Education in Ancient Rome.* Methuen & Co., London.

Branca, N.A. (1980). Communication of Mathematical Structure and Its Relationships to Achievement. *Journal for Research in Mathematics Education*, 11(1): 37-49.

Breazeale, D. (1979). *Philosophy and Truth: Selections from Nietzsche's Notebook of the Early 1870's.* Humanities Press Inc., N.J.

Brown, A.L. & Pallincsar, A.S. (1989). Guided, Cooperative Learning and Individual Knowledge Acquisition. in L.B. Resnick (ed). *Knowing, Learning and Instruction: Essays in Honor of Robert Glaser.* Lawrence Erlbaum Assoc., Publ., U.S.A.: 393-451.

Bruner, J.S. (1965). *The Process of Education.* Harvard University Press, Cambridge - Massachusetts.

Bruner, J.S. (1967). *Toward a Theory of Instruction.* Harvard University Press, U.S.A.

Bryant, P. (1974). *Perception and Understanding in Young Children: An Experimental Approach.* Methuen & Co Ltd., U.K.

Burger, W.F. & Shaugnessy, J.M. (1986). Characterizing the van Hieles Levels of development in Geometry. *Journal for Research in Mathematics Education*, 17(1): 31-45.

Burnet, J. (1905). *Aristotle on Education.* Cambridge: At the University Press, G.B.

Burton, D.M. (1991). *The History of Mathematics: An Introduction.* 2nd ed. Wm. C. Brown Publ., U.S.A.

Butron, R.R., Brown, J.S. & Fischer, G. (1984). Skiing as a model of Instruction. in B. Roggoff and J. Lave (eds). *Everyday Gognition: Its Development in Social Context.* Harvard University Press: 139-150.

Byers, V. & Herscovics, N. (1977). Understanding School Mathematics. *Mathematics Teaching,* 81: 24-27.

Capon, N. & Kuhn, K. (1979). Logical Reasoning in the Supermarket: Adult Females' use of a Proportional Strategy in an Everyday Context. *Developmental Psychology,* 15(4): 450-452; (6): 544-573.

Carraher, T.N., Carraher, D.W. & Schliemann, A.D. (1984). Can Mathematics Teachers Teach Proportions?. in Unesco/Division of Science Technology and Environmental Studies. *Mathematics for All: Problems of Cultural and Unequal Distribution of Mathematical Education and Future Perspectives on Mathematics Teaching for the Majority,* Document Series No.20: 90-91.

Centre for New Schools (1977). Ethnographic techniques in educational research. in D. Hamilton, D. Jenkins, C. King, B. McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 193-200.

Chamberlin, J.G. (9174). Phenomenological Methodology and Understanding Education. in D. Denton (ed). *Existensialism & Phenomenology in Education: Collected Essays.* Teachers College Press, Columbia, N.Y.: 119-138.

Clarkson, R. (1989). Ratio: Enlargement. in D.C. Johnson (ed). *Children's Mathematical Frameworks 8-13: A Study of Classroom Teaching.* NFER-Nelson: 191-217.

Cresswell, M. & Cubb, J. (1987). *The Second International Mathematics Study in England and Wales.* NFER-Nelson, UK.

Danassis-Afentakis, A. (1985a). *Themateki tis Pedagogikis Epistemis.* 3rd ed. Hermes, Athens, Greece.

Danassis-Afentakis, A. (1985b). *H Ekseliksi tis Pedagogikis kai Didactikis Skepsis.* Hermes, Athens, Greece.

Del Grande, J.J. (1987). Spatial Perception and Geometry. in NCTM 1987 Yearbook. *Learning and Teaching Geometry, K-12.* U.S.A.: 126-135.

Del Grande, J.J. (1990). Spatial Sense. *Arithmetic Teacher*, 37(6): 14-20.

Denmark, T. & Kepner, H.S. Jr. (1980). Basic Skills in Mathematics: A Survey. *Journal for Research in Mathematics Education*, 11(2): 104-123.

Desforges, C. (1985). Matching Tasks to Children's Attainments. in N. Bennet and C Desforges (eds). *Recent Advances in Classroom Research*. British Journal of Educational Research, Monograph Series No. 2, Scottish Academic Press, Edinburgh: 92-104.

Dewey, J. (1963). *Experience and Education*. Collier-Macmillan Co., U.S.A.

Dickson, L., Brown, M. & Gibson, O. (1991): *Children Learning Mathematics*. Cassel Educational Ltd.

Dienes, Z.P. (1960). *Building Up Mathematics*. Hutchinson Educational, London.

Dienes, Z.P. (1963). *An Experimental Study of Mathematics - Learning*. Hutchinson of London, London.

Dienes, Z.P. (1964). *The Power of Mathematics*. Hutchinson Educational, London.

Dienes, Z.P. (1973). *The Six Stages in the Process of Learning Mathematics*. NFER Publ. Co., Ltd. G.B.

Eisner, E. (1977a). Establishing a direction. in D. Hamilton, D. Jenkins, C. King, B. McDonald, & M. Parlett (eds), *Beyond the Numbers Game: A Reader in Educational Evaluation*. MacMillan Educational Ltd., London: 90.

Eisner, E. (1977b). Thick description, in D. Hamilton, D. Jenkins, C. King, B. McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation*. MacMillan Educational Ltd., London: 97-98.

Eisner, E. (1977c). Instructional and Expressive Objectives. in D. Hamilton, D. Jenkins, C. King, B. McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation*. MacMillan Educational Ltd., London: 91-94.

EMU/SEAC (1991). *APU Mathematics Monitoring. (Phase 2)*. NFER-Nelson, UK.

Entwistle, N.J. (1988). *Styles of Learning and Teaching*. David Fulton Publ., London.

Eysenk, M.W. & Keane, M.T. (1990). *Cognitive Psychology: A Student's Handbook.* Lawrence Erlbaum Assoc., Publ., U.K.

Fielker, D. (1988). Metaphors and Models. *Mathematics Teaching*, 124: 4-6.

Fowler, C.A. & Turvey, M.T. (1978). Skill Acquisition: An Event Approach With Special Reference to Searching for the Optimum of a Function of Several Variables. in G.E. Stelmach (ed) *Information Processing in Motor Control and Learning.* Academic Press, New York : 2-40.

Flavell, J.H. (1963). *The Developmental Psychology of Jean Piaget.* D. Van Nostrand Co., Inc., Princeton, N.J.

Freeman, R.B. (1985). *A Cultural History of Western Education.* 2nd ed. Mc-Graw Hill Co., Inc., N.Y.

Freudenthal, H. (1973). *Mathematics as an Educational Task.* D. Reidel Publ. Co., Dordrecht, Holland.

Garden, R.A. & Robitaille, D.F. (1989). Test Development, Scoring, and Interpretation. in D.F. Robitaille & R.A. Garden (eds). *The IEA Study of Mathematics I: Contexts and Outcomes of School Mathematics.* Pergamon Press for the IEA, UK: 84-101.

Gergen, K.J., Gloger-Tippett, G. & Berkowitz, P. (1990). The Cultural Construction of the Developing Child. in G.R. Semin and K.J. Gergen (eds). *Everyday Understanding: Social and Scientific Implications.* SAGE Publ., London: 108-124.

Gheverghese, J.G. (1992). *The Crest of the Peacock: Non-European Roots of Mathematics.* Penguin Books, G.B.

Giles, G. (1981). *Factors Affecting the Learning of Mathematics.* DIME Projects. Dep. of Education, University of Stirling, Scotland.

Goodnow, J.J. (1990). The Socialisation of Cognition: What's Involved. in J.W. Stigler, R.A. Shweder and G. Herdt (eds). *Cultural Psychology: Essays on Comparative Human Development.* Cambridge University Press. N.Y.: 259-286.

Greenfield, P.M. (1984). A Theory of the Teacher in the Learning Activities of Everyday Life, in B. Roggoff and J. Lave (eds). *Everyday Gognition: Its Development in Social Context.* Harvard University Press: 117-138.

Gregory, R.L. (1974). *Concepts and Mechanisms of Perception.* G. Duckworth & Co. Ltd., London: 65-129.

Hart, K.M. (1978). *The Understanding of Ratio in the Secondary School.* Mathematics in Schools, 7(1) : 4-6.

Hart, K.M. (1981a). Measurement. in K.M. Hart (ed). *Children's Understanding of Mathematics: 11-16.* John Murray Publ., London : 9-22.

Hart, K.M. (1981b). Ratio and Proportion. in K.M. Hart (ed). *Children's Understanding of Mathematics: 11-16.* John Murray Publ., London: 88-101.

Hart, K.M. (1981c). Fractions. in K.M. Hart (ed). *Children's Understanding of Mathematics: 11-16.* John Murray Publ., London : 66-81.

Hart, K.M. (1984). *Ratio: Children's Strategies and Errors. A Report of the Strategies and Errors in Secondary Mathematics Project.* (SESM), NFER-Nelson, UK

Hart, K.M. (1985). Mathematics Education Research. in N. Bennet & C. Desforges. *Recent Advances in Classroom Research.* Scottish Academic Press, Edinburgh.

Hart, K.M. (1987). *Children's Mathematical Frameworks: What Are Equivalent Fractions? Part Two.* Mathematics in Schools, 16(4) : 5-7.

Hart, K.M. (1988). Ratio and Proportion. in M. Behr & J. Hiebert (eds). *Number Concepts and Operations in the Middle Grades.* Vol.2. Lawrence Erlbaum Associates, NCTM: 198-219.

Hart, K.M. (1989). Fractions: Equivalence and Addition. in D.C. Johnson (ed). *Children's Mathematical Frameworks 8-13: A Study of Classroom Teaching.* NFER-Nelson: 46-75.

Hershkowitz, R. (1989). Visualization in Geometry - Two Sides of the Coin. *Focus on Learning Problems in Mathematics,* 11(1): 61-76.

Hoffer, A. (1983). Van Hieles-Based Research, in R. Lesh and M. Landau (eds). *Acquisition of Mathematics Concepts and Processes.* Academic Press, Inc. U.S.A.: 205-217.

House, E. (1977). Context and justification. in D. Hamilton, D. Jenkins, C. King, B. McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 219-221.

Hundeide, K. (1985). The Tacit Background of Children's Judgements. in J.W. Wertsch (ed). *Culture, Communication and Cognition: Vygotskian Perspectives.* Cambridge University Press, N.Y. : 306-322.

Hunt, H.E. (1912). *Auto-Education and Montessori Method.* C.W. Bardeen, Publ., Syracuse, N.Y.

Husen, T. (ed) (1967). *International Study of Achievement in Mathematics. A Comparison of Twelve Countries. Part I & II.* Almqvist & Wiksell, Stockholm.

Inhelder, B. & Piaget, J. (1958). *The Growth of Logical Thinking From Childhood to Adolescence.* Routledge & Kegan Paul, London.

James, N. (1985). *Learning Mathematics.* Open University Press., Education: A Second Level Course.

Jaworski, B. (1991). Can All Pupils be Mathematicians?. *Proceedings of Mathematics Teaching Conference.* The University of Edinburgh, Edinburgh: 8-12.

Jones, B. (1981) : The Developmental Significance of Cross-Modal Matching. in R.D. Walk and H.L. Pick, Jr. (eds). *Intersensory Perception and Sensory Integration.* Plenum Pres, New York: 109-136.

Johnson, D.C (ed) (1989). *Children's Mathematical Frameworks 8-13: A Study of Classroom Teaching.* NFER-Nelson: 218-224.

Karplus, R., Pulos, S. & Stage, E.K. (1983a). Proportional Reasoning of Early Adolescents. in R. Lesh & M. Landau (eds). *Acquisition of Mathematics Concepts and Processes.* Academic Press. Inc.

Karplus, R., Pulos, S. & Stage, E.K. (1983b). Early Adolescents' Proportional Reasoning or Rate Problems. *Educational Studies in Mathematics,* 14: 219-233.

Keele, S.W. & Summers, I.J. (1976). The Structure of Motor Programs. in G.E. Stelmach (ed) *Motor Control: Issues and Trends.* Academic Press, New York: 109-142.

Keogh, J. & Sugden, D. (1985). *Movement Skill Development.* MacMillan Publ. Co., USA.

Khale, J.B. (1987). Teachers and Students: Gender Differences in Science Classrooms. *Proceedings of the 4th GASAT Conference.*

Klein, R.M. (1976), Attention and Movement. in G.E. Stelmach (ed) *Motor Control: Issues and Trends.* Academic Press, New York : 143-173.

Kramer, R. (1976). *Maria Montessori - A Biography.* Basil Blackwell, Oxford.

Labinowicz, E. (1980). *The Piaget Primer: Thinking, Learning, Teaching.* Addison - Welsey Publ. Co., U.S.A.

Lakoff, G. & Johnson, M. (1980). *Metaphors We Live By.* The Univ. of Chicago Press, Chicago.

Lamon, S.J. (1990). Ratio and Proportion: Cognitive Foundations in Unitizing and Norming. *Paper presented at the 1990 Annual Meeting of the American Educational Research Association.* April 16-20, Boston, Massachusetts.

Lapointe, A., Mead, N. & Askew, J. (1992). *Learning Mathematics. The Second IAEP.* ETS, Report No.22-CAEP-01, USA.

Lappan, G. & Even, R. (1988). Similarity in the Middle Grades. *Arithmetic Teacher,* 35(9) : 32-35.

Laszlo, J.L. & Bairstow, P.J. (1985), *Perceptual-Motor Behaviour: Developmental Assessment and Theory.* Holt, Rinehart & Winston Ltd., GB.

Laurie, S.S. (1899). *John Amos Comenius.* 6th ed. C.J. Clay and Sons, London.

Leinhardt, G. (1986). Expertise in Mathematics Teaching. *Educational Leadership,* 43(2). 28-33.

Leinhardt, G. & Greeno J.G. (1986). The Cognitive Skill of Teaching. *Journal of Educational Psychology,* 78(2): 75-95.

Leinhardt, G. & Putnam, R.T. (1987). The Skill of Learning From Classroom Lessons. *American Educational Research Lournal,* 24(4): 557-587.

Lesh, R., Post, T. & Behr, M. (1988). Proportional Reasoning. in M. Behr & J. Hiebert (eds). *Number Concepts and Operations in the Middle Grades. Vol.2.* Lawrence Erlbaum Associates, NCTM: 93-118.

Liebeck, P. (1984). *How Children Learn Mathematics.* Pelican Books, G.B.

Lopez-Rial, F. (1989). Metaphors and Related Concepts in Mathematics: Part I. *Mathematics Teaching,* 127: 50-52.

Lopez-Rial, F. (1990). Metaphors and Related Concepts in Mathematics: Part II. *Mathematics Teaching,* 130: 34-36.

Lovell, K. & Butterworth, I.B. (1966). Abilities Underlying the Understandings of Proportionality. *Mathematics Teaching,* 37 : 5-9.

Lunzer, E.A. & Pumphrey, P.D. (1966). Understanding Proportionality. *Mathematics Teaching,* 34 : 7-12.

Marcuse, H. (1984). Remarks on a Redefinition of Culture. Translation in Greek, Z. Sarikas. *Techni kai Mazeki Coultoura*. Epsilon, Athens: 27-47.

McDonald, J.I.H. (1989). *Forum on Scottish Education*. Church of Scotland, Department of Education, Edinburgh.

McDonald, J.L. (1989). Cognitive Development and the Structuring of Geometric Content. *Journal of Research in Mathematics Education*, 20(1): 76-94.

Mason, J. (1992). Working on Awareness. *Proceedings of Mathematics Teaching Conference*. The University of Edinburgh, Edinburgh, : 10-14.

Matlin, M.W. (1988). *Sensation and Perception*. Revised Edition. Allyn and Bacon, Inc., U.S.A.

Michotte, A. (1991). Perception and Cognition. in G. Thines, A., Costall and G. Butterworth (eds). *Michotte's Experimental Phenomenology of Perception*. Lawrence Erlbaum Assoc., Publ., N.J., U.S.A.: 226-244.

Millar, S. (1981). Crossmodal and Intersensory Perception and the Blind. in R.D. Walk and H.L. Pick, Jr. (eds). *Intersensory Perception and Sensory Integration*. Plenum Pres, New York: 281-314.

Miller, P. (1989). *Theories of Developmental Psychology*. 2nd Edition. W.H. Freeman & Co, New York.

Montessori, M. (1912). *The Montessori Method*. William Heinemann, U.S.A.

Muller, D. (1978). Children's Concepts of Proportion: An Investigation Into The Claims of Bryant and Piaget. *British Journal of Educational Psychology*, V.48: 29-35.

National Centre for Education Statistics (1991). *Psychometric Report for the National Longitudinal Study: 88 Base Year Test Battery*. NCES 91-468, USA.

Nesher, P. & Sukenik, M. (1991). The Effect of Formal Representation on the Learning of Ratio Concepts. *Learning and Instruction*, 1 : 161-175.

Nietzsche, F. (1871). *Ei Genesis tis Tragodias*. translation in Greek. N.A. Kefalas, Govostis, Athens.

Noelting, G. (1980a). The Development of Proportional Reasoning and the Ratio Concept. Part I - Differentiation of Stages. *Educational Studies in Mathematics*, 11 : 217-253.

Noelting, G. (1980b). The Development of Proportional Reasoning and the Ratio Concept. Part II - Problem-Structure at Successive Stages; Problem-Solving Strategies and the Mechanisms of Adaptive Restructuring. *Educational Studies in Mathematics*, 11 : 331-363.

Ohlsson, S. (1988). Mathematical Meaning and Applicational Meaning in the Semantics of Fractions and Related Concepts. in M. Behr & J. Hiebert (eds). *Number Concepts and Operations in the Middle Grades. Vol.2.* Lawrence Erlbaum Associates, NCTM: 53-92.

Parlett, M. & Hamilton, D. (1972). *Evaluation as Illumination: a new Approach to the study of Innovatory Programs.* Center for Research in Educational Sciences, University of Edinburgh. Occasional Paper 9 [or in D. Hamilton, D. Jenkins, C. King, B.McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 6-22.]

Parzysz, B. (1988). 'Knowing' vs. 'Seeing'. Problems of the Plane Representation of Space Geometry Figures. *Educational Studies in Mathematics*, 11: 79-92.

Piaget, J. & Inhelder, B. (1956). *The Child's Conception of Space.* Routledge & Kegan Paul, London.

Pick, H.L.Jr. (1980). Tactual and Haptic Perception. in R.L. Welsh and B.B. Blasch (eds). *Foundations of Orientation and Mobility.* American Foundation for the Blind, Inc., New York.: 89-114.

Pirie, S. (1992). Watching Their Understanding Grow. *Proceedings of Mathematics Teaching Conference.* The University of Edinburgh, Edinburgh: 2-6.

Pramling, I. (1983). *The child's conception of learning.* Acta Universitatis Gothoburgensis, Kompendiet, Sweden.

Resnick, L.B. (1987). Learning in School and Out. *Educational Researcher*, 16(9): 13-20. Publ., U.S.A.

Robitaille, D.F. (1989). Students' Achievements: Population A. in D.F. Robitaille & R.A. Garden (eds). *The IEA Study of Mathematics I: Contexts and Outcomes of School Mathematics.* Pergamon Press for the IEA, UK: 102-125.

Robitaille, D.F. & Taylor, A.R. (1989). Changes in Patterns of Achievement Between the First and Second Mathematics Studies. in D.F. Robitaille & R.A. Garden (eds). *The IEA Study of Mathematics I: Contexts and Outcomes of School Mathematics.* Pergamon Press for the IEA, UK: 153-177.

Roth, I. (1986). Conceptual Categories. in I. Roth and P. Frisby. *Perception and Representation: A Cognitive Approach.* Open Univ. Press, U.K.: 18-77.

Schiff, W. (1980). *Perception: An Applied Approach.* Houghton Miffin Co., Boston, U.S.A.

Schmidt, R.A. (1976). The Schema as a Solution to Some Persistent Problems in Motor Learning Theory. in G.E. Stelmach (ed) *Motor Control: Issues and Trends.* Academic Press, New York: 41-65.

Schoenfeld, A.H. (1988). When Goog Teaching Leads to Bad Results: The Disasters of "Well Taught" Mathematics Courses. *Educational Psychologist,* 23(2): 145-166.

Schoenfeld, A.H. (1989). Explorations of Students' Mathematical Beliefs and Behaviour. *Journal for Research in Mathematical Education,* 20(4): 338-355.

Schwartz, J.L. (1988). Intensive Quantity and Referent Transforming Arithmetic Operations. M. Behr & J. Hiebert (eds). *Number Concepts and Operations in the Middle Grades. Vol.2.* Lawrence Erlbaum Associates, NCTM, 41-52.

Scriven, M. (1967). The methodology of evaluation. in R.W. Tyler, R.M. Gagne & M. Scriven (eds). *Perspectives of Curriculum Evaluation.* Rand McNally & Co, Chicago: 39-83.

Semple, McLean B./SOED, ETS. (1992). *Performance Assessment: An International Experiment.* Report No.22-CAEP-06, USA.

Shweder. R.A. (1990). Cultural psychology - what is it?. in J.W. Stigler, R.A. Shweder & G. Herdt (eds). *Cultural Psychology, Essays on Comparative human Development.* Cambridge University Press, N.Y.: 1-46.

Singer, R.N. (1980). Motor Behavior and the Role of Cognitive Processes and Learner Strategies. in G.E Stelmach & J. Requin (eds) *Tutorials in Motor Behavior.* North-Holland Publ. Co.: 591-603.

Skemp, R.R. (1976). Relational Understanding and Instrumental Understanding. *Mathematics Teaching,* 77: 20-26.

Skemp, R.R. (1979). Goals of Learning and Qualities of Understanding. *Mathematics Teaching,* 88: 44-49.

Skemp, R.R. (1986). *The Psychology of Learning Mathematics.* 2nd ed. Penguin Books Ltd., G.B.

SOED (1992). *Second IAEP. Mathematics and Science Around the World. A Scottish Perspective.* Research and Intelligence Unit.

Stake, R.E. (1977a). The countenance of educational evaluation. in D. Hamilton, D. Jenkins, C. King, B.McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 146-153.

Stake, R.E. (1977). Description versus analysis. in D. Hamilton, D. Jenkins, C. King, B.McDonald, & M. Parlett (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 161-162.

Stake, R.E. (1967). Toward a technology for the evaluation of educational programs. in R.W. Tyler, R.M. Gagne & M. Scriven (eds). *Beyond the Numbers Game: A Reader in Educational Evaluation.* MacMillan Educational Ltd., London: 107-108.

Stevenson, H.W. & Stigler, J.W. (1986). Mathematics Achievement of Chinese, Japanese and American Children. *Science*, 231: 693-699.

Stigler, J.W., Lee, S.Y. & Stevenson, H.W. (1987). Mathematics Classrooms in Japan, Taiwan and the United States. *Child Development*, 58: 1272-1285.

Stigler, J.W. & Baranes, R. (1988). Culture and Mathematics Learning. *Review of Research in Education*, 15: 253-306.

Stigler, J.W. & Perry, M. (1990). Mathematics Learning in Japanese, Chinese and American Classrooms. in J.W. Stigler, R.A. Shweder and G. Herdt (eds.. *Cultural Psychology: Essays on Comparative Human Development.* Cambridge University Press. N.Y. : 328-353.

Streefland, L. (1984). Searching for the Roots of Ratio Some Thoughts on the Long Term Learning Process (Towards a Theory). Part I: Reflections on a Teaching Experiment. *Educational Studies in Mathematics*, 15 : 327-348.

Streefland, L. (1985). Searching for the Roots of Ratio: Some Thoughts on the Long Term Learning Process (Towards a Theory). Part II: The Outline of the Long Term Learning Process. *Educational Studies in Mathematics*, 16: 75-94.

Struik, D.J. (1982). *A Concise History of Mathematics.* Translated in Greek. A. Ferentinou-Nikolakopoulou. I. Zacharopoulos, Athens, Greece.

Sugden, D.A. & Keogh, J.F. (1990). *Problems in Movement Skill Development.* Univ. of South Carolina Press, Columbia.

Sutherland, P. (1992). *Cognitive Development Today: Piaget and His Critics.* Paul Chapman Publ. Ltd., London.

Szetela, W. (1980). Calculators and the Teaching of Ratios in Grade 7. *Journal for Research in Mathematics Education*, 11(1) : 67-70.

Tourniaire, F.& Pulos, S. (1985). Proportional Reasoning: A Review of the Literature. *Educational Studies in Mathematics*, 16 : 181-204.

Turner, A.D. (1982). Mathematics-Science Links in the Secondary School. Part 2: Some Research Findings on Children's Understandings of Ratio and Proportion. *Mathematics in Schools*, 11(1) : 16-20.

Ulich, R. (1963). *Three Thousands Years of Educational Wisdom.* Selections from Great Documents. 2nd ed. Harvard University Press, Cambridge, Massachusetts.

Vallbo, Å.B. (1987). Touch, Sensory Coding of, in the Human Hand, in J.H. Wolfe (ed). Selected Readings from the Encyclopedia of Neuroscience [G. Adelman (ed)]. *Sensory Systems II: Senses Other Than Vision.* Birkhauser, Boston, Inc.: 136-138.

Van den Brink, J.& Streefland, L. (1979). Young-Children (6-8) - Ratio and Proportion. *Educational Studies in Mathematics*, 10 : 403-420.

Vergnaud, G. (1983). Multiplicative Structures. in R. Lesh & M. Landau (eds). *Acquisition of Mathematics Concepts and Processes.* Academic Press. Inc.

Vergnaud, G. (1988). Multiplicative Structures. M. Behr & J. Hiebert (eds). *Number Concepts and Operations in the Middle Grades. Vol.2.* Lawrence Erlbaum Associates, NCTM: 141-161.

Walden, R. and Walkerdine, V. (1985). *Girls and Mathematics: From Primary to Secondary Schooling.* Bedford Way Papers.

Wall, A.E. (1986). A Knowledge-Based Approach to Motor Skill Acquisition. in M.G. Wade & H.T.A. Whiting (eds). *Motor Development in Children: Aspects of Coordination and Control.* NATO Advanced Science Institutes Series, Series D - No.34, Martinus Nijhoff Publ., The Netherlands: 33-49.

Weber, E.H. (1987). *The Sense of Touch.* Translated by E.H. Ross (De Tactu) and D.J. Murray (Der Tastsinn), Academic Press, U.K.

Whiting, H.T.A. (1980), Dimensions of Control in Motor Learning, in G.E. Stelmach & J. Requin (eds) *Tutorials in Motor Behavior.* North-Holland Publ. Co.: 537-550.

Wilkins, A.S. (1974). *Roman Education.* Cambridge: At the University Press, G.B.

Williams, H.G. (1983), *Perceptual and Motor Development.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Williams, E. & Shuard, H. (1991). *Primary Mathematics Today.* Third Edition, For the Age of the Calculator, Longman Group Ltd., London.

Winne, P.H. & Marx, R.W. (1982). Students' and Teachers' Views of Thinking Processes for Classroom Learning, *Elementary School Journal,* 82(5): 493-518.

Wollman, W.T. & Lawson, A.E. (1978). The Influence of Instruction on Proportional Reasoning in Seventh Graders. *Journal of Research in Science Teaching,* 15(3): 227-232.

Woods, P. (1985). Pupil Strategies, in N. Bennet and C. Desforges (eds), *Recent Advances in Classroom Research.* British Journal of Educational Research, Monograph Series No. 2, Scottish Academoc Press, Edinburgh: 92-104.

Yannicopoulos, A.V. (1983). *H Ekpedefsi Kata ton 4o Aiona kai Kata tin Archaiotita.* Athens, Greece.

Yannicopoulos, A.V. (1989). *H Ekpedefsi Stin Klassiki kai Proclassiki Archaiotita.* Gregoris, Athens, Greece.