SELF-SIMILAR PROBLEMS IN GAS DYNAMICS

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CONTENTS

CHAPTER 1. INTRODUCTION

In this work we examine self-similar solutions of the equations governing unsteady, two-dimensional, isentropic flow of a polytropic gas, that is solutions of the equations in which the independent variables x, y and t occur only in the combinations x/t and y/t.

In Chapter 2 we develop the basic equations, showing the form which they take under the assumption that the independent variables occur only in the above combinations and some account is given of the type of problem which can be solved in terms of these special coordinates. The hodograph plane is introduced and a derivation is given of the partial differential equation satisfied by the sound speed when regarded as a function of the cartesian velocity components. The important equations providing the transformation between the physical and hodograph planes are given and the concepts of simple and mixed waves are explained. The chapter concludes with a survey of some of the existing work in the subject.

Chapter 3 gives a discussion of four exact self-similar solutions. It is pointed out that, although three of these four are well known, in some cases they take on a new aspect when viewed as solutions of the self-similar equations of gas dynamics. For example, one such case is that of steady, uniform flow. When regarded as a solution of the equations governing two-dimensional steady gas dynamics, uniform flow has associated with it no real characteristics if the flow is subsonic and two families of parallel lines if the flow is supersonic. However, when viewed as a solution of the self-similar gas dynamic equations, uniform flow has an entirely different pattern of characteristics in which there are no real characteristics inside a certain circle whilst outside the circle the two characteristics through any point are just the two tangents to the circle drawn through that point: and this regardless of whether the flow is subsonic or supersonic.

The least well known of the four solutions discussed is a remarkable exact solution obtained by Suchkov, which describes the expansion into vacuum of a wedge of gas.

In Chapter 4 are presented three problems which are tackled by linearised theory, the solutions sought being perturbations on three of the exact solutions presented in Chapter 3. The second of these problems is formulated in the hodograph plane and leads to a singular perturbation problem in which the domain in which a solution is sought vanishes as the perturbation parameter tends to zero. The problem is overcome by a boundary layer type stretching of coordinates. The results obtained are shown to be in agreement with some obtained by Powell. The same problem leads naturally to a discussion of a paper due to Anderson. It is suggested that some of Anderson's results are not correct and alternative conclusions are put forward.

Chapter 5 is concerned with two non-linearised problems. The second of these is tackled in the hodograph plane by a numerical method and results are given. These two problems involve unsteady flows of Prandtl-Meyer type and they illustrate how the classical steady Prandtl-Meyer solution describing supersonic flow into vacuum around a sharp corner can be considered as the limit as time $t \to \infty$ of the solution of an initial value problem.

CHAPTER 2. FUNDAMENTALS

2.1 The Basic Equations

The equations governing unsteady, two dimensional, isentropic gas dynamics are the momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad (2.1.1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \qquad (2.1.2)$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (2.1.3)$$

where $u(x, y, t)$, $v(x, y, t)$, $p(x, y, t)$ and $\rho(x, y, t)$ are the x and y components of velocity, the pressure and the density of the gas respectively.

If the gas is taken to be polytropic, then the density and pressure satisfy the relation

$$p = K\rho^{\gamma} \qquad (2.1.4)$$

where K is a constant and γ is the adiabatic index of the gas (which may be taken to be between 1 and 3). The local speed of sound $c(x, y, t)$ is then given by

$$c^2 = \frac{dp}{d\rho} = K\gamma\rho^{\gamma-1} = \frac{\gamma p}{\rho} \qquad (2.1.5)$$

and, since $\gamma \neq 1$, (2.1.5) may be used to eliminate both p and ρ from (2.1.1), (2.1.2) and (2.1.3) to give the system

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{c}{\kappa}\frac{\partial c}{\partial x} = 0 \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{c}{\kappa}\frac{\partial c}{\partial y} = 0 \qquad (2.1.6) \\
&\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + \kappa c\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0
\end{aligned}$$

where $\kappa = \frac{\gamma - 1}{2}$.

2.2 Similarity Solutions

One method of attacking (2.1.6) is to seek solutions in which the independent variables x, y and t do not occur truly independently but only in the combinations

$$f(x, y, t) \quad \text{and} \quad g(x, y, t) .$$

Such a solution is termed self-similar, since, if u, v and c are known for all x and y at a time $t = t_0$, during the motion, then they are known for all x and y for all subsequent $t > t_0$. If u, v and c are each functions of f(x, y, t) and g(x, y, t) then there exists a functional relation between u, v and c and we may write c = c(u, v). Conversely, if c = c(u, v) then u, v and c are each dependent upon two functions f(x, y, t) and g(x, y, t). This suggests that an alternative approach might be to consider u and v as independent variables and to seek a solution for which c = c(u, v). This is the hodograph plane method, a method which has proved very useful in two dimensional, steady gas dynamics. We shall return to describe it later.

Interesting as it may be to consider solutions of (2.1.6) in which u, v and c are dependent on x, y and t only in the combinations f(x, y, t) and g(x, y, t), consideration of many physical problems leads us to conclude that one particular choice of the functions f and g may be of greater interest than any other.

Consider the physical problem in which gas, initially at rest, at constant pressure, in the quarter plane x > 0, y > 0 is suddenly allowed to expand into vacuum by the removal of the walls x = 0 and y = 0. Since the problem contains no length or time parameter in its formulation, except in the form of a velocity, we deduce that the independent variables x, y and t can occur only in the combinations x/t and y/t. Or again, consider the problem in which gas, initially flowing uniformly in the x direction filling the half plane $y > 0$, is suddenly permitted to expand into vacuum by removal of the boundary $y = 0, x > 0$. The argument used above enables us again to assert that x, y and t can occur only as x/t and y/t.

Many problems in two dimensional, unsteady gas dynamics may be formulated without reference to any length or time parameter, except in the form of a velocity, and so must have solutions involving x, y and t only in the form x/t and y/t. The problems which we shall describe are all of this type and so henceforth we shall restrict the term 'self-similar' to solutions of (2.1.6) in which x, y and t occur only in the combinations x/t and y/t.

The substitutions

$$X = \frac{x}{c_o t}, \quad Y = \frac{y}{c_o t}, \quad U = \frac{u}{c_o}, \quad V = \frac{v}{c_o}, \quad F = \frac{c}{c_o}$$

where c_o is some reference velocity, which will, whenever appropriate, be taken to be the speed of sound in the gas at rest, reduce (2.1.6) to

$$\begin{aligned}
&(U - X)U_X + (V - Y)U_Y + \frac{1}{\kappa} FF_X = 0 \\
&(U - X)V_X + (V - Y)V_Y + \frac{1}{\kappa} FF_Y = 0 \qquad\qquad (2.2.1) \\
&(U - X)F_X + (V - Y)F_Y + \kappa F(U_X + V_Y) = 0
\end{aligned}$$

where the subscripts denote partial derivatives.

If, in addition, the flow is irrotational, as will be the case in the problems we shall consider, we can introduce a dimensionless velocity potential and derive the second order partial differential equation which it satisfies. In the rest of this section, as in the next section, on the hodograph plane, we follow the approach of Mackie (1966).

Irrotationality ensures that there exists a dimensionless velocity potential $\Phi(X, Y)$, related to the physical velocity potential $\phi(x, y, t)$ by

$$\phi(x, y, t) = c_o{}^2 t \ \Phi(X, Y) \tag{2.2.2}$$

for which

$$U = \Phi_X , \quad V = \Phi_Y \tag{2.2.3}$$

If the values of F_X and F_Y from the first and second members of (2.2.1) are substituted into the third, the equation

$$[(U - X)^2 - F^2]\Phi_{XX} + 2(U - X)(V - Y)\Phi_{XY} + [(V - Y)^2 - F^2]\Phi_{YY} = 0$$

results. Using (2.2.3) we obtain the equation

$$[(\Phi_X - X)^2 - F^2]\Phi_{XX} + 2(\Phi_X - X)(\Phi_Y - Y)\Phi_{XY} + [(\Phi_Y - Y)^2 - F^2]\Phi_{YY} = 0 \tag{2.2.4}$$

for the dimensionless potential $\Phi(X, Y)$. This equation is clearly quasi-linear. The characteristics are given by

$$\left.\frac{dY}{dX}\right|_{\mathcal{C}\pm} = \frac{(U - X)(V - Y) \pm F[(U - X)^2 + (V - Y)^2 - F^2]^{\frac{1}{2}}}{[(U - X)^2 - F^2]}$$

or equivalently (2.2.5)

$$\left.\frac{dX}{dY}\right|_{\mathcal{C}\pm} = \frac{(U - X)(V - Y) \mp F[(U - X)^2 + (V - Y)^2 - F^2]^{\frac{1}{2}}}{[(V - Y)^2 - F^2]}$$

Equation (2.2.4) is valid throughout the region of flow in the X-Y plane. It is easily shown that the unsteady form of Bernoulli's theorem

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\phi_x{}^2 + \phi_y{}^2) + \int \frac{dp}{\rho} = \text{function of } t \tag{2.2.6}$$

becomes

$$\Phi - X\Phi_X - Y\Phi_Y + \frac{1}{2}(\Phi_X{}^2 + \Phi_Y{}^2) + \frac{F^2}{2\kappa} = \frac{F_o{}^2}{2\kappa} \tag{2.2.7}$$

with F_o a constant.

2.3 The Hodograph Plane

As we remarked in 2.2 an attempt at solving (2.2.1) may be made by seeking a solution for F of the form $F = F(U, V)$. We now derive the partial differential equation satisfied by F in the hodograph plane. We have

$$dU = U_X dX + U_Y dY$$
$$dV = V_X dX + V_Y dY$$
$$dF = F_X dX + F_Y dY .$$

If the Jacobian $J = \partial(U, V)/\partial(X, Y) \neq 0$ or ∞, dX and dY may be eliminated from the third equation of this system by means of the first two, giving

$$(U_X V_Y - V_X U_Y) dF = (F_X V_Y - F_Y V_X) dU + (F_Y U_X - F_X U_Y) dV .$$

Substituting for F_X and F_Y from the first two members of (2.2.1) and using the irrotationality condition $U_Y = V_X$ we obtain

$$X = U + \frac{1}{\kappa} FF_U \quad , \quad Y = V + \frac{1}{\kappa} FF_V . \tag{2.3.1}$$

These are the important equations which provide a transformation from the U-V to the X-Y plane. Substituting these into the last member of (2.2.1) we obtain

$$(F_U U_X + F_V V_X) F_U + (F_U U_Y + F_V V_Y) F_V = \kappa^2 (U_X + V_Y).$$

The homogeneity of this equation in U_X, U_Y, V_X and V_Y allows us to replace these derivatives by Y_V, $-X_V$, $-Y_U$ and X_U respectively. Furthermore, these last mentioned quantities may be obtained from (2.3.1). Hence we obtain, finally

$$(\kappa^2 - F_V^2) F_{UU} + 2F_U F_V F_{UV} + (\kappa^2 - F_U^2) F_{VV} = \frac{\kappa}{F} [(F_U^2 + F_V^2)(1 - \kappa) - 2\kappa^2] . \tag{2.3.2}$$

This equation is clearly quasi-linear and the characteristics are given by

$$\frac{dV}{dU} = \frac{F_U F_V \pm \kappa\sqrt{F_U^2 + F_V^2 - \kappa^2}}{(\kappa^2 - F_V^2)} . \tag{2.3.3}$$

Equation (2.3.2) is valid only in regions of the U-V plane in which the Jacobian $\partial(U, V)/\partial(X, Y)$ is neither zero nor infinite.

Any solution of (2.2.1) satisfying this requirement is termed a mixed wave.

Solutions of (2.2.1) for which the Jacobian $\partial(U, V)/\partial(X, Y)$ is identically zero may belong to one of two classes. The first is the class of solutions of the form U = U(F), V = V(F); any solution belonging to this class is termed a simple wave. The second class is made up of solutions U = constant, V = constant, F = constant. This is the class of uniform flows.

It is clear that any solution of (2.2.1) must be a mixed wave, a simple wave or a uniform flow. Further, it is clear from their definitions that a mixed wave region in the X-Y plane maps into a region in the U-V plane, whilst a simple wave region in the X-Y plane maps into a curve in the hodograph plane and a uniform flow maps into a point. Thus only mixed waves may be studied in regions of the hodograph plane.

The solutions to the physical problems which we shall consider will consist of combinations of mixed waves, simple waves and uniform flows.

2.4 Summary of Existing Work

Perhaps the first example of the use of similarity variables is to be found in a paper of Busemann (1943). In applying the linearised theory of supersonic flow to conefield flow he was able to use similarity reasoning to reduce a boundary value problem involving the wave equation in three variables to one involving the same equation in two variables.

Since then a number of physical problems have been successfully tackled by the use of similarity variables: some of these problems are described below.

Lighthill (1949) used Busemann's transformation in studying the linearised theory of the diffraction of a plane shock by a corner of small angle. An extension of Lighthill's work was carried out by Chester (1954).

Powell (1957) applied linearised theory to the problem of diffraction of a complete rarefaction wave by a corner whilst Anderson (1966) studied the diffraction of an incomplete rarefaction wave by a corner. We shall discuss these last two works in greater detail in 4.2.

In a non-linearised problem, Jones, Martin and Thornhill (1951) used similarity variables to discuss the diffraction or reflection of a plane shock, travelling parallel to a rigid wall, meeting a corner.

Several Russian authors have tackled problems by use of the hodograph plane. Pogodin, Suchkov and Ianenko (1958) studied the motion of a gas, initially at rest in $x > 0$, $y > 0$ caused by the walls, $x = 0$, $y = 0$, beginning, at time $t = 0$, to move away from the gas with constant speed.

Ermolin and Sidorov (1966) and Ermolin, Rubina and Sidorov (1968) investigated the ensuing motion of a gas, originally contained at rest between two walls meeting at angle $< \pi/2$, caused by the walls beginning, at time $t = 0$, to move away from the gas with constant speed.

Suchkov (1963) studied the motion of gas, initially contained at rest between two walls meeting at an angle $< \pi$, allowed, at time $t = 0$, to expand into vacuum by the instantaneous removal of the walls. He was able to show that when a certain relation between the adiabatic index of the gas and the angle between the walls was satisfied, the solution to the problem could be given explicitly in a remarkably simple form.

Levine (1968) has given a detailed study of Suchkov's problem, including some numerical results for a range of values of γ when the angle between the walls is $\pi/2$. The same author (1969) has derived certain important properties of simple waves in two dimensional, unsteady gas dynamics.

Mackie (1966) has showed how Suchkov's results can be interpreted in the physical plane. The same paper includes a discussion of unsteady Prandtl-Meyer flows..

Gorshkova and Stanukovich (1966) have also discussed unsteady Prandtl-eyer flows, as have Greenspan and Butler (1961) in a paper dealing with everal topics involving the expansion of a gas into vacuum.

CHAPTER 3. EXACT SOLUTIONS

There are four exact solutions of the system (2.2.1) to be found in the literature. In this chapter we shall discuss in turn each of these solutions both in the X-Y plane and, where appropriate, in the hodograph plane.

3.1 Uniform Flow

Let us consider steady, uniform flow of a polytropic gas given by

$$U = U_1 \;,\quad V = V_1 \;,\quad F = F_1 \qquad U_1, V_1, F_1 \text{ constants.}$$

Substituting these values into (2.2.5) we obtain the differential equation satisfied by the characteristics in the X-Y plane.

$$\left.\frac{dY}{dX}\right|_{\mathcal{C}\pm} = \frac{(U_1 - X)(V_1 - Y) \pm F_1[(U_1 - X)^2 + (V_1 - Y)^2 - F_1^2]}{[(U_1 - X)^2 - F_1^2]} \qquad (3.1.1)$$

where the $\mathcal{C}_+$ family of characteristics is obtained by taking the positive sign and the $\mathcal{C}-$ family by taking the negative sign. The change of variable

$$R \cos \theta = \frac{X - U_1}{F_1} \;,\qquad R \sin \theta = \frac{Y - V_1}{F_1}$$

leads to

$$\frac{dR}{d\theta} = \pm R \sqrt{R^2 - 1}$$

which has the solution

$$R \cos(\theta_o \pm \theta) = 1 \qquad \text{with } \theta_o \text{ constant.}$$

Thus in the region $(X - U_1)^2 + (Y - V_1)^2 < F_1^2$ there are no real characteristics. In the region $(X - U_1)^2 + (Y - V_1)^2 > F_1^2$ the two characteristics through any point are the two tangents to the circle $(X - U_1)^2 + (Y - V_1)^2 = F_1^2$ drawn from that point. The region interior to the circle $(X - U_1)^2 + (Y - V_1)^2 = F_1^2$ is therefore an elliptic region, the region exterior to this circle a hyperbolic region and the circle itself a

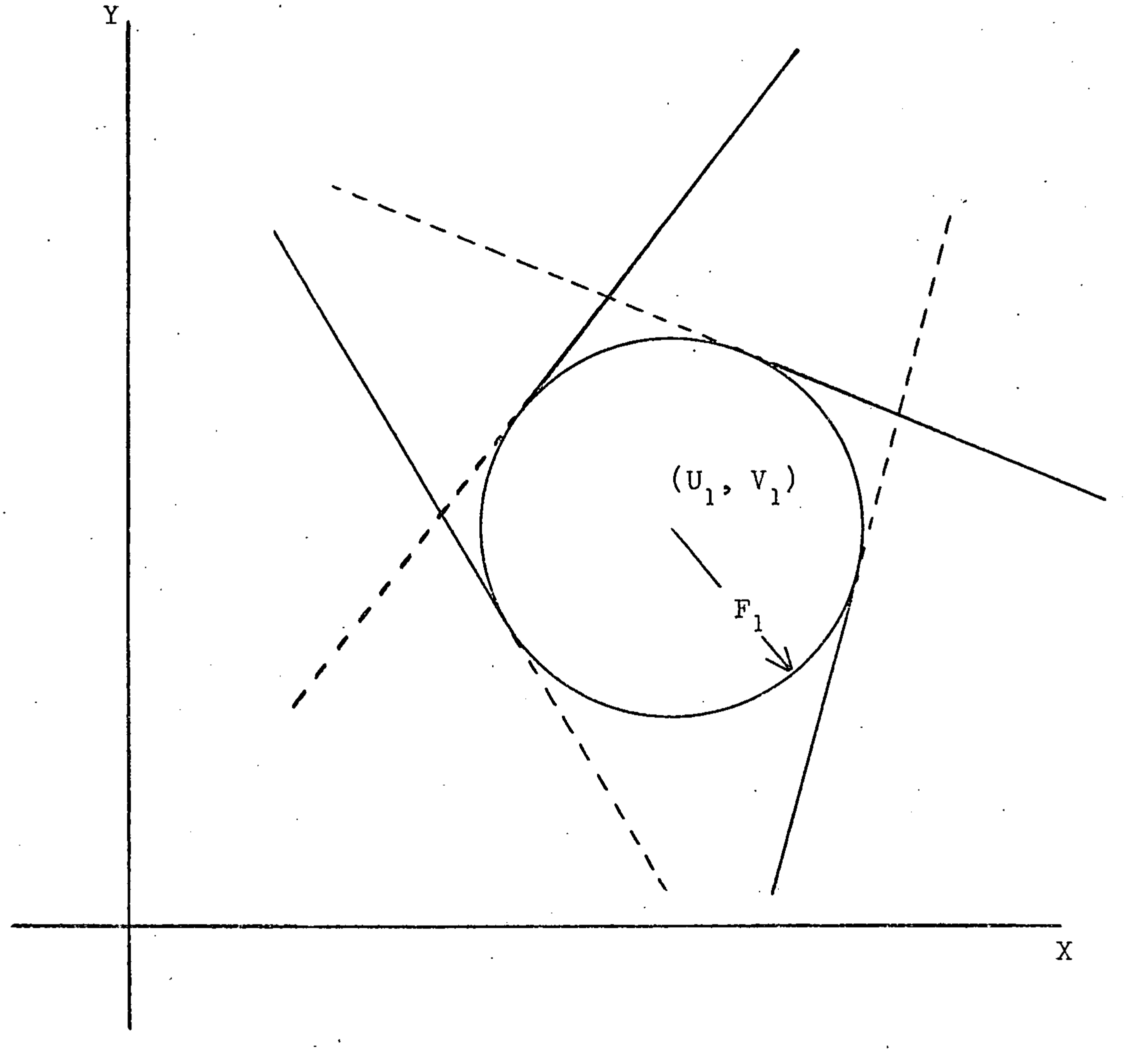


figure 1

parabolic line. The system of characteristics is shown in figure 1.

We conclude this description of the first of the four exact solutions by remaking the point which was made in Chapter 2, that a region of uniform flow in the X-Y plane maps into a single point in the hodograph plane.

3.2 Rarefaction Waves

Consider the situation shown in figure 2, in which polytropic gas at rest, at constant pressure, is contained in the half space $x > 0$. The gas is separated from the vacuum $x < 0$ by the rigid wall $x = 0$.

Suppose now that at some instant the wall $x = 0$ is suddenly withdrawn from the gas with constant speed, u_1. The resulting motion of the gas will clearly be both one-dimensional and self-similar. The system (2.2.1) reduces to the two coupled ordinary differential equations

$$(U - X)\frac{dU}{dX} + \frac{F}{\kappa}\frac{dF}{dX} = 0$$

and

$$(U - X)\frac{dF}{dX} + \kappa F\frac{dU}{dX} = 0$$

which, together with the appropriate boundary conditions are sufficient to determine the resulting motion. Writing $U_1 = u_1/c_0$, we find that there are two distinct cases to consider.

If $|U_1| \geqslant 1/\kappa$ the resulting flow is given by

$$U = \frac{X - 1}{1 + \kappa}, \quad V = 0, \quad F = \frac{1 + \kappa X}{1 + \kappa} \qquad -\frac{1}{\kappa} \leqslant X \leqslant 1. \qquad (3.2.1)$$

Such a flow is termed a complete rarefaction wave or Riemann wave.

If $|U_1| < 1/\kappa$ the resulting flow is given by

$$\begin{aligned} U &= \frac{X - 1}{1 + \kappa}, \quad V = 0, \quad F = \frac{1 + \kappa X}{1 + \kappa} && 1 - (1 + \kappa)U_1 \leqslant X \leqslant 1 \\ U &= -U_1, \quad V = 0, \quad F = 1 - \kappa U_1 && -U_1 \leqslant X \leqslant 1 - (1 + \kappa)U_1. \end{aligned} \qquad (3.2.2)$$

Such a flow is an incomplete rarefaction wave.

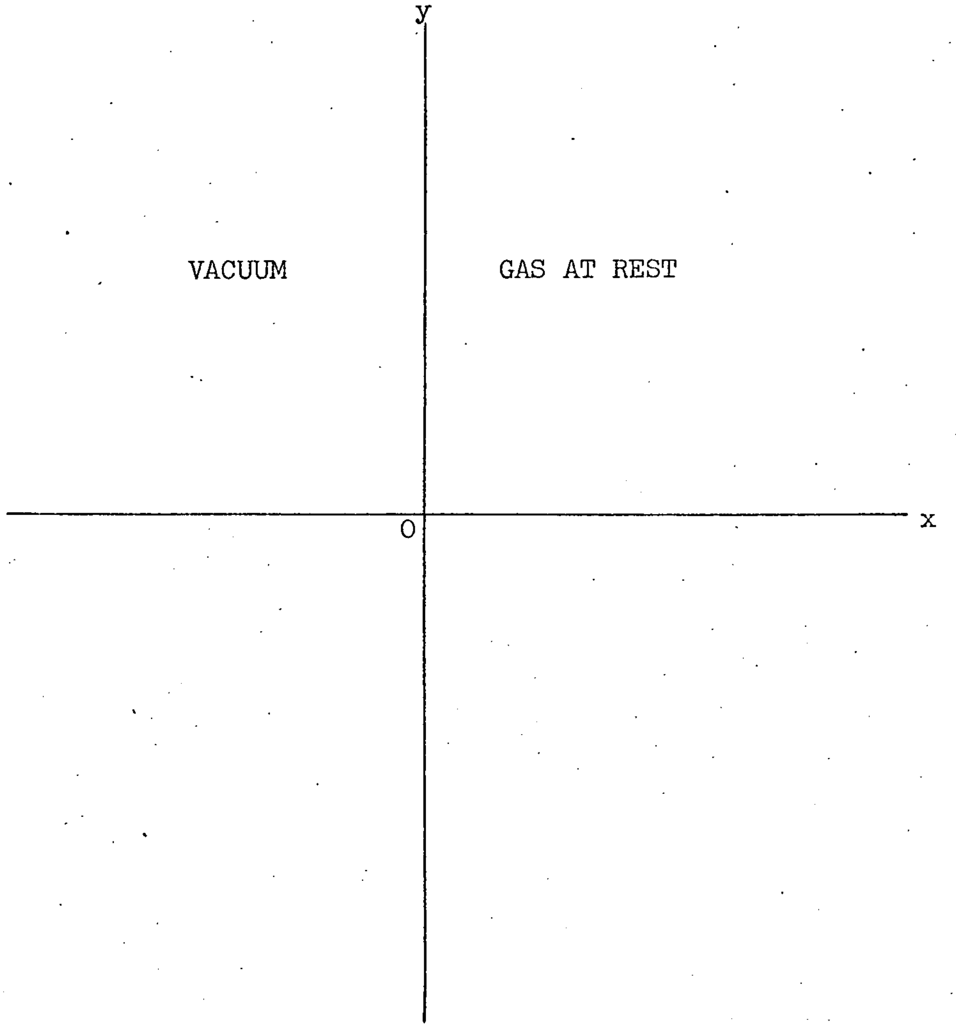


figure 2

Substituting the expressions for U, V and F from (3.2.1) into the second member of (2.2.5) we find that the characteristics of the Riemann wave satisfy the equation

$$\left.\frac{dX}{dY}\right|_{\mathcal{C}\pm} = \frac{\frac{1 + \kappa X}{1 + \kappa} Y \mp \frac{1 + \kappa X}{1 + \kappa} Y}{Y^2 - \left(\frac{1 + \kappa X}{1 + \kappa}\right)^2} . \tag{3.2.3}$$

Thus the family of $\mathcal{C}_+$ characteristics is seen to be made up of lines X = constant whilst the equation

$$\left.\frac{dX}{dY}\right|_{\mathcal{C}-} = \frac{2Y \frac{1 + \kappa X}{1 + \kappa}}{Y^2 - \left(\frac{1 + \kappa X}{1 + \kappa}\right)^2}$$

may be integrated to show that the family of $\mathcal{C}_-$ characteristics is given by

$$4(1 - \kappa^2)Y^2 = (2\kappa X + 2)^2 + A\,(2\kappa X + 2)^{\frac{1 + \kappa}{\kappa}} \tag{3.2.4}$$

where A is the parameter of the family. We observe that, for any value of κ, two of the characteristics (namely those given by $A = 0$) are straight lines. The system of characteristics of the Riemann wave for the case $\kappa = \frac{1}{2}$ $(\gamma = 2)$ is shown in figure 3. We note in passing that $Y = 0$ is a parabolic line although there is no region of ellipticity.

In the region $X > 1$ the gas is at rest which is, of course, a special case of uniform flow. The two families of characteristics in $X > 1$ are easily seen to be the two families of tangents to the circle $X^2 + Y^2 = 1$. Continuity of U, V and F across $X = 1$ ensures that the characteristics have no discontinuity of slope across this line. That the characteristics given by (3.2.4) do indeed have the required gradient at $X = 1$ may be verified by elementary means.

The characteristics of the incomplete rarefaction wave are identical to those of the complete rarefaction wave in the region $1 - (1 + \kappa)U_1 \leq X \leq 1$.

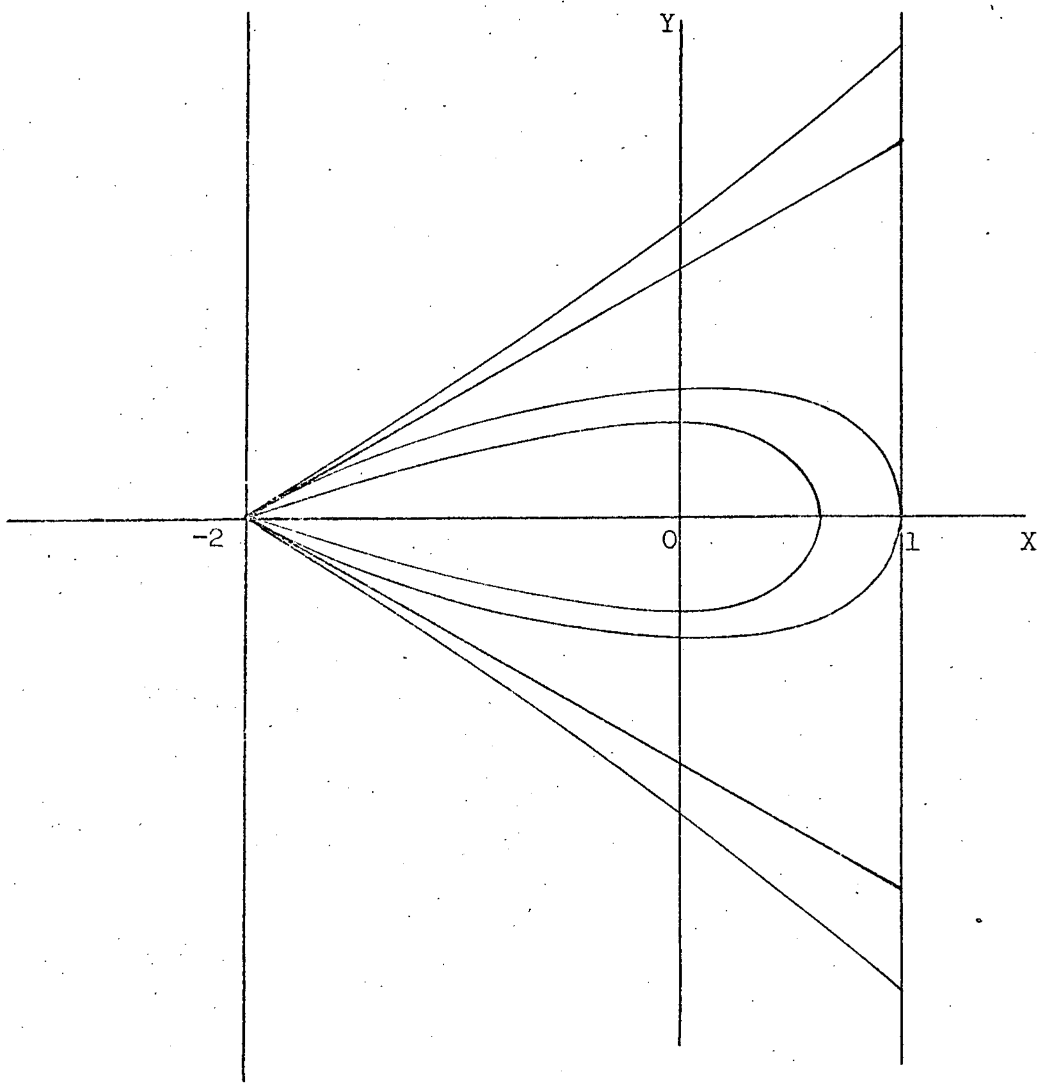


figure 3

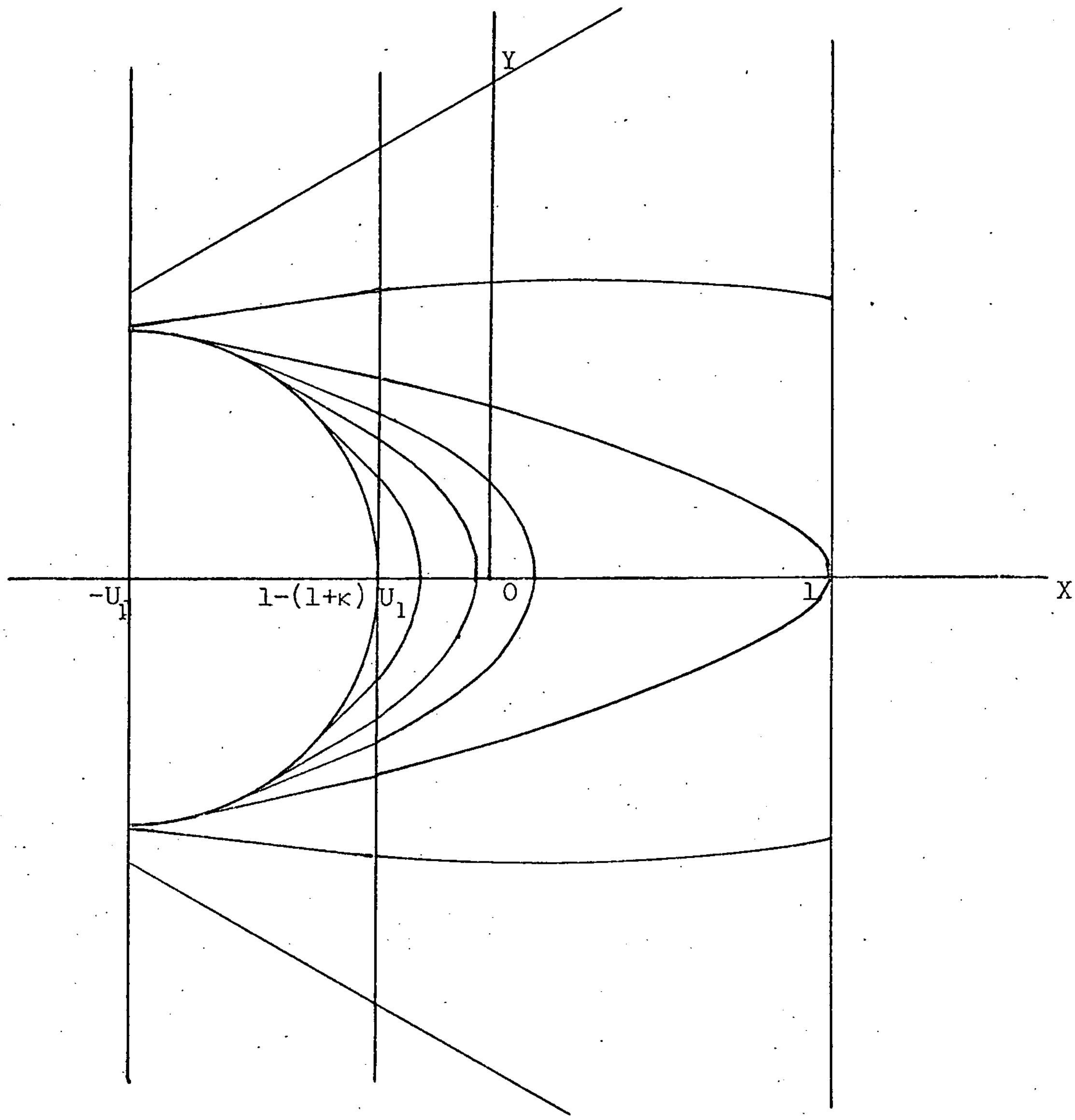


figure 4

However, in $-U_1 \leqslant X \leqslant 1-(1+\kappa)U_1$ the gas is in uniform flow and thus the characteristics in $-U_1 \leqslant X \leqslant 1-(1+\kappa)U_1$ are found to be the tangents to the circle $(X+U_1)^2+Y^2=(1-\kappa U_1)^2$. The circle $(X+U_1)^2+Y^2=(1-\kappa U_1)^2$ is a parabolic line and the region bounded by this circle and the line $X=-U_1$ a region of ellipticity. The system of characteristics of the incomplete rarefaction wave for the case $\kappa=\frac{1}{2}$ is shown in figure 4.

In the hodograph plane the Riemann wave maps into

$$-\frac{1}{\kappa} \leqslant U \leqslant 0 , \quad V=0$$

and elimination of X between the first and third members of (3.2.1) gives

$$F=1+\kappa U \quad \text{on} \quad V=0 \qquad \frac{-1}{\kappa} \leqslant U \leqslant 0 .$$

The Riemann wave is clearly a simple wave.

The incomplete rarefaction wave given by (3.2.2) maps into

$$-U_1 \leqslant U \leqslant 0 , \quad V=0$$

in the hodograph plane and the relation

$$F=1+\kappa U \quad \text{on} \quad V=0 \qquad -U_1 \leqslant U \leqslant 0$$

is obtained as before.

3.3 Suchkov's Solution

Suchkov (1963) set himself the task of determining the resulting flow when polytropic gas, initially at rest at constant pressure and contained between the infinite wall $y=x\cot\theta$ and the semi-infinite wall $x=0$, $y>0$, is allowed to expand into vacuum by the removal, at time $t=0$, of the wall $x=0, y>0$; the wall $y=x\cot\theta$ being retained as a rigid barrier. The condition of no-flow across the barrier $y=x\cot\theta$ enables us to reflect across this line the solution obtained, thus obtaining the solution to the problem of expansion into vacuum of a wedge of gas of angle 2θ.

At any finite time, t, after the start of the motion there exists a value of $y = y(t)$ sufficiently large that no disturbance originating at $y = 0$ at $t = 0$ can have reached y in time t. Thus it may be reasoned that, for sufficiently large Y, the gas is, so to speak, unaware of the existence of the corner at $Y = 0$ and expands as a Riemann wave. Therefore, in the hodograph plane, an integral part of the solution is the Riemann wave

$$F = 1 + \kappa U \ , \quad V = 0 \ , \quad -\frac{1}{\kappa} \leq U \leq 0 . \tag{3.3.1}$$

The wall $y = x \cot \theta$ is clearly also given by $Y = X \cot \theta$ and on this wall we require $V = U \cot \theta$. Substitution of this condition into the transformation equations (2.3.1) yields immediately

$$F_V = F_U \cot \theta \qquad \text{on} \qquad V = U \cot \theta . \tag{3.3.2}$$

The boundary conditions (3.3.1) and (3.3.2), together with equation (2.3.2)

$$(\kappa^2 - F_V{}^2)F_{UU} + 2F_UF_VF_{UV} + (\kappa^2 - F_U{}^2)F_{VV} = \frac{\kappa}{F}[(F_U{}^2 + F_V{}^2)(1 - \kappa) - 2\kappa^2]$$

provide the boundary value problem in the U-V plane.

The Riemann wave (3.3.1) is, of course, a characteristic of (2.3.2) and hence the normal derivative on $V = 0$, F_V, may be calculated. Substituting $F = 1 + \kappa U$, $F_U = \kappa$ and $F_{UU} = 0$ into (2.3.2) we obtain the equation

$$\kappa \frac{d}{dU}(F_V{}^2) = \frac{\kappa}{1 + \kappa U}[(1 - \kappa)F_V{}^2 - \kappa(1 + \kappa)] \tag{3.3.3}$$

which may be integrated to give

$$F_V(U, 0) = \left[\frac{1 + \kappa}{1 - \kappa} + \left(\cot^2\theta - \frac{1 + \kappa}{1 - \kappa}\right)(1 + \kappa U)^{\frac{1 - \kappa}{\kappa}}\right]^{\frac{1}{2}} \tag{3.3.4}$$

where the constant of integration is fixed by noting that (3.3.1) and (3.3.2) together imply $F_V(0, 0) = \kappa \cot \theta$.

Suchkov observed that if $\cot^2\theta = (1 + \kappa)/(1 - \kappa)$, that is if the semi-angle of the wedge, θ, and the adiabatic index of the gas, γ, satisfy the relation

$$\cot^2\theta = \frac{1 + \gamma}{3 - \gamma} , \tag{3.3.5}$$

then the above specified boundary value problem possesses the linear solution

$$F = 1 + \kappa U + \kappa\sqrt{\frac{1+\kappa}{1-\kappa}}\,V\ . \tag{3.3.6}$$

Among the values of θ and γ satisfying (3.3.5) is the pair $\theta = \pi/6$, $\gamma = 2$. It is well known that the equations governing the motion of a polytropic gas of adiabatic index $\gamma = 2$ are identical in form to those of shallow water theory. Thus (3.3.6) with $\gamma = 2$ and the variables suitably interpreted describes the subsequent motion when shallow water, initially contained at rest between two plane walls forming an infinite dihedral, is allowed to 'expand' by the removal, at time $t = 0$, of the two walls. The water (or more correctly, fluid) 'expands' only in the sense that the average height of the fluid surface decreases and the area in the physical plane covered by the fluid increases. Rather than discuss the solution (3.3.6) for arbitrary γ (and, of course, suitably related θ) we shall discuss in detail the case $\theta = \pi/6$, $\gamma = 2$. This case exhibits all the essential character of the solution for general γ.

With $\kappa = \frac{1}{2}$ then, we substitute the values of F_U and F_V obtained from (3.3.6) into (2.3.1) to obtain the equations

$$X = 1 + \frac{3U}{2} + \frac{\sqrt{3}}{2}V\ ,$$

and

$$Y = \sqrt{3} + \frac{\sqrt{3}}{2}U + \frac{5}{2}V\ .$$

These may be solved for U and V to give

$$U = \frac{1}{6}(5X - \sqrt{3}Y - 2)\ , \tag{3.3.7}$$

and

$$V = \frac{1}{6}(-\sqrt{3}X + 3Y - 2\sqrt{3}). \tag{3.3.8}$$

Substituting these last two results into (3.3.6) gives

$$F = \frac{1}{6}(X + \sqrt{3}Y + 2)\ . \tag{3.3.9}$$

So curves F = constant are given by $\sqrt{3}Y + X$ = constant and in particular the curve $F = 0$ is given by $\sqrt{3}Y + X + 2 = 0$.

Substitution of the values of U, V and F from (3.3.7), (3.3.8) and (3.3.9) respectively into (2.2.5) gives

$$\left.\frac{dX}{dY}\right|_{\mathcal{C}+} = 0 \qquad\qquad \left.\frac{dX}{dY}\right|_{\mathcal{C}-} = \sqrt{3}$$

showing that the family of $\mathcal{C}_+$ characteristics is made up of lines $X =$ constant whilst that of $\mathcal{C}_-$ characteristics is composed of lines $\sqrt{3}Y - X =$ constant.

The curve across which the mixed wave as given by (3.3.9) adjoins the Riemann wave is obtained by equation V, as given by (3.3.8), to zero. Thus the required curve is

$$\sqrt{3}Y - X - 2 = 0 \ .$$

This curve, of course, is a characteristic both of the mixed wave (3.3.9) and the Riemann wave. It is in fact one of the two straight characteristics of the Riemann wave which we noted earlier. This was pointed out by Mackie (1966).

Thus we see that in the X-Y plane the flow is made up of regions of uniform flow and of simple-wave and mixed-wave regions, the boundaries between these various regions being made up of straight lines. The X-Y plane is shown in figure 5.

In the hodograph plane the characteristics are given by

$$\left.\frac{dV}{dU}\right|_{\mathcal{C}\pm} = \frac{F_U F_V \pm \kappa\sqrt{F_U{}^2 + F_V{}^2 - \kappa^2}}{(\kappa^2 - F_V{}^2)} = \frac{\frac{\sqrt{3}}{4} \pm \frac{\sqrt{3}}{4}}{-\frac{1}{2}}$$

$$\left.\frac{dV}{dU}\right|_{\mathcal{C}+} = -\sqrt{3} \qquad\qquad \left.\frac{dV}{dU}\right|_{\mathcal{C}-} = 0$$

whilst the curves $F =$ constant are given by

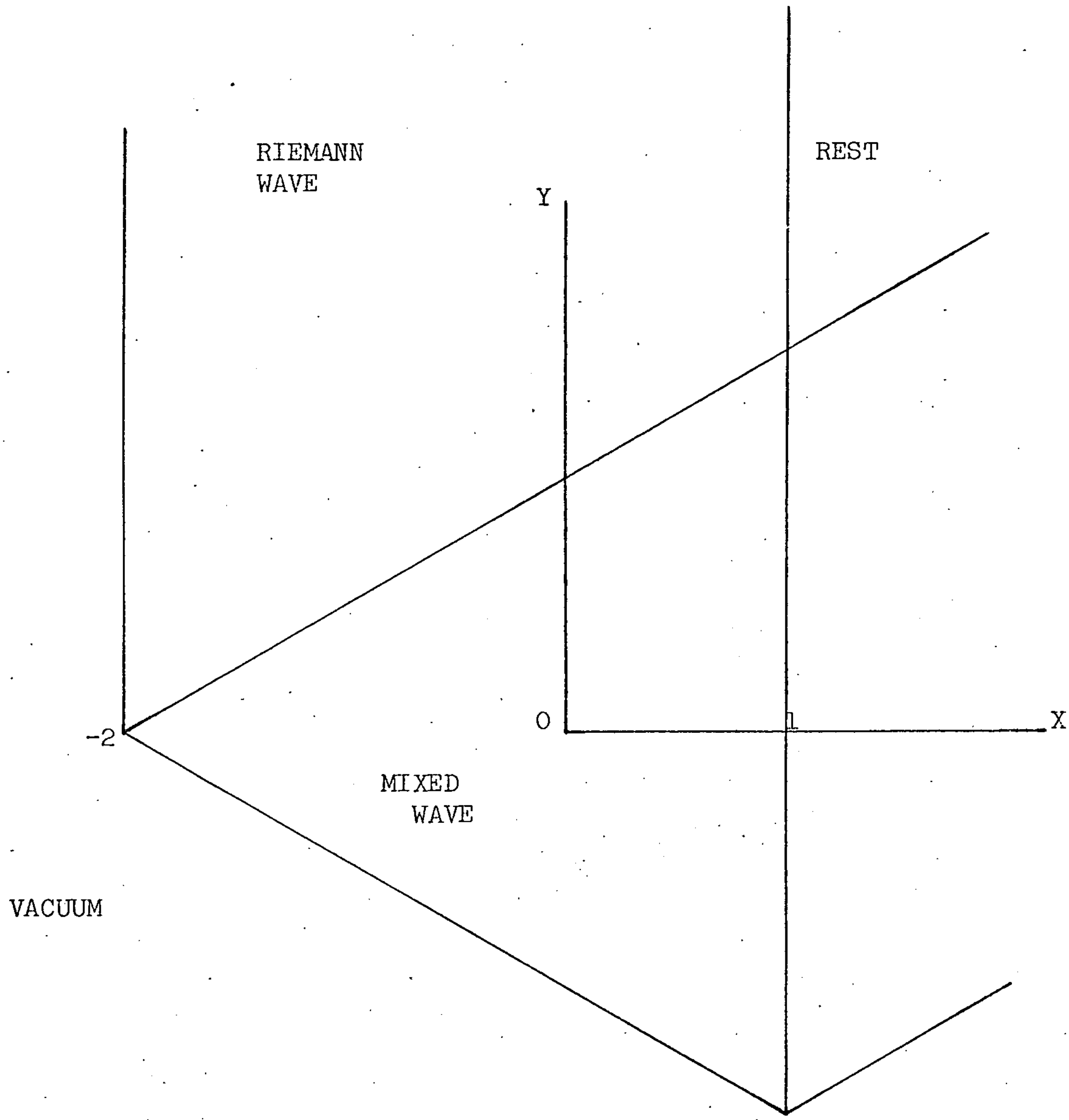


figure 5

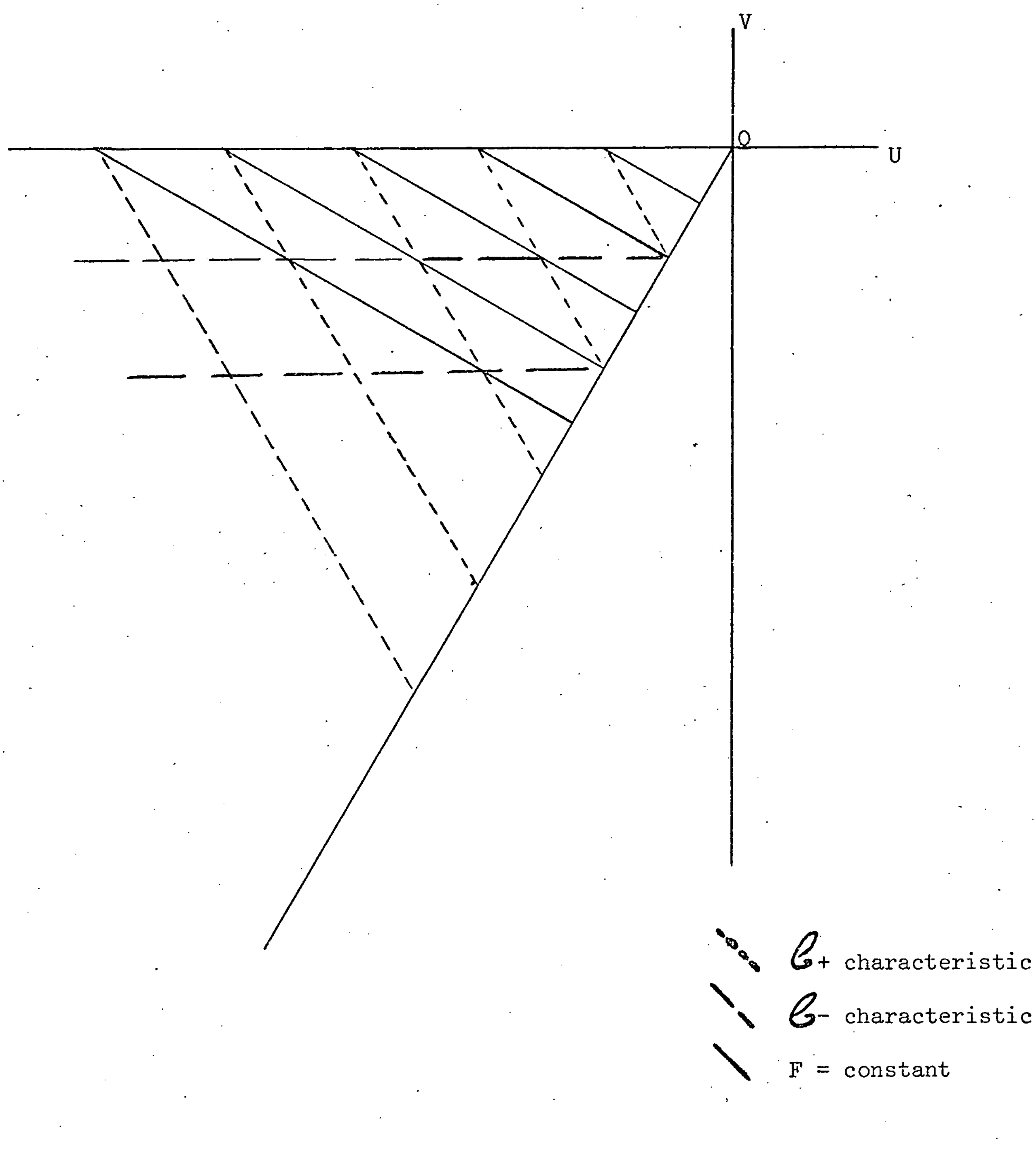

V
O
U
$\mathcal{C}_+$ characteristic
$\mathcal{C}_-$ characteristic
F = constant

figure 6

$$\left.\frac{dV}{dU}\right|_{F = \text{constant}} = -\frac{F_U}{F_V} = -\frac{1}{\sqrt{3}}$$

Therefore in the U-V plane the system of characteristics is as shown in figure 6.

We end this discussion of Suchkov's solution by noting that in cases in which θ and γ do not satisfy (3.3.5) the boundary value problem posed cannot be solved analytically and a numerical method must therefore be used. This has been done by Levine (1968) in studying the case $\theta = \pi/4$.

3.4 Prandtl-Meyer Flow

It is well known that steady, supersonic flow of a polytropic gas past the convex wall

$$y = 0 \qquad x < 0$$

$$x = -y \cot \alpha \qquad y < 0$$

is given by

$$u_r = \frac{q^*}{\lambda} \sin [\lambda\theta + \tan^{-1}(\lambda \cot \mu_1)] \tag{3.4.1}$$

$$c = u_\theta = q^* \cos [\lambda\theta + \tan^{-1}(\lambda \cot \mu_1)] \tag{3.4.2}$$

where c_1 is the sound speed far upstream, μ_1 is the upstream Mach angle, u_r, u_θ and θ are as shown in figure 7 and λ and q^* are given by

$$\lambda^2 = \frac{\kappa}{1 + \kappa}, \qquad q^* = c_1\sqrt{1 + \lambda^2\cot^2\mu_1} .$$

(3.4.1) and (3.4.2) are valid in $0 \leqslant \theta \leqslant \theta_o$ where

$$\theta_o + \tan^{-1}(\lambda \cot [\lambda\theta_o + \tan^{-1}(\lambda \cot \mu_1)]) = \mu_1 + \alpha . \tag{3.4.3}$$

The region $0 \leqslant \theta \leqslant \theta_o$ is often referred to as a Prandtl-Meyer fan. For $\theta > \theta_o$ the flow is steady, uniform, supersonic and parallel to the wall $x = -y \cot \alpha$.

Consider the case in which the wall $x = -y \cot \alpha$ is absent and the gas

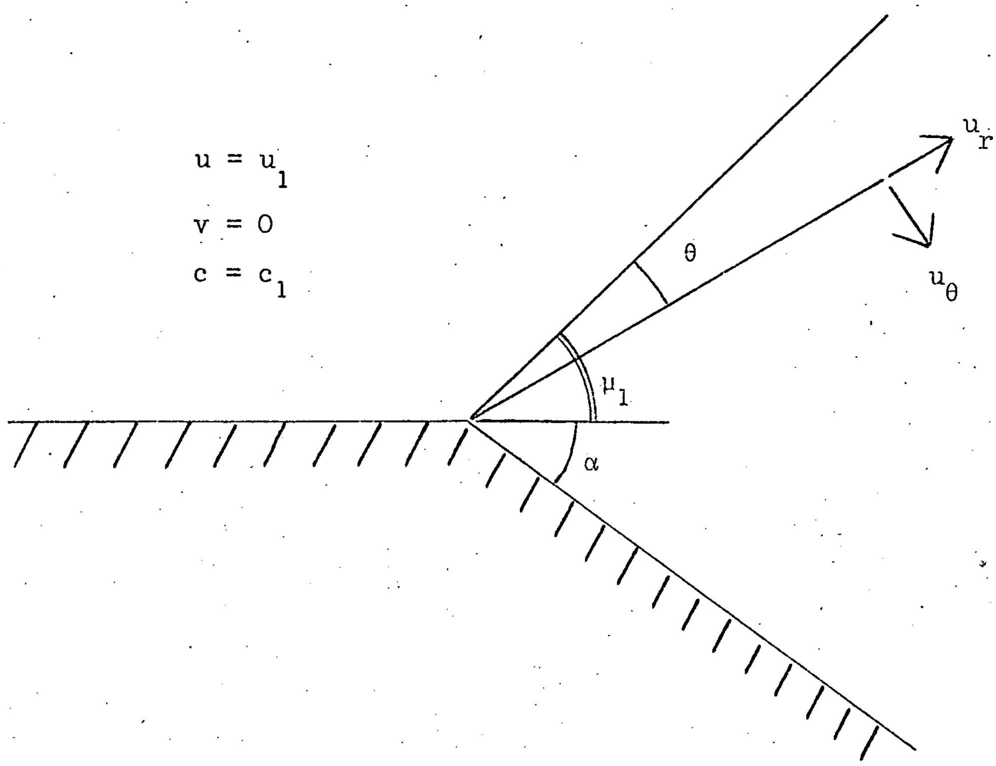


figure 7

flows past the sudden termination of the wall $y = 0$ at $x = 0$ into vacuum. The steady solution is then given by (3.4.1) and (3.4.2) but now extends from $\theta = 0$ to $\theta = \theta_1$ where θ_1 is given by

$$\lambda\theta_1 + \tan^{-1}(\lambda \cot \mu_1) = \frac{\pi}{2} . \qquad (3.4.4)$$

In the x-y plane the characteristics of the Prandtl-Meyer flow are found to satisfy

$$\left.\frac{dy}{dx}\right|_{\mathcal{C}+} = \tan(\mu_1 - \theta) = \frac{y}{x} , \qquad (3.4.5)$$

$$\left.\frac{dy}{dx}\right|_{\mathcal{C}-} = \tan(\mu_1 - \theta - 2\tan^{-1}[\lambda\cot[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]]) . \qquad (3.4.6)$$

Hence the $\mathcal{C}_+$ characteristics are straight lines through the origin. The $\mathcal{C}_-$ characteristics can be obtained from (3.4.6) only by numerical integration.

It is easily seen from (3.4.1) and (3.4.2) that

$$u = \frac{q^*}{\lambda}\sin[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\cos(\mu_1 - \theta) + q^*\cos[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\sin(\mu_1 - \theta) \qquad (3.4.7)$$

$$v = \frac{q^*}{\lambda}\sin[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\sin(\mu_1 - \theta) - q^*\cos[\lambda\theta + \tan(\lambda\cot\mu_1)]\cos(\mu_1 - \theta) \qquad (3.4.8)$$

where u and v are the cartesian velocity components. Thus the Prandtl-Meyer flow maps into a curve in the u-v plane. The curve is a part of an epicycloid, as are all characteristics in the u-v plane.

So far, in this section, our discussion has been restricted to steady flow. However, it was pointed out in a paper of Jones, Martin and Thornhill (1951) that, since (3.4.1) and (3.4.2) represent a solution of the gas dynamic equations which involves x and y only in the combination y/x, and since for all t

$$\frac{Y}{X} = \frac{y}{x} ,$$

where X and Y are our usual similarity variables, then

$$U_R = \frac{u_r}{c_o} = \frac{q^*}{c_o \lambda} \sin [\lambda\theta + \tan^{-1}(\lambda \cot \mu_1)] \tag{3.4.9}$$

and $$F = U_\theta = \frac{q^*}{c_o} \cos [\lambda\theta + \tan^{-1}(\lambda \cot \mu_1)] \tag{3.4.10}$$

where $R^2 = X^2 + Y^2$ and $\tan \theta = Y/X$, represent a solution of the gas dynamic equations in terms of similarity variables.

Now with R and θ defined above, the equation (2.2.4) governing the dimensionless potential, Φ becomes

$$\Phi_{RR}[(U_R - R)^2 - F^2] + \Phi_{R\theta}[2(U_R - R)\frac{U\theta}{R}] + \Phi_{\theta\theta}[\frac{U_\theta^2 - F^2}{R^2}]$$

$$+ \frac{U_R}{R}[U_\theta^2 + F^2] + 2U_\theta^2 = 0 \tag{3.4.11}$$

where $U_R \equiv \Phi_R$ and $RU_\theta \equiv \Phi_\theta$. The characteristics of this equation are given by

$$\left.\frac{d\theta}{dR}\right|_{\mathcal{C}\mp} = \frac{(U_R - R)U_\theta \pm F\sqrt{(U_R - R)^2 + U_\theta^2 - F^2}}{R[U_R - R)^2 - F^2]} . \tag{3.4.12}$$

Substituting U_R, U_θ and F from (3.4.9) and (3.4.10) into (3.4.12) gives

$$\left.\frac{d\theta}{dR}\right|_{\mathcal{C}+} = 0 \tag{3.4.13}$$

and $$\left.\frac{d\theta}{dR}\right|_{\mathcal{C}-} = \frac{2(U_R - R)U_\theta}{R[(U_R - R)^2 - U_\theta^2]} \quad \text{with } U_R, U_\theta \text{ as given} \tag{3.4.14}$$

by (3.4.9) and (3.4.10).

Hence we see that the $\mathcal{C}_+$ characteristics in the X-Y plane are straight lines through the origin. As was the case in the x-y plane the $\mathcal{C}_-$ characteristics may be obtained only by numerical integration. However, it is easily seen from a comparison of equations (3.4.6) and (3.4.14) that the $\mathcal{C}_-$ characteristics in the X-Y plane are completely different from those in the x-y plane, even though they are derived from the same physical flow.

It is clear that the cartesian velocity components in the self-similar plane are given by

$$U = \frac{q^*}{c_o\lambda}\sin[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\cos(\mu_1 - \theta) + \frac{q^*}{c_o}\cos[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\sin(\mu_1 - \theta) \tag{3.4.15}$$

$$V = \frac{q^*}{c_o\lambda}\sin[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\sin(\mu_1 - \theta) - \frac{q^*}{c_o}\cos[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\cos(\mu_1 - \theta) \tag{3.4.16}$$

The Prandtl-Meyer flow, when regarded as a solution of the self-similar equations, is a simple wave and its map in the U-V plane is a part of an epicycloid. However, whereas in the case of steady flow, one could assert *a priori* that the Prandtl-Meyer flow, being a simple wave (that term having the obvious interpretation in the case of steady flow) must map into one of the *already known* characteristics in the u, v plane, in the case of self-similar flow the hodograph equation (2.3.2) is non-linear and thus the characteristics not known *a priori*.

CHAPTER 4. PERTURBATIONS OF EXACT SOLUTIONS

4.1 Perturbation of Uniform Flow

Consider the problem shown in figure 8. AOB is a rigid wall consisting of the two planes $y = 0,\ x > 0$ and $y = x \tan \varepsilon,\ x < 0$ where ε is a small angle. The plane wall DC is perpendicular to OB and moves in the negative x direction with speed u_1. The space to the right of DC and above the wall AOB is supposed filled with polytropic gas moving with the wall DC, that is to say the gas has everywhere the velocity $-u_1$, and a corresponding sound speed c_1 (also supposed uniform throughout the gas). At time $t = 0$ the wall DC passes O, thereafter continuing to move according to $x = -u_1 t$ but in such a way as to remain in contact with AO. We seek to use a perturbation method to determine the resulting motion of the gas. Accordingly, with $U_1 = u_1/c_o$ and $F_1 = c_1/c_o$ we put

$$
\begin{aligned}
U(X, Y) &= -U_1 + \varepsilon U'(X, Y) + o(\varepsilon) \\
V(X, Y) &= \varepsilon V'(X, Y) + o(\varepsilon) \\
F(X, Y) &= F_1 + \varepsilon F'(X, Y) + o(\varepsilon)
\end{aligned}
\qquad (4.1.1)
$$

and substitute into (2.2.1). The linearised equations are readily seen to be

$$
\begin{aligned}
(U_1 + X)U'_X + YU'_Y - \frac{1}{\kappa} F_1 F'_X &= 0 \\
(U_1 + X)V'_X + YV'_Y - \frac{1}{\kappa} F_1 F'_Y &= 0 \\
(U_1 + X)F'_X + YF'_Y - \kappa F (U'_X + V'_Y) &= 0.
\end{aligned}
$$

With $(U_1 + X) = F_1 X_1$, $Y = F_1 Y_1$ these become

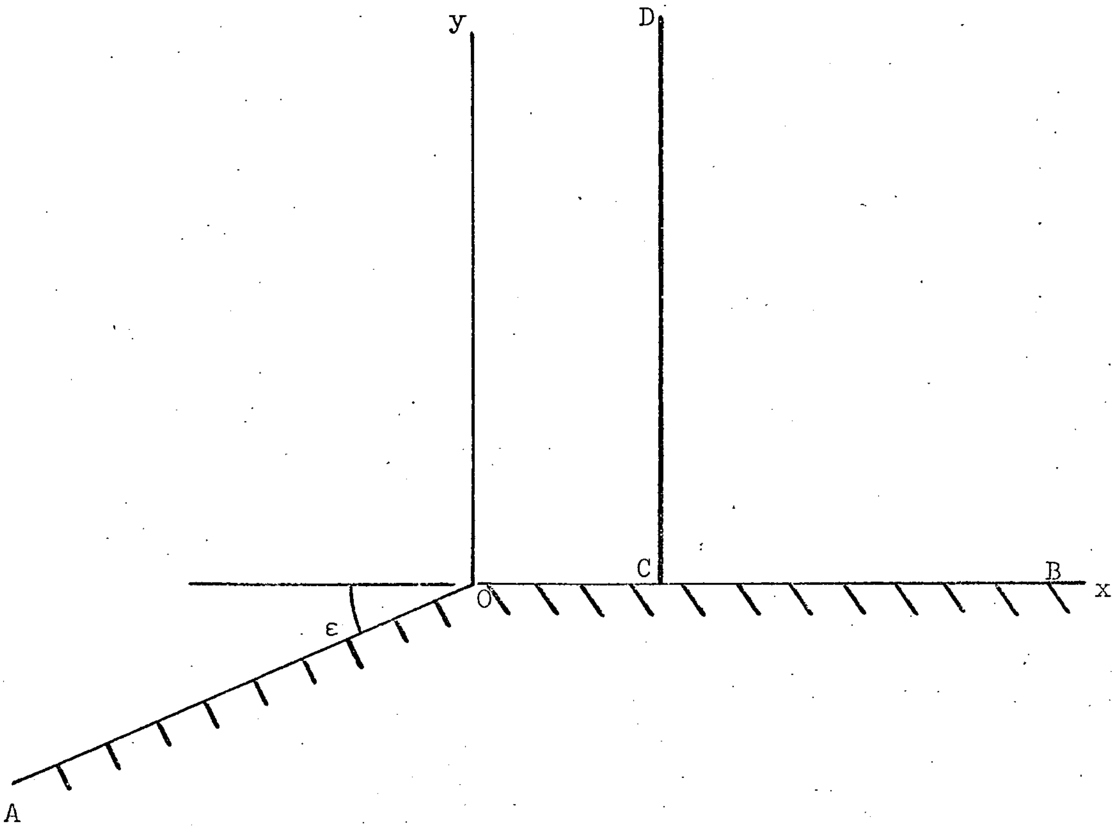


figure 8

$$X_1 \frac{\partial U'}{\partial X_1} + Y_1 \frac{\partial U'}{\partial Y_1} - \frac{1}{\kappa} \frac{\partial F'}{\partial X_1} = 0 \; ,$$

$$X_1 \frac{\partial V'}{\partial X_1} + Y_1 \frac{\partial V'}{\partial Y_1} - \frac{1}{\kappa} \frac{\partial F'}{\partial Y_1} = 0 \; , \qquad (4.1.2)$$

$$X_1 \frac{\partial F'}{\partial X_1} + Y_1 \frac{\partial F'}{\partial Y_1} - \kappa\left(\frac{\partial U'}{\partial X_1} + \frac{\partial V'}{\partial Y_1}\right) = 0 \; .$$

From (4.1.2) the equation

$$\frac{\partial^2 F'}{\partial X_1{}^2} + \frac{\partial^2 F'}{\partial Y_1{}^2} = \left(X_1 \frac{\partial}{\partial X_1} + Y_1 \frac{\partial}{\partial Y_1} + 1\right)\left(X_1 \frac{\partial F'}{\partial X_1} + Y_1 \frac{\partial F'}{\partial Y_1}\right)$$

or

$$(X_1{}^2 - 1)\frac{\partial^2 F'}{\partial X_1{}^2} + 2X_1 Y_1 \frac{\partial^2 F'}{\partial X_1 \partial Y_1} + (Y_1{}^2 - 1)\frac{\partial^2 F'}{\partial Y_1{}^2} + 2X_1 \frac{\partial F'}{\partial X_1} + 2Y_1 \frac{\partial F'}{\partial Y_1} = 0 \qquad (4.1.3)$$

is easily deduced.

It is necessary to distinguish between subsonic and supersonic unperturbed flow; that is to consider separately the cases $U_1 < F_1$ and $U_1 > F_1$. We shall deal with the subsonic case first.

At any time after the commencement of the perturbation, the situation in the X_1-Y_1 plane is as shown in figure 9. In the X_1-Y_1 plane the wall CD is $X_1 = 0$ and the wall AOB may be consistently approximated by $Y_1 = 0$. The corner maps into the point $(M_1, 0)$, where $M_1 = U_1/F_1$, and since we are considering subsonic unperturbed flow, this point lies inside the sonic circle $X_1{}^2 + Y_1{}^2 = 1$. The linear equation (4.1.3) is elliptic inside the sonic circle and hyperbolic outside it. It is clear that the perturbation quantities vanish outside the sonic circle. Thus we have the boundary condition

$$F' = 0 \quad \text{on} \quad X_1{}^2 + Y_1{}^2 = 1 \; . \qquad (4.1.4)$$

Since $U' = 0$ on $X_1 = 0$, the first member of (4.1.2) gives

$$\frac{\partial F'}{\partial X_1} = 0 \quad \text{on} \quad X_1 = 0 \; . \qquad (4.1.5)$$

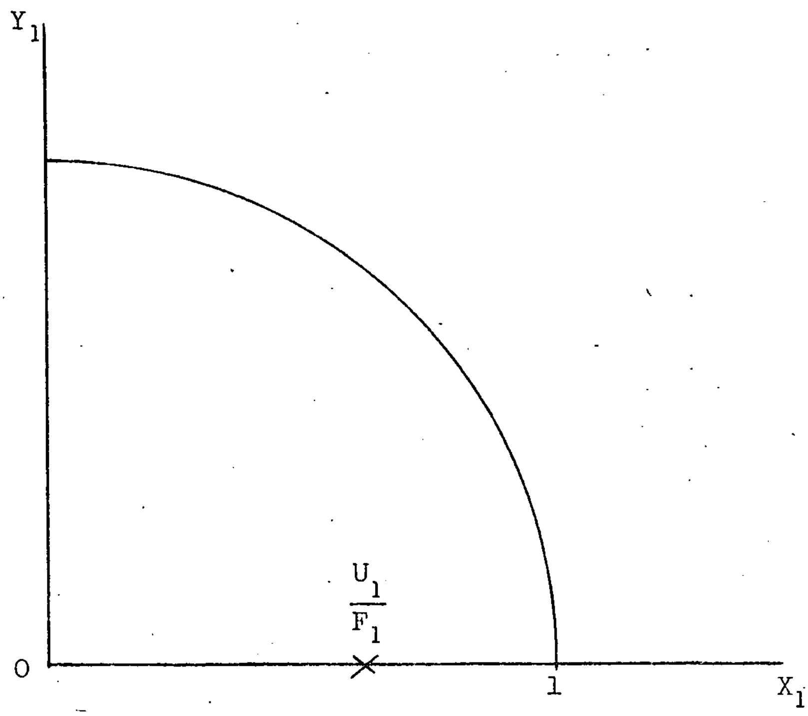


figure 9

The linearised boundary condition on the wall AOB becomes

$$\left.\begin{aligned} V' &= -U_1 \qquad && 0 < X_1 < M_1 \\ &= 0 && M_1 < X_1 \end{aligned}\right\} \quad \text{on } Y_1 = 0 .$$

Hence $\partial V'/\partial X_1$ vanishes on $Y_1 = 0$ except at $X_1 = M_1$ where it is undefined. However, we have

$$\int_{M_1-\varepsilon}^{M_1+\varepsilon} \left(\frac{\partial V'}{\partial X_1}\right)_{Y_1 = 0} dX_1 = U_1 \qquad \varepsilon > 0$$

which, together with the second member of (4.1.2) implies

$$\int_{M_1-\varepsilon}^{M_1+\varepsilon} \left(\frac{\partial F'}{\partial Y_1}\right)_{Y_1 = 0} dX_1 = \kappa M_1 U_1 . \tag{4.1.6}$$

We now solve the boundary value problem specified by (4.1.3), (4.1.4), (4.1.5) and (4.1.6) by means of Busemann's transformation

$$\begin{aligned} X_1 &= \frac{2\rho\cos\theta}{(1+\rho^2)} , \qquad & Y_1 &= \frac{2\rho\sin\theta}{(1+\rho^2)} \\ \rho\cos\theta &= \sigma & \rho\sin\theta &= \tau . \end{aligned} \tag{4.1.7}$$

By means of the transformation (4.1.7) equation (4.1.3) in the interior of the circle $X_1{}^2 + Y_1{}^2 = 1$ is transformed into

$$\frac{\partial^2 F'}{\partial \sigma^2} + \frac{\partial^2 F'}{\partial \tau^2} = 0 \tag{4.1.8}$$

whilst the boundary conditions (4.1.4), (4.1.5) and (4.1.6) become

$$F' = 0 \quad \text{on} \quad \rho = 1 \tag{4.1.9}$$

$$\frac{\partial F'}{\partial \sigma} = 0 \quad \text{on} \quad \sigma = 0 \tag{4.1.10}$$

and

$$\left(\frac{\partial F'}{\partial \tau}\right)_{\tau=0} = \frac{\kappa M_1 U_1}{\sqrt{1 - M_1{}^2}} \, \delta(\sigma - \alpha) \tag{4.1.11}$$

respectively where $\alpha = 1 - \sqrt{1 - M_1{}^2}/M_1$ and $\delta(\sigma - \alpha)$ is the Dirac delta function having singularily at $\sigma = \alpha$. The boundary value problem specified by (4.1.8) to (4.1.11) has as solution the Green's function

$$F' = \frac{-\kappa M_1{}^2 F_1}{2\pi\sqrt{1 - M_1{}^2}} \log \left[\frac{\alpha^2 - 2\alpha\sigma + \sigma^2 + \tau^2}{\alpha^2(\sigma^2 + \tau^2) - 2\alpha\sigma + 1} \cdot \frac{\alpha^2 + 2\alpha\sigma + \sigma^2 + \tau^2}{\alpha^2(\sigma^2 + \tau^2) + 2\alpha\sigma + 1} \right] . \tag{4.1.12}$$

This analysis of the subsonic case has been closely modelled on some work of Anderson (1966). In this work, Anderson asserts that a boundary value problem such as that specified by (4.1.8) to (4.1.11) can by symmetry considerations be transformed into one requiring solution of Laplace's equation in the unit circle with boundary condition $F' = 0$ on the circumference of the circle. This can indeed be done. However, he then invokes the mean value theorem to argue that $F' = 0$ at the centre of the circle. This reasoning is invalid because of the singularity at $(\alpha, 0)$ (and that at $(-\alpha, 0)$ obtained by reflection). That F' does not vanish at the origin is readily seen from (4.1.12).

It is easily shown that, to first order in ε

$$\frac{F'}{F_1} = \frac{\kappa}{\gamma} \frac{p'}{p_1}$$

where p_1 is the pressure in the unperturbed flow and p' is the perturbation pressure. Hence, from (4.1.12), the perturbation pressure is given by

$$p' = \frac{-\gamma M_1{}^2 p_1}{2\pi\sqrt{1 - M_1{}^2}} \log \left[\frac{\alpha^2 - 2\alpha\sigma + \sigma^2 + \tau^2}{\alpha^2(\sigma^2 + \tau^2) - 2\alpha\sigma + 1} \cdot \frac{\alpha^2 + 2\alpha\sigma + \sigma^2 + \tau^2}{\alpha^2(\sigma^2 + \tau^2) + 2\alpha\sigma + 1} \right] . \tag{4.1.13}$$

This last equation implies that the perturbation pressure becomes logarithmically infinite at $(\alpha, 0)$. However, we should not be surprised to find that linearised theory breaks down in the neighbourhood of the corner. The same equation, together with (4.1.7), implies that

$$\left(\frac{\partial p'}{\partial \rho}\right)_{\rho=1} = \frac{-\gamma M_1{}^2 p_1}{\pi\sqrt{1 - M_1{}^2}} \quad \frac{2(1 - \alpha^4)}{(1 + \alpha^2)^2 - 4\alpha^2\cos^2\theta}$$

so that

$$\lim_{R\to 1-} \left[\frac{\partial p}{\partial R}\right] = \frac{-\varepsilon\gamma M_1{}^2 p_1}{\pi\sqrt{1 - M_1{}^2}} \cdot \frac{2(1 - \alpha^4)}{(1 + \alpha^2)^2 - 4\alpha^2\cos^2\theta} \lim_{R\to 1-}\left[\frac{1 - \sqrt{1 - R^2}}{R^2\sqrt{1 - R^2}}\right] .$$

This singularity in the pressure gradient was found by Lighthill (1949) in a study of shock diffraction. He concluded that the true phenomenon is a shock when $\varepsilon < 0$ and a rapid but not discontinuous expansion when $\varepsilon > 0$.

We now turn our attention to the case in which the unperturbed flow is supersonic. In this case, in the X_1-Y_1 plane the corner again maps into $(M_1, 0)$ but this point now lies outside the sonic circle. There can be no disturbance upstream from the leading Mach line through O. We conclude that in the X_1-Y_1 plane the perturbation is confined to the region bounded by the Mach line through A, the arc of the sonic circle CD and the axes OX_1, OY_1, as shown in figure 10. Again we have

$$(X_1{}^2 - 1)\frac{\partial^2 F'}{\partial X_1{}^2} + 2X_1Y_1\frac{\partial^2 F'}{\partial X_1 \partial Y_1} + (Y_1{}^2 - 1)\frac{\partial^2 F'}{\partial Y_1{}^2} + 2X_1\frac{\partial F'}{\partial X_1} + 2Y_1\frac{\partial F'}{\partial Y_1} = 0.$$

As in the subsonic case we require $U'(0, Y_1) = 0$ and the first member of (4.1.2) can be used to establish the boundary condition

$$\frac{\partial F'}{\partial X_1}(0, Y_1) = 0 . \qquad (4.1.14)$$

Since the corner is at $(M_1, 0)$, we have that on OBA $V' = -U_1$ and the second member of (4.1.2) now gives

$$\frac{\partial F'}{\partial Y_1} = 0 \quad \text{on} \quad \text{OBA} . \qquad (4.1.15)$$

We clearly require $F' = 0$ on AC and on CD.

Now the substitutions

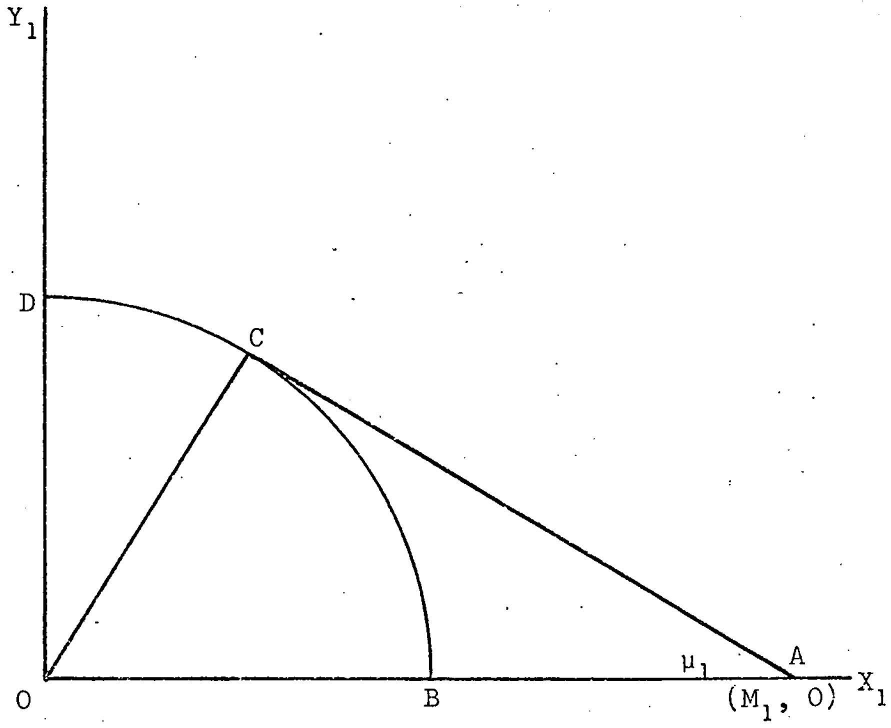


figure 10

$$X_1 = R\cos\theta \qquad Y_1 = R\sin\theta$$
$$\rho = \cos^{-1}\left(\frac{1}{R}\right) \qquad (4.1.16)$$

valid in $R \geqslant 1$ reduce (4.1.3) to

$$\frac{\partial^2 F'}{\partial \rho^2} - \frac{\partial^2 F'}{\partial \theta^2} = 0 \ . \qquad (4.1.17)$$

In BAC we have a Goursat type problem for F' which in the ρ-θ plane may be formulated as follows

AC is $(Y_1 - \cos\mu_1) = -\tan\mu_1(X - \sin\mu_1)$

where $\sin\mu_1 = 1/M_1$

AC is thus $\rho = \theta + \mu_1 - \frac{\pi}{2}$

and on this line $F' = 0$. AB is clearly $\theta = 0$ and on this line $\partial F'/\partial\theta = 0$. Thus in the ρ-θ plane we have the boundary value problem shown in figure 11. And, since the general solution of (4.1.17) is known to be $F' = G(\rho - \theta) + H(\rho + \theta)$ for arbitrary twice differentiable functions G and H, it is readily seen that the above problem possesses only the trivial solution $F' = 0$.

So we have seen that in ABC the only solution of (4.1.3) consistent with the boundary conditions is the trivial solution $F'(X_1, Y_I) = 0$. This should not surprise us for, in the exact treatment of this problem, we should certainly conclude that near the corner the solution would be a Prandtl-Meyer expansion occupying a region expanding linearly with time. Further, AC would be the first member of a family of straight characteristics passing through A making up a fan of small angular extent through which all flow variables would change continuously. In the linearised theory of steady supersonic flow past the wall $y = x\tan\varepsilon,\ x < 0;\ y = 0,\ x > 0$ the solution is found to be

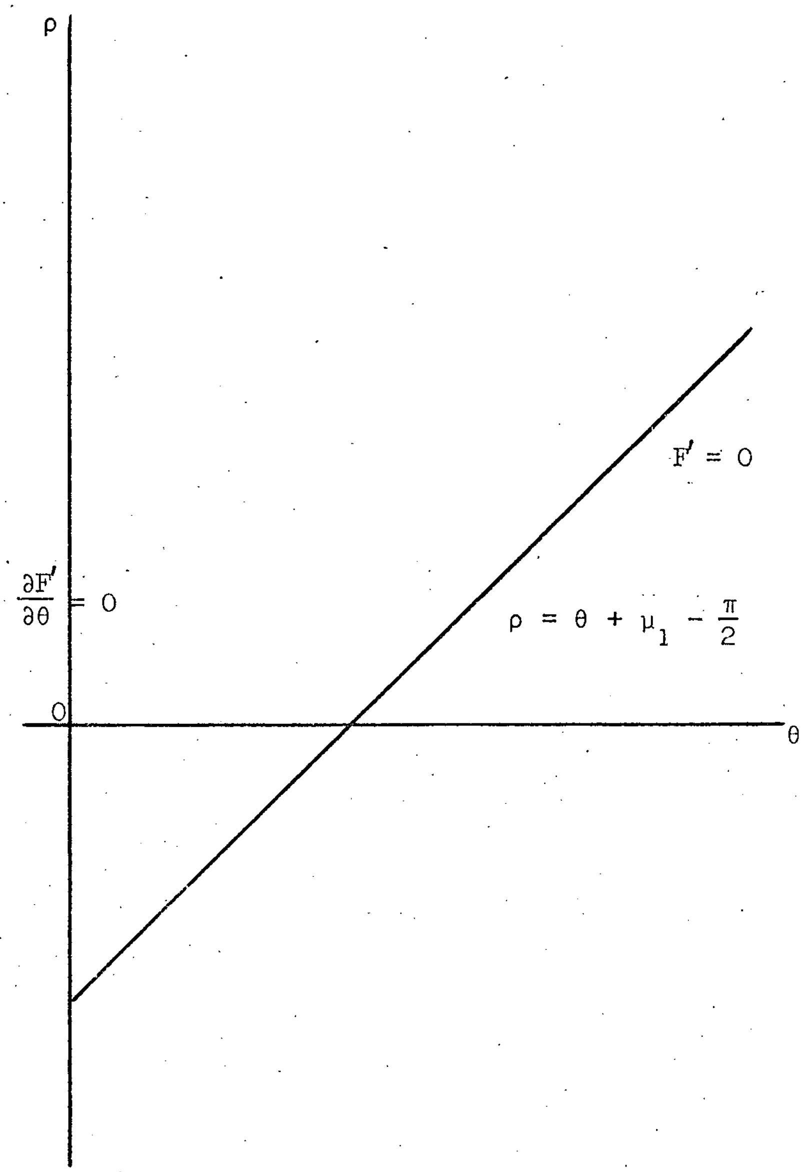


figure 11

$$F' = 0 \qquad Y_1 > \frac{-1}{\sqrt{M_1^2 - 1}}(X_1 - M_1)$$

$$F' = \frac{-2\kappa F_1 M_1^2}{\sqrt{M_1^2 - 1}} = F's \text{ , say} \qquad Y_1 < \frac{-1}{\sqrt{M_1^2 - 1}}(X_1 - M_1). \tag{4.1.18}$$

In the steady problem, then, we accept a discontinuity in F' (and, therefore, of course, in the perturbation pressure and density and in one velocity component) across the line $Y_1\sqrt{M_1^2 - 1} + (X_1 - M_1) = 0$. This discontinuity in the steady problem is a result of the Prandtl-Meyer fan's being 'collapsed' onto the line $Y_1\sqrt{M_1^2 - 1} + (X_1 - M_1) = 0$.

Bearing in mind that we require the solution of our unsteady problem to tend in some sense to the solution of the steady problem as the time $t \to \infty$, we are led to conclude that the correct approach to our unsteady problem is to accept a discontinuity in F' across $Y_1\sqrt{M_1^2 - 1} + (X_1 - M_1) = 0$ and in BAC (figure 10) to take $F' = F's$. The solution in OBCD is then readily obtained by an application of the Poisson integral formula.

By symmetry considerations we can see that the required solution in OBCD is the solution of Laplace's equation in the unit circle which takes the value $f(\theta)$ on the circumference of the circle where $f(\theta)$ is given by

$$f(\theta) = \begin{cases} F's & 0 < \theta < \frac{\pi}{2} - \mu_1 \\ 0 & \frac{\pi}{2} - \mu_1 < \theta < \frac{\pi}{2} + \mu_1 \\ F's & \frac{\pi}{2} + \mu_1 < \theta < \frac{3\pi}{2} - \mu_1 \\ 0 & \frac{3\pi}{2} - \mu_1 < \theta < \frac{3\pi}{2} + \mu_1 \\ F's & \frac{3\pi}{2} + \mu_1 < \theta < 2\pi \end{cases}$$

Hence, in OBCD F' is given by

$$F' = \frac{F's}{2\pi}\left[\int_{-\frac{\pi}{2}+\mu_1}^{\frac{\pi}{2}-\mu_1} + \int_{\frac{\pi}{2}+\mu_1}^{\frac{3\pi}{2}-\mu_1}\right] \frac{(1 - R^2)\, d\psi}{(1 + R^2) - 2R\cos(\psi - \theta)}$$

where, as before $X_1 = R\cos\theta$, $Y_1 = R\sin\theta$. Hence, for $0 < \theta < \frac{\pi}{2} - \mu_1$,

$$F' = \frac{F's}{\pi}\left[\Big[T(\psi - \theta)\Big]_{-\frac{\pi}{2}+\mu_1}^{\frac{\pi}{2}-\mu_1} + \lim_{\varepsilon\to 0+}\Big[T(\psi - \theta)\Big]_{\frac{\pi}{2}+\mu_1}^{\pi+\theta-\varepsilon} + \lim_{\varepsilon\to 0+}\Big[T(\psi - \theta)\Big]_{\pi+\theta+\varepsilon}^{\frac{3\pi}{2}-\mu_1}\right]$$

where $T(x) = \tan^{-1}\left[\frac{1+R}{1-R}\tan\frac{1}{2}x\right]$, the principal value of the inverse tangent being taken. Hence it is found that

$$F' = \frac{F's}{\pi}\left[\pi + T(\tfrac{\pi}{2} - \mu_1 - \theta) + T(\tfrac{\pi}{2} - \mu_1 + \theta) - T(\tfrac{\pi}{2} + \mu_1 + \theta) - T(\tfrac{\pi}{2} + \mu_1 - \theta)\right]. \qquad (4.1.19)$$

Similarly it is found that for $\frac{\pi}{2} - \mu_1 < \theta < \frac{\pi}{2}$ F' is given by

$$F' = \frac{F's}{\pi}\left[T(\tfrac{\pi}{2} - \mu_1 - \theta) + T(\tfrac{\pi}{2} - \mu_1 + \theta) - T(\tfrac{\pi}{2} + \mu_1 + \theta) - T(\tfrac{\pi}{2} + \mu_1 - \theta)\right]. \qquad (4.1.20)$$

It is easily established that F' as given separately in $0 < \theta < \frac{\pi}{2} - \mu_1$ and $\frac{\pi}{2} - \mu_1 < \theta < \frac{\pi}{2}$ by (4.1.19) and (4.1.20) respectively is continuous across the line $\theta = \frac{\pi}{2} - \mu$.

As in the subsonic case, the radial derivative of the perturbation pressure (or, equivalently, perturbation dimensionless sound speed) is singular on the sonic circle $X_1^2 + Y_1^2 = 1$.

4.2 Perturbation of the Rarefaction Waves

In this section we shall consider perturbations of both the complete and the incomplete rarefaction wave.

In figure 12 AOC is the plane rigid wall $y = x\tan\varepsilon$. Initially the space $y > x\tan\varepsilon$, $x > 0$ is occupied by polytropic gas at rest at constant pressure. At time $t = 0$ the plane wall OB is withdrawn from the gas with constant speed u_1 in such a way that for all subsequent time the wall OB remains in contact with AO. In the case $\varepsilon = 0$ the resulting motion of the gas would be either the complete rarefaction wave given by (3.2.1) or the incomplete rarefaction wave given by (3.2.2) according as $|U_1| >$ or

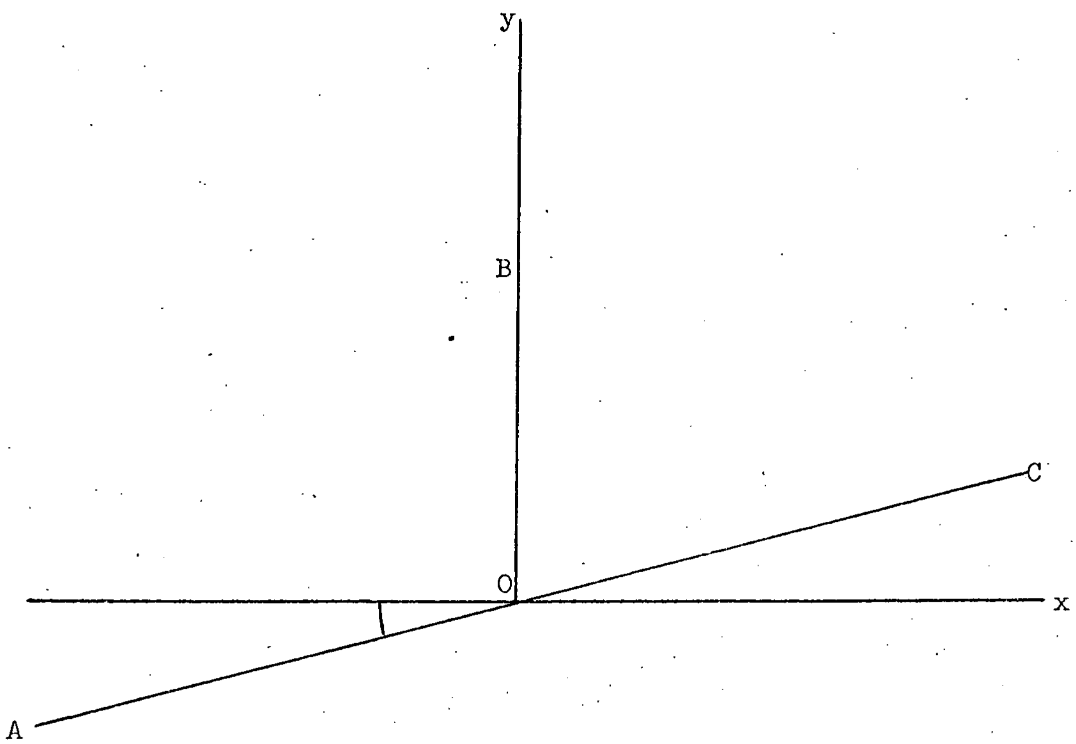


figure 12

$< 1/\kappa$ where $c_o U_1 = u_1$. With $\varepsilon \neq 0$ the resulting flow is more complicated and we seek a solution of the linearised problem.

We attempt first the case $|U_1| > 1/\kappa$: the case in which the unperturbed flow is the complete rarefaction wave or Riemann wave. Powell (1957) has studied this case and has obtained the complete solution to the problem in the X-Y plane. In fact, Powell has obtained, in the X-Y plane, the complete solution to the more general problem in which the wall AOC is no longer straight but takes the form $y = - x \tan(\delta_1 + \delta_2)$, $x \leq 0$; $y = - x \tan \delta_2$ $x > 0$. We present here an outline of Powell's work and results. We have altered the approach to the problem somewhat and have found it convenient to use notation different from that of Powell. We have also restricted our discussion to the case $\delta_1 = 0$, $\delta_2 = -\varepsilon$ for reasons which we shall amplify later.

The substitution of the expressions

$$\begin{aligned} U &= \frac{X-1}{1+\kappa} + \varepsilon U'(X, Y) + o(\varepsilon) \\ V &= \varepsilon V'(X, Y) + o(\varepsilon) \\ \Phi &= \frac{1}{2}\frac{(X-1)^2}{1+\kappa} + \varepsilon \Phi'(X, Y) + o(\varepsilon) \\ F &= \frac{1+\kappa X}{1+\kappa} + \varepsilon F'(X, Y) + o(\varepsilon) \end{aligned} \qquad (4.2.1)$$

into (2.2.4) results in the equation

$$2Y \frac{1+\kappa X}{1+\kappa} \Phi'_{XY} + \left[Y^2 - \left(\frac{1+\kappa X}{1+\kappa}\right)^2 \right] \Phi'_{YY} - 2 \frac{1+\kappa X}{1+\kappa} \Phi'_X - \frac{2\kappa}{1+\kappa} Y \Phi'_Y + \frac{2\kappa}{1+\kappa} \Phi' = 0 \qquad (4.2.2)$$

for the perturbation potential $\Phi'(X, Y)$. The characteristics of this equation are just those of the Riemann wave and can be written

$$Q = \text{constant} \qquad \text{or} \qquad Q^{\frac{1-\kappa}{2\kappa}} R = \text{constant} \qquad (4.2.3)$$

where

$$Q = \frac{1+\kappa X}{1+\kappa}, \quad R = \frac{Q}{\left[Q^2 - \frac{1-\kappa}{1+\kappa} Y^2\right]^{\frac{1}{2}}} . \qquad (4.2.4)$$

With Q, R as new independent variables and with ϕ' written in the form

$$\phi' = Yf(Q, R) \qquad (4.2.5)$$

(4.2.2) becomes $R(R^2 - 1)f_{RR} \frac{-2\kappa}{1-\kappa}(R^2 - 1)f_{RQ} + 3R^2 f_R = 0$.

which has the general solution

$$f_R = (R^2 - 1)^{-3/2}\,\psi(Q^{\frac{1-\kappa}{\kappa}} R) \qquad (4.2.6)$$

where ψ is an arbitrary function of its argument.

Powell then uses this result to argue that all perturbation quantities vanish beyond the boundary $Q^{\frac{1-\kappa}{\kappa}} R = 1$, which is just the curved characteristic through $X = 1$, $Y = 0$ in figure 3. He then uses the condition $\phi'_Y = 0$ on this boundary, together with the linearised boundary condition wall at the wall, which is readily seen to be

$$\phi'_Y(X, 0) = \frac{X - 1}{1 + \kappa} = \frac{Q - 1}{\kappa} \qquad (4.2.7)$$

to obtain an integral equation for the function ψ. Powell is then able to show that this equation reduces to an Abel integral equation which he solves for the function ψ. For later convenience we quote here some results for the case $\gamma = 2$. In this case, Powell's method yields

$$\phi' = \frac{-4\sqrt{3}}{\pi}\left[\frac{Q(Q-1)(R^2-1)^{\frac{1}{2}}}{R}\sin^{-1}\left[\frac{Q(R^2 - 1)}{1 - Q}\right]^{\frac{1}{2}} - Q\cos^{-1}Q^{\frac{1}{2}}R + Q^{3/2}\frac{(1-QR^2)^{\frac{1}{2}}}{R} - \frac{Q}{R}(Q - 1)(R^2 - 1)^{\frac{1}{2}}\frac{\pi}{2}\right]. \qquad (4.2.8)$$

From (4.2.4), (4.2.5) and (4.2.7) it can be shown that on the wall

$$\phi'(X, 0) = \frac{-4\sqrt{3}}{\pi}\left[Q^{3/2}(1 - Q)^{\frac{1}{2}} - Q\cos^{-1}Q^{\frac{1}{2}}\right] \qquad (4.2.9)$$

$$\phi'_X(X, 0) = \frac{-4\sqrt{3}}{\pi}\left[Q^{\frac{1}{2}}(1 - Q)^{\frac{1}{2}} - \cos^{-1}Q^{\frac{1}{2}}\right] \qquad (4.2.10)$$

and $\phi'_Y(X, 0) = \frac{-4}{\pi}\left[-(Q - 1)\frac{\pi}{2}\right] = 2(Q - 1)$ as required.

Confining ourselves to this case $\delta_1 = 0$, $\delta_2 = -\varepsilon$ we ask whether it is possible to obtain Powell's solution by using a perturbation technique in the hodograph plane. We first formulate in the U-V plane the exact boundary value problem to be solved.

In the X-Y plane, it is clear that for sufficiently large values of Y the resulting motion of the gas will be described by the Riemann wave solution. Therefore, in the hodograph plane one boundary condition is given by

$$F = 1 + \kappa U \quad \text{on} \quad V = 0 \quad \frac{-1}{\kappa} \leq U \leq 0. \tag{4.2.11}$$

A second boundary condition is furnished by the requirement that the velocity at the wall AOC be parallel to the wall. Thus we require

$$V = U \tan \varepsilon \quad \text{on} \quad Y = X \tan \varepsilon \,. \tag{4.2.12}$$

This condition, taken in conjunction with (2.3.1) implies

$$F_V = F_U \tan \varepsilon \quad \text{on} \quad V = U \tan \varepsilon \,. \tag{4.2.13}$$

The partial differential equation to be satisfied is (2.3.2) viz.

$$(\kappa^2 - F_V{}^2)F_{UU} + 2F_U F_V F_{UV} + (\kappa^2 - F_U{}^2)F_{VV} = \frac{\kappa}{F}\,[(F_U{}^2 + F_V{}^2)(1-\kappa) - 2\kappa^2] \,.$$

The exact boundary value problem is illustrated in figure 13.

We observe that as $\varepsilon \to 0$ the region in the U-V plane in which we seek a solution vanishes. We note also that the $\varepsilon = 0$ solution (the Riemann wave) is not a solution of (2.3.2). This result should occasion no surprise since the $\varepsilon = 0$ solution is a simple wave whereas (2.3.2) is valid only where $J = \partial(U, V)/\partial(X, Y)$ is non-zero. We are thus confronted with a singular perturbation problem.

We attempt to solve this problem by the method of stretched coordinates, introducing W by the relation $V = \delta W$ where $\delta = \tan \varepsilon$. We transform (2.3.2) to new independent variables U, W obtaining

$$[\delta^2\kappa^2 - F_W{}^2]F_{UU} + 2F_U F_W F_{UW} + (\kappa^2 - F_U{}^2)F_{WW} = \frac{\kappa}{F}[(\delta^2 F_U{}^2 + F_W{}^2)(1-\kappa) - 2\delta^2\kappa^2] \,. \tag{4.2.14}$$

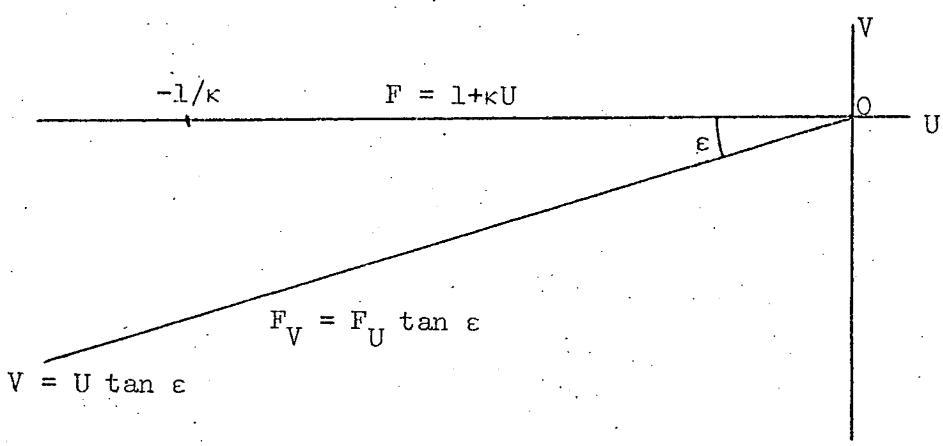


figure 13

The boundary conditions (4.2.11) and (4.2.13) become

$$F = 1 + \kappa U \quad \text{on} \quad W = 0 , \quad \frac{-1}{\kappa} \leqslant U \leqslant 0 \tag{4.2.15}$$

and $$F_W = \delta^2 F_U \quad \text{on} \quad W = U \tag{4.2.16}$$

respectively. Having now transformed our original boundary value problem to one in which the solution is sought in a domain which does not vanish as $\varepsilon \to 0$, we seek a solution for F of the form

$$F(U,W) = F_o(U, W) + \delta F_1(U, W) + \delta^2 F_2(U, W) + \ldots . \tag{4.2.17}$$

Upon substituting this form for $F(U, W)$ into (4.2.14), (4.2.15) and (4.2.16) and equating like powers of δ we find that the function $F_o(U, W)$ is required to satisfy the following conditions

$$\left(\frac{\partial F_o}{\partial W}\right)^2 \frac{\partial^2 F_o}{\partial U^2} - 2 \frac{\partial F_o}{\partial U} \frac{\partial F_o}{\partial W} \frac{\partial^2 F_o}{\partial U \partial W} - \left[\kappa^2 - \left(\frac{\partial F_o}{\partial U}\right)^2\right] \frac{\partial^2 F_o}{\partial W^2} + \frac{\kappa(1-\kappa)}{F_o} \left(\frac{\partial F_o}{\partial W}\right)^2 = 0$$

$$F_o(U, 0) = 1 + \kappa U , \qquad \frac{\partial F_o}{\partial W} (U, U) = 0 .$$

Hence we may take $F_o(U, W) = 1 + \kappa U$. (4.2.18)

The differential equation and boundary conditions to be satisfied by the second term in the expansion (4.2.17) depend upon the first term. When the expression (4.2.18) is used it is found that $F_1(U, W)$ is required to satisfy

$$(1 + \kappa U)\left[\frac{\partial F_1}{\partial W} \frac{\partial^2 F_1}{\partial U \partial W} - \frac{\partial F_1}{\partial U} \frac{\partial^2 F_1}{\partial W^2}\right] = \frac{1 - \kappa}{2}\left[\left(\frac{\partial F_1}{\partial W}\right)^2 - \kappa^2 \frac{1 + \kappa}{1 - \kappa}\right], \tag{4.2.19}$$

$$F_1(U, 0) = 0 \tag{4.2.20}$$

and $$\frac{\partial F_1}{\partial W} (U, U) = 0. \tag{4.2.21}$$

Now (4.2.19) is a quasi-linear second order equation which in passing we may note is nowhere elliptic. Its characteristics may be obtained by equating to zero the determinant of the matrix of coefficients in the system

$$\begin{bmatrix} 0 & \frac{\partial F_1}{\partial W} & -\frac{\partial F_1}{\partial U} \\ \frac{dU}{d\sigma} & \frac{dW}{d\sigma} & 0 \\ 0 & \frac{dU}{d\sigma} & \frac{dW}{d\sigma} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F_1}{\partial U^2} \\ \frac{\partial^2 F_2}{\partial U \partial W} \\ \frac{\partial^2 F_1}{\partial W^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\kappa U} \cdot \frac{1-\kappa}{2}\left[\left(\frac{\partial F_1}{\partial W}\right)^2 - \kappa^2\frac{1+\kappa}{1-\kappa}\right] \\ \frac{d}{d\sigma}\left(\frac{\partial F_1}{\partial U}\right) \\ \frac{d}{d\sigma}\left(\frac{\partial F_1}{\partial W}\right) \end{bmatrix} . \tag{4.2.22}$$

It is readily seen that this results in the equation

$$\frac{dU}{d\sigma}\left[\frac{\partial F_1}{\partial U}\frac{dU}{d\sigma} + \frac{\partial F_1}{\partial W}\frac{dW}{d\sigma}\right] = 0. \tag{4.2.23}$$

Hence one family of characteristics is the family of lines U = constant whilst the other is the family of curves upon which F_1 is constant. Thus (4.2.19) has the unusual property of having one family of characteristics independent of the solution considered (a property of the linear hyperbolic equation) and the other family dependent upon the individual solution considered. The compatability conditions upon these characteristics may be found in the usual manner (that is by equating to zero the determinant of the matrix resulting when any column of the above matrix is replaced by the column vector on the right of (4.2.22)). The compatability condition on the characteristics U = constant is non-integrable but that on the characteristics F_1 = constant can be integrated to give

$$\left(\frac{\partial F_1}{\partial W}\right)^2 - \kappa^2\,\frac{1+\kappa}{1-\kappa} = A(1+\kappa U)^{\frac{1-\kappa}{\kappa}} \quad \text{on} \quad F_1 = \text{constant} \tag{4.2.24}$$

where A is a constant. Hence A is a function only of F_1 and we may write

$$F_1(U, W) = g\left[\frac{\kappa^2\frac{1+\kappa}{1-\kappa} - \left(\frac{\partial F_1}{\partial W}\right)^2}{(1+\kappa U)^{\frac{1-\kappa}{\kappa}}}\right] \tag{4.2.25}$$

throughout the domain in which the solution for $F_1(U, W)$ is sought. Thus

(4.2.19) has been re-written as a first order equation, but at the expense of introducing the unknown function g. However, the whole apparatus of the theory of first order partial differential equations is now available to us and so the presence of the unknown function g may not prove too great a difficulty.

Since $F_1(U, 0) = 0$, the line $W = 0$ is a characteristic and it is easily deduced that on this line

$$\frac{\partial F_1}{\partial W}(U, 0) = \kappa\sqrt{\frac{1+\kappa}{1-\kappa}}\left[1 - (1 + \kappa U)^{\frac{1-\kappa}{\kappa}}\right]^{\frac{1}{2}} . \tag{4.2.26}$$

Substitution of (4.2.26) into (4.2.25), together with $F_1(U, 0) = 0$ implies

$$g\left[\kappa^2\left(\frac{1 + \kappa}{1 - \kappa}\right)\right] = 0 . \tag{4.2.27}$$

Our problem now is to find an integral surface of the equation

$$F_1 - g\left[\frac{\kappa^2\frac{1+\kappa}{1-\kappa} - \left(\frac{\partial F_1}{\partial W}\right)^2}{(1 + \kappa U)^{\frac{1-\kappa}{\kappa}}}\right] = 0$$

where g is an, as yet, unknown function of its argument, satisfying (4.2.27), which passes through the curve in U, W, F_1 space parametrised by

$$U = s \;, \quad W = 0 \;, \quad \underset{\sim}{F}_1 = 0$$
$$\frac{\partial F_1}{\partial U} = 0 \;, \quad \frac{\partial F_1}{\partial W} = \kappa\sqrt{\frac{1+\kappa}{1-\kappa}}\left[1 - (1 + \kappa s)^{\frac{1-\kappa}{\kappa}}\right]^{\frac{1}{2}} = A(s), \text{ say} \tag{4.2.28}$$

and which, in addition, satisfies the requirement that $\partial F_1/\partial W = 0$ on $W = U$. The technique for finding an integral surface of a first order partial differential equation, $L[\phi(x, y)] = 0$, which passes through a given twisted curve in $x, y, \phi(x, y)$ space is well known. Here the Lagrange-Charpit equations become

$$\frac{dU}{dt} = 0$$

$$\frac{dW}{dt} = g'(\psi)\,\frac{2F_{1W}}{(1+\kappa U)^{\frac{1-\kappa}{\kappa}}}$$

$$\frac{dF_1}{dt} = g'(\psi)\,\frac{2(F_{1W})^2}{(1+\kappa U)^{\frac{1-\kappa}{\kappa}}} \qquad (4.2.29)$$

$$\frac{dF_{1U}}{dt} = g'(\psi)\,\frac{(1-\kappa)\psi}{\kappa(1+\kappa U)} - F_{1U}$$

$$\frac{dF_{1W}}{dt} = -\,F_{1W}$$

where we have set

$$\frac{\kappa^2\,\frac{1+\kappa}{1-\kappa} - (F_{1W})^2}{(1+\kappa U)^{\frac{1-\kappa}{\kappa}}} = \psi \qquad (4.2.30)$$

and denoted $\partial F_1/\partial U$ and $\partial F_1/\partial W$ by F_{1U} and F_{1W} respectively. The first members of (4.2.28) and (4.2.29) together imply

$$U = s\ , \qquad (4.2.31)$$

whilst the fifth members of these same equations give

$$F_{1W} = A(s)\,e^{-t}\ . \qquad (4.2.32)$$

From (4.2.32) we deduce that the line $W = U$ is approached as $t \to \infty$. The same equation gives also

$$\psi = \frac{\kappa^2\,\frac{1+\kappa}{1-\kappa} - [A(s)]^2 e^{-2t}}{(1+\kappa s)^{\frac{1-\kappa}{\kappa}}} \geq 0\ . \qquad (4.2.33)$$

Substitution of (4.2.31), (4.2.32) and (4.2.33) into the second member of (4.2.29) now gives

$$\frac{dW}{dt} = g'\left[\frac{\kappa^2\,\frac{1+\kappa}{1-\kappa} - [A(s)]^2 e^{-2t}}{(1+\kappa s)^{\frac{1-\kappa}{\kappa}}}\right]\frac{2A(s)e^{-t}}{(1+\kappa s)^{\frac{1-\kappa}{\kappa}}}$$

and, since $W \to U$ as $t \to \infty$ we have,

$$s = \int_0^{\infty} g'(\psi)\frac{2A(s)e^{-t}}{(1+\kappa s)^{\frac{1-\kappa}{\kappa}}}\,dt$$

and, after some manipulation

$$\frac{1}{\kappa}\left(\frac{\kappa^2}{\beta}\ \frac{1+\kappa}{1-\kappa}\right)^{\frac{1}{2}}\left[\left(\frac{\kappa^2}{\beta}\ \frac{1+\kappa}{1-\kappa}\right)^{\frac{\kappa}{1-\kappa}} - 1\right] = \int_{\kappa^2\frac{1+\kappa}{1-\kappa}}^{\beta} \frac{g'(\psi)\,d\psi}{\sqrt{\beta-\psi}} \qquad (4.2.34)$$

where we have set $\kappa^2\,\frac{1+\kappa}{1-\kappa}\,(1+\kappa s)^{-(\frac{1-\kappa}{\kappa})} = \beta \geqslant 0.$ (4.2.35)

This is an Abel integral equation, and has the explicit solution

$$g'(\psi) = \frac{1}{\pi}\,\frac{d}{d\psi}\left[\int_{\kappa^2\frac{1+\kappa}{1-\kappa}}^{\psi} \frac{1}{\kappa}\left(\frac{\kappa^2}{\beta}\ \frac{1+\kappa}{1-\kappa}\right)^{\frac{1}{2}}\left[\left(\frac{\kappa^2}{\beta}\ \frac{1+\kappa}{1-\kappa}\right)^{\frac{\kappa}{1-\kappa}} - 1\right]\frac{d\beta}{\sqrt{\psi-\beta}}\right]. \qquad (4.2.36)$$

Therefore

$$g(\psi) = \frac{1}{\pi}\int_{\kappa^2\frac{1+\kappa}{1-\kappa}}^{\psi} \frac{1}{\kappa}\left(\frac{\kappa^2}{\beta}\ \frac{1+\kappa}{1-\kappa}\right)^{\frac{1}{2}}\left[\left(\frac{\kappa^2}{\beta}\ \frac{1+\kappa}{1-\kappa}\right)^{\frac{\kappa}{1-\kappa}} - 1\right]\frac{d\beta}{\sqrt{\psi-\beta}}\ . \qquad (4.2.37)$$

Hence the function g may be determined. When g is determined, then F_1 is known in terms of s and t. Since U and W are each known as functions of s and t (by (4.2.31) and (4.2.32) respectively), F_1 may be obtained, in principle at least, in terms of U and W by elimination of the parameters s and t.

We now take $\gamma = 2$ $(\kappa = \frac{1}{2})$ in (4.2.37) and proceed to show that the solution obtained is in agreement with that obtained by Powell. With $\kappa=\frac{1}{2}$, then, (4.2.37) becomes

$$g(\psi) = \frac{2}{\pi}\int_{\frac{3}{4}}^{\psi} \left(\frac{3}{4\beta}\right)^{\frac{1}{2}}\left(\frac{3}{4\beta} - 1\right)\frac{d\beta}{\sqrt{\psi-\beta}} \qquad (4.2.38)$$

which may be integrated to give

$$g(\psi) = \frac{\sqrt{3}}{\pi}\left[\sin^{-1}\left(\frac{3-2\psi}{2\psi}\right) + \frac{\sqrt{3}\sqrt{4\psi-3}}{2\psi} - \frac{\pi}{2}\right]. \qquad (4.2.39)$$

Now $W = U \Rightarrow \psi = \dfrac{3}{2(2+s)}$ and therefore, on $W = U$ we have

$$F_1 = \frac{\sqrt{3}}{\pi}\left[\sin^{-1}(1+s) + (2+s)\sqrt{\frac{-s}{2+s}} - \frac{\pi}{2}\right]. \qquad (4.2.40)$$

It is readily seen that, on $W = U$, F_{1U} is given by

$$F_{1U} = -U(-2U - U^2)^{\frac{1}{2}} . \qquad (4.2.41)$$

Hence, using (2.3.1), we may write

$$X = U + 2\left[1 + \frac{U}{2} + \varepsilon\frac{\sqrt{3}}{\pi}\,[(-2U - U^2)^{\frac{1}{2}} - \cos^{-1}(1+U)]\right] \times$$

$$\left[\frac{1}{2} + \varepsilon\frac{\sqrt{3}}{\pi}(-U)(-2U - U^2)^{-\frac{1}{2}}\right] + o(\varepsilon)$$

$$= 1 + \frac{3U}{2} + \varepsilon\frac{\sqrt{3}}{\pi}\left[(-2U - U^2)^{\frac{1}{2}} - \cos^{-1}(1+U) - U(2+U)(-2U-U^2)^{-\frac{1}{2}}\right] + o(\varepsilon)$$

$$= 1 + \frac{3U}{2} + \varepsilon f(U) + o(\varepsilon), \text{ say.}$$

Therefore $U = \frac{2}{3}(X-1) - \varepsilon\frac{2}{3} f[\frac{2}{3}(X-1)] + o(\varepsilon)$.

Hence, on $W = U$, we have

$$U = \frac{2}{3}(X-1) - \varepsilon\frac{2}{3}\cdot\frac{\sqrt{3}}{\pi}\left[2[\frac{2}{3}(1-X)\frac{2}{3}(2+X)]^{\frac{1}{2}} - \cos^{-1}\left(\frac{1+2X}{3}\right)\right] + o(\varepsilon)$$

$$= \frac{2}{3}(X-1) - \varepsilon\frac{4}{\sqrt{3}\pi}\left[2\left(\frac{2+X}{3}\right)^{\frac{1}{2}}\left(\frac{1-X}{3}\right)^{\frac{1}{2}} - \cos^{-1}\left(\frac{2+X}{3}\right)^{\frac{1}{2}}\right] + o(\varepsilon)$$

which is in agreement with Powell's results.

Thus the solution expressed in parametric form by (4.2.31), (4.2.32), (4.2.25) and (4.2.39) is seen to give results in accord with Powell's along the axis $Y = 0$.

Throughout this work on the diffraction of the Riemann wave, we have restricted ourselves to the case $\delta_1 = 0$, that is, the case in which the wall has no 'kink' in it. We have not imposed this restriction to lessen the labour involved, but for a more significant reason. The boundary condition at the wall gave us (4.2.13) viz. $F_V = F_U \tan \varepsilon$ on $V = U \tan \varepsilon$. However, had we taken as our wall the boundary $y = -x \tan(\delta_1 + \delta_2)$, $x < 0$; $y = -x \tan \delta_2$, $x > 0$, as does Powell, then, in the hodograph plane our boundary condition at the wall would take the form

$$F_V = -F_U \tan(\delta_1 + \delta_2) \quad \text{on} \quad V = -U \tan(\delta_1 + \delta_2) \qquad U < \frac{-1}{\kappa} FF_U$$

$$F_V = -F_U \tan \delta_2 \quad \text{on} \quad V = -U \tan \delta_2 \qquad U > \frac{-1}{\kappa} FF_U .$$

The boundary value problem specified by this condition and by (2.3.2) and (4.2.11) is in a completely different category and we do not consider it further here. Despite the limitation imposed by setting $\delta_1 = 0$, the recovery of Powell's solution by using hodograph methods is felt to be of some interest and the device of preventing the domain in the hodograph plane from shrinking to zero by a boundary layer type stretching of coordinates is one which may well have other applications.

We now turn our attention to the case $|U_1| < 1/\kappa$, the case in which the unperturbed solution is the incomplete rarefaction wave given by (3.2.2). This problem has been studied by Anderson (1966), again in the more general case in which the boundary AOC (figure 12) is given by $y = -x \tan(\delta_1 + \delta_2)$, $x < 0$; $y = -x \tan \delta_2$, $x > 0$. In this discussion of the diffraction of the incomplete rarefaction wave, we shall be concerned only with the X-Y plane and thus shall not find it necessary to make the restriction $\delta_1 = 0$. Accordingly, with the wall AOC as given above, we begin by describing briefly Anderson's work.

The system of characteristics associated with the incomplete rarefaction wave was shown in figure 4. Figure 4 is reproduced in figure 14 with some of the characteristics omitted.

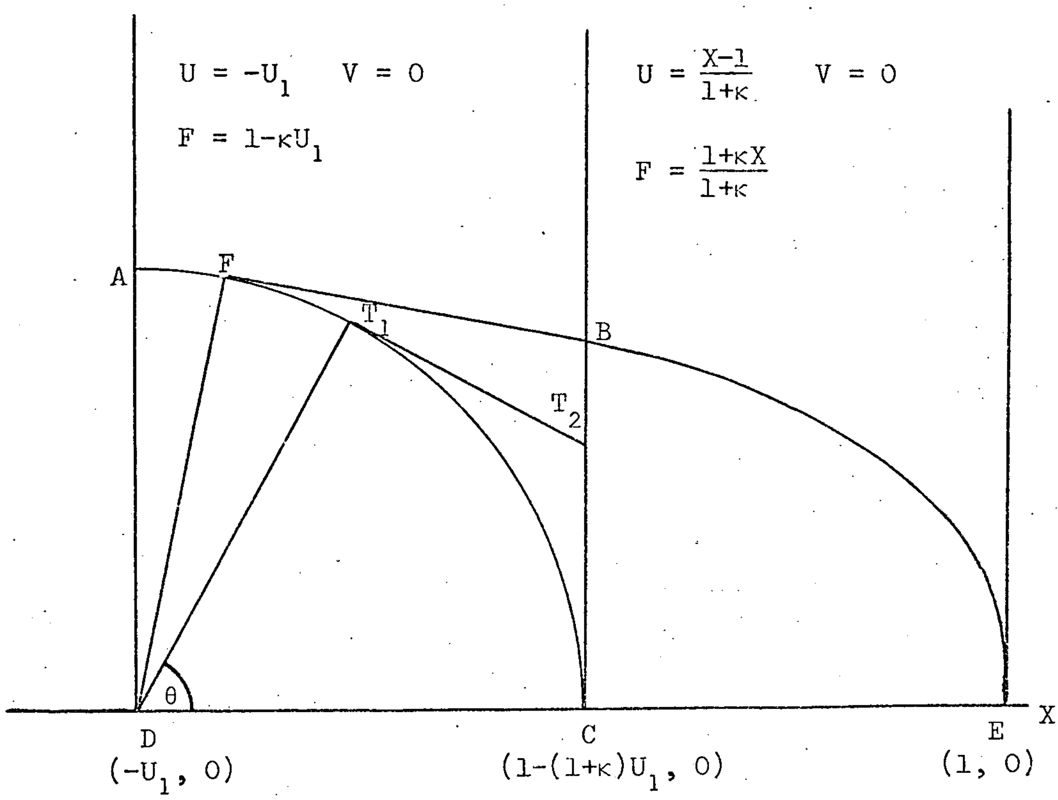


figure 14

Anderson argues that all perturbation quantities vanish beyond the boundary EBFA. EB is a part of that characteristic which, were the unperturbed flow the complete rarefaction wave, would bound the region of perturbation. The straight line FB is a characteristic of the uniform flow, which is the unperturbed solution in $-U_1 \leq X \leq 1 - (1+\kappa)U_1$. The arc AFC is, of course, part of the sonic circle $(X + U_1)^2 + Y^2 = [1 - \kappa U_1]^2$. The quadrant bounded by AFC, AD and DC is a region of ellipticity. The origin of our coordinate system $X = 0$, $Y = 0$ lies inside or outside the sonic circle according as $U_1 <$ or $> 1/(1 + \kappa)$.

Anderson then, reasoning that the boundary conditions $\phi'_Y = 0$ on the characteristic BE and $\phi'_Y = (x - 1)[\delta_2 + \delta_1 H(X)]/(1 + \kappa)$ on the non-characteristic CE together with the differential equation (4.2.2) provide a well posed Goursat type boundary value problem states that the solution for ϕ' in BEC is just Powell's solution. He then asserts that in FBC the solution is a simple wave. However, he then goes on to argue that it is possible to solve for the perturbation velocity components or the perturbation pressure in AFBCD without reference to the boundary condition on BC which Powell's solution in BEC provides. He is thus led to postulate the existence of a discontinuity across the line BC of the perturbation quantities we have listed above. We intend to put forward an alternative formulation to show that no such discontinuity need exist.

In AFBCD, the solution we seek is a perturbation on the uniform flow $U = -U_1$, $V = 0$, $F = (1 - \kappa U_1)$. The analysis embodied in equations (4.1.1), (4.1.2) and (4.1.3) is therefore valid and so with $X + U_1 = F_1 X_1$, $Y = F_1 Y_1$ we have equation (4.1.3)

$$(X_1{}^2 - 1)\frac{\partial^2 F'}{\partial X_1{}^2} + 2X_1 Y_1 \frac{\partial^2 F'}{\partial X_1 \partial Y_1} + (Y_1{}^2 - 1)\frac{\partial^2 F'}{\partial Y_1{}^2} + 2X_1 \frac{\partial F'}{\partial X_1} + 2Y_1 \frac{\partial F'}{\partial Y_1} = 0$$

valid in AFBCD. As we have remarked previously (4.1.3) is elliptic in $X_1^2 + Y_1^2 < 1$ and hyperbolic in $X_1^2 + Y_1^2 > 1$. Further, in FBC the two characteristics through any point are just the two tangents to the sonic circle $X_1^2 + Y_1^2 = 1$. Now a comparison of the situation obtaining in AFBCD with that obtaining in Tricomi's problem (see, for example, Garabedian (1964)) might lead one to conclude that boundary data for F' (in the form of the function itself or its normal derivative) on AD, DC and AFB provide in AFBCD a well posed problem for F'. However, such a conclusion would not be correct. For, in the Tricomi problem, the solution is required to be continuously differentiable across the parabolic line (otherwise it would be sufficient to let the dependent variable take, on the parabolic line, any suitable value and to solve separately a Dirichlet problem in the elliptic domain and a Goursat problem in the hyperbolic domain). However, we now show that the requirement that the solution in FBC be a simple wave in which the perturbation sound speed, F', is constant upon each characteristic of that family of which FB is a member implies that the normal derivative of F' becomes infinite as one approaches the parabolic line AFC. For, in figure 15, let F' be given on the circumference of the circle $R = 1$ by any continuous function $f(\theta)$ and let F' be constant on any tangent of the family of which AB and CD are members. The radial derivative of F' at C is given by

$$\frac{\partial F'}{\partial R} = \lim_{\delta\theta\to 0} \frac{[f(\theta + \delta\theta) - f(\theta)]\cos\delta\theta}{1 - \cos\delta\theta}$$

$$= \lim_{\delta\theta\to 0} \frac{[\delta\theta f'(\theta) + \frac{1}{2}(\delta\theta)^2 f''(\theta) + \ldots][1 - \frac{1}{2}(\delta\theta)^2 + \ldots]}{\frac{1}{2}(\delta\theta)^2[1 - \frac{1}{12}(\delta\theta)^2 + \ldots]}$$

which is finite only if $f'(\theta) = 0$. The case $f'(\theta) = 0$ corresponds to uniform flow in the region immediately exterior to the circle $R = 1$, as is

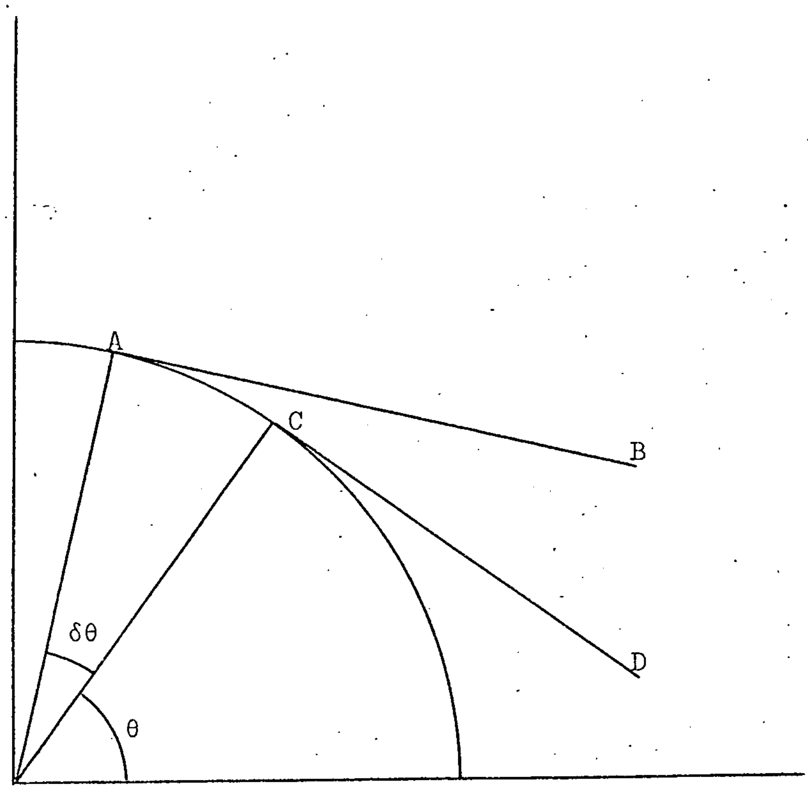


figure 15

discounted.

Thus we have shown that the requirement that the solution in FBC (figure 14) be a simple wave of the required type implies that the normal derivative of the perturbation sound speed becomes infinite as one approaches the sonic circle AFC through values of $(X_1{}^2 + Y_1{}^2)^{\frac{1}{2}}$ greater than unity. Hence the condition of continuity of normal derivative across the parabolic line cannot be used in this problem. It therefore follows that the Tricomi problem may not be taken as a model, that data on AF, FB, CD and DA do not constitute a well posed mixed-type problem in the Tricomi sense and that information on BC is necessary to determine the solution in AFBCD. This point of view is consistent with the actual data as determined by the physical problem in relation to finding a unique continuous solution.

Anderson, in claiming that the solution in FBC (figure 14) can be obtained without reference to any information on BC, makes use of a result (Anderson (1966) p.913) which, we suggest, is not correct. He claims that if T_1T_2 is any characteristic in FBC of the same family as FB and if $\delta(u_{3b})$, $\delta(v_{3b})$ and δq denote changes in the two components and magnitude of the perturbation velocity respectively across T_1T_2, then

$$\delta(u_{3b}) = \delta q \cos\theta \quad \text{and} \quad \delta(v_{3b}) = \delta q \sin\theta$$

(where the meaning of θ is seen in figure 14.) These equations imply that the change in the perturbation velocity vector in making an infinitesimal change of position across T_1T_2 is perpendicular to T_1T_2. There seems no justification for this assertion which is then used to deduce a solution in FBC without making any use of data on BC and thereby leading to an artificial discontinuity across BC. We now show how the solution in FBC can be found such that all perturbation quantities are continuous across BC. Let us deal with the dimensionless perturbation sound speed, F'.

The value of F' on BC may be obtained from an application of the linearised form of Bernoulli's theorem. Let us therefore consider F' known on BC and denote its value by $G(Y_1)$. In FBC the perturbation sound speed F' satisfies (4.1.3)

$$(X_1^2 - 1)\frac{\partial^2 F'}{\partial X_1^2} + 2X_1Y_1\frac{\partial^2 F'}{\partial X_1\partial Y_1} + (Y_1^2 - 1)\frac{\partial^2 F'}{\partial Y_1^2} + 2X_1\frac{\partial F'}{\partial X_1} + 2Y_1\frac{\partial F'}{\partial Y_1} = 0$$

which can by means of the transformations

$$\alpha = \frac{X_1Y_1 + \sqrt{X_1^2 + Y_1^2 - 1}}{Y_1^2 - 1}, \quad \beta = \frac{X_1Y_1 - \sqrt{X_1^2 + Y_1^2 - 1}}{Y_1^2 - 1} \tag{4.2.42}$$

be reduced to the equation

$$\frac{\partial^2 F'}{\partial\alpha^2} - \frac{\partial^2 F'}{\partial\beta^2} = 0 \tag{4.2.43}$$

which has the general solution

$$F' = A(\alpha) + B(\beta) \tag{4.2.44}$$

where A and B are arbitrary functions of their arguments. It should be noted that the two families of lines α = constant and β = constant are just the two families of characteristics of (4.1.3).

Now FB is a line upon which α is constant and since $F' = 0$ on this line (4.2.44) reduces to

$$F' = A(\alpha) . \tag{4.2.45}$$

Further, BC is $X_1 = 1$ and on BC

$$\alpha = \frac{2Y_1}{Y_1^2 - 1} .$$

Since $F' = G(Y_1)$ on BC , (4.2.45) gives

$$G(Y_1) = A\left[\frac{2Y_1}{Y_1^2 - 1}\right] . \tag{4.2.46}$$

If $2Y_1/(Y_1^2 - 1)$ be denoted by λ, then (4.2.46) can be inverted to give

$$A(\lambda) = G\left[\frac{1 - \sqrt{1 + \lambda^2}}{\lambda}\right] .$$

Hence, in FBC , F' is given by

$$F' = G\left[\frac{1 - \sqrt{1 + \alpha^2}}{\alpha}\right]. \qquad (4.2.47)$$

The solution as given by (4.2.47) is a simple wave since the analysis which has been applied to F' could equally have been applied to U' or V' to establish that the perturbation velocity components are constant on one family of characteristics. Further, it satisfies the boundary condition on BC and, provided that the perturbation pressure tends to zero on BC as one approaches the point B (which is the case), it satisfies the boundary condition on FB.

On the arc FC, $\alpha = -Y_1/X_1$ and (4.2.47) then gives

$$F' = G\,[\tan\,\theta/2]$$

which, together with the requirement that $F' = 0$ on AF, and the boundary conditions on AD and DC enables us to solve for F' in AFCD.

Thus we have seen that it is possible to solve for F' in AFBCD, obtaining a solution which is continuous across BC. It should be pointed out that the solution obtained by the method which we have outlined is continuous everywhere (except, in the case $\delta_1 \neq 0$,) and that this is the best that can be achieved since the normal derivative (of, say, the perturbation pressure) necessarily becomes infinite on the sonic circle.

4.3 Perturbation of Suchkov's Solution

We have seen that when the adiabatic index of the gas γ and the semi angle θ of the wedge satisfy the relation (3.3.5)

$$\cot^2\theta = \frac{1 + \gamma}{3 - \gamma} = \frac{1 + \kappa}{1 - \kappa}$$

then the problem of the expansion of a wedge of gas into vacuum has the remarkable exact linear solution (3.3.6)

$$F = 1 + \kappa U + \kappa \sqrt{\frac{1 + \kappa}{1 - \kappa}} \; V$$

where, of course, $\kappa = (\gamma - 1)/2$. With κ considered fixed, denote the value of θ for which (3.3.5) holds by θ^* and consider the problem of the expansion into vacuum of a wedge of gas of semi angle $\theta = \theta^* + \varepsilon$ where ε is a small quantity.

In the hodograph plane, we readily see that the full problem may be formulated exactly as the boundary value problem shown in figure 16. We seek now to obtain a solution by the use of first order perturbation theory. Accordingly we put

$$F = 1 + \kappa U + \kappa \sqrt{\frac{1 + \kappa}{1 - \kappa}} \, V + \varepsilon F'(U, V) + o(\varepsilon) \tag{4.3.1}$$

and substitute into (2.3.2). Retaining only the terms linear in ε we obtain

$$\frac{\kappa}{1-\kappa}\left[1 + \kappa U + \kappa\sqrt{\frac{1+\kappa}{1-\kappa}} \; V\right] F'_{UU} - \sqrt{\frac{1+\kappa}{1-\kappa}}\left(1 + \kappa U + \kappa\sqrt{\frac{1+\kappa}{1-\kappa}}\right) F'_{UV}$$
$$+ (1 - \kappa)\sqrt{\frac{1+\kappa}{1-\kappa}} \, F'_V + (1 - \kappa) F'_U = 0 \; . \tag{4.3.2}$$

We note first of all that, since the unperturbed solution is a mixed wave and therefore a solution of (2.3.2), we are not faced with the difficulties which confronted us when examining a perturbation of the Riemann wave, that is to say we are able to pose the problem in a region of the hodograph plane which does not degenerate to zero as $\varepsilon \to 0$. We note secondly that (4.3.2) is elliptic nowhere. The transformations

$$\xi = \sqrt{\frac{1+\kappa}{1-\kappa}} \; U + \frac{\kappa}{1-\kappa} V + \frac{1}{\kappa}\sqrt{\frac{1+\kappa}{1-\kappa}} \, , \qquad \eta = \frac{1}{1-\kappa} V \tag{4.3.3}$$

reduce (4.3.2) to

$$F'_{\xi\eta} + \frac{\kappa-1}{\kappa} \, \frac{F'_\xi + F'_\eta}{\xi + \eta} = 0 \tag{4.3.4}$$

which is the Euler-Poisson-Darboux equation, an equation encountered not

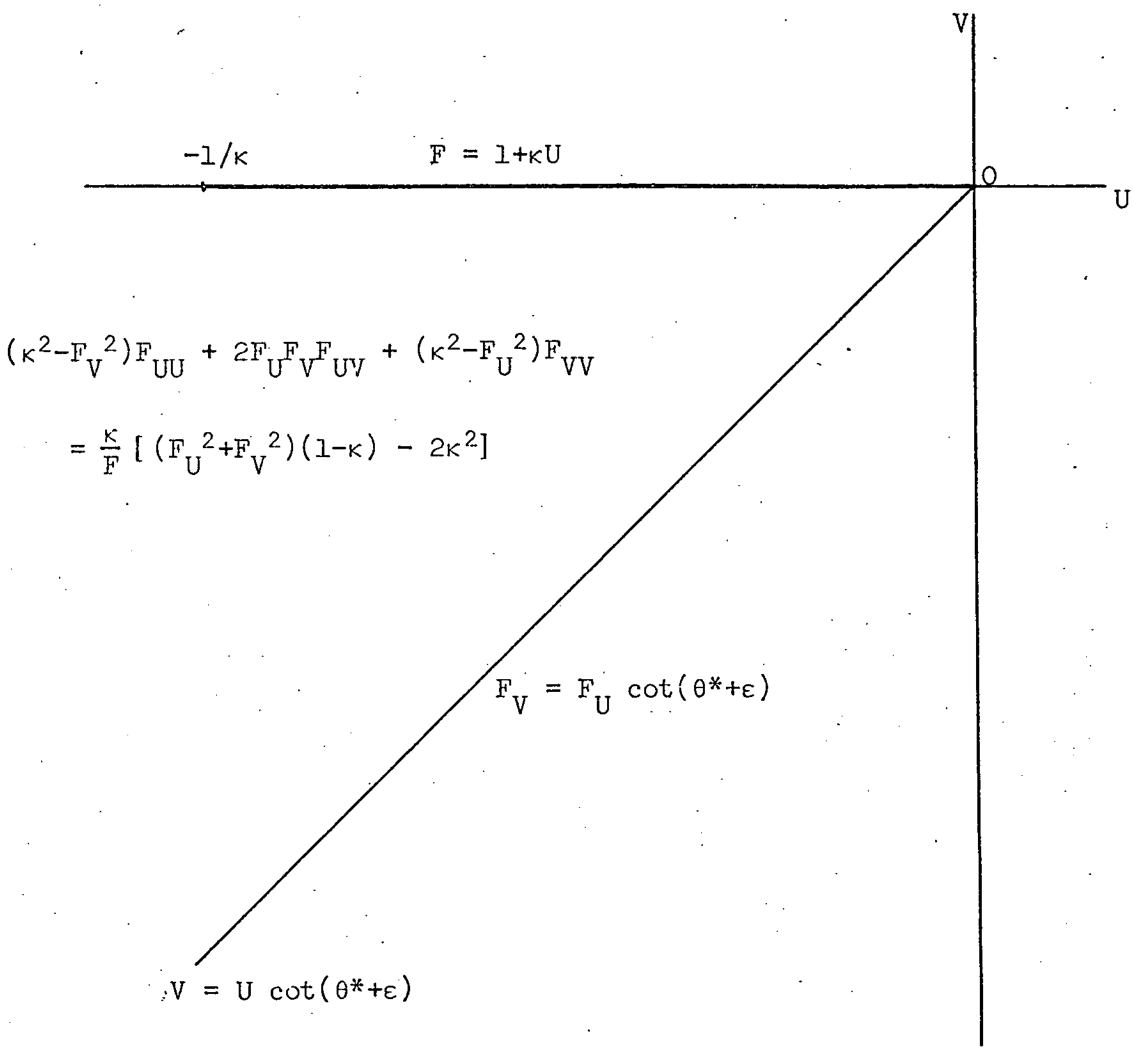

V
$-1/\kappa$
$F = 1+\kappa U$
O
U
$(\kappa^2-F_V{}^2)F_{UU} + 2F_UF_VF_{UV} + (\kappa^2-F_U{}^2)F_{VV}$
$= \frac{\kappa}{F}[(F_U{}^2+F_V{}^2)(1-\kappa) - 2\kappa^2]$
$F_V = F_U \cot(\theta^*+\varepsilon)$
$V = U \cot(\theta^*+\varepsilon)$

figure 16

infrequently in Applied Mathematics.

The boundary condition on $V = 0$ becomes

$$F' = 0 \qquad \text{on} \qquad \eta = 0 \qquad 0 \leqslant \xi \leqslant \frac{1}{\kappa}\sqrt{\frac{1+\kappa}{1-\kappa}} . \tag{4.3.5}$$

Now the boundary condition

$$F_V = F_U \cot(\theta^* + \varepsilon) \quad \text{on} \quad V = U \cot(\theta^* + \varepsilon)$$

can be linearised to

$$F'_V = F'_U \cot\theta^* - \kappa \operatorname{cosec}^2\theta^* \quad \text{on} \quad V = U\cot\theta^* \tag{4.3.6}$$

which, under the transformations (4.3.3) becomes

$$F'_\xi - F'_\eta = 2\kappa \qquad \text{on} \qquad \eta = \xi - \frac{1}{\kappa}\sqrt{\frac{1+\kappa}{1-\kappa}} . \tag{4.3.7}$$

The boundary value problem to be solved in the ξ-η plane is thus that shown in figure 17. The equation (4.3.4) is in its normal form and thus the characteristics are the lines ξ = constant and the lines η = constant. The Goursat type boundary value problem illustrated in figure 17 determines F' up to the line $\xi = 0$. We are unable to solve this problem in its full generality, but the solution is readily obtained in certain special cases. In the case $\gamma = 2$ $(\kappa = \frac{1}{2})$ the solution is found (almost by inspection) to be

$$F'(\xi, \eta) = \frac{-1}{2\sqrt{3}} \xi\eta \tag{4.3.8}$$

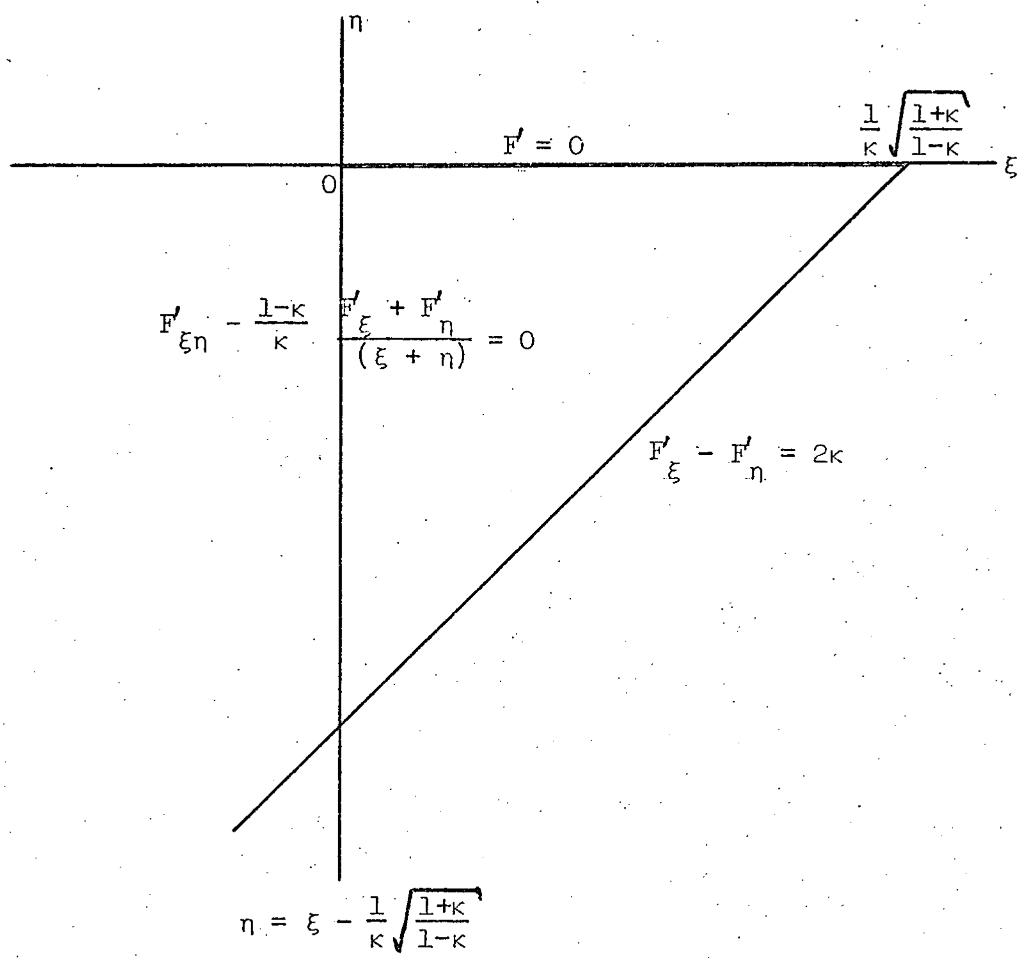
and hence the solution of the perturbed problem is given by

$$F(U, V) = 1 + \frac{U}{2} + \frac{\sqrt{3}}{2} V - \frac{\varepsilon}{2\sqrt{3}} (\sqrt{3}U + V + 2\sqrt{3})(2V) + o(\varepsilon)$$

that is $$F(U, V) = 1 + \frac{U}{2} + \frac{\sqrt{3}}{2} V - \frac{\varepsilon}{\sqrt{3}} (\sqrt{3}UV + V^2 + 2\sqrt{3}V) + o(\varepsilon). \tag{4.3.9}$$

Let us now return to the boundary value problem for general κ, as shown in figure 17, and write the Euler-Poisson-Darboux equation in the form

$$F'_{\xi\eta} + N \frac{(F'_\xi + F'_\eta)}{\xi + \eta} = 0 \tag{4.3.10}$$



η
$F' = 0$
$\frac{1}{\kappa}\sqrt{\frac{1+\kappa}{1-\kappa}}$
ξ
0
$F'_{\xi\eta} - \frac{1-\kappa}{\kappa}\,\frac{F'_\xi + F'_\eta}{(\xi + \eta)} = 0$
$F'_\xi - F'_\eta = 2\kappa$
$\eta = \xi - \frac{1}{\kappa}\sqrt{\frac{1+\kappa}{1-\kappa}}$

figure 17

where $N = \frac{\kappa-1}{\kappa} < 0$. Equation (4.3.10) has several well-known and interesting properties which may be used to solve our boundary value problem for certain values of κ (and therefore N). We illustrate this by considering the case $\kappa = \frac{1}{3}$ $(N = -2)$.

It can be shown (see, for example, Mackie (1965) p.229) that if F'_N satisfies (4.3.10) for some given value of N, then $(\xi+\eta)^{-1+2N}F'_N$ satisfies (4.3.10) with N replaced by $1-N$. It can also be shown (reference as above) that, for N a positive integer, (4.3.10) has the general solution

$$F'_N = \left[\frac{1}{\xi + \eta}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\right]^N \left[f(\xi) + g(\xi)\right] \tag{4.3.11}$$

where f and g are arbitrary functions of their arguments. These two results show that (4.3.10), with $N = -2$ has the general solution

$$F' = (\xi + \eta)^5 \left[\frac{1}{\xi + \eta}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\right]^3 \left[f(\xi) + g(\eta)\right]. \tag{4.3.12}$$

After some manipulation we obtain

$$F'(\xi, \eta) = (\xi + \eta)^2 \left[f'''(\xi) + g'''(\eta)\right] - 6(\xi + \eta)\left[f''(\xi) + g''(\eta)\right] + 12\left[f'(\xi) + g'(\eta)\right]. \tag{4.3.13}$$

Our task now is to use the boundary conditions (4.3.5) and (4.3.7) to determine the functions f and g. In fact, we shall be concerned only with $f'(\xi)$ and $g'(\eta)$ as is obvious from (4.3.13). Now (4.3.5) implies

$$\xi^2 f'''(\xi) - 6\xi f''(\xi) + 12f'(\xi) = -g_3\xi^2 + 6g_2\xi - 12g_1 \tag{4.3.14}$$

where we have set $g^{(i)}(0) = g_i$. Equation (4.3.14) is an inhomogeneous second order ordinary differential equation for $f'(\xi)$. It is readily solved by the method of complementary function and particular integral and has the general solution

$$f'(\xi) = A\xi^4 + B\xi^3 - \tfrac{1}{2}g_3\xi^2 + g_2\xi - g_1 \tag{4.3.15}$$

A, B arbitrary constants. Now the boundary condition (4.3.7) with $\kappa=1/3$ becomes

$$F'_\xi - F'_\eta = \frac{2}{3} \quad \text{on} \quad \eta = \xi - 3\sqrt{2}\,. \tag{4.3.16}$$

Differentiation of (4.3.13) gives, after some manipulation

$$F'_\xi - F'_\eta = (\xi + \eta)^2 \, [f^{(iv)}(\xi) - g^{(iv)}(\eta)] - 6(\xi + \eta)[f'''(\xi) - g'''(\eta)] + 12\,[f''(\xi) - g''(\eta)] \; .$$

This last equation, together with (4.3.16) implies

$$\frac{2}{3} = (2\eta+3\sqrt{2})^2 \, [f^{(iv)}(\eta+3\sqrt{2})-g^{(iv)}(\eta)] - 6(2\eta+3\sqrt{2})[f'''(\eta+3\sqrt{2})-g'''(\eta)] + 12[f''(\eta+3\sqrt{2})-g''(\eta)] \qquad (4.3.17)$$

which is satisfied if we write

$$\begin{aligned} g^{(iv)}(\eta) &= f^{(iv)}(\eta + 3\sqrt{2}) \\ g'''(\eta) &= f'''(\eta + 3\sqrt{2}) \\ g''(\eta) &= f''(\eta + 3\sqrt{2}) - \frac{1}{18} \end{aligned} \qquad . \qquad (4.3.18)$$

The last of (4.3.18) together with (4.3.15) now gives

$$g'(\eta) = A(\eta+3\sqrt{2})^4 + B(\eta + 3\sqrt{2})^3 - \tfrac{1}{2}g_3(\eta + 3\sqrt{2})^2 + g_2(\eta + 3\sqrt{2}) - \frac{\eta}{18} + C \qquad (4.3.19)$$

with C a constant.

If it is possible now to choose A, B and C to make (4.3.19) self-consistent, then $f'(\xi)$ and $g'(\eta)$ as given by (4.3.15) and (4.3.19) respectively with these values of A, B and C may be substituted into (4.3.13) to give the required solution.

To determine the constants A, B and C we set $\eta = 0$ in (4.3.19) to get

$$g_1 = A(3\sqrt{2})^4 + B(3\sqrt{2})^3 - \tfrac{1}{2}g_3(3\sqrt{2})^2 + g_2 3\sqrt{2} + C \; . \qquad (4.3.20)$$

Differentiating (4.3.19) once and twice and setting $\eta = 0$ gives

$$g_2 = 4(3\sqrt{2})^3A + 3(3\sqrt{2})^2B - g_3 3\sqrt{2} + g_2 - \frac{1}{18} \qquad (4.3.21)$$

and

$$g_3 = 12(3\sqrt{2})^2A + 6.3\sqrt{2}\,B - g_3 \qquad (4.3.22)$$

which can be solved for A, B and C to give

$$A = \frac{-\sqrt{2}}{12.(18)^2} \quad , \quad B = \frac{1}{18}(\sqrt{2}g_3 + \frac{1}{18}) \quad , \quad C = g_1 - 3\sqrt{2}g_2 - 3g_3 - \frac{\sqrt{2}}{12} \qquad (4.3.23)$$

$g'(\eta)$ and $f'(\xi)$ can now be written in terms only of g_1, g_2 and g_3 and since there are no more conditions to be satisfied by these three quantities we aim for simplicity and set $g_1 = g_2 = g_3 = 0$.

Then

$$f'(\xi) = \frac{-\sqrt{2}}{12.(18)^2}\,\xi^4 + \frac{1}{(18)^2}\,\xi^3 \qquad (4.3.24)$$

and

$$g'(\eta) = \frac{-\sqrt{2}}{12.(18)^2}\,(\eta + 3\sqrt{2})^4 + \frac{1}{(18)^2}(\eta + 3\sqrt{2})^3 - \frac{\eta}{18} - \frac{\sqrt{2}}{12} \qquad (4.3.25)$$

when (4.3.24) and (4.3.25) are substituted into (4.3.13) there results after some simplification the equation

$$F'(\xi, \eta) = \frac{-\sqrt{2}}{9.18}\,\xi^2\eta^2 - \frac{6}{9.18}\,\xi^2\eta + \frac{6}{9.18}\,\xi\eta^2 \; . \qquad (4.3.26)$$

That $F'(\xi, \eta)$ as given by (4.3.26) satisfies all the requirements of our boundary value problem is readily seen. Thus we have, substituting back for U, V

$$F(U, V) = 1 + \frac{1}{3}U + \frac{\sqrt{2}}{3}V + \frac{\varepsilon}{18}\left[-\frac{\sqrt{2}}{2}U^2V^2 - \frac{\sqrt{2}}{4}V^4 - \frac{5}{2}V^3U - \frac{7\sqrt{2}}{2}V^2U - 6\sqrt{2}V^2 + 12UV - 18V\right] + o(\varepsilon) \; . \qquad (4.3.27)$$

The method which has been employed here can clearly be used for other values of κ, but the amount of labour involved will in many cases militate against the use of the method.

CHAPTER 5. UNSTEADY FLOWS OF PRANDTL-MEYER TYPE

Consider the problem of determining the resulting motion when polytropic gas, initially in steady, uniform flow in the x-direction in the half plane $y > 0$, is allowed at time $t = 0$ to expand into vacuum by the instantaneous removal of the boundary $y = 0$, $x > 0$. Mackie (1966) has remarked that it ought to be possible to obtain the classical Prandtl-Meyer solution as time $t \to \infty$ provided that the initial flow is supersonic. We shall formulate the problem, both in the X-Y and hodograph planes. We shall indicate possible difficulties which might arise in the numerical work which is required to obtain a complete solution in either the X-Y or U-V plane. We shall not attempt to obtain a complete solution to the problem.

We formulate the problem first in the X-Y plane and denote by U_1 $(= u_1/c_o)$ and F_1 $(= c_1/c_o)$ respectively, the dimensionless X component of velocity and the dimensionless sound speed in the initial steady, uniform flow. Of course, in this problem we have described there is no region in which gas is at rest. Therefore, throughout this chapter, c_o is to be interpreted as any convenient velocity. In the formulation of the second problem of this chapter, for example, we shall see that a certain choice of c_o produces considerable algebraic simplification. For large values of x, the resulting flow will be essentially a one-dimensional expansion of the gas into vacuum. This flow differs only slightly from the Riemann wave and is given by

$$U = U_1 \ , \ V = \frac{1}{1+\kappa}(Y - F_1) \ , \ F = \frac{1}{1+\kappa}[F_1 + \kappa Y] \quad -\frac{F_1}{\kappa} \leqslant Y \leqslant F_1 . \quad (5.1.1)$$

We are, of course, taking the initial flow to be supersonic and therefore it seems clear that the steady, uniform flow remains undisturbed upstream of the leading Mach line through the origin and that, near the origin, the solution is just Prandtl-Meyer flow as given by (3.4.1) and (3.4.2).

Thus in the X-Y plane the flow pattern at any time after the removal of the wall $X = 0$, $Y > 0$ is that shown in figure 18. BD is that curved characteristic of the expansion wave passing through the point at which the line $Y = F_1$ is a tangent to the circle $(X - U_1)^2 + Y^2 = F_1{}^2$ and is given by

$$(1-\kappa^2)(X-U_1)^2 - [\kappa Y + F_1]^2 = -\frac{1}{4}[2\kappa + 2)F_1]^{\frac{\kappa-1}{\kappa}}[2\kappa Y + 2F_1]^{\frac{\kappa+1}{\kappa}}. \qquad (5.1.2)$$

AP is the curved characteristic of the Prandtl-Meyer flow passing through the point at which the Mach line through O is a tangent to the circle $(X - U_1)^2 + Y^2 = F_1{}^2$. That is to say, AP is a solution of (3.4.14). The straight lines OP and DE are part of the gas-vacuum interface. The remaining problem is to determine the flow in the region between BD and AP and in particular to find the portion of the gas-vacuum interface between P and D. However, we are not confronted with a normal characteristic boundary value problem since AB is a parabolic curve. This militates against using a numerical method in the X-Y plane. However, as we remarked earlier, the problem can be formulated in the hodograph plane.

The expansion wave given by (5.1.1) is clearly given by

$$F = F_1 + \kappa V, \quad U = U_1, \quad -\frac{F_1}{\kappa} \leqslant V \leqslant 0. \qquad (5.1.3)$$

The Prandtl-Meyer expansion is given by

$$U = \frac{\frac{q^*}{c_o\lambda}\sin[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]\cos(\mu_1 - \theta) + \frac{q^*}{c_o}\cos[\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)]}{\sin(\mu_1 - \theta)} \qquad (5.1.4)$$

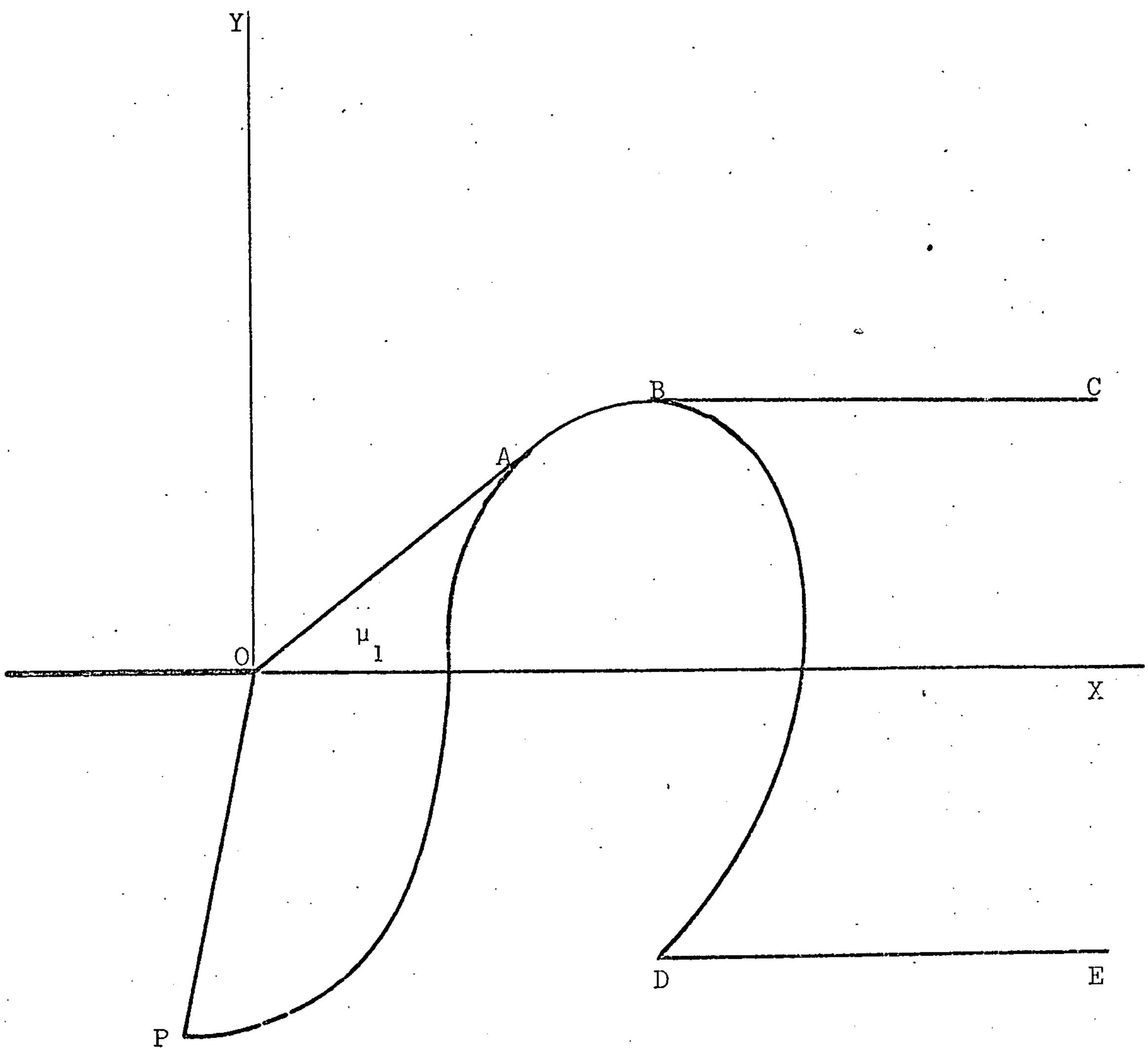


figure 18

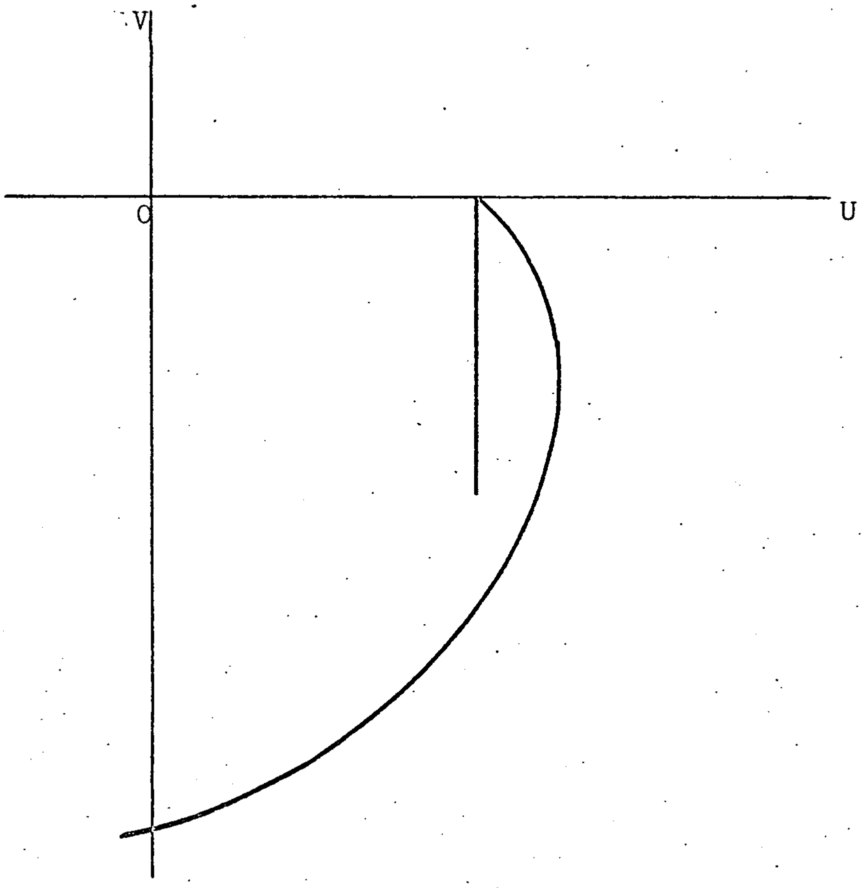


figure 19

$$V = \frac{q^*}{c_o\lambda} \sin [\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)] \sin(\mu_1 - \theta) - \frac{q^*}{c_o} \cos [\lambda\theta + \tan^{-1}(\lambda\cot\mu_1)] \cos(\mu_1 - \theta) \quad (5.1.5)$$

where the meaning of the various symbols employed is given in section 3.4. Elimination of the parameter θ between (5.1.4) and (5.1.5) shows that the Prandtl-Meyer flow maps, in the hodograph plane, into a section of an epicycloid (as has been stated previously).

The situation in the U-V plane is shown in figure 19. It is at this point that a possible difficulty becomes apparent, for there is no reason why the section of the epicycloid should not re-interest the straight line representing the expansion wave and the flow thereby map into more than one sheet of the hodograph plane. Of course it is possible to obtain a restriction on U_1 and F_1 which ensures that such a complication does not arise. However, we prefer to consider a completely different problem in which, as we shall show, these problems in the U-V and X-Y planes do not arise.

Consider now polytropic gas in steady uniform flow given by $U = U_1$, $V = 0$, $F = F_1$. Suppose the gas is travelling behind the rigid wall $X = U_1$ and is separated from vacuum by the wall. If at some instant, $t = 0$, the wall is instantaneously removed, the resulting motion of the gas is given by

$$U = \frac{1}{1+\kappa} [(F_1+\kappa U_1) + X], \; V = 0, \; F = \frac{1}{1+\kappa} [(F_1 + U_1) - \kappa X], \; U_1 - F_1 \leq X \leq \frac{(F_1+\kappa U_1)}{\kappa}. \quad (5.1.6)$$

A comparison of (5.1.6) with (3.2.1) shows that the resulting motion of the gas is given by familiar expansion wave solution provided that

$$F_1 + \kappa U_1 = 1 . \quad (5.1.7)$$

But this condition can always be satisfied by a suitable choice of the reference velocity c_o. Therefore throughout the rest of this chapter we shall consider c_o as chosen to ensure that (5.1.7) is correct. With c_o so chosen, then, imagine the gas to be ins teady, uniform motion behind the rigid plane wall $X = U_1$ as before, but the gas occupying only the space $Y > 0$, being separated from the vacuum $Y < 0$ by the fixed plane barrier $Y = 0$, $X < 0$. The moving wall is instantaneously removed at time $t = 0$ when it reaches the origin, which also marks the termination of the fixed wall $Y = 0$. Considerations similar to those of the last problem indicate that, at any time after the removal of the wall $X = U_1$ the flow pattern in the X-Y plane is as shown in figure 20. AB is that characteristic of the expansion wave which passes through A, the point at which the leading Mach line through O intersects $X = U_1 - F_1$, the line which, in the undisturbed flow, separates the expansion wave from the uniform flow. AC is simply that curved characteristic of the Prandtl-Meyer flow which passes through A. We note that, in the self-similar plane, we have a characteristic boundary value problem to solve to determine the flow in the remaining region. The difficulty which was encountered when attempting to tackle the previous problem in the X-Y plane is not met here. However, we prefer to go on to examine the present problem in the hodograph plane. The Prandtl-Meyer flow is again given by (5.1.4) and (5.1.5). The expansion wave is now given by

$$F = 1 - \kappa U \ , \ V = 0 \ , \ U_1 - F_1 \leqslant U \leqslant \frac{1}{\kappa} . \qquad (5.1.8)$$

The hodograph plane is shown in figure 21. It is clear that the difficulty which threatened in the hodograph approach to the previous problem, namely the possibility of the two characteristics intersecting in more than one point, is no longer present.

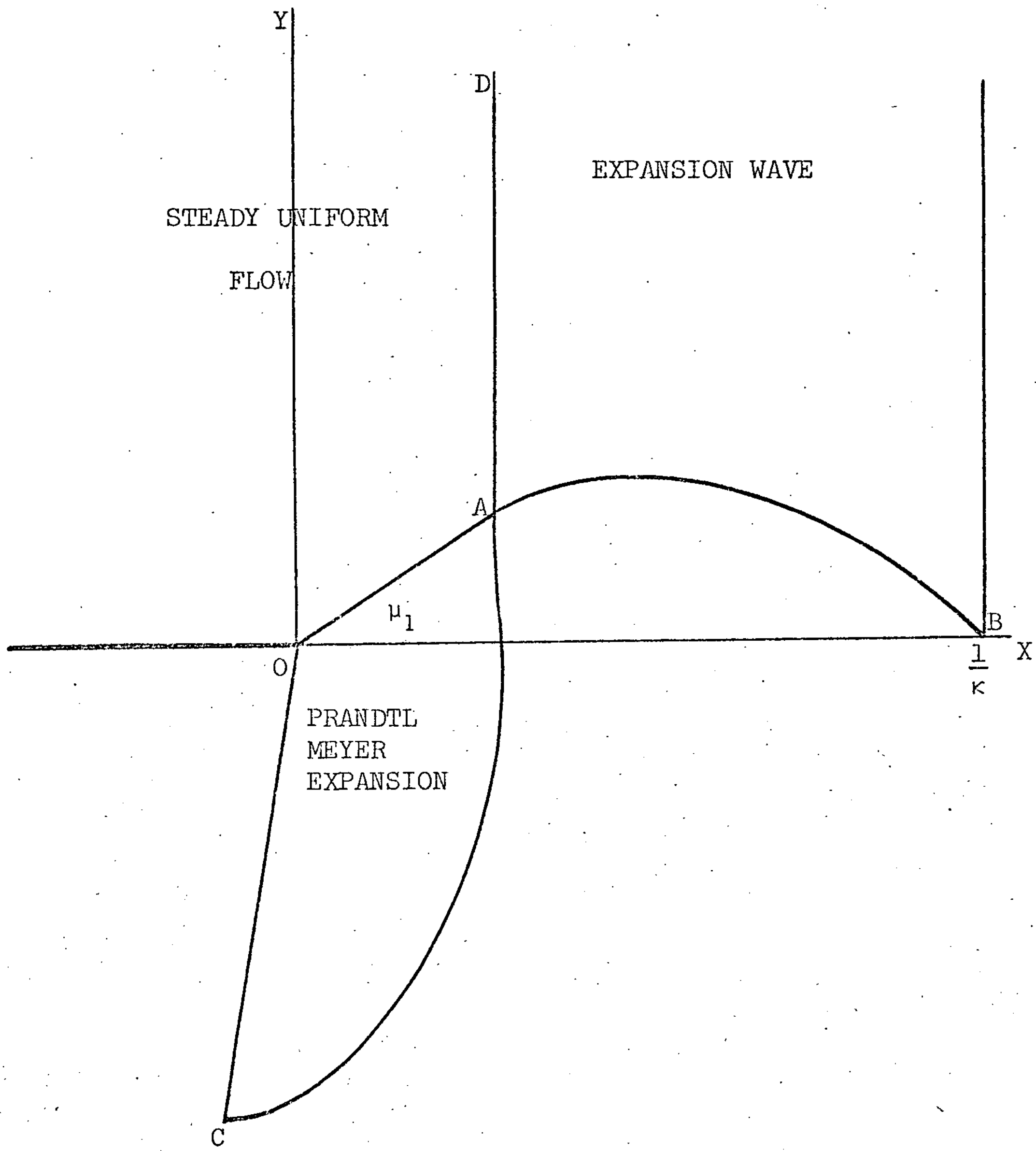

Y
D
EXPANSION WAVE
STEADY UNIFORM
FLOW
A
μ_1
B
O
$\frac{1}{\kappa}$
X
PRANDTL
MEYER
EXPANSION
C

figure 20

We have tackled numerically the boundary value problem shown in figure 21, adapting for our purposes a program given by Levine (1968). The modified program is given in the appendix and the results in the U-V and X-Y planes are shown in figures 22 and 23 respectively.

The method used by the program is described in some detail by Levine and we merely remark that the computation procedes along the (initially unknown) characteristics and is not continued across the line $F = 0$ in order that any complication which might be encountered beyond this line (such as the equation's becoming elliptic) should not halt the computation. Of course, the domain in which the solution is determined by the data on the two characteristics in general will extend beyond the line on which $F = 0$ but the solution beyond this line has no physical significance.

In fact, in the particular case computed ($\gamma = 1.4$, $M_1 = 5$) the equation becomes elliptic in a very small region near to the centre of the line $F = 0$. We have no a priori guarantee in this non-linear problem that the equation will not become elliptic but we believe that this small region of ellipticity does not affect the overall qualitative picture given in figures 22 and 23.

We conclude our discussion of this problem by observing that figure 23 shows that the remark of Mackie to which we referred earlier was quite correct; the figure does indeed show the classical Prandtl-Meyer solution as the limit as $t \to \infty$ of the solution to an initial value problem. Furthermore, figure 23 shows that, at any point at a finite distance from the origin, the classical Prandtl-Meyer solution is taken up <u>after a finite time</u>, not as a limit as time $t \to \infty$.

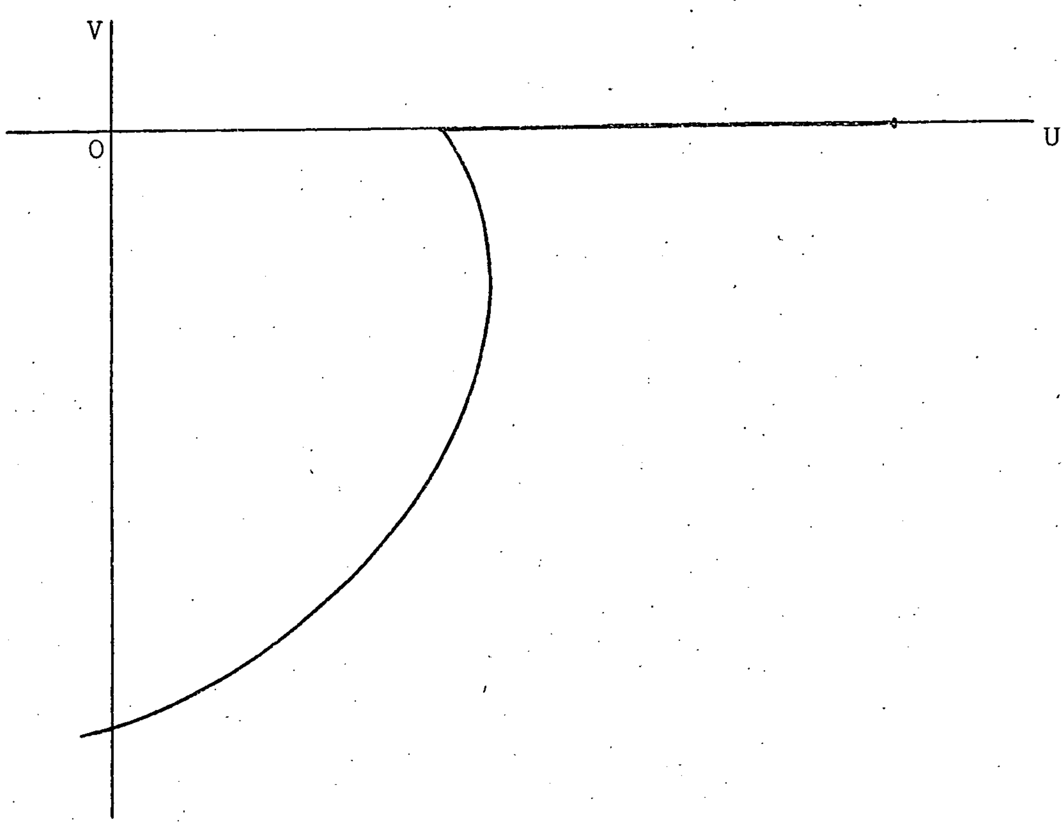


figure 21

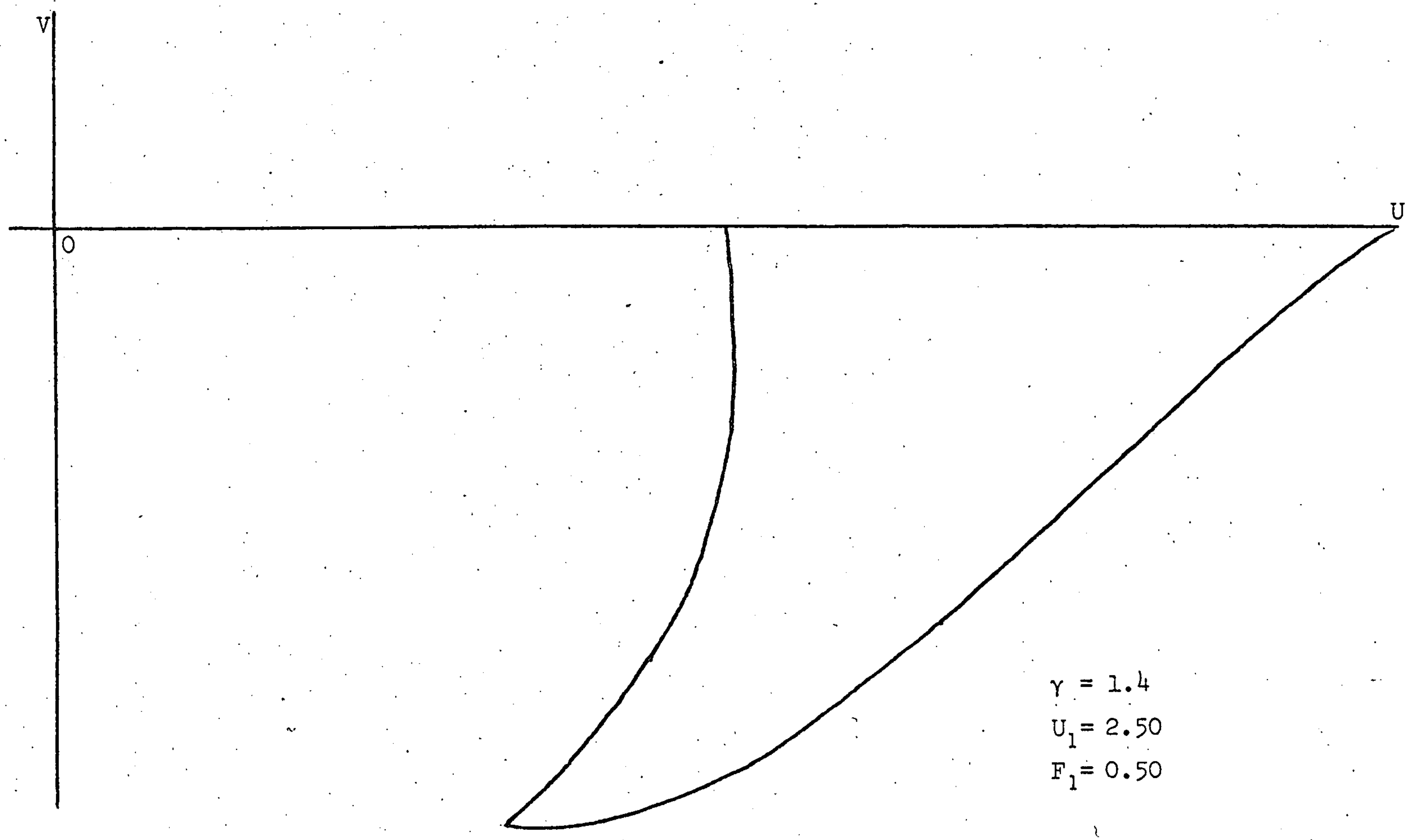


figure 22

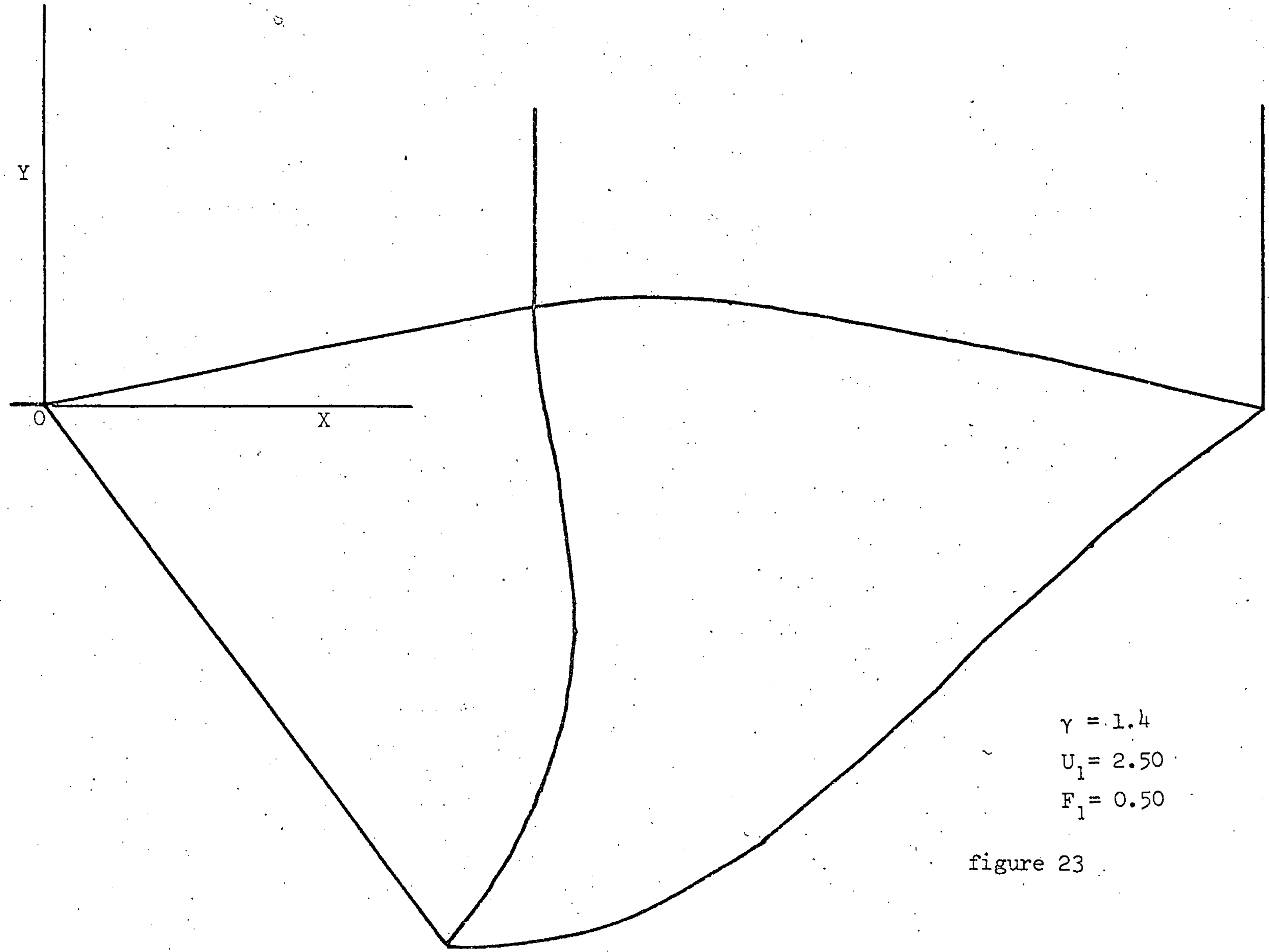


figure 23

APPENDIX

```
      DIMENSION C(400,2),C1(400,2),C2(400,2),X1(400),X2(400),
     1DU(400),DW(400),AD10V2(400,2),BD10V2(400),  TD10V2(400),
     2XD10V2(400),U1(400,2),U2(400,2),SD10V2(400),TEST(400),CB(400),
     3CS(400,2),DV(200) ,DISC(200),MG10V2(400,2),CG10V2(400)
      REAL G1,G2,G3,MU1,U10,F10,GK,TH,PI
400   READ (5,10) H,G,U10
10    FORMAT (3F10.0)
      WRITE (6,513) H
513   FORMAT (1H0 5X, 3H H=, F7.4)
      WRITE (6,514) G
514   FORMAT (1H0 5X, 6HGAMMA=, F12.8)
      WRITE (6,515) U10
515   FORMAT (1H0 5X, 4HU10=, F12.8)
      GK=(G-1.)/2.
      B=(1.+GK)/(1.-GK)
      N=(1./GK-U10)/H+1.
      PI=3.141592
      F10=1.0-(GK*U10)
      MU1=ARSIN(F10/U10)
      G2=SQRT(GK/(1.0+GK))
      G3=ATAN(G2/TAN(MU1))
      G1=F10/(G2*COS(G3))
      DO 1 J=1,N
      AJ=J-1
      U1(J,1)=-(AJ*H+U10)
      U2(J,1)=0.0
      C(J,1)=1.+GK*U1(J,1)
      CS(J,1)=C(J,1)
      C1(J,1)=+GK
      C2(J,1)=GK* SQRT(B+(((1.- SIN(MU1))/ COS(MU1))**2-B)*((1.+GK*U1
     4(J,1))/(1.-GK*U10))**((1.-GK)/GK))
      X1(J)=U1(J,1)+(C(J,1)*C1(J,1))/GK
      X2(J)=U2(J,1)+(C(J,1)*C2(J,1))/GK
1     CONTINUE
      I=1
      WRITE (6,11) I
11    FORMAT (1H0 5X, 2HI=, I3)
      WRITE (6,111) (U1(J,1),U2(J,1),J=1,N)
```

```
111     FORMAT (1H0 5X, 2HU1, 5X, 2HU2/(2E30.8))
        WRITE (6,110) (C(J,1),C1(J,1),C2(J,1),J=1,N)
110     FORMAT (1H0 5X, 1HC,5X, 2HC1, 5X,2HC2/(3E30.8))
        WRITE (6,121) (X1(J), X2(J), J=1,N)
121     FORMAT (1H0 5X, 2HX1,5X, 2HX2/(2E30.8))
333     I=I+1
        WRITE (6,11) I
        AD10V2(1,1)=+1.0/TAN(MU1)
        AI=I-1
        TH=AI*(0.5*PI-G3)/(G2*(N-1.0))
        U1(1,2)=-(G1*SIN(G2*TH+G3)*COS(MU1-TH)+G1*G2*COS(G2*TH+G3)*
       1SIN(MU1-TH))
        U2(1,2)=G1*SIN(G2*TH+G3)*SIN(MU1-TH)-G1*G2*COS(G2*TH+G3)*
       1COS(MU1-TH)
        C(1,2)=G1*G2*COS(G2*TH+G3)
        CS(1,2)=C(1,2)
        CALL CALC(TH,U10,GK,C1(1,2),C2(1,2))
        X1(1)=U1(1,2)+(C(1,2)*C1(1,2))/GK
        X2(1)=U2(1,2)+(C(1,2)*C2(1,2))/GK
        DO 2 J=2,N
        AD10V2(J,1)=(C1(J,1)*C2(J,1)+GK*SQRT(C1(J,1)**2+C2(J,1)**2-GK**2
       1))/(GK**2-C2(J,1)**2)
        TD10V2(J)=AD10V2(J,1)
        IF (I.GT.2) GO TO 2
        SD10V2(J)=AD10V2(J,1)
2       CONTINUE
        WRITE (6,12) (AD10V2(J,1), J=1,N)
12      FORMAT (1H0 5X, 6HAD10V2/(F30.8))
        BD10V2(1)=(C1(1,2)*C2(1,2)+GK* SQRT(C1(1,2)**2+C2(1,2)**2-GK**2))
       1/(GK**2-C1(1,2)**2)
        WRITE (6,77) BD10V2(1)
77      FORMAT (1H0 20X,9HBD10V2(1)/(E30.8))
        XD10V2(1)=BD10V2(1)
        M=N
        DO 3 J=2,M
        U1(J,2)=10000.0
        U2(J,2)=10000.0
        C1(J,2)=10000.0
        C2(J,2)=10000.0
        DO 4 L=1,250
        TU1=U1(J,2)
        TU2=U2(J,2)
        TC1=C1(J,2)
        TC2=C2(J,2)
        DET=-AD10V2(J,1)*BD10V2(J-1)+1.
        Z=AD10V2(J,1)*U1(J,1)-U2(J,1)
        ZZ=-BD10V2(J-1)*U2(J-1,2)+U1(J-1,2)
        U1(J,2)=(-Z*BD10V2(J-1)+ZZ)/DET
        U2(J,2)=(AD10V2(J,1)*ZZ-Z)/DET
        DU(J)=U2(J,2)-U2(J,1)
        DW(J)=U2(J,2)-U2(J-1,2)
        DV(J)=U1(J,2)-U1(J-1,2)
        E=(GK**2-C1(J,1)**2)
        F=AD10V2(J,1)*(GK**2-C2(J,1)**2)
        EE=BD10V2(J-1)*(GK**2-C1(J-1,2)**2)
        FF=(GK**2-C2(J-1,2)**2)
        DL=E*FF-EE*F
        XL=DU(J)*GK/CS(J,1)*((C1(J,1)**2+C2(J,1)**2)*(1.-GK)-2.*(GK**2))
       1+C1(J,1)*F+C2(J,1)*E
        YL=DV(J)*GK/CS(J-1,2)*((C1(J-1,2)**2+C2(J-1,2)**2)*(1.-GK)-2.*
       1(GK**2))+C1(J-1,2)*FF+C2(J-1,2)*EE
        C1(J,2)=(E*YL-EE*XL)/DL
        C2(J,2)=(XL*FF-YL*F)/DL
        DISC(J)=C1(J,2)**2+C2(J,2)**2-GK**2
```

```
      IF (M.EQ.J-1 ) GO TO 619
      AD10V2(J,2)=(C1(J,2)*C2(J,2)+GK*SQRT(C1(J,2)**2+C2(J,2)**2
     1-GK**2))/(GK**2-C2(J,2)**2)
      BD10V2(J)=(C1(J,2)*C2(J,2)+GK* SQRT(C1(J,2)**2+C2(J,2)**2-GK**2))
     1/(GK**2-C1(J,2)**2)
      AD10V2(J,1)=0.5*(TD10V2(J)+AD10V2(J,2))
      BD10V2(J-1)=0.5*(XD10V2(J-1)+BD10V2(J))
      TEST(J)= ABS(TU1-U1(J,2))+ ABS(TU2-U2(J,2))+ ABS(TC1-C1(J,2))
     1+ ABS(TC2-C2(J,2))
      IF (TEST(J).LT.0.000001   ) GO TO 55
4     CONTINUE
      WRITE (6,2001) J,TEST(J)
2001  FORMAT (1H0 5X, 2HJ= I3,5X,
     15HTEST= E20.8, 22H THE TEST DID NOT WORK)
      GO TO 56
55    WRITE (6,2011) J,L,TEST(J)
2011  FORMAT (1H0 5X,2HJ=,I3,5X,2HL=,I3,5X,5HTEST= E30.8)
56    C(J,2)=CS(J,1)+(U1(J,2)-U1(J,1))*(C1(J,1)+C1(J,2))/2.
     1+DU(J)*(C2(J,1)+C2(J,2))/2.
      CB(J)=CS(J-1,2)+DV(J)*(C1(J-1,2)+C1(J,2))/2.
     1+DW(J)*(C2(J-1,2)+C2(J,2))/2.
      CS(J,2)=(C(J,2)+CB(J))/2.
      X1(J)=U1(J,2)+(CS(J,2)*C1(J,2))/GK
      X2(J)=U2(J,2)+(CS(J,2)*C2(J,2))/GK
      XD10V2(J)=(C1(J,2)*C2(J,2)+GK* SQRT(C1(J,2)**2+C2(J,2)**2-GK**2))
     1/(GK**2-C1(J,2)**2)
3     CONTINUE
619   WRITE (6,101) (AD10V2(J,1),BD10V2(J-1),J=2,M)
101   FORMAT (1H0 5X, 6HAD10V2, 5X, 6HBD10V2/(2E30.8))
      WRITE (6,1010) (U1(J,2), U2(J,2), J=1,M)
1010  FORMAT (1H0 5X, 2HU1,5X, 2HU2/(2E30.8))
      WRITE (6,211) (C(J,2),C1(J,2),C2(J,2),DISC(J), J=1,M)
211   FORMAT (1H0 5X, 6HC(J,2),5X,7HC1(J,2),5X,7HC2(J,2),5X,7HDISC(J)
     1/(4E30.8))
      WRITE (6,1211) (CB(J),CS(J,2), J=2,M)
1211  FORMAT (1H0 5X,5HCB(J),15X, 5HCS(J)/(2E30.8))
      WRITE (6,121) (X1(J),X2(J),J=1,M)
      IF (I.GE.N) GO TO 400
      DO 90 J=1,M
      U1(J,1)=U1(J,2)
      U2(J,1)=U2(J,2)
      C(J,1)=C(J,2)
      CS(J,1)=CS(J,2)
      C1(J,1)=C1(J,2)
      C2(J,1)=C2(J,2)
90    CONTINUE
      GO TO 333
      END


      SUBROUTINE CALC(TH,U10,GK,C1,C2)
      REAL TH,U10,GK,C1,C2
      REAL G1,G2,G3,MU1,F10
      REAL U1,U2,V1,V2,F1,F2,FU1,FU2,FV1,FV2,GRAD,DEL
      INTEGER I,N
      F10=1.0-(GK*U10)
      MU1=ARSIN(F10/U10)
      G2=SQRT(GK/(1.0+GK))
      G3=ATAN(G2/TAN(MU1))
      G1=F10/(G2*COS(G3))
      U1=-U10
      V1=0.0
      F1=1.0-GK*U10
```

```
      F1=1.0-GK*U10
      FU1=GK
      FV1=GK*(1.0-SIN(MU1))/COS(MU1)
      GRAD= 1.0/TAN(MU1)
      N=   INT(TH*G2*1600.0/(0.50*3.14159-G3))
      DO 1 I=1,N
      DEL=I*TH/N
      U2=(G1*SIN(G2*DEL+G3)* COS(MU1-DEL)+G1*G2* COS(G2*DEL+G3)
     1* SIN(MU1-DEL))*(-1.0)
      V2=G1* SIN(G2*DEL+G3)* SIN(MU1-DEL)-G1*G2* COS(G2*DEL+G3)
     1* COS(MU1-DEL)
      F2=G1*G2* COS(G2*DEL+G3)
      FU2=((F2-F1)*(GK**2-FU1**2)-((V2-V1)*GK*((FU1**2+FV1**2)
     1*(1.0-GK)-2.0*GK**2)/F1+FU1*GRAD*(GK**2-FV1**2)+FV1*(GK**2-FU1**2)
     2)*(V2-V1))/((U2-U1)*(GK**2-FU1**2)-GRAD*(GK**2-FV1**2)*(V2-V1))
      FV2=((F2-F1)*GRAD*(GK**2-FV1**2)-((V2-V1)*GK*((FU1**2+FV1**2)*
     1(1.0-GK)-2.0*GK**2)/F1+FU1*GRAD*(GK**2-FV1**2)+FV1*(GK**2-FU1**2)
     2)*(U2-U1))/((V2-V1)*GRAD*(GK**2-FV1**2)-(GK**2-FU1**2)*(U2-U1))
      GRAD=(FU2*FV2+GK* SQRT(FU2**2+FV2**2-GK**2))/(GK**2-FV2**2)
      FU1=FU2
      FV1=FV2
      U1=U2
      V1=V2
      F1=F2
    1 CONTINUE
      C1=FU2
      C2=FV2
      RETURN
      END
```

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