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The Dirac Equations in Spherical Space

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Contents

Introduction and summary

a) In the following paper a new representation of the Dirac equations in spherical space is given. It arose in connection with a paper by Schrödinger on the proper vibrations of spherical space (S2). The special choice of the coordinates was suggested by the desire to obtain that repre-
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sentation in which two of the six generators of the rota-

tion-group in four real dimensions(infinitesimal rotations in two totally perpendicular planes)are diagonal.This is effected by the use of what one may naturally call "cylindric coordinates on the hypersphere".Cylindric,because the one set of coordinate surfaces is a set of co-axial cylinders in spherical space,i.e.the loci of all points that have the same distance from the points on a great circle. These hyperspherical cylinders are surfaces of revolution for <u>both</u> the other two coordinates(not only for one,as is the case with ordinary cylinders),very much similar to the <u>two</u> azimuthal angles on an ordinary <u>torus</u>.

In this representation the eigensolutions of the Dirac equations for a free electron are waves winding along great circles around these cylinders,the intensity being constant <u>on</u> any cylinder.The polarisation of the waves in general is discussed,in particular for the specially interesting types of "tube-waves" and "skin-waves",whose intensity is virtually concentrated to a one-dimensional,respectively two-dimensional,region of certain cylinders,and for the ground-vibrations.It is found that the polarisation is always <u>longitudinal</u> with respect to the direction of the current,parallel or antiparallel with the latter according to the solution in question;the correlation can be formulated in a simple way.

b) Formally our eigensolutions are of interest because of their being double-valued functions of the two azimuthal angles (φ , ψ) on the cylinders. They have this in common with Schrödinger's eigensolutions (loc.cit) whose dependence on the azimuthal angle of the polar coordinates on the hypersphere is also a double-valued one.

This departure from the usual single-valuedness of the eigenfunctions arises from having to assign half-odd eigennumbers (m , m') to the above mentioned generators of the rotation-group (essentially the momentum operators, angular and "linear"; in spherical space the difference is only a quantitative one). Consequently our eigenfunctions, depending on the two azimuthal angles by the factor $e^{i(m\varphi+m'\psi)}$, change sign if either φ or ψ is increased by a multiple of 2π. The theoretical admissibility of half-odd eigennumbers, implying double-valued wave-functions, has been secured by Schrödinger (S3). That it is they (and not the usual integral ones) which actually occur in our problem is necessitated by arguments developed and applied by Pauli in two other cases (P2). Our case, involving two azimuthal quantum numbers, is particularly appropriate for the exhibition of the efficacy of "Pauli's criterion".

Incidentally it turns out that Pauli's arguments can also conveniently be used for elimination of a continuous spectrum not yet excluded by the boundary condition of quadratic integrability of the eigenfunctions.

c) The eigenvalues of the energy of a free electron in spherical space are,of course,discrete;the main quantum number n is half-odd owing to the two azimuthal quantum numbers m and m' being half-odd.Another new and surprising feature is the degeneracy of the ground-state whose multiplicity is <u>four</u>(or rather more eight,counting the positive and negative eigenstates of the energy together).This is, however,in conformity with group-theoretical requirements (S2,p.331)from which one can deduce that the ground-state is $2(2|n'|+1)$-fold if the wave-function is of the tensor-rank $|n'|$(except for the scalar case, $n'=0$,where the ground-state is simple).In our case of the Dirac electron, we have $|n'|=\frac{1}{2}$.

d) As pointed out by Pauli,(loc. cit.),the occurrence of half-odd quantum numbers,implying double-valued wave-functions,is closely connected with the curious behaviour of the <u>general</u> Dirac equation under coordinate transformations.In general,this equation will not go over <u>directly</u> into the ordinary Dirac equation in flat space if we let the radius of curvature of space tend to infinity. It does so, however, provided that isotropic coordinates are chosen. To demonstrate this, we have, in the first chapter, written down the Dirac equation in spherical space in "coordinates of stereographic projection".

The latter coordinates are obtained by contemplating the hypersphere in Euclidean, four-dimensional, space and choosing <u>one of its points</u> ("north-pole") as a centre for projecting all its points on the three-dimensional, flat, tangential space at the <u>opposite point</u> ("south-pole"). These coordinates are then just ordinary Cartesian ones of the projected points in the tangential space.

It will be seen that, using <u>these</u> coordinates, there is no unexpected difference between the equation in spherical space and that in flat one. The spacial derivatives in the Hamiltonian are merely multiplied by a centrally symmetric, stereographic "gauge-factor" which becomes unity in the limiting case of flat space, the Hamiltonian in these coordinates thus exhibiting all the features familiar from the ordinary Dirac equation.

For the following our starting point is Schrödinger's form of the electron wave equation in a world(space-time) where the line-element is given by

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad (\mu, \nu = 1,2,3,4\,;\ x_4 = ct).$$

(Summation convention as usual.)

According to Schrödinger (S1) the general Dirac equation reads

(I.1) then:

$$\gamma^\nu \left(\frac{\partial}{\partial x_\nu} - \Gamma_\nu\right)\Psi = i\mu\Psi \quad \left(i = \sqrt{-1}\,,\ \mu = \frac{2\pi m_0 c}{h}\right).$$

The γ^ν are the analogons to Dirac's α-matrices. Now, however, they constitute matrix-<u>fields</u> depending on the metric by the relations *)

(I.2)

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\, g_{\mu\nu}\,;$$

$$\gamma^\nu = g^{\nu\mu}\, \gamma_\mu\,.$$

Also the Γ_ν are matrices depending on the coordinates(they do

(I.3) not, however, form a tensor). They are defined by the commutator

$$\Gamma_\nu\, \gamma_\mu - \gamma_\mu\, \Gamma_\nu = \frac{\partial \gamma_\mu}{\partial x_\nu} - \{\mu\nu, \sigma\}\, \gamma_\sigma\,,$$

$\{\mu\nu,\sigma\}$ denoting the Christoffel three-index symbols accoun-

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*) The fact that the matrix-equation (I.2) admits of a unitary transformation which may depend on the coordinates at each point is the root of the peculiar features of the Dirac equation (I.1) when represented in certain coordinates(cf.p.14).

ting for the parallel transport of a covariant vector, viz.

(I.4) $$\{\mu\nu,\sigma\} \equiv \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x_\nu} + \frac{\partial g_{\nu\lambda}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\lambda} \right).$$

As apparent from (I.1), the Γ_ν represent the parallel transport of the spinor Ψ, $(\frac{\partial}{\partial x_\nu} - \Gamma_\nu)\Psi$ denoting the covariant derivative of Ψ. On the other hand, the Γ_ν occur in (I.1) where in the ordinary Dirac equation the electromagnetic potentials A_ν occur. In fact, the A_ν have to be put into the diagonal elements of the matrices Γ_ν, (cf. S1), these elements not being determined otherwise, since Γ_ν is defined by a commutator. So, e.g. if Γ_4 is found to commute with all the γ^ν, and consequently equals a multiple of the unit-matrix, we shall take

(I.5) $$\Gamma_4 = -\frac{2\pi i e}{h c} V \equiv i \varphi_4 ,$$

$-V = A_4$ being the scalar potential.

Now we shall represent the general equation (I.1) by coordinates of stereographic projection and by cylindric coordinates in spherical space.

1) The Dirac Equations in Spherical Space in Stereographic Coordinates

In the Einstein universe the metric of space is that of the three-dimensional surface of a hypersphere of constant radius R, whereas the time-dimension remains uncurved (cf. El, p.155). To represent space alone, we introduce polar coordinates on the hypersphere:

(1.1)
$$\xi_1 = R\sin\chi\sin\vartheta\cos\varphi, \qquad \xi_3 = R\sin\chi\cos\vartheta,$$
$$\xi_2 = R\sin\chi\sin\vartheta\sin\varphi, \qquad \xi_4 = R\cos\chi \qquad ,$$

where
$$0 \le \chi \le \pi, \quad 0 \le \vartheta \le \pi, \quad 0 \le \varphi \le 2\pi, \quad R = \text{const}.$$

(ξ_4 is not to be confused with $x_4 = ct$ in (I.1) ; here we are concerned only with real, curved space.)

Hence
$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = R^2; \quad \xi_1^2 + \xi_2^2 + \xi_3^2 \equiv \rho^2 = R^2\sin^2\chi .$$

$\chi = 0$ is the point with coordinates $(0, 0, 0, R)$. Taking this point as the origin - this is, of course, quite arbitrary, since there is no absolute centre distinguished from other points - then in its neighbourhood ϑ, φ are ordinary polar coordinates and $R\chi$ can be regarded as radius-vector. The equator, $\chi = \vartheta = \frac{\pi}{2}$, is a great circle in the spherical distance $\frac{\pi}{2}$ from the origin. It represents the intersection of the (ξ_1, ξ_2)-plane with the hypersphere.

Differentiating (1.1), squaring and adding, we obtain the

(1.1)

line-element of the Einstein universe in Hyperspherical, polar coordinates,

$$d\sigma^2 = -\sum_1^4 d\xi_i^2 + c^2 dt^2 = -R^2 d\chi^2 - R^2 \sin^2\chi (d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) + c^2 dt^2 .$$

This is the line-element used by Schrödinger for the investigation of both the d'Alembertian and Dirac equation in spherical space.

S t e r e o g r a p h i c   c o o r d i n a t e s

Let us now perform a stereographic projection of the hypersphere on the tangential, three-dimensional, linear sub-space (R_3): $\xi_4 + R = 0$, the centre of projection being the point $(0,0,0,R)$. The transformed coordinates are then given by

$$x_1 = \frac{2R\xi_1}{R-\xi_4}, \quad x_2 = \frac{2R\xi_2}{R-\xi_4}, \quad x_3 = \frac{2R\xi_3}{R-\xi_4},$$

the new axes x_1, x_2, x_3 being parallel to ξ_1, ξ_2, ξ_3 respectively. Hence, expressing the ξ by polar coordinates (1.1),

$$x_1 = 2R\cot\frac{\chi}{2}\sin\vartheta\cos\varphi, \quad x_2 = 2R\cot\frac{\chi}{2}\sin\vartheta\sin\varphi, \quad x_3 = 2R\cot\frac{\chi}{2}\cos\vartheta,$$

whence the radius-vector in our tangential (R_3)

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} = 2R\cot\frac{\chi}{2}.$$

Conversely we find

$$\xi_1 = \frac{x_1}{1+\frac{r^2}{4R^2}} = \frac{r}{1+\frac{r^2}{4R^2}}\sin\vartheta\cos\varphi, \quad \xi_2 = \frac{x_2}{1+\frac{r^2}{4R^2}} = \frac{r}{1+\frac{r^2}{4R^2}}\sin\vartheta\sin\varphi,$$

$$\xi_3 = \frac{x_3}{1+\frac{r^2}{4R^2}} = \frac{r}{1+\frac{r^2}{4R^2}}\cos\vartheta, \quad \xi_4 = R\frac{1-\frac{r^2}{4R^2}}{1+\frac{r^2}{4R^2}}; \tag{1.2}$$

$$\rho = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2} = R\sin\chi = \frac{r}{1+\frac{r^2}{4R^2}}.$$

Differentiating (1.2), squaring, and adding, we obtain

$$\sum_1^4 d\xi_i^2 = \left(1+\frac{r^2}{4R^2}\right)^{-2}\left(dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta \, d\varphi^2\right).$$

Hence the line-element of the Einstein universe can be written:

$$ds^2 = -\frac{dx_1^2+dx_2^2+dx_3^2}{\left(1+\frac{r^2}{4R^2}\right)^2} + c^2 dt^2 \quad .$$

We may choose units such that $c=1$; our metric tensor is then clearly given by

(1.3a) $$g_{11} = g_{22} = g_{33} = -\left(1+\frac{r^2}{4R^2}\right)^{-2}, \quad g_{44} = +1 ; \quad g_{\mu\nu} = 0 \quad \text{if } \mu \neq \nu ,$$

whence the square root of minus the determinant

(1.3b) $$\sqrt{-g} = \frac{1}{\left(1+\frac{r^2}{4R^2}\right)^3} \equiv \frac{1}{K^3} \quad .$$

It is obvious from (1.3a) that our coordinates are orthogonal and <u>locally</u> isotropic,but the actual length is $\frac{1}{1+\frac{r^2}{4R^2}}$ times the Euclidean length.The points on the hypersphere are here projected into three-dimensional Euclidean space with the variable gauge factor $\frac{1}{K}$,similar to the stereographic projection of the points on an ordinary sphere into the infinite plane touching the sphere at its south-pole.

T h e   D i r a c   e q u a t i o n   Now let us see what the Dirac equation (I.1) is like in our stereographic coordinates.Owing to the local isotropy,we may expect a great resemblance to the ordinary Dirac equation in flat space in Cartesian coordinates.The only difference can be the appearance of the stereographic gauge factor $K = 1+\frac{r^2}{4R^2}$.

For orthogonal coordinates the Christoffel three-index symbols (I.4) become

(1.4') $$\{\mu\nu,\sigma\} = \frac{1}{2} g^{\sigma\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu} + \frac{\partial g_{\nu\sigma}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\sigma} \right),$$

(no sum)

whence

$\{\mu\nu,\sigma\} = 0$ unless, at least, two indices are equal.

The non-vanishing symbols work out in our case

$$-\{\mu\mu,\mu\} = -\{\mu\nu,\nu\} = \{\nu\nu,\mu\} = \frac{1}{2R^2}\frac{x_\mu}{K} \quad (\mu,\nu = 1,2,3;\ \mu\neq\nu).$$

The matrix-fields γ as found from (I.2) and (1.3) are

(1.4) $$\gamma_\ell = -\frac{i}{K}\alpha_\ell, \quad \gamma^\ell = iK\alpha_\ell, \quad \gamma_4 = \gamma^4 = \alpha_4 \quad (\ell = 1,2,3),$$

the α's forming an ordinary set of anticommuting Dirac matrices whose square equals unity,

(1.5) $$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu} \quad (\mu,\nu = 1,2,3,4).$$

Apart from their spurs, the matrices Γ_ν, describing the parallel transport of the spinor, are determined from (I.3). As said on p.8, we put $\frac{2\pi i e}{hc}A_\nu$, the potentials, into the undetermined diagonal elements of the Γ_ν. Thus we obtain

(1.6) $$\Gamma_1 = \frac{1}{4R^2}\frac{1}{K}(x_2\alpha_1\alpha_2 + x_3\alpha_1\alpha_3) + \frac{2\pi i e}{hc}A_1, \quad \Gamma_4 = -\frac{2\pi i e}{hc}V = i\varphi_4 .$$

Γ_2 and Γ_3 are obtained from Γ_1 by cyclic permutation of the suffixes 1, 2, 3.

Hence, compounding according (I.1) the Dirac equation with (1.4) and (1.6), we get

$$K\sum_{\ell=1}^{3}\left(\alpha_\ell\frac{\partial\Psi}{\partial x_\ell} - \frac{x_\ell}{2R^2K}\Psi - \frac{2\pi i e}{hcK}A_\ell\Psi\right) - i\alpha_4\left(\frac{\partial\Psi}{c\,\partial t} - i\varphi_4\Psi\right) = \frac{2\pi m_0 c}{h}\Psi .$$

Substituting

$$\Phi = \frac{1}{K}\Psi$$

and multiplying by $\frac{h}{2\pi}\alpha_4$, this reads:

$$\frac{h}{2\pi i}\left(\frac{\partial\Phi}{c\,\partial t} - i\varphi_4\Phi\right) = K\sum_{\ell=1}^{3} i\alpha_\ell\alpha_4\left(\frac{h}{2\pi i}\frac{\partial\Phi}{\partial x_\ell} - \frac{e}{c}A_\ell\Phi\right) + \alpha_4 m_0 c\,\Phi .$$

We may replace $i\alpha_4\alpha_1$, $i\alpha_4\alpha_2$, $i\alpha_4\alpha_3$, $-\alpha_4$ by α_1, α_2, α_3, α_4 respectively.(This is allowed, since both sets satisfy the same commutation relations (1.5). It amounts to a permissible S-transformation: $\Phi \to S^{-1}\Phi$ and $i\alpha_4\alpha_\ell \to S^{-1}\alpha_\ell S$, $-\alpha_4 \to S^{-1}\alpha_4 S$, with $S^{-1} = \alpha_1\alpha_2\alpha_3\alpha_4 \frac{1+i\alpha_4}{\sqrt{2}}$, $S = \frac{1-i\alpha_4}{\sqrt{2}}\alpha_1\alpha_2\alpha_3\alpha_4$. The factor $\alpha_1\alpha_2\alpha_3\alpha_4$ is to transform α_4 into $-\alpha_4$.) Assuming that the vector potential vanishes and that the scalar potential φ_4 is centrally symmetric, we obtain the equation

(1.7) $$\frac{h}{2\pi i}\left(\frac{\partial\Phi}{c\,\partial t} - i\varphi_4(r)\,\Phi\right) + \frac{h}{2\pi i} K \sum_{\ell=1}^{3} \alpha_\ell \frac{\partial\Phi}{\partial x_\ell} + \alpha_4 m_0 c\, \Phi = 0 \quad .$$

C o n s t a n t   o f   m o t i o n   As expected, the only difference between our equation (1.7) and the ordinary Dirac equation in flat space is the centrally symmetric, stereographic factor $K = 1 + \frac{r^2}{4R^2}$. Evidently, therefore, the operator

$$m_3 = \frac{h}{2\pi i}\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) = \frac{h}{2\pi i}\frac{\partial}{\partial\varphi}$$

does not commute with our Hamiltonian, since m_3 does not commute with Dirac's Hamiltonian in flat space. Exactly as in flat space, we have to add a spin-term

$$\frac{1}{2}\frac{h}{2\pi i}\alpha_1\alpha_2$$

to m_3 in order to obtain a constant of motion, m_3 thus clearly corresponding <u>here</u>, as in flat space, to the azimuthal <u>orbital</u> angular momentum.

C o m p a r i s o n   w i t h   t h e   D i r a c   e q u a t i o n   i n   b o t h   D i r a c ' s   a n d   S c h r ö d i n g e r ' s   f o r m   In Schrödinger's representation of the Dirac equation in hyperspherical, <u>polar</u> coordinates (S2) (as shown in the second part of our paper, the same is true when the equation is represented in cylindric coordinates)the operator $\frac{h}{2\pi i}\frac{\partial}{\partial\varphi}=\frac{h}{2\pi i}\left(\xi_1\frac{\partial}{\partial\xi_2}-\xi_2\frac{\partial}{\partial\xi_1}\right)$ <u>is</u> a constant of motion in spherical space as well as in the limiting case of flat one(i.e. $R\to\infty$, $\sin\chi\sim\chi\to 0$, $R\sin\chi\to\rho$ (finite)).It <u>does</u> commute with the Hamiltonian in Schrödinger's representation:

(1.8) $$-\frac{1}{c}\frac{\partial}{\partial t}=-i\varphi_4(\rho)+\frac{\alpha_1}{\rho}\left(\frac{\partial}{\partial\vartheta}+\frac{1}{2}\cot\vartheta\right)+\frac{\alpha_2}{\rho\sin\vartheta}\frac{\partial}{\partial\varphi}+\alpha_3\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)+\frac{2\pi i m_0 c}{h}\alpha_4 .$$

Evidently,therefore, $\frac{h}{2\pi i}\frac{\partial}{\partial\varphi}$ corresponds <u>there</u> to the <u>total</u> azimuthal angular momentum.

It is very peculiar that the physical meaning of the operator $\frac{h}{2\pi i}\frac{\partial}{\partial\varphi}$ can be changed by an ordinary coordinate transformation(from Cartesian to polar ones in this case).The root of this peculiarity is to be found in the fact that equation (I.2) admits of a unitary transformation which may depend on the <u>coordinates</u>.Therefore,as pointed out by Pauli,(P2,p.151), it is characteristic for the general equation (I.1) to require an additional spin transformation <u>depending on the coordinates</u>,if the original form is to be retained after coordinate transformation.(The same is also true for the general Dirac equation as derived by V.Fock,Zs.f.Phys.,<u>57</u>,1929,p.270.)

In the limiting case of flat space(i.e. $K=1+\frac{r^2}{4R^2}\to 1$)our equation in stereographic coordinates,(1.7),goes <u>directly</u> over into the ordinary Dirac equation(apart from the S-transformation with constant coefficients performed on p.13).This is not so with the equation in polar coordinates,(1.8).As shown by Pauli,(P2,p.165),the <u>variable</u> spin transformation leading from the latter to the ordinary Dirac equation is

$$S(\vartheta,\varphi)=e^{i s_2 \frac{\vartheta}{2}}\, e^{i s_3 \frac{\varphi}{2}} = \cos\frac{\vartheta}{2}\cos\frac{\varphi}{2} - i\sin\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cdot s_1 + i\sin\frac{\vartheta}{2}\cos\frac{\varphi}{2}\cdot s_2 + i\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cdot s_3 ,$$

whence

$$S^{-1}(\vartheta,\varphi)=e^{-i s_3 \frac{\varphi}{2}}\, e^{-i s_2 \frac{\vartheta}{2}} = \cos\frac{\vartheta}{2}\cos\frac{\varphi}{2} + i\sin\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cdot s_1 - i\sin\frac{\vartheta}{2}\cos\frac{\varphi}{2}\cdot s_2 - i\cos\frac{\vartheta}{2}\sin\frac{\varphi}{2}\cdot s_3 ,$$

s_1, s_2, s_3 being respectively the three Pauli matrices $-i\alpha_2\alpha_3$, $-i\alpha_3\alpha_1$, $-i\alpha_1\alpha_2$.Now,it can easily be verified that indeed

$$S^{-1}(\vartheta,\varphi)\,\frac{h}{2\pi i}\frac{\partial}{\partial\varphi}\,S(\vartheta,\varphi) = \frac{h}{2\pi i}\frac{\partial}{\partial\varphi} + \frac{1}{2}\frac{h}{2\pi i}\alpha_1\alpha_2$$

showing how the spin term $\frac{1}{2}\frac{h}{2\pi i}\alpha_1\alpha_2$ was absorbed in the operator $\frac{h}{2\pi i}\frac{\partial}{\partial\varphi}$ in Schrödinger's representation;the S -transformation between (1.7) and (1.8) is just such as to transform the azimuthal <u>orbital</u> angular momentum into the <u>total</u> one.

Using Dirac's well known method,one can now easily transform our equation (1.7) to polar coordinates r, ϑ, φ.The angular part of our equation is,then,the same as in the customary Dirac equation,whereas the radial part differs by the stereographic factor K.We shall not,however,deal here with these radial equations;we prefer,instead of it,to introduce the more interesting "cylindric coordinates on the hypersphere".

2) The Dirac Equations in Spherical Space in Cylindric Coordinates

A)

C y l i n d r i c   c o o r d i n a t e s   Now we introduce coordinates by the help of which the eigensolutions of the Dirac equations in spherical space provide that representation of the six-dimensional group of rotations in which two of the six generators(infinitesimal rotations in totally perpendicular(dual)planes)are diagonal.These coordinates are very closely related to ordinary toroidal ones.They actually are the exact analogon of cylindric coordinates in flat space;for one of the three sets of coordinate surfaces is a family of co-axial cylinders(loci of all points that have the same distance from a "straight line",i.e.from a great circle). The other two sets are two families of great spheres,the one passing through a given great circle,the other through the polar one(i.e.through the"axis"of the given circle).These coordinates have also been used by P.O.Müller (M1) dealing with the d'Alembertian in spherical space.Our notation is slightly different from his,our angle ω (measuring the radius of the cylinder)being identical with the complement of his angle ϑ .

Designating the angle of rotation in the (x_1, x_2)-plane by φ and that in the (x_3, x_4)-plane(the dual plane to the former)by ψ ,we have,instead of (1.1),

(2.1)
$$x_1 = R\cos\omega\cos\varphi, \qquad x_3 = R\sin\omega\sin\psi,$$
$$x_2 = R\cos\omega\sin\varphi, \qquad x_4 = R\sin\omega\cos\psi,$$

where,in order to comprise all points on the hypersphere just once,

$$0 \leq \omega \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi; \quad R = const.$$

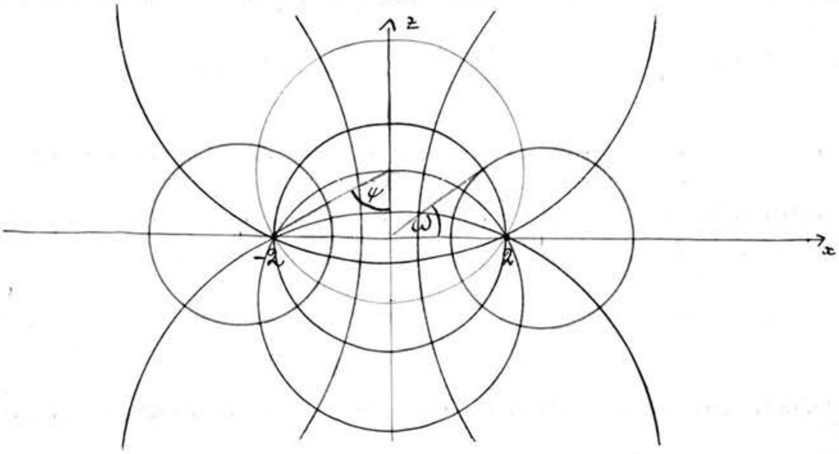
The spacial part of the line-element takes then the form

(2.2)
$$\sum_1^4 dx_i^2 = R^2 d\omega^2 + R^2\cos^2\omega\, d\varphi^2 + R^2\sin^2\omega\, d\psi^2 .$$

From this form of the line-element it is evident that the surfaces $\omega = const$, $\varphi = const$, and $\psi = const$ are orthogonal on each other and that the two-dimensional geometry on any cylinder $\omega = const$ is Euclidean.The line-element of ordinary cylindric coordinates is covered by (2.2) as the limiting case

$$R \to \infty, \ \sin\omega \sim \omega \to 0, \ \frac{R\omega}{R\sin\omega} \to \rho(\text{finite}), \ R\,d\varphi \to dz;$$
$$ds^2 = d\rho^2 + dz^2 + \rho^2 d\psi^2 .$$

Stereographic projection and relation to toroidal coordinates Performing a stereographic projection of the hypersphere (2.1) into the tangential,three-dimensional, linear sub-space $R_3(x,y,z)$: $x_4 + R = 0$,the surfaces of constant φ (the family of great spheres passing through the great circle $\omega = \frac{\pi}{2}$)become the planes through the z -axis: $\tan\varphi = \frac{y}{x}$ .The surfaces of constant ω (co-axial cylinders)and of constant ψ (great spheres passing through the great circle $\omega = 0$ which is the axis of the great circle through which the spheres $\varphi = const$ pass)are symmetric around the z -axis.Their cross-section with the (x , z)-plane(i.e. $y=0$)gives a system of ortogonal circles whose centres are situated on the x-axis and on the z-axis respectively(cf.M1,p.368).



The above figure shows the cross-section of the tori $\omega = const$ (red) and the spheres $\psi = const$ (green) with the plane $\varphi = 0$ (i.e. $y = 0$).The equation of the red circles is: $(x-\frac{2}{\cos\omega})^2+z^2=(2\tan\omega)^2$, that of the green ones: $(z-2\cot\psi)^2+x^2=(\frac{2}{\sin\psi})^2$. ω is the angle between the x-axis and the tangent drawn from the origin at the torus ω . ψ is the angle between the radius of the sphere ψ (or rather half-sphere),through $x=2$,and the z-axis.In our units $R=1$.

This picture is well known from the equipotentials($\omega = const$) and lines of constant electric field strength($\psi = const$)produced by two oppositely charged,parallel($\perp$ (x , z)-plane)wires through the two points(red)in which our null-torus($\omega = 0$:circle of radius 2 in the (x , y)-plane)crosses the (x , z)-plane.
(It can elementarily be proved that for any point on a circle

$\omega = const$ the ratio - on which the potential depends - of the distances from these two (red) points equals $\tan\frac{\omega}{2}$.) We have simply to rotate this figure around the z-axis in order to get an exact idea of the tori $\omega = const$ and the spheres $\psi = const$ in the stereographic projection of our cylindric coordinates.

The "south-pole" of the hypersphere (1.1) , $\chi = \pi$,(where our $R_3(x,y,z)$ touches the hypersphere)is,of course,the origin, $x=y=z=0$,in R_3 ,the "north-pole, $\chi = 0$,corresponding to the points at infinity in R_3 .The "equator", $\chi = \frac{\pi}{2}$,a great sphere, is represented in R_3 by the sphere of radius 2 around the origin.All tori $\omega = const$ are intersected at right angles by this sphere;its upper half($z > 0$)corresponds to $\psi = \frac{\pi}{2}$,the lower one ($z < 0$)corresponding to $\psi = \frac{3\pi}{2}$.On <u>this</u> sphere the parallels are directly given by $\frac{\pi}{2} - \omega$,our null-torus thus being the equator-circle on the equator-sphere.

The singularities of our coordinate system are obviously at the two,polar,great circles $\omega = 0$ and $\omega = \frac{\pi}{2}$ (z-axis in the stereographic projection),all planes $\varphi = const$ passing through the z-axis and all spheres $\psi = const$ passing through the null-torus.The <u>reciprocal</u> relation between these two great circles ~~on the hypersphere~~(each there playing the role of the axis of the other) is,of course,no longer maintained in the stereographic image.-

On any torus $\omega = const$ the curves $\varphi = const$ and $\psi = const$ form an orthogonal system of longitudes and latitudes, and the same holds true on the hyperspherical cylinders.

The relation between our coordinates ω, ψ, φ and the so-called torus-coordinates λ, α, φ' (cf.Rl,p.1o3) is a very close one; φ' is identical with our φ, α equals minus the supplement of our ψ, and our angle ω is the complement of the gudermannian of λ,

$$\varphi = \varphi' \ , \quad \psi = \alpha + \pi \ , \quad \frac{1}{\sin\omega} = \cosh\lambda \ .$$

T h e D i r a c e q u a t i o n Now let us write the Dirac equations (I.1) in spherical space in the cylindric coordinates (2.1). Taking units such that both R and c equal unity, the line-element of the Einstein universe reads:

$$ds^2 = -d\omega^2 - \cos^2\omega\, d\varphi^2 - \sin^2\omega\, d\psi^2 + dt^2 \ .$$

Hence the metric tensor is given by

(2.3a) $$g_{11} = -1, \quad g_{22} = -\cos^2\omega, \quad g_{33} = -\sin^2\omega, \quad g_{44} = +1; \quad g_{\mu\nu} = 0 \ \text{if}\ \mu \neq \nu,$$

whence the volum-element of spherical space

(2.3b) $$d\tau = \sqrt{-g}\, d\omega\, d\varphi\, d\psi = \sin\omega\cos\omega\, d\omega\, d\varphi\, d\psi \ .$$

Since the azimuthal angles φ and ψ do not appear in the metric (2.3a), they will not appear explicitly in the Dirac equation in the case of vanishing potential. Owing to this, the variables can then be separated in a very simple way.

For the matrix-fields γ we find from (I.2) and (2.3a)(taking the positive sign for the square root)

(2.4)
$$\gamma_1 = i\alpha_1 \,, \quad \gamma_2 = i\cos\omega\cdot\alpha_2 \,, \quad \gamma_3 = i\sin\omega\cdot\alpha_3 \,, \quad \gamma_4 = \alpha_4 \,;$$
$$\gamma^1 = -i\alpha_1 \,, \quad \gamma^2 = -\frac{i\alpha_2}{\cos\omega} \,, \quad \gamma^3 = -\frac{i\alpha_3}{\sin\omega} \,, \quad \gamma^4 = \alpha_4 \,,$$

the α'_o forming an ordinary set of anticommuting Dirac matrices,(1.5).

Due to the simple metric,the non-vanishing Christoffel three-index symbols are obviously those with two equal indices $\neq 1$ (and,of course, different from 4)whereas the other index equals 1 .According to (1.4') we have

$$\{12,2\} = -\tan\omega \,, \quad \{13,3\} = \cot\omega \,, \quad \{22,1\} = \sin\omega\cos\omega = -\{33,1\} \,,$$

all the other symbols vanishing.

The matrices Γ_ν ,as found from (I.3) and (I.5),work out

(2.5)
$$\Gamma_1 = 0 \,, \quad \Gamma_2 = \frac{1}{2}\sin\omega\cdot\alpha_1\alpha_2 \,, \quad \Gamma_3 = \frac{1}{2}\cos\omega\cdot\alpha_1\alpha_3 \,, \quad \Gamma_4 = -\frac{2\pi i\varepsilon}{hc}V \equiv i\varphi_4 \,.$$

Inserting (2.4) and (2.5) into (I.1),we obtain the Dirac equation in the Einstein universe in cylindric coordinates:

(2.6)
$$-i\alpha_1\frac{\partial\Psi}{\partial\omega} - i\frac{\alpha_2}{\cos\omega}\left(\frac{\partial\Psi}{\partial\varphi} + \frac{1}{2}\sin\omega\cdot\alpha_1\alpha_2\Psi\right) -$$
$$-i\frac{\alpha_3}{\sin\omega}\left(\frac{\partial\Psi}{\partial\psi} + \frac{1}{2}\cos\omega\cdot\alpha_3\alpha_1\Psi\right) + \alpha_4\left(\frac{\partial\Psi}{\partial t} - i\varphi_4\Psi\right) = i\mu\Psi \,.$$

Multiplying by α_4, and putting

(2.6')
$$\Phi = \Phi(\omega,\varphi,\psi,t) = \sqrt{\sin\omega\cos\omega}\;\Psi(\omega,\varphi,\psi,t) \,,$$

the Hamiltonian form of (2.6) reads:

(2.7a)
$$\frac{\partial\Phi}{\partial t} = i\alpha_4\alpha_1\frac{\partial\Phi}{\partial\omega} + \frac{i\alpha_4\alpha_2}{\cos\omega}\frac{\partial\Phi}{\partial\varphi} + \frac{i\alpha_4\alpha_3}{\sin\omega}\frac{\partial\Phi}{\partial\psi} + i\varphi_4\Phi + i\mu\alpha_4\Phi \,.$$

In ordinary units of length and time this becomes

(2.7b) $$\frac{\partial \Phi}{c\partial t} = i\alpha_4\alpha_1 \frac{1}{R}\frac{\partial \Phi}{\partial \omega} + \frac{i\alpha_4\alpha_2}{\cos\omega}\frac{1}{R}\frac{\partial \Phi}{\partial \varphi} + \frac{i\alpha_4\alpha_3}{\sin\omega}\frac{1}{R}\frac{\partial \Phi}{\partial \psi} + i\varphi_4\Phi + i\mu\alpha_4\Phi \; .$$

T h e c o n s t a n t s o f m o t i o n As expected for the case of vanishing potential, both the azimuthal angles φ and ψ do not occur in (2.7) but in form of the derivatives $\frac{\partial}{\partial\varphi}$, $\frac{\partial}{\partial\psi}$. We can, therefore, separate the spacial part of the eigenfunctions of (2.7) into three factors, each depending on only <u>one</u> of the angles ω, φ, ψ. The dependence on φ and ψ is that of the eigenfunctions of the operators $-i\frac{\partial}{\partial\varphi}$, $-i\frac{\partial}{\partial\psi}$ respectively. The stationary solutions of equ. (2.7) are therefore of the form

(2.8) $$\Phi(\omega,\varphi,\psi,t) = \Omega(\omega)\cdot e^{i(m\varphi + m'\psi + nt)} \; .$$

Here n denotes the eigenvalues of the Hamiltonian; in ordinary units we have

(2.9) $$n = \frac{2\pi\nu R}{c} =$$ number (per great circle $2\pi R$) of wave-lengths of light corresponding to the eigenfrequency ν. m and m' are respectively the eigenvalues of the operators

(2.1o) $$M_3 = -i\frac{\partial}{\partial\varphi} \quad , \quad N_3 = -i\frac{\partial}{\partial\psi} \; .$$

These operators describe dual, infinitesimal rotations in the (x_1, x_2)-plane (i.e. $\omega = 0$) and (x_3, x_4)-plane (i.e. $\omega = \frac{\pi}{2}$) respectively. They represent two of the six generators of the rotation-group in spherical space (cf. S2, p. 333). Physically they correspond to the angular and linear momentum in the x_3-direction, from the point of view of a local geometer at $x_1 = x_2 = x_3 = 0$, $x_4 = R$, cf. (2.1). In spherical space the two kinds of momenta are on equal footing, the difference being only a quantitative one, viz. how far away the axis of rotation is situated.

Alternative of double-valued eigenfunctions What are the admissible values of the eigennumbers m and m' depends, of course, on the conditions imposed upon the wave funtions. If we demand the latter to be single-valued, as is usually done, then both m and m' must be integers, since the range of the azimuthal angles φ, ψ is $0 \leq \varphi, \psi \leq 2\pi$. However, as pointed out by Schrödinger (S3; S2, p.348) and Pauli (P1, p.126; P2), (cf. also E2 and T1), there is no apriori argument to discard half-integral azimuthal quantum numbers. If, e.g., m in (2.8) equals half of an odd integer, then the eigenfunctions simply change sign if φ is increased by 2π, whereas all physical quantities (since they are bilinear in Ψ and its complex-conjugate, Ψ^*) remain single-valued. Schrödinger proved that in general only <u>either</u> single-valued eigenfunctions <u>or</u> double-valued ones can and must be admitted in any given wave-mechanical problem, the two branches of the double-valued ones differing only by the factor -1 (S3). Accordingly we have for both m and m' two alternatives:

(2.11a) either $2m = 0, \pm 2, \pm 4, \pm 6, \dots$

(2.11b) or $2m = \pm 1, \pm 3, \pm 5, \pm 7, \dots,$

and independently:

(2.12a) either $2m' = 0, \pm 2, \pm 4, \pm 6, \dots$

(2.12b) or $2m' = \pm 1, \pm 3, \pm 5, \pm 7, \dots .$

The ambiguity which of the four possibilities must actually be chosen for our eigenfunctions can be removed by means of"Pauli's criterion" (P2).The latter makes use of the existence of a group of transformations of the Hamiltonian, i.e.the six-dimensional rotation-group in our case, assuming that the potentials vanish.The criterion simply demands that the eigenfunctions transfotm as the irreducible representations of the group.When in possession of the general expression of our eigensolutions,this condition will prove to be of the same efficacy in our case as in those demonstrated by Pauli (P2).

Inserting (2.8) into (2.7a),multiplying by $i\alpha_4\alpha_1$,and remembering the relations (1.5) satisfied by the α'_s ,we obtain

$$\frac{d\Omega(\omega)}{d\omega} + \frac{m}{\cos\omega} i\alpha_1\alpha_2 \Omega(\omega) + \frac{m'}{\sin\omega} i\alpha_1\alpha_3 \Omega(\omega) - n\alpha_1\alpha_4\Omega(\omega) + \mu\alpha_1\Omega(\omega) = 0 \quad (2.7')$$

This is a four-componential equation, $\Omega(\omega)$forming a column on which the α -matrices operate.For economy of paper,however, we write it as a row instead of as a column, (2.13) $$\Omega(\omega) = \left(\Omega_1(\omega), \Omega_2(\omega), \Omega_3(\omega), \Omega_4(\omega) \right).$$

Now let us first consider the case of vanishing mass, ($\mu = 0$), (and vanishing potential, $\varphi_4 = 0$). The solutions for the case of non-vanishing mass can <u>then</u> easily be derived.

B) Provisional restriction to the case of vanishing mass

Omitting the last ~~two~~ term~~s~~ in (2.7'), we obtain

(2.14) $$\frac{d\Omega(\omega)}{d\omega} + \frac{m}{\cos\omega} i\alpha_1\alpha_2 \Omega(\omega) + \frac{m'}{\sin\omega} i\alpha_1\alpha_3 \Omega(\omega) - n\alpha_1\alpha_4 \Omega(\omega) = 0 \quad ,$$

n and $\Omega(\omega)$ <u>henceforth</u> denoting the eigenvalue of the energy, (2.9), and the wave function for <u>vanishing</u> mass. $\Omega(\omega)$ depends on ω ($0 \leq \omega \leq \frac{\pi}{2}$) and on the two spin-variables, each of which can assume only two values; they appear as suffixes of the wave function, as denoted in (2.13).

Choice of a convenient representation of α-matrices

Now we introduce van der Waerden's notation, (W1, §20), by which every suffix is replaced by a pair of suffixes, dotted and undotted ones, viz.

(2.15) $$\Omega(\omega) = \left(\Omega_1(\omega), \Omega_2(\omega), \Omega_3(\omega), \Omega_4(\omega) \right) = \left(\Omega_{1\dot{1}}(\omega), \Omega_{2\dot{1}}(\omega), \Omega_{1\dot{2}}(\omega), \Omega_{2\dot{2}}(\omega) \right).$$

Accordingly the quadratic four-row α -matrices are considered as the "direct product" of two quadfatic two-row matrices out of two"different"sets of Pauli matrices,dotted and undotted ones.These operate separately on the dotted, respectively undotted,suffix of the wave function and commute,therefore,with each other.Let us take

(2.16a) $$\beta_1 \text{ and } \dot{\beta}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta_2 \text{ and } \dot{\beta}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \beta_3 \text{ and } \dot{\beta}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These β -matrices satisfy the following relations:

(2.16b) $$\beta_1\beta_2 + \beta_2\beta_1 = 0, \quad \beta_1\beta_2 = -i\beta_3, \quad \beta_1^{\,2} = 1,$$

and the analogous equations obtained by cyclic permutation of 1 , 2 , 3 .The dotted matrices satisfy,of course,the same relations.

The "direct product" of an undotted β -matrix into a dotted one is defined as the quadratic four-row matrix which results from inserting the undotted β -matrix into the scheme of the dotted one.

Using the commutation relations (1.5), it can easily be verified that the four-row matrices $i\alpha_1\alpha_2$, $i\alpha_3\alpha_1$,occurring in (2.14),and $i\alpha_3\alpha_2$ satisfy the same equations (16b) as the two-row β -matrices.We may,therefore,represent $i\alpha_1\alpha_2$, $i\alpha_3\alpha_1$, and $i\alpha_3\alpha_2$ by the direct product of the (undotted) β-matrices (2.16a) into the (dotted) unit-matrix. Say

(2.17a)
$$i\alpha_1\alpha_2 = \beta_2\dot{1} = \begin{pmatrix} & i & & \\ -i & & & \\ & & & i \\ & & -i & \end{pmatrix}, \quad i\alpha_3\alpha_1 = \beta_3\dot{1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}, \quad i\alpha_3\alpha_2 = \beta_1\dot{1} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}.$$

(The free places in the matrices are zeros.)

For $\alpha_1\alpha_4$ we take

(2.17b)
$$\alpha_1\alpha_4 = -i\beta_1\dot{\beta}_3 = \begin{pmatrix} & -i & & \\ -i & & & \\ & & & i \\ & & i & \end{pmatrix}$$

Hence we have the following representation of α-matrices:

(2.17c)
$$\alpha_1 = \beta_1\dot{\beta}_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \quad \alpha_2 = -\beta_3\dot{\beta}_1 = \begin{pmatrix} & & -1 & \\ & & & 1 \\ -1 & & & \\ & 1 & & \end{pmatrix}, \quad \alpha_3 = -\beta_2\dot{\beta}_1 = \begin{pmatrix} & & & -i \\ & & i & \\ & -i & & \\ i & & & \end{pmatrix}, \quad \alpha_4 = 1\dot{\beta}_2 = \begin{pmatrix} & & i & \\ & & & i \\ -i & & & \\ & -i & & \end{pmatrix}, \quad \alpha_5 = \alpha_1\alpha_2\alpha_3\alpha_4 = 1\dot{\beta}_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

Now, expressing in our equation (2.14) the α-matrices by the direct products $\beta_i\dot{\beta}_k$, we obtain

(2.18)
$$\frac{d\Omega(\omega)}{d\omega} + \frac{m}{\cos\omega}\beta_2\dot{1}\,\Omega(\omega) - \frac{m'}{\sin\omega}\beta_3\dot{1}\,\Omega(\omega) + in\beta_1\dot{\beta}_3\,\Omega(\omega) = 0 \quad .$$

S p l i t t i n g o f t h e f o u r e q u a t i o n s i n t o t w o s i m i l a r s e t s o f t w o e q u a t i o n s   Since the matrix $\dot{\beta}_3$ commutes with each term in (2.18), we can split $\Omega(\omega)$ into two factors. The one factor is the function depending on ω and on the undotted spin-variable $(1, 2)$; we shall denote this function by $\Omega_0(\omega; 1, 2) \equiv \Omega_0$. The other factor is the eigenfunction of $\dot{\beta}_3$, belonging respectively to the eigenvalue $+1$ and -1 of $\dot{\beta}_3$. The eigenfunction to $+1$ vanishes "at" $\dot{2}$, the eigenfunction to

-1 vanishes "at" $\dot{1}$.

Thus, having separated the dotted spin-variable from the undotted one, we obtain for the function $\Omega_0(\omega; 1,2)$ the equation

(2.19) $$\frac{d\Omega_0}{d\omega} + \frac{m}{\cos\omega}\beta_2\Omega_0 - \frac{m'}{\sin\omega}\beta_3\Omega_0 \pm i n \beta_1 \Omega_0 = 0 .$$

The two signs in the last term refer to the two eigenvalues of the matrix $\dot{\beta}_3$, $+1$ and -1 respectively. From any one of the <u>two</u>-componential solutions Ω_0 - corresponding to a definite value of n and to, say, the upper sign - we therefore obtain two <u>four</u>-componential solutions Ω by multiplying Ω_0 by one or the other of the <u>eigenfunctions of $\dot{\beta}_3$</u>. (Obviously, one of these Ω belongs to the eigenvalue n, the other one to $-n$.) Following (2.15) this amounts to

i) taking the two components of Ω_0 to form the <u>first</u> two components (Ψ_1, Ψ_2) of Ψ (belonging to n), the <u>second</u> pair (Ψ_3, Ψ_4) being zeros, and

ii) taking the two components of Ω_0 to form the <u>second</u> pair (Ψ_3, Ψ_4) of Ψ (belonging to $-n$), the <u>first</u> pair (Ψ_1, Ψ_2) being zeros.

Putting

(2.20) $$\Omega_0(\omega; 1, 2) = (f(\omega) , g(\omega)) ,$$

and inserting the matrices (2.16a) into equation (2.19), we obtain two simultaneous differential equations determining the two functions (f, g) to a common constant factor(which can be fixed by normalisation),

$$\left.\begin{aligned} \frac{df}{d\omega} + i\frac{m}{\cos\omega}g - \frac{m'}{\sin\omega}f + ing &= 0 \\ \frac{dg}{d\omega} - i\frac{m}{\cos\omega}f + \frac{m'}{\sin\omega}g + inf &= 0 \end{aligned}\right\} \quad (2.21)$$

To fix the ideas,we have taken the positive sign in the last term of equ.(2.19).According to the above(p.28)this means that we are focussing our attention on those eigensolutions which belong to the eigenvalue $+1$ of $\dot{\beta}_3$;i.e.(Ψ_1, Ψ_2) being formed by (f, g) and (Ψ_3, Ψ_4) being zeros.

R e s t r i c t i o n t o n o n - n e g a t i v e v a l u e s o f m a n d m'. From (2.21) it is evident that if

(f, g) is the solution to m , m', n ,

then

(2.22)
a) (g, f) is the solution to $-m$, $-m'$, n ,
b) (f, $-g$) is the solution to $-m$, m', $-n$,
c) (g, $-f$) is the solution to m , $-m'$, $-n$.

Owing to this scheme,we need not solve (2.21) for both positive and negative sign of m and m' but may confine our consideration to,say,positive sign of both m and m'.No similar restriction is permissible for n .For,e.g.,with $n=-5$ and $\dot{\beta}_3=-1$ (type i)p.28)one obtains, by the above scheme b) or c), a solution Ψ belonging to $n=+5$(and $\dot{\beta}_3=-1$,as before)which is,of

2 **course**, **different** from the solution obtained with 2 and 2 (type ii) p.28) and which would therefore be dropped by dropping negative n'_0. I say it is **of course** different, for the **same** function cannot make $\dot{\beta}_3 = +1$ **and** $= -1$. For the following we have therefore

(2.23) $$m \geq 0 \;, \quad m' \geq 0 \;, \quad n \gtrless 0 \;.$$

S o l u t i o n o f t h e s i m u l t a n e o u s d i f f e r e n t i a l e q u a t i o n s (2,21)

Splitting from f and g factors containing non-negative powers of $\cos\omega$, $\cos\frac{\omega}{2}$, $\sin\frac{\omega}{2}$, the exponents are determined by the equations (2.21) and by the above convention (2.23). We arrive then at the following assumption:

(2.24a) $$f = i\cos^m\omega \,\sin^{m'-1}\omega \sqrt{1-\cos\omega}\cdot u(z) \,, \quad g = \cos^m\omega\, \sin^{m'-1}\omega \sqrt{1+\cos\omega}\cdot v(z) \;; \qquad z \equiv \cos\omega \quad (0 \leq z \leq 1) \;.$$

Inserting this intà equs. (2.21), we obtain, after an elementary though laborious calculation, the following two simultaneous differential equations for the functions $u(z)$, $v(z)$:

(2.25) $$\left.\begin{aligned} (1-z)\frac{du}{dz} + m\frac{1-z}{z}u + \left(m'-\tfrac{1}{2}\right)\frac{1-z}{1+z}u - \frac{m}{z}v - nv &= 0 \\ (1+z)\frac{dv}{dz} + m\frac{1+z}{z}v - \left(m'-\tfrac{1}{2}\right)\frac{1+z}{1-z}v - \frac{m}{z}u + nu &= 0 \end{aligned}\right\} .$$

Divide the first equation by $1-z$, the second by $1+z$ and make the further substitution

(2.24b) $$u(z) + v(z) = \mathcal{P}(z) \equiv \mathcal{P} \,, \quad u(z) - v(z) = \mathcal{Q}(z) = z\,\mathcal{Q}' \;; \qquad z^2 \equiv x \quad (0 \leq x \leq 1) \;.$$

Putting for brevity

(2.24')
$$\alpha=\tfrac{1}{2}(m+m'+\tfrac{1}{2}-n)\,,\quad \beta=\tfrac{1}{2}(m+m'-\tfrac{1}{2}+n)\,,\quad \gamma=m+\tfrac{1}{2}\,,$$

we obtain then from (2.25)

(2.26)
$$\left.\begin{aligned}(1-x)\frac{d\mathcal{P}}{dx}-\beta\mathcal{P}+\beta\mathcal{Q}'&=0\\ x(1-x)\frac{d\mathcal{Q}'}{dx}-\alpha x\mathcal{Q}'+\gamma\mathcal{Q}'-(\gamma-\alpha)\mathcal{P}&=0\end{aligned}\right\}\,.$$

<u>This</u> system of two simultaneous differential equations can easily be integrated in the usual way by power series,putting

$$\mathcal{P}=\sum_{r_0} a_r x^r\,,\quad \mathcal{Q}'=c\sum_{s_0} b_s x^s\,,$$

c denoting a constant $\neq 0$ (as regards $c=0$ cf.below)and the initial coefficients a_{r_0} and b_{s_0} being normalised to 1 ,say.We get,of course,two possible values for r_0 and s_0 ,the two simultaneous differential equations of first order being equivalent to two ordinary differential equations of second order for $\mathcal{P}$ and $\mathcal{Q}'$ separately.

Inserting the power series into the differential equations (2.26),we obtain

$$\left.\begin{aligned}\sum_{r_0}\left(r a_r x^{r-1}-(r+\beta)a_r x^r\right)&=-c\beta\sum_{s_0} b_s x^s\\ c\sum_{s_0}\left((s+\gamma)b_s x^s-(s+\alpha)b_s x^{s+1}\right)&=(\gamma-\alpha)\sum_{r_0} a_r x^r\end{aligned}\right\}\,.$$

Equating coefficients yields

(2.26')
$$\left.\begin{aligned}(s+1)a_{s+1}-(s+\beta)a_s+c\beta b_s&=0\\ (s+\gamma)c\,b_s-(s-1+\alpha)c\,b_{s-1}-(\gamma-\alpha)a_s&=0\end{aligned}\right\}\,.$$

For the initial exponents, r_0 and s_0 ,we find the following two possibilities:

$$r_0 = s_0 = 0 \, . \qquad\qquad \Big| \qquad\qquad r_0 - 1 = s_0 \, .$$

Hence from (2.26') respectively

$$\begin{array}{c|c} \beta(c-1) + a_1 = 0 & \left.\begin{array}{r} (s_0+1)\, a_{s_0+1} + c\,\beta\, b_{s_0} = 0 \\ c\,(s_0+\gamma)\, b_{s_0} = 0 \end{array}\right\} \text{i.e. } \begin{array}{l} s_0 = -\gamma \\ c = \dfrac{\gamma-1}{\beta} \end{array} . \\ c = \dfrac{\gamma-\alpha}{\gamma} \, . & \end{array}$$

From (2.26') we get then

$$\begin{array}{c|c} \dfrac{a_1}{a_0} = \dfrac{\alpha\beta}{\gamma} \, , & \dfrac{a_{r_0+1}}{a_{r_0}} = \dfrac{a_{s_0+2}}{a_{s_0+1}} = \dfrac{(\alpha-\gamma+1)(\beta-\gamma+1)}{2-\gamma} \, , \\ \dfrac{b_1}{b_0} = \dfrac{\alpha(\beta+1)}{\gamma+1} \, . & \dfrac{b_{s_0+1}}{b_{s_0}} = \dfrac{(\alpha-\gamma)(\beta-\gamma+1)}{1-\gamma} \, . \end{array}$$

Generally

$$\begin{array}{c|c} \dfrac{a_k}{a_{k-1}} = \dfrac{(\alpha+k-1)(\beta+k-1)}{k(\gamma+k-1)} \, , & \dfrac{a_{r_0+k}}{a_{r_0+k-1}} = \dfrac{(\alpha-\gamma+k)(\beta-\gamma+k)}{k(1-\gamma+k)} \, , \\ \dfrac{b_k}{b_{k-1}} = \dfrac{(\alpha+k-1)(\beta+k)}{k(\gamma+k)} \, . & \dfrac{b_{s_0+k}}{b_{s_0+k-1}} = \dfrac{(\alpha-\gamma+k-1)(\beta-\gamma+k)}{k(-\gamma+k)} \, . \end{array}$$

Hence the general solution of our simultaneous differential equations (2.26) is

(2,27)
$$\begin{aligned} \mathcal{P} &= \mathcal{F}(\alpha,\beta,\gamma;x) + c'\, x^{1-\gamma}\, \mathcal{F}(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; x) \, , \\ \mathcal{Q}' &= \frac{\gamma-\alpha}{\gamma}\, \mathcal{F}(\alpha, \beta+1, \gamma+1; x) + c'\, \frac{\gamma-1}{\beta}\, x^{-\gamma}\, \mathcal{F}(\alpha-\gamma, \beta-\gamma+1, 1-\gamma; x) \, , \end{aligned}$$

~~and~~ c' being an arbitrary constante and $\mathcal{F}(\alpha,\beta,\gamma;x)$ denoting, as usual, the hypergeometric series (cf. Whl, §14)

(2.27')
$$\mathcal{F}(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}\, x + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\cdot 2\ \gamma(\gamma+1)}\, x^2 + \cdots \, .$$

Of course, we could have obtained the same result by eliminating $\mathcal{P}$ or $\mathcal{Q}'$ from the first order equations (2.26) and identifying the thus established second order differential equations for $\mathcal{P}$ and $\mathcal{Q}'$ with the well known hypergeometric (Gaussian) differential equation (cf. Whl, loc. cit.). For determination of the constant factor c in Q' we must, however, recur to one of the first order equations.

The solutions of equs.(2.26) are not yet exhausted, the cases where <u>either</u> $\mathcal{P}$ or Q' vanishes identically not being covered by the solutions (2.27). Indeed, inspection of (2.26) shows respectively that

$$Q' \equiv 0 \,, \quad \mathcal{P} \neq 0$$

is a possible solution

if only

$$\gamma - \alpha = 0 \,.$$

The first equation

(2.26) demands then:

$$(1-x)\frac{d\mathcal{P}}{dx} - \beta\,\mathcal{P} = 0 \,,$$

whence

$$\mathcal{P} = (1-x)^{-\beta}$$

where, since $\gamma - \alpha = 0$,

$$\beta = m' - \tfrac{1}{2} \,.$$

$$\mathcal{P} \equiv 0 \,, \quad Q' \neq 0$$

is a possible solution

if only

$$\beta = 0 \,.$$

The second equation (2.26)

can then be written:

$$x(1-x)\frac{dQ'}{dx} + (\gamma-\alpha)\,x\,Q' + \gamma(1-x)\,Q' = 0 \,,$$

whence

$$Q' = x^{-\gamma}\,(1-x)^{\gamma-\alpha}$$

where, since $\beta = 0$,

$$\gamma - \alpha = -(m' - \tfrac{1}{2}) \,.$$

Compounding according (2.24a,b) the functions f, g which result from these "exceptional solutions", we find respectively

$$\begin{pmatrix} f \\ g \end{pmatrix} = \cos^{m}\omega\,\sin^{-m'}\omega \begin{pmatrix} i\sqrt{1-\cos\omega} \\ \sqrt{1+\cos\omega} \end{pmatrix} . \qquad \begin{pmatrix} f \\ g \end{pmatrix} = \cos^{-m}\omega\,\sin^{-m'}\omega \begin{pmatrix} i\sqrt{1-\cos\omega} \\ -\sqrt{1+\cos\omega} \end{pmatrix} .$$

We have now to inquire whether these solutions and the ones obtained with (2.27) conform to the boundary condition to be demanded from eigensolutions.

The boundary condition The eigenfunctions and eigenvalues of the energy are determined by the condition that $\Psi^{+}\Psi$ be integrable. However, this condition, though stringently selecting among all possible solutions, will not be sufficient to exclude a certain continuous spectrum (belonging to arbitrary eigenvalue n). We shall, however, be able to eliminate the latter by the very same argument (Pauli's criterion) by which finally the still outstanding decision between single and double-valued eigenfunctions (cf. (2.11) and (2.12)) will be achieved (cf. also note on p.42).

Since the volum-element of our space is given by (2.3b), and since f and g, (2.20), are connected with Ψ (the Dirac function properly speaking) by

$$\Psi_1 = \frac{f}{\sqrt{\sin\omega\cos\omega}}\, e^{i(m\varphi+m'\psi+nt)}, \quad \Psi_2 = \frac{g}{\sqrt{\sin\omega\cos\omega}}\, e^{i(m\varphi+m'\psi+nt)},$$

the condition of quadratic integrability of Ψ over all spherical space is equivalent to the condition that

(2.28) both f and g must be finite (or zero) at $\omega=0$ and at $\omega=\frac{\pi}{2}$.

First of all, a glance on the "exceptional solutions" on p.33, and remembering our convention (2.23), shows clearly that

$\mathcal{Q}' \equiv 0\,,\ \mathcal{P} \neq 0$	$\mathcal{P} \equiv 0\,,\ \mathcal{Q}' \neq 0$
is admissible only for	**is admissible only for**
$\underline{m' = 0}$.	$\underline{m = 0}$ **and** $\underline{m' = 0}$.
$\gamma - \alpha = 0$ **entails then**	$\beta = 0$ **entails then**
$\underline{-n = m + \frac{1}{2}}$.	$\underline{n = \frac{1}{2}}$.

Now we discuss the result of the condition (2.28) on the solutions compounded with non-identically vanishing $\mathcal{P}$ and $\mathcal{Q}'$ from (2.27).

Result from the boundary condition at $\omega = \frac{\pi}{2}$ (i. e. $x = 0$) **From (2.23) it is apparent that $\gamma = m + \frac{1}{2} > 0$. Hence the coefficient c' in $\mathcal{P}(x)$ and $\mathcal{Q}'(x)$ in (2.27) must be zero in order that (2.28) be fulfilled at $\omega = \frac{\pi}{2}$ ($x = 0$). The solutions of equs. (2.21) which are finite(or vanishing) at $\omega = \frac{\pi}{2}$ must therefore, according to (2.24a,b) and (2.27), be of the following form:**

$$\Omega_0(\omega; 1, 2) = \begin{cases} f = i \cos^m\omega \sin^{m'-1}\omega \sqrt{1-\cos\omega}\left[\mathcal{F}(\alpha,\beta,\gamma;\cos^2\omega) + \frac{\gamma-\alpha}{\gamma}\cos\omega\, \mathcal{F}(\alpha,\beta+1,\gamma+1;\cos^2\omega)\right] \\ g = \cos^m\omega \sin^{m'-1}\omega \sqrt{1+\cos\omega}\left[\mathcal{F}(\alpha,\beta,\gamma;\cos^2\omega) - \frac{\gamma-\alpha}{\gamma}\cos\omega\, \mathcal{F}(\alpha,\beta+1,\gamma+1;\cos^2\omega)\right] \end{cases} \tag{2.29}$$

Result from the boundary condition at $\omega = 0$ (i. e. $x = 1$) **In appendix 1 a detailed discussion of the behaviour of f and g at $\omega = 0$ ($x = 1$) is given. The discussion shows that, apart from a continuous spectrum(i.e. n arbitrary(real)) belonging to $m' = 0$ or to $m' = \frac{1}{2}$, the solutions**

$$\Psi_1 = {}_1\Psi^n_{mm'} = i \cos^{m-\frac{1}{2}}_{\omega} \sin^{m'-\frac{3}{2}}_{\omega} \sqrt{1-\cos\omega}\; e^{i(m\varphi+m'\psi+nt)} \cdot \left[F(\alpha,\beta,\gamma;\cos^2\omega) + \frac{\gamma-\alpha}{\gamma}\cos\omega\, F(\alpha,\beta+1,\gamma+1;\cos^2\omega)\right],$$

(2.30)
$$\Psi_2 = {}_2\Psi^n_{mm'} = \cos^{m-\frac{1}{2}}_{\omega} \sin^{m'-\frac{3}{2}}_{\omega} \sqrt{1+\cos\omega}\; e^{i(m\varphi+m'\psi+nt)} \cdot \left[F(\alpha,\beta,\gamma;\cos^2\omega) - \frac{\gamma-\alpha}{\gamma}\cos\omega\, F(\alpha,\beta+1,\gamma+1;\cos^2\omega)\right],$$

$\Psi_3 = \Psi_4 = 0$ (cf.p.29,after (2.21))

are not quadratically integrable unless either

(2.31a)
$$\alpha = \tfrac{1}{2}(m+m'+\tfrac{1}{2}-n) = -k$$

or

(2.31b)
$$\beta+1 = \tfrac{1}{2}(m+m'+\tfrac{3}{2}+n) = -k \quad ,$$

where

$$k = 0, +1, +2, \dots .$$

(Remember that for negative values of m or m' the regular solutions are obtained from those in (2.30) by help of the scheme (2.22).) The hypergeometric functions in the bracketed part of the solutions are <u>then</u> Jacobi <u>polynomials</u>.

The condition (2.31) determines <u>discrete</u> eigenvalues n of the energy. Since both m and m' can be <u>either</u> integral <u>or</u> half-integral(cf.(2.11) and (2.12);the discussion of regularity in appendix 1 does not prejudice anything in favour or against either possibility),the following <u>four</u> cases are compatible with (2.31):

m	m'	n	m	m'	n
half-odd	half-odd	half-odd	integral	half-odd	integral
integral	integral	half-odd	half-odd	integral	integral.

Besides, we must not lose sight of the above mentioned continuous spectra and of the functions resulting from the "exceptional solutions" still admissible under the conditions stated above (p.35). We shall, however, show now that it is only the polynomial solutions (2.30) with half-odd m, m' and n which satisfy the more stringent condition to be demanded; all other solutions do not conform to "Pauli's criterion".

Pauli's criterion (P2) This criterion is based on the existence of a group of transformations of the Hamiltonian H (i.e. the six-dimensional rotation-group in our case). In this case certain operators, D say, commute with H. Consequently the result of D operating on an eigenfunction $\Psi^n_{mm'}$ of the Hamiltonian must be expressible by a linear combination of the eigenfunctions belonging to the same eigenvalue n of the energy. In other words: the eigenfunctions of the Hamiltonian must transform as the irreducible representations of the group, i.e.

(2.32) $$D\Psi^n_{mm'} = \sum_{\mu}\sum_{\mu'} c^{\mu\mu'}_{mm'}\Psi^n_{\mu\mu'} .$$

In particular, (2.32) evidently says that $D\Psi$ must also be a regular eigensolution.

Now, it can be shown (cf. below) that for $m = 0$ as well as for

$m'=0$ (2.32) is not fulfilled. As in the two cases treated by Pauli (cf.P2), we also find that if (2.32) is not fulfilled then there are always solutions Ψ^{n_1} (we can write them as $D\Psi^{n_1}$) which are not orthogonal on certain solutions Ψ^{n_2}, although $n_1 \neq n_2$ and despite that D -since it commutes with H - ought to be diagonal with respect to n. This implyes that the usual connection between operator and matrix-calculus does no longer hold for <u>integral</u> values of m and m'; the latter must therefore be rejected. On the other hand, it can be shown that (2.32) is always fulfilled if both m and m' equal half-odd integers. We must therefore decide in favour of this latter possibility.

Application of Pauli's criterion Let M_i and N_i ($i = 1, 2, 3$) denote the operators of angular and "linear" momentum in spherical space; in (2.10)(p.22) we have already met one of these three dual operator-pairs, namely

$$M_3 = -i\frac{\partial}{\partial\varphi} \quad , \quad N_3 = -i\frac{\partial}{\partial\psi} \quad .$$

Our notation is that used by Schrödinger; the six operators are the generators of the rotation-group in spherical space treated by the latter author.[S2] We put

$$\xi_i = \frac{1}{2}(M_i + N_i) \quad , \quad \eta_i = \frac{1}{2}(M_i - N_i) \quad .$$

According to their commutation relations the ξ_i' and the η_i'

form two independent angular momenta(cf.p. 46).

Now we consider the following four linear combinations (each equation stands unmistakebly for two,one for the upper signs,the other for the lower ones):

(2.33)
$$D_{\pm\pm} = \xi_1 \pm i\,\xi_2 = \frac{1}{2} e^{\pm i(\varphi+\psi)}\left[\frac{\partial}{\partial\omega} \mp i\tan\omega\frac{\partial}{\partial\varphi} \pm i\cot\omega\frac{\partial}{\partial\psi} - \frac{i}{2}\beta_1 \pm \frac{1}{2}\frac{\beta_3}{\sin\omega} \mp \frac{1}{2}\frac{\beta_2}{\cos\omega}\right],$$
$$D_{\pm\mp} = \eta_1 \pm i\,\eta_2 = \frac{1}{2} e^{\pm i(\varphi-\psi)}\left[-\frac{\partial}{\partial\omega} \pm i\tan\omega\frac{\partial}{\partial\varphi} \pm i\cot\omega\frac{\partial}{\partial\psi} - \frac{i}{2}\beta_1 \pm \frac{1}{2}\frac{\beta_3}{\sin\omega} \pm \frac{1}{2}\frac{\beta_2}{\cos\omega}\right].$$

They commute with our Hamiltonian(cf.(2.7),(2.6'),(2.6),and (2.17c))

(2.34)
$$-i\frac{\partial}{\partial t} = i\beta_1\dot{\beta}_3\left(\frac{\partial}{\partial\omega} + \frac{1}{2}\cot\omega - \frac{1}{2}\tan\omega\right) - i\frac{\beta_3\dot{\beta}_3}{\cos\omega}\frac{\partial}{\partial\varphi} - i\frac{\beta_2\dot{\beta}_3}{\sin\omega}\frac{\partial}{\partial\psi}.$$

Operating with (2.33) on the eigenfunctions of the Hamiltonian,(2.30),we find the following relations(cf.appendix IIa):

(2.35)
$$\left.\begin{matrix}\text{a)}\\\text{b)}\end{matrix}\right\} D_{\pm\pm}\,{}_\rho\Psi^n_{mm'} = \text{const}\;{}_\rho\Psi^n_{m\pm1\;m'\pm1} \qquad \left.\begin{matrix}\text{c)}\\\text{d)}\end{matrix}\right\} D_{\pm\mp}\,{}_\rho\Psi^n_{mm'} = \text{const}\;{}_\rho\Psi^n_{m\pm1\;m'\mp1},$$

ρ denoting the spin-index 1 , 2.

Now,operating in particular with D_{--} on the regular solution ${}_\rho\Psi^n_{00}$,we obtain

$$D_{--}\,{}_\rho\Psi^n_{00} = \frac{1}{2}\,{}^{i}_{1}\,\cos^{-\frac{3}{2}}\omega\,\sin^{-\frac{5}{2}}\omega\sqrt{1\mp\cos\omega}\;e^{i(-\varphi-\psi+nt)}\cdot$$
$$\cdot\left[F(\alpha-1,\beta-1,\gamma-1;\cos^2\omega) \pm \frac{\gamma-\alpha}{\gamma-1}F(\alpha-1,\beta,\gamma;\cos^2\omega)\right],$$

the two signs and ${}^{i}_{1}$ referring <u>here</u> respectively to the spin-index $\rho = {}^{1}_{2}$.Because of the negative powers of $\cos$ and $\sin$ this function is clearly <u>not</u> quadratically integrable.In contradiction with (2.33) the operator D_{--} has produced this irregular function, while it ought to have produced the regular solution ${}_{\rho}\Psi^{n}_{-1-1}$ which is obtained from ${}_{\rho}\Psi^{n}_{11}$ by application of the scheme (2.22a). Analogously for the operators $\mathcal{D}_{+-}$, $\mathcal{D}_{-+}$, the functions produced by these being respectively too powerfully singular at $\omega = 0$ or at $\omega = \frac{\pi}{2}$.

Moreover, (cf. appendix 11b), we can find solutions $\Psi^{n_1}_{om'}$, $\Psi^{n_2}_{om'}$ (they can obviously be written as $\mathcal{D}\Psi$) which are not mutually orthogonal, and the same is true for solutions $\Psi^{n_1}_{mo}$, $\Psi^{n_2}_{mo}$. In fact, this is the case for all $n_1 - n_2 =$ odd number. Consequently the Hamiltonian is not self-adjoint in these solutions and the matrices of the $\mathcal{D}'_3$ are not diagonal with respect to n . These solutions, and consequently the whole system with integral values of m and m', can, therefore, not be used for a true matrix-representation of our operators. (One cannot simply omit these pathological solutions from the rest of the system, since they will be produced again by $\mathcal{D}$ -operators; the omission would spoil the completeness of the system.) The integral values of m and m' must therefore be rejected.

On the other hand, (cf. appendix 11c), it can be verified that (2.32) is always fulfilled if both m and m' equal half-odd numbers, as provided in (2.11b) and (2.12b). For example: operating with $\mathcal{D}_{--}$ on the regular solution ${}_{\rho}\Psi^{n}_{\frac{1}{2}\frac{1}{2}}$, we do get the regular solution ${}_{\rho}\Psi^{n}_{-\frac{1}{2}-\frac{1}{2}}$ obtained from ${}_{\rho}\Psi^{n}_{\frac{1}{2}\frac{1}{2}}$ by help of the scheme (2.22a). <u>We must therefore decide in favour of the half-odd numbers m and m'</u>, our eigensolutions (2.30) thus being double-valued, changing sign after a full encirclement of $\omega = 0$ or $\omega = \frac{\pi}{2}$, (cf. p. 19).

It is clear that henceforth the "exceptional solutions" from p. 35 and the continuous spectrum to $m' = 0$ from appendix 1 do not come any more, since they all belong to $m = 0$ or $m' = 0$.

- Here we must add a remark on the derivative with respect to ω of the solutions Ψ. It is readily seen that for $m' = 0$ or for $m = 0$ (this is also true for the "exceptional solutions") the derivative becomes infinite at $\omega = 0$ or at $\omega = \frac{\pi}{2}$ respectively. (This is of course intimately connected with the fact that the application of the rotation operators D leads to irregular functions in these cases.) Also in the cases $m' = \frac{1}{2}$ or $m = \frac{1}{2}$, though, the derivative becomes infinite at $\omega = 0$ or $\omega = \frac{\pi}{2}$. However, in <u>these</u> cases the discontinuity of the function is only an apparent one, since, because of the phase factors $e^{\frac{i}{2}\varphi}$, $e^{\frac{i}{2}\psi}$, the gradient is complex, its value at $\varphi + 2\pi$ (the diametrally opposite direction to φ, the full angle around each branch point being 4π) equalling <u>minus</u> the value at φ,

E l i m i n a t i o n o f t h e c o n t i n u o u s s p e c t r u m t o $m' = \frac{1}{2}$ After having decided in favour of the half-odd numbers m' (and m)by means of Pauli's criterion,we must now remember that we have obtained a <u>continuous</u> spectrum to $m'=\frac{1}{2}$(cf.end of appendix 1), no restriction having been necessary in this case for either of the hypergeometric parameters, α or β ,in order that the solutions be quadratically integrable.I have found that this continuous spectrum can easily be eliminated by the very same argument by which the integral values of m'(or m)have just been rejected.For,e.g.:applying the operator D_{++} to the sosution $_{\rho}\Psi_{m\,\frac{1}{2}}^{\frac{2}{7}}$ from this continuous spectrum,we obtain,according to (2.35a),the solution $_{\rho}\Psi_{m+1\;\frac{3}{2}}^{\frac{2}{7}}$.But we have seen in appendix 1 that in all cases $m'\neq\frac{1}{2}$ the solutions are not quadratically integrable unless α or $\beta+1$ equals a non-positive integer,i.e. (since also m is half-odd)unless the eigenvalue n is restricted to certain half-odd values,(cf.end of p.36).Hence already for $m'=\frac{1}{2}$ we must restrict n to half-odd values in order that the application of the D -operators may <u>always</u> lead to quadratically integrable functions and the condition (2.32) be fulfilled.Or in other words:the continuous spectrum must be excluded because it is not invariant under rotations admitted of by the Hamiltonian.

(Note It is perhaps noteworthy to remark that exactly the same reasoning applies for the eventual exclusion of the continuous spectrum of the quadratically integrable (logarithmic) Legendre functions of first and second kind, $P_\nu(\cos\vartheta)$ and $Q_\nu(\cos\vartheta)$, with arbitrary (real) ν from the eigenspectrum of the ordinary scalar wave equation with spherically symmetric potential. The well known rotation operators $M_x \pm i M_y$ produce there associated Legendre functions, $P_\nu^{|m|=1}(\cos\vartheta)$ and $Q_\nu^{|m|=1}(\cos\vartheta)$; the latter are in no case quadratically integrable, the former only if $|m| \leq |\nu|$ and $\nu - m$ = integer, i.e. if ν equals an integer n. Thus, the condition that (2.32) be always fulfilled singles there out the discrete set of Legendre polynomials $P_n(\cos\vartheta)$. - Alternatively one might argue that the $P_\nu(\cos\vartheta)$ and $Q_\nu(\cos\vartheta)$ are not orthogonal to the Legendre polynomials $P_n(\cos\vartheta)$ (cf. Ganesh Prasad, Proc. Benares Math. Soc., 12, 1930, pp. 33-39), thus from the point of view of the expansion theorem it being useless to consider the $P_\nu(\cos\vartheta)$ and $Q_\nu(\cos\vartheta)$ as eigenfunctions. And moreover, the fundamental assumption that the Hamiltonian be selfadjoint (i.e. $\int v^* H u\, d\tau = \int u (Hv)^* d\tau$) is not fulfilled for any $v = P_\nu(\cos\vartheta)$ and regular $u = \sum_n^\infty a_n P_n(\cos\vartheta)$. Both the latter arguments, to which Professor W. Pauli has kindly drawn my attention, apply also in the text to eliminate the continuous spectrum to $m' = \frac{1}{2}$.)

D e g r e e o f d e g e n e r a c y We have so far focused our attention on that eigenfunction of the matrix $\dot{\beta}_3$ which belongs to the eigenvalue $+1$ of $\dot{\beta}_3$ (cf.p.29, after (2.21)).Accordingly we have obtained all those four-componential eigensolutions which are of the type

(2.36a) $$(\Psi_1\,,\,\Psi_2) = \frac{(f,g)}{\sqrt{\sin\omega\cos\omega}}\, e^{i(m\varphi+m'\psi+nt)}\,,\quad \Psi_3=\Psi_4\equiv 0 \;.$$

Because of the restrictions (2.31,a,b) imposed upon the parameters α, β of the occurring hypergeometric series,these eigensolutions belong to either of the following eigenvalues of the energy:

(2.37a) $$n = 2k+m+m'+\tfrac{1}{2} \quad \text{or} \quad -n = 2k+m+m'+\tfrac{3}{2} \quad (k=0,+1,+2,\dots).$$

The <u>other</u> eigenfunction of $\dot{\beta}_3$ (namely the one which belongs to the eigenvalue -1)gives rise to <u>another</u> type of four-componential eigensolutions Ψ ,viz.

(2.36b) $$(\Psi_3\,,\,\Psi_4) = \frac{(f,g)}{\sqrt{\sin\omega\cos\omega}}\, e^{i(m\varphi+m'\psi+nt)}\,,\quad \Psi_1=\Psi_2\equiv 0 \;.$$

As stated on p. 28,case ii),the sign of the eigenvalues n to which <u>these</u> eigensolutions belong is <u>opposite</u> to that in (2.36a),viz.

(2.37b) $$-n = 2k+m+m'+\tfrac{1}{2} \quad \text{or} \quad n = 2k+m+m'+\tfrac{3}{2} \quad (k=0,+1,+2,\dots).$$

The number of eigenfunctions belonging to a definite eigenvalue n (half-odd)is four times the number of pairs (m , m')(positive,half-odd)which satisfy either (2.31a) or (2.31b) for given n .(The factor four is clearly due to the existence of the scheme (2.22) owing to which we could confine ourselves to positive values of both m and m'.)

Thus the degree of degeneracy works out

$$G = 2\left(n-\tfrac{1}{2}\right)\left(n+\tfrac{1}{2}\right) = 2\left(n^2-\tfrac{1}{4}\right).$$

This is in agreement with Schrödinger's result from treating the Dirac equations in spherical space in hyperspherical polar coordinates(cf.S2,formula (8.25))if we identify our n with Schrödinger's n''.

S y s t e m o f e i g e n v a l u e s As in Schrödinger's treatment,here too the smallest value of n equals $\frac{3}{2}$ (one must remember that in (2.31a,b) the smallest value of m and m' equals $\frac{1}{2}$).Our system of eigenvalues is:

$$\begin{aligned} n &= \qquad\quad \pm\tfrac{3}{2},\ \pm\tfrac{5}{2},\ \dots \\ m &= \pm\tfrac{1}{2},\ \pm\tfrac{3}{2},\ \pm\tfrac{5}{2},\ \dots \qquad |m| \leq |n|-1 \\ m' &= \pm\tfrac{1}{2},\ \pm\tfrac{3}{2},\ \pm\tfrac{5}{2},\ \dots \qquad |m'| \leq |n|-1 \end{aligned} \tag{2.38}$$

where either (2.37a) or (2.37b) must be satisfied.

C o m p a r i s o n w i t h t h e d e g r e e o f d e g e n e r a c y a s p o s t u l a t e d b y g r o u p - t h e o r y Our system of eigenvalues,the restrictions imposed upon them,and the resulting degree of degeneracy obtained here from the regularity condition for the eigenfunctions can be fit to those found by Schrödinger by general group-theoretical considerations,(S2,p.331).The rotation-group in four dimensions can be considered as the direct product of two ordinary, three-dimensional, independent ones, the (discrete) eigenvalues of the generators of the latter, $\xi_{i'}$ and $\eta_{i'}$ (cf. end of p.38), being well known from their commutation relations. Let the dimensions of the irreducible representations of the two independent groups be $n+n'$ and $n-n'$ respectively, where evidently either both n and n' are integers or both are ~~both are~~ half-odd, and $n > |n'|$. The dimension of the representation of the direct product

$$\mathcal{D}_{\frac{n-1+n'}{2}} \times \mathcal{D}_{\frac{n-1-n'}{2}} = \mathcal{D}_{n-1} + \mathcal{D}_{n-2} + \cdots + \mathcal{D}_{|n'|}$$

is clearly

$$(n+n')(n-n') = n^2 - n'^2 .$$

Comparing this with our result $G = 2(n^2 - \frac{1}{4})$, we see that our eigensolutions Ψ provide the representations of the rotation-group in four dimensions that belong to $n' = +\frac{1}{2}$ <u>and</u> $n' = -\frac{1}{2}$. As index of the smallest representation, $\mathcal{D}_{|n'|}$, occurring in the direct product, n' is closely connected with the tensor-rank of Ψ, i.e. with the spin. A similar consideration shows that the n here is indeed the energy quantum number.

The ground-state $n = \frac{3}{2}$ ($> |n'|$, as postulated) is degenerated in our case. Its multiplicity is four-fold (according to the four possibilities $m = \pm\frac{1}{2}$, $m' = \pm\frac{1}{2}$); the multiplicity of its negative counterpart, $n = -\frac{3}{2}$, is, of course, the same. Evidently, the ground-state in four dimensions will always be degenerated(except for the scalar case $n'=0$)and its multiplicity - as difference of the squares of the two consecutive numbers n and n' - will be the higher the larger $|n'|$ is, viz. $2(2|n'|+1)$: $2|n'|+1$ of the functions belong to $|n'|$ ($\dot{\beta}_3=+1$), the other half to $-|n'|$ ($\dot{\beta}_3=-1$); they form two irreducible families, as apparent from(233).

C) **The eigensolutions and eigenvalues in the case of non-zero rest-mass**

Now we return to equation (2.7')(p.24)where the mass-term occurs; we want to obtain the eigensolutions and eigenvalues of the Hamiltonian in the case of non-vanishing mass.

We use a bow in the notation, $\widehat{\Omega}$, $\widehat{\Psi}$, and $\widehat{n}$, to distinguish the <u>now</u> occurring functions, eigensolutions, and eigenvalues from those of the just treated case of vanishing mass, Ω, Ψ and n respectively.

Multiplying equ. (2.7') by $\alpha_4\alpha_1 = i\beta_1\dot{\beta}_3$ and again expressing in this equation all α-matrices as direct products, according (2.17)(p.27), we get

$$(2.7'') \qquad i\beta_1\dot{\beta}_3\frac{d\widehat{\Omega}}{d\omega} + \frac{m}{\cos\omega}\beta_3\dot{\beta}_3\widehat{\Omega} + \frac{m'}{\sin\omega}\beta_2\dot{\beta}_3\widehat{\Omega} = \widehat{n}\widehat{\Omega} - k_0 1\beta_2\widehat{\Omega},$$

where in ordinary units

(2.39a) $\hat{n} = \frac{2\pi R}{c}\hat{\nu}$ = number of wave-lengths of light per great circle $2\pi R$, th(e) wave length corresponding now to the eigenfrequency in case of non-zero mass. k_o is an abbreviation for

(2.39b) $k_o \equiv \mu R = \frac{2\pi R}{c}\frac{m_o c^2}{h}$ = number of wave-lengths of light (per $2\pi R$) corresponding to the Compton frequency $\nu_o = \frac{m_o c^2}{h}$.

For brevity let us put

$$\left(i\beta_1\dot{\beta}_3\frac{d}{d\omega} + \frac{m}{\cos\omega}\beta_3\dot{\beta}_3 + \frac{m'}{\sin\omega}\beta_2\dot{\beta}_3 \right) \equiv B \quad .$$

This operator B anticommutes with $\dot{\beta}_2$ since, according to (2.16b)(p.26), $\dot{\beta}_3$ does,

(2.16') $$B\dot{\beta}_2 = -\dot{\beta}_2 B \quad .$$

The equation for $\hat{\Omega}$, (2.7''), can now shortly be written:

(2.7''') $$B\hat{\Omega} = \hat{n}\hat{\Omega} - k_o\dot{\beta}_2\hat{\Omega} \quad .$$

Similarly for the equation of the corresponding <u>known</u> function Ω, (2.18)(p.27),

(2.18') $$B\Omega = n\Omega \quad .$$

We reduce $\hat{\Omega}$ to Ω putting

(2.40a) $$\Omega = (1 + b\dot{\beta}_2)\hat{\Omega} \quad ,$$

whence

(2.40b) $$\hat{\Omega} = \frac{1 - b\dot{\beta}_2}{1 - b^2}\Omega \quad ,$$

b being a constant (ordinary c-number, $\neq \pm 1$) to be determined. Hence, inserting (2.40a) into (2.18') and observing (2.16'),

$$B\widehat{\Omega} = n(1+\sigma\dot{\beta}_2)\Omega + \sigma\dot{\beta}_2 B\widehat{\Omega} \quad .$$

By help of (2.40b),(2.18') and (2.16'),(2.40a) the last term can respectively be written as

$$\sigma\dot{\beta}_2 B\frac{1-\sigma\dot{\beta}_2}{1-\sigma^2}\Omega = \frac{n\sigma\dot{\beta}_2}{1-\sigma^2}(1+\sigma\dot{\beta}_2)\Omega = \frac{n\sigma\dot{\beta}_2}{1-\sigma^2}(1+\sigma\dot{\beta}_2)(1+\sigma\dot{\beta}_2)\widehat{\Omega} \ .$$

Hence,

$$B\widehat{\Omega} = n(1+\sigma\dot{\beta}_2)\left[1+\frac{\sigma\dot{\beta}_2}{1-\sigma^2}(1+\sigma\dot{\beta}_2)\right]\widehat{\Omega} = \frac{n}{1-\sigma^2}(1+\sigma\dot{\beta}_2)^2\widehat{\Omega}$$

$$= n\frac{1+\sigma^2}{1-\sigma^2}\widehat{\Omega} + \frac{2\sigma n}{1-\sigma^2}\dot{\beta}_2\widehat{\Omega} \ .$$

Comparing this with (2.7''') we obtain

(2.41a)
$$\hat{n} = n\frac{1+\sigma^2}{1-\sigma^2} \ , \qquad k_0 = -\frac{2n\sigma}{1-\sigma^2} \ ,$$

whence

(2.41b)
$$\hat{n}+k_0 = n\frac{1-\sigma}{1+\sigma} \ , \qquad \hat{n}-k_0 = n\frac{1+\sigma}{1-\sigma} \ ;$$

$$\hat{n}^2 = n^2 + k_0^2 \ , \qquad \hat{n} = \pm\sqrt{n^2+k_0^2} \ .$$

From the quadratic equation (2.41a) and from (2.41b) we find

$$\sigma_1 = \sigma_2 = \frac{n-\hat{n}}{k_0} = -\frac{k_0}{n+\hat{n}} \ ;$$

$$1+\sigma = \frac{k_0+n-\hat{n}}{k_0} \ , \qquad 1-\sigma = \frac{k_0-n+\hat{n}}{k_0} \ ; \qquad 1-\sigma^2 = \frac{2n}{\hat{n}+n} \ .$$

Hence we finally obtain for the function $\widehat{\Omega}$,(2.40b),

(2.42)
$$\widehat{\Omega} = \frac{1}{2n}(\hat{n}+n+k_0 I\dot{\beta}_2)\Omega \ .$$

In (2.41b) the <u>sign</u> of $\hat{n}$ (i.e.of the eigenfrequency $\hat{\nu}$) is <u>not</u> connected with that of n and,therefore,the eigensolutions to <u>both</u> positive and negative sign of n (i.e.(2.36a) and (2.36b))could form the eigensolution belonging to $\hat{n}$.

In order to avoid this ambiguity we attach the sign of n to $\hat{n}$ such that the special case of vanishing mass($k_0 = 0$)is also covered by (2.42),viz.

(2.43)
$$\text{sign}\,(\hat{n}) = \text{sign}\,(n) \quad .$$

The eigenfrequency $\hat{\nu}$ is <u>then</u> determined by (2.39) and (2.41b),

$$\hat{\nu} = \text{sign}(n)\sqrt{\nu_0{}^2 + \frac{n^2c^2}{4\pi^2R^2}} \qquad \left(\nu_0 = \frac{m_0c^2}{h}\right) .$$

This is identical with Schrödinger's result (cf.S2,formula (8.21)).Since $n^2>0$, the energy of the particle will always be greater than its rest-energy.This is quite natural,since the particle is in closed space.

Owing to the matrix $1\dot{\beta}_2 = \alpha_4$,(2.17c),occurring in (2.42), all <u>four</u> components of the eigensolutions $\hat{\Psi}$ are different from zero.From the two types of eigensolutions(belonging respectively to the eigenvalue $+1$ or -1 of the matrix $\dot{\beta}_3$)in the special case of vanishing mass,(2.36a) and (2.36b),we now obtain the following two types(belonging to opposite sign of $\hat{n}$) for the <u>general</u> case of non-vanishing mass:

(2.44a)
$$\frac{1}{2n}\,\frac{e^{i(m\varphi+m'\psi+\hat{n}t)}}{\sqrt{\sin\omega\cos\omega}}\Big((\hat{n}+n)f,\,(\hat{n}+n)g,\,-ik_0f,\,-ik_0g\Big),$$

(2.44b)
$$\frac{1}{2n}\,\frac{e^{i(m\varphi+m'\psi+\hat{n}t)}}{\sqrt{\sin\omega\cos\omega}}\Big(ik_0f,\;ik_0g,\,(\hat{n}+n)f,\,(\hat{n}+n)g\Big).$$

E.g.:the solution to $n=+\frac{3}{2}$, $m=\frac{1}{2}$, $m'=\frac{1}{2}$,(where $\alpha=0$,cf.(2.31)) is of the type (2.44a), whereas the solution to $n=\frac{5}{2}$, $m=\frac{1}{2}$, $m'=\frac{1}{2}$, is of the type (2.44b), since $\beta+1=0$ for $n=-\frac{5}{2}$, $m=\frac{1}{2}$, $m'=\frac{1}{2}$. The solution to $n=\frac{5}{2}$, $m=-\frac{1}{2}$, $m'=\frac{1}{2}$, is again of the type (2.44a); it is obtained from that to $n=-\frac{5}{2}$, $m=\frac{1}{2}$, $m'=\frac{1}{2}$, (type (2.44a))by application of the scheme (2.22b).

Denoting by M the number of minus-signs in the triple (n, m, m'), ($M=0,1,2,3$; e.g. for $n=+\frac{5}{2}$, $m=-\frac{1}{2}$, $m'=+\frac{1}{2}$, we have $M=1$), we can make the general statement: a solution is of the type (2.44a) or (2.44b) according to whether $|m|+|m'|+|n|-\frac{1}{2}+M$ is even or odd.

It is apparent that, on account of (2.41b), the last, respectively first, pair of the four components (2.44a,b) is small if $k_0=\frac{2\pi m_0 c}{h} \ll n$. In the extreme case of zero rest-mass the solutions go over into the massless solutions (2.36)p.44) by help of the convention (2.43).

D) Behaviour of the eigenwaves and their polarisation

Our solutions of the Dirac equations in spherical space, (2.44), can truely be called cylinder-waves, their amplitudes varying only with the cylinder-radius ω ,whereas the two azimuthal angles φ and ψ occur only in the phase.The same kind of waves occurs and has been discussed for the scalar case by P.O.Müller (M1).<u>Our</u> waves,however,are <u>spinor</u>-waves,it thus being of additional interest to inquire into their <u>polarisation</u>,i.e.the direction of the spin with respect to that of the current.This can be done quite generally,and we shall in particular apply it to the interesting types of "skin-waves","tube-waves",and the ground-vibration,to be described now.

These simple types of waves are obtained by putting

$$2\alpha = m + m' + \tfrac{1}{2} - n = 0 \; .$$

The bracketed part of the functions f and g ,(2.29)(p.35), containing respectively the sum or difference of two polynomials,reduces then to $1-\cos\omega$ or $1+\cos\omega$ .Hence f and g assume then the following simple form:

(2.45)
$$\begin{pmatrix} f \\ g \end{pmatrix} = \cos^{m}\omega \, \sin^{m'}\omega \begin{pmatrix} i\sqrt{1+\cos\omega} \\ \sqrt{1-\cos\omega} \end{pmatrix} .$$

The maximum of the total intensity(sum over the four components (2.44)),viz. $\psi^{+}\psi = \frac{\hat{n}(\hat{n}+n)}{n^2} \frac{f^*f+g^*g}{\sin\omega\cos\omega}$,is then found to be on the cylinder $\omega = \tan^{-1}\sqrt{\frac{m'-\frac{1}{2}}{m-\frac{1}{2}}}$.

S k i n - w a v e s For large m and m' <u>and</u> small

$|m-m'|$ the maximum becomes very pronounced. In particular, if $|m|=|m'|$ then the maximum is on the cylinder $\omega=\frac{\pi}{4}$ and for large m the intensity behaves then like $e^{-m\varepsilon^2}$ in the thin cylindrical skin $\omega=\frac{\pi}{4}\pm\varepsilon$, i.e. the intensity is practically constant for $\varepsilon \ll \frac{1}{\sqrt{m}}$ and zero for $\varepsilon \gg \frac{10}{\sqrt{m}}$, say. The corresponding "skin-waves" (2.44) wind just once along great circles around a very thin cylindrical skin embedding the cylinder $\omega=\frac{\pi}{4}$. There (if the electron-<u>waves</u> were observable) a local observer, using the tangents at the coordinate lines ω , φ , ψ , as axes of reference, would find a plane wave extending indefinitely in two directions, (φ , ψ), but if he would extend his observations beyond the inner and outer surface of the thin cylindrical skin (i.e. beyond $\omega=\frac{\pi}{4}\pm\varepsilon$ respectively), he would find that the wave is rapidly loosing its intensity in the third (ω) direction. (The phase-shift between f and g -wave always equals $\frac{\pi}{2}$).

T u b e - w a v e s   In the extreme cases where either m or m' is very large ($\gg \frac{1}{2}$) and the other one is very small ($\sim\frac{1}{2}$) the intensity $\Psi^+\Psi$ is concentrated near the great circle (degenerated cylinder) $\omega=0$, or $\omega=\frac{\pi}{2}$ respectively, and its immediate neighbourhood, thus exhibiting the phenomenon of "tube-waves". The existence of the <u>latter</u> was already apparent in Schrödinger's representation of the eigenwaves in hyperspherical <u>polar</u> coordinates (S2, pp. 328, 361). A local observer at the great circle would observe a plane wave indefinitely propagating in the direction of the great circle but rapidly fading away at the end of a thin tube around it.

G r o u n d - s t a t e   The ground-state too belongs to the type (2.45). For it we have $|n| = \frac{3}{2}$, $|m| = \frac{1}{2}$, $|m'| = \frac{1}{2}$, and therefore

$$\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \frac{\hat{n}+n}{2n} e^{i(m\varphi + m'\psi + \hat{n}t)} \begin{pmatrix} i\sqrt{1+\cos\omega} \\ \sqrt{1-\cos\omega} \end{pmatrix},$$

the cosmical intensity $\psi^+\psi$ thus being constant in the ground-state(cosmical, as distinct from the local one; they differ by a factor $\sin\omega\cos\omega$(cf. below)).

P o l a r i s a t i o n   o f   o u r   e i g e n - w a v e s   For determination of the polarisation of our waves (2.44), i.e. relation between the directions of spin and current *), we observe that the tangents at our coordinate lines ω, φ, ψ, provide a local Euclidean system of reference at any point, for a local observer the increments of the coordi-

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*) Usually one determines the direction in which the spin is sharp(diagonal) with respect to a "given z-axis". It can easily be seen that in our case polar angle Θ and azimuth Φ around the local φ-direction are determined by $\frac{f}{g} = \tan\frac{\Theta}{2} e^{i\Phi}$.

nates being given from (2.2)(p.17) as

$$d\omega\,,\quad \cos\omega\, d\varphi\,,\quad \sin\omega\, d\psi\,,$$

respectively. For him equation (2.7a)(p.21) plays the role of the ordinary Dirac equation in Cartesian coordinates and from the resulting continuity equation for current and density he finds the local stream-components, with the α-matrix representation (2.17), as

$$v_\omega = \Phi^\dagger \overset{i\alpha_4\alpha_1}{\begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}} \Phi\,,\qquad v_\varphi = \Phi^\dagger \overset{i\alpha_4\alpha_2}{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}} \Phi\,,\qquad v_\psi = \Phi^\dagger \overset{i\alpha_4\alpha_3}{\begin{pmatrix} & i & & \\ -i & & & \\ & & & -i \\ & & i & \end{pmatrix}} \Phi\,.$$

(As already used, the dagger indicates the complex conjugate function taken as a row instead of as a column. Φ is connected with Ψ, the wave function proper, by (2.6'): $\Phi = \sqrt{\sin\omega\cos\omega}\,\Psi$. In our units $c = 1$.)

The remaining three components of the six-vector current-spin are formed by the local spin-components,

$$s_\omega = \Phi^\dagger \overset{i\alpha_2\alpha_3}{\begin{pmatrix} & -1 & & \\ -1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix}} \Phi\,,\qquad s_\varphi = \Phi^\dagger \overset{i\alpha_3\alpha_1}{\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}} \Phi\,,\qquad s_\psi = \Phi^\dagger \overset{i\alpha_1\alpha_2}{\begin{pmatrix} & i & & \\ -i & & & \\ & & & i \\ & & -i & \end{pmatrix}} \Phi\,.$$

For comparison with the stream-components it is convenient to write them in the following way:

$$s_\omega = \Phi^\dagger i\alpha_4\alpha_1 \tilde{\Phi}\,,\quad s_\varphi = \Phi^\dagger i\alpha_4\alpha_2 \tilde{\Phi}\,,\quad s_\psi = \Phi^\dagger i\alpha_4\alpha_3 \tilde{\Phi}\,,$$

where, owing to the relations (1.5)(p.12) and our α-representation (2.17c),

$$\widetilde{\Phi} = \alpha_1\alpha_2\alpha_3\alpha_4\Phi = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ -\Phi_3 \\ -\Phi_4 \end{pmatrix} .$$

With our solutions (2.44) we obtain then for the type (2.44a) (i.e. belonging, in the massless case, to the eigenvalue $+1$ of the matrix $\dot{\beta}_3$), using the connection (2.41b) between $\hat{n}$ and n,

$v_\omega = \frac{\hat{n}+n}{2n}(f^*g+g^*f) \equiv 0$, since f^*g is purely imaginary (cf. (2.29));

$$v_\varphi = \frac{\hat{n}+n}{2n}\left(-f^*f+g^*g\right), \quad v_\psi = -i\,\frac{\hat{n}+n}{2n}\left(f^*g-g^*f\right) ,$$

and

$$s_\omega = \frac{\hat{n}}{n}\frac{\hat{n}+n}{2n}\left(f^*g+g^*f\right) \equiv 0 , \quad s_\varphi = \frac{\hat{n}}{n}\frac{\hat{n}+n}{2n}\left(-f^*f+g^*g\right), \quad s_\psi = -i\,\frac{\hat{n}}{n}\frac{\hat{n}+n}{2n}\left(f^*g-g^*f\right).$$

Since, according to (2.43), sign ($\hat{n}$) = sign (n), current and spin of <u>these</u> solutions have evidently the <u>same</u> direction on any cylinder. The components v_ω and s_ω , normal to the cylinder, vanish.

For solutions of the type (2.44b) (i.e. belonging to $\dot{\beta}_3 = -1$ in the massless case) we obtain the same expressions as above for the spin-components, whereas the <u>sign</u> of the components of the current is now changed. In these cases, therefore, current and spin have <u>opposite</u> direction on any cylinder, the components normal to the cylinder again vanishing.

These relations between the directions of current and spin are not altered if both m and m' are negative or in the other two cases where use of the scheme (2.22a,b,c) must be made of in order to obtain the right solutions.Both current v and spin σ are thereby affected in the <u>same</u> way,namely respectively change of <u>sign</u> of a)the φ-components <u>and</u> ψ-components, b)the ψ-components only, c)the φ-components only.

Now,assuming that the solution to given n ,m ,and m' is of the type (2.44<u>a</u>),then the solution to $-n$,m ,m' or to n , $-m$, m' or to n , m , $-m'$ is of the type (2.44<u>b</u>).Since change of sign of <u>one</u> of the azimuthal eigennumbers, m <u>or</u> m',amounts to reversion of the screw-sense in which the wave winds around the cylinders,and since the sign of n is interpreted as referring to the character of the particle(electron or positron, respectively neutrino or "antineutrino" in the massless case), we may summarise the above results in the following way:

<u>the polarisation of our eigenwaves is always longitudinal</u>, parallel or antiparallel to the stream-direction.For given $|n|$, $|m|$, $|m'|$,(eight waves),this depends on whether the massless case belongs to $\dot{\beta}_3=+1$ or $\dot{\beta}_3=-1$,or,as we may say,on the screw-sense in which the waves wind over the cylinders <u>and</u> on the character of the particle.The exact correlation can,according to our statement on p.51,be expressed in the form

$$\underline{\text{direction of spin} = (-)^{|m|+|m'|+|n|-\frac{1}{2}+M} \text{ direction of current},}$$

M again denoting the number of negative signs in the triple (n, m, m').

This result holds for <u>all</u> our waves, no assumption having been made so far for explicit expressions for the functions f, g (2.29). We must, however, make sure that it has really a meaning on any cylinder, i.e. that we do not apply it to cylinders where the local intensity $\Phi^+\Phi = \frac{\Psi^+\Psi}{\sin\omega\cos\omega} = \frac{\bar{n}(\bar{n}+n)}{n^2}(f^*f+g^*g)$ vanishes altogether. Now, a glance at the general expressions (2.29) shows that but for $\omega=0$ and $\omega=\frac{\pi}{2}$ the zeros of f are different from the zeros of g. Hence f^*f+g^*g does not vanish except at $\omega=0$ and, simultaneously, at $\omega=\frac{\pi}{2}$, i.e. at the two degenerated cylinders which form a pair of polar great circles (cf. pp. 17, 19).

The expressions for the spin-components s show that on a cylinder where either f or g has a zero also the φ-component of spin (and current) is zero, the spin (and current) thus pointing there in the positive or negative φ-direction, since the ω-components are <u>always</u> zero.

Now we turn to the specially simple types of waves described above. We have simply to insert the expressions (2.45) for f and g into the components of current v and spin s and keep to that value of ω where the intensity has its maximum.For the __skin-waves__($|m|=|m'|$ and large),which vanish practically everywhere but for the immediate neighbourhood of the cylinder $\omega=\frac{\pi}{4}$,one thus finds that the values of the φ and ψ-components of current and spin are equal.Hence the current-vector(and the spin)intersects the longitudes($\varphi=const$)and latitudes($\psi=const$)of the cylinder $\omega=\frac{\pi}{4}$ under 45° .As for the __tube-waves__,concentrated near the great circle $\omega=0$ (for $|m'|=\frac{1}{2}$, $|m|\gg\frac{1}{2}$) or $\omega=\frac{\pi}{2}$ (for $|m|=\frac{1}{2}$, $|m'|\gg\frac{1}{2}$),a local observer finds there current and spin in the direction of the great circle.Finally,for the spin in the __ground-state__,where the cosmical intensity $\Psi^+\Psi$ is a constant,the local one being proportional to $\sin 2\omega$,we find $\frac{s_\psi}{s_\varphi}=\tan\omega$.

It is of interest to remark that in our representation the direction of spin(and current)depends only on the __radius__ ω of the cylinder;it does not change with the longitudes and latitudes __on__ the cylinders.This is due to the dependence of both Ψ_1 and Ψ_2 being contained in the __same__ factor $e^{i(m\varphi+m'\psi)}$.This is not so in the usual theory in flat space in polar or cylindric coordinates.There Ψ_1 differs from Ψ_2 also by a factor $e^{i\varphi}$,the direction of sharp(diagonal)spin consequently depending there also on the azimuth φ .It is,of course,again the dependence on φ of the spin-transformation connecting different representations of the general Dirac equation (I.1),(cf.p.15),which is responsible for this unusual behaviour in our case.

E) Appendices

Appendix 1

Investigation of the regularity of the solutions We have to inquire into the behaviour of the functions f and g ,(2.29)(p.35),at $\omega=0$ ($x=1$).For this purpose we use the connection between the hypergeometric series as a function of x and of $1-x$,showing the nature of the singularity of $\mathcal{F}(\alpha,\beta,\gamma;x)$ at $x=1$ (cf,Whl,§14.53),

(1) $$\mathcal{F}(\alpha,\beta,\gamma;x)=\frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,\mathcal{F}(\alpha,\beta,\alpha+\beta-\gamma+1;\,1-x)+\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}(1-x)^{\gamma-\alpha-\beta}\mathcal{F}(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;\,1-x)\,.$$

Here Γ denotes the gamma function(cf.Whl,§12.12)satisfying the difference equation

(2) $$\Gamma(\rho+1)=\rho\,\Gamma(\rho)\,.$$

If ρ equals a positive integer then $\Gamma(\rho+1)$ is identical with the factorial of ρ ,the poles being at non-positive integral values of ρ .The principal part of $\Gamma(\rho)$ at $\rho=-\ell$ ($\ell=0,+1,+2,\dots$) is $\frac{1}{\rho+\ell}$. $\Gamma(1)=1$.

It is obvious from (1) that at $x=1$ the finiteness of $F(\alpha,\beta,\gamma;x)$ depends on that of the term containing the factor

$$(1-x)^{\gamma-\alpha-\beta}=(\sin\omega)^{2(\gamma-\alpha-\beta)}=(\sin\omega)^{-2(m'-\frac{1}{2})}.$$

(Since $\gamma>0$ and $\alpha+\beta-\gamma\geqq 0$,according to (2.23) and (2.24'),poles of the Γ-functions in (1) occur only for $\gamma-\alpha-\beta=0,-1,-2,\dots$ i.e. $m'=\frac{1}{2},\frac{3}{2},\dots$.These cases are treated separately below.)Owing to the scheme (2.22)we could confine ourselves to $m'\geqq 0$,(2.23),(this has already been used in the assumption (2.24a)).Further,according to (2.12), m' can assume only <u>either</u> integral values <u>or</u> half-odd ones.To test the finiteness of f and g at $\omega=0$ it will then be useful to classify in the following way:

i) $\gamma-\alpha-\beta=-\frac{1}{2},-\frac{3}{2},\dots$ (i.e. $m'=1,2,\dots$),

ii) $\gamma-\alpha-\beta=\frac{1}{2}$ (i.e. $m'=0$),

iii) $\gamma-\alpha-\beta=0,-1,\dots$ (i.e. $m'=\frac{1}{2},\frac{3}{2},\dots$).

We start with the behaviour of (f , g)in the cases

i) $\gamma-\alpha-\beta=-\frac{1}{2},-\frac{3}{2},\dots$ (i.e. $m'=1,2,\dots$).

The factor $\sin^{m'-1}\omega$ in both f and g,(2.29),is finite in all these cases.So is the <u>first</u> expression on the right hand side of (1).From the <u>second</u> one (consequently attaching an index (a) to f and g),and observing (2),we find the following behaviour at $x=1$ ($\omega=0$):

(3a)
$$f_{(a)}=\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\sin^{-m'}\omega\sqrt{1-\cos\omega}\cdot$$
$$\cdot\left[F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;\sin^2\omega)+\frac{\gamma-\alpha}{\beta}\cos\omega\, F(\gamma-\alpha+1,\gamma-\beta,\gamma-\alpha-\beta+1;\sin^2\omega)\right],$$

(3b)
$$g_{(a)}=\frac{\Gamma(\gamma)\,\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\sin^{-m'}\omega\cdot$$
$$\cdot\left[F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1;\sin^2\omega)-\frac{\gamma-\alpha}{\beta}\cos\omega\, F(\gamma-\alpha+1,\gamma-\beta,\gamma-\alpha-\beta+1;\sin^2\omega)\right].$$

Because of the factor $\sin^{-m'}\omega$ ($m'\geq 1$) no pair (f, g) satisfies the condition (2.28)(p.34),the brackets in (3) being finite ($\neq 0$)for $\omega=0$.We cannot obtain,therefore,regular solutions for $m'=1,2,\ldots$ unless $f_{(a)}$ and $g_{(a)}$ vanish identically.This is the case <u>if,and only if,</u> α <u>or</u> $\beta+1$ <u>equals a non-positive integer</u> $0,-1,-2,\ldots$;the hypergeometric series (2,27') goes <u>then</u> over into a Jacobi polynomial.($\beta=0$ is not sufficient,since in this case the one denominator in $f_{(a)}$ and $g_{(a)}$, $\beta\,\Gamma(\beta)=\Gamma(\beta+1)=1$,is still finite.)

Now we have to consider the case

ii) $\gamma-\alpha-\beta=\frac{1}{2}$ (i.e. $m'=0$).

The contribution of the <u>first</u> expression in (1) to f and g at $x=1$ ($\omega=0$)satisfies the condition (2.28),since,according to (2.29), the leading terms are respectively

$$f_{(1)} \sim \frac{\sqrt{1-\cos\omega}}{\sin\omega} \sim \mathrm{const}, \quad g_{(1)} \sim \frac{1-\cos\omega}{\sin\omega} \sim \sqrt{1-\cos\omega} .$$

(The additional factor $1-\cos\omega$ in $g_{(1)}$ results from the difference in the bracketed part of g , use being made again of (2).) Also the contribution of the <u>second</u> expression in (1) to f and g satisfies (2.28), the behaviour of the leading terms being respectively

$$f_{(2)} \sim \sqrt{1-\cos\omega} , \quad g_{(2)} \sim \mathrm{const} .$$

Hence, in the case $m'=0$ we do get quadratically integrable(not everywhere finite, though) solutions Ψ <u>without restriction upon α or β</u>. The eigenvalue n (cf.(2.24')) may therefore assume any(real)value in <u>this</u> case without destroying the quadratic integrability of the solutions Ψ .

Lastly we have to investigate the behaviour of the functions (2.29) in the cases

iii) $\gamma-\alpha-\beta = 0, -1, \ldots$ (i.e. $m' = \frac{1}{2}, \frac{3}{2}, \ldots$)

These cases must be treated separately because of the poles $\Gamma(\gamma-\alpha-\beta)$ now occurring in formula (1); the latter must now be modified. The initial exponents of the two series in (1) differ now by zero(in case $\gamma-\alpha-\beta=0$)or by integers(in all other cases) and, therefore, logarithmic terms may appear. Moreover, these cases are distinguished in so far as we shall eventually have to choose just these half-odd numbers for m'. But previously, what regards regularity, they are on equal footing with the integral values m', as we shall see now.

Define for brevity

$$-\lambda = \gamma - \alpha - \beta$$

where, for the moment, $\lambda \neq 0, +1, \dots$, and apply the well known theorem (cf. Whl, §12.14; Jl, p.89)

$$\Gamma(-\lambda)\,\Gamma(\lambda+1) = -\frac{\pi}{\sin\lambda\pi} .$$

We may then write (1) in the following form:

(4)
$$F(\alpha,\beta,\alpha+\beta-\lambda;x) = -\frac{\pi}{\sin\lambda\pi}\,\frac{\Gamma(\alpha+\beta-\lambda)}{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\alpha-\lambda)\,\Gamma(\beta-\lambda)} \cdot$$
$$\cdot \left\{ \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\,\Gamma(\beta+r)}{\Gamma(r+1)\,\Gamma(\lambda+r+1)}(1-x)^{r} - \sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r-\lambda)\,\Gamma(\beta+r-\lambda)}{\Gamma(r+1)\,\Gamma(r-\lambda+1)}(1-x)^{r-\lambda} \right\} .$$

It can now easily be seen that in the limiting case $\lambda = \ell$ ($\ell = \alpha+\beta-\gamma = 0, +1, \dots$) the expression $\frac{\{\ \}}{\sin\lambda\pi}$ in (4) takes the undetermined form $\frac{0}{0}$. (The coefficient of the positive powers of $(1-x)$ is equal in both sums in the bracket $\{\ \}$ in (4), while, owing to the poles of the Γ-function in the denominator, the coefficients of the negative powers in the second sum are themselves zero.) We differentiate, therefore, both nominator and denominator with respect to λ and go <u>then</u> to the limit $\lambda = \ell$. Thus, denoting by $\psi(\rho)$ the logarithmic derivative of $\Gamma(\rho+1)$, (cf. Jl, p.92; the poles

of $\psi(\rho)$ are of the same kind as those of $\Gamma(\rho+1)$, the ratio $\frac{\psi(\rho)}{\Gamma(\rho+1)}$ equalling $(-)^{\rho}$.) we get

$$\mathcal{F}(\alpha,\beta,\alpha+\beta-\ell;x)=\frac{(-)^{\ell}\,\Gamma(\alpha+\beta-\ell)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha-\ell)\Gamma(\beta-\ell)}\left\{\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\Gamma(\beta+r)}{\Gamma(r+1)\Gamma(\ell+r+1)}\psi(\ell+r)(1-x)^{r}-\right.$$

(5)
$$\left.-\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r-\ell)\Gamma(\beta+r-\ell)}{\Gamma(r+1)\Gamma(r-\ell+1)}\cdot\left[\psi(\alpha+r-\ell-1)+\psi(\beta+r-\ell-1)-\psi(r-\ell)+\log(1-x)\right](1-x)^{r-\ell}\right\}.$$

Realising that, because of the $\Gamma(r-\ell+1)$ in the denominator, all coefficients of the negative powers of $(1-x)$ vanish but for those also containing the compensating $\psi(r-\ell)$, (5) can finally be written:

$$\mathcal{F}(\alpha,\beta,\alpha+\beta-\ell;x)=\frac{(-)^{\ell}\,\Gamma(\alpha+\beta-\ell)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha-\ell)\Gamma(\beta-\ell)}\left\{\sum_{\vartheta=0}^{\infty}\frac{\Gamma(\alpha+\vartheta)\Gamma(\beta+\vartheta)}{\Gamma(\vartheta+1)\Gamma(\vartheta+\ell+1)}(1-x)^{\vartheta}\cdot\right.$$

(6)
$$\left.\cdot\left[\psi(\vartheta+\ell)-\psi(\alpha+\vartheta-1)-\psi(\beta+\vartheta-1)+\psi(\vartheta)-\log(1-x)\right]+\sum_{\sigma=1}^{\ell}\frac{(-)^{\sigma}\,\Gamma(\alpha-\sigma)\Gamma(\beta-\sigma)}{\Gamma(\ell-\sigma+1)}(1-x)^{-\sigma}\right\}.$$

<u>This</u> formula can now be used for determining the behaviour of (f, g), (2.29), at $\omega=0$ $(x=1)$ in all the cases $\gamma-\alpha-\beta=-\ell=0,-1,\dots$.

First we consider all cases $m'\neq\frac{1}{2}$ and then the last remaining case of $m'=\frac{1}{2}$.

a) $\ell=1,2,\dots$ **(i.e.** $m'=\frac{3}{2},\frac{5}{2},\dots$**)**

Apart from the logarithmic term, the behaviour of (f, g) is determined by the $(-\ell)$th power of $(1-x)$ in (6),

$$f\sim\sqrt{1-\cos\omega}\,\sin^{\ell-\frac{1}{2}}\omega\,\sin^{-2\ell}\omega\sim\sin^{-\ell+\frac{1}{2}}\omega\,,\quad g\sim\sin^{\ell-\frac{1}{2}}\omega\,\sin^{-2\ell}\omega\sim\sin^{-\ell-\frac{1}{2}}\omega\,.$$

The condition (2.28) is clearly <u>not</u> fulfilled. Again, therefore, we do not obtain quadratically integrable solutions, unless the terms containing the negative powers and the logarithm in (6) vanish identically. Again, this is secured <u>if, and only if, α or $\beta+1$ equals a non-positive integer</u>. All terms not containing the compensating factor $\psi(\alpha+\nu-1)$ (or $\psi(\beta+\nu)$, as the case may be), with $\nu \leq |\alpha|$, vanish then identically because of the poles in the denominators. The condition $\nu \leq |\alpha|$ turns $\sum_{\nu=0}^{\infty}$ into $\sum_{\nu=0}^{|\alpha|}$, (6) thus becoming a (Jacobi) polynomial, as is also obvious from (2.27').

Finally, it remains to consider the case

b) $\quad \ell=\alpha+\beta-\gamma=0$ (i.e. $m'=\frac{1}{2}$).

The behaviour of (f, g), (2.29), at $\omega=0$ ($x=1$) is now solely determined by the logarithmic term in (6),

$$\begin{pmatrix} f \\ g \end{pmatrix} \sim \sin^{-\frac{1}{2}}\omega \, \log \sin^2\omega \begin{pmatrix} \sqrt{1-\cos\omega} \\ 1-\cos\omega \end{pmatrix} \sim \sin^{\frac{1}{2}}\omega \, \log \sin\omega \begin{pmatrix} 1 \\ \sin\omega \end{pmatrix} .$$

The additional factor $1-\cos\omega$ in g arises, again, from the difference in the bracketed part of g, use being made again of (2); the coefficients of <u>both</u> 1 and $-\cos\omega$ equal $\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}$, since $\gamma-\alpha=\beta$.)

Because of the positive $\sin$-powers, both f and g do satisfy the condition (2.28). Since, then, we must not restrict α or $\beta+1$ to non-positive integers (i.e. the polynomial solutions where n, the eigenvalue, is restricted to certain discrete values according (2.24'), we obtain <u>a continuous set of quadratically integrable (though not everywhere finite) solutions Ψ to $m'=\frac{1}{2}$</u>, (The same was found true for $m'=0$.) However, as will be shown in the text, it is, in fact, only the polynomial solutions which fulfil the more stringent conditions to be imposed upon our eigenfunctions.

We may summarise the result of the boundary condition at $\omega=0$: <u>apart from a continuous spectrum of solutions to $m'=0$ and to $m'=\frac{1}{2}$, only solutions with non-positive integral values of α or $\beta+1$ are admissible as eigensolutions</u>.

Appendix 11

Application of Pauli's criterion

a) Verification of the D-operator relations (2.35)(p.39) We reproduce the verification of (2.35) for Ψ_1, say, (1 denoting the spin index), the calculation being quite analogous for Ψ_2.

For brevity put

(1)
$$A = i\cos^{m-\frac{3}{2}}\omega\,\sin^{m'-\frac{5}{2}}\omega\,\sqrt{1-\cos\omega}\,,\qquad exp = e^{i(m\varphi+m'\psi+nt)}\,,$$
$$\mathcal{F} = \mathcal{F}(\alpha,\beta,\gamma;x)\,,\quad \mathcal{F}' = \mathcal{F}(\alpha,\beta+1,\gamma+1;x)\,;\quad x = \cos^2\omega\,.$$

With (2.3o) and (2.16a) we get then

(2)
$$\frac{1}{exp}\frac{1}{A}\frac{\partial \Psi_1}{\partial\omega} = \sin\omega\cos\omega\frac{d\mathcal{F}}{d\omega} + \frac{\gamma-\alpha}{\gamma}\cos\omega\left(\sin\omega\cos\omega\frac{d\mathcal{F}'}{d\omega} - \sin^2\omega\,\mathcal{F}'\right) +$$
$$+\left[(m'-\tfrac{3}{2})\cos^2\omega - (m-\tfrac{1}{2})\sin^2\omega + \tfrac{1}{2}\cos^2\omega + \tfrac{1}{2}\cos\omega\right]\left(\mathcal{F} + \frac{\gamma-\alpha}{\gamma}\cos\omega\,\mathcal{F}'\right),$$
$$i\tan\omega\frac{\partial\Psi_1}{\partial\varphi} = -A\,m\sin^2\omega\left(\mathcal{F}+\frac{\gamma-\alpha}{\gamma}\cos\omega\,\mathcal{F}'\right)exp\,,$$
$$i\cot\omega\frac{\partial\Psi_1}{\partial\psi} = -A\,m'\cos^2\omega\left(\mathcal{F}+\frac{\gamma-\alpha}{\gamma}\cos\omega\,\mathcal{F}'\right)exp\,,$$
$$\frac{i}{2}\beta_1\Psi_1 = \frac{i}{2}\Psi_2 = \frac{A}{2}(\cos\omega+\cos^2\omega)\left(\mathcal{F}-\frac{\gamma-\alpha}{\gamma}\cos\omega\,\mathcal{F}'\right)exp\,,$$
$$\frac{1}{2}\frac{\beta_3}{\sin\omega}\Psi_1 = \frac{1}{2}\frac{\Psi_1}{\sin\omega} = \frac{A}{2}\cos\omega\left(\mathcal{F}+\frac{\gamma-\alpha}{\gamma}\cos\omega\,\mathcal{F}'\right)exp\,,$$
$$\frac{1}{2}\frac{\beta_1}{\cos\omega}\Psi_1 = \frac{i}{2}\frac{\Psi_2}{\cos\omega} = \frac{A}{2}(1+\cos\omega)\left(\mathcal{F}-\frac{\gamma-\alpha}{\gamma}\cos\omega\,\mathcal{F}'\right)exp\,.$$

Compounding, then, the operator D_{--} according to (2.33), and using (2.24'), we get

(3)
$$D_{--}\,{}_1\Psi^n_{mm'} = -A\,e^{i[(m-1)\varphi+(m'-1)\psi+nt]}\left\{\left[x(1-x)\frac{d\mathcal{F}}{dx}+(\gamma-\alpha-\beta)x\mathcal{F}+(\gamma-1)(1-x)\mathcal{F}\right]+\right.$$
$$\left.+\frac{\gamma-\alpha}{\gamma}\cos\omega\left[x(1-x)\frac{d\mathcal{F}'}{dx}+(\gamma-\alpha-\beta)x\mathcal{F}'+\gamma(1-x)\mathcal{F}'\right]\right\}.$$

Now we have simply to use the following recurrence formulae (they can easily be verified, writing out the coefficients of x^ν, according (2.27')) in order to obtain the relation (2.35a):

(4)
$$(1-x)\frac{d\mathcal{F}}{dx} + (\gamma-\alpha-\beta)\mathcal{F} = \frac{(\alpha-\gamma)(\beta-\gamma)}{\gamma}\,\mathcal{F}(\alpha,\beta,\gamma+1;x)\,,$$

(5)
$$\frac{(\alpha-\gamma)(\beta-\gamma)}{\gamma}\,x\,\mathcal{F}(\alpha,\beta,\gamma+1;x) + (\gamma-1)(1-x)\mathcal{F} = (\gamma-1)\,\mathcal{F}(\alpha-1,\beta-1,\gamma-1;x)\,,$$

and the analogous equations for $\mathcal{F}'$, replacing β, γ by $\beta+1$, $\gamma+1$ respectively. Hence, remembering (1),

(6)
$$
\begin{aligned}
D_{--}\,{}_1\Psi^n_{mm'} &= (1-\gamma)\, i \cos^{m-\frac{3}{2}}\omega \sin^{m'-\frac{5}{2}}\omega \sqrt{1-\cos\omega}\; e^{i[(m-1)\varphi+(m'-1)\psi+nt]} \cdot \\
&\quad \cdot \left[\mathcal{F}(\alpha-1,\beta-1,\gamma-1;x)+\frac{\gamma-\alpha}{\gamma-1}\cos\omega\, \mathcal{F}(\alpha-1,\beta,\gamma;x)\right] \\
&= (1-\gamma)\, {}_1\Psi^n_{m-1\;m'-1}
\end{aligned}
$$

By a similar calculation we get (2.35c). First we find

$$
\begin{aligned}
D_{+-}\,{}_1\Psi^n_{mm'} &= A\, e^{i[(m+1)\varphi+(m'-1)\psi+nt]} \cdot \\
&\quad \cdot \left\{\left[x(1-x)\frac{d\mathcal{F}}{dx}+(\gamma-\alpha-\beta)x\mathcal{F}\right]+\frac{\gamma-\alpha}{\gamma}\cos\omega\left[x(1-x)\frac{d\mathcal{F}'}{dx}+(\gamma-\alpha-\beta)x\mathcal{F}'\right]\right\}.
\end{aligned}
$$

Observing (4) we obtain then

(7)
$$
\begin{aligned}
D_{+-}\,{}_1\Psi^n_{mm'} &= \frac{(\alpha-\gamma)(\beta-\gamma)}{\gamma}\, i \cos^{m+\frac{1}{2}}\omega \sin^{m'-\frac{5}{2}}\omega \sqrt{1-\cos\omega}\; e^{i[(m+1)\varphi+(m'-1)\psi+nt]} \cdot \\
&\quad \cdot \left[\mathcal{F}(\alpha,\beta,\gamma+1;x)+\frac{\gamma+1-\alpha}{\gamma+1}\cos\omega\, \mathcal{F}(\alpha,\beta,\gamma+2;x)\right] \\
&= \frac{(\alpha-\gamma)(\beta-\gamma)}{\gamma}\, {}_1\Psi^n_{m+1\;m'-1}\,.
\end{aligned}
$$

Three more formulae are necessary to verify the relation (2.35d), viz.

(8)
$$
(1-x)\frac{d\mathcal{F}}{dx} = \beta\left(\mathcal{F}-\frac{\gamma-\alpha}{\gamma}\mathcal{F}'\right), \quad x\frac{d\mathcal{F}}{dx}+(\gamma-1)\mathcal{F} = (\gamma-1)\,\mathcal{F}(\alpha,\beta,\gamma-1;x),
$$

(9)
$$
x\frac{d\mathcal{F}'}{dx}+(\gamma-1)\mathcal{F}'-\frac{\gamma}{\beta(\gamma-\alpha)}\frac{d\mathcal{F}}{dx} = \frac{\gamma-\alpha-1}{\gamma-\alpha}\,\mathcal{F}(\alpha,\beta+1,\gamma;x).
$$

Then we obtain

(10)
$$
\begin{aligned}
D_{-+}\,{}_1\Psi^n_{mm'} &= (\gamma-1)\, i \cos^{m-\frac{3}{2}}\omega \sin^{m'-\frac{1}{2}}\omega \sqrt{1-\cos\omega}\; e^{i[(m-1)\varphi+(m'+1)\psi+nt]} \cdot \\
&\quad \cdot \left[\mathcal{F}(\alpha,\beta,\gamma-1;x)+\frac{\gamma-\alpha-1}{\gamma-1}\cos\omega\, \mathcal{F}(\alpha,\beta+1,\gamma;x)\right] \\
&= (\gamma-1)\, {}_1\Psi^n_{m-1\;m'+1}\,.
\end{aligned}
$$

In the same way we find

$$D_{++}\,{}_{1}\Psi^{n}_{mm'} = \frac{-\alpha(\beta+1)}{\gamma}\, i \cos^{m+\frac{1}{2}}\omega \sin^{m'-\frac{1}{2}}\omega \sqrt{1-\cos\omega}\; e^{i[(m+1)\varphi+(m'+1)\psi+nt]} \cdot$$
$$\cdot\left[F(\alpha+1,\beta+1,\gamma+1;x)+\frac{\gamma-\alpha}{\gamma+1}\cos\omega\, F(\alpha+1,\beta+2,\gamma+2;x)\right]$$
$$= \frac{-\alpha(\beta+1)}{\gamma}\,{}_{1}\Psi^{n}_{m+1\;m'+1} \quad .$$

The latter formula will not be used in the text for the actual application of Pauli's criterion. We have, however, derived it here to make sure that the matrix-element of D_{++} vanishes identically if $\alpha=0$ or if $\beta+1=0$, as it must be, since no regular eigenfunction exists for $\alpha=1$ or for $\beta+1=1$.

b) Non-orthogonality of solutions to $m=0$ (or to $m'=0$) We show that for $m=0$ (the same consideration holds almost literally for $m'=0$) there exist two groups of solutions, the members of the one group not being orthogonal on those of the other one. They belong, say, to the eigenvalues n_1 and n_2 respectively.

Remembering (2.6')(p.21) and that the volum-element is given by (2.3b), we have clearly to evaluate the expression

(11) $$\int_0^{\frac{\pi}{2}} (f^{n_1}\bar{f}^{n_2}+g^{n_1}\bar{g}^{n_2})\,d\omega = \frac{i}{n_2-n_1}\left(f^{n_1}\bar{g}^{n_2}+g^{n_1}f^{n_2}\right)\Big|_0^{\frac{\pi}{2}} \quad ,$$

the bar denoting the complex canjugate quantity. The latter equation is obtained in the usual way from the differential equations (2.21)(p.29), with $m=0$ and equal m' in both solutions under consideration. n_1 and $-n_2$ are such that α, respectively $\beta+1$, equals a non-positive integer (cf.(2.31)p.36),

$$n_1 = m'+\tfrac{1}{2}+2k \ , \quad -n_2 = m'+\tfrac{3}{2}+2k' \ ; \quad k, k' = 0, 1, 2, \cdots \ .$$

Now, from the function belonging to $-n_2$, $m=0$, m', we obtain the function to n_2, $m=0$, m', by application of the scheme (2.22b),

$$f^{\,n_2}_{0\,m'} = C\, f^{\,-n_2}_{0\,m'} \ , \quad g^{\,n_2}_{0\,m'} = C(-g)^{\,-n_2}_{0\,m'} \ ,$$

C denoting a constant factor. The right hand side of (11) becomes then:

$$i(n_1-n_2)\int_0^{\frac{\pi}{2}}(f^{n_1}\bar{f}^{n_2}+g^{n_1}\bar{g}^{n_2})\,d\omega = C\left(-f^{n_1}\bar{g}^{-n_2}+g^{n_1}\bar{f}^{-n_2}\right)\Big|_0^{\frac{\pi}{2}} \ .$$

This does <u>not</u> vanish. According to (2.29) the constant term in $F(\alpha,\beta,\gamma;\cos^2\omega)$ gives rise to

$$\left(f^{n_1}_{m=0}\right)\Big|_0^{\frac{\pi}{2}} = i = -\left(\bar{f}^{-n_2}_{m=0}\right)\Big|_0^{\frac{\pi}{2}} \ , \quad \left(g^{n_1}_{m=0}\right)\Big|_0^{\frac{\pi}{2}} = 1 = \left(\bar{g}^{-n_2}_{m=0}\right)\Big|_0^{\frac{\pi}{2}} \ ,$$

the right hand side of (11) thus taking the value($\neq 0$) $2iC$.

The functions of the two groups $n_1-n_2 = 2(k-k')-1$ = odd number are therefore <u>not</u> mutually orthogonal.

c) Lastly it remains to make sure that the result of D operating on a regular solution to $m=\frac{1}{2}$ or to $m'=\frac{1}{2}$ is indeed again a regular solution of our system, as postulated by (2.32).

At first sight it would seem from (6),(7),and(1o)that also in these cases irregular functions are being produced and that, therefore,the condition (2.32) could not be fulfilled,neither with integral nor with half-odd quantum numbers m , m' .That this is **not** so is due to the fact that the recurrence formulae (4),(5),(8),and (9) ,leading to (6),(7),and (1o) respectively, hold for all values m , m' except just for $m=\frac{1}{2}$ (i.e. $\gamma-1=0$) and $m'=\frac{1}{2}$ (i.e. $\gamma-\alpha-\beta=0$).For example:in the case $m=\frac{1}{2}$ and $m'=\frac{1}{2}$ we obtain,instead of (3),

$$D_{--}\,{}_1\Psi^{n}_{\frac{1}{2}\frac{1}{2}} = -i\cos^{-1}\omega\sqrt{1-\cos\omega}\left[x\frac{dF}{dx}+\frac{\gamma-\alpha}{\gamma}\cos\omega\left(x\frac{dF'}{dx}+F'\right)\right]e^{i(-\frac{1}{2}\varphi-\frac{1}{2}\psi+nt)}.$$

Writing out the coefficients of x^{ν} and of $\cos\omega\cdot x^{\nu}$,it can easily be verified that this function is indeed identical with the **regular** solution ${}_1\Psi^{n}_{-\frac{1}{2}-\frac{1}{2}}$ which is obtained from $\Psi^{n}_{\frac{1}{2}\frac{1}{2}}$ by help of the scheme (2.22)(apart from a constant factor $i\beta=i(1-\alpha)$).

Similarly,understanding the right hand sides to be the regular eigensolutions obtained from those in (2.3o)(p.36) by help of the scheme (2.22),we obtain

$$D_{--}\Psi^{n}_{\frac{1}{2}\,m'} = (\alpha-1)\,\Psi^{n}_{-\frac{1}{2}\,m'-1}\ ,\ (m'\neq\tfrac{1}{2})$$

$$D_{--}\Psi^{n}_{m\,\frac{1}{2}} = -i(\gamma-1)\,\Psi^{n}_{m-1\,-\frac{1}{2}}\ ,\ (m\neq\tfrac{1}{2})$$

$$D_{+-}\Psi^{n}_{m\,\frac{1}{2}} = i\frac{\alpha\beta}{\alpha+\beta}\,\Psi^{n}_{m+1\,-\frac{1}{2}}\ ,$$

$$D_{-+}\Psi^{n}_{\frac{1}{2}\,m'} = -\alpha\,\Psi^{n}_{-\frac{1}{2}\,m'+1}\ .$$

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