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On Numerical Approximations for Stochastic Differential Equations

Xīlíng Zhāng

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Xīlíng Zhāng*)

To Zhènnán Zhāng and Shūqín Lǐ,
who never stopped believing in their son.

Acknowledgements

It all dates back to the 12th of April, 2013.

I had no idea how I had ended up selling door to door in the dullest neighbourhood of Northampton; I only knew that I was losing my last chance of continuing studying mathematics, as well as hope for future. Then at about 20:30 I received an unexpected phone call from Professor István Gyöngy, asking me if I was "interested in doing a PhD".

I will *never* forget that phone call.

Neither will I forget Dr. Sotirios Sabanis who recruited me in the very beginning, or Dr. Łukasz Szpruch who gave me the chance to come back. There was clearly some unspoken effort in securing me a full scholarship, though I never asked about it.

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Abstract

This thesis consists of several problems concerning numerical approximations for stochastic differential equations, and is divided into three parts. The first one is on the integrability and asymptotic stability with respect to a certain class of Lyapunov functions, and the preservation of the comparison theorem for the explicit numerical schemes. In general, those properties of the original equation can be lost after discretisation, but it will be shown that by some suitable modification of the Euler scheme they can be preserved to some extent while keeping the strong convergence rate maintained. The second part focuses on the approximation of iterated stochastic integrals, which is the essential ingredient for the construction of higher-order approximations. The coupling method is adopted for that purpose, which aims at finding a random variable whose law is easy to generate and is close to the target distribution. The last topic is motivated by the simulation of equations driven by Lévy processes, for which the main difficulty is to generalise some coupling results for the one-dimensional central limit theorem to the multi-dimensional case.

Lay Summary

Stochastic differential equations are common mathematical tools to model various systems and mechanisms in physical and natural sciences, financial activities and population growth, etc. with random behaviour. Given that those mathematical models are well-defined, in practice one needs to know how to approximate them, and particularly how to simulate them on a computer. However, there is an important middle layer bridging these two ends together, that is, the theoretical guarantee that an approximation method will work and perform well in a reasonable sense. This thesis reviews several important questions that appear on this level, and presents a few attempts to answer them.

Contents

Chapter 1

Introduction

This thesis is a compilation of quantitative investigations of numerical approximations for stochastic differential equations (SDEs), concerning different problems such as whether the moment bounds, asymptotic stability and comparison properties of some SDEs can be preserved by their numerical approximations to some extent, whether there is a way to approximate a general SDE faster than Euler's method, and whether SDEs with jumps can be approximated in an efficient way.

Approximations for SDEs Driven by a Wiener Process

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions, W_t be an q-dimensional $(\mathcal{F}_t)_{t\geqslant 0}$-adapted Wiener process, and consider a d-dimensional SDE for $t \in [0,T]$ for some $T > 0$:

$$X_t = X_0 + \int_0^t b(s, X_s)\mathrm{d}s + \int_0^t \sigma(s, X_s)\mathrm{d}W_s, \tag{1.1}$$

where the functions $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma =: [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times q}$ are locally Lipschitz continuous. The (explicit) Euler's approximation with step size $h \in (0,1]$,

$$\begin{aligned} \widehat{X}_{k+1} =& \widehat{X}_k + b(t_k, \widehat{X}_k)h + \sigma(t_k, \widehat{X}_k)\Delta W_{k+1}, \\ \widehat{X}_0 =& X_0, \end{aligned} \tag{1.2}$$

where $\Delta W_{k+1} := W_{t_{k+1}} - W_{t_k},\ t_k := kh$, is well-studied in the literature. In particular, one can construct a strong solution of the equation (1.1) via the scheme (1.2) under mild conditions (see [15]). What is more of practical interest is its strong-L^p convergence for some $p > 1$. Standard calculation shows that $(\mathbb{E} \max_k |X_{t_k} - \widehat{X}_k|^p)^{1/p} = O(h^{1/2})$ when the coefficients b, σ are Lipschitz and have linear growth on the entire interval $[0,T]$ and $\mathbb{E}|X_0|^p < \infty$. A slightly weaker formulation $\max_k(\mathbb{E}|X_{t_k} - \widehat{X}_k|^p)^{1/p}$ is also widely used in the literature.

Most of the topics in this thesis are directed towards or extended from the question of the strong L^p-convergence for explicit numerical schemes. The second and the third chapters concern SDEs of the type (1.1), which is the general formulation of many models in physics, finance, weather forecast, etc. The implicit schemes, on the other hand, will not be considered. Solving an implicit equation at each iteration of the algorithm requires a high level of computational cost, and therefore they are not very practical to implement compared to explicit ones.

Tamed Euler Schemes

The linear growth condition turns out to be somewhat important for the standard method (1.2) to work. As is shown by Hutzenthaler, Jentzen and Kloeden [25] (Theorem 2.1), when the coefficients have polynomial growth the Euler scheme (1.2) may not have finite moments and hence diverge in L^p. Later on, assuming the global Lipschitz condition for the diffusion matrix σ, the authors [26] managed to recover the strong-L^p convergence by modifying the drift b so that the new numerical scheme has bounded moments. Such a modification of explicit schemes is conventionally called *"taming"*. A tamed Euler scheme is usually of the following form:

$$\bar{X}_{k+1} = \bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1}, \ k \in \mathbb{N}, \tag{1.3}$$

where usually a taming coefficients b^h, σ^h are chosen s.t. $b^h(t,x) \to b(t,x), \ \sigma^h(t,x) \to \sigma(t,x)$ as $h \to 0$ uniformly in (t,x). This resembles the treatment for the stiff problem when approximating ODEs.

Several different taming method have been proposed by many authors, e.g. [24,27,48,49,54], etc, and the proofs of their convergence results all rely on one key step - to show certain moment bounds for their numerical schemes, which is the motivation to introduce the V-integrability property in Chapter 1.

Consider a non-negative, C^2 function V on $\mathbb{R}^d$. Both integrability and asymptotic stability of the equation (1.1) w.r.t. V can be deduced by examining the generator

$$\mathcal{L}_t V(x) = \langle \nabla V(x), b(t,x)\rangle + \frac{1}{2}\mathrm{tr}\left[\sigma(t,x)\mathrm{D}^2 V(x)\sigma(t,x)^\top\right],$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$, where $\mathrm{D}^2 V$ is the Hessian matrix of V. For $T > 0$ fixed, one knows from classical results [31] that if there is a constant $\rho > 0$ s.t. $\forall t \in [0,T], \ x \in \mathbb{R}^d$,

$$\mathcal{L}_t V(x) \leqslant \rho V(x), \tag{1.4}$$

then one has a uniform bound:

$$\mathbb{E}V(X_t) \leqslant e^{\rho T}\mathbb{E}V(X_0), \ \forall t \in [0,T]. \tag{1.5}$$

In the context of *asymptotic stability*, instead of a finite interval $[0,T]$ one considers the SDE (1.1) on $[0,\infty)$ and coefficients satisfying $b(t,0) \equiv 0, \ \sigma(t,0) \equiv 0, \ \forall t \geqslant 0$ (see [37,39]). Given the well-posedness of the SDE (1.1) one sees that the system has trivial solution (equilibrium) $X_t \equiv 0, \ \forall t \geqslant 0$ a.s. when $X_0 \equiv 0$ a.s. The question of stability concerns the behaviour of the solution X_t as $t \to \infty$ when the initial condition X_0 is perturbed. Similar to the Lyapunov technique used for ODEs, one considers a function $V \in \mathcal{C}^2(\mathbb{R}^d)$ that takes value 0 at the origin and is strictly positive elsewhere (e.g. $V(\cdot) = |\cdot|^p$ for some $p \in \mathbb{Z}^+$). Instead of (1.4), a sufficient condition for $X_t \to 0$ a.s. as $t \to \infty$, regardless of the value of X_0, is that

$$\mathcal{L}_t V(\cdot) \leqslant -z(\cdot), \tag{1.6}$$

for some non-negative $z \in \mathcal{C}(\mathbb{R}^d)$ such that $\ker(z) = \{0\}$. Moreover if $z(\cdot) \geqslant \rho V(\cdot)$ for some constant $\rho > 0$, then instead of (1.5) one has

$$\mathbb{E}V(X_t) \leqslant e^{-\rho t}\mathbb{E}V(X_0) \to 0, \tag{1.7}$$

as $t \to \infty$, given that $\mathbb{E}V(X_0) < \infty$. Conditions of the type (1.6) with $z(\cdot) \geqslant \rho V(\cdot)$

also play a crucial role in establishing ergodicity properties of SDEs - see [42].

We also introduce the "tamed" generator corresponding to a tamed Euler scheme of the form (1.3):

$$\mathcal{L}_t^h V(x) := \langle \nabla V(x), b^h(t,x)\rangle + \frac{1}{2}\mathrm{tr}\left[\sigma^h(t,x)\mathrm{D}^2 V(x)\sigma^h(t,x)^\top\right].$$

It will be shown in Chapter 1 (Theorem 2.20) that if the tamed coefficients b^h, σ^h satisfy certain growth assumptions and the tamed generator $\mathcal{L}_t^h$ satisfies a similar condition as (1.6), then for fixed h the tamed scheme $\bar{X}_k$ also goes to 0 as $k \to \infty$ in the corresponding sense.

In addition, it will be shown in Chapter 1 that, in the one-dimensional case, the tamed scheme $\bar{X}_k$ can preserve the non-negativity or the comparison property of the the original SDE (1.1) using a suitable truncation of the noise.

The Coupling Method for Higher-Order Approximations

Chapter 2 concerns higher-order approximations for the equation (1.1) on $[0,T]$. One can derive numerical schemes that converge in the strong-L^p sense of order greater than 1/2 from stochastic Taylor expansions, as is shown in [32]. For simplicity consider the case where b and σ do not depend on t. Then, for example, by applying Itô's formula to the coefficients b and σ, one obtains the Itô-Taylor expansion of length 2: for each component $i = 1,\cdots,d$ on the interval $[s,t]$,

$$\begin{aligned} X_t^i =& X_s^i + b_i(X_s)(t-s) + \sum_{j=1}^q \sigma_{ij}(X_s)(W_t^j - W_s^j) \qquad (1.8)\\ &+ \int_s^t\int_s^r \mathcal{L}b_i(X_u)\mathrm{d}u\mathrm{d}r + \sum_{j=1}^q \int_s^t\int_s^r \sum_{k=1}^d \sigma_{kj}(X_u)\partial_k b_i(X_u)\mathrm{d}W_u^j\mathrm{d}r \\ &+ \sum_{j=1}^q \int_s^t\int_s^r \mathcal{L}\sigma_{ij}(X_u)\mathrm{d}u\mathrm{d}W_r^j + \sum_{j,k=1}^q \int_s^t\int_s^r \sum_{l=1}^d \sigma_{lk}(X_u)\partial_l\sigma_{ij}(X_u)\mathrm{d}W_u^k\mathrm{d}W_r^j, \end{aligned}$$

where $\partial_k = \partial_{x_k}$ is the partial derivative w.r.t. the k-th coordinate. The last term in (1.8) involves an iterated stochastic integral, and it gives rise to Milstein's method: for each component $i = 1,\cdots,d$,

$$\widetilde{X}_{k+1}^i = \widetilde{X}_k^i + b_i(\widetilde{X}_k)h + \left(\sum_{j=1}^q \sigma_{ij}(\widetilde{X}_k)\Delta W_{k+1}^j + \sum_{j,l=1}^q \varsigma_{ijl}(\widetilde{X}_k)A_k(j,l)\right), \qquad (1.9)$$

where $\varsigma_{ijl}(x) := \sum_{m=1}^d \sigma_{mj}(x)\partial_m\sigma_{il}(x)$ and

$$A_k(j,l) := \int_{t_k}^{t_{k+1}} (W_t^j - W_{t_k}^j)\mathrm{d}W_t^l.$$

The scheme (1.9) has strong-L^2 convergence rate $O(h)$ according to Kloeden and Platen [32] (Section 10.3), but the problem lies in the generation of the double integral $I_{jl} = \int_0^h W_t^j \mathrm{d}W_t^l$, which is non-trivial for $q \geqslant 2$.

As mentioned by Wiktorsson [56] and Davie [8] (Section 2), if the diffusion matrix satisfies the commutativity condition $\varsigma_{ijl}(x) = \varsigma_{ilj}(x)$ for all $x \in \mathbb{R}^d$ and all

$i = 1, \cdots, d$, $j, l = 1, \cdots, q$, one only needs to generate the Wiener increments ΔW_{k+1} to achieve the order-1 convergence. But this is not always the case: using only the Wiener increments ΔW_{k+1} to implement a numerical method will, in general, result in a convergence rate no more than $O(h^{1/2})$, according to [7].

One attempt to generate the double integral I_{jl} was made by Lyons and Gaines [36], but their method only works for $q = 2$. Recently a strong result for any dimension has been proved by Davie [8] (Theorem 4) under the condition that the diffusion matrix σ has rank q everywhere, and it provides a way to approximate the SDE (1.1) up to an arbitrary order. This is a significant improvement concerning higher-order approximations. The idea is that, rather than generating the double integrals at each step k, one approximates the quantity inside the big parentheses in (1.9) as a whole. This is a completely different approach than the usual ones, as Davie's arguments are based on the coupling method, quantifying the strong-L^p convergence in terms of the Vaserstein[1] metrics.

For probability measures $\mathbb{P}, \mathbb{Q}$ on $\mathbb{R}^q$ and $p \geqslant 1$, the **Vaserstein p-distance** is defined by

$$\mathbb{W}_p(\mathbb{P}, \mathbb{Q}) := \inf_{\pi \in \Pi(\mathbb{P}, \mathbb{Q})} \left(\int_{\mathbb{R}^q \times \mathbb{R}^q} |x - y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{1/p},$$

where $\Pi(\mathbb{P}, \mathbb{Q})$ is the set of all joint probability measures on $\mathbb{R}^q \times \mathbb{R}^q$ with marginal laws $\mathbb{P}$ and $\mathbb{Q}$. In general $\mathbb{P}$ and $\mathbb{Q}$ need not be defined on the same probability space, but this definition is enough for the purpose of this thesis. The notation $\mathbb{W}_p(X, Y)$ will not cause any confusion for random variables X and Y having laws $\mathbb{P}$ and $\mathbb{Q}$, respectively. If one can show a bound for the distance between the two laws, we then say there is a **coupling** between X and Y (or $\mathbb{P}$ and $\mathbb{Q}$).

The significance of using the Vaserstein distances instead of other ones is that, when generating numerical schemes for an SDE, the convergence in the Vaserstein-type distance $\mathbb{W}_{p,\infty}$ (replacing $|x-y|^p$ in the definition above by $\max_k |x_k - y_k|^p$) is equivalent to the usual strong L^p-convergence, for the purpose of simulation at least. Too see this, suppose we have found a coupling between the solution $X = \{X_{t_k}\}$ and a numerical scheme $\bar{X} = \{\bar{X}_k\}$ with $\mathbb{W}_{p,\infty}(X, \bar{X}) \leqslant Ch^\gamma$ for some $\gamma > 0$. Then by definition, $\forall \varepsilon > 0$ there is a random vector Y^ε on the same probability space as the solution X, having the same distribution as $\bar{X}$, s.t. $(\mathbb{E} \max_k |X_{t_k} - Y_k|^p)^{1/p} \leqslant \mathbb{W}_{p,\infty}(X, \bar{X}) + \varepsilon$. Choose $\varepsilon = h^\gamma$ and in practice one generates Y instead of $\bar{X}$ to approximate X. The reader is referred to Section 12 in [8] for a detailed discussion on the contexts where such a substitution holds or fails.

Although there is no general formulas for the quantity $\mathbb{W}_p(\mathbb{P}, \mathbb{Q})$, if $\mathbb{P}$ and $\mathbb{Q}$ have densities f and g, respectively, then there is the elementary and yet important inequality

$$\mathbb{W}_p(\mathbb{P}, \mathbb{Q}) \leqslant C_p \left(\int_{\mathbb{R}^q} |x|^p |f(x) - g(x)| \mathrm{d}x \right)^{1/p}, \tag{1.10}$$

for all $p \geqslant 1$, as a variant of Proposition 7.10 in [55]. This inequality serves as a main tool to give an $\mathbb{W}_2$-estimate in [8] and [9], and will be used for all the coupling results in this thesis.

The more difficult situation is that σ has rank less than q, which could well happen. In Section 9 in [8] a different approach based on the Fourier expansion introduced in Section 5.8 in [32] is proposed, giving a coupling for the double integral I_{jl}. Chapter 3 in this thesis presents an attempt to generalise the that method to the iterated integral

[1] Also spelt as "Wasserstein".

of length 3:

$$I_{jkl} := \int_0^1 \int_0^t W_s^j \mathrm{d}W_s^k \mathrm{d}W_t^l.$$

Some partial results analogous to those of Davie [8] will be given in detail, followed by a discussion on the remaining obstacles towards a similar coupling result.

Approximating SDEs Driven by a Lévy Process

In Chapter 4 we return to the Euler approximation, but for SDEs with jumps.

For $x_0 \in \mathbb{R}^q$ and a bounded Lipschitz function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times q}$, consider the d-dimensional SDE,

$$x_t = x_0 + \int_0^t \sigma(x_{s-})\mathrm{d}Z_s,$$

driven by a q-dimensional Lévy process on $[0,T]$. Just like SDEs driven by a Wiener process, it is known that the standard Euler's approximation,

$$X_{k+1} := X_k + \sigma(X_k)\left(Z_{t_{k+1}} - Z_{t_k}\right), X_0 = x_0,$$

converges with rate 1/2 to the solution in mean-square as $h \to 0$ - see e.g. [33], [29] and [28]. Although the increments $Z_{t_k} - Z_{t_{k-1}}$ are hard to generate, one may simply ignore the small jumps

$$Z_t^\epsilon := \int_0^t \int_{0<|z|\leqslant\epsilon} z\tilde{N}(\mathrm{d}z, \mathrm{d}s),$$

for some $\epsilon \in (0,1)$, and show that the mean-square convergence rate is preserved. However, that is not a very economical way of simulation, as pointed out by Fournier [11]. Indeed, when the small jumps are completely ignored, the computational cost, that is, the total number of Wiener increments and the big jumps to be generated, is of order $O\left(h^{-1} + \nu\left(\{|z| > \epsilon\}\right)\right)$, which can be considerably large.

This happens, e.g., when the Lévy measure ν behaves like α-stable near 0, i.e. there exist $\tau > 0$ and $\alpha \in (0,2)$ s.t. $\nu(\mathrm{d}z) \simeq |z|^{-q-\alpha}\mathrm{d}z,\ \forall 0 < |z| \leqslant \tau$. In this case the set of big jumps has measure $\nu(\{|z| > \epsilon\}) \simeq \epsilon^{-\alpha}$, and one has to choose $\epsilon = h^{1/(2-\alpha)}$ to ensure the order 1/2 of mean-square convergence. As a result the computational cost becomes $O\left(h^{-1} + h^{\alpha/(\alpha-2)}\right)$, and hence explodes when α is close to 2.

As a remedy, one may consider approximating the small jumps (4.3) with a normal random variable using the central limit theorem, on which some classical theorems can be found in several books such as [44] and [2]. Asmussen and Rosiński [1] adopted this idea and derived some Berry-Esseen bounds for the normal approximation of the small jumps Z_1^ϵ; they also gave conditions for the weak convergence in the Skorohod space. But their method only works for $q = 1$, and the Berry-Esseen-type bounds are not very useful for the strong L^p-approximation of Lévy-SDEs as they only concern the uniform distance between the c.d.f's. Aiming at the Euler approximation of (4.2), Fournier [11] proved that by adding this normal random variable to the Euler scheme the expected computational cost can be controlled (no explosion of the computational cost near $\alpha = 2$), while the 1/2 convergence rate is still preserved. However, as pointed out himself, the method is also restricted to the case $q = 1$.

Such a restriction of dimension only emerged at a key step in [11] (Corollary 4.2), borrowed from a result by Rio [45] (Corollary 4.2) on the central limit theorem. The latter ensures that, for a sequence of i.i.d., mean-0 random variables $X_j \in \mathbb{R}$ and

$Y_m := m^{-1/2}\sum_{j=1}^m X_j$ for any $m \in \mathbb{Z}^+$, there is an absolute constant C s.t.

$$\mathbb{W}_2\left(\mathbb{P}_m,\ \mathcal{N}(0, \mathrm{var}X_1)\right) \leqslant C\left(\frac{\mathbb{E}|X_1|^4}{\mathrm{var}X}\right)^{\frac{1}{2}} m^{-\frac{1}{2}}, \tag{1.11}$$

where $\mathbb{P}_m$ denotes the distribution of Y_m. Rio [45] (Theorem 4.1) in fact only assumed the independence of $\{X_j\}$, but regarding central limit approximations and the simulation of Lévy processes one only considers the i.i.d. case. The constant C in (1.11) would vary in p for a bound in $\mathbb{W}_p$ and is later optimised in [46]. Apart from the restriction $q = 1$, Rio's effective bounds only hold for $p \leqslant 4$. But this has been improved by Bobkov [3] (Theorem 1.1), allowing the $\mathbb{W}_p$-convergence of order $O(m^{-1/2})$ for any $p \geqslant 1$.

The dimensional restriction in Rio and Bobkov's results comes from the fact that when $q = 1$, for $p \geqslant 1$ the $\mathbb{W}_p$ distance between two probability measures $\mathbb{P}, \mathbb{Q}$ on $\mathbb{R}$ is explicitly given (see Theorem 2.18 and Remarks 2.19 in [55]):

$$\mathbb{W}_p(\mathbb{P}, \mathbb{Q}) = \left(\int_0^1 \left|F^{-1}(t) - G^{-1}(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}, \tag{1.12}$$

where $p \geqslant 1$, F, G are the c.d.f's of $\mathbb{P}, \mathbb{Q}$, and F^{-1}, G^{-1} are their generalised inverses, respectively. For $p = 1$ there is a further equality $\mathbb{W}_1(\mathbb{P}, \mathbb{Q}) = \int_{\mathbb{R}} |F(x) - G(x)|\mathrm{d}x$. But these formulas do not apply to the multi-dimensional case.

The main results of Chapter 4 are the generalisation of the one-dimensional coupling (1.11) and the normal approximation for the small jumps Z_t^ϵ for $q \geqslant 2$ using the bound (1.10), giving a positive answer to Fournier's question.

Notation. Throughout this thesis $\mathbb{Z}^+, \mathbb{N}$ denote the sets of positive integers and non-negative integers, repectively. Unless specified separately, the generic positive constants $C_\cdot$ and $c_\cdot$ may change their values, with subscripts indicating their dependence of parameters. The notations $\lesssim$ and $\gtrsim$ indicate inequalities that hold with a factor C_q, and $\simeq$ means that both inequalities hold. The symbol $|\cdot|$, depending on the object it acts on, stands for the modulus of vectors on $\mathbb{R}^q$, the absolute value for scalars, and the 1-norm of multi-indices on $\mathbb{N}^q$. In the context of matrices, I stands for the identity matrix and $\|\cdot\|$ denotes any matrix norm. In the context of derivatives, ∂^α stands for the mixed partial derivatives w.r.t. a multi-index $\alpha \in \mathbb{N}$, and $\mathrm{D}^n f = (\partial^\alpha f)_{|\alpha|=n}$ is the n-th derivative matrix or block of a sufficiently smooth multi-variate function f, and $\|\mathrm{D}^n f\|$ denotes its Hilbert-Schmidt norm. For a non-negative real number x, its integer part is denoted by $[x]$.

Chapter 2

On Certain Properties of Tamed Euler Schemes

This chapter is a revised version of the author's joint work with Szpruch [52]. The main goal is to extend the applicability of Lyapunov function techniques of Khasminskii [31] to various numerical approximations taking the form (1.3). In particular, we investigate the integrability and asymptotic stability of numerical approximations of SDEs, paying particular attention to SDEs with non-globally Lipschitz drift and diffusion.

Much of the research on integrability or stability of the numerical schemes relies on simple Lyapunov functions, typically $V(x) = |x|^p,\ p \geqslant 2$, see e.g. [26, 32, 43, 48], with the exception of [24, 27]. Here we aim at handling more general cases, particularly polynomials of the general form

$$V(x) = \sum_{i=1}^{d} c_i x_i^{p_i}, \quad c_1, \cdots, c_d \in \mathbb{R}, \tag{2.1}$$

where the (non-negative) p_i's are not necessarily identical. This is necessary if one wishes to analyse many important SDEs in literature, see [24, 27][1] and Example 2.31 in this chapter. It turns out that for a special class of Lyapunov functions $V(x) = |x|^p,\ p \geqslant 2$, the drift-implicit Euler scheme admits a discrete-time analogue of (1.5), without the global Lipschitz condition - see [19, 40, 41].

The main challenge is to preserve condition (1.4) or (1.6) for the tamed generator $\mathcal{L}_t^h$ and to benefit from some extra control on the growth of the tamed coefficients. Although integrability results have been established in the literature for some specific explicit schemes of the form (1.3), it is not clear how property (1.5) can be inherited (possibly with a different ρ) under simple assumptions. For example, in [24] the authors showed some criteria for moment bounds (Proposition 2.7) and one can indeed recover (1.5), but an a priori estimate is needed: $\sup_h \max_k \|V(\bar{X}_k)\|_{L^p(\Omega)} h^{(\alpha-1)(1-1/p)} < \infty$ for some $\alpha > 1$. We will show in Section 2.1 that such a property can be preserved by controlling the generator $\mathcal{L}_t^h$ and the coefficients b^h, σ^h. We will also propose a type of projected schemes (2.2) that preserve the strong convergence rate $1/2$ and a uniform bound of the form (1.5), with respect to a larger class of Lyapunov functions.

On the other hand, the problem of asymptotic stability has received less attention in the literature so far and to the best of our knowledge the asymptotic stability of explicit numerical schemes beyond the Lipschitz setting is entirely new. Nonetheless,

[1] In [24, 27] authors investigated integrability, but not asymptotic stability of explicit schemes allowing Lyapunov functions of the form (2.1)

considerable effort has been made in this direction (mainly for implicit schemes) in [16–18, 20, 21, 40, 42, 57]. We will extend these results in two ways: a) we allow a bigger class of Lyapunov functions; b) we consider explicit Euler-type schemes. The idea seems similar to that of integrability - the main difference, however, lies in the recovery of condition (1.6). The issue here is that the strictly negative bound for the original generator,

$$\mathcal{L}_t V(\cdot) \leqslant -z(\cdot),$$

is not immediately preserved for the tamed one; one usually can only deduce that

$$\mathcal{L}_t^h V(\cdot) \leqslant -\rho^h(\cdot) z(\cdot),$$

for some $\rho^h(\cdot) \geqslant 0$ and finds no strictly positive lower bound for $\rho^h(\cdot)z(\cdot)$. The same problem would occur if one tries to recover the ergodicity of the underlying SDE using scheme (1.3) -see [42]). Nevertheless, explicit schemes of type (1.3) can recover the almost-sure stability if $\ker(\rho^h) = \{0\}$, but the exponential stability (1.7) seems not to hold. This, however, can be resolved by schemes of the form:

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{2.2}$$

where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ is a projection function that can be customised. The advantage of this method lies in that $\rho^h(\cdot) \geqslant c$ for some $c > 0$.

In Section 2.3 we will investigate the preservation of non-negativity and the comparison theorem for explicit numerical schemes. This is aimed at some one-dimensional SDEs whose solutions, for example, only stay in $[0, \infty)$. We will see that the condition $b(t, 0) \gg 0,\ \sigma(t, 0) \equiv 0$ is enough to guarantee $X_t \geq 0$ a.s., but not necessarily the case for numerical schemes. We will show that simply by truncating the noise as is done in Section 1.3.4 in [43], one can easily recover non-negativity of the tamed Euler scheme. The same method can readily be used to preserve the comparison theorem for SDEs with non-globally Lipschitz coefficients.

2.1 V-Integrability of Tamed Euler Schemes

In this section we investigate the integrability of tamed Euler schemes $\{\bar{X}_k\}$, (1.3) or (2.2), for an SDE driven by an $\mathcal{F}_t$-Wiener martingale W_t on a fixed interval $[0, T]$:

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t. \tag{2.3}$$

Following [24], let $p, d \in \mathbb{N}^+$, $\gamma \in (0, 1/p]$ and consider the following class of Lyapunov functions $\mathcal{V}_\gamma^p \subset \mathcal{C}^p(\mathbb{R}^d)$, where for $\mathbb{N} \ni p \geqslant 2$ and $0 < \gamma \leqslant \frac{1}{p}$,

$$\begin{aligned}\mathcal{V}_\gamma^p := \{V \geq 0 :\ & \ker(V) = \{0\},\ \exists c > 0 \text{ s.t.} \\ & \|\mathrm{D}^s V(\cdot)\| \leqslant c(1 + V(\cdot))^{1-s\gamma},\ \forall s \in \mathbb{N} \cap [0, p]\}.\end{aligned} \tag{2.4}$$

Note that the set $\mathcal{V}_\gamma^p$ not only covers power functions $|\cdot|^p,\ p > 0$, but also covers polynomials of the form (2.1). Hence it is rich enough for one to choose suitable Lyapunov functions for many of important SDEs (see [24] for more details). The property $\ker(V) = \{0\}$ is in fact not necessary for integrability, but is needed for stability results in Section 2.2. We introduce this definition here rather than later for the simplicity of presentation: if a non-negative function U only satisfies the growth condition of its derivatives as in (2.4), then $V(x) := U(x) - U(0) \in \mathcal{V}^p_\gamma$ and $U(x)$ is thus equivalent to $1 + V(x)$.

Remark 2.1. *The function* $|\cdot|^p$ *for some even number* p *is a candidate in the subset* $\bar{\mathcal{V}}^p_{1/p} := \mathcal{V}^p_{1/p} \cap \left\{ \mathrm{D}^{p+1}V \equiv 0,\ \exists c > 0 \ s.t.\ \|\mathrm{D}^s V(\cdot)\|_{HS} \leqslant cV(\cdot)^{1-s/p},\ \forall s \in \mathbb{N} \cap [0,p] \right\}.$

Once we fix a Lyapunov function $V \in \mathcal{V}^p_\gamma$ it will be useful if the growth conditions of the coefficients of the SDE (2.3) can be expressed in terms of V.

Assumption 2.2. *There exists a Lyapunov function* $V \in \mathcal{V}^p_\gamma$ *and constants* $K, \kappa > 0$, *s.t.* $\forall t \in [0,T],\ x \in \mathbb{R}^d$,

$$|b(t,x)| \vee \|\sigma(t,x)\| \leqslant K\left(1 + V(x)^{\kappa\gamma}\right).$$

Take $V(\cdot) = |\cdot|^p \in \bar{\mathcal{V}}^p_{1/p}$, then Assumption 2.2 essentially imposes the polynomial growth condition on the coefficients of the SDE (2.3). Indeed, we may observe that if there exists $L > 0$ such that $\forall t, x,\ |b(t,x)| \leqslant L(1 + |x|^{\kappa_1})$, one can find $K > 0$ such that $|b(t,x)| \leqslant K(1 + V(x))^{\kappa_1/p}$. The same applies to the diffusion coefficient with polynomial growth of degree κ_2 and let $\kappa = \kappa_1 \vee \kappa_2$. Expressing all estimates in terms of the chosen Lyapunov function[2] makes all calculations convenient.

Definition 2.3. *Let* V *be a non-negative Borel function on* $\mathbb{R}^d$. *The solution to the SDE* (2.3) *is said to be integrable with respect to* V, *or* V*-integrable, if*

$$\sup_{t \in [0,T]} \mathbb{E}V(X_t) < \infty.$$

A time-discretisation $\{\bar{X}_k\}$, *with step size* $h \in (0,1]$, *of the SDE* (2.3) *is said to be* V*-integrable, if*

$$\sup_{h>0} \max_{0 \leqslant k \leqslant [T/h]} \mathbb{E}V(\bar{X}_k) < \infty.$$

To clarify the idea of this section without going into too much technical detail let us consider a motivational example.

Example 2.4. *Let* $(X_t)_{t \in [0,T]}$ *be the solution of the* 1*-d autonomous SDE*

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \tag{2.5}$$

with $\mathbb{E}|X_0|^2 < \infty$ *and* b *and* σ *satisfying Assumption 2.2 and monotonicity condition:*

$$2xb(x) + |\sigma(x)|^2 \leqslant \rho(1 + |x|^2) \quad \forall x \in \mathbb{R}. \tag{2.6}$$

Note that (2.6) corresponds to the special case of the Lyapunov function $V(x) = |x|^2 \in \bar{\mathcal{V}}^2_{1/2}$, and it immediately follows that $\forall t \geq 0,\ \mathbb{E}V(X_t) \leqslant e^{\rho t}\mathbb{E}(1 + V(X_0))$. We are seeking some condition under which the tamed Euler scheme

$$\bar{X}_{k+1} = \bar{X}_k + b^h(\bar{X}_k)h + \sigma^h(\bar{X}_k)\Delta W_{k+1},$$

is also $|\cdot|^2$-integrable. Let us first square both sides of the scheme to get

$$\mathbb{E}_k|\bar{X}_{k+1}|^2 = |\bar{X}_k|^2 + \left(2\bar{X}_k b^h(\bar{X}_k) + |\sigma^h(\bar{X}_k)|^2\right)h + |b^h(\bar{X}_k)|^2 h^2, \tag{2.7}$$

[2] This corresponds to the Lyapunov-type functions $\tilde{V}(\cdot) := 1 + V(\cdot)$ defined in [24].

where $\mathbb{E}_k(\cdot) := \mathbb{E}(\cdot|\mathcal{F}_{t_k})$. If a taming method is chosen such that $\exists\mu > 0$,

$$2xb^h(x) + |\sigma^h(x)|^2 \leq \rho(1 + V(x)) \quad \text{and} \quad |b^h(x)|^2 h \leqslant \mu(1 + V(x)),\ \forall x \in \mathbb{R}, \tag{2.8}$$

then $\forall 1 \leq k \leqslant [T/h]$,

$$\begin{aligned} &\mathbb{E}_k(1 + V(\bar{X}_{k+1})) \leq 1 + V(\bar{X}_k) + (\rho + \mu)(1 + V(\bar{X}_k))h \\ \Rightarrow\ &\mathbb{E}V(\bar{X}_{[T/h]}) \leqslant e^{(\rho+\mu)T}\mathbb{E}(1 + V(X_0)). \end{aligned}$$

One can use taming method, e.g.,

$$b^h(t,x) := \frac{b(t,x)}{1 + G_b(x,h)},\ \sigma^h(t,x) := \frac{\sigma(t,x)}{1 + G_\sigma(x,h)},\ \forall t \in [0,T],\ x \in \mathbb{R}^d, \tag{2.9}$$

for some $G_b(\cdot,\cdot), G_\sigma(\cdot,\cdot) \geqslant 0$. Then the first condition in (2.8) holds if $1+G_b(x,h) \leqslant (1+G_\sigma(x,h))^2$. Furthermore for the second condition in (2.8) take $G_\sigma(x,h) = G_b(x,h) := CV(x)^{\kappa_0/2}h^\beta$, with $C = K/\sqrt{\mu},\ k_0 = (\kappa-1)^+$ and $\beta = 1/2$, so that

$$|b^h(x)|h^{1/2} = \frac{|b(x)|h^{1/2}}{1 + CV(x)^{\kappa_0/2}h^{1/2}} \leqslant \frac{KV(x)^{\kappa/2}h^{1/2}}{1 + CV(x)^{\kappa_0/2}h^{1/2}} \leqslant \sqrt{\mu}V(x)^{1/2},$$

as required.

2.1.1 Taming Conditions for V-Integrability

The V-integrability of numerical schemes can be studied by applying Taylor's theorem. It will be shown below that if the coefficients b and σ are appropriately modified (tamed), one can recover the integrability property by controlling the remainder term of the Taylor expansion - this is the essential idea of Theorem 2.5.

In the first part of this section we focus on another subset of $\mathcal{V}^p_\gamma$ denoted by $\widehat{\mathcal{V}}^p_\gamma = \mathcal{V}^p_\gamma \cap \{V^{(p+1)} \equiv 0\}$ (this class contains almost all examples of polynomial Lyapunov functions presented in [24]). As an example one may consider the most common choice $V(x) = |x|^p,\ p \geqslant 2$, which allows one to exploit the so-called one-sided Lipschitz property of the drift coefficient of the SDE (1.1). Later on we will show that integrability results can be extended to the whole family $\mathcal{V}^p_\gamma$.

Theorem 2.5. *Suppose for the tamed coefficients* (b^h, σ^h) *as in* (1.3) *there is a Lyapunov function* $V \in \widehat{\mathcal{V}}^p_\gamma,\ p \geq 2$ *s.t.* $\mathbb{E}V(X_0) < \infty$ *and*

$$\mathcal{L}^h_t V(x) \leqslant \rho(1 + V(x)),\ \forall (t,x) \in [0,T] \times \mathbb{R}^d, \tag{2.10}$$

for some $\rho > 0$. *Also assume that* $\exists\mu > 0$ *s.t.*

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu(1 + V(x))^\gamma. \tag{2.11}$$

Then there exists a constant $\widetilde{\rho} = O(\mu^2)$ *s.t.*

$$\mathbb{E}V(\bar{X}_k) \leqslant e^{(\rho+\widetilde{\rho})T}\mathbb{E}(1 + V(X_0)) < \infty, \quad \forall 0 \leqslant k \leqslant [T/h].$$

Remark 2.6. *V-integrability of numerical schemes has already been studied in [24] (Section 2.2), but the results are based on a weaker "semi-stability" condition. Here condition* (2.11) *ensures full "V-stability" defined therein.*

Proof. Since $V \in \widehat{\mathcal{V}}_\gamma^p$, one has the following finite Taylor expansion:

$$\mathbb{E}_k(1+V(\bar{X}_{k+1})) = 1+V(\bar{X}_k)+\mathbb{E}_k \sum_{1\leqslant|\alpha|\leqslant p} \frac{\partial^\alpha V(\bar{X}_k)}{\alpha!}(\bar{X}_{k+1}-\bar{X}_k)^\alpha. \tag{2.12}$$

For the convenience of notation denote $\bar{b}_k := b^h(t_k,\bar{X}_k)$, $\bar{\sigma}_k := \sigma^h(t_k,\bar{X}_k)$, and S_s the summation with index $|\alpha| = s$, $s = 1,\cdots,p$. It is easy to see that the conditional expectation of the first two terms of the summation in (2.12) are:

$$\begin{aligned}
\mathbb{E}_k S_1 :=& \mathbb{E}_k \sum_{|\alpha|=1} \frac{\partial^\alpha V(\bar{X}_k)}{\alpha!}(\bar{b}_k h+\bar{\sigma}_k\Delta W_{k+1})^\alpha = \left\langle \bar{b}_k, \nabla V(\bar{X}_k)\right\rangle h,\\
\mathbb{E}_k S_2 =& \frac{1}{2}\sum_{i,j=1}^{d}\sum_{l=1}^{m}\frac{\partial^2 V}{\partial x_i\partial x_j}(\bar{X}_k)\bar{\sigma}_k^{(il)}\bar{\sigma}_k^{(jl)}h+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 V}{\partial x_i\partial x_j}(\bar{X}_k)\bar{b}_k^{(i)}\bar{b}_k^{(j)}h^2\\
=& \frac{1}{2}\sum_{l=1}^{m}\left\langle \bar{\sigma}_k^{(\cdot,l)}, \mathrm{D}^2V(\bar{X}_k)\bar{\sigma}_k^{(\cdot,l)}\right\rangle h+\frac{1}{2}\left\langle \bar{b}_k, \mathrm{D}^2V(\bar{X}_k)\bar{b}_k\right\rangle h^2\\
\leqslant& \frac{1}{2}\mathrm{tr}\left[\mathrm{D}^2V(\bar{X}_k)\bar{\sigma}_k\bar{\sigma}_k^\top\right]h+\frac{1}{2}\left\|\mathrm{D}^2V(\bar{X}_k)\right\|\left|\bar{b}_k\right|h^2.
\end{aligned}$$

We can now analyse the rest of the expansion for $|\alpha| = s \geqslant 3$ by rewriting the sum

$$S_s = \frac{1}{s!}\sum_{|\alpha|=s}\binom{s}{\alpha}\left(\bar{X}_{k+1}^{(1)}-\bar{X}_k^{(1)}\right)^{\alpha_1}\cdots\left(\bar{X}_{k+1}^{(d)}-\bar{X}_k^{(d)}\right)^{\alpha_d}\frac{\partial^s}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}V(\bar{X}_k),$$

where for $i = 1,\cdots,d$, each $\left(\bar{X}_{k+1}^{(i)}-\bar{X}_k^{(i)}\right)^{\alpha_i}$ is equal to

$$\left(\bar{b}_k^{(i)}h+\bar{\sigma}_k^{(i,\cdot)}\Delta W_{k+1}\right)^{\alpha_i} = \sum_{r=0}^{\alpha_i}\binom{\alpha_i}{r}\left(\bar{b}_k^{(i)}h\right)^{\alpha_i-r}\left(\bar{\sigma}_k^{(i,\cdot)}\Delta W_{k+1}\right)^r. \tag{2.13}$$

Due to the independence and the law of the Wiener increments $\Delta W_{k+1}^{(j)}$, the terms with odd r's are zero under $\mathbb{E}_k$. Therefore, with some relabelling,

$$\begin{aligned}
\mathbb{E}_k S_s \leqslant& \left\|\mathrm{D}^s V(\bar{X}_k)\right\|\frac{d^{s-1}}{s!}\sum_{r=0}^{[s/2]}\binom{s}{2r}\left|\bar{b}_k\right|^{s-2r}\left\|\bar{\sigma}_k\right\|^{2r}h^{s-r}\\
\leqslant& \phi_s\left\|\mathrm{D}^s V(\bar{X}_k)\right\|\sum_{r=0}^{[s/2]}\left|\bar{b}_k\right|^{s-2r}\left\|\bar{\sigma}_k\right\|^{2r}h^{s-r},
\end{aligned}$$

where the positive constants

$$\phi_s := \frac{d^{s-1}}{s!}\max_{r=0,\cdots,s}\binom{s}{r} \leqslant \frac{d^{s-1}}{([s/2]!)^2}, \tag{2.14}$$

for each s. Returning to (2.12) and using the above estimates, we obtain

$$\mathbb{E}_k(1+V(\bar{X}_{k+1})) = 1+V(\bar{X}_k)+\mathcal{L}_{t_k}^h V(\bar{X}_k)h+R^hV(\bar{X}_k), \tag{2.15}$$

where, by relabelling the indices (with $i, j \in \mathbb{N}$) in the summation,

$$R^h V(\bar{X}_k) \leqslant \frac{1}{2} \left\|\mathrm{D}^2 V(\bar{X})\right\| \left|\bar{b}_k\right|^2 h^2 + \sum_{3 \leqslant i+2j \leqslant p} \phi_{i+2j} \left\|\mathrm{D}^{i+2j} V(\bar{X}_k)\right\| \left|\bar{b}_k\right|^i \|\bar{\sigma}_k\|^{2j} h^{i+j}. \tag{2.16}$$

Now given (2.11) and the estimates of $V^{(i+2j)}$ as in (2.4), we have

$$\begin{aligned} R^h V(\bar{X}_k) \leqslant & \frac{1}{2} c\mu^2 (1 + V(\bar{X}_k)) h + \sum_{3 \leq i+2j \leq p} \phi_{i+2j} c\mu^{i+2j} (1 + V(\bar{X}_k)) h^{\frac{i+j}{2}} \\ = & \left(\frac{1}{2} c\mu^2 + \sum_{s=3}^{p} \sum_{i+2j=s} \phi_s c\mu^s h^{\frac{s}{2}-1} \right) (1 + V(\bar{X}_k)) h \\ \leqslant & \left(\frac{1}{2} c\mu^2 + c \sum_{s=3}^{p} \left[\frac{s+1}{2}\right] \phi_s \mu^s \right) (1 + V(\bar{X}_k)) h. \end{aligned}$$

Set $\widetilde{\rho} := \frac{1}{2} c\mu^2 + \frac{1}{2} c(p+1) \sum_{s=3}^{p} \phi_s \mu^s h^{s/2-1}$, and from (2.15) we get

$$\begin{aligned} \mathbb{E}(1 + V(\bar{X}_{k+1})) \leqslant & (1 + (\rho + \widetilde{\rho})h) \mathbb{E}(1 + V(\bar{X}_k)) \leqslant (1 + (\rho + \widetilde{\rho})h)^k \mathbb{E}(1 + V(X_0)) \\ \leqslant & e^{(\rho + \widetilde{\rho})T} \mathbb{E}(1 + V(X_0)), \end{aligned}$$

and the result follows by removing 1 from the left-hand-side. $\square$

Remark 2.7. *For $p = 2$ one only needs to check condition* (2.11) *for $b^h(\cdot, \cdot)$.*

Remark 2.8. *In practice one can take $\mu \leqslant 1$ and choose $\widetilde{\rho} := c(p^2 - 1)d^{p-1}\mu^2$ since $\sup_{3 \leqslant s \leqslant p} \phi_s \leqslant d^{p-1}$. Therefore $\widetilde{\rho}$ can be arbitrarily small by a suitable choice of the parameter μ. E.g. the choice $\mu = O(h^{\varepsilon})$ for some $\varepsilon > 0$ will lead to the generalisation of Proposition 2.7 in [24], where asymptotically $\widetilde{\rho} \to 0$ as $h \to 0$, but the authors proved the result only on a suitable subset of $\mathbb{R}^d$.*

In a similar way we extend applicability of tamed Euler schemes to all Lyapunov functions from $\mathcal{V}_\gamma^p$. It turns out that the smoothness of V affects the rate of taming of the diffusion coefficient.

Proposition 2.9. *Let $V \in \mathcal{V}_\gamma^p$, $p \geqslant 3$. Suppose there is a constant $\rho > 0$ s.t. $\mathcal{L}^h V(\cdot) \leqslant \rho V(\cdot)$, and a constant $\mu > 0$ s.t.*

$$\left|b^h(t,x)\right| h^{\beta_1} \vee \left\|\sigma^h(t,x)\right\| h^{\beta_2} \leqslant \mu(1 + V(x))^{\gamma}, \ \forall t, x, \tag{2.17}$$

for some $\beta_1 \leqslant 1/2$ and $\beta_2 \leqslant 1/2 - 1/(p \wedge 4)$. Then $\exists \widetilde{\rho} := \widetilde{\rho}(\mu)$ s.t.

$$\mathbb{E}V(\bar{X}_k) \leqslant e^{(\rho + \widetilde{\rho})T} \mathbb{E}(1 + V(X_0)), \ \forall 0 \leqslant k \leqslant [T/h].$$

Proof. The proof is very similar to the proof of Theorem 2.5. We write

$$\begin{aligned} \mathbb{E}_k(1 + V(\bar{X}_{k+1})) = & 1 + V(\bar{X}_k) + \sum_{1 \leqslant |\alpha| \leqslant p-1} \frac{\partial^\alpha V(\bar{X}_k)}{\alpha!} \mathbb{E}_k(\bar{X}_{k+1} - \bar{X}_k)^\alpha \\ & + p \sum_{|\alpha| = p} \mathbb{E}_k \frac{(\bar{X}_{k+1} - \bar{X}_k)^\alpha}{\alpha!} \int_0^1 (1-t)^{p-1} \partial^\alpha V(\bar{X}_k + t(\bar{X}_{k+1} - \bar{X}_k)) \mathrm{d}t. \end{aligned} \tag{2.18}$$

It therefore suffices to look at the remainder term for $p \geqslant 2$. Denote the last term above by $\widetilde{R}^h$ and one has

$$\begin{aligned}\widetilde{R}^h &\leqslant p \sum_{|\alpha|=p} \mathbb{E}_k \frac{|(\bar{X}_{k+1}-\bar{X}_k)^\alpha|}{\alpha!} \int_0^1 (1-t)^{p-1} \left\| \mathrm{D}^p V(\bar{X}_k + t(\bar{X}_{k+1}-\bar{X}_k)) \right\| \mathrm{d}t \\ &\leqslant cp \sum_{|\alpha|=p} \mathbb{E}_k \frac{|(\bar{X}_{k+1}-\bar{X}_k)^\alpha|}{\alpha!} \int_0^1 (1-t)^{p-1} \left(1 + V(\bar{X}_k + t(\bar{X}_{k+1}-\bar{X}_k))\right)^{1-p\gamma} \mathrm{d}t.\end{aligned}$$

By Lemma 2.12 in [24] we have

$$1 + V(x+y) \leqslant c^{\frac{1}{\gamma}} 2^{\frac{1}{\gamma}-1} \Big(1 + V(x) + |y|^{\frac{1}{\gamma}}\Big), \ \forall x, y \in \mathbb{R}^d,$$

which leads to

$$\begin{aligned}\Big(1+V(x+y)\Big)^{1-p\gamma} \leqslant & c^{\frac{1}{\gamma}-p} 2^{\left(\frac{1}{\gamma}-p\right)(1-\gamma)} \Big(1 + V(x) + |y|^{\frac{1}{\gamma}}\Big)^{1-p\gamma} \\ \leqslant & (2c)^{\frac{1}{\gamma}-p} \Big((1+V(x))^{1-p\gamma} + |y|^{\frac{1}{\gamma}-p}\Big),\end{aligned}$$

for $\gamma \in (0, 1/p]$. Consequently,

$$\begin{aligned}\widetilde{R}^h &\leqslant cp \sum_{|\alpha|=p} \mathbb{E}_k \frac{|(\bar{X}_{k+1}-\bar{X}_k)^\alpha|}{\alpha!} (2c)^{\frac{1}{\gamma}-p} \Big((1+V(\bar{X}_k))^{1-p\gamma} + \left|\bar{X}_{k+1}-\bar{X}_k\right|^{\frac{1}{\gamma}-p}\Big) \\ &\leqslant p \frac{c^{\frac{1}{\gamma}-p+1} 2^{\frac{1}{\gamma}-p}}{p!} \mathbb{E}_k \left(\sum_{i=1}^d |\bar{X}_{k+1}^{(i)} - \bar{X}_k^{(i)}|\right)^p \Big((1+V(\bar{X}_k))^{1-p\gamma} + \left|\bar{X}_{k+1}-\bar{X}_k\right|^{\frac{1}{\gamma}-p}\Big) \\ &\leqslant \frac{d^{p-1} c^{\frac{1}{\gamma}-p+1} 2^{\frac{1}{\gamma}-p}}{(p-1)!} \mathbb{E}_k \left|\bar{X}_{k+1}-\bar{X}_k\right|^p \Big((1+V(\bar{X}_k))^{1-p\gamma} + \left|\bar{X}_{k+1}-\bar{X}_k\right|^{\frac{1}{\gamma}-p}\Big) \\ &\leqslant c\widetilde{\psi} \Big(\left|\bar{b}_k\right|^p h^p + \|\bar{\sigma}_k\|^p h^{p/2}\Big) (1+V(\bar{X}_k))^{1-p\gamma} + c\widetilde{\psi}\Big(\left|\bar{b}_k\right|^{\frac{1}{\gamma}} h^{\frac{1}{\gamma}} + \|\bar{\sigma}_k\|^{\frac{1}{\gamma}} h^{\frac{1}{2\gamma}}\Big),\end{aligned}$$

where, similar to the proof of Theorem 2.5, $\widetilde{\psi} := (d(m+1))^{\frac{1}{\gamma}-1} (2c)^{\frac{1}{\gamma}-p}/(p-1)!$.

Now given (2.17), $\exists \widetilde{\rho} = \widetilde{\rho}(\mu) > 0$ s.t. one has $R^h V(\bar{X}_k) \leqslant \widetilde{\rho}(1+V(\bar{X}_k))h$ for R^h defined in (2.15). This is obtained by the following estimate (with $i, j \in \mathbb{N}$):

$$\begin{aligned}R^h V(\bar{X}_k) \leqslant & \frac{1}{2} \left\|\mathrm{D}^2 V(\bar{X})\right\| \left|\bar{b}_k\right|^2 h^2 \\ & + \sum_{3 \leqslant i+2j \leqslant p-1} \phi_{i+2j} \left\|\mathrm{D}^{i+2j} V(\bar{X}_k)\right\| \left|\bar{b}_k\right|^i \|\bar{\sigma}_k\|^{2j} h^{i+j} + \widetilde{R}^h \\ \leqslant & \left(\frac{1}{2} c\mu^2 h^{1-2\beta_1} + \sum_{3 \leqslant i+2j \leqslant p-1} \phi_{i+2j} c\mu^{i+2j} h^{(1/2-\beta_1)i + (1/2-2\beta_2)j}\right) (1+V(\bar{X}_k))h \\ & + c\mu^p \widetilde{\psi} (1+V(\bar{X}_k)) \Big(h^{p(1-\beta_1)-1} + h^{p(1/2-\beta_2)-1}\Big) h \\ & + c\mu^{\frac{1}{\gamma}} \widetilde{\psi} (1+V(\bar{X}_k)) \left(h^{\frac{1-\beta_1}{\gamma}-1} + h^{\frac{1-2\beta_2}{2\gamma}-1}\right) h \\ \leqslant & \widetilde{\rho}(1+V(\bar{X}_k))h,\end{aligned}$$

for $\beta_1 \leqslant 1/2$ and $\beta_2 \leqslant 1/2 - 1/(p \wedge 4)$, and

$$\widetilde{\rho} := \frac{1}{2}c\mu^2 + \frac{1}{2}c(p+1)\sum_{s=3}^{p-1}\mu^s\phi_s + 2c\mu^p\widetilde{\psi},$$

where $\{\phi_s\}$ are the same positive constants as in (2.14). □

2.1.2 Taming Choices

The results in the previous subsection give us some general integrability conditions for the tamed Euler scheme (1.3). A natural question would be if the assumptions in Theorem 2.5 and Proposition 2.9 can be satisfied for specific taming methods, i.e., for $V \in \mathcal{V}_\gamma^p$ whether $\forall(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\mathcal{L}_t V(x) \leqslant \rho(1 + V(x)) \implies \mathcal{L}_t^h V(x) \leqslant \bar{\rho}(1 + V(x)), \tag{2.19}$$

for some $\rho, \bar{\rho} > 0$, and $\forall(t,x) \in [0,T] \times \mathbb{R}^d$,

$$|b^h(t,x)|h^{\beta_1} \vee \|\sigma^h(t,x)\|h^{\beta_2} \leqslant \mu(1 + V(x))^\gamma, \tag{2.20}$$

for some $\beta_1 \leqslant 1/2$ and $\beta_2 \leqslant 1/2 - 1/(p \wedge 4)$ hold.

Balanced Schemes

Let us first look at the balanced schemes proposed in [26, 49, 54], which in general are of the form

$$b^h(t,x) := \frac{b(t,x)}{1 + G_b(x,h)}, \ \sigma^h(t,x) := \frac{\sigma(t,x)}{1 + G_\sigma(x,h)}, \ \forall t, x, \tag{2.21}$$

where $G_b, G_\sigma \geqslant 0$ and $G_b(\cdot,h), G_\sigma(\cdot,h) \to 0$ as $h \to 0$. In this case requirement (2.19) is interpreted as

$$\begin{aligned}\mathcal{L}_t^h V(x) &:= \nabla V(x) \cdot b^h(t,x) + \frac{1}{2}\mathrm{tr}\left[\mathrm{D}^2 V(x)\sigma^h(\sigma^h)^\top(t,x)\right] \\ &= \frac{\nabla V(x) \cdot b(t,x)}{1 + G_b(x,h)} + \frac{1}{2}\frac{\mathrm{tr}\left[\mathrm{D}^2 V(x)\sigma\sigma^\top(t,x)\right]}{(1 + G_\sigma(x,h))^2} \leqslant \rho(1 + V(x)).\end{aligned}$$

Hence, condition (2.19) holds if either of the following conditions is satisfied:

i) $1 + G_b(x,h) = (1 + G_\sigma(x,h))^2, \ \forall x, h$;

ii) $1 + G_b(x,h) \leqslant (1 + G_\sigma(x,h))^2, \ \forall x, h$, if $\mathrm{tr}\left[\mathrm{D}^2 V(x)\sigma\sigma^\top(t,x)\right] > 0, \ \forall x \in \mathbb{R}^d$ (this is the case for most Lyapunov functions).

One may consider case i) and let, e.g.,

$$G_b(x,h) := 2CV(x)^{\kappa^*\gamma}h^{\beta_2} + C^2V(x)^{2\kappa^*\gamma}h^{2\beta_2} \quad \text{and} \quad G_\sigma(x,h) := CV(x)^{\kappa^*\gamma}h^{\beta_2}.$$

In order for (2.20) to hold we take $\beta_1 = 2\beta_2$, $C \geqslant K/\mu$ and $\kappa^* \geqslant \kappa - 1$ so that

$$\|\sigma^h(t,x)\|h^{\beta_2} = \frac{\|\sigma(t,x)\|h^{\beta_2}}{1 + CV(x)^{\kappa^*\gamma}h^{\beta_2}} \leqslant \frac{K(1 + V(x))^{\kappa\gamma}h^{\beta_2}}{1 + CV(x)^{\kappa^*\gamma}h^{\beta_2}} \leqslant \mu(1 + V(x))^\gamma,$$

by Assumption 2.2. We also need to choose $C^2 \geqslant K/\mu$ so that

$$|b^h(t,x)|h^{\beta_1} \leqslant \frac{K(1+V(x))^{\kappa\gamma}h^{2\beta_2}}{1+2CV(x)^{\kappa^*\gamma}h^{\beta_2}+C^2V(x)^{2\kappa^*\gamma}h^{2\beta_2}} \leqslant \mu(1+V(x))^{\gamma},$$

as $2\kappa^* \geqslant \kappa-1$. Therefore we choose $\kappa^* \geqslant \kappa-1$ and $C \geqslant (K/\mu)\vee 1$, which gives a reasonable taming method for the scheme to be bounded with respect to V.

Projected Schemes

Motivated by a different type of projected scheme introduced in [6], where the authors considered 1-d SDEs with strong solutions on $[0,\infty)$, we propose a new type of Euler schemes:

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b(t_k,\bar{X}_k)h + \sigma(t_k,\bar{X}_k)\Delta W_{k+1}\right), \tag{2.22}$$

where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ defined s.t. $|\Pi(x)| \leqslant h^{-r}$, $\forall x$ and some $r>0$ to be chosen. For example one can define $\Pi(x) = (\Pi_i(x_i))_{i=1}^d$ as a truncation, where $\Pi_i(x_i) = (-h^{-r} \vee x_i \wedge h^{-r})/\sqrt{d}$, or as a scaling: $\Pi(x) = \min\{1, h^{-r}|x|^{-1}\}x$. In order to ensure $|\bar{X}_k| \leqslant h^{-r}$ for all $k \geq 0$ we may assume $|X_0| \leqslant h^{-r}$, otherwise send in $\Pi(X_0)$ for the first iteration. Integrability of this scheme becomes straightforward for Lyapunov functions V satisfying $V\circ\Pi(\cdot) \leqslant V(\cdot)$. This additional condition does not significantly narrow the set $\mathcal{V}_\gamma^p$ of choices; in particular, it is usually satisfied for polynomials of the general form (2.1). In Section 2.2 we will show that these schemes preserve the exponential stability, which balanced schemes may fail to achieve.

Theorem 2.10. *Consider a projected scheme $\{\bar{X}_k\}$ defined by (2.22). Let Assumption 2.2 hold and $V \in \mathcal{V}_\gamma^p$ s.t. $\forall x \in \mathbb{R}^d$, $V(\Pi(x)) \leqslant V(x) \leqslant \nu(1+|x|^q)$ for some constants $\nu>0$, $q \geqslant 1$. If $\exists \rho > 0$ s.t.*

$$\mathcal{L}_t V(x) \leqslant \rho(1+V(x)),\ \forall (t,x) \in [0,T]\times\mathbb{R}^d,$$

and $\mathbb{E}V(X_0) < \infty$, then $\{\bar{X}_k\}$ is V-integrable for $r \leqslant (1/2 - 1/(p\wedge 4))/((\kappa-1)q\gamma)$.

Proof. The same arguments in the proofs of Theorem 2.5 and Proposition 2.9 imply

$$\begin{aligned}\mathbb{E}_k V(\bar{X}_{k+1}) =& V\left(\Pi(\bar{X}_k + b(t_k,\bar{X}_k)h + \sigma(t_k,\bar{X}_k)\Delta W_{k+1})\right)\\ \leqslant& V(\bar{X}_k + b(t_k,\bar{X}_k)h + \sigma(t_k,\bar{X}_k)\Delta W_{k+1})\\ =& V(\bar{X}_k) + \mathcal{L}_{t_k}V(\bar{X}_k)h + R^hV(\bar{X}_k) + M_{k+1},\end{aligned} \tag{2.23}$$

where M_{k+1} is a local martingale, as the expression given in (2.15). This immediately shows that one need only work with $\mathcal{L}_tV(x)$, $b(t,x)$ and $\sigma(t,x)$ directly for $|x| \leqslant h^{-r}$. Thus (2.19) is redundant and we have

$$\begin{aligned}|b(t,x)|h^{\frac{1}{2}} \vee \|\sigma(t,x)\|h^{\frac{1}{2}-\frac{1}{p\wedge 4}} \leqslant& K(1+V(x))^{\kappa\gamma}h^{\frac{1}{2}-\frac{1}{p\wedge 4}}\\ \leqslant& 2K\nu\left(1+|x|^{q(\kappa-1)\gamma}\right)(1+V(x))^{\gamma}h^{\frac{1}{2}-\frac{1}{p\wedge 4}}\\ \leqslant& 4K\nu h^{\frac{1}{2}-\frac{1}{p\wedge 4}-r(\kappa-1)q\gamma}(1+V(x))^{\gamma}\\ =:& \mu(1+V(x))^{\gamma},\end{aligned} \tag{2.24}$$

by choosing $r \leqslant (1/2 - 1/(p\wedge 4))/((\kappa-1)q\gamma)$, which achieves (2.20). The result thus follows by Theorem 2.9. $\square$

Strong Convergence

Now given the integrability (in particular, bounded moments) of the scheme we can explain how in general one may establish the strong convergence of (1.3) based on the results in [24] (Definition 3.1 and Corollary 3.12) and [54] (the proof of Lemma 3.2 and Theorem 2.1). Roughly speaking, both results state that provided that appropriate moment bounds ($V(\cdot) = |\cdot|^p$) for the tamed Euler scheme (1.3) are achieved, and that the strong and weak one-step differences against the standard Euler scheme are given by appropriate rates, then the tamed Euler scheme (1.3) converges to the solution of the SDE (1.1) in L^p. Precise statements are made in Appendix A.1.

Proposition 2.11. *Under appropriate assumptions (more precisely, let Assumption A.1 in Appendix A.1 hold for $p = 2$ and some even number $p_0 > 2$ sufficiently large), the projected schemes* (2.22) *converge to the solution to the SDE* (2.3) *in L^2 with rate $1/2$ for $r < 1/(2(\kappa - 1))$.*

Corollary 2.12. *If a tamed Euler scheme* (1.3) *already satisfies the conditions for L^2-convergence (see Theorem A.2 in Appendix A.1), then the composed scheme*

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{2.25}$$

with an appropriate value of r chosen, also converges in L^2 with the same rate.

The proofs of both claims above can be found in Appendix A.2.

2.2 Asymptotic Stability of Equilibrium

Suppose for all $\mathcal{F}_0$-measurable X_0, there exists a unique (strong or weak) solution to the SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,\ t \geqslant 0, \tag{2.26}$$

with drift and diffusion satisfying $b(t, x^*) \equiv 0,\ \sigma(t, x^*) \equiv 0,\ \forall t \geq 0$ for some $x^* \in \mathbb{R}^d$. When almost surely $X_0 = x^*$, the SDE has trivial solution $X_t = x^*$ a.s. Analogous to the concept of equilibria of ODEs, one can re-write the SDE as

$$Y_t := X_t - x^* = \int_0^t b(s, Y_s + x^*)\mathrm{d}s + \sigma(s, Y_s + x^*)\mathrm{d}W_s =: \int_0^t \tilde{b}(s, Y_s)\mathrm{d}s + \tilde{\sigma}(s, Y_s)\mathrm{d}W_s,$$

and therefore assume, without loss of generality, the equilibrium $x^* = 0$ and

$$b(t, 0) \equiv 0,\ \sigma(t, 0) \equiv 0,\ \forall t \geqslant 0. \tag{2.27}$$

In the context of stability one still needs to model the growth of b and of σ in terms of the selected Lyapunov function in the class V_γ^p. But instead of $1 + V$ as in the integrability discussion before, we need a different assumption than Assumption 2.2 to model the growth conditions of b and σ, due to (2.27) and the possibility of V taking the form (2.1). More precisely,

Assumption 2.13. *There is a $V \in \mathcal{V}_\gamma^p$ and a non-negative function $U \in \mathcal{C}(\mathbb{R}^d)$, $\ker(U) = \{0\}$, s.t. $V(\cdot) \leqslant U(\cdot)$, and constants $K > 0,\ \kappa_{1,2} \geqslant 1$ s.t.*

$$|b(t, x)| \leqslant KU(x)^{\kappa_1\gamma},\ \|\sigma(t, x)\| \leqslant KU(x)^{\kappa_2\gamma},\ \forall t \geqslant 0,\ x \in \mathbb{R}^d.$$

In most cases the function U can be reasonably assumed to have polynomial growth in the sense

$$U(\cdot) \lesssim |\cdot|^{q_1} + |\cdot|^{q_2},$$

with $0 < q_1 \leqslant q_2$, which gives polynomial growth for b and σ - see Example 2.30.

Definition 2.14. *The solution to the SDE* (2.26) *is said to be almost surely stable, if $X_t \to 0$ a.s. as $t \to \infty$, regardless of the value of X_0. A time-discretisation $\{\bar{X}_k\}$, with step size $h \in (0, 1]$, of the solution to the SDE* (2.26) *is said to be almost surely stable, if for fixed step size $h > 0$, $\bar{X}_k \to 0$ a.s. as $k \to \infty$, regardless of the value of X_0.*

Definition 2.15. *Let $V \in \mathcal{V}^p_\gamma$. The solution to the SDE* (2.26) *is said to be exponentially stable with respect to V, or V-exponentially stable, with rate ρ, if $\mathbb{E}V(X_0) < \infty$ and $\exists \rho > 0$ s.t.*

$$\mathbb{E}V(X_t) \leqslant e^{-\rho t}\mathbb{E}V(X_0), \ \forall t \geqslant 0.$$

A time-discretisation $\{\bar{X}_k\}$, with step size $h \in (0, 1]$, of the solution to the SDE (2.26) *is said to be V-exponentially stable with rate $\widetilde{\rho}$, if for fixed time-step $h > 0$, $\exists \widetilde{\rho} > 0$ s.t.*

$$\mathbb{E}V(\bar{X}_k) \leqslant e^{-\widetilde{\rho} kh}\mathbb{E}V(X_0), \ \forall k \geqslant 0.$$

Remark 2.16. *By the Borel-Cantelli lemma, V-exponential stability implies almost-sure stability.*

First we check the conditions for stability of equilibrium on the SDE level. We first quote a simplified version of stochastic LaSalle theorem regarding the almost-sure stability of SDE (2.26) from [38, 41, 50]:

Theorem 2.17. *Let b and σ be locally Lipschitz in x and $V \in \mathcal{C}^2(\mathbb{R}^d)$ be non-negative. If $V(X_0) < \infty$ a.s. and there is a non-negative $z \in \mathcal{C}(\mathbb{R}^d)$ s.t.*

$$\mathcal{L}_t V(x) \leqslant -z(x), \ \forall (t, x) \in [0, \infty) \times \mathbb{R}^d, \tag{2.28}$$

then almost surely we have

$$\varlimsup_{t\to\infty} V(X_t) < \infty, \ \lim_{t\to\infty} z(X_t) = 0,$$

regardless of the value of X_0. In addition, if $\ker(z) = \{0\}$, then $X_t \to 0$ a.s. as $t \to \infty$. Moreover, when $z(\cdot) \geqslant \rho V(\cdot)$ for some constant $\rho > 0$, then the solution X_t is V-exponentially stable.

One can use Theorem 2.17 to determine whether a system is almost surely stable. In particular, mean-square stability, i.e. $V(\cdot) = |\cdot|^2$, is the most popular choice. Before introducing stability results for tamed Euler schemes let us consider the following simple case.

Example 2.18. *The solution to*

$$\mathrm{d}X_t = -|X_t|^2 X_t \mathrm{d}t + |X_t|^2 \mathrm{d}W_t, \ |X_0|^2 < \infty \ a.s.$$

is almost surely stable at 0.

Indeed one finds $\mathcal{L}|x|^2 = -2|x|^4 + |x|^4 = -|x|^4 =: -z(x)$, where $z(x) \geqslant 0$ and $z(x) = 0 \Leftrightarrow x = 0$. Note that in this case the solution is not necessarily mean-square exponentially stable, but Theorem 2.17 still holds. Nevertheless, the stability property of numerical schemes is not immediate. One may, for example, consider the following balanced scheme:

$$b^h(x) = \frac{b(x)}{1+G(x)h^\alpha},\ \sigma^h(x) = \frac{\sigma(x)}{1+G(x)h^\alpha},\ 0<\alpha\leqslant 1. \tag{2.29}$$

This is a simple version of (2.21). Notice that before taking expectation in (2.7),

$$|\bar{X}_{k+1}|^2 = |\bar{X}_k|^2 + \mathcal{L}^h|\bar{X}_k|^2 h + |b^h(\bar{X}_k)|^2h^2 + M_{k+1},$$

where $M_{k+1} = 2(\bar{X}_k + b^h(\bar{X}_k)h)\cdot\sigma^h(\bar{X}_k)\Delta W_{k+1}$. For the tamed generator,

$$\mathcal{L}^h|x|^2 = 2\frac{x\cdot b(x)}{1+G(x)h^\alpha} + \frac{\|\sigma(x)\|^2}{(1+G(x)h^\alpha)^2} \leqslant \frac{1}{1+G(x)h^\alpha}\mathcal{L}|x|^2 = -\frac{z(x)}{1+G(x)h^\alpha}.$$

One can choose $\alpha\leqslant 1$ and $G(x) := 2|x|^2$, s.t.

$$\begin{aligned} A^h(x) :=& \frac{z(x)}{1+G(x)h^\alpha} - \frac{|b(x)|^2h}{(1+G(x)h^\alpha)^2} = \frac{|x|^4}{1+2|x|^2h^\alpha} - \frac{|x|^6h}{(1+2|x|^2h^\alpha)^2} \\ \geqslant& \frac{2|x|^6h^\alpha - |x|^6h}{(1+2|x|^2h^\alpha)^2} \geqslant \frac{|x|^6h}{(1+2|x|^2h^\alpha)^2} \geqslant 0, \end{aligned}$$

and $A^h(x) = 0 \Leftrightarrow x = 0$. Thus one arrives at, for all k,

$$|\bar{X}_{k+1}|^2 \leqslant |\bar{X}_k|^2 - A^h(\bar{X}_k)h + M_{k+1} \leqslant |\bar{X}_0|^2 - \sum_{l=0}^{k} A^h(\bar{X}_l)h + \sum_{l=0}^{k} M_{l+1}.$$

Note that each M_{l+1} is $\mathcal{F}_{t_{l+1}}$-adapted and $\mathbb{E}_l M_{l+1} = 0$, implying that the process $S_{k+1} := \sum_{l=0}^{k} M_{l+1}$ with $S_0 := 0$ is an $\mathcal{F}_{t_{k+1}}$-martingale. One can then deduce that $A^h(\bar{X}_l) \to 0$ a.s. and hence $\bar{X}_l \to 0$ a.s. as $l\to\infty$. This can be seen by applying the following lemma (see [39], Theorem 1.3.9) to the non-negative process

$$V_k := V_0 - \sum_{l=0}^{k-1} A^h(\bar{X}_l)h + \sum_{l=1}^{k} M_l,\ V_0 := |X_0|^2.$$

Lemma 2.19. *Consider a non-negative stochastic process $\{V_k\}$ with representation*

$$V_k = V_0 + H_k^1 - H_k^2 + S_k,$$

where $\{H_k^1\}$ and $\{H_k^2\}$ are almost surely increasing, predictable processes with $H_0^1 = H_0^2 = 0$, and $\{S_k\}$ is an $\mathcal{F}_{t_k}$-local martingale with $S_0 = 0$. Then with probability 1,

$$\left\{\lim_{k\to\infty} H_k^1 < \infty\right\} \subset \left\{\lim_{k\to\infty} H_k^2 < \infty\right\} \cap \left\{\lim_{k\to\infty} V_k < \infty \text{ exsits}\right\}.$$

This is in fact a discrete version of Theorem 2.6.7 in [35] for special semimartingales. Now we investigate the stability conditions for a general tamed explicit Euler scheme

$$\bar{X}_{k+1} = \bar{X}_k + b^h(t_k,\bar{X}_k)h + \sigma^h(t_k,\bar{X}_k)\Delta W_{k+1}. \tag{2.30}$$

We first remark that a result on the preservation of almost-sure stability for the drift-implicit Euler scheme has been studied in [41], where only $V = |\cdot|^2$ is considered.

Theorem 2.20. *Let $V \in \widehat{\mathcal{V}}^p_\gamma := \mathcal{V}^p_\gamma \cap \{\mathrm{D}^{p+1}V \equiv 0\}$ be dominated by a non-negative function U and $\mathbb{E}V(X_0) < \infty$. Suppose there is a non-negative function $z^h \in \mathcal{C}(\mathbb{R}^d)$, s.t. $\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$,*

$$\mathcal{L}^h_t V(x) \leqslant -z^h(x), \tag{2.31}$$

and a constant $0 < \mu \leq 1$ s.t.

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu \frac{(1+U(x))^\gamma z^h(x)}{1+U(x)+z^h(x)}. \tag{2.32}$$

Then for $\mu < \sqrt{2}/\sqrt{c + cd^{p-1}(p^2-1)}$, the scheme (2.30) satisfies:

$$\varlimsup_{k\to\infty} V(\bar{X}_k) < \infty, \quad \varlimsup_{k\to\infty} z^h(\bar{X}_k) = 0, \ \ a.s.,$$

and hence if $\ker(z^h) = \{0\}$ then $\bar{X}_k \to 0$ a.s. as $k \to \infty$.

Moreover, in the particular case where $z^h(\cdot) \geqslant \rho V(\cdot)$ for some $\rho > 0$, if $\exists \mu > 0$ s.t. $\forall t \geqslant 0,\ \forall x \in \mathbb{R}^d$,

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu V(x)^\gamma, \tag{2.33}$$

then the scheme (2.30), with $\mu < \sqrt{2\rho}/\sqrt{c + cd^{p-1}(p^2-1)}$, admits V-exponential stability with a rate $\widetilde{\rho} \in (0,\rho)$, $\rho - \widetilde{\rho} = O(\mu^2)$.

Proof. The proof is almost identical to that of Theorem 2.5. However, by the estimate for the remainder (2.16), instead of (2.15) we have the following estimate (with $i, j \in \mathbb{N}$):

$$\begin{aligned} V(\bar{X}_{k+1}) =& V(\bar{X}_k) + \mathcal{L}^h_{t_k} V(\bar{X}_k) h + R^h V(\bar{X}_k) + M_{k+1} \\ \leqslant& V(\bar{X}_k) - \mathcal{L}^h_{t_k} V(\bar{X}_k) h + \frac{1}{2} \left\|\mathrm{D}^2 V(\bar{X}_k)\right\| \left|\bar{b}_k\right|^2 h^2 \\ &+ \sum_{3 \leq i+2j \leqslant p} \phi_{i+2j} \left\|\mathrm{D}^{i+2j} V(\bar{X}_k)\right\| \left|\bar{b}_k\right|^i \left\|\bar{\sigma}_k\right\|^{2j} h^{i+2j} + M_{k+1}, \end{aligned} \tag{2.34}$$

where M_{k+1} corresponds to the odd terms in (2.13), and is hence $\mathcal{F}_{t_{k+1}}$-measurable with $\mathbb{E}_k M_{k+1} = 0$. Notice that all derivatives of V have upper bounds as defined in (2.4). Now apply (2.31) and (2.32) and we get (recall that $\gamma \leq 1/p$ and that $V \leqslant U$):

$$\begin{aligned} V(\bar{X}_{k+1}) \leqslant& V(\bar{X}_k) - z^h(\bar{X}_k) h + \frac{1}{2} c\mu^2 (1+V(\bar{X}_k))^{1-2\gamma} \left(\frac{(1+U(\bar{X}_k))^\gamma z^h(\bar{X}_k)}{1+U(\bar{X}_k)+z^h(\bar{X}_k)} \right)^2 h \\ &+ \sum_{3 \leqslant i+2j \leqslant p} \phi_{i+2j} c\mu^{i+2j} (1+V(\bar{X}_k))^{1-(i+2j)\gamma} \left(\frac{(1+U(\bar{X}_k))^\gamma z^h(\bar{X}_k)}{1+U(\bar{X}_k)+z^h(\bar{X}_k)} \right)^{i+2j} h^{\frac{i+j}{2}} \\ &+ M_{k+1} \\ \leqslant& V(\bar{X}_k) - z^h(\bar{X}_k) h + \frac{1}{2} c\mu^2 \frac{1+U(\bar{X}_k)}{\left(1+(1+U(\bar{X}_k))/z^h(\bar{X}_k)\right)^2} h \\ &+ \sum_{3 \leqslant i+2j \leqslant p} \phi_{i+2j} c\mu^{i+2j} \frac{1+U(\bar{X}_k)}{\left(1+(1+U(\bar{X}_k))/z^h(\bar{X}_k)\right)^{i+2j}} h^{\frac{i+j}{2}} \\ &+ M_{k+1}, \end{aligned}$$

where, again, the summations are over integral indices i, j. By the trivial fact that the terms in the denominators above are no less than 1,

$$\begin{aligned}V(\bar{X}_{k+1}) \leqslant & V(\bar{X}_k) - z^h(\bar{X}_k)h + \frac{1}{2}c\mu^2 z^h(\bar{X}_k)h + \sum_{s=3}^{p}\sum_{i+2j=s}\phi_s c\mu^s z^h(\bar{X}_k)h^{\frac{s}{2}} + M_{k+1}\\ \leqslant & V(\bar{X}_k) - z^h(\bar{X}_k)h + \frac{1}{2}c\mu^2 z^h(\bar{X}_k)h + \sum_{s=3}^{p}\left[\frac{s+1}{2}\right]\phi_s c\mu^s z^h(\bar{X}_k)h^{\frac{s}{2}} + M_{k+1}.\end{aligned}$$

This implies that, $\forall k$,

$$V(\bar{X}_{k+1}) \leqslant V(X_0) - \sum_{l=0}^{k} a(\mu,h)z^h(\bar{X}_l)h + \sum_{l=0}^{k} M_{l+1}, \tag{2.35}$$

where

$$a(\mu,h) := 1 - \frac{1}{2}c\mu^2 - \frac{1}{2}c(p+1)\sum_{s=3}^{p}\phi_s\mu^s h^{\frac{s}{2}-1}.$$

One can find a taming method with μ and h sufficiently small a.s. $a(\mu,h) > 0$, so that $H_{k+1} := \sum_{l=0}^{k} a(\mu,h)z^h(\bar{X}_l)h$ is an increasing, predictable process with $H_0 = 0$. Now the same argument used at the end of Example 2.18 applies: $S_{k+1} := \sum_{l=0}^{k} M_{l+1}$ is an $\mathcal{F}_{t_{k+1}}$-martingale with $S_0 = 0$, and so by Lemma 2.19, both $V(\bar{X}_k)$ and H_k converge a.s. as $k \to \infty$, implying that $z^h(\bar{X}_k) \to 0$ a.s.

Moreover, when $z^h(x) = 0$ iff $x = 0$ one concludes that $\bar{X}_k \to 0$ a.s. In fact, assuming $\mu, h \leqslant 1$, by Remark 2.8 one just needs to choose $\mu < 1/\sqrt{c/2 + cd^{p-1}(p^2-1)/2}$.

If in addition $z^h(\cdot) \geqslant \rho V(\cdot)$ for some $\rho > 0$ and condition (2.33) holds, then instead of (2.35) one runs the same calculation to get

$$V(\bar{X}_{k+1}) \leqslant V(\bar{X}_k) - (\rho - 1 + a(\mu,h))V(\bar{X}_k)h + M_{k+1}.$$

One can then choose μ and h sufficiently small s.t. $\widetilde{\rho} := \rho - 1 + a(\mu,h) > 0$. Finally, by taking expectation on both sides, one arrives at

$$\begin{aligned}\mathbb{E}V(\bar{X}_{k+1}) \leqslant & (1-\widetilde{\rho}h)\mathbb{E}V(\bar{X}_k) \leqslant (1-\widetilde{\rho}h)^{k+1}\mathbb{E}V(X_0)\\ \leqslant & e^{-\widetilde{\rho}(k+1)h}\mathbb{E}V(X_0).\end{aligned}$$

Assuming again $\mu, h \leqslant 1$, one can choose $\mu < \sqrt{\rho}/\sqrt{c/2 + cd^{p-1}(p^2-1)/2}$. □

Remark 2.21. *In analogy to Proposition 2.9, Theorem 2.20 also holds for* $V \in \mathcal{V}_\gamma^p$.

Remark 2.22. *By* (2.34)*, condition* (2.32) *can be weakened to*

$$\left\|\mathrm{D}^{i+2j}V(x)\right\| \left|b^h(t,x)\right|^i \left\|\sigma^h(t,x)\right\|^{2j} h^{\frac{i+j}{2}} \leqslant \mu z^h(x), \ \forall t \geqslant 0, \ x \in \mathbb{R}^d, \tag{2.36}$$

for $i = 2, j = 0$ *and all* $i, j \in \mathbb{N}$ *s.t.* $3 \leqslant i + 2j \leqslant p$.

Remark 2.23. *For* $V \in \bar{\mathcal{V}}_\gamma^p$ *condition* (2.32) *can be simplified to*

$$\left|b^h(t,x)\right| h^{1/2} \vee \left\|\sigma^h(t,x)\right\| h^{1/4} \leqslant \mu\frac{U(x)^\gamma z^h(x)}{U(x) + z^h(x)}, \ \forall t \geqslant 0, \ x \in \mathbb{R}^d, \tag{2.37}$$

which also implies (2.36) *for* $0 < \mu \leq 1$.

Notice that (2.37) is reasonable since from (2.31) we have

$$
\begin{aligned}
z^h(x) \leqslant & \|\nabla V(x)\| \left| b^h(t,x) \right| + \frac{1}{2} \left\| V^{(2)}(x) \right\| \left\| \sigma^h(t,x) \right\|^2 \\
\leqslant & KU(x)^{1+(\kappa_1-1)\gamma} + KU(x)^{1+2(\kappa_2-1)\gamma},
\end{aligned} \tag{2.38}
$$

which ensures no singularity in the right-hand-side term in (2.37).

2.2.1 Balanced Schemes

Now with Theorem 2.20 one can determine whether a certain type of taming methods can preserve stability. For this we may derive some general conditions with respect to Lyapunov functions in $\mathcal{V}_\gamma^p$. Although most practically relevant Lyapunov functions can be found in the subset $\bar{\mathcal{V}}_\gamma^p$ defined in Remark 2.1, we may treat them as a special case. Let us first investigate the following type of tamed schemes adopted by [26, 49, 54]:

$$
b^h(t,x) = \frac{b(t,x)}{1+G(x)h^\alpha}, \ \sigma^h(t,x) = \frac{\sigma(t,x)}{1+G(x)h^\alpha}, \tag{2.39}
$$

for some $G(\cdot) \geqslant 0 < \alpha \leqslant 1$. Given the growth condition (2.38), which also holds for $z(\cdot)$, it turns out that by imposing some lower bounds on z one can recover almost-sure stability for (2.39).

Proposition 2.24. *Let Assumption 2.13 hold for $V \in \mathcal{V}_\gamma^p$ s.t. the coefficients of the SDE (2.26) satisfy*

$$
\mathcal{L}_t V(x) \leqslant -z(x), \ \forall (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{2.40}
$$

for some $0 \leqslant z \in \mathcal{C}(\mathbb{R}^d)$ satisfying

$$
z(x) \geqslant \lambda (1+U(x))^{1-\gamma} \left(U(x)^{\kappa_1 \gamma} \vee U(x)^{\kappa_2 \gamma} \right), \ \forall x \in \mathbb{R}^d, \tag{2.41}
$$

for some $\lambda > 0$. Then, by choosing $h < (\mu\lambda/K)^4$ and $G(x) = C(U(x)^{(\kappa_1-1)\gamma} \vee U(x)^{(\kappa_2-1)\gamma})$, $C \gg 1/\left(\mu/K - h^{1/4}/\lambda\right)$, $\alpha \leqslant 1/4$, the Euler scheme (2.30) with tamed coefficients (2.39) preserves almost-sure stability for the trivial solution, where μ satisfies the requirement in Theorem 2.20.

Proof. First one calculates

$$
\begin{aligned}
\mathcal{L}_t^h V(x) = & \nabla V(x) \cdot \frac{b(t,x)}{1+G(x)h^\alpha} + \frac{1}{2(1+G(x)h^\alpha)^2} \mathrm{tr} \left[\nabla^2 V(x) \sigma\sigma^\top (t,x) \right] \\
\leqslant & \frac{1}{1+G(x)h^\alpha} \mathcal{L}|x|^2 \\
\leqslant & -\frac{z(x)}{1+G(x)h^\alpha} =: -z^h(x),
\end{aligned} \tag{2.42}
$$

which satisfies $z^h(x) = 0 \Leftrightarrow x = 0$. Now one only needs to select appropriate $G(\cdot)$ and α s.t. condition (2.32) is satisfied, i.e.,

$$
\begin{aligned}
\frac{|b(t,x)|h^{\frac{1}{2}} \vee \|\sigma(t,x)\|h^{\frac{1}{4}}}{1+G(x)h^\alpha} \leqslant & \mu \frac{(1+U(x))^\gamma}{1+U(x)+\frac{z(x)}{1+G(x)h^\alpha}} \frac{z(x)}{1+G(x)h^\alpha} \\
\Leftrightarrow \ |b(t,x)|h^{\frac{1}{2}} \vee \|\sigma(t,x)\|h^{\frac{1}{4}} \leqslant & \frac{\mu(1+U(x))^\gamma}{\frac{1+U(x)}{z(x)} + \frac{1}{1+G(x)h^\alpha}}.
\end{aligned}
$$

One has an upper bound for the left-hand-side above by Assumption 2.13 and a lower bound for the right-hand-side by (2.41). Hence for the above inequality to hold, one can require

$$
\begin{aligned}
K\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right) h^{1/4} &\leqslant \frac{\mu(1+U(x))^\gamma}{\left(\frac{(1+U(x))^\gamma}{\lambda(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma})} + \frac{1}{1+G(x)h^\alpha}\right)} \\
\Leftrightarrow\ \mu(1+U(x))^\gamma &\geqslant \frac{K}{\lambda} h^{1/4}(1+U(x))^\gamma + \frac{K\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right)}{1+G(x)h^\alpha} h^{1/4} \\
\Leftrightarrow\ 1+G(x)h^\alpha &\geqslant \frac{K\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right)}{(\mu - Kh^{1/4}/\lambda)(1+U(x))^\gamma} h^{1/4},
\end{aligned}
$$

where for fixed $\mu \leqslant 1$ we choose $h \leqslant h_0 < (\mu\lambda/K)^4$. Thus by choosing $\alpha = 1/4$ and $G(x) := C\left(U(x)^{(\kappa_1-1)\gamma} \vee U(x)^{(\kappa_2-1)\gamma}\right)$, the taming condition (2.37) is satisfied for $\mu \geqslant K\left(1/C + h^{1/4}/\lambda\right)$. Hence by Remark 2.23 and Theorem 2.20, the scheme (2.39) is almost surely stable when C and h are chosen sufficiently large and small, respectively. $\square$

When $U(\cdot) = |\cdot|^{q_1} + |\cdot|^{q_2}$, $0 < q_1 \leqslant q_2$, one sees $U(\cdot)^{\kappa_1\gamma} \vee U(\cdot)^{\kappa_2\gamma} = |\cdot|^{(\kappa_1\wedge\kappa_2)q_1\gamma} + |\cdot|^{(\kappa_1\vee\kappa_2)q_2\gamma}$.

Corollary 2.25. *In the special case where $V(\cdot) = |\cdot|^p$ and $z(x) \gtrsim |x|^{\kappa_1+p-1} + |x|^{\kappa_2+p-1}$, one just needs to choose $\alpha = 1/4$ and $G(x) := C(|x|^{\kappa_1-1} + |x|^{\kappa_2-1})$ with C sufficiently large.*

2.2.2 Projected Schemes

In general there is no evident clue that the balanced scheme (2.39) can preserve moment-exponential stability, since the factor $1/(1+G(x)h^\alpha)$ has no positive lower bound. However, this can be resolved if at every step the scheme is projected onto a bounded range:

$$
\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{2.43}
$$

where $\Pi : \mathbb{R}^d \to \mathbb{R}^d$ is a function such that $|\Pi(x)| = |x| \wedge h^{-r}$ for some $r > 0$, $\forall x \in \mathbb{R}^d$, and b^h, σ^h are as in (2.39). By adopting this scheme one can immediately have z^h in (2.42) replaced by just z itself (with scaling):

$$
\begin{aligned}
z^h(x) =& \frac{z(x)}{1+G(x)h^\alpha} = \frac{z(x)}{1+C|x|^{\kappa^*}h^\alpha} \\
\geqslant& \frac{z(x)}{1+Ch^{\alpha - rq\kappa^*}} \geqslant \frac{1}{1+C} z(x),\ \forall x \in \mathbb{R}^d,
\end{aligned}
$$

by choosing $r < \alpha/(q\kappa^*)$, where $G(\cdot)$ is, for instance as in Example 2.18, chosen to be $C|\cdot|^{\kappa^*}$ for some $C, \kappa^* > 0$. This motivates the idea that (2.43) can remedy the shortcoming of the balanced scheme (2.39). Indeed, when $z(\cdot) \geqslant \rho V(\cdot)$, for the balance schemes one has

$$
\mathcal{L}_t^h V(x) \leqslant -\rho \frac{V(x)}{1+G(x)h^\alpha},
$$

where one sees that $z^h(\cdot) \gtrsim V(\cdot)$ is violated due to the unboundedness of $G(\cdot)$. However, this can be avoided by using projection (2.43).

Proposition 2.26. *Let Assumption 2.13 hold with* $U(\cdot) = V(\cdot) \leqslant \nu(1+|\cdot|^q)$ *for some* $\nu, q > 0$, *and*

$$V(\Pi(x)) \leqslant V(x),\ \forall x \in \mathbb{R}^d, \tag{2.44}$$

for a chosen projection Π. *Suppose* $\exists \rho > 0$ *s.t.* $\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\mathcal{L}_t V(x) \leqslant -\rho V(x).$$

Then, with $G(x) := C(1+|x|^{(\check{\kappa}-1)q\gamma}),\ C\gamma K\nu^{(\check{\kappa}-1)\gamma}/\mu,\ \alpha \leqslant 1/4,\ r < \alpha/((\check{\kappa}-1)q\gamma)$, *the scheme* (2.43) *is* V*-exponentially stable, where* $\check{\kappa} = \kappa_1 \vee \kappa_2$ *and* μ *satisfies the requirement in Theorem 2.20.*

Proof. Notice that by the same argument as in the proof of Theorem 2.10, we treat $\mathcal{L}_t^h(b^h, \sigma^h)$ as $\mathcal{L}_t(b^h, \sigma^h)$ restricted on $\{|x| \leqslant h^{-r}\}$, and b^h, σ^h in Theorem 2.20 are just as in (2.39). We first verify condition (2.33) by finding a sufficient condition:

$$\begin{aligned}
&\frac{|b(t,x)|\,h^{1/2} \vee \|\sigma(t,x)\|\,h^{1/4}}{1+G(x)h^\alpha} \leqslant \mu V(x)^\gamma \\
\Leftarrow\ & K(V(x)^{\kappa_1\gamma} \vee V(x)^{\kappa_2\gamma})h^{1/4} \leqslant \mu V(x)^\gamma G(x) h^\alpha,
\end{aligned}$$

which is achieved by choosing $\alpha \leqslant 1/4,\ G(x) := C(1+|x|^{(\check{\kappa}-1)q\gamma}),\ C\gamma K\nu^{(\check{\kappa}-1)\gamma}/\mu$, assuming $\nu \geqslant 1$ without loss of generality. Also for $x \in \{|x| \leqslant h^{-r}\}$, we have $G(x) \leqslant C + Ch^{-r(\check{\kappa}-1)q\gamma}$, and thus $\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\begin{aligned}
\mathcal{L}_t^h V(x) &\leqslant -\frac{\rho}{1+G(x)h^\alpha} V(x) \\
&\leqslant -\frac{1}{1+Ch^\alpha + Ch^{\alpha - r(\check{\kappa}-1)q\gamma}} V(x) =: -\widetilde{\rho} V(x),
\end{aligned}$$

if we choose $r < \alpha/((\check{\kappa}-1)q\gamma)$. Note that there is no restriction on the step size h. $\square$

In fact, one can show that projecting the standard Euler scheme - with the original drift and diffusion:

$$\bar{X}_{k+1} = \Pi\left(\bar{X}_k + b(t_k, \bar{X}_k)h + \sigma(t_k, \bar{X}_k)\Delta W_{k+1}\right), \tag{2.45}$$

is enough to inherit V-exponential stability under suitable conditions. This has been introduced earlier in (2.22), which by Proposition 2.11 is well-defined.

Proposition 2.27. *Let Assumption 2.13 hold with* $U = V$ *satisfying* (2.44) *for a chosen projection* Π *and* $V(\cdot) \leqslant \nu(1+|\cdot|^q)$ *for some* $\nu, q > 0$. *If* $\exists \rho > 0$ *s.t.* $\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$,

$$\mathcal{L}_t V(x) \leqslant -\rho V(x),$$

then with $r < 1/(4(\check{\kappa}-1)q\gamma),\ h < (\mu/(2K\nu^{(\check{\kappa}-1)\gamma}))^\beta$, *the scheme* (2.45) *preserves* V*-exponential stability, where* $\beta = 1/4 - r(\check{\kappa}-1)q\gamma$ *and* μ *satisfies the requirement in Theorem 2.20.*

Proof. As shown in (2.23) condition (2.31) is redundant and one only needs to verify condition (2.33) for b and σ, i.e.

$$|b(t,x)|\,h^{1/2} \vee \|\sigma(t,x)\|\,h^{1/4} \leqslant \mu V(x)^\gamma,\ \forall t, x. \tag{2.46}$$

The left-hand-side term has upper bound $K\left(V(x)^{\kappa_1\gamma}h^{1/2}\right) \vee \left(V(x)^{\kappa_2\gamma}h^{1/4}\right)$, and for scheme (2.45) we know $|\bar{X}_k| \leqslant h^{-r}$. Since $V(\cdot) \leqslant \nu\,(1 + |\cdot|^q)$, one can require

$$\begin{aligned}
&\mu V(x)^\gamma \geqslant KV(x)^\gamma \left(V(x)^{(\kappa_1-1)\gamma} h^{1/2}\right) \vee \left(V(x)^{(\kappa_2-1)\gamma} h^{1/4}\right)\\
\Leftarrow\;\; &\mu \geqslant K\nu^{(\check{\kappa}-1)\gamma}\left(1 + |x|^{(\kappa_1-1)q\gamma}\right) h^{1/2} \vee \left(1 + |x|^{(\kappa_2-1)q\gamma}\right) h^{1/4}\\
\Leftarrow\;\; &\mu \geqslant 2K\nu^{(\check{\kappa}-1)\gamma}\left(h^{1/2-r(\kappa_1-1)q\gamma} \vee h^{1/4-r(\kappa_2-1)q\gamma}\right)\\
\Leftarrow\;\; &\mu \geqslant 2K\nu^{(\check{\kappa}-1)\gamma} h^\beta. \qquad (2.47)
\end{aligned}$$

Note that one can immediately let inequality (2.47) hold by choosing

$$r < \frac{1}{2(\kappa_1-1)q\gamma} \wedge \frac{1}{4(\kappa_2-1)q\gamma},\;\; h < h_0 \leqslant \left(\frac{\mu}{2K\nu^{(\check{\kappa}-1)\gamma}}\right)^{1/\beta}, \qquad (2.48)$$

for fixed μ. Therefore, the scheme (2.45) preserves V-exponential stability when such r is chosen and h is sufficiently small. $\square$

Moment exponential stability immediately follows when $V(\cdot) = U(\cdot) = |\cdot|^p$, $q = p = 1/\gamma$. On the other hand, scheme (2.45), as expected, also admits almost-sure stability given the same conditions as for scheme (2.39).

Proposition 2.28. *Let Assumption 2.13 hold with V satisfying (2.44) for a chosen projection Π. Suppose $\exists 0 \leq z \in \mathcal{C}(\mathbb{R}^d)$ satisfying (2.41), s.t. $\forall (t,x) \in [0,\infty) \times \mathbb{R}^d$, $\mathcal{L}_t V(x) \leqslant -z(x)$. If $\exists \nu, q > 0$ s.t. $U(\cdot) \leqslant \nu(1 + |\cdot|^q)$, then, with $r < (4(\check{\kappa}-1)q\gamma)^{-1}$, $h < \left(\mu\lambda/(K + 2\lambda K\nu^{(\check{\kappa}-1)\gamma})\right)^{1/\beta}$, the scheme (2.45) is almost-surely stable, where $\beta = 1/4 - r(\check{\kappa}-1)q\gamma$ and μ satisfies the requirement in Theorem 2.20.*

Proof. Again one only needs to check condition (2.32) for b and σ for scheme (2.45), which satisfies $|\bar{X}_k| \leqslant h^{-r}$, $\forall k \geqslant 1$, with $z^h(\cdot) = z(\cdot)$. Indeed for all x (regardless of X_0 since we are only interested in the long-term behaviour),

$$|b(t,x)|h^{1/2} \vee \|\sigma(t,x)\|h^{1/4} \leqslant \mu\frac{(1+U(x))^\gamma z(x)}{1 + U(x) + z(x)},$$

where, the left-hand-side term above has upper bound $Kh^{1/4}\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2)\gamma}\right)$, and the right-hand-side term minimizes when $z(x)$ reaches its lower bound in (2.41). Thus, due to $|x| \leqslant h^{-r}$, one can require

$$\begin{aligned}
&Kh^{1/4}\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2)\gamma}\right) \leqslant \mu\frac{\lambda(1+U(x))^\gamma\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2)\gamma}\right)}{(1+U(x))^\gamma + \lambda\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right)}\\
\Leftrightarrow\;\; &Kh^{1/4}\left(U(x)^{\kappa_1\gamma} \vee U(x)^{\kappa_2\gamma}\right) \leqslant \left(\mu - \frac{K}{\lambda}h^{1/4}\right)(1+U(x))^\gamma\\
\Leftarrow\;\; &\nu^{(\check{\kappa}-1)\gamma}Kh^{1/4}(1 + |x|^{(\check{\kappa}-1)q\gamma}) \leqslant \mu - \frac{K}{\lambda}h^{1/4}\\
\Leftarrow\;\; &\left(\frac{K}{\lambda} + \nu^{(\check{\kappa}-1)\gamma}K\right)h^{1/4} + \nu^{(\check{\kappa}-1)\gamma}Kh^{1/4-r(\check{\kappa}-1)q\gamma} \leqslant \mu.
\end{aligned}$$

Set $r < (4(\check{\kappa}-1)q\gamma)^{-1}$ s.t. $\beta = 1/4 - r\check{\kappa}q\gamma > 0$. One can then choose $h < \left(\mu\lambda/(K + 2\lambda K\nu^{(\check{\kappa}-1)\gamma})\right)^{1/\beta}$, and hence almost-sure stability is achieved. $\square$

In most cases $V(\cdot) = U(\cdot) = |\cdot|^p$ is chosen, then $q = p = 1/\gamma$ and the conditions become much simpler:

Corollary 2.29. *In the special case where $V(\cdot) = |\cdot|^p$ and $z(x) \gtrsim |x|^{\kappa_1+p-1} + |x|^{\kappa_2+p-1}$, one just needs to choose r and h sufficiently small.*

2.2.3 Other Examples

Example 2.30. *Consider the Stochastic Lorenz Equation [24] in $\mathbb{R}^3$ driven by a 3-d Wiener process:*

$$b(x) = \begin{pmatrix} \alpha_1(x_2 - x_1) \\ -\alpha_1 x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \alpha_2 x_3 \end{pmatrix}, \ \sigma(x) = \begin{pmatrix} \beta_1 x_1 & 0 & 0 \\ 0 & \beta_2 x_2 & 0 \\ 0 & 0 & \beta_3 x_3 \end{pmatrix}, \tag{2.49}$$

where $2\alpha_1 > \beta_1^2,\ \beta_2^2 < 2,\ 2\alpha_2 > \beta_3^2$.

One can immediately check for the Lyapunov function $V(\cdot) = |\cdot|^2 \in \bar{\mathcal{V}}^2_{1/2}$:

$$\mathcal{L}|x|^2 = -(2\alpha_1 - \beta_1^2)x_1^2 - (2 - \beta_2^2)x_2^2 - (2\alpha_2 - \beta_3^2)x_3^2 \leqslant -\rho|x|^2,$$

where $\rho := (2\alpha_1 - \beta_1^2) \wedge (2 - \beta_2^2) \wedge (2\alpha_2 - \beta_3^2)$. According to Theorem 2.17 the system (2.30) is mean-square stable for the equilibrium. One can thus choose taming method (2.45) to preserve mean-square stability for the tamed Euler scheme. One observes

$$\begin{aligned} |b(x)| &= \sqrt{\alpha_1^2(x_2 - x_1)^2 + (\alpha_1 x_1 + x_2 + x_3)^2 + (x_1 x_2 - \alpha_2 x_3)^2} \leqslant K(|x| + |x|^2), \\ \|\sigma(x)\| &= \sqrt{\beta_1^2 x_1^2 + \beta_2^2 x_2^2 + \beta_3 x_3^2} \leqslant K|x|, \end{aligned}$$

where $K = \sqrt{5\alpha_1^2 + 4\alpha_1 + \alpha_2^2 + 4} \vee \sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}$. Then one can choose $U(x) = |x| + |x|^2$, $\kappa_1 = 2$, $\kappa_2 = 1$ for Assumption 2.13 to hold. Note that due to $p = 2$ in this case, one only needs the requirement on $b(t, x)$ as in (2.46). Hence according to Proposition 2.27, one needs to choose $r < 1/2$ and $h < (2K)^{-1/(1/2-r)}$ sufficiently small.

Example 2.31. *Consider the following 2-d SDE with drift and diffusion similar to the Stochastic Duffing-van der Pol Oscilator [24]:*

$$b(x) = \begin{pmatrix} x_2 - \alpha_1 x_1 \\ -\alpha_2 x_2 - x_1^3 \end{pmatrix}, \ \sigma(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta x_2 & 0 \end{pmatrix}, \tag{2.50}$$

where $\alpha_1 > 0,\ 2\alpha_2 > \beta^2$.

In this case one can set the Lyapunov function to be

$$V(x) = x_1^4 + 2x_2^2,$$

which is from a broader class $\widehat{\mathcal{V}}^4_{1/4}$. Then one observes that

$$\mathcal{L}V(x) = -4\alpha_1 x_1^4 - (4\alpha_2 - 2\beta^2)x_2^2 \leqslant -\rho V(x),$$

where $\rho := 4 \wedge (4\alpha_2 - 2\beta^2)$. According to Theorem 2.17, the trivial solution of (2.50) is V-exponentially stable. Therefore we consider using the projected scheme (2.45), for which all conditions regarding (b^h, σ^h, z^h) are reduced to those of (b, σ, z) on the set

$\{x: \ |x| \leqslant h^{-r}\}$. In this 2-d case one can, for example, define

$$\Pi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -h^{-r} \vee x_1 \wedge h^{-r} \\ -h^{-r} \vee x_2 \wedge h^{-r} \end{pmatrix},$$

s.t. $|\Pi x| \leqslant h^{-r}$ and (2.44) is satisfied. Hence in order to verify condition (2.33), one only needs to check for the points (x_1, x_2) satisfying $|x_1| \vee |x_2| \leqslant h^{-r}/\sqrt{2}$:

$$\begin{aligned} |b(x)|h^{1/2} =& \left((\alpha_2+1)|x_2| + \alpha_1|x_1| + |x_1|^3\right) h^{1/2} \\ \leqslant& \frac{\alpha_2+1}{\sqrt[4]{2}}|x_2|^{1/2}h^{1/2-r/2} + \frac{\alpha_1+1}{2}|x_1|h^{1/2-2r} \\ \leqslant& \frac{\alpha_1 \vee \alpha_2 + 1}{2} h^{1/2-2r}(|x_1| + 2|x_2|^{1/2}) \leqslant \mu V(x)^{1/4}, \\ \|\sigma(x)\|h^{1/4} =& |\beta||x_2|h^{1/4} \leqslant \frac{|\beta|}{\sqrt[4]{2}} h^{1/4-r/2}|x_2|^{1/2} \leqslant \mu V(x)^{1/4}, \end{aligned}$$

where we choose $r < 1/4$ and $\mu := \max\{4(\alpha_1 \vee \alpha_2 + 1)h^{1/2-2r}/2, |\beta|h^{1/4-r/2}/\sqrt[4]{2}\} \leqslant 1$. Thus according to Theorem 2.20, the projected scheme (2.45) is exponentially stable with respect to V when h is chosen sufficiently small.

2.3 Non-Negativity and The Comparison Theorem

Apart from integrability and stability, there are some other properties on the SDE level that can be preserved via taming. For example, some SDEs have solution only in a bounded region, and especially in 1-d case two SDEs with the same diffusion can be compared, subject to some conditions.

2.3.1 Non-Negativity

The issue of non-negativity preservation can be seen from the following 1-d linear SDE with non-zero constants μ and σ:

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t. \tag{2.51}$$

The solution $X_t = X_0 \exp\left\{\left(\mu - \sigma^2/2\right)t + \sigma W_t\right\} \geqslant 0$ a.s. if $X_0 \geqslant 0$ a.s. However this may not be the case for the standard Euler scheme

$$\bar{X}_{k+1} = (1+\mu h)\bar{X}_k + \sigma \bar{X}_k \Delta W_{k+1}.$$

More precisely, suppose that $\bar{X}_k \geqslant 0$ a.s., then for $\sigma > 0$,

$$\mathbb{P}(\bar{X}_{k+1} < 0) = \mathbb{P}\left(\Delta W_{k+1} < -\frac{1+\mu h}{\sigma}\right) > 0;$$

the same applies for $\sigma < 0$ due to the symmetry of the Gaussian distribution. However, one can avoid this situation by simply truncating the Wiener process. For SDEs with super-linear growth coefficients a little bit more work is needed to preserve non-negativity. Non-negativity of the SDE can be regarded as a corollary of the comparison theorem to be mentioned later (Theorem 2.34). However, it turns out that for non-negativity the requirement on the drift is slightly weaker than that for the comparison theorem.

Lemma 2.32. *Given a 1-d SDE*

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \tag{2.52}$$

with $X_0 \geq 0$ a.s. and $\mathbb{E}X_0 < \infty$ Suppose

i) *there exists a unique, $|\cdot|^\kappa$-integrable, strong solution of (2.52) for some $\kappa \geq 1$;*

ii) $|b(t,x)| \vee |\sigma(t,x)|^2 \lesssim 1 + |x|^\kappa,\ \forall (t,x) \in [0,\infty) \times \mathbb{R}$, *and b satisfies the one-sided Lipschitz condition:*

$$(x-y)(b(t,x) - b(t,y)) \leqslant K|x-y|^2,\ \forall x, y \in \mathbb{R},\ \forall t \geq 0; \tag{2.53}$$

iii) $b(t,0) \geqslant 0,\ \sigma(t,0) = 0,\ \forall t \geqslant 0.$

Then $X_t \geqslant 0$ a.s. $\forall t$.

This has been mentioned and heuristically explained in [20]. We give a proof of it in Appendix A.3. Now consider a tamed Euler scheme for (2.52):

$$\hat{X}_{k+1} = \hat{X}_k + b^h(t_k, \hat{X}_k)h + \sigma^h(t_k, \hat{X}_k)\sqrt{h}\xi, \tag{2.54}$$

where $\xi \sim N(0,1)$. Non-negativity generally does not hold any more for $\hat{X}_k$, but one can recover this property by truncating the noise:

$$\zeta_h = (-A_h) \vee \xi \wedge A_h, \tag{2.55}$$

where one takes $A_h = \sqrt{2|\log h|}$. This idea is introduced in Section 1.3.4 in [43] for mean-square convergence of the implicit Euler scheme. We would like to point out that such a truncation can be used to preserve non-negativity.

Theorem 2.33. *Let the assumptions in Lemma 2.32 hold. If one can find a taming method (b^h, σ^h) such that $b^h(\cdot, 0) \geq 0$ and $\exists \mu, \alpha > 0$,*

$$|b^h(t,x) - b^h(t,0)|h^\alpha \vee |\sigma^h(t,x)|h^{\alpha/2} \leqslant \mu|x|,\ \forall (t,x) \in [0,\infty) \times \mathbb{R}, \tag{2.56}$$

then the tamed Euler scheme

$$\bar{X}_{k+1} = \bar{X}_k + b^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\sqrt{h}\zeta_h, \tag{2.57}$$

is almost surely non-negative for $\alpha < 1$ and h, μ sufficiently small.

Proof. Rewrite the scheme (2.57) and inductively assume $\bar{X}_k \geqslant 0$ a.s.,

$$\begin{aligned}\bar{X}_{k+1} =& \bar{X}_k + b^h(t_k, 0)h + (b^h(t_k, \bar{X}_k) - b^h(t_k, 0))h + \sigma^h(t_k, \bar{X}_k))\sqrt{h}\zeta_h \\ \geqslant& \bar{X}_k\left(1 - \mu h^{1-\alpha} - \mu h^{1/2-\alpha/2}A_h\right),\end{aligned} \tag{2.58}$$

as $b^h(t,0) \geqslant 0$. In order for (2.58) to stay nonnegative, we set $\alpha < 1$ and $h^{1-\alpha} + h^{1/2-\alpha/2}A_h \leqslant 1/\mu$. □

If $|b(\cdot,x) - b(\cdot,0)| \lesssim |x| + |x|^m$ for some $m \geqslant 1$, then (2.56) can be realised by a suitable balanced scheme as discussed in Subsection 2.1.2, for which the constant μ can be arbitrarily small. Under the same assumption, condition (2.56) can also be realised by the projected scheme (2.45) by choosing an appropirate r. In fact, in this case one need not truncate the noise via (2.55). Instead one need only define a reasonable projection:

$$\Pi(x) = \left(0 \vee x_i \wedge h^{-r}\right)_{i=1,\cdots,d}, \tag{2.59}$$

where r is chosen s.t. Proposition 2.11 holds. This is similar to what is suggested in [6], where the authors ensure the approximation stay strictly positive. For that one just replaces the 0 above with h^r.

2.3.2 Comparison Result

As an extension of non-negativity preservation, one can preserve comparison result for SDEs by applying taming techniques. It is known that two SDEs with the same diffusion and noise can be compared by the comparison theorem:

Theorem 2.34. *Consider two* 1*-d SDEs:*

$$\begin{aligned}
\mathrm{d}X_t =& \nu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \\
\mathrm{d}Y_t =& \lambda(t, Y_t)\mathrm{d}t + \sigma(t, Y_t)\mathrm{d}W_t,
\end{aligned}$$

with $X_0 \leqslant Y_0$ a.s. and $\mathbb{E}|Y_0| \vee \mathbb{E}|X_0| < \infty$. Assume the following conditions:

(i) *each SDE has a unique, $|\cdot|^\kappa$-integrable, strong solution for some $\kappa \geq 1$;*

(ii) $|\nu(t,x)| \vee |\lambda(t,x)| \vee |\sigma(t,x)|^2 \lesssim 1 + |x|^\kappa,\ \forall (t,x) \in [0,\infty) \times \mathbb{R}$*;*

(iii) *σ is locally Hölder in x with exponent $\alpha \geqslant 1/2$;*

(iv) $\nu(t,x) \leqslant \lambda(t,x),\ \forall (t,x) \in [0,\infty) \times \mathbb{R}$*;*

(v) *either λ or ν satisfies one-sided Lipschitz condition* (2.53)*.*

Then $X_t \leqslant Y_t$ a.s., $\forall t \geqslant 0$.

Although condition (v) is weaker than usually stated in the literature, e.g. Proposition 5.2.18 in [30], one still applies Itô's formula to the process $(Y_t - X_t)^-$ via smooth approximation (for which (iii) is needed), and the result follows from the same arguments adopted in Appendix A.3. Now consider the Euler scheme for each equation:

$$\begin{aligned}
\hat{X}_{k+1} =& \hat{X}_k + \nu(t_k, \hat{X}_k)h + \sigma(t_k, \hat{X}_k)\sqrt{h}\xi, \\
\hat{Y}_{k+1} =& \hat{Y}_k + \lambda(t_k, \hat{Y}_k)h + \sigma(t_k, \hat{Y}_k)\sqrt{h}\xi,
\end{aligned}$$

where $\xi \sim N(0,1)$. In general the comparison property does not necessarily hold for $\hat{X}_k$ and $\hat{Y}_k$, but by truncating the noise using (2.55) it can be recovered.

Theorem 2.35. *Let the assumptions in Theorem 2.34 hold with λ satisfying one-sided Lipschitz condition* (2.53)*. If there is a taming method (λ^h, σ^h) s.t. $\exists \mu, \alpha > 0,\ \forall x, y \in \mathbb{R},\ t \geq 0$,*

$$|\lambda^h(t,x) - \lambda^h(t,y)|h^\alpha \vee |\sigma^h(t,x) - \sigma^h(t,y)|h^{\alpha/2} \leqslant \mu|x-y|, \tag{2.60}$$

and $\nu^h(t,x) \leqslant \lambda^h(t,x)$, then, for $\alpha < 1$, ζ_h defined as in (2.55) *and h, μ sufficiently small, the tamed Euler schemes*

$$\begin{aligned}
\bar{X}_{k+1} =& \bar{X}_k + \nu^h(t_k, \bar{X}_k)h + \sigma^h(t_k, \bar{X}_k)\sqrt{h}\zeta_h, \\
\bar{Y}_{k+1} =& \bar{Y}_k + \lambda^h(t_k, \bar{Y}_k)h + \sigma^h(t_k, \bar{Y}_k)\sqrt{h}\zeta_h,
\end{aligned}$$

preserve the comparison property: $\bar{X}_k \leqslant \bar{Y}_k$ *a.s.* $\forall k \in \mathbb{N}$.

Proof. Inductively suppose $\bar{Y}_k \geqslant \bar{X}_k$ a.s. and take the difference of the two SDEs:

$$\begin{aligned}\bar{Y}_{k+1} - \bar{X}_{k+1} \geqslant &(\bar{Y}_k - \bar{X}_k)(1 - \mu h^{1/2-\alpha/2} A_h) + (\lambda^h(\bar{Y}_k) - \nu^h(\bar{X}_k))h \\ \geqslant &(\bar{Y}_k - \bar{X}_k)(1 - \mu h^{1/2-\alpha/2} A_h) + (\lambda^h(\bar{Y}_k) - \lambda^h(\bar{X}_k))h \\ \geqslant &(\bar{Y}_k - \bar{X}_k)(1 - \mu h^{1-\alpha} - \mu h^{1/2-\alpha/2} A_h).\end{aligned}$$

Require $\alpha < 1$ and $h^{1-\alpha} + h^{1/2-\alpha/2} A_h \leqslant 1/\mu$, and the result follows. □

Condition $\nu^h(t,x) \leqslant \lambda^h(t,x)$ is usually immediately satisfied given $\nu(t,x) \leqslant \lambda(t,x)$ for all t, x. Now let us investigate whether (2.60) is achievable. If $\lambda(t,x)$ is differentiable in x and $|\partial_x \lambda(t,x)| \vee |\lambda(t,x)| \leqslant K(1+|x|^m)$ for some constants $K > 0, m \geqslant 1$, one multiplies the taming factor $(1 + G(x)h^\alpha)^{-1}$ with λ for $G(x) = C|x|^{m-1}$ for some constant $C \geqslant 1$, and by the mean value theorem, $|\lambda^h(t,x) - \lambda^h(t,y)| \leqslant |\partial_x \lambda^h(t,\xi)||x-y|$ for some ξ between x and y. Then by the chain rule,

$$\begin{aligned}|\partial_x \lambda^h(t,\xi)| \leqslant &\frac{|\partial_x \lambda^h(t,\xi)|(1 + Ch^\alpha|\xi|^{m-1}) + C|\lambda(t,\xi)|h^\alpha(m-1)|\xi|^{m-2}}{(1+Ch^\alpha|\xi|^{m-1})^2} \\ \leqslant &Km\frac{(1+|\xi|^{m-1})(1+Ch^\alpha|\xi|^{m-1}) + C(1+|\xi|^m)h^\alpha|\xi|^{m-2}}{(1+Ch^\alpha|\xi|^{m-1})^2} \\ = &Km\frac{1 + Ch^\alpha|\xi|^{m-2} + (1+Ch^\alpha)|\xi|^{m-1} + 2Ch^\alpha|\xi|^{2m-2}}{1 + 2Ch^\alpha|\xi|^{m-1} + C^2h^{2\alpha}|\xi|^{2m-2}} \\ \leqslant &2Km\frac{1 + 2|\xi|^{m-1} + h^\alpha|\xi|^{2m-2}}{Ch^\alpha(1 + 2|\xi|^{m-1} + h^\alpha|\xi|^{2m-2})} = \frac{2Km}{C}h^{-\alpha},\end{aligned}$$

where the last inequality holds for $Ch^\alpha \leq 1$. Thus $|\lambda^h(x) - \lambda^h(y)| \leqslant \mu|x-y|h^{-\alpha}$ where, by choosing a large C, the constant $\mu = 2Km/C$ can be arbitrarily small.

Chapter 3

The Fourier Method for Higher-Order Approximations

Higher-order approximations can be derived from stochastic Taylor expansions, and Davie showed (Theorem 4 in [8]) that there exists a coupling for the Taylor approximation that is arbitrarily close, giving a numerical approximation for the solution of an SDE up to any order. However, this is proved under the assumption that the diffusion matrix σ admits a right inverse everywhere, which is rather restrictive.

The degenerate case, where the matrix σ has rank less than d, is much harder to handle. Davie [8] (Section 9) found a coupling for the double integral (Theorem 15 therein), allowing the Milstein method with step size h to have an $O(h)$-convergence in general, whereas the case of longer iterated integrals is still an open problem.

The motivation of this chapter is to provide a feasible approximation for SDEs of a higher order. For simplicity consider the following autonomous SDE on the interval $[0,T]$:

$$X_t = X_0 + \int_0^t b(X_s)\mathrm{d}s + \int_0^t \sigma(X_s)\mathrm{d}W_s, \tag{3.1}$$

where W_t is a q-dimensional Wiener process and $b : \mathbb{R}^d \to \mathbb{R}^d,\ \sigma : \mathbb{R}^d \to \mathbb{R}^{d\times q}$ are sufficiently smooth functions. By applying Itô's formula again to the term $\sigma_{kl}(X_u)\partial_k\sigma_{ij}(X_u)$ in (1.8), one obtains, for each component $i = 1, \cdots, d$ on the interval $[s,t]$,

$$\begin{aligned}
X_t^i =& X_s^i + b_i(X_s)(t-s) + \sigma_{ij}(X_s)(W_t^j - W_s^j) + \sigma_{kl}(X_s)\partial_k\sigma_{ij}(X_s)\int_s^t\int_s^r \mathrm{d}W_u^l\mathrm{d}W_r^j \\
&+ \int_s^t\int_s^r \mathcal{L}b_i(X_u)\mathrm{d}u\mathrm{d}r + \int_s^t\int_s^r \sigma_{kl}(X_u)\partial_k b_i(X_u)\mathrm{d}W_u^l\mathrm{d}r \\
&+ \int_s^t\int_s^r \mathcal{L}\sigma_{ij}(X_u)\mathrm{d}u\mathrm{d}W_r^j + \int_s^t\int_s^r\int_s^u \mathcal{L}\left(\sigma_{kl}(X_v)\partial_k\sigma_{ij}(X_v)\right)\mathrm{d}v\mathrm{d}W_u^l\mathrm{d}W_r^j \\
&+ \int_s^t\int_s^r\int_s^u \partial_m\left(\sigma_{kl}(X_v)\partial_k\sigma_{ij}(X_v)\right)\sigma_{mn}(X_v)\mathrm{d}W_v^n\mathrm{d}W_u^l\mathrm{d}W_r^j,
\end{aligned}$$

where the summation signs over repeated indices are omitted. From this expression one can obtain a suitable numerical scheme (formula (10.4.6) in [32]) with strong convergence order $O(h^{3/2})$. Just as the Milstein scheme, the crucial ingredient to achieve such a higher-order convergence is the generation of the triple integrals $I_{jkl}(s,t) := \int_s^t\int_s^r\int_s^u \mathrm{d}W_v^j\mathrm{d}W_u^k\mathrm{d}W_r^l,\ j,k,l = 1,\cdots,q$.

Similar to the way the double stochastic integral is treated in [8], one would expect the same method to be extended to treat triple integrals. For the simplicity of formulation, the Stratonovich triple integral $I^\circ_{jkl}(s,t) := \int_s^t \int_s^r \int_s^u \mathrm{d}W_v^j \circ \mathrm{d}W_u^k \circ \mathrm{d}W_r^l$ will be considered instead of the Itô version, since the Fourier representation of the former has a relatively simpler form. This is due to the fact that the product of two Stratonovich integrals is a shuffle product - see Proposition 2.2 in [12]. In other words, an iterated Stratonovich integral of longer length can be represented by shorter ones in a much simpler way compared its Itô counterpart.

The goal is to find a random variable $\bar{I}_{jkl}$ whose law is close to that of I°_{jkl} in the Vaserstein distance, which in turn gives a feasible $O(h^{3/2})$-approximation for the SDE (3.1). In order to understand the logic of this chapter let us briefly review Davie's Fourier method (Section 9 in [8]). According to [32] (Section 5.8), the Brownian bridge process $W_t - tW_1$ has Fourier expansion

$$W_t^j - tW_1^j = \frac{1}{2\sqrt{2}\pi} x_{j0} + \frac{1}{\sqrt{2}\pi} \sum_{r=1}^{\infty} x_{jr} \cos(2\pi r t) + \frac{1}{\sqrt{2}\pi} \sum_{r=1}^{\infty} y_{jr} \sin(2\pi r t), \tag{3.2}$$

where $x_{jr},\ y_{jr}$ are $\mathcal{N}(0,1)$-random variables mutually independent for different values of $j = 1, \cdots, q$ or $r \in \mathbb{Z}^+$, all independent of W_1. Then the double integral $I^\circ_{jk} = \int_0^t W_s^j \mathrm{d}W_s^k$ has Fourier representation

$$I^\circ_{jk} = \frac{1}{2} W_1^j W_1^k + \frac{1}{\sqrt{2}\pi} \left(W_1^j z_k - W_1^k z_k \right) + \frac{1}{2\pi} \lambda_{jk}, \tag{3.3}$$

where $\lambda_{jk} = \sum_{r \geqslant 1} r^{-1}(x_{jr} y_{kr} - y_{jr} k_{jr})$ and $z_j = \sum_{r \geqslant 1} r^{-1} x_{jr}$. One then needs to approximate each λ_{jk} and z_j by their partial sums $\lambda_{jk} = \sum_{r=1}^p r^{-1}(x_{jr} y_{kr} - y_{jr} k_{jr})$ and $z_j = \sum_{r=1}^p r^{-1} x_{jr}$. Denote $\widetilde{\lambda}_{jk}^{(p)} = \lambda_{jk} - \lambda_{jk}^{(p)}$, $\widetilde{z}_j^{(p)} = z_j - z_j^{(p)}$ and $U := (\lambda, z)$, $U_p := (\lambda^{(p)}, z^{(p)})$, $\widetilde{U}_p := (\widetilde{\lambda}^{(p)}, \widetilde{z}^{(p)})$.

Davie's result states that if there is a random variable $\bar{U}$, independent of U_p, having the same moments as $\widetilde{U}_p$ up to order $m-1$ and satisfying $\mathbb{E}\exp(a\sqrt{p}|\bar{U}|) \leqslant b$ for some positive constants a, b, then $\mathbb{W}_2(U,\ U_p + \bar{U}) = O(p^{-m/2})$ for p sufficiently large. The idea is to estimate the densities $g(\zeta)$ of U and $h(\zeta)$ of $U_p + \bar{U}$. If f_p is the density of U_p, then $g(\zeta) = \mathbb{E} f_p(\zeta - \widetilde{U}_p)$ and $h(\zeta) = \mathbb{E} f_p(\zeta - \bar{U})$. By Taylor's theorem, for all $\zeta, w \in \mathbb{R}^d$,

$$\begin{aligned} f_p(\zeta - w) = &\sum_{|\beta|=0}^{m-1} \frac{(-1)^{|\beta|}}{\beta!} \partial^\beta f_p(\zeta) w^\beta \\ &+ \sum_{|\beta|=m} \frac{|\beta|(-1)^{|\beta|}}{\beta!} w^\beta \int_0^1 (1-\theta)^{|\beta|-1} \partial^\beta f_p(\zeta - \theta w) \mathrm{d}\theta. \end{aligned} \tag{3.4}$$

Since up to the $(m-1)$-th moments of $\widetilde{U}_p$ and $\bar{U}$ match, when taking the difference $g(\zeta) - h(\zeta)$ the first summation vanishes, and hence $\forall \zeta \in \mathbb{R}^d$,

$$\begin{aligned} &g(\zeta) - h(\zeta) = \\ &\quad \sum_{|\beta|=m} C_\beta \int_0^1 (1-\theta)^{m-1} \left(\mathbb{E} \widetilde{U}_p^\beta \partial^\beta f_p(\zeta - \theta \widetilde{U}_p) - \mathbb{E} \bar{U}^\beta \partial^\beta f_p(\zeta - \theta \bar{U}) \right) \mathrm{d}\theta. \end{aligned} \tag{3.5}$$

If one can give a uniform bound for some higher derivatives of f_p in terms of p, then using an interpolation argument one can show a reasonable decay for the m-th derivative of f_p, and finally one finds a coupling between U and $U_p + \bar{U}$ by the inequality (1.10).

From this calculation one sees that the key step towards a good coupling result depends on how well the behaviour of f_p is understood. Davie's result is a significant improvement to the existing rate of approximation - see the discussion following the proof of Theorem 15 therein. This is due to some careful estimates (Lemma 12, 13 and 14 in [8]) for the density f_p. For the triple integral I°_{jkl}, however, showing similar estimates becomes much more complicated as the Fourier coefficients for I°_{jkl} have summands that are not independent of each other - see the definition of the random variable Δ_{jkl} below. The main purpose of this chapter is to show the boundedness of the derivatives of the density f_p in the triple integral case, as a partial result leading to a conjectured coupling result; some remaining obstacles will be discussed at the end of the chapter.

Throughout this chapter we will denote by ϕ the standard normal density of dimension 1, by $B(x,r)$ the open ball of radius r centred at x, and by Λ^d the Lebesgue measure on $\mathbb{R}^d$. The notation C_0^∞ stands for the set of functions that are infinitely times continuously differentiable with compact support.

3.1 The Fourier Representation for Triple Stochastic Integrals

For the simplicity of presentation let us consider the triple integral on the unit interval $[0,1]$. Following Section 5.8 in [32], from the Fourier expansion (3.2) the triple Stratonovich integral

$$I^\circ_{jkl} = \int_0^1 \int_0^t W_s^j \circ \mathrm{d}W_s^k \circ \mathrm{d}W_t^l,$$

for each $(j,k,l) \in \{1, \cdots, q\}^3$ has the following representation:

$$\begin{aligned} I^\circ_{jkl} =& \frac{1}{6} W_1^j W_1^k W_1^l - \frac{1}{2\sqrt{2}\pi} W_1^j W_1^k \left(z_l - \frac{1}{\pi} u_l \right) - \frac{1}{2\sqrt{2}\pi} W_1^k W_1^l \left(z_j - \frac{1}{\pi} u_j \right) \\ &- \frac{1}{\sqrt{2}\pi^2} W_1^j W_1^l u_k - \frac{1}{2\pi^2} z_j \left(W_1^k z_l - W_1^l z_k \right) + \frac{1}{2\pi} W_1^l \left(\frac{1}{2}\lambda_{jk} + \frac{1}{\pi}\nu_{kj} \right) \\ &+ \frac{1}{2\pi} W_1^j \left(\frac{1}{2}\lambda_{kl} - \frac{1}{\pi}\nu_{kl} \right) + \frac{1}{4\pi^2} \left(W_1^j \mu_{kl} - W_1^k \mu_{jl} \right) - \frac{1}{2\sqrt{2}\pi^2} z_j \lambda_{kl} \\ &+ \frac{1}{4\sqrt{2}\pi} \Delta_{jkl}, \end{aligned}$$

where the coefficients z, u, λ, μ, ν are defined as

$$\begin{aligned} z_j &= \sum_{r=1}^{\infty} \frac{1}{r} x_{jr}, \; u_j = \sum_{r=1}^{\infty} \frac{1}{r^2} y_{jr}, \\ \lambda_{jk} &= \sum_{r=1}^{\infty} \frac{1}{r} \left(x_{jr} y_{kr} - y_{jr} x_{kr} \right), \; \mu_{jk} = \sum_{r=1}^{\infty} \frac{1}{r^2} \left(x_{jr} x_{kr} + y_{jr} y_{kr} \right), \\ \nu_{jk} &= \sum_{\substack{r,s=1 \\ r \neq s}}^{\infty} \frac{1}{r^2 - s^2} \left(\frac{r}{s} x_{jr} x_{ks} - y_{jr} y_{ks} \right), \end{aligned}$$

with x_{jr}, y_{jr}, again, being $\mathcal{N}(0,1)$-random variables independent for different indices $j = 1, \cdots, q$, $r \in \mathbb{Z}^+$ and all independent of W_1^j, and the last coefficient Δ is given by

$$\begin{aligned}\Delta_{jkl} = \sum_{r,s=1}^{\infty} \Big\{ &-\frac{1}{r(r+s)} [(x_{jr}y_{ks} + y_{jr}x_{ks})x_{l,r+s} + (-x_{jr}x_{ks} + y_{jr}y_{ks})y_{l,r+s}] \\ &+\frac{1}{rs} [(x_{jr}y_{ls} + y_{jr}x_{ls})x_{k,r+s} + (-x_{jr}x_{ls} + y_{jr}y_{ls})y_{k,r+s}] \\ &+\frac{1}{s(r+s)} [(-x_{kr}y_{ls} + y_{kr}x_{ls})x_{j,r+s} + (x_{kr}x_{ls} + y_{kr}y_{ls})y_{j,r+s}] \Big\}.\end{aligned}$$

For an integer $p > 0$, write $z^{(p)}$ as the p-th partial sum of z and $\widetilde{z}^{(p)} = z - z^{(p)}$. Similar notations are applied to u, λ and μ. Let $\nu^{(p)}$ be the partial sum of ν over $r, s \leqslant p$, $r \neq s$ and $\widetilde{\nu}^{(p)} = \nu - \nu^{(p)}$, whilst $\Delta^{(p)}$ denotes the partial sum of Δ up to $r + s \leqslant p$ and $\widetilde{\Delta}^{(p)} = \Delta - \Delta^{(p)}$.

From the definition of the variables $\nu_{jk}^{(p)}$ one observes that, if $\mu_{jk}^{(p)}$ is split into two parts as $\mu_{jk}^{(1,p)} := \sum_{r=1}^{p} r^{-2} x_{jr}x_{kr}$ and $\mu_{jk}^{(2,p)} := \sum_{r=1}^{p} r^{-2} y_{jr}y_{kr}$, then one only needs to generate $\nu_{jk}^{(p)}$ for $j < k$ since

$$\nu_{jk}^{(p)} + \nu_{kj}^{(p)} = z_j^{(p)} z_k^{(p)} - \mu_{jk}^{(1,p)}.$$

Equivalent notations for the infinite sums are used by omitting the superscript (p) and the identity still holds. Therefore one need only consider ν_{jk} for $j < k$.

Another observation is that one need not consider all choices of the 3-tuple $(j, k, l) \in \{1, \cdots, q\}^3$ for Δ; it suffices to focus on those terms with (j, k, l) being a **Lyndon word** - a word that is strictly less than all of its proper right factors in the lexicographic order. This is due to the fact that all triple Stratonovich integrals I_{jkl}° can be expressed by the Lyndon words of length at most 3 - see Corollary 3.3 in [12].

For a word w in a totally ordered set A, if it is the concatenation of two non-empty words $u, v \in A$, i.e. $w = uv$, then v is called a proper right factor of w. For example, $(1, 1, 2)$ and $(1, 3, 2)$ are both Lyndon words but $(1, 2, 1)$ is not. By definition, a triple (j, k, l) is a Lyondon word if and only if $j < k \wedge l$ or $j = k < l$. According to [12], there are $(q^3 - q)/3$ Lyndon words of length 3.

As an analogue of the work by Davie [8] (Section 9), one seeks to approximate the variable $V = (z, u, \lambda, \mu, \nu, \Delta)$ by studying the distribution of the partial sums

$$V_p = (z^{(p)}, u^{(p)}, \lambda^{(p)}, \mu^{(p)}, \nu^{(p)}, \Delta^{(p)}),$$

and that of the remainder $\widetilde{V}_p := (\widetilde{z}^{(p)}, \widetilde{u}^{(u)}, \widetilde{\lambda}^{(p)}, \widetilde{\mu}^{(p)}, \widetilde{\nu}^{(p)}, \widetilde{\Delta}^{(p)})$. Note that for an $O(h^{3/2})$-approximation of the SDE (3.1), one also needs to simulate the double integrals (3.3) along with the triple ones. But they are determined by the variables (z, λ), which are already included in V.

By definition the characteristic function $\psi_p(\xi)$ of V_p is given by

$$\begin{aligned}\psi_p(\xi) &= \int_{\mathbb{R}^{2pq}} e^{i|\xi|\Phi_p(x,y)} \prod_{j=1}^{q} \prod_{r=1}^{p} \phi(x_{jr})\phi(y_{jr}) \mathrm{d}x\mathrm{d}y \\ &=: \int_{\mathbb{R}^{2pq}} e^{i|\xi|\Phi_p(v)} \phi_p(v) \mathrm{d}v,\end{aligned}$$

where ϕ is the density function of $\mathcal{N}(0,1)$, and the phase function is defined by

$$\Phi_p(v) = \sum_{j<k} \left(\alpha_{jk}\lambda_{jk}^{(p)} + \gamma_{jk}\nu_{jk}^{(p)}\right) + \sum_{j\leqslant k} \left(\beta_{jk}^{(1)}\mu_{jk}^{(1,p)} + \beta_{jk}^{(2)}\mu_{jk}^{(2,p)}\right)$$
$$+ \sum_{j=1}^{q} \left(a_j z_j^{(p)} + b_j u_j^{(p)}\right) + \sum_{j,k,l=1}^{q} \rho_{jkl}\Delta_{jkl}^{(p)},$$

where $(\alpha, \beta^{(1)}, \beta^{(2)}, \gamma, a, b, \rho) = \xi/|\xi|$ is a unit vector. Observe that the matrices λ and μ are skew-symmetric and symmetric, respectively, so it would be convenient to extend the values of the coefficients α, $\beta^{(1)}$, $\beta^{(2)}$ to their lower-triangles by setting $\alpha_{kj} = -\alpha_{jk}$, $\beta_{kj}^{(1)} = \beta_{jk}^{(1)}$, $\beta_{kj}^{(2)} = \beta_{jk}^{(2)}$ for all $j,k = 1,\cdots,q$. Set $\gamma_{jk} = 0$ for all $j \geqslant k$. Regarding the last summation above, since one need only generate the triple integrals with Lyndon-word subscripts, set $\rho_{jkl} = 0$ if (j,k,l) is not a Lyndon word.

In order to give a good estimate for magnitude of the oscillatory integral $\psi_p(\xi)$ one resorts to the method of stationary phase, and for that one needs to study the derivatives of the phase function Φ_p.

To find the gradient $\nabla\Phi_p$, one can make use the extended definitions of α, β, γ and write down the partial derivatives. For each $j = 1,\cdots,q$ and $r = 1,\cdots,p$, differentiating w.r.t. x_{jr} and y_{jr} gives

$$\begin{aligned}
\partial_{x_{jr}}\Phi_p(x,y) = {} & \frac{1}{r}\alpha_{jk}y_{kr} + \frac{1}{r^2}(1+\delta_{kj})\beta_{jk}^{(1)}x_{kr} + \sum_{\substack{s=1\\ s\neq r}}^{p} \frac{1}{r^2-s^2}\left(\frac{r}{s}\gamma_{jk} - \frac{s}{r}\gamma_{kj}\right)x_{ks} + \frac{1}{r}a_j \\
& + \sum_{s=1}^{p-r}\Bigg[\left(\frac{-\rho_{jkl}+\rho_{lkj}}{r(r+s)} - \frac{\rho_{kjl}}{s(r+s)}\right)y_{ks}x_{l,r+s} + \left(\frac{\rho_{jkl}+\rho_{lkj}}{r(r+s)} + \frac{\rho_{kjl}}{s(r+s)}\right)x_{ks}y_{l,r+s} \\
& \qquad + \left(\frac{\rho_{jkl}+\rho_{lkj}}{rs} - \frac{\rho_{kjl}}{s(r+s)}\right)(y_{ls}x_{k,r+s} - x_{ls}y_{k,r+s})\Bigg] \\
& + \sum_{s=1}^{r-1}\Bigg[\left(-\frac{\rho_{jkl}}{rs} + \frac{\rho_{kjl}}{(r-s)s}\right)x_{k,r-s}y_{ls} + \left(\frac{\rho_{jkl}}{rs} + \frac{\rho_{kjl}}{(r-s)s}\right)y_{k,r-s}x_{ls} \\
& \qquad - \frac{\rho_{lkj}}{(r-s)r}(x_{l,r-s}y_{ks} + y_{l,r-s}x_{ks})\Bigg],
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\partial_{y_{jr}}\Phi_p(x,y) = {} & -\frac{1}{r}\alpha_{jk}x_{kr} + \frac{1}{r^2}(1+\delta_{kj})\beta_{jk}^{(2)}y_{kr} - \sum_{\substack{s=1\\ s\neq r}}^{p} \frac{1}{r^2-s^2}(\gamma_{jk} - \gamma_{kj})y_{ks} + \frac{1}{r^2}b_j \\
& + \sum_{s=1}^{p-r}\Bigg[\left(-\frac{\rho_{jkl}+\rho_{lkj}}{r(r+s)} - \frac{\rho_{kjl}}{s(r+s)}\right)x_{ks}x_{l,r+s} + \left(\frac{-\rho_{jkl}+\rho_{lkj}}{r(r+s)} - \frac{\rho_{kjl}}{s(r+s)}\right)y_{ks}y_{l,r+s} \\
& \qquad + \left(\frac{\rho_{jkl}+\rho_{lkj}}{rs} + \frac{\rho_{kjl}}{s(r+s)}\right)(x_{ls}x_{k,r+s} + y_{ls}y_{k,r+s})\Bigg] \\
& + \sum_{s=1}^{r-1}\Bigg[\left(\frac{\rho_{jkl}}{(r-s)r} - \frac{\rho_{kjl}}{(r-s)s}\right)x_{k,r-s}x_{ls} + \left(\frac{\rho_{jkl}}{rs} + \frac{\rho_{kjl}}{(r-s)s}\right)y_{k,r-s}y_{ls} \\
& \qquad + \frac{\rho_{lkj}}{(r-s)r}(x_{l,r-s}x_{ks} - y_{l,r-s}y_{ks})\Bigg],
\end{aligned} \tag{3.7}$$

where δ_{jk} is the Krönecker delta, the summation signs over the repeated indices $k,l = 1,\cdots,q$ are omitted, and all the x,y terms second subscripts outwith the interval $[1,p]$

are assumed to vanish. The Hessian matrix of Φ_p takes the form

$$\mathrm{D}^2\Phi_p(x,y) = \begin{pmatrix} H_{xx}(1,1) & \cdots & H_{xx}(1,q) & H_{xy}(1,1) & \cdots & H_{xy}(1,q) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{xx}(q,1) & \cdots & H_{xx}(q,q) & H_{xy}(q,1) & \cdots & H_{xy}(q,q) \\ H_{yx}(1,1) & \cdots & H_{yx}(1,q) & H_{yy}(1,1) & \cdots & H_{yy}(1,q) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ H_{yx}(q,1) & \cdots & H_{yx}(q,q) & H_{yy}(q,1) & \cdots & H_{yy}(q,q) \end{pmatrix}, \tag{3.8}$$

where for each pair $(j,k) \in \{1,\cdots,q\}^2$ the blocks $H_{xx}(j,k),\ H_{xy}(j,k),\ H_{yy}(j,k)$ are $p \times p$ matrices, e.g.,

$$H_{xx}(j,k) = \begin{pmatrix} \partial^2_{x_{j1}x_{k1}} & \partial^2_{x_{j1}x_{k2}} & \cdots & \partial^2_{x_{j1}x_{kp}} \\ \partial^2_{x_{j2}x_{k1}} & \partial^2_{x_{j2}x_{k2}} & \cdots & \partial^2_{x_{j2}x_{kp}} \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2_{x_{jp}x_{k1}} & \partial^2_{x_{jp}x_{k2}} & \cdots & \partial^2_{x_{jp}x_{kp}} \end{pmatrix} \Phi_p(x,y), \tag{3.9}$$

and the rest are similarly defined. From the gradient of Φ_p one can compute the second derivative $D^2\Phi_p$ by finding the mixed derivatives for each pair (j,k) and (r,s). The (r,s)-th entries of the blocks $H_{xx}(j,k),\ H_{yy}(j,k)$ and $H_{xy}(j,k)$ are given by

$$\begin{aligned}
\partial^2_{x_{jr}x_{ks}}\Phi_p(x,y) =& \frac{1}{r^2}(1+\delta_{jk})\beta^{(1)}_{jk}\delta_{rs} + \frac{1}{r^2-s^2}\left(\frac{r}{s}\gamma_{jk} - \frac{s}{r}\gamma_{kj}\right)(1-\delta_{rs}) \\
&+ \left(\frac{\rho_{jkl}+\rho_{lkj}}{r(r+s)} + \frac{\rho_{kjl}+\rho_{ljk}}{s(r+s)} - \frac{\rho_{jlk}+\rho_{klj}}{rs}\right) y_{l,r+s} \\
&+ \left(\frac{-\rho_{jlk}+\rho_{klj}}{rs} - \frac{\rho_{ljk}+\rho_{kjl}}{(s-r)s} + \frac{\rho_{jkl}+\rho_{lkj}}{r(s-r)}\right) y_{l,s-r} \\
&+ \left(\frac{-\rho_{klj}+\rho_{jlk}}{rs} + \frac{\rho_{kjl}+\rho_{ljk}}{(r-s)s} - \frac{\rho_{jkl}+\rho_{lkj}}{(r-s)r}\right) y_{l,r-s},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
\partial^2_{y_{jr}y_{ks}}\Phi_p(x,y) =& \frac{1}{r^2}(1+\delta_{jk})\beta^{(2)}_{jk}\delta_{rs} - \frac{1}{r^2-s^2}(\gamma_{jk}-\gamma_{kj})(1-\delta_{rs}) \\
&+ \left(\frac{-\rho_{jkl}+\rho_{lkj}}{r(r+s)} + \frac{-\rho_{kjl}+\rho_{ljk}}{s(r+s)} + \frac{\rho_{jlk}+\rho_{klj}}{rs}\right) y_{l,r+s} \\
&+ \left(\frac{-\rho_{jlk}+\rho_{klj}}{rs} + \frac{-\rho_{ljk}+\rho_{kjl}}{(s-r)s} + \frac{\rho_{jkl}+\rho_{lkj}}{r(s-r)}\right) y_{l,s-r} \\
&+ \left(\frac{\rho_{jlk}-\rho_{klj}}{rs} + \frac{\rho_{ljk}+\rho_{kjl}}{(r-s)s} + \frac{\rho_{jkl}-\rho_{lkj}}{(r-s)r}\right) y_{l,r-s},
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\partial^2_{x_{jr}y_{ks}}\Phi_p(x,y) =& \frac{1}{r}\alpha_{jk}\delta_{rs} + \left(\frac{-\rho_{jkl}+\rho_{lkj}}{r(r+s)} - \frac{\rho_{kjl}+\rho_{ljk}}{s(r+s)} + \frac{\rho_{jlk}+\rho_{klj}}{rs}\right) x_{l,r+s} \\
&+ \left(\frac{\rho_{jlk}+\rho_{klj}}{rs} + \frac{\rho_{ljk}+\rho_{kjl}}{(s-r)s} - \frac{\rho_{jkl}+\rho_{lkj}}{r(s-r)}\right) x_{l,s-r} \\
&+ \left(-\frac{\rho_{jlk}+\rho_{klj}}{rs} + \frac{\rho_{ljk}+\rho_{kjl}}{(r-s)s} + \frac{-\rho_{lkj}+\rho_{jkl}}{(r-s)r}\right) x_{l,r-s},
\end{aligned} \tag{3.12}$$

where, again, the summation sign over the repeated index $l = 1,\cdots,q$ is omitted, and all x, y terms with second subscripts outwith the interval $[1,p]$ are assumed to vanish.

3.2 Estimates for the Derivatives of the Joint Density

With the gradient and the Hessian matrix of the phase function $\Phi_p(v)$ given above, one can apply the method of stationary phase to study the asymptotic behaviour of the oscillatory integral $\psi_p(\xi)$. A useful tool is provided in [51] (Lemma 0.4.7), and the first estimate given in the following lemma is a more quantitative version of that.

Lemma 3.1. *Let Ψ and φ belong to $C^\infty(\mathbb{R}^d)$ with* $\mathrm{supp}\varphi = \Omega$ *bounded. Then for all $\delta > 0$ and $K > 0$,*

$$\left|\int_\Omega e^{i|\xi|\Psi(x)}\varphi(x)\mathrm{d}x\right| \leqslant C|\varphi|_{K,\infty}|\Psi|_{K,\infty}^{-K}\delta^{-2K}|\xi|^{-K} + \int_{\Omega\setminus\Omega_\delta}|\varphi(x)|\mathrm{d}x,$$

where the constant C depends on d, K and $\Lambda^d(\Omega)$, $\Omega_\delta := \{x \in \Omega :\ |\nabla\Psi(x)| > \delta\}$, and

$$|\varphi|_{K,\infty} := \max_{n\leqslant K}\sup_{x\in\Omega}\|\mathrm{D}^n\varphi(x)\|.$$

Proof. It suffices to show that the integral on Ω_δ is bounded by the first term on the right hand side. For any fixed $K > 0$ write $M = |\Psi|_{K,\infty} \vee 1$ and further divide the set Ω_δ into several level sets of the gradient:

$$\Omega_r := \{x \in \Omega_\delta :\ 2^{-r}M \leqslant |\nabla\Psi(x)| \leqslant 2^{-r+1}M\},$$

for $r = 1, \cdots, r_0 := [\log_2(M/\delta)]$; there are at most $[\log_2(M/\delta)]+1$ non-empty Ω_r's. On each Ω_r, which is bounded, choose $\varepsilon_r = 2^{-r}M/(M+1)$ and let $N_r = N_r(d, \varepsilon_r)$ be the maximum number s.t. there are $x_1, \cdots, x_{N_r} \in \Omega_r$ so that the balls $B(x_j, \varepsilon_r/2)$ are all disjoint. Then the balls $\{B(x_j, \varepsilon_r)\}_j$ must cover Ω_r: if there is $x_* \in \Omega_r$ s.t. $|x_*-x_j| > \varepsilon_r$ for all j, then $B(x_*, \varepsilon_r/2)$ is disjoint from all other balls $B(x_j, \varepsilon_r)$ or those with half radius, which contradicts the maximality of N_r. Note that $\bigcup_{j=1}^{N_r} B(x_j, \varepsilon_r/2) \subset \Omega_r^{\varepsilon_r/2}$, the $\varepsilon_r/2$-neighbourhood of Ω_r, and therefore

$$N_r \leqslant \frac{\Lambda^d\left(\Omega_r^{\varepsilon_r/2}\right)}{\Lambda^d\left(B(x_j, \varepsilon_r/2)\right)} \leqslant C2^d\varepsilon_r^{-d}\Lambda^d\left(\Omega^{1/4}\right) \leqslant C\varepsilon_r^{-d},$$

where C is a constant depending on d and the size of Ω. This provides a finite open cover for the entire Ω_δ, and there exist non-negative functions $\alpha_{j,r} \in C_0^\infty(B(x_j, \varepsilon_r))$ that give a partition of unity (Theorem 1.4.5 in [22]): $\forall x \in \Omega_\delta$,

$$\sum_r\sum_{j=1}^{N_r}\alpha_{j,r}(x) = 1,$$

with derivatives satisfying $|\alpha_{j,r}|_{K,\infty} \leqslant C_{d,K}\varepsilon_r^{-K}$ for all K, j, r. For each j and r let $\widetilde{\Psi}_{j,r}(y) := M^{-1}\varepsilon_r^{-2}(\Psi(\varepsilon_r y+x_j)-\Psi(x_j))$. Then for each $y \in B(0,1)$, the point $\varepsilon_r y+x_j \in B(x_j, \varepsilon_r)$, and by Taylor's theorem, there is some $x' \in B(x_j, \varepsilon_r)$ s.t.

$$\begin{aligned}\left|\nabla\widetilde{\Psi}_{j,r}(y)\right| = M^{-1}\varepsilon_r^{-1}|\nabla\Psi(\varepsilon_r y + x_j)| &\geqslant M^{-1}\varepsilon_r^{-1}|\nabla\Psi(x_j)| - \frac{1}{2}M^{-1}\|\mathrm{D}^2\Psi(x')\| \\ &\geqslant \varepsilon_r^{-1}2^{-r} - \frac{1}{2} > \frac{1}{2}.\end{aligned}$$

Since each $x_j \in \Omega_r$, one applies Taylor's theorem again to get, for all $y \in B(0,1)$ and some $x'' \in B(x_j, \varepsilon_r)$,

$$\left|\widetilde{\Psi}_{j,r}(y)\right| \leqslant M^{-1}\varepsilon_r^{-1}\left|\nabla\Psi(x_j)\right| + \frac{1}{2}M^{-1}\left\|\mathrm{D}^2\Psi(x'')\right\| \leqslant \varepsilon_r^{-1}2^{-r+1} + \frac{1}{2} \leqslant \frac{9}{2};$$

the same argument gives the same upper bound for $|\nabla\widetilde{\Psi}_{j,r}(y)|$. For all $n \geqslant 2$, one also has $\|\mathrm{D}^n\widetilde{\Psi}_{j,r}(y)\| \leqslant M^{-1}\varepsilon_r^{n-2}\|\mathrm{D}^n\Psi(x_j)\| \leqslant 1$. Therefore $\widetilde{\Psi}_{j,r}$ is in a bounded subset of $C^\infty(B(0,1))$.

Now that each function $\varphi_{j,r} := \alpha_{j,r}\varphi$ is supported on the ball $B(x_j, \varepsilon_r)$, the function $\psi_{j,r}(y) := \varphi_{j,r}(\varepsilon_r y + x_j)$ is then supported on $B(0,1)$, satisfying $|\psi_{j,r}|_{d,K} \leqslant C_{d,K}$ for all K, j, r. Hence using the same arguments as in the proof of Lemma 0.4.7 in [51], one arrives at:

$$\begin{aligned}\left|\int_{B(x_j,\varepsilon_r)} e^{i|\xi|\Psi(x)}\varphi_{j,r}(x)\mathrm{d}x\right| =& \varepsilon_r^d\left|\int_{B(0,1)} e^{iM\varepsilon_r^2|\xi|\widetilde{\Psi}_{j,r}(y)}\varphi_{j,r}(\varepsilon_r y + x_j)\mathrm{d}y\right| \\ \leqslant& C_{d,K}|\varphi|_{K,\infty}M^{-K}\varepsilon_r^{d-2K}|\xi|^{-K}.\end{aligned}$$

Finally, since $\mathrm{supp}\varphi = \Omega$, by the triangle inequality one deduces,

$$\begin{aligned}\left|\int_{\Omega_\delta} e^{i|\xi|\Psi(x)}\varphi(x)\mathrm{d}x\right| \leqslant& \sum_r\sum_{j=1}^{N_r}\left|\int_{B(x_j,\varepsilon_r)} e^{i|\xi|\Psi(x)}\varphi_{j,r}(x)\mathrm{d}x\right| \\ \leqslant& C|\varphi|_{K,\infty}M^{-K}\sum_r N_r\varepsilon_r^{d-2K}|\xi|^{-K} \\ \leqslant& C|\varphi|_{K,\infty}M^{-K}|\xi|^{-K}\sum_r \varepsilon_r^{-2K} \\ \leqslant& C|\varphi|_{K,\infty}M^{-K}\delta^{-2K}|\xi|^{-K},\end{aligned}$$

where C is a constant depending on d, K and $\Lambda^d(\Omega)$. The estimate on $\Omega \setminus \Omega_\delta$ is trivial. $\square$

The result above is to be applied to $\Psi = \Phi_p$ and $\Omega_\delta = \{v \in \Omega,\ |\nabla\Phi_p(v)| > \delta\}$. For the characteristic function ψ_p to have an appropriate rate of decay, one needs to show that the measure $\Lambda^d(\Omega \setminus \Omega_\delta)$ is also small, but it is more intricate to give an explicit estimate. One can start with the special case where the Hessian matrix $\mathrm{D}\Phi_p$ has certain eigenvalues that are not too small, by using the following general fact.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $f : \Omega \to \mathbb{R}^{d'}$ be a C^1 function. For each x, let $\sigma_1(x) \geqslant \sigma_2(x) \geqslant \cdots \geqslant \sigma_{d\wedge d'}(x)$ be the singular values of its derivative $\mathrm{D}f(x)$. For any $n \in [1, d \wedge d'] \cap \mathbb{N}$ and $\eta > 0$, define $G_{n,\eta}(f) := \{x \in \Omega : \sigma_n(x) > \eta\}$. If $\mathrm{D}f$ is Lipschitz continuous with Lipschitz constant L, then $\forall\delta > 0$,*

$$\Lambda^d(G_{n,\eta}(f) \cap \{|f| \leqslant \delta\}) \leqslant C\eta^{-2n}\delta^n,$$

where the constant C depends on d, d', L and $\Lambda^d(\Omega)$.

Proof. For fixed n, η and any $z \in G_{n,\eta}$, by definition the matrix $\mathrm{D}f(z)$ has rank n. This implies that for each z there are n-dimensional subspaces E_z of $\mathbb{R}^d$ and F_z of $\mathbb{R}^{d'}$ s.t., with $g_z(\cdot) := \pi_{F_z} \circ f|_{E_z}(\cdot)$ and $\pi_\cdot$ being the orthogonal projection, the linear map $\mathrm{D}g_z(z)$ is invertible. Denote by $E_z^\perp$ the orthogonal complement of E_z for each z.

By the continuity of $\mathrm{D}f$ the set $G_{n,\eta}(f)$ is open, and the inverse function theorem implies that g_z is a diffeomorphism in some neighbourhood[1] $B^{(n)}(z,\varepsilon) \subset E_z$. Moreover, in the proof of the inverse function theorem (see, e.g., Theorem 9.24 in [47] or Theorem 1.1.7 in [22]), the ball $B^{(n)}(z,\varepsilon)$ can be typically constructed with radius $\varepsilon \leqslant 1/(2L\|(\mathrm{D}g_z(z))^{-1}\|) \leqslant \|\mathrm{D}f(z)\|/(2L)$. As $z \in G_{n,\eta}(f)$, one can choose e.g. $\varepsilon = \eta/(4L) \wedge 1$.

Since $G_{n,\eta}(f)$ is bounded, similar to the proof of Lemma 3.1 there are finitely many points $z_1, \cdots, z_{N_\varepsilon} \in G_{n,\eta}(f)$ s.t. $G_{n,\eta}(f) \subset \bigcup_{j=1}^{N_\varepsilon} B(z_j,\varepsilon)$, with the number of balls satisfying

$$N_\varepsilon \leqslant \frac{\Lambda^d\left(G_{n,\eta}^{\varepsilon/2}(f)\right)}{\Lambda^d\left(B(z_j,\varepsilon/2)\right)} \leqslant C2^d\varepsilon^{-d}\Lambda^d\left(\Omega^{1/2}\right) \leqslant C\varepsilon^{-d},$$

for some constant C depending on d and $\Lambda^d(\Omega)$. Write $B_j = B(z_j,\varepsilon) \cap G_{n,\eta}(f)$ and let $B_{j,\delta}^{(n)}$, $B_{j,\delta}^{(d-n)}$ be the projections of $B_j \cap \{|f| \leqslant \delta\}$ onto E_{z_j}, $E_{z_j}^{\perp}$, respectively. Notice that all the eigenvalues of $\mathrm{D}g_z$ are greater than η on $B_{j,\delta}^{(n)}$. Then, by a change of coordinates and variables,

$$\begin{aligned}
&\Lambda^d\left(G_{n,\eta}(f) \cap \{|f| \leqslant \delta\}\right) \leqslant \sum_{j=1}^{N_\varepsilon} \int_{B_j \cap \{|f| \leqslant \delta\}} \mathrm{d}x_1 \cdots \mathrm{d}x_d \\
&= \sum_{j=1}^{N_\varepsilon} \int_{B_{j,\delta}^{(d-n)}} \mathrm{d}x_{k+1} \cdots \mathrm{d}x_d \int_{B_{j,\delta}^{(n)}} \mathrm{d}x_1 \cdots \mathrm{d}x_k \\
&= \sum_{j=1}^{N_\varepsilon} \int_{B_{j,\delta}^{(d-n)}} \mathrm{d}x_{k+1} \cdots \mathrm{d}x_d \int_{g_{z_j}(B_j) \cap \{|y| \leqslant \delta\}} \left|\mathrm{det}\mathrm{D}g_{z_j}^{-1}(y)\right| \mathrm{d}y_1 \cdots \mathrm{d}y_n \\
&\leqslant C\delta^n \left(\min_j \inf_{x \in B_{j,\delta}^{(n)}} \left|\mathrm{det}\mathrm{D}g_{z_j}(x)\right|\right)^{-1} \sum_{j=1}^{N_\varepsilon} \Lambda^{d-n}\left(B^{(d-n)}(z_j,\varepsilon)\right) \\
&\leqslant C\eta^{-n}\delta^n N_\varepsilon \varepsilon^{d-n},
\end{aligned}$$

where the constant C depends on d, d' and $\Lambda^d\Omega$. Then the result follows from the bound for N_ε and the choice of ε. □

Now write $G_{n,\eta} = G_{n,\eta}(\nabla\Phi_p)$ as defined in Lemma 3.2 with $d = d' = 2qp$, and one needs to estimate the measure of the complement $\Lambda^{2qp}(\Omega \setminus G_{n,\eta})$ for suitable values of η and $n \leqslant 2qp$. However, the behaviour of the second derivatives, according to (3.10), (3.11) and (3.12), depends on the magnitude of the parameter ρ. One may first deal with the case where ρ is not too small.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^{2qp}$ be bounded and $\mathbb{Z}^+ \ni n \leqslant \sqrt{2p}/4$. If $\|\rho\| > \varepsilon$ for some fixed $\varepsilon > 0$, then one has $\Lambda^{2qp}(\Omega \setminus G_{n,\eta}) \leqslant C\varepsilon^{-n}n^{1+n/2}\eta^n$, where C is a constant depending on q, p and the size of Ω.*

Proof. It suffices to focus on a submatrix of $\mathrm{D}^2\Phi_p$ since $\widetilde{G}_{n,\eta} \subset G_{n,\eta}$ where $\widetilde{G}_{n,\eta}$ is similarly defined by the singular values of the submatrix. Since $\|\rho\| > \varepsilon$, fix the pair (j,k) for which $|\rho_{jkl}| \geqslant \varepsilon\sqrt{3/(q^3-q)}$. For a particular pair (r,s), observe from (3.10) that $\partial^2_{x_{jr}x_{ks}}\Phi_p(x,y)$ contains all the permutations of the (Lyndon) word (j,k,l); since all non-Lyndon entries of ρ are defined to be 0, only one of ρ_{kjl} and ρ_{jlk} may not

[1]The superscript (n) indicates that it is a ball in the $\mathbb{R}^n$. Balls without superscripts lie in the whole space $\mathbb{R}^d$.

vanish. Notice that, in the coefficients of $y_{l,r+s}$, $y_{l,r-s}$ and $y_{l,s-r}$, the denominator of either ρ_{kjl} or ρ_{jlk} cannot simultaneously coincide with that of ρ_{jkl}, so the coefficients of the y-terms for each l are not all zero. The summation in $l = 1, \cdots, q$ in (3.10) then gives a linear combination of q different entries of $y_{l,r+s}, y_{l,r-s}$ and $y_{l,s-r}$.

For integers $n \leqslant m \leqslant \sqrt{p/2} - 1$, one can choose $r_1, \cdots, r_m, s_1, \cdots, s_m \leqslant p$ s.t. the integers $r_a + s_b$ and $|r_c - s_d|$ are different for all choices of $a, b, c, d = 1, \cdots, m$. For example, one can choose $r_a = a$, $s_a = a(2m+1)$. In this case, the only choice of (a, b, c, d) s.t. $r_a + s_b = r_c + s_d$, i.e. $(c-a) + (d-b)(2m+1) = 0$, is that $a = c$ and $b = d$; it is the same for $r_a - s_b = r_c - s_d$; there is no choice of (a, b, c, d) for the equation $(a+c) + (b-d)(2m+1) = 0$ to hold so $r_a + s_b = s_c - r_d$ is never satisfied. Since we require all $r_a + s_b$ and $|r_c - s_d|$ are no greater than p, it is necessary that $\max_{a,b}(r_a + s_b) = 2m(m+1) \leqslant p$.

Thus one obtains an $m \times m$ submatrix $Q_m(y)$ of $H_{xx}(j,k)$ whose entries take the form (3.10) involving m^2 different linear combinations of distinct entries of the vector y. Denote the rows of $Q_m(y)$ by $q_1(y), \cdots, q_m(y)$, and define

$$F_j := \{y : \ \mathrm{dist}(q_j, \ \mathrm{span}\{q_l : \ l = 1, \cdots, n, \ l \neq j\}) > n^{1/2}\eta\}, \ j = 1, \cdots, n.$$

Then the exceptional set has measure $\Lambda^{qp}(\Omega \setminus F_j) \leqslant C(\varepsilon^{-1}n^{1/2}\eta)^{m-n+1}$ where C depends on q, p and the size of Ω, and $Q_m(y)$ has rank at least n for $y \in \bigcap_{j=1}^n F_j$.

For each $y \in \bigcap_{j=1}^n F_j$ and $|a| = |(a_1, \cdots, a_n)| = 1$, consider any linear combination $a \cdot (q_1(y), \cdots, q_n(y))$ of the rows. Choose j s.t. $|a_j| = \max\{|a_1|, \cdots, |a_n|\} \geqslant 1/\sqrt{n}$, then

$$|a_1 q_1(y) + \cdots + a_n q_n(y)| = |a_j| \left| q_j(y) + \sum_{l \neq j} a_j^{-1} a_l q_l(y) \right| \geqslant \eta.$$

Thus, the $n \times m$ submatrix $\widetilde{Q}_n(y) := (q_1(y)^\top, \cdots, q_n(y)^\top)^\top$ has a right inverse $R_n(y)$ on a n-dimensional subspace E of $\mathbb{R}^m$, and

$$\|R_n(y)\| = \sup_{|a|=1} |R_n(y)a| \leqslant \left(\inf_{|a|=1} |R_n(y)a| \right)^{-1} \leqslant \eta^{-1}.$$

Then the matrix $\widetilde{Q}_n(y)$ has singular values bounded from below by $\|R_n(y)^{-1}\|^{-1} \geqslant \eta$, which in turn gives an estimate for the measure of the exceptional set:

$$\Lambda^{2qp}(\Omega \setminus G_{n,\eta}) \leqslant \Lambda^{qp}\left(\Omega \setminus \widetilde{G}_{n,\eta}\right) \leqslant \Lambda^{qp}\left(\bigcup_{j=1}^n (\Omega \setminus F_j)\right) \leqslant Cn(\varepsilon^{-1}n^{1/2}\eta)^{m-n+1},$$

and the result then follows by taking $m = 2n - 1$. □

The result of Lemma 3.3 is meaningful for small values of η. It remains to show that the measure $\Lambda^{2qp}(\Omega \setminus G_{n,\eta})$ is also small when ρ is small.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^{2qp}$ be bounded and $n \in \mathbb{Z}^+$. Then, depending on q, p, n and the size of Ω, one can choose $\varepsilon, \delta, \eta > 0$ sufficiently small s.t. for $\|\rho\| \leqslant \varepsilon$, either $\Omega_\delta = \Omega$ or $G_{n,\eta} = \Omega$.*

Proof. For $\varepsilon \in (0,1)$ define $\varepsilon' = \sqrt{1 - \varepsilon^2}$, and assume $\mathrm{diam}(\Omega) = 1$ w.l.o.g., otherwise replace ε with $\varepsilon/(1 \vee \mathrm{diam}(\Omega))$ for all the arguments below. First of all that $\|\rho\| \leqslant \varepsilon$ implies that $|(\alpha, \beta^{(1)}, \beta^{(2)}, \gamma, a, b)| \geqslant \varepsilon'$. If $|(\alpha, \beta^{(1)}, \beta^{(2)}, \gamma)| \leqslant \varepsilon\varepsilon'$, then the constants

(a,b) are dominant with $|(a,b)| \geqslant (\varepsilon')^2$, and immediately from the first derivatives (3.6) one sees that

$$\begin{aligned}\left|\partial_{x_{jr}}\Phi_p(v)\right|^2 \geqslant & \frac{1}{r^2}a_j^2 - \frac{2}{r}|a_j||Q_{x_{jr}}(\rho,v)| - 2\left(\frac{1}{r}|a_j| + |Q_{x_{jr}}(\rho,v)|\right) \\ & \cdot \left(\frac{1}{r}|\alpha_{jk}||y_{kr}| + \frac{2}{r^2}|\beta^{(1)}_{jk}||x_{kr}| + \frac{p}{2r-1}(|\gamma_{jk}| + |\gamma_{kj}|)\sum_{s\neq r}|x_{ks}|\right),\end{aligned}$$

where $Q_{x_{jr}}(\rho,v)$ denotes the quadratic terms in (3.6) and the summation over the repeated indices k is omitted. Since x and y are bounded, one has

$$|Q_{x_{jr}}(\rho,v)| \leqslant C_q|\rho|\frac{1}{r}\left(\sum_{s=1}^{p-r}\frac{1}{s} + \sum_{s=1}^{r-1}\frac{1}{s}\right) \leqslant C_q\frac{\varepsilon}{r}\log p,$$

and hence one derives

$$|\partial_{x_{jr}}\Phi_p(v)|^2 \geqslant \frac{1}{r^2}a_j^2 - C_q\frac{\varepsilon^2\log p}{r^2}|a_j| - C_q\varepsilon\varepsilon'\frac{1}{r}(|a_j| + \varepsilon\log p),$$

and a similar inequality for $|\partial_{y_{jr}}\Phi_p|^2$ with a_j/r replaced with b_j/r^2 as per (3.7). Thus,

$$\begin{aligned}|\nabla\Phi_p(v)|^2 \geqslant & |(a,b)|^2 - C_q\log p\left(\varepsilon^2 + \varepsilon\varepsilon'\right)|(a,b)| - C_q\varepsilon^2\varepsilon'\log p \\ \geqslant & (\varepsilon')^4 - C_q(\varepsilon^2(\varepsilon')^2 + \varepsilon(\varepsilon')^3 + \varepsilon^2\varepsilon')\log p \\ \geqslant & (1-\varepsilon^2)^2 - C_q\varepsilon\log p,\end{aligned}$$

which is close to 1 for ε sufficiently small. Then for small values of δ, $\Omega\setminus\Omega_\delta = \varnothing$.

Now suppose that $|(a,b)| \leqslant \varepsilon\varepsilon'$, then the entries $|(\alpha,\beta^{(1)},\beta^{(2)},\gamma)| \geqslant (\varepsilon')^2$ are dominant, corresponding to the constant terms in the second derivative $\mathrm{D}^2\Phi_p(v)$. Write $\mathrm{D}^2\Phi_p(v) = A_p + L_p(v)$ according to (3.10), (3.11) and (3.12), where A_p and $L_p(v) = (L_{x_{jr}x_{ks}}, L_{y_{jr}y_{ks}}, L_{x_{jr}y_{ks}})(v)$ are the constant and linear parts, respectively. Then for each (j,k) and (r,s),

$$\sup_{v\in\Omega}|L_{x_{jr}x_{ks}}(v)| \leqslant C_q\|\rho\|\left(\frac{1}{rs} + \frac{\delta_{rs}}{r|r-s|} + \frac{\delta_{rs}}{s|r-s|}\right) \leqslant C_q\varepsilon,$$

and the same bound holds for $L_{y_{jr}y_{ks}}$ and $L_{x_{jr}y_{ks}}$. Let $H_n(v) = A_n + L_n(v)$ be an $n\times n$ submatrix of $\mathrm{D}^2\Phi_p(v)$ with constant part A_n and linear part $L_n(v)$, $\|L_n(v)\| \leqslant C_q\varepsilon$ for all $v\in\Omega$. By definition, $\det H_n(v) = \det A_n + P_n(\rho,v)$, where $P_n(\rho,v)$ is a polynomial of which each monomial has positive degrees in the components of ρ and v. Then one has $|\det H_n(v)| \geqslant |\det A_n| - C_q\varepsilon$ for all $v\in\Omega$. If A_n is invertible with $\|A_n^{-1}\| \leqslant 1/\eta$, then $\|L_n(v)\| < \|A_n^{-1}\|^{-1}$ for $\varepsilon < \eta$ and

(a) $|\det A_n| \geqslant \|A_n^{-1}\|^{-n} \geqslant \eta^n$, and so H_n is invertible for $\varepsilon \lesssim \eta^n$ sufficiently small;

(b) $\|H_n^{-1}(v)\| \leqslant \|A_n^{-1}\|\|(I + A_n^{-1}L_n(v))^{-1}\| \leqslant \|A_n^{-1}\|/(1 - \|A_n^{-1}\|\|L_n(v)\|) \leqslant 1/\eta$ for all $v\in\Omega$.

This implies that $\mathrm{D}^2\Phi_p(v)$ has at least n singular values no less than η for all $v\in\Omega$, in other words, $\Omega\setminus G_{n,\eta} = \varnothing$. Henceforth, one looks for an invertible $n\times n$ submatrix A_n of A_p and chooses appropriate values of η and ε so that $\|A_n^{-1}\| \leqslant 1/\eta$.

Write $D_n = \mathrm{diag}(1, 1/2, \cdots, 1/n)$, $n \leqslant p$ for simplicity. If the entries of α are dominant with $\|\alpha\| \geqslant (\varepsilon')^3$, choose the largest entry $|\alpha_{jk}| \geqslant c_q(\varepsilon')^3$ where $c_q = \sqrt{2/q(q-1)}$.

Then the constant part of the n-th principle submatrix of the block $H_{xy}(j,k)$ is $A_n = \alpha_{jk} D_n$, and $\|A_n^{-1}\| \leqslant |\alpha_{jk}|^{-1} n \leqslant c_q^{-1}(\varepsilon')^{-3} n$. For this case choose $\eta \leqslant c_q(\varepsilon')^3/n$.

For the case where the diagonal terms of $(\beta^{(1)}, \beta^{(2)}, \gamma)$ are dominant with $|\beta_{jj}^{(i)}| \geqslant c_q(\varepsilon')^3,\ i = 1,2$ (recall that $\gamma_{jj} = 0$), the constant part of the n-th principle submatrix of the block $H_{xx}(j,j)$ or the block $H_{yy}(j,j)$ is $A_n^{(i)} = 2\beta_{jj}^{(i)} D_n^2,\ i = 1,2$. Then choose $\eta \leqslant 2c_q(\varepsilon')^3/n^2$.

When the off-diagonal terms of the components $(\beta^{(1)}, \beta^{(2)}, \gamma)$ are dominant, choose the dominant pair (j,k) as before and assume $j < k$ w.l.o.g. Then the constant part of the n-th principle submatrix of the block $H_{yy}(j,k)$ takes the form

$$A_n^{(2)} = \begin{pmatrix} \beta_{jk}^{(2)} & \frac{1}{3}\gamma_{jk} & \frac{1}{8}\gamma_{jk} & \cdots & \frac{1}{n^2-1}\gamma_{jk} \\ -\frac{1}{3}\gamma_{jk} & \frac{1}{4}\beta_{jk}^{(2)} & \frac{1}{5}\gamma_{jk} & \cdots & \frac{1}{n^2-4}\gamma_{jk} \\ -\frac{1}{8}\gamma_{jk} & -\frac{1}{5}\gamma_{jk} & \frac{1}{9}\beta_{jk}^{(2)} & \cdots & \frac{1}{n^2-9}\gamma_{jk} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{n^2-1}\gamma_{jk} & -\frac{1}{n^2-4}\gamma_{jk} & -\frac{1}{n^2-9}\gamma_{jk} & \cdots & \frac{1}{n^2}\beta_{jk}^{(2)} \end{pmatrix} = \beta_{jk}^{(2)} D_n^2 + \gamma_{jk} S_n,$$

where S_n is the skew-symmetric matrix with (r,s)-th entry $(s^2 - r^2)^{-1},\ r \neq s$ and 0 on the diagonal. Similarly, the constant part of the n-th principle submatrix of the block $H_{xx}(j,k)$ takes the form $A_n^{(1)} = \beta_{jk}^{(1)} D_n^2 + \gamma_{jk} S_n'$ where S_n' is the matrix with (r,s)-th entry $(r^2 - s^2)^{-1} r/s$.

If $|\beta_{jk}^{(2)}| \geqslant c_q(\varepsilon')^3$, then $A_n^{(2)}$ has full rank. To see this, notice that the matrix $\bar{S}_n := D_n^{-1} S_n D_n^{-1}$ is also skew-symmetric, whose eigenvalues are all purely imaginary. Then all the eigenvalues of the scaled matrix $\bar{A}_n^{(2)} := I + \gamma_{jk}\bar{S}_n/\beta_{jk}^{(2)}$ have real parts 1, implying that $\|(\bar{A}_n^{(2)})^{-1}\| \leqslant 1$. Therefore one has

$$\left\|(A_n^{(2)})^{-1}\right\| = \left\|\left(\beta_{jk}^{(2)} D_n \bar{A}_n^{(2)} D_n\right)^{-1}\right\| \leqslant |\beta_{jk}^{(2)}|^{-1} n^{-2},$$

and chooses $\eta \leqslant c_q(\varepsilon')^3/n^2$.

The same argument applies to the case where $|\beta_{jk}^{(1)}| \geqslant c_q(\varepsilon')^3$, since $S_n' = D_n S_n D_n^{-1}$ and all the eigenvalues of the scaled matrix $\bar{A}_n^{(1)} := I + \gamma_{jk} S_n D_n^{-2}/\beta_{jk}^{(1)}$ also have real parts 1.

Finally, if $|(\alpha, \beta^{(1)}, \beta^{(2)})| \leqslant \varepsilon(\varepsilon')^2$, i.e. there is a $|\gamma_{jk}| \geqslant c_q(\varepsilon')^3$, then $A_n = \gamma_{jk} S_n$. Since S_n is skew-symmetric, $\det S_n = 0$ for all odd n. If n is even, by definition the determinant of S_n is given by the expansion

$$\det S_n = \sum_{\sigma \in \Pi_n} \operatorname{sgn}(\sigma) \frac{1}{1 - \sigma_1^2} \frac{1}{4 - \sigma_2^2} \cdots \frac{1}{n^2 - \sigma_n^2},$$

where Π_n is the symmetric group of order n. Notice that this summation includes the product of all the entries on the reflected diagonal $r + s = n + 1$, each of which has denominator divisible by $n+1$. Clearly, out of all the permutations of the set $\{1, \cdots, q\}$, this product is the only term in the above expansion whose denominator is divisible by $(n+1)^n$ if $n+1$ is prime. Then it follows from the fundamental theorem of arithmetic that $\det S_n \neq 0$. Denote $\text{þ}_n := \|S_n^{-1}\|$ and then one can choose $\eta \leqslant c_q(\varepsilon')^3 \text{þ}_n^{-1}$.

Combining all the criteria above, one can choose $\varepsilon \lesssim \text{þ}_n^{-n} \wedge n^{-2n}$ with $\varepsilon' > 1/2$, and then the result holds for $\eta \lesssim \text{þ}_n^{-1} \wedge n^{-2}$ and δ sufficiently small. $\square$

To get a global estimate for $|\psi_p(\xi)|$, first choose a non-negative, smooth cut-off function $\zeta_0 \in C_0^\infty(B(0,2))$ s.t. $\zeta_0 \equiv 1$ on $B(0,1)$ and its derivatives are bounded on $B(0,2) \setminus B(0,1)$. Divide the rest of $\mathbb{R}^{2qp}$ by the annuli

$$A_r := \{u \in \mathbb{R}^{2qp} : \; 2^{r-1} \leqslant |u| < 2^r\},$$

for $r \in \mathbb{N}$, and define $A'_r := \{2^{r-2} \leqslant |u| < 2^{r+1}\}$. Choose another non-negative, smooth cut-off $\zeta_1 \in C_0^\infty(A'_1)$ taking value 1 on A_1 and bounded derivatives on $A'_1 \setminus A_1$, and define $\zeta_r(u) := \zeta_1(2^{-r+1}u)$, $\forall r \geqslant 2$. Then for each $r \geqslant 1$, the smooth function ζ_r is supported on $A'_r := \{2^{r-2} \leqslant |u| < 2^{r+1}\}$ with value 1 on A_r and bounded derivatives on $A'_r \setminus A_r$; the sum $\sigma(u) := \sum_{r=0}^\infty \zeta_r(u)$ is then supported on the whole of $\mathbb{R}^{2qp}$.

If one further sets $\widetilde{\zeta}_r(u) := \zeta_r(u)/\sigma(u)$, then each $\widetilde{\zeta}_r$ has the same properties as ζ_r, and $\sum_{r=0}^\infty \widetilde{\zeta}_r \equiv 1$ trivially. Therefore, one can write

$$\begin{aligned}\psi_p(\xi) &= \int_{\mathbb{R}^{2qp}} e^{i|\xi|\Phi_p(u)}\phi_p(u)\sum_{r=0}^\infty \widetilde{\zeta}_r(u)\mathrm{d}u \\ &= \int_{B(0,2)} e^{i|\xi|\Phi_p(u)}\phi_p(u)\widetilde{\zeta}_0(u)\mathrm{d}u + \sum_{r=1}^\infty \int_{A'_r} e^{i|\xi|\Phi_p(u)}\phi_p(u)\widetilde{\zeta}_r(u)\mathrm{d}u,\end{aligned}$$

where the first integral can be readily estimated by the lemmas above, since the vector $(\alpha, \beta, \gamma, a, b, \rho)$ is normalised and all the derivatives of Φ_p and ϕ_p uniformly are bounded on $\Omega = B(0,2)$. By choosing $\eta = \delta^{1/4}$ and $\delta = |\xi|^{-1/4}$, for $\varepsilon \lesssim \mathrm{þ}_n^{-n} \wedge n^{-2n}$ one achieves

$$\begin{aligned}\left|\int_{B(0,2)} e^{i|\xi|\Phi_p(u)}\phi_p(u)\widetilde{\zeta}_0(u)\mathrm{d}u\right| &\leqslant C_{q,p}\left(|\xi|^{-\frac{1}{2}K} + |\xi|^{-\frac{1}{8}n} + C_n|\xi|^{-\frac{1}{16}n}\right) \\ &\leqslant C_{q,p,n}|\xi|^{-\frac{1}{16}K},\end{aligned}$$

for $|\xi|$ sufficiently large and $n > K$, and hence for $p > 8K^2$ by choosing $n = [\sqrt{2p}/4]$.

For each $r \geqslant 1$, let $\widetilde{\Phi}_p(v) := 2^{-16r}\Phi_p(2^r v)$ and one has $|\widetilde{\Phi}_p|_{K,\infty} \leqslant 1$ over the annulus $A'_0 \subset B(0,2)$ for any $K \geqslant 0$, as it is a cubic polynomial. Thus, by the rapid decay of the Gaussian density ϕ_p,

$$\begin{aligned}\left|\int_{A'_r} e^{i|\xi|\Phi_p(u)}\phi_p(u)\widetilde{\zeta}_r(u)\mathrm{d}u\right| &= 2^{2qpr}\left|\int_{A'_0} e^{i2^{16r}|\xi|\widetilde{\Phi}_p(v)}\phi_p(2^r v)\widetilde{\zeta}_r(2^r v)\mathrm{d}v\right| \\ &\leqslant C_{q,p}\left|\int_{A'_0} e^{i2^{16r}|\xi|\widetilde{\Phi}_p(v)}\frac{\zeta_1(2v)}{\sigma(2^r v)}\mathrm{d}v\right|.\end{aligned}$$

This integral can be again estimated by the lemmas above, but with

$$|\varphi|_{K,\infty} = |\zeta_1(2\cdot)/\sigma(2^r\cdot)|_{K,\infty} \simeq 2^{rK}.$$

Then by the previous estimate, one gets a bound $C_{q,p}2^{rK}(2^{16r}|\xi|)^{-\frac{1}{16}K} = C_{q,p}|\xi|^{-\frac{1}{16}K}$ for $p > 8K^2$ and $|\xi|$ sufficiently large.

Combining all the estimates above together, for any $K > 0$, one concludes that $|\psi_p(\xi)| \leqslant C_{q,p}|\xi|^{-\frac{1}{16}K}$ for $p > 8K^2$. Then for any given $N > 0$, by the inversion formula, the density f_p of $\zeta^{(p)}$ has continuous and bounded derivatives up to order N for $p > p_0 = 8(N + 2q^2 + 2q + (q^3 - q)/3)^2$. The question remains whether those bounds necessarily depend on p instead of p_0 only.

Write $v_p = \{(x_{jr}, y_{ks}) : \; j,k = 1,\cdots,q; \; r,s = 1,\cdots,p\}$ and similarly v_{p_0}. Denote $p' = p - p_0$ and write $v_{p'} := \{(x_{jr}, y_{ks}) : \; j,k = 1,\cdots,q; \; r,s = p_0+1,\cdots,p\}$, then conditional on $v_{p'}$ the characteristic function ψ_p can be written as

$$\begin{aligned}\psi_p(\xi) &= \int_{\mathbb{R}^{2qp'}} \widetilde{\phi}_{p'}(v_{p'})\mathrm{d}v_{p'} \int_{\mathbb{R}^{2qp_0}} e^{i|\xi|\left(\Phi_{p_0}(v_{p_0})+\widetilde{\Phi}_{p|p'}(v_{p_0},v_{p'})\right)}\phi_{p_0}(v_{p_0})\mathrm{d}v_{p_0} \\ &=: \int_{\mathbb{R}^{2qp'}} \psi_{p|p'}(\xi)\widetilde{\phi}_{p'}(v_{p'})\mathrm{d}v_{p'},\end{aligned}$$

where $\widetilde{\phi}_{p'}(v_{p'}) = \phi_p(v_p)/\phi_{p_0}(v_{p_0}) = \prod_{j=1}^{q}\prod_{r=p_0+1}^{p}\phi(x_{jr})\phi(y_{jr})$ and $\widetilde{\Phi}_{p|p'}(v_{p_0}, v_{p'}) = \Phi_p(v_p) - \Phi_{p_0}(v_{p_0})$. The function $\widetilde{\Phi}_{p|p'}$ is then a quadratic polynomial in v_{p_0}, i.e. $\mathrm{D}^2\widetilde{\Phi}_{p|p'} \equiv C_{p'}$ is a constant depending on $v^{(p')}$. If one can show that $|\xi|^{p_0}|\psi_{p|p'}(\xi)|$ is bounded by a constant independent of $p > p_0$ then so is $|\xi|^{p_0}|\psi_p(\xi)|$ by the rapid decay of the Gaussian density $\widetilde{\phi}_{p'}$.

Using the same cut-off arguments, it suffices to focus on the case where ϕ_p is compactly supported on $\Omega = B(0,2)$, and Lemma 3.1 can be readily applied to $\psi_{p|p'}$ with the first bound only dependent on p_0; for the estimate for $\Lambda^{2qp_0}(\Omega \cap \{|\nabla\Phi_p| \leqslant \delta\})$, Lemma 3.2 also applies directly and gives a bound depending only on p_0, since the Lipschitz constant of $\mathrm{D}f = \mathrm{D}^2\Phi_p$ remains unchanged (and hence the ε therein) when adding a constant $C_{p'}$ to $\mathrm{D}^2\Phi_{p_0}$. Finally, the estimate given by Lemma 3.3 should also be independent of p. The difference here is that in the proof of Lemma 3.3 a constant vector $c_{j,p'}$ is added to each row $q_j(y)$ of the submatrix $Q_m(y)$, and the sets F_j are replaced by

$$F'_j = \{y : \; \mathrm{dist}(q_j + c_{j,p'}, \; \mathrm{span}\{q_l + c_{l,p'} : \; j \neq l = 1,\cdots,n\}) > n^{1/2}\eta\}.$$

Then geometrically each F'_j is just a translated copy of F_j, whose volume remains the same. And hence one claims the following:

Theorem 3.5. *The density f_p of V_p has continuous and uniformly bounded derivatives up to order N if $p > p_0 = 8(N + 2q^2 + 2q + (q^3 - q)/3)^2$ is an even integer s.t. $p+1$ is prime.*

This is an analogue of part (1) of Lemma 11 in [8]. It is not clear whether part (2) of that lemma is also true. In fact, whether the moments of the variables V_p and $\widetilde{V}_p$ are bounded is not clear, either. Some potential implications of Theorem 3.5 will be discussed in the next section.

3.3 Remaining Difficulties and Limitations

For simplicity denote the dimension of V by $d = 2q^2 + 2q + (q^3 - q)/3$. Following Davie's idea presented in Section 9 in [8], one needs to give some suitable estimates for the moments of V_p and $\widetilde{V}_p$, and an analogue of Theorem 15 therein would give a coupling for I°_{jkl} up to some appropriate order.

Lemma 3.6. *For any $m \geqslant 2$, the m-th moments of the random variables $\widetilde{z}^{(p)}$, $\widetilde{\lambda}^{(p)}$ and $\widetilde{\nu}^{(p)}$ are of order $O(p^{-m/2})$, and those of the terms $\widetilde{u}^{(p)}$, $\widetilde{\mu}^{(1,p)}$ and $\widetilde{\mu}^{(2,p)}$ are of order $O(p^{-3m/2})$.*

Proof. The moment bounds for $\widetilde{z}^{(p)}$ and $\widetilde{\lambda}^{(p)}$ are implied by part (2) of Lemma 11 in [8]. For the other terms, one simply derives such bounds for each component. Notice that each of them is an infinite sum of independent random variables. Consider $\widetilde{u}_j^{(p)}$ for instance: for $m \geqslant 2$ and any $N > p$, one applies Rosenthal's inequality (see e.g. Lemma 1 in [14]) to get

$$\begin{aligned}\mathbb{E}\left|\sum_{r=p+1}^{N} \frac{1}{r^2} y_{jr}\right|^m \leqslant & C_m \left(\sum_{r=p+1}^{N} \frac{1}{r^{2m}} \mathbb{E}|y_{jr}|^m + \left(\sum_{r=p+1}^{N} \frac{1}{r^4} \mathbb{E}|y_{jr}|^2\right)^{m/2}\right) \\ \leqslant & C_m \left((p+1)^{1-2m} + (p+1)^{-3m/2}\right).\end{aligned} \tag{3.13}$$

By Kolgomorov's three-series theorem, the infinite sum $\widetilde{u}_j^{(p)}$ converges almost surely, and therefore by Fatou's lemma $\widetilde{u}_j^{(p)} \leqslant C_m p^{-3m/2}$ for all $j = 1, \cdots, q$. The same argument leads to the same bound for $\mathbb{E}|\widetilde{\mu}_{jk}^{(1,p)}|^m$ and $\mathbb{E}|\widetilde{\mu}_{jk}^{(2,p)}|^m$ for all $j, k = 1, \cdots, q$. For $\widetilde{\nu}_{jk}^{(p)}$, notice that if $r \geqslant 2s \geqslant s^2/(s-1)$, then $r \leqslant (r-s)s$ and applying Rosenthal's inequality again the summand in the second summation in (3.13) would be bounded by

$$\frac{2}{(r+s)^2|r-s|^2}\left(\frac{r^2}{s^2}\mathbb{E}|x_{jr}|^2\mathbb{E}|x_{ks}|^2 + \mathbb{E}|y_{jr}|^2\mathbb{E}|y_{ks}|^2\right) \leqslant \frac{4}{(r+s)^2};$$

this also holds trivially for the case where $s \neq r \leqslant 2s$, and the result follows. □

The proof above is rather simple because of the summands (with different r) of $z_j, u_j, \lambda_{jk}, \mu_{jk}$ and ν_{jk}, respectively, are all independent with one another. The same argument cannot be applied immediately to the moments of $\widetilde{\Delta}_{jkl}^{(p)}$: there are many different pairs (r, s) having the same value of $r + s$. One needs to use conditional arguments to estimate the moments, and it is already quite complicated for $m = 2$.

Nevertheless, assuming that $\mathbb{E}|\widetilde{\Delta}_{jkl}^{(p)}|^m = O(p^{-m/2})$, one should expect an analogue of Davie's result in [8] (Theorem 15): for $m \in \mathbb{Z}^+$ and $p_0 = 8(m+1+d)^2$, if there exists an $\mathbb{R}^d$-random variable $\bar{V}$ s.t. $\mathbb{E}\bar{V}^\beta = \mathbb{E}\widetilde{V}_p^\beta$ for all $|\beta| \leqslant m-1$ and $\mathbb{E}|\bar{V}|^m \leqslant C_{q,m} p^{-m/2}$, then for any even integer $p > p_0$ s.t. $p+1$ is prime,

$$\mathbb{W}_2(V,\ V_p + \bar{V}) \leqslant C_{q,m} p^{-m/4}.$$

However, this conjecture is potentially subject to some additional assumptions.

To give an estimate for the Vaserstein distances presumably one would resort to the inequality (1.10), but the random variables V and $V_p + \bar{V}$ do not necessarily have densities. This is different from the double integral case introduced in the beginning of the chapter: for $I_{jk}^\circ = \int_0^t W_s^j \mathrm{d}W_s^k$, its Fourier representation only involves $U = (\lambda, z)$, and the independence of the summands of λ and z ensures that U has a smooth density (as the convolution of the density f_p of U_p and the law of $\widetilde{U}_p$), which significantly simplifies the analysis. More importantly, the characteristic function of U_p can be explicitly calculated - see formula (32) in the proof of Lemma 11 in [8]. This provides some convenience for investigating the global and local behaviour of the density f_p (Lemma 12, 13 and 14). In particular, Lemma 14 therein gives a lower bound for f_p, which is the essential reason why one can achieve a coupling for U of order $O(p^{-m/2})$ in the $\mathbb{W}_2$ distance.

Without Lemma 14, as a compromise approach one could simplify the proof of Davie's result by directly showing a decay of the difference $|g(\zeta) - h(\zeta)|$. For p sufficiently large, one has $\mathrm{D}^{2m} f_p$ uniformly bounded everywhere due to part (1) of Lemma 11 in [8]. Also by Lemma 12 therein, one has $f_p(\zeta) \leqslant e^{-c_q|\zeta|}$ for $|\zeta|$ sufficiently large.

Then one can apply Lemma 9 therein to get a rapid decay for $\mathrm{D}^m f_p(\zeta)$. To see this, consider $|\zeta| > p$ sufficiently large and the ball $B(\zeta, 1)$ that is disjoint with $B(0, p)$. Then $\sup_{y\in B(\zeta,1)} f_p(y) \leqslant e^{-c_q(|\zeta|-1)}$, and by applying Lemma 9 to the ball $B(\zeta, 1)$ one has

$$\|\mathrm{D}^m f_p(\zeta)\| \leqslant C_q \max\left\{ \sup_{y\in B(\zeta,1)} \sqrt{f_p(y)} \sup_{y\in B(\zeta,1)} \sqrt{\|\mathrm{D}^{2m} f_p(y)\|}, \sup_{y\in B(\zeta,1)} f_p(y) \right\}.$$

This yields $\|\mathrm{D}^m f_p(\zeta)\| \leqslant C_q e^{-c_q|\zeta|}$. Therefore from (3.5) and part (2) of Lemma 11 in [8] one has, by the Cauchy-Schwartz inequality, that for all $\zeta \in \mathbb{R}^{q(q+1)/2}$,

$$\begin{aligned}|g(\zeta) - h(\zeta)| \leqslant& C_{d,m} \sum_{|\beta|=m} \left(\mathbb{E}|\widetilde{U}_p^\beta \partial^\beta f_p(\zeta - \widetilde{U}_p)| + \mathbb{E}|\bar{U}^\beta \partial^\beta f_p(\zeta - \bar{U})| \right) \\ \leqslant& C_{d,m} p^{-m/2} \left(\sqrt{\mathbb{E}\|\mathrm{D}^m f_p(\zeta - \widetilde{U}_p)\|^2} + \sqrt{\mathbb{E}\|\mathrm{D}^m f_p(\zeta - \bar{U})\|^2} \right).\end{aligned}$$

Notice that, on the set $\{\omega : |\widetilde{U}_p| \leqslant 1\}$ one has $\|\mathrm{D}^m f_p(\zeta - \widetilde{U}_p)\|^2 \leqslant C_q e^{-c_q|\zeta|}$ by the rapid decay of $\mathrm{D}^m f_p$; on the complement $\{\omega : |\widetilde{U}_p| > 1\}$, part (2) of Lemma 11 and Chebyshev's inequality imply that $\mathbb{P}(|\widetilde{U}_p| > 1) \leqslant C_M p^{-M}$ for any $M > 0$. The same argument works for the second term above involving $\bar{U}$, and so by the inequality (1.10) for the quadratic distance,

$$\mathbb{W}_2(U,\ \widetilde{U}_p + \bar{U}) \leqslant C \left(\int_{\mathbb{R}^{q(q+1)/2}} |\zeta|^2 |g(\zeta) - h(\zeta)| \mathrm{d}\zeta \right)^{1/2} \leqslant C_{q,m} p^{-m/4},$$

which agrees with the conjectured rate for the triple integrals above.

However, this method cannot be immediately applied to the case of triple integrals here. Suppose, by mollification, the random variable V has density g and $V_p + \bar{V}$ has density h, and let κ_a, η_a be the densities of $\widetilde{V}_p$ and $\bar{V}$ conditional on that $V_p = a$, respectively. Then one has

$$g(z) = \int_{\mathbb{R}^d} f_p(z - w)\kappa_{z-w}(w)\mathrm{d}w,\ h(z) = \int_{\mathbb{R}^d} f_p(z - w)\eta_{z-w}(w)\mathrm{d}w,$$

and by (3.4), for all $z \in \mathbb{R}^d$ one arrives at

$$\begin{aligned}|g(z) - h(z)| \leqslant& C_{d,m} \sum_{|\beta|=0}^{m-1} \left| \int_{\mathbb{R}^d} \left(w^\beta \kappa_{z-w}(w) - w^\beta \eta_{z-w}(w) \right) \mathrm{d}w \right| \\ &+ C_{d,m} \sum_{|\beta|=m} \int_{\mathbb{R}^d} \left| w^\beta \kappa_{z-w}(w) - w^\beta \eta_{z-w}(w) \right| \mathrm{d}w.\end{aligned}$$

One then sees the complication of estimating the integrands above, compared to the proof of Theorem 15 in [8]: in the double integral case, due to the independence the first integral above will just be $\mathbb{E}\widetilde{U}_p^\beta - \mathbb{E}\bar{U}^\beta$, which vanishes by assumption, and the rest is of order $O(p^{-m/2})$ by Lemma 3.6 (or Lemma 11 in [8]). However, here $\int_{\mathbb{R}^d} w^\beta \kappa_{z-w} \mathrm{d}w$ is not even the conditional moment of $\widetilde{V}_p$ due to the appearance of w in the subscript of κ. Further investigation is therefore needed to tackle these problems.

Finally we remark that the rate $O(p^{-m/4})$ is the best one can expect simply from Theorem 3.5 alone using the aforementioned simplified apporach. This is the because the particular forms of the derivatives of the phase function Φ_p are not fully exploited.

In fact we have only used the fact that the phase function $\Phi_p(v)$ is a cubic polynomial in Lemma 3.3 and Lemma 3.4. The numerical scheme based on a coupling of order $O(p^{-m/4})$ is computationally equivalent to the Milstein scheme based on Wiktorsson's result [56] with step size $h^{3/2}$- see the discussion following the proof of Theorem 15 in [8].

Despite that the conjectured rate above might not bring a genuine improvement, to my best knowledge what is presented in this chapter is the first attempt to find a coupling for the triple integrals. I believe that this limitation could be improved if analogues of Lemma 12, 13 and 14 in [8] can be shown, but the question is still open.

Chapter 4

Approximating Lévy-SDEs via the Central Limit Theorem

This chapter presents the results and discussions from the author's work [60]. Given $d, q, q_1 \in \mathbb{Z}^+$, let $a \in \mathbb{R}^q$, $B \in \mathbb{R}^{q \times q_1}$ and $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\{\mathscr{F}_t\}_{t \geqslant 0}$ generated by a q_1-dimensional Wiener process $\{W_t\}$ and an independent Poisson random measure $N(\mathrm{d}z, \mathrm{d}s)$ on $\mathbb{R}^q \setminus \{0\} \times [0, \infty)$ with intensity $\nu(\mathrm{d}z)\mathrm{d}s$. Consider the q-dimensional Lévy process on $[0, T]$:

$$Z_t = at + BW_t + \int_0^t \int_{\mathbb{R}^q \setminus \{0\}} z\widetilde{N}(\mathrm{d}z, \mathrm{d}s), \tag{4.1}$$

where $\widetilde{N}(\mathrm{d}z, \mathrm{d}s)$ is the compensated Poisson measure. Assume the second moment of the Lévy measure $\int_{\mathbb{R}^q \setminus \{0\}} |z|^2 \nu(\mathrm{d}z) < \infty$. For $x_0 \in \mathbb{R}^q$ and a bounded Lipschitz function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times q}$, consider the d-dimensional SDE driven by the Lévy process above:

$$x_t = x_0 + \int_0^t \sigma(x_{s-})\mathrm{d}Z_s. \tag{4.2}$$

As mentioned in the introduction, the small jumps of the Lévy process (4.1) of size up to some $\epsilon \in (0, 1)$,

$$Z_t^\epsilon := \int_0^t \int_{0<|z|\leqslant\epsilon} z\tilde{N}(\mathrm{d}z, \mathrm{d}s), \tag{4.3}$$

play an important role in controlling the computational cost when simulating the solution of the equation (4.2). Fournier [11] showed that if Z_t^ϵ is completely ignored, a potential blow-up can happen even when the Lévy measure ν satisfies some typical stable-like conditions, such as

Assumption 4.1. *There exist $\tau > 0$ and $\alpha \in (0, 2)$ s.t. $\forall 0 < |z| \leqslant \tau$,*

$$\nu(\mathrm{d}z) \simeq |z|^{-q-\alpha}\mathrm{d}z.$$

Similar to the one-dimensional treatment done by Fournier, one can also apply central limit arguments to handle the case where $q > 1$. The idea is to generalise Rio [45] and Bobkov's [3] results to the multi-dimensional case first, and then apply it to small jumps Z_t^ϵ due to the infinite divisibility of its law.

Consider i.i.d. $\mathbb{R}^q$-random variables $X_1, X_2, \cdots$ with mean 0 and covariance Σ, and the weighted sum $Y_m = m^{-1/2} \sum_{j=1}^m X_j$ for $m \in \mathbb{Z}^+$. Davie [9] sketched an asymptotic approach via Edgeworth expansion of the density of Y_m, and proved (as a corollary to Proposition 2 therein) the rate $O(m^{-1/2})$ under the assumption that all moments of X are bounded[1]. Moreover, he in fact showed a coupling between Y_m and the normal distribution perturbed by polynomials using the inequality

$$\mathbb{W}_p(X,Y) \leqslant C_p \left(\int_{\mathbb{R}^q} |x|^p |f(x) - g(x)| \mathrm{d}x \right)^{1/p}, \tag{4.4}$$

for $p \geqslant 1$ and $\mathbb{R}^q$-random variables X and Y having densities f and g, respectively. Section 4.1 basically follows Davie's approach, but expounds detailed calculations and specify the range of p and precisely how many moments of X are needed.

The rate of convergence for the multi-dimensional central limit theorem has been studied using different methods. A strong result by Zaitsev (summarised as Theorem 2 in [59] and proved as Theorem 1.3 in [58]) gives a sharp Chernoff-type bound, and by Chebyshev's inequality the central limit theorem follows in a stronger sense: for independent $\{X_j\}$ each having identity covariance and independent standard Gaussian $\{\xi_j\}$ with partial sums $\Upsilon_m := m^{-1/2} \sum_{j=1}^m \xi_j$, if the law of each X_j satisfies certain analyticity conditions (see the definition of the class $\mathcal{A}_q(\tau)$ in [59]), then the distance $\max_{k \leqslant m} |Y_k - \Upsilon_k|$ is of order $O(m^{-1/2} \log m)$ in probability. The logarithmic factor emerges because the method is based on the dyadic approximation by Komlós, Major and Tusnády (KMT) [34]. The KMT method is much stronger than the usual central limit theorem since it considers the simultaneous approximation between $Y_1, Y_2, \cdots, Y_m$ and $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_m$. Einmahl [10] generalised the original KMT method to the multi-dimensional case, and Zaitsev's theorem [58] is an improved version of that, albeit it requires the local existence of the moment generating function.

Since the central limit theorem only concerns the coupling between Y_m and Υ_m, one should expect the $\log m$ factor to be removed as in the one-dimensional result of Rio. This has indeed been achieved by Bobkov [4] (Theorem 6.1) under the assumption that $\mathbb{E}|X|^5 < \infty$; given only $\mathbb{E}|X|^4 < \infty$, his result is weakened to $O(m^{-1/2} (\log m)^{q/4-1})$. It is worth mentioning here that, shortly after [60], using Stein's method Bonis [5] (Theorem 8) managed to achieve the optimal rate $O(m^{-1/2})$ given only $\mathbb{E}|X|^4 < \infty$, which is a significant improvement. However, both approaches only work for $p = 2$ since their arguments rely on some entropic transport inequalities for the $\mathbb{W}_2$ distance. In this special case (normal approximation for Y_m in $\mathbb{W}_2$) the result derived in this article is not optimal, as it requires $\mathbb{E}|X|^{4+\tau} < \infty$ for some $\tau \in (0,1)$ and Cramér's condition $\overline{\lim}_{|s| \to \infty} |\mathbb{E} \exp(isX)| < 1$.

Nevertheless, given that $\mathbb{E}|X|^{6+\tau} < \infty$ and Cramér's condition, the result here would give a coupling for Y_m of order $O(m^{-1})$ in $\mathbb{W}_p$ for a positive even ingeter p, if one perturbs the normal distribution with a cubic Edgeworth polynomial. The Edgeworth expansion is used by Bobkov [3] (Corollary 9.2) in the one-dimensional case for higher-order approximations for Y_m, but in return Cramér's condition and some higher moments are needed. Theorem 4.9 here can be regarded as a generalisation of that.

In Section 4.2, the central limit bound in $\mathbb{W}_p$ is applied to the normal approximation for the small jumps (4.3). This is done by viewing Z_t^ϵ as a compound Poisson process, assuming Cramér's condition and that the Lévy measure ν is sufficiently singular at 0 (Theorem 4.13). A desired coupling $\mathbb{W}_p(Z_t^\epsilon, \sqrt{t}\mathcal{N}(0, \Sigma_\epsilon)) = O(\epsilon)$ is then achieved for $t = \epsilon$ and $\Sigma_\epsilon = \int_{0<|z|\leqslant\epsilon} zz^\top \nu(\mathrm{d}z)$, which covers the case of Assumption 4.1. However, those assumptions can all be removed if one compromises for a suboptimal rate, as

[1]When only the distribution of the X_j's is considered, the subscript j is omitted for simplicity.

is proved in the appendices of Godinho's paper [13] (Proposition A.2), where only bounded jumps are considered. Again, there is a logarithmic factor because the proof directly uses the aforementioned result of Zaitsev.

In this chapter the notation ξ_Σ always stands for an $\mathcal{N}(0,\Sigma)$-random variable on $\mathbb{R}^q$ and ϕ_Σ stands for its density if Σ is non-singular. For any multi-index $\rho \in \mathbb{N}^q$, apart from $|\rho| = \sum_{j=1}^q \rho_j$ it would also be convenient to introduce the notation $|\rho|_* := \sum_{j=1}^q j\rho_j$. The notation for the Lebesgue measure Λ^q will be simplified as just Λ.

4.1 A Coupling for the Central Limit Theorem

This section follows Davie's asymptotic approach via Edgeworth expansion briefly sketched in [9], and elaborates each step rigorously. The goal is to achieve a good $\mathbb{W}_p$ bound using (4.4), and for that one may first approximate the Fourier transform.

4.1.1 Asymptotic Estimates of the Characteristic Function

Denote by χ the characteristic function of X, and by ψ_m and $\mathbb{P}_m$ the characteristic function and distribution of Y_m, respectively. Then one has asymptotic expansion

$$\log\chi(s) \sim -\frac{1}{2}s\cdot\Sigma s + \sum_{|\alpha|\geqslant 3}\frac{i^{|\alpha|}}{\alpha!}\mu_\alpha s^\alpha,$$

where $\mu_\alpha = \mu_\alpha(X) = i^{-|\alpha|}\partial^\alpha \log\chi(0)$ is the α-th cumulant of X. This gives a formal expansion for $\log\psi_m(z) = m\log\chi(m^{-1/2}z) \sim -\frac{1}{2}z\cdot\Sigma z + \sum_{|\alpha|\geqslant 3}\frac{i^{|\alpha|}}{\alpha!}m^{1-|\alpha|/2}\mu_\alpha z^\alpha$, and

$$\psi_m(z) \sim e^{-\frac{1}{2}z\cdot\Sigma z}\left(1+\sum_{k=1}^{\infty} m^{-\frac{k}{2}}P_k(z)\right), \tag{4.5}$$

where $P_k(z)$ is a polynomial whose monomials have highest degree $3k$ and lowest degree $k+2$, with coefficients bounded by $C_k(\mathbb{E}|X|^{k+2})^k$ - see Lemma 7.1 in [2]. The inverse Fourier transform of (4.5) gives the Edgeworth expansion for the density f_m of Y_m, if it exists. Detailed derivation for $q=1$ can also be found in [44] (Chapter VI).

In this section the shorthand notations $\varepsilon := m^{-1/2}$, $\mathcal{P}_{\varepsilon,r} := 1+\sum_{k=1}^r \varepsilon^k P_k$, $\forall r\in\mathbb{Z}^+$, and $\mathcal{P}_\varepsilon := \mathcal{P}_{\varepsilon,\infty}$ are used, and ε and m may be frequently interchanged. Denote by $\lambda_1 \leqslant \cdots \leqslant \lambda_q$ the eigenvalues of Σ, and assume $\lambda_1 \leqslant 1 \leqslant \lambda_q$ without loss of generality. Furthermore, $\forall M>0$ denote $\kappa_M := 1\vee\mathbb{E}|X|^M$, then $\kappa_M^{1/M}$ increases in M by Hölder's inequality, and so does κ_M. By Lemma 6.3 in [2], $|\mu_\alpha| \leqslant C_\alpha\kappa_{|\alpha|}$, $\forall\alpha\in\mathbb{N}^q$.

Lemma 4.2. *Suppose Σ is non-singular and $\mathbb{E}|X|^{n+\tau}<\infty$ for a fixed integer $n\geqslant 3$ and $\tau\in(0,1)$. Let $\beta\in(0,1/3)$ and $\delta := \min\{\lambda_1/\kappa_3, \kappa_n^{-1/n}/2\}$. Then,*

(i) *for $|z|\leqslant m^{1/2}\delta,\ m\in\mathbb{Z}^+,\ |\psi_m(z)|\leqslant \exp\left(-\frac{1}{4}z\cdot\Sigma z\right)$;*

(ii) *for $|z|\leqslant m^{\beta/2}$ and $m>(\kappa_3/\lambda_1)^3\vee\kappa_{n+\tau}^{\max\{4,\ 6/(n(1-3\beta))\}}$,*

$$\left|\psi_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_{\varepsilon,n-2}(z)\right| \leqslant C_{n,\tau}\kappa_{n+\tau}^{n-2}\left(|z|^{n+1}+|z|^{3(n-1)}\right)e^{-\frac{1}{4}z\cdot\Sigma z}\varepsilon^{n-1}. \tag{4.6}$$

Proof. First of all Taylor's theorem gives the identity

$$\chi(s) = 1-\frac{1}{2}s\cdot\Sigma s + \mathbb{E}\int_0^1 \frac{1}{2}e^{i\theta(s\cdot X)}(1-\theta)^2(is\cdot X)^3\mathrm{d}\theta. \tag{4.7}$$

Then for $|s| \leqslant \delta_1 := \lambda_1/\kappa_3 \leqslant \sqrt{2/\lambda_q}$, the inequality $\log u \leqslant u - 1$, $\forall u > 0$, implies that

$$
\begin{aligned}
\log|\chi(s)| \leqslant& \log\left(1 - \frac{1}{2}s\cdot\Sigma s + \frac{1}{6}\mathbb{E}|X|^3|s|^3\right) \leqslant -\frac{1}{2}s\cdot\Sigma s + \frac{1}{6}\delta_1\mathbb{E}|X|^3|s|^2 \\
\leqslant& -\frac{1}{2}s\cdot\Sigma s + \frac{1}{4}\lambda_1|s|^2 \leqslant -\frac{1}{4}s\cdot\Sigma s,
\end{aligned}
$$

and the first claim $|\psi_m(z)| \leqslant \exp\left(-\frac{1}{4}z\cdot\Sigma z\right)$ holds for $|z| \leqslant m^{1/2}\delta_1$.

On the other hand, for $|s| \leqslant \kappa_3^{-1/3}/2 \leqslant \lambda_q^{-1/2}/2$, from (4.7) one sees that

$$
\mathrm{Re}\chi(s) \geqslant 1 - \frac{1}{2}\lambda_q|s|^2 - \frac{1}{6}\mathbb{E}|X|^3|s|^3 > \frac{1}{2},
$$

and hence the principle branch of $\log\chi(s)$ is well-defined, and $|\chi(s)| > 1/2$. For fixed $n \geqslant 3$, define, $\forall s \in \mathbb{R}^q$,

$$
S_n(s) := \sum_{|\alpha|=2}^{n} \frac{i^{|\alpha|}}{\alpha!}\mu_\alpha s^\alpha,\ T_n(s) := \sum_{|\alpha|=2}^{n} \frac{i^{|\alpha|}}{\alpha!}s^\alpha\mathbb{E}X^\alpha = \sum_{j=2}^{n}\frac{1}{j!}\mathbb{E}(is\cdot X)^j.
$$

Then using the inequality $|e^{iu} - 1| \leqslant 2 \wedge |u| \leqslant 2^{1-\tau}|u|^\tau$, $\forall \tau \in (0,1)$, and the identity

$$
e^{iu} = \sum_{k=0}^{n}\frac{(iu)^k}{k!} + \frac{i^n}{(n-1)!}\int_0^1 (1-\theta)^{n-1}u^n\left(e^{i\theta u} - 1\right)\mathrm{d}\theta,
$$

for all $u \in \mathbb{R}$, one deduces $|\chi(s) - 1 - T_n(s)| \leqslant C_{n,\tau}\kappa_{n+\tau}|s|^{n+\tau}$ by the substitution $u = s\cdot X$. Meanwhile one can write (with Taylor remainder $R_n(s)$):

$$
\log\left(1 + T_n(s)\right) = \sum_{l=1}^{n}\frac{(-1)^{l+1}}{l}T_n^l(s) + R_n(s) = S_n(s) + \widetilde{S}_n(s) + R_n(s),
$$

where $\widetilde{S}_n(s)$ is a polynomial of which each monomial has degree at least $n+1$. The fact that the first few terms agree with $S_n(s)$ is due to the relation between the cumulants μ_α and the moments $\mathbb{E}X^\alpha$ - see Section 6 (page 46) in [2]. By the multinomial theorem, for $l = 1, \cdots, n$ each monomial in $T_n^l(s)$ takes the form

$$
\sigma_{\rho,l}(s) = C_{n,l,\rho}\prod_{j=1}^{n-1}\left(\mathbb{E}(s\cdot X)^{j+1}\right)^{\rho_j},
$$

for some $\rho \in \mathbb{N}^{n-1}$, $|\rho| = l$. Then the monomials $\widetilde{\sigma}_{\rho,l}(s)$ of $\widetilde{S}_n$ correspond to those with $\sum_{j=1}^{n-1}(j+1)\rho_j = |\rho|_* + l \geqslant n+1$. If one further chooses $\delta_2 := \kappa_n^{-1/n}/2 < 1$, then for $|s| \leqslant \delta_2$,

$$
\begin{aligned}
|\widetilde{\sigma}_{\rho,l}(s)| \leqslant& C_{n,l}|s|^{|\rho|_*+l}\prod_{j=1}^{n-1}\kappa_{j+1}^{\rho_j} \leqslant C_{n,l}|s|^{n+1}\kappa_n^{-(|\rho|_*+l-(n+1))/n}\prod_{j=1}^{n-1}\kappa_{j+1}^{(|\rho|_*+l)/(j+1)} \\
=& C_{n,l}|s|^{n+1}\kappa_n^{(n+1)/n}\prod_{j=1}^{n-1}\left(\kappa_n^{-1/n}\kappa_{j+1}^{1/(j+1)}\right)^{|\rho|_*+l} \leqslant C_{n,l}\kappa_n^{1+1/n}|s|^{n+1},
\end{aligned}
$$

where Hölder's inequality is used in the last step. Therefore $|\widetilde{S}_n(s)| \leqslant C_n\kappa_n^{1+1/n}|s|^{n+1}$.

Also notice that, for $|s| \leqslant \delta_2$ and $j = 2, \cdots, n$, one has $|s|^{j-1}\kappa_j \leqslant \kappa_n^{1/n}(\kappa_n^{-1/n}\kappa_j^{1/j})^j \leqslant \kappa_n^{1/n}$. This implies that

$$|T_n(s)| \leqslant \sum_{j=2}^{n} \frac{1}{j!}|s|^{j-1}\kappa_j|s| \leqslant \frac{(e-2)}{2}\kappa_n^{1/n}|s| < \frac{1}{2},$$

and that $|1 + \theta T_n(s)| \geqslant 1/2$ for any $\forall\theta \in [0, 1]$. Therefore

$$|R_n(s)| \leqslant \int_0^1 (1-\theta)^n \left|\frac{T_n(s)}{1+\theta T_n(s)}\right|^{n+1} \mathrm{d}\theta \leqslant C_n|T_n(s)|^{n+1} \leqslant C_n\kappa_n^{1+1/n}|s|^{n+1}. \quad (4.8)$$

Thus $|\log(1 + T_n(s)) - S_n(s)| \leqslant C_n\kappa_n^{1+1/n}|s|^{n+1}$. Since $|\chi(s)| \wedge |1 + T_n(s)| \geqslant 1/2$ for $|s| < \delta_2$, the triangle inequality implies that

$$\begin{aligned}|\log\chi(s) - S_n(s)| \leqslant& 2\,|\chi(s) - 1 - T_n(s)| + |\log(1 + T_n(s)) - S_n(s)|\\ \leqslant& C_{n,\tau}\kappa_{n+\tau}^{1+1/n}|s|^{n+1}.\end{aligned}$$

Returning to ψ_m, as $\log\psi_m(z) = \varepsilon^{-2}\log\chi(\varepsilon z)$, from the estimate above one has

$$\left|\log\psi_m(z) - \varepsilon^{-2}S_n(\varepsilon z)\right| \leqslant C_{n,\tau}\varepsilon^{n-1}|z|^{n+1}\kappa_{n+\tau}^{1+1/n}. \quad (4.9)$$

Moreover, writing $U_n(z) := \frac{1}{2}z\cdot\Sigma z + \varepsilon^{-2}S_n(\varepsilon z)$, one can apply Taylor's theorem again to the exponential $\exp(U_n(z))$ (recall the notation $\mathcal{P}_{\varepsilon,\cdot}$):

$$\begin{aligned}\exp\left(\sum_{|\alpha|=3}^{n}\frac{i^{|\alpha|}}{\alpha!}\varepsilon^{|\alpha|-2}\mu_\alpha z^\alpha\right) =& 1 + U_n(z) + \frac{1}{2!}U_n^2(z) + \cdots + \frac{1}{(n-2)!}U_n^{n-2}(z) + V(z)\\ =& 1 + \mathcal{P}_{\varepsilon,n-2}(z) + \widetilde{P}(z) + V(z),\end{aligned}$$

where $\widetilde{P}(z) = 0$ for $n = 3$ (i.e. $P_1(z) = U_3(z)$ contains all the cubic terms) and otherwise a polynomial of degree $n(n-2)$ with complex coefficients that contain products of the cumulants μ_α up to $|\alpha| = n$ and powers of ε at least $n-1$; the Taylor remainder $V(z)$ is given by

$$V(z) = \frac{1}{(n-2)!}\int_0^1 (1-\theta)^{n-2}U_n^{n-1}(z)e^{\theta U_n(z)}\mathrm{d}\theta.$$

For $|z| \leqslant m^{1/6} = \varepsilon^{-1/3}$, one claims the following bound:

$$\left|\tilde{P}(z)\right| \leqslant C_n\kappa_n^{n-2}\varepsilon^{n-1}(|z|^{n+3} + |z|^{3(n-1)}).$$

This can be seen by checking the powers of ε and z in each $U_n^l(z)$, $l = 1, \cdots, n-2$. For each l, the multinomial theorem gives (with multi-indices $\rho \in \mathbb{N}^{n-2}$, $\alpha \in \mathbb{N}^q$)

$$U_n^l(z) = (-1)^l\sum_{|\rho|=l}\binom{l}{\rho}(i\varepsilon)^{|\rho|_*}\prod_{j=1}^{n-2}\left(\sum_{|\alpha|=j+2}\frac{1}{\alpha!}\mu_\alpha z^\alpha\right)^{\rho_j}.$$

Then each monomial of $U_n^l(z)$ is bounded by $C_{n,l}\kappa_n^l\varepsilon^{|\rho|_*}|z|^{|\rho|_*+2l}$, and the monomials $\widetilde{p}_{\rho,l}(z)$ of $\widetilde{P}(z)$ correspond to those with $|\rho|_* \geqslant n-1$ and $l \geqslant 2$. When $|\rho|_*+2l \leqslant 3(n-1)$ the claim follows immediately from interpolating the powers of $|z|$; when $|\rho|_* + 2l >$

$3(n-1)$, note that $|\rho|_* > |\rho| = l$, and so for $|z| \leqslant \varepsilon^{-1/3}$,

$$|\widetilde{p}_{\rho,l}(z)| \leqslant C_{n,l}\kappa_n^l \varepsilon^{\frac{2}{3}(|\rho|_*-l)+n-1}|z|^{3(n-1)} \leqslant C_n\kappa_n^{n-2}\varepsilon^{n-1}|z|^{3(n-1)}.$$

Regarding the Taylor remainder $V(z)$, notice that for $|z| \leqslant \varepsilon^{-\beta}$, $\forall\beta \in (0,1/3)$, and $\varepsilon < \kappa_n^{-1}$,

$$\begin{aligned}|U_n(z)| &\leqslant \sum_{j=1}^{n-2}\varepsilon^j|z|^{j+2}\kappa_{j+2} \leqslant \sum_{j=1}^{n-2}\varepsilon^{j-\beta(j-1)}|z|^3\kappa_{j+2} \leqslant \sum_{j=1}^{n-2}\varepsilon^{\frac{2}{3}(j-1)}\kappa_n^{(j+2)/n}\varepsilon|z|^3\\ &\leqslant \sum_{j=1}^{n-2}\kappa_n^{\frac{3}{n}+\left(\frac{1}{n}-\frac{2}{3}\right)(j-1)}\varepsilon|z|^3 \leqslant (n-2)\kappa_n^{3/n}\varepsilon|z|^3,\end{aligned}$$

and furthermore $|U_n(z)| \leqslant (n-2)\kappa_n^{3/n}\varepsilon^{1-3\beta}$. Thus one arrives at

$$|V(z)| \leqslant C_n\kappa_n^3 \exp\left((n-2)\varepsilon^{1-3\beta}\kappa_n^{3/n}\right)\varepsilon^{n-1}|z|^{3(n-1)}.$$

Combining with (4.9) one deduces, for $|z| \leqslant \varepsilon^{-\beta}$,

$$\begin{aligned}&\left|\psi_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_\varepsilon^{(n-2)}(z)\right|\\ &\leqslant \left|e^{\log\psi_m(z)} - e^{-\frac{1}{2}z\cdot\Sigma z+U_n(z)}\right| + \left|e^{-\frac{1}{2}z\cdot\Sigma z+U_n(z)} - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_{\varepsilon,n-2}(z)\right|\\ &\leqslant |\psi_m(z)|\left|1-\exp\left(-\log\psi_m(z) - \frac{1}{2}z\cdot\Sigma z + U_n(z)\right)\right| + e^{-\frac{1}{2}z\cdot\Sigma z}\left(|\widetilde{P}(z)| + |V(z)|\right)\\ &\leqslant C_{n,\tau}|\psi_m(z)|\exp\left(\varepsilon^{2(n-2)/3}\kappa_{n+\tau}^{1+1/n}\right)\varepsilon^{n-1}|z|^{n+1}\kappa_{n+\tau}^{1+1/n}\\ &\quad + C_n\kappa_n^{n-2}\exp\left((n-2)\varepsilon^{1-3\beta}\kappa_n^{3/n}\right)\varepsilon^{n-1}(|z|^{n+3} + |z|^{3(n-1)})e^{-\frac{1}{2}z\cdot\Sigma z},\end{aligned}$$

where in the last step the inequality $|1-e^u| \leqslant e^{|u|}|u|$, $\forall u \in \mathbb{C}$, is used for the first term.

Now with $\delta := \delta_1 \wedge \delta_2$ fixed, for m large one has $m^{\beta/2} < m^{1/2}\delta$. Also, for fixed $\beta \in (0,1/3)$ and $\tau \in (0,1)$, one may further choose $m > \kappa_{n+\tau}^{3(1+1/n)/(n-2)} \vee \kappa_n^{6/(n(1-3\beta))}$ s.t. the exponents in coefficients above are bounded by 1. This is satisfied when $m > \kappa_{n+\tau}^{\max\{4,6/(n(1-3\beta))\}}$. For $m > \delta^{-3} > \delta^{2/(\beta-1)}$ the first claim still holds, and so the second claim follows. ☐

In order to bound the integral of the left-hand side term in (4.6) over all of $\mathbb{R}^q$, one may assume **Cramér's condition**:

$$\overline{\lim_{|s|\to\infty}}\,|\chi(s)| < 1,$$

or equivalently,

Assumption 4.3. *There exist $\rho > 0$ and $\gamma \in (0,1)$ s.t. $|\chi(s)| \leqslant \gamma$, $\forall|s| \geqslant \rho$.*

As explained in [2] (page 207), if χ satisfies Cramér's condition, then $|\chi(s)| < 1$, $\forall s \neq 0$; it is satisfied when X has a density by the Riemann-Lebesgue theorem. Discrete distributions are excluded, but some singular and yet non-lattice distributions are also allowed, such as the distribution on the Cantor middle-third set that gives mass 2^{-j} to each interval on the j-th level.

Given the X_j's satisfying Cramér's condition, the following lemma shows that it is also satisfied for the weighted sum Y_m.

Lemma 4.4. *Let χ satisfy Assumption 4.3 with ρ, γ explicitly known and $\delta \in (0, \rho \wedge 1)$. Then $\exists \bar{\gamma} = \bar{\gamma}(\rho, \gamma, \delta) \in (0,1)$ s.t. $|\psi_m(z)| < \bar{\gamma}^m$ for $|z| > m^{1/2}\delta$.*

Proof. Let $N \in \mathbb{Z}^+$ and write $\chi(Ns) = |\chi(Ns)|e^{i\theta_1}$, $\chi(s) = |\chi(s)|e^{i\theta_0}$, where θ_1, θ_0 depend on s. Then, with F being the distribution of X, one gets $\int_{\mathbb{R}^q} \sin(s \cdot x - \theta_0) F(\mathrm{d}x) = 0$ and

$$
\begin{aligned}
Iwaslosing 1 - |\chi(s)| = & \int_{\mathbb{R}^q} \left(1 - \cos(s \cdot x - \theta_0)\right) F(\mathrm{d}x) = \int_{\mathbb{R}^q} 2 \sin^2 \frac{1}{2}(s \cdot x - \theta_0) F(\mathrm{d}x) \\
\geqslant & \frac{1}{N^2} \int_{\mathbb{R}^q} 2 \sin^2 \frac{N}{2}(s \cdot x - \theta_0) F(\mathrm{d}x) \\
= & \frac{1}{N^2} \int_{\mathbb{R}^q} \left(1 - \cos(Ns \cdot x - N\theta_0)\right) F(\mathrm{d}x),
\end{aligned}
$$

where the inequality $|\sin(N\phi)| \leqslant N|\sin\phi|$, $\forall N \in \mathbb{N}$, $\phi \in \mathbb{R}$, is used. Meanwhile,

$$
|\chi(Ns)| = e^{-i\theta_1} \int_{\mathbb{R}^q} e^{iNs \cdot x} F(\mathrm{d}x) = e^{i(N\theta_0 - \theta_1)} \int_{\mathbb{R}^q} e^{i(Ns \cdot x - N\theta_0)} F(\mathrm{d}x),
$$

which implies

$$
\begin{aligned}
1 - |\chi(s)| \geqslant & \frac{1}{N^2} - \frac{1}{N^2} \mathrm{Re} \int_{\mathbb{R}^q} e^{i(Ns \cdot x - N\theta_0)} F(\mathrm{d}x) \\
\geqslant & \frac{1}{N^2} - \frac{1}{N^2} \left| \int_{\mathbb{R}^q} e^{i(Ns \cdot x - N\theta_0)} F(\mathrm{d}x) \right| = \frac{1}{N^2} - \frac{1}{N^2} |\chi(Ns)|.
\end{aligned}
$$

Choose $N = [(\rho+1)/\delta] > \rho/\delta$, then $|\chi(s)| \leqslant 1 - (1-\gamma)\delta^2/(\rho+1)^2 =: \bar{\gamma}$ for $\delta < |s| < \rho$. Clearly $\bar{\gamma} \geqslant \gamma$, and $|\psi_m(z)| = |\chi(m^{-1/2}z)|^m < \bar{\gamma}^m < 1$ for $|z| > m^{1/2}\delta$. $\square$

From now on the following bounds will be frequently used: $\forall M, c > 0$,

$$
\begin{aligned}
\int_{\mathbb{R}^q} |x|^M e^{-cx \cdot \Sigma x} \mathrm{d}x = & \int_{\mathbb{R}^q} \left| \Sigma^{-\frac{1}{2}} y \right|^M e^{-c|y|^2} \det\left(\Sigma^{-\frac{1}{2}}\right) \mathrm{d}y \\
\leqslant & C_{q,c,M} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{M}{2}},
\end{aligned}
\tag{4.10}
$$

and

$$
\int_{\mathbb{R}^q} |x|^M \phi_\Sigma(x) \mathrm{d}x = C_q \int_{\mathbb{R}^q} \left| \Sigma^{\frac{1}{2}} y \right|^M e^{-\frac{1}{2}|y|^2} \mathrm{d}y \leqslant C_{q,M} \lambda_q^{\frac{M}{2}},
\tag{4.11}
$$

where the inverse and the square root of Σ are well-defined since it is positive definite.

Although Cramér's condition gives some restriction on the law of X, it does not require the smoothness or the existence of the density f_m of Y_m. In order to see how close the law of Y_m is to the perturbed normal distributions from polynomial expansions, one may use a smoothing argument. Let $\widetilde{f}_m$ and $\widetilde{\psi}_m$ be the density and characteristic function of the mollified measure $\mathbb{P}_m * \theta_m$, where θ_m is a measure with smooth density, still denoted by θ_m or θ_ε:

$$
\theta_\varepsilon(x) = \varepsilon^{-q(n+1)} h(\varepsilon^{-n-1} x),
\tag{4.12}
$$

for some function $0 \leqslant h \in C_0^\infty(\mathbb{R}^q)$ supported on the open unit ball and $\int_{\mathbb{R}^q} h(x) \mathrm{d}x = 1$. Thus θ_ε is a probability density supported on $\{|x| < \varepsilon^{n+1}\}$. Write $\hat{h}$ and $\hat{\theta}_\varepsilon$ as their respective Fourier transforms.

Proposition 4.5. *Under the assumptions in Lemma 4.2 and Lemma 4.4, for any integer $n \geqslant 3$, $\tau \in (0,1)$, $\beta \in (0,1/3)$ and m sufficiently large, it holds true that*

$$\int_{\mathbb{R}^q} \left|\tilde{\psi}_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_{\varepsilon,n-2}(z)\right| \mathrm{d}z \leqslant C_{q,n,\tau}(\det\Sigma)^{-\frac{1}{2}}\lambda_1^{-\frac{n-1}{2\beta}}\kappa_{n+\tau}^{n-2}\varepsilon^{n-1}.$$

Proof. Note that $\widetilde{\psi}_m = \psi_m\hat{\theta}_\varepsilon$, and for $|z| \leqslant m^{1/2}\delta$,

$$\begin{aligned}\left|\tilde{\psi}_m(z) - \psi_m(z)\right| =& |\psi_m(z)| \left|\hat{\theta}_\varepsilon(z) - 1\right| \leqslant |\psi_m(z)| \int_{|x|<\varepsilon^{n+1}} \left|e^{iz\cdot x} - 1\right| \theta_\varepsilon(x)\mathrm{d}x \\ \leqslant& |\psi_m(z)||z|\varepsilon^{n+1} \leqslant \varepsilon^{n+1}|z|e^{-\frac{1}{4}z\cdot\Sigma z},\end{aligned}$$

and hence by Lemma 4.2 and triangle inequality,

$$\left|\tilde{\psi}_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_\varepsilon^{(n-2)}(z)\right| \leqslant C_{n,\tau}\varepsilon^{n-1}\kappa_{n+\tau}^{n-2}\left(|z|^{n+1} + |z|^{3(n-1)}\right)e^{-\frac{1}{4}z\cdot\Sigma z},$$

for $|z| \leqslant m^{\beta/2}$. Also for all $z \in \mathbb{R}^q$,

$$\begin{aligned}\left|\hat{\theta}_\varepsilon(z)\right| =& \left|\int_{|x|<\varepsilon^{n+1}} e^{iz\cdot x}\theta_\varepsilon(x)\mathrm{d}x\right| = \left|\int_{|x|<\varepsilon^{n+1}} e^{iz\cdot x}\varepsilon^{-q(n+1)}h(\varepsilon^{-n-1}x)\mathrm{d}x\right| \\ =& \left|\int_{|y|<1} e^{i\varepsilon^{n+1}z\cdot y}h(y)\mathrm{d}y\right| = \left|\hat{h}(\varepsilon^{n+1}z)\right| \leqslant C_q\varepsilon^{-K(n+1)}|z|^{-K}, \qquad (4.13)\end{aligned}$$

for any $K > 0$, since $h \in C_0^\infty(\mathbb{R}^q)$ with all the derivatives in $L^1(\mathbb{R}^q)$. One may choose $K = q+1$ for convenience and $|\widetilde{\psi}_m(z)| \leqslant \bar{\gamma}^m \min\{1, C_q\varepsilon^{-(q+1)(n+1)}|z|^{-q-1}\}$ for $|z| > m^{1/2}\delta$. For $|z| \leqslant m^{1/2}\delta$ one still has $|\widetilde{\psi}_m(z)| \leqslant \exp(-\frac{1}{4}z\cdot\Sigma z)$.

Given all the estimates for $\widetilde{\psi}_m(z)$ on different domains, one can split the integral in question into three parts:

$$\begin{aligned}\widetilde{I} :=& \int_{\mathbb{R}^q} \left|\tilde{\psi}_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_\varepsilon^{(n-2)}(z)\right| \mathrm{d}z \\ =& \left(\int_{|z|\leqslant m^{\beta/2}} + \int_{m^{\beta/2}<|z|\leqslant m^{1/2}\delta} + \int_{|z|>m^{1/2}\delta}\right) \left|\tilde{\psi}_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_\varepsilon^{(n-2)}(z)\right| \mathrm{d}z.\end{aligned}$$

Then by virtue of Lemma 4.2, Lemma 4.4,

$$\begin{aligned}\widetilde{I} \leqslant& C_{n,\tau}\kappa_{n+\tau}^{n-2}\varepsilon^{n-1}\int_{|z|\leqslant m^{\beta/2}} \left(|z|^{n+1} + |z|^{3(n-1)}\right)e^{-\frac{1}{4}z\cdot\Sigma z}\mathrm{d}z + \int_{m^{\beta/2}<|z|\leqslant m^{\frac{1}{2}}\delta} e^{-\frac{1}{4}z\cdot\Sigma z}\mathrm{d}z \\ &+ \int_{|z|>m^{1/2}\delta} \bar{\gamma}^m\left(1 \wedge C_q\varepsilon^{-(q+1)(n+1)}|z|^{-q-1}\right)\mathrm{d}z + \int_{|z|>m^{\beta/2}} e^{-\frac{1}{2}z\cdot\Sigma z}|\mathcal{P}_{\varepsilon,n-2}(z)|\mathrm{d}z.\end{aligned}$$

Use (4.10) for the first integral, combine the second and the fourth, and split the third into the set where $|z|$ is large and its complement to get (Λ denotes the Lebesgue measure on $\mathbb{R}^q$)

$$\begin{aligned}\widetilde{I} \leqslant & C_{q,n,\tau}(\det\Sigma)^{-\frac{1}{2}}\lambda_1^{-\frac{3}{2}(n-1)}\kappa_{n+\tau}^{n-2}\varepsilon^{n-1} + \bar{\gamma}^m \Lambda\left(\{|z| \leqslant C_q\varepsilon^{-n-1}\}\right) \\ & + C_q\bar{\gamma}^m\varepsilon^{-(q+1)(n+1)}\int_{|z|>C_q\varepsilon^{-n-1}} |z|^{-q-1}\mathrm{d}z \\ & + 2\int_{|z|>m^{\beta/2}} e^{-\frac{1}{4}z\cdot\Sigma z}\left(1+\sum_{k=1}^{n-2}\varepsilon^k\left|P_k(z)\right|\right)\mathrm{d}z \\ \leqslant & C_{q,n,\tau}(\det\Sigma)^{-\frac{1}{2}}\lambda_1^{-\frac{3}{2}(n-1)}\kappa_{n+\tau}^{n-2}\varepsilon^{n-1} + C_q\bar{\gamma}^m\varepsilon^{-(q+1)(n+1)} \\ & + C_{q,n}\int_{|z|>m^{\beta/2}} e^{-\frac{1}{4}z\cdot\Sigma z}\kappa_n^{n-2}\left(1+\sum_{k=1}^{n-2}\varepsilon^k|z|^{3k}\right)\mathrm{d}z.\end{aligned}$$

The second term can be absorbed by the first term if m is sufficiently large s.t. it satisfies the criterion of Lemma 4.2 and that

$$\bar{\gamma}^m m^{\frac{1}{2}(q+1)(n+1)} \leqslant (\det\Sigma)^{-\frac{1}{2}}\lambda_1^{-\frac{3}{2}(n-1)}\kappa_{n+\tau}^{n-2}. \tag{4.14}$$

For the third term, notice that $|z|>1$ and that $1<\varepsilon|z|^{1/\beta},\ \forall\beta\in(0,1/3)$. Thus

$$\widetilde{I} \leqslant C_{q,n,\tau}(\det\Sigma)^{-\frac{1}{2}}\lambda_1^{-\frac{3}{2}(n-1)}\kappa_{n+\tau}^{n-2}\varepsilon^{n-1} + C_{q,n}\kappa_n^{n-2}\int_{\mathbb{R}^q} e^{-\frac{1}{4}z\cdot\Sigma z}(\varepsilon|z|^{1/\beta})^{n-1}\mathrm{d}z,$$

and the result follows from (4.10) again. □

4.1.2 Perturbed Normal Distributions

Now given Proposition 4.5, one can approximate the density $\widetilde{f}_m$ by the inverse Fourier transform $\mathcal{F}^{-1}$ of $\exp\left(-\frac{1}{2}z\cdot\Sigma z\right)\mathcal{P}_\varepsilon(z)$. Define, $\forall x\in\mathbb{R}^q$, the Edgeworth polynomials $\{Q_k\}$ by

$$\phi_\Sigma(x)Q_k(x) := \mathcal{F}^{-1}\left\{e^{-\frac{1}{2}z\cdot\Sigma z}P_k(z)\right\}(x),\ \forall k\in\mathbb{Z}^+, \tag{4.15}$$

and accordingly $\mathcal{Q}_{\varepsilon,r} := 1+\sum_{k=1}^r \varepsilon^k Q_k,\ \forall r\in\mathbb{Z}^+$. Then each monomial of Q_k has the same degree as that of P_k. In fact, if $\Sigma=\mathrm{diag}(\lambda_1,\cdots,\lambda_q)$, one can explicitly show that

$$Q_k(x) = \sum_{|\alpha|=k+2}^{3k}(-1)^{|\alpha|}b_\alpha\prod_{j=1}^q \lambda_j^{-\alpha_j/2}H_{\alpha_j}\left(\lambda_j^{-1/2}x_j\right), \tag{4.16}$$

where $b_\alpha = b_\alpha(\mu_\beta :\ |\beta|\leqslant k+2)$ is the real coefficient of $(iz)^\alpha$ in $P_k(z)$ satisfying $|b_\alpha|\leqslant\kappa_{k+2}^k$, and H_j is the Hermite polynomial of degree j. See [44] (Chapter VI §1) for the precise values.

Remark 4.6. *Since* $\exp(-\frac{1}{2}z\cdot\Sigma z)\mathcal{P}_{\varepsilon,n-2}(z)$ *and* $\psi_m(z)$ *have the same derivatives at* 0 *up to order* n*, the Edgeworth sum* $\phi_\Sigma\mathcal{Q}_{\varepsilon,n-2}$ *and* Y_m *have the same moments up to order* n*.*

For a positive-definite $q\times q$ matrix Σ, let $\mathscr{P}_\Sigma$ be the set of polynomials $S:\mathbb{R}^q\to\mathbb{R}$ s.t. $\int_{\mathbb{R}^q}S_j(x)\phi_\Sigma(x)\mathrm{d}x = 0$ and $\mathscr{P}_G$ be the set of polynomials $U:\mathbb{R}^q\to\mathbb{R}^q$ s.t. $U=\nabla u$ for some polynomial $u:\mathbb{R}^q\to\mathbb{R}$. Furthermore let $\mathscr{P}_\Sigma^\infty$ be the set of sequences $(S_1,S_2,\cdots),\ S_j\in\mathscr{P}_\Sigma$, and $\mathscr{P}_G^\infty$ be the set of sequences $(U_1,U_2,\cdots),\ U_j\in\mathscr{P}_G$.

Given polynomials $U_j : \mathbb{R}^q \to \mathbb{R}^q,\ j = 1, \cdots, k$, define $\forall \varepsilon > 0$,

$$\mathbf{U}_{\varepsilon,k}(x) := x + \sum_{j=1}^{k} \varepsilon^j U_j(x).$$

Then for a ξ_Σ following $\mathcal{N}(0, \Sigma)$, the random variable $\mathbf{U}_{\varepsilon,k}(\xi_\Sigma)$ is said to have a perturbed normal distribution, whose density can be formally expressed as

$$\zeta_{\varepsilon,k}(y) = \det\left(\mathrm{D}\mathbf{U}_{\varepsilon,k}^{-1}(y)\right) \phi_\Sigma\left(\mathbf{U}_{\varepsilon,k}^{-1}(y)\right).$$

Davie [9] (Section 2) showed, using a recursive construction, that one can approximate $\zeta_{\varepsilon,k}(y)$ by the perturbed normal density $\phi_\Sigma(y)\mathcal{S}_{\varepsilon,l}(y) := \phi_\Sigma(y)\left(1 + \sum_{j=1}^{l} \varepsilon^j S_j(y)\right)$ up to order $O(\varepsilon^{l+1})$, where for each $j \leqslant l,\ S_j : \mathbb{R}^q \to \mathbb{R}$ is a polynomial uniquely determined by $U_1, \cdots, U_j$ only. Since l is arbitrary, for each k the polynomials $U_1, \cdots, U_k$ uniquely determine a sequence $(S_1, S_2, \cdots)$, and hence the map $\mathfrak{S}_\Sigma : (U_1, U_2 \cdots) \mapsto (S_1, S_2, \cdots)$ is well-defined. Moreover, each $S_j \in \mathscr{P}_\Sigma$ by Lemma 1 in [9].

A given sequence $(S_1, S_2, \cdots) \in \mathscr{P}_\Sigma^\infty$ can have several preimages under $\mathfrak{S}_\Sigma$. But according to Lemma 2 in [9], if one restricts $\mathfrak{S}_\Sigma$ on $\mathscr{P}_G^\infty$ then it is a bijection[2]. As is shown in the preceding paragraphs therein, this follows from the bijectivity of the linear map

$$\mathscr{L}_\Sigma : \mathscr{P}_G \to \mathscr{P}_\Sigma,\ U(x) \mapsto \nabla \cdot U(x) - x \cdot \Sigma^{-1} U(x).$$

The preimages of the bijection $\mathfrak{S}_\Sigma$ are defined inductively in the following way: given a sequence $(S_1, S_2, \cdots) \in \mathscr{P}_\Sigma^\infty$, suppose $U_1, \cdots, U_k \in \mathscr{P}_G$ are found with

$$\mathfrak{S}_\Sigma(U_1, \cdots, U_k) = (S_1, \cdots, S_k, \widetilde{S}_{k+1}, \cdots),$$

then adding an additional U_{k+1} gives a different sequence

$$\mathfrak{S}_\Sigma(U_1, \cdots, U_k, U_{k+1}) = (S_1, \cdots, S_k, \widetilde{S}_{k+1} - \mathscr{L}_\Sigma U_{k+1}, \cdots).$$

This means that $U_{k+1} \in \mathscr{P}_G$ is determined by the equation $\widetilde{S}_{k+1} - \mathscr{L}_\Sigma U_{k+1} = S_{k+1}$. Writing $U_{k+1} = \nabla u_{k+1}$, one looks for a polynomial u_{k+1} that solves the Hermite-type equation

$$-\Delta u_{k+1}(x) + x \cdot \Sigma^{-1} \nabla u_{k+1}(x) = S_{k+1}(x) - \widetilde{S}_{k+1}(x),\ x \in \mathbb{R}^q. \tag{4.17}$$

For the initial step set $\widetilde{S}_1 \equiv 0$ and solve the PDE by induction on the degree of u_1; at each step, first compute $\widetilde{S}_{k+1}$ from $u_1, \cdots, u_k$ and then solve the PDE again by induction on the degree of u_{k+1} - see similar arguments presented in the proof of Lemma 1 in [8].

The computation of $\widetilde{S}_{k+1}(x)$ can be done in the following formal way. First write

$$\phi_\Sigma(x) = \zeta_{\varepsilon,k}\left(\mathbf{U}_{\varepsilon,k}(x)\right) \det\left(\mathrm{D}\mathbf{U}_{\varepsilon,k}(x)\right), \tag{4.18}$$

by a change of variables. With $U_j = \nabla u_j,\ \mathrm{D}\mathbf{U}_{\varepsilon,k}(x) = I + \sum_{j=1}^{k} \varepsilon^j \mathrm{D}^2 u_j(x)$, and so the determinant above can be expressed as $1 + \varepsilon v_1(x) + \cdots + \varepsilon^{qk} v_{qk}(x)$, where for each $l \leqslant qk,\ v_l$ is the sum of $(\partial^2_{i_1 j_1} u_1)^{\rho_1} \cdots (\partial^2_{i_k j_k} u_k)^{\rho_k}$ over all the second derivatives and all

[2]This is motivated by Brenier's transport theorem for the quadratic cost - see Theorem 2.12 in [55] for the general statement and Lemma 5 in [9] for a special case.

multi-indices $\rho \in \mathbb{N}^k$ s.t. $|\rho|_* = l$. Then by formally writing $\zeta_{\varepsilon,k}(y) = \phi_\Sigma(y)\widetilde{\mathcal{S}}_\varepsilon(y)$ with $y = \mathbf{U}_{\varepsilon,k}(x)$ and $\widetilde{\mathcal{S}}_\varepsilon(y) = 1 + \sum_{j=1}^k \varepsilon^j S_j(y) + \sum_{j=k+1}^\infty \varepsilon^j \tilde{S}_j(y)$, one can rearrange (4.18) to get

$$\begin{aligned}
&1 + \varepsilon S_1(y) + \cdots + \varepsilon^k S_k(y) + \varepsilon^{k+1}\widetilde{S}_{k+1}(y) + \cdots \\
&= \frac{\exp\left\{\sum_{j=1}^k \varepsilon^j x \cdot \Sigma^{-1}\nabla u_j(x) + \frac{1}{2}\sum_{j_1,j_2=1}^k \varepsilon^{j_1+j_2}\nabla u_{j_1}(x)\cdot\Sigma^{-1}\nabla u_{j_2}(x)\right\}}{1 + \varepsilon v_1(x) + \cdots + \varepsilon^{qk} v_{qk}(x)} \\
&= 1 + \varepsilon T_1(x) + \varepsilon^2 T_2(x) + \cdots, \qquad (4.19)
\end{aligned}$$

where the series on the right-hand side is obtained by multiplying out the Maclaurin series for e^z and $1/(1+z)$. Since differentiating a polynomial only changes its coefficients by a constant and reduces its degree, one has

$$|T_{k+1}(x)| \leqslant C_{q,k}\|\Sigma^{-1}\|^{k+1}\sum_{|\rho|_*=k+1}(1+|u_1(x)|)^{\rho_1}\cdots(1+|u_k(x)|)^{\rho_k}.$$

On the left-hand side in (4.19), each polynomial $S_j(y)$ with degree $d_j \geqslant 1$ can be expressed as $S_j(x) + \varepsilon w_{j,1}(x) + \cdots + \varepsilon^{d_j k} w_{j,d_j k}(x)$ by its Taylor expansion about x. Since all the derivatives of $S_j(x)$ have their norms bounded by $C_{q,j}(1+|S_j(x)|)$, one has, for each $j \leqslant k$ and $l \leqslant d_j k$, that

$$|w_{j,l}(x)| \leqslant C_{q,j,l}\sum_{s=1}^{l}\sum_{|\rho|_*=l}(1+|S_j(x)|)|U_1(x)|^{\rho_1}\cdots|U_s(x)|^{\rho_s}.$$

Thus, by expanding out $\widetilde{S}_{k+1}(y)$ in terms of x and matching the coefficients of ε^{k+1} on both sides, one gets

$$\widetilde{S}_{k+1}(x) = T_{k+1}(x) - w_{k,1}(x) - w_{k-1,2}(x) - \cdots - w_{2,k-1}(x) - w_{1,k}(x). \qquad (4.20)$$

Although the calculation for $\widetilde{S}_{k+1}$ above is completely formal, it is equivalent to Davie's construction in [9] due to the uniqueness of the power series expansion. For a rigorous proof of such an approximation of $\zeta_{\varepsilon,k}$, the reader is referred to Proposition 1 in [9].

Remark 4.7. *The set $\mathscr{P}_G$ is invariant under orthogonal transformation: given $U(x) \in \mathscr{P}_G$ and an orthogonal matrix A, the polynomial $G(x) = A^{-1}U(Ax)$ also lies in $\mathscr{P}_G$.*

To see this, notice that if $U(x) = \nabla u(x)$ and A is a $q\times q$ matrix, then $g(x) := u(Ax)$ has gradient $A^\top U(Ax)$ and so $G(x) = \nabla u(Ax)$ if A is orthogonal.

The following lemma is a quantitative application of Proposition 1 in [9].

Lemma 4.8. *The real polynomials $\{Q_k\}_{k=1}^\infty$ uniquely determine a sequence of polynomials $\{p_k\}_{k=1}^\infty \in \mathscr{P}_G^\infty$ s.t. $\forall r \in \mathbb{Z}^+$ and ε sufficiently small,*

(i) $|p_k(x)| \leqslant C_{q,k}\lambda_1^{-5k(k+2)}\lambda_q^{1+\frac{5}{2}k(k+2)}\kappa_{r+2}^{k^2}\left(1+|x|^{3k}\right)$ *for all $k = 1,\cdots,r$ and $x \in \mathbb{R}^q$;*

(ii) *The random variable $\mathbf{p}_{\varepsilon,r}(\xi_\Sigma) := \xi_\Sigma + \sum_{k=1}^r \varepsilon^k p_k(\xi_\Sigma)$ has density*

$$\zeta_{\varepsilon,r}(x) = \phi_\Sigma(x)\mathcal{Q}_{\varepsilon,r}(x) + R_{\varepsilon,r}(x),$$

where for any $M \geqslant 1$,

$$\int_{\mathbb{R}^q}|x|^M|R_{\varepsilon,r}(x)|\mathrm{d}x \leqslant C_{q,r,M}\lambda_1^{-5(r+1)(r+2)}\lambda_q^{\frac{5}{2}(r+1)(r+3)+\frac{M}{2}}\kappa_{r+2}^{(r+1)^2}\varepsilon^{r+1}.$$

Proof. First of all, the Edgeworth polynomials $\{Q_k\}$ defined by (4.15) are orthogonal to ϕ_Σ:

$$\int_{\mathbb{R}^q} \phi_\Sigma(x) Q_k(x) \mathrm{d}x = \widehat{\phi_\Sigma Q_k}(0) = 1 \cdot P_k(0) = 0.$$

Thus $\{Q_k\} \in \mathscr{P}_\Sigma^\infty$, and hence $\{p_k\} := \mathfrak{S}_\Sigma^{-1}(\{Q_k\})$ gives the sequence sought after; for a fixed r, $p_1, \cdots, p_r$ are determined by $Q_1, \cdots, Q_r$ only. Moreover, if $\mathfrak{S}_\Sigma(p_1, \cdots, p_r) = (Q_1, \cdots, Q_r, \widetilde{Q}_{r+1}, \cdots)$, then the density $\zeta_{\varepsilon,r}$ of $\mathbf{p}_{\varepsilon,r}(\xi_\Sigma)$ can be approximated by the expansion $\phi_\Sigma(\mathcal{Q}_{\varepsilon,r} + \varepsilon^{r+1}\widetilde{Q}_{r+1})$ according to Proposition 1 in [9]. More precisely, $\forall M \geqslant 1$,

$$\int_{\mathbb{R}^q} |x|^M |\zeta_{\varepsilon,r}(x) - \phi_\Sigma(x)(\mathcal{Q}_{\varepsilon,r}(x) + \varepsilon^{r+1}\widetilde{Q}_{r+1}(x))| \mathrm{d}x \leqslant C_{q,r,M} K_r^{N_r} \varepsilon^{r+2}, \tag{4.21}$$

where K_r is an upper bound for $\|\Sigma\|, \|\Sigma^{-1}\|$ and the absolute value of the coefficients of $p_1, \cdots, p_r$, and $N_r = N_r(q, M) > 0$ is a constant depending on the maximum degree of $p_1, \cdots, p_r$. Then for $\varepsilon \leqslant K_r^{-N_r}$ this bound can be brought down to $C_{q,r,M}\varepsilon^{r+1}$, and it remains to find an upper bound for $\widetilde{Q}_{r+1}$ to derive the estimates in question.

For all $k \leqslant r$, write $p_k = \nabla u_k$ where u_k satisfies (4.17) with $S_k \equiv Q_k$ and $\widetilde{S}_k \equiv \tilde{Q}_k$. Assume that Σ is diagonal. Then by (4.16), $\forall k, x$ one has $|Q_k(x)| \leqslant C_{q,k} \lambda_1^{-3k} \kappa_{k+2}^k (1 + |x|^{3k})$. Now one can bound the polynomials $\{\widetilde{Q}_k\}$ and $\{u_k\}$ inductively. For each $k \leqslant r - 1$ suppose that (i) holds true for all $j \leqslant k$:

$$|u_j(x)| \leqslant C_{q,j} \lambda_1^{-5j(j+2)} \lambda_q^{1+\frac{5}{2}j(j+2)} \kappa_{r+2}^{j^2} (1 + |x|^{3j}).$$

From the construction of T_{k+1} and $\{w_{j,l}\}$ one sees that,

$$\begin{aligned}
|T_{k+1}(x)| \leqslant & C_{q,k} \|\Sigma^{-1}\|^{k+1} \lambda_1^{-5\sum^* j(j+2)\rho_j} \lambda_q^{k+1+\frac{5}{2}\sum^* j(j+2)\rho_j} \kappa_{r+2}^{\sum^* j^2 \rho_j} \left(1 + |x|^{\sum^* 3j\rho_j}\right) \\
\leqslant & C_{q,k} \lambda_1^{-(k+1)(5k+11)} \lambda_q^{(k+1)\left(\frac{5}{2}k+6\right)} \kappa_{r+2}^{(k+1)^2} \left(1 + |x|^{3(k+1)}\right), \qquad (4.22) \\
|w_{j,l}(x)| \leqslant & C_{q,j,l} \lambda_1^{-3j-5\sum^\dagger s(s+2)\rho_s} \lambda_q^{l+\frac{5}{2}\sum^\dagger s(s+2)\rho_s} \kappa_{r+2}^{j+\sum^\dagger s^2\rho_s} \left(1 + |x|^{3j+\sum^\dagger 3s\rho_s}\right) \\
\leqslant & C_{q,j,l} \lambda_1^{-3j-5l(l+2)} \lambda_q^{l+\frac{5}{2}l(l+2)} \kappa_{r+2}^{j+l(l+1)} \left(1 + |x|^{3(j+l)}\right),
\end{aligned}$$

where $\sum^*$ denotes the summation over $j = 1 \cdots, k$ and all multi-indices $\rho \in \mathbb{N}^k$ s.t. $|\rho|_* = k + 1$, and $\sum^\dagger$ denotes the summation over $s = 1, \cdots, l$ and all $\rho \in \mathbb{N}^l$ s.t. $|\rho|_* = l$. Then $\left|\sum_{j+l=k+1} w_{j,l}(x)\right| \leqslant C_{q,k} \lambda_1^{-5(k+1)(k+2)} \lambda_q^{k\left(\frac{5}{2}k+6\right)} \kappa_{r+2}^{(k+1)^2} \left(1 + |x|^{3(k+1)}\right)$, which is no more than (4.22), and hence by (4.20) $\widetilde{Q}_{k+1}$ has the same bound as (4.22).

For each $\alpha \in \mathbb{N}^q$, it is known that the Hermite-type polynomial

$$H_{\alpha,\Sigma}(x) = \frac{1}{\sqrt{\alpha!}} \prod_{j=1}^q H_{\alpha_j}(\lambda_j^{-1/2} x_j)$$

is the eigenfunction of the differential operator of the equation (4.17) corresponding to the eigenvalue $\nu_\alpha := \sum_{j=1}^q \alpha_j \lambda_j^{-1} \leqslant |\alpha|/\lambda_q$. Since $\{H_{\alpha,\Sigma}\}$ form an orthonormal basis for the Hilbert space $L^2(\mathbb{R}^q, \phi_\Sigma \mathrm{d}\Lambda)$, the polynomial u_{k+1} can be expressed as

$$u_{k+1}(x) = \sum_{|\alpha| \leqslant 3(k+1)} c_\alpha \nu_\alpha^{-1} H_{\alpha,\Sigma}(x),$$

where $c_\alpha = \int_{\mathbb{R}^q}(Q_{k+1}(z) - \widetilde{Q}_{k+1}(z))H_{\alpha,\Sigma}(z)\phi_\Sigma(z)\mathrm{d}z$. Then by the Cauchy-Schwartz inequality and (4.11), the above estimate for $\widetilde{Q}_{k+1}$ implies that

$$
\begin{aligned}
|u_{k+1}(x)| \leqslant & C_{q,k} \sum_{|\alpha|\leqslant 3(k+1)} C_\alpha \left(\int_{\mathbb{R}^q} |Q_{k+1}(z) - \widetilde{Q}_{k+1}(z)|^2 \phi_\Sigma(z)\mathrm{d}z\right)^{\frac{1}{2}} \lambda_q \lambda_1^{-\frac{|\alpha|}{2}} \left(1 + |x|^{|\alpha|}\right) \\
\leqslant & C_{q,k}\lambda_1^{-(k+1)(5k+11)-\frac{3}{2}(k+1)} \lambda_q^{(k+1)\left(\frac{5}{2}k+6\right)+\frac{3}{2}(k+1)+1} \kappa_{r+2}^{(k+1)^2} \left(1 + |x|^{3(k+1)}\right) \\
\leqslant & C_{q,k}\lambda_1^{-5(k+1)(k+3)} \lambda_q^{1+\frac{5}{2}(k+1)(k+3)} \kappa_{r+2}^{(k+1)^2} \left(1 + |x|^{3(k+1)}\right),
\end{aligned}
$$

which agrees with the induction hypothesis; the initial step for u_1 also holds true as $\widetilde{Q}_1 \equiv 0$. Therefore the bound in (i) holds true for each u_k, and it holds true for its gradient p_k, too. The induction step also gives the bound (4.22) for $\widetilde{Q}_{r+1}$, and hence (ii) follows from the triangle inequality and (4.11) again.

For a general positive-definite Σ, one diagonalises it with an orthogonal matrix A and applies the same arguments above to the scaled polynomials $p_k^*(x) := A^\top p_k(Ax)$. By Remark 4.7 the p_k^*'s still lie in $\mathscr{P}_G$, and the results still hold. $\square$

The proof above takes a compromise approach by introducing $\widetilde{Q}_{r+1}$ in (4.21): the condition "ε sufficiently small" is not needed for Lemma 4.8, as Proposition 1 in [9] allows an $O(\varepsilon^{r+1})$ estimate for $\int_{\mathbb{R}^q} |x|^M |\zeta_{\varepsilon,r}(x) - \phi_\Sigma(x)\mathcal{Q}_{\varepsilon,r}(x)|\mathrm{d}x$ for all $\varepsilon > 0$. However, whilst practically $K_r = \lambda_1^{-5r(r+2)}\lambda_q^{1+5r(r+2)/2}\kappa_{r+2}^{r^2}$ by (i), it is rather complicated to compute N_r explicitly.

Before proceeding to the main result, given fixed parameters $\beta \in (0, 1/3)$ and $\bar{\gamma}, \tau \in (0, 1)$, it would be convenient to combine all the criteria for ε together: for any integer $r \geqslant 3$ the statement "m **sufficiently large w.r.t.** r" or "ε **sufficiently small w.r.t.** r" refers to that $m > \kappa_{r+\tau}^{\max\{4,\ 6/(r(1-3\beta))\}} \vee K_{r-3}^{2N_{r-3}}$ with $K_0, N_0 := 1$ and that (4.14) holds for $n = r$.

4.1.3 Main Result and Some Special Cases

Given Lemma 4.8, it will be shown in the following theorem that the normal distribution $\mathcal{N}(0, \Sigma)$ perturbed by the polynomials $\{p_k\}$ is close to the law $\mathbb{P}_m$ in the Vaserstein distances. The proof is a more detailed and quantitative version of what is exhibited in Section 4 in [9], and specifies the dependence on Σ and certain moments of X.

Theorem 4.9. *Suppose Σ is non-singular and χ satisfies Assumption4.3. Fix an integer $n \geqslant 3$, an even integer $p \in 2\mathbb{Z}^+$ and $\beta \in (0, 1/3)$. If $\mathbb{E}|X|^{p(n-2)+2+\tau} < \infty$ for some $\tau \in (0, 1)$, then for m sufficiently large w.r.t. $p(n-2)+2$,*

$$
\mathbb{W}_p(Y_m,\ \mathbf{p}_{m,n-3}(\xi_\Sigma)) \leqslant C_{p,q,n,\tau}\Xi_X m^{-(n-2)/2},
$$

where $\mathbf{p}_{m,n-3}$ is the polynomial defined by Lemma 4.8, and Ξ_X is a constant depending on $p, n, \beta, \eta, \Sigma,\ \mathbb{E}|X|^{p(n-2)+1}$ and $\mathbb{E}|X|^{p(n-2)+2+\tau}$.

Proof. Denote $r = p(n-2)+2$. Taking the inverse Fourier transform, Proposition 4.5 implies that for all $x \in \mathbb{R}^q$ and for m sufficiently large w.r.t. r,

$$
\begin{aligned}
|F_{r-2}(\varepsilon, x)| := \left|\tilde{f}_m(x) - \phi_\Sigma(x)\mathcal{Q}_{\varepsilon,r-2}(x)\right| \leqslant & C_q \int_{\mathbb{R}^q} \left|\tilde{\psi}_m(z) - e^{-\frac{1}{2}z\cdot\Sigma z}\mathcal{P}_{\varepsilon,r-2}(z)\right| \mathrm{d}z \\
\leqslant & C_{q,r,\tau}(\det\Sigma)^{-\frac{1}{2}}\lambda_1^{-\frac{r-1}{2\beta}}\kappa_{r+\tau}^{r-2}\varepsilon^{r-1}.
\end{aligned} \tag{4.23}
$$

The goal is to use the inequality (4.4) to bound the $\mathbb{W}_p$ distance, for which one first writes

$$\int_{\mathbb{R}^q} |x|^p |F_{r-2}(\varepsilon,x)| \mathrm{d}x \leqslant \int_{\mathbb{R}^q} |x|^p \left(\tilde{f}_m(x) + \phi_\Sigma(x)\,|\mathcal{Q}_{\varepsilon,r-2}(x)|\right) \mathrm{d}x \leqslant I_1 + 2I_2 + I_3,$$

where, for any $\eta \in (0,1)$,

$$I_1 := \int_{|x|\leqslant \varepsilon^{-\eta/(p+q)}} |x|^p |F_{r-2}(\varepsilon,x)| \mathrm{d}x,\ I_2 := \int_{|x|>\varepsilon^{-\eta/(p+q)}} |x|^p \phi_\Sigma(x)\,|\mathcal{Q}_{\varepsilon,r-2}(x)|\,\mathrm{d}x,$$
$$I_3 := \int_{|x|>\varepsilon^{-\eta/(p+q)}} |x|^p F_{r-2}(\varepsilon,x) \mathrm{d}x.$$

For any fixed $p \geqslant 2$ and $\eta \in (0,1)$, one finds, by virtue of (4.23),

$$I_1 \leqslant C_{q,r,\tau} (\det \Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} \kappa_{r+\tau}^{r-2} \varepsilon^{r-1-\eta}.$$

For the integral on the complement $\{x:\ |x| > \varepsilon^{-\eta/(p+q)}\} = \{1 < \varepsilon |x|^{(p+q)/\eta}\}$, one gets

$$\begin{aligned} I_2 \leqslant & \int_{|x|>\varepsilon^{-\eta/(p+q)}} |x|^p \phi_\Sigma(x) \kappa_r^{r-2} \left(1 + \sum_{k=1}^{r-2} \varepsilon^k |x|^{3k}\right) \mathrm{d}x \\ \leqslant & C_r \int_{|x|>\varepsilon^{-\eta/(p+q)}} |x|^p \phi_\Sigma(x) \kappa_r^{r-2} \varepsilon^{r-1} |x|^{\frac{p+q}{\eta}(r-1)} \mathrm{d}x \leqslant C_r \lambda_q^{\frac{p}{2}+\frac{p+q}{2\eta}(r-1)} \kappa_r^{r-2} \varepsilon^{r-1}, \end{aligned}$$

due to the fact that $(p+q)/\eta > 3$ and (4.11). Also observe that

$$I_3 \leqslant \int_{\mathbb{R}^q} |x|^p \left(\tilde{f}_m(x) - \phi_\Sigma(x) \mathcal{Q}_{\varepsilon,r-2}(x)\right) \mathrm{d}x + I_1 =: I_4 + I_1,$$

by the triangle inequality. In order to get a good estimate for I_4, first observe that $\forall p \geqslant 2$ by Rosenthal's inequality - see e.g. Lemma 1 in [14],

$$\int_{\mathbb{R}^q} |x|^p \mathbb{P}_m(\mathrm{d}x) = \mathbb{E}|Y_m|^p = m^{-\frac{p}{2}} \mathbb{E}\left|\sum_{j=1}^m X_j\right|^p \leqslant C_p \left(m^{1-\frac{p}{2}} \mathbb{E}|X|^p + (\mathbb{E}|X|^2)^{\frac{p}{2}}\right). \quad (4.24)$$

Also, from the construction of θ_ε (4.12) (now supported on $\{|x| < \varepsilon^{r+1}\}$),

$$\int_{\mathbb{R}^q} |x|^p \theta_\varepsilon(\mathrm{d}x) = \int_{|y|<1} |y|^p \varepsilon^{p(r+1)} h(y) \mathrm{d}y < \varepsilon^{p(r+1)}, \quad (4.25)$$

by a change of variables. For an even $p \geqslant 4$, as $p < r$ observe that all the moments up to p of the expansion $\phi_\Sigma \mathcal{Q}_{\varepsilon,r-2}$ match those of Y_m by Remark 4.6. Hence by (4.24) and (4.25),

$$\begin{aligned} I_4 \leqslant & \int_{\mathbb{R}^q}\int_{\mathbb{R}^q} (|x+y|^p - |x|^p)\, \mathbb{P}_m(\mathrm{d}x)\theta_\varepsilon(\mathrm{d}y) \leqslant C_{p,q} \sum_{k=1}^p \int_{\mathbb{R}^q}\int_{\mathbb{R}^q} |x|^{p-k}|y|^k \mathbb{P}_m(\mathrm{d}x)\theta_\varepsilon(\mathrm{d}y) \\ \leqslant & C_{p,q}\varepsilon^{r+1} \int_{\mathbb{R}^q} |x|^{p-1} \mathbb{P}_m(\mathrm{d}x) \leqslant C_{p,q} \varepsilon^{r+1} \left(\varepsilon^{p-3} \mathbb{E}|X|^{p-1} + \lambda_q^{\frac{p-1}{2}}\right); \end{aligned}$$

for $p = 2$ the bound is reduced to $C_{p,q}\varepsilon^{r+1}(\mathbb{E}|X|^2)^{1/2}$ by (4.24) and Hölder's inequality.

Therefore, altogether one arrives at, for $p \leqslant r$,

$$\int_{\mathbb{R}^q} |x|^p |F_{r-2}(\varepsilon, x)| \mathrm{d}x \leqslant C_{p,q,r,\tau} \left((\det\Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} + \lambda_q^{\frac{p}{2} + \frac{p+q}{2\eta}(r-1)} \right) \kappa_{r+\tau}^{r-2} \varepsilon^{r-1-\eta}.$$

Finally by the triangle inequality one removes the $(r-2)$-th term in $\mathcal{Q}_{\varepsilon,r-2}$:

$$\begin{aligned}\int_{\mathbb{R}^q} |x|^p |F_{r-2}(\varepsilon, x)| \mathrm{d}x \geqslant & \int_{\mathbb{R}^q} |x|^p |F_{r-3}(\varepsilon, x)| \mathrm{d}x \\ & - C_{q,r} \varepsilon^{r-2} \kappa_r^{r-2} \int_{\mathbb{R}^q} |x|^p \left(|x|^r + |x|^{3(r-2)} \right) \phi_\Sigma \mathrm{d}x,\end{aligned}$$

and uses (4.11) again to deduce the following estimate:

$$\int_{\mathbb{R}^q} |x|^p |F_{r-3}(\varepsilon, x)| \mathrm{d}x \leqslant C_{p,q,r} \left((\det\Sigma)^{-\frac{1}{2}} \lambda_1^{-\frac{r-1}{2\beta}} + \lambda_q^{\frac{p}{2} + \frac{p+q}{2\eta}(r-1)} \right) \kappa_{r+\tau}^{r-2} \varepsilon^{r-2}.$$

Since the smooth measure θ_ε is also supported on $\{x : |x| < \varepsilon^{r-2}\}$, the estimate above implies that the Edgeworth polynomials $\{Q_k\} \in \mathscr{P}_\Sigma$ form an $\mathcal{A}_\Sigma$-sequence for the family of probability measures $\{\mathbb{P}_m\}$ - see Definition 1 in [9]. Then one can extend the expansion $\mathcal{Q}_{\varepsilon,r-3}$ to a larger value of r and take the p-th root to get a $\mathbb{W}_p$ estimate, as in done in the proof of Theorem 4 in [9].

If ς_ε is a random variable having law θ_ε, independent of Y_m, then $\widetilde{Y}_m := Y_m + \varsigma_\varepsilon$ has law $\mathbb{P}_m * \theta_\varepsilon$, and $\mathbb{W}_p(\widetilde{Y}_m, Y_m) \leqslant (\mathbb{E}|\varsigma_\varepsilon|^p)^{1/p} \leqslant \varepsilon^{p(n-2)}$. Now with the polynomials $\{p_k\} = \mathfrak{S}_\Sigma^{-1}(\{Q_k\})$ and $\mathbf{p}_{\varepsilon,r-3}$, $R_{\varepsilon,r-3}$ defined as in Lemma 4.8, using the triangle inequality and the inequality (4.4), one can deduce the following estimate for an integer $n \geqslant 3$ by replacing $r = p(n-2)+2$ in the estimate:

$$\begin{aligned}&\mathbb{W}_p \left(\widetilde{Y}_m,\ \mathbf{p}_{\varepsilon,p(n-2)-1}(\xi_\Sigma) \right) \\ \leqslant & C_p \left(\int_{\mathbb{R}^q} |x|^p |F_{p(n-2)-1}| \mathrm{d}x + \int_{\mathbb{R}^q} |x|^p \left| R_{\varepsilon,p(n-2)-1}(x) \right| \mathrm{d}x \right)^{1/p} \\ \leqslant & C_{p,q,n,\tau} \left((\det\Sigma)^{-\frac{1}{2p}} \lambda_1^{-\frac{1}{2\beta}\left(n-2+\frac{1}{p}\right)} + \lambda_q^{\frac{1}{2} + \frac{p+q}{2\eta}\left(n-2+\frac{1}{p}\right)} \right) \kappa_{p(n-2)+2+\tau}^{n-2} \varepsilon^{n-2} \\ & + C_{p,q,n} \lambda_1^{-5(n-2)(p(n-2)+1)} \lambda_q^{\frac{1}{2} + \frac{5}{2}(n-2)(p(n-2)+1)} \kappa_{p(n-2)+1}^{p(n-2)^2} \varepsilon^{n-2},\end{aligned}$$

whilst the excess terms from $n-2$ to $p(n-2)-1$ can be handled by part (i) of Lemma 4.8 and the inequality (4.11) again:

$$\begin{aligned}\mathbb{W}_p \left(\mathbf{p}_{\varepsilon,p(n-2)-1}(\xi_\Sigma),\ \mathbf{p}_{\varepsilon,n-3}(\xi_\Sigma) \right) & \leqslant C_p \left(\mathbb{E} \left| \sum_{k=n-2}^{p(n-2)-1} \varepsilon^k p_k(\xi_\Sigma) \right|^p \right)^{1/p} \\ & \leqslant C_{p,q,n} \lambda_1^{-5p^2(n-2)^2+5} \lambda_q^{\frac{5}{2}p^2(n-2)^2 + \frac{3}{2}p(n-2) - 3} \kappa_{p(n-2)+1}^{(p(n-2)-1)^2} \varepsilon^{n-2}.\end{aligned}$$

Thus the claimed result follows from the triangle inequality. □

Remark 4.10. *The number of moments of X needed for Theorem 4.9 is independent of the dimension q.*

The optimal result for the central limit theorem for $q = 1$ is already given in [3], which is not fully recovered by Theorem 4.9 as the inequality (4.4) is rather crude compared to (1.12). For $q \geqslant 2$, Theorem 4.9 immediately implies the following (by choosing $n = 3$):

Corollary 4.11. *Suppose the i.i.d. random variables $\{X_j\}$ have non-singular covariance Σ and satisfy Assumption 4.3, and let $p \in 2\mathbb{Z}^+$. If $\mathbb{E}|X|^{p+2+\tau} < \infty$ for some $\tau \in (0,1)$, then by taking e.g. $\beta = 1/6,\ \eta = 1/2$, the following holds for m sufficiently large w.r.t. $p+2$:*

$$\begin{aligned}\mathbb{W}_p(Y_m,\ \xi_\Sigma) \leqslant & C_{p,q,\tau}\left((\det\Sigma)^{-1/(2p)}\lambda_1^{-3(1+1/p)} + \lambda_q^{(p+q)(1+1/p)+1/2}\right)\kappa_{p+2+\tau}m^{-1/2} \\ & + C_{p,q}\lambda_1^{-5p^2+5}\lambda_q^{(5p^2+3p)/2-3}\kappa_{p+1}^{2\vee(p-1)^2}m^{-1/2}.\end{aligned}$$

As mentioned in the beginning of the chapter, for the special case $p = 2$ this corollary is weaker than the results of Bobkov [4] and Bonis [5]. Although the condition $\mathbb{E}|X|^{4+\tau} < \infty$ is slightly better than that of Bobkov, he does not require Cramér's condition as he used Talagrand's transport inequality [53] and aimed at estimating the relative entropy $\mathbb{H}(\mathbb{P}_m \| \mathcal{N}(0,\Sigma))$. On the other hand, Bonis' optimal result relies on a differential estimate in terms of the Fisher information. Compared to those special properties of the $\mathbb{W}_2$ distance, the inequality (4.4) is rather crude.

However, Theorem 4.9 can potentially give higher-order convergence if one considers a non-trivial expansion ($n > 3$). For example, when choosing $n = 4$, one gets a rate $O(m^{-1})$ under Cramér's condition and that $\mathbb{E}|X|^{6+\tau} < \infty$. The task is to find the polynomial p_1 using the method described in the discussion before Lemma 4.8: given $Q_1(x)$ defined in (4.16), one looks for the unique (up to an additive constant) polynomial solution $u_1 : \mathbb{R}^q \to \mathbb{R}$ satisfying (4.17) for the initial step:

$$-\Delta u_1(x) + x \cdot \Sigma^{-1}\nabla u_1(x) = Q_1(x).$$

To illustrate that consider the simplest case where $q = 2$ and $\Sigma = I$. The cubic polynomial $6iP_1(z) = \mu_{(3,0)}z_1^3 + 3\mu_{(2,1)}z_1^2 z_2 + 3\mu_{(1,2)}z_1 z_2^2 + \mu_{(0,3)}z_2^3$ gives

$$6Q_1(x) = \mu_{(3,0)}H_3(x_1) + 3\mu_{(2,1)}H_2(x_1)H_1(x_2) + 3\mu_{(1,2)}H_1(x_1)H_2(x_2) + \mu_{(0,3)}H_3(x_2).$$

Notice that $x \cdot \nabla u(x) = ku(x)$ for any monomial u of degree k, and so the polynomial solution to the PDE above is cubic with no quadratic terms. Then using the property $H_j' = jH_{j-1}$ and matching the coefficients on both sides of the equation, one gets

$$\begin{aligned}u_1(x) =& \frac{1}{18}\mu_{(3,0)}H_3(x_1) + \frac{1}{6}\mu_{(2,1)}H_2(x_1)H_1(x_2) + \frac{1}{6}\mu_{(1,2)}H_1(x_1)H_2(x_2) \\ & + \frac{1}{18}\mu_{(0,3)}H_3(x_2) + \frac{1}{3}(\mu_{(3,0)} + \mu_{(1,2)})H_1(x_1) + \frac{1}{3}(\mu_{(0,3)} + \mu_{(2,1)})H_1(x_2) + C,\end{aligned}$$

and hence the perturbing polynomial $p_1 = \nabla u_1$ is found.

Under certain stronger conditions, one can also obtain higher-order convergence without specifying the perturbing polynomials p_k. For $q = 1$, Bobkov [3] (Theorem 1.3) proved that if $\mathbb{E}X^k = \mathbb{E}\xi_\Sigma^k$ for $k = 1, \cdots, n-1,\ 3 \leqslant n \in \mathbb{Z}^+$, and $\mathbb{E}|X|^{p(n-2)+2} < \infty$, then under Cramér's condition one has $\mathbb{W}_p(Y_m, \xi_\Sigma) = O(m^{-(n-2)/2})$. This can be readily generalised to $q \geqslant 2$ by Theorem 4.9: if the first $n-1$ moments match those of $\mathcal{N}(0,\Sigma)$, the cumulants $\mu_\alpha(X) = \mu_\alpha(\xi_\Sigma) = 0$ for all $|\alpha| = 3, \cdots, n-1$, implying that $P_k = Q_k \equiv 0$. This immediately implies that $\mathcal{L}_\Sigma p_k \equiv 0$ in (4.17) for all $k = 1, \cdots, n-3$, forcing $p_k \equiv 0$. Therefore one asserts the following:

Corollary 4.12. *Suppose the i.i.d. random variables $\{X_j\}$ with non-singular covari-*

ance Σ satisfy Cramér's condition and let $p \in 2\mathbb{Z}^+$. If $\exists 3 \leqslant n \in \mathbb{Z}^+$ s.t. $\mathbb{E}X^\alpha = \mathbb{E}\xi_\Sigma^\alpha$ for all $|\alpha| = 1, \cdots, n-1$, and $\mathbb{E}|X|^{p(n-2)+2+\tau} < \infty$ for some $\tau \in (0,1)$, then $\mathbb{W}_p(Y_m, \xi_\Sigma) = O(m^{-(n-2)/2})$ for m sufficiently large w.r.t. $p(n-2)+2$.

4.2 Application to Euler's Method for Lévy-SDEs

Consider the d-dimensional SDE (4.2) driven by a q-dimensional Lévy process (4.1). Assume that the Lévy measure ν has finite second moment, and the function $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times q}$ is bounded and Lipschitz. It will be shown in this section that the q-dimensional small jumps (4.3) can also be approximated by a normal random variable with rate 1 while the computational cost $E_\nu(h)$ remains controlled for ν satisfying certain stable-like conditions, in particular Assumption 4.1.

4.2.1 Normal Approximation of the Small Jumps

The way both Fournier [11] and Godinho [13] applied the central limit theorem for the small jumps Z_t^ϵ is to split the time interval into m subdivisions and view Z_t^ϵ as the sum of the i.i.d. random variables $\int_{(j-1)t/m}^{jt/m}\int_{0<|z|\leqslant\epsilon} z\tilde{N}(\mathrm{d}z,\mathrm{d}s)$, $j = 1, \cdots, m$. Here an alternative approach is considered: one may decompose the range of the jumps $\{0 < |z| \leqslant \epsilon\}$ into countably many annuli and represent the small jumps as a sum:

$$Z_t^\epsilon = \sum_{r=r_0}^{\infty} \int_0^t \int_{\Omega_r} z\tilde{N}(\mathrm{d}z,\mathrm{d}s) =: \sum_{r=r_0}^{\infty} V_t^r, \tag{4.26}$$

where $\Omega_r = \{2^{-r-1} < |z| \leqslant 2^{-r}\}$ and $r_0 = -\log_2 \epsilon > 0$. Assume ν to be σ-finite and denote $\nu_r := \nu(\Omega_r)$. By the Lévy-Itô decomposition one knows that each V_t^r is a compensated compound Poisson process:

$$V_t^r = \sum_{j=1}^{N_t^r} X_j^r - t\nu_r \mathbb{E}X_j^r, \tag{4.27}$$

where $\{X_j^r\}$ are i.i.d. random variables bounded within Ω_r and N_t^r follows $\mathrm{Poi}(t\nu_r)$.

Instead of directly working with V_t^r, one may first consider a general compound Poisson process V_t of the form (4.27) with $N_t \sim \mathrm{Poi}(t\mu)$ and the jumps X_j on the interval $[0,1]$. Expecting μ to be large, one can write

$$Y = \mu^{-\frac{1}{2}} V_1 = \mu^{-\frac{1}{2}} \sum_{j=1}^{N_1} X_j - \mu^{\frac{1}{2}} \mathbb{E}X_j,$$

and approximate it by Edgeworth-type polynomials using the same recipe just as before.

Let ψ and χ be the characteristic functions of Y and the X_j's, respectively, and Σ_X be the covariance of X, with eigenvalues $\lambda_{1,X} \leqslant \cdots \leqslant \lambda_{q,X}$, and similarly $\kappa_{M,X} = 1 \vee \mathbb{E}|X|^M$, $\forall M > 0$. Then one has the following simple relation between the distributions of X and Y:

$$\psi(z) = \exp\left\{\mu\left(\chi\left(\mu^{-\frac{1}{2}}z\right) - 1\right) - i\mu^{\frac{1}{2}} z \cdot \mathbb{E}X\right\}.$$

Given this convenient expression, instead of taking the logarithm one may directly apply Taylor expansion to χ and have, instead of (4.5), a formal expansion

$$\psi(z) \sim e^{-\frac{1}{2}z\cdot\Sigma_X z}\left(1+\sum_{k=1}^{\infty}\mu^{-\frac{k}{2}}P_k(z)\right),$$

whose $(n-2)$-th truncation leads to the same bound as in Lemma 4.2 with μ in place of m and $\varepsilon = \mu^{-1/2}$ for $|z| \leqslant \mu^{\beta/2}$, $\beta \in (0, 1/3)$. Note that the P_k here are slightly different (in fact simpler): since no logarithm is taken, the cumulants μ_α are replaced with just $\mathbb{E}X^\alpha$. Also for $|z| \leqslant \mu^{1/2}\delta = \mu^{1/2}\min\{\lambda_{1,X}/\kappa_{3,X},\ \kappa_{n,X}^{-1/n}/2\}$, one still has

$$\begin{aligned}|\psi(z)| =& \left|\exp\left\{\mu\left(\chi\left(\mu^{-\frac{1}{2}}z\right)-1\right)-i\mu^{\frac{1}{2}}z\cdot\mathbb{E}X\right\}\right|\\ \leqslant& e^{-\frac{1}{2}z\cdot\Sigma_X z+\frac{1}{6}\mu^{-\frac{1}{2}}\mathbb{E}|X|^3|z|^3} \leqslant e^{-\frac{1}{4}z\cdot\Sigma_X z}.\end{aligned}$$

Moreover, by imposing Cramér's condition (Assumption 4.3) on the distribution of X, one can still achieve a similar bound for $|\psi|$:

$$\begin{aligned}|\psi(z)| =& \left|\exp\left\{\chi\left(\mu^{-\frac{1}{2}}z\right)-1-i\mu^{-\frac{1}{2}}z\cdot\mathbb{E}X\right\}\right|^{\mu}\\ =& \left(\exp\left\{\mathrm{Re}\chi\left(\mu^{-\frac{1}{2}}z\right)-1\right\}\right)^{\mu} \leqslant \left(e^{\bar{\gamma}-1}\right)^{\mu} \in (0,1),\end{aligned} \tag{4.28}$$

for $|z| > \mu^{1/2}\delta$ and some $\bar{\gamma} \in (0,1)$ according to Lemma 4.4. Thus one arrives at virtually the same estimate as in Proposition 4.5, and therefore Theorem 4.9 still holds true for $\varepsilon = \mu^{-1/2}$ sufficiently small w.r.t. $p+2$, and Corollary 4.11 applies with μ in place of m and $\exp(\bar{\gamma}-1)$ in place of $\bar{\gamma}$. For the normal approximation $(n = 3)$, since no perturbing polynomials p_k are concerned, one can scale the jumps $\widehat{X} := \Sigma_X^{-1/2}X$ and deduce, $\forall\tau \in (0,1)$,

$$\mathbb{W}_p\left(V_1, \mu^{\frac{1}{2}}\xi_{\Sigma_X}\right) \leqslant \left\|\Sigma_X^{\frac{1}{2}}\right\|\mathbb{W}_p\left(\Sigma_X^{-\frac{1}{2}}V_1, \mu^{\frac{1}{2}}\xi_I\right) \leqslant C_{p,q,\tau}\kappa_{p+2+\tau,\widehat{X}}^{2\vee(p-1)^2}\lambda_{q,X}^{1/2}. \tag{4.29}$$

One can apply the above arguments to the jump process (4.27) by scaling the jump sizes. For the jumps X_j^r on each annulus Ω_r, define $X_j := 2^r X_j^r$ and $\widehat{X}_j := \Sigma_X^{-1/2}X_j$ accordingly. For each fixed r, the X_j^r's are i.i.d. with characteristic function

$$\chi^r(s) = \nu_r^{-1}\int_{\Omega_r} e^{is\cdot x}\nu(\mathrm{d}x).$$

This implies that X has scaled covariance $\Sigma_X = \nu_r^{-1}2^{2r}\int_{\Omega_r} xx^\top\nu(\mathrm{d}x)$ with eigenvalues $\lambda_{j,X} = \nu_r^{-1}2^{2r}\lambda_{j,r}$, where $\lambda_{1,r} \leqslant \cdots \leqslant \lambda_{q,r}$ are the eigenvalues of $\Sigma_r := \int_{\Omega_r} xx^\top\nu(\mathrm{d}x)$. Also notice that $\mathbb{E}|X|^M = \nu_r^{-1}2^{rM}\int_{\Omega_r}|x|^M\nu(\mathrm{d}x) \leqslant 1$, $\forall M > 0$, implying that $\mathbb{E}|\widehat{X}|^M \leqslant \lambda_{1,X}^{-M/2}$.

Thus, if Σ_r is non-singular for each r, then (assuming $\lambda_{1,X} \leqslant 1$ w.l.o.g.) one can apply (4.29) with parameter $\mu = t\nu_r$:

$$\begin{aligned}\mathbb{W}_p\left(V_t^r, \sqrt{t}\xi_{\Sigma_r}\right) =& 2^{-r}\mathbb{W}_p\left(2^r V_t^r, \sqrt{t\nu_r}\xi_{\Sigma_X}\right)\\ \leqslant& C_{p,q,\tau}\lambda_{1,X}^{-(1\vee\frac{(p-1)^2}{2})(p+2+\tau)}\lambda_{q,r}^{1/2}\nu_r^{-1/2}.\end{aligned} \tag{4.30}$$

Denote further $\Sigma_\epsilon := \int_{0<|x|\leqslant\epsilon} xx^\top\nu(\mathrm{d}x)$, then from this bound one can find a coupling between Z_t^ϵ and $\mathcal{N}(0, t\Sigma_\epsilon)$ under suitable conditions.

Theorem 4.13. *Suppose $\xi_r(s) := \chi^r(2^r s)$ satisfies Cramér's condition uniformly for all $r \geqslant r_0$, i.e. Assumption 4.3 holds for each ξ_r with ρ, γ independent of r. If $\nu_r^{-1} = o(2^{-r})$ as $r \to \infty$, then $\forall p \in 2\mathbb{Z}^+,\ t \geqslant \epsilon$ and ϵ sufficiently small,*

$$\mathbb{W}_p\left(Z_t^\epsilon, \sqrt{t}\xi_{\Sigma_\epsilon}\right) \leqslant C_{p,q}\epsilon.$$

Proof. Note that on each Ω_r it is always true that $\lambda_{q,r} \leqslant \mathrm{tr}\Sigma_r \leqslant 2^{-2r}\nu_r$ and $\lambda_{q,r} \geqslant q^{-1}\mathrm{tr}\Sigma_r \geqslant q^{-1}2^{-2(r+1)}\nu_r$. Write $\xi_r(s) = |\xi_r(s)|e^{i\theta}$, where $\theta = \theta(r,s)$ satisfies

$$\int_{\Omega_r} \sin(2^r s \cdot x - \theta)\nu(\mathrm{d}x) = 0.$$

Then $\int_{\Omega_r} \sin(2^r s{\cdot}x)\nu(\mathrm{d}x) = \tan\theta \int_{\Omega_r} \cos(2^r s{\cdot}x)\nu(\mathrm{d}x)$ if $\theta \not\equiv \pm\pi/2 \bmod \pi$, and otherwise $\int_{\Omega_r} \cos(2^r s \cdot x)\nu(\mathrm{d}x) = 0$. By the uniform Cramér's condition for $\xi_r(s)$, there exist $\rho > 0,\ \gamma \in (0,1)$ s.t. $\forall r \geqslant r_0$ and $|s| \geqslant \rho$,

$$|\xi_r(s)| = \nu_r^{-1}\int_{\Omega_r} \cos(2^r s \cdot x - \theta)\nu(\mathrm{d}x) \in [0, \gamma].$$

If $\theta \not\equiv \pm\pi/2 \bmod \pi$, expand out the integrand using the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,\ \forall \alpha,\beta \in \mathbb{R}$, replace the term $\int_{\Omega_r} \sin(2^r s \cdot x)\nu(\mathrm{d}x)$ and rearrange to get

$$(\nu_r\cos\theta)^{-1}\int_{\Omega_r} \cos(2^r s \cdot x)\nu(\mathrm{d}x) \in [0, \gamma].$$

Therefore, regardless of the values of θ, one always has $\left|\int_{\Omega_r} \cos(2^r s \cdot x)\nu(\mathrm{d}x)\right| \leqslant \gamma\nu_r$ for $|s| \geqslant \rho$. Write $s = |s|v$ where $v \in \mathbb{S}^{q-1}$ is a unit vector. Then for $|s| \geqslant \rho$,

$$\begin{aligned} v \cdot \Sigma_r v &= \int_{\Omega_r} |v \cdot x|^2\nu(\mathrm{d}x) \geqslant 2^{-2r+2}|s|^{-2}\int_{\Omega_r} \sin^2(2^{r-1}s \cdot x)\nu(\mathrm{d}x) \\ &= 2^{-2r+1}|s|^{-2}\int_{\Omega_r} \left(1 - \cos(2^r s \cdot x)\right)\nu(\mathrm{d}x) \geqslant 2^{-2r+1}\rho^{-2}(1-\gamma)\nu_r. \end{aligned}$$

This means $\lambda_{1,r} \gtrsim 2^{-2r}\nu_r$ by choosing v to be the eigenvector of $\lambda_{1,r}$. Hence $\lambda_{1,r} \simeq \lambda_{q,r} \simeq 2^{-2r}\nu_r$ and $\lambda_{1,X} = \nu_r^{-1}2^{2r}\lambda_{1,r} \simeq 1,\ \forall r \geqslant r_0$.

Since $\xi_r(s)$ is the characteristic function of $X = 2^r X^r$, the uniform Cramér's condition validates the bound (4.28) with a uniform $\bar{\gamma} = \bar{\gamma}(\rho, \gamma)$ and (4.29) holds with $\mu = t\nu_r \geqslant \epsilon\nu_r \geqslant 2^{-r}\nu_r$ sufficiently large w.r.t. $p+2$. More precisely, one can choose ϵ sufficiently small s.t. for all $r \geqslant r_0$, similar to (4.14),

$$\left(e^{\bar{\gamma}-1}\right)^{2^{-r}\nu_r}\left(2^{-r}\nu_r\right)^{(q+1)(p+3)/2} \lesssim 1.$$

Thus, all the arguments leading towards (4.30) are justified, which is immediately reduced to the bound $\mathbb{W}_p\left(V_t^r, \sqrt{t}\xi_{\Sigma_r}\right) \leqslant C_{p,q}2^{-r}$, and therefore

$$\mathbb{W}_p\left(Z_t^\epsilon, \sqrt{t}\xi_{\Sigma_\epsilon}\right) = \mathbb{W}_p\left(\sum_{r=r_0}^{\infty} V_t^r, \sum_{r=r_0}^{\infty}\sqrt{t}\xi_{\Sigma_r}\right) \leqslant C_{p,q}\sum_{r=r_0}^{\infty} 2^{-r} = C_{p,q}\epsilon.$$

$\square$

Together with the finite second moment of ν, the theorem above requires the order of ν_r is between $O(2^{r+})$ and $O(2^{2r})$ as $r \to \infty$, i.e. the Lévy measure needs to be sufficiently singular near 0. The uniform Cramér condition requires certain comparability between ν and the Lebesgue measure Λ, conditional on Ω_r. The following lemma gives a sufficient condition.

Lemma 4.14. *If there exist $a, b \in (0,1)$ s.t. for each $r \geqslant r_0$, any measurable subset Γ_r of Ω_r satisfying $\Lambda(\Gamma_r)/\Lambda(\Omega_r) \geqslant a$ must have that $\nu(\Gamma_r)/\nu(\Omega_r) \geqslant b$, then $\xi_r(s) = \chi^r(2^r s)$ satisfies Assumption 4.3 uniformly for all $r \geqslant r_0$.*

Proof. For any $a' \in (0,1)$ denote $b' = \sin^2 \frac{\pi}{2}(1-a') \in (0,1)$. For any $\theta \in \mathbb{R}$, $v \in \mathbb{R}^q$, consider, for each $k \in \mathbb{Z}$, the set

$$D_k = D_k(v,\theta) := \left\{x \in \Omega_r : \; 2k\pi + (1-a')\pi \leqslant v \cdot x - \theta \leqslant 2(k+1)\pi - (1-a')\pi\right\},$$

on each of which $\sin^2 \frac{1}{2}(v \cdot x - \theta) \geqslant b'$. They are parallel "stripes" across the annulus Ω_r with width $2a'\pi/|v|$ equidistantly away from each other by $2(1-a')\pi/|v|$. This can be seen by rotating so that v lies on one axis. Thus for $|v| > \pi 2^{r+1}$ there is at least one non-empty D_k. Denote $\Gamma_r = \bigcup_{k\in\mathbb{Z}} D_k$, then the ratio $\Lambda(\Gamma_r)/\Lambda(\Omega_r)$ approaches a' as $|v| \to \infty$, regardless of the translation θ. Therefore one can find some constants $\rho > 0$ and $\gamma' = \gamma'(\rho, q) \in (0, a')$ s.t. for all $|v| \geqslant 2^r\rho$, $\Lambda(\Gamma_r)/\Lambda(\Omega_r) \geqslant \gamma'$. Choose $\gamma' = a$ as given in the assumption, then $\nu(\Gamma_r)/\nu(\Omega_r) \geqslant b$ for all $|v| \geqslant 2^r\rho$. Write $\xi_r(s) = |\xi_r(s)|e^{i\theta}$ for some $\theta = \theta(r,s)$, then

$$1 - |\xi_r(s)| \geqslant 2\nu_r^{-1} \int_{\Gamma_r} \sin^2 \frac{1}{2}(2^r s \cdot x - \theta)\nu(\mathrm{d}x) \geqslant 2b'\nu(\Gamma_r)/\nu(\Omega_r) \geqslant 2b'b,$$

for all $r \geqslant r_0$, and the result follows by setting $v = 2^r s$ and $\gamma = 1 - 2b'b \in (0,1)$. □

Corollary 4.15. *If ν satisfies Assumption 4.1 with some $\alpha \in (1,2)$, then Theorem 4.13 holds for $\epsilon \in (0,\tau)$ sufficiently small.*

Proof. One just needs to check that the assumptions in Theorem 4.13 are satisfied. First of all, for $\alpha \in (1,2)$ and $\forall r \geqslant r_0$,

$$\nu_r \simeq \int_{\Omega_r} |x|^{-q-\alpha}\mathrm{d}x = C_q \frac{2^\alpha - 1}{\alpha} 2^{\alpha r}.$$

Then $2^r \nu_r^{-1} = o(1)$ for $\alpha \in (1,2)$. For any measurable subset Γ_r of Ω_r,

$$\frac{\nu(\Gamma_r)}{\nu(\Omega_r)} \geqslant \frac{\int_{\Gamma_r} |x|^{-q-\alpha}\mathrm{d}x}{\int_{\Omega_r} |x|^{-q-\alpha}\mathrm{d}x} \geqslant 2^{-q-\alpha}\frac{\Lambda(\Gamma_r)}{\Lambda(\Omega_r)},$$

which validates Lemma 4.14. □

It is worth mentioning that if Assumption 4.1 is assumed a priori, then one could directly use the Lévy-Khintchine formula to study the global behaviour of the characteristic function of Z_t^ϵ, which would greatly simplify the analysis of Section 4.1, but the same arguments used in the proof of Theorem 4.9 would still be needed for the coupling result. Detailed derivation can be found in Appendix B.

4.2.2 A Coupling for Euler's Approximation

Given the coupling result above, one finally arrives at the stage of recovering Fournier's results [11] on the Euler approximation of the SDE (4.2) driven by the Lévy process (4.1):

$$x_t = x_0 + \int_0^t \sigma(x_{s-})\mathrm{d}Z_s,\ t \in [0,T], \tag{4.31}$$

$$Z_t = at + BW_t + \int_0^t \int_{\mathbb{R}^q\setminus\{0\}} z\tilde{N}(\mathrm{d}z, \mathrm{d}s). \tag{4.32}$$

For fixed $h, \epsilon \in (0,1)$ introduce the following random variable

$$\bar{\Delta}_1 := \bar{a}h + \bar{B}\sqrt{h}\xi_I + \sum_{j=1}^{N_h^\epsilon} Y_j^\epsilon, \tag{4.33}$$

and take independent copies $\bar{\Delta}_2, \cdots, \bar{\Delta}_{[T/h]}$, where $\{Y_j^\epsilon\}$ are i.i.d. random variables having distribution $\mathbb{1}_{|z|>\epsilon}\nu(\mathrm{d}z)/\nu(|z|>\epsilon)$, N_h^ϵ is Poisson with parameter $h\nu(\{|z|>\epsilon\})$, and the coefficients $\bar{a} = a - \int_{|z|>\epsilon} z\nu(\mathrm{d}z)$, $\bar{B} := \left(BB^\top + \Sigma_\epsilon\right)^{1/2}$, $\Sigma_\epsilon = \int_{0<|z|\leqslant\epsilon} zz^\top \nu(\mathrm{d}z)$. For $t_k = kh,\ k = 1, \cdots, [T/h]$, write the increments $\Delta_k := Z_{t_k} - Z_{t_{k-1}}$. Then one may attempt to find a coupling between the standard Euler's approximation

$$X_{k+1} := X_k + \sigma(X_k)\Delta_{k+1},\ X_0 = x_0,$$

and the numerical scheme

$$\bar{X}_{k+1} := \bar{X}_k + \sigma(\bar{X}_k)\bar{\Delta}_{k+1},\ \bar{X}_0 = x_0. \tag{4.34}$$

For that one claims the following statement as an analogue to Lemma 5.2 in [11]:

Proposition 4.16. *If ν satisfies the conditions of Theorem 4.13, then for ϵ sufficiently small there exist on the same probability space two sequences of i.i.d. random variables $\{\Delta_k'\}$, $\{\bar{\Delta}_k'\}$, with the same distributions as $\{\Delta_k\}$ and $\{\bar{\Delta}_k\}$ respectively, s.t.*

$$\left(\mathbb{E}\left|\Delta_k' - \bar{\Delta}_k'\right|^p\right)^{\frac{1}{p}} \leqslant C_q\epsilon,$$

for all $k \in \mathbb{Z}^+$ and $p \in 2\mathbb{Z}^+$, and $\mathbb{E}\Delta_k' = \mathbb{E}\bar{\Delta}_k' = ah$, $\mathrm{var}\Delta_k' = \mathrm{var}\bar{\Delta}_k' = \left(BB^\top + \Sigma_\epsilon\right)h$.

Proof. By Theorem 4.13, for ϵ sufficiently small there is a standard normal random variable ξ_I' on the same probability space s.t.

$$\left(\mathbb{E}\left|\int_0^h \int_{0<|z|\leqslant\epsilon} z\tilde{N}(\mathrm{d}z,\mathrm{d}s) - h\Sigma_\epsilon^{\frac{1}{2}}\xi_I'\right|^p\right)^{\frac{1}{p}} \leqslant C_q\epsilon,$$

according to the definition of the $\mathbb{W}_p$ distance. If one sets

$$\begin{aligned}\Delta_1' &:= ah + BW_h + \int_0^h \int_{0<|z|\leqslant\epsilon} z\tilde{N}(\mathrm{d}z,\mathrm{d}s) + \int_0^h \int_{|z|>\epsilon} z\tilde{N}(\mathrm{d}z,\mathrm{d}s),\\ \bar{\Delta}_1' &:= ah + BW_h + h\Sigma_\epsilon^{\frac{1}{2}}\xi_I' + \int_0^h \int_{|z|>\epsilon} z\tilde{N}(\mathrm{d}z,\mathrm{d}s),\end{aligned}$$

then Δ_1' has the same law as Δ_1, and $\bar{\Delta}_1'$ has the same law as $\bar{\Delta}_1$. Thus the result follows by taking independent copies. $\square$

Proposition 4.16 can be immediately used to partially recover the main results in [11] (Theorem 2.2): the proof is independent of the key coupling result (Lemma 5.2), so one can replace the latter with the proposition above. Hence one can restate those results as follows:

Theorem 4.17. *Suppose $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times q}$ is bounded and Lipschitz, and the Lévy measure ν for the Lévy process* (4.32) *satisfies conditions of Theorem 4.13. Let $\epsilon, h \in (0,1)$, and $\{x_t\}$ be the unique solution to the SDE* (4.31) *for $t \in [0,T]$. Then for $\rho_h(t) = [t/h]h$ and ϵ sufficiently small, there exists a coupling between $\{x_t\}$ and $\{\bar{X}_{\rho_n(t)}\}$ defined by* (4.34) *and* (4.33) *s.t.*

$$\mathbb{E} \sup_{t\in[0,T]} \left|x_t - \bar{X}_{\rho_h(t)}\right|^2 \leqslant C_1(h+\epsilon).$$

Moreover, if $\nu(\{|z| > \epsilon\}) = 0$, i.e. $Z_t = Z_t^\epsilon$ as in (4.32)*, and $\{\widetilde{x}_t^\epsilon\}$ is the unique solution to the continuous SDE $\widetilde{x}_t^\epsilon = x_0 + \int_0^t \sigma(\widetilde{x}_t^\epsilon)\mathrm{d}\tilde{Z}_t^\epsilon$ where $\widetilde{Z}_t^\epsilon = at + (BB^\top + \Sigma_\epsilon)^{1/2}W_t$, then there exists a coupling between x_t and $\widetilde{x}_t^\epsilon$ s.t.*

$$\mathbb{E} \sup_{t\in[0,T]} |x_t - \widetilde{x}_t^\epsilon|^2 \leqslant C_2\epsilon.$$

The constants C_1, C_2 depend on $d, q, T, |a|, \|B\|, \|\sigma\|_\infty, \Sigma_\epsilon$.

Instead of repeating the same arguments of Fournier [11], the reader is referred to the proof of Theorem 2.2 therein. Note that Proposition 4.16 above allows one to replace the $\beta_\epsilon(\nu)$ in Lemma 5.2 with ϵ^2, and the rest of the calculations can be readily generalised to the multi-dimensional case. In particular, under Assumption 4.1 for some $\alpha \in (1,2)$, by choosing $\epsilon = h$ one recovers the mean-square convergence rate $O(h)$ and the computational cost $E_\nu(h) = O(h^{-1} + h^{-\alpha})$ is controlled. The second statement corresponds to Corollary 3.2 in [11]. For that, one simply takes $\bar{\Delta}_1 = \bar{a}h + \bar{B}\sqrt{h}\xi_I$ instead of (4.33) and $h = \epsilon$, and runs the same argument as in Proposition 4.16, omitting the big-jump part.

The general case where σ is locally Lipschitz with linear growth and only $\int_{\mathbb{R}^q\setminus\{0\}} 1 \wedge |z|^2\nu(\mathrm{d}z) < \infty$ is assumed can be treated by the same localisation argument as in Theorem 7.1 in [11], and the mean-square convergence could be generalised to the strong L^p-convergence for $p \in 2\mathbb{Z}^+$ without much trouble. Nevertheless, it needs to be pointed out that the rate of convergence here is optimal for coupling the small jumps only - it might not be so if one can couple the entire Lévy increment. For the same reason the results achieved here cannot be applied to recover Theorem 3.1 in [11]. Finally, I believe the conditions of Theorem 4.13 can be relaxed to some extent. E.g., one may take a hint from Proposition A.2 in [13] that it possibly suffices for ν to give a suitable portion of mass to the biggest annulus Ω_{r_0}.

Appendix A

Appendices to Chapter 2

A.1 V-Integrability Applied to Strong Convergence

Strong L^p-convergence of explicit numerical methods of an SDE

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t,\ t \in [0, T], \tag{A.1}$$

has been well studied in the literature. Although this is not the main topic, we still summarise the framework of it in order for this thesis to be self-contained. For simplicity we may consider L^2 convergence of an explicit numerical scheme $\bar{X}$. A typical proof adopted in [54] is based on splitting the one-step difference into two:

$$\begin{aligned}
&X_{t_k,X(t_k)}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) \\
&= X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}) + X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}).
\end{aligned}$$

The first difference is the one-step perturbation[1] of the solution X given different initial conditions, which by Lemma 2.2 in [54] can be handled provided that Assumption A.1 below holds. The second difference is the one-step error between $\bar{X}$ and X starting from the same initial condition, and that, as seen from the proof of Lemma 3.2 in [54], can be studied by further decomposing the error as

$$\begin{aligned}
&X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) \\
&= X_{t_k,\bar{X}_k}(t_{k+1}) - \widehat{X}_{t_k,\bar{X}_k}(t_{k+1}) + \widehat{X}_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}),
\end{aligned} \tag{A.2}$$

where $\widehat{X}$ is the standard Euler scheme

$$\widehat{X}_{t,x}(t+h) = x + b(t,x)h + \sigma(t,x)(W_{t+h} - W_t). \tag{A.3}$$

As is shown in [54], one can achieve optimal rates for the one-step error of (A.3) against the solution X_t without additional assumptions.

Alternatively, one can regard the local estimates for one-step perturbation and one-step error as special cases of what is stated in Theorem 1.2 in [23], which holds for two processes at a stopping time.

Assumption A.1. *For SDE* (A.1)*, there exist* $p_0 \geqslant 2$ *and* $\kappa \geqslant 1$*, s.t.* $\forall t, s \in [0, T],\ x, y \in \mathbb{R}^d$,

i) $\langle x - y, b(t,x) - b(t,y)\rangle + \frac{p_0 - 1}{2}\|\sigma(t,x) - \sigma(t,y)\|^2 \lesssim |x - y|^2$*;*

[1] Or one-step stability, not to be confused with the asymptotic stability of equilibrium.

ii) $|b(t,0)| \vee \|\sigma(t,0)\| \vee \sup_{\gamma>0} \mathbb{E}|X_0|^\gamma < \infty$;

iii) $|b(t,x)-b(t,y)| \lesssim \left(1+|x|^{\kappa-1}+|y|^{\kappa-1}\right)|x-y|$ *and*
$\|\sigma(t,x)-\sigma(t,y)\| \lesssim \left(1+|x|^{(\kappa-1)/2}+|y|^{(\kappa-1)/2}\right)|x-y|$;

iv) $|b(t,x)-b(s,x)| \lesssim (1+|x|^\kappa)\,|t-s|$ *and*
$\|\sigma(t,x)-\sigma(s,x)\| \lesssim \left(1+|x|^{(\kappa+1)/2}\right)|t-s|$.

Note that i) and iii) above provides convenience for the strong and weak estimates of one-step perturbation $X_{t,x}(t+h)-X_{t,y}(t+h)$ for the SDE. If we let $V(\cdot)=|\cdot|^{p_0} \in \bar{\mathcal{V}}^{p_0}_{1/p_0}$, then by i) and ii),

$$\mathcal{L}V(x) = |x|^{p_0-2}\left(\langle x, b(t,x)\rangle + \frac{p_0-1}{2}\|\sigma(t,x)\|^2\right) \lesssim 1+V(x), \tag{A.4}$$

which together with the growth condition implied by ii) and iii),

$$|b(t,x)| \lesssim 1+|x|^\kappa,\ \ \|\sigma(t,x)\| \lesssim 1+|x|^{(\kappa+1)/2},\ \ \forall t\in[0,T],\ x\in\mathbb{R}^d, \tag{A.5}$$

can make it possible for the tamed Euler scheme to achieve Theorem 2.5.

Although the argument (A.2) is hidden in the proof of the main result in [54], here we reformulate it as the following:

Theorem A.2. *Let Assumption A.1 hold for some even $p_0 \in \mathbb{N}^+$. If there is a real number $p_1 \geqslant 1$ s.t. a numerical scheme $\{\bar{X}_k\}$ with step size h is $|\cdot|^{p_1}$-integrable and its one-step error against the standard Euler scheme* (A.3) *satisfies, $\forall \eta \geqslant 1$,*

$$\begin{aligned} \mathbb{E}\left|\bar{X}_{t,x}(t+h)-\widehat{X}_{t,x}(t+h)\right|^\eta &\lesssim (1+|x|^\alpha)\,h^{\delta\eta},\\ \left|\mathbb{E}\bar{X}_{t,x}(t+h)-\mathbb{E}\widehat{X}_{t,x}(t+h)\right| &\lesssim \left(1+|x|^{\alpha'}\right)h^{\delta+1/2}, \end{aligned}$$

for some $\alpha,\alpha'>0$ and $\delta>1/2$, then for some $p\in[1,p_1]$,

$$\max_k (\mathbb{E}|\bar{X}_k - X_{t_k}|^p)^{1/p} = O(h^{\delta-1/2}).$$

Regarding moment bounds, Theorem 2.5 plays an essential role in controlling the highest (p_1) moments of $\{\bar{X}_k\}$ needed for L^p convergence. The relation between p_0, p_1 and p depends on what specific taming method one adopts and how one decomposes the global error. This has been studied for various balanced schemes in [26, 49, 54].

A.2 Proof of Proposition 2.11 and Corollary 2.12

Proof of Proposition 2.11.

Proof. Since both drift and diffusion are Lipschitz in t, we may assume $b(t,x) = b(x)$, $\sigma(t,x)=\sigma(x)$, $\forall t,x$. Notice that using a more precise growth condition (A.5) rather than Assumption 2.2, we can estimate $|b|h^{1/2}$ and $\|\sigma\|h^{1/4}$ separately in (2.24) and need only choose $r<1/(2(\kappa-1))$, $q\gamma=1$.

One only needs to check if $\delta=1$ in Theorem A.2. Indeed the weak one-step error has estimate, by the Cauchy-Schwartz inequality and Chebyshev's inequality (denote

$\Delta W := W_{t+h} - W_t$),

$$\begin{aligned}
\left|\mathbb{E}\bar{X}_{t,x} - \mathbb{E}\widetilde{X}_{t,x}\right| =& \left|\mathbb{E}\Pi(x + b(x)h + \sigma(x)\Delta W) - \mathbb{E}\left(x + b(x)h + \sigma(x)\Delta W\right)\right| \\
\leqslant& 2\mathbb{E}\left|x + b(x)h + \sigma(x)\Delta W\right| \mathbb{1}_{|x+b(x)h+\sigma(x)\Delta W|>h^{-r}} \\
\leqslant& K\left(\mathbb{E}|x + b(x)h + \sigma(x)\Delta W|^{2+\frac{3}{r}}\right)^{\frac{1}{2}} h^{\frac{3}{2}} \\
\leqslant& K\left(|x|^{1+\frac{3}{2r}} + \left((1+|x|)h^{\frac{1}{2}}\right)^{1+\frac{3}{2r}} + \left((1+|x|)h^{\frac{1}{4}}\right)^{1+\frac{3}{2r}}\right) h^{\frac{3}{2}} \\
\leqslant& K\left(1 + |x|^{1+\frac{3}{2r}}\right) h^{\frac{3}{2}},
\end{aligned}$$

where we used (2.24) for $|x| \leqslant h^{-r}$. Similarly,

$$\begin{aligned}
\mathbb{E}\left|\bar{X}_{t,x} - \widehat{X}_{t,x}\right|^2 =& \mathbb{E}\left|\Pi(x + b(x)h + \sigma(x)\Delta W) - x - b(x)h - \sigma(x)\Delta W\right|^2 \\
\leqslant& K\mathbb{E}\left|x + b(x)h + \sigma(x)\Delta W\right|^2 \mathbb{1}_{|x+b(x)h+\sigma(x)\Delta W|>h^{-r}} \\
\leqslant& K\left(\mathbb{E}|x + b(x)h + \sigma(x)\Delta W|^{4+\frac{4}{r}}\right)^{\frac{1}{2}} h^2 \\
\leqslant& K\left(|x|^{2+\frac{2}{r}} + \left((1+|x|)h^{\frac{1}{2}}\right)^{2+\frac{2}{r}} + \left((1+|x|)h^{\frac{1}{4}}\right)^{2+\frac{2}{r}}\right) h^2 \\
\leqslant& K\left(1 + |x|^{2+\frac{2}{r}}\right) h^2.
\end{aligned}$$

This validates the L^2 convergence of (2.22). □

It is worth mentioning that if set $r = 1/(2\kappa)$, in the end (involving the Cauchy-Schwartz inequality) we have $p_1 = 8\kappa + 4$, which is almost the same p_1 needed for the specific balanced scheme introduced in [54]. However, as shown in Lemma 3.1 therein, $p_0 \geqslant O(p_1\kappa)$, whereas for the projected scheme proposed here we have $p_0 = p_1$. We leave the details of this calculation to the reader.

Proof of Corollary 2.12.

Proof. Suppose we already have a numerical scheme (b^h, σ^h) satisfying the conditions of Theorem A.2. For the composed scheme (2.25) to converge in L^2, one uses the same arguments adopted above to give the one-step estimates

$$\begin{aligned}
\left|\mathbb{E}\Pi\left(x + b^h(x)h + \sigma^h(x)\Delta W\right) - \mathbb{E}\left(x + b^h(x)h + \sigma^h(x)\Delta W\right)\right| = O(h^{\frac{3}{2}}), \\
\mathbb{E}\left|\Pi\left(x + b^h(x)h + \sigma^h(x)\Delta W\right) - x - b^h(x)h - \sigma^h(x)\Delta W\right|^2 = O(h^2),
\end{aligned}$$

and the result follows from the triangle inequality. □

A.3 Proof of Lemma 2.32

Proof. Consider $f(x) = x^- = \max(0, -x)$. Take a monotone sequence of smooth functions $\phi_n(x)$ s.t.

$$\phi_n(x) \to f(x),\ \phi_n'(x) \to -\mathbb{1}_{\{x<0\}}(x),\ \phi_n''(x) \to 0,$$

uniformly as $n \to \infty$, and the derivatives satisfy $|\phi_n'(x)| \le 1$, $\phi_n''(x) \lesssim n^{-1}|x|^{-\kappa}$, $\forall x \in \mathbb{R}$. Existence of such approximation can be found in, e.g. section 5.2.C in [30]. By Itô's formula,

$$\begin{aligned}\phi_n(X_t) = \phi_n(X_0) + \int_0^t \left(\phi_n'(X_s)b(s,X_s) + \frac{1}{2}\phi_n''(X_s)\sigma^2(s,X_s)\right)\mathrm{d}s \\ + \int_0^t \phi_n'(X_s)\sigma(s,X_s)\mathrm{d}W_s.\end{aligned} \tag{A.6}$$

From (2.53) one can show that $b(t,x) = b_1(t,x)+b_2(t,x)$, where $b_1(t,x)$ is monotonically decreasing in x, and $b_2(t,x)$ is Lipschitz. One can choose e.g. $b_2(t,x) = Kx$ and hence

$$\begin{aligned}(x-y)(b_1(t,x) - b_1(t,y)) =&(x-y)(b(t,x) - Kx - b(t,y) + Ky)\\ =&(x-y)(b(t,x) - b(x,y)) - K|x-y|^2 \leqslant 0.\end{aligned}$$

Taking expectation on both sides of (A.6) and letting $n \to \infty$, by the monotone and dominated convergence theorems we find that only one term remains:

$$\begin{aligned}\mathbb{E}X_t^- \leqslant&\mathbb{E}\int_0^t -\mathbb{1}_{\{X_s<0\}}b(s,X_s)\mathrm{d}s = \mathbb{E}\int_0^t -\mathbb{1}_{\{X_s<0\}}\left(b_1(s,X_s) + b_2(s,X_s)\right)\mathrm{d}s\\ \leqslant&\mathbb{E}\int_0^t -\mathbb{1}_{\{X_s<0\}}\left(b_1(s,0) + b_2(s,0) - K|X_s|\right)\mathrm{d}s\\ =&\mathbb{E}\int_0^t \mathbb{1}_{\{X_s<0\}}\left(-b(s,0) + K|X_s|\right)\mathrm{d}s.\end{aligned}$$

Note that $b(s,0) \geqslant 0$, thus

$$\mathbb{E}X_t^- \leqslant \int_0^t K\mathbb{E}X_s^-\mathrm{d}s \;\Rightarrow\; \mathbb{E}X_t^- = 0,\; \forall t \ge 0,$$

by Grönwall's inequality, which is validated by checking, for all $t \ge 0$,

$$\begin{aligned}\mathbb{E}X_t^- =&\mathbb{E}\mathbb{1}_{X_t<0}\left|X_0 + \int_0^t b(s,X_s)\mathrm{d}s + \sigma(s,X_s)\mathrm{d}W_s\right|\\ \leqslant&\mathbb{E}X_0 + \mathbb{E}\int_0^t |b(s,X_s)|\mathrm{d}s + C\left(\mathbb{E}\int_0^t \sigma^2(s,X_s)\mathrm{d}s\right)^{\frac{1}{2}} < \infty,\end{aligned}$$

for some constant $C > 0$, due to polynomial growth of b and σ^2 and bounded moments of X_t up to the same order. Thus we conclude that $X_t \geqslant 0$ a.s. $\square$

Appendix B

A Direct Approach via the Lévy-Khintchine Formula

If ν is assumed to satisfy Assumption 4.1 a priori, then, instead of viewing Z_t^ϵ as a compound Poisson process and applying the central limit theorem, one may directly use the Lévy-Khinchine formula to derive a coupling for the normal approximation. For the simplicity of calculation we only consider the $\mathbb{W}_2$ distance.

Denote by ψ_ϵ and f_ϵ the characteristic function and density function of Z_t^ϵ, and by $\bar{\psi}_\epsilon$ and $\bar{f}_\epsilon$ those of the scaled process

$$\bar{Z}_t^\epsilon := t^{-\frac{1}{2}} \Sigma_\epsilon^{-\frac{1}{2}} Z_t^\epsilon.$$

Then $\mathbb{W}_2(Z_t^\epsilon, \sqrt{t}\xi_{\Sigma_\epsilon}) \leqslant t^{\frac{1}{2}} \|\Sigma_\epsilon^{1/2}\| \mathbb{W}_2\left(\bar{Z}_t^\epsilon, \xi_I\right)$, and by the inversion formula,

$$\mathbb{W}_2(Z_t^\epsilon, \sqrt{t}\xi_{\Sigma_\epsilon}) \leqslant C t^{\frac{1}{2}} \lambda_{q,\epsilon}^{\frac{1}{2}} \left(\int_{\mathbb{R}^q} |y|^2 \left| \int_{\mathbb{R}^q} e^{-iz\cdot y} \left(\bar{\psi}_\epsilon(z) - \widehat{\phi_I}(z) \right) \mathrm{d}z \right| \mathrm{d}y \right)^{1/2}. \tag{B.1}$$

When ν satisfies Assumption 4.1, one asserts that $\lambda_{1,\epsilon}, \lambda_{q,\epsilon} \simeq \epsilon^{2-\alpha}$ according to the proof of Corollary 4.15. Thus, directly from the Lévy-Khintchine formula, one can, using the notation $\bar{z} := t^{-1/2}\Sigma_\epsilon^{-1/2} z \in \mathbb{R}^q$, formally expand the characteristic function of the scaled jump process by Edgeworth-type polynomials:

$$\begin{aligned}
\bar{\psi}_\epsilon(z) = \psi_\epsilon(\bar{z}) = \exp\left(t \int_{0<|x|\leqslant\epsilon} e^{i\bar{z}\cdot x} - 1 - i\bar{z}\cdot x \nu(\mathrm{d}x) \right) \qquad (\mathrm{B.2})\\
= \exp\left(-\frac{1}{2}|z|^2 + t \sum_{|\beta|\geqslant 3} \frac{i^{|\beta|}}{\beta!} \bar{z}^\beta \int_{0<|x|\leqslant\epsilon} x^\beta \nu(\mathrm{d}x) \right) \sim e^{-\frac{1}{2}|z|^2} \left(1 + \sum_{k=1}^\infty \left(t^{-1}\epsilon^\alpha\right)^{\frac{k}{2}} P_k(z) \right),
\end{aligned}$$

where each $P_k(z)$ is a polynomial of which each monomial has highest degree $3k$ and lowest degree $k+2$, with coefficients independent of t and ϵ. This agrees with $\mu \simeq t\epsilon^{-\alpha}$ as shown in the first approach. Note that P_1 contains all the cubic terms in the expansion.

In order to find a coupling between Z_t^ϵ and $\mathcal{N}(0, t\Sigma_\epsilon)$ the following fact is useful:

Lemma B.1. *If ν satisfies Assumption 4.1 for $\tau > \epsilon$, then for $|z| > \pi\epsilon^{-1}$ and some $c_q > 0$:*

$$A_\epsilon(z) := \int_{0<|x|\leqslant\epsilon} 2\sin^2\left(\frac{1}{2} z\cdot x\right) \nu(\mathrm{d}x) \geqslant c_q |z|^\alpha.$$

Proof. Similar to the proof of Lemma 4.14 consider a slightly different family of sets:

$$D_k := \{x: \ 2k\pi + \pi/2 \leqslant z\cdot x \leqslant 2k\pi + 3\pi/2\} \cap \{|x| \leqslant \epsilon\}, \ k \in \mathbb{Z},$$

on each of which $\sin^2(z\cdot x/2) \geqslant 1/2$. They are parallel "stripes" across the ball $\{|x| \leqslant \epsilon\}$ with width $\pi/|z|$, equidistantly away from each other by $\pi/|z|$. Since the density of ν is singular at the origin, it suffices to find a subset of $\bigcup_k D_k$ where the majority of mass of ν is given. For example, one may only look at the cube inside D_0 closest to the origin with edge width $\pi/|z|$. Then one deduces, for $\epsilon < \delta$,

$$A_\epsilon(z) \geqslant c_q \int_{D_0} |x|^{-q-\alpha}\mathrm{d}x \geqslant c_q \left(\frac{\pi}{2|z|}\right)^{-q-\alpha}\left(\frac{\pi}{|z|}\right)^q = c_q|z|^\alpha.$$

□

The lower bound above provides convenience for investigating the global behaviour of $\bar\psi_\epsilon$ since it controls the exponent in (B.2) for z large.

Theorem B.2. *Suppose the Lévy measure ν satisfies Assumption 4.1 for $\tau > \epsilon$ and $\alpha \in (1,2)$. Then for all $t \geqslant \epsilon$, the following holds for ϵ sufficiently small:*

$$\mathbb{W}_2\left(Z_t^\epsilon, \sqrt{t}\xi_{\Sigma_\epsilon}\right) \leqslant C_q\sqrt{t\epsilon}.$$

In particular, $\mathbb{W}_2(Z_\epsilon^\epsilon, \mathcal{N}(0,\epsilon\Sigma_\epsilon)) \leqslant C_q\epsilon$.

Proof. Using the same idea as of the proof of Theorem 4.9, one starts from estimating the difference between $\bar\psi_\epsilon(z)$ and the characteristic function of $\mathcal{N}(0,I)$ perturbed by the cubic terms P_1. Then for $|z| \leqslant t^{1/2}\epsilon^{-\alpha/2} =: \sqrt{\mu}$ with $\alpha \in (1,2)$, using Taylor's theorem (twice) for the expansion of (B.2) and the fact that $|\exp(P_1(z))| \equiv 1$, one has

$$\begin{aligned}
&\int_{|z|\leqslant\sqrt{\mu}} \left|\psi_\epsilon\left(t^{-\frac12}\Sigma_\epsilon^{-\frac12}z\right) - e^{-\frac12|z|^2}\left(1 + t^{-\frac12}\epsilon^{\frac{\alpha}{2}}P_1(z)\right)\right|\mathrm{d}z \\
&\leqslant \int_{|z|\leqslant\sqrt{\mu}} e^{-\frac12|z|^2}\left|\exp\left(\int_0^1\int_{0<|x|\leqslant\epsilon}\frac{1}{24}(1-\theta)^3(\bar z\cdot x)^4 e^{i\theta\bar z\cdot x}\nu(\mathrm{d}x)\mathrm{d}\theta\right) - 1\right|\mathrm{d}z \\
&\quad + \int_{|z|\leqslant\sqrt{\mu}} e^{-\frac12|z|^2}\left|\exp\left(t^{-\frac12}\epsilon^{\frac{\alpha}{2}}P_1(z)\right) - \left(1 + t^{-\frac12}\epsilon^{\frac{\alpha}{2}}P_1(z)\right)\right|\mathrm{d}z \\
&\leqslant \frac{1}{24}\int_{|z|\leqslant\sqrt{\mu}} \exp\left(\left(-\frac12 + \frac{1}{24}t^{-1}\lambda_{1,\epsilon}^{-1}\epsilon^2|z|^2\right)|z|^2\right)t^{-1}\lambda_{1,\epsilon}^{-1}\epsilon^2|z|^4\mathrm{d}z \\
&\quad + C_q\int_{|z|\leqslant\sqrt{\mu}} e^{-\frac12|z|^2}t^{-1}\lambda_{q,\epsilon}^{-1}\epsilon^2|z|^6\left|e^{-i\int_0^1\int_{0<|x|\leqslant\epsilon}\frac16(1-\theta)^2(\theta\bar z\cdot x)^3\nu(\mathrm{d}x)\mathrm{d}\theta}\right|\mathrm{d}z \\
&\leqslant C_q\mu^{-1}\int_{\mathbb{R}^q} e^{-\frac{11}{24}|z|^2}|z|^4\mathrm{d}z + C_q\mu^{-1}\int_{\mathbb{R}^q} e^{-\frac14|z|^2}|z|^6\mathrm{d}z \leqslant C_q\epsilon^{\alpha-1},
\end{aligned} \tag{B.3}$$

where again the inequality $|e^u - 1| \leqslant e^{|u|}|u|$, $\forall u \in \mathbb{C}$, is used in the second step and the choice $t \geqslant \epsilon$ is considered. For $|z| > \epsilon^{(1-\alpha)/2}$, first observe that the following holds for arbitrary large $K > 0$ provided that ϵ is sufficiently small:

$$\int_{|z|>\sqrt{\mu}} e^{-\frac12|z|^2}\mathrm{d}z \leqslant e^{-\frac14\epsilon^{1-\alpha}}\int_{\mathbb{R}^q} e^{-\frac14|z|^2}\mathrm{d}z \leqslant C_q\epsilon^K. \tag{B.4}$$

Also for $|z| > \sqrt{\mu}$ one has $|\bar{z}| = \left|t^{-1/2}\Sigma_\epsilon^{-1/2} z\right| \gtrsim \epsilon^{-1}$. Although this bound is not exactly $\pi\epsilon^{-1}$, Lemma B.1 can still be satisfied by multiplying a constant factor in the domain of integration. Thus the following integral is also small for $\alpha \in (1,2)$:

$$\begin{aligned}\int_{|z|>\sqrt{\mu}} |\psi_\epsilon(\bar{z})| \,\mathrm{d}z &= \int_{|z|>\sqrt{\mu}} \exp\left(-tA_\epsilon(\bar{z})\right) \mathrm{d}z \\ &\leqslant \int_{|z|>\sqrt{\mu}} \exp\left(-C_q t^{1-\alpha/2} \lambda_{q,\epsilon}^{-\alpha/2} |z|^\alpha\right) \mathrm{d}z \\ &\leqslant \exp\left(-\frac{1}{2} C_q \epsilon^{1-\alpha}\right) \int_{\mathbb{R}^q} \exp\left(-\frac{1}{2} C_q \epsilon^{(\alpha-1)(\alpha-2)/2} |z|^\alpha\right) \mathrm{d}z \leqslant C_q \epsilon^K,\end{aligned}$$

for any $K > 0$ and ϵ sufficiently small. So altogether one arrives at, for $\alpha \in (1,2)$,

$$\int_{\mathbb{R}^q} \left|\psi_\epsilon\left(t^{-\frac{1}{2}} \Sigma_\epsilon^{-\frac{1}{2}} z\right) - e^{-\frac{1}{2}|z|^2}\left(1 + \sqrt{\mu} P_1(z)\right)\right| \mathrm{d}z \leqslant C_q \epsilon^{\alpha-1}.$$

Use the notation $\mathcal{Q}_{t,\epsilon,\cdot} := 1 + \sum_{k=1}^{\cdot} \mu^{k/2} Q_k$ for the Edgeworth-type expansion of $\bar{f}_\epsilon$. Then given the estimate above, one can first bound the integral in (B.1) over the ball $\{y : |y| \leqslant \epsilon^{-\eta/(q+2)}\}$ with $\eta := (\alpha-1)/2 \in (0,1)$:

$$\begin{aligned}I_1 :=& C_q \int_{|y| \leqslant \epsilon^{-\eta/(q+2)}} |y|^2 \mathrm{d}y \int_{\mathbb{R}^q} \left|\psi_\epsilon\left(t^{-\frac{1}{2}} \Sigma_\epsilon^{-\frac{1}{2}} z\right) - e^{-\frac{1}{2}|z|^2}\left(1 + \sqrt{\mu} P_1(z)\right)\right| \mathrm{d}z \\ \leqslant& C_q \epsilon^{\alpha-1-\eta} = C_q \epsilon^{(\alpha-1)/2},\end{aligned}$$

and then over the complement by matching the second moment of $\bar{f}_\epsilon$:

$$\begin{aligned}&\int_{|y|>\epsilon^{-\eta/(q+2)}} |y|^2 \left|\bar{f}_\epsilon(y) - \phi_I(y) \mathcal{Q}_{t,\epsilon,1}(y)\right| \mathrm{d}y \\ \leqslant& \int_{|y|>\epsilon^{-\eta/(q+2)}} |y|^2 \left(\bar{f}_\epsilon(y) + \phi_I(y) \mathcal{Q}_{t,\epsilon,1}(y)\right) \mathrm{d}y \\ \leqslant& \int_{\mathbb{R}^q} |y|^p \left(\bar{f}_\epsilon(y) - \phi_I(y) \mathcal{Q}_{t,\epsilon,1}(y)\right) \mathrm{d}y + \int_{|y| \leqslant \epsilon^{-\eta/(q+2)}} |y|^2 \left|\bar{f}_\epsilon(y) - \phi_I(y) \mathcal{Q}_{t,\epsilon,1}(y)\right| \mathrm{d}y \\ &+ 2 \int_{|y|>\epsilon^{-\eta/(q+2)}} |y|^2 \phi_I(y) \mathcal{Q}_{t,\epsilon,1}(y) \mathrm{d}y \\ \simeq& \int_{\mathbb{R}^q} |y|^2 \left(\bar{f}_\epsilon(y) - \phi_I(y) \mathcal{Q}_{t,\epsilon,p-2}(y)\right) \mathrm{d}y + I_1 + C_q \epsilon^\eta \int_{\mathbb{R}^q} \phi_I(y) |y|^{q+4} \left(1 + \sqrt{\mu} |y|^3\right) \mathrm{d}y \\ \leqslant& 0 + C_q \sqrt{\mu} + C_q \epsilon^{(\alpha-1)/2} \leqslant C_q \epsilon^{(\alpha-1)/2}.\end{aligned}$$

Thus, removing the cubic terms $\sqrt{\mu} \int_{\mathbb{R}^q} |y|^2 \phi_I(y) Q_1(y) \mathrm{d}y = O\left(\epsilon^{(\alpha-1)/2}\right)$ one arrives at

$$\int_{\mathbb{R}^q} |y|^2 \left|\bar{f}_\epsilon(y) - \phi_I(y)\right| \mathrm{d}y \leqslant C_q \epsilon^{(\alpha-1)/2}.$$

Finally by the same argument as in the last step in the proof of Theorem 4.9, one concludes

$$\mathbb{W}_2\left(Z_t^\epsilon, \mathcal{N}(0, t\Sigma_\epsilon)\right) \leqslant C_q t^{1/2} \lambda_{q,\epsilon}^{1/2} \epsilon^{(\alpha-1)/2} \leqslant C_q t^{1/2} \epsilon^{1/2}.$$

□

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