A Set-Theoretical Definition of Application

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Preface

This paper is in two parts. Part 1 is the previously unpublished 1972 memorandum [41], with editorial changes and some minor corrections. Part 2 presents what happened next, together with some further development of the material. The first part begins with an elementary set-theoretical model of the $\lambda\beta$-calculus. Functions are modelled in a similar way to that normally employed in set theory, by their graphs; difficulties are caused in this enterprise by the axiom of foundation. Next, based on that model, a model of the $\lambda\beta\eta$-calculus is constructed by means of a natural deduction method. Finally, a theorem is proved giving some general properties of those non-trivial models of the $\lambda\beta\eta$-calculus which are continuous complete lattices.

In the second part we begin with a brief discussion of models of the λ-calculus in set theories with anti-foundation axioms. Next we review the model of the $\lambda\beta$-calculus of Part 1 and also the closely related—but different!—models of Scott [51, 52] and of Engeler [19, 20]. Then we discuss general frameworks in which elementary constructions of models can be given. Following Longo [36], one can employ certain Scott-Engeler algebras. Following Coppo, Dezani-Ciancaglini, Honsell and Longo [13], one can obtain filter models from their Extended Applicative Type Structures. We give an extended discussion of various ways of constructing models of the $\lambda\beta\eta$-calculus, and the connections between them. Finally we give extensions of the theorem to complete partial orders. Throughout we concentrate on means of constructing models. We hardly consider any analysis of their properties; we do not at all consider their application.

Part1: Introduction

There seem to be three main difficulties in the way finding a reasonable concept of application which allows self-application. First there is the cardinality difficulty: if a set contains at least two elements then the set of functions from that set to itself has a greater cardinality than the set itself. So one cannot expect to find a set containing all functions from itself to itself (other than a trivial one).

So one has to pick out just some of the functions. But, as a version of Russell's paradox shows, it is not obvious which are the correct ones. It seems reasonable that a set with self-application should contain a function, f, say, with no fixed point. That is, $f(x) \neq x$ for any x in the set. On the other hand, given f, one can define a function, g, from the set to itself by setting $g(x) = f(xx)$. But if g is in the set then $g(g) = f(g(g))$—a contradiction.

Scott's answer (see [47, 48, 49]) is that such a set should be a complete lattice, and each function should be continuous. Then every function has a fixed point. He constructs such lattices as certain inverse limits.

We too will find such lattices, but *via* the third difficulty: what kind of object is a function? With the usual definition of a function as a certain kind of set of ordered pairs and the usual definition of application, the axiom of foundation precludes self-application. We search for variants of these definitions within ZF set theory which allow self-application and can also be used in the same way as the conventional ones.

To avoid confusion, we use primes to distinguish non-standard from standard concepts of function, application and mapping. An operation on sets called "application$'$ "is defined by:

$$x[y] = \bigcup\{w \mid \exists z \subseteq y\,(< z, w >\in x \text{ and } z \text{ is finite})\}$$

This is as an instance of a more general definition given relative to a fixed relation R between sets:

$$x[y] = \bigcup\{w \mid \exists z\,(< z, w >\in x \text{ and } R(z, y))\}$$

This reduces to a mild variant of the standard case, if we take R to be equality and define x to be a function$'$ iff it is a function and the second component of every ordered pair in x is a singleton. We have "collected" the "outputs" of the various members of x since, in general, more than one may be given.

If we define R by: $R(z, y) \equiv (z \subseteq y$ and z is finite), then the first definition of application$'$ is obtained. To show that nothing has been "lost" by this definition, let f be a standard function from X to Y. Let

$$\hat{f} = \{< \{x\}, \{y\} > | < x, y > \in f\}$$

Then, if $x \in X$, $\hat{f}[\{x\}] = \{f(x)\}$. So $\hat{f}$ has the "same" behaviour as f. However we now have better possibilities of self-application$'$. For example, given x, let $f = \{< \{1\}, \{y\} > | \ y \in x\} \bigcup \{1\}$. Then $f[f] = x$.

Further, following the Scottian precept, application$'$ is continuous in its second argument place. To see what this means in the present context, a *directed* set is defined to be one that is non-empty and given two of its members, there is a third member including them both. A function, f, from sets to sets is *continuous* iff, for any directed set, X, $f(\bigcup X) = \bigcup_{x \in X} f(x)$. The reader can easily check that application$'$ is indeed continuous in this sense at its second argument. It has a stronger property at its first one: it is completely additive there. That is, if X is any set, $(\bigcup X)[y] = \bigcup_{x \in X} x[y]$.

Unfortunately we do not have a good definition of function$'$ which allows both ordinary uses, as outlined above, and good collections of functions$'$ with self-application$'$. We will outline some of the difficulties and then show how a good collection of sets (not functions$'$) with self-application$'$ can be obtained.

The problem is that extensionality fails. For if, say, $< y, z >$ is in x, and w is any finite set, x and $x \bigcup \{< y \bigcup w, z >\}$ are extensionally equivalent. So a definition of function$'$ would have to select one member from each extensional equivalence class. It is natural to choose either minimal or maximal members. Let us briefly examine the first alternative. Say that x is a function$'$ iff

- Every member of x is an ordered pair.
- If $< y, z > \in x$ then z is a singleton.
- If $< y, z >, < y', z > \in x$ and $y \supseteq y'$ then $y' = y$.

It is not known if there is a way to use this definition to obtain a good domain with self-application.

The other alternative suffers from a major defect since the example of how extensionality can fail shows also that there is no maximal set in any equivalence class. However one can try to define mappings$'$ instead. Let us say that a set, f, is *from* X *to* Y iff $f \subseteq \mathcal{P}_f(\bigcup X) \times \mathcal{P}_f(\bigcup Y)$, where for any X, $\mathcal{P}_f(X)$ is the set of finite subsets of X. One then says that a *mapping$'$* from X to Y is a set from X to Y which includes any extensionally equivalent set from X to Y. This is in fact a good definition. But now no non-trivial set can be a set of mappings$'$ from itself to itself!

For suppose X is a set of mappings$'$ from itself to itself. Let $T = \bigcup X$. As T is a union of mappings$'$, we get that $T \subseteq \mathcal{P}_f(T) \times \mathcal{P}_f(T)$. We prove by induction on the depth of t that, if t is in T then, for every f in X, t is in f. Given such a t, let $t =< x, x' >$, and choose an f in X. By induction, $x' \subseteq g$, for every g in X. But then f is extensionally equivalent to $f \cup \{t\}$, and so, by the maximality of mappings$'$, we get that t is in f.

We turn now to finding a good collection of sets with self-application$'$, neglecting any question as to whether the sets can be regarded as functions$'$ or mappings$'$. First we try to find a lattice of sets, T^*_C, which obeys the *comprehension axiom*: if $f : T^*_C \to T^*_C$ is continuous then for some $\hat{f}$ in T^*_C and all x in T^*_C, $f(x) = \hat{f}[x]$. Only then will we worry about extensionality.

Now if $T_C = \bigcup T^*_C$, then $x \in T^*_C$ implies that $x \subseteq T_C$. So the simplest choice making T^*_C a lattice is to take $T^*_C = \mathcal{P}(T_C)$, and we only have to decide the nature of T_C. Now the function $\lambda x : T^*_C\, (x \supseteq y \to z, \emptyset)$ is a continuous function from T^*_C to T^*_C where y and z are finite subsets of T^*_C. (Here $(x \supseteq y \to z, \emptyset)$ is an example of McCarthy's *conditional expression*; it denotes z, if $x \supseteq y$ holds, and $\emptyset$ otherwise.) Now $\{< y, z >\}$ is extensionally equivalent to this function and so we may as well assume that if y and z are finite subsets of $T_C, < y, z >$ is in T_C, that is that:

$$\mathcal{P}_f(T_C) \times \mathcal{P}_f(T_C) \subseteq T_C$$

If equality holds here then the above argument suggests that difficulties might later arise with obtaining a model of extensionality and comprehension. So we will take T_C to contain some set ι which is not an ordered pair. Specifically, if we take T_C to be the least set such that:

$$T_C = \{\iota\} \cup \mathcal{P}_f(T_C) \times \mathcal{P}_f(T_C)$$

then, as we verify in the next section, T_C^* obeys the axiom of comprehension.

Extensionality is obtained by a process which effectively identifies ι with some other members of T_C and then obtains maximal elements using a operation on T_C^* which, in turn, is specified by means of a natural deduction system. It seems easier to delay more extensive explanations till the actual construction in the next section. Surprisingly it turns out that for certain choices of the identification of ι, one obtains models isomorphic to some obtained by Scott [48, p.33].

Variants on this construction are possible. For example one could build up T_C from any number of "atoms" like ι. Or one could insist that if y were a finite subset of T_C and z were any subset of cardinality less than κ, say, then $< y, z >$ is in T_C. This would give rather larger collections and the natural deduction system mentioned above would have to be infinitary.

As regards the possibility of choosing minimal rather than maximal elements, we suspect that one could have similar difficulties with the definition of mapping′. Perhaps one way of overcoming the general difficulty would be to regard ordered pairs of ordered pairs of ..... as being trees of finite depth and consider replacing them by trees of arbitrary depth. In this way one would avoid those difficulties whose existence depends on the axiom of foundation.

The construction of models for the λ-calculus given in the next section seems less general than the Scott construction. In future work we hope to give a construction generalising them both.

Other definitions of application′ are also of some interest. Let us define κ-λ-application, where κ and λ are cardinals and $\kappa < \lambda$ by:

$$x[y]^\lambda_\kappa = \bigcup\{w \mid \exists z \subseteq y\, (< z, w > \in x \text{ and } \kappa \leq \mid z \mid < \lambda\}$$

This application′ is completely additive in its first argument and what may be called κ-λ-continuous in its second. A function from sets to sets is κ-λ-*continuous* iff:

$$f(\bigcup X) = \bigcup\{f(\bigcup Y) \mid Y \subseteq X \text{ and } \kappa \leq \mid Y \mid < \lambda\}$$

Ordinary continuity is 0-$\aleph_0$-continuity. Complete additivity is 0-2-continuity and what is called additivity, 1-2-continuity. One can obtain complete lattices comprehending all κ-λ-continuous functions and satisfying the axiom of extensionality by similar methods to the above in the cases of complete additivity and additivity. We have not investigated the other cases.

More interestingly, define application$'$ by:

$$x[y]_N = \bigcup\{w \mid \exists z, z' \, (z \subseteq y \text{ and } z' \cap y = \emptyset \text{ and } << z, z' >, w > \in x)\}$$

Now functions can have no fixed points. Define T_N by:

- $\iota \in T_N$
- If x, y, z are finite subsets of T_N then $<< x, y >, z > \in T_N$.

Let comp $= \{<< \emptyset, x >, x >| \ x \in T_N\}$. Then comp $\in \mathcal{P}(T_N)$ and if $x \in \mathcal{P}(T_N)$, then comp$[x]_N = (T_N \setminus x) \neq x$. This is particularly interesting in view of the version given above, of Russell's paradox, for one of the legs it stands on is the existence of precisely such a function$'$ as comp.

Finally we give some technical definitions and facts taken from the work of Scott [47, 48, 49]. A *continuous lattice* is a structure $< D, \sqsubseteq, \ll, \bigsqcup >$ satisfying the following axioms:

1 $x = y \equiv (x \sqsubseteq y \wedge y \sqsubseteq x)$

2 $x \sqsubseteq y \equiv \forall z \, (z \ll x \supset z \ll y)$

3 $x \ll y \equiv \exists z \, (x \ll z \ll y)$

4 $\bigsqcup X \sqsubseteq y \equiv \forall x \in X \, (x \sqsubseteq y)$

5 $\bigsqcup X \ll y \equiv \forall x \in X \, (x \ll y)$ (when X is finite)

6 $x \ll \bigsqcup Y \equiv \exists y \in Y \, (x \ll y)$ (when Y is directed)
(Y is *directed* iff it is non-empty and $x, y \in Y$ implies that $x \sqsubseteq z$ and $y \sqsubseteq z$ for some $z \in Y$.)

These lattices can be given a topological characterisation [48].

Let $\perp = \bigsqcup \emptyset$ and $\top = \bigsqcup D$. The following facts taken from unpublished lecture notes of Scott are worth knowing:

1 $x \ll y \ll z \supset x \ll z$

2 $x \sqsubseteq y \ll z \supset x \ll z$

3 $x \ll z \supset x \sqsubseteq y$

4 $\perp \ll x$

5 $(x \ll z \wedge y \ll z) \supset (x \sqcup y) \ll z$

6 $\forall z\,(z \ll x \supset z \sqsubseteq y) \supset x \sqsubseteq y$

7 $\forall x\,(x = \bigsqcup\{z \mid z \ll x\})$

8 $x \ll y \equiv \forall Y\,((\mathit{Directed}(Y) \wedge y = \bigsqcup Y) \supset \exists y' \in Y.x \sqsubseteq y')$

A simple example of a continuous lattice is $< \mathcal{P}(X), \subseteq, \ll, \bigcup >$ where X is any set and, given $x, y \in \mathcal{P}(X), x \ll y$ iff $x \subseteq y$ and x is finite.

An element x of D is *isolated* iff $x \ll x$. If for any x in D,

$$x = \bigsqcup\{z \mid z \ll x \text{ and } z \text{ is isolated}\}$$

then D is an algebraic lattice [28]. For example, a function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is a *closure operation* iff it is continuous (with respect to the subset ordering), idempotent and $c(x) \supseteq x$ when $x \subseteq X$. Then $< D, \subseteq, \ll, \bigcup >$ is an algebraic lattice if $D = \{c(x) \mid x \subseteq X\}, x \ll z$ iff for some finite $y \subseteq X, x \subseteq c(y) \subseteq z$, and, given $Y \subseteq \mathcal{P}(X), \bigsqcup Y = c(\bigcup Y)$. It follows from the continuity of c, that for any directed subset Y of D, $\bigsqcup Y = \bigcup Y$.

Suppose, now, that $< D, \sqsubseteq>$ and $< E, \sqsubseteq>$ are complete lattices and f is a function from D to E. Then f is *continuous* iff for any directed set $X \subseteq D$,

$$f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$$

It is *additive* iff for any non-empty set $X \subseteq D$,

$$f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$$

It is *completely additive* iff for any $X \subseteq D$,

$$f(\bigsqcup X) = \bigsqcup_{x \in X} f(x)$$

2 A Continuous Domain

Let T_C be the smallest set satisfying:

- $\iota \in T_C$.
- If μ, ν are finite subsets of T_C, then $< \mu, \nu > \in T_C$.

Generally we shall use $\mu \to \nu$ as an abbreviation for $< \mu, \nu >$. We shall use $\tau, \tau', \ldots$ for members of T_C and $\mu, \nu, \ldots$ for finite subsets of T_C and x, y for arbitrary subsets.

Let $T_C^* = \mathcal{P}(T_C)$. Define $x \ll y$ to hold iff $x = \mu \subseteq y$ for some μ. Then $< T_C^*, \subseteq, \ll, \bigcup >$ is a continuous lattice.

Let x, y be in T_C^*. Application is defined by:

$$x[y] = \bigcup\{\nu \mid \exists \mu \subseteq y\,(\mu \to \nu \in x)\}$$

Application is completely additive in its first argument and continuous in its second one. For the first of these assertions, calculate that:

$$\begin{aligned}
(\bigcup X)[y] &= \bigcup\{\nu \mid \exists \mu \subseteq y\,(\mu \to \nu \in \bigcup X)\} \\
&= \bigcup\{\bigcup_{x \in X}\{\nu \mid \exists \mu \subseteq y\,(\mu \to \nu \in x)\}\} \\
&= \bigcup_{x \in X}\{\bigcup\{\nu \mid \exists \mu \subseteq y\,(\mu \to \nu \in x)\}\} \\
&= \bigcup_{x \in X} x[y]
\end{aligned}$$

For the second of these assertions, suppose Y is directed. Then $\mu \subseteq \bigcup Y$ iff $\exists y \in Y\,(\mu \subseteq y)$. So:

$$\begin{aligned}
x[\bigcup Y] &= \bigcup\{\nu \mid \exists \mu \subseteq \bigcup Y\,(\mu \to \nu \in x)\} \\
&= \bigcup\{\nu \mid \exists y \in Y \exists \mu \subseteq y\,(\mu \to \nu \in x)\} \\
&= \bigcup_{y \in Y}\{\bigcup\{\nu \mid \exists \mu \subseteq y\,(\mu \to \nu \in x)\}\} \\
&= \bigcup_{y \in Y} x[y]
\end{aligned}$$

Next we demonstrate comprehension. Let $f : T_C^* \to T_C^*$ be continuous. Let $\hat{f} = \{\mu \to \nu \mid \nu \subseteq f(\mu)\}$. If $x \in T_C^*$ then:

$$\begin{aligned}
\hat{f}[x] &= \bigcup\{\nu \mid \exists \mu \subseteq x\,(\mu \to \nu \in \hat{f})\} \\
&= \bigcup\{\nu \mid \exists \mu \subseteq x\,(\nu \subseteq f(\mu))\} \\
&= \bigcup_{\mu \subseteq x} f(\mu) \\
&= f(\bigcup_{\mu \subseteq x} \mu) \text{ (as } \{\mu \mid \mu \subseteq x\} \text{ is directed)} \\
&= f(x)
\end{aligned}$$

We now have a model for the λ-calculus without extensionality.

Rather than characterise how extensionality fails, we give some examples.

- For any x, x and $x \cup \{\emptyset \to \emptyset\}$ are extensionally equivalent.
- For any x, x and $x \cup \{\iota\}$ are extensionally equivalent.
- For any μ, μ', ν, ν', $\{\mu \to \nu \cup \nu'\}$ and $\{\mu \to \nu \cup \nu', \mu \cup \mu' \to \nu\}$ are extensionally equivalent.
- For any μ, ν, μ', ν',$\{\mu \to \nu, \mu' \to \nu'\}$ and $\{\mu \to \nu, \mu' \to \nu', \mu \cup \mu' \to \nu \cup \mu'\}$ are extensionally equivalent.

Extensionality is obtained by considering only a subset of T_A^*. These subsets are maximal members of the equivalence classes generated by the extensional equivalence relation and satisfy certain other conditions which we shall explain later.

To obtain them we specify a natural deduction system where the set of formulas is T_C. Natural deduction systems are described by Prawitz in [44]. Let Th be the smallest subset of T_C such that:

- If $\nu \subseteq Th$ then $\mu \to \nu \in Th$.

Note that $\emptyset \to \emptyset \in Th$. Choose $\omega_\iota = \{\tau'_1, \ldots, \tau'_n\}$ $(n > 0)$ such that $\omega_\iota \cap Th$ is empty and ι is not in ω_ι.

The axioms and rules are:

Axioms

1. $\emptyset \to \emptyset$.

Rules

1. For $1 \le i \le n$,
$$\frac{\iota}{\tau_i^\iota}$$

2.
$$\frac{\tau_1^\iota \ldots \tau_n^\iota}{\iota}$$

3.
$$\frac{\mu \to (\nu \cup \nu')}{(\mu \cup \mu') \to \nu}$$

4.
$$\frac{\mu \to \nu, \mu' \to \nu'}{(\mu \cup \mu') \to (\nu \cup \nu')}$$

5.
$$\frac{\mu \to \nu\ ,\ \begin{matrix}[\mu'_1] \\ \tau_1\end{matrix} \cdots \begin{matrix}[\mu'_n] \\ \tau_n\end{matrix}\ ,\ \begin{matrix}[\nu_1] \\ \tau'_1\end{matrix} \cdots \begin{matrix}[\nu_1] \\ \tau'_n\end{matrix}}{\mu' \to \nu'}$$

where $\mu' = \bigcup \mu'_i$; $\nu = \bigcup \nu_i$; $\mu = \{\tau_1, \ldots, \tau_n\}$; $\nu' = \{\tau'_1, \ldots, \tau'_{n'}\}$, for $n \geq 0$, $n' \geq 0$; μ'_i is the entire set of assumptions for τ_i $(1 \leq i \leq n)$; and ν_i is the entire set of assumptions for τ'_i $(1 \leq i \leq n')$—see the Notes.

Rules 1 and 2 are sometimes displayed as $\frac{\iota}{\omega_\iota}$ and $\frac{\omega_\iota}{\iota}$ respectively. Rule 5 is sometimes displayed as:

$$\frac{\mu \to \nu\ ,\ \overset{[\mu']}{\mu}\ ,\ \overset{[\nu]}{\nu'}}{\mu' \to \nu'}$$

Notes 1. In natural deduction systems, derivations are trees whose top formulas are the *assumptions*. Other formulas result from the ones above them by means of the rules of inference. The assumptions are either *open* or *closed*. The rules show how, as a derivation is built up, other assumptions are closed. This is indicated by the square brackets; it is intended that the formulas in the brackets include *all* the open assumptions of the corresponding branch. *Main branches* are those depending from open assumptions. The formula at the root of the tree is the *conclusion* of the derivation and *follows* from the open assumptions. Axioms yield trees with the axiom as conclusion and with no assumptions.

2. The set of theorems of the system will prove to be Th. The restriction that $\omega_\iota \cap Th = 0$ allows a clearer development, eliminating some trivial cases and redundancies.

3. The members of our extensional domain will be those subsets of T_C closed under deduction. The axiom and rules 3 and 4 are justified by our requirement that the subsets are maximal members of the extensional equivalence classes. (See examples 1, 3 and 4 above.)

Rules 1 and 2 are intended to, as it were, make ι behave like a function. Since example 1 above is analogous to example 2, one might expect, instead, that ι would be an axiom. However it would then follow that there would be exactly one set closed under deduction, *viz* T_C. (See also the discussion below of fixed points of K.)

Rule 5 can be split into two parts, which, under the same conventions as Rule 5, may be displayed as:

5a

$$\frac{\overset{[\mu']}{\mu} \;,\; \mu \to \nu}{\mu' \to \nu}$$

5b

$$\frac{\mu \to \nu \;,\; \overset{[\nu]}{\nu'}}{\mu \to \nu'}$$

Rule 5b is intended to ensure that if x is closed under deduction, so is $x[y]$, for any y. If we did not have Rule 5a, extensionality would fail: suppose μ', μ, ν are as in Rule 5a and $(\mu \to \nu) \in x$ but $(\mu' \to \nu) \notin x$. Then x and $x \cup \{\mu' \to \nu\}$ are extensionally equal (in the proposed domain). One would certainly like to have a less *ad hoc* explanation of the rules.

4. It is extremely useful to obtain a simple normal form theorem (see [45]). Evidently if one has a derivation tree with a sub-derivation of either of the forms,

$$\frac{\overset{\iota}{\omega_\iota}}{\iota}$$

or

$$\frac{\overset{\omega_\iota}{\iota}}{\tau_i^\iota}$$

(where $1 \leq i \leq n$) then there is one with the same conclusion and open assumptions, but no such sub-derivations. This is called a *normal* derivation. Notice that sub-derivations of normal derivations are normal and that ι can only occur on a main branch of a derivation if it is either an assumption or else is the conclusion. In fact stronger results are obtainable, although unnecessary. One can give what Prawitz calls a normalization theorem and show that the system is decidable.

We write $\mu \vdash \nu$ iff $\forall \tau \epsilon \nu \exists \mu' \subseteq \mu\,(\tau$ follows from $\mu')$. Since we are dealing with a natural deduction system, $\vdash$ is a quasi-order.

Let $Cl = \{\mu \to \nu \mid \mu \vdash \nu\}$. Then $Cl[x] = \bigcup\{\nu \mid \exists \mu \subseteq x\,(\mu \vdash \nu)\}$.

Let $T^*_{CE} = \{Cl[x] \mid x \epsilon T^*_A\} \subseteq T^*_A$.

As $Cl[\cdot]$ is a closure operation, $< T^*_{CE}, \subseteq, \ll, \bigsqcup >$ is a continuous lattice if we define $\ll$ and $\bigsqcup$ by:

$x \ll y$ iff $\exists \mu . x \subseteq Cl[\mu] \subseteq y$.

$\bigsqcup X = Cl[\bigcup X]$.

Application is a well-defined operation on T^*_{CE}. For let $\nu \subseteq Cl[x][y]$. There are $\mu_i \subseteq y\ \ (i = 1, n)$ and $\nu_i\,(i = 1, n)$ such that $\mu_i \to \nu_i \in Cl[x]$ and $\bigcup \nu_i \supseteq \nu$. Therefore, by Rule 4, $\bigcup \mu_i \to \bigcup \nu_i$ is in $Cl[x]$, and so $\bigcup \mu_i \to \nu$ is in $Cl[x]$ by Rule 3, This shows that: $\exists \mu \subseteq y\,(\mu \to \nu \in Cl[x])$ (take $\mu = \bigcup \mu_i$). Now, if $\nu \vdash \nu'$ and $\nu' \neq \emptyset$, let τ be in ν' and choose $\nu'' \subseteq \nu$ such that there is a derivation of τ from ν''. Then, taking μ as above we see that $\mu \to \nu \vdash \mu \to \nu'' \vdash \mu \to \tau$ by Rules 3 and 5. Hence $\tau \in Cl[x][y]$ and so $\nu' \subseteq Cl[x][y]$. Therefore $Cl[x][y]$ is closed and so application is indeed well-defined. It further follows that:

$$Cl[Cl[x][y]] = Cl[x][y]$$

.

Application is continuous in both arguments. For the first argument, let X be a directed subset of T^*_{CE} and choose $y \in T^*_{CE}$. Then:

$$\begin{aligned}
(\bigsqcup X)[y] &= Cl[\bigcup X][y] \\
&= (\bigcup_{x \in X} Cl[x])[y]\ \text{(as } X \text{ is directed)} \\
&= (\bigcup_{x \in X} x)(y)\ \text{(as } X \subseteq T^*_{CE}) \\
&= (\bigcup_{x \in X} x[y])\ \text{(additivity with respect to } < T^*_C, \bigcup >) \\
&= (\bigsqcup_{x \in X} x[y])\ \text{(as } \{x[y] \mid x \in X\} \text{ is directed)}
\end{aligned}$$

For the second argument, let Y be a directed subset of T^*_{CE} and let x be in T^*_{CE}. Then:

$$\begin{array}{rcl}
x[\bigsqcup Y] & = & x[\bigcup Y]\,(\text{as } Y \text{ is directed}) \\
& = & \bigcup_{y \in Y} x[y]\,(\text{by continuity with respect to} \\
& & \quad < T^*_C, \bigcup > \text{ and directedness of } X) \\
& = & (\bigsqcup_{x \in X} x[y])\,(\text{as } \{x[y] \mid y \in Y\} \text{ is directed})
\end{array}$$

It will turn out later that application is actually completely additive in its first argument.

First we need some proof theory, which we begin by describing the theorems.

Lemma 1 $\emptyset \vdash \tau$ *iff* $\tau \in Th$
Proof Suppose $\tau \in Th$. We prove by induction on the structure of τ that $\emptyset \vdash \tau$. For some μ and ν, $\tau = \mu \to \nu$ and $\nu \subseteq Th$. By induction hypothesis, $\emptyset \vdash \nu$. Then, by Axiom 1 and Rule 5, $\emptyset \vdash \emptyset \to \nu$; so, by Rule 3 we get that $\emptyset \vdash \mu \to \nu$.

Suppose $\emptyset \vdash \tau$. We proceed by induction on the size of the derivation of τ from $\emptyset$. The proof divides into cases, according to the last axiom or rule applied.

Axiom 1 Here $\tau = \emptyset \to \emptyset$ and so $\tau \in Th$.

Rule 1 Here τ is some τ_i^ι, and there is a smaller proof of ι from $\emptyset$. But then, by induction, we get that ι is in Th, which is a contradiction. So this case cannot arise.

Rule 2 Here we get a smaller proof of each τ_i^ι, contradicting the conditions on ω_ι.

Rule 3 Here $\mu \to (\nu \cup \nu') \in Th$ implies $\nu \cup \nu' \subseteq Th$ implies $\tau \in Th$.

Rule 4 Here $\mu \to \nu, \mu' \to \nu' \in Th$ implies $\nu, \nu' \subseteq Th$ implies $\tau \in Th$.

Rule 5 here $\mu \to \nu \in Th$ implies $\nu \subseteq Th$ implies (by transitivity of $\vdash$) $\nu' \subseteq Th$ implies $\tau \in Th$

$\square$

Note that since $\iota \notin Th$, T^*_{CE} is non-trivial.

Lemma 2 *If $\omega \vdash \mu \to \nu$ and $\iota \notin \omega$ then either $\mu \to \nu \in Th$ or else there are $\mu_j \to \nu_j$ $(j = 1, m)\,(m \neq 0)$ in ω such that: $\mu \vdash \bigcup \mu_j$ and $\bigcup \nu_j \vdash \nu$.*

Proof By induction on the size of normal derivations of $\mu \to \nu$ from ω, supposing that $\mu \to \nu \notin Th$. Different cases correspond to the different axioms or rules last used.

Axiom 1 Inapplicable.

Rule 1 Here the derivation has the form

$$\begin{array}{c} \omega' \\ \\ \dfrac{\iota}{\tau_i^\iota} \end{array}$$

where $\omega' \subseteq \omega$. As the derivation is normal, it then follows that $\omega' = \iota$. But by assumption $\iota \notin \omega$, and so this case cannot arise.

Rule 2 Inapplicable.

Rule 3 Here the derivation has the form

$$\begin{array}{c} \omega' \\ \\ \dfrac{\mu' \to (\nu \cup \nu')}{(\mu' \cup \mu'') \to \nu} \end{array}$$

where $\mu = \mu' \cup \mu''$ and $\omega' \subseteq \omega$. As $\mu \to \nu \notin Th$, $\mu' \to \nu \cup \nu' \notin Th$. So we find, by induction, $\mu_j \to \nu_j$ $(j = 1, m)$ in ω such that $\mu = (\mu' \cup \mu'') \vdash \mu' \vdash \bigcup \mu_j$ and $\bigcup \nu_j \vdash \nu \cup \nu' \vdash \nu$.

Rule 4 Here the derivation has the form

$$\begin{array}{c} \omega' \qquad\quad \omega'' \\ \\ \dfrac{\mu_1 \to \nu_1 \;\; \mu_1' \to \nu_1'}{\mu \to \nu} \end{array}$$

where $\omega' \cup \omega'' \subseteq \omega$ and $\mu = \mu_1 \cup \mu_1'$ and $\nu = \nu_1 \cup \nu_1'$. Both $\mu_1 \to \nu_1$ and $\mu_1' \to \nu_1'$ cannot be in Th, for then $\mu \to \nu$ would be. There are three cases of which we consider only one: $\mu_1 \to \nu_1$ not in Th and $\mu'_1 \to \nu'_1$ in Th. The others are similar. Then we find $\mu_{1j} \to \nu_{1j}$ in $\omega' \subseteq \omega$ such that $\mu = \mu_1 \cup \mu'_1 \vdash \mu_1 \vdash \bigcup \mu_{1j}$ and $\bigcup \nu_{1j} \vdash \nu_1 \vdash \nu_1 \cup \nu_2 = \nu$ (since $\nu_2 \subseteq Th$).

Rule 5

$$\frac{\begin{array}{cc} \omega' & [\mu]\,[\nu'] \\ (\mu' \to \nu') & \mu' \quad \nu \end{array}}{\mu \to \nu}$$

where $\omega' \subseteq \omega$. If $\mu' \to \nu' \in Th$, $\nu \subseteq Th$ contradicting the fact that $\mu \to \nu \notin Th$. So, by induction, we find $\mu_j \to \nu_j \in \omega$ so that $\mu \vdash \mu' \vdash \bigcup \mu_j$ and $\bigcup \nu_j \vdash \nu' \vdash \nu$.

□

Suppose x is in T^*_C. We say that x *types* ι iff $\iota \in x$ implies there is an $\omega \subseteq x$ such that $\omega \vdash \iota$ and $\iota \notin \omega$.

Lemma 3 *If x types ι and y is in T^*_{CE} then $Cl[x[y]] = Cl[x][y]$.*

Proof $Cl[x[y]] \subseteq Cl[Cl[x][y]] = Cl[x][y]$, by a previous remark. Suppose $\nu \subseteq Cl[x][y]$ Then, by a previous remark, for some $\mu \subseteq y$, $\mu \to \nu \in Cl[x]$. So for some $\omega \subseteq x, \omega \vdash \mu \to \nu$. If $\mu \to \nu \in Th, \nu \subseteq Cl[x[y]]$. As x types ι, we may assume that $\iota \notin \omega$. So by Lemma 2, we find $\mu_j \to \nu_j$ in ω such that $\mu \vdash \bigcup \mu_j$ and $\bigcup \nu_j \vdash \nu$. As y is closed, $\bigcup \mu_j \subseteq y$, so $\bigcup \nu_j \subseteq x[y]$ and hence $\nu \subseteq Cl[x[y]]$.

□

It is now easy to verify the comprehension axiom. Suppose $f : T^*_{CE} \to T^*_{CE}$ is continuous. Let $x_f = \{\mu \to \nu \mid \nu \subseteq f(Cl[\mu])\}$ and let $\hat{f} = Cl[x_f]$. Evidently x_f types ι. We calculate, given y in T^*_{CE}:

$$\begin{aligned}
\hat{f}[y] &= Cl[x_f[y]]\,(\text{ by Lemma 3}) \\
&= Cl[\bigcup\{\nu \mid \exists \mu \subseteq y\,(\mu \to \nu \in x_f\}] \\
&= Cl[\bigcup\{\nu \mid \exists \mu \subseteq y\,(\nu \subseteq f(Cl[\mu]))\}] \\
&= Cl[[\bigcup_{\mu \subseteq y}\{\bigcup\{\nu \mid \nu \subseteq f(Cl[\mu])\}\}]
\end{aligned}$$

$$
\begin{aligned}
&= Cl[\bigcup_{\mu \subseteq y} f(Cl[\mu])] \\
&= \bigsqcup_{\mu \subseteq y} f(Cl[\mu]) \\
&= f(\bigsqcup_{\mu \subseteq y} Cl[\mu])\ (\text{as } \{Cl[\mu] \mid \mu \subseteq y\} \text{ is directed, and } f \text{ is continuous}) \\
&= f(y)
\end{aligned}
$$

To see that the lattice ordering agrees with the induced pointwise function ordering suppose, given x, y in T^*_{CE} that $x[z] \subseteq y[z]$ for all z in T^*_{CE}. We wish to prove that $x \subseteq y$. If $\mu \to \nu$ is in x then $\nu \subseteq x[Cl[\mu]] \subseteq y[Cl[\mu]]$. So there is a $\mu' \subseteq Cl[\mu]$ such that $\mu' \to \nu$ is in y. Since $\mu \vdash \mu'$, it follows by Rules 5 and 3 that $\mu \to \nu$ is in y. Again, if $\iota \in x$ then $\omega_\iota \subseteq x$ (by Rule 1), so $\omega_\iota \subseteq y$ (by the above) and finally, $\iota \in y$ (by Rule 2).

□

Some of the properties of the T^*_{CE}'s obtained by varying ω_ι, can be established in a general axiomatic way that applies also to the domains constructed by Scott [48]. Suppose $< D, \sqsubseteq, \ll, \bigsqcup, \cdot[\cdot] >$ satisfies these axioms:

Axiom 1 $< D, \sqsubseteq, \ll, \bigsqcup >$ is a continuous lattice.

Axiom 2 Application, $\cdot[\cdot]$, is continuous in its second argument.

Axiom 3 Every continuous function is comprehended by $\cdot[\cdot]$.

Axiom 4 The lattice ordering agrees with the induced pointwise function ordering.

Axiom 5 $\mid D \mid \geq 2$.

These axioms are satisfied by $< T^*_{CE}, \subseteq, \ll, \bigsqcup, \cdot[\cdot] >$ as shown above. Comparing these axioms to the last set of axioms given by Scott in [47], our axiom 1 strengthens his requirement that $< D, \sqsubseteq, \bigsqcup >$ is a complete lattice and axiom 5 is much weaker than his axiom of substance. The other axioms are also asserted by him.

It is a trivial consequence of axiom 4 that application is monotonic in its first argument and that extensionality holds. On occasion the brackets will be omitted when writing applications. For example, we may write xy for $x[y]$. From axioms 3 and 4 it follows that we can define $\Rightarrow : D \times D \to D$ by:

$$(x \Rightarrow y)[z] = (z \gg x \to y, \perp)$$

This is $\vec{e}(x, y)$ in the notation of [48]. We will always use the infix notation for $\Rightarrow$. The function $\Rightarrow$ is completely additive in its second argument, and antimonotonic in its first argument. We define the *combinators* K and S to be the elements of D such that, respectively:

$$Kxy = x$$

$$Sxyz = xz[yz]$$

where the missing brackets are associated to the left. By extensionality, these equations define K and S uniquely, if they exist.

A set $B \subseteq D$ is *dense* iff when $x \ll y$ there is a $b \in B$ such that $x \ll b \ll y$. As a consequence of this definition, if B is dense then for every x in D, $\{b \in B \mid b \ll x\}$ is directed and $x = \bigsqcup\{b \in B \mid b \ll x\}$.

The usefulness of these axioms is demonstrated by:

Theorem 1 *1. Application is completely additive in its first argument. (This does not depend on the fact that D is actually a continuous lattice—only that it is complete.)*

2. The combinator K exists, and if B is dense in D,

$$K = \bigsqcup_{b \in B}\{b \Rightarrow (\perp \Rightarrow b)\}$$

.

3. The combinator S exists, and if B is dense in D,

$$S = \bigsqcup_{d,e,e',f \in B}\{(d \Rightarrow (e \Rightarrow f)) \Rightarrow ((d \Rightarrow e') \Rightarrow (d \Rightarrow f)) \mid e \ll e'\}$$

.

4. *If, in addition, D is algebraic then,*

$$S = \bigsqcup_{d,e,f \in B} \{(d \Rightarrow (e \Rightarrow f)) \Rightarrow ((d \Rightarrow e) \Rightarrow (d \Rightarrow f))\}$$

.

5. $\mid D \mid \geq 2^{\aleph_0}$.

6. *Suppose T is a suitable second-order theory whose axioms are the formal counterparts of 1-5. Let T^* be the extension of T by the equational definitions of the combinators S and K. Then the ordering $\ll$ is definable in the theory T^* by means of a first-order formula involving only application (and the equality predicate).*

7. *Let $CL + Ext$ be the standard first-order theory of combinatory logic with the principle of extensionality. Then T^* is not a conservative extension of $CL + Ext$. In particular,*

$$\exists xy \forall z(x[z] = x \wedge y[z] = y \wedge x \neq y)$$

is provable in the one, but not the other.

The proof is delayed until the Appendix. Note that it follows from part 6 that if D_1 and D_2 satisfy axioms 1-5 then they have an isomorphism of their structures iff they have isomorphisms of the functional part of their structures alone.

Corollary 1 1. $|T^*_{CE}| = 2^{\aleph_0}$

2. *In T^*_{CE},*

$$K = \bigsqcup \{Cl[\{\mu \to \{\emptyset \to \mu\}\}]\}$$

and

$$S = \bigsqcup \{Cl[\{\mu \to \{\nu \to \omega\}\} \to \{\{\mu \to \nu\} \to \{\mu \to \omega\}\}]\}$$

.

Proof 1.

$$\begin{aligned} 2^{\aleph_0} &\leq |T^*_{CE}| \text{ (by Theorem 1.5)} \\ &\leq |T^*_C| \\ &= 2^{\aleph_0} \end{aligned}$$

2. Evidently $B = \{Cl[\mu]\}$ is dense in T^*_{CE} and T^*_{CE} is algebraic. Further $Cl[\mu] \Rightarrow Cl[\nu] = Cl[\{\mu \to \nu\}]$. So,

$$\begin{aligned} K &= \bigsqcup_\mu \{Cl[\mu] \Rightarrow (Th \Rightarrow Cl[\mu])\} \text{ (by Theorem 1.2).} \\ &= \bigsqcup_\mu \{Cl[\mu] \Rightarrow Cl[\{\emptyset \to \mu\}]\}. \\ &= \bigsqcup_\mu Cl[\{\mu \to \{\emptyset \to \mu\}\}] \end{aligned}$$

The formula for S follows in a similar way from Theorem 1.4.

$\Box$

The reader will notice the similarity between the formulas for S and K and the corresponding formulas occurring in the well-known Curry-Feys connection between combinatory logic and minimal implicational logic, which arose through their theory of functionality. It was, in fact, an attempt to make type symbols (in the usual sense) form a model that led to the present work.

Next we would like to compare our models with those obtained by Scott by trying to see which of ours are isomorphic to which of his. Notice that, because of Theorem 1.7, it makes no difference whether we consider just functional isomorphisms or isomorphisms of the entire structures. First, however, it is necessary to find the fixed points of K in T^*_{CE} and in some of Scott's models.

Suppose $x \in T^*_{CE}$ is a fixed point of K. Now

$$K[x] = (\bigsqcup_\mu \{Cl[\{\mu \to \{\emptyset \to \mu\}\}])[x] \text{ (by Corollary 1.2)}$$

$$
\begin{aligned}
&= \bigsqcup_{\mu}(Cl[\{\mu \to \{\emptyset \to \mu\}\}][x]) \text{ (by Theorem 1.1)} \\
&= \bigsqcup_{\mu} Cl[\{\mu \to \{\emptyset \to \mu\}\}[x]] \text{ (by Lemma 3)} \\
&= \bigsqcup_{\mu}\{Cl[\emptyset \to \mu] \mid \mu \subseteq x\}
\end{aligned}
$$

Therefore if $\tau \in x, \emptyset \to \{\tau\} \in K[x] = x$. Conversely suppose $\mu \to \nu$ is in x. Now $x = K[x] = \bigsqcup_{\mu}\{Cl(\emptyset \to \mu) \mid \mu \subseteq x\}$, which is the closure of a directed set. So for some $\nu_i \subseteq x\,(i = 1, n)$, $\{\emptyset \to \nu_i \mid i = 1, n\} \vdash \mu \to \nu$. So, if $\mu \to \nu \notin Th, \bigcup \nu_i \vdash \nu$ by Lemma 2, and then $\nu \subseteq x$.

Suppose now that $x \neq Th\,(= \perp)$. Then some τ is in $x \backslash Th$. Choose such a τ of lowest complexity (say complexity = number of arrows). If $\tau = \mu \to \nu$ then $\nu \subseteq x$, by the above, and $\nu \backslash Th \neq \emptyset$ as $\tau \notin Th$. This contradicts the minimal complexity of τ. So $\tau = \iota$. We now show that $x = T_C$. Certainly $\iota \in x$. Suppose $\nu \subseteq x$. Then, by the above, $\emptyset \to \nu$ is in x and so $\mu \to \nu \in x$. So by induction $x = T_C$.

We have therefore shown that if x is a fixed-point of K, then it is either $\perp$ $(= Th)$, or $\top$ $(= T_C)$. Conversely $\perp[x] = \perp = K[\perp][x]$ and so $K[\perp] = \perp$ by extensionality. Further if $x = \top$, then $\top[x] = K[\top][x]$ and so $K[\top] = \top$. Therefore K has exactly two fixed points, $\perp$ and $\top$, in T^*_{CE}.

Scott describes a general class of models of the λ-calculus with extensionality in [48], to which we refer the reader for definitions and notation. These models are obtained as inverse limits of systems $< D_n, \psi_n >_{n=0}^{\infty}$ of complete lattices D_n and projections $\psi_n : D_{n+1} \to D_n$ where $D_{n+1} = D_n \to D_n$ (the complete lattices of continuous functions from D_n to D_n) and ψ_{n+1} is determined by ψ_n. More specifically, the ψ_n have partial inverses $\varphi_n : D_n \to D_{n+1}$ and the following formulas hold for $n > 0$:

$$\psi_{n+1}(g) = \varphi_n \circ g \circ \psi_n$$

and

$$\varphi_{n+1}(f) = \psi_n \circ f \circ \varphi_n$$

.

So his models are determined by the choice of ψ_0 and D_0. If we want the limit to be a continuous lattice, then D_0 must be one. However we need not assume this for the moment. (Neither is it assumed by Scott in [47], but there ψ_0 *is* restricted—but, as remarked by David Park, this is also unnecessary.) We will restrict ourselves to finding the fixed points of K when $\psi_0 = \lambda x : D_1.x(t)$, where t is an isolated member of D_1. This ψ_0 has partial inverse $\varphi_0 : D_0 \to D_1$, where $\varphi_0 = \lambda x : D_0.\lambda y : D_0.(y \sqsupseteq t \to x, \perp)$. For each $d_0 \in D_0$ we define a vector $d = < d_n >_{n=0}^{\infty}$ by $d_{n+1} = \lambda x : D_n.d_n$.

Now $\psi_0(d_1) = d_1(t) = d_0$ and, proceeding inductively, if $x_n \in D_n$,

$$\begin{aligned}
\psi_{n+1}(d_{n+2})(x_n) &= \psi_n \circ d_{n+2} \circ \varphi_n(x_n) \\
&= \psi_n(\lambda x : D_n.d_{n+1}(\varphi_n(x_n))) \\
&= \psi_n(d_{n+1}) \\
&= d_n \text{ (by induction hypothesis)}
\end{aligned}$$

Therefore $\psi_{n+1}(d_{n+2}) = \lambda x : D_n.d_n = d_{n+1}$.

So d is in D_∞. Now if $e \in D_\infty$,

$$d(e) = \bigsqcup_{n=0}^{\infty} d_{n+1}(e_n) = \bigsqcup_{n=0}^{\infty} d_n = d$$

showing that d is a fixed point of K (for then $Kde = d = de$).

Conversely, if d is a fixed point of K, in [47] Scott proves $d_{n+1} = \lambda x : D_n.d_n$ by an argument which covers this case just as well as the one considered there. So in D_∞, the combinator K has exactly $|\, D_0 \,|$ fixed points.

As, in T^*_{CE}, the combinator K has two fixed points it could only be isomorphic to a D_∞ obtained from a ψ_0 and D_0 as described above if $|\, D_0 \,| = 2$, when $D_0 = \{\top, \perp\}$. In this case there are two possible D_∞'s, obtained by taking $t = \perp, \top$ respectively and in fact this gives all the projections from $D_0 \to D_0$ to D_0. These models are discussed in [48, p.33]. Surprisingly, if we take $\omega_\iota = \{\emptyset \to \iota\}, T^*_{CE}$ is isomorphic to the first and if ω_ι is $\{\iota \to \iota\}$, it is isomorphic to the second. We delay the proof to a later memorandum.

However, any T^*_{CE} can certainly be obtained from some D_0 and ψ_0, even if not in the way considered above. For one can always take $T^*_{CE} = D_0$ and let $\psi_0 : (D_0 \to D_0) \to D_0$ be the appropriate isomorphism. However this is not a very interesting characterisation!

The problem of characterisation really exists: we show, in outline, that there are infinitely many non-isomorphic lattices obtainable by varying ω_ι.

Let us consider some T^*_{CE}. An element $d \in T^*_{CE}$ is said to be *periodic* iff it is isolated and occurs infinitely often in the sequence

$$d, d[T_C], d[T_C][T_C], d[T_C][T_C][T_C], \ldots$$

In T^*_{CE}, d is isolated if it is $Cl[\mu]$, for some μ. Define $URS : T_C \to \wp(T_C)$ (for ultimate right-hand-side) by: $URS(\iota) = 0, URS(\mu \to \nu) = \nu \cup (\bigcup_{\tau \in \nu} URS(\tau))$.

Let $U_1 = \{\mu \mid \mu \subseteq \bigcup_i URS(\tau_i^\iota)\}$. One can prove first that if $\mu \in U_1$ then $Cl[\mu][T_C] = Cl[\mu']$ for some $\mu' \in U_1$ and second that for any μ some member of the sequence $Cl[\mu], Cl[\mu][T_C], Cl[\mu][T_C][T_C], \ldots$ is $Cl[\mu']$ for some $\mu' \in U_1$. So if d is periodic then it is $Cl[\mu']$ for some μ' in U_1 (and as a matter of fact there are always at least two periodic elements of D). So $P(\omega_\iota) = \mid \{d \in T^*_{CE} \mid d \text{ is periodic}\} \mid$ is a well-defined integer (which is actually greater than 1), which is an isomorphism invariant.

Now define τ_n $(n \geq 0)$ by:

$\tau_1 = \emptyset \to \{\iota\}$

$\tau_{n+1} = \emptyset \to \{\tau_n\}$.

Then it is not hard to show that $P(\{\tau_n\}) = 2^n$ giving the required infinite collection of non-isomorphic models.

Appendix: The Proof of Theorem 1

1.Application is completely additive in its first argument

Suppose $X \subseteq D$. Define $f : D \to D$ by: $f(y) = \bigsqcup_{x \in X} x[y]$. This function is continuous, for if Y is directed then,

$$f(\bigsqcup Y) \quad = \quad \bigsqcup_{x \in X} x[\bigsqcup Y]$$

$$
\begin{aligned}
&= \bigsqcup_{x\in X} \bigsqcup_{y\in Y} x[y] \\
&= \bigsqcup_{y\in Y} \bigsqcup_{x\in X} x[y] \\
&= \bigsqcup_{y\in Y} f(y)
\end{aligned}
$$

So by axiom 3, there is an $\hat{f}$ in D such that $f(y) = \hat{f}[y]\,(y \in D)$. On the one hand, given y in D,

$$
\begin{aligned}
\hat{f}[y] &= \bigsqcup_{x\in X} x[y] \\
&\sqsubseteq \bigsqcup_{x\in X} (\bigsqcup X)[y] \\
&= (\bigsqcup X)[y] \text{ (by monotonicity)}
\end{aligned}
$$

So $\hat{f} \sqsubseteq \bigsqcup X$ (by axiom 4).

On the other hand, since $\hat{f}[y] \sqsupseteq x[y]\,(x \in X)$, given $y \in D$, $\hat{f} \sqsupseteq x\,(x \in X)$, by axiom 4, and so $\hat{f} \sqsupseteq \bigsqcup X$. So, $\hat{f} = \bigsqcup X$ and for y in D we have:

$$(\bigsqcup X)[y] = \hat{f}[y] = \bigsqcup_{x\in X} x[y]$$

2. The combinator K exists, and if B is dense $K = \bigsqcup_{b\in B}\{b \Rightarrow (\perp\Rightarrow b)\}$

For any dense set B, we may calculate that:

$$
\begin{aligned}
(\bigsqcup_{b\in B}\{b \Rightarrow (\perp\Rightarrow b)\})xy &= (\bigsqcup_{b\in B}\{\perp\Rightarrow b \mid b \ll x\})y \text{ (by part 1.)} \\
&= \bigsqcup_{b\in B}\{b \ll x\} \\
&= x
\end{aligned}
$$

This also yields the existence of K as there is always at least one dense set, *viz* D.

3. The combinator S exists, and if B is dense,

$$S = \bigsqcup_{d,e,e',f \in B} \{(d \Rightarrow (e \Rightarrow f)) \Rightarrow ((d \Rightarrow e') \Rightarrow (d \Rightarrow f)) \mid e \ll e'\}$$

Lemma 4 *Suppose B is dense and that $z \ll x[y]$. Then there are $y', y'' \in B$ such that $y' \ll y'' \ll y$ and $(y' \Rightarrow z) \ll x$.*

Proof Note that:

$$\begin{aligned} x[y] &= \bigsqcup\{x' \in B \mid x' \ll x\}[\bigsqcup\{y' \in B \mid y' \ll y\}] \\ &= \bigsqcup\{x'[y'] \mid x', y' \in B, x' \ll x \text{ and } y' \ll y\} \text{ (by part 1)} \end{aligned}$$

This equation expresses $x[y]$ as the least upper bound of a directed set. So there are x', y' in B such that $x' \ll x, y' \ll y$ and $z \sqsubseteq x'[y']$. Then $(y' \Rightarrow z) \sqsubseteq x'$; for given t in D, if $(y' \Rightarrow z)[t] = z$ then $y' \ll t$ and so $(y' \Rightarrow z)[t] = z \sqsubseteq x'[y'] \sqsubseteq x'[t]$. Then $(y' \Rightarrow z) \sqsubseteq x'$ follows by axiom 4. So $(y' \Rightarrow z) \ll x$. The existence of a y'' in B satisfying $y' \ll y'' \ll y$ is guaranteed by the definition of density.

□

Now, as in the proof of part 2, we find that:

$$S'xyz = \bigsqcup_{d,e,e',f \in B} \{f \mid (d \Rightarrow (e \Rightarrow f)) \ll x \wedge (d \Rightarrow e') \ll y \wedge d \ll z \wedge e \ll e'\}$$

where

$$S' = \bigsqcup_{d,e,e',f \in B} \{(d \Rightarrow (e \Rightarrow f)) \Rightarrow ((d \Rightarrow e') \Rightarrow (d \Rightarrow f)) \mid e \ll e'\}$$

Suppose $(d \Rightarrow (e \Rightarrow f)) \ll x, (d \Rightarrow e') \ll y, d \ll z$ and $e \ll e'$. Then $(e \Rightarrow f) \sqsubseteq x[z], e' \sqsubseteq y[z]$ and so $f \sqsubseteq xz[yz]$. Therefore $S'xyz \sqsubseteq xz[yz]$.

Conversely suppose $f \ll xz[yz]$. By Lemma 4, there are e, e' in B such that $(e \Rightarrow f) \ll x[z]$ and $e \ll e' \ll y[z]$. By Lemma 4, we can now find d', d''

in B such that $(d' \Rightarrow (e \Rightarrow f)) \ll x$ and $d' \ll d'' \ll z$. Similarly there are d''', d'''' in B such that $(d''' \Rightarrow e') \ll y$ and $d''' \ll d'''' \ll z$. As $d'' \sqcup d''' \ll z$, $d'' \sqcup d''' \ll d \ll z$ for some d in B. Then $(d \Rightarrow (e \Rightarrow f)) \sqsubseteq (d' \Rightarrow (e \Rightarrow f)) \ll x, (d \Rightarrow e') \sqsubseteq (d''' \Rightarrow e') \ll y$, $d \ll z$ and $e \ll e'$ and so $f \ll S'xyz$. But now, as $xz[yz] = \bigsqcup\{f \in b \mid f \ll xz[yz]]\}$, $xz[yz] \sqsubseteq S'xyz$.

The existence of S follows as before.

4. If B is dense and D is algebraic, then:

$$S = \bigsqcup_{d,e,f \in B} \{(d \Rightarrow (e \Rightarrow f)) \Rightarrow ((d \Rightarrow e) \Rightarrow (d \Rightarrow f)) \mid e \ll e\}$$

Every isolated element in D is in B for if $d \ll d$ then $d \ll e \ll d$ for some e in B. So $b = d$. One can then, using the fact that D is algebraic, strengthen Lemma 4 so that $y' = y''$. The proof of 4 is then analogous to that of 3, but uses this stronger version of Lemma 4.

5. $\mid D \mid \geq 2^{\aleph_0}$

Lemma 5 *I is non-isolated.*

Proof As $(\bigsqcup_{d \in D}(d \Rightarrow d))[x] = \bigsqcup\{d \mid d \ll x\} = x$, given x in D, by part 1, it follows from axiom 4 that $I = \bigsqcup_{d \in D}(d \Rightarrow d)$. Then, if I were isolated, we would have $I = \bigsqcup_{d \in D_0}(d \Rightarrow d)$, for some finite subset, D_0 of D. Then $\mid D \mid = \mid \{I[d] \mid d \in D\} \mid$ would also be finite. But it is well known that no nontrivial (axiom 5) model of the λ-calculus can be finite.

□

Now, form a chain $d_0 \ll d_1 \ll \ldots \ll I$ such that $d_{i+1} \not\sqsubseteq d_i$, for all i, starting with $d_0 = \perp$. If we have defined $d_0, \ldots, d_n$, suppose, for the sake of contradiction, that $d_n \ll e \ll I$ implies $e \sqsubseteq d_n$, given e. Then

$$I = \bigsqcup\{e \mid e \ll I\} = \bigsqcup\{e \mid d_n \ll e \ll I\} = d_n$$

and so $I \ll I$, contradicting Lemma 5. So for some d_{n+1}, $d_n \ll d_{n+1} \ll I$, but $d_{n+1} \not\sqsubseteq d_n$. So such a chain exists.

With each $f : \boldsymbol{N} \to \boldsymbol{N}$ which is a strictly increasing function from the natural numbers to the natural numbers, and such that $f^0(0) = 0$, we associate a member x_f of D by:

$$x_f = \bigsqcup_{n \geq 0} (d_{f^n(0)} \Rightarrow d_{f^{n+1}(0)})$$

Suppose f, f' are two such functions such that $f(0) \geq 2$ and $f'(0) \geq 2$, and for some n, $f^n(0) \neq f'^n(0)$. We show that $x_f \neq x_{f'}$. Since one can find a set of $2^{\aleph_0}$ such functions such that any two different members satisfy these conditions on f and f', this will conclude the proof.

Let n_0 be the smallest integer such that $f^{(n_0+1)}(0) \neq f'^{(n_0+1)}(0)$. Then $f^{n_0}(0) = f'^{n_0}(0)$ (possibly $n_0 = 0$) and

$$\begin{array}{rcl} x_f[d_{f^{n_0}(0)+1}] & = & \bigsqcup_{0 \leq n \leq n_0} d_{f^{(n+1)}(0)} \\ & = & d_{f^{(n_0+1)}(0)} \\ & \neq & d_{f'^{(n_0+1)})(0)} \\ & = & x_{f'}[d_{f'^{n_0}(0)+1}] \\ & = & x_{f'}[d_{f^{n_0}(0)+1}] \end{array}$$

By extensionality $x_f \neq x_{f'}$.

6.Suppose T is a suitable second-order theory whose axioms are the formal counterparts of 1-5. Let T^* be the extension of T by the equational definitions of the combinators S and K. Then the ordering $\ll$ is definable in the theory T^* by means of a first-order formula involving only application (and the equality predicate).

First we need to show that a weaker version of Lemma 5 is provable in T^*.

Lemma 6 *The formula $\exists z.z \not\ll z$is provable in T^*.*

Proof We give an informal proof although it will be obvious that a more rigorous formulation is possible.

By the axiom of comprehension of sets, there is a least set X_0 such that:

$$K \in X_0 \wedge \forall x \in X_0(K[x] \sqcup K \in X_0)$$

There is an admissible rule of single induction for X_0:

$$\frac{\phi(K)\ ,\ \forall x \in X_0\,(\phi(x) \supset \phi(K[x] \sqcup K))}{\forall x \in X_0\,\phi(x)}$$

This is proven by a standard method. There is then derivable a rule of double induction:

$$\frac{\forall x \in X_0\,\phi(x,K)\ ,\forall y \in X_0\,\phi(K,y)\ ,(\forall x,y \in X_0\,\phi(x,y) \supset \phi(K[x] \sqcup x, K[y] \sqcup y))}{\forall x,y \in X_0\,\phi(x,y)}$$

where x, y are distinct variables.

One easily proves by single induction that $\forall x \in X_0\,(x \sqsupseteq K)$. Then by double induction, $\forall x, y \in X_0(x \sqsubseteq y \vee y \sqsubseteq x)$. As $K \in X_0$, one now sees that X_0 is directed. Now we show by single induction that $\forall x \in X_0\,(x \not\sqsubseteq x[\perp])$. If $K \sqsubseteq K[\perp]$ then $x = Kxy \sqsubseteq K \perp xy = \perp$, contradicting axiom 5. Suppose $x \not\sqsubseteq x[\perp]$ for a given x in X_0. If $K[x] \sqcup K \sqsubseteq (K[x] \sqcup K) \perp = x \sqcup K[\perp]$, then $x = Kx \perp \sqsubseteq (x \perp) \sqcup (K \perp\perp) = (x \perp)$, a contradiction.

We can now prove that $\bigsqcup X_0 \not\ll \bigsqcup X_0$. For otherwise $\bigsqcup X_0 \sqsubseteq x$ for some x in X_0, as X_0 is directed. Then as $(K[x] \sqcup K) \in X_0$, $(K[x] \sqcup K) \sqsubseteq \bigsqcup X_0 \sqsubseteq x$. So $K[x] \sqsubseteq x$ and $x = Kx \perp \sqsubseteq x \perp$, a contradiction

□

Next, let $x \sqsubseteq_z y$ be an abbreviation for: $\forall w\ (w[x] = z \supset w[y] = z)$

Lemma 7 *It is provable in T^* that:*

1. $x \sqsubseteq_\perp y$ *iff* $x \sqsupseteq y$.
2. $x \sqsubseteq_\top y$ *iff* $x \sqsubseteq y$.

3. If $z \neq \bot$ and $z \neq \top$ then $x \sqsubseteq_z y$ iff $x = y$

Proof We again proceed informally, using the fact that the existence, as a continuous function, of $x \Rightarrow y$ can be demonstrated in T^*.

1. Suppose $x \sqsubseteq_\bot y$ and $y' \ll y$. Then $(y' \Rightarrow \top)[y] = \top \neq \bot$. So $(y' \Rightarrow \top)[x] \neq \bot$ so $y' \ll x$. Therefore $y \sqsubseteq x$. Conversely, suppose $y \sqsubseteq x$ and $w[x] = \bot$. Then $\bot \sqsubseteq w[y] \sqsubseteq w[x] = \bot$.

2. Suppose $x \sqsubseteq_\top y$ and $x' \ll x$. Then $(x' \Rightarrow \top)[x] = \top = (x' \Rightarrow \top)[y]$. So $x' \ll y$ also. Conversely, suppose $x \sqsubseteq y$ and $w[x] = \top$, then $\top \sqsupseteq w[y] \sqsupseteq w[x] = \top$.

3. Suppose $z \neq \bot, \top$ and $x \sqsubseteq_z y$. If $x' \ll x$, $(x' \Rightarrow z)[x] = z$ and so $(x' \Rightarrow z)[y] = z$. But $z \neq \bot$. Therefore $x' \ll y$, showing that $x \sqsubseteq y$. Suppose $y' \ll y$. Then, $((y' \Rightarrow \top) \sqcup (\bot \Rightarrow z))[y] = \top \neq z$, by part 1. So $((y' \Rightarrow \top) \sqcup (\bot \Rightarrow z))[x] \neq z$. Therefore, by part 1, $y' \ll z$ and we conclude that $y \sqsubseteq x$. The converse is evident.

□

Let $x \ll_\top y$ be an abbreviation for:

$$\exists z\, (\forall w\, ((z[w] = \top \vee z[w] = \bot) \wedge (z[w] = \top \supset w \sqsupseteq x)) \wedge z[y] = \top)$$

Let $x \ll_\bot y$ be an abbreviation for:

$$\exists z\, (\forall w\, ((z[w] = \top \vee z[w] = \bot) \wedge (z[w] = \bot \supset w \sqsubseteq x)) \wedge z[y] = \bot)$$

Lemma 8 *It is provable in T^* that:*

1. $x \ll_\top y \equiv x \ll y$.

2. $x \ll_\bot y \equiv x \sqsupseteq y$.

Proof

1. It is provable in T^* that:

$$x \ll y \equiv \forall Y\, ((\mathrm{Directed}(Y) \wedge y = \bigsqcup Y) \supset \exists y' \in Y . x \sqsubseteq y')$$

where Directed(Y) is an obvious formula. Now suppose $x \ll_\top y$, Y is directed and $y = \bigsqcup Y$. Then some z exists as guaranteed by the condition $x \ll_\top y$. For this z, it is the case that $z[y] = z[\bigsqcup Y] = \bigsqcup_{y' \in Y} z[y']$. Now for every y' in Y, $z[y']$ is $\top$ or $\perp$. If it is $\perp$ for every such y', $z[y] =\perp$. So for some y' in Y, $z[y'] = \top$ and so $x \sqsubseteq y'$ by the properties of z. This shows that $x \ll_\top y$ implies $x \ll y$.

Conversely, suppose $x \ll y$ and take $z = (x \Rightarrow \top)$. Evidently $z[w]$ is $\top$ or $\perp$ for any w. Further, given w, $(z[w] = \top) \supset (w \gg x) \supset (w \sqsupseteq x)$. Finally, as $y \gg x$, $z[y] = \top$.

2. Evidently if $x \ll_\perp y$ then $x \sqsupseteq y$. Conversely, set:

$$z = \bigsqcup\{(x' \Rightarrow \top) \mid x' \not\ll x\}$$

It is obvious that $z[w]$ is $\top$ or $\perp$ for any w. Now, by part 1 we have that $z[w] = \bigsqcup\{\top \mid \exists x' \not\ll x\,(x' \ll w)\}$. So $z[w] =\perp$ iff whenever $x' \not\ll x$ then $x' \not\ll w$, that is iff $w \sqsubseteq x$. From this we also see that $z[y] =\perp$, since $y \gg x'$ implies $x \gg x'$.

□

We can now prove that in T^* the ordering $\ll$ is first-order definable in terms of application (and the equality predicate). Define Extreme(e), Is(e, x) and Strict(e, x, y) by:

$$\begin{aligned}
\text{Extreme}(e) &\equiv \exists x, y\,(x \sqsubseteq_e y \wedge x \neq y) \\
\text{Is}(e, x) &\equiv \forall y\,(y \sqsubseteq_e x) \\
\text{Strict}(e, x, y) &\equiv \exists z\,(\forall w(\text{Extreme}(z[w]) \wedge (\text{Is}(e, z[w]) \supseteq x \sqsubseteq_e w)) \\
&\qquad \wedge\ \text{Is}(e, z[y]))
\end{aligned}$$

Now, by Lemma 7, the three formulas $\text{Extreme}(e) \equiv (e =\perp \vee e = \top)$, and $\text{Is}(\perp, x) \equiv (x =\perp)$ and $\text{Is}(\top, x) \equiv (x = \top)$ are provable in T^*. So too, therefore, are $\text{Strict}(\top, x, y) \equiv x \ll_\top y$ and $\text{Strict}(\perp, x, y) \equiv x \ll_\perp y$. Now, by Lemma 6, we see that $\exists z \neg\, \text{Strict}(\top, z, z)$ is provable in T^*. So as $\forall z Strict(\perp, z, z)$ is also provable in T^*, we get that:

$$(x = T) \equiv \text{Extreme}(x) \wedge \exists z \neg \text{Strict}(x, z, z)$$

is provable in T^*.That is, $\top$ is first-order definable in T^* in terms of application alone. That $\ll$ is also so definable now follows from Lemma 8.1.

7. Let $CL + Ext$ be the standard first-order theory of combinatory logic with the principle of extensionality. Then T^* is not a conservative extension of $CL + Ext$. In particular,

$$\exists x, y \forall z\, ((x[z] = x) \wedge (y[z] = y) \wedge (x \neq y))$$

is provable in one but not the other.

It is easy to show that in T, for any z, $\perp [z] =\perp$, $\top[z] = \top$ and $\perp \neq \top$. Existential introduction gives the required theorem.

On the other hand, in [6], Barendregt has given a model of $CL + Ext$ possessing only one fixed point (that is, an element x such that $x[y] = x$, for all y in the model). So the given sentence cannot be a theorem of $CL + Ext$.

Part 2: Introduction

In Part 2 we discuss and expand on developments in the subject since 1972, keeping close to the topic of elementary constructions of models of the lambda-calculus. We begin with a discussion of models in set theories with an anti-foundation axiom. One is interested in solving the domain equation $D \cong [D \to D]$ up to equality. It would be interesting to look at other domain equations, and also to consider constructive set theories with an anti-foundation axiom, where it may be possible to find sets equal to their own function space. Next we present the non-extensional model T_C^* of Part 1, Scott's $\mathcal{P}(\omega)$ model, Engeler's model $\mathcal{P}(B_A)$, and variants; here, and throughout the paper, we emphasise the consideration of models in the categorical sense, mainly within the category of algebraic complete lattices.

Next we consider an idea of Longo's, and introduce Scott-Engeler algebras. Each such algebra gives rise to a model of the $\lambda\beta$-calculus. Both Scott and Engeler's models fit in this general framework; T_C^* does not. A wider class can be obtained from the Extended Applicative Type Structures, by means of the filter model construction of Coppo, Dezani-Ciancaglini, Honsell and Longo; EATSs arose originally in connections with the intersection type discipline, studied in a long series of papers by Coppo and his co-workers. It turns out that all models of the $\lambda\beta\eta$-calculus in the category of algebraic complete lattices can be obtained from these structures. Thus they yield a representation theory for such models. However we lack a similar theory for models of the $\lambda\beta$-calculus; for example the model considered by Scott in [52] is not obtainable as such a filter model.

There are various ways to obtain explicit constructions of models of the $\lambda\beta\eta$-calculus. One is to construct the EATS as a free algebra and then show that it satisfies conditions, presented below, for the filter construction to yield a model. It turns out that the natural deduction method used in Part 2 can be seen as a way of presenting and analysing such free algebras; for any of a wide class of equations one always obtains a non-trivial model of the $\lambda\beta\eta$-calculus. Another—and previous—method is Scott's well-known D_∞ construction. Here one constructs a model starting from an embedding of any complete lattice (or even cpo) in its own function space and then taking the inverse limit of a derived chain of higher type function spaces.

We show that all the models obtainable from the above class of equations can be constructed in this way too, starting from a simply presented initial algebraic complete lattice and embedding.

Unfortunately, we do not have a converse showing how from a given initial lattice and embedding to obtain a corresponding set of equations. Coppo, Dezani-Ciancaglini, Honsell and Longo [14, 16] have shown how to find a set of equations, but they do not fall within the class considered here; it would be interesting to find a more inclusive class without this defect. We also consider a construction of Scott, starting from a certain kind of model of the $\lambda\beta$-calculus and then applying a closure operation to obtain an extensional model. Theorem 1 of Part 1 concerns general properties of any continuous lattice which forms a model of the $\lambda\beta\eta$-calculus. The paper concludes by considering generalisations to complete partial orders (cpos) obtaining particularly that for cpo models of the $\lambda\beta$-calculus, the partial order is first-order definable in terms of application alone, and provably so in a suitable second-order theory. (In this paper, complete partial orders are taken to be partial orders with a least element and lubs of all directed sets.) Interesting completeness problems arise: what equations hold between terms in all models of the $\lambda\beta$-calculus (or the $\lambda\beta\eta$-calculus) in, say, the category of cpos and continuous functions? Such problems were also considered in [30].

As remarked in the Preface, we do not consider the application of the models or the analysis of their properties. For example, Scott used his $\mathcal{P}(\omega)$ model [51] as the basis for a very concrete exposition of his programme of computation theory. Again, much of the work of Coppo *et al* concerns the connections with type theory, especially the theory of intersection types (see *e.g.* [12] and the references contained therein). The connection plays a double rôle, yielding both a greater understanding of type theory and a tool for analysing filter models. Even with all the omissions, some selection has still remained necessary. For example we have concentrated on models in the categorical sense; sometimes weaker structures arise: λ-algebras or combinatory algebras.

There are a number of directions for possible further research. An evident one is to generalise to a wider class of cpos, such as Scott domains. Some work of this kind can be found in [15] using Scott's information systems [54] as a more general version of EATSs; see also [12]. Presumably models such as T^ω [42, 11] would fit into this framework. In a more esoteric direction, one can consider weaker notions of continuity, such as preservation only of ω_1-lubs. An extreme possibility was suggested by Scott [53]; he considered taking the class of all sets as a model, and gave versions of T^*_C and Engeler's model employing "set-sized" continuity.

Next, one might consider models of related untyped λ-calculi. For example, the call-by-name and call-by-value (or partial) λ-calculi [1, 38] correspond to slightly different domain equations. For the first, one works with the equation $D \cong [D \to D]_\perp$ where $(\cdot)_\perp$ is the lifting construction, which adds a new least element. For the second, one works with the equation $D \cong [D \to_\perp D]_\perp$ where $[\cdot \to_\perp \cdot]$ is the strict function space construction. These will result in slightly amended notions of EATS and filter models. See [8, 31, 18] for work in this direction for the call-by-value λ-calculus. More broadly still, one might look for a notion of EATS or information system corresponding to any given domain equation. By contrast, in [2] it is shown how to construct particular information systems, the ones corresponding to their "standard" solution (by an inverse limit construction). The idea of the present work is rather to consider a class of such systems. That is, one is studying all structures which are solutions to domain equations rather than a single standard one. This is an interesting enterprise for the untyped λ-calculus where it was an achievement to obtain even one (non-syntactical) model. The mathematical interest in studying a wide class of models extends to other domain equations; it is less clear that the computational interest does.

One can also consider models of "substructural" λ-calculi, such as the λI-calculus or the linear or affine λ-calculi; see [32] for work in this direction. These possibilities lead one beyond "traditional" domain theory. Now one might think of investigating Girard's qualitative models [26] or, more generally, models in categories of stable functions [9]; for work of this kind on the qualitative models see [29]. Interesting analogues of the graph-theoretic models arise in the quantitative models of Girard [27, 39]. It seems very likely from the work in [35, 17, 31] that an understanding of generalised graph-theoretic models, EATSs, and so on, can be gained in the context of (categorical models of) linear logic.

The book by Aczel [3] can be consulted for historical and mathematical information on set theories with an anti-foundation axiom. For accounts, in great depth, of continuous lattices and of the more general continuous partial orders see [25, 33]; both books contain extensive historical discussion. For accounts of constructions of models of the λ-calculus, whether set-theoretic or by inverse limits see [6], and for more on inverse limits, *e.g.* for finding solutions of domain equations, see [43, 25].

2 Non-Well-Founded Models

We consider *applicative structures* $< X, \cdot >$ where $\cdot$ is a binary function over X; this function is called *application* and written using infix notation. Such a structure is *extensional* iff

$$\forall f, g\ (\forall x\ f{\cdot}x = g{\cdot}x) \supset f = g$$

It is *functional* if X is a set of functions from X to X and application is the usual set-theoretic application, that is for all f, x in X: $f{\cdot}x = f(x)$. As long as we are using well-founded set theory, there are no functional applicative structures. Boffa has remarked (see [6, p495]) that with the weak axiom BA_1 [3, p51] every extensional applicative structure is isomorphic to a functional one. Now, the usual D_∞ construction [6, p 477] can be carried out without using the axiom of foundation, yielding an isomorphism of cpos,

$$\phi : D \cong [D \to D]$$

(where, in general, $[D \to E]$ is the cpo of all continuous functions from D to E). So, applying Boffa's remark to $< D_\infty, \cdot >$, where $x \cdot y = \phi(x)(y)$, we obtain an isomorphic functional applicative structure $< E, \cdot >$. Then by transferring the partial order structure of D along the isomorphism, E becomes a cpo isomorphic to D such that

$$E = [E \to E]$$

The situation changes according to the anti-foundation axiom considered. Say that an applicative structure is *rigid* if it has only one automorphism, the identity. Aczel has remarked that with Finsler's axiom [3, p46] one has rather that any rigid extensional applicative structure is isomorphic to a functional one. It was—essentially—noted by Scott [47] that the continuous automorphisms of $< D_\infty, \cdot >$ are in 1-1 correspondence with the order-theoretic automorphisms of D_0 (a proof can be based on Exercise 18.4.18 of [6]). But by the extension of Theorem 1.6 of Part 2 given below, every automorphism of D_∞ is continuous. So D_∞ is rigid iff the only automorphism of the partial-order D_0 is the identity (that is, D_0 is rigid in the order-theoretic sense). There are many such cpos; the simplest non-trivial one is the two element complete lattice.

Let us now change the anti-foundation axiom under consideration to AFA. This was originally introduced under the name of X_1 by Forti and Honsell in [21], and is the main axiom considered by Aczel in [3]. It then turns out that the only functional structures are trivial, being either the empty one, or the one whose only element is x where:

$$x = \{< x, x >\}$$

For if $< X, \cdot >$ is a non-empty functional applicative structure, then any two of its elements are bisimilar. It would be very interesting to know what happens with other domain equations. For example, using AFA, is there a unique cpo D such that:

$$D = [D \to D] + \mathbf{N}$$

where $\mathbf{N}$ is the "flat" cpo of the natural numbers?

In another direction, Aczel has suggested considering non-well-founded *intuitionistic* set theories [37, 22]. It is then possible for *all* functions from one cpo to another to be continuous. One would conjecture, for example, that there were non-trivial solutions to the equation

$$X = X \to X$$

if such a theory were based on Boffa's or Finsler's axioms, "defeating" the difficulties caused by the foundation axiom and cardinality considerations.

3 The Models of Plotkin, Scott and Engeler

It was stated above that the applicative structure $< T_C^*, \cdot[\cdot] >$ is a model for the λ-calculus without extensionality. However no interpretation of the calculus was given modeling β-conversion. Since [41] was written, definitions have been given of the notion of a model of the $\lambda\beta$-calculus; one of these—syntactical λ-models [6, p.101]—incorporates the idea of interpreting the calculus (see [6, Chapter 5]).

We will also be interested in the categorical organisation of the our models. Say that a *λ-structure* in a cartesian closed category $\mathbf{K}$ is a triple $< D, \lambda, \phi >$ where D is an object in the category and $\lambda : D^D \to D$ and $\phi : D \to D^D$ are morphisms. It is a *model of the $\lambda\beta$-calculus* in $\mathbf{K}$ if $\phi \circ \lambda = id_{D^D}$; if, in addition, $\lambda \circ \phi = id_D$ then it is a *model of the $\lambda\beta\eta$-calculus* in $\mathbf{K}$. If $< D, \lambda, \phi >$ is such a model of the $\lambda\beta$-calculus then, in the terminology of [6], D is a reflexive object; also, to each such model there is canonically associated a syntactical λ-model. Note that giving a morphism $\phi : D \to D^D$ is equivalent to giving an *applicative structure* in the category, by which we mean a structure $< D, \cdot >$ where $\cdot : D \times D \to D$. We will mainly consider the category, **ALG**, the cartesian closed category of algebraic complete lattices and continuous functions.

Let us consider applicative structures of the form $< T_C^*, \cdot[\cdot] >$, but starting with an arbitrary set of "atoms", A, instead of just one, ι; this possibility was already noted above. Let T_A be the least set such that:

$$T_A = A \cup (\mathcal{P}_f(T_A) \times \mathcal{P}_f(T_A))$$

Define application on $\mathcal{P}(T_A)$ by:

$$x \cdot y = \bigcup \{\nu \mid \exists \mu \subseteq y. \mu \to \nu \in x\}$$

where $\mu \to \nu$ denotes $< \mu, \nu >$. Then, as before, application is completely additive in its first argument and continuous in its second. Also as before, every continuous function f is comprehended by $\hat{f}$ where:

$$\hat{f} = \{\mu \to \nu \mid \nu \subseteq f(\mu)\}$$

Since application is continuous in both arguments we get a continuous map:

$$\phi : \mathcal{P}(T_A) \to [\mathcal{P}(T_A) \to \mathcal{P}(T_A)]$$

Since, as one easily verifies, the passage $f \mapsto \hat{f}$ is continuous, we also get a continuous *abstraction* map:

$$\lambda : [\mathcal{P}(T_A) \to \mathcal{P}(T_A)] \to \mathcal{P}(T_A)$$

where $\lambda(f) = \hat{f}$. As $\hat{f}$ comprehends f, it follows that $\phi \circ \lambda = id_{\mathcal{P}(T_A)}$. That is, $< \mathcal{P}(T_A), \lambda, \phi >$ is a model of the λ-calculus in **ALG**.

The applicative structure $< \mathcal{P}(T_A), \cdot >$ can be made into a syntactical λ-model by defining a syntactical interpretation [6, p.101]. The definition is by induction on the structure of λ-terms, augmented with constants for elements of $\mathcal{P}(T_A)$. It yields the syntactical λ-model associated to $< \mathcal{P}(T_A), \lambda, \phi >$. All cases other than abstraction are determined by the applicative structure; for abstraction one has:

$$[\![\lambda x.M]\!]_\rho = \{\mu \to \nu \mid \nu \subseteq [\![M]\!]_{\rho(x:=\mu)}\}$$

Very similar models were given by Scott and Engeler. Scott [51] considered an applicative structure on $\mathcal{P}(\omega)$, with application given by:

$$x \cdot y = \{m \mid \exists e_n \subseteq y.(n,m) \in x\}$$

where (n,m) is a standard enumeration of all pairs of integers, and e_n is a standard enumeration of all finite subsets of integers. For abstraction one puts:

$$\lambda(f) = \{(n,m) \mid m \in f(e_n)\}$$

For the associated syntactical λ-model the clause for abstraction is:

$$[\![\lambda x.M]\!]_\rho = \{(n,m) \mid m \in [\![M]\!]_{\rho(x:=e_n)}\}$$

A variant is provided by the use of non-standard pairings; this possibility was used to good effect by Baeten and Boerboom in [5].

Engeler [19, 20] considered B_A, the least set such that:

$$B_A = A \cup (\mathcal{P}_f(B_A) \times B_A)$$

He obtains an applicative structure on $\mathcal{P}(B_A)$ by putting:

$$x \cdot y = \{b \mid \exists \mu \subseteq y.\mu \to b \in x\}$$

where $\mu \to b$ denotes $< \mu, b >$. For abstraction one puts:

$$\lambda(f) = \{\mu \to b \mid b \in f(\mu)\}$$

and we again obtain a reflexive object in **ALG**. For the syntactical λ-model, the clause for abstraction is:

$$[\![\lambda x.M]\!]_\rho = \{\mu \to b \mid b \in [\![M]\!]_{\rho(x:=\mu)}\}$$

Finally, in [52], Scott showed that models could be based on finite sequences, rather than on sets, and gave a construction which provides a model of the λ-calculus given any set S such that $S^+ \subset S$. (Actually, he also assumed that ϵ was in S, but—as he remarked—this is not necessary.) Here is an example, presented rather in the style of Engeler's model. Let S_A be the least set such that:

$$S_A = A \cup (S_A^* \times S_A)$$

Application is given by:

$$x{\cdot}y = \{b \mid \exists n\, \exists a_1, \ldots, a_n \in y. << a_1, \ldots, a_n >, b >\in x\}$$

and abstraction by:

$$\lambda(f) = \{<< a_1, \ldots, a_n >, b >\mid b \in f(\{a_1, \ldots, a_n\})\}$$

One interest of this model is that it does not fall within the ambit of either of the general frameworks we consider below, Scott-Engeler algebras or EATSs.

In the following two sections, proofs—or at least indications of proofs—will be given, that the only isomorphisms that hold between all these models are the evident ones, arising from two sets of atoms of the same cardinality or in case the models are trivial. Scott pointed out that idea behind these models is already implicit in the work on enumeration operators in the recursion-theoretic literature (see, for example, [46, Chapter 9.7]). For example, to each integer n is associated an enumeration operator Φ_n, where for any y in $\mathcal{P}(\omega)$:

$$\Phi_n(y) = W_n{\cdot}y$$

Here W_n is the standard enumeration of r.e. sets. That is the definition of *application* was already (essentially) known. Note, by the way, that it is straightforward to turn the collection of r.e. sets into a model of the $\lambda\beta$-calculus: one simply interprets the above definition of application and interpretation of abstraction in the context of the r.e. sets.

4 Scott-Engeler Algebras

One can treat models such as $\mathcal{P}(\omega)$ and $\mathcal{P}(B_A)$ systematically by following an idea originating with Longo [36] and developed further by others [34, 55, 31]. Say that a *Scott-Engeler* algebra (SE-algebra) is a structure $< X, \rightarrow >$ where:

$$\rightarrow : \mathcal{P}_f(X) \times X \rightarrow X$$

Longo considered *extensional* SE-algebras, where $< X, \rightarrow >$ is extensional iff for all μ,ν in $\mathcal{P}_f(X)$ and x,y in X:

$$(\mu \rightarrow x) = (\nu \rightarrow y) \text{ implies } \mu = \nu \text{ and } x = y$$

Note the use of infix notation $(\mu \rightarrow x)$ here. Krivine [34] considered a more general concept of *ordered* SE-algebras (but not using the terminology); we will not develop their theory here, except implicitly *via* our treatment of EATSs in the following sections.

SE-algebras can be presented as algebras in the usual sense of universal algebra. One considers structures $< X, \rightarrow_n (n \geq 0) >$ where the functions $\rightarrow_n : X^{n+1} \rightarrow X$ are subject to evident axioms. The induced notion of morphism $h :< X, \rightarrow_X > \rightarrow < Y, \rightarrow_Y >$ is that of a function $h : X \rightarrow Y$ such that $h(\mu \rightarrow_X a) = h(\mu) \rightarrow_Y h(a)$. Equipped with the evident $\rightarrow$, B_A is the free SE-algebra over A. Another example is provided by the $\mathcal{P}(\omega)$ model: ω can be made into a SE-algebra by putting $(e_n \rightarrow m) = (n, m)$; replacing the pairing function by an arbitrary recursive function, one may obtain any effective SE-algebra over ω; the extensional ones are those where the function is 1-1.

Say that a Scott-Engeler algebra is *set-theoretic* if the "step function" $\rightarrow$ is pairing. In well-founded set theory the set-theoretic algebras are just the B_A. Other natural examples arise from the perspective of non-well-founded set theory. Working with the axiom AFA one can consider the maximal set B'_A such that

$$B'_A = A \cup (\mathcal{P}_f(B'_A) \times B'_A)$$

Now one has available the sub-algebras of the B'_A; these possibilities will, however, not be considered further.

Let $< X, \rightarrow >$ be a SE-algebra. Then $\mathcal{P}(X)$ can be made into an applicative structure by defining:

$$x{\cdot}y = \{b \mid \exists \mu \subseteq y.(\mu \rightarrow b) \in x\}$$

There are two obvious possible abstraction functions:

$$\lambda(f) = \{(\mu \rightarrow b) \mid b \in f(\mu)\}$$

and:

$$\lambda^+(f) = \{(\mu \rightarrow b) \mid b \in f(\mu)\} \cup \mathrm{At}_X$$

Here At_X is the set of atoms of $< X, \rightarrow >$: an element of an SE-algebra is said to be *atomic* iff it is not of the form $(\mu \rightarrow z)$; otherwise it is said to be *functional*.

Other than the necessity of extensionality, all of the following theorem can be found in [36]:

Theorem 2 *1. $< \mathcal{P}(X), \cdot >$ can be made into a model of the $\lambda\beta$-calculus iff $< X, \rightarrow >$ is extensional. In that case λ' is an abstraction function iff $\lambda \leq \lambda' \leq \lambda^+$.*

2. $< \mathcal{P}(X), \cdot >$ can be made into a model of the $\lambda\beta$-calculus in exactly one way iff $< X, \rightarrow >$ is extensional and X contains no atoms.

Proof

1. Suppose that λ' is an abstraction function for the applicative structure $< \mathcal{P}(X), \cdot >$. Then:

   (i) If $(\mu \rightarrow a) \in \lambda'(f)$ then $a \in f(\mu)$.

   (ii) If $a \in f(\mu)$ then for some $\nu \subset \mu$, $(\nu \rightarrow a) \in \lambda'(f)$.

   (iii) $(\mu \rightarrow a) \in \lambda'(\mu \Rightarrow \{a\})$

   For part (i), if $(\mu \rightarrow a) \in \lambda'(f)$, then $a \in \lambda'(f){\cdot}\mu = f(\mu)$. For part (ii), if $a \in f(\mu)$, then $a \in \lambda'(f){\cdot}\mu$ and so, by the definition of application, for some $\nu \subset \mu$, $(\nu \rightarrow a) \in \lambda'(f)$. For part (iii) as $a \in (\mu \Rightarrow \{a\})(\mu)$

we get by part (ii) that for some $\nu \subset \mu$, $(\nu \to a) \in \lambda'(\mu \Rightarrow \{a\})$. So by part (i), we have that $a \in (\mu \Rightarrow \{a\})(\nu)$ and so that $\mu \subset \nu$. Therefore as we already know that $\nu \subset \mu$ and $(\nu \to a) \in \lambda'(\mu \Rightarrow \{a\})$, part (iii) follows.

We can now prove extensionality. Suppose that $(\mu \to a) = (\nu \to b)$. By part (iii) we then have that $(\nu \to b) \in \lambda'(\mu \Rightarrow a)$. So by part (i) we have that $b \in (\mu \Rightarrow \{a\})(\nu)$. Therefore $\mu \subset \nu$ and $a = b$; by symmetry we also have that $\nu \subset \mu$.

The statement that $\lambda \leq \lambda' \leq \lambda^+$ is equivalent to the statement:

$$(\mu \to a) \in \lambda'(f) \text{ iff } a \in f(\mu)$$

The implication from left to right is just part (i) above. In the other direction, if $a \in f(\mu)$ then $(\mu \Rightarrow \{a\}) \leq f$ and so by part (iii) and the monotonicity of λ', $(\mu \to a) \in \lambda'(f)$.

Conversely, it is easy to check that if $\lambda \leq \lambda' \leq \lambda^+$ then λ' is an abstraction function.

2. Immediate from part 1.

□

The minimal abstraction function seems somehow the most natural choice. Let us try to make the choice functorial. First we define a category of λ-structures in a cartesian closed category. A morphism from one such structure $< D, \lambda_D, \phi_D >$ to another $< E, \lambda_E, \phi_E >$ is a pair $< f, g >$ where $f : D \to E$ and $g : E \to D$ and the following diagrams commute:

$$\begin{tikzcd} D^D \arrow[r, "\lambda_D"] \arrow[d, "f^g"'] & D \arrow[d, "f"] \\ E^E \arrow[r, "\lambda_E"'] & E \end{tikzcd} \qquad \begin{tikzcd} D^D & D \arrow[l, "\phi_D"'] \\ E^E \arrow[u, "g^f"] & E \arrow[l, "\phi_E"] \arrow[u, "g"'] \end{tikzcd}$$

Composition is defined by: $< f', g' > \circ < f, g > = < f' \circ f, g \circ g' >$, and there is an evident identity. The idea of this definition is to regard λ-structures as *dialgebras* in the sense of Freyd [23]

Now we can set:

$$\mathcal{P}_\lambda(< X, \rightarrow >) = < \mathcal{P}(X), \lambda, \phi >$$

on objects and

$$\mathcal{P}_\lambda(h) = < h_*, h^{-1} >$$

on morphisms $h :< X, \rightarrow > \rightarrow < Y, \rightarrow >$ (where $h_*(x) = \{h(a) \mid a \in x\}$ and $h^{-1}(y) = \{a \in X \mid h(a) \in y\}$). Note that h_* is left adjoint to h^{-1}. Unfortunately, this does not define a morphism. The difficulty is that while the second diagram does commute, the first does not. In general one only has the inclusion:

$$h_*(\lambda_{\mathcal{P}_\lambda(X)}(f)) \leq \lambda_{\mathcal{P}_\lambda(Y)}(h_* \circ f \circ h^{-1})$$

for the left-hand-side is $\{h(\mu) \rightarrow h(b) \mid b \in f(\mu)\}$, whereas the right-hand-side is $\{\mu' \rightarrow h(b) \mid \exists \mu.\, b \in f(\mu) \wedge h(\mu) \subset \mu'\}$. Oddly, if one restricts the morphisms one can obtain a *contravariant* functor by putting instead:

$$\mathcal{P}_\lambda(h) = < h^{-1}, h_* >$$

Say that h is *strong* iff whenever $h(x) = \mu' \rightarrow b'$ then:

1. There are μ, b such that $x = \mu \rightarrow b$, $h(\mu) \supset \mu'$ and $h(b) = b'$
2. There are μ, b such that $x = \mu \rightarrow b$, $h(\mu) \subset \mu'$ and $h(b) = b'$

Then one can show that $\mathcal{P}_\lambda(h)$ is a morphism iff h is strong, obtaining a contravariant functor on the subcategory of strong morphisms.

Finally, we consider when two models of the $\lambda\beta$-calculus obtained from SE-algebras are isomorphic. It turns out that all the obvious notions of isomorphism coincide. The next theorem was (essentially) proved by Schellinx in [55]. His proof used the special case of theorem 7 (see below) of models of the $\lambda\beta$-calculus constructed from extensional SE-algebras; this result was proved by Bethke.

Theorem 3 *Let X and Y be extensional SE-algebras. Then the following are equivalent:*

1. *The λ-structures $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are isomorphic.*
2. *The applicative structures $< \mathcal{P}(X), \cdot >$ and $< \mathcal{P}(Y), \cdot >$ are isomorphic (either in* **ALG** *or in* **SET***).*
3. *X and Y are isomorphic.*

Proof Since X and Y are extensional, the λ-structures $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are models of the $\lambda\beta$-calculus. So by theorem 7 any isomorphism of the applicative structures $< \mathcal{P}(X), \cdot >$ and $< \mathcal{P}(Y), \cdot >$ is continuous. The only other non-trivial implication is that 2 implies 3. So, suppose that $g :< \mathcal{P}(X), \cdot > \cong < \mathcal{P}(Y), \cdot >$ is a continuous isomorphism. Then g is a continuous isomorphism of the complete lattices $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. So there is a bijection $h : X \cong Y$ such that for any subset u of X, $g(u) = \{h(x) \mid x \in u\}$. Now calculate for any finite $\mu \subset X$, finite $\nu \subset Y$ and x in X that:

$$\begin{aligned}
\{h(\mu \to x)\} \cdot \nu &= g(\{\mu \to x\}) \cdot g(h^{-1}(\nu)) \\
&= g(\{\mu \to x\} \cdot h^{-1}(\nu)) \\
&\quad \text{(as } g \text{ is a morphism of applicative structures)} \\
&= g(\{x \mid \mu \subset h^{-1}(\nu)\}) \\
&= \{h(x) \mid h(\mu) \subset \nu\} \\
&= \{h(\mu) \to h(x)\} \cdot \nu
\end{aligned}$$

But then, as Y is extensional we have that $h(\mu \to x) = h(\mu) \to h(x)$, showing that h is a homomorphism.

□

So, if two such models are isomorphic, the corresponding EATSs must have the same number of atoms. So, as remarked by Schellinx, two applicative structures $< \mathcal{P}(B_A), \cdot >$ and $< \mathcal{P}(B_{A'}), \cdot >$ are isomorphic iff A and A' have the same cardinality. Further, no $< \mathcal{P}(B_A), \cdot >$ is isomorphic to

$< \mathcal{P}\omega, \cdot >$. For then by Theorem 3 we would have B_A and $< \omega, \rightarrow >$ isomorphic, which cannot be as the latter has no atoms. This result was first proved by Longo [36], by a different method. By analogous arguments to that used in the above proof, one can show that $< \mathcal{P}(T_A), \cdot >$ and $< \mathcal{P}(T_{A'}), \cdot >$ are isomorphic iff A and A' have the same cardinality, and similarly for the S_A. We also have that no $< \mathcal{P}(T_A), \cdot >$ is isomorphic to any $< \mathcal{P}(X), \cdot >$ (where X is a SE-algebra) as there is always an atom u in any $\mathcal{P}(T_A)$ (in the order-theoretic sense) which on application can yield a non-bottom non-atom. For further work, *e.g.* on elementary equivalence or embeddings, see [36, 13, 55].

5 Extended Applicative Type Structures

A yet more general approach is available following ideas of Coppo, Dezani-Ciancaglini, Honsell and Longo [13]. The "arrow" functions of Scott-Engeler algebras have two aspects: they combine groups, by forming finite sets, and they also form "step-functions." Separating these two aspects, and abstracting on the formation of finite sets we can consider a semilattice with a binary function.

Say that an *EATS (Extended Applicative Type Structure)* is a structure $< X, \wedge, \top, \rightarrow >$ where $< X, \wedge, \top >$ is an lower semi-lattice and $\rightarrow$ is a binary function on X; we will use the evident infix notation $(a \rightarrow b)$. This definition generalises that in [13]; there the following additional conditions were imposed linking $\rightarrow$ and the partial order:

C1 If $a' \leq a$ and $b \leq b'$ then $(a \rightarrow b) \leq (a' \rightarrow b')$

C2 $\top \leq (\top \rightarrow \top)$

C3 $(a \rightarrow b) \wedge (a \rightarrow c) \leq a \rightarrow (b \wedge c)$

These conditions can be amalgamated into the single condition:

C If $a \leq \bigwedge_{j \in J} a_j$ and $\bigwedge_{j \in J} b_j \leq b$ then $\bigwedge_{j \in J}(a_j \rightarrow b_j) \leq (a \rightarrow b)$

(where J is a finite set). Condition C1 can be rewritten as the inequation:

C1′ $a \to (b \wedge b') \leq (a \wedge a') \to b$

and the last two conditions can be strengthened to:

C2′ $\top = (x \to \top)$

C3′ $(a \to b) \wedge (a \to c) = a \to (b \wedge c)$

which are equivalent to C2 and C3, respectively, in the presence of C1.

EATSs form a variety of universal algebras and so we have available a standard notion of morphism: a function is a morphism iff it preserves finite meets and the $\to$ operation. The set T_A considered above yields an EATS with carrier $\mathcal{P}_f(T_A)$; the semilattice operations are taken to be set theoretic unions and the $\to$ operation is taken to be pairing. This EATS is the free EATS over A.

Every Scott-Engeler algebra $< X, \to >$ yields an EATS with carrier $\mathcal{P}_f(X)$. The semilattice operations are again taken to be set theoretic unions; the $\to$ operation is given by:

$$(\mu \to \nu) = \{\mu \to b \mid b \in \nu\}$$

The construction yields a functor $\mathcal{E}$ whose action on morphisms is given by $\mathcal{E}(h)(\mu) = \{h(a) \mid a \in \mu\}$; it is left adjoint to an evident forgetful functor. As an example, $\mathcal{E}(B_A)$ is the free EATS over A satisfying C2′ and C3′.

To each EATS $< X, \wedge, \top, \to >$ we will associate a λ-structure, $\mathcal{F}_\lambda(X)$. First, though, we consider—see also [25]—how to associate to any semilattice $< X, \wedge, \top >$ an algebraic complete lattice $\mathcal{F}(X)$. It consists of all filters over $< X, \top, \wedge >$, partially ordered by subset, where a *filter* over $< X, \top, \wedge >$ is a subset of X closed under finite meets and also closed upwards in the partial order. We write x^* for the least filter containing x; it is given by:

$$x^* = \{a \mid \text{there is a finite set } J \text{ and } a_j \text{ in } x \ (j \in J) \text{ such that } \bigwedge_{j \in J} a_j \leq a\}$$

Least upper bounds in $\mathcal{F}(X)$ are given by the formula:

$$\bigvee(A) = (\bigcup A)^*$$

and when A is directed

$$\bigvee(A) = \bigcup A$$

as directed unions of filters are filters. The finite elements of $\mathcal{F}(X)$ have the form

$$a{\uparrow} = \{b \in X \mid a \leq b\}$$

Note that $a{\uparrow} \leq b{\uparrow}$ iff $b \leq a$. This representation of algebraic complete lattices by lower semilattices is *complete* in that any one can be so represented, up to isomorphism. Given an algebraic complete lattice D, construct the semilattice $< B, \wedge, \top >$, taking B to be the set of finite elements of D, $(a \wedge b) = (a \vee_D b)$ and $\top = \bot_D$. There is an isomorphism $\theta : D \to \mathcal{F}(B)$ where $\theta(x) = \{a \in B \mid a \leq x\}$ and $\theta^{-1}(y) = \bigvee_D y$.

Now we set $\mathcal{F}_\lambda(X) =< \mathcal{F}(X), \lambda, \phi >$ where the functions λ and ϕ are defined by:

$$\lambda(f) = \{a \to b \mid b \in f(a{\uparrow})\}^*$$

and

$$\phi(x) = \bigvee\{a{\uparrow} \Rightarrow b{\uparrow} \mid (a \to b) \in x\}$$

The associated application is given by:

$$x{\cdot}y = \{b \mid \exists a \in y.(a \to b) \in x\}^*$$

Note that, unlike the case of SE-algebras, we are just considering one abstraction function; it would be interesting to investigate the range of possible abstraction functions.

We have that $\phi \circ \lambda \geq id_{[\mathcal{F}(X) \to \mathcal{F}(X)]}$. For if $b \in f(x)$ then for some a in x, $b \in f(a{\uparrow})$. But then $(a \to b) \in \lambda(f)$ and so $b \in \lambda(f){\cdot}x$. When Condition C holds, application is given by the simpler formula:

$$x{\cdot}y = \{b \mid \exists a \in y.(a \to b) \in x\}$$

and we also have that:

$$x{\cdot}(a{\uparrow}) = \{b \mid (a \to b) \in x\}$$

To see the first of these, suppose b is in the left hand side. Then there is a finite set J and a_j,b_j ($j \in J$) such that $\bigwedge_{j \in J} b_j \leq b$ and $a_j \in y$ and $(a_j \to b_j) \in x$. Setting $a = \bigwedge_{j \in J} a_j$, we get that $a \in y$ and, by Condition C, that $(a \to b) \in x$; so b is in the right hand side. For the second of these, suppose b is in the left hand side. Then, from what we have just seen, there is an a' in $a\uparrow$ such that $(a' \to b) \in x$. But then $a \leq a'$ and so by Condition C, $(a \to b) \in x$.

As may be expected, translating Scott-Engeler algebras to EATSs and then obtaining λ-structures yields isomorphic results to obtaining the structures directly. One has an isomorphism $\theta : \mathcal{P}_\lambda(X) \cong \mathcal{F}_\lambda(\mathcal{E}(X))$ where $\theta(x) = \mathcal{P}_f(x)$ and $\theta^{-1}(y) = \bigcup y$.

In the work of Cardone and Coppo [12] a slightly different approach is taken to the description of filter models which are complete algebraic lattices. They consider an inequational theory of type schemes closed under rules corresponding to the semi-lattice conditions and Condition C, and form the model as a collection of filters of type schemes. Starting from a given collection of inequations between type schemes one can take the least such theory, and form the filter model. For example, from the empty set one obtains the original BCD model [10]. It should be noted that Cardone and Coppo actually consider a more general scheme, constructing filter models which are Scott domains.

This construction of a filter model amounts to the same thing as first forming an EATS from the equivalence classes of the type schemes and then taking the filter λ-structure as given above. The process of forming the least theory can also be described in standard universal algebraic terms. One regards type schemes over a set of type parameters A as terms in the signature with ω and the elements of A as constants and binary function symbols for intersection and arrow. Then one adds equational axioms for: the semi-lattice structure; the given inequations (writing $t = t \wedge u$ for $t \leq u$); and Condition C. These last can be taken as the evident transcriptions of C1$'$, C2$'$ and C3$'$. Lastly one obtains the required filter model as the filter model of the initial algebra satisfying all the equations. From this point of view the BCD model appears as the filter model formed from the free EATS over A that satisfies Condition C. An explicit description of this EATS is given below.

The construction yields a model of the $\lambda\beta$-calculus in **ALG** if a certain converse form of Condition C holds:

B If $\bigwedge_{j\in J}(a_j \to b_j) \leq (a \to b)$ then $a \leq \bigwedge_{i\in I} a_i$ and $\bigwedge_{i\in I} b_i \leq b$ for some finite subset I of J,

(where J is a finite set); the consequent can be equivalently written in the form: $\bigwedge\{b_j \mid a \leq a_j\} \leq b$. An equivalent version of this condition appears in [13].

Theorem 4 *Let $< X, \wedge, \top, \to >$ be an EATS. Then the following are equivalent:*

1. *$\mathcal{F}_\lambda(X)$ is a model of the $\lambda\beta$-calculus in* **ALG**
2. *Condition B holds.*
3. *For any finite set J and a_j, b_j (j in J):*

$$\phi(\bigvee_{j\in J}(a_j \to b_j)\uparrow) = \bigvee_{j\in J}(a_j\uparrow \Rightarrow b_j\uparrow)$$

Proof Let us prove that 1 implies 2. So, suppose that $< \mathcal{F}(X), \lambda, \phi >$ is a model of the $\lambda\beta$-calculus in **ALG**. Suppose that $\bigwedge_{j\in J}(a_j \to b_j) \leq (a \to b)$. Set f equal to $\bigvee_{j\in J}(a_j\uparrow \Rightarrow b_j\uparrow)$. Then $a \to b$ is in $\lambda(f)$, as each $a_j \to b_j$ is. But then $b \in \lambda(f)\cdot(a\uparrow)$ which is equal to $f(a\uparrow)$ by the assumption. Therefore $\bigvee_{j\in J}(a_j\uparrow \Rightarrow b_j\uparrow) = f \geq (a\uparrow \Rightarrow b\uparrow)$ and it follows that $a \leq \bigwedge_{i\in I} a_i$ and $\bigwedge_{i\in I} b_i \leq b$ for some finite subset I of J, as required.

Next, we prove that 2 implies 3. In general, $b \in \phi((a \to b)\uparrow)(a\uparrow)$, and so $\phi((a \to b)\uparrow) \geq (a\uparrow \Rightarrow b\uparrow)$. Therefore, $\phi(\bigvee_{j\in J}(a_j \to b_j)\uparrow) \geq \bigvee_{j\in J}(a_j\uparrow \Rightarrow b_j\uparrow)$. In the other direction, we calculate for any x in $\mathcal{F}(X)$ that:

$$\begin{aligned}
\phi(\bigvee_{j\in J}(a_j \to b_j)\uparrow)(x) &= \{b \mid \exists a \in x.\ \bigwedge_{j\in J}(a_j \to b_j) \leq (a \to b)\}^* \\
&\leq \{b \mid \exists a \in x, I \subset J.\ \bigwedge_{i\in I} b_i \leq b \text{ and } a \leq \bigwedge_{i\in I} a_i\}^*
\end{aligned}$$

$$\begin{array}{rl} & \text{(by Condition B)} \\ = & \bigvee_{j \in J} (a_j{\uparrow} \Rightarrow b_j{\uparrow})(x) \end{array}$$

Finally we show that 3 implies 1. For any continuous $f : \mathcal{F}(X) \to \mathcal{F}(X)$, calculate:

$$\begin{array}{rcl} \phi(\lambda(f)) & = & \phi(\bigvee\{(a \to b){\uparrow} \mid b \in f(a{\uparrow})\}) \\ & = & \bigvee\{a{\uparrow} \Rightarrow b{\uparrow} \mid b \in f(a{\uparrow})\} \text{ (by 3 and the continuity of } \phi) \\ & = & f \end{array}$$

□

The equivalence of parts 1 and 2 is proved in [13] under the assumption of Condition C; however the proof is (essentially) that just given.

There is a pleasant characterisation of those EATSs which satisfy the additional conditions of [13]. We write $\lambda \dashv \phi$ to mean that λ,ϕ are an adjoint pair of maps.

Proposition 1 *Let* $< X, \top, \wedge, \to >$ *be an EATS. Then the following are equivalent:*

1. $\lambda \dashv \phi$
2. λ *preserves finite sups and for all a,b in* X, $\lambda(a{\uparrow} \Rightarrow b{\uparrow}) = (a \to b){\uparrow}$
3. *Condition C holds*

Proof To show 1 implies 2, assume λ,ϕ are an adjoint pair of maps. Then λ preserves finite sups as left adjoints preserve all existing sups. Next, for a,b in X, $\lambda(a{\uparrow} \Rightarrow b{\uparrow}) \geq (a \to b){\uparrow}$ holds in general as $(a \to b) \in \lambda(a{\uparrow} \Rightarrow b{\uparrow})$. For the converse, as $(a{\uparrow} \Rightarrow b{\uparrow}) \leq \phi((a \to b){\uparrow})$ holds in general, we have $\lambda(a{\uparrow} \Rightarrow b{\uparrow}) \leq (a \to b){\uparrow}$, as $\lambda \dashv \phi$.

To show 2 implies 3, assume that λ preserves finite sups and for all a,b in X, $\lambda(a{\uparrow}\Rightarrow b{\uparrow}) = (a \rightarrow b){\uparrow}$. Suppose that we have a finite set J and a,b,a_j,b_j ($j \in J$) such that $a \leq \bigwedge_{j\in J} a_j$ and $\bigwedge_{j\in J} b_j \leq b$. Then

$$\begin{array}{rcl}
(\bigwedge_{j\in J}(a_j \rightarrow b_j)){\uparrow} & = & \bigvee_{j\in J} \lambda(a_j{\uparrow}\Rightarrow b_j{\uparrow}) \\
& = & \lambda(\bigvee_{j\in J}(a_j{\uparrow}\Rightarrow b_j{\uparrow})) \\
& \leq & \lambda(a{\uparrow}\Rightarrow b{\uparrow}) \quad (\text{as } a \leq \bigwedge_{j\in J} a_j \text{ and } \bigwedge_{j\in J} b_j \leq b) \\
& = & (a \rightarrow b)
\end{array}$$

To show 3 implies 1, assume that Condition C holds. Since, in general, $\phi\circ\lambda \geq id_{[\mathcal{F}(X)\rightarrow\mathcal{F}(X)]}$, we have only to show that $\lambda\circ\phi \leq id_{\mathcal{F}(X)}$. For this it is enough to show that if $b \in x\cdot(a{\uparrow})$ then $(a \rightarrow b) \in x$. This is immediate from the above remarks on application and Condition C.

□

That Condition C implies adjointness was already noted in [13].

We can also characterise when $< \lambda, \phi >$ is a closure pair (that is when it is an adjoint pair such that $\lambda\phi = id$). Say that an element of X is *functional* if it has the form:

$$\bigwedge_{j\in J}(b_j \rightarrow c_j)$$

where J is a finite set.

Proposition 2 *Let $< X, \top, \wedge, \rightarrow>$ be an EATS. Then $< \lambda, \phi >$ is a closure pair iff Condition C holds and every element of X is functional.*

Proof First, suppose that $< \lambda, \phi >$ is a closure pair. By Proposition 1, Condition C holds. Next, choose a in X. As $\lambda\circ\phi \geq id_{\mathcal{F}(X)}$, we have that a is in $\lambda(\phi(a{\uparrow}))$. So there is a finite set J and b_j,c_j ($j \in J$) such that $\bigwedge_{j\in J}(b_j \rightarrow c_j) \leq a$, and $c_j \in (a{\uparrow})\cdot(b_j{\uparrow})$. By the second of these and Condition C we get that $(b_j \rightarrow c_j) \in (a{\uparrow})$, and so $a = \bigwedge_{j\in J}(b_j \rightarrow c_j)$.

For the converse, suppose Condition C holds and every element of X is functional. By Proposition 1 we only need to show that $\lambda\circ\phi \geq id_{\mathcal{F}(X)}$. For this, it is enough to show that for every a in X, $a \in \lambda(\phi(a\uparrow))$. Now, as every element of X is functional, a has the form $\bigwedge_{j\in J}(b_j \to c_j)$ where J is a finite set. But then $c_j \in (a\uparrow)\cdot(b_j\uparrow)$; so $(b_j \to c_j)$ is in $\lambda(\phi(a\uparrow))$, and therefore a is too.

□

Note that the second half of the proof shows (without using Condition C) that if every element is functional then $\lambda\circ\phi \geq id_{\mathcal{F}(X)}$. A result of Coppo *et al* [13] states that any EATS satisfying Condition C yields an extensional applicative structure iff all its elements are functional. Since a pair of adjoint maps between partial orders is a closure pair iff the right adjoint is 1-1, Proposition 2 is equivalent, given Proposition 1, to the result of Coppo *et al.*

Putting Proposition 2 together with Theorem 4, we can characterise when the construction yields a model of the $\lambda\beta\eta$-calculus in **ALG**.

Corollary 2 *$\mathcal{F}_\lambda(X)$ is a model of the $\lambda\beta\eta$-calculus iff conditions B and C hold and every element of X is functional.*

We now have conditions under which various classes of λ-structures are represented by EATSs, in particular models of the $\lambda\beta$-calculus, or models of the $\lambda\beta\eta$-calculus. It is natural to ask if all such structures are represented.

Theorem 5 *Let $< D, \lambda, \phi >$ be a λ-structure. If λ,ϕ are an adjoint pair of maps, then it is represented by an EATS satisfying Condition C.*

Proof Suppose $< D, \lambda, \phi >$ is a λ-structure and $< \lambda, \phi >$ is an adjoint pair. Define an EATS $< B, \wedge, \top, \to>$ by taking $< B, \wedge, \top >$ to be the lower semilattice set of finite elements of D, as discussed above, and setting $(a \to b) = \lambda(a\uparrow \Rightarrow b\uparrow)$—which is a good definition as left adjoints preserve finiteness. (And recall the isomorphism $\theta : D \to \mathcal{F}(B)$.)

Let us show that the EATS satisfies Condition C. So suppose that we have a finite set J and a,b,a_j,b_j $(j \in J)$ such that $a \leq \bigwedge_{j\in J} a_j$ and $\bigwedge_{j\in J} b_j \leq b$.

Then

$$\begin{aligned}
\bigwedge_{j \in J} (a_j \to b_j) &= \bigvee_D \{\lambda(a_j\!\uparrow \Rightarrow b_j\!\uparrow) \mid j \in J\} \\
&= \lambda(\bigvee_{j \in J} (a_j\!\uparrow \Rightarrow b_j\!\uparrow)) \quad \text{(left adjoints preserve sups)} \\
&\leq \lambda(a\!\uparrow \Rightarrow b\!\uparrow) \quad (\text{as } a \leq \bigwedge_{j \in J} a_j \text{ and } \bigwedge_{j \in J} b_j \leq b) \\
&= (a \to b)
\end{aligned}$$

Finally we prove that $< \theta, \theta^{-1} >$ and $< \theta^{-1}, \theta >$ are morphisms of λ-structures. Since they are isomorphisms in **ALG** it is enough to prove that $< \theta, \theta^{-1} >$ is a morphism, and since $\lambda \dashv \phi$ and $\lambda_B \dashv \phi_B$ (as B satisfies Condition C), it is enough to prove that either one of the diagrams commute. Again using that θ is an isomorphism in **ALG**, we see that we need only check that it preserves application, and calculate:

$$\begin{aligned}
\theta(x) \cdot \theta(y) &= \{b \mid \exists a \in \theta(y).\ (a \to b) \in \theta(x)\} \text{ (as } B \text{ satisfies Condition C)} \\
&= \{b \mid \exists a \leq y.\ \lambda(a\!\uparrow \Rightarrow b\!\uparrow) \leq x\} \\
&= \{b \mid \exists a \leq y.\ a\!\uparrow \Rightarrow b\!\uparrow \leq \phi(x)\} \text{ (as } \lambda \dashv \phi) \\
&= \phi(x)(y)
\end{aligned}$$

□

In [13] this theorem is proved assuming also that $< D, \lambda, \phi >$ is a model of the $\lambda\beta$-calculus.

By the theorem, all models of the $\lambda\beta\eta$-calculus in **ALG** are represented. Unfortunately, as we shall see, this is not the case for the $\lambda\beta$-calculus as, for example, the models introduced by Scott in [52] cannot be so represented, in general. Evidently, one would wish for a more general representation theory which would allow the representation of all such models.

As before we may try to make the construction of $\mathcal{F}_\lambda$ functorial. It is again possible to obtain a contravariant functor by taking a suitable notion of strong morphism, but we prefer to obtain a covariant functor by restricting the objects to those EATSs satisfying Condition C, and thereby obtain a categorical view of Theorem 5. We begin by making $\mathcal{F}$ functorial, setting, for any semilattice morphism $h : X \to Y$, $\mathcal{F}(h) = h_*$ where for any x in X, $h_*(x) = \{a' \mid \exists a \in x.\, h(a) \leq a'\}$. This has a right adjoint, given by: $h^{-1}(y) = \{a \in X \mid h(a) \in y\}$. Now a map between complete upper semilattices is a left adjoint iff it preserves all lubs. Therefore $\mathcal{F}$ is a functor from the category of EATSs to the category of complete algebraic lattices and completely additive maps. It is clearly faithful, and it is not hard to see that it is full. (For if $f : \mathcal{F}(X) \to \mathcal{F}(Y)$ is a left adjoint then it preserves finiteness and so we may define $h : X \to Y$ by $h(a)\!\uparrow = f(a\!\uparrow)$; then h is a morphism and f is h_*.) As we also know the representation is complete, we have shown that $\mathcal{F}$ is an equivalence of categories.

Returning to $\mathcal{F}_\lambda$, put

$$\mathcal{F}_\lambda(h) = < h_*, h^{-1} >$$

on EATS morphisms $h : X \to Y$.

Lemma 9 *1. If Y satisfies Condition C, then $\mathcal{F}_\lambda(h)$ is a morphism of λ-structures.*

2. Let X and Y be EATSs satisfying Condition C. Then every morphism from $\mathcal{F}_\lambda(X)$ to $\mathcal{F}_\lambda(Y)$ that is an adjoint pair is represented.

Proof

1. First, for any $f : X \to Y$ we have

$$\begin{array}{rcl} h_*(\lambda_X f) & = & h_*(\bigvee\{(a \to b)\!\uparrow \mid b \in f(a\!\uparrow)\}) \\ & = & \bigvee\{(h(a) \to h(b))\!\uparrow \mid b \in f(a\!\uparrow)\} \\ & & \text{(as } h_* \text{ is a left adjoint and } h_*(c\!\uparrow) = h(c)\!\uparrow) \end{array}$$

and also

$$\begin{array}{rcl} \lambda_Y(h_* \circ f \circ h^{-1}) & = & \{a' \to b' \mid b' \in h_*(f(h^{-1}(a'\!\uparrow)))\}^* \\ & = & \{a' \to b' \mid \exists b \in f(h^{-1}(a'\!\uparrow)).h(b) \leq b'\}^* \\ & = & \{a' \to b' \mid \exists a, b.a' \leq h(a) \wedge h(b) \leq b' \wedge b \in f(a)\}^* \end{array}$$

and as Y satisfies Condition C, the two are equal.

For the other diagram we have to show that for all v in Y and u in X, $h^{-1}(v \cdot h_*(u)) = h^{-1}(v) \cdot u$. Calculation shows that the left-hand-side is $\{b \mid \exists a \in u, a'.\ h(a) \leq a' \wedge (a' \to h(b)) \in v\}^*$, and that the right-hand-side is $\{b \mid \exists a \in u.\ (h(a) \to h(b)) \in v\}$. And again using the fact that Y satisfies Condition C, we see that the two are equal.

2. Let X and Y be EATSs satisfying Condition C, and let $< f, g >$ be a morphism from $\mathcal{F}_\lambda(X)$ to $\mathcal{F}_\lambda(Y)$ that is an adjoint pair. Then, by the above remarks we may define a semilattice morphism, $h : X \to Y$ by $h(a)\!\uparrow = f(a\!\uparrow)$ and f is h_* and g is h^{-1}. That h preserves $\to$ is shown by the following calculation:

$$
\begin{array}{rcl}
h(a \to_X b)\!\uparrow & = & f((a \to b)\!\uparrow) \\
& = & f(\lambda_X(a\!\uparrow \Rightarrow b\!\uparrow)) \quad \text{(by Proposition1)} \\
& = & \lambda_Y(f \circ (a\!\uparrow \Rightarrow b\!\uparrow) \circ g) \quad \text{(as } < f, g > \text{ is a morphism)} \\
& = & \lambda_Y(h_* \circ (a\!\uparrow \Rightarrow b\!\uparrow) \circ h^{-1}) \\
& = & \lambda_Y(h(a)\!\uparrow \Rightarrow h(b)\!\uparrow) \\
& = & h(a) \to_Y h(b) \quad \text{(by Proposition1)}
\end{array}
$$

□

It is easy to see that $\mathcal{F}_\lambda$ is faithful. It follows from Theorem 5 and Lemma 9 that if we restrict the domain of $\mathcal{F}_\lambda$ to EATSs satisfying Condition C, and the range to the subcategory with objects the λ-structures which are adjoint pairs and morphisms which are adjoint pairs, then $\mathcal{F}_\lambda$ becomes an equivalence of categories. In particular this give a good representation theory for models of the $\lambda\beta\eta$-calculus in **ALG**.

Finally we prove that not all models of the $\lambda\beta$-calculus can be represented by an EATS.

Fact 1 *Let A be a set with at least two members. Then there is no EATS X such that the λ-structure $\mathcal{P}(S_A)$ is isomorphic to $\mathcal{F}_\lambda(X)$*

Proof Suppose, for the sake of contradiction, that X is an EATS such that $f : \mathcal{P}(S_A) \cong \mathcal{F}_\lambda(X)$. Then $f : \mathcal{P}(S_A) \cong \mathcal{F}(X)$. But $\theta : \mathcal{P}(S_A) \cong \mathcal{F}(\mathcal{P}_f(S_A))$ where $\theta(x) = \mathcal{P}_f(x)$. So $\mathcal{F}(\mathcal{P}_f(S_A)) \cong \mathcal{F}(X)$ and there is a semilattice isomorphism $h : \mathcal{P}_f(S_A) \cong X$ such that $f = \mathcal{F}(h) \circ \theta$. We may therefore assume that the EATS X has the form $< \mathcal{P}_f(S_A), \rightarrow>$ and f is θ. From this it follows that the applicative structure on $\mathcal{P}(S_A)$ is given by the formula:

$$u \cdot v = \bigcup \{\nu \in \mathcal{P}_f(S_A) \mid \exists \mu \subset v.(\mu \rightarrow \nu) \subset u\}$$

Now let a and b be two distinct elements of A. Set $x = \{a, b\} \rightarrow \{a\}$. Then as $\{<< a, b >, a >\} \cdot \{a, b\} = \{a\}$ we have that $x \subset \{<< a, b >, a >\}$; similarly $x \subset \{<< b, a >, a >\}$. But as $a \neq b$, it follows that $x = \emptyset$, which is impossible.

$\square$

6 Extensional Models

In Part 1, we provided an elementary "natural deduction" method for constructing models of the $\lambda\beta\eta$-calculus. The λ-structure, $< T^*_{CE}, \lambda, \phi >$ considered there is a model of the $\lambda\beta\eta$-calculus in **ALG** where:

$$\lambda(f) = \{\mu \rightarrow \nu \mid \nu \subset f(\mu^*)\}^*$$

and

$$\phi(x)(y) = x[y]$$

This construction provided an alternative to Scott's well-known inverse limit construction [47, 48, 49, 6]. Another elementary "closure" method of constructing models was given by Scott in [51, 52]. The idea is to start with a model of the $\lambda\beta$-calculus $< D, \lambda, \phi >$ such that $< \phi, \lambda >$ is a closure pair and then take the fixed points of a certain associated closure operation on D. A final method was introduced by Coppo *et al* [13]: one constructs EATSs satisfying the conditions of Theorem 2.

Here we compare the different methods. Each one can be used to obtain any model of the $\lambda\beta\eta$-calculus in **ALG**. This is trivially the case for the inverse limit construction or Scott's closure method; one just starts off with the model to be constructed. For the EATS method it is a consequence of Theorem 5, and for the natural deduction method it is an immediate consequence of remarks below linking it to the EATS method. What is important is, rather, to relate the ways made available by the different methods of *constructing* or *presenting* models.

6.1 EATSs and Natural Deduction

According to Theorem 2 a model can be constructed if we can find an EATS satisfying conditions B and C and such that every element is functional. One way to try to do this is to start from $\mathcal{P}_f(T_A)$, the free EATS over a set of atoms A, and then divide out by the least congruence that equates each atom with a specified non-atom and that satisfies Condition C (in the evident sense); it should be proved that the resulting EATS satisfies Condition B.

By contrast, the natural deduction method works directly with T_A, providing a consequence relation $\vdash$ over T_A. The link between the two is provided by a notion of a *consequence relation* on a semilattice $< X, \wedge, \top >$. This is a relation $\vdash$ over X such that:

$$1.\ \frac{a \vdash b \quad b \vdash c}{a \vdash c} \qquad 2.\ \frac{a \vdash b \quad a \vdash c}{a \vdash b \wedge c} \qquad 3.\ a \wedge b \vdash a$$

To each such relation $\vdash$ one can associate a semilattice congruence $\equiv_\vdash$ by: $a \equiv_\vdash b$ iff $a \vdash b \vdash a$; conversely to each semilattice congruence $\equiv$ on X one can associate a consequence relation $\vdash_\equiv$ by: $a \vdash_\equiv b$ iff $(a \wedge b) \equiv a$. This puts the consequence relations and the congruences into monotone bijective correspondence. Let $\vdash$ be a consequence relation on X. We write $[a]$ for the equivalence class of a relative to $\equiv_\vdash$; note that $a \vdash b$ holds iff $[a] \leq [b]$ holds in $X/\equiv_\vdash$. Say that a *theory* is a subset x of X closed under finite meets and upper closed under $\vdash$ (the latter meaning that if a is in x and $a \vdash b$ then b is in x). Let $Th_\vdash(X)$ be the collection of theories partially ordered by inclusion. Then there is an isomorphism $\theta : \mathcal{F}(X/\equiv_\vdash) \cong Th_\vdash(X)$ where $\theta(x) = \{a \mid [a] \in x\}$; its inverse is $\theta^{-1}(y) = \{[a] \mid a \in y\}$.

In the particular case of a free semilattice $\mathcal{P}_f(L)$, consequence relations correspond to consequence relations on L [50, 4], that is relations $\vdash$ between

$\mathcal{P}_f(L)$ and L such that

$$1.\ \Gamma, \phi \vdash \phi \qquad 2.\ \frac{\Gamma \vdash \phi \quad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

where Γ,Δ range over finite subsets of L; ϕ,ψ range over elements of L and we follow a standard convention in using commas for union and confusing elements of L with their singleton sets. To each consequence relation $\vdash$ over $\mathcal{P}_f(L)$ we associate a consequence relation over L, also denoted by $\vdash$, by: $\Gamma \vdash \phi$ iff $\Gamma \vdash \{\phi\}$; conversely to each consequence relation $\vdash$ over L we associate a consequence relation over $\mathcal{P}_f(L)$, again also denoted by $\vdash$, by: $\Gamma \vdash \Delta$ iff $\Gamma \vdash \phi$ for every ϕ in Δ. This puts the two classes of consequence relation in monotone bijective correspondence. Given a consequence relation $\vdash$ on L, a *theory* is a subset x of L closed under $\vdash$, meaning that if $\Gamma \subset x$ and $\Gamma \vdash \phi$ then ϕ is in x. Let $Th_\vdash(L)$ be the collection of theories partially ordered by inclusion. Then there is an isomorphism $\delta : Th_\vdash(\mathcal{P}_f(L)) \cong Th_\vdash(L)$ given by: $\delta(x) = \{\phi \mid \{\phi\} \in x\}$; its inverse is $\delta^{-1}(y) = \{\Gamma \mid \Gamma \subset y\}$.

Now let us consider congruences and consequence relations on an EATS $< X, \wedge, \top, \rightarrow>$. Let $\vdash$ be a consequence relation on $< X, \wedge, \top >$. We say that $\vdash$ *satisfies Condition B* iff whenever $\bigwedge_{j\in J}(a_j \rightarrow b_j) \vdash (a \rightarrow b)$ then $a \vdash \bigwedge_{i\in I} a_i$ and $\bigwedge_{i\in I} b_i \vdash b$ for some finite subset I of J (where J is a finite set); we say that $\vdash$ *satisfies Condition C* iff whenever $a \vdash \bigwedge_{j\in J} a_j$ and $\bigwedge_{j\in J} b_j \vdash b$ then $\bigwedge_{j\in J}(a_j \rightarrow b_j) \vdash (a \rightarrow b)$ (where J is a finite set). If $\equiv_\vdash$ is an EATS congruence, $\vdash$ satisfies Condition B (respectively C) iff $X/\equiv_\vdash$ does. Note that if $\vdash$ satisfies Condition C then $\equiv_\vdash$ is an EATS congruence. We can define a λ-structure on $Th_\vdash(X)$ by: $\lambda_\vdash(f) = \{a \rightarrow b \mid b \in f(a\uparrow)\}^*$ and $\phi_\vdash(x)(y) = \{b \mid \exists a \in y.(a \rightarrow b) \in x\}^*$ (where x^* is the least theory containing x, and $a\uparrow$ is $\{a\}^*$). One can show that if $\equiv_\vdash$ is an EATS congruence, then $< \theta^{-1}, \theta >$ is an isomorphism of λ-structures.

Let us now consider $< \mathcal{P}_f(T_A), \cup, \emptyset, \rightarrow>$ the free EATS over A (where $\mu \rightarrow \nu$ is $< \mu, \nu >$). Let $\vdash$ be a consequence relation on T_A. We say that $\vdash$ *satisfies Condition B* iff whenever $\{\mu_j \rightarrow \nu_j \mid j \in J\} \vdash (\mu \rightarrow \nu)$ then $\mu \vdash \bigcup_{i\in I} \mu_i$ and $\bigcup_{i\in I} \nu_i \vdash \nu$ for some finite subset I of J (where J is a finite set); we say that $\vdash$ *satisfies Condition C* iff whenever $\mu \vdash \bigcup_{j\in J} \mu_j$ and $\bigcup_{j\in J} \nu_j \vdash \nu$ then $\{\mu_j \rightarrow \nu_j \mid j \in J\} \vdash (\mu \rightarrow \nu)$ (where J is a finite set). Then $\vdash$ satisfies Condition B (respectively C) iff the associated consequence relation on $\mathcal{P}_f(T_A)$ does. A λ-structure on $Th_\vdash(T_A)$ can be defined, setting $\lambda_\vdash(f) = \{\mu \to \nu \,|\, \nu \subset f(\mu^*)\}^*$ and $\phi_\vdash(x)(y) = (\bigcup\{\nu| \; \exists \mu \subset y.(\mu \to \nu) \in x\})^*$ (where x^* is the least theory containing x). One can show that $< \delta^{-1}, \delta >$ is an isomorphism of λ-structures.

Natural deduction is a useful way of presenting consequence relations on T_A. Given a set of pairs $< \mu_i, \nu_i >$ of finite subsets of T_A (with i ranging over a given index set I) consider the natural deduction system whose set of formulas is T_A and whose axioms and rules are as in Part 1, except that the first two rules are replaced by:

1 For i in J and τ in ν_i,

$$\frac{\sigma \quad (\sigma \in \mu_i)}{\tau}$$

2 For i in J and σ in μ_i,

$$\frac{\tau \quad (\tau \in \nu_i)}{\sigma}$$

Define $\vdash$ by: $\mu \vdash \tau$ iff there is a proof of τ from a subset of μ. Then $\vdash$ is the least consequence relation satisfying Condition C and such that for all i in I, $\mu_i \equiv_\vdash \nu_i$. It follows, by the above remarks, that $\equiv_\vdash$ is the least congruence on $\mathcal{P}_f(T_A)$ satisfying Condition C and such that for all i in I, $\mu_i \equiv_\vdash \nu_i$.

By employing a free EATS, the axioms for semilattices are "built-in" to the natural deduction method. Matters are further simplified if we also build-in some of Condition C; this can be done by working with Scott-Engeler algebras. Let $< X, \to >$ be a SE-algebra and let $\vdash$ be a consequence relation over X. We say that $\vdash$ *satisfies Condition B* iff if $\{\mu_j \to b_j \mid j \in J\} \vdash (\mu \to b)$ then $\mu \vdash \bigcup_{i \in I} \mu_i$ and $\{b_i \mid i \in I\} \vdash b$ for some finite subset I of J (where J is a finite set); we say that $\vdash$ *satisfies Condition C* iff if $\mu \vdash \bigcup_{j \in J} \mu_j$ and $\{b_i \mid i \in I\} \vdash b$ then $\{\mu_j \to b_j \mid j \in J\} \vdash (\mu \to b)$ (where J is a finite set). Then $\vdash$ satisfies Condition B (respectively C) iff the associated consequence relation on $\mathcal{E}(X)$ does. A λ-structure on $Th_\vdash(X)$ can be defined, setting $\lambda_\vdash(f) = \{\mu \to b \,|\, b \in f(\mu^*)\}^*$ and $\phi_\vdash(x)(y) = \{b \,|\, \exists \mu \subset y.(\mu \to b) \in x\}^*$ (where x^* is the least theory containing x). One can again prove $< \delta^{-1}, \delta >$ is an isomorphism of λ-structures.

Let us now consider a natural deduction system whose set of formulas is B_A, the free SE-algebra over A. Let $< \mu_i, \nu_i >$ be a set of pairs of finite subsets of B_A (with i ranging over a given index set I). The rules are:

1 For i in J and τ in ν_i,

$$\frac{\sigma \quad (\sigma \in \mu_i)}{\tau}$$

2 For i in J and σ in μ_i,

$$\frac{\tau \quad (\tau \in \nu_i)}{\sigma}$$

3

$$\frac{(\mu \to b_1)\,, \ldots, (\mu \to b_n)\,,\ \ \begin{array}{c}[\mu']\\ \mu\end{array}\ ,\ \ \begin{array}{c}[b_1, \ldots, b_n]\\ b'\end{array}}{\mu' \to b'}$$

where $n \geq 0$.

Defining $\vdash$ as before, $\mu \equiv_\vdash \nu$ iff μ is congruent to ν in the least congruence $\equiv$ over $\mathcal{E}(B_A)$ satisfying Condition C and equating all the pairs μ_i and ν_i. Note that if we define normal derivations to be those in which rules 1 and 2 do not occur in immediate succession, then all derivations can be put in normal form.

For example, if I is empty $\mathcal{E}(B_A)/\equiv_\vdash$ is the free EATS over A satisfying Condition C; its filter model is the BCD model [10]. A straightforward induction on derivations shows that if $\mu \vdash b$ then $a \vdash b$ for some a in μ. With this one can easily show that $\vdash$ is the least consequence relation over B_A such that if $\mu' \vdash \mu$ and $b \vdash b'$ then $(\mu \to b) \vdash (\mu' \to b')$.

Now suppose instead that A is non-empty, I is A, each μ_a is $\{a\}$ and each ν_a is a non-empty set of functional elements of B_A. Then every element of $\mathcal{E}(B_A)/\equiv_\vdash$ is functional. Further a straightforward induction on derivations in normal form shows that $\vdash$ satisfies Condition B. Another induction on derivations in normal form shows that there are no theorems, and so, as A is non-empty, there are at least two distinct theories. It follows, using Corollary 2, that $\mathcal{F}(\mathcal{E}(B_A)/\equiv_\vdash)$ is a non-trivial model of the $\lambda\beta\eta$-calculus.

A construction of Hoofman and Schellinx [31] (and see also [7, 34]) fits within this framework. They defined a preorder $\preceq_{f\epsilon}$ over B_A, given any 1-1 map $f : A \to A$ and arbitrary $\epsilon : A \to \mathcal{P}_f(A)$. They then obtained a model of the $\lambda\beta\eta$-calculus as (essentially) the complete lattice of lower-$\preceq_{f\epsilon}$-closed subsets of B_A (under the subset ordering). This amounts to applying the above construction to the A-indexed system $< \{a\}, \{\epsilon(a) \to f(a)\} >$.

There is an intimate connection between the two natural deduction approaches; it allows us to greatly generalise the construction given in Part 1 of models of the $\lambda\beta\eta$-calculus using natural deduction systems over T_A. Let $h : \mathcal{P}(T_A) \to \mathcal{E}(B_A)$ be the unique EATS homomorphism such that $h(\{a\}) = \{a\}$ $(a \in A)$. Consider the natural deduction system over T_A with axioms given as above from a set of pairs $< \mu_i, \nu_i >$ of finite subsets of T_A $(i \in I)$. Then a straightforward induction on normal derivations shows that $\mu \vdash \nu$ iff $h(\mu) \vdash h(\nu)$, where we are now considering the natural deduction system over B_A with axioms given from the set of pairs $< h(\mu_i), h(\nu_i) >$ $(i \in I)$. It follows that the λ-structure of theories over T_A is isomorphic to the corresponding λ-structure of theories over B_A. Suppose now that A is non-empty, that I is A, μ_a is $\{a\}$ and that ν_a has empty intersection with A but that $\nu_a \backslash Th$ is non-empty, where Th is the set of theorems considered in Part 1. This generalises the situation considered there for models of the $\lambda\beta\eta$-calculus. One can show that $Th = h^{-1}(\{\emptyset\})$ and so $h(\nu_a)$ is a non-empty set of functional elements. Therefore, by the above result on natural deduction systems over B_A, one has that the λ-structure of theories over T_A is a non-trivial model of the $\lambda\beta\eta$-calculus.

6.2 The D_∞ Construction

One can consider the D_∞ construction as providing a way of presenting models of the $\lambda\beta\eta$-calculus. Given an embedding $\phi : D_0 \to D_0^{D_0}$ the construction provides a model *via* the inverse limit construction. In [13], (see also [16]) Coppo *et al* showed (in our terms) that provided D_0 is an algebraic complete lattice, D_∞ can be obtained as the filter model of a free EATS satisfying Condition C and identifying a certain explicitly given set of pairs of elements. Unfortunately, these identifications are not of the general form we have considered above, where we equate pairs a and ν_a. We will show that the converse does hold: given any presentation of a model as considered above, we can obtain the model also *via* the D_∞ construction, starting from a rather simply presented initial embedding. It would be interesting to improve our results by finding a more general class of equations which always yield models of the $\lambda\beta\eta$-calculus and for which there is a theorem of the kind proved by Coppo *et al.*

So let us take a non-empty set of atoms A, and consider a set of pairs of the form $< \{a\}, \nu_a >$ where ν_a is a non-empty set of functional elements of B_A. Write $\vdash_A$ for the associated consequence relation on $\mathcal{E}(B_A)$, and $\equiv_A$ for the associated congruence relation. We know that $\mathcal{F}_\lambda(\mathcal{E}(B_A)/\equiv_A)$ is a non-trivial model of the $\lambda\beta\eta$-calculus. In order to find $\phi : D_0 \to D_0^{D_0}$ we need the set of pairs to satisfy the condition that every element of each ν_a is *first-order*, that is the element has the form $\mu \to b$ where μ is a set of atoms and b is an atom.

If this is not the case, it can be made so *via* a transformation. First let $t : B_A \to T$ put B_A in bijective correspondence with a set of atoms T. Write t_b for $t(b)$, and t_μ for $\{t_b \mid b \in \mu\}$. Consider the set of pairs

$$\{< \{t_a\}, \{t_\mu \to t_b \mid (\mu \to b) \in \nu_a\} >\} \cup \{< \{t_{\mu \to b}\}, \{t_\mu \to t_b\} >\}$$

and let $\vdash_T$ and $\equiv_T$ be, respectively, the associated consequence and congruence relations on $\mathcal{E}(B_T)$. Evidently the first-order condition is now satisfied, and $\mathcal{F}_\lambda(\mathcal{E}(B_T)/\equiv_T)$ is a non-trivial model of the $\lambda\beta\eta$-calculus; we will see that it is isomorphic to $\mathcal{F}_\lambda(\mathcal{E}(B_A)/\equiv_A)$. Define $h : A \to B_T$ and $k : T \to B_A$ by, respectively, $h(a) = t_a$ and $k(t_b) = b$; we will also write h for the extension to B_A and even for $\mathcal{E}(h)$, and similarly for k. An easy inductive proof on the size of b in B_A shows that $\{h(b)\} \equiv_T \{t_b\}$. With this one can show that if $\mu \vdash_A \nu$ then $h(\mu) \vdash_T h(\nu)$; further it is straightforward to show that if $\mu \vdash_T \nu$ then $k(\mu) \vdash_A k(\nu)$. But one has for any a in A that $k(h(a)) = a$ and so $k(h(\mu)) \equiv_A \mu$; finally as one has $\{h(k(t_b))\} = \{h(b)\} \equiv_T \{t_b\}$, we have that $h(k(\mu)) \equiv_T \mu$. Putting the four facts together we get that $\mathcal{E}(B_A)/\equiv_A$ and $\mathcal{E}(B_T)/\equiv_T$ are isomorphic, and so too, therefore, are the corresponding models of the the $\lambda\beta\eta$-calculus.

We may therefore assume that every element of each ν_a is first-order, and proceed to construct an initial embedding $\phi_0 : D_0 \to D_0^{D_0}$ for the D_∞ construction. Recall that we can regard $\mathcal{F}(\mathcal{E}(B_A)/\equiv_A)$ as the collection

$Th_{\vdash}(B_A)$ of theories over B_A of the consequence relation $\vdash$ induced by the above natural deduction system; we abbreviate $Th_{\vdash}(B_A)$ to $Th_{\vdash}$. The λ-structure is given by:

$$\phi(y)(z) = \{b \mid \exists \mu \subset z.(\mu \to b) \in y\}^*$$

and abstraction is given by

$$\lambda(f) = \{\mu \to b \mid b \in f(\mu^*)\}^*$$

where $(\cdot)^*$ is the closure operation associated to $\vdash$. Now let $\vdash_0$ be the restriction of $\vdash$ to the set of atoms and take D_0 to be the set of theories of $\vdash_0$. There is an embedding $\alpha_0 : D_0 \to Th_{\vdash}$ with right adjoint β_0 where $\alpha_0(x) = x^*$ and $\beta_0(y) = (y \cap A)$.

To construct ϕ_0 we make use of the following fact, whose easy proof is omitted:

Fact 2 *Suppose $A \xrightarrow{i} C \xleftarrow{j} B$ is a pair of embeddings with respective right adjoints i^R and j^R. Then i factors through j iff $i = jj^Ri$. In that case j^Ri is an embedding with right adjoint i^Rj.*

In the present case, we have two embeddings: $D_0 \overset{\phi\alpha_0}{\to} Th_{\vdash}^{Th_{\vdash}} \overset{\alpha_0^{\beta_0}}{\leftarrow} D_1$ where D_1 is $D_0^{D_0}$. So to find an embedding $\phi_0 : D_0 \to D_1$ we have to show that $\phi\alpha_0 = \alpha_0^{\beta_0}\beta_0{}^{\alpha_0}\phi\alpha_0$, which is equivalent to showing that $\phi\delta_0 = \delta_0^{\delta_0}\phi\delta_0$ where $\delta_0 = \alpha_0\beta_0$. This, in its turn, is equivalent to showing that for all x in D_0 and y in $Th_{\vdash}$, $x^* \cdot y = \delta_0(x^* \cdot \delta_0(y))$ and that is an immediate consequence of the following lemma:

Lemma 10 *For all x in D_0 and y in $Th_{\vdash}$,*

$$x^* \cdot y = \{b \mid \exists a \in x, \mu \subset y.(\mu \to b) \in \nu_a\}^*$$

Proof Clearly the right hand side is included in the left hand side. For the converse inclusion, suppose that b is in $x^* \cdot y$. Then there is a finite subset μ of y such that $\mu \to b$ is in x^*. So there are $a_1, \ldots, a_m$ and $\mu_i \to b_i$ in ν_{a_i} such that $(\mu_1 \to b_1), \ldots, (\mu_m \to b_m) \vdash (\mu \to b)$. Taking a minimal such m, we find by Condition B that $\mu \vdash_A \mu_i$ and $b_1, \ldots, b_m \vdash b$. But then each μ_i is a subset of y, and so $b_1, \ldots, b_m$ are in the right hand side, and therefore b is too.

$\square$

We therefore have an embedding $\phi_0 : D_0 \to D_1$ with right adjoint ψ_0, where $\phi_0 = \beta_0^{\alpha_0}\phi\alpha_0$ and $\psi_0 = \beta_0\lambda\alpha_0^{\beta_0}$. Since $\phi_0(u)(v) = \beta_0(u^* \cdot v^*)$, we have by Lemma 10 that

$$\phi_0(u)(v) = \{b \mid \exists a \in u, \mu \subset v.(\mu \to b) \in \nu_a\}^*$$

where we are now taking closures with respect to $\vdash_0$.

Now we can construct a sequence $D_n \overset{\phi_n}{\to} D_{n+1} \overset{\psi_n}{\leftarrow} D_n$ of embedding-projection pairs in the usual way, with $D_{n+1} = D_n^{D_n}$, and $\phi_{n+1} = \phi_n^{\psi_n}$ and $\psi_{n+1} = \psi_n^{\phi_n}$, and D_∞ is $\varinjlim < D_n, \phi_n >$. There is a colimiting cone of embeddings $\rho_n :< D_n, \phi_n > \to D_\infty$, whose right adjoints we denote by σ_n. Setting

$$\lambda_\infty = \bigvee_{n \geq 0} \rho_{n+1}\sigma_n^{\rho_n}$$

and

$$\phi_\infty = \bigvee_{n \geq 0} \rho_n^{\sigma_n}\sigma_{n+1}$$

we obtain $< D_\infty, \lambda_\infty, \phi_\infty >$, the model of the $\lambda\beta\eta$-calculus provided by the D_∞ method.

Theorem 6 *$< D_\infty, \lambda_\infty, \phi_\infty >$ is isomorphic to $< Th_\vdash, \lambda, \phi >$.*

Proof We can define embeddings $\alpha_n : D_n \to Th_\vdash$ with right adjoints β_n by taking α_0 and β_0 as above and putting $\alpha_{n+1} = \lambda\alpha_n^{\beta_n}$ and $\beta_{n+1} = \beta_n^{\alpha_n}\phi$. Let us prove that the α_n form a cone, *i.e.* for all n, $\alpha_n = \alpha_{n+1}\phi_n$. For $n = 0$ we have:

$$\begin{aligned} \alpha_1\phi_0 &= (\lambda\alpha_0^{\beta_0})(\beta_0^{\alpha_0}\phi\alpha_0) \\ &= \lambda\phi\alpha_0 \text{ (by the above discussion of } \phi_0) \\ &= \alpha_0 \end{aligned}$$

For $n+1$ we have:

$$\begin{aligned}
\alpha_{n+2}\phi_{n+1} &= \lambda\alpha_{n+1}^{\beta_{n+1}}\phi_n^{\psi_n} \\
&= \lambda(\alpha_{n+1}\phi_n)^{(\psi_n\beta_{n+1})} \\
&= \lambda\alpha_n^{\beta_n} \\
&= \alpha_{n+1}
\end{aligned}$$

Next we show that the cone is colimiting. For this by Theorem 2 of [43] it is enough to show that

$$id = \bigvee_{n\geq 0}\delta_n$$

where $\delta_n = \alpha_n\beta_n$. Let us show that for every b in B_A there is an n such that $b \in \delta_n(\{b\}^*)$; the equation for the identity will then follow. The proof is by induction on b. If b is in A, we can take $n = 0$. Otherwise b has the form $\mu \to c$, and by the induction hypothesis there is an n such that $\mu \subset \delta_n(\mu^*)$ and $c \in \delta_n(\{c\}^*)$. But then $c \in \delta_n(\{\mu \to c\}^* \cdot \delta_n(\mu^*))$, and so $(\mu^* \Rightarrow \{c\}^*) \leq \delta_n^{\delta_n}(\phi(\{\mu \to c\}^*))$. But as $\delta_{n+1} = \lambda\delta_n^{\delta_n}\phi$ we now see that $(\mu \to c) \in \delta_{n+1}(\{\mu \to c\}^*)$, as required.

Since the α_n form a colimiting cone, we can assume that D_∞ is $Th_\vdash$ and $\rho_n = \alpha_n$. We will show that the identity on $Th_\vdash$ is an isomorphism of the λ-structures, *i.e.* that $\phi = \phi_\infty$ and $\lambda = \lambda_\infty$. As both λ-structures are models of the $\lambda\beta\eta$-calculus, the second equation follows from the first. For the first equation we first need the fact that for all f in D_{n+1} and x in D_n, $\alpha_n(fx) = \alpha_{n+1}(f) \cdot \alpha_n(x)$, which is proved by a calculation:

$$\begin{aligned}
\alpha_{n+1}(f)\cdot\alpha_n(x) &= (\lambda\alpha_n^{\beta_n})(f)\cdot\alpha_n(x) \\
&= \lambda(\alpha_n f\beta_n)\cdot\alpha_n(x) \\
&= \alpha_n(f(\beta_n(\alpha_n(x)))) \\
&= \alpha_n(fx)
\end{aligned}$$

Now we can prove that $\phi = \phi_\infty$ by the following calculation:

$$\begin{aligned}
\phi_\infty(x)(y) &= \bigvee_{n\geq 0}\alpha_n^{\beta_n}(\beta_{n+1}(x))(y) \\
&= \bigvee_{n\geq 0}(\alpha_n\circ(\beta_{n+1}x)\circ\beta_n)(y)
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{n \geq 0} \alpha_n((\beta_{n+1}x)(\beta_n y)) \\
&= \bigvee_{n \geq 0} (\delta_{n+1}x) \cdot (\delta_n y) \text{ (by the previous calculation)} \\
&= x \cdot y \text{ (as } \bigvee_{n \geq 0} \delta_n = id)
\end{aligned}
$$

□

As an example, if we take A to be $\{\iota\}$, and $\nu_\iota = \{\emptyset \to \iota\}$, D_0 is the two-point complete lattice $\{\bot, \top\}$ and ϕ_0 is the standard embedding given by: $\phi_0(x) = (\bot \Rightarrow x)$; on the other hand, if we take $\nu_\iota = \{\{\iota\} \to \iota\}$, we get the embedding considered by Park [40, 30] where $\phi_0(x) = (\top \Rightarrow x)$. By the above discussion on the relation with natural deduction systems over T_A, these models are obtained in that framework if we take ν_a to be, respectively, $\{\emptyset \to \{\iota\}\}$ or $\{\{\iota\} \to \{\iota\}\}$. This establishes a connection with the D_∞ construction asserted in Part 1.

A wider class of Park models can be obtained by taking any non-empty set A, choosing a finite subset μ of A and setting $\nu_a = \{\mu \to a\}$. Then D_0 is $\mathcal{P}(A)$ and $\phi(x) = (\mu \Rightarrow x)$. The model D^*_∞ of [16] can be obtained by taking A to be $\{*, t\}$ (corresponding to the authors' $\{\varphi_*, \varphi_\top\}$) and setting $\nu_* = \{\{t\} \to *\}$ and $\nu_t = \{\{*\} \to t, \{t\} \to *\}$. It remains to understand the full scope of the method; the question is which pairs of algebraic complete lattices and embeddings can be represented.

6.3 Scott's Closure Method

Suppose we are given a λ-structure $< D, \lambda, \phi >$ and a continuous closure operation $c : D \to D$. Then c splits as $D \xrightarrow{i} \mathrm{Fix}_c \xrightarrow{j} D$ where Fix_c is the algebraic complete lattice of fixed points of c. We can then define an associated λ-structure $< \mathrm{Fix}_c, \lambda_c, \phi_c >$ by taking

$$\lambda_c = i\lambda j^i$$

and

$$\phi_c = i^j \phi j$$

Scott discovered that if one starts with a model of the $\lambda\beta$-calculus such that $\lambda\phi \geq id$ then by an appropriate choice of closure operation, this construction produces a model of the $\lambda\beta\eta$-calculus. It is interesting to consider the more general case where $\phi \dashv \lambda$; we suppose from now on that this is the case.

Proposition 3 *1.* $< \phi_c, \lambda_c >$ *is an embedding-projection pair iff* $\lambda c^c \phi = c$.

2. Suppose that $< \phi, \lambda >$ *is a closure pair. Then* $< Fix_c, \lambda_c, \phi_c >$ *is a model of the* $\lambda\beta\eta$*-calculus iff* $\lambda c^c \phi = c$.

Proof

1. First, suppose that $\lambda\phi \geq id$. Then:

$$\begin{array}{lll} \lambda_c\phi_c = id & \text{iff} & i\lambda j^i i^j \phi j = id \\ & \text{iff} & c\lambda c^c \phi c = c \quad (\text{as } i \dashv j) \\ & \text{iff} & \lambda c^c \phi \leq c \quad (\text{as } c \text{ is a closure operation and } \lambda\phi \geq id) \end{array}$$

   Next, suppose that $\phi \dashv \lambda$. Then:

$$\begin{array}{lll} \phi_c\lambda_c \leq id & \text{iff} & i^j \phi j i \lambda j^i \leq id \\ & \text{iff} & c^c \phi c \lambda c^c \leq c^c \quad (\text{as } i \dashv j) \\ & \text{iff} & \phi c \lambda \leq c^c \quad (\text{as } c \text{ is a closure operation}) \\ & \text{iff} & c \leq \lambda c^c \phi \quad (\text{as } \phi \dashv \lambda) \end{array}$$

   The result is then an immediate consequence.

2. Suppose that $< \phi, \lambda >$ is a closure pair. By part 1 it is enough to show that if $\lambda c^c \phi = c$ then $\phi_c \lambda_c \geq id$. Calculating, we find: $\phi_c\lambda_c = i^j \phi c \lambda j^i \geq i^j \phi \lambda j^i = id$.

□

There is a least closure operation $c : D \to D$ such that:

$$\lambda \circ c^c \circ \phi \leq c$$

It can be found as a lub $\bigvee_{n \geq 0} c_n$ of a sequence of iterates, where $c_0 = id_D$ and $c_{(n+1)} = \lambda \circ {c_n}^{c_n} \circ \phi$. To see this one uses that $\lambda \dashv \phi$ to show that c_n is an increasing sequence of closure operations; then c is a closure operation too. For minimality one shows by induction on n that for any closure operation $\bar{c}$, if $\lambda \bar{c}^{\bar{c}} \phi \leq \bar{c}$ then $c_n \leq \bar{c}$. One then has, by a standard argument, that:

$$\lambda \circ c^c \circ \phi = c$$

and so if $< D, \lambda, \phi >$ is a model of the $\lambda\beta$-calculus such that $\lambda\phi \geq id$, then by Proposition 6.3, $< \mathrm{Fix}_c, \lambda_c, \phi_c >$ is a model of the $\lambda\beta\eta$-calculus.

We can find such a model of the $\lambda\beta$-calculus by dividing out a free EATS $\mathcal{P}_f(T_A)$ by an appropriate congruence forcing all its elements to be functional. We use a natural deduction system over T_A to do this. Let E be an I-indexed collection $< \mu_i, \nu_i >$ of pairs of finite subsets of T_A, (i in I), and consider the natural deduction system whose formulas range over T_A and whose axioms and rules are:

1 For i in J and τ in ν_i,

$$\frac{\sigma \quad (\sigma \in \mu_i)}{\tau}$$

2 For i in J and σ in μ_i,

$$\frac{\tau \quad (\tau \in \nu_i)}{\sigma}$$

3

$$\frac{(\mu \to \nu)\,, \quad \overset{[\mu']}{\mu} \quad , \overset{[\mu]}{\mu'} \;, \quad \overset{[\nu']}{\nu} \quad , \overset{[\nu]}{\nu'}}{\mu' \to \nu'}$$

Then $\mu \vdash \nu \vdash \mu$ iff μ is congruent to ν in the least EATS congruence $\equiv_E$ over $\mathcal{P}_f(T_A)$ equating all the pairs μ_i and ν_i; here $\vdash$ is defined as before. There is an evident normal form for derivations.

Suppose now that A is non-empty, that I is A, μ_a is $\{a\}$ and that $\nu_a \backslash Th$ is non-empty, and has empty intersection with A. Then the system has no theorems, and $\vdash$ satisfies Condition B; the proofs are by induction on normal derivations. So $\mathcal{P}_f(T_A)/\equiv_E$ is non-trivial, satisfies Condition B and every element is functional. It follows, using Theorem 4, that $\mathcal{F}_\lambda(\mathcal{P}(T_A)/\equiv_E)$ is a non-trivial model of the $\lambda\beta$-calculus, and by the remark after Proposition 2 that $\lambda\phi \geq id$; we can then apply Scott's method to obtain a model of the $\lambda\beta\eta$-calculus.

We now wish to relate the models to those obtained by directly constructing an EATS satisfying the criterion of Corollary 2. The first task is to relate consequence relations $\vdash$ on a semilattice $< X, \wedge, \top >$ to continuous closure operations c on the associated algebraic complete semilattice of filters $\mathcal{F}(X)$. To each such $\vdash$ we associate a continuous closure operation $c_\vdash$ by: $c_\vdash(x) = x^*$ $(= \{b \mid \exists a \in x.a \vdash b\})$; conversely to each such c we associate a consequence relation $\vdash_c$ by: $a \vdash_c b$ iff $b \in c(a\uparrow)$. In this way consequence relations and continuous closure operations are in 1-1 correspondence; if consequence relations are ordered by inclusion and continuous closure relations by the usual pointwise ordering, then the correspondences are monotonic.

Each such closure operation, c, splits as $\mathcal{F}(X) \xrightarrow{i} \mathrm{Fix}_c \xrightarrow{j} \mathcal{F}(X)$ where Fix_c is the algebraic complete lattice of fixed points of c. Let $\equiv_c$ be the semilattice congruence associated to the consequence relation $\vdash_c$. Then there is an isomorphism $\alpha : \mathrm{Fix}_c \cong \mathcal{F}(X/\equiv_c)$ where $\alpha(x) = \{[a]_{\equiv_c} \mid\ a \in x\}$ (and $[a]_{\equiv_c}$ is the $\equiv_c$-equivalence class of a). Suppose now that $\equiv_c$ is an EATS congruence (we do not know a corresponding condition on c). Then $< \alpha, \alpha^{-1} >$ is an isomorphism of the λ-structures $< \mathrm{Fix}_c, \lambda_c, \phi_c >$ and $\mathcal{F}_\lambda(X/\equiv_c)$.

Lemma 11 *Let $< X, \wedge, \top, \rightarrow >$ be an EATS satisfying Condition B, in which every element is functional. Suppose too that c is a continuous closure operation on (the complete semilattice) $\mathcal{F}(X)$. Then $\vdash_c$ satisfies Condition C iff $\lambda c^c \phi \leq c$.*

Proof Suppose that $\vdash_c$ satisfies Condition C. It is enough to show for any a,b in X and finite x in $\mathcal{F}(X)$, that if $b \in c^c(\phi(x))(a\uparrow)$ then $(a \rightarrow b) \in c(x)$. In this case $b \in c(\phi(x)(c(a\uparrow)))$, and so, by continuity, there are a' in $c(a\uparrow)$, and b' in $c(b\uparrow)$ such that $b' \in \phi(x)(a'\uparrow)$. As every element is functional, there is a finite set I and a_i, b_i (for i in I) such that $x = \bigvee_{i\in I}(a_i \to b_i)\!\uparrow$ and so, as Condition B holds, by Theorem 4 we have that $\phi(x) = \bigvee_{i\in I}(a_i\!\uparrow \Rightarrow b_i\!\uparrow)$. So, as $b' \in \phi(x)(a'\!\uparrow)$, there is a $J \subset I$ such that $a_j\!\uparrow \subset a'\!\uparrow$ (for j in J) and $\bigwedge_{j\in J} b_j \leq b'$. But then as $\vdash_c$ satisfies Condition C, we have $(a \to b)$ in $c(x)$ as required.

Conversely, suppose that $\lambda c^c\phi \leq c$. There are three cases to show that $\vdash_c$ satisfies Condition C. First, we must show that $\top \vdash_c (\top \to \top)$, *i.e.* that $(\top \to \top) \in c(\top\!\uparrow) = c(\bot)$. But $\top \in \bot(\top\!\uparrow)$; so $(\top \to \top) \in \lambda(\bot) \leq (\lambda c^c\phi)(\bot) \leq c(\bot)$. Second we have to show that for any a, b and b' in X: $(a \to b) \wedge (a \to b') \vdash_c a \to (b \wedge b')$, for which it is enough to show that $a \to (b \wedge b') \in \lambda\phi(((a \to b) \wedge (a \to b'))\!\uparrow)$. But $b \in \phi((a \to b)\!\uparrow)(a\!\uparrow) \leq \phi(((a \to b) \wedge (a \to b'))\!\uparrow)(a\!\uparrow)$, and similarly for b'. Therefore, we have that: $(b \wedge b') \in \phi(((a \to b) \wedge (a \to b'))\!\uparrow)(a\!\uparrow)$, and the conclusion follows.

Third, we must show that if $a' \vdash_c a$ and $b \vdash_c b'$, then $(a \to b) \vdash_c (a' \to b')$. We have that: $b \in \phi((a \to b)\!\uparrow)(a\!\uparrow)$, $a \in c(a'\!\uparrow)$ and $b' \in c(b\!\uparrow)$. Therefore $b' \in c(\phi((a \to b)\!\uparrow)(c(a'\!\uparrow)))$, and so $(a' \to b') \in \lambda c^c\phi((a \to b)\!\uparrow)$.

$\square$

With all this we have enough information to show the two approaches to the construction of models of the $\lambda\beta\eta$-calculus equivalent. In both approaches we take an A-indexed collection ν_a of finite subsets of T_A where $\nu_a\backslash Th$ is non-empty and has empty intersection with A. In one, we construct the least congruence $\equiv$ on $\mathcal{P}_f(T_A)$ equating each a with the corresponding ν_a and satisfying Condition C. Then $\mathcal{F}_\lambda(\mathcal{P}_f(T_A)/\equiv)$ is a non-trivial model of the $\lambda\beta\eta$-calculus. In the other, we take the least congruence $\equiv_E$ on $\mathcal{P}_f(T_A)$ equating each a with the corresponding ν_a, and then take the least continuous closure operation c on $\mathcal{F}_\lambda(\mathcal{P}_f(T_A)/\equiv_E)$ such that $\lambda c^c\phi \leq c$, and apply Scott's method to obtain a model $< \mathrm{Fix}_c, \lambda_c, \phi_c >$ of the $\lambda\beta\eta$-calculus.

Now, by Lemma 11 and the discussion of the relation between closure operations and consequence relations, $\vdash_c$ is the least semi-lattice consequence relation on $\mathcal{P}_f(T_A)/\equiv_E$ satisfying Condition C. It follows that $\equiv_c$ is an EATS congruence, and $< \mathrm{Fix}_c, \lambda_c, \phi_c >$ is isomorphic to $\mathcal{F}_\lambda((\mathcal{P}_f(T_A)/\equiv_E)/\equiv_c)$. But $(\mathcal{P}_f(T_A)/\equiv_E)/\equiv_c$ is isomorphic to $\mathcal{P}_f(T_A)/\equiv$, and so we have that $< \mathrm{Fix}_c, \lambda_c, \phi_c >$ is isomorphic to $\mathcal{F}_\lambda(\mathcal{P}_f(T_A)/\equiv)$, as desired.

7 General Properties of λ-Calculus Models

A variety of properties of T^*_{CE} were established in Corollary 1; the method is to give axioms on a continuous complete lattice that T^*_{CE} satisfies and then establish general versions of these properties for all continuous complete lattices satisfying the axioms. Here we see how to relax some of these axioms; the main new result is an extension of part 6 of Theorem 1 to any model of the $\lambda\beta$-calculus in the cartesian closed category **CPO** of cpos and continuous functions.

One can easily show that the first four axioms are equivalent to saying that D and $[D \to D]$ have an isomorphism of their partial orders. (Such isomorphisms are necessarily completely additive, yielding part 1 of Theorem 1.) Parts 2, 3 and 4 of theorem 1 give formulas for the combinators K and S. These also hold if one only assumes that D is a continuous cpo (or an algebraic one, as appropriate); the proofs are the same as in the case of a complete lattice.

Part 5 of Theorem 1 can be very much generalised.

Fact 3 *Let $< D, \lambda, \varphi >$ be a non-trivial model of the $\lambda\beta$-calculus in* **CPO**. *Then $|D| \geq 2^{\aleph_0}$.*

Proof First, to finite sequences u of 0s and 1s we assign elements a_u of D, setting $a_\epsilon = \perp$ and $a_{iu} = [b_i, a_u]$. Here b_0 and b_1 are distinct elements of D, $[\cdot, \cdot]$ is the standard pairing combinator [6, Chapter 6] and we confuse λ-terms with their denotations in D. By induction on the length of u one has that if u is a prefix of u', then $a_u \leq a_{u'}$; that is, the assignment is monotonic in the prefix ordering. But then, to infinite sequences v of 0s and 1s we can assign elements a_v by:

$$a_v = \bigvee \{a_u \mid u \text{ is a finite prefix of } v\}$$

Define $\boldsymbol{\pi}_i$ for $i \geq 0$ by setting $\boldsymbol{\pi}_0 = (\cdot)_0$ and $\boldsymbol{\pi}_{i+1} = \lambda x.\boldsymbol{\pi}_i((x)_1)$, where the $(\cdot)_j$ are as in [6, Chapter 6]. Then if v is a finite or infinite sequence of length $\geq (i+1)$, $\boldsymbol{\pi}_i(v) = b_j$, where j is the ith element of v; this is proved by induction for finite v, and then by continuity for infinite v. It follows that the assignment $v \mapsto a_v$ is 1-1.

$\Box$

This proof was (essentially) suggested by Paul Taylor. A different proof can be given, following ideas of Honsell reported in Exercise 5.8.5 of [6].

Turning next to part 7 of Theorem 1, one can consider the evident generalisation to continuous cpos; however the proof given above does not immediately generalise as it depends on the existence of a top element. Nonetheless we conjecture that the generalisation would hold. Part 7 showed that, in the case of non-trivial models which are continuous complete lattices, the natural second-order theory of such models was not conservative over the natural first-order theory of non-trivial models of the lambda calculus, even for Σ_1-sentences (the universal quantifier can be absorbed by a λ-abstraction). One can ask what happens with simpler sentences, particularly equations. Now a very interesting completeness question arises. In [24] Harvey Friedman showed that two terms of the typed λ-calculus are $\beta\eta$-convertible iff they denote the same elements in the full type hierarchy over the natural numbers. One would like similar results for the untyped $\lambda\beta$-calculus and $\lambda\beta\eta$-calculus. In particular we conjecture that two terms of the untyped λ-calculus are β-convertible iff they are equal in all models in **CPO** of the untyped $\lambda\beta$-calculus, and similarly for the $\lambda\beta\eta$-calculus. Continuous models in the sense of [6, page 508] equate all unsolvable terms; Honsell and Ronchi della Rocca have shown (private communication) that in Park's models [40] all terms of order 0 are equated. Perhaps the techniques of Baeten and Boerboom [5] could be of some use here. Such questions have also been considered by Honsell and Ronchi della Rocca in [30]; in particular, they proved that there is a λ-theory which is not the theory of any model of the $\lambda\beta$-calculus in **CPO**.

Finally we consider part 6 and will prove:

Theorem 7 *Let $< D, \lambda, \varphi >$ be a model of the $\lambda\beta$-calculus in* **CPO**. *Then $\sqsubseteq_D$ is first-order definable from application and equality.*

This can be strengthened to show that the definition is provable in a suitable second-order theory; however, being a routine matter, that is left to the interested reader. An important corollary of this theorem is that if D and E are both models of the $\lambda\beta$-calculus in **CPO** then $\theta : D \to E$ is an isomorphism of both the partial-order structure and the applicative structure iff it is an isomorphism of the applicative structure alone. In the case of models of the $\lambda\beta\eta$-calculus, isomorphisms of the applicative structure also yield isomorphisms of the entire λ-structure. This fails for models of the $\lambda\beta$-calculus as is evidenced by the above examples of non-λ-categorical models constructed from Scott-Engeler algebras.

The above proof of Theorem 1.6 can be adapted to the present more general situation. One mainly needs another supply of continuous functions, as there may no longer be a sufficient supply of step functions. Let E,F be cpos and suppose x is in E and y,z are in F with $y \sqsubseteq z$. Then there is a continuous function, $n^x_{y,z} : E \to F$ where:

$$n^x_{y,z}(u) = \begin{cases} z & (\text{if } u \not\sqsubseteq x) \\ y & (\text{otherwise}) \end{cases}$$

The formulas used in the proof are obtained by considering transcriptions into the language of application and equality of topological ideas. In any cpo we have that the partial order is the same as the so-called *specialisation* order of the Scott topology:

$$x \sqsubseteq y \text{ iff } \forall V.x \in V \supset y \in V$$

where V ranges over all Scott open sets. (A subset is open in this topology iff it is an upper set, inaccessible under directed lubs.) Transcribing this to the language at hand one obtains:

$$x \sqsubseteq y \text{ iff } \forall w(w[x] \neq \perp \supset w[y] \neq \perp)$$

But we do not (yet!) have a definition of $\perp$ so it is natural to abstract on $\perp$ and consider the relation:

$$x \sqsubseteq_e y \equiv \forall w(w[x] \neq e \supset w[y] \neq e)$$

Lemma 12 *1. If e is $\perp$, then $x \sqsubseteq_e y \equiv x \sqsubseteq y$*

2. *If e is maximal, then $x \sqsubseteq_e y \equiv y \sqsubseteq x$*

3. *If e is neither $\perp$ nor maximal, then $x \sqsubseteq_e y \equiv x = y$*

Proof

1. Suppose that e is $\perp$. In one direction, if $x \sqsubseteq_e y$, take w to be $\lambda(n^y_{\perp,I})$. If $x \not\sqsubseteq y$ then $w[x] \neq \perp$ and so $w[y] \neq \perp$ and then $y \not\sqsubseteq y$, which is a contradiction. So $x \sqsubseteq y$.

   In the other direction suppose that $x \sqsubseteq y$. Then if $w[x] \neq \perp$, it follows by monotonicity that $w[y] \neq \perp$.

2. Suppose that e is maximal. In one direction, if $x \sqsubseteq_e y$, take w to be $\lambda(n^x_{\perp,e})$. Then $w[x] = \perp \neq e$ and so $w[y] \neq e$ and so $y \sqsubseteq x$.

   In the other direction suppose that $y \sqsubseteq x$, and that $w[x] \neq e$. If $w[y] = e$ then by monotonicity and the maximality of e we get that $w[x] = e$, which is a contradiction.

3. Suppose that e is neither $\perp$ nor maximal. Assume that $x \sqsubseteq_e y$. Take w to be $\lambda(n^y_{e,e'})$ where e' is strictly above e. If $x \not\sqsubseteq y$ then $w[x] \neq e$ and so $w[y] \neq e$ and so $y \not\sqsubseteq y$, a contradiction. Therefore $x \sqsubseteq y$. Next, take w to be $\lambda(n^x_{\perp,e})$. Then $w[x] \neq e$ and so $y \sqsubseteq x$. Thus we have proved that $x = y$.

   In the other direction, it is evident that if $x = y$ then $x \sqsubseteq_e y$.

□

With this we are in a symmetric situation: we cannot distinguish between D and D^{op}. Let us consider the relation:

$$x \prec y \equiv \exists V.(V \subseteq x\uparrow) \wedge y \in V$$

If D were a continuous cpo, this would be the same as the relation $\ll$; without the assumption of continuity one has that $\prec$ is a sub-relation of $\ll$ (see [25] for a fuller discussion). If we again transcribe into the language at hand, and abstract on $\perp$ we obtain the relation:

$$x \prec_e y \equiv \exists w((\forall z.w[z] \neq e \supset x \sqsubseteq_e z) \wedge w[y] \neq e)$$

Lemma 13 *If e is maximal, then $x \prec_e y \equiv y \sqsubseteq x$*

Proof Suppose that e is maximal. In one direction assume that $x \prec_e y$. With w as guaranteed by the assumption, we have that $w[y] \neq e$ and so $x \sqsubseteq_e y$. Therefore, by Lemma 12.2, $y \sqsubseteq x$. Conversely, if $y \sqsubseteq x$ then to show that $x \prec_e y$ we can take w to be $\lambda(n^x_{\perp,e})$.

□

With this we are in a position to break the symmetry:

Lemma 14 *There is an element c of D such that $c \not\prec_\perp c$*

Proof Let C_* be the combinator denoted by $\lambda z \lambda x.xz$. Let Z be the least set containing $\perp$ and closed under C_*. Then Z is directed and for any z in Z, one has that $C_*(z) \not\sqsubseteq z$. Let c be $\bigvee Z$. Suppose, for the sake of contradiction, that $c \prec_\perp c$, and let w be as guaranteed by the assumption. Then $w[c] \neq \perp$ and so by continuity, $w[z] \neq \perp$ for some z in Z. But then $c \sqsubseteq_\perp z$ and, by Lemma 12 we get: $C_*(z) \sqsubseteq c \sqsubseteq z$ yielding the required contradiction.

□

Using all three lemmas, we can now define $\perp$ by the formula:

$$\mathrm{Bot}(e) \equiv (\exists a.a \not\sqsubseteq_e a) \wedge (\exists b.b \not\prec_e b)$$

and so the partial order can be defined by the formula:

$$x \sqsubseteq y \equiv \exists e.\mathrm{Bot}(e) \wedge x \sqsubseteq_e y$$

and Theorem 7 is proved.

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