Fundamental Domains of Infinite Cyclic Covers

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Abstract

A fundamental domain, of a free properly discontinuous group action on a space, is a compact subspace with boundary, which is the closure of a subspace containing one point for each orbit of the action. A knot $k : S^n \to S^{n+2}$ is the simplest example of co-dimension 2 embedding of compact manifolds. The complement of k has an infinite cyclic cover, with fundamental domain obtained geometrically by cutting along a Seifert surface.

This thesis concerns this geometric construction and its algebraic analogues for chain complexes over polynomial rings, with applications to the classification of high-dimensional knots. Algebraic Transversality relates the Seifert Form and the Alexander Module of a Simple odd-dimensional knot.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Adam Hughes*)

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Introduction

Topology is the study of shape. Algebraic Topology uses some of the vast formal machinery of Algebra to discover things about shape which are otherwise to complex to fit into our imagination.

The classification of embeddings of one space within another is a typical problem of algebraic topology; the sort of problem which can be simply formulated but when studied turns out to be such a huge endeavour as to (so far) defy description. Embeddings of co-dimension 3 or more were essentially shown to be topologically trivial in the sixties by Stallings and Zeeman. Smooth embeddings were not so easily dismissed, but Haefliger reduced the classification of all smooth co-dimension k embeddings into n-dimensional spaces to homotopy theory, for $k > 2$ and $n > 4$.

Knot theory, the study of co-dimension 2 embeddings, is an area of topology which has been studied since the nineteenth century. Classical knots were first imagined to explain the periodic table of elements and quickly discarded by scientists, but the fascination remained with mathematicians who had glimpsed a world which seems to be ordered but refused to bow to their methods of classification. Mathematical knots differ from real world knots in that a mathematical knot is imagined to exist in a loop of string and as such is impossible to untie. The question "How many different knots are there?" very soon became "How can you tell if two knots are the same?", since schemes of labelling knots are not obvious.

Tesselation was known to the Greeks, and Islamic artists had discovered all 17 wallpaperings of the plane before the European Renaissance had first whispered around Italy. Tesselations of 3, 4 and even 5 dimensions are well studied, but in higher dimensions still the problem is open. To a topologist, tiles are too rigid, but the idea of group actions, orbits and fundamental domains has been well studied in the twentieth century.

This thesis presents a small area where these two subjects overlap, looking at coverings and knot invariants derived from coverings.

An attempt has been made to present the material in a straight forward way, to enable a non-specialist the chance to follow the arguments here. In order to understand the entire thesis, the reader needs to understand modules, chain complexes and exact sequences, have a working knowledge of the homotopy and homology groups of a space, and at least familiarity with the theorems of Hurewicz, Seifert-VanKampen and Mayer-Vietoris. All this can be found in an Algebraic Topology text such as [18].

The definition and description of coverings is given in Chapter 1 along with a definition of the action of the fundamental group of a space on its coverings. This definition leads to a method for discovering if a covering of a connected space is connected or not.

Chapter 2 investigates the questions of isomorphism ("How do you tell if two covers are the same?") and classification ("How many different covers of a space are there?") of coverings. It begins by generalising the result from Chapter 1 to find the number of connected components in a cover, and defining the deck transformations of the covering. It contains a formula for the group of deck transformations of any cover and finishes with the classification of all covers of a space in terms of conjugacy classes of the fundamental group.

Chapter 3 discusses the construction of fundamental domains for coverings, and singles out the infinite cyclic cover as the one associated with knots. Geometric transversality is discussed, and Seifert's algorithm for finding a Seifert Surface for classical knots is used to find a map from a knot complement to the circle for classical knots. The Seifert Surface is transverse to this map.

Chain complexes are defined, and the idea of a fundamental domain for certain classes of chain complexes is described in Chapter 4, again picking out the infinite cyclic group as special. It is shown that any chain complex over a polynomial ring has a fundamental domain, which leads to a linear resolution for the chain complex in terms of its fundamental domain.

Chapter 5 amalgamates the ideas of chapters 3 and 4 giving a theory of fundamental domains for CW-complexes, which combines the geometric and algebraic properties. The chapter looks again at infinite cyclic coverings, and shows that manifolds con be considered to be "linear" in some sense. It goes on to conclude that any CW-complex is homotopic to a complex which is "linear".

The work in chapters 3, 4 and 5 develops ideas in geometry and algebra along parallel courses. The real tying up of these loose ends is beyond the scope of this thesis.

Knots are finally described in detail in Chapter 6, which goes on to the work of Seifert on spanning surfaces of knots. The big result of the chapter is Trotter's S-equivalence of Seifert Matrices.

Finally, Chapter 7 brings together ideas from coverings and algebra to define the Blanchfield Form, an invariant of a certain class of knots, and relate that to the Seifert Form.

Chapter 1

Coverings

The first chapter of this thesis lays out the fundamental properties of covering spaces. Most of this can be found in basic texts, for instance Armstrong, Fulton or Fenn ([1], [8], [7]).

1.1 What is a covering?

Definition 1.1. Let X be a topological space, then a *covering* of X is a pair (p, Y), such that $p : Y \to X$ is a continuous map, with the property that each point of X has a neighbourhood N, such that $p^{-1}(N)$ is a disjoint union of open sets in Y, each of which is mapped homeomorphically by p onto N. The map p is known as the covering map. (If N is connected then these must be the components of $p^{-1}(N)$.)

Example 1.1.1. Any space X is its own covering space, with the identity as the covering map. The identity $1_X : X \to X$ maps any neighbourhood N of a point in X onto itself homeomorphically. Obviously the inverse image of N is open.

Example 1.1.2. If Y_1 and Y_2 are both covering spaces of a space X, with covering maps $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$, then the disjoint union $Y_1 \sqcup Y_2$ is a covering space of X too, with $p_1 \sqcup p_2$ as the covering map.

Example 1.1.3. We can see $p : \mathbb{R} \to S^1$; $x \to e^{2\pi i x}$ is a covering map, because a neighbourhood, N, of a point in S^1, in the subspace topology from $\mathbb{C}$, is an open interval in the circle. The inverse image of any open interval is a union of countably many open intervals in $\mathbb{R}$, each of which is homeomorphic to N.

In order to understand more about covers, a few more results are needed. The following concepts will be used later, and give a feeling of what coverings are.

Let $p : Y \to X$ be a covering. First look at the properties of "lifting" a map from the base space to the covering space.

Definition 1.2. A *lift* of a map $m : Z \to X$, from some arbitrary space Z, is a map $\widetilde{m} : Z \to Y$ which commutes with the covering $p\widetilde{m} = m$. i.e.the following diagram commutes

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "\widetilde{m}"] \arrow[r, "m"'] & X \end{tikzcd}$$

Theorem 1.3 (The Unique Path Lifting Property). *Given a path, $\alpha : I \to X$ with $\alpha(0) = x_0$, then there is a unique lift of α to a covering space Y, $\widetilde{\alpha} : I \to Y$, with $\widetilde{\alpha}(0) = y_0 \in p^{-1}(x_0)$, for each y_0.*

Proof. This property follows from the homeomorphism from a neighbourhood in Y to a neighbourhood in X; in a neighbourhood of x_0, we can find a homeomorphism to a neighbourhood of any $y_0 \in p^{-1}(x_0)$, and as the path heads out of that neighbourhood, we can find another neighbourhood in X which contains the next section of the path and lift it to Y. This approach can lift the whole path piece by piece, provided we know that we can break the path down piecewise. This is true for any space which admits a metric (Lebesgue's Lemma). □

Theorem 1.4 (The Unique Homotopy Lifting Property). *Given a homotopy of paths, $h : I \times I \to X$ with $h(0,t) = \alpha(t)$, and $h(1,t) = \beta(t)$, then there is a unique lift of h to a covering space Y, $\widetilde{h} : I \times I \to Y$, with $\widetilde{h}(0,t) = \widetilde{\alpha}(t)$, and $\widetilde{h}(1,t) = \widetilde{\beta}(t)$.*

Proof. Analogous to the proof of Theorem 1.3. It can be found in Armstrong [1]. □

Definition 1.5. A path $\alpha : I \to X$ lifts to a path in Y with $\widetilde{\alpha}(0) = y_0$. The point $y_0 * \alpha$ is defined to be the end point of the path, $\widetilde{\alpha}(1)$. It is well defined, because of the Unique Path Lifting Property.

Since neighbourhoods in the base space are covered by a number of disjoint neighbourhoods in the covering space, the next important idea to consider is that of the "fibre".

Definition 1.6. The *fibre* over a point $x \in X$ is the inverse image space $F_x = p^{-1}(x)$. It has the discrete topology.

The fibre over each point in the cover is the same, under certain circumstances. The fibre of a cover becomes a useful phrase when every point has the same fibre!

Definition 1.7. A space, X, is *path connected* if for any two points $x_1, x_2 \in X$ there is a continuous path $\alpha : I \to X$ with $\alpha(0) = x_1, \alpha(1) = x_2$.

Proposition 1.8. *There is a homeomorphism $F_{x_1} \cong F_{x_2}$ for every x_1, x_2, points in a single path connected component of X. The fibre can be different for two points in different path components.*

Proof. This can be demonstrated using the unique path lifting property. Suppose $p : Y \to X$ is a cover, and x_1, x_2 are arbitrary points in X. If there is a path $\alpha : I \to X$ where $\alpha(0) = x_1$, $\alpha(1) = x_2$, then for each element of the fibre F_{x_1}, α lifts to a unique path $\widetilde{\alpha}$ ending at a unique point in F_{x_2}. This lift defines a homeomorphism $F_{x_1} \to F_{x_2}$. The reverse path $(-\alpha)(t) = \alpha(1-t)$ can be lifted also to a unique path $\widetilde{-\alpha}$ for every element in F_{x_2}, which similarly defines a homeomorphism $F_{x_2} \to F_{x_1}$. Hence each fibre has the same cardinality, and since $\widetilde{\alpha}(0) = \widetilde{-\alpha}(1)$ if and only if $\widetilde{\alpha}(1) = \widetilde{-\alpha}(0)$, these two functions are bijections. So the fibres over any two points in a path connected component of X are homeomorphic (in the discrete topology). □

So it is meaningful to talk about the fibre over a path connected component, or if the covered space is path connected, to talk about the fibre of the cover. If the fibre of the cover is a finite set $|F| = n$, then the cover is an n-sheeted, or an n-fold cover. Coverings are a special case of fibre bundles with discrete fibre.

There are two simple examples which will keep arising in this work. The mapping torus is a way of creating a space with an obvious map to the circle out of any space and a self homeomorphism. The pullback cover is a method of getting "new covers for old". Most commonly this will be used to find infinite cyclic covers, of spaces with maps to the circle.

Construction 1.9 (Mapping Torus). If $h : X \to X$ is an automorphism of a topological space X, then the *Mapping Torus* $T(h)$ is the quotient space

$$T(h) = X \times I/(x,1) \sim (h(x),0)$$

There is a map form $T(h)$ to the circle,

$$p : T(h) \to S^1 \quad ; \quad (x,t) \to e^{2\pi it}$$

If X has the discrete topology, then the mapping torus is a covering space of the circle with covering map p and fibre X.

This can be seen to be a cover, because any neighbourhood of a point in S^1 is an open interval in the circle and its inverse image in $T(h)$ falls into two cases. Either the inverse image is an open set entirely contained in each interval I, or it is split between two adjacent intervals, glued together by the relation. In either case this can be seen to be an open set, and homeomorphic to the original interval. Hence p is a covering.

It can also be seen that the example cover of the circle S^1 by the real line $\mathbb{R}$ is a special case of the mapping torus, with $X = \mathbb{Z}$ and $h(x) = x + 1$. This will turn out to be an important cover.

The fibre of the mapping torus as a cover is X, because the covering map is the projection of the second coordinate from $X \times I$ onto the circle, so $p^{-1}(e^{2\pi it}) = X \times \{t\}$.

Construction 1.10 (Pullback Cover). An *Infinite Cyclic Cover* is a cover with fibre $\mathbb{Z}$. It is of interest since it has uses in knot theory. It is possible to define an infinite cyclic cover for any space.

The first and most important infinite cyclic cover is the cover the circle by the real line. This is not the only infinite cyclic cover of the circle. The disjoint union of a countable number of copies of S^1 also covers the circle and has fibre $\mathbb{Z}$.

It is possible to obtain a number of infinite cyclic covers for any space X with a map to S^1. This is done by constructing a *pullback cover*, which constructs a cover from an already existing one. How do we do that?

Given a cover $p : Y \to X$, and a continuous map $f : Z \to X$ we can construct a space $f^*Y =$ as

$$f^*Y = \{(y, z) \in Y \times Z | f(z) = p(y) \in X\}$$

Defining the map q to be projection on the second element of the product, it can be seen that $q : f^*Y \to Z$ is a cover, with the following diagram commuting:

$$\begin{tikzcd}
f^*Y \arrow[r] \arrow[d, "q"'] & Y \arrow[d, "p"] \\
Z \arrow[r, "f"'] & X
\end{tikzcd}$$

The space f^*Y is know as the pullback of the space Y by the map f. The cover is likewise called a pullback cover.

Infinite cyclic covers are defined as the pullback covers of the real line by maps to the circle. We can see that these covers all have $\mathbb{Z}$ for their fibre because of the following property of pullback covers:

Proposition 1.11. *The fibre of a pullback cover is isomorphic to the fibre of the original cover.*

Proof. As usual, let $p : Y \to X$ be a cover. Let the fibre over a point $x \in X$ in the cover Y be $F^p_x = p^{-1}(x)$. Let the fibre over a point $z \in Z$ in the cover f^*Y

be $F^q_z = q^{-1}(z)$. Consider the point $x = f(z) \in X$. The fibre over this point is $F^p_{f(z)}$. This fibre over $z \in Z$ is

$$\begin{aligned} F^q_z &= \{(y,z) \in Y \times \{z\} | p(y) = f(z)\} \\ &= \{(y,z) \in Y \times \{z\} | y \in F^p_{f(z)}\} \\ &= (F^p_{f(z)}, z) \end{aligned}$$

so there is a natural bijection from the F^q_z to F^p_x; projection on the first element of F^q_z. $\square$

So since all infinite cyclic covers (over reasonable spaces, e.g. countable CW complexes) come from pullbacks of the following commutative diagram,

$$\begin{tikzcd} f^*\mathbb{R} \arrow[r] \arrow[d, "q"'] & \mathbb{R} \arrow[d, "e^{2\pi i t}"] \\ Z \arrow[r, "f"'] & S^1 \end{tikzcd}$$

the fibre over these covers will always be the same as the fibre of the real line over the circle. We have already seen this cover is the mapping torus, $\mathbb{R} = T(h : \mathbb{Z} \to \mathbb{Z}; x \mapsto x+1)$, and so has fibre $\mathbb{Z}$.

1.2 Fundamental Group

From now on, we will restrict our attention to covers of connected spaces, so it is reasonable to talk about the fibre of a cover. It is possible to have a connected cover, that is a cover $p : Y \to X$ where Y is connected. Are there any easy ways of spotting a connected cover?

The next few sections will show that the interaction of the fibre and the fundamental group is vital in determining whether a cover is connected.

If $p : Y \to X$ is a cover, then there is an induced map:

$$p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$$

where, $p(y_0) = x_0$. This map is well defined because any loop in Y based at y_0 is mapped to a loop in X based at x_0. From now on let $[\sigma]$ denote an element (homotopy class) of $\pi_1(X, x_0)$ where $\sigma : S^1 \to X$ is a representative loop based at x_0,

Proposition 1.12. *The map p_* is injective.*

Proof. By the Unique Path Lifting Property, any element of $\pi_1(X, x_0)$ lifts to a unique path starting at the basepoint in Y. Also, by the Unique Homotopy Lifting Property, two paths homotopic relative to their end points in X lift to paths homotopic relative to their end points in Y. So, if two loops in X based at x_0 are homotopic in X, i.e.represent the same element of the fundamental group, then they both lift to paths homotopic relative to their end points. If we know that these two paths are loops in Y, based at y_0, then they are homotopic loops.

So let $[\alpha], [\beta]$ be two elements in $\pi_1(Y, y_0)$. Then $p\alpha$ and $p\beta$ are loops in X, based at x_0, i.e.$[p\alpha], [p\beta] \in \pi_1(X, x_0)$. Suppose $[p\alpha] = [p\beta]$, so $p_*[\alpha] = p_*[\beta]$. Then by the argument above, $[\alpha] = [\beta]$, so p_* is injective. □

Since this means the fundamental group of Y is a subgroup of the fundamental group of X, we can look at properties of this subgroup.

Definition 1.13. A covering $p : Y \to X$ is *regular* if Y is connected and the image of the fundamental group of Y is normal in the fundamental group of X Recall: A subgroup H of G is normal iff

$$g^{-1}Hg = H, \quad \forall g \in G$$

Definition 1.14. Two subgroups H, K of G are *conjugate* if there exists an element g such that:

$$g^{-1}Hg = K$$

Conjugacy is an equivalence class on subgroups of G. A normal subgroup is only conjugate to itself.

So what does the fundamental group have to do with the connectedness of the cover? To begin with there must be a way for the group and space to interact with the covering space. The next section defines how groups "act" on spaces, which turns out to be the important property.

1.3 Group Actions - a short detour

Beardon describes group actions in some detail in [2], except for free actions. Actions, including free ones, are described in Dicks and Dunwoody [6]

Definition 1.15. An action of a group G on a space A is a continuous function

$$G \times A \to A \ ; \ (g, a) \to ga$$

such that:

$$1_G a = a \quad ; \quad g(ha) = (gh)a \ \forall g, h \in G$$

The *orbit* of a point $a \in A$ is the subset

$$Ga = \{b \in A \mid b = ga, g \in G\} \subseteq A$$

The *stabiliser* of a point is the subgroup

$$G_a = \{g \in G \mid ga = a\} \subseteq G$$

An action is *free* if the stabilizer of each point is the trivial subgroup

$$G_a = \{1\} \quad \forall a \in A$$

An action is *discontinuous* if for every element $a \in A$, there is a small neighbourhood U containing a for which

$$gU \cap U = \emptyset$$

for all except finitely many $g \in G$. An action is *properly discontinuous* if every $g \in G$ satisfies the condition above, except for the identity.

An *orbit space* of a free properly discontinuous action is the quotient space obtained by identifying all points in an orbit of the action, with topology induced from the topology at the points in the orbit. It is denoted A/G

Example 1.15.1. The cyclic group of n elements, C_n, acts on the complex plane $\mathbb{C}$ by multiplication by ω, a primitive nth root of unity:

$$C_n \times \mathbb{C} \to \mathbb{C} \quad ; \quad (x, z) \to \omega^x z$$

The orbits, $C_n z$, are all vertex sets of regular n-gons inscribed in a circle of radius $|z| > 0$, and the origin. The stabiliser of the origin is C_n, and the stabiliser of any other point is the trivial subgroup. This action is not free, since the origin has stabilizer C_n. The action is discontinuous, since the action is properly discontinuous on all points except the origin, and there only finitely many elements of the group (all of them) do not act discontinuously.

Example 1.15.2. The circle group S^1 acts on the complex plane $\mathbb{C}$ by rotations

$$S^1 \times \mathbb{C} \to \mathbb{C} \quad ; \quad (\theta, z) \to e^{i\theta} z$$

The orbits, $S^1 z$, are the circles centred on the origin radius $|z| > 0$ and the origin itself. The stabiliser of the origin, S^1_0 is S^1, and of all other points is trivial. This action is neither free nor discontinuous. Notice that C_n is a discrete subgroup of S^1, and its action is a special case of the action of the circle.

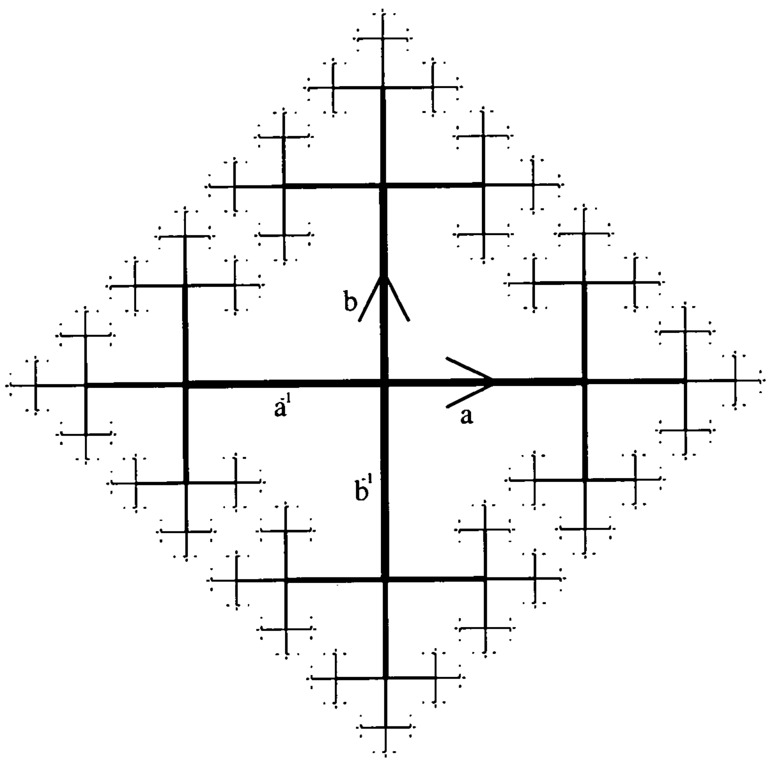

Figure 1.1: The infinite T.V. aerial, T

Example 1.15.3. The free group on two generators, $\mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$ acts on the infinite (directed) 4-valent tree, T, by translation. Each edge in T is considered to be in one of two disjoint subsets, A, B, and at each vertex there are two edges from each subset, one entering and one leaving. Pick a vertex of T to be the origin, and label each vertex by the path from the origin. The path from the origin is a word, l, where each letter corresponds to a choice of edge at a vertex, a for moving forward along an edge in A, a^{-1} for moving backward along an edge in A, and b or b^{-1} similarly with edges in B. At each stage the extra letter is added by right multiplication. Label the edges as $l + \mu s$ where l is the vertex of the edge closest to the origin (having the shortest word) and $s \in \{a, a^{-1}, b, b^{-1}\}$ is the generator which you have to multiply l by, to get the other end of the edge. $\mu \in (0, 1)$ is the distance of the point from the vertex l, in the direction of s.

There is a natural action of $\langle a, b \rangle$ on this tree. We can consider the action to move the origin, and then relabel the other vertices from this new starting point. It can be seen that the action on the edges is defined by the action on the vertices. Each point (except the origin 1) will appear to jump from branch to branch in the action. This is the left action,

$$\langle a, b\rangle \times T \to T \quad ; \quad \begin{cases} (w, l) \to wl & l \in V(T) \\ (w, l+\mu s) \to wl + \mu s & l + \mu s \in E(T) \end{cases}$$

The orbit of the vertex set $\langle a, b\rangle l$ is the vertex set, and the orbit of an edge of the tree is one of the two edge subsets. The stabiliser $\langle a, b\rangle_T = \{1\}$ for every edge or vertex. The action is both free and properly discontinuous.

Definition 1.16. An action is transitive if it has only one orbit, ie for any pair of elements $a, b \in A$ there is an element $g \in G$ which satisfies $b = ga$.

Example 1.16.1. None of the previous examples are transitive, since each has more than one orbit.

Example 1.16.2. Any group G acting on itself $(g_1, g_2) \to g_1g_2$ is a transitive action. Because the group has inverses, the element hg^{-1} will act on the element g, taking it to h, for any choice of $g, h \in G$

Example 1.16.3. The free group on two generators acts transitively on the vertex set of the infinite 4-valent tree, T. This is because it is one of the orbits of the action above.

1.4 Connected Covers

Given a cover $p : Y \to X$, X path connected, is there an action of $\pi_1(X, x_0)$ on Y? We shall see that the answer to this question is "sometimes", but that there is always an action on the fibre. This action provides a way to find out if Y is path connected. Assume that every space X is path connected.

Theorem 1.17. *The fundamental group of X acts on the fibre of a cover F,*

$$\pi_1(X, x_0) \times F \to F \quad ; \quad \sigma y = y * \sigma$$

Proof. If $\alpha : I \to X$ is a path in X with $\alpha(0) = x_1, \alpha(1) = x_2$, then by the unique path lifting property, there is a unique path $\widetilde{\alpha} : I \to Y$ with $\widetilde{\alpha}(0) = y_1$ for each $y_1 \in p^{-1}(x_1)$.

If a path $\beta : I \to X$ is homotopic to α relative to its end points, then the unique homotopy lifting property of covering spaces means that the lift of β to $\widetilde{\beta} : I \to Y$ with $\beta(0) = \alpha(0) = y_1$ will also lift the end point of the paths to the same point $\beta(1) = \alpha(1) \in p^{-1}(x_2)$.

So if we take a homotopy class of loops $[\sigma] \in \pi_1(X, x_0)$, given a base point $y_0 \in Y, p(y_0) = x_0$, then the class lifts to a homotopy class of paths $[\widetilde{\sigma}]$ in Y, starting at y_0 and ending at the same point in the set $p^{-1}(x_0)$.

We denote this unique end point of the paths, $[\widetilde{\sigma}]$, starting at y_0, by $y_0 * \sigma = \widetilde{\sigma}(1)$. Notice $y_0 * [\sigma] = y_0 * \sigma$ because all the paths in the homotopy class lift to a path ending at the same point.

So the action as written in the theorem above is well defined. Next check it satisfies the axioms for an action:

1. If $e \in \pi_1(X, x_0)$ is the constant map, then $\widetilde{e}$ is a constant path in Y. So the end of a constant path is obviously $y * e = y$ for all $y \in p^{-1}(x_0)$

2. Given $\sigma, \tau \in \pi_1(X, x_0)$, and $y \in p^{-1}(x_0)$, let $\tau y = y * \tau = y'$. Then:

$$\begin{aligned} \sigma y' &= y' * \sigma \\ &= (y * \tau) * \sigma \end{aligned}$$

   But, since the end of a path started at the end of another path is the end of the concatenation of those paths, this is just:

$$\sigma y' = y * \sigma\tau$$

   So $\sigma(\tau y) = (\sigma\tau)y$.

□

For interest, we can note that in certain cases, the action can be extended to the total space in the following way:

Proposition 1.18. *If $p : Y \to X$ is a regular covering, then the action of $\pi_1(X, x_0)$ can be extended from the fibre to the total space:*

$$\pi_1(X, x_0) \times Y \to Y \ ; \ [\sigma].y = y * (p\gamma_y)\sigma(p\gamma_y^{-1})$$

Where $\gamma_y : I \to Y$ is a path with $\gamma_y(1) = y$ and $\gamma_y(0) = y_0$, the base point of the fundamental group of that component of Y.

Proof. We must show the expression in the hypothesis is well defined. Suppose we choose γ_y and δ_y distinct paths in Y starting at the basepoint, and ending at y as required. We must show

$$y * (p\gamma_y)\sigma(p\gamma_y^{-1}) = y * (p\delta_y)\sigma(p\delta_y^{-1})$$

Notice $\gamma_y^{-1}\delta_y$ is a loop in Y, based at y_0, so $[\gamma_y^{-1}\delta_y] \in \pi_1(Y, y_0)$. Because it is a loop, showing that the two paths above end at the same point reduces to showing

$$y_0 * \sigma(p\gamma_y^{-1}) = y_0 * \sigma(p\delta_y^{-1})$$

If we want these two end points equal, then this is the same as wanting the path going out along one of the paths and back along the other to be a loop, i.e.

$$\widetilde{\sigma}\gamma_y^{-1}\ (\widetilde{\sigma}\delta_y^{-1})^{-1} = \widetilde{\sigma}\ \gamma_y^{-1}\delta_y\ \widetilde{\sigma}^{-1} \in \pi_1(Y, y_0)$$

Now if we push the loop into X, using p, we get

$$\sigma\ (p\gamma_y^{-1}\delta_y)\ \sigma^{-1} \in p_*(\pi_1(Y, y_0))$$

which, since $\gamma_y^{-1}\delta_y \in \pi_1(Y, y_0)$, is requiring that $p_*(\pi_1(Y, y_0))$ is a normal subgroup of $\pi_1(X, x_0)$, i.e. the cover is regular.

With this well defined, the group action axioms are satisfied by a parallel argument to the one in the previous theorem. So the fundamental group of X acts on the total space Y. □

If there is an action on all the path connected components of a disconnected cover, i.e. all path components are regular, it is easy to see that the actions can be assembled to an action on the disjoint union.

Notice also, in the sense that the action of each element of $\pi_1(X, x_0)$ is a homeomorphism $[\sigma] : Y \to Y$, this action commutes with the covering map.

Example 1.18.1. Consider the torus $T^2 = S^1 \times S^1$, and its cover

$$p : \mathbb{R} \times S^1 \to S^1 \times S^1 \quad ; \quad (x, e^{i\theta}) \to (e^{2\pi i x}, e^{2i\theta})$$

by the infinite cylinder. In order to consider the action of the fundamental group, $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$, we must choose a base point in T^2. Choose the point $p(0,1)$ for convenience. The fibre over that point will be

$$F_{(0,1)} = \{(k, \pm 1) | k \in \mathbb{Z}\}$$

The action of $\pi_1(T^2, p(0,1))$ on $\mathbb{R} \times S^1$ as described above is then

$$(a, b).(x, e^{i\theta}) = (x + a, (-1)^b.e^{i\theta})$$

Example 1.18.2. The infinite 4-valent tree, T, is a cover of the figure-eight, $S^1 \vee S^1$ with covering map taking oriented elements of the two edge subsets to the two oriented loops of the figure eight, and the elements of the vertex set to the intersection point of the two circles.

The action of the fundamental group of the figure eight, $\langle a, b\rangle$ is the right action defined in the example in Section 4.

Now let us turn our attention to whether the cover is connected.

Theorem 1.19. *If $p : Y \to X$ is a cover and X is path connected, Y is path connected if and only if $\pi_1(X, x_0)$ acts transitively on the fibre.*

Proof. Suppose Y is path connected. For every $y \in p^{-1}(x_0)$ there is a path, $\alpha : I \to Y$, with $\alpha(0) = y_0, \alpha(1) = y$. This path has the property that $p\alpha \in \pi_1(X, x_0)$, since $p\alpha(0) = p\alpha(1) = x_0$.

So there is an element of $\pi_1(X, x_0)$ which lifts to a path ending at y for all $y \in p^{-1}(x_0)$, ie there is an element of $\pi_1(X, x_0)$ taking $y_0 \in F$ to any other element of F, by our action, i.e.the action is transitive.

Now suppose the action is transitive. Thus any element of F may be taken to any other element of F by the action of an element of $\pi_1(X, x_0)$, i.e.there is a lift of a homotopy class of loops, $[\sigma]$ in X, starting at some $y_0 \in Y, p(y_0) = x_0$, going to any element of the fibre $p^{-1}(x_0)$.

Since X is path connected, for any two points $y_1, y_2 \in Y$, there are paths, τ_1 from $p(y_1)$ to x_0 and, τ_2 from $p(y_2)$ to x_0. Hence, lifting τ_1 to $\widetilde{\tau_1}$ starting at y_1 and τ_2 to $\widetilde{\tau_2}$ starting at y_2, there is a path $\widetilde{\tau_2}^{-1}\sigma\widetilde{\tau_1}$, where σ joins the point of the fibre $p^{-1}(x_0)$ in $\widetilde{\tau_1}$ to that in $\widetilde{\tau_2}$, which joins y_1 to y_2.

Hence Y is path connected. □

Now let us apply this to the mapping torus and the infinite cyclic cover.

Example 1.19.1. The mapping torus gives a supply of covers of the circle. Can we determine which are connected? Consider the mapping torus of an automorphism h of a finite set F. The fibre over any point in the covering is F. The action of the fundamental group $\mathbb{Z}$ of the circle on the fibre over the base point is

$$\mathbb{Z} \times F \to F \quad ; \quad (n, x) \to h(h(\cdots h(x) \cdots)) = h^n(x)$$

To determine if the cover is connected, we must check if the action is transitive. h is a way of permuting $|F|$ elements, i.e.is an element of the group $\Sigma_{|F|}$. Pick a point $x \in F$ and look at its orbit $\mathbb{Z}x$. There is a bijection,

$$\langle h \rangle \to \mathbb{Z}x \; ; \; h^n \to h^n(x)$$

So in a transitive action, with F finite, the order of h in $\Sigma_{|F|}$ must be $|F|$.

In the case where $F = S^0$, $\Sigma_{|F|}$ is the group with two elements, so we have two permutations to consider.

- $h = 1$, the identity element has order 1, so the mapping torus $T(1 : S^0 \to S^0)$ is disconnected.
- $h = -1$, the exchanging element has order 2, so the mapping torus $T(-1 : S^0 \to S^0)$ is connected.

Example 1.19.2. How can you tell if an infinite cyclic cover found as a pullback from a map $f : Z \to S^1$ is connected or not?

We are pulling back the cover from the cover of the circle by the real line. In this case, since $\mathbb{R}$ is connected, we know the fundamental group of the circle acts transitively on the fibre, $\mathbb{Z}$.

In an infinite cyclic cover we know that the fibre is $\mathbb{Z}$, and we are trying to find a transitive action of $\pi_1(Z)$ on the fibre $\mathbb{Z}$. The action requires loops in Z which are not homotopic to the constant map to take one point in the fibre to another.

If we can find a loop in Z which is pushed by f onto a loop in the circle, then the action of the two loops will be homeomorphic on the fibre. So if we can find loops in Z sent by f to all loops in $\pi_1(S^1)$, then those loops in the fundamental group of Z will have the same action in the fibre as the fundamental group of the circle has on the fibre of the real line.

So the cover is connected, if and only if $f_* : \pi_1(Z) \to \pi_1(S^1) = \mathbb{Z}$ is surjective.

In fact, this result will be generalised in Lemma 3.4.

Studying the orbits determines whether the path is connected or not, but how many path components are there in the cover of a space? We can generalise Theorem 1.19 in the following way.

Theorem 1.20. *Given a cover $p : Y \to X$, with X path connected, there is a bijection between the set of path components of a cover $\pi_0(Y)$ and the set of orbits of the action of the fundamental group of the base space on the fibre of the cover.*

$$\pi_1(X, x_0)y \leftrightarrow [y] \in \pi_0(Y)$$

Proof. If the action of the fundamental group on the fibre of the cover is transitive, we know that there is only one path component. When the action is not transitive, there must be more path components, because at least two points of the fibre will not be taken to each other by the action of any group element, i.e.they lie in different orbits.

If we can show there is a path to a point in the fibre over the basepoint from any point in Y, then determining which path component that point lies in reduces to determining which path component the point in the fibre lies in.

So consider a point $y \in Y$, obviously a path to an element in the fibre over the base point exists since X is path connected; any path from $p(y)$ to x_0 lifts to a path from y to a point in $p^{-1}(x_0)$. There may be many homotopy classes (rel end points) of such paths, but since each lifts to a path, they must all remain in the same path connected component.

So finally, we look at points in the fibre over the base point. We know there is a path between points in a single fibre if and only if there is a loop in $\pi_1(X, x_0)$ which lifts to one, i.e.if there is an element of $\pi_1(X, x_0)$ which takes one point in the fibre to the other. So every orbit of the fibre is contained within a paths component of the action.

Conversely, if two points in the fibre are not in the same paths component, then there is not path between them, and so there is no element in $\pi_1(X, x_0)$ which takes one to the other, so they are not in the same orbit.

Hence the map above is well defined, and bijective. □

Example 1.20.1. For a cover of the circle by a mapping torus, $p : T(h) \to S^1$, the fundamental group $\mathbb{Z}$ acts on the fibre by $x \to h^n(x)$ as above. This is a subgroup of the group of permutations of $|F|$ elements, $\Sigma_{|F|}$, generated by $h \in \Sigma_{|F|}$. Recall cycle notation for permutations, writing brackets $(a_1, a_2, \dots, a_k)$ with $a_{i+1} = h(a_i)$ and $a_1 = h(a_k)$ with enough brackets that all elements of F are written in one and only one.

If we write h in cycle notation, then each cycle is an orbit, and so the number of path connected components of the cover is the number of cycles (orbits) of h.

If the fibre of the cover has five points, and $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$ then we write it in cycle notation as $h = (1254)(3)$, so there are two cycles, two orbits, two path connected components in the cover.

Example 1.20.2. In the case of the infinite cyclic cover we must look again at $f_* : \pi_1(Z) \to \pi_1(S^1)$. The loops in the image in $\pi_1(S^1) = \mathbb{Z}$ must be one of the subgroups, either $\{0\}$ or $n\mathbb{Z}$, $n \in \mathbb{Z}$. If the image is trivial, then each point in the fibre is isolated, and so the cover is a countably infinite disjoint union of copies of Z. If the image is $n\mathbb{Z}$ than any two points in the fibre are in the same orbit if they are separated by exactly a multiple of n. So the cover is n components each with fibre $\mathbb{Z}$.

Proposition 1.21. *If $p : Y \to X$ is a regular cover, then the orbits of the action of $\pi_1(X, x_0)$ on the total space Y are the fibres. Further, the orbit space of the action is homeomorphic to X.*

Proof. Since the definition of the extended action of $\pi_1(X, x_0)$ on Y as found in Proposition 1.18 is:

$$\pi_1(X, x_0) \times Y \to Y \ ; \ [\sigma].y = y * (p\gamma_y)\sigma(p\gamma_y^{-1})$$

then the σ affects the place in the fibre, and the γ_y decides which fibre the point is in (i.e. the same one as y). So the orbits must be contained in the fibres, and since there is an element of G which takes the base point to each point in the fibre over the basepoint, the whole fibre over every other point must be reached by elements of G, so the orbits are the fibres.

Consider the map h, from the orbit space $Y/\pi_1(X, x_0)$

$$h : Y/\pi_1(X, x_0) \to X \quad ; \quad h(F_x) = x$$

Since the orbits are the fibres, and each fibre is the fibre over one point in X, and each neighbourhood of a point in Y is homeomorphic to its image in X, then the orbit space must be homeomorphic to X. $\square$

Chapter 2

Classifying Covers

This second chapter details some further properties of covering spaces. Assume X is a path connected space.

2.1 Isomorphism of Covers

How can we tell if two covers are different?

Definition 2.1. Let $p_1 : Y_1 \to X$ and $p_2 : Y_2 \to X$ be two covers of the same space. These covers are *isomorphic* if there is a homeomorphism $h : Y_1 \cong Y_2$ which commutes with the covering maps $p_2 h = p_1$.

If the cover has more than one component, we can also determine whether two of those components are isomorphic by considering them to be separate covers with a restricted covering map.

Notice isomorphic covers have isomorphic fibre and the actions of $\pi_1(X, x_0)$ on two isomorphic covers are homeomorphic.

Proposition 2.2. *Isomorphism of covers is an equivalence relation.*

Proof. Isomorphism of covers is reflexive, since the identity homeomorphism commutes with the covering map. Homeomorphisms have inverses, so the isomorphism of covers is symmetric. The homeomorphisms can be composed, so if h and k are homeomorphisms of covers, and $p_1 h = p_2$ and $p_2 h' = p_3$, then $p_1(hh') = p_3$ and the relation is transitive. □

Example 2.2.1. Consider two automorphisms of a set, $a, b : F \to F$, and the mapping tori $T(a)$ and $T(bab^{-1})$. Each has fibre F and each is a cover of the circle. Are they isomorphic? The map $b \times 1 : T(a) \to T(bab^{-1})$ is a homeomorphism between the two covers, and obviously preserves the covering map. Hence these two are isomorphic covers.

Example 2.2.2. Look at the torus $T^2 = S^1 \times S^1$. Let $f : T^2 \to S^1$; $(x, y) \mapsto x$ and $g : T^2 \to S^1$; $(x, y) \mapsto y$ be maps from the torus to the circle. Use the pullback construction to make the infinite cyclic covers $f^*\mathbb{R}$, $g^*\mathbb{R}$.

These two covers are both cylinders, so they are homeomorphic. However, consider a circle in $f^*\mathbb{R}$ which goes around the cylinder. The image of this circle under the homeomorphism to $g^*\mathbb{R}$ must also be a circle round the cylinder. Now use the covering map to push the circle and its image into the torus. The circles both go round the torus, but if the one pushed from $f^*\mathbb{R}$ goes around longitudinally, then the other goes round meridionally. So this homeomorphism doesn't commute with the covering maps. In fact, no homeomorphism from $f^*\mathbb{R}$ to $g^*\mathbb{R}$ can commute with the covering maps. So these two covers are not isomorphic.

This is a special case of the following general property of pullback covers.

Proposition 2.3. *Let $p : Y \to X$ be a cover, and $f, g : Z \to X$ be two continuous maps. Then the pullback covers f^*Y, g^*Y are isomorphic if and only if f and g are homotopic.*

Proof. Suppose f, g homotopic. Let $H : Z \times I \to X$ be a homotopy with $H(z, 0) = f(z)$ and $H(z, 1) = g(z)$. Now construct the pullback map, H^*Y, which is a cover of $Z \times I$. We know

$$H^*Y = \{(y, z, t) \in Y \times Z \times I \mid H(z, t) = p(y) \in X\}$$

Look at the "end" of H^*Y with $t = 0$. At $t = 0$, H takes the values of f, so f^*Y embeds in H^*Y as the hyperplane $t = 0$. Likewise g^*Y embeds in H^*Y as the hyperplane $t = 1$.

The path $\alpha_{(y,z)}(t) : I \to H^*Y$ given by

$$\alpha_{(y,z)}(0) = (y, z, 0)$$
$$\alpha_{(y,z)}(t) = (\hat{y}(t), z, t)$$

and where $\hat{y}$ is given by

$$p\hat{y}(t) = H(z, t)$$

has its start point in f^*Y and ends in g^*Y. Each such path has a unique start point and end point. Define a map $\alpha : Z \times Y \times \{0\} \to Z \times Y \times \{1\}$

$$\alpha_{(y,z)}(0) \mapsto \alpha_{(y,z)}(1)$$

This map is a bijection, and is continuous because H is continuous. Similarly h^{-1} is continuous and hence α is a homeomorphism.

Does it commute with the covering map (projection into Z)? Since the paths $\alpha_{(y,z)}$ have only one value of z along the whole path, they project into z as points, and so the homeomorphism α commutes with the covering map also.

To see the converse, suppose f^*Y, g^*Y isomorphic. Let $h : g^*Y \to f^*Y$ be the homeomorphism. The map h commutes with the covering map, projection on the second element of $Y \times Z$, in each case. Consider the projection map on the first element $\rho_1 : Y \times Z \to Y$ and compose that with p the covering map into X. By definition $p(y) = f(z)$ for any point in f^*Y, and $p(y) = g(z)$ for any point in g^*Y, so if f and g are homotopic in X, then $\rho_1 : f^*Y \to Y$ must be homotopic to $\rho_1 : g^*Y \to Y$ in Y.

Since h is an isomorphism of covers, it must be an element of the group of covering transformations. Hence there must be a loop in Z for each path component of the cover, which lifts to a path in f^*Y which moves the cover to g^*Y. This path defines a homotopy $k : f^*Y \times I \to g^*Y$ with $k(y, z, 0) = i$ the identity and $k(y, z, 1) = h$. So $\rho_1 : f^*Y \times I \to Y$ is a homotopy with $\rho_1(z, y, 0) = \rho_1 : f^*Y \to Y$ and $\rho_1(z, y, 1) = \rho_1 : g^*Y \to Y$. Hence $p\rho_1 : f^*Y \times I \to X$ is a homotopy between the images of f and g in X. □

2.2 Covering Transformations

Since a cover is isomorphic to itself, how many different automorphisms of a cover are there? These automorphisms compose, and form a group under composition, called the *covering transformations* or *deck transformations.* Deck transformations can be found in [8] and briefly in [1].

Definition 2.4. The group of *covering transformations* G of a cover $p : Y \to X$ is the set of automorphisms, $h : Y \to Y$ for which $ph = p$.

How do we find the group of covering transformations? If the cover is regular, then since the cover is connected and the action of any element of the fundamental group of X commutes with p on the whole space, then $\pi_1(X, x_0) \leqslant G$.

Let $f : \pi_1(X, x_0) \to G$. Since regular covers are connected, the orbits of the action of $\pi_1(X, x_0)$ are the fibres of the cover. So for any $y_1, y_2 \in F_x$ there is a $[\sigma] \in \pi_1(X, x_0)$ s.t. $\sigma y_1 = y_2$, and so if $g \in G$ is composed with σ then there is a $[\tau] \in \pi_1(X, x_0)$ s.t. $g\sigma = \tau$. i.e.

$$gf(\pi_1(X, x_0)) = f(\pi_1(X, x_0))$$

so f is onto.

Is f injective also? The kernel of f is the set of loops in $\pi_1(X, x_0)$ which take all elements of the fibre back to themselves, ie loops which lift to loops in Y. So, taking Y connected and choosing some y_0 in the fibre

$$\begin{aligned}\ker f &= \{[\sigma] \in \pi_1(X, x_0) \mid \widetilde{\sigma} \text{ a loop}\} \\ &= \{[\sigma] \in \pi_1(X, x_0) \mid [p^{-1}\sigma \text{ at } y_0] = p_*^{-1}[\sigma] \in \pi_1(Y, y_0)\} \\ &= \{[\sigma] \in \pi_1(X, x_0) \mid [\sigma] \in p_*(\pi_1(Y, y_0))\} \\ &= p_*\pi_1(Y, y_0)\end{aligned}$$

So f is not injective unless $p_*\pi_1(Y, y_0)$ is trivial. But we can get an isomorphism if we take the quotient of $\pi_1(X, x_0)$ by the kernel of f, which we can find if $p_*\pi_1(Y, y_0)$ is normal in $\pi_1(X, x_0)$.

Proposition 2.5. *For a regular cover $p : Y \to X$ of a connected space X, the group of covering transformations G is the quotient group:*

$$G \cong \pi_1(X, x_0)/p_*\pi_1(Y, y_0)$$

Proof. From the argument above. □

If the cover is connected, but not regular, then Proposition 1.18 says that not all the elements of $\pi_1(X, x_0)$ act in a well defined way on Y. Only elements $[\sigma] \in \pi_1(X, x_0)$ which satisfy the condition $[\sigma]p_*\pi_1(Y, y_0)[\sigma]^{-1} = p_*\pi_1(Y, y_0)$ will act on Y.

Definition 2.6. Let H be a group with a subgroup K. The normaliser of K in H is the largest subgroup N in which K is a normal subgroup. ie

$$N = \{h \in H | hkh^{-1} \in K \ \forall\ k \in K\}$$

If K is normal in H, then the normaliser subgroup is the whole of H.

So, if we denote the normaliser of $p_*\pi_1(Y, y_0)$ in $\pi_1(X, x_0)$ by N_Y, then the arguments leading up to Proposition 2.5 all hold for the normaliser, and so

Proposition 2.7. *For any connected cover $p : Y \to X$, the group of covering transformations G is given by*

$$G \cong N_Y/p_*\pi_1(Y, y_0)$$

Proof. Follows from the argument above. □

In the case of disconnected Y, $f : \pi_1(X, x_0) \to G$ is not onto in general, since if Y contains two disjoint isomorphic components, then while no element of $\pi_1(X, x_0)$ will map one to the other, it is required that there be a map in G which takes each to the other.

However, for each connected component of Y, we can apply the same formula for the group of covering transformations of that component. The groups of covering transformations, G_1, G_2, of two non-isomorphic components of a cover combine independently, so

$$G_{total} = G_1 \times G_2$$

Any components which are homeomorphic to each other, preserving p, can be mapped to each other by the covering transformations. So the group of covering transformations must reflect this. We want to be able to allow covering transformations which swap any two isomorphic components, and allow each component to be acted on by its own group of covering transformations. We use the wreath product.

Definition 2.8. The *wreath product* of a group G and the permutation group Σ_n is

$$G \wr \Sigma_n = G \times G \times \ldots \times G \times \Sigma_n$$

with n copies of G on the right hand side. If we consider an element of $G \wr \Sigma_n$ to be in two parts, the G parts and then the permutation, then the multiplication of two elements $(a_1, a_2, \ldots, a_n, h), (b_1, b_2, \ldots, b_n, k) \in G \wr \Sigma_n$ must be defined so the following diagram commutes. Let $J = G_1 \times G_2 \times \ldots \times G_n$ be the ordered direct product of n copies of G. Elements h, k of Σ_n act on J by permuting the ordering of the Gs.

$$\begin{tikzcd}
J \arrow[r, "b_1 \times b_2 \times \ldots \times b_n"] & J \arrow[r, "k"] \arrow[d, "a_{k^{-1}(1)} \times a_{k^{-1}(2)} \times \ldots \times a_{k^{-1}(n)}"'] & J \arrow[d, "a_1 \times a_2 \times \ldots \times a_n"] \\
& J \arrow[r, "k"] & J \arrow[d, "h"'] \\
& & J
\end{tikzcd}$$

Hence the multiplication is explicitly given by:

$$(a_1, a_2, \ldots, a_n, h)(b_1, b_2, \ldots, b_n, k) = (a_{k^{-1}(1)}b_1, a_{k^{-1}(2)}b_2, \ldots, a_{k^{-1}(n)}b_n, hk)$$

The wreath product $G \wr \Sigma_n$ is a nonabelian group for $n > 2$ with $n!|G|^n$ elements. There are n conjugate subgroups isomorphic to G, one for each G in the direct product. There is one subgroup isomorphic to Σ_n.

If we have a cover $p : Y \to X$, with a finite number of path connected components, label the isomorphism classes of path components, Y_j. Define n_j as the number of components of Y isomorphic to Y_j.

Now let G_j be the group of covering transformations for a component Y_j of the cover and then we can calculate the group of covering transformations of a disconnected cover as

$$G = \prod_j G_j \wr \Sigma_{n_j}$$

or by considering the formula for G_j, we can write this as follows.

Theorem 2.9. *If $p : Y \to X$ is a cover of a connected space, with n_j copies of each path component Y_j, and the restricted covering map p_j on each Y_j, then the group of covering transformations G can be written*

$$G = \prod_j \left(N_{Y_j}/p_{j*}\pi_1(Y_j)\right) \wr \Sigma_{n_j}$$

Proof. By Proposition 2.7, $N_{Y_j}/p_{j*}\pi_1(Y_j)$ is the group of covering transformations of the cover restricted to one of the Y_j. Let a_j be an element of this group. Let G be defined as above, and let C be the group of covering transformations of Y. We define a map $f : G \to C$ so an element $\prod_j(a_j^1, a_j^2, \dots, a_j^{n_j}, h_{n_j})$ of G, where h_{n_j} is an element of Σ_{n_j} goes to the element of C consisting of each a_j^i acting on the ith copy of Y_j first and then h_{n_j} permuting the n_j copies of Y_j. This is obviously well defined. The map f is injective, since the only element of G which maps to the identity in C must consist of the identity in each of the component's covering transformations and the identity permutation in the isomorphism class, which is the identity in G. Hence the kernel of the map is trivial. The map f is surjective because for any element of C, we can find an element of G, by breaking its action on the cover into the action on the components and the action on the isomorphism classes of components. So the map is a group isomorphism. $\square$

Example 2.9.1. Suppose we have $s \in \Sigma_n$ which is written as the product of disjoint cycles. Suppose that there are m_ℓ cycles of length ℓ, where ℓ runs over $\{1, 2, \dots, n\}$. So we know that the Mapping Torus $T(s)$ has $\sum_\ell m_\ell$ components, all of which are circles, and that as a cover of a circle, m_ℓ of the circles are ℓ-fold covers.

It is easy to see that the group of covering transformations of a circle as an ℓ-fold cover of the circle is the cyclic group C_ℓ. So the group of covering transformations of $T(s)$ as a cover of the circle is

$$\prod_{\ell=1}^{n} C_\ell \wr \Sigma_{m_\ell}$$

Example 2.9.2. We know there are only a few possibilities for the structure of the infinite cyclic cover, from Example 1.20.2. So if the cover is connected, then we know the fibre is $\mathbb{Z}$ and so we know the group of covering transformations is $\mathbb{Z}$ also. If the cover is not connected there are two possibilities, either there are a countable number of copies of the space, in which case the group of covering transformations is the infinite symmetric group, Σ_∞. If there are n components, then each has fibre $\mathbb{Z}$, so the group of covering transformations is $\mathbb{Z} \wr \Sigma_n$.

2.3 The Universal Cover

Before attempting the classification of all covers, one further piece of information is needed. The formula for the group of covering transformations of a connected cover hints at an interesting idea.

This idea is that since a group of covering transformations appears to exist for each subgroup of $\pi_1(X, x_0)$, is there a cover for each subgroup, for the covering transformations to act on? It will turn out that this all hinges on the existence of a cover with the group of covering transformations $\pi_1(X, x_0)$, that is one with $p_*\pi_1(X, x_0)$ trivial.

So how do you construct a space with $\pi_1(X, x_0)$ as the group of covering transformations? The space must have its own fundamental group trivial because p_* is injective, so it will be simply connected. It needs one more property not yet discussed too.

Definition 2.10. Suppose there is a neighbourhood of a point $x \in X$, and all loops in that neighbourhood are homotopic in X to the constant map at x. A space which has this property for every point is *semi-locally simply connected*

Definition 2.11. For any space X, let $P(X, x_0)$ be the space of homotopy classes (relative to end points) of paths in X based at x_0. The topology of $P(X, x_0)$ is the quotient of the compact open topology on the space of paths in X.

Proposition 2.12. *Let X be a path connected, semi-locally simply connected topological space. The space $P(X, x_0)$ is simply connected and is a cover of X.*

Proof. The space of paths from x_0 is obviously simply connected, because each path is homotopic to the constant map at its basepoint, via all 'shorter' paths. Hence any loop at the basepoint can be contracted pointwise to a constant map, so the fundamental group is trivial.

The space is a cover, with covering map the projection of the end point into X, because each neighbourhood of a point (path) in $P(X, x_0)$ is a neighbourhood of the end points of the paths, and so is a homeomorphic to a neighbourhood in X. □

This space is known as the *Universal Cover.* It is denoted $\widetilde{X}$.

The fibre of the cover is $\pi_1(X, x_0)$, because up to homotopy there is one path in $\widetilde{X} = P(X, x_0)$ ending at any fixed point in X for each element of the fundamental group.

The group of covering transformations of the space is $\pi_1(X, x_0)$ also, by Theorem 2.9 because $\widetilde{X}$ has only one component and its fundamental group is trivial, so the image $p_*\pi_1(\widetilde{X})$ is trivial. The universal cover is a regular cover.

The useful thing about the universal cover, is that it allows other covers to be constructed from it, like this:

Proposition 2.13. *Taking the canonical action of $\pi_1(X, x_0)$ on the regular cover $\widetilde{X}$*

1. *The orbit space $\widetilde{X}/H$ of the action of a subgroup H of $\pi_1(X, x_0)$ on $\widetilde{X}$ is a cover of X.*
2. *The fundamental group of $\widetilde{X}/H$ is H.*
3. *If H is normal in $\pi_1(X, x_0)$ then $\widetilde{X}/H$ is a regular cover of X,*
4. *$\widetilde{X}$ is the universal cover of $\widetilde{X}/H$.*

Proof.

1. Define the map $p_H : \widetilde{X}/H \to X$, using $p : \widetilde{X} \to X$; p commutes with all the elements of the subgroup. Since identifying points in the same fibre will identify the neighbourhoods of those points too, any neighbourhood of a point in $\widetilde{X}/H$ will map onto a neighbourhood of X. So p_H is the covering map.

2. Consider loops in $\pi_1(X, x_0)$ which lift to loops in $\widetilde{X}/H$, at a base point x_H. They will be elements which take the basepoint to itself in the action of $\pi_1(X, x_0)$ on the fibre. The elements which take the base point to itself are the same ones which, when lifted from X to $\widetilde{X}$ take the base point (of $\widetilde{X}$) to a point which the quotient by H pushes onto the base point (of $\widetilde{X}/H$). These are simply elements in the subgroup H. So $\pi_1(\widetilde{X}/H, x_H) = H$.

3. If H is normal, then $\pi_1(\widetilde{X}/H, x_H)$ projects onto a normal subgroup of $\pi_1(X, x_0)$, i.e. $p_{H*}\pi_1(\widetilde{X}/H, x_H) = H$. So $\widetilde{X}/H$ is a regular cover of X.

4. Define the map $p/H : \widetilde{X} \to \widetilde{X}/H$ as $x \mapsto Hx$ the orbit of x. Neighbourhoods of $\widetilde{X}/H$ are orbits of neighbourhoods of $\widetilde{X}$, so each one in $\widetilde{X}$ must

be mapped homeomorphically onto its image in the quotient. So p/H is the covering map. We know $\widetilde{X}$ is simply connected, so it must be the universal cover too.

$\square$

2.4 Classification of Covers

We now have sufficient information to answer the question, how many covers of a space are there up to isomorphism?

Obviously there are an infinite number, since disjoint unions of trivial covers of a space by itself give a cover for each positive integer. However if we consider first only connected covers of spaces which admit a universal cover, we can say more.

Proposition 2.14. *Let $Con(X)$ be the set of isomorphism classes of connected covers of a space X with a universal cover $\widetilde{X}$. Let $C(\pi_1(X, x_0))$ be the set of conjugacy classes of subgroups of $\pi_1(X, x_0)$. There is a bijection between these two, given by:*

$$
\begin{gathered}
Con(X) \cong C(\pi_1(X, x_0)) \\
\widetilde{X}/H \longleftrightarrow [H]
\end{gathered}
$$

Proof. Conjugate subgroups obviously give isomorphic covers, because conjugation by a path corresponds to moving the basepoint of the covering space. By Proposition 2.13 every subgroup of $\pi_1(X, x_0)$ gives us a cover of X, the quotient of $\widetilde{X}$.

The covering transformations give us the map in the opposite direction. The group of covering transformations are a quotient of the fundamental group, or of some smaller group contained in $\pi_1(X, x_0)$ (the normaliser), by a subgroup. Hence there can be at most one cover for each subgroup of $\pi_1(X, x_0)$, since connected covers with isomorphic groups of covering transformations are isomorphic covers.

So, each isomorphism class of connected covers corresponds to exactly one conjugacy class of subgroups. $\square$

Example 2.14.1. The connected covers of the torus, T^2, are easily classified. The universal cover of the torus is $\mathbb{R}^2$, with fibre the lattice $\mathbb{Z} \oplus \mathbb{Z} = \pi_1(T^2, 0)$. So, consider the subgroups of the abelian group with two generators; they fall into three kinds, 0-dimensional (the trivial subgroup), 1-dimensional (eg $\langle(1,5)\rangle$) and 2-dimensional (eg $\langle(0,1),(1,0)\rangle$).

These correspond to sublattices generated by pairs of lattice vectors in the fibre lattice, with the zero vector as the "dummy" to pad the generating sets out to two vectors.

Take the quotient of $\mathbb{R}^2$ by the lattices generated by these vectors, and you get $\mathbb{R}^2$, cylinders, and tori, for 0, 1 and 2 dimensional sublattices. These are all the connected covers of the torus.

So how many covers (connected or otherwise) of a connected space X are there? Well, we have seen we are allowed to take disjoint unions of covers to get a new one. This brings us to:

Proposition 2.15. *Let $Cover(X)$ denote the set of isomorphism classes of covers of a connected space X with a finite number of components. Then $Cover(X)$ is the free abelian monoid generated by $Con(X)$, less the zero element.*

$$Cover(X) \cong \mathbb{N}[Con(X)] \setminus 0$$

Proof. Disjoint union is obviously commutative, and associative too. The zero of the monoid is the empty space, which is not a cover of any space except itself. □

Example 2.15.1. Now we know the covers of the circle are disjoint unions of circles (covers with finite fibre) and copies of the real line (cover with infinite fibre). How does this relate to the mapping torus?

Let $p : T(h) \to S^1$ be a cover of S^1, with fibre F. Consider the action of the fundamental group $\mathbb{Z}$ on F. It takes each element of the fibre to another in that path component. Let $z : F \to F$ be the map from the fibre to itself, corresponding to the action of the generating element of $\mathbb{Z}$. It is an automorphism by the definition of an action. Hence the mapping torus $T(z)$ is a cover of the circle. It is easy to see that $h = z$ and hence this mapping torus is isomorphic as a cover to the cover we used to construct it. So each cover of the circle is a mapping torus, by the construction we have just seen. We also know that the mapping torus of any automorphism of a discrete set is a cover of the circle.

Mapping Tori of automorphisms of discrete sets are all the covers of a circle, and they are the only covers of a circle.

Chapter 3

Geometric Fundamental Domains

Tessellation and patterns have been the subject of mathematical study since the earliest times, and the idea of a shape which fits together with itself, repeated at intervals is just as interesting in more abstract settings. Given an action of a group on a space, can you find a subset whose translates under the action fill out the space? The "simplest" example is that the whole space will fill itself out under the action of a group.

Definition 3.1. A *generating domain* of a free properly discontinuous (right) action of a group G on a set X, is a subset $D \subseteq X$ such that

$$\bigcup_{g \in G} Dg = X \tag{3.1}$$

Example 3.1.1. The whole space X is a generating space for any action on it.

Example 3.1.2. The action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by addition gives the classic tessellation of the plane by a square. A generation domain for the action would be any shape large enough to contain a unit square, for instance Figure 3.1 shows a disc with radius $\sqrt{2}/2$.

We need to refine this notion a bit in order to find an analogue to tessellation.

Definition 3.2. A *fundamental domain* of a free properly discontinuous (right) action of a group G on a space X, is a subset $U \subseteq X$ such that

$$\bigcup_{g \in G} Ug = X \tag{3.2}$$

and U is the closure of a subset $O \subseteq X$ which contains one point for each of the orbits of the action.

Example 3.2.1. There is a unique fundamental domain $U = X$ if and only if G acts trivially on X, because for a trivial action each point in X is an orbit on its own.

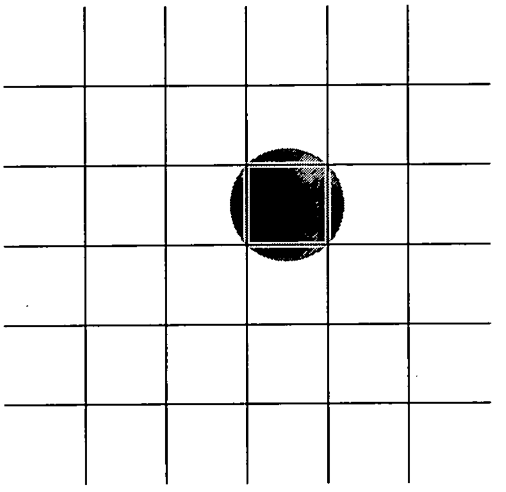

Figure 3.1: A circle radius $\sqrt{2}/2$ contains a unit square, and hence generates $\mathbb{R}^2$.

Example 3.2.2. A fundamental domain of the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ by addition is the unit square. In fact, for any tessellation of the plane, there is an action of a group for which the initial shape is a fundamental domain.

Example 3.2.3. Let S^1 be parametrised as $I/(0 = 1)$, and let C_n act on it by addition of $\frac{1}{n}$. The action is free and properly discontinuous, and for each point $x \in [0, 1)$ there is a fundamental domain

$$U_x = \begin{cases} [x, x + \frac{1}{n}] & \text{if } 0 \leqslant x \leqslant \frac{n-1}{n} \\ [x, 1] \cup [0, x + \frac{1}{n} - 1] & \text{if } \frac{n-1}{n} < x < 1 \end{cases}$$

These are not the only fundamental domains - although these are the only connected ones.

Construction 3.3. Voronoi [23] gives a construction of fundamental domains. There follows an adaptation of his method to our situation.

Given a free properly discontinuous (right) action of a group G on a space X, take a single point $a \in X$, and label its orbit as $aG = \{ag : g \in G\}$. Since G acts freely and properly discontinuously, all the ag are distinct. Next suppose that there is a continuous metric $d : X \times X \to \mathbb{R}_{\geqslant 0}$ which is compatible with the action, ie:

$$d(x_1, x_2) = d(x_1 g, x_2 g) \quad \forall\, g \in G\, x_1, x_2 \in X$$

and for which there exists a value $R \in \mathbb{R}$ such that for all $x \in X$ $d(x, ag) < R$ for some ag.

To each point ag assign a set, $\Pi(ag)$ of all points $x \in X$ whose distance $d(x, ag)$ is equal to the minimum of all $d(x, ah)$, $h \in G$. Then $\Pi(ag)$ is the intersection of all half spaces given by the inequality:

$$d(x, ag) \leqslant d(x, ah) \quad g \neq h$$

This construction doesn't seem to easily satisfy the definition given above, but we can show that any $\Pi(ag)$ is a fundamental domain, for any $a \in X$ and any $g \in G$. To do so we must consider the properties of the metric. Suppose y is a point in $\Pi(ag)$, and $h \in G$ acts on it. Then by the way the metric was defined

$$d(y, ag) = \min\{d(y, ak) : k \in G\} \implies d(yh, agh) = \min\{d(yh, ak) : k \in G\}$$

and so $\Pi(ag)h = \Pi(agh)$. Hence since every point in X is in one of the $\Pi(ag)$s:

$$\bigcup_{g \in G} \Pi(a1)g = X$$

Now consider the interior of $\Pi(ag)$. We know all its translates are distinct, hence it contains only one point in each orbit. The boundary of $\Pi(ag)$ is another matter. Any point $x \in \partial\Pi(ag)$ is also in another set, $\Pi(agh)$ for some $h \in G$ dependent on x, since

$$d(x, ag) = d(x, agh) = min\{d(x, ak) | k \in G\}$$

Points in the boundary must be in the same orbit as other points (namely x and xh^{-1}) in the boundary of $\Pi(ag)$. However, all orbits have at least one representative in the boundary or the interior. Let B be a set of representatives of each of the orbits in the boundary, and so there is a set $\mathrm{Int}(\Pi(ag)) \cup B$ which contains one point from each fibre and whose closure is the whole of $\Pi(ag)$.

Example 3.3.1. There is an action of $\mathbb{Z}$ on $\mathbb{R}$ given by addition, which is free and properly discontinuous.

$$\mathbb{R} \times \mathbb{Z} \to \mathbb{R} \quad ; \quad (x, n) \mapsto x + n$$

Let us use the Voronoi Construction to find a fundamental domain for this action. Take the point 0 as our base point, and so the labeled points are the integers. Now if we choose the standard metric on $\mathbb{R}$ as our metric, ie

$$d(x_1, x_2) = |x_1 - x_2|$$

Then this has the property that $d(x_1+n, x_2+n) = d(x_1, x_2)$ and taking $R = 1$, we can see that there is no point in $\mathbb{R}$ which is more than distance 1 from a labeled point.

So, now construct $\Pi(0)$. The half space given by the inequality

$$|x-n| \leqslant |x-m|$$

is the set of all $x \geqslant \frac{m+n}{2}$ if $n > m$ or $x \leqslant \frac{m+n}{2}$ if $n < m$. Hence

$$\Pi(0) = \bigcap_{n\in\mathbb{N}}[-\frac{n}{2},\infty] \cap \bigcap_{n\in\mathbb{N}}[\frac{n}{2},-\infty] = [-\frac{1}{2},\frac{1}{2}]$$

In general, $\Pi(x) = [x-\frac{1}{2}, x+\frac{1}{2}]$, which shouldn't be too much of a surprise.

We know that the fundamental group of a space acts on its regular covers, so the question arises "does a covering have a fundamental domain?" If so, can it be easily constructed? If not, can that be detected in some way? Even for a manifold, this problem is not easily solved by the tools to hand - the Voronoi construction relies on finding a metric with very specific properties.

In the case of surfaces Voronoi's construction is sufficient, since any surface can be given a metric induced from its universal cover, euclidean in the case of the torus or klein bottle, spherical for the sphere and real projective plane, or hyperbolic otherwise.

For other spaces we must resort to indirect means. Suppose there is a fundamental domain for one cover. Is it possible to construct a cover of a different space using the first to give a template for the fundamental domain? Pullback covers give a way of making one cover from another, and it will be shown that fundamental domains of a cover do induce fundamental domains in pullback covers.

For the rest of this chapter, let $p : \widetilde{Y} \to Y$ be a connected regular covering with group of covering transformations $G = \pi_1(Y, y_0)/p_*\pi_1(\widetilde{Y}, \widetilde{y}_0)$. For any $y \in Y$ there is defined a bijection between the fibre $F_y = p^{-1}(y)$ and G

$$F_y \to G \quad ; \quad z \to h(y,z)$$

where $h(y,z) : \widetilde{Y} \to \widetilde{Y}$ is the unique covering translation sending y to z. Let $f : X \to Y$ be a continuous map. The pullback cover $f^*\widetilde{Y}$ is constructed from the diagram:

$$\begin{tikzcd} f^*\widetilde{Y} \arrow[r, "\widetilde{f}"'] \arrow[d, "q"'] & \widetilde{Y} \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It can be described in the following way (Construction 1.10):

$$f^*\widetilde{Y} = \{(x,\widetilde{y}) \in X \times \widetilde{Y} | f(x) = p(\widetilde{y}) \in Y\}$$

with the covering map q just projection on the first factor.

The fibres of these two covers are in bijective correspondence, by Proposition 1.11, and the bijection respects the actions of the fundamental group of the base space. Denote the fibres by F_X and F_Y, and the bijection by $\rho : F_X \to F_Y$.

We begin by strengthening Example 1.19.2 to a generalisation about all pullback covers.

Lemma 3.4. If $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective then the pullback cover $f^*\widetilde{Y}$ is connected.

Proof. Theorem 1.19 yields two useful facts

- since $\widetilde{Y}$ is assumed connected, $\pi_1(Y, y_0)$ must act transitively on the fibre of $\widetilde{Y}$,
- it is sufficient to show that the action of $\pi_1(X, x_0)$ on the fibre of $f^*\widetilde{Y}$ is transitive if f_* is surjective.

So write the action of $\pi_1(Y, y_0)$ on F_Y as

$$\pi_1(Y, y_0) \times F_Y \to F_Y \quad ; \quad ([\sigma], y) \mapsto [\sigma]y$$

and then define an action of $\pi_1(X, x_0)$ on F_Y by

$$\pi_1(X, x_0) \times F_Y \to F_Y \quad ; \quad ([\tau], y) \mapsto f_*([\tau])y$$

Since $\mathrm{Im}(f_*) = \pi_1(Y, y_0)$ this action is transitive too. Now use $\rho : F_X \to F_Y$ to write an action of $\pi_1(X, x_0)$ on F_X.

$$\pi_1(X, x_0) \times F_X \to F_X \quad ; \quad ([\tau], x) \mapsto \rho^{-1}(f_*([\tau])\rho(x))$$

This is transitive too, since ρ is a bijection. There is a commutative diagram from the action:

$$\begin{tikzcd}
\pi_1(X, x_0) \times F_X \arrow[r] \arrow[d, "(f_* \times \rho)"'] & F_X \arrow[d, "\rho"] \\
\pi_1(Y, y_0) \times F_Y \arrow[r] & F_Y
\end{tikzcd}$$

□

Lemma 3.5. If $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective then the pullback cover $f^*\widetilde{Y}$ is regular and has the same group, G, of covering transformations as $\widetilde{Y}$.

Proof. Define $u : \pi_1(Y, y_0) \to G, v : \pi_1(X, x_0) \to G$ as projections onto G. The following diagram commutes:

$$\begin{tikzcd}
\pi_1(f^*\widetilde{Y}, \widetilde{x}_0) \arrow[r, "\widetilde{f}_*"'] \arrow[d, "q_*"'] & \pi_1(\widetilde{Y}, \widetilde{y}_0) \arrow[d, "p_*"] \\
\pi_1(X, x_0) \arrow[r, "f_*"'] \arrow[d, "v"'] & \pi_1(Y, y_0) \arrow[d, "u"] \\
G \arrow[r, equal] & G
\end{tikzcd} \tag{3.3}$$

The right hand column is exact, since $p : \widetilde{Y} \to Y$ is regular, that is, G is a quotient. The lemma will be proved by showing that the left hand column is exact also.

Firstly, q_* is injective, by Proposition 1.12 and v is a surjection since f_* and u are surjective by hypothesis. It only remains to show that:

$$\ker(v : \pi_1(X, x_0) \to G) = \mathrm{Im}(q_* : \pi_1(f^*\widetilde{Y}, \widetilde{x}_0) \to \pi_1(X, x_0))$$

Using commuting diagram 3.3, any $[\tau] \in \pi_1(f^*\widetilde{Y}, \widetilde{x}_0)$ goes to $q_*[\tau] \in \pi_1(X, x_0)$, and hence to $f_*q_*[\tau] \in \pi_1(Y, y_0)$. But $f_*q_* = p_*\widetilde{f}_*$, so mapping into G gives:

$$\begin{aligned}
vq_*[\tau] &= uf_*q_*[\tau] \\
&= up_*\widetilde{f}_*[\tau] \\
&= 0
\end{aligned}$$

by exactness on the right. So $\ker(v) \supseteq \mathrm{Im}(q_*)$.

Next suppose there is a $[\beta] \in \pi_1(X, x_0)$, such that $v[\beta] = 0 \in G$. Since $f_*[\beta] \in \pi_1(Y, y_0)$ goes to $0 \in G$, it must come from an element of $\pi_1(\widetilde{Y}, \widetilde{y}_0)$. So lift $f_*[\beta]$ to $[\alpha] \in \pi_1(\widetilde{Y}, \widetilde{y}_0)$ to get elements which agree in $\pi_1(Y, y_0)$: $p_*[\alpha] = f_*[\beta]$. Using Theorem 1.3, the unique path lifting property of covering spaces, lift β into $f^*\widetilde{Y}$ at the base point $\widetilde{x}_0$. Then $\widetilde{f}(\beta * \widetilde{x}_0)$ will be homotopic to α, by commuting diagram 3.3 and Theorem 1.4, the unique homotopy lifting property of covering spaces.

Hence, it is possible to construct a loop γ in $f^*\widetilde{Y}$ and hence a $[\gamma] \in \pi_1(f^*\widetilde{Y}, \widetilde{x}_0)$ where

$$\begin{tikzcd}
S^1 \arrow[dr, dotted, "\gamma=(\alpha,\beta)"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& f^*\widetilde{Y} \arrow[r, "\widetilde{f}"'] \arrow[d, "q"'] & \widetilde{Y} \arrow[d, "p"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

In other words $\gamma(t) = (\alpha(t), \beta(t))$. Hence for each $[\beta]$ there is a $[\gamma]$ and so $\ker(v) \subseteq \mathrm{Im}(q_*)$.

Thus $\ker(v) = \mathrm{Im}(q_*)$ and so the sequence is exact and $q_*\pi_1(f^*\widetilde{Y}, \widetilde{x}_0)$ is a normal subgroup of $\pi_1(X, x_0)$. □

Proposition 3.6. *Let $U \subset \widetilde{Y}$ be a fundamental domain for the action of the group of covering translations G.*

Proof. Given $U \subset \widetilde{Y}$ a fundamental domain, construct its pre-image in $f^*\widetilde{Y}$.

$$f^*U = \{X \times U | f(x) = p(\widetilde{y}) \in Y, x \in X, \widetilde{y} \in U\}$$

Since U is a fundamental domain, there is an open subset $O \subset U$ with U the closure of O. O has one point in each orbit of the action of G, that is, $pO = Y$. $\widetilde{f}$ is continuous, so there is an open set f^*O, which contains one point for each orbit because the action of G respects the map $\widetilde{f}$. Note that f^*U is the closure of f^*O.

Because the fibres of $\widetilde{Y}$ and $f^*\widetilde{Y}$ are the same, and G acts in the same way on them

$$\bigcup_{g \in G} f^*Ug = f^*\widetilde{Y}$$

□

Example 3.6.1. The universal cover of the torus can be written as $p : \mathbb{R}^2 \to S^1 \times S^1$, and is, of course, regular.

Take $A : S^1 \vee S^1 \to S^1 \times S^1$ mapping one circle to the meridian of the torus, and the other to the longitude. Then

$$A^*\mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 | x \text{ or } y \in \mathbb{Z}\}$$

which is a grid of lines. A fundamental domain of $\mathbb{R}^2$ is a unit square, and its lift into the grid is the intersection of the grid with the square. In this case the group of covering translations is the same for both covers.

Example 3.6.2. Let X be a topological space and $f : X \to S^1$ be a map. It is possible to construct a pullback cover with fibre $\mathbb{Z}$ which is denoted $\overline{X}$. The homeomorphism type of $\overline{X}$ is determined by the homotopy type of f.

$$\begin{tikzcd}
\overline{X} \arrow[r] \arrow[d, "p_X"'] & \mathbb{R} \arrow[d, "p_{S^1}"] \\
X \arrow[r, "f"] & S^1
\end{tikzcd}$$

A fundamental domain for $\overline{X}$ is

$$f^*[0,1] = \{X \times \mathbb{R} | f(w) = p_{S^1}(r), w \in X, r \in [0,1]\}$$

This construction gives a fundamental domain which will not necessarily have any special properties; particularly, if X is a manifold, it will be useful to look at fundamental domains which are manifolds (with boundary). In Example 3.6.2, the only thing preventing $f^*[0,1]$ from being a manifold is that the map may put cusps and unpleasantness into $f^*[0,1]$. If we chose X a manifold and f to be a smooth map, then the interior of the fundamental domain is a manifold, but the boundary may not be. In order to ensure that this is nice too, we must use transversality.

Transversality is a result of the work of Sard and Brown, which culminated in Sard's Theorem. This formulation of the important lemma is taken from Milnor [13], where separate proofs can be found for manifolds with and without boundary.

Definition 3.7. Let $f : X \to Y$ be a smooth map of manifolds. A point $\eta \in X$ is regular if the derivative is nonsingular. $f(\eta) \in Y$ is a regular value of f if η is a regular point.

Lemma 3.8. If $f : X \to Y$ is a smooth map of manifolds (with or without boundary) of dimensions $x \geqslant y$, and if $\xi \in Y$ is a regular value of f and the restriction $f|_{\partial X}$ then the set $f^{-1}(\xi) \subset X$ is a smooth manifold (with or without boundary) of dimension $x - y$. Furthermore the boundary $\partial(f^{-1}(\xi))$ is precisely the intersection of $f^{-1}(\xi)$ with ∂X.

Let M be an oriented manifold with boundary. To construct the fundamental domain for an infinite cyclic cover similar to 3.6.2, first choose a point $\xi \in S^1$, which is not critical and so $N = f^{-1}(\xi)$ is a co-dimension-1 sub-manifold. Since both M and S^1 are orientable, N is also orientable. It is not much more work to see that if $[\xi - \delta, \xi + \delta]$ contains no critical values, transversality also gives a bi-collar neighbourhood $N \times [-1,1] \subset M$. If the point ξ is lifted into $\mathbb{R}$ to $\{\xi + \mathbb{Z}\}$, it forms boundaries of a fundamental domain and its translates. By analogy lifting N to $N \times \mathbb{Z}$ in $\overline{M}$ should give us the boundaries of a fundamental domain and its translates for $\overline{M}$. The following commuting diagram illustrates this.

$$\begin{tikzcd}
& N \times \mathbb{Z} \arrow[rr] \arrow[dl, hook] \arrow[dd] & & \{\xi + \mathbb{Z}\} \arrow[dl, hook] \arrow[dd] \\
\overline{M} \arrow[rr] \arrow[dd] & & \mathbb{R} \arrow[dd] & \\
& N \arrow[rr] \arrow[dl, hook] & & \xi \arrow[dl, hook] \\
M \arrow[rr] & & S^1 &
\end{tikzcd}$$

Hence we can describe a fundamental domain for $\overline{M}$ as M_N

$$M_N = \{M \times \mathbb{R} | f'(w) = p_{S^1}(r), w \in M, r \in [\xi, \xi + 1]\}$$

Now, it is also possible to construct finite cyclic covers of M, by the pullback construction on a finite cover $n : S^1 \to S^1$. Fundamental domains can be found in just the same way, since f hasn't changed, we can use the same ξ and find the same $N \subset M$. Hence we can think of the cyclic covers as being constructed by gluing fundamental domains together. And, since M is the simplest cyclic cover of itself, it is easy to see that M_N is just the closure of M cut along N. Hence, cyclic covers can be built by gluing, choosing as building blocks the original space cut along some non-separating orientable co-dimension-1 submanifold.

Example 3.8.1. Let $k : S^n \to S^{n+2}$ be a knot,and let M be the closure of $S^{n+2} \setminus k(S^n) \times D^2$. See Theorem 6.4)for the construction of a map inducing a surjection $f_* : \pi_1(M) \to \mathbb{Z}$.

In the classical case (n=1), M is homeomorphic to a solid sphere with a knotted tunnel inside it, by considering S^3 as $\mathbb{R}^3 \cup \{\text{point}\}$, and making sure that point is in the image of k. See Figure 3.2. Using the pullback construction as

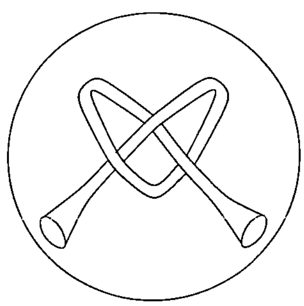

Figure 3.2: The complement of a trefoil knot 3_1

above the space has a connected infinite cyclic cover.

We find a fundamental domain of $\overline{M}$ by using transversality to find its boundary. We know part of its boundary lies in the boundary of M, since f restricts to the boundary of the knot complement as projection onto the meridian. Hence the remaining part of the boundary will be a spanning surface of the knot. It is easy to see that any spanning surface will do. In fact, given an oriented co-dimension-1 submanifold with a bi-collar neighbourhood, it is possible to construct a map to the circle which can be used for the pullback construction of an infinite cyclic cover.

Construction 3.9. Seifert [17] developed an algorithm for finding a spanning surface of a knot. It relies on finding a knot diagram; a projection of the knot into the euclidean plane, with finitely many intersections, all transverse and of multiplicity 2.

Given a knot $k : S^1 \to S^3$ and its knot diagram the algorithm can be stated as follows. For each crossing point in the knot diagram, add an arc between the two points in S^3. This gives a set of oriented simple closed curves, which have the arcs we just added in common, in opposite orientations. To each closed curve, you can add an oriented disc embedded in S^3. Figure 3.3 shows the two discs added to the knot complement, with the edges in common perpendicular to the page, between the under and overpasses of the crossings. The resulting surface is

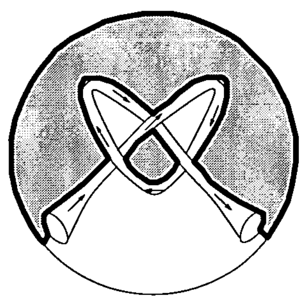

Figure 3.3: Two Seifert discs in the complement of a trefoil knot 3_1

orientable with boundary k. This is a Seifert Surface.

A more concrete description of the algorithm would be:

1. Let $k : S^1 \to S^3$ be a (smooth) knot, and take a projection of k onto $\mathbb{R}^2$, by removing a single point in the complement of k, projecting stereographically from that point onto $\mathbb{R}^3$, and then projecting onto a plane. By choosing a "nice" representative from the isometry class of k, it is possible to make the projection $Pk : S^1 \to \mathbb{R}^2$ a closed curve with finite intersections, all of multiplicity 2. Denote the set of intersection points by $I \subset \mathbb{R}^2$. Let $J = Pk^{-1}(I)$ be the pre-image points of the intersections. Let A be the set of arcs of Pk after removing the intersection points, and label them $\{e_1, \dots, e_{|J|}\}$.

2. The four lines at each crossing in the knot diagram Pk can be oriented by S^1 and labeled with the labels of the arcs in A, as in Figure 3.4. Define an action of $\mathbb{Z}$ on A, using the labeled knot diagram. 1 acts on an arc by taking it to the arc which leaves the crossing it enters, but isn't the continuation of that arc embedded in S^1. That is, looking at Figure 3.4

$$e_1 \mapsto e_3$$

$$e_2 \mapsto e_4$$

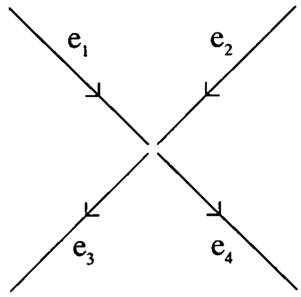

Figure 3.4: A typical crossing in $Pk \setminus I$

and e_3, e_4 are sent to other arcs by the crossing at their other ends.

3. The orbits of that action correspond to the *Seifert Circles* of the knot diagram. It is possible to add the discs across these circles and the resulting surface is called the Seifert surface of the knot diagram.

This Seifert spanning surface has a bi-collar neighbourhood, which can be seen from the fact that it is piecewise linear, orientable and embedded in an orientable space. Hence it can be used to define a map $f : T \to S^1$ for use as the map to pullback an infinite cyclic cover.

So, given a knot diagram, it is possible to construct a map $f : T \to S^1$ and hence an infinite cyclic cover and fundamental domain.

For classical knots $S^1 \to S^3$, this is fine, and even for a general manifold there is some method of construction of a fundamental domain, but these rely heavily on geometric intuition and on *general position* arguments. Infinite cyclic covers are simple enough to study easily but rich enough that they are are used in many different settings as well as knot theory. In fact, for higher dimensional knots, the infinite cyclic cover is a well used invariant, but Seifert's algorithm will not work for higher dimensions.

In order to allow a more concrete method of construction of fundamental domains, it is worth using *CW-complexes.* It would be worth considering simplicial complexes, but simplicial complexes often have very many simplices, making algebraic calculations complicated and tedious. There is usually a homotopically equivalent CW-complex with fewer cells, reducing the complexity of algebraic calculations. CW-complexes are described fully in Chapter 5.

Chapter 4

Algebraic Fundamental Domains

In this chapter a definition of fundamental domain of a chain complex is given, along with a justification in from geometry. The work presented here is based on that of Waldhausen and Ranicki ([24],[15]).

Let A be any associative ring with a 1. Let $A[t, t^{-1}]$ denote the ring of polynomials in t and t^{-1} with coefficients in A.

Definition 4.1. A *based chain complex* $\mathcal{C}$ of dimension n over a ring A is a finite sequence of based free (right)-A-modules $\mathcal{C}_n, \ldots, \mathcal{C}_0$ and A-module morphisms $d_i^{\mathcal{C}} : \mathcal{C}_i \to \mathcal{C}_{i-1}$ called *differentials* such that composition of two successive differentials is zero

$$d_i^{\mathcal{C}} d_{i+1}^{\mathcal{C}} = 0 : \mathcal{C}_{i+1} \to \mathcal{C}_{i-1}.$$

The basis of each $\mathcal{C}_i$ is denoted $\underline{c}_i$. If $\mathcal{C}$ is finitely generated, then we use the notation $\underline{c}_i = \{c_{ij} : 1 \leqslant j \leqslant N_i\}$ where N_i is the A-dimension of $\mathcal{C}_i$. $\mathcal{C}_i$ is taken to be zero outside the range $0 \leqslant i \leqslant n$. The zero chain complex is the chain complex which has only zero modules in all dimensions.

Example 4.1.1. Any exact sequence (of A-modules) is a chain complex (over A).

Definition 4.2. Given a based chain complex $\mathcal{C}$ over a ring A and subsets $\underline{b}_i \subseteq \underline{c}_i$ with $d_i^{\mathcal{C}}(\underline{b}_i) \subseteq \operatorname{span}(\underline{b}_{i-1})$, let $\mathcal{B} \subseteq \mathcal{C}$ be the subcomplex generated by the $\underline{b}_i$. We call $\mathcal{B}$ a *based subcomplex*.

Example 4.2.1. Given a cellular chain complex $\mathcal{W}$ of a CW complex X, then any subcomplex of X determines a based subcomplex of $\mathcal{W}$.

Definition 4.3.

1. The *restriction* of a finitely generated based (right)-$A[t, t^{-1}]$-module M with basis $\{m_1, \ldots, m_N\}$ is a based A-module $M^!$ with infinite basis

$$\{m_j t^k | 1 \leqslant j \leqslant N, k \in \mathbb{Z}\}$$

There is a bijective morphism $\rho_M : M \to M^!$ which takes an element

$$M \ni \sum_{i=1}^{N} \left(\sum_{k=-\infty}^{\infty} a_{ik} t^k \right) m_i \mapsto \sum_{i=1}^{N} \sum_{k=-\infty}^{\infty} a_{ik}(m_i t^k) \in M^!$$

I shall call this the restriction identification map.

2. Given a morphism of based $A[t,t^{-1}]$-modules, $f : M \to Q$ there is an induced morphism $f^! : M^! \to Q^!$ which commutes with the two maps ρ_M, ρ_Q.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"'] & Q \arrow[d, "\rho_Q"] \\ M^! \arrow[r, "f^!"'] & Q^! \end{tikzcd}$$

Hence $(gf)^! = g^! f^!$.

3. The *restriction* $\mathcal{C}^!$ of a finitely generated based chain complex $\mathcal{C}$ over $A[t,t^{-1}]$ is a (infinitely generated) based chain complex over A, where $(\mathcal{C}^!)_i = (\mathcal{C}_i)^!$, and the $d_i^{\mathcal{C}^!}$ are induced.

Example 4.3.1. A 1-dimensional chain complex $\mathcal{U}$ over the polynomial ring $A[t,t^{-1}]$ can be formed from any polynomial $p(t) \in A[t,t^{-1}]$ in this way:

$$d_i^{\mathcal{U}} : \mathcal{U}_1 = A[t,t^{-1}] \to \mathcal{U}_0 = A[t,t^{-1}] \quad ; \quad u(t) \mapsto p(t)u(t)$$

Hence, denoting $p(t) = \sum_{i=-\infty}^{\infty} p_i t^i$ (with finitely many $p_i \neq 0$), $\mathcal{U}^!$ is an infinitely generated based chain complex

$$d_1^{\mathcal{U}^!} : \mathcal{U}_1^! = \bigoplus_{-\infty}^{\infty} A \to \mathcal{U}_0^! = \bigoplus_{-\infty}^{\infty} A \quad ;$$

$$\begin{pmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{pmatrix} \mapsto \begin{pmatrix} & \vdots & \vdots & \vdots & \\ \cdots & p_0 & p_{-1} & p_{-2} & \cdots \\ \cdots & p_1 & p_0 & p_{-1} & \cdots \\ \cdots & p_2 & p_1 & p_0 & \cdots \\ & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ u_{-1} \\ u_0 \\ u_1 \\ \vdots \end{pmatrix}$$

Definition 4.4.

1. It is possible to *extend* an A-module L to an $A[t,t^{-1}]$-module $L[t,t^{-1}]$ by a tensor product;

$$L[t,t^{-1}] = L \otimes_A A[t,t^{-1}].$$

An element of $L[t,t^{-1}]$ is the finite polynomial $\lambda(t) = \sum_{i=D^-}^{D^+} \lambda_i t^i, \lambda_i \in L$. There is an inclusion map

$$\eta_L : L \to L[t,t^{-1}] \quad ; \quad l \mapsto \sum_{i=D^-}^{D^+} \lambda_i t^i \text{ where } \lambda_0 = l \text{ and } \lambda_i = 0, i \neq 0$$

2. Given a map of A-modules $g : L \to P$ there is an induced map $g_t : L[t,t^{-1}] \to P[t,t^{-1}]$ of $A[t,t^{-1}]$-modules commuting with the extension inclusion maps η_L, η_P.

$$\begin{tikzcd} L \arrow[r, "g"] \arrow[d, "\eta_L"'] & P \arrow[d, "\eta_P"] \\ L[t,t^{-1}] \arrow[r, "g_t"'] & P[t,t^{-1}] \end{tikzcd}$$

3. It is possible to extend a chain complex, by extending its modules. The differentials of the new complex are induced from the old.

Definition 4.5. Let M be a based free (right)-$A[t,t^{-1}]$-module.

$$M = \sum_{j=1}^{N} m_j A[t,t^{-1}]$$

A submodule Q of $M^!$ is *interval* if,

$$Q = \sum_{j=1}^{N} \sum_{k=q_j^-}^{q_j^+} m_j t^k A$$

that is, for each basis element m_j of M and some interval $q_j^- \leqslant 0 \leqslant q_j^+$, the basis of Q contains only $m_j t^k$ for each integer $q_j^- \leqslant k \leqslant q_j^+$. Neither q_j^- or q_j^+ need be finite. Likewise, if $\mathcal{C}$ is a finitely generated based chain complex over $A[t,t^{-1}]$, then a subcomplex $\mathcal{B}$ of $\mathcal{C}^!$ is said to be interval if every module $\mathcal{B}_i$ is an interval submodule of $\mathcal{C}_i^!$ for some range $b_{ij}^- \leqslant 0 \leqslant b_{ij}^+$.

Example 4.5.1. Consider $A[t,t^{-1}]$ as a module. A is the smallest interval submodule of $A[t,t^{-1}]$. The submodule $t^{-1}A + A + tA$ is also interval.

Lemma 4.6. Let M be a based free $A[t,t^{-1}]$-module. If P, Q are interval submodules of $M^!$, then $P+Q$ and $P \cap Q$ are an interval submodule of $M^!$ also.

Proof. Let $P+Q$ be the submodule

$$\begin{aligned} P+Q &= \sum_{j=1}^{N} \sum_{k=p_j^-}^{p_j^+} m_j t^k A + \sum_{j=1}^{N} \sum_{k=q_j^-}^{q_j^+} m_j t^k A \\ &= \sum_{j=1}^{N} \sum_{k=\min(p_j^-,q_j^-)}^{\max(p_j^+,q_j^-)} m_j t^k A \end{aligned}$$

which is interval. Let $P \cap Q$ be the submodule

$$P \cap Q = \sum_{j=1}^{N} \sum_{k=p_j^-}^{p_j^+} m_j t^k A \cap \sum_{j=1}^{N} \sum_{k=q_j^-}^{q_j^+} m_j t^k A$$
$$= \sum_{j=1}^{N} \sum_{k=\max(p_j^-, q_j^-)}^{\min(p_j^+, q_j^-)} m_j t^k A$$

which is interval. ◻

Lemma 4.7. Let $h : M \to N$ be a morphism of based $A[t, t^{-1}]$-modules, and P be an interval submodule of $M^!$. Then there is an minimal interval submodule Q of $N^!$ such that $h^!(P)$ is contained in Q.

Proof. Let h be a finite polynomial with matrix coefficients:

$$h_{i,j} = \sum_{\ell=D^-}^{D^+} h_\ell t^\ell$$

Take P to be the interval submodule

$$P = \sum_{j=1}^{N} \sum_{k=p_j^-}^{p_j^+} m_j t^k A$$

Let $s \leqslant \max(0, p_j^+ + D^+)$ and $r \geqslant \max(0, p_j^- + D^-)$. $h^!(P)$ is contained in any submodule:

$$\sum_{j=1}^{N} \sum_{k=s}^{r} n_j t^k A$$

which is obviously interval. Let Q be the submodule defined with the limiting values of s, r. Then $h^!(P) \subseteq Q$ and Q is minimal as required. ◻

Example 4.7.1. Consider a one dimensional chain complex $\mathcal{U}$, as in example 4.3.1, with differential $d_1 = p(t) = 1 - t^2$ Then A is an interval submodule of $\mathcal{U}_1^!$, and its image in $\mathcal{U}_0^!$ under the differential $d_1^!$ is $A + t^2 A$.

$A + tA + t^2 A$ is the minimal interval submodule of $\mathcal{U}_0^!$, which contains $d_1^!(A)$.

Definition 4.8. A *fundamental domain* of a finitely generated based chain complex $\mathcal{C}$ over $A[t, t^{-1}]$ is an interval subcomplex $\mathcal{K}$ of $\mathcal{C}^!$ over A.

Theorem 4.9. *Every finitely generated free based chain complex over $A[t, t^{-1}]$ has a finitely generated fundamental domain.*

Proof. Let $\mathcal{C}$ be a n-dimensional finitely generated free based chain complex over $A[t,t^{-1}]$, with

$$\mathcal{C}_i = \sum_{j=1}^{N_i} c_{ij} A[t,t^{-1}]$$

and differentials $d_i^{\mathcal{C}} : \mathcal{C}_i \to \mathcal{C}_{i-1}$.

Construct $\mathcal{K}$ to be a fundamental domain. Let $\mathcal{K}_n$ be

$$\mathcal{K}_n = \sum_{j=1}^{N_n} c_{nj} A \subset \mathcal{C}_n^!$$

Let $\mathcal{K}_{i-1}$ be a minimal interval submodule of the image of $\mathcal{K}_i$, under the differential of $\mathcal{C}^!$, defining successively for $n \geqslant i \geqslant 1$.

Let $d_i^{\mathcal{K}}$ be the restriction of $d_i^{\mathcal{C}^!}$ to $\mathcal{K}$.

Hence, $\mathcal{K}$ is a finitely generated free based chain complex over A, and is a interval sub-complex of $\mathcal{C}^!$. $\square$

Now suppose L to be a finitely generated free based (right)-A-module. Let P be an interval based submodule of $L[t,t^{-1}]^!$. Let $P \cap Pt^{-1}$ be defined in terms of its generators in the obvious way. Then it is possible to define two maps $f, g : P \cap Pt^{-1} \to P$ by:

$$f : x \mapsto xt$$
$$g : x \mapsto x$$

each commuting with the identification map.

Claim 4.10.

$$0 \to (P \cap Pt^{-1})[z,z^{-1}] \xrightarrow{f-gz} P[z,z^{-1}] \xrightarrow{h} L[z,z^{-1}] \to 0$$

where h sets $t = z$, is an exact sequence of $A[z,z^{-1}]$-modules.

Proof. Since L is free we need only prove this for a one generator case, all other cases being direct sums of the one generator case. The simplest proof is by induction on n^-, n^+ of the interval. We can write $P = \bigoplus_{n^-}^{n^+} A$ and $P \cap Pt^{-1} = \bigoplus_{n^-}^{n^+-1} A$, and $L = A$. In the case $n^+ = n^- = 0$, we find $P = A$, and $P \cap Pt^{-1} = 0$. It is straightforward to see that the sequence:

$$0 \to 0 \xrightarrow{f-gz} A[z,z^{-1}] \xrightarrow{h} A[z,z^{-1}] \to 0$$

is exact.

Now suppose the result is true for $0, n^+$ and construct a commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_0^{n^+-1} A[z,z^{-1}] \arrow[r, "f-gz"] \arrow[d] & \bigoplus_0^{n^+} A[z,z^{-1}] \arrow[r, "h"] \arrow[d] & A[z,z^{-1}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_0^{n^+} A[z,z^{-1}] \arrow[r, "f-gz"] \arrow[d] & \bigoplus_0^{n^++1} A[z,z^{-1}] \arrow[r, "h"] \arrow[d] & A[z,z^{-1}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A[z,z^{-1}] \arrow[r, "-z"] \arrow[d] & A[z,z^{-1}] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns of the diagram are exact and split since the middle row is the direct sum of the entries in the top and bottom rows. The top row is exact by inductive hypothesis. The last row is an isomorphism. The bottom left square commutes because the vertical maps are projections of the last factor, and $f - gz$ takes $1 \mapsto -z$ in the last factor. To prove that the middle row is exact, it must be shown that $f - gz$ is injective, h is surjective and $\ker(h) = \mathrm{Im}(f - gz)$.

Let x, y, a be elements of the modules in the first column, in the top, middle and bottom rows respectively. Further, choose x, a so that $y = x \oplus a$, and so then $(f - gz)(y) = (f - gz)(x) \oplus -z(a)$. Let x', y', a' be other elements of the same modules, with $y' = x' \oplus a'$, and suppose $(f - gz)(y) = (f - gz)(y')$. Hence

$$(f - gz)(x) \oplus -z(a) = (f - gz)(x') \oplus -z(a')$$

But, since the top and bottom rows are exact,

$$\begin{aligned} & (f - gz)(x) = (f - gz)(x') && \implies x = x' \\ \text{and} \quad & -z(a) = -z(a') && \implies a = a' \end{aligned}$$

and so

$$x \oplus a = x' \oplus a' \qquad \implies y = y'$$

So $f - gz$ is injective.

Let a_1, a_2 be elements of the modules in the third column, in the first two rows respectively. Let y_1 be an element in the module in the middle of the top row, with $h(y_1) = a_1$. Let a_3 be any element of the module in the middle of the bottom row. Hence there are elements of the middle-middle module, $y_1 \oplus a_3$, which are mapped by h to a_2. Hence h is onto.

Let w, v, b be elements of the modules in the middle column, in the top, middle and bottom rows respectively, such that $v = w \oplus b$. Suppose x, y, a are elements of the modules in the first column, such that $w = (f - gz)(x)$, $v = (f - gz)(y)$ and $b = -z(a)$. Then since the tops row is exact and the right column is an isomorphism, $h(w) = 0$ and $h(v) = h(w) = 0$. Suppose instead that $h(v) = 0$. Then $h(w) = 0$, and so there is an element x such that $w = (f - gz)(x)$. Then take any element of the bottom-left module a, and this gives an element $x \oplus a$ such that $(f - gz)(x \oplus a) = v$. Hence $\ker(h) = \operatorname{Im}(f - gz)$.

Finally the proof for the interval n^-, n^+ is symmetric to the one given. $\square$

Corollary 4.11. *Let $\mathcal{C}(t)$ be a finitely generated free chain complex over $A[t, t^{-1}]$ and let $\mathcal{K}$ be an algebraic fundamental domain of $\mathcal{C}(t)$. Define two maps f, g : $\mathcal{K} \cap \mathcal{K}t^{-1} \to K$ by:*

$$f_i : x \mapsto xt$$

$$g_i : x \mapsto x$$

each commuting with the identification map and the differentials. Then

$$0 \to (\mathcal{K} \cap \mathcal{K}t^{-1})[z, z^{-1}] \xrightarrow{f-gz} \mathcal{K}[z, z^{-1}] \xrightarrow{h} \mathcal{C}(z) \to 0$$

is exact.

Proof. The previous result directly gives the result for the component modules. The only complications may arise from the differentials of $\mathcal{K} \cap \mathcal{K}t^{-1}$, $\mathcal{K}$ and $\mathcal{C}(t)$. The map $f - gz$ obviously commutes with the differentials by its definition. The map h is the map which sets $t = z$ in $\mathcal{K}[z, z^{-1}]$ which also commutes with the differentials. $\square$

So there is a neat connection between a chain complex $\mathcal{C}(t)$ over $A[t, t^{-1}]$ and a fundamental domain $\mathcal{K}$. At a slight stretch of the imagination, this is reminiscent of the connection between a covering and a geometric fundamental domain, thinking of $f - gz$ as some kind of gluing. In order to explore this a bit further, some notion of gluing chain complexes together is needed, for use in the next chapter.

Definition 4.12. Let $\mathcal{M}, \mathcal{N}$ be chain complexes over A and let $f : \mathcal{M} \to \mathcal{N}$ be a chain map, that is a sequence of ring morphisms $f_0, \ldots$ where $f_i : \mathcal{M}_i \to \mathcal{N}_i$. The *algebraic mapping cone* of f is the chain complex $\mathcal{C}(f : \mathcal{M} \to \mathcal{N})$, where:

$$\mathcal{C}(f)_i = \mathcal{M}_{i-1} \oplus \mathcal{N}_i$$

and

$$d_i^{\mathcal{C}(f)} = \begin{pmatrix} d_{i-1}^{\mathcal{M}} & f \\ 0 & -d_i^{\mathcal{N}} \end{pmatrix}$$

Algebraic mapping cones have much the same properties as geometric mapping cones. Many of the standard results can be found in Spanier [18], chapter 4.2.

Particularly important to Chapter 7 is:

Theorem 4.13. *Let*

$$0 \to \mathcal{A} \xrightarrow{f} \mathcal{B} \to \mathcal{C} \to 0$$

be a short exact sequence of chain complexes. The homology of $\mathcal{C}$ is the homology of the mapping cone of $f : \mathcal{A} \to \mathcal{B}$.

Proof. Consider the split short exact sequence of chain complexes for the mapping cone:

$$0 \to \mathcal{B} \to \mathcal{C}(f) \to \mathcal{A}^{-1} \to 0$$

where $\mathcal{A}^{-1}$ is the chain complex with $\mathcal{A}_i^{-1} = \mathcal{A}_{i-1}$. Now the two long exact sequences in homology:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_i(A) & \longrightarrow & H_i(B) & \longrightarrow & H_i(C) & \longrightarrow & H_{i-1}(A) & \longrightarrow & H_{i-1}(B) & \longrightarrow & \cdots \\ & & \| & & \| & & \downarrow & & \| & & \| & & \\ \cdots & \longrightarrow & H_i(A) & \longrightarrow & H_i(B) & \longrightarrow & H_i(\mathcal{C}(f)) & \longrightarrow & H_{i-1}(A) & \longrightarrow & H_{i-1}(B) & \longrightarrow & \cdots \end{array}$$

and apply the five lemma. $\square$

Chapter 5

Homotopy Theoretic Fundamental Domains

This chapter discusses the possibility of finding a fundamental domain for the infinite cyclic cover of a CW-complex.

CW-complexes were first defined by J.H.C. Whitehead in [26]. A standard reference may be found in G.W. Whitehead [25], whence came much of the detail below. There is also a quick introduction to CW-complexes in Hirsch [11]. Brown [4] gives a clear explanation of adjunction spaces and the point-set properties used (and abused) in this chapter.

It will be shown that it is possible to define fundamental domains for infinite cyclic covers of CW-complexes, but that a fundamental domain does not exist in many cases (example 5.6.2). However, it will also be shown that all CW-complexes are homotopy equivalent to a CW-complex with an infinite cyclic cover which has a fundamental domain.

Definition 5.1. A topological space B is said to be an *i-cellular extension* of a (compactly generated) subspace $A \subseteq B$ if there is a set of i-discs indexed by a set J, and a gluing map $j : \bigsqcup_J \partial D^i \to A$ such that the natural map $A \cup_j (\bigsqcup_J D^i) \to B$ is a homeomorphism. The image of the discs in B are known as i-cells, denoted e^i.

Definition 5.2. A *CW-complex* X is topological space made by gluing together discs, known as cells. We think of X as a sequence of subspaces

$$X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \ldots \subseteq X$$

where the *i-skeleton*, $X^{(i)}$, is an i-cellular extension of $X^{(i-1)}$ by a gluing map j_i on a set of cells (discs) indexed by J_i:

$$X^{(i)} = X^{(i-1)} \cup_{j_i} \left(\bigcup_{J_i} e^i \right)$$

The 0-skeleton is just the set of distinct points indexed by $J_0 \neq \emptyset$.

It is possible to write X as

$$\begin{aligned} X &= \bigcup_{i=0}^{\infty} X^{(i)} \\ &= \left(\bigcup_{J_0} e^0\right) \cup_{j_1} \left(\bigcup_{J_1} e^1\right) \cup_{j_2} \left(\bigcup_{J_2} e^2\right) \cup \dots \end{aligned}$$

The topology on CW-complexes is described at length in [26] and briefly on page 50 of [25]

The dimension of X is the smallest n such that $X = X^{(n)}$. A CW-complex is *finite* if the total number of cells is finite. A *subcomplex* $A \subseteq X$ is closed subspace of X which is a union of cells of X.

Example 5.2.1. Any simplicial complex is a CW-complex, with one i-cell for each i-simplex.

Example 5.2.2. Any differentiable manifold is triangulable (Cairns [5]) and hence is (homeomorphic to) a CW-complex.

Example 5.2.3. The simplest CW-complex homeomorphic to a circle is a 0-cell with both ends of a single a 1-cell attached to it. It is finite and of dimension 1. An n-sphere, in fact, is homeomorphic to a 0-cell with and n-cell attached to it by all its boundary, that is a one point compactification of an open n-disc.

Example 5.2.4. The real line is homeomorphic to an open 1-disc, but that is not a CW-complex. Hence the simplest CW-complex homeomorphic to the real line, $\mathbb{R}$, is the infinite complex made of $\mathbb{Z} \times e^0$ with $\mathbb{Z} \times e^1$ with a gluing map that attaches the ends of each 1-cell to adjacent 0-cells. In fact, this is the unique CW-complex (up to labelling the cells) homeomorphic to $\mathbb{R}$.

Definition 5.3. A *cell map* is a map between two CW-complexes $f : X \to Y$ which takes each $X^{(i)}$ into $Y^{(i)}$. f can be considered to be (in some sense) the limit of a sequence $f^{(0)}, f^{(1)}, \dots$ of maps between the i-skeletons of X and Y.

Example 5.3.1. The identity map from a space to itself is a cell map.

Example 5.3.2. The simplest CW-complex homeomorphic to $S^1 \times S^2$ is just the direct product of their simplest CW-complexes:

$$\begin{array}{ccccccc} e^0 & \cup_{j_1} & e^1 & \cup_{j_2} & e^2 & \cup_{j_3} & e^3 \\ \| & & \| & & \| & & \| \\ e^0 \times e^0 & \cup_{j_1} & e^1 \times e^0 & \cup_{j_2} & e^0 \times e^2 & \cup_{j_3} & e^1 \times e^2 \end{array}$$

with gluing maps induced from the product. The projection map $p_1 : S^1 \times S^2 \to S^1$ is a cell map since it takes the 0-cell and 2-cell onto the 0 cell and the 1-cell and 3-cell onto the 1-cell. Similarly the other projection map p_2 is also a cell map. In fact all projection maps are cell maps.

Example 5.3.3. Let $p : \widetilde{X} \to X$ be the universal covering of a CW-complex X. By the definition of the universal cover (Proposition 2.12) a triangulable space must have a triangulable universal cover, and the covering map respects that triangulation. Hence $\widetilde{X}$ is a CW-complex and p is a cell map. Pointwise, this is a projection of $\pi_1(X, x_0) \times X \to X$.

Definition 5.4. The *cellular chain complex* $C(X)$ of a CW-complex X is the based $\mathbb{Z}$-module chain complex (Definition 4.1) with

$$C(X)_i = H_i(X^{(i)}, X^{(i-1)}) = \mathbb{Z}^{J_i}$$

and differentials induced from the boundary map.

$$H_i(X^{(i)}, X^{(i-1)}) \xrightarrow{\partial} H_i(X^{(i-1)}, X^{(i-2)})$$

It is assumed that $C(X)$ is over $\mathbb{Z}$ unless otherwise stated. Thus each $C(X)_i$ is a free $\mathbb{Z}$-module generated by J_i, labelling the i-cells of X.

If $X = \widetilde{Y}$ is a covering of a CW-complex Y, with group of covering transformations G, then $C(X)$ has the extra structure of a based $\mathbb{Z}[G]$-module chain complex, with one basis element for each cell of Y.

Notice that the cellular chain complex of the base space can easily be recovered from the cellular chain complex of the cover since:

$$C(Y) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} C(\widetilde{Y})$$

Given a cell map of CW-complexes $h : X \to Y$, there is an induced *cellular chain map* $h_* : C(X) \to C(Y)$ defined such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "h"] & C(X) \arrow[d, "h_*"] \\ Y \arrow[r] & C(Y) \end{tikzcd}$$

Example 5.4.1. Let $p : \overline{X} \to X$ be an infinite cyclic cover of a space, and let the generator of the group of covering transformations be $t : \overline{X} \to \overline{X}$. Then $C(\overline{X})$ is a cellular chain complex over the Laurent polynomials $\Lambda = \mathbb{Z}[t, t^{-1}]$.

Notice that the $t_* : C(\overline{X}) \to C(\overline{X})$ is just multiplication by t, so the lower star is often omitted. Also notice $p_* : C(\overline{X}) \to C(X)$ is the map setting $t = 1$.

Example 5.4.2. The most obvious infinite cyclic covering is $p : \mathbb{R} \to S^1$. Then the cellular chain complex, $C(S^1)$ is

$$C(S^1)_1 = \mathbb{Z} \xrightarrow{0} C(S^1)_0 = \mathbb{Z}$$

By labelling the CW-complex of $\mathbb{R}$ by the group of covering transformations, $\mathbb{Z}$ written multiplicatively as $\langle t \rangle$, we can write the cellular chain complex of $\mathbb{R}$ as:

$$C(\mathbb{R})_1 = \Lambda = \mathbb{Z}[t, t^{-1}] \xrightarrow{1-t} C(\mathbb{R})_0 = \Lambda = \mathbb{Z}[t, t^{-1}]$$

The standard (geometric) fundamental domain for this covering is the 0-cells labeled 1 and t and the 1-cell joining them. It is a subcomplex, and its cellular chain complex is a fundamental domain for $C(\mathbb{R})$.

Example 5.4.3. Let V be a CW-complex, with a cell in each of the first three dimensions, $0, 1, 2$, where the 1-cell is glued to the 0-skeleton to make a circle, and then the 2-cell is glued to the 1-skeleton onto the 0-cell to make a one point union of a circle and a 2-sphere, $S^1 \vee S^2$ (figure 5.1).

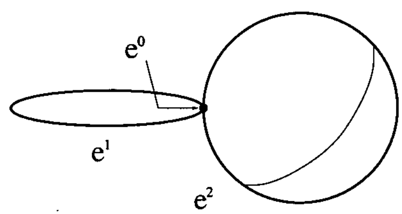

Figure 5.1: $V = S^1 \vee S^2$

The cellular chain complex of V is

$$C(V)_2 = \mathbb{Z} \xrightarrow{0} C(V)_1 = \mathbb{Z} \xrightarrow{0} C(V)_0 = \mathbb{Z}$$

By Van Kampen's theorem, the fundamental group of a one-point union is the free product of the fundamental groups of its components, and so $\pi_1(V, 0) = \pi_1(S^1) \star \pi_1(S^2) = \mathbb{Z}$. Hence the universal cover of V is infinite cyclic, a "string of balloons" (figure 5.2).

The cellular chain complex of $\overline{V}$ is

$$C(\overline{V})_2 = \Lambda \xrightarrow{0} C(\overline{V})_1 = \Lambda \xrightarrow{1-t} C(\overline{V})_0 = \Lambda$$

Notice that $\overline{V}$ has an obvious (geometric) fundamental domain which is also a subcomplex, (figure 5.3), and the cellular chain complex of which is an (algebraic) fundamental domain in $C(\overline{V})$.

It is obvious that there is an inclusion $i : \mathbb{R} \hookrightarrow V$. It is equally obvious that the cellular chain inclusion i_* is given by the identity on dimensions 0 and 1, and the zero map in all other dimensions.

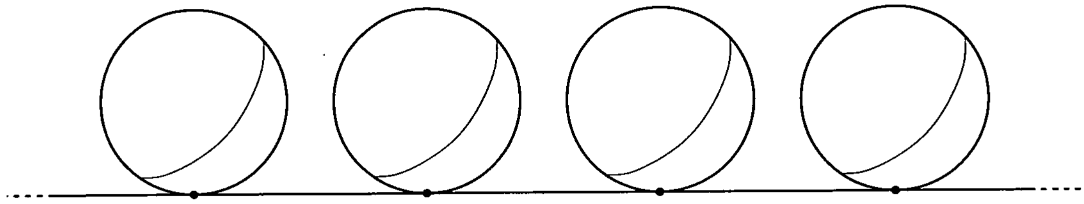

Figure 5.2: $\overline{V} = \mathbb{R} \cup_{\mathbb{Z}} (\mathbb{Z} \times S^2)$

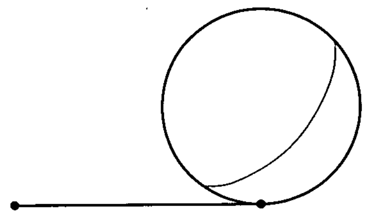

Figure 5.3: A (geometric) fundamental domain of $\overline{V}$, which is also a subcomplex.

Example 5.4.4. The next example extends the previous, demonstrating just how many CW-complexes there can be with very few cells.

Let W be a CW-complex, containing a cell in each of the first 4 dimensions, $0, 1, 2, 3$, and glued together as

$$W = (S^1 \vee S^2) \cup_{p(t)} e^3$$

where $p(t)$ is a gluing map. By Hurewicz Theorem, $p(t)$ must be a Laurent polynomial, because it is a map $p(t) : \partial e^3 = S^2 \to S^1 \vee S^2 = V$, so $p(t) \in \pi_2(V, 0) = H_2(\overline{V}) = \Lambda = \mathbb{Z}[t, t^{-1}]$ where t is the generator of the group of covering translations of the universal cover of $S^1 \vee S^2$.

The cellular chain complex of W is

$$C(W)_3 = \mathbb{Z} \xrightarrow{p(1)} C(W)_2 = \mathbb{Z} \xrightarrow{0} C(W)_1 = \mathbb{Z} \xrightarrow{0} C(W)_0 = \mathbb{Z}$$

By Van Kampen's Theorem, the addition of a 3-cell by the whole of it's boundary gives $\pi_1(W, 0) = \pi_1(S^1 \vee S^2) \star_{p(t)} \pi_1(e^3) = \mathbb{Z}$, Hence the universal cover of W is infinite cyclic and can be constructed as a pullback. There is a map $f : W \to S^1$, given by the identity on the 0-cell and 1-cell, by collapsing the 2-cell to the 0-cell, and collapsing the 3-cell to a line wrapped around the whole of S^1 by the map with the same degree as $p(t)$. Hence there is a pullback covering:

$$\begin{tikzcd} \overline{W} = f^*\mathbb{R} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ W \arrow[r, "f"] & S^1 \end{tikzcd}$$

The cellular chain complex of $\overline{W}$ is

$$C(\overline{W})_3 = \Lambda \xrightarrow{p(t)} C(\overline{W})_2 = \Lambda \xrightarrow{0} C(\overline{W})_1 = \Lambda \xrightarrow{1-t} C(\overline{W})_0 = \Lambda$$

What can be said about fundamental domains for W? The most important thing to do is to construct a definition of a fundamental domain which is consistent with the geometric fundamental domain of the space and the algebraic fundamental domain of the cellular chain complex.

Claim 5.5. *Let X be a finite CW-complex and $\overline{X}$ be a CW-complex, such that $p : \overline{X} \to X$ is an infinite cyclic covering. Moreover, suppose $U \subset \overline{X}$ is a subcomplex, and that U is a connected geometric fundamental domain of $\overline{X}$. Then, $C(U) \subset C(\overline{X})$ is an algebraic fundamental domain for the cellular chain complex.*

Proof. Definition 4.8 requires $C(\overline{X})$ to be finitely generated over $\mathbb{Z}[t, t^{-1}]$, which it is since $C(X)$ is a finitely generated complex over $\mathbb{Z}$.

$C(U)$ is a subcomplex of $C(\overline{X})$ because U is a subcomplex of $\overline{X}$.

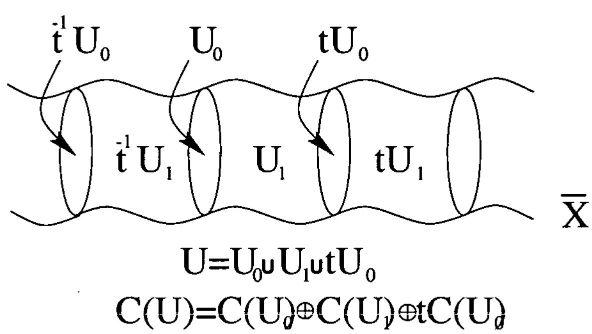

Figure 5.4: $C(U)$ is interval because U is a geometric fundamental domain

Let $U_0 = t^{-1}U \cap U$, so the boundary of U is $\partial U = U_0 \cup tU_0$. Call the interior of U, $U_1 = U \setminus \partial U$. $C(U)$ is finitely generated over $\mathbb{Z}$, because U is a geometric fundamental domain, hence $U_0 \cup U_1$ contains one point for each fibre of $\overline{X}$. The number of cells in $U_1 \cup U_0$ is the number of cells in X. So the number of cells in U is the number in X plus the number in U_0, which can be at most the same number as in X. Further, $C(U)$ is interval, as each generator of $C(\overline{X})$ over $\mathbb{Z}[t, t^{-1}]$ corresponds one generator in $C(X)$, so U_1 gives one generator in $C(U)$ per generator of $C(\overline{X})$ while ∂U gives two generators in $C(U)$ per generator of $C(\overline{X})$, where one is mapped to the other by t. Figure 5.4 shows subcomplexes U_0, U_1 pictorially in $\overline{X}$. Since each of these is finitely generated, as they are both subcomplexes of a finitely generated complex X, U is finitely generated also. Also $C(U)$ is interval. $\square$

Definition 5.6. A *CW-fundamental domain* of an infinite cyclic covering of a CW-complex $p : \overline{X} \to X$ is a subcomplex $U \subset \overline{X}$ which is a geometric fundamental domain of $\overline{X}$.

Example 5.6.1. Example 5.4.2, of the covering of the circle by the real line has a CW-fundamental domain, U:

$$C(U)_1 = \mathbb{Z} \xrightarrow{1-t} C(U)_0 = \mathbb{Z} \oplus t\mathbb{Z}$$

Example 5.6.2. In example 5.4.3, there is already a CW-fundamental domain for V. Is there one for W?

Since $\overline{W}$ is defined by a pullback of a map $f : W \to S^1$, the easiest (geometric) way to find a fundamental domain would be to pull back $[0,1]$, the fundamental domain for the covering of the circle by the real line.

If d_p, the degree of the polynomial $p(t)$, is more than 1, then it is clear that $f^*[0,1]$ is not a subcomplex of $\overline{W}$ since the 3-cells are glued to d_p+1 2-cells in $\overline{W}$, and to d_p 1-cells, and so the pull back contains in total one 3-cell, but split into d_p sections, each attached to the 2-cell pulled back from 0 and the 2-cell pulled back from 1.

Claim 5.7. *The Infinite cyclic cover of any mapping torus has a CW-fundamental domain.*

Proof. Recall the definition of a mapping torus (Construction 1.9). Let X be a CW-complex, and $h : X \to X$ a self-cell-map.

Then $T(h)$ is a CW-complex, the adjunction space $X \times I \cup_{1,h} X \times I$ which glues $X \times \{0\}$ to $X \times \{1\}$ by the identity on one end and $X \times \{1\}$ to $X \times \{0\}$ by the map h at the other (figure 5.5). There is an obvious homeomorphism to the standard (geometric) mapping torus.

The cellular chain complex of $T(h)$ is:

$$C\left(T(h)\right)_i = C(X)_i \oplus C(X)_{(i-1)} \oplus C(X)_i \oplus C(X)_{(i-1)}$$

with differential

$$d_i^{T(h)} = \begin{pmatrix} d_i^X & 1 & 0 & -1 \\ 0 & -d_{(i-1)}^X & 0 & 0 \\ 0 & -h_* & d_i^X & 1 \\ 0 & 0 & 0 & -d_{(i-1)}^X \end{pmatrix}$$

where $h_* : C(X) \to C(X)$ is the cellular chain map of h.

There is a map $f : T(h) \to S^1$ by suppression of the X coordinate, and this map gives an infinite cyclic cover,

$$\overline{T(h)} = X \times I \times \langle t \rangle \cup_{t^{-1},h} X \times I \times \langle t \rangle,$$

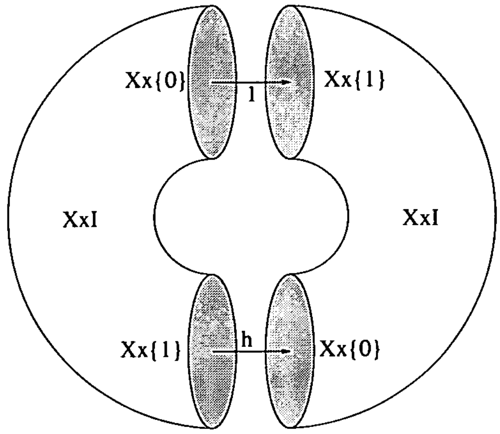

Figure 5.5: Symbolic diagram of CW-mapping torus T(h)

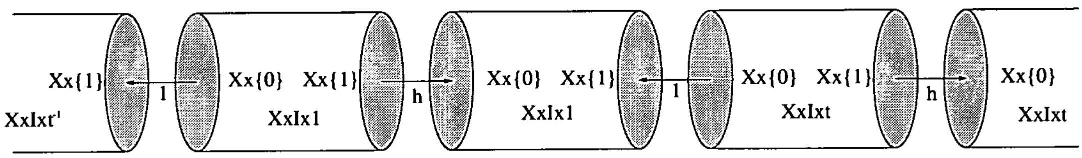

Figure 5.6: Symbolic diagram of $\overline{T(h)}$

with group of covering translations $\mathbb{Z} = \langle t \rangle$ (figure 5.6).

The cellular chain complex of $\overline{T(h)}$ is:

$$C\left(\overline{T(h)}\right)_i = C(X)_i[t,t^{-1}] \oplus C(X)_{(i-1)}[t,t^{-1}] \oplus C(X)_i[t,t^{-1}] \oplus C(X)_{(i-1)}[t,t^{-1}]$$

with differential

$$d_i^{\overline{T(h)}} = \begin{pmatrix} d_i^X & 1 & 0 & -t \\ 0 & -d_{(i-1)}^X & 0 & 0 \\ 0 & -h_* & d_i^X & 1 \\ 0 & 0 & 0 & -d_{(i-1)}^X \end{pmatrix}$$

There is a subcomplex geometric fundamental domain U for $\overline{T(h)}$, made from two sections of $X \times I$ joined together by the identity. Its cellular chain complex is:

$$C(U)_i = C(X)_i \oplus C(X)_{(i-1)} \oplus C(X)_i t \oplus C(X)_{(i-1)} t \oplus C(X)_i t^2$$

with differential

$$d_i^U = \begin{pmatrix} d_i^X & 1 & 0 & 0 & 0 \\ 0 & -d_{(i-1)}^X & 0 & 0 & 0 \\ 0 & -1 & d_i^X & 1 & 0 \\ 0 & 0 & 0 & -d_{(i-1)}^X & 0 \\ 0 & 0 & 0 & -h & d_i^X \end{pmatrix}$$

This is a subcomplex of $C\left(\overline{T(h)}\right)$ and trivially interval, hence it is an algebraic fundamental domain.

Notice that the differential of $C\left(\overline{T(h)}\right)$ is linear (that is, the difference between the highest and lowest powers of t in the matrix is 1) and that setting $t = 1$ regains $C(T(h))$. □

Claim 5.8. *Any infinite cyclic cover of any triangulable, orientable manifold has a CW-fundamental domain.*

Proof. Let M be an n-dimensional orientable manifold, hence there is an orientable n-dimensional CW-complex homeomorphic to M. Let $f : M \to S^1$ be a smooth map of manifolds.

By the arguments in Chapter 3, M has an infinite cyclic cover, $\overline{M}$ from the pullback of f, and by transversality, we can find a regular value ξ of f and hence an $n-1$-dimensional submanifold $N = f^{-1}(\xi)$ transverse to f. Again, write $M_N = f^*[0, 1]$ for the geometric fundamental domain of the covering $\overline{M} = f^*\mathbb{R}$. From the diagram, it is easy to see that $N = t^{-1}M_N \cap M_N$ and that if there were

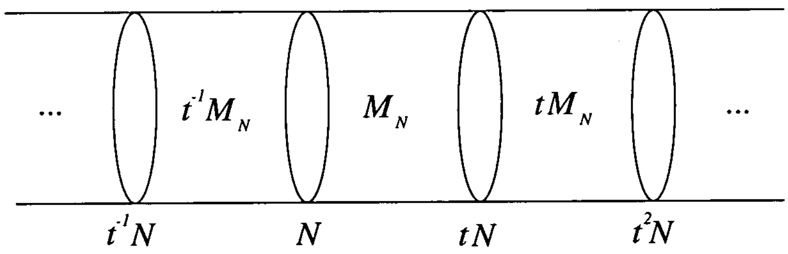

Figure 5.7: Symbolic diagram of $\overline{M}$ divided into fundamental domains

cellular chain complexes for N and M_N then the cellular chain complex for M would be regained by setting $t = 1$.

In fact since f is smooth, N is differentiable, and so Example 5.2.2 tells us that it is a CW complex. Likewise M_N is a triangulable manifold with triangulable boundary, and so it is a CW-complex. Hence, by using this CW structure, M has an infinite cyclic cover with a CW-fundamental domain. □

Claim 5.9. *The cellular chain complex of $\overline{M}$ has differential polynomial of degree at most* 1, *in all dimensions.*

Proof. Since $N = t^{-1}M_N \cap M_N$, one way to see this is to follow the example of the mapping torus above, and construct a space homeomorphic to M as:

$$M = M_N \cup_{j^+, j^-} N \times I$$

where $j_+, j_- : N \hookrightarrow M_N$ are the two cell-map inclusions. Then the cellular chain complex of M can be written as $C(M)_i = C(M_N)_i \oplus C(N)_{i-1}$ with differential

$$d_i^M = \begin{pmatrix} d_i^{M_N} & 0 \\ j_*^+ - j_*^- & -d_{i-1}^N \end{pmatrix}$$

where j_*^+, j_*^- are the cellular chain inclusions of $C(N)$ into $C(M_N)$.

Again by analogy to the mapping torus case, the infinite cyclic cover $\overline{M}$ is homeomorphic to the infinite cyclic cover of the space defined above:

$$\overline{M} = M_N \times \langle t \rangle \cup_{j^+t, j^-} N \times I \times \langle t \rangle$$

The cellular chain complex of this infinite cyclic cover is:

$$C(\overline{M})_i = C(M_N)_i[t, t^{-1}] \oplus C(N)_{i-1}[t, t^{-1}]$$

with differential

$$d_i^{\overline{M}} = \begin{pmatrix} d_i^{M_N} & 0 \\ j_*^+ t - j_*^- & -d_{i-1}^N \end{pmatrix}$$

This differential is obviously of degree 1, unless j_*^+ is the zero map (that is N has less than two inclusions in M_N) in which case the degree is 0 (and $\overline{M} = \mathbb{Z} \times M$). □

So it more or less means something to say that (smooth) manifolds are "linear", and that there are more complicated CW-complexes which are "of higher degree", for example W for $p(t)$ non-linear. Those spaces with non-linear differential have no CW-fundamental domain, since example 5.6.2 shows how the geometric fundamental domain is splintered by the degree of the differential.

Theorem 5.10. *If X is a finite CW-complex with infinite cyclic cover $\overline{X}$, then it is homotopy equivalent to a finite CW-complex Y with a CW fundamental domain for its infinite cyclic cover $\overline{Y}$.*

Proof. Two proofs:

1. Every CW-complex can be embedded in a Euclidean Space with sufficiently large number of dimensions. A thickening of this embedding, the closure of a neighbourhood of X, is Y a manifold with boundary. By Claim 5.8, there is a CW-fundamental domain for the infinite cyclic cover of Y.

2. By analogy to 4.9.

□

Chapter 6

The Seifert Form in Knot Theory

This chapter discusses some attempts at classifying both classical knots and knots in higher dimensions. The most important algebraic invariant of a knot, the S-equivalence class of its Seifert Matrices, is described.

6.1 Knots and Knot groups

An n-knot is a simple closed curve S^n in S^{n+2}. It is usual to restrict knots to PL or smooth categories. Since spaces in both categories are triangulable ([5]), knots and knot complements will be taken to be simplicial complexes for this thesis. It is also usual for knots to be considered up to ambient isotopy, so an ambient isotopy class of knots is usually called a knot.

For instance, a sheet bend and a bowline are two well known knots, tied by boyscouts everywhere. They become mathematical knots only by fusing the ends of the rope together to make an embedding of $S^1 \hookrightarrow R^3 \subset S^3$. They both belong to the same ambient isotopy class, 6_2.

Definition 6.1. Two oriented knots can be combined by *knot addition.* Let $k_1, k_2 : S^n \to S^{n+2}$ be two knots, then the knot $k_1 \# k_2$ is made by removing the neighbourhood of a point on the knot from each S^{n+2}, and then gluing the two $n+2$-discs together along their S^{n+1} boundary, joining the two knots. There are two (homotopy classes of) possibilities for the gluing map

$$(S^{n-1}, S^{n+1}) \to (S^{n-1}, S^{n+1})$$

but if the knots have an orientation, then only one of the choices respects the orientation.

Addition of 1-knots can be visualised as taking the first knot, cutting the loop, and tying the second, then rejoining the ends.

It is obvious that it is a commutative operation, and that the sum of any two knots is a knot also. Knot addition allows no inverses, but the unknot, the plain loop, is a zero for the addition. Hence knots form a monoid under knot addition.

Example 6.1.1. Figure 6.1 shows the addition of two trefoil knots, $3_1\#3_1$.

a) Two trefoil knots. Each is oriented in the same way.

b) The removal of the neighbourhood of a point.

c) A homeomorphism of (b), aligning the two spherical neighbourhoods in S^3 so the gluing map is obvious.

d) The glued space.

e) An ambient isotopy of (d), just to clean up the result.

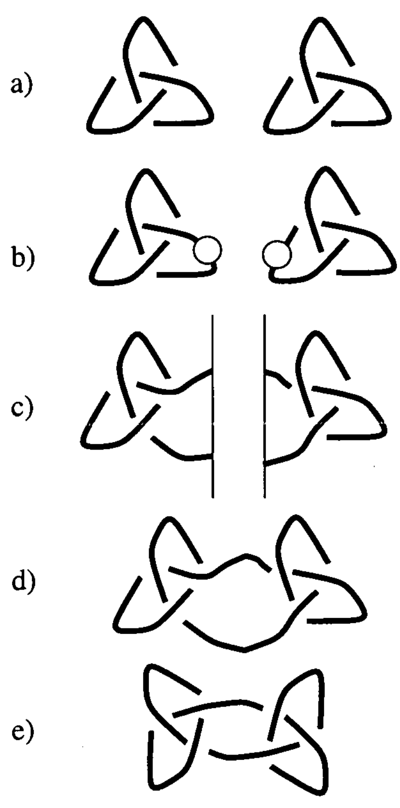

Figure 6.1: The addition $3_1\#3_1$

It was known to Tait [19] that many 1-knots could be broken down into sums of smaller knots, and he was the first to begin the classification of *prime knots.* He began by investigating the smallest number of crossings a knot could be displayed with, giving us the classification system used to this day - the bowline, 6_2 is the second knot found by Tait with six crossings.

The history of knot theory is well presented in Gordon's *Some Aspects of Classical Knot Theory* [9].

Another early attempt at classification of knots is to use the fundamental group of the knot complement, called the *knot group*. For 1-knots, there is an algorithm for finding a presentation of the knot group depending on a knot Diagram, attributed to Wirtinger.

Construction 6.2 (Wirtinger Presentation). Given an oriented diagram of a 1-knot, the group contains a generator for each arc of the diagram and a relation for each crossing. Suppose at a crossing the overpassing arc is the generator a, while the underpassing arcs are b and c. There are two ways a crossing can be oriented, giving different relations. These are shown in figure 6.2.

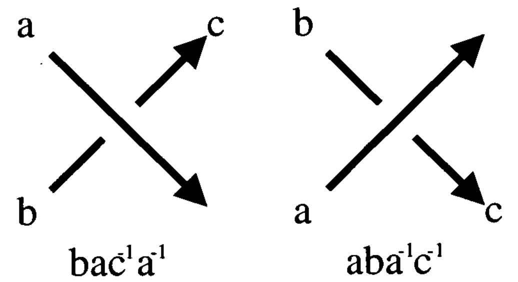

Figure 6.2: Two oriented crossings, and their relations

Example 6.2.1. A Wirtinger presentation is specific to a particular knot diagram. Figure 6.3 is a knot diagram for a trefoil knot, 3_1. Crossing 1 has arc a

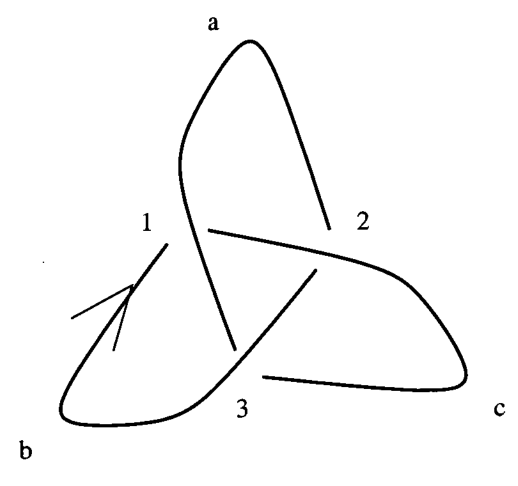

Figure 6.3: A trefoil knot, 3_1, with labeled arcs and crossings

for the overpass, and b and c for the underpass, so the relation for crossing 1 is $aba^{-1}c^{-1}$. So the Wirtinger presentation of the knot group for this diagram is:

$$\langle a, b, c | aba^{-1}c^{-1}, cac^{-1}b^{-1}, bcb^{-1}a^{-1} \rangle$$

A small amount of algebra shows that it is possible to express one of the three generators in terms of the other two, and hence reduce the relations to only one. To see that the knot group is in fact invariant between knot diagrams, it is easy to check that in fact this is the fundamental group of the space $S^3 \setminus 3_1$.

In fact the Wirtinger presentation is constructed from an application of the Seifert-Van Kampen Theorem to "top" and "bottom" half-spaces easily constructed from the knot diagram. However, even for 1-knots this is a patchy invariant at best. Trefoil knots can be left and right handed, but the knot group is the same for both. Worse, the knot groups of non-prime knots are independent of the orientation of the knot, so a granny knot and a reef (square) knot have the same knot group.

In the search for knot invariants however, we are not entirely lost. Seifert surfaces (3.9) are no good as an invariant, since there is one for each knot diagram, so there is more than one per knot, and worse, two knots can have the same Seifert surface. However the embedding of the Seifert surface in the space does lead to a useful (almost)-invariant.

6.2 The Seifert Manifold and Linking

Seifert invented an algebraic structure for classical knots, describing how the Seifert Surface is embedded in the knot complement. It turns out that there is an equivalence relation which makes this a good invariant for certain kinds of knot.

Let $k : S^n \to S^{n+2}$ be an n-knot. Construct a space M by removing a tubular neighbourhood of k from S^{n+2}. M is an $n+2$-manifold with boundary $S^1 \times k$.

Definition 6.3. A *Seifert Manifold* of a knot k is an orientable $n+1$-dimensional sub-manifold of M with boundary k.

To ensure a Seifert Manifold F exists, M must have the homology of a circle. Then transversality 3.8 of the homology map $M \to S^1$ can be used to find F. Also the map $M \to S^1$ can be used to construct $\overline{M}$ as a pullback cover.

Theorem 6.4. *M has the homology of a circle.*

Proof. Notice that

$$M \subset S^{n+2}$$

and hence there is an exact sequence of chain complexes (excision)

$$0 \to C(M) \to C(S^{n+2}) \to C(S^{n+2}, M) \to 0$$

Hence there is a long exact sequence in homology (over $\mathbb{Z}$)

$$\ldots \to H_i(M) \to H_i(S^{n+2}) \to H_i(S^{n+2}, M) \to \ldots$$

But,

$$H_i(S^{n+2}) = \begin{cases} \mathbb{Z} & i = 0, n+2 \\ 0 & \text{otherwise} \end{cases}$$

and also (by excision)

$$H_i(S^{n+2}, M) = H_i(S^n \times D^2, S^n \times S^1) = \begin{cases} \mathbb{Z} & i = 2, n+2 \\ 0 & \text{otherwise} \end{cases}$$

The map $H_i(S^{n+2}) \to H_i(S^{n+2}, M)$ is induced by the map which crushes M to a point. This map is the composite

$$S^{n+2} \xrightarrow{projection} \frac{S^n \times D^2}{S^n \times S^1} \simeq S^{n+2} \vee S^2$$

This composite is just the inclusion in the first summand.

Hence in intermediate dimensions

$$H_{i+1}(S^{n+2}, M) = 0 \to H_i(M) \to H_i(S^{n+2}) = 0$$

and so $H_i(M) = 0$ for $2 \leqslant i \leqslant n$.

For $i = n+2, n+1$ the long exact sequence becomes

$$0 \to H_{n+2}(M) \to H_{n+2}(S^{n+2}) \to H_{n+2}(S^{n+2}, M) \to H_{n+1}(M) \to H_{n+1}(S^{n+2}) = 0$$

but the composite map above shows that $H_{n+2}(S^{n+2}) \to H_{n+2}(S^{n+2}, M)$ is an isomorphism, and hence $H_{n+2}(M) = H_{n+1}(M) = 0$.

For $i = 0, 1$ the long exact sequence becomes

$$0 \to H_2(S^{n+2}, M) \to H_1(M) \to H_1(S^{n+2}) = 0 \to \ldots$$

So $H_2(S^{n+2}, M) \cong H_1(M)$, but $H_2(S^{n+2}, M) = \mathbb{Z}$, so $H_1(M) = \mathbb{Z}$.

$$\ldots \to 0 \to H_1(S^{n+2}, M) \to H_0(M) \to H_0(S^{n+2}) = \mathbb{Z} \to H_0(S^{n+2}, M) \to 0$$

$H_0(M) = \mathbb{Z}$ because M is connected.

Hence $H_*(M) = H_*(S^1)$.

The projection $\partial M = k(S^n) \times S^1 \to S^1$ extends to a map $f : M \to S^1$ since the obstruction to such an extension is an element of $H^2(M, \partial M) = 0$. Any such extension induces an isomorphism $f_* : H_*(M) \to H_*(S^1)$ (and a surjection $f_* : \pi_1(M) \to \mathbb{Z}$). $\square$

Hence for any knot, $k : S^n \to S^{n+2}$, there is an orientable $n+1$-submanifold $F \subset S^n + 2$ with $k = \partial F$. Furthermore there is an infinite cyclic cover of the knot complement M.

F has a simplicial structure induced from k and its complement. Construct a space M_F by cutting M along F, and including both boundary copies of F. M_F an $n+2$-manifold, with boundary $F \cup_+ (I \times k) \cup_- F$. Both M and M_F have simplicial structures. There are two inclusions $i^+, i^- : F \hookrightarrow M_F$ induced from this cutting. This gives a geometric construction of $\overline{M}$, the infinite cyclic cover

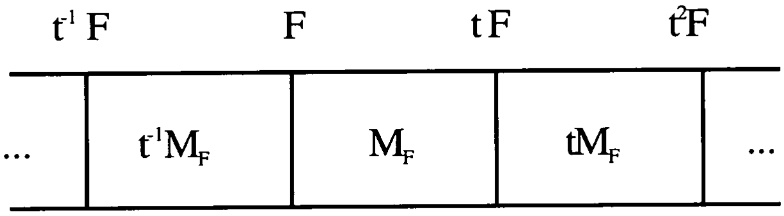

Figure 6.4: Infinite cyclic cover of a knot complement M

of M.

Lemma 6.5. *M_F is a geometric fundamental domain for $\overline{M}$.*

Proof. Since F was chosen by transversality to be a simplicial complex satisfying $f(F) = 0$ for some map $f : M \to S^1$, $\overline{M} = f^*\mathbb{R}$ the pullback covering, and hence $M_F = f^*[0,1]$ is a geometric fundamental domain. □

A class of knots which will be used to a large extent later are simple knots.

Definition 6.6. An odd dimensional knot $k^{2r-1} \subset S^{2r+1}$ which has an $(r-1)$-connected Seifert Manifold F, that is $H_i(F) = 0$ for $1 \leqslant i < r$ and $\pi_1(F) = 0$, is *simple.*

The definition of the Seifert Matrix requires the definition of "linking". Suppose there are two maps $x, y : S^1 \to S^3$. If y is not homologous to zero in $S^3 \setminus x$ then x and y are linked, in the sense that if the embeddings x and y were realised by rubber bands in space, then there would be no way to separate then - one would pass through the other.

Definition 6.7. The *linking number*, $lk(x, y)$, of x, y an oriented m-sphere and an oriented n-sphere disjointly embedded in S^{m+n+1} is the degree of the map

$$y : S^n \to (S^{m+n+1} \setminus x(S^m)) \simeq_H S^n$$

where $\simeq_H$ means homology equivalence. This is equivalent to the evaluation of y as an element of $H_n(S^{m+n+1} \setminus x(S^m)) = \mathbb{Z}$. Further definitions of linking number, and their equivalence can be found in Rolfsen, [16].

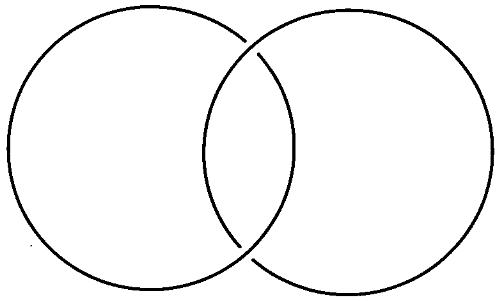

Figure 6.5: Linked circles

Lemma 6.8. 1. $lk(x, y) = (-1)^{mn-1} lk(y, x)$

2. $lk(x, -y) = -lk(x, y)$

3. $lk(x + y, z) = lk(x, z) + lk(y, z)$

Proof. See Rolfsen [16]. □

So far it has been enough to consider F as the boundary of M_F, and as a co-dimension-1 submanifold in M. The information about how F is embedded in M comes from allowing i_+ to move one of the representations of the homology loops slightly away from F, by a small amount δ. This δ-displacement is allowed since the oriented co-dimension-1 submanifold F lies in a regular neighbourhood $F \times [\delta, -\delta]$, by transversality [13], [17].

Since this has boundary

$$F \times \{\delta\} \cup k \times [\delta, -\delta] \cup F \times \{-\delta\}$$

its complement within M is (homeomorphic to) M_F. But $F \times [\delta, -\delta]$ is homeomorphic to $F \times I$, so it is possible (and fruitful) to consider $F = F \times \{\frac{1}{2}\}$, and then i^+, i^- are end points of homotopies which slide $F \times \{\frac{1}{2}\}$ to $F \times \{1\}, F \times \{0\}$ respectively.

6.3 The Seifert Matrix

For the remainder of this chapter, let k be an odd dimensional simple knot. Let $n = 2r - 1$.

Definition 6.9. Let $k : S^{2r-1} \to S^{2r+1}$ be a simple knot with a $2r$-dimensional Seifert manifold F. Since F is $r-1$-connected, $H_r(F) = \pi_r(F)$, and so every element in $H_r(F)$ is represented by an embedding $S^r \to F$. The *Seifert form* is defined on the integral homology of a Seifert Manifold $H_r(F; \mathbb{Z})$:

$$\sigma : H_r(F; \mathbb{Z}) \times H_r(F; \mathbb{Z}) \to \mathbb{Z} \quad ; \quad (x, y) \to lk(i^+x, y)$$

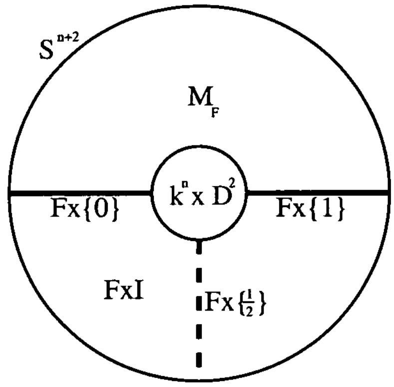

Figure 6.6: A symbolic picture of S^{n+2}

sending $([x],[y])$ to the linking number of the representative of y with the representative of i^+x in $F \times I \subset M \subset S^{2r+1}$.

Lemma 6.10.

$$lk(i^+x, y) = lk(x, i^-y)$$

Proof. There is an isotopy between the identity map $S^{n+2} \to S^{n+2}$ and a homeomorphism which locally takes

$$(F \setminus \partial F) \times [0, \frac{1}{2}] \to (F \setminus \partial F) \times [\frac{1}{2}, 1]$$

Hence the linking of (i^+x, y) and (x, i^-y) are the same. This is an application of the Isotopy extension theorem, which can be found in Hirsch ([11] Thm 1.3,p180). $\square$

The homology of the Seifert Manifold, $H_r(F)$ is a finitely generated free $\mathbb{Z}$-module.

Definition 6.11. A *Seifert Matrix V* is the matrix of the Seifert form of a Seifert Manifold of a knot with respect to some basis for $H_r(F;\mathbb{Z})$.

Example 6.11.1. The unknot is the obvious embedding of $S^{2r-1} \hookrightarrow S^{2r+1}$, so that it bounds a $2r$-disc, which is its most obvious Seifert manifold F_0. $H_r(F_0) = 0$, so the only possible Seifert form is the zero form. This is not very exciting.

Example 6.11.2. The trefoil knot 3_1 is the simplest classical knot, which is actually knotted. Using the Seifert Algorithm (Construction 3.9) a Seifert Surface F_{3_1} can be found which is homeomorphic to the punctured torus. One picture of this Seifert surface can be found in figure 3.3, but to see the homology generators, and work out the Seifert matrix, the picture in figure 6.7 is clearer.

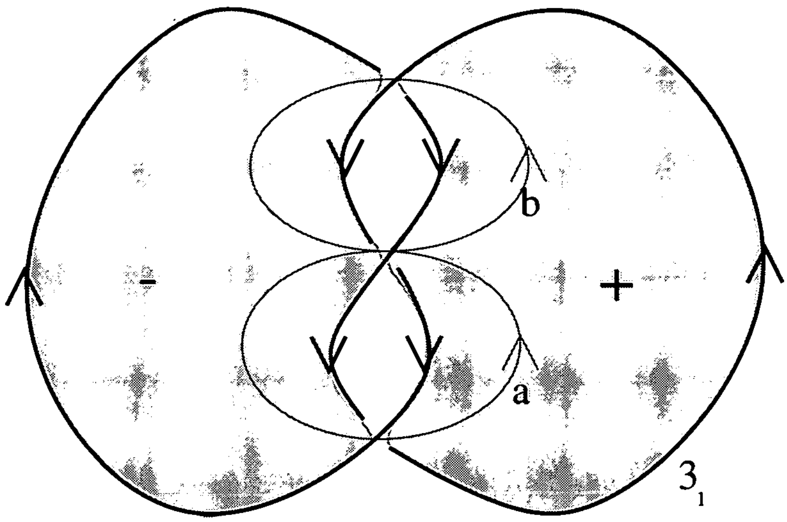

Figure 6.7: Seifert surface in the complement of a trefoil knot 3_1

This is just an ambient isotopy of figure 3.3. The right hand lobe has its positive side (defined by a right-hand convention) facing the reader, while the left hand lobe has its negative side facing us. The two most obvious homology generators a, b are marked, although it would be equally obvious to take $a, a+b$. Their orientation is chosen arbitrarily.

The Seifert matrix can be determined from the figure by visualising i^+a and i^+b and how they link a and b. Since linking is additive, determining the linking on the basis determines the form.

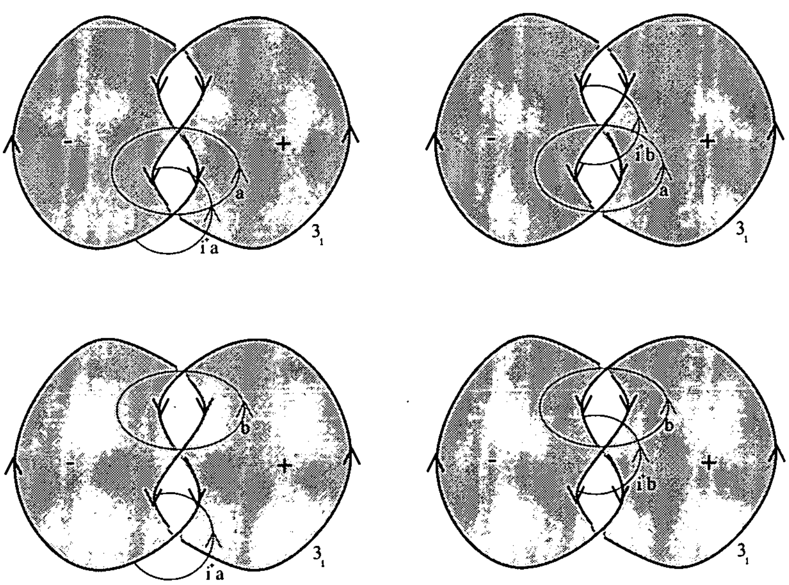

Figure 6.8: Linking the homology generators for $H_1(F_{3_1})$

Figure 6.8 shows the linking, and (taking a right-hand convention again) gives

Seifert matrix:

$$V_{F_{3_1}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Notice that taking a left-hand convention would have given $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and that using a left-hand convention for orienting F would have changed i^+, so the matrices would have been $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ for respectively right-hand, left-hand conventions. These are transposes (changing the handedness for induced orientation) or negatives (changing the handedness for linking) of our initial choice. Changing the orientation of the knot would affect the induce orientation of the Seifert surface, and transpose the Seifert matrix. Changing the orientation of a or b or both, is a just basis change for $H_r(F_{3_1})$.

Example 6.11.3. The knot 5_2 is interesting because it is the simplest knot with a non-invertible Seifert Matrix (over $\mathbb{Z}$). Geometrically, this is equivalent to the knot not being fibred, that is, M_{5_2} is not just $F_{5_2} \times S^1$. See Rolfsen for details of fibred knots [16].

Figure 6.9 shows 5_2 with a Seifert Surface, F_{5_2}, found using Seifert's algorithm, and its two homology generators. Like the trefoil knot, 3_1, in the previous

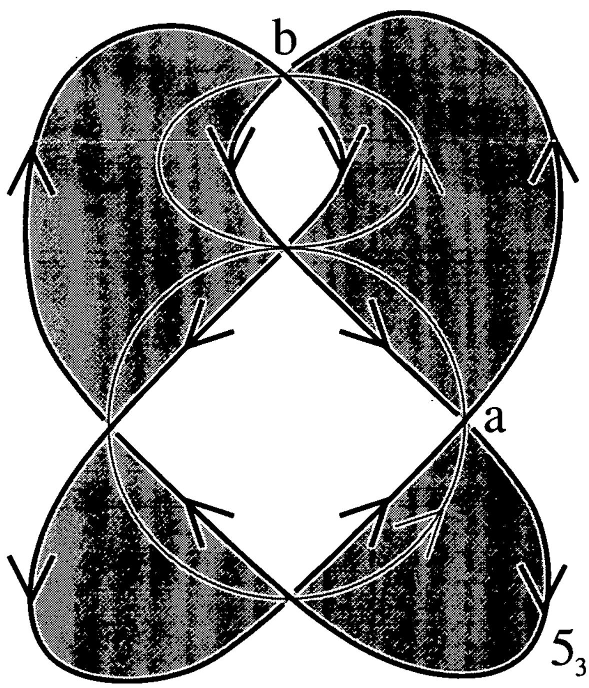

Figure 6.9: Linking the homology generators for $H_1(F_{3_1})$

example, the Seifert surface shown is homeomorphic to a punctured torus.

Choosing a right-handed convention for both inducing the orientation of F_{5_2} and defining linking, the Seifert Matrix is:

$$V_{F_{5_2}} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

These examples have shown Seifert matrices for some specific Seifert Surfaces of classical knots. But there are relationships between different Seifert Manifolds on the same knot.

Lemma 6.12. Suppose V, V' are Seifert matrices with respect to different bases b, b' for $H_r(F;\mathbb{Z})$. The change of base matrix is an invertible matrix over $\mathbb{Z}$, P such that $V = P^T V' P$

Definition 6.13. For the purpose at hand, a surgery on F is the process of adding an r-handle to F.

Remove a $S^{r-1} \times D^{r+1}$ from the interior of F, and glue back $D^r \times S^r$. The boundary of both $S^{r-1} \times D^{r+1}$ and $D^r \times S^r$ is $S^{r-1} \times S^r$. The $D^r \times S^r$ must be embedded smoothly or PL into M (depending on whether your knot is smooth or PL).

$$\begin{tikzcd} S^{r-1} \times D^{r+1} \arrow[r, hook] \arrow[d, hook] & F \arrow[d, hook] & \\ D^r \times D^{r+1} \arrow[r, hook] & M \arrow[r, hook] & S^{2r+1} \end{tikzcd}$$

This kind of surgery does not change the boundary of F.

Lemma 6.14. Suppose F, F' are two Seifert manifolds for the same knot. Either $F \simeq F'$ or there is a sequence of surgeries changing F to F'

Lemma 6.15. Suppose F, F' are two Seifert manifolds, and V, V' are the Seifert matrices for F, F' with respect to bases b, b' (without loss of generality, assume $|b| \leqslant |b'|$) for $H_r(F), H_r(F')$ such that the surgeries which change F to F' change b to b' also. Then those surgeries can each be represented by adding 2 rows and columns to V:

$$V \mapsto \begin{pmatrix} & & & & 0 & 0 \\ & & & & . & . \\ & & V & & . & . \\ & & & & . & . \\ & & & & 0 & 0 \\ x_1 & . & . & x_{|b|} & 0 & 0 \\ 0 & . & . & 0 & 1 & 0 \end{pmatrix}$$

where $\{x_k\}$ record the way the handle links into the other (existing) handles of F in its embedding in S^{n+2}.

Definition 6.16 (S-equivalence). Lemmas 6.12,6.15 give two equivalence relations between Seifert matrices V, V'. These two in combination are called *S-equivalence.*

Example 6.16.1. From example 6.11.2, there are four seemingly interchangeable Seifert Matrices for the trefoil knot, as show in figure 6.7.

$$V_1 = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$
$$V_3 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad V_4 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

There is, to be sure, a basis change which converts V_1 to V_4, and V_2 to V_3. However, since the signature of the matrix V_1 is different for that of V_2, there is no change of basis which will transform one to the other.

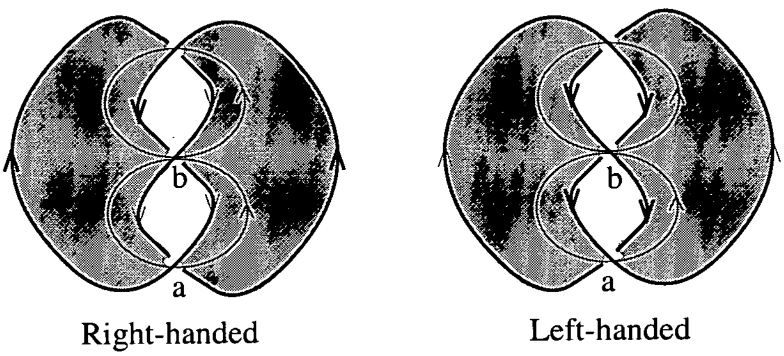

Figure 6.10: Two different trefoil knots.

In fact this shows that there are two different trefoil knots, left and right handed as in figure 6.10, with Seifert Matrices:

$$V_{3_1^R} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad V_{3_1^L} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The right and left hand labeling comes from looking at the crossings of the knot. All three crossings in the right-handed trefoil follow a right-hand rule, while the three in the left-handed trefoil follow the left and rule.

This subtle difference is not exposed by the knot group.

Example 6.16.2. Recall from example 6.11.2 that the Seifert Matrix for the Seifert surface shown in figure 6.7 is:

$$V_{F_{3_1}} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Let F'_{3_1} be the Seifert Surface for 3_1, which is shown in Figure 6.11.

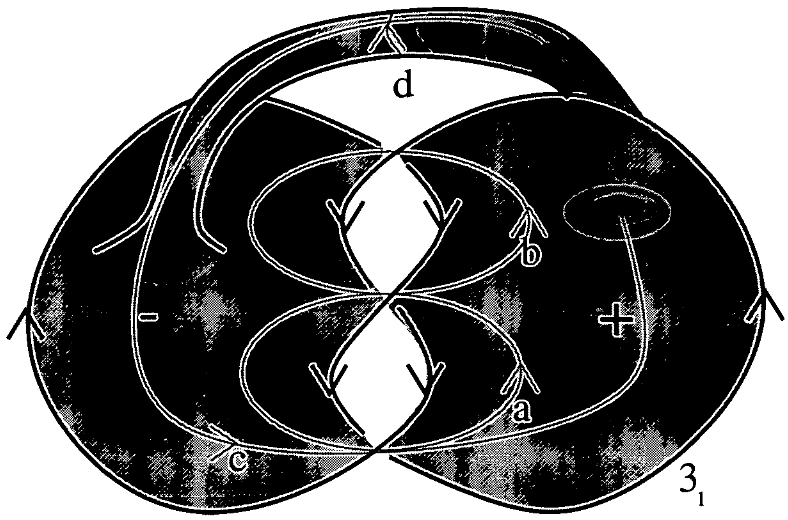

Figure 6.11: Another Seifert surface in the complement of a trefoil knot 3_1

The Seifert Matrix for this surface is:

$$V_{F'_{3_1}} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This is obviously equivalent to $V_{F_{3_1}}$ in the sense of lemma 6.15.

Theorem 6.17 (Trotter [20]). *All Seifert matrices calculated from Seifert Manifolds of a specific knot are members of the same S-equivalence class.*

Proof. Lemma 6.12 obviously defines an equivalence, and Lemma 6.15 defines an equivalence if we allow symmetry of the relation. These two relations taken together define a larger equivalence class.

Suppose V, V' are Seifert matrices for Seifert Manifolds F, F' of a knot k. By lemma 6.14 there is a sequence of surgeries connecting F to F', and so there is a Seifert matrix V'' which is equivalent to V' is the sense of lemma 6.15. Then since V, V'' are defined on the same Seifert Manifold (up to homotopy), there is a lemma 6.12 equivalence between them. Hence V is S-equivalent to V'. □

So the Seifert matrix (under S-equivalence) is certainly a way to tell if two knots are different. However it is obtained in a rather arbitrary way, since it depends on the choice of Seifert Manifold, and S-equivalence is quite a hefty piece of machinery to wield to discover if two knots are different.

The Seifert form can be used to define an intersection pairing for representatives of elements of $H_r(F)$.

In using $lk(i^+x, y)$ as the definition of the Seifert Form, a small oversight has been made. It is true that by lemma 6.10, $lk(i^+x, y) = lk(x, i^-y)$, but what about the linking $lk(i^- x, y)$. Since i^+, i^- are somewhat arbitrarily labeled, depending on the convention for inducing the orientation of F from that of k, does it matter which we chose?

Geometrically, the two linkings have an interesting interpretation. Suppose that $i^- x$ links y, but that $i^+ x$ does not link y (see figure 6.12). This means that x and y intersect in F. In fact it is stronger than that, since x, y are representatives of elements of $H_r(F)$ - it means that $[x]$ and $[y]$ intersect transversely in F.

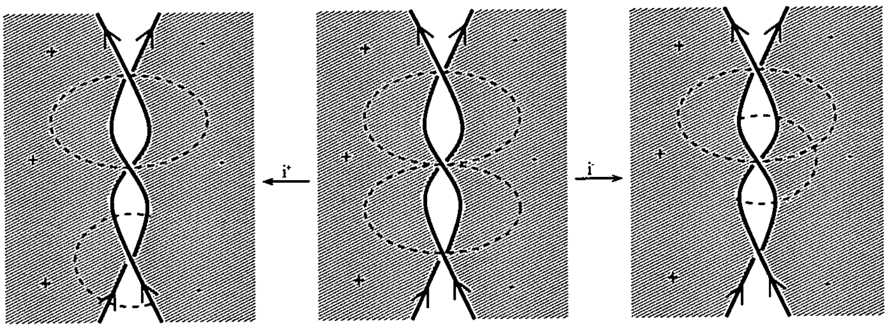

Figure 6.12: Pushed back it links, pushed forward it doesn't.

Definition 6.18. The *intersection form I* for middle dimension homology of a $2r$-dimensional manifold

$$I : H_r(F) \times H_r(F) \to \mathbb{Z}$$

is given by $I(x, y) = lk(i^+ x, y) - lk(i^- x, y)$ using the Hurewicz theorem to identify elements of $\pi_r(F)$ as representative cycles for x, y. [16].

In Definition 6.1, knot addition was described, and it was stated that Tait knew that many knots were sums of "prime" knots. The Seifert Matrix is a nice invariant, since we shall see that it is additive.

Theorem 6.19. *Let k_1, k_2 be $2r-1$-knots. Suppose there is a Seifert Manifold for each knot, F_1, F_2, then there is a Seifert Manifold for $k_1 \# k_2$ which is the connect sum of F_1 and F_2. The homology group of the new Seifert Manifold is*

$$H_r(F_1 \# F_2) = H_r(F_1) \oplus H_r(F_2)$$

If the Seifert Matrices for F_1, F_2 are V_1, V_2 respectively, then inducing a basis for $H_r(F_1 \# F_2)$ from the bases of the individual homology groups, the Seifert Matrix for $F_1 \# F_2$ is:

$$V_{1\#2} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Proof. The connect sum of F_1 and F_2 is a $2r$-dimensional manifold with boundary. Remove a $2r-1$-disc from the boundary of each of F_1, F_2, and glue in an oriented $I \times D^{2r-1}$ by the ends, $S_0 \times D^{2r-1}$, so the orientation matches on all three regions of the new space.

$$F_1 \# F_2 = F_1 \cup I \times D^2 r - 1 \cup F_2$$

The orientation of the boundary is consistent with the orientation of $k_1 \# k_2$.

The Mayer-Vietoris exact sequence for the homology of $F_1 \# F_2$ is clearly

$$\ldots \to 0 \to H_i(F_1) \oplus H_i(F_2) \xrightarrow{\cong} H_i(F_1 \# F_2) \to \ldots$$

since the gluing of F_1 to F_2 is done by a disc.

Since the definition of knot sum requires the two knots to effectively embed into separate half spaces, there is no chance of any linking between homology generators of the two joined surfaces.

Hence the linking form on $F_1 \# F_2$ will be the direct sum of the linking forms on F_1 and F_2. Hence the Seifert Matrix will be as above. □

Example 6.19.1. From example 6.16.1, Seifert Matrices for the right and left handed trefoil knots are:

$$V_{3_1^R} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad V_{3_1^L} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The reef (or square) knot is the connect sum of a right-handed trefoil and a left-handed trefoil.

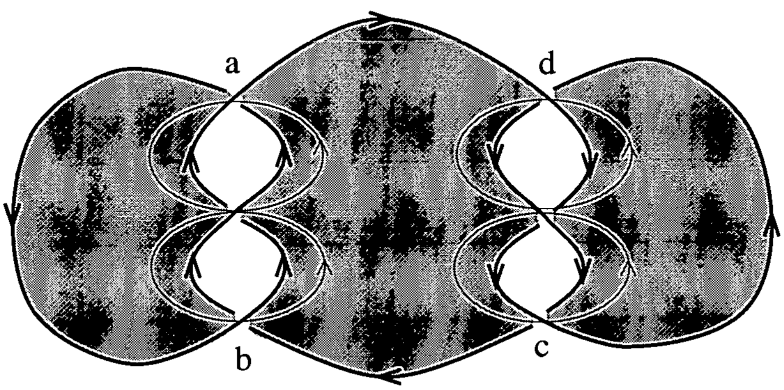

Figure 6.13: The Reef (or Square) Knot, $3_1^R \# 3_1^L$

Its Seifert Matrix for the Seifert Surface in figure 6.13 is:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is clearly the direct sum of $V_{3_1^R}$ and $V_{3_1^L}$.

6.4 The Seifert Endomorphism

It would be useful to turn the Seifert form into a map. Let's indulge in a little homotopy equivalence juggling. Let k be a simple knot, and F be an $(r-1)$-connected Seifert manifold for k. Denote the Seifert form defined by linking using i^+ by σ^+ and its matrix by V^+ and the form defined using i^- by σ^- and its matrix by V^-. Using figure 6.6, define a map

$$j : F \to F \times \{\frac{1}{2}\} \subset F \times I \subset S^{n-2} \setminus M_F$$

which leads to an isomorphism

$$j_* : H_r(F) \to H_r(S^{n+2} \setminus M_F)$$

and hence there is a commutative diagram:

$$\begin{array}{ccclcl}
H_r(F) \times H_r(F) & \xrightarrow{\sigma^+} & \mathbb{Z}; & ([x],[y]) \longmapsto lk(i^+x, y) & & (6.1) \\
\downarrow{\scriptstyle i_*^+ \times 1} & & \| & & & \\
H_r(M_F) \times H_r(F) & \xrightarrow{\sigma'} & \mathbb{Z}; & ([x],[y]) \longmapsto lk(x, jy) & & \\
\cong \downarrow{\scriptstyle 1 \times j_*} & & \| & & & \\
H_r(M_F) \times H_r(S^{2r+1} \setminus M_F) & \xrightarrow{\sigma''} & \mathbb{Z}; & ([x],[y]) \longmapsto lk(x, y) & &
\end{array}$$

and a similar one for σ^-.

Lemma 6.20. $\mathrm{adj}(\sigma'') : H_r(M_F) \cong H_r(S^{2r+1} \setminus M_F)^*$

Proof. Let A be the closure of $S^{2r+1} \setminus M_F$. Then A is the closure of $(F \times I) \cup (k \times D^2)$. Since $\partial F = k$, $A \simeq F$. Then also

$$\begin{aligned} A \cup M_F &= S^{2r+1} \\ A \cap M_F &= \partial M_F \simeq \partial(F \times I) \end{aligned}$$

Note that A is a regular neighbourhood of F in S^{2r+1}, with a handlebody structure consisting of one r-handle for each generator in $H_r(F)$. Let $\{a_i\}$ denote the representative chains of the homology generators in $H_r(A) = H_r(F)$.

The boundary of M_F is the boundary of A. The boundary of the $2r+1$-dimensional handlebody A with r-handles has homology only in the rth dimension. There are twice as many homology generators in $H_r(\partial A)$ as there are in $H_r(A)$. Each handle of A gives rise to one generator in $H_r(A)$, and two in $H_r(\partial A)$, so that for each handle the natural inclusion $\partial A \hookrightarrow A$ sends one generator of $H_r(\partial A)$ to zero (that is, it bounds an $r+1$-disc in A) and the other to the generator in $H_r(A)$. Denote the generators or $H_r(\partial A)$ by $\{a_i', b_i'\}$ so that the natural inclusion sends $a_i' \mapsto a_i$ and $b_i' \mapsto 0$.

The Mayer-Vietoris exact sequence for the homology of S^{2r+1} in terms of the homologies of M_F and A says that

$$\ldots \to H_{r+1}(S^{2r+1}) \to H_r(\partial M_F) \to H_r(A) \oplus H_r(M_F) \to H_r(S^{2r+1}) \to \ldots$$

but $H_{r+1}(S^{2r+1}) = H_r(S^{2r+1}) = 0$, so

$$H_r(\partial M_F) \cong H_r(A) \oplus H_r(M_F)$$

Hence $H_r(M_F) \cong \bigoplus_{\{b_i'\}} \mathbb{Z}$. Denote the generators of $H_r(M_F)$, $\{b_i\}$, so that the natural inclusion of $\partial A \hookrightarrow M_F$ sends the $b_i' \mapsto b_i$ and $a_i' \mapsto 0$.

So, there is a natural bilinear form

$$\zeta : H_r(M_F) \times H_r(A) \to \mathbb{Z}$$

given by the linking pairing, which sends $([b_i], [a_j]) \mapsto \delta_{ij}$ (the Kronecker delta) and extends linearly.

Hence there is a map, the adjoint of ζ

$$\mathrm{adj}(\zeta) : H_r(A) \to \mathrm{Hom}(H_r(M_F); \mathbb{Z}) = H_r(M_F)^*$$

which is an isomorphism. Since σ'' is defined to be

$$\sigma'' : H_r(M_F) \times H_r(A) \to \mathbb{Z} \ ; \ ([x], [y]) \mapsto lk(x, y)$$

there are changes of bases $\phi : H_r(A) \to H_r(A), \psi : H_r(M_F) \to H_r(M_F)$ so that $\mathrm{adj}(\zeta) = \phi^{-1}\,\mathrm{adj}(\sigma'')\psi$. □

By looking at the adjoints of the forms in the lower commuting square in commuting diagram (6.1), there is another commuting diagram:

$$\begin{tikzcd} & H_r(F)^* \arrow[d, "j_*^*"] \\ H_r(M_F) \arrow[ur, "\mathrm{adj}(\sigma')"] \arrow[r, "\mathrm{adj}(\sigma'')"'] & H_r(S^{2r+1} \setminus M_F)^* \end{tikzcd} \tag{6.2}$$

Since it is known that j_* is an isomorphism, then j_*^* must be also, and it has just been proven that $\mathrm{adj}(\sigma'')$ is an isomorphism, so $\mathrm{adj}(\sigma')$ must be an isomorphism also.

Considering the upper commuting square of (6.1) and its equivalent for σ^- gives two more commuting diagrams:

$$\begin{tikzcd} H_r(F) \arrow[d, "i_*^+"'] \arrow[dr, "\mathrm{adj}(\sigma^+)"] & \\ H_r(M_F) \arrow[r, "\cong", "\mathrm{adj}(\sigma')"'] & H_r(F)^* \end{tikzcd} \qquad \begin{tikzcd} H_r(F) \arrow[d, "i_*^-"'] \arrow[dr, "\mathrm{adj}(\sigma^-)"] & \\ H_r(M_F) \arrow[r, "\cong", "\mathrm{adj}(\sigma')"'] & H_r(F)^* \end{tikzcd}$$

It is possible to go further in this direction, but first another property of i^+, i^- is needed.

For any knot, there is a Mayer-Vietoris sequence of the homology of $\overline{M}$ using the construction shown in figure 6.4:

$$\overline{M} = \left(\bigcup_{q \text{ even}} \{q\} \times M_F \right) \cup_{i^-, i^+} \left(\bigcup_{q \text{ odd}} \{q\} \times M_F \right) ;$$
$$(\{q\} \times M_F) \cap (\{q+1\} \times M_F) = \{q\} \times F$$

gluing M_F to M_F in an orientation preserving way, by i^+, i^-.

It is sensible at this stage to introduce a group ring notation for $\mathbb{Z}$, writing $M_F[t, t^{-1}]$ for $\mathbb{Z} \times M_F$ so the construction looks like:

$$\overline{M} = \left(\bigcup_{q \text{ even}} t^q M_F \right) \cup \left(\bigcup_{q \text{ odd}} t^q M_F \right) ; \qquad t^q M_F \cap t^{q+1} M_F = t^q F$$

It is straightforward to see that

$$\left(\bigcup_{q \text{ even}} t^q M_F \right) \cap \left(\bigcup_{q \text{ odd}} t^q M_F \right) = F[t, t^{-1}]$$

Hence the Mayer-Vietoris sequence for the homology of $\overline{M}$ is:

$$\ldots \to H_i(F)[t, t^{-1}] \xrightarrow{i_*^+ - t i_*^-} H_i(M_F)[t, t^{-1}] \to H_i(\overline{M}) \to \ldots \tag{6.3}$$

Theorem 6.21. *If k is an n-knot then there is an isomorphism*

$$i_*^+ - i_*^- : H_r(F; \mathbb{Z}) \to H_r(M_F; \mathbb{Z})$$

for $r \geqslant 1$.

Proof. This theorem has striking parallels to Claim 4.10.

The exact sequence (6.3) can be used to calculate the homology of M, by "setting $t = 1$". In other words, by gluing one copy of M_F to itself making the smallest cyclic cover of M, which is M itself.

The Mayer-Vietoris sequence which results looks like:

$$\ldots \to H_r(F) \xrightarrow{i_*^+ - i_*^-} H_r(M_F) \to H_r(M) \to \ldots$$

But the homology of M has already (theorem 6.4) been shown to have the homology of a circle. That is:

$$H_r(M) = \begin{cases} \mathbb{Z} & r = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Thus for $i \geqslant 2$,

$$i_*^+ - i_*^- : H_i(F) \to H_i(M_F)$$

is an isomorphism.

At the lower end of the Mayer-Vietoris sequence,

$$0 \to H_1(F) \xrightarrow{i_*^+ - i_*^-} H_1(M_F) \to H_1(M) \to H_0(F) \xrightarrow{i_*^+ - i_*^-} H_0(M_F) \to H_0(M) \to 0$$

But both F and M_F are known to be path connected, and so $H_0(F) = H_0(M_F) = \mathbb{Z}$. Hence

$$0 \to H_1(F) \xrightarrow{i_*^+ - i_*^-} H_1(M_F) \to \mathbb{Z} \to \mathbb{Z} \xrightarrow{i_*^+ - i_*^-} \mathbb{Z} \to \mathbb{Z} \to 0$$

and so $i_*^+ - i_*^- = 0 : H_0(F) \to H_0(M_F)$ and

$$i_*^+ - i_*^- : H_1(F) \to H_1(M_F)$$

is an isomorphism. □

Let k be a $2r-1$-knot with $(r-1)$-connected Seifert Manifold F. Commuting diagram (6.4) follows from commuting diagram 6.1:

$$\begin{tikzcd} H_r(F) \times H_r(F) \arrow[r, "\sigma^+ - \sigma^-"] \arrow[d, "(i_*^+ - i_*^-)\times 1"'] & \mathbb{Z}; \arrow[d, equal] & ([x],[y]) \arrow[r, mapsto] & lk(i^+x, y) - lk(i^-x, y) \\ H_r(M_F) \times H_r(F) \arrow[r, "\sigma'"] & \mathbb{Z}; & ((i_*^+ - i_*^-)[x],[y]) \arrow[r, mapsto] & lk((i^+ - i^-)x, y) \end{tikzcd} \tag{6.4}$$

Which leads to commuting diagram 6.5.

Theorem 6.22. *The diagram*

$$\begin{tikzcd} H_r(F) \arrow[d, "i_*^+ - i_*^-"'] \arrow[dr, "\mathrm{adj}(\sigma^+) - \mathrm{adj}(\sigma^-)"] & \\ H_r(M_F) \arrow[r, "\cong", "\mathrm{adj}(\sigma')"'] & H_r(F)^* \end{tikzcd} \tag{6.5}$$

commutes and each map is an isomorphism.

Proof. From commuting diagram (6.2), σ' is an isomorphism, and Theorem 6.21 proves that $i_*^+ - i_*^-$ is an isomorphism. Hence $\mathrm{adj}(\sigma^+) - \mathrm{adj}(\sigma^-)$ is also an isomorphism. □

Lemma 6.23. $V^+ = (-1)^{r-1}(V^-)^T$

Proof. V^- is the matrix of the form which sends $([x],[y])$ to $lk(i^-x,y)$. The transpose of V^- is the matrix of the form which sends $([x],[y])$ to $lk(i^-y,x)$. However, by lemma 6.10, $lk(i^+x,y) = lk(x,i^-y)$. So, using lemma 6.8, this gives:

$$lk(i^-y,x) = (-1)^{r^2-1} lk(i^+x,y)$$

Hence their matrices will agree up to sign. □

Lemma 6.24. Let A be a $\mathbb{Z}$-module, with elements $x = \sum_s (x_s a_s)$ with respect to a basis $\underline{a}$, and let $\psi : A \times A \to \mathbb{Z}$ a bilinear form with a matrix Ψ with respect to $\underline{a}$. The adjoint of ψ, $\operatorname{adj}(\psi) : A \to A^* = \operatorname{Hom}(A,\mathbb{Z})$ can be written as:

$$\operatorname{adj}(\psi)(x) = \left(y \mapsto \sum_{s_2} \left(\sum_{s_1} \Psi_{(s_1,s_2)} x_{s_1} \right) y_{s_2} \right)$$

The point is that the matrix is the same for the form and the adjoint.

Hence $V + (-1)^r V^T$ is the matrix of an isomorphism from $H_r(F)$ to $H_r(F)^*$, for both $V = V^+, V^-$.

Definition 6.25. The *Seifert Endomorphism* e is the map:

$$e = (i_*^+ - i_*^-)^{-1} i_*^+ : H_r(F) \to H_r(F)$$

Lemma 6.26.

$$(i_*^+ - i_*^-)^{-1} i_*^+ = (\operatorname{adj}(\sigma^+) - \operatorname{adj}(\sigma^-))^{-1} \operatorname{adj}(\sigma^+)$$

Proof. It merely needs to be shown that the following diagram commutes:

$$\begin{tikzcd}
H_r(F) \arrow[r, "i_*^+"] \arrow[d, "\operatorname{adj}(\sigma^+)"'] & H_r(M_F) \\
H_r(F)^* & H_r(F) \arrow[l, "\operatorname{adj}(\sigma^+)-\operatorname{adj}(\sigma^-)"] \arrow[u, "i_*^+-i_*^-"']
\end{tikzcd}$$

Clearly by comparison to commuting diagram 6.5 it is possible to add the map $\operatorname{adj}(\sigma')$

$$\begin{tikzcd}
H_r(F) \arrow[r, "i_*^+"] \arrow[d, "\operatorname{adj}(\sigma^+)"'] & H_r(M_F) \arrow[dl, "\operatorname{adj}(\sigma')"'] \\
H_r(F)^* & H_r(F) \arrow[l, "\operatorname{adj}(\sigma^+)-\operatorname{adj}(\sigma^-)"] \arrow[u, "i_*^+-i_*^-"']
\end{tikzcd}$$

and hence both the top-left and bottom-right triangles commute. □

Theorem 6.27. *The Seifert endomorphism has a matrix* $(V + (-1)^r V^T)^{-1} V^+$

Proof. By Lemma 6.26 the two maps are identical, hence the matrix representation of one will be that of the other. □

Chapter 7

The Alexander Module of a Knot

The Alexander Module is a module over $\mathbb{Z}[t, t^{-1}]$. It is an invariant of the infinite cyclic cover of a knot. Blanchfield [3] defined a pairing on the Alexander module which Levine [12] showed was related to the Seifert form for knots.

Definition 7.1. Let k be an odd dimensional simple (simplicial) knot,

$$k : S^{2r-1} \to S^{2r+1}$$

and let M be the knot complement. By Theorem 6.4, there is a homology equivalence map $h : M \to S^1$. There is a pullback cover $\overline{M}$ defined by this map:

$$\begin{tikzcd}
\overline{M} \arrow[r] \arrow[d, "p"'] & \mathbb{R} \arrow[d, "e^{2\pi i x}"] \\
M \arrow[r, "h"'] & S^1
\end{tikzcd}$$

$\overline{M}$ is the infinite cyclic cover of M.

The *Alexander Module* of k is $H_r(\overline{M})$

Lemma 7.2. $\mathbb{Z}[t, t^{-1}]$ acts on $H_r(\overline{M})$ by the generator of the group of covering transformation $t : \overline{M} \to \overline{M}$.

Proof. Let the fibre of $\overline{M}$ be denoted by $\{t^q : q \in \mathbb{Z}\}$. The map h is a homology equivalence so the induced map $h_* : \pi_1(M) \to \pi_1(S^1)$ is surjective. Hence by Lemma 3.5 the cover is regular and the group of covering translations of $\overline{M}$ is $\mathbb{Z} \cong \langle t \rangle$, writing the group in multiplicative form.

Since M is a simplicial complex, there is a cellular chain complex for M over $\mathbb{Z}$, $C(M)$ (definition 5.4). Further $\overline{M}$ has a cellular chain complex $C(\overline{M})$ over $\mathbb{Z}[t, t^{-1}]$.

The action of $\mathbb{Z}[t, t^{-1}]$ on $H_r(\overline{M})$ is induced from this. $\square$

The homology of $\overline{M}$ was described in Chapter 6, using a Mayer-Vietoris sequence for constructing the infinite cyclic cover from the Seifert manifold and the knot complement. The exact sequence given in (6.3) was:

$$\ldots \to H_i(F)[t,t^{-1}] \xrightarrow{i_*^+ - ti_*^-} H_i(M_F)[t,t^{-1}] \to H_i(\overline{M}) \to \ldots \tag{7.1}$$

However, when $1 \leqslant i \leqslant r-1$, $H_i(F) = 0$ for a simple knot and so $H_i(M_F) = 0$, and so $H_i(\overline{M}) = 0$ in those cases. Hence:

$$\ldots \to H_r(F)[t,t^{-1}] \xrightarrow{i_*^+ - ti_*^-} H_r(M_F)[t,t^{-1}] \to H_r(\overline{M}) \to 0 \tag{7.2}$$

is exact.

Lemma 7.3. $i_*^+ - ti_*^- : H_r(F)[t,t^{-1}] \to H_r(M_F)[t,t^{-1}]$ is an injection.

Proof. $H_r(F)$ is a finitely generated free abelian module, and since there is an isomorphism $i_*^+ - i_*^- : H_r(F) \to H_r(M_F)$, so is $H_r(M_F)$. Given a basis for $H_r(F)$, and hence by isomorphism for $H_r(M_F)$ there is a matrix λ for $i_*^+ - ti_*^- : H_r(F)[t,t^{-1}] \to H_r(M_F)[t,t^{-1}]$. This matrix has determinant $p(t)$, where $p(1) = 1$ because $i_*^+ - i_*^-$ is an isomorphism. Let $\lambda^\#$ denote the transpose of the matrix of co-factors of λ. This matrix has the property that

$$\lambda^\# \lambda = p(t)I$$

Hence taking $x = (x_0, \ldots) \in H_r(F)$, if $\lambda x = 0$ then

$$\lambda^\# \lambda x = p(t)Ix = 0$$

and hence since $p(t) \neq 0$, it follows that $x = 0$.

Hence $i_*^+ - ti_*^- : H_r(F)[t,t^{-1}] \to H_r(M_F)[t,t^{-1}]$ is an injection. □

Proposition 7.4. *The following sequence is exact:*

$$0 \to H_r(F)[t,t^{-1}] \xrightarrow{e-(e-1)t} H_r(F)[t,t^{-1}] \to H_r(\overline{M}) \to 0$$

Proof. Sequence (7.2) is exact, and since $i_*^+ - ti_*^-$ is an injection, the following sequence is exact also:

$$0 \to H_i(F)[t,t^{-1}] \xrightarrow{i_*^+ - ti_*^-} H_i(M_F)[t,t^{-1}] \to H_i(\overline{M}) \to 0 \tag{7.3}$$

The following diagram commutes, and hence the new sequence is exact:

$$\begin{tikzcd}
0 \arrow[r] & H_r(F)[t,t^{-1}] \arrow[r, "i_*^+ - ti_*^-"] & H_r(M_F)[t,t^{-1}] \arrow[r] & H_r(\overline{M}) \arrow[r] & 0 \\
0 \arrow[r] & H_r(F)[t,t^{-1}] \arrow[r, "w"'] \arrow[u, equal] & H_r(F)[t,t^{-1}] \arrow[r] \arrow[u, "i_*^+ - i_*^-"'] & H_r(\overline{M}) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

where $w = (i_*^+ - i_*^-)^{-1}(i_*^+ - ti_*^-)$. But $(i_*^+ - i_*^-)^{-1}i_*^+ = e$, the Seifert Endomorphism, and since

$$\begin{aligned} 1 = (i_*^+ - i_*^-)^{-1}(i_*^+ - i_*^-) &= e - (i_*^+ - i_*^-)^{-1}i_*^- \\ \implies (i_*^+ - i_*^-)^{-1}i_*^- &= e - 1 \end{aligned}$$

the new exact sequence can be written as

$$0 \to H_r(F)[t, t^{-1}] \overset{e-(e-1)t}{\to} H_r(F)[t, t^{-1}] \to H_r(\overline{M}) \to 0$$

as required. □

Definition 7.5. The *Alexander Polynomial* $A_k(t)$ of a knot, k, is the determinant of $e - (e - 1)t$.

Lemma 7.6. The Alexander polynomial annihilates the Alexander Module.

Proof. The Alexander module has a resolution in terms of $e - (e - 1)t$:

$$0 \to H_r(F)[t, t^{-1}] \xrightarrow{e-(e-1)t} H_r(F)[t, t^{-1}] \xrightarrow{\zeta} H_r(\overline{M}) \to 0$$

Hence, any element of $H_r(\overline{M})$ comes from an element of $H_r(F)[t, t^{-1}]$. Suppose $y \in H_r(F)[t, t^{-1}]$

$$\begin{aligned} A_k(t)\zeta(y) &= \zeta(A_k(t)y) \\ &= \zeta(A_k(t)Iy) \\ &= \zeta(\lambda\lambda^{\#}y) \\ &= \zeta(\lambda(\lambda^{\#}y)) \end{aligned}$$

but then any element of the form λx is sent to 0 by ζ, so $A_k(t)\zeta(y) = 0$. □

Theorem 7.7. *Given any resolution of the Alexander module, in terms of a finitely generated free $\mathbb{Z}$-module Γ, and a matrix $\gamma(t)$:*

$$0 \to \Gamma[t, t^{-1}] \overset{\gamma(t)}{\to} \Gamma[t, t^{-1}] \to H_i(\overline{M}) \to 0$$

the Alexander polynomial is given by: $A_k(t) = \det \gamma(t)$

Proof. See Milnor's paper, [14]. □

Example 7.7.1. The Alexander polynomial of the trefoil knot 3_1 can be calculated from its Seifert matrix. Recalling that

$$0 \to H_i(F)[t, t^{-1}] \xrightarrow{e-(e-1)t} H_i(F)[t, t^{-1}] \to H_i(\overline{M}) \to 0$$

and that e has matrix $(V+(-1)^r V^T)^{-1}V^+$ by theorem 6.27, and $1-e$ has matrix

$$(V+(-1)^r V^T)^{-1}V^- = (-1)^{r+1}(V+(-1)^r V^T)^{-1}(V^+)^T$$

by Lemma 6.23, so $e-(e-1)t$ has matrix

$$(V+(-1)^r V^T)^{-1}V^+ + (-1)^r(V+(-1)^r V^T)^{-1}(V^+)^T t$$

But $V+(-1)^r V^T$ has determinant 1 since it is invertible over $\mathbb{Z}$.

Hence given that $r=1$ for classical knots:

$$\begin{aligned}
A_{3_1}(t) &= \det(V^+ - (V^+)^T t) \\
&= \det\left(\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} t\right) \\
&= \det\begin{pmatrix} t-1 & 1 \\ -t & t-1 \end{pmatrix} \\
&= (t-1)^2 + t \\
&= 1 - t + t^2
\end{aligned}$$

using $V^+ = V_{F_{3_1}}$ from example 6.11.2.

It is straight forward to see that since $\overline{M}$ does not depend on the choice of a Seifert Surface F, then neither should the Alexander polynomial, so S-equivalent Seifert Matrices will give the same Alexander polynomial.

Example 7.7.2. The Alexander polynomial calculated by using $V^+ = V_{F'_{3_1}}$ is

$$\begin{aligned}
A'_{3_1}(t) &= det\left(\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} t\right) \\
&= \det\begin{pmatrix} t-1 & 1 & -t & 0 \\ -t & t-1 & 0 & 0 \\ -1 & 0 & 0 & -t \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
&= -(-1)(-1(0+0) + 0 + t((t-1)^2 + t)) \\
&= t - t^2 + t^3
\end{aligned}$$

but since t is a unit of $\mathbb{Z}[t, t^{-1}]$, this is equivalent to $1 - t + t^2$.

Suppose next, that $H_r(\overline{M})$ is known, and that it has a resolution in terms of some finitely generated $\mathbb{Z}$-module Γ:

$$0 \to \Gamma[t, t^{-1}] \xrightarrow{\gamma(t)} \Gamma[t, t^{-1}] \to H_r(\overline{M}) \to 0 \qquad (7.4)$$

where up to multiplication by units in $\mathbb{Z}[t,t^{-1}]$, $\gamma(t) = \sum_{\ell=0}^{\mu} \gamma_\ell t^\ell$, where $\gamma_\ell \in M_{dim\Gamma}$ and setting $t=1$ gives $\gamma(1)=1$. For instance,

$$\Gamma = \mathbb{Z}, \quad \gamma = A_k(t)$$

gives a resolution of the homology of the infinite cyclic cover of the knot complement of any knot k.

Theorem 7.8 (The Higman Trick [10]). *This resolution (7.4) can be linearised, by application of Corollary 4.11 and Theorem 4.13, to find a new resolution:*

$$0 \to \Gamma^{\oplus\mu+1}[t,t^{-1}] \xrightarrow{ft-g} \Gamma^{\oplus\mu+1}[t,t^{-1}] \to H_r(\overline{M}) \to 0$$

where μ is the polynomial degree of $\gamma(t)$.

Proof. Consider $\gamma(t) : \Gamma[t,t^{-1}] \to \Gamma[t,t^{-1}]$ to be a one dimensional chain complex, $\mathcal{G}(t)$. Then $H_r(\overline{M})$ is $H_0(\mathcal{G})$.

By Theorem 4.9, $\mathcal{G}(t)$ has a fundamental domain, $\mathcal{K}$:

$$d_1^{\mathcal{K}} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \cdot \\ \cdot \\ \cdot \\ \gamma_\mu \end{pmatrix} : \mathcal{K}_1 = \Gamma \longrightarrow \mathcal{K}_0 = \Gamma^{\oplus\mu+1}$$

Let τ represent the action of t on $\mathcal{G}^!$, so $\Gamma^{\oplus\mu}$ is written

$$\sum_{\ell=0}^{\mu} \Gamma\tau^\ell$$

By Corollary 4.11 there is a resolution for $\mathcal{G}(t)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_1[t,t^{-1}] \arrow[r, "f_1-g_1t"] \arrow[d] & \mathcal{K}_1[t,t^{-1}] \arrow[r] \arrow[d, "d_1^{\mathcal{K}}"] & \mathcal{G}(t)_1 \arrow[r] \arrow[d, "\gamma(t)"] & 0 \\
0 \arrow[r] & \mathcal{L}_0[t,t^{-1}] \arrow[r, "f_0-g_0t"] & \mathcal{K}_0[t,t^{-1}] \arrow[r] & \mathcal{G}(t)_0 \arrow[r] & 0
\end{tikzcd}$$

Where $\mathcal{L} = \mathcal{K} \cap \mathcal{K}\tau^{-1}$. But since $\mathcal{K}_1 = \Gamma$, $\mathcal{L}_1 = \mathcal{K}_1 \cap \mathcal{K}_1\tau^{-1} = 0$.

By Theorem 4.13, $\mathcal{G}$ has the same homology as the mapping cone $\mathcal{C}(f-gt)$:

$$\mathcal{L}_0[t,t^{-1}] \oplus \mathcal{K}_1[t,t^{-1}] \xrightarrow{\begin{pmatrix} f-gt \\ -d_1^{\mathcal{K}} \end{pmatrix}} \mathcal{K}_0[t,t^{-1}]$$

which is linear, as required.

$\mathcal{K}_1, \mathcal{K}_0$ are, as noted above, $\Gamma, \sum_{\ell=0}^{\mu} \Gamma\tau^\ell$ respectively. Hence $\mathcal{L}_0$ is $\Gamma^{\oplus(\mu)}$, so $(\mathcal{K}_0 \cap \mathcal{K}_0 t^{-1}) \oplus \mathcal{K}_1$ is $\Gamma^{\oplus\mu+1}$. ☐

Setting $t = 1$ will still give $f - g$ an isomorphism. Replacing f, g by $f' = (f - g)^{-1}f$ and $g' = (f - g)^{-1}g$ makes $f' - g' = 1$, the identity.

Definition 7.9. The *Algebraic Seifert Endomorphism, e_A* is

$$e_A = (f - g)^{-1}f : \Gamma^{\oplus\mu+1} \to \Gamma^{\oplus\mu+1}$$

Proposition 7.10. *If the resolution of $H_r(\overline{M})$ is a Mayer-Vietoris sequence construction of the homology in terms of a Seifert Surface, then $e_A = e$. That is the geometric and algebraic Seifert endomorphisms agree.*

Proof. If the resolution used is: .

$$0 \to H_i(F)[t, t^{-1}] \xrightarrow{e-(e-1)t} H_i(F)[t, t^{-1}] \to H_i(\overline{M}) \to 0$$

then $e_A = (e - (e - 1))^{-1}e = e$. □

Definition 7.11. The *Blanchfield Form* of a knot k is a bilinear form on the Alexander module.

$$\beta : H_r(\overline{M}, \partial\overline{M}) \times H_r(\overline{M}) \to \mathbb{K}$$

For simple knots $H_r(\overline{M}, \partial\overline{M}) = H_r(\overline{M})$. $\mathbb{K}$ is the quotient by $\mathbb{Z}[t, t^{-1}]$ of the localisation $P^{-1}\mathbb{Z}[t, t^{-1}]$, inverting the subset $P \subset \mathbb{Z}[t, t^{-1}]$ of polynomials $p(t)$ with $p(1) = 1$. Then

$$\mathbb{K} = \frac{P^{-1}\mathbb{Z}[t, t^{-1}]}{\mathbb{Z}[t, t^{-1}]}$$

Blanchfield [3] defined β geometrically in terms of linking in $\overline{M}$, which needs care to define it usefully.

One of the definitions of linking is in terms of an intersection pairing. If two circles are linked then they each pierce the spanning surface of the other. Blanchfield used this idea to define his pairing.

Let x, y be r-spheres (or more generally r-chains) embedded in $\overline{M}$. Each has an associated element of $H_r(\overline{M})$. We need to find a spanning manifold for x (i.e.an $r + 1$-dimensional manifold embedded in $\overline{M}$ with boundary x) and then use an intersection pairing to discover if they intersect. That will tell us if x and y link. The trouble is that it is usually impossible to find a spanning manifold for x, since having a spanning manifold implies that $x = 0$ in the homology.

Blanchfield looked at annihilators of the Alexander module, and found that since the Alexander polynomial is an annihilator, $A_k(t)x = 0 \in H_r(\overline{M})$ has a spanning manifold. Let z be the spanning surface of $A_k(t)x$. So the Blanchfield pairing was defined to be:

$$\beta(x,y) = \sum_{\ell=-\infty}^{\infty} \frac{I(z, yt^{\ell})}{A_k(t)} t^{\ell}$$

Where $I(z,y)$ is the oriented intersection pairing on $\overline{M}$. See page 14 of Levine [12] for a more detailed description of the chain definition of the Blanchfield pairing.

Theorem 7.12 (Levine [12]). *The Blanchfield Form on the Alexander module*

$$H_r(\overline{M}) = coker\left(e - (e-1)t : H_r(F)[t,t^{-1}] \to H_r(F)[t,t^{-1}]\right)$$

is given by

$$\beta(x,y) = (1-t)(V + (-1)^r tV^T)^{-1}(x)(y) \in \mathbb{K}$$

where V is the matrix of the Seifert Form with respect to the generators of $H_r(F)$.

Proof. See Levine's paper, [12], particularly page 14 and pages 44 – 47. □

It is possible to recover the Seifert matrix from the Blanchfield Form. Trotter [22, 21] defines a *trace map, s,* and a modified trace χ to do this.

Definition 7.13. The *trace map,* $s : \mathbb{K} \to \mathbb{Q}$, takes a proper fraction over $\mathbb{Z}[t,t^{-1}]$, v, to a rational, $s(v)$. $s(v)$ is the coefficient of t^{-1} in the Laurent expansion of v at infinity.

That is, substitute $u = t^{-1}$ in v, and look at the u term in the Taylor expansion.

Define $\chi : \mathbb{K} \to \mathbb{Z}$, to be $\chi(v) = 0$ if the denominator of v is entirely multiples of $t, 1-t$ and s otherwise.

Theorem 7.14. *Using the inclusion of $H_r(F)$ into $H_r(\overline{M})$, induced from the inclusion of F in $\overline{M}$, the following diagram commutes.*

$$\begin{tikzcd} H_r(F) \times H_r(F) \arrow[r, hook] \arrow[d, "\sigma"'] & H_r(\overline{M}) \times H_r(\overline{M}) \arrow[d, "\beta"] \\ \mathbb{Z} & \mathbb{K} \arrow[l, "\chi"] \end{tikzcd} \tag{7.5}$$

Proof. See Proposition 32.45 of [15]. □

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