Counting and Sampling Problems on Eulerian Graphs

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Abstract

In this thesis we consider two sets of combinatorial structures defined on an Eulerian graph: the Eulerian orientations and Euler tours. We are interested in the computational problems of *counting* (computing the number of elements in the set) and *sampling* (generating a random element of the set). Specifically, we are interested in the question of when there exists an efficient algorithm for counting or sampling the elements of either set.

The Eulerian orientations of a number of classes of planar lattices are of practical significance as they correspond to configurations of certain models studied in statistical physics. In 1992 Mihail and Winkler showed that counting Eulerian orientations of a general Eulerian graph is #P-complete and demonstrated that the problem of sampling an Eulerian orientation can be reduced to the tractable problem of sampling a perfect matching of a bipartite graph. We present a proof that this problem remains #P-complete when the input is restricted to being a planar graph, and analyse a natural algorithm for generating random Eulerian orientations of one of the afore-mentioned planar lattices. Moreover, we make some progress towards classifying the range of planar graphs on which this algorithm is rapidly mixing by exhibiting an infinite class of planar graphs for which the algorithm will always take an exponential amount of time to converge.

The problem of counting the Euler tours of undirected graphs has proven to be less amenable to analysis than that of Eulerian orientations. Although it has been known for many years that the number of Euler tours of any directed graph can be computed in polynomial time, until recently very little was known about the complexity of counting Euler tours of an undirected graph. Brightwell and Winkler showed that this problem is #P-complete in 2005 and, apart from a few very simple examples, e.g., series-parellel graphs, there are no known tractable cases, nor are there any good reasons to believe the problem to be intractable. Moreover, despite several unsuccessful attempts, there has been no progress made on the question of approximability. Indeed, this problem was considered to be one of the more difficult open problems in approximate counting since long before the complexity of exact counting was resolved. By considering a randomised input model, we are able to show that a very simple algorithm can sample or approximately count the Euler tours of almost every d-in/d-out directed graph in expected polynomial time. Then, we present some partial results towards showing that this algorithm can be used to sample or approximately count the Euler tours of almost every $2d$-regular graph in expected polynomial time. We also provide some empirical evidence to support the unproven conjecture required to obtain this result. As a side-result of this work, we obtain an asymptotic characterisation of the distribution of the number of Eulerian orientations of a random $2d$-regular graph.

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Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified. Most of the material in Chapter 2 and some of Section 3.3 of this thesis have been published in:

- Páidí Creed, "Sampling Eulerian orientations of the triangular lattice". *Journal of Discrete Algorithms*, **7** (2009), 168–180.
- Páidí Creed and Mary Cryan, "The number of Euler tours of a random d-in/d-out graph", In *Proceedings of Analysis of Algorithms 2010 (AofA 2010)*, pp. 117–130.

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Chapter 1

Introduction

Counting problems are a fundamental class of problems in the fields of *combinatorics* and *algorithms and complexity*. Moreover, solutions to such problems are of great practical value across the sciences because many real problems can be modelled as counting problems, particularly those related to the estimation of probabilities. For example, evaluating thermodynamic properties of physical systems [6], calculating the output probabilities of Bayesian networks [24], and calculating the probability of correct reconstruction of DNA from fragments using *sequencing by hybridisation* [4] are all counting problems.

Many counting problems, including those studied in this thesis, are defined on *graphs*. Before proceeding, we summarise the definitions of graphs and the graph-theoretic properties we will use later in the thesis. For a thorough introduction to the theory of graphs see, for example, [90].

Definition 1.1. An *undirected graph* $G = (V, E)$ is a set of vertices V and a set of edges E; each $e \in E$ is an unordered pair of vertices, $e = \{u, v\}$, where $u, v \in V$. The *degree* of a vertex $v \in V$ is the number of edges $e \in E$ containing v. We denote the degree of a vertex by $deg(v)$.

Definition 1.2. A *directed graph* $\vec{G} = (V, A)$ is a set of vertices V and a set of arcs A; each $a \in A$ is an ordered pair of vertices, $a = (u, v)$, where $u, v \in V$. We say an arc $a = (u, v)$ is directed *towards* v and *away from* u, and call u and v the *source* and *target* of a, respectively. The *out-degree* (resp. *in-degree*) of a vertex $v \in V$ is the number of arcs directed away from (resp. towards) v in A. We denote the in-degree and out-degree of v by $indeg(v)$ and $outdeg(v)$, respectively.

Definition 1.3. We say a directed graph $\vec{G} = (V, A)$ is an *orientation* of a graph

$G = (V,E)$ if there exists a bijection from E to A in which each $\{u,v\} \in E$ is mapped to (u,v) or (v,u).

Typically, we do not allow loops (edges or arcs where $u = v$) or multiple copies of the same edge or arc. However, in some cases we wish to allow duplicates, in which case we speak of *multigraphs* and *directed multigraphs*.

Definition 1.4. An *undirected multigraph* $G = (V,E)$ is a set of vertices V and a multiset of edges E.

Definition 1.5. A *directed multigraph* $G = (V,A)$ is a set of vertices V and a multiset of arcs A.

Remark 1.6. A graph or directed graph which does not contain any loops or multiple edges is often said to be *simple*. A simple directed graph is not necessarily an orientation of a simple graph, since (u,v) and (v,u) do not count as the same arc.

Often, we are interested in some particular substructure of a graph or directed graph G.

Definition 1.7. Let $G = (V,E)$ be a graph or directed graph. A *subgraph* of G is a graph $H = (V',E')$, where $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$, we say H is a *spanning subgraph* of G. If $H \neq G$ we say G is a *proper subgraph* of G.

Definition 1.8. A graph $G = (V,E)$ with vertex set $V = \{v_0, v_1, \ldots, v_{k-1}\}$ and edge set

$$E = \{\{v_i, v_{i+1 \bmod k}\} : i = 0, 1, \ldots, k-1\},$$

is called a *cycle*. The subgraphs of a graph G which are cycles are called the cycles of G.

Definition 1.9. A directed graph $G = (V,A)$ with vertex set $V = \{v_0, v_1, \ldots, v_{k-1}\}$ and arc set

$$A = \{(v_i, v_{i+1 \bmod k}) : i = 0, 1, \ldots, k-1\},$$

is called a *directed cycle*. The subgraphs of a directed graph $\vec{G}$ which are directed cycles are called the directed cycles of $\vec{G}$, or sometimes the cycles of $\vec{G}$.

Definition 1.10. A connected graph is called a *tree* if it contains no cycles. The spanning subgraphs of G which are trees are called the *spanning trees* of G.

Definition 1.11. A *rooted tree* is a tree $T = (V,E)$ with a single distinguished root vertex r. For each $v \in V$, the first node on the unique path from v to r is called the *parent* of v; all other neighbours of v are called the *children* of v.

Definition 1.12. An *arborescence* with root v is a rooted tree in which every edge has been directed towards v.

Let $\vec{G} = (V,A)$ be a directed graph and let $v \in V$. The set of arborescences of $\vec{G}$ rooted at v, denoted $\mathrm{ARB}(\vec{G},v)$, is the set of spanning subgraphs of $\vec{G}$ which are arborescences rooted at v. We denote by $\mathrm{ARB}(\vec{G})$ the set of all arborescences of G.

There are several classes of graphs in which some additional conditions are placed on E or A.

Definition 1.13. A graph $G = (V,E)$ is said to be *bipartite* if there exists a partition of V into $A \cup B$ such that for every $e = \{u,v\} \in E$, either $u \in A$ and $v \in B$ or $v \in A$ and $u \in B$.

Definition 1.14. Let $G = (V,E)$ be an undirected graph. We say G is *d-regular* if we have $deg(v) = d$ for every vertex $v \in V$.

Definition 1.15. Let $\vec{G} = (V,A)$ be a directed graph. We say $\vec{G}$ is *d-in/d-out* if we have $indeg(v) = outdeg(v) = d$ for every vertex $v \in V$.

Definition 1.16. A graph G is called *planar* if it can be drawn in the plane such that

1. each vertex is mapped to a unique point in $\mathbb{R}^2$;
2. each edge $\{u,v\}$ is a line whose endpoints are the points for u and v;
3. the interior of an edge $\{u,v\}$ contains no vertex and no point of any other edge.

This drawing is called an *embedding* of G. An embedding of G can be viewed as a decomposition of the plane $\mathbb{R}^2$ into a number of regions, exactly one of which is unbounded. We refer to the sets of edges bounding these regions as the *faces* of G. The unbounded region is referred to as the *unbounded face* of G.

Generally, when we speak of a graph G being planar we are considering the graph G and a fixed embedding of G. Assuming we have a fixed embedding of G, we have the notion of *dual graph* of G.

Definition 1.17. Let G be a planar graph with some fixed embedding in the plane. The *dual graph* of G, denoted $G^\star$, is the graph whose vertices are the faces of G and which has an edge joining each pair of faces η and σ which share an edge in G.

The classes of graphs and directed graphs which are the focus of this thesis are those which satisfy the *Eulerian condition*.

Definition 1.18. An undirected graph $G = (V,E)$ is said to be *Eulerian* if all vertices have even degree. A directed graph $\vec{G} = (V,A)$ is said to be Eulerian if we have $indeg(v) = outdeg(v)$ for every vertex $v \in V$.

Note that for all $d \geq 1$, every $2d$-regular undirected graph, and every d-in/d-out directed graph, satisfies the Eulerian condition. We are interested in two closely related structures which can only be defined on Eulerian graphs: *Eulerian orientations* and *Euler tours*.

Definition 1.19. Let $G = (V,E)$ be an undirected graph and let $\alpha : V \to \mathbb{Z}$ satisfy

$$\sum_{v \in V} \alpha(v) = 0\,.$$

We say an orientation of G is an α-*orientation*[1] of G if for every $v \in V$,

$$outdeg(v) - indeg(v) = \alpha(v)\,.$$

If $\alpha(v) = 0$ for all $v \in V$, then each α-orientation of G is an Eulerian directed graph. In this case, we refer to the α-orientations as *Eulerian orientations*, and denote this set by $\mathrm{EO}(G)$.

Remark 1.20. The set $\mathrm{EO}(G)$ is non-empty if and only if G satisfies the Eulerian condition (Definition 1.18).

Definition 1.21. Let $G = (V,E)$ be a graph. We define a tour of G to be a sequence of vertices $v_0 v_1 v_2 \ldots v_{k-1}$ in which $\{v_i, v_{i+1 \bmod k}\} \in E$ for $i = 0,1,\ldots,k-1$. An *Euler tour* of G is a tour that uses every edge exactly once. Two Euler tours are equivalent if one is a cyclic permutation of the other. We denote the set of all Euler tours of a graph G by $\mathrm{ET}(G)$.

Definition 1.22. Let $\vec{G} = (V,A)$ be a directed graph. We define a tour of G to be a sequence of vertices $v_0 v_1 v_2, \ldots, v_{k-1}$ in which $(v_i, v_{i+1 \bmod k}) \in A$ for $i = 0,1,\ldots,m-1$. An *Euler tour* is a tour that uses every arc exactly once. Two Euler tours are equivalent if one is a cyclic permutation of the other. We denote the set of all Euler tours of a directed graph $\vec{G}$ by $\mathrm{ET}(\vec{G})$.

[1] In [32], α-orientations were defined as orientations satisfying $outdeg(v) = \alpha(v)$. However, for our purposes, it is more convenient to work with this definition

Remark 1.23. We can also define Euler tours of multigraphs. In this case, we treat the duplicate edges as distinct so a sequence $v_0v_1\dots v_{m-1}$ can correspond to multiple Euler tours, as we assign each occurrence of uv or vu to a distinct instance of the duplicated edge $\{u,v\}$.

Remark 1.24. The set $\mathrm{ET}(G)$ is non-empty if and only if G connected and satisfies the Eulerian condition (Definition 1.18).

The remainder of this chapter is organised as follows. In §1.1.1 we summarise the theory of the complexity of counting problems. In §1.1.2 we describe the relationship between counting and sampling that is the key tool for finding approximate solutions to counting problems. In §1.2 we summarise the basic techniques used to construct sampling algorithms. Finally, in §1.3 we survey previous work on the problems of counting/sampling Eulerian orientations and Euler tours and put the work of this thesis into its correct context.

1.1 Counting and Sampling

1.1.1 Computational Complexity of Counting

In theoretical computer science, computational problems are often viewed as questions about relations between *problem instances* and sets of *feasible solutions*. Formally, if we consider instances and solutions to be encoded in the same finite alphabet Σ then a computational problem can be viewed as a relation $R \subseteq \Sigma^\star \times \Sigma^\star$ which maps each instance $x \in \Sigma^\star$ to a (finite) set of solutions

$$\Omega(x) = \{y \in \Sigma^\star : (x,y) \in R\}.$$

The most widely studied computational problems of this form are *decision problems*, where we are asking if a particular input has a solution.

Definition 1.25. Let Σ be a finite alphabet and $R \subset \Sigma^\star \times \Sigma^\star$. For a particular instance $x \in \Sigma^\star$, the *decision problem* R asks whether there exists some $y \in \Sigma^\star$ such that $(x,y) \in R$. We say a decision problem R is in the class NP (*non-deterministic polynomial time*) if

1. there exists a polynomial $p(\cdot)$ such that for any instance $x \in \Sigma^\star$ we have

$$|y| \leq p(|x|) \quad \forall y \in \Sigma^\star \text{ such that } (x,y) \in R;$$

2. there exists a polynomial time algorithm for testing the predicate $(x, y) \in R$.

The subset of NP consisting of problems for which there exist polynomial time algorithms is known as P. Although it is not known whether or not there exists a polynomial time algorithm for all problems in NP, it is well known [20, 60] (see [89] for an English translation of the second article and [3, Chapter 2] for a general reference) that there exists another subset of NP, known as the NP-complete problems, which are the "hardest" problems in NP; that is, if we had a polynomial time algorithm for any NP-complete problem then we could construct a polynomial time algorithm for any problem in NP. The canonical problem in this class is the *Boolean satisfiability* problem, defined below.

Example 1.26. The *Boolean satisfiability* problem (SAT) is a decision problem whose instances are Boolean expressions written using only $\wedge$, $\vee$, and $\neg$. Given a Boolean expression ψ, we want to determine whether there exists an assignment to the variables in ψ that will make ψ true. Given any particular assignment to the variables, we can quickly check whether or not the expression evaluates to true, so the problem is in NP. It was shown by Cook [20] (and, simultaneously, by Levin [60]) that for any instance x of an NP relation R we can construct an instance of SAT, ψ_{R}, such that there exists some $(x, y) \in \mathrm{R}$ if and only if ψ_{R} has a satisfying assignment. Hence, SAT is NP-complete.

There are two possible scenarios: either $\mathrm{P} = \mathrm{NP}$, in which case every algorithm in NP has a polynomial time algorithm; or $\mathrm{P} \subset \mathrm{NP}$, in which case there does not exist a polynomial time algorithm for any NP-complete problem.

Counting problems are a natural generalisation of decision problems. A decision problem asks whether some set of objects is non-empty, whereas a counting problem asks for something stronger: the exact number of objects in the set. Valiant formalised the computational complexity of counting problems in his seminal 1979 paper [92], in which he defined the complexity class #P.

Definition 1.27. Let Σ be a finite alphabet. Given a relation $R \subset \Sigma^\star \times \Sigma^\star$ we can define the *counting problem* for R as the problem of computing a function $N_R : \Sigma^\star \to \mathbb{N}$ with values

$$N_R(x) = \left|\{y \in \Sigma^\star : (x, y) \in R\}\right|.$$

We say the counting problem N_R is in the class #P if

1. there exists a polynomial $p(\cdot)$ such that for any instance $x \in \Sigma^\star$ we have

$$|y| \leq p(|x|) \quad \forall y \in \Sigma^\star \text{ such that } (x,y) \in R;$$

2. there exists a polynomial time algorithm for testing the predicate $(x,y) \in R$.

In general, for any reasonable function $f : \Sigma^\star \to \mathbb{N}$ there exists some relation $R \subseteq \Sigma^\star \times \Sigma^\star$ such that $f = N_R$. Thus, we will usually refer to counting problems as functions from $\Sigma^\star$ to $\mathbb{N}$, without referring to their defining relation R.

An algorithm that counts the number of solutions to a problem can also determine whether or not there exists a solution, so we cannot expect every counting problem to be efficiently solvable unless P = NP. We denote by FP the subset of #P which consists of all functions $f : \Sigma^\star \to \mathbb{N}$ which can be computed in polynomial time. As in the decision case, the set of problems which are "hardest" for #P are called #P-complete.

Definition 1.28. Given two counting problems f and g we say f is *polynomial-time Turing reducible* to g, denoted $f \leq_{PT} g$, if there exists a Turing machine with an oracle for f that computes $g(x)$ in time polynomial in $|x|$. We say a problem f is #P-complete if every problem in #P is polynomial-time Turing reducible to f.

Remark 1.29. #P-completeness is accepted to be strong evidence of computational intractability: exhibiting a polynomial time algorithm for all problems in #P would imply the collapse of the polynomial-time hierarchy [88] (for a general reference, see [3, Chapter 17]).

The construction used in [20] to show that SAT is NP-complete can be used to construct an instance of #SAT with the same number of solutions as any #P problem [92, Lemma 2]. Hence, every #P problem is reducible to #SAT; that is to say, #SAT is #P-complete. This is the observation used by Valiant in [92], in which #SAT is taken to be the defining problem for the class of #P-complete problems. The interesting result in [92] was that there exist #P-complete problems for which the decision problem is in P. Specifically, Valiant showed that every problem in #P can be reduced to counting perfect matchings in bipartite graphs; that is, counting perfect matchings of a bipartite graph is #P-complete. But the question of whether or not a bipartite graph has a perfect matching can be solved by computing a maximum flow in a network, something which can be achieved in polynomial time [98, §3.8].

There are some non-trivial counting problems for which polynomial-time algorithms exist. For example, counting perfect matchings in a bipartite planar graph G can

be achieved by evaluating the determinant of a special matrix, known as the Pfaffian [54]. Similarly, the number of spanning trees of a graph can be computed by evaluating the determinant of another special matrix, known as the Laplacian [55] (see §1.3.2 for a description of a related result). It is well-known that the determinant of any matrix can be computed in polynomial time [85]; hence, both these problems are in FP. See [47, Chapter 1] for a nice presentation of both these results.

However, the majority of interesting counting problems tend to be #P-complete. For example, counting perfect matchings [92], counting Eulerian orientations [69], counting solutions to propositional formulae in DNF [93], and estimating the partition function of several important models in statistical physics [97] are all examples of #P-complete problems. In all these cases, the best known algorithms for *exact counting* have exponential running time, despite the corresponding decision problems lying in P. This has shifted the focus towards designing polynomial algorithms for *approximate counting*. Many of these algorithms use randomisation to produce a close estimate of the true answer with high probability, but run in polynomial time. That is, at some points during the execution of the algorithm a "fair coin" is tossed and the next step of the algorithm depends on the outcome. This leads to algorithms which can produce different outputs on different executions with the same input and sometimes produce an output that does not match the requirements of the problem. The goal in analysing this type of algorithm is not only to prove that they run in polynomial time but also that the probability of the answer being correct is sufficiently high. Formally, we are interested in designing algorithms with the following specifications:

Definition 1.30. Let Σ be a finite alphabet and let $f : \Sigma^* \to \mathbb{N}$ be a counting problem on Σ. A *randomised approximation scheme* for f with confidence parameter δ is a randomised algorithm that takes as input an instance $x \in \Sigma^*$ and an error parameter ε, and outputs a number $N \in \mathbb{N}$ (this is a random variable of the "coin tosses" made by the algorithm),

$$\mathbb{P}[(1-\varepsilon)f(x) \leq N \leq (1+\varepsilon)f(x)] \geq 1-\delta. \tag{1.1}$$

We call this a *fully polynomial randomised approximation scheme*, or an *fpras*, if the algorithm runs in time bounded by a polynomial in $|x|$, ε^{-1} and $\log(\delta^{-1})$ for every instance x.

Remark 1.31. We cannot expect an *fpras* to exist for every problem in #P. For example, suppose we had an *fpras* for #SAT. That is, we have an algorithm $\mathcal{A}$ that takes as input

a propositional formula ψ and returns a number N satisfying

$$(1-\varepsilon)\#\mathrm{SAT}(\psi) \leq N \leq (1+\varepsilon)\#\mathrm{SAT}(\psi)$$

with probability $3/4$. Then, if ψ has no satisfying assignments $\mathcal{A}$ returns 0 with probability $3/4$. But if ψ has one or more satisfying assignments $\mathcal{A}$ returns some number $N > 0$ with probability $3/4$. Hence, $\mathcal{A}$ can determine whether or not ψ is satisfiable with $3/4$ accuracy. It is widely believed [3, Chapter 7] that such an algorithm cannot exist for any NP-complete problem; hence, it is unlikely that an *fpras* for #SAT exists.

In light of Remark 1.31 we only search for an *fpras* for a problem $f \in \#\mathrm{P}$ if we know that the corresponding decision problem is in P.

1.1.2 The relationship between counting and sampling

In almost all known approximate counting algorithms, the use of randomness referred to in Definition 1.30 is necessary. That is to say, apart from a few notable cases, e.g., [96, 8], the only known approximate counting algorithms are randomised algorithms. One of the reasons for this fact is that most approximation algorithms for #P-complete problems take advantage of a close relationship between approximate counting and random generation discovered in 1986 by Jerrum, Valiant, and Vazirani [52]. Before we describe this relationship, we define the sampling problem for a relation R.

Definition 1.32. Let Σ be a finite alphabet and let $\mathrm{R} \subset \Sigma^\star \times \Sigma^\star$. The *sampling problem* for R asks for a randomised algorithm that, given an instance $x \in \Sigma^\star$, generates uniformly at random $y \in \Sigma^\star$ such that $(x,y) \in \mathrm{R}$.

Often, it is too much to ask for an algorithm that samples exactly from the uniform distribution. However, when this is the case we are sometimes able to come up with an algorithm that generates samples from a distribution that is close enough to uniform for practical purposes. In order to be able to speak about the accuracy of a particular sampling algorithm we need a notion of distance between probability distributions; the *variation distance* (defined below) is a standard measure of distance between two probability distributions [41, Chapter 2].

Definition 1.33. The *total variation distance* $\| \mu_1 - \mu_2 \|_{TV}$ between two distributions μ_1, μ_2 on a set Ω is defined as

$$\| \mu_1 - \mu_2 \|_{TV} = \frac{1}{2} \sum_{\omega \in \Omega} |\mu_1(\omega) - \mu_2(\omega)| = \max_{A \subseteq \Omega} |\mu_1(A) - \mu_2(A)| .$$

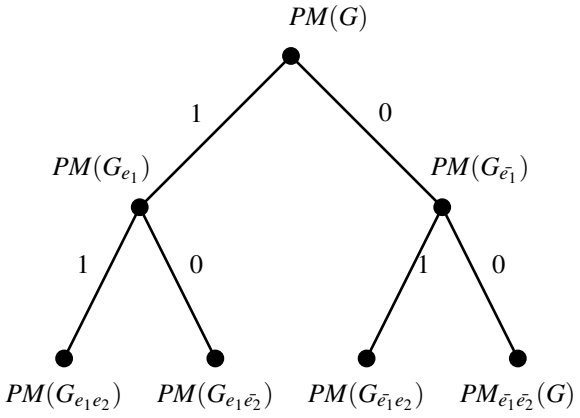

Figure 1.1: Top three levels of a decomposition tree for the perfect matching problem

Now that we have a notion of distance between probability distributions, we are able to define what it means for an algorithm to be an *almost uniform sampler*.

Definition 1.34. Let Σ be a finite alphabet and $\mathrm{R} \subset \Sigma^\star \times \Sigma^\star$. An *almost uniform sampler* for R is a randomised algorithm which, for instance $x \in \Sigma^*$ and sampling tolerance $\delta > 0$, outputs a random variable $W \in \{y \in \Sigma^\star : (x,y) \in \mathrm{R}\}$ such that the variation distance between the distribution of W and the uniform distribution is no more than δ. We call this a *fully polynomial almost uniform sampler*, or an *fpaus*, if the algorithm runs in time polynomial in $|x|$ and $\log \delta^{-1}$ for all instances x.

There is a class of relations for which the existence of an *fpras* for the counting problem is equivalent to the existence of a *fpaus* for the sampling problem [52]. The relations for which this is the case are those which satisfy a property known as *self-reducibility*.

Definition 1.35. A relation $\mathrm{R} \subset \Sigma^\star \times \Sigma^\star$ is said to be *self-reducible* if and only if

1. there exists a polynomial time computable function $g : \Sigma^\star \to \mathbb{N}$ such that

$$(x,y) \in R \Rightarrow |y| = g(|x|)\,;$$

2. there exist polynomial time computable functions $\psi : \Sigma^\star \times \Sigma^\star \to \Sigma^\star$ and $\sigma : \Sigma^\star \to \mathbb{N}$ satisfying

$$\sigma(x) = O(\log(|x|))\,, \tag{1.2}$$

$$g(x) > 0 \to \sigma(x) > 0 \quad \forall x \in \Sigma^\star\,, \tag{1.3}$$

$$|\psi(x,w)| < |x| \quad \forall x,w \in \Sigma^\star\,, \tag{1.4}$$

and such that, for all $x \in \Sigma^\star$ and all $y = y_1 \cdots y_{g(|x|)} \in \Sigma^\star$,

$$(x, y_1 y_2 \cdots y_{g(|x|)}) \in \mathrm{R} \Leftrightarrow (\psi(x, y_1 \cdots y_{\sigma(x)}), y_{\sigma(x)+1} \cdots y_{g(|x|)}) \in \mathrm{R}\,. \tag{1.5}$$

The above definition requires that if we choose the first $\sigma(x)$ bits of a solution to x we can construct another smaller instance of the same problem $x' = \psi(x, y_1 \cdots y_{\sigma(x)})$ such that the solutions of x starting with $y_1 \cdots y_{\sigma(x)}$ are obtained by concatenating $y_1 \cdots y_{\sigma(x)}$ with a solution y' of x', and that every solution of x' gives a solution to x in this way.

Intuitively, this says that, given any instance x, we can construct a number of smaller instances of the same problem such that the solutions of x are obtained by extending solutions to the smaller instances. Thus, we can think of the function ψ as decomposing the set of solutions to x into a tree structure such that

1. each vertex is a subset of the solutions to x, with all solutions at the root, and the sets at the children of a node partitioning the set at that node ;
2. each edge is labelled with a string $w \in \Sigma^\star$;
3. each leaf is of the form $S = \{y\}$, with y equal to the concatenation of the labels on the path from the root to S.

We will now demonstrate the concept of self-reducibility with a concrete example.

Example 1.36. Let $G = (V, E)$ be a graph. A *matching* of G is a set $M \subset E$ such that each vertex is contained in no more than one $e \in M$. We say a matching M is *perfect* if every vertex in V is contained in some edge $e \in M$. We denote by $PM(G)$ the set of perfect matchings of G. Now, suppose we have a relation R where $(x, y) \in \mathrm{R}$ if and only if x encodes a graph $G = (V, E)$ and y encodes a perfect matching of G. Each solution is a string $y_1 y_2 \cdots y_m$ where $y_i = 1$ if $e_i \in M$ and $y_i = 0$ if $e_i \notin M$ (we are assuming some order on the edges $E = \{e_1, e_2, \ldots, e_m\}$).

Given any edge $e = \{u, v\}$ of G we can construct two graphs G_e, in which we remove u and v and all edges incident with them from G, and $G_{\bar{e}}$, in which we only remove e from the set of edges. Every perfect matching of $G_{\bar{e}}$ is a perfect matching of G; in fact, the perfect matchings of $G_{\bar{e}}$ are exactly the set of perfect matchings of G that do not contain e. The perfect matchings of G_e are not perfect matchings of G, since neither u nor v is contained in any perfect matching of G_e. But we can add e to every perfect matching of G_e to obtain a perfect matching of G; hence, the perfect

matchings of G_e correspond exactly to the perfect matchings of G that contain e. By repeating this process, we obtain a tree decomposition of the set of perfect matchings of G in the form of Figure 1.1

Suppose we have a self-reducible relation R and an *fpras*, COUNT, for the counting problem of R. The *fpaus* implied by the result of [52] is given (roughly) in Algorithm 1. If we had an algorithm for computing N_{R} exactly, then Algorithm 1 would be an exact sampler. However, since we only have an *fpras* we must make do with approximations to each of the $N_{\mathrm{R}}(x_i)$ when we compute the probabilities at Step 4. However, we are able to obtain close enough approximations to the values $N_{\mathrm{R}}(x_i)$ to ensure that the algorithm is an *fpaus*. See [52] for details.

Algorithm 1: GEN

Input: $x \in \Sigma^\star$ and self-reducible $\mathrm{R} \subset \Sigma^\star \times \Sigma^\star$

Output: An almost uniformly random $y \in \Sigma^\star$ such that $(x,y) \in \mathrm{R}$

1 **begin**
2 Obtain $x_1, x_2, \ldots, x_n$ from x by choosing the first $\sigma(|x|)$ bits of the solution;
3 **for** $i \leftarrow 1$ **to** n **do**
4 $\rho_i \leftarrow \mathrm{COUNT}(x_i)$;
5 **end**
6 Set $x' = x_i$ with probability proportional to ρ_i;
7 $y_i \leftarrow \mathrm{GEN}(x')$;
8 **return** the $y \in \Sigma^\star$ with $(x,y) \in R$ corresponding to y_i;
9 **end**

To go in the other direction, turning an *fpaus* into an *fpras* we use Algorithm 2. Assuming we have an *fpaus* for a self-reducible R, GEN, Algorithm 2 is an *fpras* for N_{R}. Again, see [52] for the details. In §2.7, we describe the application of Algorithm 2 to a concrete example.

Fortunately, for those of us interested in algorithms for counting and sampling, a great deal of #P-complete problems are self-reducible. Moreover, in the cases where they are not, e.g., transposition tables [22, 23], the problems often satisfy a slightly weaker self-reducible property which means we can still use Algorithm 2 to construct approximate counting algorithms.

Algorithm 2: COUNT

Input: $x \in \Sigma^\star$ and self-reducible $\mathrm{R} \subset \Sigma^\star \times \Sigma^\star$ and a constant $\varepsilon > 0$

Output: A number N such that $(1-\varepsilon)N \leq N_{\mathrm{R}}(x) \leq (1+\varepsilon)N$

```
begin
    Π := 1, m := g(x), t := 180|x|^{3c};
    while g(x) > 0 do
        Let S = {y_1, ..., y_t} be the set of results obtained by making t calls to
        GEN with input (x, ε/11m);
        Let w be the most common prefix of elements of S;
        ρ := |{y ∈ S : w is an initial segment of y}|/|S|;
        x := ψ(x, w);
        Π := Π/ρ;
    end
    return Π
end
```

1.1.3 Applications of counting/sampling algorithms

An important class of counting/sampling problems come from the field of statistical mechanics (also known as statistical physics) [7]. Statistical mechanics provides a framework for relating the microscopic (local) properties of individual atoms and molecules in some physical object to the macroscopic (global) properties that can be observed in the real world; in particular, it provides an interpretation of the thermodynamic properties of an object, such as free energy and entropy, in terms of the microscopic properties of the configuration of the particles within that object. These systems will, in general, contain a large number of particles, thus rendering exact computation impractical due to an unfeasibly large number of possible configurations. However, if we have an *fpaus* for generating random configurations, we can generate a large number of samples and then estimate the thermodynamic property of interest by averaging over all the samples generated. Moreover, an important quantity known as the *partition function* is no more than the solution to the counting problem which enumerates the number of configurations (possibly with some weight associated with each configuration). In many cases, evaluating the partition function is #P-complete [45, 97] and many algorithms for approximating the partition function use *Markov chain Monte Carlo* [50, 62]. One of the problems studied in this thesis,

counting and sampling Eulerian orientations of planar graphs, is equivalent to counting and sampling configurations in so-called *ice-type models* [7, Chapter 8].

1.2 Markov chain Monte Carlo

In this section we define *Markov chains* and explain how they are used in the construction of algorithms for almost uniform sampling. General references for the material of this section are, e.g., [41, Chapter 6], [70, Chapters 7, 10 and 11] and [47].

Definition 1.37. A discrete-time *Markov chain* $\mathcal{M}$ with finite state space Ω is a stochastic process $(X_t)_{t\geq 0}$, with $X_t \in \Omega$ for all $t = 1, 2, \ldots$, that satisfies the *Markov property*:

$$\mathbb{P}[X_{t+1} = y \,|\, X_t = x_t, X_{t-1} = x_{t-1}, \ldots, X_0 = x_0] = \mathbb{P}[X_{t+1} = y | X_t = x_t]\,; \tag{1.6}$$

that is, the probability of being in a particular state at the $(t+1)$-th step depends only on the state at the t-th step.

In the applications of Markov chains to sampling problems we are only ever interested in *time-homogeneous* Markov chains, where the probability in (1.6) depends only on the state and not on the time t. In this situation, we can define a Markov chain by its *transition probability matrix* P:

$$P(x,y) = \mathbb{P}[X_{t+1} = y | X_t = x] \quad \forall x, y \in \Omega\,. \tag{1.7}$$

For $t \geq 0$, we then define the t-step probability distribution of $\mathcal{M}$ by the following inductive formula:

$$P^t(x,y) = \begin{cases} I(x,y)\,, & \text{if } t = 0\,; \\ \sum_{z\in\Omega} P^{t-1}(x,z)P(z,y)\,, & \text{otherwise}\,, \end{cases}$$

where $I(x,y) = 1$ if $x = y$ and 0 otherwise. Every Markov chain has a number of *stationary distributions* which are the distributions $P^t(x,\cdot)$ can converge to as $t \to \infty$.

Definition 1.38. Let $\mathcal{M}$ be a time-homogeneous Markov chain with state space Ω and transition probability matrix P. A distribution $\pi : \Omega \to [0,1]$ is called a *stationary distribution* of $\mathcal{M}$ if

$$\sum_{x\in\Omega} \pi(x)P(x,y) = \pi(y) \quad \forall y \in \Omega\,. \tag{1.8}$$

An arbitrary Markov chain can have any number of stationary distributions, but in the application of Markov chains to designing sampling algorithms we will want chains that always converge to the distribution we are interested in. We now define the *ergodic* property, which ensures that a chain has a unique stationary distribution.

Definition 1.39. Let $\mathcal{M}$ be a time-homogeneous Markov chain with state space Ω and transition probability matrix P. $\mathcal{M}$ is said to be *ergodic* if $\mathcal{M}$ is *aperiodic* and *irreducible*:

$$\mathcal{M} \text{ is aperiodic if } \forall x \in \Omega\,,\ \gcd\{t : P^t(x,x) > 0\} = 1\,; \tag{1.9}$$

$$\mathcal{M} \text{ is irreducible if } \forall x,y \in \Omega,\ \exists t \text{ such that } P^t(x,y) > 0\,. \tag{1.10}$$

Remark 1.40. A sufficient (and easily testable) condition for aperiodicity is:

$$\forall x \in \Omega\,,\ P(x,x) > 0\,. \tag{1.11}$$

The following well-known [41] theorem shows that ergodic Markov chains always converge to a unique stationary distribution.

Theorem 1.41. *Every ergodic Markov chain $\mathcal{M}$ has a unique stationary distribution π; moreover, $\mathcal{M}$ converges to π in the sense that $P^t(x,y) \to \pi(y)$ for all $x,y \in \Omega$.*

We say a Markov chain is *time-reversible* if it satisfies the *detailed balance condition*

$$\forall x,y \in \Omega \quad \pi(x)P(x,y) = \pi(y)P(x,y),\,. \tag{1.12}$$

If P is symmetric then (1.12) is true if and only if π is uniform over Ω.

A very simple example of a Markov chain is a *random walk* on a graph or directed graph.

Example 1.42. Let $G = (V,E)$ be a graph. The random walk on G is a Markov chain with state space V and probability transition matrix

$$P(u,v) = \begin{cases} \frac{1}{deg(u)} & \text{if } \{u,v\} \in E\,; \\ 0 & \text{otherwise} \end{cases}\,. \tag{1.13}$$

This is just the random process on V which, at each step, moves to a random neighbour of the current vertex. We observe that this Markov chain is ergodic if and only if G is connected and non-bipartite. Obviously, the graph must be connected for the random walk to be irreducible. On the other hand, if G is bipartite then the random walk

is periodic with period 2: after t steps we are in the same part of the bipartition we started in if and only if t is even. The presence of a single odd cycle ensures that the random walk is aperiodic, since this gives an odd length path by which the chain can return to any vertex on the cycle from itself. The stationary distribution of this chain is not uniform; if we take π to denote the stationary distribution of the random walk, we have

$$\pi(v) = \frac{deg(v)}{2m},$$

where $2m = \sum_{v \in V} deg(v)$. It is easy to check that this distribution satisfies (1.8) since, for all $v \in V$,

$$\sum_{u \in V} \pi(u)P(u,v) = \frac{1}{2m}|\{u \in V : P(u,v) > 0\}| = \frac{deg(v)}{2m} = \pi(v).$$

We can also define random walks on directed graphs. For a directed graph $G = (V,A)$ we replace (1.13) by

$$P(u,v) = \begin{cases} \frac{1}{outdeg(u)} & \text{if } (u,v) \in A; \\ 0 & \text{otherwise} \end{cases}.$$

In this case, at each step we follow a random outgoing arc of current vertex to reach the next state. However, in this case we do not always have a nice stationary distribution as we do for random walks on undirected graphs.

Given a relation $R \subset \Sigma^\star \times \Sigma^\star$ we can construct an almost uniform sampler for R by defining a class of Markov chains such that for each instance we have an ergodic chain $\mathcal{M}$ that converges to the uniform distribution on the set of feasible solutions.

If we run a Markov chain $\mathcal{M} = (\Omega, P)$ for an infinite number of steps, and then take the current state as our output, we know that this will be be sampled from the stationary distribution of $\mathcal{M}$. In practice, we can only run the chain for a finite number steps. A *Markov chain Monte Carlo (MCMC)* algorithm generates random samples from a set Ω by starting at some state $x_0 \in \Omega$ and then running a Markov chain for sufficiently many steps that the probability of any particular $y \in \Omega$ being the current state is (approximately) $\pi(y)$. The quality of the sample improves as a function of the number of steps. However, given an arbitrary Markov chain it is far from clear how long we need to run it for before the distribution is "close enough" to the stationary distribution (i.e., within ε in total variation distance), and this can vary hugely even across Markov chains defined on the same state space. The time it takes for this convergence to occur is known as the *mixing time* of the Markov chain, defined below.

In the following, we assume we are dealing with an infinite family of Markov chains defined by the instances of some sampling problem. The properties described are asymptotic properties for all chains in such a class.

Definition 1.43. Let $\mathcal{M}$ be a finite, discrete time Markov chain with state space Ω. For $\varepsilon > 0$, the *mixing time* of $\mathcal{M}$ from initial state $x \in \Omega$, $\tau_x(\varepsilon)$, is defined as

$$\tau_x(\varepsilon) = \min\{t : ||P^t(x,\cdot) - \pi||_{TV} \leq \varepsilon\}.$$

We define the *mixing time* of $\mathcal{M}$, $\tau(\varepsilon)$, to be the maximum over the mixing times from each state:

$$\tau(\varepsilon) = \sup_{x\in\Omega} \tau_x(\varepsilon).$$

Definition 1.44. A Markov chain $\mathcal{M}$ on state space Ω is said to be *rapidly mixing* if, for all for all initial states $x \in \Omega$ and all $\varepsilon > 0$, $\tau_x(\varepsilon)$ is bounded above by some function which is polynomial in $|x|$ and ε^{-1}. A Markov chain is said to be *torpidly mixing* if there exist some $\varepsilon > 0$ and $x \in \Omega$ for which $\tau_x(\varepsilon)$ is bounded below by some function exponential in $|x|$.

For example, if the state space of the Markov chain is the set of perfect matchings (Example 1.36), then the mixing time must be bounded in terms of the number of vertices and edges of G.

Suppose R is a relation defining a self-reducible counting problem. If we have a Markov chain $\mathcal{M}$ which is rapidly mixing on the set of solutions,

$$\Omega(x) = \{y \in \Sigma^\star : (x,y) \in R\},$$

for all $x \in \Sigma^\star$, then $\mathcal{M}$ is an *fpaus* for R, and we can use the Algorithm COUNT (Algorithm 2) to construct an *fpras* for the counting problem N_{R}. So, the problem of finding an *fpras* can be reduced to the problem of finding a rapidly mixing Markov chain.

There are two principal techniques used in the analysis of the mixing times of Markov chains: *coupling* and *conductance*. Coupling is used to show that chains are rapidly mixing, whereas conductance can be used to show that chains are rapidly mixing or torpidly mixing.

1.2.1 Coupling

Coupling is a standard technique for proving upper bounds on the mixing time of Markov chains [1, 16].

Definition 1.45. A *coupling* $\mathcal{C}$ of a Markov chain $\mathcal{M}$ is a stochastic process $(X_t, Y_t)_{t\geq 0}$ on $\Omega \times \Omega$ such that each of the marginal distributions, $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, is a faithful copy of the original Markov chain.

One approach to bounding the mixing time of a Markov chain is to use the *coupling inequality* of Aldous [1].

Lemma 1.46 ([1])**.** *Let $\mathcal{M} = (\Omega, P)$ be a time-homogeneous, finite state Markov chain. For every state $x \in \Omega$, the variation distance between π and the t-step distribution at time t, $P^t(x,\cdot)$, is bounded above by the probability of any coupling coalescing by time t, i.e.,*

$$\max_{x\in\Omega} \| P^t - \pi \|_{TV} \leq \sup_{X_0, Y_0} \mathbb{P}[X_t \neq Y_t]\,. \tag{1.14}$$

Thus, in order to obtain a polynomial bound on the mixing time it suffices to construct a coupling which will have coalesced (with high probability) after a polynomial number of steps. In general, couplings can be difficult to describe and analyse, so it is common to restrict attention to *Markovian couplings*.

Definition 1.47. A coupling $\mathcal{C}$ of a Markov chain $\mathcal{M} = (\Omega, P)$ is said to be *Markovian* if $\mathcal{C}$ is itself a Markov chain on $\Omega \times \Omega$. That is, the distribution of the pair of states (X_{t+1}, Y_{t+1}) depends only on the values of (X_t, Y_t).

The advantage of Markovian couplings over general couplings is that they allow us to analyse the behaviour of the coupling by considering only single steps at a time. We now give an illustrative example of a Markovian coupling.

Example 1.48. Let $G = (V, E)$ be a graph. A *k-colouring* of G is a mapping

$$\sigma : V \to \{1, 2, \ldots, k\}\,.$$

We say σ is a *valid k-colouring* if $\sigma(u) \neq \sigma(v)$ for every $\{u, v\} \in E$. Let Ω be the set of all *valid k-colourings* of G. There is a very simple Markov chain $\mathcal{M}$ on Ω. Let $X_t \in \Omega$ denote the current state of the chain. We now define how one step of the Markov chain obtains X_{t+1} from X_t.

One step of the chain $\mathcal{M}$

1. Choose $v \in V$ and $c \in \{1, 2, \ldots, k\}$ uniformly at random;
2. Let X' denote the colouring in which $X'(u) = X_t(u)$ for all $u \neq v$ and $X'(v) = c$;
3. If $X' \in \Omega$ then set $X_{t+1} = X'$. Otherwise, set $X_{t+1} = X_t$.

This chain is ergodic when the number of colours k is greater than the minimum degree of G plus 2, see, e.g., [70, §11.5], and converges to the uniform distribution on Ω. The simplest coupling of $\mathcal{M}$ is the one which chooses the same vertex v and colour c for both chains. If we can re-colour v to c in both chains, then we have increased the number of vertices that the two copies of the chain agree on. However, if we can re-colour v in one copy of the chain but not the other, and v has the same colour in the current state of both chains, then the number of disagreements will increase,

It has been shown that the simple coupling described above can be used to prove rapid mixing of $\mathcal{M}$ whenever we have $k \geq 2\Delta + 1$, where Δ is the maximum degree of G [46]. It is known that the Markov chain $\mathcal{M}$ is rapidly mixing for graphs with fewer colours but these results require more sophisticated couplings, some of which are non-Markovian. See [34] for a survey of these results.

We will give more details of coupling and a technique called *path coupling* [16, 13] when we analyse the mixing time of a particular Markov chain in Chapter 2.

1.2.2 Conductance

The coupling technique can only be used to show that a Markov chain is rapidly mixing and, in the case of Markovian couplings at least, cannot always be applied [58]. An alternative approach is to estimate the *conductance* of the chain.

Definition 1.49. Let $\mathcal{M} = (\Omega, P)$ be a discrete state, time-homogeneous, Markov chain with stationary distribution π. For any non-empty set $S \subset \Omega$ the *conductance* $\Phi(S)$ of S is defined to be

$$\Phi(S) = \frac{\sum_{x \in S, y \in \Omega \setminus S} \pi(x) P(x,y)}{\pi(S)}.$$

We define the conductance of $\mathcal{M}$ as being the minimum over all sets:

$$\Phi(\mathcal{M}) = \min_{\substack{S \subset \Omega \\ 0 < \pi(S) \leq 1/2}} \Phi(S).$$

In 1989, Sinclair and Jerrum [79] showed how conductance could be used to analyse the mixing times of Markov chains by exhibiting the following close relationship between the conductance and mixing time of a Markov chain.

Theorem 1.50 ([79])**.** *Let $\mathcal{M} = (\Omega, P)$ be a time-homogeneous, reversible Markov chain and suppose $P(x,x) \geq 1/2$ for all $x \in \Omega$. Then, the mixing time, $\tau(\varepsilon)$, and conductance, $\Phi(\mathcal{M})$, satisfy*

1. *For all* $x \in \Omega$, $\tau_x(\varepsilon) \leq \frac{2}{\Phi(\mathcal{M})^2}\left(\log(\varepsilon^{-1}) + \log(\pi(x)^{-1})\right)$;

2. $\max_{x\in\Omega} \tau_x(\varepsilon) \geq \frac{1-2\Phi(\mathcal{M})}{2\Phi(\mathcal{M})}\log(\varepsilon^{-1})$.

Thus, if one can show that the conductance of $\mathcal{M}$ is "large", i.e., bounded below by an inverse polynomial in the size of the elements of Ω, then one can deduce that $\mathcal{M}$ is rapidly mixing. On the other hand, if the conductance of $\mathcal{M}$ is exponentially small then $\mathcal{M}$ is torpidly mixing. Moreover, to show torpid mixing we only need to show a bound on the mixing time for some $\varepsilon > 0$. Defining the boundary of a set S as

$$\partial S = \{x \in S : P(x,y) > 0 \text{ for some } y \in \Omega\backslash S\}$$

we get

$$\Phi(S) \leq \pi(\partial S)/\pi(S)\,. \tag{1.15}$$

Hence, to show that a chain is torpidly mixing it suffices to find a set for which the right-hand side of (1.15) is bounded above by some exponentially small function. This is encapsulated in the following theorem, taken from [63]:

Theorem 1.51. *If, for some* $S \subset \Omega$ *satisfying* $0 < |S| \leq |\Omega|/2$*, the ratio* $\pi(\partial S)/\pi(S)$ *is exponentially small in the size of the elements of* Ω*, then the Markov chain is torpidly mixing.*

Conductance can also be used to find upper bounds on the mixing time, particularly using a technique known as *canonical paths*, due to Jerrum and Sinclair [49]. This technique has been successfully applied to analysing many Markov chains, e.g., [22, 51], but will not be used in this thesis. However, in Chapter 2 we will use a technique known as *comparison*, which is also based on conductance, to analyse the mixing time of a Markov chain (see §2.6.2 for details).

1.2.3 Dart throwing

An alternative approach that is sometimes applicable is *dart throwing*. Suppose we have a set S whose elements we want to count or sample, but do not know of a polynomial time algorithm for doing so. Now suppose that we have a polynomial time algorithm for testing $x \in S$ and that there exists a set $S' \supseteq S$ for which we have an *fpaus* and which satisfies

$$\frac{|S|}{|S'|} \geq \frac{1}{p(n)}\,, \tag{1.16}$$

where n is the size of the elements of S and $p(\cdot)$ is a polynomial. Then, using our *fpaus* for S', we generate elements of S' until we obtain an element of S. Each element of S' is generated in polynomial time and (1.16) means we will only have to generate a polynomial number of elements of S' before we get an element of S. Moreover, we can approximate

$$\frac{|S|}{|S'|} \tag{1.17}$$

by generating a polynomial number of samples from S' and then taking the proportion of them which lie in S as an estimation of (1.17). Then, if we can also calculate or approximate $|S'|$, we can use this to obtain an approximation of $|S|$. The *fpaus* for S' could be a rapidly mixing Markov chain, e.g., [36], or S' could be some simple set that we can sample from without recourse to MCMC, e.g., [27].

In Chapter 3, we describe how this approach can be used to sample Euler tours of a graph or directed graph. The algorithms resulting from this are shown to run in expected polynomial time for almost every d-in/d-out graph and we conjecture that they run in expected polynomial time for almost every $2d$-regular graph. These results are proven using *random graphs*.

1.3 Eulerian orientations and Euler tours

Recall that we denote the set of Eulerian orientations of a graph G by $\mathrm{EO}(G)$ and the set of Euler tours by $\mathrm{ET}(G)$. It is a classical graph-theoretic result that the Eulerian graphs are the only graphs that admit Eulerian orientations and that Eulerian graphs and directed graphs are the only graphs and directed graphs to admit Euler tours. In this thesis, we are interested in the complexity of the following two problems:

Name.	#EO.
Instance.	An undirected graph G.
Output.	The number of Eulerian orientations of G.

and

Name.	#ET.
Instance.	An undirected graph G.
Output.	The number of Euler tours of G.

We are also interested in these problems under various restrictions, specifically:

Name.	#PlanarEO.
Instance.	A planar graph G.
Output.	The number of Eulerian orientations of G.

and

Name.	#DirectedET.
Instance.	A directed graph $\vec{G}$.
Output.	The number of Euler tours of $\vec{G}$.

Recall the conditions for $\mathrm{EO}(G)$ $\mathrm{ET}(G)$ to be non-empty (Remarks 1.20 and 1.24). Hence, to solve the decision problem $\mathrm{EO}(G) \neq \emptyset$ all we need to do is check that each vertex satisfies the Eulerian condition (Definition 1.18). To solve the decision problem $\mathrm{ET}(G) \neq \emptyset$ we need to check that each vertex satisfies the Eulerian condition and that G is connected. Checking that each vertex has even degree and that a graph is connected can both be achieved in polynomial time, so both decision problems are in P. However, both #EO and #ET are #P-complete. Exact counting of Eulerian orientations was shown to be #P-complete by Mihail and Winkler in 1992 [68, 69]. In [68, 69], Mihail and Winkler also described a method to construct an *fpras* for #EO that works for every undirected graph. In the case of Euler tours, the complexity of counting Euler tours remained open until 2005, when it was shown to be #P-complete by Brightwell and Winkler [15]. More recently, Ge and Stefankovic [37] have shown that counting Euler tours is #P-complete for 4-regular planar graphs. The status of approximately counting or sampling the Euler tours of an undirected graph is still unresolved, even for simple classes, e.g., 4-regular planar graphs. Indeed, the only positive result to date is the recent proof of Chebolu et al. [17] that we can exactly count the Euler tours of series-parallel graphs in polynomial time. In the following two sections, we present a summary of what is known about these two problems.

1.3.1 Eulerian orientations

The number of Eulerian orientations of an undirected graph is a subject that has received attention in statistical physics [61, 6, 29], combinatorics [78, 95, 65, 33], and theoretical computer science [68, 69, 62, 38]. The problem of sampling/counting Eulerian orientations in a general Eulerian graph is almost entirely understood: the complexity of exact counting resolved by Mihail and Winkler in 1992 [68, 69]; Mihail and Winkler (also in [68, 69]) provided a reduction for constructing an *fpaus* or an *fpras* for any Eulerian graph, which we will describe in a moment.

The Eulerian orientations of certain planar lattices have special significance in statistical physics as the configurations of so-called *"ice models"* [61, 6, 7, 29], motivating studying the complexity of counting/sampling Eulerian orientations restricted to planar graphs or planar lattices. Lieb [61] and Baxter [6] have derived asymptotic expressions for the number of Eulerian orientations of the square and triangular lattice.

Definition 1.52. The *infinite square lattice* is the 4-regular infinite graph with vertex set $\{(i,j) : i,j \in \mathbb{Z}\}$ and such that for every $i,j \in \mathbb{Z}$ there is an edge joining (i,j) to $(i,j\pm 1)$ and $(i\pm 1,j)$.

Definition 1.53. The *infinite triangular lattice* is the 6-regular infinite graph with vertex set $\{(i,j) : i,j \in \mathbb{Z}\}$ and such that for every $i,j \in \mathbb{Z}$ there is an edge joining (i,j) to $(i\pm 1,j)$, $(i,j\pm 1)$, $(i-1,j+1)$, and $(i+1,j-1)$.

The ice models defined on two further Eulerian lattices are studied in [29] but we will not be considering them in this thesis. The lattices defined in Definition 1.52 and 1.53 are both infinite, so finite sub-lattices are used in order to be able to perform computational analysis; *boundary conditions* are used to simulate the influence of the infinite lattice.

Definition 1.54. A *solid subgraph* of an infinite lattice is subgraph which can be obtained by taking some finite cycle and everything lying on its interior.

Suppose, for example, we are taking the subgraph of the square lattice induced by the set

$$\{(i,j) : 0 \leq i,j < n\},$$

which we denote by $G(n,n)$.

Each of the $(n-2)^2$ internal vertices of $G(n,n)$ (those in which $i,j \notin \{0,n-1\}$) has degree 4 and we enforce the Eulerian condition on each of these. The boundary vertices (those with at least one of i and j equal to 0 or $n-1$) have one of the following boundary conditions applied to them.

Fixed Suppose we have fixed an orientation for the rest of the infinite lattice, which extends to an Eulerian orientation of $G(n,n)$. Then, sampling an Eulerian orientation consistent with the fixed boundary condition is equivalent to sampling an α-orientation of $G(n,n)$ for a specific $\alpha : V \to \mathbb{Z}$, which we will now define. For each (i,j) with $0 < i,j < n-1$ we set $\alpha(v_{i,j}) = 0$. Each of the

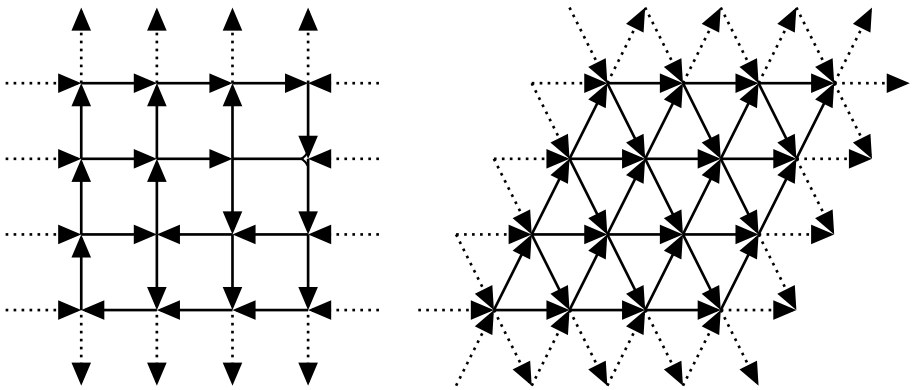

Figure 1.2: Square and triangular ice

non-corner boundary vertices has $\alpha(v) = \pm 1$ and each of the corner vertices has $\alpha(v) \in \{-2, 0, 2\}$ such that

$$\sum_{i=0}^{n-1} \alpha(v_{0,i}) + \alpha(v_{n-1,i}) + \alpha(v_{i,0}) + \alpha(v_{i,n-1}) = 0\,.$$

Clearly, any α-orientation of $G(n,n)$ can be extended to an Eulerian orientation of the infinite lattice. Typically, a fixed boundary condition will be of a regular form similar to that given by the dotted arrows in Figure 1.2;

Free Enforce the Eulerian condition on all even degree vertices and insist that

$$|indeg(v) - outdeg(v)| = 1$$

for all vertices v of odd degree. The Eulerian orientations with free boundary conditions could be viewed as a generalisation of α-orientations. Here we are giving a set, $\alpha(v) = \{-1, +1\}$, of allowed values for $indeg(v) - outdeg(v)$;

Torus Assume that the graph is a torus, so the edge leaving a boundary vertex wraps around and attaches to another boundary vertex. In the case of the grid we have edges connecting $(0, j)$ to $(n-1, j)$ and connecting $(j, 0)$ to $(n-1, 0)$.

The ice models defined on the square lattice and the triangular lattice are considered *solved* [61, 6, 7], in the sense that an asymptotic expression is known for the number of Eulerian orientations of the subgraphs induced by the set $\{(i, j) : 0 \leq i, j < n\}$ for a toroidal boundary condition. Specifically, in each case the authors obtain the exponential growth rate of the number of Eulerian orientations of an $n \times n$ section of the lattice with toroidal boundary condition. This is achieved using the *transfer matrix method* [61, 6, 7, 39].

We now sketch the transfer matrix method, as applied to counting Eulerian orientations of the square grid; this work is originally due to Lieb [61]. Let $\mathcal{C}$ denote a configuration of the arcs in the k-th column of an Eulerian orientation of the $n \times n$ grid; that is, an assignment of *up* or *down* to each arc. Each configuration can be defined by choosing the set of *down* arcs, so there are 2^n possible configurations. For each configuration $\mathcal{C}'$ of arcs for the $(k+1)$-th column, let $A(\mathcal{C}, \mathcal{C}')$ denote the number of ways to configure the horizontal arcs joining both columns so that the Eulerian condition can be satisfied at each vertex. Note that this is dependent only on $\mathcal{C}$ and $\mathcal{C}'$ so $A(\mathcal{C}, \mathcal{C}')$ depends only on n and not k. Then, the number of Eulerian orientations of the $n \times n$ grid with toroidal boundary conditions can be written as

$$\sum_{\mathcal{C}_0} \sum_{\mathcal{C}_1} \cdots \sum_{\mathcal{C}_{n-1}} \prod_{i=0}^{n} A(\mathcal{C}_i, \mathcal{C}_{i+1 \bmod n}) = \mathrm{Trace}(A^n)\,.$$

In [61] Lieb analyses the structure of A to find the maximum eigenvalue of A, which can be used to estimate the asymptotic growth rate of $\mathrm{Trace}(A^n)$. In [6] Baxter performs a similar, though more complicated, analysis to obtain a corresponding result for the triangular lattice.

Remark 1.55. The transfer matrix method is a general schema for counting structures defined on grids and similar *recursively definable* graphs, see [39] for a general description. When applicable, it yields an algorithm for exact counting. However, these algorithms are often inefficient as they require the multiplication of large matrices. This approach only becomes practical when some useful structure exists that allows us to either (a) improve the running time of the algorithm or (b) obtain an asymptotic estimate for the quantity in question. In §3.6, we discuss how the transfer matrix method can be applied to counting Euler tours of the grid.

The research of Lieb and Baxter was carried out in the 1960s. Despite their analyses of the asymptotic growth rate of the EOs of these lattices, the question of whether there existed efficient algorithms for counting (exactly or approximately) was still of interest. The estimates of [61] and [6] only capture the exponential growth rate and so are not useful for the construction of an *fpaus* for sampling Eulerian orientations either, so the question of whether one could sample Eulerian orientations of these lattices in polynomial time remained open until the result of Mihail and Winkler [68, 69] in 1992. Being able to generate random configurations of models is important in statistical physics as physicists often want to compute average properties of configurations; hence, this was an important open problem in the area of sampling algorithms.

In a related line of research, McKay [65] has obtained asymptotic estimates for the total number of Eulerian directed graphs and Eulerian orientations on n vertices, as well as the number of Eulerian directed graphs and Eulerian orientations on n vertices with exactly m edges[2]. The results of §3.4 of this thesis can be viewed as complementing those of [65] by providing an asymptotic estimate of the number of Eulerian orientations of random $2d$-regular graphs, when d is a constant. In fact, we provide a stronger result: an asymptotic expression for the distribution of the number of Eulerian orientations of a random $2d$-regular graph. We are also able to show that *almost every* $2d$-regular graph has *few* Eulerian orientations, where few is taken to mean within a polynomial factor of Schrijver's lower bound [78] (see Theorem 1.56 below).

The flavour of the calculations in [65] is similar to that of §3.4, in the sense that both obtain the asymptotic estimate by the so-called *saddle-point method* [25]. However, the actual approaches are quire different. McKay defines a generating function whose constant term is equal to the required quantity and then applies Cauchy's theorem to obtain an integral expression for this value. In §3.4 we take a more direct combinatorial approach, using the *configuration model* for graphs with fixed degree sequence. The approach of McKay seems to be best suited to the situation where we are enumerating all objects in a set, without regard to the vertex degrees. Although a similar approach has been applied to estimating the number of graphs with fixed degree sequence [67], we do not know of any result applying this approach to counting Eulerian orientations or Eulerian directed graphs with fixed degree sequences. Moreover, McKay's method is not suited to the extra analysis required to obtain the concentration result mentioned above. However, one distinct advantage of the approach used in [65] and [67] is that the results hold for larger degrees. Results using the configuration model only hold when if the degrees of the vertices are very small compared to n. Although beyond the scope of this thesis, it would be interesting to see if McKay's approach could be extended to obtain similar results (to those of §3.4) for graphs with larger degrees.

Several authors have derived upper and lower bounds on the number of Eulerian orientations of certain classes of Eulerian graphs. The number of Eulerian orientations of a regular graph has been studied by Schrijver [78] and Las Vergnas [95]. Schrijver proved the following theorem.

Theorem 1.56 (Schrijver [78])**.** *Let d be some fixed integer and let G be a $2d$-regular*

[2]Recall from the discussion in Remark 1.6 that not every Eulerian directed graph is an Eulerian orientation.

graph on n vertices. Then,

$$\left(\frac{\binom{2d}{d}}{2^d}\right)^n \leq |\mathrm{EO}(G)| \leq \binom{2d}{d}^{n/2}.$$

Las Vergnas [95] was able to generalise the bound of Theorem 1.56 by considering also the girth and the number of edge-disjoint cycles in the graphs.

For the planar case, Zickfeld and Felsner [33] have, in the more general context of α-orientations, obtained upper and lower bounds on the number of Eulerian orientations of several classes of planar graphs. In particular, they showed that the number of Eulerian orientations of any planar Eulerian graph G is never less than the number of α-orientations of G, for any α.

The number of Eulerian orientations of a graph G is closely related to the number of perfect matchings of a specially constructed bipartite graph. For any Eulerian graph $G = (V,E)$ we can construct a bipartite graph H_G such that

$$\prod_{v\in V}(deg(v)/2)!|\mathrm{EO}(G)| = |PM(H_G)|, \tag{1.18}$$

where $\mathrm{EO}(G)$ denotes the set of Eulerian orientations of G and $PM(H_G)$ denotes the set of perfect matchings of H_G. This reduction was first made explicit by Mihail and Winkler [68, 69] but is implicit in the earlier work of Schrijver [78].

We now describe the reduction and explain how it can be used to construct an *fpras* or an *fpaus* for $\mathrm{EO}(G)$, despite the fact that Eulerian orientations are not self-reducible. Let $G = (V,E)$ be an Eulerian graph. Let $H_G = (A,B,F)$ be the bipartite graph with bipartition

$$A = \bigcup_{v\in V} X_v, \quad X_v = \{x_{v,e} : e = \{u,v\} \in E\},$$

and

$$B = \{w_e : e \in E\} \cup \bigcup_{v\in V} Y_v, \quad Y_v = \{y_{v,i} : 1 \leq i \leq deg(v)/2\}.$$

and edge set

$$F = \{(x_{u,e}, w_e), (x_{v,e}, w_e) : e = \{u,v\} \in E\} \cup \bigcup_{v\in V} X_v \times Y_v.$$

Suppose we have an Eulerian orientation $\mathcal{E} \in \mathrm{EO}(G)$. We construct a perfect matching $M \in PM(H_G)$ as follows: for each edge $e = \{u,v\} \in E$ we include $(x_{v,e}, w_e)$ in M if e is oriented (u,v) in $\mathcal{E}$ and include $(x_{u,e}, w_e)$ in M if e is oriented (v,u) in $\mathcal{E}$; complete M by, for each $v \in V$, choosing any perfect matching on Y_v and the unmatched vertices

in X_v. Every edge in G receives an orientation in $\mathcal{E}$, so every vertex in $\{w_e : e \in E\}$ is matched with a vertex in $\bigcup_{v\in V} X_v$ in the first step. This uses up half of the vertices in $\bigcup_{v\in V} X_v$. The remaining vertices are all matched with Y_v in the second step, so the construction always yields a perfect matching. Moreover, each Eulerian orientation gives rise to exactly

$$\prod_{v\in V}(deg(v)/2)!$$

perfect matchings; this is the number of ways to choose the edges in the second step.

Now, suppose we have a perfect matching $M \in PM(H_G)$. We construct an Eulerian orientation $\mathcal{E}$ of G as follows: for each edge $e = \{u, v\} \in E$ we orient e towards u if $(x_{u,e}, w_e) \in M$ and orient e towards v if $(x_{v,e}, w_e) \in M$. Exactly half of the vertices in X_v must be matched to vertices in Y_v in any perfect matching. Thus, exactly half of the edges incident with v are oriented towards v in this construction. Moreover, the perfect matching M exactly determines $\mathcal{E} \in \mathrm{EO}(G)$, so there is exactly one Eulerian orientation of G corresponding to each perfect matching of H_G. The ratio (1.18) follows from the fact that the procedure just described is exactly the reversal of the mapping from Eulerian orientation to perfect matching described in the previous paragraph.

The relationship between $\mathrm{EO}(G)$ and $PM(G)$ has two important consequences: the existence of an *fpaus* and an *fpras* for $\mathrm{EO}(G)$. Given a uniformly random perfect matching $M \in PM(H_G)$, applying the reduction described above gives a uniformly random Eulerian orientation $\mathcal{E} \in \mathrm{EO}(G)$, since each $\mathcal{E} \in \mathrm{EO}(G)$ arises as a result of an equal number of perfect matchings. Thus, any *fpaus* for $PM(H_G)$ can be turned into an *fpaus* for $\mathrm{EO}(G)$. The celebrated Jerrum-Sinclair chain [49] (see [51, 10] for improvements) can be used to sample a perfect matching of any bipartite graph in time $O(n^7(\log n)^4)$.

Similarly, an *fpras* for counting perfect matchings of bipartite graphs can be used to construct an *fpras* for the number Eulerian orientations of any graph G: multiplying an ε-approximation for $PM(H_G)$ by

$$\prod_{v\in V}(deg(v)/2)!$$

gives an ε-approximation for $\mathrm{EO}(G)$. In [51], Jerrum et al. describe how to use the Jerrum-Sinclair chain to construct an *fpras* for the number of perfect matchings of any bipartite graph that runs in polynomial-time. Thus, by using the construction described in [69], we can use the results of [51] (with an improvement to the running time from

[10]) to show the existence of an *fpaus* and *fpras* for the Eulerian orientations of any Eulerian graph.

Several authors [62, 38, 29] have considered other sampling schemes for the Eulerian orientations of the Eulerian planar lattices, due to their connection to the ice models of statistical physics. Eloranta [29] has studied dynamics for generating random configurations of four different ice models, including the two described above (see Figure 1.2). Although called *probabilistic cellular automata* in [29], these dynamics are nothing more than the face reversing Markov chain studied in [62, 38] and in Chapter 2 of this thesis.

Luby et al. [62] and Goldberg et al. [38] have shown that the face reversing Markov chain mixes rapidly on the Eulerian orientations of the square lattice with fixed and free boundary conditions, respectively. In this thesis, we extend current knowledge by showing that counting Eulerian orientations remains #P-complete for planar graphs and show that the Markov chain studied in [62] and [38] can be used to construct an *fpras* or *fpaus* for the Eulerian orientations of the triangular lattice. Specifically, in Chapter 2 of this thesis we will show that this chain is rapidly mixing on the triangular lattice under any fixed boundary condition. However, we also show that there exists an infinite family of planar graphs for which this chain is torpidly mixing.

1.3.2 Euler tours

Euler tours are a classical graph theoretical structure, studied in every undergraduate graph theory class. Indeed, the result believed to be the first theorem in graph theory is the one that introduced Euler tours: Euler's solution to the Königsberg bridge problem [30]. It was in order to solve this problem that Euler defined the concept of a graph and stated what is now known as the *Eulerian condition* (Definition 1.18). The complexity of counting/sampling Euler tours is quite different depending on whether the input is a directed graph or an undirected graph. For any Eulerian directed graph $\vec{G}$ it is possible to exactly count the number of Euler tours of $\vec{G}$ in polynomial time [94, 47] and there exist many polynomial time sampling algorithms, e.g., [19, 72]. In the undirected case the counting problem is #P-complete in general and, apart from the special case of series-parallel graphs [17] there are no known polynomial cases. Moreover, apart from the cases of the complete graph [66] and series-parallel graphs [17] there are no known polynomial time sampling algorithms.

1.3.2.1 Euler tours of directed graphs

In the case of Eulerian directed graphs, it is possible to compute $\#\mathrm{ET}(\vec{G})$ in polynomial time using a relationship between the Euler tours of $\vec{G}$ and the arborescences of $\vec{G}$ (see Definition 1.12).

For any Eulerian directed graph $\vec{G} = (V,A)$, the so-called **BEST** theorem (due to de **B**ruijn and van Aardenne-**E**hrenfest [94], extending a result of **S**mith and **T**utte [80]) reduces the problem of computing $|\mathrm{ET}(\vec{G})|$ to the problem of computing $|\mathrm{ARB}(\vec{G},v)|$ for any vertex $v \in V$.

Theorem 1.57 ([94]). *Let $\vec{G} = (V,A)$ be an Eulerian directed graph and let $d_u = outdeg(u)$ for each $u \in V$. For any $v \in V$, we have*

$$|ET(\vec{G})| = \left[\prod_{u \in V} (d_u - 1)!\right] |\mathrm{ARB}(\vec{G},v)|. \tag{1.19}$$

We now sketch the proof of Theorem 1.57. Let $\vec{G} = (V,A)$ be an Eulerian directed graph and let $v \in V$. Suppose we have an Euler tour $\mathcal{T} = (a_0, a_1, \ldots, a_{m-1})$ of $\vec{G}$ and further suppose that v is the source of a_0. Given T we can construct a unique $\mathcal{A} \in \mathrm{ARB}(\vec{G},v)$: for each vertex $u \neq v$ we take the last arc leaving u on $\mathcal{T}$ to be the unique arc leaving u in $\mathcal{A}$. To see that this construction always gives an arborescence, suppose we have a cycle $C = (v_0, v_1, \ldots, v_{k-1})$ such that $(v_i, v_{(i+1) \bmod k})$ is the last arc leaving v_i on $\mathcal{T}$ for each $i = 0,1,\ldots,k-1$. Consider how $\mathcal{T}$ behaves on $\vec{G}$. Each time $\mathcal{T}$ reaches a vertex v_i on C the next arc on $\mathcal{T}$ is from $\vec{G} - C$, unless we have reached the last occurrence of v_i on $\mathcal{T}$. But this cannot occur until $\mathcal{T}$ has passed all paths from v_i to v in $\vec{G} - C$, in which case $\mathcal{T}$ would define an Euler tour on $\vec{G} - C$, contradicting the assumption that $\mathcal{T}$ is an Euler tour of G. Thus, choosing a_0 from the d_v arcs leaving v in $\vec{G}$ is sufficient to induce a unique arborescence $\mathcal{A} \in \mathrm{ARB}(\vec{G},v)$.

Next, suppose we have an arborescence $\mathcal{A}$ of $\vec{G}$ with root v. Moreover, suppose we have an ordering on the arcs leaving each vertex of $\vec{G}$ such that for each $u \neq v$ the unique arc leaving u in $\mathcal{A}$ is the last arc in the ordering. For each such ordering we can construct a unique Euler tour of $\vec{G}$ as follows. Starting at v we choose the least-most arc in the ordering of the out-arcs of v. Then, at each subsequent vertex we choose the least arc which has not yet been used. This process terminates when we reach v and there are no more unused arcs left to choose, and at this point we have constructed an Euler tour of $\vec{G}$.

Thus, there are d_v arborescences with root v associated to each Euler tour, one for

each arc leaving v, and

$$(d_v)! \prod_{u \neq v} (d_u - 1)!$$

Euler tours associated to each arborescence with root v. Taken together, these two facts imply (1.19).

One immediate corollary to Theorem 1.57 is that there are an equal number of arborescences rooted at each vertex of any Eulerian directed graph. However, more importantly, Tutte [90] showed that we can calculate the number of arborescences rooted at a vertex of any directed graph by computing the determinant of *minor* of a specially constructed matrix.

Definition 1.58. Let M be a $n \times n$ matrix. For any row/column index i, we denote by M_{ii} the matrix obtained by removing the row and column indexed by i. This is known as the *i-th minor* of M.

Theorem 1.59 (Tutte [90])**.** *Let $\vec{G} = (V,A)$ be a directed graph. We define the* Laplacian *of $\vec{G}$ to be the matrix L with entries given by*

$$L_{u,v} = \begin{cases} outdeg(u) & \text{if } u = v; \\ -d_{u,v} & \text{if } u \neq v \text{ and there are } d_{u,v} \text{ arcs from } u \text{ to } v \text{ in } \vec{G}. \end{cases}$$

For any vertex $v \in V$, the number of arborescences rooted at v is equal to the determinant of the v-th minor of L.

We can calculate the determinant of any $n \times n$ matrix in time polynomial in n by, e.g., Gaussian elimination [84]; hence, the above theorem tells us that we can count the Euler tours of any Eulerian directed graph in time polynomial in the number of vertices.

Remark 1.60. For the special case of 2-in/2-out directed graphs, Macris & Pulé [64] claimed that one could obtain the number of Euler tours by computing the determinant of an alternative matrix; Lauri [59] claimed to have a combinatorial proof of this result. However, the result in [64] is incorrect. Indeed, we can construct an example with 5 vertices for which this method gives the wrong value. This counter-example is contained in Appendix A.

Remark 1.61. The importance of the BEST theorem goes beyond the fact that it allows us to count Euler tours of directed graphs in polynomial time. Indeed, almost every result about counting or sampling Euler tours of any graph (directed or undirected) depends on the relationship implied by this theorem, e.g., [19, 66, 72, 15].

There exist several schemes for generating random Euler tours of an Eulerian directed graph $\vec{G}$, all of them based on the relationship between tours and arborescences described above. These can be grouped into two types: those based on the fact that we can count the number of arborescences in any directed graph in polynomial time [42, 18, 57, 19]; and those based on a random walk in $\vec{G}$ [53, 99, 72]. In the next two paragraphs we will briefly describe algorithms of both types.

In the algorithms based on Theorem 1.59, the arborescence is constructed one arc at a time. At each step the algorithm considers a vertex u which has not yet been assigned its unique out-arc in $\mathcal{A}$. Let $Out(u)$ denote the set of arcs leaving u in $\vec{G}$ and suppose we want to compute the probability $a = (u,w) \in Out(u)$ is contained in a uniformly random arborescence of $\vec{G}$. Let $\vec{G}_a$ denote the directed graph in which we have replaced u and w by a single vertex uw which has the same outgoing arcs as w in $\vec{G}$ and has the incoming arcs of both u and w in $\vec{G}$. The arborescences in $\mathrm{ARB}(\vec{G},v)$ containing (u,w) are in one-to-one correspondence with the arborescences in $\mathrm{ARB}(\vec{G}_a,v)$. Hence, the probability (u,w) is contained in a uniformly random element of $\mathrm{ARB}(\vec{G},v)$ is

$$\frac{|\mathrm{ARB}(\vec{G}_a,v)|}{|\mathrm{ARB}(\vec{G},v)|}. \tag{1.20}$$

By Theorem 1.59, we can compute (1.20) in polynomial time. We need to choose $n-1$ arcs, so the total computational cost of choosing a random arborescence is a polynomial in n. Implemented naively, this algorithm will have running time $O(n^3)$ but some clever tricks used in [19] reduce this to the amount of time taken to multiply $n \times n$ matrices (currently $O(n^{2.376})$ [21][3]).

Remark 1.62. The astute reader will notice that the algorithm just described is no more than an instantiation of the general scheme described in Algorithm 1.

The second, and perhaps better-known, class of algorithms are based on random walks [72]. The most efficient of these algorithms is believed to be the Propp-Wilson algorithm [72], which is based on the concept of *loop-erased random walks*. The running times of these algorithms are in terms of properties of the random walk, e.g., cover time or hitting time (defined below), so a direct comparison of running times is difficult. For some cases, the running time of a random walk based algorithm could be exponential, in which case algorithms based on Theorem 1.57 could be more

[3] It is worth mentioning that the constant term hidden by the $O(\cdot)$ here is so large that the algorithm is only worthwhile when used for matrices that are so large modern computers would struggle to handle them. Hence, the best *practical* algorithm is still Strassen's algorithm [85] (see [98] for a general reference), which has running time $O(n^{2.807})$

efficient. However, it is believed that for most graphs, the Propp-Wilson algorithm will outperform all known algorithms based on Theorem 1.57 [72].

Definition 1.63. Let $\vec{G} = (V,A)$ be a directed graph and let $u, v \in V$. A random walk γ from u to v is obtained by setting $\gamma_1 = u$ and then setting γ_{i+1} to be the target of a randomly chosen arc leaving γ_i until we reach $\gamma_t = v$. A *loop erased random walk*, γ', is constructed from γ by removing all loops in the order in which they appear on γ. We define the indices of the positions in γ that are not on loops, i.e., the positions that do not occur between two occurrences of some vertex on γ, inductively by

$$\begin{aligned} i_1 &= 1\,; \\ i_j &= \max\{i : \gamma_i = \gamma_{i_{j-1}}\} + 1\,. \end{aligned}$$

Then γ', the loop erased version of γ, is defined by $\gamma'_j = \gamma_{i_j}$.

We define a random variable $LERW(u,v)$ whose value is a loop erased random walk from u to v. Similarly, $LERW(u,U)$ is a loop erased random walk from u to any vertex in U; that is, the random walk γ terminates as soon as we reach a vertex from U.

Example 1.64. Suppose we have

$$\gamma = 1,2,3,1,4,2,5\,.$$

Then $i_1 = 1$, $i_2 = 5$, $i_3 = 6$ and $i_4 = 7$, so

$$\gamma' = 1,4,2,5\,.$$

Algorithm 3: Propp-Wilson

Input: A directed graph $\vec{G} = (V,A)$ and a vertex $v \in V$

Output: A uniformly random arborescence $\mathcal{A}$ of $\vec{G}$ rooted at v

begin
  $U \longleftarrow \{v\}$, $\mathcal{A} \longleftarrow \emptyset$;
  while $U \neq V$ **do**
    Choose $u \in V - U$;
    $\gamma \longleftarrow LERW(u,U)$;
    Add the edges of γ to T and the vertices to U;
  end
end

The Propp-Wilson algorithm is presented as Algorithm 3. Propp and Wilson proved that this algorithm generates a random arborescence from the uniform distribution on $\mathrm{ARB}(\vec{G}, v)$ for any directed graph, and that the running time is bounded by the *hitting time*.

Definition 1.65. Let $\vec{G} = (V, A)$ be a directed graph. The *hitting time* of $u, v \in V$ is the random variable recording the amount of time for the random walk on $\vec{G}$, $(X_t)_{t \geq 0}$, to reach v given that it started at u:

$$H_{u,v} = \inf\{t : X_t = v : X_0 = u\}.$$

The *hitting time* of $\vec{G}$ is the maximum expected hitting time over all pairs of vertices:

$$h(\vec{G}) = \max_{u,v \in V} \mathbb{E}(H_{u,v}).$$

Remark 1.66. The Propp-Wilson algorithm can be regarded as a Markov chain, $\mathcal{M}_{PW}$, on $\mathrm{ARB}(\vec{G})$, which operates as follows. Let X_t denote the t-th state of the chain and let r_t be the root of X_t. We use $X_t(v)$ to denote the unique arc leaving v in X_t, for each $v \neq r_t$. We now describe how $\mathcal{M}_{PW}$ generates the next state of the chain, X_{t+1}, given X_t.

One step of the Markov chain $\mathcal{M}_{PW}$

1. Choose a random arc $(r_t, r_{t+1}) \in A$.

2. Obtain X_{t+1} from X_t by removing $X_t(r_{t+1})$ and adding (r_t, r_{t+1}).

Consider what happens when the sequence of root vertices of states, $X_t, \ldots, X_{t+k}$, forms a cycle $(v_0, v_1, \ldots, v_{k-1})$. On the transition (X_t, X_{t+1}) we remove (v_0, v_1) from the arborescence and add (v_1, v_2). Then, on the transition (X_{t+1}, X_{t+2}) we remove (v_1, v_2) and add (v_2, v_3), and so on. Eventually, the transition (X_{t+k-1}, X_{t+k}) removes (v_{k-1}, v_0) and adds (v_0, v_1). Hence, $X_{t+k} = X_t$. Thus, at any particular step t of the chain, the arcs which have been used by the random walk but are not on a loop are contained in X_t, so this chain is equivalent to the Propp-Wilson algorithm [72]. The result of Propp-Wilson [72] was to show that this chain has reached its stationary distribution once all vertices have been visited by the random walk.

1.3.2.2 Euler tours of undirected graphs

Next, we turn to the problem of generating and counting Euler tours of an *undirected* graph $G = (V, E)$. Two alternative representations have been used to reason about the set of Euler tours of an undirected graph: *transition systems* and *orbs*.

The first, and more direct representation, is to define a tour $\mathcal{T}$ according to its component 2-paths: the pairs of edges which are visited consecutively on $\mathcal{T}$.

Definition 1.67. Let $G = (V,E)$ be an Eulerian graph and let $v \in V(G)$. We define a *transition system* at v to be a decomposition of the set of edges incident with v into pairs. A *transition system* of G is a function T that maps each v to a *transition system* on v. We denote by $\mathrm{TS}(G)$ the set of transition systems of G.

We can also define a *transition system* of an Eulerian directed graph by adding the restriction that an in-arc of v must be paired with an out-arc of v.

Suppose $G = (V,E)$ is an Eulerian graph on n vertices such that $deg(v) = d_v$, for each $v \in V$. The number of ways to choose a pairing of the edges at v is

$$\frac{\binom{2d_v}{d_v} d_v!}{2^{d_v}};$$

hence, the number of transition systems of any graph satisfying $deg(v) = d_v$ is

$$\prod_{v \in V} \frac{\binom{2d_v}{d_v} d_v!}{2^{d_v}}.$$

In particular, every $2d$-regular graph has

$$\left(\frac{\binom{2d}{d} d!}{2^d} \right)^n$$

transition systems. Similarly, a directed graph in which $outdeg(v) = d_v$, for every $v \in V$, has

$$\prod_{v \in V} d_v!$$

transition systems, and every d-in/d-out directed graph has

$$d!^n$$

transition systems.

Each transition system T of G corresponds to a decomposition of the edges of G into a set of edge-disjoint cycles, denoted by $\mathcal{C}(T)$, where the edges that are paired together by T lie on the same cycle. Hence, the Euler tours of G are equivalent to the set of transition systems for which $\mathcal{C}(T)$ contains exactly one component. It seems the first method to exactly count the number of Euler tours of any Eulerian graph was developed by Tarry [86] in 1887. This approach chooses a transition system vertex

by vertex, ensuring that the current transition system can always be extended to the transition system of an Euler tour. Suppose G is an Eulerian graph and v is a vertex of G. Let $\mathcal{P}(v)$ denote the set of pairings of the edges incident with v. For each $\rho \in \mathcal{P}(v)$, let G_ρ denote the graph obtained by removing v and adding the edges u,w for each $\{\{u,v\},\{v,w\}\} \in \rho$. Then, for any $v \in V$, we have

$$\#\mathrm{ET}(G) = \sum_{\substack{\rho \in \mathcal{P}(v) \\ G_\rho \text{ is connected}}} \#\mathrm{ET}(G_\rho)\,. \tag{1.21}$$

The equation in (1.21) can be expanded recursively to compute each of the $\#\mathrm{ET}(G_\rho)$, stopping the recursion when we reach a graph for which we can easily calculate the number of Euler tours, e.g., a graph with 2 vertices. This procedure is not practical for large graphs as the running time is proportional to the number of Euler tours of G, which is almost certainly exponential in the number of vertices. Therefore, Tarry's approach is not useful for developing polynomial-time algorithms for counting Euler tours.

Kötzig [56] showed that any Eulerian transition system can be transformed into another by performing a sequence of local changes, each one altering the transition system at exactly one vertex.

Definition 1.68. Let $G = (V,E)$ be an Eulerian graph, let $\mathcal{T}$ be an Euler tour of G and let T denote the transition system corresponding to $\mathcal{T}$. A κ-transformation at v takes two pairs of edges $\{e,f\}$ and $\{e',f'\}$ in $T(v)$ and obtains T' by replacing them by $\{e,f'\}$ and $\{e',f\}$. If T' induces an Euler tour $\mathcal{T}'$ we call this an *allowed transition* and denote this by $\mathcal{T} \Rightarrow_v \mathcal{T}'$.

An allowed κ-transformation corresponds to choosing two distinct occurrences of v on $\mathcal{T}$ and reversing the segment of $\mathcal{T}$ between them. In [87] Tetali and Vempala used this result to define an ergodic Markov chain that converges to the uniform distribution on the set of Euler tours of any Eulerian graph. The transitions in this chain are very simple. Let X_t denote the t-th state of the chain. We now describe how a step of the Kötzig move chain obtains X_{t+1}, given X_t.

One step of the Kötzig chain

1. Choose $v \in V$ uniformly at random and choose a κ-transformation uniformly at random from the set of potential transformations;

2. If the chosen transformation is allowed in X_t then apply it to obtain X_{t+1} as the next state;
3. Otherwise, set $X_{t+1} = X_t$.

In [87], Tetali and Vempala attempted to use the multi-commodity flow technique [49] to prove this chain was rapidly mixing, in the case where the given graph is either 4-regular or 6-regular. To apply this technique one needs to define certain *canonical paths* between states in the Markov chain. The construction for these paths in [87] does not give paths with the claimed properties; hence, the proof does not work. Even now, several years later, no one has been able to fix the argument of [87]. The reason for this is that the structure of the state space of the chain, upon which proofs of rapid mixing depend, is very complicated. Although the change we are making in an individual transition is local, we need to look at the whole tour to determine whether or not a particular change is legal. Generally speaking, the only Markov chains whose mixing times are amenable to analysis by current proof techniques are those in which the moves are very local; that is, we can determine which changes we can apply to a part of the structure by considering the parts of the structure *close* to the part we want to change. So, even in the simple cases of 4-regular and 6-regular graphs, the problem of sampling Euler tours of undirected graphs in polynomial time remains open.

McKay and Wormald [67] have obtained an asymptotic expression for the number of Euler tours of K_n for n odd. One corollary of this result is that a simple dart throwing algorithm can be used to generate random Euler tours of K_n in polynomial time. This algorithm, described in Chapter 3 of this thesis, does not run in polynomial time for every Eulerian graph: the graph B_n (see Example 1.70 below) is one example of a graph for which it will take exponential time; see §3.6 for more interesting examples. Thus, it is an interesting open problem to determine on which classes of graphs this algorithm is effective. In Chapter 3 of this thesis we present some partial results towards showing that the simple algorithm studied by McKay and Wormald will run in expected polynomial time for almost all $2d$-regular graphs. Whether or not this is the case depends on an unproven conjecture, a conjecture which is supported by empirical evidence.

For many years it was not even known whether or not there might exist a polynomial-time algorithm for exactly counting the Euler tours of an undirected graph. This question was finally resolved by Brightwell and Winkler [15] in 2005, when they showed that the problem was indeed #P-complete[4]. Brightwell and Winkler

[4] As one might have expected, given the difficultly in finding a polynomial-time algorithm.

considered a second representation for the Euler tours of a graph G.

Definition 1.69. Let $G = (V,E)$ be an Eulerian graph. We define an *orb* of G rooted at v to be a pair $(\mathcal{E},\mathcal{A})$, where $\mathcal{E} \in \mathrm{EO}(G)$ and $\mathcal{A} \in \mathrm{ARB}(\mathcal{E},v)$. We use $\mathrm{ORB}(G,v)$ to denote the set of orbs of G rooted at a particular vertex v, and $\mathrm{ORB}(G)$ to denote the set of all orbs of G.

From Theorem 1.57, we know that the number of Euler tours of a particular Eulerian orientation $\mathcal{E} \in \mathrm{EO}(G)$ is

$$\left[\prod_{u \in V(G)} (deg(u)/2-1)!\right] |\mathrm{ARB}(\mathcal{E},v)|,$$

for an arbitrary vertex $v \in V$. Hence, the number of Euler tours of G satisfies

$$|\mathrm{ET}(G)| = \frac{1}{n}\left[\prod_{u \in V(G)} (deg(u)/2-1)!\right] |\mathrm{ORB}(G)|.$$

Brightwell and Winkler [15] showed that counting orbs is #P-complete via a reduction from #EO; hence, #ET is also #P-complete.

Although the reason Brightwell and Winkler came up with the concept of an orb was to prove #P-completeness of #ET, the connections between orbs and Euler tours also suggests ideas that might allow us to come up with polynomial-time algorithms for sampling Euler tours. Let $(\mathcal{E},\mathcal{A}) \in \mathrm{ORB}(G,v)$. We can generate a random Euler tour associated with $(\mathcal{E},\mathcal{A})$ in polynomial time: all we need to do is choose a random ordering on the non-arborescence out-arcs of each vertex and then follow the deterministic procedure described in the proof of Theorem 1.57. Hence, if we could sample from the uniform distribution on the orbs of G, we could also sample from the uniform distribution on the Euler tours of G. Unfortunately, this is no easy task. The algorithms for sampling arborescences do not generalise to sampling orbs as most known algorithms for sampling arborescences are heavily dependent on the fact that the input graph is directed.

To illustrate why the problem of sampling orbs of an undirected graph is so different from that of sampling arborescences of a directed graph, we consider the following naive idea for a sampling algorithm. Suppose we try to choose the arborescence part of the orb step by step, in a similar fashion to the algorithms based on Theorem 1.59, e.g., [19]. At each step we want to calculate the probability that an edge incident with a vertex u is oriented away from u and contained in the arborescence part of a random orb. But this would require being able to count the

number of orbs containing a particular set of arcs in their arborescence, and this is clearly #P-complete! Attempts to generalise the random walk algorithms to the undirected case suffer from similar problems, since we need an algorithm that changes both the orientation part and the arborescence part simultaneously.

We know that we can generate random Eulerian orientations of any graph in polynomial time, and that we can generate Euler tours of any Eulerian directed graph in polynomial time. However, simply choosing a uniformly random Eulerian orientation $\mathcal{E} \in \mathrm{EO}(G)$ and then choosing a random Euler tour $\mathcal{T} \in \mathrm{ET}(\mathcal{E})$ will not give a uniformly random Euler tour, since different Eulerian orientations of a graph can have very different numbers of Euler tours. For example, consider the graph obtained by duplicating each edge of the n-cycle.

Example 1.70. Let B_n be the multigraph obtained by duplicating each edge in the n-cycle $C_n = (0,1,2,\ldots,n-1)$. The Eulerian orientations of B_n can be grouped into two equivalence classes. For each i we label the edges joining i to $(i+1) \bmod n$ as e_i and f_i. We say two Eulerian orientations $\mathcal{E}, \mathcal{E}' \in \mathrm{EO}(B_n)$ are *equivalent* if either $\mathcal{E}'$ is the reverse of $\mathcal{E}$ or $\mathcal{E}'$ can be obtained from $\mathcal{E}$ by switching the orientations of e_i and f_i for each i in some set $S \subset \{1,2,\ldots,n\}$. This relation partitions $\mathrm{EO}(G)$ into two classes: the orientations in which e_i and f_i are both oriented in the same direction for every i and the orientations in which e_i and f_i are oriented in opposite directions for every i. There are exactly two Eulerian orientations in the first class and 2^n in the second.

Both of the orientations in the first class have 2^{n-1} arborescences rooted at each vertex v, and each of the orientations in the second class have n arborescences rooted at each vertex v. Hence, if we choose a random orb as suggested above, the probability of obtaining a particular $(\mathcal{E}, \mathcal{A})$ with $\mathcal{E}$ in the the first class is

$$\frac{1}{2^n+2}\frac{1}{n2^{n-1}},$$

but the probability of obtaining a particular $(\mathcal{E}, \mathcal{A})$ with $\mathcal{E}$ in the second class is

$$\frac{1}{2^n+2}\frac{1}{n^2}.$$

1.4 Contents of the thesis

The problems #EO and #ET have been shown to #P-complete for general graphs by Mihail and Winkler [69] and Brightwell and Winkler [15], respectively. We extend current knowledge by showing that the complexity of counting Eulerian orientations

remains #P-complete for planar graphs. This result is contained in the first section of Chapter 2.

The Eulerian orientations of any graph can be sampled in polynomial time, by combining the reduction of Mihail and Winkler [69] with the results of Jerrum et al. [49, 51] (improvements in [10]). A more natural chain on the Eulerian orientations of planar graphs has been studied in [62, 38], in which the mixing time was analysed on the Eulerian orientations of the square lattice. In Chapter 2 we show that this chain is rapidly mixing on the triangular lattice, a planar lattice that which is of interest in the statistical physics community [6, 29]. However, we also show that there exist planar graphs for which it will always take an exponential time to converge.

In Chapter 3 we describe a simple algorithm that can be used to generate uniform Euler tours or approximately count the number of Euler tours of any Eulerian graph or Eulerian directed graph. It is known that this algorithm runs in expected polynomial time for the complete graph [66], but also that it can be expected to take an exponential amount of time for other graphs. We use the theory of random graphs to show that the algorithm runs in expected polynomial time for almost every Eulerian directed graph.

Although we are not able to show that this algorithm runs in expected polynomial time on almost every $2d$-regular graph, we obtain some partial results towards showing that this is the case. This includes an analysis of the number of Eulerian orientations of a random $2d$-regular graph that is of independent interest. Whether or not the algorithm runs in expected polynomial time for almost every $2d$-regular graph depends on an unproven conjecture regarding an estimate for the second moment of the number of orbs of a random $2d$-regular graph. In Chapter 3, we provide some empirical evidence to support our conjecture.

Chapter 2

Eulerian orientations of planar graphs

2.1 Introduction

Recall that the Eulerian orientations of certain planar graphs are of particular practical interest as they correspond to configurations of the *ice-type models* studied in the statistical physics community, e.g., the square lattice [61], the triangular lattice [6], the Kagomé lattice [29], and the 3.4.6.4 lattice [29]. While the #P-completeness of counting the Eulerian orientations of a general Eulerian graph (#EO) was established in [68, 69], it is not known if the problem remains #P-complete under the restriction that the input be a planar graph (#PlanarEO). The first result of this chapter answers this question by showing that #EO is polynomial-time Turing reducible to #PlanarEO.

Given the computational hardness of exact counting, we turn our attention to approximation. Recall from §1.1.2 that one approach to approximate counting is via random sampling. It is known that we can generate (almost) uniformly random Eulerian orientations of any Eulerian graph in polynomial time (by combining the result of Mihail and Winkler [68, 69] described in §1.3.1 with the rapidly mixing Markov chain of [49, 51, 10]). In this chapter, we are interested in a different approach, which uses a natural Markov chain on the α-orientations of a planar graph. The reason for using a chain on the set of α-orientations is that sampling Eulerian orientations of a solid subgraph of an infinite lattice with some fixed boundary condition requires one to sample α-orientations of a non-Eulerian graph.

The Markov chain we study is in the style of the "Glauber dynamics" often used to generated random configurations of physical models. Although the approach of Mihail and Winkler detailed in §1.3.1 gives an *fpaus* for the set of Eulerian orientations of any graph, this chain is still of interest as it is the method most commonly used in

practice [28, 2, 29]. The main contribution of this chapter is to show that this Markov chain is rapidly mixing for any solid subgraph of the triangular lattice. However, we are also able to find an infinite family of planar graphs for which this Markov chain will take an exponential number of steps to converge.

The contents of this chapter is as follows. In §2.2 we describe the reduction that shows #PlanarEO is #P-complete. In §2.3, we define the *face-reversal Markov chain* and show that the stationary distribution is the uniform distribution on the set of α-orientations of any planar graph. In this section, we also describe a result of Felsner [32] that is useful in the analysis of this Markov chain. In §2.4, we describe *path coupling*, a technique for constructing couplings that we will use to analyse the mixing time of the Markov chain. Path coupling is not applicable directly to the face-reversal chain, so, in §2.5, we first extend the chain with additional moves, in the style of [62, 38]. Then, in §2.6, we show that the face-reversal chain is rapidly mixing when the input is a solid section of the triangular lattice with a fixed boundary condition. This analysis is broken into two parts. In §2.6.1 we use path coupling to analyse the mixing time of the extended chain. Then, in §2.6.2, we use a technique called *comparison* to infer that the face-reversal chain is rapidly mixing on this class of graphs. In §2.7 we show how the face-reversal chain can be used to construct an *fpras* for counting Eulerian orientations of the triangular lattice. Finally, in §2.8 we show that there exist planar graphs for which the face-reversal chain is torpidly mixing, using a distributive lattice structure (due to Felsner [32]) on the set of α-orientations of a planar graph.

2.2 Computational Complexity of #PlanarEO

Before discussing algorithms for sampling Eulerian orientations of planar graphs, we provide a polynomial-time reduction from #EO to #PlanarEO. This shows that #PlanarEO is #P-complete since the #P-completeness of #EO is already known [68, 69]. Our reduction uses a recursive gadget (suggested by Mark Jerrum [48]) and can be seen as an application of the so-called Fibonacci method of Vadhan [91].

The proof is as follows. Given a non-planar Eulerian graph G with an embedding in the plane, we construct a sequence of planar Eulerian graphs G_k, $k = 0, 1, \ldots, \ell + 1$, where ℓ is the number of crossing edges in the embedding of G. We then show that the values $\#\mathrm{EO}(G_k)$ give evaluations of a degree ℓ polynomial $p(x)$ at a sequence of distinct points. This polynomial, $p(x)$, has the value $\#\mathrm{EO}(G)$ as one of its coefficients. Hence, we can obtain the value $\#\mathrm{EO}(G)$ from the values $\#\mathrm{EO}(G_k)$, using the *Lagrange*

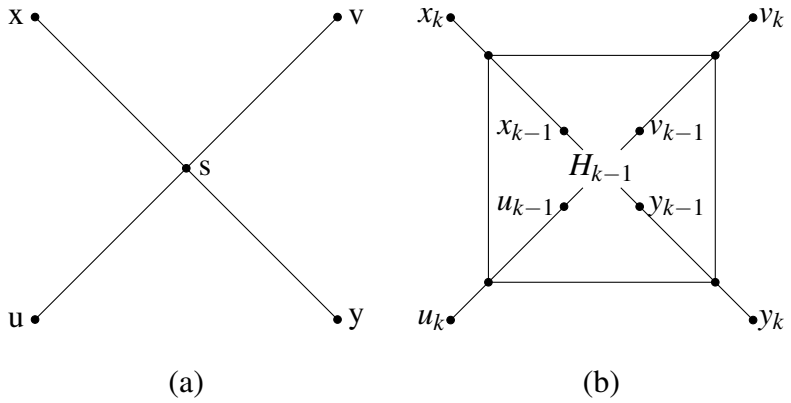

Figure 2.1: The crossover box

Interpolation Formula:

Theorem 2.1 (Lagrange)**.** *Given $d+1$ distinct real values $\beta_0, \beta_1, \ldots, \beta_d$, $d \geq 2$, and $d+1$ points $(\alpha_0, \beta_0), (\alpha_1, \beta_1), \ldots, (\alpha_d, \beta_d)$, there is a unique polynomial of the form*

$$f(x) = \sum_{i=0}^{d-1} a_i x^i$$

passing through these points, specified by

$$f(x) = \sum_{i=0}^{d} \beta_i \prod_{j \neq i} \frac{x - \alpha_j}{\alpha_i - \alpha_j}. \tag{2.1}$$

Theorem 2.2. #PlanarEO *is* #P*-complete.*

Proof. We prove the theorem by exhibiting a polynomial time reduction from #EO to #PlanarEO. That, is given an oracle for computing #PlanarEO we can construct an algorithm that can compute #EO for any non-planar graph.

Let G be any non-planar graph and suppose we have an embedding in the plane with ℓ crossings. We turn G into a planar graph G_0 by removing each pair of crossing edges $\{x,y\}$ and $\{u,v\}$ and replacing them by a pair of paths (x,s,y) and (u,s,v), where s is a vertex not contained in G, as in Figure 2.1(a). Every Eulerian orientation of G induces an Eulerian orientation of G_0; however, not all Eulerian orientations of G_0 correspond to Eulerian orientations of G, e.g., $\{(u,s),(v,s)\}$ may be present in an Eulerian orientation of G_0. We say $\mathcal{E} \in \mathrm{EO}(G_0)$ is *valid* at s if the orientation of the edges incident with s is consistent with an orientation of G, i.e., exactly one of $\{u,s\}$ and $\{v,s\}$, and of $\{x,s\}$ and $\{y,s\}$ is directed towards s in $\mathcal{E}$.

We now define a sequence of graphs H_k, each of which has 4 vertices of degree 1, labelled u_k, v_k, x_k, and y_k. Let H_0 be the graph Figure 2.1(a). For $k \geq 1$, we define H_k recursively in terms of H_{k-1} and a 4-cycle. Each vertex of the cycle connects to a pair of vertices from $\{(u_k,u_{k-1}),(v_k,v_{k-1}),(x_k,x_{k-1}),(y_k,y_{k-1})\}$, as illustrated in Figure 2.1(b). We then define G_k to be the graph obtained from G by replacing each pair of crossing edges $\{x,y\},\{u,v\}$ with H_k. To do this we delete the edges $\{x,y\}$ and $\{u,v\}$ and identify x, y, u and v with x_k, y_k, u_k, and v_k. We call the copy of H_k used to replace $\{x,y\}$ and $\{u,v\}$ the *crossover box* for the crossing $\{x,y\}$ and $\{u,v\}$. We say an orientation of G_k is valid at particular crossover box if the orientation of the edges (of H_k) incident with u_k, v_k, x_k, and y_k are consistent with an orientation of G.

Let a_k (resp. b_k) denote the number of orientations of the edges of H_k satisfying the Eulerian condition which correspond to *valid* (resp. *invalid*) orientations. These values satisfy

$$a_k = 4a_{k-1} + 2b_{k-1} \tag{2.2}$$

$$b_k = 4a_{k-1} + 3b_{k-1} \tag{2.3}$$

with $a_0 = b_0 = 1$. Now, let N_i denote the number of Eulerian orientations of G_0 which are valid at exactly i crossover boxes, so $N_\ell = \#\mathrm{EO}(G)$. Each Eulerian orientation of G_0 counted by N_i corresponds to exactly $a_k^i b_k^{\ell-i}$ Eulerian orientations of G_k, so we can write

$$\#\mathrm{PlanarEO}(G_k) = \sum_{i=0}^{\ell} N_i a_k^i b_k^{\ell-i}.$$

Hence, computing $\#\mathrm{PlanarEO}(G_k)/b_k^\ell$ corresponds to evaluating the polynomial

$$p(z) = \sum_{i=0}^{\ell} N_i z^i$$

at the point a_k/b_k. It is straightforward to check that a_k/b_k is a non-repeating sequence (either by basic calculus or an application of [91, Lemma 6.2]). Hence, by evaluating $\#\mathrm{PlanarEO}(G_k)/b_k^\ell$ at the value a_k/b_k, for $k = 0,1,\ldots,\ell+1$, we obtain enough information to recover the values N_i, $i = 0,1,\ldots,\ell$, by the Lagrange interpolation formula (Theorem 2.1).

Letting n and m denote the number of vertices and edges in G we have

$$|V(G_k)| = n + (4k+1)\ell$$

and

$$|E(G_k)| = m + (8k+2)\ell\,.$$

Hence, we can compute $\#\mathrm{EO}(G_k)$ in time polynomial in n, m, and ℓ, for each k. We can embed a graph with m edges in the plane with at most $\binom{m}{2}$ crossings: if we lay the vertices out on a circle each pair of edges can cross at most once. Thus, the number of steps required for the whole reduction is bounded by a polynomial in m and n. □

2.3 A natural Markov chain for sampling α-orientations

Before we define the natural Markov chain on the set of α-orientations of a planar graph, we introduce some structural results of Felsner [32] which will be useful in the definition and subsequent analysis of the chain. Let $G = (V,E)$ be a planar graph and let $\mathcal{F}(G)$ denote the set of bounded faces in some planar embedding of G. We will use f to denote the number of elements in $\mathcal{F}(G)$. A face is said to be directed in an orientation of G if its boundary edges form a directed cycle. We say a directed face γ is clockwise oriented (resp. counter-clockwise oriented) if the orientation of the edges of γ is clockwise (resp. counter-clockwise) on γ. We recall the definition of α-orientation.

Definition 2.3. Let $G = (V,E)$ be a graph and let $\alpha : V \to \mathbb{Z}$. An α-*orientation* of G is an orientation in which $outdeg(v) - indeg(v) = \alpha(v)$ for all $v \in V$. Let $\alpha\text{-}\mathrm{O}(G)$ denote the set of α-orientations of G.

Note that the Eulerian orientations of a graph are the α-orientations where $\alpha(v) = 0$ for all $v \in V$.

Definition 2.4. If an edge $e \in E$ has the same orientation in every α-orientation of G, we say e is α-*rigid*.

Felsner defined a special set of cycles in a planar graph, with respect to α, which he calls the *essential cycles*.

Definition 2.5 ([32])**.** A cycle C of G is *essential* (with respect to α) if

1. C is simple and chord free;
2. all edges on the interior of C are α-rigid;
3. there exists an α-orientation of G in which C is a directed cycle.

We denote by $\mathcal{F}_\alpha(G)$ the set of essential cycles of G with respect to α.

Remark 2.6. The set of essential cycles are exactly the faces of the planar graph obtained by removing all rigid edges. In particular, for the case of Eulerian orientations we have that $\mathcal{F}(G)$ is the set of essential cycles.

From here on we will assume that all rigid edges have been removed, with α updated accordingly, so the essential cycles of G, with respect to α, will always be $\mathcal{F}(G)$.

In [32], Felsner has shown that it is possible to convert any α-orientation of a planar graph G into another by performing a sequence of reversals of the edges of directed faces. Furthermore, Felsner defined a partial order on α-$\mathrm{O}(G)$.

Definition 2.7. Let $G = (V, E)$ be a planar graph and suppose we have some $\alpha : V \to \mathbb{N}$. We define a partially ordered set $Fels(G) = (\alpha\text{-}\mathrm{O}(G), \prec)$, where $\mathcal{E} \prec \mathcal{E}'$ if $\mathcal{E}'$ can be obtained from $\mathcal{E}$ by performing a sequence of reversals of clockwise oriented directed faces.

This order has a unique maximum element and minimum element: the unique α-orientation with no clockwise cycles, $\mathcal{E}_{\max}$, and the unique α-orientation with no counter-clockwise cycles, $\mathcal{E}_{\min}$. In [32], Felsner proved this order forms a *finite distributive lattice*.

Definition 2.8. A *distributive lattice* is a partially ordered set $(L, \prec)$ which also satisfies:

1. every pair of elements $x, y \in L$ has a unique least upper bound, $x \vee y$, called the *join*;
2. every pair of elements $x, y \in L$ has a unique greatest lower bound, $x \wedge y$, called the *meet*;
3. the join and meet operations are distributive

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

That $Fels(G)$ is a distributive lattice is established by giving a bijection between the α-orientations of a planar graph and a set of functions called α-*potentials*: for each $\mathcal{E} \in \alpha\text{-}\mathrm{O}(G)$ the α-potential associated with $\mathcal{E}$ is a function

$$\wp_{\mathcal{E}} : \mathcal{F}(G) \to \mathbb{N}.$$

To define these functions Felsner used a partial order on $\mathcal{F}(G)$: $\sigma \prec_{\mathcal{F}} \rho$ if σ and ρ share an edge and that edge is counter-clockwise on σ in the minimum orientation. The α-potentials are then defined as the set of functions of the form $\wp : \mathcal{F}(G) \to \mathbb{N}$ satisfying

$$\sigma \text{ and } \rho \text{ share an edge} \Rightarrow |\wp(\sigma) - \wp(\rho)| \leq 1 \tag{2.4}$$

$$\sigma \text{ is on the boundary} \Rightarrow \wp(\sigma) \leq 1 \tag{2.5}$$

$$\sigma \prec_{\mathcal{F}} \rho \Rightarrow \wp(\sigma) \leq \wp(\rho) \tag{2.6}$$

The bijection is given as follows: for each $\mathcal{E} \in \alpha\text{-O}(G)$, we define $\wp_{\mathcal{E}}$ to be function $\mathcal{F}(G) \to \mathbb{N}$ where $\wp_{\mathcal{E}}(\sigma)$ is equal to the number of times σ is reversed on any shortest path from $\mathcal{E}_{\min}$ to $\mathcal{E}$. See Figure 2.2 for an example. In the figure, each face is labelled with its potential value.

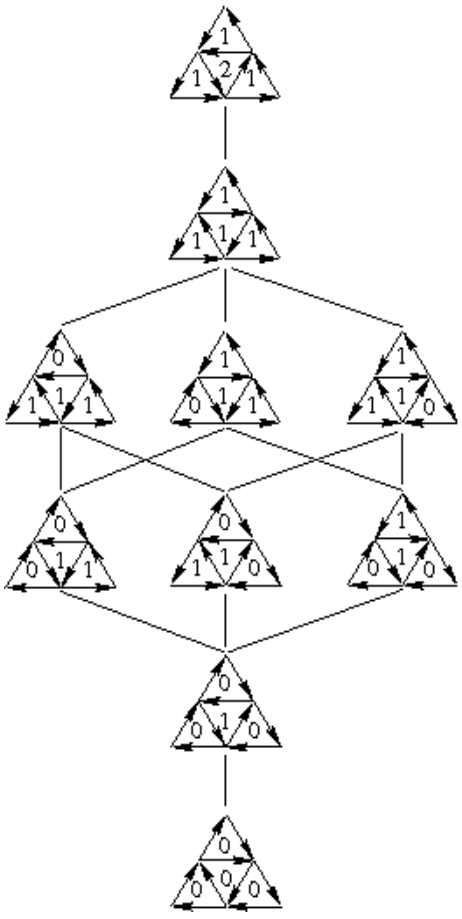

Figure 2.2: The Felsner lattice for a small graph

Any partial order $\prec$ of a set L induces a graph with vertex set L called the *cover graph* of L.

Definition 2.9. The *cover graph* of a partially ordered set $(L, \prec)$ is the graph with vertex set L and an edge joining $x, y \in L$ if there is no $z \in L$ such that $x \prec z \prec y$.

In the case of the Felsner lattice, two orientations are adjacent in the cover graph if they disagree on the orientation of a single directed face. Felsner's result suggests a natural Markov chain $\mathcal{M}_{\mathcal{F}}$ on the set of α-orientations of any planar graph G: a random walk on the cover graph of $Fels(G)$. Each transition in this chain reverses the orientation of a random directed face, so we call this the *face-reversal* Markov chain.

We now formally define our face-reversal chain, by specifying how X_{t+1} is generated from the current state X_t. We use *cw* and *ccw* to denote clockwise and counter-clockwise, respectively.

One step of the chain $\mathcal{M}_{\mathcal{F}}$

1. Choose $\kappa \in \mathcal{F}(G)$ and $dir \in \{cw, ccw\}$ uniformly at random.
2. If κ is *dir*-directed then obtain X_{t+1} from X_t by reversing the orientation of all the edges in κ.
3. Otherwise, set $X_{t+1} = X_t$.

Proposition 2.10. *For any planar graph $G = (V, E)$ and any $\alpha : V \to \mathbb{Z}$, the chain $\mathcal{M}_{\mathcal{F}}$ is ergodic and converges to the uniform distribution on α-$\mathrm{O}(G)$.*

Proof. By choosing the direction (cw or ccw) in which to perform the reversal as well as choosing a face, we automatically ensure that the probability of X_{t+1} being equal to X_t is at least $1/2$. Hence, $\mathcal{M}_{\mathcal{F}}$ is aperiodic.

Felsner's result [32] (described above) ensures the chain is connected, and so ergodic. The transition probability matrix of $\mathcal{M}_{\mathcal{F}}$ is

$$P(\mathcal{E}, \mathcal{E}') = \begin{cases} \frac{1}{2|\mathcal{F}(G)|} & \text{if } \mathcal{E} \oplus \mathcal{E}' = \gamma, \text{ for } \gamma \in \mathcal{F}(G) \\ 1 - \sum_{\mathcal{E}'' \neq \mathcal{E}} P(\mathcal{E}, \mathcal{E}'') & \text{if } \mathcal{E} = \mathcal{E}' \\ 0 & \text{otherwise} \end{cases}.$$

Thus, the chain is ergodic and symmetric, and so converges to the uniform distribution on α-$\mathrm{O}(G)$. □

Felsner's result has more use than proving ergodicity. We use the distributive lattice structure as part of our proofs of rapid mixing (§2.6), and torpid mixing (§2.8), of $\mathcal{M}_{\mathcal{F}}$ on different classes of graphs.

2.4 Path Coupling

Suppose $\mathcal{M}$ is an ergodic time-homogeneous Markov chain on finite state space Ω with probability transition matrix P and stationary distribution π. Recall the definition of the mixing time of a Markov chain from §1.2: a function $\tau(\varepsilon)$ such that for any $\varepsilon > 0$ and $t > \tau(\varepsilon)$,

$$\max_{x\in\Omega} \| P^t(x,\cdot) - \pi(\cdot) \|_{TV} \leq \varepsilon .$$

We say a chain is *rapidly mixing* if there is a polynomial upper bound on $\tau(\varepsilon)$ and *torpidly mixing* if there exists some $\varepsilon > 0$ such that $\tau(\varepsilon)$ is exponentially bounded from below.

Path Coupling

Recall from §1.2 that *coupling* is a common method used to show a Markov chain is rapidly mixing. The *path coupling* method, due to Bubley and Dyer [16], simplifies the process of defining and analysing a coupling. Bubley and Dyer showed that, if we define and analyse a Markovian coupling on the adjacent states in the chain then we can also construct a coupling for nonadjacent X and Y by composing the couplings of the pairs of states along a path $X = Z_0, Z_1, \ldots, Z_r = Y$, where (Z_i, Z_{i+1}) is a pair of states adjacent in the chain, for $i = 0, 1, \ldots, r$. This is encapsulated in the following theorem:

Theorem 2.11 (Bubley and Dyer [16])**.** *Let $\mathcal{M}$ be an ergodic Markov chain with state space Ω and let δ be an integer valued metric defined on $\Omega \times \Omega$ which takes values in $\{0, 1, \ldots, D\}$. Let S be a subset of $\Omega \times \Omega$ such that for all $(X,Y) \in \Omega \times \Omega$ there exists a path*

$$X = Z_0, Z_1, \ldots, Z_r = Y$$

between X and Y such that $(Z_i, Z_{i+1}) \in S$ for $0 \leq i < r$ and

$$\sum_{i=0}^{r-1} \delta(Z_i, Z_{i+1}) = \delta(X,Y) .$$

Now suppose (X_t, Y_t) is a coupling of $\mathcal{M}$ defined on S. If there exists $\beta \leq 1$ such that for all $(X,Y) \in S$

$$\mathbb{E}[\delta(X_{t+1}, Y_{t+1}) | (X_t, Y_t) = (X,Y)] \leq \beta\delta(X_t, Y_t)$$

then this coupling can be extended to a coupling (X_t, Y_t) defined on the whole of $\Omega \times \Omega$ such that

$$\mathbb{E}[\delta(X_{t+1}, Y_{t+1})] \leq \beta\delta(X_t, Y_t) .$$

Moreover, if $\beta < 1$ *then* $\tau(\varepsilon) \leq \frac{\log(D\varepsilon^{-1})}{1-\beta}$.

Remark 2.12. Although Theorem 2.11 does not require it, in most applications we have that S is the set of pairs of states adjacent in the chain:

$$S = \{(X,Y) \in \Omega : P(X,Y) > 0\}.$$

If $\beta = 1$ in Theorem 2.11 then in order to use the Bubley-Dyer path coupling technique of Theorem 2.11 it must be shown that the variance of the distance between any two states (not just the pairs in S) after one step of the coupling can be bounded away from 0, see, e.g., [16, 38]. This situation is undesirable, as the power of the path coupling technique is that it allows one to avoid considering all pairs of states. A recent result has relaxed this condition:

Theorem 2.13 (Bordewich and Dyer [13])**.** *Suppose we have a path coupling* $\mathcal{C} = (X_t, Y_t)$ *for an ergodic Markov chain* $\mathcal{M}$ *with distance metric* $\delta : \Omega \times \Omega \to \{0,\ldots,D\}$*, where* $S \subset \Omega \times \Omega$ *is the set of pairs of states at distance* 1*. Let* p *denote the minimum transition probability between pairs of states in* S*. We define a new chain, the* lazy chain $\mathcal{M}^\star$*, with probability transition matrix*

$$P^\star(X,Y) = \begin{cases} \frac{P(X,Y)+p}{1+p} & X = Y \\ \frac{P(X,Y)}{1+p} & \text{otherwise} \end{cases}.$$

If $\beta \leq 1$ *for the coupling* $\mathcal{C}$ *then the mixing time of the chain* $\mathcal{M}^\star$ *satisfies*

$$\tau^\star(\varepsilon) \leq \lceil p^{-1} e D^2 \rceil \lceil \log \varepsilon^{-1} \rceil.$$

Moreover, if $\tau'(\varepsilon)$ *denotes the random time* $\mathrm{Bin}(\tau^\star(\varepsilon), (1+p)^{-1})$*, where* Bin *denotes the binomial distribution, then the distribution of* $X_{\tau'(\varepsilon)}$ *is within* ε *of the stationary distribution of the original chain* $\mathcal{M}$*.*

Observation 2.14. For any Markov chain $\mathcal{M}$ which has a uniform stationary distribution, the lazy chain $\mathcal{M}^\star$ will also have a uniform stationary distribution.

The effect of the lazy chain $\mathcal{M}^\star$ is to adjust the transition probabilities slightly, so as to introduce a small additional probability that the distance between the two chains will change at any particular step. In many circumstances where $\beta = 1$, this is sufficient to ensure that the lazy chain is rapidly mixing. The second part of the above theorem means that, for all practical purposes, we can consider the mixing time of the chain $\mathcal{M}$ to be the same as the mixing time of $\mathcal{M}^\star$.

Applying Path Coupling to $\mathcal{M}_{\mathcal{F}}$

To apply path coupling to bounding the mixing time of the face-reversal chain $\mathcal{M}_{\mathcal{F}}$, we need to define a metric $\delta : \Omega^2 \to \mathbb{N}$ on the set of α-orientations of a planar graph G. The most natural choice is to define $\delta(\mathcal{E}, \hat{\mathcal{E}})$, for each $\mathcal{E}, \hat{\mathcal{E}} \in \alpha\text{-O}(G)$, to be the length of the shortest path between $\mathcal{E}$ and $\hat{\mathcal{E}}$ in the cover graph of $Fels(G)$. In particular, $\delta(\mathcal{E}, \hat{\mathcal{E}}) = 1$ if $\mathcal{E}$ and $\hat{\mathcal{E}}$ differ on a single face. Let $\mathcal{E}_{\min}$ and $\mathcal{E}_{\max}$ denote the unique minimum and maximum elements of $Fels(G)$. The fact that $Fels(G)$ is a distributive lattice implies that the maximum value the metric δ can take on pairs from $\alpha\text{-O}(G)$ is $D = \delta(\mathcal{E}_{\min}, \mathcal{E}_{\max})$, since we can always find a path of length $\leq D$ between two states $\mathcal{E}$ and $\hat{\mathcal{E}}$ either by going down from $\mathcal{E}$ to $\mathcal{E}_{min}$ and then up to $\hat{\mathcal{E}}$, or by taking a similar path through $\mathcal{E}_{\max}$.

Let S denote the set of pairs of Eulerian orientations which disagree on the orientation of a single directed face:

$$S = \{(\mathcal{E}, \hat{\mathcal{E}}) \in \alpha\text{-O}(G) \times \alpha\text{-O}(G) : \mathcal{E} \oplus \hat{\mathcal{E}} = \gamma, \text{ for some } \gamma \in \mathcal{F}(G)\}.$$

To apply path coupling (Theorem 2.11 or Theorem 2.13), we need to define a one-step Markovian coupling $\mathcal{C}$ on S and show that, in expectation, the distance between any pair of states in S does not increase after one step of $\mathcal{C}$. The natural choice is the coupling that chooses the same face and direction in both chains. This is a maximal coupling for our chosen distance metric. That is, if this coupling does not contract (for our chosen metric δ), we cannot expect any other one-step Markovian coupling to contract.

Proposition 2.15. *There is no one-step Markovian coupling for $\mathcal{M}_{\mathcal{F}}$ that contracts under the metric δ.*

Proof. Consider the situation in Figure 2.3. This is the subgraph of some graph G incident with a face γ, which consists of the only edges on which a pair of α-orientations, $\mathcal{E}$ and $\hat{\mathcal{E}}$, disagree. Any face in G not contained in the part of the graph shown in Figure 2.3 will either be directed in both $\mathcal{E}$ and $\hat{\mathcal{E}}$ or will be directed in neither $\mathcal{E}$ nor $\hat{\mathcal{E}}$. Hence, the move available, if any, resulting from choosing any face not included in the subgraph shown will be the same for $\mathcal{E}$ and $\hat{\mathcal{E}}$. By choosing one of these faces and the same direction in both chains, we can ensure that the distance between the two copies of the chain does not increase over one step of the coupling.

By choosing γ and a different direction in both copies of the chain, we can ensure coalescence with probability $1/f$. However, each neighbour of γ will be reversed with

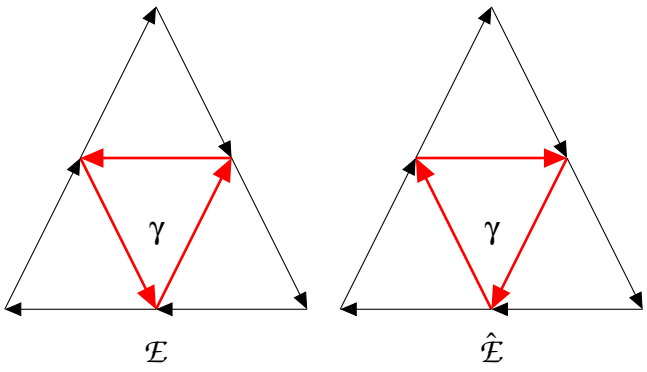

Figure 2.3: A configuration that gives rise to a non-contracting coupling for $\mathcal{M}_{\mathcal{F}}$

probability $1/2f$ in $\mathcal{E}$ but cannot be reversed in $\hat{\mathcal{E}}$. Hence, the distance will increase by 1 with probability $3/2f$, giving a total expected increase of $1/2f$. □

2.5 Extending the chain

In §2.4, we saw that no one-step Markovian coupling is sufficient to show rapid mixing of $\mathcal{M}_{\mathcal{F}}$ using the metric δ. However, we will see that if we extend the chain with extra moves in the style of [62, 38], we will then be able to design an appropriate path coupling for the extended chain. It is these additional *"tower moves"* that will allow us to couple with $\beta \leq 1$.

Definition 2.16. Let G be a planar graph, $\mathcal{E}$ an α-orientation of G, and γ a face of G. We say γ is *almost-directed* in $\mathcal{E}$ if all but one of the edges of γ have a common direction on γ. We call the edge oriented in the opposite direction the *blocking edge* of γ. For faces η and σ which are, respectively, almost-directed and directed in the orientation $\mathcal{E}$, we say there is a *tower* starting at η and ending at σ if there is a sequence of faces $\eta = \gamma_1, \ldots, \gamma_h = \sigma$ such that

- γ_i is almost directed in $\mathcal{E}$, and the blocking edge of γ_i is shared with γ_{i+1} for $1 \leq i \leq h-1$.
- γ_h is directed in $\mathcal{E}$

We say h is the *height* of the tower.

Observe that the definition of a tower implies that $C = \bigoplus_{1 \leq i \leq h} \gamma_i$ is a directed cycle in $\mathcal{E}$. We say that a tower is clockwise (resp. counter-clockwise) in an orientation if

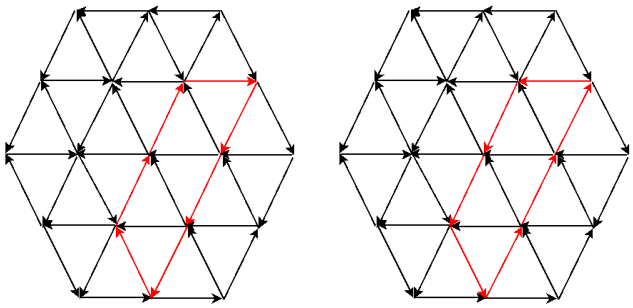

Figure 2.4: Reversing a tower

this cycle is clockwise (resp. counter-clockwise) in the orientation. We call γ_h and γ_1 the top and bottom of the tower, respectively, and refer to the right and left sides of the towers in terms of a walk from the bottom to the top. It follows that in a clockwise tower the internal edges are all directed from the right to the left, and vice-versa for counter-clockwise towers. Moreover, if $T = (\gamma_1, \ldots, \gamma_h)$ is a clockwise tower in $\mathcal{E}$ and $\mathcal{E}'$ is obtained by reversing C, then $T' = (\gamma_h, \gamma_{h-1}, \ldots, \gamma_1)$ is a counter-clockwise tower in $\mathcal{E}'$, see, e.g., Figure 2.4.

Let $\mathcal{E} \in \alpha\text{-O}(G)$ and suppose $\eta \in \mathcal{F}(G)$ is almost directed in $\mathcal{E}$. If there is a tower in $\mathcal{E}$ starting at η we can find it by walking along the faces of G, starting at η and choosing the face sharing the blocking edge with the current face at each step. If at any point we reach a directed face then we have found a tower. On the other hand, we can be certain there is no tower starting at η if we reach a face γ_i satisfying one of the following:

- γ_i is not directed but not almost directed[1];
- the blocking edge of γ_i lies on the boundary of the graph;
- the blocking edge of γ_i is the same edge as the blocking edge of γ_{i-1}.

To see that the process of forming a tower terminates (i.e., does not wrap around on itself), note that reaching some already explored face implies the existence of a cycle C such that every edge joining C to a vertex on its interior is directed towards C. But then none of these vertices can be involved in a directed cycle in $\mathcal{E}$, or, indeed, in any other α-orientation of G. Hence, these edges must have the same orientation in every α-orientation of G; that is, all edges joining C to its interior are rigid. We

[1]This situation cannot occur in the subgraphs of the triangular lattice we consider in §2.6

have assumed that all rigid edges have already been removed from G, and α has been adjusted accordingly, so this situation cannot occur. Furthermore, we observe that when $\alpha(v) = 0$, v cannot be incident with more than $deg(v)/2$ faces in any tower.

We now define the tower-moves chain, $\mathcal{M}_{\mathrm{T}}$, by describing how to obtain X_{t+1} from X_t. The definition includes an undetermined probability p_T which will be fixed later.

One step of the Markov chain $\mathcal{M}_{\mathrm{T}}$

1. Choose $dir \in \{cw, ccw\}$ and $\kappa \in \mathcal{F}(G)$ u.a.r.
2. If κ is a *dir*-cycle then obtain X_{t+1} from X_t by reversing the orientation of all the edges in κ.
3. If there is a *dir*-tower $T = (\gamma_i)_{1 \le i \le h}$ with $\gamma_1 = \kappa$ then let $C = \bigoplus_{1 \le i \le h} \gamma_i$. With probability p_T obtain X_{t+1} from X_t by reversing all the edges of C.
4. Otherwise, set $X_{t+1} = X_t$

This type of chain has been used to extend the face reversal chain in the past, see [62, 38]. The irreducibility of this chain is inherited from the irreducibility of $\mathcal{M}_{\mathcal{F}}$, since every transition in $\mathcal{M}_{\mathcal{F}}$ is also a transition in $\mathcal{M}_{\mathrm{T}}$. Again, the fact they we choose the direction of reversal for each transition ensures that there is a self-loop probability of $1/2$. Hence, $\mathcal{M}_{\mathrm{T}}$ is aperiodic and so ergodic. As long as p_T is chosen to be independent of whether T is a clockwise or a counter-clockwise tower, $\mathcal{M}_{\mathrm{T}}$ converges to the uniform distribution. To see this suppose $\mathcal{E}$ can be obtained from $\hat{\mathcal{E}}$ by reversing a clockwise tower T. Then we can obtain $\hat{\mathcal{E}}$ from $\mathcal{E}$ by reversing a counter-clockwise tower containing the same faces as T. Hence, $P(\mathcal{E}, \hat{\mathcal{E}}) = P(\hat{\mathcal{E}}, \mathcal{E})$ for any $\mathcal{E}, \hat{\mathcal{E}} \in \Omega$, and the stationary distribution of $\mathcal{M}_{\mathrm{T}}$ is uniform.

2.6 Rapid mixing on the triangular lattice

Up to now, we have not placed any restriction on G or α, beyond the fact that G is a planar graph and α is a function assigning an integer to the vertices of G, such that G has no α-rigid edges. That is to say, we have shown that both $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\mathrm{T}}$ converge to the uniform distribution on the set of α-orientations of any planar graph $G = (V, E)$, for any $\alpha : V \to \mathbb{N}$. In this section, we will prove that $\mathcal{M}_{\mathcal{F}}$ is rapidly mixing on the set of Eulerian orientations, possibly with some fixed boundary condition, of any *solid subgraph* G of the triangular lattice. These are the α-orientations of solid subgraph of the triangular lattice G, for some α with $\alpha(v) = 0$ for every internal vertex v; the boundary condition determines the values for $\alpha(v)$ on boundary vertices. Hereafter,

we are assuming we have some fixed boundary condition, and use $\mathrm{EO}(G)$ to refer to the set of Eulerian orientations of G with this fixed boundary condition.

First, in §2.6.1, we show that $\mathcal{M}_\Gamma$ (or, more precisely, the lazy version of $\mathcal{M}_\Gamma$) is rapidly mixing on the Eulerian orientations of any *solid subgraph* G of the *triangular lattice*, possibly with some fixed boundary condition. Then, in §2.6.2, we use the *comparison technique* of Diaconis and Saloff-Coste [26] to obtain rapid mixing of $\mathcal{M}_\mathcal{F}$.

2.6.1 Rapid mixing of $\mathcal{M}_\Gamma^\star$

In this section we use the path coupling technique of Bubley and Dyer [16] to analyse the mixing time of the Markov chain $\mathcal{M}_\Gamma$ on $\mathrm{EO}(G)$ when G is a solid subgraph of the triangular lattice with some fixed boundary condition, and p_T is chosen appropriately; more precisely, we use the result of Bordewich and Dyer [13] to show that the *lazy chain* $\mathcal{M}_\Gamma^\star$ (see Theorem 2.13) mixes rapidly.

This is not the first result regarding the mixing time of this type of chain. $\mathcal{M}_\Gamma$ has been shown to be rapidly mixing on the square lattice (using different p_T values to ones we shall use) for the case of a particular fixed boundary condition [62] and for the case of a free boundary [38]. Fehrenbach and Rüschendorf [31] attempted to give a proof of rapid mixing for a related Markov chain (on the triangular lattice) in which only towers of height 2 are used. However, the path coupling defined for the chain in [31] does not contract as claimed. In fact, it is possible to show that no one-step path coupling, using the natural distance metric, can prove rapid mixing for the chain in [31]. It could be that a one-step coupling will work with a more complicated metric [14], but this has not been attempted.

Rapid mixing proofs for $\mathcal{M}_\Gamma$ are dependent on the correct choice of the probabilities p_T. For example, the proof of [38] sets p_T to $1/4h$ if the tower runs along the boundary, and $1/2h$ otherwise, where h is the height of the tower T. We now state the main result of this section but defer the proof until we have presented some useful lemmas.

Theorem 2.17. *Let G be a solid section of the triangular lattice with any fixed boundary condition. Then, $\mathcal{M}_\Gamma^\star$, the lazy version of the Markov chain $\mathcal{M}_\Gamma$, is rapidly mixing with mixing time $\tau_{\mathcal{M}_\Gamma^\star}(\varepsilon)$ satisfying*

$$\tau_{\mathcal{M}_\Gamma^\star}(\varepsilon) \in O(f^4 \log \varepsilon^{-1}),$$

when $p_T = 1/3h$ for all towers T of height h.

Consequentially, the chain $\mathcal{M}_{\mathrm{T}}$ can be considered to mix in this time, for all practical purposes (see [13]). Note that, with $p_T = 1/3h$, the transition probability matrix of the chain, P, is given by

$$P(\mathcal{E},\mathcal{E}') = \begin{cases} \frac{1}{2f} & \text{if } \mathcal{E}\oplus\mathcal{E}' \text{ is a single face} \\ \frac{1}{6fh} & \text{if } \mathcal{E}\oplus\mathcal{E}' \text{ is a single tower of height } h \\ 1-\sum_{\mathcal{E}''\neq\mathcal{E}} P(\mathcal{E},\mathcal{E}'') & \text{if } \mathcal{E}=\mathcal{E}' \\ 0 & \text{otherwise}. \end{cases}$$

We first bound the maximum distance between any pair of α-orientations of the triangular lattice.

Lemma 2.18. *Let G be any solid subgraph of the triangular lattice with f bounded faces. The maximum distance between any pair of α-orientations of G in the Felsner lattice is $O(f^{3/2})$.*

Proof. The maximum distance between any two vertices in the cover graph of the Felsner lattice is equal to the distance between the maximum and minimum elements. Thus, the problem reduces to bounding $\delta(\mathcal{E}_{\max},\mathcal{E}_{\min})$.

We write $\wp_{\max}$ for $\wp_{\mathcal{E}_{\max}}$, the potential function corresponding to the maximum element of $Fels(G)$. Then, from the definition of the bijection between α-orientations and α-potentials given in §2.3 we can conclude that the distance between the maximum and minimum orientations is $\sum_{\gamma\in\mathcal{F}(G)}\wp_{\max}(\gamma)$.

Conditions (2.4) and (2.5) of the definition of α-potentials imply that $\wp_{\max}(\gamma)$ is exactly the minimum number of edges in a shortest path in the dual graph of G from γ to the unbounded face. Let G_k be the smallest graph which contains a face γ with $\wp_{\max}(\gamma)=k$. We can construct G_k inductively, starting with $G_1 = K_3$. To extend G_k to G_{k+1} we add a triangular face onto each edge of the boundary of G_k; see Figure 2.5 for an example of what these look like. Let γ be the single face which is contained in G_1. Every path in the dual graph of G_k from γ to the unbounded face has length k, so $\wp_{\max}(\gamma)=k$ in the maximum α-potential of G_k. Moreover, removing any boundary face will introduce a path (in the dual graph) from the γ to the unbounded face of length $k-1$, so any graph smaller than G_k will not have any face which has value k in the maximum α-potential.

A simple inductive argument shows that the number of faces added at each step is $3k$. This implies $|\mathcal{F}(G_k)| \in \Theta(k^2)$, so $\wp_{\max}(\gamma) \in O(\sqrt{f})$ for any face γ. □

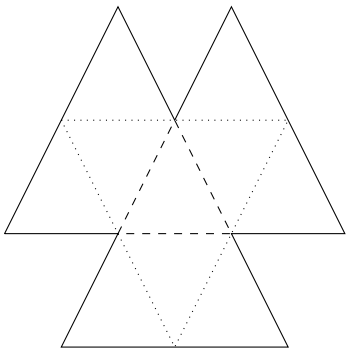

Figure 2.5: The graph G_3 from Lemma 2.18. The edges from G_1 and $G_2 \setminus G_1$ are marked by dashed and dotted lines respectively

Let S be the set of pairs of orientations at distance 1 in the Felsner lattice. We want to define a coupling $\mathcal{C}$ for every pair $(\mathcal{E}, \hat{\mathcal{E}}) \in S$. Suppose $(\mathcal{E}, \hat{\mathcal{E}}) \in S$, let γ be the face on which they disagree, and let $N(\gamma)$ denote the set of faces which share an edge with γ. Suppose $(X_t, Y_t) = (\mathcal{E}, \hat{\mathcal{E}})$. Our path coupling $\mathcal{C}$ for $\mathcal{M}_{\mathrm{T}}$ chooses the same face κ in both chains and couples the transitions as follows:

- if $\kappa = \gamma$ then reverse κ in $\mathcal{E}$ but not $\hat{\mathcal{E}}$ with probability $1/2$, and reverse κ in $\hat{\mathcal{E}}$ but not in $\mathcal{E}$ with probability $1/2$;
- if the same transition is available for the choice of κ in both $\mathcal{E}$ and $\hat{\mathcal{E}}$ then apply this change to both orientations with the relevant probability and leave both unchanged otherwise;
- if different transitions are available for κ in both orientations (with the transition in $\mathcal{E}$ happens with higher probability) then apply both transitions with the lower probability, apply the transition in $\mathcal{E}$ alone with its remaining probability, and leave both orientations unchanged otherwise.

In order to apply Bordewich and Dyer's path coupling theorem (Theorem 2.13) we need to show that the expected distance between the two copies of the chain does not increase after a single step of the coupling. To do this we need to consider which choice of faces will cause the distance to increase, which choices will leave the distance unchanged, and which choices will cause the distance to decrease.

Definition 2.19. Suppose $(\mathcal{E}, \hat{\mathcal{E}})$ is the current state, (X_t, Y_t), of the coupling $\mathcal{C}$. We say the move resulting from choosing κ *involves* $\sigma \in N(\gamma)$ if $\kappa = \sigma$ or there is a tower (in either $\mathcal{E}$ or $\hat{\mathcal{E}}$) starting at κ that contains σ.

The distance between the coupled chains can change if and only if $\kappa = \gamma$ or the move at κ involves some $\sigma \in N(\gamma)$. Note that no move can involve more than one element of $N(\gamma)$, so, in the analysis of our path coupling, we can treat each $\sigma \in N(\gamma)$ separately. In the following analysis, for each $\sigma \in N(\gamma)$, we use δ_σ to denote the contribution to the change to the distance between the two Eulerian orientations after a single step of the coupling resulting from moves involving the face σ. We will now analyse the expected value of δ_σ, conditioned on whether σ is a directed face in $\mathcal{E}$ or $\hat{\mathcal{E}}$, or is directed in neither.

Lemma 2.20. *Suppose $\sigma \in N(\gamma)$ is a directed face in $\mathcal{E}$ or $\hat{\mathcal{E}}$. Then $\mathbb{E}[\delta_\sigma] = \frac{1}{3f}$.*

Proof. We assume that σ is directed in $\mathcal{E}$; the case when σ is directed in $\hat{\mathcal{E}}$ is symmetric. Let $\kappa \in \mathcal{F}(G)$ such that selecting κ gives a move which involves σ in at least one of the coupled chains. We have two cases to consider:

Case $\kappa = \sigma$: Since σ is a neighbour of γ, and σ is directed in $\mathcal{E}$, it follows that the blocking edge of σ in $\hat{\mathcal{E}}$ is the edge shared with γ. Then $T = \{\sigma, \gamma\}$ is a tower of height 2 in $\hat{\mathcal{E}}$ with a reversal probability of $\frac{1}{12f}$. The coupling reverses σ in $\mathcal{E}$ and T in $\hat{\mathcal{E}}$ with probability $\frac{1}{12f}$, and reverses σ in $\mathcal{E}$ but leaves $\hat{\mathcal{E}}$ unchanged with probability $\frac{5}{12f}$. The former results in coalescence, whereas the latter yields a pair of orientations which are distance 2 apart. Hence, the contribution to $\mathbb{E}[\delta_\sigma]$ from this case is

$$\frac{5}{12f} - \frac{1}{12f} = \frac{1}{3f}.$$

Case $\kappa \neq \sigma$: Since γ and σ are both directed in $\mathcal{E}$ it follows that there must be a tower T_1 starting at κ and ending at σ in $\mathcal{E}$ that does not contain γ, and a tower $T_2 = T_1 \cup \{\gamma\}$ starting at κ in $\hat{\mathcal{E}}$. Let h be the height of T_1, so T_1 is reversed in $\mathcal{E}$ with probability $\frac{1}{6fh}$ and T_2 is reversed in $\hat{\mathcal{E}}$ with probability $\frac{1}{6f(h+1)}$. Observe that if we reverse T_1 in $\mathcal{E}$ we obtain an orientation which is distance $h+1$ from $\hat{\mathcal{E}}$, but if we also reverse T_2 in $\hat{\mathcal{E}}$ then we have the same orientation in both chains. Therefore, the contribution to $\mathbb{E}[\delta_\sigma]$ from this case is

$$h\left(\frac{1}{6fh} - \frac{1}{6f(h+1)}\right) - \frac{1}{6f(h+1)} = 0.$$

Thus, the only face whose selection will result in a move which has non-zero contribution to $\mathbb{E}[\delta_\sigma]$ is σ itself, so $\mathbb{E}[\delta_\sigma] = 1/3f$. □

Lemma 2.21. *If $\sigma \in N(\gamma)$ is not directed in $\mathcal{E}$ or $\hat{\mathcal{E}}$ then $\mathbb{E}[\delta_\sigma]$ is no more than $\frac{1}{3f}$.*

Proof. Observe that, since σ is not directed in $\mathcal{E}$ or $\hat{\mathcal{E}}$, the blocking edge of σ will not be shared with γ in either orientation and the blocking edge of σ is different in both orientations. Hence, no tower can include both σ and γ. Assuming there is a tower containing σ in at least one of the two orientations we have two disjoint cases to consider. Suppose $\mathcal{E}$ has a tower starting at some $\eta \in N(\sigma)\backslash\{\gamma\}$. Then there will be no tower containing σ in $\hat{\mathcal{E}}$ as the procedure for constructing towers described in §2.3 is guaranteed to reach a pair of consecutive faces sharing the same blocking edge (σ and η). Thus, when we are in this situation we can assume that one of the orientations will be unchanged after one step of the coupling. In the second case, when there is no tower starting at any $\eta \in N(\sigma)\backslash\{\gamma\}$ in $\mathcal{E}$ or $\hat{\mathcal{E}}$, there may be a tower starting at σ in either orientation.

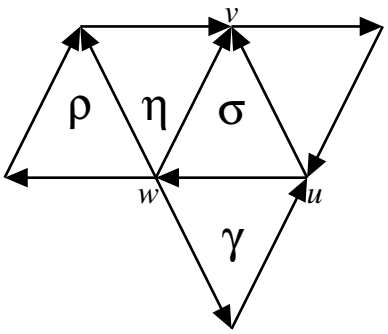

Figure 2.6: Example from Lemma 2.21

$\exists$ **a tower starting at** $\eta \in N(\sigma)\backslash\gamma$ **in** $\mathcal{E}$ **or** $\hat{\mathcal{E}}$**:** We can assume, without loss of generality, that $\mathcal{E}$ is the orientation with a tower starting at some $\eta \in N(\sigma)\backslash\gamma$. Since no move involving σ is possible in $\hat{\mathcal{E}}$ we only need to bound the expected distance between $\mathcal{E}$ and all $\mathcal{E}'$ which can be obtained by making a move involving σ in $\mathcal{E}$.

We begin by showing that any tower containing σ in $\mathcal{E}$ must start at σ or a neighbour of σ. To see this suppose we have a tower in $\mathcal{E}$ containing ρ, η, and σ where η is a neighbour of σ and $\rho \in N(\eta)\backslash\{\sigma\}$. Let u, v, w be the vertices of σ, and suppose that the edges of σ are oriented (u,v), (u,w), and (w,v) and that $\{u,w\}$ is the edge shared between γ and σ. Then σ must share $\{v,w\}$ with η. Recall that no vertex can be incident with more then 3 faces in any tower. Therefore, v cannot belong to the edge shared between ρ and η, so ρ must contain w (as illustrated in Figure 2.6). To satisfy the definition of a tower, both edges of ρ incident with w must be oriented away from w. But this implies that there are 4 edges oriented away from w in $\mathcal{E}$, a contradiction (e.g., see Figure 2.6). An identical argument holds if the edges of σ are

oriented (v,u), (u,w), and (v,w). Hence, any tower containing σ must start at η or σ.

Let h be the height of the tower starting at σ. Then, the move in which σ is the chosen face is made with probability $\frac{1}{6fh}$, and the move in which η is the chosen face is made with probability $\frac{1}{6f(h+1)}$ (if it exists). Since these moves increase the distance by h and $h+1$, respectively, we have

$$\mathbb{E}[\delta_\sigma] \leq \frac{h}{6fh} + \frac{h+1}{6f(h+1)} = \frac{1}{3f}.$$

$\nexists$ a tower starting at $\eta \in N(\sigma)\backslash\gamma$ in $\mathcal{E}$ or $\hat{\mathcal{E}}$: In the worst case, we could have a tower starting at σ in both orientations. If this occurs, then the two towers will have opposite orientations in $\mathcal{E}$ and $\hat{\mathcal{E}}$, and each will be the largest tower containing σ in the orientation. Let T_1 and T_2 denote the towers in $\mathcal{E}$ and $\hat{\mathcal{E}}$, respectively, and let h_1 and h_2 be the heights of each tower. If $h_1 \leq h_2$ then the coupling reverses the towers in $\mathcal{E}$ and $\hat{\mathcal{E}}$ with probability $\frac{1}{6fh_2}$, and reverses the tower in $\mathcal{E}$ alone with probability $\frac{1}{6fh_1} - \frac{1}{6fh_2}$, conditional on σ being the chosen face. The first situation yields a pair orientations which are distance $h_1 + h_2 + 1$ apart, and the second a pair orientations which are $h_1 + 1$ apart. Hence,

$$\mathbb{E}[\delta_\sigma] \leq (h_1+h_2)\frac{1}{6fh_2} + h_1\left(\frac{1}{6fh_1} - \frac{1}{6fh_2}\right) = \frac{1}{3f}.$$

The analysis is identical if $h_2 \leq h_1$.
Since two cases are disjoint, we see that $\mathbb{E}[\delta_\sigma]$ is no more than $\frac{1}{3f}$. □

We now apply Theorem 2.13 to obtain a bound on the mixing time.

Proof of Theorem 2.17. We use the coupling defined earlier and analysed in Lemma 2.20 and Lemma 2.21. With probability $1/f$, γ is reversed in one or the other of the two orientations, causing the two chains to coalesce. Combining this fact with the results of Lemmas 2.20 and 2.21 we find that for all $\mathcal{E}$ and $\hat{\mathcal{E}}$ differing on the orientation of a single face

$$\mathbb{E}[\delta(X_{t+1},Y_{t+1}) - \delta(X_t,Y_t) | (X_t,Y_t) = (\mathcal{E},\hat{\mathcal{E}})] \leq 3\frac{1}{3f} - \frac{1}{f} = 0,$$

where $(X_t,Y_t)_{t\geq 1}$ is the coupling $\mathcal{C}$ defined on S, the set of pairs of Eulerian orientations which are adjacent in the cover graph of the Felsner lattice. The transition probability for any pair of states in S is $1/2f$. Moreover, since we have just shown that $\beta = 1$ for the path coupling $\mathcal{C}$, we can apply Theorem 2.13 with $p = 1/2f$ and $D \in O(f^{\frac{3}{2}})$ (from Lemma 2.18) to obtain that the lazy version of the chain, $\mathcal{M}_\Gamma^\star$, mixes in time $\tau(\varepsilon) \in O(f^4 \log \varepsilon^{-1})$. □

2.6.2 Rapid mixing of $\mathcal{M}_{\mathcal{F}}$

In the previous section, we applied Theorem 2.13 to show that the lazy version of the tower chain, $\mathcal{M}_{\mathrm{T}}^{\star}$, is rapidly mixing. Since the smallest transition probability of any pair in the set S used to define the path coupling is $1/2f$, it follows that the transition probability matrix of this chain, $P^{\star}$, is given by

$$P^{\star}(\mathcal{E},\mathcal{E}') = \begin{cases} \frac{1}{2f+1} & \text{if } \mathcal{E}\oplus\mathcal{E}' = \text{ is a single face} \\ \frac{1}{3(2f+1)h} & \text{if } \mathcal{E}\oplus\mathcal{E}' \text{ is a single tower of height } h \\ 1-\sum_{\mathcal{E}''\neq\mathcal{E}} P^{\star}(\mathcal{E},\mathcal{E}'') & \text{if } \mathcal{E}=\mathcal{E}' \\ 0 & \text{otherwise} \end{cases}$$

Although this chain has different transition probabilities, it also converges to the uniform distribution on $\Omega = \mathrm{EO}(G)$. Hence, we can apply the *comparison method* of Diaconis and Saloff-Coste [26] to obtain a bound on the mixing time of $\mathcal{M}_{\mathcal{F}}$.

The Comparison Method

If we cannot find a bound on the mixing time of a Markov chain $\mathcal{M}$ by direct analysis of $\mathcal{M}$, but can find a bound on the mixing time of another chain $\widetilde{\mathcal{M}}$, which has the same state space as $\mathcal{M}$, we can often use the so-called *comparison theorem* of Diaconis and Saloff-Coste [26] to obtain a bound on the mixing time of $\mathcal{M}$. We will use the formulation of the Diaconis and Saloff-Coste result from [73], restated here for convenience. Note that we are using $E(P)$ to denote the set of edges corresponding to moves between adjacent states in the Markov chain with transition matrix P.

Theorem 2.22. ([73, Proposition 4]) *Suppose P and $\widetilde{P}$ are the transition matrices of two reversible Markov chains, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, both with the state space Ω and stationary distribution π, and let $\pi_{\star} = \min_{x\in\Omega}\pi(x)$. For each pair $(u,v)\in E(\widetilde{P})$, define a path γ_{uv} which is a sequence of states $u = u_0, u_1, , \ldots, u_k = v$ with $(u_i,u_{i+1})\in E(P)$ for all i. For $(x,y)\in E(P)$, let $\Gamma(x,y) = \{(u,v)\in E(\widetilde{P}) : (x,y)\in\gamma_{uv}\}$. Let*

$$A = \max_{(x,y)\in E(P)} \left\{ \frac{1}{\pi(x)P(x,y)} \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|\pi(u)\widetilde{P}(u,v) \right\}.$$

Suppose that the second largest eigenvalue, λ_1, of $\widetilde{P}$ satisfies $\lambda_1 \geq 1/2$. Then for any $0<\varepsilon<1$

$$\tau_{\mathcal{M}}(\varepsilon) \in O(A\tau_{\widetilde{\mathcal{M}}}(\varepsilon)\log 1/\pi_{\star}).$$

To obtain a polynomial bound on the mixing time of $\mathcal{M}$, we want to show that the simulation of the transitions of $\widetilde{\mathcal{M}}$ by transitions of $\mathcal{M}$ does not overload any edge in

$E(P)$. That is, we want the value of A in Theorem 2.22 to be bounded by a polynomial in the size of the elements of Ω.

Comparing $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\mathrm{T}}^{\star}$

We first show that every move of $\mathcal{M}_{\mathrm{T}}^{\star}$ can be simulated by moves of the chain $\mathcal{M}_{\mathcal{F}}$. Suppose $T = \{\gamma_1, \gamma_2, \ldots, \gamma_h\}$ is a tower in $\mathcal{E}$ and that $\hat{\mathcal{E}}$ is the orientation obtained by reversing T. Observe that by the definition of a tower γ_h is a directed cycle in $\mathcal{E}$. Then we can apply the transition in $\mathcal{M}_{\mathcal{F}}$ that reverses γ_h to $\mathcal{E}$ and obtain a new Eulerian orientation $\mathcal{E}''$. But there is now a tower $T' = \{\gamma_1, \gamma_2, \ldots, \gamma_{h-1}\}$ in $\mathcal{E}'$ which can be reversed to obtain $\hat{\mathcal{E}}$. Repeating this process until we reach $\hat{\mathcal{E}}$ gives a decomposition of the tower move reversing T in $\mathcal{E}$ into moves of the chain $\mathcal{M}_{\mathcal{F}}$. Hence, we are able to apply the comparison method (Theorem 2.22) to obtain a bound on the mixing time of $\mathcal{M}_{\mathcal{F}}$.

We are now ready to prove our rapid mixing result for $\mathcal{M}_{\mathcal{F}}$. We bound the mixing time in terms of f, the number of unbounded faces, and h, the height of the largest tower. Then, we derive mixing times for several different cases by considering the values h can take.

Theorem 2.23. *Suppose G is a solid subgraph of the triangular lattice and let h be the maximum height of any tower in an Eulerian orientation of G (possibly with respect to some fixed boundary condition). Then the mixing time of the face-reversal Markov chain $\mathcal{M}_{\mathcal{F}}$ satisfies*

$$\tau_{\mathcal{M}_{\mathcal{F}}}(\varepsilon) \in O(h^2 f^5 \log \varepsilon^{-1}).$$

Proof. Note that $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\mathrm{T}}^{\star}$ both have a self-loop probability of $1/2$, ensuring that the second-largest eigenvalue of either chain will be at least $1/2$, so Theorem 2.22 can be applied.

Let P and $P^{\star}$ denote the transition matrices of $\mathcal{M}_{\mathcal{F}}$ and $\mathcal{M}_{\mathrm{T}}^{\star}$. For each pair of states (x, y) that differ on the orientation of exactly one face (i.e., each $(x, y) \in E(P)$), we define $\Gamma(x, y)$ to be the set of all transitions in $\mathcal{M}_{\mathrm{T}}^{\star}$ containing the transition $t = (x, y)$ as a sub-move. Each pair $(u, v) \in \Gamma(x, y)$ corresponds to a pair of orientations which

differ on exactly one tower; let γ_{uv} denote this tower. For each $(x,y) \in E(P)$ we have

$$\begin{aligned} A_{x,y} &= \frac{1}{\pi(x)P(x,y)} \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|\pi(u)P^\star(u,v) \\ &= 2f \sum_{(u,v)\in\Gamma(x,y)} |\gamma_{uv}|P^\star(u,v)\,, \qquad (2.7) \\ &\leq 1 + \frac{1}{3}(|\Gamma(x,y)|-1)\,, \qquad (2.8) \end{aligned}$$

where (2.7) is due to the fact that all transition probabilities in P are $\frac{1}{2f}$ and that π is uniform, and (2.8) is due to the fact that $P^\star(u,v) = \frac{1}{2f+1}$ if $(u,v) = (x,y)$, and $P^\star(u,v) = \frac{1}{3(2f+1)|\gamma_{uv}|}$ if (u,v) is the reversal of a tower.

We will now find an upper bound for $|\Gamma(x,y)|$. Let γ be the face that is reversed in the transition $t = (x,y)$. We need to consider the different cases in which t can feature as part of the decomposition of a tower move $(u,v) \in E(P^\star)$. Observe that there are three different directions in which a tower can pass through γ and contain t as a sub-move (one for each pair of edges of γ). Let h_i be the height of the maximum length tower passing through γ in direction i (over all orientations $\mathcal{E} \in \Omega$), let $(\gamma_j)_{1\leq j\leq h_j}$ denote the sequence of faces that make up this tower, and suppose $\gamma_k = \gamma$. Any other tower which passes through γ in this direction and whose encoding uses t must be subtower of the maximum length tower. Moreover, the bottom of any such tower must be contained in $\{\gamma_j : 1 \leq j < k\}$ and the top of any such tower must be contained in $\{\gamma_j : k \leq j \leq h_i\}$. To see this, observe that x is the orientation obtained by reversing the tower $(\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_{h_i})$ (in the orientation containing the maximum length tower). Hence, reversing the tower $(\gamma_j, \gamma_{j+1}, \ldots, \gamma_{h_i})$, for any j satisfying $k < j \leq h_i$ will yield an orientation containing a tower whose encoding uses t. Moreover, each of these orientations has a tower whose encoding uses t for each element of $\{\gamma_1, \ldots, \gamma_{k-1}\}$. Thus, the number of towers using t that pass through γ in direction i is $(k-1)(h_i - k + 1)$, which could be as large as $h_i^2/4$. Hence, we have

$$|\Gamma(x,y)| \leq O(h_0^2) + O(h_1^2) + O(h_2^2) = O(h^2)\,.$$

Thus, we have $A_{x,y} \in O(h^2)$ for all $(x,y) \in E(P)$. Finally, we need to find an upper bound on the value $1/\pi_\star$, where $\pi_\star$ is the minimum value of the stationary distribution of $\mathcal{M}_\mathcal{F}$. Since this is the uniform distribution, $1/\pi$ has the same value on all $x \in \Omega$: $|\Omega|$. The number of edges in G is no more than $3f$, so 2^{3f} provides an upper bound on the number of orientations of G, and so also on $1/\pi_\star$. Combining all this with Theorems 2.22 and 2.17 we get

$$\tau_{\mathcal{M}_\mathcal{F}}(\varepsilon) \in O(h^2 f^5 \log \varepsilon^{-1})\,. \qquad \square$$

As a corollary to Theorem 2.23, we can obtain a bound on the mixing time of $\mathcal{M}_{\mathcal{F}}$ for any solid subgraph of the triangular lattice and arbitrary boundary condition (Corollary 2.24). However, this bound is overly pessimistic and we can improve it by a factor of f for two important special cases (Corollary 2.25 and Corollary 2.26).

Corollary 2.24. *Let G be a solid subgraph of the triangular lattice and let α be an arbitrary boundary condition. Then the mixing time of $\mathcal{M}_{\mathcal{F}}$ satisfies*

$$\tau_{\mathcal{M}_{\mathcal{F}}}(\varepsilon) \in O(f^7 \log \varepsilon^{-1})$$

Proof. In general, when we consider an arbitrary solid subgraph of the triangular lattice with an arbitrary fixed boundary condition, we can have towers of height $O(f)$ and so, in this case, Theorem 2.23 gives a bound on the mixing time of

$$\tau_{\mathcal{M}_{\mathcal{F}}}(\varepsilon) \in O(f^7 \log \varepsilon^{-1}). \qquad \square$$

Corollary 2.25. *Suppose G is an Eulerian solid subgraph of the triangular lattice. Then, the mixing time of $\mathcal{M}_{\mathcal{F}}$ satisfies*

$$\tau_{\mathcal{M}_{\mathcal{F}}}(\varepsilon) \in O(f^6 \log \varepsilon^{-1})$$

Proof. We need to bound the height of any tower in an Eulerian orientation of G, where G is a solid subgraph of the triangular lattice.

Let $T = (\gamma_1, \gamma_2, \ldots, \gamma_h)$ denote a tower in an Eulerian orientation of G. We assume that T is a clockwise tower; the argument is identical for counter-clockwise towers. Recall that all the internal edges of a clockwise tower are directed towards the vertices on the left, so any Eulerian orientation of G must contain a set of $h-1$ edge-disjoint directed paths linking the left-side vertices to the right-side vertices. Each of these paths must go around the top or the bottom of the tower. But each path that goes around the bottom (resp. top) contributes 1 to the distance from γ_1 (resp. γ_h) to the boundary. Hence, by (2.4), (2.5) and (2.6), $\max(\wp_{\max}(\gamma_1), \wp_{\max}(\gamma_h)) \geq h/2$. But $\wp_{\max}(\gamma) \in O(\sqrt{f})$ for any γ (see proof of Lemma 2.18), whence $h \in O(\sqrt{f})$. $\square$

Corollary 2.26. *Let G be the subgraph of triangular lattice induced by the set of vertices*

$$\{(i,j) : 0 \leq i < n, 0 \leq j < n\}.$$

Then, the mixing time of $\mathcal{M}_{\mathcal{F}}$ satisfies

$$\tau_{\mathcal{M}_{\mathcal{F}}}(\varepsilon) \in O(f^6 \log \varepsilon^{-1})$$

Proof. Recall that no vertex can be incident with more than 3 faces in a tower in an orientation of G. That is, towers can only extend in straight lines in the natural embedding of G in the plane. Thus, the longest sequence of any faces which can form a tower is n, which is equal to $\sqrt{f}$. □

2.7 Approximating $\#\mathrm{EO}(G)$

Suppose G is a solid subgraph of the triangular lattice and α some fixed boundary condition. If we choose an edge $e = \{u, v\}$ from the unbounded face for the above reduction, then the resulting graph will be a solid subgraph of the triangular lattice and the pair of functions $\alpha_{(u,v)}$ and $\alpha_{(v,u)}$ will both correspond to boundary conditions on $G - e$. In other words, the problem of counting Eulerian orientations with fixed boundary conditions is self-reducible (Definition 1.35) for solid subgraphs of the triangular lattice. Thus, we can use our rapidly mixing Markov chain with Algorithm 2 to approximate the number of Eulerian orientations with fixed boundary condition of any solid subgraph of the triangular lattice.

We now sketch how the algorithm operates. We construct a sequence of graphs $G_1, \ldots, G_m$, where m is the number of edges, defined inductively as

$$\begin{aligned} G_1 &= G; \\ G_i &= G_{i-1} - e_{i-1} \text{ for } i \geq 2, \end{aligned}$$

where e_i is an edge from the boundary face of G_i for each $i = 1, 2, \ldots, m-1$, and a sequence of boundary conditions

$$\begin{aligned} \alpha_1 &= \alpha \\ \alpha_i &= (\alpha_{i-1})_{a_i} \text{ for } i \geq 2, \end{aligned}$$

where a_i is an orientation of the edge e_i. For $i = 1, 2, \ldots, m-1$, we estimate

$$\rho_i = \frac{\#\alpha_i\text{-}\mathrm{O}(G_{i+1})}{\#\alpha_i\text{-}\mathrm{O}(G_i)},$$

by generating random α_i-orientations of G_i and counting how many can be extended to α_{i+1}-orientations of G_{i+1}. The value returned by Algorithm 2 is an approximation to

$$\#\alpha\text{-}\mathrm{O}(G) = \left(\prod_{i=1}^{m-1} \rho_i \right)^{-1}.$$

2.8 Torpid mixing

Given the small collection of positive results regarding the mixing time of the face-reversal chain $\mathcal{M}_{\mathcal{F}}$ (Theorem 2.23 and [62, 38]) and given that the reduction of [69] allows us to sample from $\mathrm{EO}(G)$ in polynomial time for any graph, one might hope that $\mathcal{M}_{\mathcal{F}}$ would be rapidly mixing on the set of Eulerian orientations of any planar graph. In fact, this is not true and in this section we exhibit a family of planar graphs for which $\mathcal{M}_{\mathcal{F}}$ is torpidly mixing. Consider the family of graphs H_N, of which H_2 is shown in Figure 2.7. Formally, H_N is a graph with vertex set

$$V = \{v_i : 1 \le i \le 12N\} \cup \{u_i : 1 \le i \le 12N+6\} \cup \{w_i : 1 \le i \le 6N\}.$$

The edges of H_N consists of the disjoint union of three large cycles:

$$\begin{aligned} E_1 &= (v_1, \ldots, v_{12N}, v_1); \\ E_2 &= (u_1, \ldots, u_{12N+6}, u_1); \\ E_3 &= (v_1, u_2, w_1, u_3, v_3, \ldots, u_{12N+6}, v_1). \end{aligned}$$

It is the large face in the centre of each of these graphs that creates the bottleneck in the Markov chain we will use to show torpid mixing. We label this face C and its neighbours η_i (for $1 \le i \le 6N$). The face that is adjacent to both η_i and η_{i+1} is labelled σ_i, and the face that is only adjacent to η_i is labelled ρ_i. Note that there are $6N$ σ_i faces and $6N$ ρ_i faces.

The reason C causes torpid mixing is that we are able to partition the state space of the chain into two parts S and $\bar{S}$, such that the only transitions crossing from S into $\bar{S}$ are transitions reversing C. Because C has so many edges (linear in the number of vertices) we are able to show that the set of orientations in which C is directed is exponentially small. Hence, by Theorem 1.51, the face-reversal chain will be torpidly mixing.

Theorem 2.27. *The face-reversal chain $\mathcal{M}_{\mathcal{F}}$ is torpidly mixing on* $\mathrm{EO}(H_N)$ *for $N \ge 3$.*

Proof. From Theorem 1.51 we know that $\mathcal{M}_{\mathcal{F}}$ is torpidly mixing on a set of Eulerian orientations Ω if there exists some $S \subset \Omega$, with $0 < |S| \le |\Omega|/2$, such that $|\partial S|/|S|$ is exponentially small in f.

Recall the definition of α-potentials and the Felsner lattice from §2.3; in particular, recall that for any $\gamma \in \mathcal{F}(G)$ the value $\wp_{\mathcal{E}}(\gamma)$ is equal to the number of times γ is reversed on any path from $\mathcal{E}_{\min}$, the unique minimum element of the Felsner lattice,

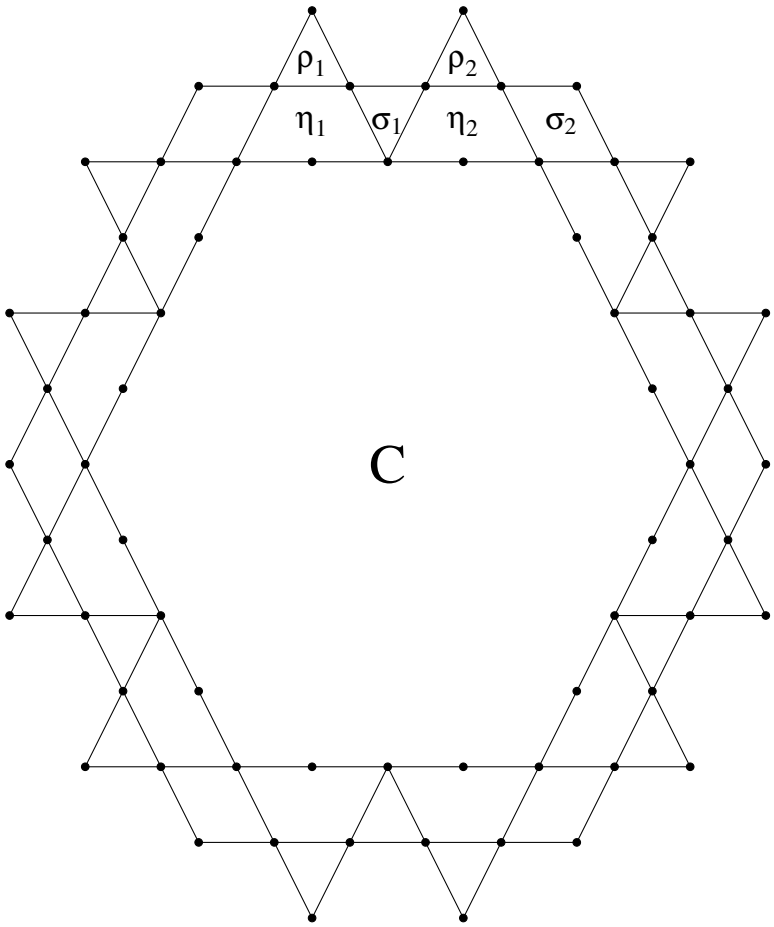

Figure 2.7: The graph H_2

to $\mathcal{E}$. Let $\wp_{\max} = \wp_{\mathcal{E}_{\max}}$, where $\mathcal{E}_{\max}$ is the unique maximum element of the Felsner lattice.

Let S be the set of all Eulerian orientations $\mathcal{E}$ on H_N satisfying $\wp_{\mathcal{E}}(C) \leq 1$. From the conditions (2.4), (2.5), and (2.6), we can deduce that the value of $\wp_{\max}(C)$ is 3. Hence, we can define a bijection between S and $\Omega \backslash S$ by mapping $\wp_{\mathcal{E}}$ to $\wp_{\max} - \wp_{\mathcal{E}}$ for each $\mathcal{E} \in S$ (this corresponds to mapping $\mathcal{E}$ to the orientation with all edges reversed), so $|S| = |\Omega|/2$.

An Eulerian orientation $\mathcal{E}$ is an element of ∂S if and only if C is a counter-clockwise directed cycle in $\mathcal{E}$ and $\wp_{\mathcal{E}}(C) = 1$. For this to occur we must have $\wp_{\mathcal{E}}(\eta_i) = 1$ for each η_i. Hence, the number of Eulerian orientations satisfying this condition is exactly 2^{2k} since each of the σ_i and ρ_i can take potential value 0 or 1, where $k = 6N$.

We note that C is the only directed cycle in $\mathcal{E}_{\min}$ so $|S| = |S'| + 1$, where

$$S' = \{\mathcal{E} \in \Omega : \wp_{\mathcal{E}}(C) = 1\}.$$

We can partition S' as $\bigcup_{I\subset[k]} S_I$, where

$$S_I = \{\mathcal{E}\in\Omega : \wp_{\mathcal{E}}(C)=1 \wedge \wp_{\mathcal{E}}(\eta_i)=1 \Leftrightarrow i\in I\}.$$

We can find the size of each of the S_I by counting the number of potential functions which correspond to members of S_I. If $\mathcal{E}\in S_I$ then there are two possible values for $\wp_{\mathcal{E}}(\sigma_i)$ for each i with $\wp_{\mathcal{E}}(\eta_i)=1$ and $\wp_{\mathcal{E}}(\eta_{i+1})=1$, and two possible values for $\wp_{\mathcal{E}}(\rho_i)$ for each i with $\wp_{\mathcal{E}}(\eta_i)=1$. All of the other σ_i and ρ_i must have potential value 0. Hence,

$$|S_I| = 2^{|I|+c(I)},$$

where $c(I)$ counts the number of circular successions in I. The number of j-subsets of $\{1,\ldots,k\}$ containing m circular successions is given by the following expression[2]:

$$c(k,j,m) = \begin{cases} 0 & \text{if } j=0, j>k, \text{ or } m<2j-k \\ \frac{k}{j}\binom{j}{m}\binom{k-j-1}{j-m-1} & \text{otherwise} \end{cases}. \tag{2.9}$$

Then,

$$\begin{aligned} |S| &= 1+\sum_{j=0}^{k} 2^j \sum_{m=0}^{j} 2^m c(k,j,m) && (2.10)\\ &= \sum_{j=1}^{k-1} \sum_{m=\max(0,2j-k)}^{j-1} \frac{k}{j}\binom{j}{m}\binom{k-j-1}{j-m-1} 2^{j+m} + 1 + 2^{2k} && (2.11)\\ &> \sum_{j=1}^{k-1} \binom{j}{2j-k} 2^{3j-k} && (2.12)\\ &> \binom{\lfloor \frac{16}{17}k\rfloor}{\lceil \frac{1}{17}k\rceil} 2^{\frac{31}{17}k-3} \quad \text{if } k\geq 17 && (2.13)\\ &\geq 2^{(2+\frac{1}{17})k-3} && (2.14) \end{aligned}$$

The last line of this follows from the fact that

$$\binom{\lfloor \frac{16}{17}k\rfloor}{\lceil \frac{1}{17}k\rceil} \geq 2^{\frac{4k}{17}}$$

when $k\geq 17$. Hence,

$$|\partial S|/|S| < 8\cdot 2^{-\frac{1}{17}k} \in O(2^{-\frac{1}{51}f}).$$ □

Remark 2.28. Observe that we can obtain H_N as a subgraph of the infinite triangular lattice (see the embedding of H_2 in Figure 2.7, for an example). Hence, the requirement that the input is a solid subgraph is necessary for the rapid mixing result of §2.6 to hold.

[2]The expression (2.9) can be obtained by standard generating function calculations, see e.g., [40, §2.3.22]

Chapter 3

Euler tours of Random Graphs

3.1 Introduction

For situations where we do not have a rapidly mixing Markov chain (or some other *fpaus*) for every instance of a sampling problem, we relax the problem and search for a sampling algorithm that works for almost all inputs. Formally, we want to show, for some particular sampling problem on an alphabet Σ, that there exists an algorithm $\mathcal{A}$ such that for x chosen uniformly at random from $\Sigma^\star$, the probability that $\mathcal{A}$ behaves like an a *fpaus* for $\Omega(x)$ tends to 1 as $|x| \to \infty$.

When we first started looking at Euler tours, our goal was to find a rapidly mixing Markov chain on the set of Euler tours of an undirected Eulerian graph. This task proved too difficult, even restricted to special classes of graphs such as planar graphs or 4-regular graphs. Inspired by results showing that, with high probability, one could sample or approximately count Hamiltonian cycles of a random d-regular graph in polynomial time [36], we turned our attention in this direction. Still, the problem proved too difficult. In this chapter we describe results that we believe to be milestones on the road to showing we can, with high probability, sample or approximately count the Euler tours of a random $2d$-regular graph, using very simple algorithms.

When studying the behaviour of algorithms on random inputs, it is typical for the algorithm itself to be very simple; the hard work goes into the analysis. A natural and simple algorithm for generating a random Euler tour is given in Algorithm 4. In Algorithm 4, we use $E(v)$ to denote the set of edges incident with v and $\mathcal{T}$ for the partial tour, or set of edges which have already been used. We use $\mathcal{T} \cdot e$ to represent the concatenation of e with the sequence of edges $\mathcal{T}$. Recall the definition of a transition system of an Eulerian graph G from §1.3.2 (Definition 1.67), and that we use $\mathcal{C}(T)$ to

Algorithm 4: Generate

Input: An Eulerian graph or directed graph G

Output: A uniformly random Euler tour $\mathcal{T}$ of G

repeat
  $u \in V(G)$;
  $e = (u,v) \in E(u)$;
  $\mathcal{T} = (e)$;
  while $E(v) \setminus \mathcal{T} \neq \emptyset$ **do**
    $e = (v,w) \in_{u.a.r.} E(v) \setminus \mathcal{T}$;
    $v \leftarrow w$;
    $\mathcal{T} \leftarrow \mathcal{T} \cdot e$;
  end
until $\mathcal{T} = E(G)$;
return $\mathcal{T}$;

denote the decomposition of G into a set of edge-disjoint cycles induced by a particular transition system T of G. One way of looking at Algorithm 4 is that it generates a random transition system of G by choosing one random edge pairing at a time. If this closes a cycle, i.e., forms a component in $\mathcal{C}(\mathcal{T})$, before it has visited every edge, then the algorithm restarts; eventually the algorithm will generate the transition system of an Euler tour $\mathcal{T}$. The algorithm starts at a particular vertex u and chooses an edge incident with it, setting the "current vertex" v to be the opposite end-point of this edge. Then, at each subsequent step we randomly choose an edge incident with v that has not been used before and pair it with the edge chosen at the previous step, updating the "current vertex" v to be the opposite endpoint of the chosen edge. Eventually, we will return to the initial vertex with no more edges available. At this point we have either constructed a uniformly random Euler tour, or generated a partial transition system that does not extend to a transition system of a tour; i.e., we have generated the transition system of an Euler tour of an Eulerian proper subgraph of G.

We will now show that Algorithm 4 generates Euler tours with equal probability. Suppose G is a graph with degree sequence $2\mathbf{d} = (2d_1, 2d_2, \ldots, 2d_n)$, let $\mathcal{T}$ be some Euler tour of G, and let u be the initial vertex in Algorithm 4. Since Euler tours are equivalent up to circular rotation, we can choose any edge incident with u to be the first edge of the tour. The probability that this edge is directed away from u, and so valid as the first edge of the tour, is $1/2$. The second time we reach u the probability

of choosing the edge-pairing prescribed by $\mathcal{T}$ is $\frac{1}{2d_u-2}$, then $\frac{1}{2d_u-4}$ for the third time, and so on. At all other vertices v, the probability of choosing the correct edge-pairing the first time we pass v is $\frac{1}{2d_v-1}$, then $\frac{1}{2d_v-3}$, and so on. Hence, the probability that this procedure constructs the particular Euler tour $\mathcal{T}$ is

$$\frac{1}{2^{2d_u+1}} d_u \binom{2d_u}{d_u} \left(\prod_{v=1}^{n} \frac{\binom{2d_v}{d_v} d_v!}{2^{d_v}} \right)^{-1} = \frac{d_u \binom{2d_u}{d_u}}{2^{2d_u+1}} \frac{1}{|\mathrm{TS}(G)|},$$

where $\mathrm{TS}(G)$ is the set of transition systems of G. Thus, the probability of Algorithm 4 terminating, i.e., generating an Euler tour, on any iteration is

$$\frac{d_u \binom{2d_u}{d_u}}{2^{2d_u+1}} \frac{|\mathrm{ET}(G)|}{|\mathrm{TS}(G)|} \tag{3.1}$$

We can also apply Algorithm 4 to sampling Euler tours of Eulerian directed graphs. Suppose G is an Eulerian directed graph on vertex set $V = \{1, 2, \ldots, n\}$ with out-degree sequence $\mathbf{d} = (d_1, d_2, \ldots, d_n)$. If we now take $E(v)$ to be the set of arcs directed away from v in G, for each $v \in V$, then Algorithm 4 returns a uniformly random Euler tour of G. When G is directed, the probabilities of choosing the correct arc at each pass of u are $\frac{1}{d_u-1}$, $\frac{1}{d_u-2}$, etc., and the probabilities of choosing the correct arc at each pass of $v \neq u$ are $\frac{1}{d_v}$, $\frac{1}{d_v-1}$, etc. Hence, the probability of Algorithm 4 terminating on any iteration is

$$d_u \frac{|\mathrm{ET}(G)|}{|\mathrm{TS}(G)|}. \tag{3.2}$$

We can use the same idea to construct an algorithm for approximating the number of Euler tours of any Eulerian graph or directed graph. Algorithm 5 iterates the process of Algorithm 4 a number of times and takes the proportion of these iterations which yield Euler tours as an estimation of the probability (3.1) or (3.2), depending on whether G is a graph or directed graph. Since we have an exact expression for the number of transition systems of a graph or directed graph, we can use this to obtain an approximation of the number of Euler tours of G.

The time taken to generate a random Euler tour, and the time required to get a good quality approximation of the number of Euler tours, is strongly dependent on the ratio of the number of transition systems to the number of Euler tours. In particular, if $|\mathrm{ET}(G)|/|\mathrm{TS}(G)|$ is bounded from below by an inverse polynomial in the number of vertices of G, then Algorithm 4 and Algorithm 5 will run in expected polynomial time, i.e., the algorithms will behave like an *fpaus* and an *fpras*, respectively. One consequence of the results in [66] is that the ratio $|\mathrm{ET}(G)|/|\mathrm{TS}(G)|$ is bounded below

Algorithm 5: Count

Input: An Eulerian graph or directed graph G, an integer N

Output: An approximation to (3.1) or (3.2)

```
count ← 0;
u ∈ V(G);
d ← deg(u);
for t ← 1 to N do
    e = (v,u) ∈ E(v);
    T ← (e);
    while E(v)\T ≠ ∅ do
        e = (v,w) ∈_{u.a.r.} E(v)\T;
        v ← w;
        T ← T·e;
    end
    if T = E(G) then
        count ← count + 1;
    end
end
return count/N;
```

by an inverse polynomial when G is the complete graph on K_{2n+1} or a tournament on $2n+1$ vertices (an Eulerian orientation of K_{2n+1}). However, there exist graphs for which this will not be the case. For example, consider the graph from Example 1.70. This graph has 3^n transition systems, but only $(n+1)2^n$ Euler tours, so both algorithms will require exponential time. That example, however, is of a particularly contrived multigraph, so we might hope the algorithms will perform better on a typical, i.e., random, graph. That is to say, we are looking for a result of the following form:

Let $\mathcal{C}_1, \mathcal{C}_2, \ldots$, be a sequence of sets of Eulerian graphs. There exists some constant $\alpha > 0$ such that the probability that $|\mathrm{ET}(G)|/|\mathrm{TS}(G)| \geq n^{\alpha}$ for G chosen randomly from $\mathcal{C}_n$ tends to 1 as $n \to \infty$

In this chapter we investigate the above statement for the cases when $\mathcal{C}_n$ is taken to be equal to the class of d-in/d-out graphs or the class of $2d$-regular graphs. In the case of d-in/d-out graphs, we are able to obtain a complete proof.

Definition 3.1. Let d be some fixed positive integer and suppose we have a vector

$\mathbf{d} = (d_1, d_2, \cdots, d_n)$ such that

$$d \geq d_n \geq d_{n-1} \geq \cdots \geq d_1 \geq 1\,,$$

and let $m = \sum_v d_v$. We use $\vec{\mathbb{G}}(n, \mathbf{d})$ to denote the set of Eulerian directed graphs with out-degree sequence $\mathbf{d}$, and $\vec{\mathbb{G}}^\star(n, \mathbf{d})$ to denote the set of Eulerian directed multigraphs with out-degree sequence $\mathbf{d}$. We use $\vec{\mathbb{G}}(n, d)$ and $\vec{\mathbb{G}}^\star(n, d)$ to denote the set of d-in/d-out graphs and multigraphs, respectively.

Theorem 3.2. *Let d be some fixed positive integer greater than 1, let $n \in \mathbb{N}$. If $\vec{G}$ is chosen uniformly at random from $\vec{\mathbb{G}}(n, d)$ we have*

$$\mathbb{P}\left[\frac{|\mathrm{ET}(G)|}{|\mathrm{TS}(G)|} \in \Omega(n^{-2})\right] \to 1 \text{ as } n \to \infty\,.$$

In the case of $2d$-regular graphs we are only able to make a conjecture but support this conjecture with a mix of empirical and theoretical evidence.

Definition 3.3. Let d be some fixed positive integer and suppose we have a vector $\mathbf{d} = (d_1, d_2, \cdots, d_n)$ such that

$$d \geq d_n \geq d_{n-1} \geq \cdots \geq d_2 \geq d_1 \geq 1\,,$$

and let $2m = \sum_{v=1}^{n} d_v$. We use $\mathbb{G}(n, \mathbf{d})$ to denote the set of all undirected graphs with degree sequence $\mathbf{d}$ and $\mathbb{G}^\star(n, \mathbf{d})$ to denote the set of multigraphs with degree sequence $\mathbf{d}$. We use $\mathbb{G}(n, d)$ and $\mathbb{G}^\star(n, d)$ to denote the sets of d-regular graphs and d-regular multigraphs with n vertices, respectively.

Conjecture 3.4. *Let d be some fixed positive integer greater than 1, let $n \in \mathbb{N}$. If G is chosen uniformly at random from $\mathbb{G}(n, 2d)$ we have*

$$\mathbb{P}\left[\frac{|\mathrm{ET}(G)|}{|\mathrm{TS}(G)|} \in \Omega(n^{-1})\right] \to 1 \text{ as } n \to \infty\,.$$

The contents of this chapter are as follows. In §3.2, we describe the models we use to study random graphs and the techniques used in our analysis. Recall that, by the BEST Theorem (Theorem 1.57), counting arborescences is equivalent to counting Euler tours for any Eulerian directed graph. In §3.3 we analyse the distribution of the number of arborescences of a random d-in/d-out graph, and use this to infer results about the number of Euler tours of a random d-in/d-out graph. In §3.4, we analyse the distribution of the number of Eulerian orientations of a random $2d$-regular graph. Each one of these orientations is a d-in/d-out graph, so, recalling the definition of an

orb of an Eulerian graph G (Definition 1.69) as a pair $(\mathcal{E},\mathcal{A})$, where $\mathcal{E}$ is an Eulerian orientation of G and $\mathcal{A}$ is an arborescence of $\mathcal{E}$, we would hope that the results of §3.3 and S3.4 could be combined to say something about the number of orbs of a random $2d$-regular graph. As observed by Brightwell and Winkler [15], the BEST theorem can be used to relate the number of orbs of an Eulerian graph to the number of Euler tours, so this would be equivalent to analysing the number of Euler tours of a random $2d$-regular graph. Unfortunately, we were unable to bring this line of research to completion. In §3.5 we provide some initial results, and state a conjecture which, if true, will show that we can, with high probability, sample and approximately count the Euler tours of a random $2d$-regular graph in expected polynomial time. Furthermore, we present some empirical evidence to support our conjecture. Finally, in §3.6, we give some examples of graphs on which Algorithm 4 and Algorithm 5 will not be effective.

3.2 Random graphs

In this chapter, we are interested in the random variables representing the number of Euler tours or Eulerian orientations of G when G is chosen uniformly at random from $\mathbb{G}(n,\mathbf{d})$ or $\vec{\mathbb{G}}(n,\mathbf{d})$. In order to be able to study properties of random elements of $\mathbb{G}(n,\mathbf{d})$ or $\vec{\mathbb{G}}(n,\mathbf{d})$, we need a model for generating random elements of both sets. In the next section we describe the *configuration model* [9, 11], which we will use to generate random graphs with a fixed degree sequence.

3.2.1 The configuration model for random graphs

The model we use to generate random graphs is the *configuration model* of Bollobás [11] (implicit in the earlier work of Bender and Canfield [9]). See the survey of Wormald [100] or the textbooks of Bollobás [12] and Janson, Luczak, and Rucinski [44] for more details of the contents of this section.

Definition 3.5. Suppose we have $\mathbf{d}$ as in Definition 3.3. For every $v = 1,2,\ldots,n$ we define a set W_v containing d_v points. Each one of these represents an endpoint of an edge incident with v. We define the set W to be formed as the union of these n disjoint sets. We call a perfect matching on W a *configuration* and we call a matching on W a *partial configuration*. Let $\Omega_{n,\mathbf{d}}$ denote the sets of *configurations* for particular $\mathbf{d}$.

For each $\mathrm{F} \in \Omega_{n,\mathbf{d}}$, we define the *projection* of F, $\sigma(\mathrm{F})$, to be the multigraph in which each set W_v is contracted to a single vertex: a pair $\{x,y\} \in \mathrm{F}$ with $x \in W_u$ and

$y \in W_v$ corresponds to an edge $\{u,v\}$ in $\sigma(\mathrm{F})$. For each partial configuration F', we define the *projection* $\sigma(\mathrm{F}')$ to be the graph obtained by applying the same operation to the pairs in F'; that is, $\sigma(\mathrm{F}')$ is the subgraph of $\sigma(\mathrm{F})$ corresponding to the pairs in F', for any $\mathrm{F} \supseteq \mathrm{F}'$.

We say that F contains a *loop* if it contains a pair $\{x,y\} \in \mathrm{F}$ with $x,y \in W_v$ for some v, and that F contains a *double edge* if there are pairs $\{x,y\}$ and $\{x',y'\}$ in F with $x,x' \in W_u$ and $y,y' \in W_v$ for some $u \neq v$. If F does not contain any loops or double edges then $\sigma(\mathrm{F})$ is a *simple graph*, i.e., $\sigma(\mathrm{F}) \in \mathbb{G}(n,\mathbf{d})$. Furthermore each graph $G \in \mathbb{G}(n,\mathbf{d})$ is the projection of exactly $\prod_{v=1}^{n} d_v!$ different configurations; (non-simple) multigraphs are obtained as the projection of fewer configurations, with this number depending on the number of loops and double edges.

Hence, we can generate a uniformly random graph $G \in \mathbb{G}(n,\mathbf{d})$ by generating uniformly random configurations $\mathrm{F} \in \Omega_{n,\mathbf{d}}$ until we obtain an F with no loops or double edges, and then returning $G = \sigma(\mathrm{F})$. The probability of a configuration not containing any loops or double edges is, asymptotically,

$$\exp(-\lambda/2 - \lambda^2/4), \tag{3.3}$$

where $\lambda = \frac{1}{m}\sum_{1\leq v\leq n}\binom{d_v}{2}$ [12, Theorem 2.16]. Since each d_v is bounded by the constant d, it follows that this probability will always be a small constant. However, as we increase d this probability gets smaller and smaller, so this approach will not be so useful for practical generation of random graphs. For example, if $d_u = d_v = 10$ for all u and v, then (3.3) is already $e^{-99/4} < 10^{-10}$.

Our main interest in the model is as a tool for analysing properties of random graphs, and for this task the inefficiency of the random graph generation algorithm is irrelevant. In §3.5, we describe the result of some computational experiments. For this task we used one of the more practical algorithms for generating random regular graphs, due to Steger and Wormald [83]. This algorithm generates random graphs with a probability that is close to the uniform distribution, sacrificing accuracy for efficiency.

Note that the distribution on $\mathbb{G}^\star(n,\mathbf{d})$ that results from taking the projection of uniformly random $\mathrm{F} \in \Omega_{n,\mathbf{d}}$ is not uniform. This is because the probability of obtaining a particular element of $\mathbb{G}^\star(n,\mathbf{d})$ depends on the number of loops and double edges.

We use the *directed configuration model* for generating random Eulerian directed graphs with fixed out-degree sequence.

Definition 3.6. Suppose we have $\mathbf{d}$ as in Definition 3.1. For every $v = 1, 2, \ldots, n$ we define a pair of sets S_v and T_v, each containing d_v points. The points in S_v represent the endpoints of arcs for which v is the source, and the points in T_v represent the endpoints of arcs for which v is the target. We call a perfect matching from S to T a *directed configuration*, and we call a matching a *partial directed configuration*. We use $\Phi_{n,\mathbf{d}}$ to denote the set of all *directed configurations*, for particular $\mathbf{d}$.

For each $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$, we define the *projection* of $\vec{\mathrm{F}}$, $\sigma(\vec{\mathrm{F}})$, to be the directed multigraph in which the sets $S_v \cup T_v$ have been contracted to a single vertex. That is, we have an arc from u to v in $\sigma(\vec{\mathrm{F}})$ whenever we have a pair (x, y) from $S_u \times T_v$ in $\vec{\mathrm{F}}$. For each partial configuration $\vec{\mathrm{F}}'$, we define the *projection* $\sigma(\vec{\mathrm{F}}')$ to be graph obtained by applying the same operation to the pairs in $\vec{\mathrm{F}}$; that is, $\sigma(\vec{\mathrm{F}}')$ is the subgraph of $\sigma(\vec{\mathrm{F}})$ corresponding to the pairs in $\vec{\mathrm{F}}'$, for any $\vec{\mathrm{F}} \supseteq \vec{\mathrm{F}}'$.

We say a directed configuration $\vec{\mathrm{F}}$ has a *loop* if it contains a pair $(x, y) \in S_v \times T_v$ and a *double arc* if it contains pairs (x, y) and (x', y') from $S_u \times T_v$ for $u \neq v$. Note that we do not consider $(x, y) \in S_u \times T_v$ and $(x', y') \in S_v \times T_u$ to be a double arc. As in the undirected case, each simple directed graph is the projection of the same number of configurations, $\prod_v (d_v!)^2$, and the probability that a uniformly random $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ projects to a simple directed graph is, asymptotically,

$$\exp\left(-\frac{m_2}{m} - \frac{(m_2 - m)^2}{2m^2}\right), \tag{3.4}$$

where $m_2 = \sum_v d_v^2$.

Remark 3.7. The probability (3.4) follows as a result of arguments given in Theorem 3.30. We believe this is a new result, though the calculations are similar to those used to prove (3.3). The directed configuration model has been studied in the literature [4], also in the context of Euler tours, but only for the case $d_v = 2$ for all v.

3.2.2 Asymptotic Distributions

Any numerical characteristic of a graph G, e.g., the number of Euler tours, gives rise to a random variable when we consider G as being drawn at random from some class of graphs. In this work, we would like to know when the probability of Algorithm 4 returning an Euler tour is bounded below by an inverse polynomial. For example, we want to show there exists some $\alpha > 0$ such that

$$\mathbb{P}\left[\frac{|\mathrm{ET}(G)|}{|\mathrm{TS}(G)|} \geq n^{-\alpha} : G \in \mathbb{G}(n, 4)\right] \to 1, \tag{3.5}$$

as $n \to \infty$. That is, we want to show that the probability Algorithm 4 terminates on any particular iteration is at least $n^{-\alpha}$ for almost every $G \in \mathbb{G}(n,4)$. Let $\mathcal{T}_{n,4}$ denote the random variable counting the number of Euler tours of a random 4-regular graph. In Section 3.5.1, we show that the expected number of Euler tours of a 4-regular graph satisfies

$$\mathbb{E}[\mathcal{T}_{n,4}] \to e^{3/4}\sqrt{\frac{\pi}{2n}}3^n.$$

There are 3 possible transition systems at each vertex of a 4-regular graph, so $|TS(G)| = 3^n$ for each $G \in \mathbb{G}(n,4)$. Thus, to prove (3.5) all we need to do is show $\mathcal{T}_{n,4}$ is concentrated around its mean. In many cases, Chebyshev's inequality (see below) is sufficient to prove concentration results and it has seen a great deal of application to the analysis of randomised algorithms, see, e.g., [70].

Theorem 3.8 (Chebyshev's Inequality)**.** *Let X be a random variable. For any $a > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq a] \leq \frac{\mathrm{Var}[X]}{a^2}.$$

However, in some cases Chebyshev's inequality is not strong enough and we need to look harder at the distribution of the random variable of interest.

Exact formulas for the distributions of interesting combinatorial quantities are rare and even when they exist are usually too complicated to be of use. Hence, the focus is shifted to the search for an *asymptotic distribution* that holds with increasing accuracy.

Definition 3.9. Let $X_1, X_2, \ldots$ be a sequence of integer-valued random variables. We say $(X_n)_{n\geq 0}$ *converges in distribution* to a random variable Z, or Z is the *asymptotic distribution* of X_n, if

$$\mathbb{P}[X_n = x] \to \mathbb{P}[Z = x], \quad \text{as } n \to \infty.$$

We use $X_n \xrightarrow{d} Z$ to denote convergence in distribution.

If $X_n \xrightarrow{d} Z$ then, for large values of n, we can consider X_n to have the same distribution as Z. A particularly important instance of this that arises in the study of random graphs with fixed degree sequence is *convergence to a Poisson distribution*.

Definition 3.10. A non-negative integer-valued random variable X is said to have a *Poisson distribution with mean* λ if

$$\mathbb{P}[X = x] = e^{-\lambda}\frac{\lambda^x}{x!}, \quad x \in \mathbb{N}.$$

Definition 3.11. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be arbitrary non-negative real constants and let $X_{1,n}$, $X_{2,n}, \ldots, X_{k,n}$ be a set of non-negative integer random variables defined on the same probability space Ω_n, for each n. We say these random variables are *asymptotically independent Poisson random variables with means* λ_i, if their joint distribution converges to that of independent Poisson variables with means λ_i. That is, for every fixed set of non-negative integers $\{x_i : 1 \leq i \leq k\}$ we have

$$\mathbb{P}[X_{i,n} = x_i, \forall 1 \leq i \leq k] \to \prod_{i=1}^{k} e^{-\lambda_i} \frac{\lambda_i^{x_i}}{x_i!},$$

as $n \to \infty$.

It is well-known that a Poisson random variable is completely determined by the values of its *factorial moments*.

Definition 3.12. For any non-negative integers n and k, we define the *factorial power* (falling factorial) $(n)_k$ by

$$(n)(n-1)\cdots(n-k+1).$$

If $k > n$ we have $(n)_k = 0$.

Definition 3.13. Let X be a random variable. The *k-th factorial moment* of X is given by $\mathbb{E}[(X)_k]$.

The following lemma shows how the asymptotic values of the factorial moments can be used to show that a set of sequences of random variables converge to independent Poisson random variables.

Theorem 3.14 (Theorem 1.23 [12])**.** *Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be arbitrary non-negative real constants and let $X_{1,n}$, $X_{2,n}, \ldots, X_{k,n}$ be a set of non-negative integer random variables defined on the same probability space Ω_n, for each n. If*

$$\mathbb{E}\left[(X_{1,n})_{j_1}(X_{2,n})_{j_2}\cdots(X_{k,n})_{j_k}\right] \to \prod_{i=1}^{k} \lambda_i^{j_i}.$$

holds for each fixed set of non-negative integers $j_1, j_2, \ldots, j_k$ then the variables $X_{1,n}, \ldots, X_{k,n}$ are asymptotically independent Poisson random variables with means λ_i.

Often, when we apply Theorem 3.14 the random variables $X_{i,n}$ will be counting the number of occurrences of some sort of substructure. For example, suppose $X_{i,n}(G)$ counts the number of i-cycles in a random graph on n vertices and let $\mathcal{C}_{in}$ denote the set

of different i-cycles you can form on n vertices. Then, we can write $X_{i,n} = \sum_{C \in C_{in}} I_C$, where I_C is the indicator variable for the event $C \in G$. That is,

$$I_C = \begin{cases} 1 & C \in CYCLES(G) \\ 0 & C \notin CYCLES(G) \end{cases}.$$

In this case, computing the factorial power $(X_{i,n})_k$ amounts to summing over products of indicator functions for ordered tuples of distinct i-cycles, e.g.,

$$(X_{i,n})_3 = \sum_{\substack{(C_1,C_2,C_3) \in C_{in}^3 \\ C_1 \neq C_2 \neq C_3}} I_{C_1} I_{C_2} I_{C_3}\,.$$

Theorem 3.14 can be used to prove the following theorem, due to Bollobás [11] (or [12, Theorem 2.16]).

Theorem 3.15 (Bollobás [11])**.** *Let $\lambda = \frac{1}{m}\sum_v \binom{d_v}{2}$ and for each integer i let $\lambda_i = \lambda^i/2i$ and let $X_{i,n}$ count the number of i-cycles in a multigraph G obtained as the projection of a uniformly random $\mathrm{F} \in \Omega_{n,\mathbf{d}}$. For any fixed positive integer k, the set of variables $X_{i,n}$ for $i < k$ are asymptotically Poisson independent random variables with means λ_i.*

The asymptotic independence of the random variables $X_{i,n}$ follows from a well-known property of random regular graphs [44, 100].

Lemma 3.16. *Let d be some fixed, non-negative integer, and let H be some graph with more edges than vertices. Then, for G chosen uniformly at random from $\mathbb{G}(n,d)$ we have*

$$\mathbb{P}[H \text{ is a subgraph of } G] \to 0\,,$$

as $n \to \infty$.

Recall that a simple graph is a multigraph with no loops or double edges. Hence, when F is chosen uniformly at random from $\Omega_{n,\mathbf{d}}$, we have

$$\mathbb{P}[\sigma(F) \text{ is simple }] = \mathbb{P}[X_{1,n} = X_{2,n} = 0]\,,$$

and thus we can infer (3.3) from Theorem 3.15 and Definition 3.11.

3.2.3 Conditioning on short cycle counts

There are many combinatorial quantities which do not have asymptotic Poisson distributions, e.g., the number of Hamiltonian cycles, the number of perfect matchings,

or the number of Euler tours of a random $2d$-regular graph. However, it is still possible to compute asymptotic distributions for many of these quantities and Theorem 3.15 is the foundation of many of these results.

We will now discuss one of these results in more detail. As this result is on random regular graphs, where $d_v = d$ for all v, for the rest of this section we will speak of the set of d-regular graphs, $\mathbb{G}(n,d)$, and the regular configuration model, $\Omega_{n,d}$.

Definition 3.17. Let $G = (V,E)$ be a graph. We say a cycle in G is a *Hamiltonian cycle* if it visits each vertex $v \in V$ exactly once.

Definition 3.18. Let $G = (V,E)$ be a graph. We say a subgraph H of G is a *2-factor* if each vertex $v \in V$ has degree 2 in H; that is, H is a set of *vertex-disjoint cycles* which contains every vertex of G.

In 1988, Frieze [35] showed that one could find a Hamiltonian cycle in almost every d-regular graph, for $d \geq 85$, though it was widely believed that this held for all $d \geq 3$. Then, Robinson and Wormald [75] showed that, for $d = 3$, the number of Hamiltonian cycles of a random 3-regular graph, H_n, satisfied

$$\frac{\mathbb{E}[H_n^2]}{\mathbb{E}[H_n]^2} \to \frac{3}{e}\,. \tag{3.6}$$

Applying Chebyshev's inequality with $a = \mathbb{E}[H_n]$, we find

$$\mathbb{P}[H_n = 0] \leq \mathbb{P}[|H_n - \mathbb{E}[H_n]| \geq \mathbb{E}[H_n]] \leq \left(\frac{3}{e} - 1\right).$$

Hence, from (3.6) we can deduce that at least a $2 - 3e^{-1}$ fraction of 3-regular graphs are Hamiltonian. However, proving the result people expected,

$$\mathbb{P}[G \text{ is Hamiltonian} : G \in \mathbb{G}(n,d)] \to 1\,,$$

for any $d \geq 3$, was beyond the powers of Chebyshev's inequality.

Then, in a pair of papers from the early 90's [76, 77] Robinson and Wormald managed to prove that for all $d \geq 3$, a random d-regular graph will contain a Hamiltonian cycle with high probability, using a technique we call *conditioning on short cycle counts*. Their proof involved partitioning $\Omega_{n,d}$ into a family of sets indexed by vectors of the numbers of occurrences of cycles of various lengths, namely

$$\Omega_{\mathbf{x}} = \{\mathrm{F} \in \Omega_{n,d} : X_{i,n}(\mathrm{F}) = x_i \text{ for all } 1 \leq i \leq k\}\,.$$

Robinson and Wormald approached this result indirectly, by analysing the number of perfect matchings in a random d-regular graph. Subsequent work of Frieze et al. [36]

obtained the same result using the same technique, but this time working directly with the number of Hamiltonian cycles. Specifically, Frieze et al. [36] showed that the variance of the number of Hamiltonian cycles can be divided into two parts: the variance between the groups $\Omega_{\mathbf{x}}$, $\mathbf{x} \in \mathbb{N}^k$; and the variance within each of the different groups. The calculations in [76, 77, 36] showed that almost of all the variance comes from the variance between the groups, and so the variance within any particular group is negligible. Thus, with high probability the number of Hamilton cycles in a graph G chosen randomly from $\Omega_{\mathbf{x}}$ will be close to the expected number of Hamilton cycles for graphs in that group,

$$\mathbb{E}_{\mathbf{x}} = \mathbb{E}[H_n(F) : F \in \Sigma_{\mathbf{x}}],$$

Furthermore, with high probability the group F lies in will have $\mathbb{E}_{\mathbf{x}}$ large enough to imply that the number of Hamilton cycles in $\sigma(F)$ is > 0.

In fact, Frieze et al. [36] proved something stronger. The main contribution of their paper was to adapt the argument of Robinson and Wormald to show that, with high probability, the number of Hamiltonian cycles in a random d-regular graph was at most a linear factor less than the number of 2-factors. Since there exists an *fpaus* for the 2-factors of any graph, using the Jerrum-Sinclair chain [49], it follows that we can sample (and so approximately count) the Hamiltonian cycles of almost every d-regular graph in polynomial time. Their proof is comprised of the following two steps:

1. They show that the expected number of Hamiltonian cycles in a random d-regular graph G is within a linear factor of the number of 2-factors of G;

2. They show that the random variable counting the Hamiltonian cycles is concentrated around its mean.

A general version of this method was presented by Molloy et al. [71].

Janson [43] showed that the result of [76, 77, 36] really amounts to obtaining an asymptotic distribution for the number of Hamiltonian cycles. He streamlined the general approach of Molloy et al. [71] to obtain the following general theorem.

Theorem 3.19 (Janson [43])**.** *Let $\lambda_i > 0$ and $\delta_i \geq -1$, $i = 1, 2, \ldots$, be constants and suppose that for each n there are random variables $X_{i,n}$, $i = 1, 2, \ldots$, and Y_n (defined on the same probability space Ω_n) such that $X_{i,n}$ is non-negative integer valued and $\mathbb{E}[Y_n] \neq 0$ (at least for large n) and furthermore the following conditions are satisfied*

1. *$X_{i,n} \xrightarrow{d} Z_i$ as $n \to \infty$, jointly for all i, where Z_i is a Poisson random variable with mean λ_i;*

2. *For any finite sequence $x_1, \dots x_k$ of non-negative integers*

$$\frac{\mathbb{E}[Y_n | X_{1,n} = x_1, \dots X_{k,n} = x_k]}{\mathbb{E}[Y_n]} \to \prod_{i=1}^{k} (1+\delta_i)^{x_i} e^{-\lambda_i \delta_i} \quad as\, n \to \infty;$$

3. $\sum_i \lambda_i \delta_i^2 < \infty$;

4. $\lim_{n\to\infty} \frac{\mathbb{E}[Y_n^2]}{\mathbb{E}[Y_n]^2} \leq \exp(\sum_i \lambda_i \delta_i^2) + o(1)$.

Then

$$\frac{Y_n}{\mathbb{E}[Y_n]} \xrightarrow{d} W = \prod_{i=1}^{\infty} (1+\delta_i)^{Z_i} e^{-\lambda_i \delta_i},$$

as $n \to \infty$. Moreover, this and the convergence of the $X_{i,n}$ to the Z_i in 1 hold jointly.

Remark 3.20. In the proof of Theorem 3.19 given in [43], it is shown that Conditions 1 and 2 imply an asymptotic lower bound of $\exp(\sum_i \lambda_i \delta_i^2)$ on $\mathbb{E}[Y_n^2]/\mathbb{E}[Y_n]^2$. Hence, whenever the conditions of Theorem 3.19 hold, we actually have

$$\frac{\mathbb{E}[Y_n^2]}{\mathbb{E}[Y_n]^2} \to \exp(\sum_i \lambda_i \delta_i^2),$$

as $n \to \infty$.

Remark 3.21. The convergence of $X_{i,n}$ to Z_i and $Y_n/\mathbb{E}[Y_n]$ to W implied by Theorem 3.19 holds jointly. Hence, if the conditions of Theorem 3.19 hold, and Y_n' has the distribution of Y_n conditioned on $X_{1,n} = X_{2,n} = 0$, then we can conclude that

$$\frac{\mathbb{E}[Y_n']}{\mathbb{E}[Y_n]} \to \exp(-\lambda_1\delta_1 - \lambda_2\delta_2),$$
$$\frac{\mathbb{E}[(Y_n')^2]}{\mathbb{E}[Y_n']^2} \to \exp(-\lambda_1\delta_1^2 - \lambda_2\delta_2^2)\frac{\mathbb{E}[Y_n^2]}{\mathbb{E}[Y_n]^2}.$$

We will use the properties mentioned in Remark 3.20 and Remark 3.21 to simplify the proofs of §3.4. The following lemma gives a useful reformulation of Condition 2 of Theorem 3.19 that will be used several times in the proofs of this chapter.

Lemma 3.22 (Janson [43])**.** *Suppose Condition 1 of Theorem 3.19 holds, $Y_n \geq 0$, and that*

$$\frac{\mathbb{E}[Y_n (X_{1,n})_{j_1} (X_{2,n})_{j_2} \cdots (X_{k,n})_{j_k}]}{\mathbb{E}[Y_n]} \to \prod_{i=1}^{k} \mu_i^{j_i},$$

for some $\mu_i \geq 0$ and every finite sequence of non-negative integers $j_1, j_2, \dots, j_k$. Then, Condition 2 of Theorem 3.19 holds with $\delta_i = \mu_i/\lambda_i - 1$.

As in the work of Robinson and Wormald [76, 77] and Frieze et al. [36], the proof of Theorem 3.19 is basically an *analysis of variance*. The state space Ω_n is partitioned by conditioning on fixed values for the *auxiliary variables* $X_{i,n}$. The first three conditions of the theorem imply a lower bound on the variance between the partitions; the fourth condition gives a matching upper bound on the variance in the unconditioned space. Since this upper bound matches the lower bound, we can conclude that the variance within each component is negligible. Hence, the values of Y_n are almost surely determined by the values of the auxiliary variables.

For example, in the problem studied by Frieze et al. [36], we have Y_n equal to the number of Hamiltonian cycles in a random $G \in \mathbb{G}(n,d)$, for some fixed $d > 2$, and $X_{i,n}$ equal to the number of i-cycles in a random $G \in \mathbb{G}(n,d)$. It is shown in [36] that, with high probability, the value of W is sufficiently large so that for all n sufficiently large we have $Y_n > 0$.

In applications of Theorem 3.19 to random regular graphs, $X_{i,n}$ tends to be the number of i-cycles in a random configuration $\mathrm{F} \in \Omega_{n,d}$, Z_i a Poisson random variable with mean $\lambda_i = (d-1)^i/2i$, and Y_n the combinatorial quantity we are interested in. For example, this is the case in the original line of work investigating the number of Hamiltonian cycles [76, 77, 36], and also in subsequent work characterising the asymptotic distributions for several numerical characteristics of random d-regular graphs, e.g., the number of perfect matchings [43, 71], the number of 2-factors [74], and the number of 3-star factors [5]. In most cases, the chief goal of the analysis is to show that the particular structure occurs with probability approaching 1, in a random $G \in \mathbb{G}(n,d)$. However, as is the case in [36], this type of result may also have algorithmic implications.

In the remainder of this chapter, we derive asymptotic distributions for the number of Euler tours in a random d-in/d-out graph (§3.3), and the number of Eulerian orientations in a random $2d$-regular Eulerian graph (§3.4), for small, fixed, d. By arguing along the lines of Frieze et al. [36], we are able to show that the number of Euler tours in random d-in/d-out graph is close to the number of transition systems, and consequently that Algorithm 4 and Algorithm 5 run in expected polynomial time on almost every d-in/d-out graph. We conjecture that a similar result holds for the Euler tours of a random $2d$-regular graph and, in §3.5, we put forward some arguments to support this conjecture.

In §3.4, we also show that most $2d$-regular graphs have few Eulerian orientations, in the sense that the probability that $\mathrm{EO}(G)$ is within a linear factor of the lower bound

of Schrijver (Theorem 1.56) tends to 1 as $n \to \infty$.

3.3 Euler tours of random d-in/d-out graphs

In this section we analyse the number of Euler tours of random Eulerian directed graphs, using the directed configuration model as our model for generating random graphs. Recall that, given an integer vector $\mathbf{d} = (d_1, d_2, \ldots, d_n)$, this can be used to study properties of random elements of the set of directed graphs in which vertex $v \in \{1, 2, \ldots, n\}$ has $indeg(v) = outdeg(v) = d_v$.

Although the quantity we are interested in is the number of Euler tours of a random directed graph, it is easier to enumerate arborescences and, by the BEST theorem (Theorem 1.57), this is equivalent to enumerating Euler tours. Let $\mathcal{A}_{n,\mathbf{d}}$ count the number of arborescences of a directed graph G chosen uniformly at random from $\vec{\mathbb{G}}(n, \mathbf{d})$. In §3.3.1 we obtain asymptotic expressions for the first two moments of $\mathcal{A}_{n,\mathbf{d}}$. Then, in §3.3.2, we use Janson's theorem (Theorem 3.19) to characterise the asymptotic distribution of $\mathcal{A}_{n,d}$, for the special case when $d_v = d$ for all v.

Given the asymptotic distribution of $\mathcal{A}_{n,d}$ it is straightforward to show that the value of $\mathcal{A}_{n,d}$ is almost always close to the mean, $\mathbb{E}[\mathcal{A}_{n,d}]$ (Theorem 3.35), from which we can immediately infer Theorem 3.2. Although we believe a similar result holds for Eulerian directed graphs with arbitrary fixed degree sequence $\mathbf{d}$, proving this seems to be beyond the method used to prove Theorem 3.35.

3.3.1 Estimating the moments of $\mathcal{A}_{n,\mathbf{d}}$

We obtain asymptotic estimates for the moments of $\mathcal{A}_{n,\mathbf{d}}$ by first computing the moments of a random variable counting arborescences of non-uniform directed multigraphs and then conditioning on the graph being simple. We will use the following two facts several times in the proofs of this section.

Fact 3.23. Falling factorials of sums obey the well known *multinomial theorem*

$$(x_1 + x_2 + \cdots + x_l)_k = \sum_{\substack{\sum \delta_i = k \\ \delta_i \geq 0}} \binom{k}{\delta_1, \ldots, \delta_l} \prod_{i=1}^{l} (x_i)_{\delta_i}.$$

Definition 3.24. A rooted *forest* is a set of disjoint trees with distinguished root vertices. We call a rooted forest a rooted *k-forest* if it contains exactly k components.

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Fact 3.25. Let $V = \{1,2,\dots,n\}$. The number of rooted k-forests on V in which v has δ_v children is

$$\binom{n-1}{k-1}\binom{n-k}{\delta_v : v \in V}.$$

See, e.g., [81, Theorem 5.3.4], for a proof of Fact 3.25. We also consider the natural generalisation of the configuration model to non-Eulerian directed graphs.

Definition 3.26. Suppose $\mathbf{s} = (s_1, s_2, \dots, s_n)$ and $\mathbf{t} = (t_1, t_2, \dots, t_n)$ are a pair of positive vectors. Now, suppose we have disjoint sets of points S_v and T_v satisfying $|S_v| = s_v$ and $|T_v| = t_v$, for each $v = 1,2,\dots,n$ and let $S = \bigcup_{v=1}^{n} S_v$ and $T = \bigcup_{v=1}^{n} T_v$. We call the set of perfect matchings from S to T *directed configurations* and denote the set of all directed configurations by $\Phi_{n,\mathbf{s},\mathbf{t}}$. As before, we call the matchings from S to T the *partial directed configurations* of $\Phi_{n,\mathbf{s},\mathbf{t}}$.

We define projection in this generalised configuration model in the same way as in the original directed configuration model.

In Lemma 3.27 below, we use Fact 3.23 and Fact 3.25 to count the number of partial configurations that project to a directed forest in $\Phi_{n,\mathbf{s},\mathbf{t}}$.

In the proof of the following lemma, and throughout this section we take the term directed forest to mean a forest in which every edge in a component has been directed towards the the root. Furthermore, we will often speak of a configuration for an arborescence or directed forest. We take this to mean a partial configuration that projects to an arborescence or directed forest.

Lemma 3.27. *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be a pair of positive* n*-vectors and let* $V = \{1,2,\dots,n\}$*. The number of ways to choose a configuration for a directed forest rooted at* $R \subseteq V$ *in* $\Phi_{n,\mathbf{s},\mathbf{t}}$ *is*

$$\prod_{v \notin R} s_v \left(\sum_{v \in R} t_v\right)\left(\sum_{v \in V} t_v - 1\right)_{n-|R|-1}. \tag{3.7}$$

Proof. In counting the number of partial configurations that project to forests, we need to enumerate the forests that can occur in some directed graph $\vec{G} \in \vec{\mathbb{G}}(n,\mathbf{s},\mathbf{t})$, and also the number of partial configurations in $\Phi_{n,\mathbf{s},\mathbf{t}}$ that project to each forest. The first part of our analysis counts the number of such forests in which each vertex $v \in V$ has a particular number of children, δ_v.

Let $\mathcal{F}$ be a forest on V rooted at R and, for each $v \in V$, let δ_v be the number of children of v in $\mathcal{F}$. The number of ways to choose a point for the source and target of

each arc in $\mathcal{F}$ is

$$\prod_{v \notin R} s_v \prod_{v=1}^{n} (t_v)_{\delta_v}, \tag{3.8}$$

since we must choose one of the points in S_v for the source of the arc leaving each $v \notin R$ and choose one of the points in T_v as the target of each arc entering v.

We now count the number of forests on V which have roots at R, and which agree with the vector $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$ of child counts. One way to approach this task is to observe that we can construct a forest rooted at R by first choosing a k-forest on $V - R$, where $k = \sum_{v \in R} \delta_v$, and then attaching each root of this forest as a child of some $v \in R$. By Fact 3.25, the number of k-forests on $V - R$ in which $v \in V - R$ has exactly δ_v children is

$$\binom{n - |R| - 1}{k - 1} \binom{n - |R| - k}{\delta_v : v \in V - R}. \tag{3.9}$$

The number of ways we can divide the roots of this forest amongst the members of R so that each $v \in R$ has δ_v children is

$$\binom{k}{\delta_v : v \in R}. \tag{3.10}$$

Combining (3.9) and (3.10), we see that the number forests rooted at R that agree with child count vector $\boldsymbol{\delta}$ is

$$\binom{n - |R| - 1}{k - 1} \binom{n - |R| - k}{\delta_v : v \in V - R} \binom{k}{\delta_v : v \in R}.$$

Hence, by (3.8), the number of partial configurations projecting to these forests is

$$\binom{n - |R| - 1}{k - 1} \binom{n - |R| - k}{\delta_v : v \in V - R} \binom{k}{\delta_v : v \in R} \prod_{v \notin R} s_v \prod_{v \in V} (t_v)_{\delta_v}. \tag{3.11}$$

Now, summing (3.11) over all possible vectors $\boldsymbol{\delta}$ gives

$$\prod_{v \notin R} s_v \sum_{k=1}^{n-|R|} \binom{n - |R| - 1}{k - 1} \left(\sum_{\sum_{v \in R} \delta_v = k} \binom{k}{\delta_v : v \in R} \prod_{v \in R} (t_v)_{\delta_v} \right) \times \left(\sum_{\sum_{v \notin R} \delta_v = n - |R| - k} \binom{n - |R| - k}{\delta_v : v \notin R} \prod_{v \notin R} (t_v)_{\delta_v} \right). \tag{3.12}$$

We can use Fact 3.23 to simplify (3.12). The two sums over the different δ_v in (3.12) are the multinomial expansions of the falling factorial powers $(\sum_{v \in R} t_v)_k$ and $(\sum_{v \notin R} t_v)_{n-|R|-k}$, respectively. Hence, (3.12) is equal to

$$\prod_{v \notin R} s_v \sum_{k=1}^{n-|R|} \binom{n - |R| - 1}{k - 1} \left(\sum_{v \in R} t_v \right)_k \left(\sum_{v \notin R} t_v \right)_{n-|R|-k},$$

which is itself the multinomial expansion of (3.7). □

We now use Lemma 3.27 to analyse the expectation and variance of the number of arborescences in $\sigma(F)$, when F is chosen uniformly at random from $\Phi_{n,\mathbf{d}}$. In the following, we say $\mathcal{A} \subset \vec{\mathrm{F}}$ is an arborescence of $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ if $\sigma(\mathcal{A})$ is an arborescence of $\sigma(F)$, and denote the set of arborescences of $\vec{\mathrm{F}}$ by $\mathrm{ARB}(\vec{\mathrm{F}})$. In the following proofs, we will abuse terminology slightly and switch between speaking of arborescences of configurations and directed graphs arbitrarily.

Theorem 3.28. *Let d be some fixed constant, $n \in \mathbb{N}$, and let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ be a sequence of integers satisfying*

$$d \geq d_n \geq d_{n-1} \geq \cdots \geq d_1 \geq 1\,,$$

and let $m = \sum_{v=1}^{n} d_v$. We define $\mathcal{A}^\star_{n,\mathbf{d}}$ to be the random variable counting the arborescences of uniformly random $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$. Then,

$$\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}}] = \frac{n}{m}\prod_{v=1}^{n} d_v;$$
$$\frac{\mathbb{E}[(\mathcal{A}^\star_{n,\mathbf{d}})^2]}{\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}}]^2} = \frac{m}{m-n+1}\,.$$

Proof. We first consider the first moment of $\mathcal{A}^\star_{n,\mathbf{d}}$. To calculate the first moment of $\mathcal{A}^\star_{n,\mathbf{d}}$ we need to enumerate pairs $(\vec{\mathrm{F}}, \mathcal{A})$, where $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ and $\mathcal{A}$ is an arborescence of $\vec{\mathrm{F}}$, and then divide this quantity by $|\,\Phi_{n,\mathbf{d}}\,|$. Given $\mathcal{A}$, it is easy to count the number of configurations $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ for which $\mathcal{A} \subset \vec{\mathrm{F}}$. In any directed graph $\vec{G}$ with m arcs, there are exactly $m-n+1$ arcs not contained in any particular element of $\mathrm{ARB}(\vec{G})$. Hence, if we have a configuration for an arborescence, there are $(m-n+1)!$ ways to extend this to a complete configuration. Applying Lemma 3.27 with $\mathbf{s} = \mathbf{t} = \mathbf{d}$, we see that the number of arborescences rooted at any particular vertex v is

$$d_v \left(\prod_{u \neq v} d_u\right) (m-1)_{n-2}\,. \tag{3.13}$$

By the BEST theorem (Theorem 1.57), there are an equal number of arborescences rooted at each vertex of any $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$. Hence, multiplying (3.13) by by $n(m-n+1)!$ gives the number of pairs $(\vec{\mathrm{F}}, \mathcal{A})$ with $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ and $\mathcal{A} \in \mathrm{ARB}(\vec{\mathrm{F}})$:

$$n(m-1)! \left(\prod_{v \in V} d_v\right).$$

Finally, dividing by the total number of configurations in $\Phi_{n,\mathbf{d}}$, which is $m!$, gives the claimed value for $\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}}]$.

To compute the second moment of $\mathcal{A}^{\star}_{n,\mathbf{d}}$ we need to evaluate the following expression

$$\frac{1}{m!}\sum_{\vec{\mathrm{F}}\in\Phi_{n,\mathbf{d}}} |\,\mathrm{ARB}(\vec{\mathrm{F}})|^2\,. \tag{3.14}$$

We observe that the term $|\,\mathrm{ARB}(\vec{\mathrm{F}})|^2$ in (3.14) is equal to the number of elements in the set

$$\{(\mathcal{A},\mathcal{A}') : \mathcal{A},\mathcal{A}' \in \mathrm{ARB}(\vec{\mathrm{F}})\}\,.$$

That is,

$$\mathbb{E}[(\mathcal{A}^{\star}_{n,\mathbf{d}})^2] = \frac{|\widetilde{\Phi}_{n,\mathbf{d}}|}{|\,\Phi_{n,\mathbf{d}}\,|}\,,$$

where

$$\widetilde{\Phi}_{n,\mathbf{d}} = \{(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}') : \vec{\mathrm{F}}\in\Phi_{n,\mathbf{d}}, \mathcal{A},\mathcal{A}'\in\mathrm{ARB}(\vec{\mathrm{F}})|\}\,.$$

Hence, to evaluate $\mathbb{E}[(\mathcal{A}^{\star}_{n,\mathbf{d}})^2]$ we need to count the number of elements of $\widetilde{\Phi}_{n,\mathbf{d}}$.

We compute $|\widetilde{\Phi}_{n,\mathbf{d}}|$ as follows. First, we count the number of ways to choose the intersection of a pair of arborescences $\mathcal{A}$ and $\mathcal{A}'$. Then, we count the number of ways to extend this intersection to $\mathcal{A}$ and $\mathcal{A}'$. Finally, we count the number of ways to choose the remainder of $\vec{\mathrm{F}}$ so that $\mathcal{A}$ and $\mathcal{A}'$ are both in $\mathrm{ARB}(\vec{\mathrm{F}})$.

The last step is the easiest. Suppose we have a pair of arborescences $(\mathcal{A},\mathcal{A}')$ of some configuration $\vec{\mathrm{F}}\in\Phi_{n,\mathbf{d}}$ and suppose $\mathcal{F}=\mathcal{A}\cap\mathcal{A}'$ is a forest rooted at $R\subseteq V$. Since we need to add $|R|-1$ arcs to $\mathcal{F}$ for each arborescence, there will be $n+|R|-2$ edges in $\mathcal{A}\cup\mathcal{A}'$ and, hence, there are $(m-n-|R|+2)!$ ways to choose the remaining edges for $\vec{\mathrm{F}}$.

We now proceed to enumerate the different pairs $(\mathcal{A},\mathcal{A}')$ with $\mathcal{F}=\mathcal{A}\cap\mathcal{A}'$ rooted at R. In fact, we overcount slightly, with the number of times $(\mathcal{A},\mathcal{A}')$ is counted depending on the roots of $\mathcal{A}$ and $\mathcal{A}'$. We use the BEST Theorem (Theorem 1.57) to get back to the correct number at the end of the proof.

We start by counting the number of ways we can choose $\mathcal{F}$, the edges in both arborescences, and then count the number of ways to choose the edges which are in one or the other arborescence. By Lemma 3.27, the number of ways to choose $\mathcal{F}$ rooted at R is

$$\left(\prod_{v\notin R} d_v\right)\left(\sum_{v\in R} d_v\right)(m-1)_{n-|R|-1}\,. \tag{3.15}$$

For each $v\in R$, let $\mathcal{F}_v$ denote the component of $\mathcal{F}$ with root v, and let x_v be the number of points in $\bigcup_{u\in\mathcal{F}_v} T_u$ not used by arcs in $\mathcal{F}$. That is,

$$x_v = \sum_{u\in\mathcal{F}_v} d_u - |\mathcal{F}_v| + 1\,.$$

Note that this is the number of points available to add incoming arcs to vertices of $\mathcal{F}_v$ when we are completing $\mathcal{A}$ and $\mathcal{A}'$. Moreover, we have

$$\sum_{v\in R} x_v = m - n + |R|\,.$$

We now turn our attention to the number of ways to choose $\mathcal{A}\setminus\mathcal{A}'$ and $\mathcal{A}'\setminus\mathcal{A}$. Choosing the remaining arcs for $\mathcal{A}$ and $\mathcal{A}'$ is equivalent to choosing a pair of disjoint configurations for trees on R in which there are x_v points available for the targets of arcs entering v and d_v points available for the sources of the arcs leaving v, for each $v\in R$.

Suppose we have already chosen $\mathcal{A}\setminus\mathcal{A}'$ such that the root of $\mathcal{A}$ is r and suppose that there are δ_v additional arcs directed towards vertices in $\mathcal{F}_v$, for each $v\in R$. Now, suppose we want to choose $\mathcal{A}'\setminus\mathcal{A}$ such that the root of $\mathcal{A}'$ is r', and, for the moment, suppose $r\neq r'$. Choosing $\mathcal{A}'\setminus\mathcal{A}$ amounts to choosing a tree on R rooted at r' in which there are $x_v-\delta_v$ points available for arcs directed towards each v, d_v-1 points available for the source of the arc directed away from each $v\neq r$, and d_r points available for the source of the arc directed away from r. Hence, by Lemma 3.27, we see that the number of ways to choose $\mathcal{A}'\setminus\mathcal{A}$ is

$$\frac{(x_{r'}-\delta_{r'})d_r}{(d_r-1)(d_{r'}-1)}\left(\prod_{v\in R}(d_v-1)\right)(m-n)_{|R|-2}\,.$$

Using Fact 3.25, we can deduce that the number of ways to choose $\mathcal{A}$ is

$$\frac{\prod_{v\in R} d_v}{d_r}\sum_{\substack{|\boldsymbol{\delta}|=|R|-1\\ \delta_r\geq 1}}\binom{|R|-2}{\delta_r-1;\delta_v : v\in R\setminus\{r\}}\prod_{v\in R}(x_v)_{\delta_v}\,.$$

Therefore, the number of ways to complete $\mathcal{F}$ to $\mathcal{A}\cup\mathcal{A}'$ is equal to

$$\frac{\prod_{v\in R} d_v(d_v-1)}{(d_r-1)(d_{r'}-1)}(m-n)_{|R|-2}$$

times

$$\sum_{\substack{|\boldsymbol{\delta}|=|R|-1\\ \delta_r\geq 1}}(x_{r'}-\delta_{r'})\binom{|R|-2}{\delta_r-1;\delta_v : v\in R\setminus\{r\}}\prod_{v\in R}(x_v)_{\delta_v}\,. \tag{3.16}$$

We can divide (3.16) into two sums:

$$x_{r'}\sum_{\substack{|\boldsymbol{\delta}|=|R|-1\\ \delta_r\geq 1}}\binom{|R|-2}{\delta_r-1;\delta_v : v\in R\setminus\{r\}}\prod_{v\in R}(x_v)_{\delta_v} \tag{3.17}$$

and

$$-\sum_{\substack{|\boldsymbol{\delta}|=|R|-1\\ \delta_r\geq 1}} \delta_{r'}\binom{|R|-2}{\delta_r-1;\delta_v : v\in R\setminus\{r\}}\prod_{v\in R}(x_v)_{\delta_v}. \tag{3.18}$$

Applying Fact 3.23, we see that (3.17) and (3.18) are equal to $x_r x_{r'}(m-n+|R|-1)_{|R|-2}$ and $-x_r x_{r'}(|R|-2)(m-n+|R|-2)_{|R|-3}$, respectively. Hence, the number of ways to complete $\mathcal{F}$ to $\mathcal{A}\cup\mathcal{A}'$ is

$$\frac{x_r x_{r'}}{(d_r-1)(d_{r'}-1)}\left(\prod_{v\in R} d_v(d_v-1)\right)(m-n+|R|-2)_{2|R|-4}. \tag{3.19}$$

If $r=r'$, we can apply an almost identical argument to show that the number of ways to complete $\mathcal{F}$ to $\mathcal{A}\cup\mathcal{A}'$ is

$$\frac{x_r(x_r-1)}{d_r(d_r-1)}\left(\prod_{v\in R} d_v(d_v-1)\right)(m-n+|R|-2)_{2|R|-4}. \tag{3.20}$$

Multiplying (3.19) and (3.20) by $(d_r-1)(d_{r'}-1)$ and $d_r(d_r-1)$, respectively, and summing over r and r' gives

$$\left(\sum_{r\neq r'} x_r x_{r'}+\sum_r x_r(x_r-1)\right)\left(\prod_{v\in R} d_v(d_v-1)\right)(m-n+|R|-2)_{2|R|-4}. \tag{3.21}$$

Since $\sum_{r\in R} x_r = m-n+|R|$, we have

$$\begin{aligned}\sum_{r\neq r'} x_r x_{r'}+\sum_r x_r(x_r-1) &= \sum_{r\in R} x_r\left(x_r-1+\sum_{r'\neq r}(x_{r'})\right)\\ &= (m-n+|R|)(m-n+|R|-1).\end{aligned}$$

Hence, (3.21) is equal to

$$\left(\prod_{v\in R} d_v(d_v-1)\right)(m-n+|R|)_{2|R|-2}.$$

Multiplying by the number of ways to choose $\mathcal{F}$, given in (3.15), and the number of ways to choose the portion of $\vec{\mathrm{F}}$ not contained in $\mathcal{A}\cup\mathcal{A}'$, which is $(m-n-|R|+2)!$, yields the following expression

$$\left(\prod_{v\in V} d_v\right)(m-1)!\left(\prod_{v\in R}(d_v-1)\right)\left(\sum_{v\in R} d_v\right). \tag{3.22}$$

The expression (3.22) over-counts the number of triples $(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')$ in which the intersection $\mathcal{A}\cap\mathcal{A}'$ is a forest rooted at R. Each triple $(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')$ in which $\mathcal{A}$ and $\mathcal{A}'$ are

rooted at different vertices u and v is counted $(d_u-1)(d_v-1)$ times, and each triple $(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')$ in which $\mathcal{A}$ and $\mathcal{A}'$ are rooted at the same vertex v is counted $d_v(d_v-1)$ times.

Only the second two factors of (3.22) depend on R. Summing these over all $R\subseteq V$ gives

$$\sum_{R\subseteq V}\left(\sum_{v\in R}d_v\right)\left(\prod_{v\in R}(d_v-1)\right), \tag{3.23}$$

We can evaluate (3.23) by separating it into n separate sums, each corresponding to the sum over $R\ni v$, for $v\in V$,

$$d_v\sum_{R\ni v}\prod_{u\in R}(d_u-1)=(d_v-1)\left(\prod_{u\in V}d_u\right). \tag{3.24}$$

Then, summing the right-hand side of (3.24) over each $v\in V$, and combining with the rest of (3.22), gives

$$\left(\prod_{v\in V}d_v\right)^2(m-n)(m-1)!\,. \tag{3.25}$$

We cannot immediately obtain the quantity we are looking for from (3.25) as it over-counts different triples by different amounts. However, by the BEST theorem (Theorem 1.57), we know that the number of triples $(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')$ in which $\mathcal{A}$ is rooted at u and $\mathcal{A}'$ is rooted at v does not depend on u or v, since the projection $\sigma(\vec{\mathrm{F}})$ is always an Eulerian directed graph. Thus, it follows that the factor by which (3.25) over-counts the number of triples is

$$\frac{1}{n^2}\left(\sum_{u\neq v}(d_u-1)(d_v-1)+\sum_v d_v(d_v-1)\right)=\frac{(m-n+1)(m-n)}{n^2}\,. \tag{3.26}$$

Then, dividing (3.25) by (3.26) and $m!$ gives

$$\mathbb{E}[(\mathcal{A}^\star_{n,\mathbf{d}})^2]=\frac{n^2}{m(m-n+1)}\left(\prod_{v\in V}d_v\right)^2. \qquad \square$$

An Eulerian directed graph is simple if and only if it does not contain any loops or double arcs; we do, however, allow the arcs (u,v) and (v,u) to be present. In Theorem 3.30 below, we estimate the first and second moments of $\mathcal{A}_{n,\mathbf{d}}$ using Theorem 3.28 and conditioning on there being no loops or double arcs in $\vec{\mathrm{F}}$.

Before proceeding with the proof of Theorem 3.30, we state a useful property of the directed configuration model. A similar property is used, implicitly or explicitly, in almost all results on the undirected configuration model [100].

Lemma 3.29. *For any fixed directed graph H with more arcs than vertices, and $\vec{\mathrm{F}}$ chosen uniformly at random from $\Phi_{n,\mathbf{d}}$, the probability $\sigma(\vec{\mathrm{F}})$ contains H as a subgraph tends to 0, as $m \to \infty$.*

Proof. The reason for this is as follows. Suppose H has i vertices and j edges. For each set $S \subset \{1,2,\dots,n\}$ with $|S| = i$, the number of ways we can choose configuration pairs that will give H as a subgraph of $\sigma(F)$ on the vertices S is bounded by a constant. The probability of one of these sets of configuration pairs occurring in a random $F \in \Phi_{n,\mathbf{d}}$ is

$$\frac{(m-j)!}{m!} \to m^{-j},$$

since there are $(m-j)!$ ways to choose the remainder of a configuration, given a set of j edges. But the number of ways to choose S is

$$\binom{n}{i} \to n^i.$$

Hence, we can, rather crudely, bound the probability of H occurring as a subgraph of $\sigma(\vec{\mathrm{F}})$ by $O(m^{i-j})$. □

We now proceed with the proof of Theorem 3.30. We will use (the ideas of) Lemma 3.29 to show that the factorial moments of various random variables converge.

Theorem 3.30. *Let d be some fixed constant, $n \in \mathbb{N}$, and let $\mathbf{d}$ be a sequence of integers satisfying*

$$d \geq d_n \geq d_{n-1} \geq \cdots \geq d_1 \geq 1,$$

and let $m = \sum_{v=1}^{n} d_v$. Furthermore, suppose $m - n \to \infty$. Let $\mathcal{A}_{n,\mathbf{d}}$ denote the number of arborescences of an Eulerian directed graph chosen randomly from $\vec{\mathbb{G}}(n,\mathbf{d})$. Then,

$$\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}] \to e^1 \frac{n}{m} \prod_v d_v;$$

$$\frac{\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}^2]}{\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]^2} \to e^{-n/m} \frac{m}{m-n},$$

Proof. In the following we will use m_2 to denote $\sum_v d_v^2$.

The proof is as follows. Recall that $\vec{\mathrm{F}}$ contains a loop at v if there is an edge from $S_v \times T_v$ in $\vec{\mathrm{F}}$ and that $\vec{\mathrm{F}}$ contains a double arc from u to v if there is a pair of edges from $S_u \times T_v$ in $\vec{\mathrm{F}}$, for some pair of vertices $u \neq v$. Let L and D denote the number of loops and double arcs in a random $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$. Then, the event "$\vec{\mathrm{F}}$ is simple" is equivalent to the event $\{L = D = 0\}$. We first analyse the distributions of L and D, which we can

use to estimate the probability that $\vec{\mathrm{F}}$ is simple. Then, we consider two new random variables, $L^{(1)}$ and $D^{(1)}$, which count the number of loops and double arcs in $\vec{\mathrm{F}}$ when $(\vec{\mathrm{F}}, \mathcal{A})$ is chosen randomly from the set

$$\overline{\Phi}_{n,\mathbf{d}} = \{(\vec{\mathrm{F}}, \mathcal{A}) : \vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}},\, \mathcal{A} \in \mathrm{ARB}(\vec{\mathrm{F}})\}\,. \tag{3.27}$$

By analysing the distributions of $L^{(1)}$ and $D^{(1)}$ we can estimate

$$\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}] = \frac{\mathbb{P}[L^{(1)} = D^{(1)} = 0]}{\mathbb{P}[L = D = 0]}\,\mathbb{E}[\mathcal{A}^{\star}_{n.\mathbf{d}}]\,.$$

Finally, we consider random variables, $L^{(2)}$ and $D^{(2)}$, which count the number of loops and double arcs in $\vec{\mathrm{F}}$ when $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ is chosen randomly from the set

$$\widetilde{\Phi}_{n,\mathbf{d}} = \{(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}') : \vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}},\, \mathcal{A}, \mathcal{A}' \in \mathrm{ARB}(\vec{\mathrm{F}})\}\,. \tag{3.28}$$

By analysing the distributions of $L^{(2)}$ and $D^{(2)}$ we can estimate

$$\mathbb{E}[(\mathcal{A}_{n,\mathbf{d}})^2] = \frac{\mathbb{P}[L^{(2)} = D^{(2)} = 0]}{\mathbb{P}[L = D = 0]}\,\mathbb{E}[(\mathcal{A}^{\star}_{n.\mathbf{d}})^2]\,.$$

We first compute the expectation of L and D. Suppose we have a loop edge $e \in S_v \times T_v$ in $\vec{\mathrm{F}}$ and let I_e be the indicator variable for the event $e \in F$. Then, we can write $L = \sum_{v \in V} \sum_{e \in S_v \times T_v} I_e$ and, by linearity of expectation, we have

$$\mathbb{E}[L] = \sum_{v \in V} \sum_{e \in S_v \times T_v} \mathbb{E}[I_e] = \sum_{v \in V} \sum_{e \in S_v \times T_v} \mathbb{P}[e \in F]\,. \tag{3.29}$$

Given e, the number of ways to choose $\vec{\mathrm{F}}$ with $e \in \vec{\mathrm{F}}$ is $(m-1)!$, so the probability of a random $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ containing e is $1/m$. For each $v \in V$, there are d_v^2 ways to choose an edge from $S_v \times T_v$. Hence,

$$\mathbb{E}[L] = \frac{1}{m}\sum_{v} d_v^2 = \frac{m_2}{m}\,. \tag{3.30}$$

Next, we compute the expectation of D. Here, for every pair of edges $e, f \in S_u \times T_v$, for some $u \neq v$, we define an indicator variable $I_{e,f}$ for the event $e, f \in \vec{\mathrm{F}}$. Then $D = \sum_{u \neq v} \sum_{e,f \in S_u \times T_v} I_{e,f}$. By linearity of expectation, we have

$$\mathbb{E}[D] = \sum_{u \in V} \sum_{v \in V \setminus \{u\}} \sum_{e,f \in S_u \times T_v} \mathbb{P}[e, f \in \vec{\mathrm{F}}]\,. \tag{3.31}$$

The probability of a particular pair of edges e and f occurring in a random configuration $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ is, asymptotically, $1/m^2$. Moreover, the number of ways to choose

$e, f \in S_u \times T_v$ is $2\binom{d_u}{2}\binom{d_v}{2}$. Hence, the sum in (3.31) becomes

$$\begin{aligned}\mathbb{E}[D] &\to \frac{2}{m^2}\sum_{u\in V}\sum_{v\in V\setminus\{u\}}\binom{d_u}{2}\binom{d_v}{2}\\ &= \frac{1}{2m^2}\left(\sum_{u\in V}(d_u)_2\right)^2 - \frac{1}{2m^2}\sum_{u\in V}(d_u)_2^2\,. \end{aligned} \tag{3.32}$$

To finish the calculation we observe that the numerator of the negative term in (3.32) is $O(m)$ (each d_u is bounded above by a constant d, so $\sum_u (d_u)_2^2 \le d^3 m$). Hence, this part of the sum disappears as $m \to \infty$ and

$$\mathbb{E}[D] \to \frac{(m_2-m)^2}{2m^2}\,. \tag{3.33}$$

We would like to use Theorem 3.14 to show that L and D are asymptotically Poisson, so we can use (3.30) and (3.33) to estimate $\mathbb{P}[L = D = 0]$. To apply Theorem 3.14, we need to show that, for every pair of non-negative integers i and j, we have

$$\mathbb{E}[(L)_j(D)_k] \to \mathbb{E}[L]^j\,\mathbb{E}[D]^k\,. \tag{3.34}$$

Consider ordered tuples of the form $\mathbf{t} = (e_1, e_2, \ldots, e_j, (f_1, g_1), (f_2, g_2), \ldots, (f_k, g_k))$ where e_i is a loop, for $i = 1, 2, \ldots, j$, (f_i, g_i) is a double arc for $i = 1, 2, \ldots, k$, and all edges are distinct. For each such $\mathbf{t}$, we define an indicator variable $I_{\mathbf{t}}$ for the event $\mathbf{t} \in \vec{\mathrm{F}}$ ($\vec{\mathrm{F}}$ contains all components of $\mathbf{t}$). Then, we can write

$$\mathbb{E}[(L)_j(D)_k] = \sum_{\mathbf{t}}\mathbb{E}[I_{\mathbf{t}}] = \sum_{\mathbf{t}}\mathbb{P}[\mathbf{t} \in \vec{\mathrm{F}}]\,. \tag{3.35}$$

First, we consider the contribution to $\mathbb{E}[(L)_j(D)_k]$ from tuples where no vertex occurs more than once. We can write this as

$$\frac{1}{2^k}\sum_{v_1\in V_1}\frac{d_{v_1}^2}{m}\sum_{v_2\in V_2}\frac{d_{v_2}^2}{m-1}\cdots\sum_{v_{j+1}\in V_{j+1}}\frac{(d_{v_{j+1}})_2}{m-j}\cdots\sum_{v_{2k}\in V_{j+2k}}\frac{(d_{v_{2k}})_2}{m-2k}\,, \tag{3.36}$$

where $V_i = V - \{v_j : j < i\}$. We can re-write the last sum as

$$\frac{1}{m-2k}\left(m_2 - m - \sum_{v\in V_{2k}}(d_{v_{j+2k}})_2\right). \tag{3.37}$$

Since k and d are both fixed, it follows that (3.37) is, asymptotically, $(m_2-m)/m$. We can apply the same reasoning to show that each of the sums over V_i, for $i = j+1, \ldots, 2k$, converge to $(m_2-m)/m$. Similarly, each of the sums over V_i, for $i = 1, 2, \ldots, j$, converges to m_2/m. Hence, the asymptotic value of (3.36) is

$$\mathbb{E}[L]^j\,\mathbb{E}[D]^k\,.$$

By Lemma 3.29, we know that the probability of $\vec{\mathrm{F}}$ containing any set of cycles on j vertices, for fixed j, in which any vertex occurs more than once, is (asymptotically) 0. Hence, the contribution to (3.35) from ordered tuples in which any vertex features more than once is negligible and, therefore, (3.34) holds.

We have shown that L and D converge to independent Poisson random variables and, therefore, the probability that $\vec{\mathrm{F}}$ is simple when $\vec{\mathrm{F}}$ is chosen uniformly at random from $\Phi_{n,\mathbf{d}}$ is (asymptotically)

$$\exp\left(-\frac{m_2}{m}-\frac{(m_2-m)^2}{2m^2}\right). \tag{3.38}$$

Next, we consider the distributions of $L^{(1)}$ and $D^{(1)}$. We first estimate $\mathbb{E}[L^{(1)}]$. Suppose we have a loop edge $e \in S_v \times T_v$, for some $v \in V$. A loop edge cannot be contained in any arborescence, and, thus, the number of pairs $(\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}$ where $e \in \vec{\mathrm{F}}$, is equal to the number of pairs $(\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}'}$, where $\mathbf{d}'$ is equal to $\mathbf{d}$ with d_v replaced by $d_v - 1$. Hence, from Theorem 3.28, we can see that the number of elements of $\overline{\Phi}_{n,\mathbf{d}}$ with $e \in \vec{\mathrm{F}}$ is equal to

$$n(d_v-1)\prod_{u\neq v} d_u (m-2)!\,. \tag{3.39}$$

Dividing (3.39) by the total number of elements in $\overline{\Phi}_{n.\mathbf{d}}$, which we can also obtain from Theorem 3.28, gives the probability

$$\mathbb{P}[e \in \vec{\mathrm{F}} : (\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}] = \frac{d_v - 1}{d_v(m-1)} \tag{3.40}$$

Evaluating (3.29) with this probability in the place of $\mathbb{P}[e \in F]$ gives

$$\mathbb{E}[L^{(1)}] = \frac{1}{m-1}\sum_v d_v(d_v-1) \to \frac{m_2-m}{m}\,.$$

Next, we evaluate $\mathbb{E}[D^{(1)}]$. Suppose we have a pair of edges $e, f \in S_u \times T_v$ for some $u \neq v$. By Lemma 3.27, the number of arborescences rooted at u in which each $w \notin \{u, v\}$ has d_w points available for its incoming and outgoing arcs, u has d_u points available for incoming arcs, and v has $d_v - 2$ points available for incoming arcs and d_v available for outgoing arcs is

$$\left(\prod_{w\in V} d_w\right)(m-3)_{n-2}\,. \tag{3.41}$$

The expression in (3.41) counts the number of partial configurations which consist of the edges e and f along with $n-1$ configuration edges that project to an arborescence

rooted at u. There are $(m-n-1)!$ ways to extend each of these partial configurations to some $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$. Hence, the following expression counts the number of pairs $(\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}$ with $e, f \in \vec{\mathrm{F}}$ and $\mathcal{A}$ rooted at u.

$$\left(\prod_{w=1}^{n} d_w\right)(m-3)!\,. \tag{3.42}$$

By the BEST Theorem (Theorem 1.57), we know that each $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ has the same number of arborescences rooted at each vertex, so (3.42) counts exactly $1/n$ of the pairs $(\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}$ with $e, f \in \vec{\mathrm{F}}$. Multiplying (3.42) by n and dividing by $|\overline{\Phi}_{n,\mathbf{d}}|$ gives

$$\mathbb{P}[e, f \in \vec{\mathrm{F}} : (\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}] \to \frac{1}{m^2}\,. \tag{3.43}$$

This is the same probability as when $\vec{\mathrm{F}}$ is chosen uniformly at random from $\Phi_{n,\mathbf{d}}$, so evaluating (3.33) with (3.43) in place of $\mathbb{P}[e, f \in F]$ does not change the (asymptotic) value and we have

$$\mathbb{E}[D^{(1)}] \to \mathbb{E}[D]\,.$$

As described in Lemma 3.29, the probability that a random $\mathrm{F} \in \Phi_{n,\mathbf{d}}$ contains a particular k-vertex subgraph with more arcs than vertices tends to 0. This is also true when we are sampling $(\tilde{\mathrm{F}}, \mathcal{A})$ uniformly at random from $\overline{\Phi}_{n,\mathbf{d}}$. Hence, the contribution to $\mathbb{E}[(L^{(1)})_j (D^{(1)})_k]$, from ordered tuples of loops and double arcs which overlap at any vertex is negligible. We can thus conclude that the factorial moments converge to

$$\mathbb{E}[(L^{(1)})_j (D^{(1)})_k] \to \mathbb{E}[L^{(1)}]^j\, \mathbb{E}\, D^{(1)}]^k\,.$$

Hence, the probability of $\vec{\mathrm{F}}$ being simple in a random $(\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}$ is

$$\exp\left(-\frac{m_2 - m}{m} - \frac{(m_2-m)^2}{2m^2}\right). \tag{3.44}$$

Together (3.38) and (3.44) give the claimed estimate for $\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]$.

Finally, we consider the distributions of $L^{(2)}$ and $D^{(2)}$. Suppose we have a loop edge $e \in S_v \times T_v$. The number of elements of $\widetilde{\Phi}_{n,\mathbf{d}}$ with $e \in \vec{\mathrm{F}}$ is equal to the number of elements of $\widetilde{\Phi}_{n,\mathbf{d}'}$, where $\mathbf{d}'$ is the out-degree vector we used to compute $\mathbb{E}[L^{(1)}]$. By Theorem 3.28, we have

$$|\widetilde{\Phi}_{n,\mathbf{d}}| = \frac{(d_v-1)^2}{(d_v)^2}\frac{n^2}{m-n}\left(\prod_{w\in V} d_w\right)^2 (m-2)!\,.$$

Dividing by the number of elements in $\widetilde{\Phi}_{n,\mathbf{d}}$, which know from Theorem 3.28, we see that

$$\mathbb{P}[e \in \vec{\mathrm{F}} : (\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}') \in \widetilde{\Phi}_{n,\mathbf{d}}] \to \frac{(d_v-1)^2}{(d_v)^2 m}\,.$$

Evaluating (3.30) with this probability in the place of $\mathbb{P}[e \in F]$ gives

$$\mathbb{E}[L^{(2)}] \to \frac{m_2 - 2m + n}{m}. \tag{3.45}$$

We now evaluate $\mathbb{E}[D^{(2)}]$. Suppose we have a pair of edges $e, f \in S_u \times T_v$ for some $u \neq v$. There are three cases to consider: $e, f \in \mathcal{A} \cup \mathcal{A}'$; $e, f \notin \mathcal{A} \cup \mathcal{A}'$; or exactly one of e and f is in $\mathcal{A} \cup \mathcal{A}'$. We estimate $\mathbb{E}[D^{(2)}]$ as follows. Using slightly more general arguments than those used to compute the second moment in Theorem 3.28, we count the number of triples $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ for each of these three cases, obtaining expressions which overcount in the same way as (3.25). Then, since the way in which triples are over-counted is the same in each of the three analyses, i.e., the number of times each triple $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ is counted is determined by the out-degrees of the roots of $\mathcal{A}$ and $\mathcal{A}'$, we can add these three expressions together, apply the BEST theorem, and proceed as we did in the proof of Theorem 3.28.

Note that the fact that the in-degree and out-degree of each vertex v are equal is only used at the last step of the analysis of the second moment of $\mathcal{A}^{\star}_{n,\mathbf{d}}$ (in Theorem 3.28). That is, if we are working in the general directed configuration model of Definition 3.3 (with the added condition that $\sum s_v = \sum t_v = m$) and follow the arguments of the second part of Theorem 3.28 we find that, for each $R \subseteq V$, the expression over-counting triples $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ where $\mathcal{A} \cap \mathcal{A}'$ is a forest rooted at R (given by (3.22) in the proof of Theorem 3.28) becomes

$$\left(\prod_{w \in V} s_w\right)(m-1)! \left(\sum_{w \in R} t_w\right)\left(\prod_{w \in R}(s_w - 1)\right). \tag{3.46}$$

The factor by which (3.46) over-counts $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ is $(s_r - 1)(s_{r'} - 1)$ if $\mathcal{A}$ and $\mathcal{A}'$ are rooted at different vertices $r, r' \in R$, and is $s_r(s_r - 1)$ if both are rooted at the same vertex $r \in R$.

As we did for (3.22) in the proof of Theorem 3.28, we can separate (3.46) into two parts: the part that does not depend on R,

$$\left(\prod_{w \in V} s_w\right)(m-1)!,$$

and the part that does

$$\left(\sum_{w \in R} t_w\right)\left(\prod_{w \in R}(s_w - 1)\right). \tag{3.47}$$

Summing (3.47) over all possibilities for R gives

$$\left(\sum_{w \in V} \frac{t_w(s_w - 1)}{s_w}\right)\left(\prod_{w \in V} s_w\right).$$

Hence, in the general directed configuration model, the expression which over-counts triples of the form $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ is

$$\left(\sum_{w\in V} \frac{t_w(s_w-1)}{s_w}\right)\left(\prod_{w\in V} s_w\right)^2 (m-1)! \tag{3.48}$$

Now, suppose $e, f \notin \mathcal{A}\cup\mathcal{A}'$. In this case we want to count the number of triples $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ where $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{s},\mathbf{t}}$, for $\mathbf{s} = \mathbf{d}$, except $s_u = d_u - 2$, and $\mathbf{t} = \mathbf{d}$, except $t_v = d_v - 2$. Adding e and f to the $\vec{\mathrm{F}}$ part of each element of this set of triples gives the set of triples $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}') \in \widetilde{\Phi}_{n,\mathbf{d}}$ with $e, f \in \vec{\mathrm{F}} - (\mathcal{A}\cup\mathcal{A}')$. Evaluating (3.48) for the stated values of $\mathbf{s}$ and $\mathbf{t}$ gives

$$\left(m - n - \frac{d_u}{d_u-2} - \frac{d_v-2}{d_v}\right)\frac{(d_u-2)^2}{d_u^2}\left(\prod_{w\in V} d_w\right)^2 (m-3)!\,,$$

or, asymptotically, as $m-n\to\infty$,

$$(m-n)\frac{(d_u-2)^2}{d_u^2}\left(\prod_{w\in V} d_w\right)^2 (m-3)!\,, \tag{3.49}$$

Next, suppose $e, f \in \mathcal{A}\cup\mathcal{A}'$. Since there can be at most one arc leaving u in $\mathcal{A}$ or $\mathcal{A}'$ it follows that we have an arc (u,v) in both $\mathcal{A}$ and $\mathcal{A}'$. Hence, when we are choosing the pair of arborescences we must assume that (u,v) is always present. This corresponds to replacing u and v by a single vertex v' which has d_v points available for outgoing arcs and $d_u + d_v - 2$ points available for incoming arcs. That is, in this instance we want to count triples $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ with $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{s},\mathbf{t}}$, where $\mathbf{s}$ and $\mathbf{t}$ are vectors indexed by $V\backslash\{u,v\}$ and v' satisfying $s_w = t_w = d_w$ for $w \in V\backslash\{u,v\}$, $s_{v'} = d_v$, and $t_{v'} = d_u + d_v - 2$. Evaluating (3.48) with these values for $\mathbf{s}$ and $\mathbf{t}$ yields, asymptotically,

$$(m-n)\frac{1}{(d_u)^2}\left(\prod_{w\in V} d_w\right)^2 (m-3)!\,. \tag{3.50}$$

We can extend each of the triples counted by (3.50) to an element of $\widetilde{\Phi}_{n,\mathbf{d}}$ in four ways, since we can put e or f into either arborescence. Hence, the expression over-counting triples $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}') \in \widetilde{\Phi}_{n,\mathbf{d}}$ with $e, f \in \mathcal{A}\cup\mathcal{A}'$ is, asymptotically,

$$(m-n)\frac{4}{(d_u)^2}\left(\prod_{w\in V} d_w\right)^2 (m-3)!\,. \tag{3.51}$$

Finally, suppose exactly one of e and f is in $\mathcal{A}\cup\mathcal{A}'$. This case is a little bit more complicated. Suppose $e \in \mathcal{A}$. We contract u and v to a single vertex v', as in the

previous case, and choose a forest on $V\setminus\{u,v\}$ and v', with root R. Then, we proceed as in the proof of Theorem 3.28, except we consider u to be one of the roots of the components of $\mathcal{F}$ when choosing $\mathcal{A}'$; that is, when choosing $\mathcal{A}\setminus\mathcal{A}'$ we choose a tree on R, but when choosing $\mathcal{A}'\setminus\mathcal{A}$ we choose a tree on $R\cup\{u\}$, where the vertices in the component of our initial forest have now been divided between a component rooted at u and a component containing v. The final expression counting the number of ways to complete $\mathcal{F}$ to $\mathcal{A}\cup\mathcal{A}'$ does not depend on how the vertices are distributed amongst the components of $\mathcal{F}$, so it is safe to do this. In this way, we obtain the expression

$$(d_u-2)\left(\sum_{w\in R}t_w\right)\left(\prod_{w\in R}(s_w-1)\right)\left(\prod_{w\in V\setminus\{u\}}s_w\right)(m-1)! \tag{3.52}$$

which over-counts the number of triples $(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')$ where $R\cup\{u\}$ are the roots of the components of $\mathcal{A}\cap\mathcal{A}'$, $e\in\mathcal{A}$, and $f\in\vec{\mathrm{F}}\setminus(\mathcal{A}\cup\mathcal{A}')$.

Proceeding as before, by summing over all possibilities for R, gives, asymptotically,

$$(m-n)\frac{d_u-2}{(d_u)^2}\left(\prod_{w\in V}d_w\right)^2(m-1)!\,.$$

The cases where $e\in\mathcal{A}'$, $f\in\mathcal{A}$, and $f\in\mathcal{A}'$ are all equivalent, so the expression over-counting triples $(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')$ with exactly one of e and f in $\mathcal{A}\cup\mathcal{A}'$ is, asymptotically,

$$(m-n)\frac{4(d_u-2)}{(d_u)^2}\left(\prod_{w\in V}d_w\right)^2(m-1)!\,. \tag{3.53}$$

Adding (3.49), (3.51), and (3.53) gives

$$(m-n)\left(\prod_{w\in V}d_w\right)^2(m-3)!\,. \tag{3.54}$$

Finally, we observe that the triples are over-counted consistently in the three separate constructions given above. Hence, we can conclude, by the same reasoning as was used in Theorem 3.28, that (3.54) over-counts the elements of $\widetilde{\Phi}_{n,\mathbf{d}}$ by a factor of

$$\frac{(m-n-1)(m-n-2)}{n^2}\,. \tag{3.55}$$

Dividing (3.54) by (3.55) and the number of elements in $\widetilde{\Phi}_{n,\mathbf{d}}$, which we can obtain using Theorem 3.28, gives

$$\mathbb{P}[e,f\in\vec{\mathrm{F}}:(\vec{\mathrm{F}},\mathcal{A},\mathcal{A}')\in\widetilde{\Phi}_{n,\mathbf{d}}]\to\frac{1}{m^2}\,.$$

This is the same probability for $e, f \in \vec{\mathrm{F}}$ when $\vec{\mathrm{F}}$ is chosen uniformly at random from $\Phi_{n,\mathbf{d}}$, so we can conclude

$$\mathbb{E}[D^{(2)}] \to \mathbb{E}[D]\,.$$

The factorial moments converge, again because we only need to consider the contribution from ordered tuples of non-overlapping loops and double arcs. Hence, the probability that $\vec{\mathrm{F}}$ is simple, when $(\vec{\mathrm{F}}, \mathcal{A}, \mathcal{A}')$ is chosen uniformly at random from $\widetilde{\Phi}_{n,\mathbf{d}}$, is

$$\exp\left(-\frac{m_2 - 2m + n}{m} - \frac{(m_2 - m)^2}{2m^2}\right) \tag{3.56}$$

Combining (3.38) and (3.56) gives the claimed estimate for $\mathbb{E}(\mathcal{A}_{n,\mathbf{d}})^2$. □

3.3.2 The asymptotic distribution of $\mathcal{A}_{n,d}$

The primary goal in this section is to prove that $\vec{\mathcal{T}}_{n,\mathbf{d}}$ is concentrated around its mean. By the BEST Theorem, this is equivalent to proving a concentration result for $\mathcal{A}_{n,\mathbf{d}}$. The obvious first step is to try to apply Chebyshev's inequality (Theorem 3.8). Suppose we want to show

$$\mathbb{P}\{\mathcal{A}_{n,\mathbf{d}} \leq n^{-\alpha}\,\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]\} \to 0\,.$$

We can bound this probability by

$$\begin{aligned}\mathbb{P}\{\mathcal{A}_{n,\mathbf{d}} \leq n^{-\alpha}\,\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]\} &\leq \mathbb{P}\left\{|\,\mathcal{A}_{n,\mathbf{d}} - \mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]| \geq (1 - n^{-\alpha})\,\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]\right\} \\ &\leq \left(\frac{n^{\alpha}-1}{n^{\alpha}}\right)^2 \frac{\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}^2] - \mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]^2}{\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]^2}\,,\end{aligned}$$

where the last step is obtained by applying Chebyshev with $a = (1 - n^{-\alpha})\,\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]$, and using the fact that $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ for any random variable X. Then, by Theorem 3.30, we can deduce that this upper bound is, asymptotically, equal to

$$\frac{e^{-n/m}m}{m-n} - 1\,.$$

This will be a constant, since we are assuming that $d_v \leq d$ for each $v \in V$. Thus, to prove the result we want, we will need something stronger.

In this section, we show how we can apply Janson's Theorem (Theorem 3.19) to determine the asymptotic distribution of $\mathcal{A}_{n,d}$ (the special case where $d_v = d$ for all $v \in V$). In our application of Theorem 3.19 we will have $Y_n = \mathcal{A}_{n,d}$, and $X_{i,n}$ counting the number of directed i-cycles in $\sigma(\vec{\mathrm{F}})$ for $\vec{\mathrm{F}}$ chosen uniformly at random from $\Phi_{n,d}$.

In the following two lemmas (Lemma 3.31 and Lemma 3.32), we establish that Condition 1 and Condition 2 of Theorem 3.19 hold when we take $X_{i,n}$ to count directed i-cycles in a random $\vec{F} \in \Phi_{n,\mathbf{d}}$ and Y_n to be equal to $\mathcal{A}^{\star}_{n,\mathbf{d}}$.

Lemma 3.31. *For each fixed positive integer i, let $X_{i,n}$ count the number of directed i-cycles in a directed graph $\vec{G}$ chosen uniformly at random from $\vec{\mathbb{G}}(n,\mathbf{d})$, and let $\lambda_i = \lambda^i/i$, where $\lambda = m_2/m$. Then, for any fixed set of integers $j_2,\dots,j_k$ we have*

$$\mathbb{E}\left[\prod_{i=2}^{k}(X_i)_{j_i}\right] \to \prod_{i=1}^{k}\lambda_i^{j_i}\,.$$

Proof. We say a set of i edges $e_1,e_2,\dots,e_i$ in a directed configuration is an i-cycle if there is a sequence of distinct vertices $v_0,v_1,\dots,v_{i-1}$ such that $e_j \in S_{v_j} \times T_{v_{j+1 \bmod i}}$ for $j = 0,1,\dots,i-1$. Let $\mathcal{C}_{i,\mathbf{d}}$ denote the set of all possible i-cycles for a particular out-degree vector $\mathbf{d}$. For each $C \in \mathcal{C}_{i,\mathbf{d}}$, we define an indicator variable I_C for the event $C \in \vec{\mathrm{F}}$, so $X_{i,n} = \sum_{C\in\mathcal{C}_{i,n}} I_C$. Then, by linearity of expectation, we can write $\mathbb{E}[X_{i,n}]$ as

$$\begin{aligned}\mathbb{E}[X_{i,n}] &= \sum_{C\in\mathcal{C}_{i,\mathbf{d}}} \mathbb{E}[I_C] \\ &= \sum_{C\in\mathcal{C}_{i,\mathbf{d}}} \mathbb{P}[C \in \vec{\mathrm{F}}]\,. \end{aligned} \tag{3.57}$$

There are $(m-i)!$ configurations containing each i-cycle, and, therefore, for any i-cycle C,

$$\mathbb{P}[C \in \vec{\mathrm{F}}] = \frac{(m-i)!}{m!} \to \frac{1}{m^i}\,.$$

So, to estimate $\mathbb{E}[X_{i,n}]$, we only need to compute $|\mathcal{C}_{i,\mathbf{d}}|$ and then divide by m^i. Let $V = \{1,2,\dots,n\}$ and suppose S is an i-subset of V. There are $(i-1)!$ different ways to arrange S into an i-cycle $(v_0,v_1,\dots,v_{i-1})$. Given a particular i-cycle $(v_0,v_1,\dots,v_{i-1})$ on S, there are $\prod_{v\in S} d_v^2$ ways to choose edges $e_j \in S_{v_j} \times T_{v_{(j+1) \bmod i}}$, for $j = 0,1,\dots,i-1$. Hence, (3.57) becomes

$$\mathbb{E}[X_{i,n}] \to \frac{(i-1)!}{m^i} \sum_{\substack{S\subset V\\ |S|=i}} \prod_{v\in S} d_v^2\,. \tag{3.58}$$

Now, observe that

$$\begin{aligned}\left(\frac{m_2}{m}\right)^i &= \frac{1}{m^i}\left(\sum_{v\in V}(d_v)^2\right)^i \\ &= \frac{1}{m^i}\sum_{\sum s_v = i}\binom{i}{s_v : v\in V}\prod_{v\in V} d_v^{2s_v}\,. \end{aligned} \tag{3.59}$$

Consider the contribution to (3.59) from terms in which $s_v = 2$ for some $v \in V$. This can be crudely bounded as

$$\frac{(i)_2}{2m^i} \sum_{v \in V} (d_v)^4 \left(\sum_{u \in V \setminus \{v\}} d_u^2 \right)^{i-2} \leq \frac{(i)_2}{2} d^4 m^{-1} \to 0 .$$

Similarly, the contribution from all terms with $s_v = j > 2$ for some $v \in V$ can be shown to be $O(m^{i-j+1})$. Hence, we have

$$\frac{i!}{m^i} \sum_{\substack{S \subset V \\ |S|=i}} \prod_{v \in S} (d_v)^2 \to \left(\frac{m_2}{m} \right)^i . \tag{3.60}$$

Plugging the approximation given in (3.60) into (3.58) gives

$$\mathbb{E}[X_{i,n}] \to \frac{1}{i} \left(\frac{m_2}{m} \right)^i = \lambda_i .$$

Suppose we have a sequence of fixed non-negative integers $j_1, j_2, \ldots, j_k$. To evaluate the factorial moment

$$\mathbb{E}\left[\prod_{i=1}^{m} (X_{i,n})_{j_i} \right] \tag{3.61}$$

we need to enumerate ordered tuples of $j = j_1 + j_2 + \cdots + j_k$ directed cycles where the first j_1 cycles are loops, the next j_2 are 2-cycles, and so on. By Lemma 3.29, we know that the probability of any particular partial configuration with more edges than vertices occurring in a random $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ tends to 0. Hence, the only tuples of directed cycles which have non-negligible contribution to (3.61) are those in which the cycles are vertex-disjoint. Given a set $S \subset V$ with $|S| = j$, there are

$$\frac{j!}{\prod_{i=1}^{k} i^{j_i}} \prod_{v \in S} (d_v)^2$$

ways to choose configurations for an ordered tuple of vertex-disjoint j-cycles, with j_i being the number of i-cycles. The probability of any one of these sets of cycles occurring in a random configuration is, asymptotically, $1/m^j$. Hence, the contribution to (3.61) from tuples of vertex-disjoint cycles, is, asymptotically,

$$\frac{j!}{m^j \prod_{i=1}^{k} i^{j_i}} \sum_{\substack{S \subset V \\ |S|=j}} \prod_{v \in S} (d_v)^2 \tag{3.62}$$

By the same arguments used to prove (3.60), we can show that (3.62) is (asymptotically) equal to

$$\frac{1}{\prod_{i=1}^{k} i^{j_i}} \left(\frac{m_2}{m} \right)^j = \prod_{i=1}^{k} \lambda_i^{j_i} . \qquad \square$$

Lemma 3.32. *For each i, let $X_{i,n}$ be as in Lemma 3.31, and let $\mu_i = (\lambda^i - 1)/i$, where $\lambda = m_2/m$. Then, for any fixed set of integers $j_1, j_2, \ldots, j_k$ we have*

$$\frac{\mathbb{E}\left[\mathcal{A}^\star_{n,\mathbf{d}} \prod_{i=1}^k (X_{i,n})_{j_i}\right]}{\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}}]} \to \prod_{i=1}^k \mu_i^{j_i}.$$

Proof. We first establish

$$\frac{\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}} X_{i,n}]}{\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}}]} \to \mu_i.$$

The factorial moments then hold for the same reason as was given in Lemma 3.31: the contribution from ordered tuples of loops and double arcs with repeated vertices is asymptotically insignificant.

Let $\overline{\Phi}_{n,\mathbf{d}}$ be the set of pairs $(\vec{\mathrm{F}}, \mathcal{A})$ where $\vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}$ and $\mathcal{A}$ is an arborescence of $\vec{\mathrm{F}}$. We let $\mathcal{C}_i(F)$ and $\mathcal{C}_{i,\mathbf{d}}$ have the same meaning as in the proof of Lemma 3.31. Then, as we did for $\mathbb{E}[X_{i,n}]$, we can expand

$$\begin{aligned}
\frac{\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}} X_{i,n}]}{\mathbb{E}[\mathcal{A}^\star_{n,\mathbf{d}}]} &= \frac{1}{|\overline{\Phi}_{n,\mathbf{d}}|} \sum_{(\vec{\mathrm{F}},\mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}} \sum_{C \in \mathcal{C}_{i,\mathbf{d}}} I_C(F) \\
&= \frac{|\{(\vec{\mathrm{F}}, \mathcal{A}, C) : \vec{\mathrm{F}} \in \Phi_{n,\mathbf{d}}, \mathcal{A} \in \mathrm{ARB}(\vec{\mathrm{F}}), C \in \mathcal{C}_i(F)|}{|\overline{\Phi}_{n,\mathbf{d}}|} \\
&= \sum_{C \in \mathcal{C}_{i,\mathbf{d}}} \mathbb{P}[C \subset \vec{\mathrm{F}} : (\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}].
\end{aligned}$$

Hence, we only need to estimate the probability that a particular $C \in \mathcal{C}_{i,\mathbf{d}}$ is contained in $\vec{\mathrm{F}}$ when $(\vec{\mathrm{F}}, \mathcal{A})$ is chosen uniformly at random from $\overline{\Phi}_{n,\mathbf{d}}$. To do this, we enumerate the elements of $\overline{\Phi}_{n,\mathbf{d}}$ with $C \subset \vec{\mathrm{F}}$. This is done by first choosing which of the edges of C are going to be in $\mathcal{A}$, then choosing the remaining edges for $\mathcal{A}$, and finally choosing the remaining edges for $\vec{\mathrm{F}}$.

Let S denote the set of vertices of C and let $R \subseteq S$ be the vertices $v \in S$ for which the arc leaving v on C is not contained in $\mathcal{A}$. Then, $C \cap \mathcal{A}$ consists of a set of disjoint directed paths, each one ending at some $v \in R$. For each $v \in R$, let P_v denote the directed path in $C \cap \mathcal{A}$ ending at v. Choosing the remainder of $\mathcal{A}$ is then equivalent to choosing an arborescence on $(V \setminus S) \cup R$, where we have collapsed each path to a single vertex. Each $v \in V \setminus S$ has d_v points for arcs entering and leaving v. For each $v \in R$, the number of points available for arcs entering v is equal to the number of points in P_v not used by C. Therefore, for each $v \in R$, there are $\sum_{u \in P_v}(d_u - 1)$ points for arcs entering v and $d_v - 1$ points for arcs leaving v. Using Lemma 3.27, we can deduce that the number of ways to choose the remainder of $\mathcal{A}$, and then $\vec{\mathrm{F}}$, is

$$n \frac{\prod_{v \in R}(d_v - 1)}{\prod_{v \in S} d_v} \left(\prod_v d_v\right) (m - i - 1)!.$$

Summing over R and dividing by $|\overline{\Phi}_{n,\mathbf{d}}|$, which we know from Theorem 3.28, gives

$$\begin{aligned}\mathbb{P}[C \subset \vec{\mathrm{F}} : (\vec{\mathrm{F}}, \mathcal{A}) \in \overline{\Phi}_{n,\mathbf{d}}] &\to \frac{1}{m^i} \sum_{\substack{R \subseteq S \\ |R| \geq 1}} \frac{\prod_{v \in R}(d_v - 1)}{\prod_{v \in S} d_v} \\ &= \frac{1}{m^i}\left(1 - \frac{1}{\prod_{v \in S} d_v}\right).\end{aligned}$$

For each $S \subset V$ with $|S| = i$, there are $(i-1)!\prod_{v \in S} d_v^2$ ways to form an i-cycle on S. Hence,

$$\frac{\mathbb{E}[\mathcal{A}^{\star}_{n,\mathbf{d}} X_{i,n}]}{\mathbb{E}[\mathcal{A}^{\star}_{n,\mathbf{d}}]} \to \frac{(i-1)!}{m^i} \sum_{\substack{S \subset V \\ |S|=i}} \left(\prod_{v \in S} d_v^2 - \prod_{v \in S} d_v\right)$$

By a similar argument as was used in Lemma 3.31 we can show

$$\sum_{\substack{S \subset V \\ |S|=i}} \prod_{v \in S} d_v \to \frac{m^i}{i!},$$

and so conclude

$$\frac{\mathbb{E}[\mathcal{A}^{\star}_{n,\mathbf{d}} X_{i,n}]}{\mathbb{E}\,\mathcal{A}^{\star}_{n,\mathbf{d}}} \to \frac{1}{i}\left(\frac{m_2}{m}\right)^i - \frac{1}{i} = \mu_i. \qquad \square$$

We now have sufficient ammunition to apply Janson's theorem and obtain an asymptotic distribution for the number of arborescences of a random d-in/d-out graph.

Theorem 3.33. *Let $d \geq 2$ be some fixed integer, let $n \in \mathbb{N}$, and let $\mathcal{A}_{n,d}$ denote the number of arborescences in a random directed graph $\vec{G}$ chosen uniformly at random from $\vec{\mathbb{G}}(n,d)$. Then,*

$$\frac{\mathcal{A}_{n,d}}{\mathbb{E}[\mathcal{A}_{n,d}]} \to \prod_{i=2}^{\infty}\left(1 - \frac{1}{d^i}\right)^{Z_i} e^{1/i},$$

where the Z_i are independent Poisson random variables with means d^i/i.

Proof. Let $X_{i,n}$ be the random variable counting i-cycles studied in Lemma 3.31 and Lemma 3.32. To apply Janson's Theorem (Theorem 3.19), we need to show that the four conditions in the hypothesis of the theorem are satisfied by $\mathcal{A}_{n,d}$ and $\{X_{i,n} : i \geq 2\}$. Restricted to the case where $d_u = d_v = d$ for all u, v, Lemma 3.31 provides Condition 1, by showing that the random variables $X_{i,n}$ converge jointly to independent Poisson random variables with means

$$\lambda_i = \frac{d^i}{i}.$$

Lemma 3.32 tells us that the random variables $X_{i,n}$, taken with $\mathcal{A}^{\star}_{n,d}$, satisfy

$$\frac{\mathbb{E}\left[\mathcal{A}^{\star}_{n,\mathbf{d}}\prod_{i=1}^{k}(X_{i,n})_{j_i}\right]}{\mathbb{E}[\mathcal{A}^{\star}_{n,\mathbf{d}}]} \to \prod_{i=1}^{k}\mu_i^{j_i},$$

where

$$\mu_i = \frac{d^i-1}{i}.$$

Hence, by Lemma 3.22, Condition 2 is satisfied for $Y_n = \mathcal{A}^{\star}_{n,d}$ with

$$\delta_i = -\frac{1}{d^i}.$$

Since the $X_{i,n}$ are all considered to be independent, we can condition on $X_{1,n} = 0$ to infer that Condition 2 of Theorem 3.19 is also satisfied, with the same δ_i values, when we take $Y_n = \mathcal{A}_{n,d}$ and $X_{i,n}$, $i \geq 2$. Evaluating the sum in Condition 3 gives

$$\sum_{i=2}^{\infty}\frac{1}{id^i} = -\frac{1}{d} + \log\left(\frac{d}{d-1}\right),$$

since

$$\log\left(\frac{d}{d-1}\right) = \log\left(\frac{1}{1-\frac{1}{d}}\right) = \sum_{i=1}^{\infty}\frac{1}{id^i}.$$

Finally, Theorem 3.30 provides Condition 4. □

The short cycle conditioning method fails when we allow vertices of different degrees. In general, we have

$$\frac{\mathbb{E}[\mathcal{A}^2_{n,\mathbf{d}}]}{\mathbb{E}[\mathcal{A}_{n,\mathbf{d}}]^2} = e^{-n/m}\frac{m}{m-n}.$$

Consider the class of graphs containing $n/2$ vertices of out-degree 2 and $n/2$ vertices of out-degree 3, which we denote $\vec{\mathbb{G}}(n,2,3)$. If $\mathcal{A}_{n,2,3}$ counts the number of Euler tours of G chosen uniformly at random from $\vec{\mathbb{G}}(n,2,3)$ then, by Theorem 3.30, we have

$$\frac{\mathbb{E}[\mathcal{A}^2_{n,2,3}]}{\mathbb{E}[\mathcal{A}_{n,2,3}]^2} \to e^{-2/5}\frac{5}{3} \approx 1.172,$$

since we have $m = 5n/2$. From Lemma 3.31 and Lemma 3.32 we obtain

$$\lambda_i = \frac{1}{i}\left(\frac{13}{5}\right)^i \quad \text{and} \quad \delta_i = -\frac{1}{i}\left(\frac{5}{13}\right)^i.$$

Hence,

$$\sum_{i=2}^{\infty}\lambda_i\delta_i^2 = -\frac{5}{13} + \log(13/8) \approx 1.106,$$

and therefore Condition 4 of Theorem 3.19 is not satisfied. Proofs using the small subgraph conditioning method depend strongly on the fact that these values match, since it is this (asymptotic) equality that implies the variance within each class is negligible. We do not have a good explanation for why this breaks down when we allow different degrees, nor do we believe it to be strong evidence that a concentration result cannot be obtained for a more general case than $d_u = d_v = d$. Indeed, we can still conclude that a large proportion of directed graphs with fixed degree sequence have a large number of arborescences.

3.3.3 Bounding $\mathcal{T}_{n,d}$ with high probability

By Theorem 3.30 of §3.3.1, we know that the expected number of arborescences of a random d-in/d-out graph is

$$\mathbb{E}[\mathcal{A}_{n,d}] \to e^1 d^{n-1}\,.$$

Hence, if we let $\mathcal{T}_{n,d}$ denote the number of Euler tours of a random d-in/d-out graph we have

$$\mathbb{E}[\mathcal{T}_{n,d}] = n^{-1}\,\mathbb{E}[\mathcal{A}_{n,d}](d-1)!^n \to \frac{e^1}{dn}(d!)^n\,.$$

Let $\mathrm{TS}(n,d)$ denote the number of transition systems of a d-in/d-out graph. There are $d!$ ways to choose a pairing on in-arcs with out-arcs at each vertex, so

$$\mathrm{TS}(n,d) = (d!)^n\,.$$

Hence, to prove Theorem 3.2, we just need to show that the number of arborescences of a random d-in/d-out graph is concentrated around $\mathbb{E}[\mathcal{A}_{n,d}]$ (Theorem 3.35 below). We prove Theorem 3.35 by arguing along the lines of [36]. In the following proof we will use a Lemma from [77], (which was used in the proof of [36]).

Lemma 3.34 (Lemma 3 [77])**.** *Let $\eta_1, \eta_2, \ldots$ be given, Suppose that $\eta_1 > 0$ and, for some $c > 0$, $\eta_{i+1}/\eta_i > c$, for all $i \geq 1$. Then, uniformly over $x \geq 1$,*

$$\sum_{i=1}^{\infty} \sum_{t=\eta_i(1+y_i)}^{\infty} \frac{\eta_i^t}{t!e^{\eta_i}} = O(e^{-c_0 x})\,,$$

where $y_i = x\eta_i^{-1/3}$ and $c_0 = \min\{\eta_1^{1/3}, \eta_1^{2/3}\}/4$

Theorem 3.35. *Let $d \geq 2$ be some fixed integer, let $n \in \mathbb{N}$, and suppose $\vec{G}$ is a directed graph chosen uniformly at random from $\vec{\mathbb{G}}(n,d)$. Then,*

$$\mathbb{P}\left[\frac{|\mathrm{ET}(\vec{G})|}{|\mathrm{TS}(\vec{G})|} \geq \frac{e^1}{d} n^{-2}\right] \to 1\,,$$

as $n \to \infty$.

Proof. Let $\mathcal{A}_{n,d}$ denote the number of arborescences of a directed graph $\vec{G}$ chosen uniformly at random from $\vec{\mathbb{G}}(n,d)$. We have shown, in Theorem 3.30, that the expectation of $\mathcal{A}_{n,d}$ satisfies

$$\mathbb{E}[\mathcal{A}_{n,d}] \to e^1 d^{n-1}\,.$$

The number of Euler tours of any directed graph $\vec{G} \in \mathbb{G}(n,d)$ is equal to

$$\frac{1}{n}|\operatorname{ARB}(\vec{G})|(d-1)!^n\,,$$

and the number of transition systems of $\vec{G}$ is equal to $d!^n$. Hence, the statement of the theorem is equivalent to saying

$$\mathbb{P}\left[\frac{\mathcal{A}_{n,d}}{\mathbb{E}[\mathcal{A}_{n,d}]} \geq \frac{1}{n}\right] \to 1\,,$$

as $n \to \infty$.

For $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ we define $\mathcal{G}_{\mathbf{x}}$ to be the set of all d-regular Eulerian directed graphs containing exactly x_i directed cycles of length i for each $i \leq k$, and

$$W^{(k)}(\mathbf{x}) = \prod_{i=2}^{k} \left(1 - \frac{1}{d^i}\right)^{x_i} e^{1/i}\,.$$

For each fixed $\gamma > 0$ we define

$$S(\gamma) = \{\mathbf{x} : x_i \leq \lambda_i + \gamma\lambda_i^{2/3} \text{ for } 2 \leq i \leq k\}\,.$$

From Lemma 3.31, the probability that $\vec{G}$ chosen uniformly at random from $\vec{\mathbb{G}}(n,d)$ has exactly x_i i-cycles is

$$\frac{\lambda_i^{x_i}}{e^{\lambda_i} x_i!}\,.$$

Hence, by Lemma 3.34, the probability that a random $\vec{G}$ is not contained in $\mathcal{G}_{\mathbf{x}}$ for some $\mathbf{x} \in S(\gamma)$ is

$$\begin{aligned}
\sum_{\mathbf{x} \notin S(\gamma)} \mathbb{P}[\vec{G} \in \mathcal{G}_{\mathbf{x}}] &\leq \sum_{i=2}^{\infty} \mathbb{P}[X_{i,n} \geq \lambda_i + \gamma\lambda_i^{2/3}] \\
&= \sum_{i=2}^{\infty} \sum_{t=\lambda_i(1+\gamma\lambda_i^{-1/3})}^{\infty} \frac{\lambda_i^t}{e^{\lambda_i} t!} \\
&\in e^{-a\gamma},
\end{aligned}$$

where $a = d^{1/3}/4$; i.e., a is an absolute constant greater than 0. Hence, to verify the theorem all we need do is show that

$$W^{(k)}(\mathbf{x}) \geq e^{-(b+c\gamma)} \quad \forall \mathbf{x} \in S(\gamma)\,, \tag{3.63}$$

where b and c are absolute constants independent of γ. Then, by Theorem 3.33, we have

$$\mathbb{P}\left[\frac{\mathcal{A}_{n,d}}{\mathbb{E}[\mathcal{A}_{n,d}]} \geq e^{-(b+c\gamma)}\right] \geq 1 - e^{-a\gamma}.$$

For any $\varepsilon > 0$, we can choose γ so that $e^{-a\gamma} < \varepsilon$. Then, we can choose n large enough so that $e^{-(b+c)\gamma} > n^{-1}$. That is, if we can show (3.63) holds, we will have shown that for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\mathbb{P}\left[\frac{\mathcal{A}_{n,d}}{\mathbb{E}[\mathcal{A}_{n,d}]} \geq \frac{1}{n}\right] > 1 - \varepsilon\,,$$

So, it remains to prove (3.63). For $x \in S(\gamma)$ we have $W^{(k)}(\mathbf{x}) \geq AB^{\gamma}$, where

$$A = \prod_{i=2}^{k}\left(1 - \frac{1}{d^i}\right)^{\lambda_i} e^{1/i}$$
$$B = \prod_{i=2}^{k}\left(1 - \frac{1}{d^i}\right)^{\lambda_i^{2/3}}.$$

Using the elementary inequality $1 - x \geq e^{-x/(1-x)}$, we can bound A and B as

$$\begin{aligned} A &\geq \prod_{i=2}^{\infty} \exp\left(-\frac{d^i}{i(d^i-1)} + \frac{1}{i}\right) \\ &= \exp\left(\sum_{i=2}^{k} -\frac{1}{i(d^i-1)}\right), \end{aligned}$$

and

$$\begin{aligned} B &\geq \prod_{i=2}^{\infty} \exp\left(-\frac{d^{2i/3}}{i^{2/3}(d^i-1)}\right) \\ &\geq \exp\left(-\sum_{i=2}^{\infty} \frac{1}{i^{2/3}d^{i/3}}\right). \end{aligned}$$

The sums inside the exponentials are clearly convergent, so we can find absolute constants b and c so that $A \geq e^{-b}$ and $B \geq e^{-c}$. □

3.4 The number of Eulerian orientations of random regular graphs

In this section we turn our attention to the distribution of the number of Eulerian orientations of a random regular graph. Recalling the relationship between the number of orbs and Euler tours, we would hope that this will take us closer to our main goal: obtaining an asymptotic distribution for the number of Euler tours of a random regular graph. Although we did not manage to achieve this, we believe the analysis in this section, particularly the analysis of the second moment of the number of Eulerian orientations of a random regular graph, will be a key part of the analysis required to obtain the corresponding result for orbs. Moreover, as the number of Eulerian orientations is a quantity of interest in its own right, the results of this section are of independent interest.

In this section we restrict our attention to regular graphs. The reason for this is that we cannot find a closed form for the number of Eulerian orientations of a random configuration $\mathrm{F} \in \Omega_{n,\mathbf{d}}$, as we could for arborescences of $\Phi_{n,\mathbf{d}}$, and the asymptotic analysis carried out in the proof of Theorem 3.47 does not immediately carry over to the general configuration model.

For fixed positive integer d, and $n \in \mathbb{N}$, we define $\mathcal{E}_{n,2d}$ to be the random variable counting Eulerian orientations of a random $2d$-regular graph. In §3.4.2 we analyse the number of Eulerian orientations of a random $2d$-regular graph, obtaining asymptotic results for constant d. As we did for arborescences in §3.3, we first analyse the moments of an auxiliary variable, $\mathcal{E}^{\star}_{n,2d}$, which counts Eulerian orientations of random configurations $\mathrm{F} \in \Omega_{n,2d}$, and then condition on F projecting to a simple graph. The asymptotic analysis of §3.4.2 uses *Laplace's method* for estimating integrals. We briefly describe the details of this method in §3.4.1.

We apply these estimates in two ways. Firstly, in §3.4.3, we use these estimates and Theorem 3.19 to characterise the *asymptotic distribution* of $\mathcal{E}_{n,2d}$. Then, in §3.4.4, we use Chebyshev's inequality (Theorem 3.8) to show that *almost every* $2d$-regular graph has *few* Eulerian orientations; that is, we show that the probability the number of Eulerian orientations of a random $G \in \mathbb{G}(n,2d)$ exceeds Schrijver's lower bound [78] (see Theorem 1.56) by more than a linear factor tends to 0 as n goes to infinity.

In the previous section, the asymptotic distribution of the number of arborescences was used to show a simple algorithm for sampling Euler tours runs in expected polynomial time for almost every directed graph $\vec{G} \in \vec{G}(n,d)$. We remark that we

cannot obtain a similar result here. Firstly, it is not clear if the set of Eulerian orientations of a graph G sit nicely inside some other set of orientations, apart from the set of all orientations of G. Moreover, our results imply that most regular graphs have few Eulerian orientations, i.e., close to the minimum possible, so even if we had a "nice" set of orientations that the Eulerian orientations sat inside, it is still unlikely that the relative size of the set of Eulerian orientations (of a particular graph) within this set would be large enough for a dart-throwing approach to work.

3.4.1 Laplace's Method

In our analysis of the second moment of the number of Eulerian orientations, we will need to compute an asymptotic estimate of an integral. The approach we use is *Laplace's method* [25, Chapter 4], also known as *saddle-point analysis*. This is a general approach for estimating an integral of the form $\int e^{nf(\mathbf{x})}d\mathbf{x}$ whose value is concentrated around a single maximum term. The first step in an application of this method is to locate the maximum.

Definition 3.36. Let $f(x)$ be a real-valued function. We say f has a *critical point* at a, if the value of the first derivative of f at a is 0: $f'(a) = 0$.

A *critical point* a is a local maximum if and only if the second derivative of f is negative at a, $f''(a) < 0$, and a critical point a is a local minimum if and only if the second derivative is positive at a, $f''(a) > 0$.

The local maxima and minima of a multi-dimensional function can be defined in a similar way. First, we need to define *positive definite* and *negative definite* matrices.

Definition 3.37. We say an $n \times n$ symmetric, real-valued matrix A is *negative definite* if

$$\mathbf{x}^T A\mathbf{x} < 0\,, \quad \forall \mathbf{x} \in \mathbb{R}^n\,.$$

We say A is *positive definite* if $-A$ is *negative definite*.

Definition 3.38. Let $f(x_1, x_2, \ldots, x_d)$ be an d-ary real-valued function. A point $\mathbf{a} \in \mathbb{R}^d$ is a *critical point* of f if all first order partial derivatives of f are 0 at $\mathbf{a}$:

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) = 0\,, \quad \text{for } i = 1, 2, \ldots, d\,.$$

We define the *Hessian matrix* of second order partial derivatives of f:

$$D^2 f(\mathbf{x})_{i,j} = \frac{\partial f}{\partial x_i x_j}\,, \quad \text{for } i, j = 1, 2, \ldots, d\,.$$

A critical point $\mathbf{a}$ is a local maximum of f if and only if $D^2 f(\mathbf{a})$ is a negative definite matrix, and a local minimum if and only if $D^2 f(\mathbf{a})$ is a positive definite matrix.

To apply Laplace's method to estimate an integral, $\int e^{nf(\mathbf{x})} d\mathbf{x}$, we search for the local maxima on the range of integration and consider the value of $f(\mathbf{x})$ in the neighbourhoods of these points. We want to show that the asymptotic value of the integral is determined by the value within a small neighbourhood of the maximum value $\mathbf{a}$, and that in this neighbourhood $f(\mathbf{x})$ can be approximated by a *Gaussian integral*; i.e., an integral of the form

$$\int_{\mathbf{R}^n} e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x}} d\mathbf{x}, \tag{3.64}$$

for positive definite $n \times n$ matrix A.

Definition 3.39. Let A be an $n \times n$ matrix. The determinant of A, denoted $|A|$, is the quantity

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)},$$

where S_n is taken to be the set of all permutations of $\{1,2,\ldots,n\}$. The *signature* of a permutation is $+1$ (resp. -1) if σ can be obtained from the identity permutation,

$$\sigma(i) = i, \text{ for all } i \in \{1,2,\ldots,n\},$$

by performing an even (resp. odd) number of *transpositions* of adjacent numbers.

Example 3.40. The determinant of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $ad - bc$. This is the only determinant we will deal with explicitly in the proofs of this section.

The following result, which can be found in, e.g., [25, Chapter 4], shows how we can compute the value of (3.64) by computing the determinant of A.

Theorem 3.41. *Let A be a positive definite $n \times n$ matrix. Then,*

$$\int_{\mathbf{R}^n} e^{-\frac{1}{2}\mathbf{x}^T A \mathbf{x}} d\mathbf{x} = \frac{(2\pi)^{d/2}}{\sqrt{|A|}},$$

where $|A|$ denotes the determinant of A.

3.4.2 Estimating the moments of $\mathcal{E}_{n,d}$

Our goal in this section is to find asymptotic estimates of the first and second moments of the number of Eulerian orientations of a random graph $G \in \mathbb{G}(n,2d)$. We start by estimating the moments of $\mathcal{E}^{\star}_{n,2d}$, the number of Eulerian orientations of $\sigma(\mathrm{F})$ where F is chosen uniformly at random from $\Omega_{n,2d}$. These estimates can then be turned into estimates for the moments of $\mathcal{E}_{n,2d}$ by conditioning on F not containing any loops or double edges.

Before proving the theorem, we provide some technical lemmas. We make no claim to the novelty of the identities presented in Lemma 3.44 or Lemma 3.45 but include proofs here due to not knowing a suitable reference.

For a square matrix A we denote the determinant of A by $|A|$ and write $A[\alpha,\beta]$ for the submatrix with rows indexed by α and columns indexed by β. In Lemma 3.44, we show how to compute the determinant of a sum of two matrices, using Laplace's formula. Recall the definition of a matrix minor.

Definition 3.42. Let M be a $n \times n$ matrix. For any row and column indices i and j, we denote by M_{ij} the matrix obtained by removing the row indexed by i and the column indexed by j. This is known as the (i,j)-*minor* of M. We define the (i,j)-*cofactor* of M to be $(-1)^{i+j}|M_{ij}|$.

One approach to computing determinants of matrices, is to use *Laplace Expansion*, which expresses the value of the determinant of a matrix in terms of its cofactors.

Fact 3.43 (Laplace, see Chapter 5 of [84])**.** Let A be an $n \times n$ matrix and let a_{ij} denote the entry in row i and column j. Then,

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|,$$

for any $j \in \{1,2,\ldots,n\}$ (expanding along column j) and

$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|,$$

for any $i \in \{1,2,\ldots,n\}$ (expanding along row i).

Lemma 3.44. *Let A,B be $n \times n$ square matrices. Then*

$$|A+B| = \sum_{k=0}^{n} \sum_{\alpha,\beta \in S_{n,k}} (-1)^{s(\alpha)+s(\beta)} |A[\alpha,\beta]||B[\bar{\alpha},\bar{\beta}]|, \tag{3.65}$$

where $S_{n,k}$ is the set of k-subsequences of $(1,2,\ldots,n)$, $\bar{\alpha}$ is the subsequence obtained by removing α from $(1,2,\ldots,n)$, and $s(\alpha) = \sum \alpha_i$.

Proof. We prove by induction. For $n = 2$ the lemma can be checked by expanding all the determinants involved. Let $n > 2$ and suppose the lemma is true for all $k < n$. Using Fact 3.43 to expand $|A+B|$ gives

$$|A+B| = \sum_{i=1}^{n} (-1)^{i+1}(a_{1,i}+b_{1,i})|A_{1,i}+B_{1,i}|\,.$$

Each term in the sum is itself a sum of two $(n-1)\times(n-1)$ matrices, and so, by induction, we can write

$$|A+B| = \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{\alpha,\beta\in S_{n,k}\\ \alpha\ni 1,\beta\ni i}} (-1)^{s(\alpha)+s(\beta)+|\{j\in\beta: j<i\}|}(a_{1,i}+b_{1,i})|A[\alpha_{-1},\beta_{-i}]||B[\bar{\alpha},\bar{\beta}]|\,, \tag{3.66}$$

where γ_{-k} is the sequence γ with k removed. We will now explain where the terms in (3.66) come from. Consider the expansion of $|A_{1,i}+B_{1,i}|$ by applying the Lemma:

$$|A_{1,i}+B_{1,i}| = \sum_{k=0}^{n-1}\sum_{\substack{\alpha',\beta'\in S_{n,k}\\ 1\notin\alpha', i\notin\beta'}} (-1)^{s_1(\alpha')+s_i(\beta')}|A[\alpha',\beta']||B[\bar{\alpha'}_{-1},\bar{\beta'}_{-i}]|\,,$$

where $s_i(\gamma) = s(\gamma) - |\{j\in\gamma : j > i\}$. The functions s_i come from the fact that any $j > 1$ in α' is actually indexing the $(j-1)$-th row of $A_{1,i}+B_{1,i}$, and any $j > i$ in β' is actually indexing the $(j-1)$-th column of $A_{1,i}+B_{1,i}$. For the sake of presentation, we have mapped the subsequences α' and β' used in the inductive application of (3.65) to $|A_{1,i}+B_{1,i}|$ to subsequences α and β of $(1,2,\ldots,n)$ satisfying $1\in\alpha$ and $i\in\beta$. By considering this mapping, we can see that the power of -1 in the term corresponding to α and β in the inductive application of (3.65) to $|A_{1,i}+B_{1,i}|$ is

$$\begin{aligned}(-1)^{s(\alpha)-k+s(\beta)-i-|\{j\in\beta:j>i\}|} &= (-1)^{s(\alpha)-k+s(\beta)-i-(k-1-|\{j\in\beta:j<i\}|)}\\ &= (-1)^{s(\alpha)+s(\beta)-i+1+|\{j\in\beta:j<i\}|}\,.\end{aligned} \tag{3.67}$$

Multiplying (3.67) by $(-1)^{i+1}$ gives the power of -1 used in (3.66).

We will now show the equivalence of (3.66) and the RHS of (3.65) by showing that we can obtain (3.66) by expanding each term in the RHS of (3.65) using Laplace expansion. Each term in (3.65) is expanded as follows: if $1\in\alpha$ then we expand $|A[\alpha,\beta]|$ along row 1; if $1\notin\alpha$ then we expand $|B[\bar{\alpha},\bar{\beta}]|$ along row 1.

Let $i\in[n]$ and suppose we have a pair of sequences α and β with $1\in\alpha$ and $i\in\beta$. We obtain the term $a_{1,i}A[\alpha_{-1},\beta_{-i}]B[\bar{\alpha},\bar{\beta}]$ by expanding $A[\alpha,\beta]B[\bar{\alpha},\bar{\beta}]$ and obtain the term $b_{1,i}A[\alpha_{-1},\beta_{-i}]B[\bar{\alpha},\bar{\beta}]$ by expanding $A[\alpha_{-1},\beta_{-i}]B[\overline{\alpha_{-1}},\overline{\beta_{-i}}]$, as described above.

Thus, the terms in the sum obtained by expanding each term of (3.65) are identical to the terms in the RHS of (3.66). It remains to show that each term has the correct sign.

We first consider terms of the form $a_{1,i}|A[\alpha_{-1},\beta_{-i}]||B[\bar{\alpha},\bar{\beta}]|$. Using Fact 3.43, expanding $|A[\alpha,\beta]|$ along row 1 gives

$$\begin{aligned}(-1)^{s(\alpha)+s(\beta)}|A[\alpha,\beta]| &= \sum_{i\in\beta}(-1)^{s(\alpha)+s(\beta)+|\{j\in\beta:j\le i\}|+1}a_{1,i}|A[\alpha_{-1},\beta_{-i}]| \\ &= \sum_{i\in\beta}(-1)^{s(\alpha)+s(\beta)+|\{j\in\beta:j<i\}|}a_{1,i}|A[\alpha_{-1},\beta_{-i}]|\,, \end{aligned} \tag{3.68}$$

so $a_{1,i}|A[\alpha_{-1},\beta_{-i}]||B[\bar{\alpha},\bar{\beta}]|$ has the same sign in (3.66) and the sum obtained by applying Laplace expansion to each term in (3.65)..

Next, we consider terms of the form $b_{1,i}|A[\alpha_{-1},\beta_{-i}]||B[\bar{\alpha},\bar{\beta}]|$. Again using Fact 3.43, we expand $|B[\overline{\alpha_{-1}},\overline{\beta_{-i}}]|$ along row 1. Consider the term in this expansion involving $b_{1,i}$:

$$(-1)^{s(\alpha_{-1})+s(\beta_{-i})+1+|\{j\notin\beta:j\le i\}|}b_{1,i}|B[\bar{\alpha},\bar{\beta}]|\,. \tag{3.69}$$

The sign of $b_{1,i}|B[\bar{\alpha},\bar{\beta}]|$ in (3.69) is

$$(-1)^{s(\alpha)+s(\beta)-i+|\{j\notin\beta:j\le i\}|}\,.$$

Note that

$$|\{j\notin\beta : j\le i\}|+|\{j\in\beta : j<i\}| = i\,;$$

this implies that $|\{j\in\beta : j<i\}|$ is even if and only if $-i+|\{j\notin\beta : j\le i\}$ is even. Hence, $b_{1,i}|A[\alpha_{-1},\beta_{-i}]||B[\bar{\alpha},\bar{\beta}]|$ has the same sign in (3.66) and the sum obtained by applying Laplace expansion to each term in (3.65). □

Lemma 3.45. *For each positive integer d,*

$$\begin{aligned}\sum_{i=0}^{d}\binom{d}{i}^2 &= \binom{2d}{d}\,;\\ \sum_{i=1}^{d} i\binom{d}{i}^2 &= \frac{d}{2}\binom{2d}{d}\,;\\ \sum_{i=1}^{d} i^2\binom{d}{i}^2 &= \frac{d^3}{2(2d-1)}\binom{2d}{d}\,.\end{aligned}$$

Proof. We prove the first identity by observing that the number of ways to choose d values from $\{1,2,\ldots,2d\}$, is the same as the total number of ways to choose i values from $\{1,2,\ldots,d\}$ and $d-i$ values from $\{d+1,d+2,\ldots,2d\}$, for $i=0,1,\ldots,d$. Hence,

$$\binom{2d}{d} = \sum_{i=0}^{d}\binom{d}{i}\binom{d}{d-i} = \sum_{i=0}^{d}\binom{d}{i}^2\,.$$

The other two follow by straightforward calculations:

$$\sum_{i=1}^{d} i\binom{d}{i}^2 = d\sum_{i=1}^{d}\binom{d-1}{i-1}\binom{d}{d-i} = d\binom{2d-1}{d-1} = \frac{d}{2}\binom{2d}{d};$$
$$\sum_{i=1}^{d} i^2\binom{d}{i}^2 = d^2\sum_{i=0}^{d-1}\binom{d-1}{i} = \frac{d^3}{2(2d-1)}\binom{2d}{d}. \qquad \square$$

We will now use Lemma 3.44 and Lemma 3.45 to prove the following Lemma, which gives some properties of a particular matrix. This matrix will arise in the proof of Theorem 3.47 as the Hessian matrix of a function.

Lemma 3.46. *Suppose d is an integer greater than 1 and let M be the $d \times d$ symmetric matrix given by*

$$M_{ij} = \begin{cases} \frac{4ij}{d} - \binom{2d}{d} & i \neq j \\ \frac{4i^2}{d} - \binom{2d}{d} - \frac{\binom{2d}{d}}{\binom{d}{i}^2} & i = j \end{cases}$$

This matrix is negative definite and has determinant

$$|M| = (-1)^d \frac{\binom{2d}{d}^{d+1}}{\prod_{i=1}^{d}\binom{d}{i}^2}\frac{d-1}{2d-1}.$$

Proof. We first show that M is negative definite. Recall that a symmetric real-valued matrix is negative definite if and only if $\mathbf{x}^T M\mathbf{x} < 0$ for every $\mathbf{x} \in \mathbb{R}^d$. In other words, we can prove M is negative definite by showing the following strict inequality holds for all $\mathbf{x} \in \mathbb{R}^d$:

$$\left(\sum_{i=1}^{d} 2ix_i\right)^2 < d\binom{2d}{d}\left(\sum_{i=1}^{d} x_i\right)^2 + d\binom{2d}{d}\sum_{i=1}^{d}\frac{x_i^2}{\binom{d}{i}^2}.$$

We will use the following elementary inequality,

$$(a+b)^2 \leq \frac{2d-1}{d}a^2 + \frac{2d-1}{d-1}b^2,$$

which holds for any a and b. Then, by writing

$$a = \sum_{i=0}^{d}(2i-d)x_i, \text{ and } b = \sum_{i=0}^{d} dx_i,$$

we have

$$\left(\sum_{i=1}^{d} 2ix_i\right)^2 \leq \frac{2d-1}{d}\left(\sum_{i=0}^{d}(2i-d)x_i\right)^2 + \frac{d^2(2d-1)}{d-1}\left(\sum_{i=0}^{d} x_i\right)^2.$$

Next, we apply the Cauchy-Schwarz inequality [82],

$$\left(\sum_{i=1}^{d} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{d} a_i^2\right)\left(\sum_{i=1}^{d} b_i^2\right),$$

giving

$$\left(\sum_{i=1}^{d}(2i-d)x_i\right)^2 \leq \left(\sum_{i=1}^{d}(2i-d)^2\binom{d}{i}^2\right)\left(\sum_{i=1}^{d}\frac{x_i^2}{\binom{d}{i}^2}\right).$$

We can use Lemma 3.45 to show

$$\sum_{i=0}^{d}(2i-d)^2\binom{d}{i}^2 = \frac{d^2}{2d-1}\binom{2d}{d}.$$

Hence, we have shown

$$\left(\sum_{i=1}^{d} 2ix_i\right)^2 \leq \frac{d^2(2d-1)}{d-1}\left(\sum_{i=1}^{d} x_i\right)^2 + d\binom{2d}{d}\sum_{i=1}^{d}\frac{x_i^2}{\binom{d}{i}^2}.$$

Finally, we observe that for any $d \geq 2$, we have $\frac{d^2(2d-1)}{d-1} < d\binom{2d}{d}$.

We now turn our attention to evaluating the determinant. To evaluate $|M|$ using Lemma 3.44, we write M as $A+B$ where

$$A_{i,j} = \begin{cases} -\frac{\binom{2d}{d}}{\binom{d}{i}^2} & i = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$B_{i,j} = \frac{4ij}{d} - \binom{2d}{d}.$$

We first show that the determinant of every square sub-matrix of B with more than 2 rows is 0. Recall that a square matrix has determinant 0 if we can write any row as a linear combination of other rows [84]. Let B_k denote the k-th row of B,

$$B_k = \left(\frac{4k}{d} - \binom{2d}{d}, \frac{8k}{d} - \binom{2d}{d}, \ldots, \frac{4dk}{d} - \binom{2d}{d}\right).$$

It is straightforward to see that B_k can be written as the sum

$$B_k = \frac{k-j}{i-j}B_i + \frac{i-k}{i-j}B_j,$$

for any i and j. Hence, any square submatrix of B with more than two rows will have determinant 0. Moreover, the fact that A is a diagonal matrix means that $|A_{\alpha,\beta}| \neq 0$ if

and only if $\alpha = \beta$. Thus, using Lemma 3.44 and Lemma 3.45, we can compute the determinant of M as follows

$$\begin{aligned}
|M| &= \left(\prod_{j=1}^{d} A_{jj}\right)\left(1+\sum_{j=1}^{k}\frac{B_{jj}}{A_{jj}}+\sum_{1\le i<j\le d}\frac{(B_{ii}B_{jj}-B_{ij}^2)}{A_{ii}A_{jj}}\right)\\
&= (-1)^d\left(\frac{\binom{2d}{d}^d}{\prod_{i=1}^{d}\binom{d}{i}^2}\right)\left(\sum_{j=0}^{d}\binom{d}{j}^2-\frac{2}{d\binom{2d}{d}}\sum_{0\le i,j\le d}(j-i)^2\binom{d}{i}^2\binom{d}{j}^2\right)\\
&= (-1)^d\left(\frac{\binom{2d}{d}^d}{\prod_{i=1}^{d}\binom{d}{i}^2}\right)\left(\binom{2d}{d}-\frac{d}{2d-1}\binom{2d}{d}\right)\\
&= (-1)^d\frac{\binom{2d}{d}^{d+1}}{\prod_{i=1}^{d}\binom{d}{i}^2}\frac{d-1}{2d-1}\,.
\end{aligned}$$

□

In the following theorem, Theorem 3.47, we estimate the first and second moment of the number of Eulerian orientations of G, when G is obtained as the projection of a random $\mathrm{F}\in\Omega_{n,2d}$. In the following, we will refer to Eulerian orientations of graphs and configurations. We define an Eulerian orientation of a configuration $\mathrm{F}\in\Omega_{n,2d}$ to be a partitioning of W into a pair of equally sized disjoint subsets S and T, in which each W_v is partitioned into a pair of d-element subsets, S_v and T_v, such that each edge in F has one endpoint in S and the other in T. For each vertex v, S_v corresponds to the endpoints of arcs for which v is the *source* and T_v corresponds to the endpoints of arcs for which v is the *target*. We will use $\mathrm{EO}(\mathrm{F})$ to denote the set of Eulerian orientations of F. The following proof, and the proofs of several subsequent theorems, use Stirling's formula [25] to approximate factorials: as $n\to\infty$,

$$n!\to\left(\frac{n}{e}\right)^2\sqrt{2\pi n}\,. \tag{3.70}$$

Theorem 3.47. *Let $n\in\mathbb{N}$ and let d be some fixed positive integer satisfying $d\ge 2$. Let $\mathcal{E}^\star_{n,2d}$ be the number of Eulerian orientations in a random $2d$-regular multigraph obtained as the projection of a uniformly random $\mathrm{F}\in\Omega_{n,2d}$. Then, as $n\to\infty$,*

$$\mathbb{E}[\mathcal{E}^\star_{n,2d}]\to\left(\frac{\binom{2d}{d}}{2^d}\right)^n\sqrt{\pi dn}\,;$$

$$\frac{\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]^2}\to\sqrt{\frac{2d-1}{2d-2}}\,.$$

Proof. We first evaluate $\mathbb{E}[\mathcal{E}^\star_{n,2d}]$. The value of $\mathbb{E}[\mathcal{E}^\star_{n,2d}]$ is given by

$$\mathbb{E}[\mathcal{E}^\star_{n,2d}]=\frac{|\overline{\Omega}_{n,2d}|}{|\,\Omega_{n,2d}\,|}\,,$$

where

$$\overline{\Omega}_{n,2d} = \{(\mathrm{F}, \mathcal{E}) : \mathrm{F} \in \Omega_{n,2d}, \mathcal{E} \in \mathrm{EO}(\mathrm{F})\}.$$

Let $V = \{1, 2, \ldots, n\}$. We can enumerate the elements of $\overline{\Omega}_{n,2d}$ as follows: for each vertex $v \in V$ partition the set W_v into two equally sized sets S_v and T_v; choose a perfect matching from $S = \bigcup_{v \in V} S_v$ to $T = \bigcup_{v \in V} T_v$. There are

$$\binom{2d}{d}^n$$

different ways to choose the sets S and T and $(dn)!$ ways to choose a perfect matching from S to T, for each choice of S and T. Hence, the number of elements in $\overline{\Omega}_{n,2d}$ is

$$|\overline{\Omega}_{n,2d}| = \binom{2d}{d}^n (dn)!.$$

Dividing this by the number of configurations in $\Omega_{n,2d}$,

$$|\,\Omega_{n,2d}\,| = \binom{2dn}{dn} (dn)! 2^{-dn},$$

and applying Stirling's formula (3.70) we obtain

$$\mathbb{E}[\mathcal{E}^\star_{n,2d}] = \frac{\binom{2d}{d}^n 2^{dn}}{\binom{2dn}{dn}} \to \left(\frac{\binom{2d}{d}}{2^d} \right)^n \sqrt{\pi dn}.$$

We now turn our attention to computing $\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2] / \mathbb{E}[\mathcal{E}^\star_{n,2d}]^2$; in fact, we only compute an asymptotic upper bound. The corresponding lower bound follows from the results in §3.4.3 and the comments in Remark 3.20. The second moment of $\mathcal{E}^\star_{n,2d}$ can be written as

$$\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2] = \frac{|\widetilde{\Omega}_{n,2d}|}{|\,\Omega_{n,2d}\,|},$$

where

$$\widetilde{\Omega}_{n,2d} = \{(\mathrm{F}, \mathcal{E}, \mathcal{E}') : \mathrm{F} \in \Omega_{n,2d}, \mathcal{E}, \mathcal{E}' \in \mathrm{EO}(\mathrm{F})\}$$

We now describe how we can enumerate $\widetilde{\Omega}_{n,2d}$. Any pair of Eulerian orientations $(\mathcal{E}, \mathcal{E}')$ of a graph G defines a decomposition of the edges of G into two sets: $\mathcal{E} \cap \mathcal{E}'$, the edges that have the same orientation in $\mathcal{E}$ and $\mathcal{E}'$; and $\mathcal{E} \oplus \mathcal{E}'$, the edges that are oriented differently in $\mathcal{E}$ and$\mathcal{E}'$. Given this decomposition and the orientation $\mathcal{E}$, we can immediately determine the orientation $\mathcal{E}'$. Hence, to enumerate triples $(\mathrm{F}, \mathcal{E}, \mathcal{E}')$, we start by partitioning W into S and T, as we did for the first moment. Then, for each $v \in V$ we choose an equal number of vertices from S_v and T_v to be swapped between the two sets. This gives two new sets S' and T', where S'_v corresponds to endpoints

of arcs for which v is the source in $\mathcal{E}'$ and T'_v corresponds to the endpoints of arcs for which v is the target in $\mathcal{E}'$. Choosing $(\mathrm{F}, \mathcal{E}, \mathcal{E}')$ is then equivalent to choosing two perfect matchings, one from $S \cap S'$ to $T \cap T'$ and one from $S \backslash S'$ to $T \backslash T'$.

We start by choosing the out-degree in $\mathcal{E} \cap \mathcal{E}'$ for each $v \in V$; that is, we choose the number of points that will be contained in $S_v \cap S'_v$. Suppose $d_v = |S_v \cap S'_v|$. The number of ways to choose S and T is, as before,

$$\binom{2d}{d}^n.$$

Given S and T we choose, for each $v \in V$, a subset of size d_v from each of S_v and T_v; these will be the points in $S_v \cap S'_v$ and $T_v \cap T'_v$, respectively. The number of ways in which we can do this is

$$\prod_{v \in V} \binom{d}{d_v}^2.$$

Once we have chosen S, T, S', and T' the number of ways to choose a pair of perfect matchings from $S \cap S'$ to $T \cap T'$ and from $S \backslash S'$ to $T \backslash T'$ is

$$\left(\sum_{v \in V} d_v\right)! \left(dn - \sum_{v \in V} d_v\right)!.$$

For each $i = 0, 1, \ldots, d$, we define $V_i = \{v \in V : |S_v \cap S'_v| = i\}$ and let $c_i = |V_i|$. Then, the number of ways to choose $\mathcal{E}$ and $\mathcal{E}'$, satisfying $|S_v \cap S'_v| = d_v$ for all $v \in V$, is

$$\binom{2d}{d}^n \left(\prod_{i=0}^{d} \binom{d}{i}^{2c_i}\right) \left(\sum_{i=1}^{d} ic_i\right)! \left(dn - \sum_{i=1}^{d} ic_i\right)!. \tag{3.71}$$

Summing over all choices for d_v gives $|\overline{\Omega}_{n,2d}|$, and then dividing by the total number of configurations in $\Omega_{n,d}$ gives an expression for $\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]$:

$$\begin{aligned}
\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2] =& \frac{\binom{2d}{d}^n}{\binom{2dn}{dn}(dn)!2^{-dn}} \\
&\times \sum_{c_0 + c_1 + \cdots + c_d = n} \binom{n}{c_0, c_1, \ldots, c_d} \left(\prod_{i=0}^{d} \binom{d}{i}^{2c_i}\right) (\sum_{i=1}^{d} ic_i)!(dn - \sum_{i=1}^{d} ic_i)! \\
=& \frac{\binom{2d}{d}^n 2^{dn}}{\binom{2dn}{dn}} \sum_{c_0 + c_1 + \cdots + c_d = n} \left(\prod_{i=0}^{d} \binom{d}{i}^{2c_i}\right) \frac{n!(\sum_{i=1}^{d} ic_i)!(dn - \sum_{i=1}^{d} ic_i)!}{(dn)! \prod_{i=0}^{d} c_i!}.
\end{aligned} \tag{3.72}$$

By setting $c_0 = n - \sum_{i=1}^{d} c_i$ and applying Stirling's formula (3.70) to the factorials, $(dn)!$, $(\sum ic_i)!$, and $(dn - \sum ic_i)!$, we can approximate the summand corresponding to

$\mathbf{c} = (c_1, c_2, \ldots, c_d)$ in (3.72) by

$$\left(\prod_{i=0}^{d}\binom{d}{i}^{2c_i}\right)\frac{n^n(\sum_{i=1}^d ic_i)^{\sum_{i=1}^d ic_i}(dn-\sum_{i=1}^d ic_i)^{dn-\sum_{i=1}^d ic_i}}{(dn)^{dn}(n-\sum_{i=1}^d c_i)^{n-\sum_{i=1}^d c_i}\prod_{i=1}^d c_i^{c_i}} \times\left(\frac{(\sum_{i=1}^d ic_i)(dn-\sum_{i=1}^d ic_i)}{(2\pi)^{d-1}d(n-\sum_{i=1}^d c_i)\prod_{i=1}^d c_i}\right)^{1/2}. \tag{3.73}$$

We can also use Stirling's formula (3.70) to approximate the first part of (3.72) as

$$\left(\frac{\binom{2d}{d}}{2^d}\right)^n\sqrt{\pi dn}.$$

Next, we divide the top and bottom of the upper term in (3.73) by $n^{dn+1}$to get

$$\left[\left(\prod_{i=1}^{d}\binom{d}{i}^{2c_i/n}\right)\frac{(\sum_{i=1}^d ic_i/n)^{\sum_{i=1}^d ic_i/n}(d-\sum_{i=1}^d ic_i/n)^{d-\sum_{i=1}^d ic_i/n}}{d^d(1-\sum_{i=1}^d c_i/n)^{1-\sum_{i=1}^d c_i/n}\prod_{i=1}^d(c_i/n)^{c_i/n}}\right]^n.$$

Similarly, we replace c_i by c_i/n in the lower term to get

$$\left(\frac{(\sum_{i=1}^d ic_i/n)(d-\sum_{i=1}^d ic_i/n)}{(2\pi)^{d-1}dn^{d-1}(1-\sum_{i=1}^d c_i/n)\prod_{i=1}^d(c_i/n)}\right)^{1/2}.$$

Hence, we can write

$$\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2] \to \left(\frac{\binom{2d}{d}}{2^d}\right)^{2n}\frac{1}{(2\pi n)^{d/2-1}}\sum_{c_1+c_2+\cdots c_d\le n} e^{ng(\mathbf{c}/n)}h(\mathbf{c}/n), \tag{3.74}$$

where

$$\begin{aligned} g(\mathbf{x}) =& d\log(2) - d\log(d) - \log\binom{2d}{d} \\ &+ \left(\sum_{i=1}^d ix_i\right)\log\left(\sum_{i=1}^d ix_i\right) + \left(d-\sum_{i=1}^d ix_i\right)\log\left(d-\sum_{i=1}^d ix_i\right) \\ &+ \sum_{i=1}^d x_i\log\binom{d}{i}^2 - \sum_{i=1}^d x_i\log(x_i) - \left(1-\sum_{i=1}^d x_i\right)\log\left(1-\sum_{i=1}^d x_i\right). \end{aligned}$$

and

$$h(\mathbf{x}) = \left(\frac{(\sum_{i=1}^d x_i)(d-\sum_{i=1}^d x_i)}{2(\prod_{1\le i\le d}x_i)(1-\sum_{i=1}^d x_i)}\right)^{1/2}.$$

As we are only looking for an asymptotic upper bound on $\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]/\mathbb{E}[\mathcal{E}^\star_{n,2d}]^2$, we reduce our problem to estimating the following upper bound

$$\frac{\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]^2} \le \frac{2}{d(2\pi n)^{d/2}}\int_{\mathbf{x}\in S(d)} e^{ng(\mathbf{x})}h(\mathbf{x})d\mathbf{x}, \tag{3.75}$$

where $S(d) = \{\mathbf{x} \in [0,1]^d : |\mathbf{x}| \leq 1\}$; we ensure $g(\mathbf{x})$ and $h(\mathbf{x})$ are both defined for all of $S(d)$ by setting $g(\mathbf{x}) = h(\mathbf{x}) = 0$ if any of the logarithms would evaluate to $\log(0)$ (or any of the terms in the denominator of $h(\mathbf{x})$ are 0). We observe that the contribution to individual terms of the integral in (3.75) from $h(\mathbf{x})$ is very small in comparison with $e^{ng(\mathbf{x})}$. Hence, the value of the integral is dominated by the terms where $g(\mathbf{x})$ is close to the maxima of g on $S(d)$.

We will now estimate the integral in (3.75) using Laplace's method. First, we show that $g(\mathbf{x})$ has a unique local maximum at a point $\mathbf{a}$; hence, the value of $e^{ng(\mathbf{x})}$ is negligible, except in a small neighbourhood of $\mathbf{a}$. By expanding $g(\mathbf{x})$ around $\mathbf{a}$ using a Taylor series, we are able to approximate (3.75) by a Gaussian integral. We can then evaluate this using Theorem 3.41 and Lemma 3.46 to obtain an asymptotic estimate for (3.75).

First, we find the unique local maximum of g in $S(d)$. We write $g(\mathbf{x})$ as the sum

$$d\log(2) - \log\binom{2d}{d} - d\log(d) + g_1(\mathbf{x}) + g_2(\mathbf{x}),$$

where

$$g_1(\mathbf{x}) = \left(\sum_{i=1}^{d} ix_i\right)\log\left(\sum_{i=1}^{d} ix_i\right) + \left(d - \sum_{i=1}^{d} ix_i\right)\log\left(d - \sum_{i=1}^{d} ix_i\right),$$

and

$$g_2(\mathbf{x}) = \sum_{i=1}^{d} x_i \log\binom{d}{i}^2 - \sum_{i=1}^{d} x_i\log(x_i) - \left(1 - \sum_{i=1}^{d} x_i\right)\log\left(1 - \sum_{i=1}^{d} x_i\right).$$

The value of $g_1(\mathbf{x})$ is

$$m\log(m) + (d-m)\log(d-m)$$

for $m = \sum_i ix_i$, and, hence, the maxima of g_1 are given by the set of points satisfying

$$\sum_{i=1}^{d} ix_i = d - \sum_{i=1}^{d} ix_i. \tag{3.76}$$

Taking the partial derivatives of g_2 gives

$$\frac{\partial g_2}{\partial x_i} = \log\binom{d}{i}^2 - \log x_i + \log\left(1 - \sum_{j=1}^{d} x_j\right),$$

for each $i = 1,2,\ldots,d$. Hence, any critical point of g_2 must satisfy

$$x_i = \binom{d}{i}^2\left(1 - \sum_{j=1}^{d} x_j\right), \quad i = 1,2,\ldots,d \tag{3.77}$$

This is a system of d equations in d unknowns, so g_2 has a unique critical point. Using Lemma 3.45, we can verify that a particular point $\mathbf{a}$, defined by

$$\mathbf{a} = \left(\frac{\binom{d}{i}^2}{\binom{2d}{d}} : i = 1 \dots d \right),$$

satisfies (3.77). Hence, $\mathbf{a}$ is the unique critical point of g_2.

Recall the condition for $\mathbf{a}$ to be a local maximum of g_2: we must have that the Hessian matrix, $D^2 g_2(\mathbf{a})$, of second order partial derivatives evaluated at $\mathbf{a}$ is negative definite. The second order derivatives of $g_2(\mathbf{x})$ are

$$\frac{\partial g}{\partial x_i x_j} = -\frac{1}{1-\sum_{i=1}^d x_i}, \qquad i \neq j, \tag{3.78}$$

$$\frac{\partial g}{\partial x_i x_j} = -\frac{1}{1-\sum_{i=1}^d x_i} - \frac{1}{x_i} \qquad i = j. \tag{3.79}$$

Evaluating (3.78) and (3.79) at $\mathbf{a}$ gives the values for the Hessian matrix

$$D^2 g_2(\mathbf{a})_{i,j} = \begin{cases} -\binom{2d}{d} & i \neq j \\ -\binom{2d}{d} - \frac{\binom{2d}{d}}{\binom{d}{i}^2} & i = j \end{cases}.$$

$D^2 g_2(\mathbf{a})$ is certainly negative definite, so $\mathbf{a}$ is the unique local maximum of g_2 on $S(d)$.

Moreover, since $\mathbf{a}$ satisfies (3.76), $\mathbf{a}$ is also a local maximum of g_1. Recall that $g(\mathbf{x})$ is equal to the sum of $g_1(\mathbf{x})$ and $g_2(\mathbf{x})$ (plus some terms not dependent on $\mathbf{x}$). Since $\mathbf{a}$ is the unique point which is a local maximum of both functions, it follows that $\mathbf{a}$ must be the unique local maximum of $g(\mathbf{x})$. Furthermore, using Lemma 3.45, we can see that $g(\mathbf{a}) = 0$.

Next we consider the region

$$\bar{S}(d) = \{\mathbf{x} \in [0,1]^d : |x_i - a_i| \leq n^{-1/2}\log(n), \text{ for each } i = 1, 2, \dots, d\}.$$

We note that within $\bar{S}(d)$, we have that $ng(\mathbf{x})$ is approximated by its second order Taylor expansion

$$ng(\mathbf{x}) \to \frac{n}{2}(\mathbf{x}-\mathbf{a})^T D^2 g(\mathbf{a})(\mathbf{x}-\mathbf{a}).$$

To see this consider the contribution from any third order term to $ng(\mathbf{x})$:

$$n(x_i - a_i)(x_j - a_j)(x_k - a_k)\frac{\partial g}{\partial x_i x_j x_k}(\mathbf{a}). \tag{3.80}$$

Within the region $\bar{S}(d)$, the absolute value of (3.80) is $O(n^{-1/2}\log(n)^3)$, which is equal to 0 in the limit.

Moreover, the contribution to (3.75) from points outside $\bar{S}(d)$ will be negligible. To see this, we note that the point $\mathbf{x}$ with $|x_i - a_i| \geq n^{-1/2}\log(n)$ that maximizes $g(\mathbf{x})$ will, necessarily, be the point closest to $\mathbf{a}$, i.e., one of the two points with $|x_i - a_i| = n^{-1/2}\log(n)$ and $x_j = a_j$ for $j \neq i$. Then, by considering the Taylor expansion of $g(\mathbf{x})$ around $\mathbf{a}$, we can see that, for all points with $x_i - a_i \geq n^{-1/2}\log(n)$, for some $i = 1, 2, \ldots, d$, we have

$$e^{ng(\mathbf{x})} \leq e^{-\gamma \log(n)^2},$$

where γ is some absolute constant independent of n. That is, in the limit, we will have $e^{ng(\mathbf{x})}h(\mathbf{x}) \to 0$ for $\mathbf{x} \notin \bar{S}(d)$. Hence, we can approximate (3.75) as

$$\frac{2}{d(2\pi n)^{d/2}} \int_{a_1 - n^{-1/2}\log(n)}^{a_1 + n^{-1/2}\log(n)} \cdots \int_{a_d - n^{-1/2}\log(n)}^{a_d + n^{-1/2}\log(n)} e^{ng(\mathbf{x})} h(\mathbf{x}) d\mathbf{x}. \tag{3.81}$$

We want to replace the integral in (3.81) by a Gaussian. To achieve this, we apply the change of variable

$$\mathbf{x} = \mathbf{a} + \mathbf{z}n^{-1/2},$$

and integrate over $\mathbf{z} \in [-\log(n), \log(n)]^d$. As $n \to \infty$, we have

$$h(\mathbf{a} + \mathbf{z}n^{-1/2}) \to h(\mathbf{a}) = \frac{d}{2} \left(\frac{\binom{2d}{d}^{d+1}}{2\prod_{i=0}^{d} \binom{d}{i}^2} \right)^{1/2},$$

and

$$g(\mathbf{a} + \mathbf{z}n^{-1/2}) \to \frac{1}{2}\mathbf{z}^T M \mathbf{z},$$

where $M = -D^2 g(\mathbf{a})$. Furthermore, we have

$$d\mathbf{x} = n^{d/2} d\mathbf{z}.$$

Hence, by dominated convergence[1], (3.81) is asymptotically equal to

$$\left(\frac{\binom{2d}{d}^{d+1}}{2\prod_{i=0}^{d} \binom{d}{i}^2} \right)^{1/2} \frac{1}{(2\pi)^{d/2}} \int_{[-\log(n), \log(n)]^d} e^{-\frac{1}{2}\mathbf{x}^T M \mathbf{z}} d\mathbf{z}. \tag{3.82}$$

[1] If f_n converges point-wise on S and is dominated by some integrable function F, then

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S \lim_{n\to\infty} f_n d\mu.$$

We can extend the range of integration to $\mathbb{R}^d$ without changing the asymptotic value of (3.82), since the contribution from all values outside of $[-\log(n), \log(n)]^d$ will be negligible, giving

$$\left(\frac{\binom{2d}{d}^{d+1}}{2\prod_{i=0}^{d}\binom{d}{i}^2}\right)^{1/2} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{1}{2}\mathbf{z}^T M \mathbf{z}} d\mathbf{z}. \tag{3.83}$$

Recall that $D^2 g(\mathbf{a})$ is a negative definite matrix. Hence, the integral in (3.83) is a d-dimensional Gaussian, which we can evaluate using Theorem 3.41 as

$$\frac{(2\pi)^{d/2}}{\sqrt{|M|}}.$$

From Lemma 3.46, we see that

$$|M| = \frac{\binom{2d}{d}^{d+1}}{\prod_{i=1}^{d}\binom{d}{i}^2} \frac{d-1}{2d-1}.$$

and so we can conclude that

$$\lim_{n\to\infty} \frac{\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]^2} \leq \sqrt{\frac{2d-1}{2d-2}}. \qquad \square$$

Remark 3.48. It would also be possible to obtain an exact asymptotic estimate for $\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]$ by using the Euler-Maclaurin summation formula [25, Chapter 3]. This is a technique that can be used to approximate certain, smooth, summations by integrals. The terms in the summation (3.74) vary smoothly enough for this to be applicable. However, the asymptotic upper bound computed in Theorem 3.47 is sufficient for our purposes, as we can obtain the matching asymptotic lower bound as a side-effect of our application of Theorem 3.19 in §3.4.3, thus avoiding the need to introduce more analytical machinery.

The next theorem estimates the expected number of Eulerian orientations in a random $2d$-regular graph. We use the same idea as the proof of Theorem 3.30, except we are now conditioning on there being no loops or double edges. However, where we had to do a tedious analysis of the number of loops and double arcs that can occur on configurations sampled in weighted distributions in the proof of Theorem 3.30, we can automatically infer Theorem 3.49 from the joint convergence of the random variables in Janson's Theorem (Theorem 3.19; see also Remark 3.21) and the results presented in the next section.

When we were analysing the moments of the number of arborescences of a random d-in/d-out graph, we needed to condition on a random variable not involved in the application of Janson's Theorem, the number of double arcs. Hence, we could not use the joint convergence of the random variables implied by Janson's Theorem to obtain Theorem 3.30.

Theorem 3.49. *Let $n \in \mathbb{N}$ and let d be some fixed positive integer. Let $\mathcal{E}_{n,2d}$ denote the number of Eulerian orientations of a graph G chosen uniformly at random from $\mathbb{G}(n,2d)$. Then*

$$\mathbb{E}[\mathcal{E}_{n,2d}] \to e^{-3/4}\left(\frac{\binom{2d}{d}}{2^d}\right)^n \sqrt{\pi d n}, \text{ and}$$

$$\frac{\mathbb{E}[(\mathcal{E}_{n,2d})^2]}{\mathbb{E}[\mathcal{E}_{n,2d}]^2} \to \exp\left(-\frac{1}{2(2d-1)} - \frac{1}{4(2d-1)^2}\right)\sqrt{\frac{2d-1}{2d-2}}.$$

Proof. Follows from Theorem 3.47, Remark 3.21, Theorem 3.15, and Lemma 3.51.

In §3.4.3 we will prove that $\mathcal{E}^\star_{n,2d}$ and $X_{i,n}$ (the random variables counting the number of i-cycles in $\sigma(\mathrm{F})$ when F is chosen uniformly at random from $\Omega_{n,2d}$) satisfy the conditions of Janson's Theorem (Theorem 3.19) with $\lambda_i = (2d-1)^i/2i$ and $\delta_i = (2d-1)^{-i}$. Hence, in light of Remark 3.21, we have

$$\frac{\mathbb{E}[\mathcal{E}_{n,2d}]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to \exp(-1/2 - 1/4),$$

and

$$\frac{\mathbb{E}[\mathcal{E}^2_{n,2d}]}{\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]} \to \exp\left(-\frac{3}{2} - \frac{1}{2(2d-1)} - \frac{1}{4(2d-1)^2}\right). \qquad \square$$

3.4.3 The asymptotic distribution of $\mathcal{E}_{n,2d}$

In this section we characterise the asymptotic distribution of the number of Eulerian orientations of a random $2d$-regular graph.

Theorem 3.50. *Let d be some fixed integer greater than 1, and let $\mathcal{E}_{n,2d}$ count the number of Eulerian orientations of a random $G \in \mathbb{G}(n,2d)$. Then*

$$\frac{\mathcal{E}_{n,2d}}{\mathbb{E}[\mathcal{E}_{n,2d}]} \to \prod_{i=3}^{\infty}\left(1+\frac{1}{(2d-1)^i}\right)^{Z_i} e^{-1/2i},$$

as $n \to \infty$, where Z_i are independent Poisson random variables with $\mathbb{E}[Z_i] = \frac{(2d-1)^i}{2i}$.

Although we do not have an immediate algorithmic application of this result, we believe it to be a milestone on the road to characterising the asymptotic distribution of the number of Euler tours of a random $2d$-regular graph, a result that would have algorithmic value (see §3.5).

To obtain an asymptotic distribution of the number of Eulerian orientations, $\mathcal{E}_{n,2d}$, we will use Janson's Theorem (Theorem 3.19). In our application of Janson's Theorem, we will have $Y_n = \mathcal{E}_{n,2d}$, and $X_{i,n}$ counting simple cycles of length i in random $G \in \mathbb{G}(n,2d)$. For Condition 1 of Theorem 3.19, we need to show that the random variables $X_{i,n}$ are asymptotic to Poisson random variables. It is a classic result in the theory of random graphs (due to Bollobás [11]; given as Theorem 3.15 in this thesis) that, for any fixed k, the random variables counting i-cycles, for $i \leq k$, are asymptotic to independent random variables with means

$$\lambda_i = \frac{(2d-1)^i}{2i}.$$

The following lemma (see also Lemma 3.22) establishes Condition 2 of Theorem 3.19.

Lemma 3.51. *Let d be some fixed positive integer, let $\mathcal{E}^\star_{n,2d}$ denote the number of Eulerian orientations in $\sigma(\mathrm{F})$ for $\mathrm{F} \in \Omega_{n,2d}$, and let $X_{i,n}$ denote the number of cycles of length i in $\sigma(\mathrm{F})$ when F is chosen uniformly at random from $\Omega_{n,2d}$. Then, for any fixed set of non-negative integers $j_1, j_2, \ldots, j_k$ we have*

$$\frac{\mathbb{E}\left[\mathcal{E}^\star_{n,2d}(X_{1,n})_{j_1}(X_{2,n})_{j_2}\cdots(X_{k,n})_{j_k}\right]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to \prod_{i=1}^{k}\mu_i^{j_i},$$

where $\mu_i = \frac{(2d-1)^i+1}{2i}$.

Proof. We start by establishing

$$\frac{\mathbb{E}[\mathcal{E}^\star_{n,2d}X_{i,n}]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to \mu_i \tag{3.84}$$

for $i \geq 1$. Convergence of the factorial moments then follows for similar reasons as before (see Lemma 3.29 or Lemma 3.16).

In this proof, we consider cycles in configurations $\mathrm{F} \in \Omega_{n,2d}$. We say a partial configuration $\mathrm{F}' \subseteq \mathrm{F}$ is a cycle, if the projection of F', $\sigma(\mathrm{F}')$, is a cycle in $\sigma(\mathrm{F})$. That is, for each $i \geq 1$, we call a set of edges $e_0, e_1, \ldots, e_{i-1}$ in $\mathrm{F} \in \Omega_{n,d}$ an i-cycle if there is a sequence of vertices $v_0, v_1, \ldots, v_{i-1}$ such that e_j has one endpoint in W_{v_j} and one in $W_{v_{j+1 \bmod i}}$ for $i = 0, 1, \ldots, i-1$. We will use $\mathcal{C}_i(\mathrm{F})$ to denote the set of i-cycles in F, and use $\mathcal{C}_{i,n}$ to denote the set of all i-cycles that can occur in some $\mathrm{F} \in \Omega_{n,2d}$.

Recalling the definition of $\overline{\Omega}_{n,2d}$ from Theorem 3.47, we can expand the left-hand side of (3.84) as

$$\begin{aligned}\frac{\mathbb{E}[\mathcal{E}^\star_{n,2d} X_{i,n}]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} &= \frac{1}{|\overline{\Omega}_{n,2d}|} \sum_{\mathrm{F}\in\Omega_{n,2d}} |\mathrm{EO}(\mathrm{F})||C_i(\mathrm{F})| \\ &= \sum_{\mathrm{F}\in\Omega_{n,2d}} \sum_{\mathcal{E}\in\mathrm{EO}(\mathrm{F})} \sum_{C\in\mathcal{C}_i(\mathrm{F})} \frac{1}{|\overline{\Omega}_{n,2d}|} \\ &= \sum_{C\in\mathcal{C}_{i,n}} \frac{|\{(\mathrm{F},\mathcal{E})\in\overline{\Omega}_{n,2d} : C\subseteq \mathrm{F}\}|}{|\overline{\Omega}_{n,2d}|} \\ &= |\mathcal{C}_{i,n}|\,\mathbb{P}[C\subseteq \mathrm{F} : (\mathrm{F},\mathcal{E})\in\overline{\Omega}_{n,2d}]\,,\end{aligned}$$

since the probability $\mathbb{P}[C\subseteq \mathrm{F} : (\mathrm{F},\mathcal{E})\in\overline{\Omega}_{n,2d}]$ is the same for each $C\in\mathcal{C}_{i,n}$. Hence, to prove (3.84), it suffices to estimate $|\mathcal{C}_{i,n}|$ and $\mathbb{P}[C\subseteq \mathrm{F} : (\mathrm{F},\mathcal{E})\in\overline{\Omega}_{n,2d}]$.

For $i\geq 3$, the number of ways to choose an i-cycle on V is $\frac{(n)_i}{2i}$, since there are $(n)_i$ ways to choose a sequence of i vertices, and sequences which are the reverse of one another or equivalent under cyclic permutation are considered to be the same cycle. For each i-cycle C, there are $((2d)_2)^i$ ways to choose a partial configuration that projects to C, so

$$|\mathcal{C}_{i,n}| = \frac{(n)_i(2d)^i(2d-1)^i}{2i} \to \frac{(n)^i(2d)^i(2d-1)^i}{2i}\,, \tag{3.85}$$

for $i\geq 3$. In fact, (3.85) also describes the number of elements in $\mathcal{C}_{i,n}$ for $i=1$ and $i=2$. There are n ways to choose a loop vertex v, and $\binom{2d}{2}$ ways to choose the endpoints for a loop at v. There are $\binom{n}{2}$ ways to choose a pair of vertices u and v and $2\binom{2d}{2}^2$ ways to choose a partial configuration projecting to a double edge between u and v.

To estimate $\mathbb{P}[C\subseteq \mathrm{F} : (\mathrm{F},\mathcal{E})\in\overline{\Omega}_{n,2d}]$ we need to count the number of elements of $\overline{\Omega}_{n,2d}$ with $C\subseteq \mathrm{F}$. Recall how we enumerate the elements of $\overline{\Omega}_{n,2d}$ when computing $\mathbb{E}[\mathcal{E}^\star_{n,2d}]$ in the proof of Theorem 3.47. First, we choose a partition of W into S and T by dividing each W_v into equally sized disjoint subsets S_v and T_v. Then, we choose a perfect matching from S to T. We follow the same approach here, by first assigning the endpoints of the edges in C to S and T and then choosing the remainder of S and T. A vertex v on the cycle C is said to be balanced if the edges incident with W_v on C contain one point from S_v and one point from T_v, negatively imbalanced if the edges contain two points from S_v and positively imbalanced if the edges contain two points from T_v. Positively and negatively imbalanced vertices must alternate around C; therefore there must be an equal number of each. Hence, assigning the endpoints of the edges in C to S and T is equivalent to choosing an even subset of the vertices of C to be imbalanced, and then fixing one vertex in that set to be positively imbalanced.

Given an assignment of the endpoints of the edges of C to S and T, we now want to partition the rest of the points in each W_v into $S_v \cup T_v$. If v is an imbalanced vertex of C, then we have already chosen two points from W_v to lie in S_v or in T_v, and, hence, there are $\binom{2d-2}{d-2}$ ways to complete the partitioning of W_v into $S_v \cup T_v$. If v is a balanced vertex on C then there are $\binom{2d-2}{d-1}$ ways to complete the partition of W_v into $S_v \cup T_v$, since we have already chosen one element of S_v and one element of T_v. Finally, if v is not on C there are $\binom{2d}{d}$ ways to partition W_v into $S_v \cup T_v$. Once we have chosen S and T, there are $(dn-i)!$ ways to choose the remainder of the perfect matching from S to T. Hence, the number of pairs $(\mathrm{F}, \mathcal{E}) \in \overline{\Omega}_{n,2d}$ where F contains a particular cycle $C \in \mathcal{C}_{i,n}$ with $2j$ imbalanced vertices is

$$2\binom{i}{2j}\binom{2d}{d}^{n-i}\binom{2d-2}{d-2}^{2j}\binom{2d-2}{d-1}^{i-2j}(dn-i)!\,.$$

We note that

$$\binom{2d-2}{d-2} = \frac{(d-1)}{2(2d-1)}\binom{2d}{d}, \text{ and } \binom{2d-2}{d-1} = \frac{d}{2(2d-1)}\binom{2d}{d}.$$

Hence, the number of pairs $(\mathrm{F}, \mathcal{E}) \in \overline{\Omega}_{n,2d}$ where F contains a particular cycle $C \in \mathcal{C}_{i,n}$ with $2j$ imbalanced vertices is

$$2\binom{2d}{d}^{n}(dn-i)!\left(\frac{d}{2(2d-1)}\right)^{i}\binom{i}{2j}\left(\frac{d-1}{d}\right)^{2j}.$$

Using the following identity

$$\sum_{j=0}^{\lfloor i/2\rfloor}\binom{i}{2j}x^{2j} = \frac{(1+x)^i}{2} + \frac{(1-x)^i}{2},$$

we see that

$$\sum_{j=0}^{\lfloor i/2\rfloor}\binom{i}{2j}\left(\frac{d-1}{d}\right)^{2j} = \frac{1}{2}\left(\frac{2d-1}{d}\right)^{i} + \frac{1}{2d^i}.$$

Hence, we can see that the number of pairs $(\mathrm{F}, \mathcal{E}) \in \overline{\Omega}_{n,2d}$ with $C \subseteq \mathrm{F}$ is equal to

$$\binom{2d}{d}^{n}\frac{(dn-i)!}{2^i}\left(1+\frac{1}{(2d-1)^i}\right).$$

Dividing by the number of elements in $\overline{\Omega}_{n,2d}$, which we know to be $\binom{2d}{d}^{n}(dn)!$, gives

$$\begin{aligned}\mathbb{P}[C \subseteq \mathrm{F} : (\mathrm{F}, \mathcal{E}) \in \overline{\Omega}_{n,2d}] &= \frac{1}{2^i(dn)_i}\left(1+\frac{1}{(2d-1)^i}\right)\\ &\to \frac{1}{2^i d^i n^i}\left(1+\frac{1}{(2d-1)^i}\right).\end{aligned}$$

Finally, multiplying by the number of elements of $C_{i,n}$, which is given in (3.85), we obtain

$$\frac{\mathbb{E}[\mathcal{E}^\star_{n,2d} X_{i,n}]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to \frac{(2d-1)^i + 1}{2i}.$$

The factorial moments converge as stated for similar reasons as were given in Lemma 3.29, see also, e.g, [100, Lemma 2.7] (given as Lemma 3.16 of this thesis). Suppose we have a graph H on r vertices with $r+s$ edges, for r and s fixed and $s > 0$. The number of ways to choose a partial configuration F' projecting to H is $O(n^r)$, and the probability of $\mathrm{F}' \subseteq \mathrm{F}$ when $(\mathrm{F}, \mathcal{E})$ is chosen uniformly at random from $\overline{\Omega}_{n,2d}$ is $O(n^{-r-s})$. Thus, we can conclude that the contribution to

$$\frac{\mathbb{E}\left[\mathcal{E}^\star_{n,2d} (X_{1,n})_{j_1} (X_{2,n})_{j_2} \cdots (X_{k,n})_{j_k}\right]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]}$$

from sets of cycles in which any vertex occurs on more than one cycle is $O(n^{-1})$, and so can be treated as 0 as $n \to 0$. Hence, cycles occur (asymptotically) independently and we can conclude

$$\frac{\mathbb{E}\left[\mathcal{E}^\star_{n,2d} (X_{1,n})_{j_1} (X_{2,n})_{j_2} \cdots (X_{k,n})_{j_k}\right]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to \prod_{i=1}^{k} \mu_i^{j_i}. \qquad \square$$

We will now prove Theorem 3.50 by applying Janson's Theorem (Theorem 3.19) with Lemma 3.51, Theorem 3.49, and Theorem 3.15 providing the required values.

Proof of Theorem 3.50. Our goal is to show

$$\frac{\mathcal{E}_{n,2d}}{\mathbb{E}[\mathcal{E}_{n,2d}]} \to \prod_{i=3}^{\infty} \left(1 + \frac{1}{(2d-1)^i}\right)^{Z_i} e^{-1/2i},$$

as $n \to \infty$, where Z_i are independent Poisson random variables with $\mathbb{E}[Z_i] = \frac{(2d-1)^i}{2i}$. We first apply Theorem 3.19 with $Y_n = \mathcal{E}^\star_{n,2d}$ (from Theorem 3.47) and $X_{i,n}$ the random variables counting i-cycles from Lemma 3.51 and Theorem 3.15 to obtain

$$\frac{\mathcal{E}^\star_{n,2d}}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to W = \prod_{i=1}^{\infty} \left(1 + \frac{1}{(2d-1)^i}\right)^{Z_i} e^{-1/2i},$$

as $n \to \infty$. The convergence of $\mathcal{E}^\star_{n,2d} / \mathbb{E}[\mathcal{E}^\star_{n,2d}]$ to W implied by Theorem 3.19 is joint with the convergence of the random variables $X_{i,n}$ to Z_i. Hence, the distribution of $\mathcal{E}^\star_{n,2d}$ conditioned on $X_{1,n} = X_{2,n} = 0$ is asymptotic to that of W conditioned on $Z_1 = Z_2 = 0$ (see Remark 3.21). Thus, it suffices to show that Theorem 3.19 holds with $Y_n = \mathcal{E}^\star_{n,2d}$.

To apply Theorem 3.19, we need to verify the conditions 1 to 4 (of Theorem 3.19). Theorem 3.15, due to Bollobás [11, 12], tells us that the $X_{i,n}$ converge jointly to Poisson random variables with means

$$\lambda_i = \frac{(2d-1)^i}{2i}.$$

Hence, Condition 1 holds.

Lemma 3.51 tells us that the random variables $X_{i,n}$, taken together with $\mathcal{E}^\star_{n,2d}$, satisfy, for any sequence of non-negative integers $j_1, j_2, \ldots, j_k$,

$$\frac{\mathbb{E}[\mathcal{E}^\star_{n,2d}(X_{1,n})_{j_1}(X_{2,n})_{j_2}\cdots(X_{k,n})_{j_k}]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]} \to \prod_{i=1}^{k}\mu_i^{j_i},$$

as $n \to \infty$. Hence, by Lemma 3.22, we see that Condition 2 is satisfied with

$$\delta_i = \frac{\mu_i}{\lambda_i} - 1 = \frac{1}{(2d-1)^i}.$$

For Condition 3, we evaluate

$$\sum_{i=1}^{\infty}\lambda_i\delta_i^2 = \frac{1}{2}\log\left(\frac{2d-1}{2d-2}\right),$$

since

$$\log\left(\frac{2d-1}{2d-2}\right) = \log\left(\frac{1}{1-\frac{1}{2d-1}}\right) = \sum_{i=1}^{\infty}\frac{1}{i(2d-1)^i}.$$

Finally, Theorem 3.49 gives

$$\lim_{n\to\infty}\frac{\mathbb{E}[(\mathcal{E}^\star_{n,2d})^2]}{\mathbb{E}[\mathcal{E}^\star_{n,2d}]^2} \le \sqrt{\frac{2d-1}{2d-2}},$$

so Condition 4 is also satisfied. □

3.4.4 Bounding $\mathcal{E}_{n,2d}$ with high probability

Recall that Schrijver ([78], given as Theorem 1.56 of this thesis) has shown that the number of Eulerian orientations in any $2d$-regular graph is at least

$$\left(\frac{\binom{2d}{d}}{2^d}\right)^n.$$

We can use Theorem 3.49 to infer that *almost every* $2d$-regular graph has *few* Eulerian orientations, in the sense that the number of Eulerian orientations is at most a linear factor larger than Schrijver's lower bound.

Theorem 3.52. *Let d be a fixed positive integer satisfying $d \geq 2$. Suppose G is a random $2d$-regular graph. Then, with probability tending to 1 as $n \to \infty$, we have*

$$\mathbb{P}\left[\#\mathrm{EO}(G) \leq O(n)\left(\frac{\binom{2d}{d}}{2^d}\right)^n\right] \to 1\,,$$

as $n \to \infty$.

Proof of Theorem 3.52. Let $\mathcal{E}_{n,2d}$ denote the number of Eulerian orientations of a graph chosen uniformly at random from $\mathbb{G}(n,2d)$. We have shown, in Theorem 3.49, that the expectation of $\mathcal{E}_{n,2d}$ satisfies

$$\mathbb{E}[\mathcal{E}_{n,2d}] \to e^{-3/4}\sqrt{\pi d n}\left(\frac{\binom{2d}{d}}{2^d}\right)^n.$$

Hence, the statement of the theorem is equivalent to saying

$$\mathbb{P}\left\{\mathcal{E}_{n,2d} \leq \sqrt{n}\,\mathbb{E}[\mathcal{E}_{n,2d}]\right\} \to 1\,,$$

as $n \to \infty$.

Recall Chebyshev's inequality (Theorem 3.8): for any random variable X,

$$\mathbb{P}\{|X - \mathbb{E}[X]| \geq a\,\mathbb{E}[X]\} \leq \frac{\mathrm{Var}[X]}{a^2\,\mathbb{E}[X]^2}\,,$$

By Theorem 3.49, we can assume the existence of a function $c_d(n)$ such that

$$\mathbb{E}[\mathcal{E}_{n,2d}^2] = c_d(n)\,\mathbb{E}[\mathcal{E}_{n,2d}]^2\,,$$

and

$$c_d(n) \to \exp\left(-\frac{4d-1}{4(2d-1)^2}\right)\sqrt{\frac{2d-1}{2d-2}}$$

Then, we have

$$\begin{aligned}\mathbb{P}\{\mathcal{E}_{n,2d} \geq \sqrt{n}\,\mathbb{E}[\mathcal{E}_{n,2d}]\} &\leq \mathbb{P}\{|\,\mathcal{E}_{n,2d} - \mathbb{E}[\mathcal{E}_{n,2d}]| \geq (\sqrt{n}-1)\,\mathbb{E}[\mathcal{E}_{n,2d}]\}\\ &\leq \frac{c_d(n)-1}{(\sqrt{n}-1)^2}\\ &\to 0\,,\end{aligned}$$

where we are using Chebyshev's inequality with $a = \sqrt{n}-1$ and the fact that, for any random variable X, $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

□

Theorem 3.52 tells us that *almost every* $2d$-regular graph has *few* Eulerian orientations, in the sense that the number of Eulerian orientations is within a linear factor of Schrijver's lower bound. An example of a regular graph G for which the number of Eulerian orientations greatly exceeds Schrijver's lower bound is the toroidal grid (see §1.3.1). Lieb [61] showed that the number of Eulerian orientations of the $\sqrt{n} \times \sqrt{n}$ toroidal grid is

$$\left(\frac{4}{3}\right)^{3n/2} >> \left(\frac{\binom{4}{2}}{2^2}\right)^n .$$

In §3.6 we show a similar (but weaker) result for the number of Euler tours of toroidal grids.

Let G be a particular $2d$-regular graph, let $\vec{G}$ be a random orientation of G, and let I_v denote the indicator variable for v satisfying the Eulerian condition in $\vec{G}$. There are 2^{2d} different orientations for the edges incident with v, of which $\binom{2d}{d}$ satisfy the Eulerian condition. Hence,

$$\mathbb{P}(I_v = 1) = \frac{\binom{2d}{d}}{2^{2d}} .$$

Now, suppose all the I_v were independent. Then the total number of Eulerian orientations of G would be

$$2^{dn}\left(\frac{\binom{2d}{d}}{2^{2d}}\right)^n = \left(\frac{\binom{2d}{d}}{2^d}\right)^n .$$

Theorem 3.52 tells us that this rather crude estimate actually manages to capture the exponential factor in the number of Eulerian orientations of almost every $2d$-regular graph. Of course, in reality the orientations of the arcs at different vertices are not independent since choosing an orientation for the arcs incident with v implies a particular orientation on the arc v shares with each of its neighbours. However, Theorem 3.52 confirms the intuition that this dependence cannot be very strong, at least for large graphs.

3.5 The number of Euler tours of random Eulerian graphs

As was the case in the previous section, we restrict our attention to the class of graphs with fixed even degrees, and generate our random graphs using the configuration model. In this section, as was the case in §3.4, we restrict ourselves to the regular case. That is, we are sampling graphs from $\mathbb{G}(n, 2d)$. Recall the BEST Theorem

(Theorem 1.57). This gives the following relationship between the sizes of the sets of Euler tours and arborescences of an Eulerian directed graph $\vec{G}$:

$$|\mathrm{ET}(\vec{G})| = \frac{1}{n}(d-1)!^n |\mathrm{ARB}(\vec{G})|.$$

Brightwell and Winkler used this relationship to count Euler tours of an undirected graph by counting the *orbs* of G. We recall (Definition 1.69) that the set of orbs of a graph is the set of pairs

$$\mathrm{ORB}(G) = \{(\mathcal{E}, \mathcal{A}) : \mathcal{E} \in \mathrm{EO}(G), \mathcal{A} \in \mathrm{ARB}(\mathcal{E})\}.$$

In previous sections, we were able to obtain asymptotic estimates for the second moments of random variables counting arborescences (and so also Euler tours) of random d-in/d-out graphs and Eulerian orientations of random $2d$-regular graphs. Unfortunately, finding an estimate for the second moment of the number of orbs (and so also the number of tours) of undirected graphs has proved elusive. However, we are able to make a conjecture, and in §3.5.1 we provide some evidence, both empirical and theoretical, as to why we think it to be true. We then explain, in §3.5.2, how, if this conjecture holds, we will be able to obtain an asymptotic distribution for the number of orbs of a random $2d$-regular graph. Then, by almost identical arguments to those used to prove Theorem 3.35, we would be able to show the probabilistic inequality of Conjecture 3.4 holds.

3.5.1 Estimating the moments of $O_{n,2d}$

Recall the definition of an orb: the set of orbs of a graph G is the set of pairs $(\mathcal{E}, \mathcal{A})$ where $\mathcal{E}$ is an Eulerian orientation of G and $\mathcal{A}$ is an arborescence of $\mathcal{E}$. It is more straightforward to analyse the moments of the number of orbs of a random regular graph than the number of Euler tours. Given an estimate of any moment of the number of orbs of a random $2d$-regular graph we can immediately infer an estimate of the corresponding moment for the number of Euler tours.

Theorem 3.53. *Let d be some fixed positive integer and let $n \in \mathbb{N}$. Let $O^\star_{n,2d}$ be the random variable counting the orbs of a $2d$-regular multigraph obtained as the projection of a random $\mathrm{F} \in \Omega_{n,2d}$. Then,*

$$\mathbb{E}[O^\star_{n,2d}] \to \sqrt{\frac{\pi n}{d}} \left(\frac{\binom{2d}{d} d}{2^d} \right)^n.$$

Proof. As in the proof of Theorem 3.47, we define an Eulerian orientation of a configuration $\mathrm{F} \in \Omega_{n,2d}$ to be a partition $W = S \cup T$, where each W_v is split into equally sized sets S_v and T_v, such that each edge in F has one endpoint in S and one endpoint in T. Hence, each Eulerian orientation of a configuration F corresponds to a directed configuration $\vec{\mathrm{F}}$ from an instance of the directed configuration model $\Phi_{n,d}$. Recall, from §3.3.1, that we defined an arborescence of a directed configuration $\vec{\mathrm{F}}$ to be a partial directed configuration $\vec{\mathrm{F}}' \subset \vec{\mathrm{F}}$ such that $\sigma(\vec{\mathrm{F}}')$ is an arborescence of $\sigma(\vec{\mathrm{F}})$. We now define an orb of a configuration to be a pair $(\mathcal{E}, \mathcal{A})$, where $\mathcal{E} \in \mathrm{EO}(\mathrm{F})$ and $\mathcal{A} \in \mathrm{ARB}(\mathcal{E})$. The expected number of orbs of a random configuration can be computed as

$$\mathbb{E}[O^{\star}_{n,2d}] = \frac{|\overline{\Omega}_{n,2d}|}{|\,\Omega_{n,2d}\,|},$$

where

$$\overline{\Omega}_{n,2d} = \{(\mathrm{F}, \mathcal{E}, \mathcal{A}) : \mathrm{F} \in \Omega_{n,2d}, \mathcal{E} \in \mathrm{EO}(\mathrm{F}), \mathcal{A} \in \mathrm{ARB}(\mathcal{E})\}.$$

To enumerate the elements of $\overline{\Omega}_{n,2d}$, we first partition W into S and T. Then, we enumerate the triples $(\mathrm{F}, \mathcal{E}, \mathcal{A})$ corresponding to that partitioning. Each partitioning of W into S and T gives rise to an instance of the directed configuration model $\Phi_{n,d}$, and, therefore, we can compute the number of triples $(\mathrm{F}, \mathcal{E}, \mathcal{A})$ with $\mathcal{E}$ consistent with S and T using Lemma 3.27. Note that we never need to consider the generation of F, as this is implicit in $\mathcal{E}$. From Lemma 3.27, we see that the number of partial directed configurations that project to arborescences in an instance of $\Phi_{n,d}$ is

$$nd^n(dn-1)_{n-2}.$$

Given an arborescence $\mathcal{A}$, there are $dn-n+1$ arcs that need to be added to complete $\mathcal{E}$, and $(dn-n+1)!$ ways to choose configuration edges for these arcs. Finally, we note that there are $\binom{2d}{d}^n$ ways to partition W into S and T. Hence, the number of elements in $\overline{\Omega}_{n,2d}$ is

$$|\overline{\Omega}_{n,2d}| = \binom{2d}{d}^n nd^n(dn-1)!.$$

Dividing by

$$|\,\Omega_{n,2d}\,| = \frac{(2dn)!}{(dn)!2^{dn}}$$

and applying Stirling's formula (3.70) we find

$$\mathbb{E}[O^{\star}_{n,2d}] \to \left(\binom{2d}{d}\frac{d}{2^d}\right)^n \sqrt{\frac{\pi n}{d}}. \qquad \square$$

As in the case of Eulerian orientations, we can infer an estimation for $\mathbb{E}[O_{n,2d}]$ from the estimation of $\mathbb{E}[O^{\star}_{n,2d}]$ by conditioning on there being no loops or double edges.

Theorem 3.54. *Let d be some fixed positive integer and let $n \in \mathbb{N}$. Let $O_{n,2d}$ denote the random variable counting the orbs of a random $G \in \mathbb{G}(n,2d)$. Then,*

$$\mathbb{E}[O_{n,2d}] \to e^{3/4}\sqrt{\frac{\pi n}{d}}\left(\frac{\binom{2d}{d}d}{2^d}\right)^n .$$

Proof. The theorem follows from the result of Bollobás [11] (see Theorem 3.15) and from calculations already carried out in the proofs of Theorem 3.30 and Lemma 3.32.

Let L and D count the number of loops and double edges in a random configuration $\mathrm{F} \in \Omega_{n,2d}$, and let $L^{(1)}$ and $D^{(1)}$ count the number of loops and double edges in F when $(\mathrm{F},\mathcal{E},\mathcal{A})$ is chosen randomly from

$$\overline{\Omega}_{n,2d} = \{(\mathrm{F},\mathcal{E},\mathcal{A}) : \mathrm{F} \in \Omega_{n,2d}, \mathcal{E} \in \mathrm{EO}(\mathrm{F}), \mathcal{A} \in \mathrm{ARB}(\mathcal{E})\}.$$

We can estimate $\mathbb{E}[O_{n,2d}]$ as

$$\mathbb{E}[O_{n,2d}] = \frac{\mathbb{P}[L^{(1)} = D^{(1)} = 0]}{\mathbb{P}[L = D = 0]}\,\mathbb{E}[O^{\star}_{n,2d}]$$

From the classic result of Bollobás [11] (Theorem 3.15), we have

$$\mathbb{P}[L = D = 0] \to \exp\left(-\frac{2d-1}{2} - \frac{(2d-1)^2}{4}\right) = \exp(-d^2 + 1/4). \tag{3.86}$$

Thus, it remains to estimate $\mathbb{P}[L^{(1)} = D^{(1)} = 0]$.

Recall how we enumerated the elements of $\overline{\Omega}_{n,2d}$ in Theorem 3.47 by first partitioning W into S and T. Each partition of W into S and T gives rise to an instance of the directed configuration model $\Phi_{n,d}$, and an equal number of elements of $\overline{\Omega}_{n,2d}$ arise from each choice of S and T. Hence, the expected number of loops in F, when $(\mathrm{F},\mathcal{E},\mathcal{A})$ is chosen uniformly at random from $\overline{\Omega}_{n,2d}$, is equal to the expected number of loops in $\vec{\mathrm{F}}$, when $(\vec{\mathrm{F}},\mathcal{A})$ is chosen randomly from

$$\overline{\Phi}_{n,d} = \{(\vec{\mathrm{F}},\mathcal{A}) : \vec{\mathrm{F}} \in \Phi_{n,d}, \mathcal{A} \in \mathrm{ARB}(\mathcal{E})\},$$

as defined in the proof of Theorem 3.30 and Lemma 3.32. Hence, by arguments already presented in the proof of Theorem 3.30

$$\mathbb{E}[L^{(1)}] \to \frac{m_2 - m}{m} = d - 1,$$

since now $m_2 = d^2 n$ and $m = dn$. Similarly, the expected number of double edges can be obtained as the expected number of double arcs and directed 2-cycles in $\vec{\mathrm{F}}$ when $(\vec{\mathrm{F}}, \mathcal{A})$ is chosen uniformly at random from $\overline{\Phi}_{n,d}$. Hence, by arguments presented in Theorem 3.30 and Lemma 3.32, we have

$$\begin{aligned}\mathbb{E}[D^{(1)}] &\to \frac{(m_2 - m)^2}{2m^2} + \frac{(m_2/m)^2 - 1}{2} \\ &= d(d-1)\,.\end{aligned}$$

It is straightforward to check that the factorial moments of $L^{(1)}$ and $D^{(1)}$ converge jointly. Hence, $L^{(1)}$ and $D^{(1)}$ converge to a pair of independent Poisson random variables with means $d-1$ and $d(d-1)$, respectively. We can now estimate

$$\mathbb{P}[L^{(1)} = D^{(1)} = 0] \to \exp(-d^2 + 1)\,.$$

Combining our estimate of $\mathbb{P}[L = D = 0]$ and $\mathbb{P}[L^{(1)} = D^{(1)} = 0]$ gives

$$\frac{\mathbb{E}[O_{n,2d}]}{\mathbb{E}[O^{\star}_{n,2d}]} = \frac{\mathbb{P}[L^{(1)} = D^{(1)} = 0]}{\mathbb{P}[L = D = 0]} \to \exp(3/4)\,. \qquad \square$$

Let $\mathcal{T}_{n,2d}$ be the random variable counting Euler tours of a random $2d$-regular graph. Applying the BEST theorem to Theorem 3.53 and Theorem 3.54, i.e., multiplying by $n^{-1}(d-1)!^n$, we get

$$\begin{aligned}\mathbb{E}[\mathcal{T}^{\star}_{n,2d}] &\to \sqrt{\frac{\pi}{dn}} \left(\frac{\binom{2d}{d} d!}{2^d} \right)^n ; \\ \mathbb{E}[\mathcal{T}_{n,2d}] &\to e^{3/4} \sqrt{\frac{\pi}{dn}} \left(\frac{\binom{2d}{d} d!}{2^d} \right)^n\end{aligned}$$

We have not been able to obtain an asymptotic estimate for the second moment, but make the following conjecture.

Conjecture 3.55. *Let d be a fixed integer satisfying $d \geq 2$. For each $n \in \mathbb{N}$, we define $O^{\star}_{n,2d}$ to be the random variable counting the number of orbs of a graph G obtained as the projection of a random $\mathrm{F} \in \Omega_{n,2d}$ and define $O_{n,2d}$ to be the random variable counting the number of orbs of a random $2d$-regular graph. Then,*

$$\begin{aligned}\frac{\mathbb{E}[(O^{\star}_{n,2d})^2]}{\mathbb{E}[O^{\star}_{n,2d}]^2} &\to \sqrt{\frac{2d-1}{2d-2}}\,; \\ \frac{\mathbb{E}[O^2_{n,2d}]}{\mathbb{E}[O_{n,2d}]^2} &\to \exp\left(-\frac{1}{2(2d-1)} - \frac{1}{4(2d-1)^2} \right).\end{aligned}$$

We now discuss the motivation behind Conjecture 3.55, and present some empirical evidence that seems to support it. The principal reason we believe (or, rather, hope) Conjecture 3.55 to be true comes from the next section: if Conjecture 3.55 holds then we can apply Janson's Theorem (Theorem 3.19) with $Y_n = \mathcal{T}_{n,2d}$ and $X_{i,n}$ counting i-cycles to obtain an asymptotic distribution for $\mathcal{T}_{n,2d}$ (and, hence, confirm Conjecture 3.4).

The key to establishing Conjecture 3.55 is enumerating the elements of the set

$$\widetilde{\Omega}_{n,2d} = \{(\mathrm{F}, \mathcal{E}, \mathcal{E}', \mathcal{A}, \mathcal{A}') : \mathrm{F} \in \Omega_{n,2d}, \mathcal{E}, \mathcal{E}' \in \mathrm{EO}(\mathrm{F}), \mathcal{A} \in \mathrm{ARB}(\mathcal{E}), \mathcal{A}' \in \mathrm{ARB}(\mathcal{E}')\}.$$

This should be achievable by combining the proofs of the two previous sections. First, we partition W into $S \cup T$ and $S' \cup T'$, as per the proof of Theorem 3.47. Then, we want to enumerate the set of tuples $(\mathrm{F}, \mathcal{E}, \mathcal{E}', \mathcal{A}, \mathcal{A}')$ with S, T, S', and T' defining $\mathcal{E}$ and $\mathcal{E}'$. For example, by Theorem 3.28, we can compute that the number of elements of $\widetilde{\Omega}_{n,2d}$ in which $\mathcal{E} \cap \mathcal{E}'$ or $\mathcal{E} \oplus \mathcal{E}'$ is empty is

$$\frac{d^{2n-1}}{d-1}(2n)!\,.$$

As another example, consider the number of elements of $\widetilde{\Omega}_{n,2d}$ satisfying $\mathcal{A} \cup \mathcal{A}' \subseteq \mathcal{E} \cap \mathcal{E}'$. That is, the case when both the arborescences are contained in the portion of edges the Eulerian orientations agree on. Suppose we have $|S_v \cap S'_v| = d_v$, for some vector of non-negative integers $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ satisfying

$$d \geq d_n \geq d_{n-1} \geq \cdots \geq 1\,.$$

Since $\mathcal{A} \cup \mathcal{A}' \subseteq \mathcal{E} \cap \mathcal{E}'$, the number of ways we can choose $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{E} \cap \mathcal{E}'$ is, by Theorem 3.28,

$$\frac{n^2}{m(m-n)}\left(\prod_v d_v\right)^2 m!\,,$$

where $m = \sum_v d_v$. Then, since there are no edges from $\mathcal{A}$ or $\mathcal{A}'$ in $\mathcal{E} \oplus \mathcal{E}'$, the number of ways we can choose the remaining edges of $\mathcal{E}$ (and so also F and $\mathcal{E}'$) is $(dn-m)!$. Letting $c_i = |\{v : d_v = i\}|$, we can see that the number of elements of $\widetilde{\Omega}_{n,2d}$ satisfying $\mathcal{A} \cup \mathcal{A}' \subseteq \mathcal{E} \cap \mathcal{E}'$ is

$$\binom{2d}{d}^2 \sum_{c_0+c_1+\cdots+c_d=n} \binom{n}{c_0, c_1, \ldots, c_d} \prod_{i=1}^{d} \left(i^{2c_i} \binom{d}{i}^{2c_i} \right) \times \frac{n^2 (\sum_i i c_i)! (dn - \sum_i i c_i)!}{(\sum_i i c_i)! (n - \sum_i c_i)!}$$

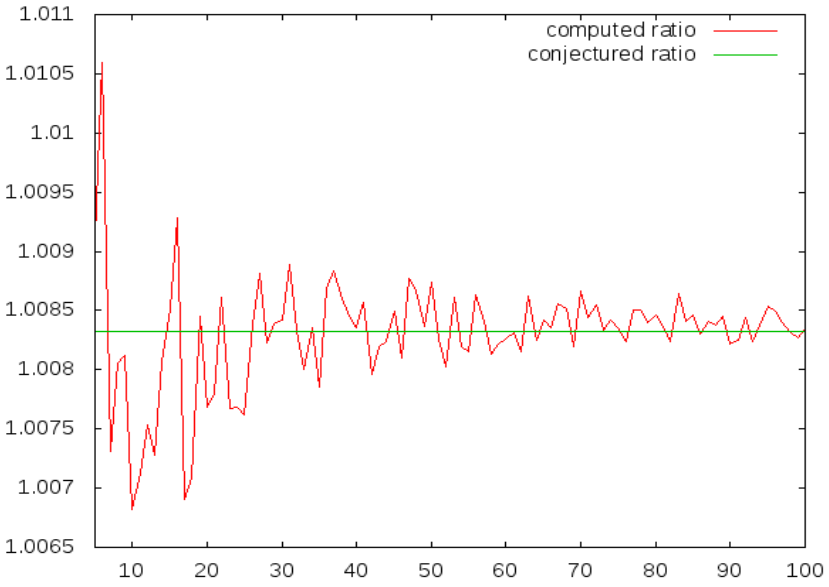


Table 3.1: Critical ratio for $d = 2$

It is clear that computing this value alone, which could be seen as one of the "easy cases" in the computation of $|\widetilde{\Omega}_{n,2d}|$ takes as much work as proving Theorem 3.28 and Theorem 3.47.

We make the conjecture that, in general, the number of triples, for a particular $\mathbf{c}$ is something like

$$d^{2n}(\sum_{i=1}^{d} ic_i)!(dn - \sum_{i=1}^{d} ic_i)!f(\mathbf{c}), \tag{3.87}$$

for some rational function $f(\mathbf{c})$. If this is the case, then Conjecture 3.55 could be proven (or disproven) by the argument used to prove Theorem 3.47 with $f(\mathbf{c})$ incorporated into $h(\mathbf{c})$. However, we have not been able to derive a nice expression in the form of (3.87). Once we allow edges from $\mathcal{A}$ and $\mathcal{A}'$ to occur in $\mathcal{E} \cap \mathcal{E}'$ or $\mathcal{E} \oplus \mathcal{E}'$, the situation grows more complicated as we now have two types of edges (those in $\mathcal{E} \cap \mathcal{E}'$ and those in $\mathcal{E} \oplus \mathcal{E}'$) and the argument used to prove Theorem 3.28 does not work. Despite this, it seems an expression of the form given in (3.87) should exist.

Although we have not been able to prove Conjecture 3.55 we have obtained some empirical evidence to support it, see Table 3.1. This data was generated as follows: for each n we generated n^2 different 4-regular graphs using the random generation algorithm of Steger and Wormald [83]; then, for each G we computed an estimate for

$|\mathrm{ET}(G)|$, Z_G, using Algorithm 5 with $N = n^2$. We then computed two values

$$Z^{(1)} = \frac{1}{n^2}\sum_G Z_G;$$
$$Z^{(2)} = \frac{1}{n^2}\sum_G Z_G^2.$$

The ratio $Z_2/(Z_1)^2$ was taken as an approximation of the *critical ratio* of $O_{n,4}$,

$$\frac{\mathbb{E}[O_{n,4}^2]}{\mathbb{E}[O_{n,4}]^2}.$$

In Table 3.1 we have graphed it against the number of vertices n, with the conjectured critical ratio marked by a green line at

$$\exp\left(-\frac{1}{6}-\frac{1}{36}\right)\sqrt{\frac{3}{2}} \approx 1.0083225.$$

3.5.2 The asymptotic distribution of $O_{n,2d}$

The main result of this section is contingent on a positive answer to Conjecture 3.55. Assuming Conjecture 3.55, we proceed to show how we can obtain an asymptotic distribution for the number of orbs (or Euler tours) of a random $2d$-regular graph. Let $X_{i,n}$ denote the number of i-cycles in a random $\mathrm{F} \in \Omega_{n,d}$ and let $\lambda_i = \frac{(2d-1)^i}{2i}$. As stated previously, it is well known that the variables $X_{i,n}$ are asymptotically independent Poisson random variables with means λ_i. Hence, Condition 1 of Theorem 3.19 is satisfied by $X_{i,n}$. The next lemma establishes Condition 2 of Theorem 3.19 for $Y_n = O_{n,2d}$.

Lemma 3.56. *Let $X_{i,n}$ count the number of directed i-cycles in a graph $G \in \mathbb{G}^\star(n,2d)$. Then, for any fixed set of integers $j_1, j_2, \ldots, j_k$ we have*

$$\frac{\mathbb{E}\left[O_{n,2d}^\star \prod_{i=1}^k (X_{i,n})_{j_i}\right]}{\mathbb{E}[O_{n,2d}^\star]} \to \prod_{i=1}^k \mu_i^{j_i},$$

where $\mu_i = \frac{(2d-1)^i-1}{2i}$.

Proof. We start establishing

$$\frac{\mathbb{E}[O_{n,2d}^\star X_{i,n}]}{\mathbb{E}[O_{n,2d}^\star]} \to \mu_i \tag{3.88}$$

for $i \geq 1$. Convergence of the factorial moments then follows for similar reasons as before (see Lemma 3.51).

Let $C_{i,n}$ denote the number of i-cycles which can occur in a configuration $\Omega_{n,2d}$, as defined in Lemma 3.51. Then, for the same reasons as were given in the proof of Lemma 3.51 we have

$$\frac{\mathbb{E}[O^{\star}_{n,2d} X_{i,n}]}{\mathbb{E}[O^{\star}_{n,2d}]} = |C_{i,n}|\,\mathbb{P}[C \subseteq \mathrm{F} : (\mathrm{F}, \mathcal{E}, \mathcal{A}) \in \overline{\Omega}_{n,2d}]\,,$$

where we now define $\overline{\Omega}_{n,2d}$ to be the set

$$\overline{\Omega}_{n,2d} = \{(\mathrm{F}, \mathcal{E}, \mathcal{A}) : \mathrm{F} \in \Omega_{n,2d}, \mathcal{E} \in \mathrm{EO}(\mathrm{F}), \mathcal{A} \in \mathrm{ARB}(\mathcal{E})\}\,.$$

From the proof of Lemma 3.51, we already know

$$|C_{i,n}| = \frac{(n)_i (2d)^i (2d-1)^i}{2i} \to \frac{(n)^i (2d)^i (2d-1)^i}{2i}\,.$$

Thus, it remains to estimate the probability $\mathbb{P}[C \subseteq \mathrm{F} : (\mathrm{F}, \mathcal{E}, \mathcal{A}) \in \overline{\Omega}_{n,2d}]$.

Let $C \in C_{i,n}$. We want to count the number of elements of $\overline{\Omega}_{n,2d}$ with $C \subset \mathrm{F}$. We can achieve this by following a similar approach to that used in the proof of Lemma 3.51. We start by choosing a partition of W into S and T as per the proof of Lemma 3.51. First, we assign the endpoints of the edges of C to S and T. A vertex v on C is said to be balanced if one of the edges incident with v on C has a point from S_v and one has a point from T_v; we say a vertex v is negatively (resp. positively) imbalanced if both edges incident with v on C contain a point from S_v (resp. T_v). Given a particular choice of j positively and negatively imbalanced vertices, the number of ways to choose S and T is (see proof of Lemma 3.51)

$$\binom{2d}{d}^n \left(\frac{d}{2(2d-1)}\right)^i \left(\frac{d-1}{d}\right)^{2j}\,. \tag{3.89}$$

Next, we choose which of the edges of C are to lie in $\mathcal{A}$. For each balanced vertex we have two choices: the edge with a point from S_v is either contained in $\mathcal{A}$ or it is not. For each negatively balanced vertex there are three choices: we could have exactly one of the two edges contained in $\mathcal{A}$, or we could have that neither of them are. Finally, the edge corresponding to the arc leaving any positively imbalanced vertex in $\mathcal{A}$ cannot be contained in C. The intersection of any arborescence $\mathcal{A}$ with C is a set of directed paths. Suppose this intersection contains k paths ending in negatively imbalanced vertices (these must necessarily be paths of length 1), and l paths ending in balanced vertices. Then $\mathcal{A} \cap C$ contains $j+k+l$ components, since each positively imbalanced vertex must necessarily be the endpoint of a path in the intersection. Suppose R is the set of endpoints of these paths and let P_v denote the path ending in v, for each $v \in R$.

Next, we want to choose the remaining edges in $\mathcal{A}$. The argument is similar to that in Lemma 3.32: choosing the remainder of $\mathcal{A}$ is equivalent to choosing a arborescence on $(V\backslash C)\cup R$, where we have contracted the path ending in v to a single vertex, for each $v \in R$. For $v \in R$, let x_v denote the points in $\bigcup_{u\in P_v} T_u$ that are not used by edges in C and let y_v be equal to the number of points from $\bigcup_{u\in P_v} S_u$ not used in edges of C. Then, for each $v \in R$, we have

$$x_v = (d-1)|P_v|\,,$$

if the first and last vertices of P_v are both balanced or the first vertex of P_v is negatively imbalanced and the last vertex is positively imbalanced,

$$x_v = (d-1)|P_v|+1\,,$$

if the first vertex of P_v is negatively imbalanced and either $|P_v| = 1$ or the last vertex is balanced, and

$$x_v = (d-1)|P_v|-1\,,$$

if the last vertex is positively imbalanced and either $|P_v| = 1$ or the first vertex of P_v is balanced. We have y_v equal to either $d-2$, $d-1$, or d, depending on whether v is negatively imbalanced, balanced, or positively imbalanced.

Choosing the remainder of $\mathcal{A}$ is equivalent to choosing a tree on $(V\backslash C)\cup R$ in which there are d points available for incoming and outgoing arcs of each $v \notin C$. For each $v \in R$, there are x_v points available for incoming arcs and y_v points available for outgoing arcs.

By Lemma 3.27, the number of ways to choose the remaining edges for $\mathcal{A}$ and $\mathcal{E}$ such that R gives the roots of the components of $\mathcal{A}\cap C$, is

$$n(d-2)^k(d-1)^l d^{n-i+j}(dn-i-1)!\,,$$

since we have $\sum_{v\in R} x_v + d(n-i) = dn-i$. Hence, the number of arborescences $\mathcal{A}$ with R as the roots of the components in $\mathcal{A}\cap C$ is

$$n2^{j-k}(d-2)^k(d-1)^l d^{n-i+j-1}(dn-i)!\,,$$

since we must choose which of the two arcs leaving each imbalanced vertex in R is contained in $\mathcal{A}$.

Suppose $j = 0$. That is, C is a directed cycle in $\mathcal{E}$. We note that $C\cap\mathcal{A}$ cannot be equal to C. Hence, in this case, we must have $l > 0$. Summing over all choices for R

gives

$$\sum_{l=1}^{i}\binom{i}{l}(d-1)^l d^{n-i}(dn-i-1)! = (d^n-d^{n-i})(dn-i-1)!\,.$$

Then, by (3.89), the number of elements of $\overline{\Omega}_{n,2d}$ with $C \subset \mathrm{F}$ in which C has no imbalanced vertices in $\mathcal{E}$ is

$$2n\binom{2d}{d}^n\left(\frac{d}{2(2d-1)}\right)^i(d^n-d^{n-i})(dn-i-1)! \tag{3.90}$$

On the other hand, suppose that $j>0$. Each of the positively imbalanced vertices of C will be the endpoint of a path in $C\cap\mathcal{A}$. Hence, in this case we can have $l=0$. First, we sum over l, to get

$$\sum_{l=0}^{i-2j}\binom{i-2j}{l}(d-1)^l = d^{i-2j}\,.$$

Then, we sum over k to get

$$2^j\sum_{k=0}^{j}\binom{j}{d}\left(\frac{d-2}{2}\right)^k = d^j\,.$$

Hence, the number of elements of $\overline{\Omega}_{n,2d}$ with $C\subset\mathrm{F}$ in which C has a particular set of j imbalanced vertices in $\mathcal{E}$ is, for each $j>0$,

$$2n\binom{2d}{d}^n\left(\frac{d}{2(2d-1)}\right)^i\left(\frac{d-1}{d}\right)^{2j}d^n(dn-i-1)!\,. \tag{3.91}$$

Combining (3.90) and (3.91), and summing over all choices for the sets of imbalanced vertices we get that the number of $(\mathrm{F},\mathcal{E},\mathcal{A})\in\overline{\Omega}_{n,2d}$ with $C\subset\mathrm{F}$ is

$$2n\binom{2d}{d}^n\left(\frac{d}{2(2d-1)}\right)^i d^n(dn-i-1)!\left(\sum_{j=0}^{\lfloor i/2\rfloor}\binom{i}{2j}\left(\frac{d-1}{d}\right)^{2j}-\frac{1}{d^i}\right).$$

We have shown, in Lemma 3.51, that

$$\sum_{j=0}^{\lfloor i/2\rfloor}\binom{i}{2j}\left(\frac{d-1}{d}\right)^{2j} = \frac{1}{2}\left(\frac{2d-1}{d}\right)^i+\frac{1}{2d^i}\,.$$

Hence, the number of $(\mathrm{F},\mathcal{E},\mathcal{A})\in\overline{\Omega}_{n,2d}$ with $C\subset\mathrm{F}$ is

$$\binom{2d}{d}^n\frac{d^n n(dn-i-1)!}{2^i}\left(1-\frac{1}{(2d-1)^i}\right).$$

Dividing by the number of elements in $\overline{\Omega}_{n,2d}$,

$$|\overline{\Omega}_{n,2d}| = \binom{2d}{d}^n nd^n(dn-1)!\,,$$

gives

$$\mathbb{P}[C \subseteq \mathrm{F} : (\mathrm{F}, \mathcal{E}, \mathcal{A}) \in \overline{\Omega}_{n,2d}] \to \frac{1}{2^i d^i n^i} \left(1 - \frac{1}{(2d-1)^i}\right).$$

Finally, multiplying by $|\mathcal{C}_{i,n}|$ gives (3.88). □

Now, if Conjecture 3.55 was true, we could apply Janson's theorem to obtain an asymptotic distribution for $\mathcal{T}_n$. For $i \geq 3$ we have

$$\delta_i = -\frac{1}{(2d-1)^i}.$$

Hence,

$$\sum_{i\geq 3} \lambda_i \delta_i^2 = \frac{1}{2} \sum_{i\geq 3} \frac{1}{i(2d-1)^i} = \frac{1}{2} \log\left(\frac{2d-1}{2d-2}\right) - \frac{1}{2(2d-1)} - \frac{1}{4(2d-1)^2}.$$

Thus, $O_{n,2d}$ and $X_{i,n}$ would satisfy the conditions of Janson's Theorem if Conjecture 3.55 was true, in which case we would have

$$\frac{O_{n,2d}}{\mathbb{E}[O_{n,2d}]} \to \prod_{i\geq 3} \left(1 + \frac{1}{(2d-1)^i}\right)^{Z_i},$$

where the Z_i are Poisson random variables with mean $\lambda_i = \frac{(2d-1)^i}{2i}$. This could then be used to infer a concentration result of the form

$$\mathbb{P}\left[\frac{O_{n,2d}}{\mathbb{E}[O_{n,2d}]} \geq n^{-1/2}\right] \to 1,$$

as $n \to \infty$, of which Conjecture 3.4 would be an immediate corollary.

3.6 Euler tours of the grid

We close this chapter by presenting some examples of simple 4-regular graphs for which Algorithm 4 and Algorithm 5 have exponential running time. The graphs we consider are the $m \times n$ *square grids* with *toroidal boundaries*.

Definition 3.57. The $m \times n$ toroidal grid, denoted $G(m,n)$, is the 4-regular graph with vertex set $\{(i,j) : 0 \leq i < m, 0 \leq j < n\}$ and an edge joining each (i,j) to $(i \pm 1 \bmod m, j \pm 1 \bmod n)$.

Remark 3.58. In the following, for ease of presentation, we assume all addition of row and column indices is done modulo m and n, respectively. Hence, the set of edges is $G(n,m)$ is given by $\{\{(i,j),(i \pm 1, j \pm 1)\} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$.

Golin et al. [39] claim that we can exactly count Euler tours of $G(m,n)$ using the *transfer matrix* method. This is the same method used in [61] and [6] to obtain the exponential growth rate of the number of Eulerian orientations of the square and triangular lattices, respectively. This is certainly true – we can define a transfer matrix for counting many combinatorial structures defined on $G(m,n)$ – but Golin et al. do not provide any calculations or even a description of the transfer matrix used. As we shall see, just defining an appropriate transfer matrix is not exactly straightforward. Moreover, the (naive) algorithm based on the transfer matrix we define is sure to be very inefficient, even for relatively small values of m.

We now discuss the problem of counting the Euler tours of $G(m,n)$ and devise a transfer matrix for the problem. We illustrate how this approach can be used to derive exact expressions for $|\mathrm{ET}(G(m,n))|$ by computing the number of Euler tours for $G(2,n)$ and $G(3,n)$. In general, this leads to a polynomial-time algorithm for counting the Euler tours of $G(m,n)$, for fixed m, albeit one with an exponential dependence on m. Finally, we discuss why Algorithms 4 and Algorithm 5 will not behave like an *fpaus* or an *fpras* for $\mathrm{ET}(G(m,n))$, even when one of m and n is bounded.

Recall the definition of a *transition system* from §1.3.2. Each transition system T of $G(m,n)$ defines a decomposition of $G(m,n)$ into a set of disjoint cycles, which we denote by $\mathcal{C}(T)$. If $\mathcal{C}(T)$ has a single element, then T defines an Euler tour of $G(m,n)$. Let $V_k = \{(i,j) : 0 \leq i \leq m-1, 0 \leq j \leq k-1\}$. We say a partial transition system, defined only on V_k, is *legal* if it can be extended to a transition system defining an Euler tour of $G(m,n)$. Each partial transition system T defined on V_k decomposes the subgraph induced by V_k into a set of disjoint paths and cycles. The paths in this decomposition will include the trailing horizontal edges joining column 0 and column $n-1$, and column $k-1$ and column k. A necessary condition for a partial transition system T defined on V_k to be legal is for there to be no cycles in the decomposition of V_k induced by T. Hence, each legal partial transition system T decomposes V_k into a set of m paths, where each one of the end-edges of these paths is contained in the set

$$\{\{(i,0),(i,n-1)\} : 0 \leq i \leq m-1\} \cup \{\{(i,k-1),(i,k)\} : 0 \leq i \leq m-1\}.$$

We classify the legal transition systems on V_k by the set of perfect matchings on $\{l_0, l_1, \ldots, l_{m-1}, r_0, r_1, \ldots, r_{m-1}\}$, denoted $\mathcal{P}(m)$, by identifying l_i with the edge $\{(i,0),(i,n-1)\}$ and r_i with the edge $\{(i,k-1),(i,k)\}$. Then, to each legal T we assign the class $\mathcal{C} \in \mathcal{P}(m)$, where the edges of $\mathcal{C}$ correspond to the endpoints of the paths in the decomposition of V_k induced by T. Clearly, the set of classes is the same

for any k.

For $\mathcal{C}, \mathcal{C}' \in \mathcal{P}(m)$, we define $A(\mathcal{C}, \mathcal{C}')$ to be the the number of transition systems on $V_{k+1} \setminus V_k = \{(i,k) : 0 \leq i \leq m-1\}$ which extend a transition system on V_k with class $\mathcal{C}$ to a transition system on V_{k+1} with class $\mathcal{C}'$. Then A is a $P(2m) \times P(2m)$ matrix, where $P(2m)$ is the number of perfect matchings on K_{2m}:

$$P(2m) = \binom{2m}{m} \frac{m!}{2^m}.$$

Consider the effect of choosing a transition system on

$$V_{k+1} \setminus V_k = \{(i,j) : 0 \leq i \leq m-1\}.$$

The only illegal transition system on $V_{k+1} \setminus V_k$ is the one in which $\{(i,k),(i+1 \bmod m)\}$ is paired with $\{(i,k),(i-1 \bmod m,k)\}$, for each $i = 0,1,\ldots,m-1$. Each other transition system defines a set of edge-disjoint paths, where each path has its end-edges in the set

$$\{\{(i,k-1),(i,k)\} : 0 \leq i \leq m-1\} \cup \{\{(i,k),(i,k+1)\} : 0 \leq i \leq m-1\}$$

Hence, we can identify the legal transition systems on $V_{k+1} \setminus V_k$ with the set of perfect matchings on $\{l'_0, l'_1, \ldots, l'_{m-1}, r'_0, r'_1, \ldots, r'_{m-1}\}$, where l'_i corresponds to the edge $\{(i,k-1),(i,k)\}$ and r'_i corresponds to the edge $\{(i,k),(i,k+1)\}$. Suppose the perfect matching contains $l'_i r'_j$ for $i > j$. Then exactly one of the following must hold (see Figure 3.1 for an example):

1. The horizontal edge $\{(i,k-1),(i,k)\}$ is paired with the upwards vertical edge $\{(i,k),(i+1,k)\}$ at (i,k), the downwards vertical edge $\{(i'-1,k),(i',k))\}$ is paired with the horizontal edge $\{(i',k),(i',k+1\}$ at (i',k), and the horizontal edge $\{(t,k-1),(t,k)\}$ is paired with the horizontal edge $\{(t,k),(t,k+1)$ at (t,k) for each $t \in [0..j-1] \cup [i+1..m-1]$;

2. The horizontal edge $\{(i,k-1),(i,k)\}$ is paired with the downwards vertical edge $\{(i,k),(i-1,k)\}$ at (i,k), the downwards vertical edge $\{(i'+1,k),(i',k))\}$ is paired with the horizontal edge $\{(i',k),(i',k+1\}$ at (i',k), and the horizontal edge $\{(t,k-1),(t,k)\}$ is paired with the horizontal edge $\{(t,k),(t,k+1)$ at (t,k) for each $t \in [j+1..i-1]$;

A similar situation holds for $i < j$ or edges $l'_i l'_j$ or $r'_i r'_j$. Hence, we can enumerate the set of matchings which can correspond to transition systems on a particular column

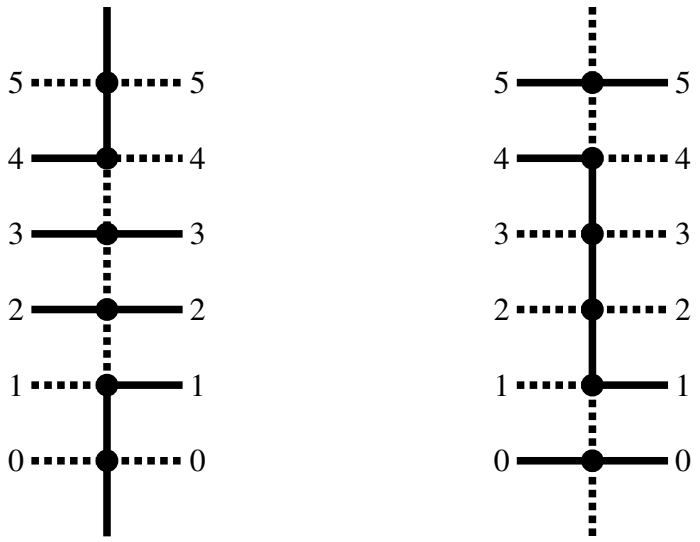


Figure 3.1: Two possibilities for transition systems extending V_k to V_{k+1} in $G(n,6)$ with edge $l'_4r'_1$. Solid (resp. broken) edges are paired together.

of $G(n,m)$ as follows. Let M represent an arbitrary matching on $\{l'_0,l'_1,\ldots,l'_{m-1}\}$ and $\{r'_0,r'_1,\ldots,r'_{m-1}\}$. Choose a decomposition of the cycle $(0,1,\ldots,m-1)$ into a number of edge disjoint paths. Then, for each i which is not the endpoint of one of these paths we set $l'_ir'_i \in M$. Each other i occurs as the endpoint of exactly two paths: label one of these l'_i and the other r'_i. Collapsing each of these paths to a single edge-pairing gives the remaining edges in M.

Given $\mathcal{C} \in \mathcal{P}(m)$ for V_k and M, a matching for $V_{k+1}\setminus V_k$, we obtain $\mathcal{C}'$ by identifying r_i with l'_i in M, and renaming r'_i as r_i. The result is either a set of 2-paths that contract to give the perfect matching $\mathcal{C}'$, or some set of edges including at least one cycle. In this latter case, the transition system defining M does not extend a legal transition system on V_k of class $\mathcal{C}$ to a legal transition system on V_{k+1}. Finally, we note that each M yields a different $\mathcal{C}'$ when applied to $\mathcal{C}$, so $A(\mathcal{C},\mathcal{C}')$ is just the number of transition systems on $V_{k+1}\setminus V_k$ that yield the M which extends $\mathcal{C}$ to $\mathcal{C}'$, if it exists, and 0 otherwise.

Suppose $l'_ir'_i \in M$ for all i. There are $2m$ distinct transition systems which yield M: m ways to decompose $(0,1,\ldots,m-1)$ into a single path and then 2 ways to label the endpoints. Alternatively, for each i, there is a pair of transition systems where l'_i is connected to r'_i by a path that uses all the vertical edges, and such that for $j \neq i$, l'_i is connected to r'_i by a path consisting of two vertical edges. These are the only transition systems that gives rise to the matching M with $l'_ir'_i \in M$ for all i. Hence, $A(\mathcal{C},\mathcal{C}) = 2m$.

Now, suppose there is exactly one pair $i \neq j$ with $l_ir_j, l_jr_i \in M$. This can arise as a

result of two transition systems: choose a decomposition of $(0,1,\ldots,m-1)$ into two paths by breaking the cycle at i and j and then label the endpoints so that l_i and r_j are opposite endpoints of one path. Similarly, there are two transition systems yielding each matching M with $l_i l_j, r_j r_i \in M$ for exactly one pair $i \neq j$. Hence, in this case we have $A(\mathcal{C},\mathcal{C}') = 2$.

In all other cases, there is a single transition system on $V_{k+1}\backslash V_k$ that gives rise to M. Hence, $A(\mathcal{C},\mathcal{C}') \in \{0,1,2,2m\}$.

Let $x(\mathcal{C}) = 1$ if $l_i r_i \in \mathcal{C}$ for all i and $x(\mathcal{C}) = 0$ otherwise, and let $y(\mathcal{C}) = 1$ if identifying l_i with r_i in $\mathcal{C}$ gives rise to a single m-cycle and $y(\mathcal{C}) = 0$ otherwise. Then,

$$|\mathrm{ET}(G(m,n))| = xA^n y^T .$$

For any fixed constant m, A is a constant-sized matrix so, in theory, we can compute $|\mathrm{ET}(G(m,n))|$ in polynomial time. However, the number of rows in A,

$$\frac{(2m)!}{m!2^m},$$

grows very quickly, e.g., for $m = 10$, A has 654729075 rows and the same number of columns, making computation impractical. It may be possible to exploit structure in A to simplify the computation, as Lieb [61] and Baxter [6] did for the transfer matrix counting Eulerian orientations.

For example, if $l_i l_j \in \mathcal{C}$, then $A(\mathcal{C},\mathcal{C}') \neq 0$ only if $l_i l_j \in \mathcal{C}'$. Hence, we could exploit the use of an ordering on $\mathcal{P}(m)$ such that $\mathcal{C} \prec \mathcal{C}'$ if there are fewer edges of the form $l_i l_j$ in $\mathcal{C}$ than are in $\mathcal{C}'$. If the rows and columns of A satisfy such an ordering then A has a structure similar to a block-diagonal matrix; see Proposition 3.59 and Proposition 3.59 below for examples. With such a matrix, we know there will be large contiguous blocks consisting entirely of 0s. Indeed, it was a structure similar to this that enabled Lieb and Baxter to compute asymptotic estimates of the number of Eulerian orientations of the grid and triangular lattice.

In the following we use $A[S]$ to denote the square sub-matrix of S with rows and columns indexed by S. We use I_n to denote the $n \times n$ matrix which has value 1 in each entry on the diagonal, and is 0 everywhere else, and use J_n to denote the $n \times n$ matrix with all entries equal to 1.

Proposition 3.59. *The number of Euler tours of the $2 \times n$ toroidal grid, $G(2,n)$, is, asymptotically,*

$$(2n+3)6^{n-1} - 2^{n-1} .$$

Proof. There are 3 perfect matchings on K_4, so the transfer matrix for counting Euler tours of the $2 \times n$ toroidal grid has 3 entries. These are

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 0 & 0 & 6 \end{bmatrix}.$$

It is straightforward to calculate A^n. Firstly, we observe that $A^n[0,1] = (A[0,1])^n$ and

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}^n = \begin{bmatrix} 2^{n-1}(3^n+1) & 2^{n-1}(3^n-1) \\ 2^{n-1}(3^n-1) & 2^{n-1}(3^n+1) \end{bmatrix}.$$

Moreover, it is clear that we also have $A^n_{3,3} = 6^n$ since $A_{3,3}$ is the only non-zero value in the third row. Finally, we can calculate $A^n_{3,1} = A^n_{3,2} = 2n6^{n-1}$. Hence, we have

$$A^n = \begin{bmatrix} 2^{n-1}(3^n+1) & 2^{n-1}(3^n-1) & 2n6^{n-1} \\ 2^{n-1}(3^n-1) & 2^{n-1}(3^n+1) & 2n6^{n-1} \\ 0 & 0 & 6^n \end{bmatrix}.$$

Finally, $x = (1,0,0)$ and $y = (0,1,1)$, from which we can deduce the claimed value for $|\mathrm{ET}(G(2,n))|$. □

The number of transition systems of $G(2,n)$ is 9^n. Hence, the probability that Algorithm 4 will generate an Euler tour on a particular orientation is $\sim \left(\frac{2}{3}\right)^n$.

Proposition 3.60. *The number of Euler tours of the* $3 \times n$ *toroidal grid* $G(3,n)$ *is*

$$\frac{4}{3}20^n - 14^n - \frac{4}{3}5^n + 2^n.$$

Proof. There are 15 pairings on $\{l_1, l_2, l_3, r_1, r_2, r_3\}$, each of which can arise from the transition system on a single column of $G(n,3)$. Table 3.2 enumerates these pairings, along with the number of transition systems on $V_{k+1} \backslash V_k$ that each arise from, and whether each of them extends to an Euler tour. With the ordering given in Table 3.2 we can write the transfer matrix A as

$$A = \begin{bmatrix} A_1 & B_1 & C_1 & C_1 & C_1 \\ B_1 & A_1 & C_1 & C_1 & C_1 \\ 0 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & D_1 & 0 \\ 0 & 0 & 0 & 0 & D_1 \end{bmatrix},$$

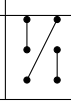
No.	σ	#TS	ET	No.	σ	#TS	ET
1		6	No	2		1	Yes
3		1	Yes	4		2	No
5		2	No	6		2	No
7		2	No	8		1	Yes
9		1	Yes	10		2	No
11		1	Yes	12		1	Yes
13		2	No	14		1	Yes
15		1	Yes				

Table 3.2: Pairings for $m = 3$

where $A_1 = 5I_3 + J_3$, $B_1 = 2J_3$, $C_1 = I_3 + J_3$ and $D_1 = 2I_3 + 6J_3$. Moreover, this block structure is preserved over taking powers of A; that is, there exist 3×3 matrices A_n, B_n, C_n and D_n such that

$$A^n = \begin{bmatrix} A_n & B_n & C_n & C_n & C_n \\ B_n & A_n & C_n & C_n & C_n \\ 0 & 0 & D_n & 0 & 0 \\ 0 & 0 & 0 & D_n & 0 \\ 0 & 0 & 0 & 0 & D_n \end{bmatrix}.$$

Observe that

$$\begin{bmatrix} A_n & B_n \\ B_n & A_n \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix}^n \quad \text{and} \quad D_n = (D_1)^n,$$

so we can calculate the values of these entries independently.

Claim 3.61.

$$A_n = 5^n I_3 + \frac{14^n + 2^n - 2 \cdot 5^n}{6} J_3 \quad \text{and} \quad B_n = \frac{14^n - 2^n}{6} J_3$$

Proof. We write

$$\begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} = 5I_6 + \begin{bmatrix} J_3 & 2J_3 \\ 2J_3 & J_3 \end{bmatrix}$$

and expand the powers of this sum by the binomial theorem[2]. Observing that

$$\begin{bmatrix} J_3 & 0 \\ 0 & J_3 \end{bmatrix}^{k-j} \begin{bmatrix} 0 & 2J_3 \\ 2J_3 & 0 \end{bmatrix}^{j} = \begin{cases} 2^j \begin{bmatrix} J_3^k & 0 \\ 0 & J_3^k \end{bmatrix} & \text{if } j \bmod 2 = 0 \\ 2^j \begin{bmatrix} 0 & J_3^k \\ J_3^k & 0 \end{bmatrix} & \text{if } j \bmod 2 = 1 \end{cases},$$

we find

$$\begin{bmatrix} J_3 & 2J_3 \\ 2J_3 & J_3 \end{bmatrix}^k = \begin{bmatrix} \frac{3^k+(-1)^k}{2} J_3^k & \frac{3^k-(-1)^k}{2} J_3^k \\ \frac{3^k-(-1)^k}{2} J_3^k & \frac{3^k+(-1)^k}{2} J_3^k \end{bmatrix},$$

from which the claimed values for A_n and B_n follow. □

By a similar calculation we can obtain

$$D_n = 2^n I_3 + \frac{20^n - 2^n}{3} J_3$$

Now, observe that $C_n = A_1 C_{n-1} + B_1 C_{n-1} + C_1 D_{n-1}$. Hence, if we let $C_n = x_n I_3 + y_n J_3$ we can derive linear recurrences for x_n and y_n, which can be solved for

$$x_n = \frac{1}{3}(5^n - 2^n);$$
$$y_n = \frac{1}{9}(2 \cdot 20^n - 2 \cdot 14^n - 5^n + 2^n).$$

From Table 3.2 we obtain

$$y = (0,1,1,0,0,0,0,1,1,0,1,1,0,1,1),$$

from which we can calculate $|\mathrm{ET}(G(3,n))|$. □

The number of transition systems of $G(3,n)$ is 27^n. Hence, the probability of Algorithm 4 generating an Euler tour of any iteration is $\sim \left(\frac{20}{27}\right)^n$. For $m = 4$ the transfer matrix has 105 rows and 105 columns, making direct computations of the n-th power impractical. In both [61] and [6] the authors obtain the exponential growth rate of $|EO(G(n,n))|$ by calculating the maximum eigenvalue of the transfer matrix A. This is made possible by the fact that the number of Eulerian orientations is equal to the trace of A^n. Moreover, the transfer matrix for Eulerian orientations is block diagonal, so it is sufficient to restrict analysis to a single block. In the case of the transfer matrix for Euler tours things become more difficult because (a) the number of

[2]This is possible since all the matrices involved commute multiplicatively

Euler tours is not obtainable in terms of something simple like the trace and (b) the matrix is not block diagonal so we need to compute all/more entries of A^n. Although this complexity prevented us from deriving exact values or asymptotic estimates for $|\mathrm{ET}(G(m,n))|$, we were able to obtain a crude upper bound on the number of Euler tours of $G(m,n)$, which is sufficient to show that Algorithm 4 and Algorithm 5 require exponential running time on $G(m,n)$, for any m and n.

Proposition 3.62.

$$|\mathrm{ET}(G(n,m)| \leq 80^{nm/4}.$$

Proof. The set of cycles of the form $((2i,2j),(2i+1,2j),(2i+1,2j+1),(2i,2j+1))$ for $0 \leq i \leq m/2$ and $0 \leq j \leq n/2$ is a set of $nm/4$ vertex-disjoint 4-cycles. The number of transition systems containing any particular k-subset of these cycles is 3^{nm-4k}. Hence, by inclusion-exclusion, the number of transition systems containing none of these cycles is

$$\sum_{k=0}^{nm/4} \binom{nm/4}{k} (-1)^k 3^{nm-4k} = 80^{nm/4},$$

and this is certainly greater than the number of Euler tours of $G(n,m)$. □

There are 3 choices for the transition system at each vertex of $G(n,m)$, so $|\mathrm{TS}(G(n,m))|$ is equal to 3^{nm}. Hence, we can bound the probability Algorithm 4 generates an Euler tour of $G(n,m)$ on any particular iteration as

$$\frac{|\mathrm{ET}(G(m,n))|}{|\mathrm{TS}(G(m,n))|} \leq \left(\frac{80}{81}\right)^{nm/4}, \tag{3.92}$$

which tends to 0 as either m or n goes to infinity. Hence, we can conclude that Algorithm 4 does not have the behaviour of an *fpaus*, and Algorithm 5 does not have the behaviour of an *fpras*, on $G(m,n)$.

In light of the conjectured asymptotic distribution of $\mathcal{T}_{n,4}$ it is not wholly surprising that $G(m,n)$ has exponentially fewer Euler tours than transition systems. There are at least nm 4-cycles in $G(m,n)$: for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$ we have the cycle

$$((i,j),(i+1,j),(i+1,j+1),(i,j+1)),$$

where addition of the first and second coordinate is taken modulo n and modulo m, respectively. Then, the factor of W corresponding to $i = 4$ is

$$\left(\frac{80}{81}\right)^{nm}$$

for the class $\mathcal{G}_{\mathbf{x}}$ containing $G(m,n)$, and thus we can expect that the value of $\mathcal{T}_{n,4}$ for graphs in this class will almost surely be exponentially smaller than $\mathbb{E}[\mathcal{T}_{n,4}]$.

Chapter 4

Conclusions

In this thesis, we addressed the complexity of generating and counting Eulerian orientations and Euler tours. Firstly, we analysed the complexity of exactly counting random Eulerian orientations of planar graphs. We then investigated the running time of a particular Markov chain Monte Carlo algorithm for generating random Eulerian orientations of planar graphs with a distribution that is close to uniform. Secondly, we studied the problem of generating and counting Euler tours of Eulerian graphs and digraphs. Although there are many positive results for the directed case of this problem, almost nothing is known for the undirected case. Indeed, the structure of this problem seems to make it particularly difficult to analyse Markov chain Monte Carlo algorithms. Hence, we shifted our focus to investigating the running time of a simple algorithm on random Eulerian graphs and digraphs.

In Chapter 2 we showed that the complexity of counting Eulerian orientations of planar graphs is #P-complete. Although there already exists an algorithm which can approximately count the Eulerian orientations of any Eulerian graph, we considered an alternative approach to this problem for planar graphs. We showed that a different approach, using a well-known Markov chain (the *face-reversal* Markov chain $\mathcal{M}_{\mathcal{F}}$, a.k.a. Glauber Dynamics) mixes rapidly on the Eulerian orientations of triangular lattice, which are of practical importance as configurations of an *ice-type model* studied in statistical physics. These results complement existing results for the square lattice and are of a practical value as it is this algorithm that physicists tend to use in practice. However, we also showed that this chain is torpidly mixing on the set of Eulerian orientations of certain planar graphs containing a bounded face with a large number of edges (linear in the size of the graph). The only rapid mixing results for this chain are on the Eulerian orientations of the square and triangular lattices; however, one

would expect that it mixes rapidly for a larger class of graphs. We were not able to determine the class of planar graphs for which it should be rapidly mixing. However, we were able to develop some intuition. For example, suppose that, in every planar graph G, the cut with the worst (smallest) conductance is of the form used in the proof of Theorem 2.27. That is, there exists some face $\gamma \in \mathcal{F}(G)$, such that we can partition $\Omega = \mathrm{EO}(G)$ into two sets S and $\bar{S}$ satisfying

1. $\forall \mathcal{E} \in S$, $\wp_{\mathcal{E}}(\gamma) \leq \wp_{\max}(\gamma)/2$;
2. Every transition (of $\mathcal{M}_{\mathcal{F}}$) from S to $\bar{S}$ reverses the edges of γ;
3. $\Phi(S) = \Phi(\mathcal{M}_{\mathcal{F}})$.

Let $k = |\eta|$. There are 2^k orientations of the edges of η, and, therefore, the fraction of Ω in which γ is clockwise oriented is $> 2^{-k}$. Thus, if k is a constant, we would expect the conductance of S to be at least bounded below by some inverse polynomial. Thus, we make the following conjecture:

Conjecture 4.1. *Let k be some fixed constant and suppose G is a planar graph that can be embedded in the plane such that none of the bounded faces have more than k edges. Then the face-reversal chain is rapidly mixing.*

In Chapter 3 we analysed the number of Eulerian orientations and Euler tours of random graphs. In particular, we were able to obtain asymptotic distributions for the number of Euler tours of a random d-in/d-out directed graph and the number of Eulerian orientations of a random $2d$-regular graph. Intuitively, one would expect that these results would combine to enable us to find an asymptotic characterisation of the distribution of the number of Euler tours of a random $2d$-regular graph. However, this result proved elusive. We did, however, make a conjecture regarding the ratio of the second moment and the square of the first (for the random variable counting Euler tours of a random $2d$-regular graph). In §3.5, we showed how, if this conjecture is true, we would be able to obtain an asymptotic distribution for the number of Euler tours of a random $2d$-regular graph. Although we did not prove this conjecture, we did perform an empirical investigation which provided some evidence to support it; see Table 3.1. Not only do our empirical studies support the conjecture upon which the proof of the asymptotic distribution depends, but the corresponding ratio of the random variable counting Eulerian orientations of $2d$-regular graphs (obtained in §3.4) matches the conjectured ratio for the random variable counting Euler orientation of random $2d$-regular graphs.

In the case of Euler tours of d-in/d-out directed graphs we were able to show that almost every digraph $G \in \vec{\mathbb{G}}(n,d)$ has close to the maximum number of Euler tours possible. A consequence of this is that a pair of simple algorithms, Algorithm 4 and Algorithm 5, satisfy the conditions of an *fpras* and an *fpaus* respectively, for the Euler tours of *almost every* d-regular Eulerian digraph. It is already known that we can sample Euler tours of every Eulerian orientation in polynomial time so this result is not so interesting in itself. However, unlike other polynomial time algorithms for this problem, Algorithm 4 and Algorithm 5 generalise naturally to the undirected case. That is, if the conjectured asymptotic distribution for the number of Euler tours of undirected $2d$-regular graphs is true, and we have provided (reasonably) convincing evidence of this, then Algorithm 4 and Algorithm 5 will be an *fpaus* and an *fpras* for the Euler tours of almost every $2d$-regular graph.

The asymptotic distributions of Chapter 3 suggest a strong connection between the number of short cycles of different lengths and the number of Euler tours of graphs. This matches the results of §3.6, where we showed that the toroidal grid, which has many short cycles, has much less Euler tours than the expected value. This also relates to the work of Lieb [61] and Baxter [6] on the Eulerian orientations of the square and triangular lattices. Our analysis of the number of Eulerian orientations of random $2d$-regular graphs, presented in §3.4 shows that the number of Eulerian orientations of each of these lattices is very far from typical. It would be interesting to try to prove a non-probabilistic result along the lines of the asymptotic distribution results, e.g., showing that graphs which do not have many short cycles, or possibly graphs with large girth, have a number of Euler tours (resp. Eulerian orientations) that is close the expected value we can obtain from Theorem 3.53 (resp. Theorem 3.49).

To conclude, we have, in the work presented this thesis, extended the body of knowledge regarding the problems of counting Euler tours and Eulerian orientations in several ways. We have analysed the complexity of exactly counting Eulerian orientations of planar graphs. We have made some further progress towards classifying the planar graphs on which the natural *face-reversal* Markov chain is rapidly mixing. Finally, in the most significant part of this thesis, we have made progress towards obtaining an algorithmic result on the complexity of sampling and approximately counting Euler tours of random graphs. The problem of sampling (and approximately counting) Euler tours in polynomial time is, as explained in §1.3.2, a hard open problem in the area of sampling algorithms. We hope that the work presented in Chapter 3 of this thesis will prove useful to future students of this problem.

Appendix A

The interlace Matrix approach to counting Euler tours

An alternative approach to the determinant method (Theorem 1.57 and Theorem 1.59) for counting Euler tours of directed graphs was provided by Macris and Pulé in [64]. The justification for the claimed result in [64] was by complicated analytic calculation, but Lauri attempted to provide a combinatorial proof in [59]. Unfortunately, both these attempted proofs are false. In this section we explain the approach and provide a counter-example.

Definition A.1. Let G be a 2-regular Eulerian digraph on $V = \{1,2,\ldots,n\}$ and let $\mathcal{T}$ be some arbitrary Euler tour of G. We say u and v interlace on $\mathcal{T}$ if they alternate on $\mathcal{T}$; that is, $\mathcal{T} = (u\ldots v\ldots u\ldots v\ldots)$. We use $u \sim v$ to denote *"u and v interlace on $\mathcal{T}$"* The interlace matrix of $\mathcal{T}$ is then defined as

$$I(\mathcal{T})_{u,v} = \begin{cases} 1 & \text{if } u \sim v \text{ and } u < v; \\ -1 & \text{if } u \sim v \text{ and } u > v; \\ 0 & \text{otherwise}. \end{cases}$$

Macris and Pulé make following claim.

Claim A.2. *Let G be a 2-regular Eulerian digraph on $V = \{1,2,\ldots,n\}$, and let I_n be the $n \times n$ identity matrix. Then, for any Euler tour $\mathcal{T}$ of G,*

$$|ET(G)| = |I_n + I(\mathcal{T})|.$$

However, there exists a simple example for which the above claim does not hold! Suppose we have the Eulerian orientation $\mathcal{E}$ of K_5 given in Figure A.1. The Laplacian

Figure A.1: Counter-example to Claim A.2

matrix of this Eulerian orientation is

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

By the BEST theorem, we can easily compute the number of Euler tours of $\mathcal{E}$ to be 11. Now, consider the Euler tour $\mathcal{T} \in ET(\mathcal{E})$: $(1,5,4,2,3,4,1,2,5,3)$. The interlace matrix of $\mathcal{T}$ is

$$I(\mathcal{T}) = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 \\ 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Calculating the determinant $|I_5 + I(\mathcal{T})|$ gives 15! Thus, even for this small example Claim A.2 does not hold.

Bibliography

[1] David Aldous. Random walks on finite groups and rapidly mixing Markov chains. In *Seminar on Probability XVII*, pp. 243–297. Springer, 1983.

[2] David Allison and Nicolai Reshetikhin. Numerical study of the 6-vertex model with domain wall boundary conditions. *Annales de L'institut Fourier* **55** (2005), 1847–1869.

[3] Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.

[4] Richard Arratia, Béla Bollobás, Don Coppersmith, and Gregory B. Sorkin. Euler circuits and DNA sequencing by hybridization. *Discrete Applied Mathematics* **104** (2000), 63–96.

[5] Hilda Assiyatun and Nicholas C. Wormald. 3-star factors in random d-regular graphs. *European Journal of Combinatorics* **27** (2006), 1249–1262.

[6] R. J. Baxter. F model on a Triangular Lattice. *Journal of Mathematical Physics* **10** (1969), 1211–1216.

[7] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic Press, 1982.

[8] Mohsen Bayati, David Gamarnik, Dimitriy A. Katz, Chandra Nair, and Prasad Tetali. Simple deterministic approximation algorithms for counting matchings. In *Proceedings of the 39th Annual ACM Symposium on Theory of Computing (STOC 2007)*, pp. 122–127.

[9] Edward A. Bender and E. Rodney Canfield. The Asymptotic Number of Labeled Graphs with Given Degree Sequences. *Journal of Combinatorial Theory, Series A*, **24** (1978), 296–307.

[10] Ivona Bezáková, Daniel Stefankovic, Vijay V. Vazirani, and Eric Vigoda. Accelerating simulated annealing for the permanent and combinatorial counting problems. In *Proceedings of the 17th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2006)*, pp. 900–907.

[11] Béla Bollobás. A probabilistic proof of an asymptotic formula for the number of labelled regular graphs. *European Journal of Combinatorics* **1** (1980), 311–316.

[12] Béla Bollobás. *Random Graphs*. Number 73 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, second edition, 2001.

[13] Magnus Bordewich and Martin E. Dyer. Path coupling without contraction. *Journal of Discrete Algorithms* **5** (2007), 280–292.

[14] Magnus Bordewich, Martin E. Dyer, and Marek Karpinski. Stopping Times, Metrics and Approximate Counting. In *Proceedings of Automata, Languages and Programming, 33rd International Colloquium (ICALP 2006)*, pp. 108–119.

[15] Graham Brightwell and Peter Winkler. Counting Eulerian Circuits is #P-Complete. In *Proceedings of the 2nd Workshop on Analytic Algorithmics and Combinatorics (ANALCO 2005)*, pp. 259–262.

[16] Russ Bubley and Martin E. Dyer. Path Coupling: A Technique for Proving Rapid Mixing in Markov Chains. In *Proceedings of the 38th Annual Symposium on Foundations of Computer Science (FOCS 1997)*, pp. 223–231.

[17] Prasad Chebolu, Mary Cryan, and Russell Martin. Exact counting of Euler tours for series-parallel graphs. Technical report, arxiv:1005.3477 (2010)

[18] Charles J. Colbourn, Robert P. J. Day, and Louis D. Nel. Unranking and Ranking Spanning Trees of a Graph. *Journal of Algorithms* **10** (1989), 271–286.

[19] Charles J. Colbourn, Wendy J. Myrvold, and Eugene Neufeld. Two Algorithms for Unranking Arborescences. *Journal of Algorithms* **20** (1996), 268–281.

[20] Stephen A. Cook. The Complexity of Theorem-Proving Procedures. In *Proceedings of 3rd Annual ACM Symposium on Theory of Computing (STOC 1971)*, pp. 151–158.

[21] Don Coppersmith and Shmuel Winograd. Matrix Multiplication via Arithmetic Progressions. *Journal of Symbolic Computation* **9** (1990), 251–280.

[22] Mary Cryan and Martin E. Dyer. A polynomial-time algorithm to approximately count contingency tables when the number of rows is constant. *Journal of Computer System Sciences* **67** (2003), 291–310.

[23] Mary Cryan, Martin E. Dyer, and Dana Randall. Approximately counting integral flows and cell-bounded contingency tables. In *Proceedings of 37th Annual ACM Symposium on Theory of Computing (STOC 2005)*, pp. 413–422.

[24] Paul Dagum and Michael Luby. Approximating Probabilistic Inference in Bayesian Belief Networks is NP-Hard. *Artificial Intelligence* **60** (1993), 141–153.

[25] Nicolaas Govert de Bruijn. *Asymptotic Methods in Analysis*. North Holland Publishing Co., 1958.

[26] Persi Diaconis and Laurent Saloff-Coste. Comparison theorems for reversible Markov chains. *Annals of Applied Probability* **3** (1993), 696–730.

[27] Martin E. Dyer. Approximate counting by dynamic programming. In *Proceedings of the 35th Annual ACM Symposium on Theory of Computing (STOC 2003)*, pp. 693–699.

[28] Kari Eloranta. Diamond ice. *Journal of Statistical Physics* **96** (1999), 1091–1109.

[29] Kari Eloranta. Archimedean Ice. Technical report, arXiv:0909.4007 (2009).

[30] Leonhard Euler. Solutio problematis ad geometriam situs pertinetis (The solution of a problem relating to the geometry of position) . *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, **8** (1736), 128–140.

[31] Johannes Fehrenbach and Ludger Rüschendorf. A Markov chain algorithm for Eulerian orientations of planar triangular graphs. In *Proceedings of the 3rd Colloqium on Mathematics and Computer Sciences, Algorithms, Trees, Combinatorics, and Probabilities (MathInfo 2004)*, pp. 429–440.

[32] Stefan Felsner. Lattice Structures from Planar Graphs. *Electronic Journal of Combinatorics* **11** (2004).

[33] Stefan Felsner and Florian Zickfeld. On the Number of α-Orientations. In *Proceedings of Graph-Theoretic Concepts in Computer Science, 33rd International Workshop (WG 2007)*, pp. 190–201.

[34] Alan Frieze and Eric Vigoda. A Survey on the use of Markov Chains to Randomly Sample Colorings, in *Combinatorics, Complexity and Chance: A Tribute to Dominic Welsh*, G. Grimmett and C. McDiarmaid *eds*, pp. 53–71. Oxford Lecture Series in Mathematics and Its Applications **34**. Oxford University Press, 2007.

[35] Alan M. Frieze. Finding Hamilton cycles in sparse random graphs. *Journal of Combinatorial Theory, Series B*, **44** (1988), 230–250.

[36] Alan M. Frieze, Mark Jerrum, Michael Molloy, Robert W. Robinson, and Nicholas C. Wormald. Generating and Counting Hamilton Cycles in Random Regular Graphs. *Journal of Algorithms* **21** (1996), 176–198.

[37] Qi Ge and Daniel Stefankovic. The Complexity of Counting Eulerian Tours in 4-Regular Graphs. In *Proceedings of the 9th Latin American Theoretical Informatics Symposium (LATIN 2010)*, pp. 638–649.

[38] Leslie Ann Goldberg, Russell A. Martin, and Mike Paterson. Random sampling of 3-colorings in $\mathbb{Z}^2$. *Random Structures & Algorithms* **24** (2004), 279–302.

[39] Mordecai J. Golin, Yiu-Cho Leung, Yajun Wang, and Xuerong Yong. Counting Structures in Grid Graphs, Cylinders and Tori Using Transfer Matrices: Survey and New Results. In *Proceedings of the 2nd Workshop on Analytic Algorithmics and Combinatorics (ANALCO 2005)*, pp. 250–258.

[40] Ian P. Goulden and David M. Jackson. *Combinatorial Enumeration*. Dover, 2000.

[41] Geoffrey Grimmett and David Stirzaker. *Probability and Random Processes*. Oxford University Press, 2001.

[42] A. Guénoche. Random Spanning Tree. *Journal of Algorithms* **4** (1983), 214–220.

[43] Svante Janson. Random Regular Graphs: Asymptotic Distributions and Contiguity. *Combinatorics, Probability & Computing* **4** (1995), 369–405.

[44] Svante Janson, Tomasz Łuczak, and Andrzej Riciński. *Random Graphs*. Wiley, 2000.

[45] Mark Jerrum. Two-dimensional monomer-dimer systems are computationally intractable. *Journal of Statistical Physics* **48** (1987), 121–134.

[46] Mark Jerrum. A very simple algorithm for estimating the number of k-colourings of a low-degree graph. *Random Structures and Algorithms* **7** (1995), 157–165.

[47] Mark Jerrum. *Counting, Sampling and Integrating: Algorithms and Complexity*. Lectures in Mathematics. ETH Zürich. Birkhäuser, 2003.

[48] Mark Jerrum. Personal communication, 2006.

[49] Mark Jerrum and Alistair Sinclair. Approximating the Permanent. *SIAM Journal on Computing* **18** (1989), 1149–1178.

[50] Mark Jerrum and Alistair Sinclair. Polynomial-Time Approximation Algorithms for the Ising Model. *SIAM Journal on Computing* **22** (1993), 1087–1116.

[51] Mark Jerrum, Alistair Sinclair, and Eric Vigoda. A polynomial-time approximation algorithm for the permanent of a matrix with nonnegative entries. *Journal of the ACM* **51** (2004), 671–697.

[52] Mark Jerrum, Leslie G. Valiant, and Vijay V. Vazirani. Random Generation of Combinatorial Structures from a Uniform Distribution. *Theoretical Computer Science* **43** (1986), 169–188.

[53] Denise B. Kandel, Yossi Matias, Ron Unger, and Peter Winkler. Shuffling Biological Sequences. *Discrete Applied Mathematics* **71** (1996), 171–185.

[54] Piet W. Kasteleyn. Graph theory and crystal physics. In *Graph Theory and Theoretical Physics*, F. Harary *ed*, pp. 43–110. Academic Press, 1997.

[55] Gustav Kirchoff. Über die Auflüsung der Gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer Ströme geführt wird. *Annalen der Physik und Chemie* **72** (1847), 497–508.

[56] Anton Kötzig. Eulerian lines in finite 4-valent graphs and their transformations. In *Theory of Graphs*, P. Erdös and G. Katona *eds*, pp. 219–230. Academic Press, 1966.

[57] Vidyadhar G. Kulkarni. Generating Random Combinatorial Objects. *Journal of Algorithms* **11** (1990), 185–207.

[58] V. S. Anil Kumar and H. Ramesh. Coupling vs. conductance for the Jerrum-Sinclair chain. *Random Structures & Algorithms* **18** (2001), 1–17.

[59] J. Lauri. On a formula for the number of Euler trails for a class of digraphs. *Discrete Mathematics* **163** (1997), 307–312.

[60] Leonid Levin. Universal Search Problems. *Problems of Information Transmission* **9** (1973), 265–266.

[61] Elliot H. Lieb. Residual Entropy of Square Ice. *Physical Review* **162** (1967), 162–172.

[62] Michael Luby, Dana Randall, and Alistair Sinclair. Markov Chain Algorithms for Planar Lattice Structures. *SIAM Journal on Computing* **31** (2001), 167–192.

[63] Tomasz Luczak and Eric Vigoda. Torpid mixing of the Wang-Swendsen-Kotecký algorithm for sampling colorings. *Journal of Discrete Algorithms* **3** (2005), 92–100.

[64] N. Macris and J. V. Pulé. An alternative formula for the number of Euler trails for a class of digraphs. *Discrete Mathematics* **154** (1996), 301–305.

[65] Brendan D. McKay. The asymptotic numbers of regular tournaments, Eulerian digraphs and Eulerian oriented graphs. *Combinatorica* **10** (1990), 367–377.

[66] Brendan D. McKay and Robert W. Robinson. Asymptotic Enumeration of Eulerian Circuits in the Complete Graph. *Combinatorics, Probability & Computing* **7** (1998), 437–449.

[67] Brendan D. McKay and Nicholas C. Wormald. Asymptotic enumeration by degree sequence of graphs with degress $o(n^{1/2})$. *Combinatorica* **11** (1991), 369–382.

[68] Milena Mihail and Peter Winkler. On the number of Eulerian orientations of a graph. In *Proceedings of the 3rd Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 1992)*, pp. 138–145.

[69] Milena Mihail and Peter Winkler. On the Number of Eulerian Orientations of a Graph. *Algorithmica* **16** (1996), 402–414.

[70] Michael Mitzenmacher and Eli Upfal. *Probability and Computing: Randomized Algorithms and Probabilistic Analysis*. Cambridge University Press, 2005.

[71] Michael S. O. Molloy, Hanna D. Robalewska, Robert W. Robinson, and Nicholas C. Wormald. 1-Factorizations of random regular graphs. *Random Structures and Algorithms* **10** (1997), 305–321.

[72] James Gary Propp and David Bruce Wilson. How to Get a Perfectly Random Sample From a Generic Markov Chain and Generate a Random Spanning Tree of a Directed Graph. *Journal of Algorithms* **27** (1998), 170–217.

[73] Dana Randall and Prasad Tetali. Analyzing Glauber dynamics by comparison of Markov chains. *Journal of Mathematical Physics* **41** (2000), 1598–1615.

[74] Hanna D. Robalewska. 2-factors in random regular graphs. *Journal of Graph Theory* **23** (1996), 215–224.

[75] Robert W. Robinson and Nicholas C. Wormald. Existence of long cycles in random cubic graphs. In *Enumeration and Design*, D. M. Jackson and S. M. Vanstone *eds*, pp. 251–270. Academic Press, 1984.

[76] Robert W. Robinson and Nicholas C. Wormald. Almost All Cubic Graphs Are Hamiltonian. *Random Structures & Algorithms* **3** (1992), 117–126.

[77] Robert W. Robinson and Nicholas C. Wormald. Almost All Regular Graphs Are Hamiltonian. *Random Structures & Algorithms* **5** (1994), 363–374.

[78] Alexander Schrijver. Bounds on the number of Eulerian orientations. *Combinatorica* **3** (1983), 375–380.

[79] Alistair Sinclair and Mark Jerrum. Approximate Counting, Uniform Generation and Rapidly Mixing Markov Chains. *Information and Compututation* **82** (1989), 93–133.

[80] C. A. B. Smith and W. T. Tutte. On unicursal paths in networks of degree 4. *American Mathematical Monthly* **48** (1941), 233–237.

[81] Richard P. Stanley. *Enumerative Combinatorics*, Volume 2. Cambridge University Press, 2001.

[82] J. Michael Steele. *"The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities"*. MAA Problem Books Series, Cambridge University Press, 2004.

[83] Angelika Steger and Nicholas C. Wormald. Generating Random Regular Graphs Quickly. *Combinatorics, Probability & Computing* **8** (1999), 377–396.

[84] Gilbert Strang. *Introduction to Linear Algebra*, 3rd edition. Wellesley-Cambridge Press, 2003.

[85] Volker Strassen. Gaussian Elimination is not Optimal. *Numerical Mathematics* **13** (1969), 354–356.

[86] Gaston Tarry. Géométrie de situation: Nombre de manières distinctes de parcourir en une seule course toutes les allées dâun labyrinthe rentrant, en ne passant quâune seule fois par chacune des allées. Comptes Rendus Assoc. Franc. Avance. Sci. **15** (1887), 29–53.

[87] Prasad Tetali and Santosh Vempala. Random Sampling of Euler Tours. *Algorithmica* **30** (2001), 376–385.

[88] Seinosuke Toda. PP is as hard as the polynomial-time hierarchy. *SIAM Journal on Computing* **20** (1991), 865–877.

[89] Boris A. Trakhenbrot. A survey of Russian approaches to perebor (brute-force searches) algorithms. *Annals of the History of Computing* **6** (1984), 384–400.

[90] W.T. Tutte. *Graph Theory*, volume 21 of *Encyclopedia of Mathematics and it Applications*. Addison-Wesley Publishing Company, 1984.

[91] Salil P. Vadhan. The Complexity of Counting in Sparse, Regular, and Planar Graphs. *SIAM Journal on Computing* **31** (2001), 398–427.

[92] Leslie G. Valiant. The complexity of computing the permanent. *Theoretical Computer Science* **8** (1979), 189–201.

[93] Leslie G. Valiant. The complexity of enumeration and reliability problems. *SIAM Journal on Computing* **8** (1979), 410–421.

[94] Tanja van Aardenne-Ehrenfest and Nicholas G. de Bruijn. Circuits and trees in oriented linear graphs. *Simon Stevin* **28** (1951), 203–217.

[95] Michel Las Vergnas. An upper bound for the number of Eulerian orientations of a regular graph. *Combinatorica* **10** (1990), 61–65.

[96] Dror Weitz. Counting independent sets up to the tree threshold. In *Proceeedings of the 38th Annual ACM Symposium on Theory of Computing (STOC 2006)*, pp. 140–149.

[97] D. J. A. Welsh. The Computational Complexity of Some Classical Problems from Statistical Physics. In *Disorder in Physical Systems: A Volume in Honour of John M. Hammersley*, G. Grimmett and D. Welsh *eds*, pp. 307–321. Oxford University Press, 1990.

[98] Herbert S. Wilf. *Algorithms and Complexity*. A. K. Peters, 2003.

[99] David Bruce Wilson and James Gary Propp. How to Get an Exact Sample From a Generic Markov Chain and Sample a Random Spanning Tree From a Directed Graph, Both Within the Cover Time. In *Proceedings of the 7th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 1996)*, pages 448–457.

[100] Nicholas C. Wormald. Models of Random Regular Graphs In *Surveys in Combinatorics 1999*, volume 276 of *London Mathematical Society Lecture Note Series*, pp. 239–298. Cambridge University Press, 1999.