Calculation of Natural Frequencies of

VIBRATION OF THIN CYLINDRICAL SHELLS

OF VARIABLE THICKNESS

by

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CONTENTS.

frequencies of non-uniform cylinders with various relative values of length of sections and their thicknesses, and (ii) investigation of the actual amplitude distribution function existing in a cylinder of variable thickness. Sets of representative results are included in the work.

In this thesis the theoretical and experimental work is concerned with cylinders consisting of two sections of different thickness. It is claimed, however, that the method of solution developed is perfectly generally applicable to cylinders with any number of sections.

The nature of the method permits it to be considered as nothing more than a good approximation, but the experimental findings illustrate how nearly accurate the calculated results are. It yet has the merit of both simplicity and brevity, and as such is of no small value in practical engineering calculations. Due to the presence of the sudden change of thickness of the cylinder, an exact mathematical solution of the problem would be most difficult and cumbersome, and as a working method would present no real value.

where the deformations were purely inextensional or flexural; the problem under such conditions reduces to one of two dimensions only. Love, in his "Theory of Elasticity" (3), investigated the problem of the extensional vibrations of such a cylinder, and also derived the complete general equations of vibration of cylindrical shells, on which the theory presented in this work is based.

A condition of ends other than free was first examined in more detail by Flügge (4); he considered the ends to be freely-supported and in fact deduced the frequency equation for this case. Immediately, of course, a restraint is imposed upon the ends of the shell, it is impossible to take the deformation as being purely flexural, as some extension must necessarily take place.

Flügge's frequency equation was a cubic in p^2 * where the displacements are proportional to a harmonic function of time cos pt, giving the frequency as $p/2\pi$; no experimental investigation was carried out, although to illustrate his theory Flügge included a numerical example.

The present work originated directly in the

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* A list of symbols used is to be found in Appendix I.

recent researches of Arnold and Warburton (1), who examined in detail the vibrations of thin cylindrical shells with freely-supported ends. They confirmed the fact that for any particular nodal configuration the frequency equation is a sextic, or strictly speaking, a cubic in p^2, for no odd powers of p are present; thus three natural frequencies correspond to any one arrangement of nodes, depending on the relative magnitudes of the displacements in the three component directions. However, only one of these - when the prevalent motion is in the radial direction - is of immediate practical interest, the remaining two lying well beyond the aural range.

Among many practical applications of this theory one that readily assumes a position of first importance is that to the cutting of engineering components which have the form of cylindrical shells. In this connexion it must be realized that more often than not the cylinder will not be uniform but will have a change of thickness at some point along its length. It was from this practical point of view that the problem was tackled, and the emphasis placed on the development of a method whereby a natural frequency of a cylindrical shell of variable thickness might be calculated with the minimum of complication and maximum accuracy.

It would be vain for this purpose to attempt an exact analytical solution of the problem: this would involve the handling of some cumbersome mathematical expressions and the precise statement of boundary conditions (including those at the change of thickness) is not simple; the presence of a series solution which is necessitated by the discontinuity in the cylinder section profile would lead to a frequency equation too complicated to be of value in practical computations. Consequently the mathematical treatment of the problem is far from rigorous, but insofar as it represents a justification for the method of solution itself - which is necessarily of an approximate nature - the author judges it to be adequate.

2. GENERAL THEORY.

It must be emphasised at the very outset that it is not the intention of this work to present a rigorous mathematical solution of the problem. The primary difficulty of such an undertaking lies in the statement of the function of x in the expressions for the displacements. Due to the presence of the sudden change of section it is impossible to obtain this function from the general equations of vibration. The only available procedure would be to employ the energy method in obtaining the frequency equation for the non-uniform shell, and that entails the assumption of the vibration form.

The required function could then be expressed as a Fourier series, but the solution obtained in this way would not lend itself to numerical calculations, and in any case some approximation would eventually become unavoidable, as only a limited small number of terms of the series would be taken into account. Thus now a method of calculating the natural frequencies of non-uniform shells is derived on the basis of some considerations of the physical aspect of the theory of vibrations of thin cylinders.

Some assumptions have been made in devising such a method, and the author does not claim more than to have endeavoured to provide some mathematical justification for these. Thus the complete proof of the method of solution consists of the analytical treatment of its basic assumptions together with the evidence in support of the general results obtained by the theory, - as supplied in the course of the extensive experimental investigations carried out.

The mathematical operations involved in the construction of the method are recorded in detail in Appendix II; here we shall confine ourselves simply to quoting the relevant results.

Let us first briefly consider the general equations of vibration of a thin cylindrical shell, as derived by Love (Theory of Elasticity, p. 543 et seq.). The general solution of these, apart from simple harmonic functions of ϕ and t , gives the form of the component displacements as:-

$$\begin{aligned} u &= \sum_{r=1}^{4} \left(\xi_r e^{k_r x} + \xi'_r e^{-k_r x} \right), \\ v &= \sum_{r=1}^{4} \left(\eta_r e^{k_r x} + \eta'_r e^{-k_r x} \right), \\ w &= \sum_{r=1}^{4} \left(\zeta_r e^{k_r x} + \zeta'_r e^{-k_r x} \right), \end{aligned} \qquad (I)$$

Where the Greek letters represent arbitrary constants and k is a root of a determinantal equation of the eighth degree, or really a quartic in k^2, for it contains no odd powers of k.

Depending then on its coefficients, the roots of the quartic can be real, imaginary or complex. Boundary conditions will determine the exact shape of the displacement function in the axial direction of the cylinder; generally we shall expect it to contain both hyperbolic (or exponential) and harmonic functions, except in some simple symmetrical cases, - e.g. for a uniform cylinder with freely-supported ends the displacements are proportional to sines or cosines of x. It is unfortunately impracticable to state the conditions for the existence of real and imaginary roots of the quartic; when the discriminant is finally formed, it is so complicated that only extremely laborious numerical calculations for particular cases can be performed.

Apart from the geometric and elastic constants of the cylinder, the coefficients of the quartic will contain powers of p^2 up to the third power. Thus the discriminant may be regarded as giving k^2 as a function of p^2. The significance of this dependence of the form of the displacement function in the x - direction on the frequency of natural vibration is of fundamental interest in the treatment of the present problem and will be made clear in the later part of this chapter.

In the case of a uniform thin cylindrical shell, as considered by Arnold and Warburton, it is most convenient to obtain the frequency equation not from the general equations of vibration but by equating the limiting values of strain and kinetic energies. The condition of freely-supported ends is obtained by imposing no additional restraint beyond maintaining the circularity of the supported section.

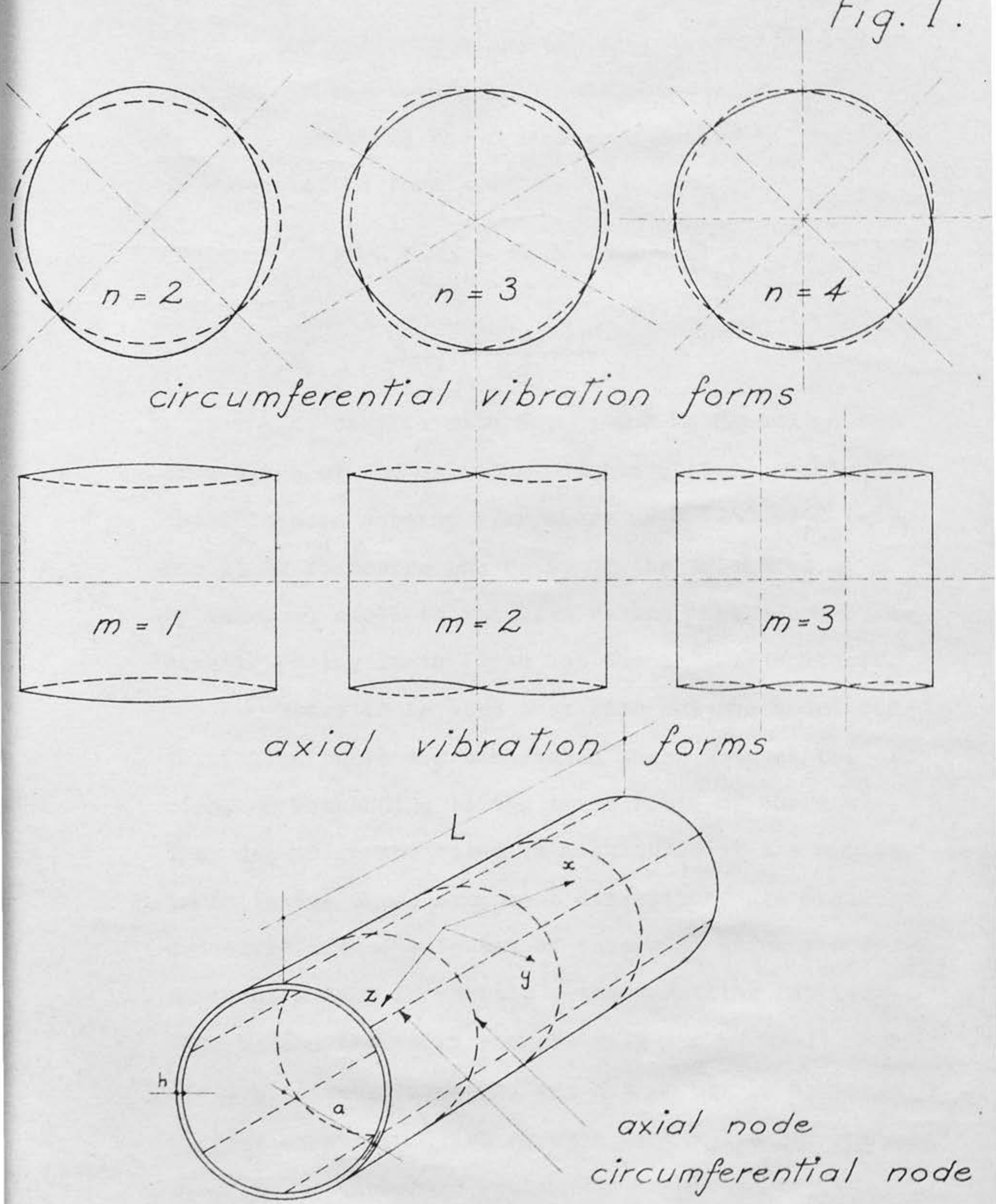
We take:-

$$\begin{aligned} u &= U \cos n\phi \cos \frac{m\pi x}{L} , \\ v &= V \sin n\phi \sin \frac{m\pi x}{L} , \\ w &= W \cos n\phi \sin \frac{m\pi x}{L} , \end{aligned} \qquad (2)$$

where U, V and W are functions of time only, proportional to cos pt ; n and $\frac{1}{2}m$ represent the number of circumferential and axial wavelengths respectively. These expressions satisfy the boundary conditions and the general equations of vibration.

The manner in which the cylinder vibrates under these conditions is shown in fig. I.

Fig. 1.



nodal arrangement :- n = 3; m = 4

The strain and kinetic energies can be stated in terms of the component displacements; applying the Lagrange equations the frequency equation is eventually obtained in the form:-

$$\Delta^3 - K_2\Delta^2 + K_1\Delta - K_0 = 0 , \qquad (3)$$

where

$$\Delta = \frac{\rho a^2 (1-\sigma^2) p^2}{Eg}$$

The coefficients K_2, K_1 and K_0 depend on the dimensions of the cylinder and its elastic constants; the thickness appears everywhere as $\frac{h}{a}$ $(=\alpha)$ or $\frac{h^2}{12a^2}$ $(=\beta)$, and it is therefore the ratio of the thickness of a cylindrical shell to its mean radius that becomes the discriminating factor, and not the thickness itself.

Hence it is seen that with any one nodal configuration there are associated three natural frequencies, corresponding to the three roots of the cubic. They depend on the relative magnitudes of the amplitudes in the three component directions; we shall be concerned with the lowest of these, at which the predominant motion is radial, - the remaining two lie well beyond the aural range: they are of small immediate practical interest, and are as yet to be investigated experimentally, as such work would necessitate special and elaborate equipment.

Thus, for any particular thickness ratio of a cylinder $\alpha = \frac{h}{a}$, we can calculate its complete frequency spectrum, knowing its length L and elastic constants E and σ. If we plot frequency to a base of $\lambda = \frac{m\pi a}{L}$, the resultant curve is as shown in fig. 2; there are drawn here two curves for some given value of n and for two different ratios α_1, and α_2, say, or, if we keep the mean radius constant, for two different thicknesses h_1, and h_2, where h_1 h_2. Natural frequencies correspond to values of λ for which m is an integer, and the frequency for $\lambda = 0$ is calculated from Lord Rayleigh's formula for a cylinder of infinite length:-

$$f_R = \frac{1}{2\pi}\sqrt{\left\{\left[\frac{Eh^2}{12\rho a^4(1-\sigma^2)}\right]\left[\frac{n^2(n^2-1)^2}{n^2+1}\right]\right\}} \qquad (4)$$

For any particular frequency there are 2n circumferential and m + 1 axial nodes (including two at the freely-supported ends); The latter are disposed symmetrically along the length of the cylinder, the distance between them being $\frac{L}{m}$ units.

Obviously the frequencies corresponding to integral values of m other than unity will also be natural frequencies of cylinders of the same thickness ratio and length $\frac{L}{m}$ vibrating with nodes at the two ends - or the fundamental mode for the particular value of n considered.

Fig. 2

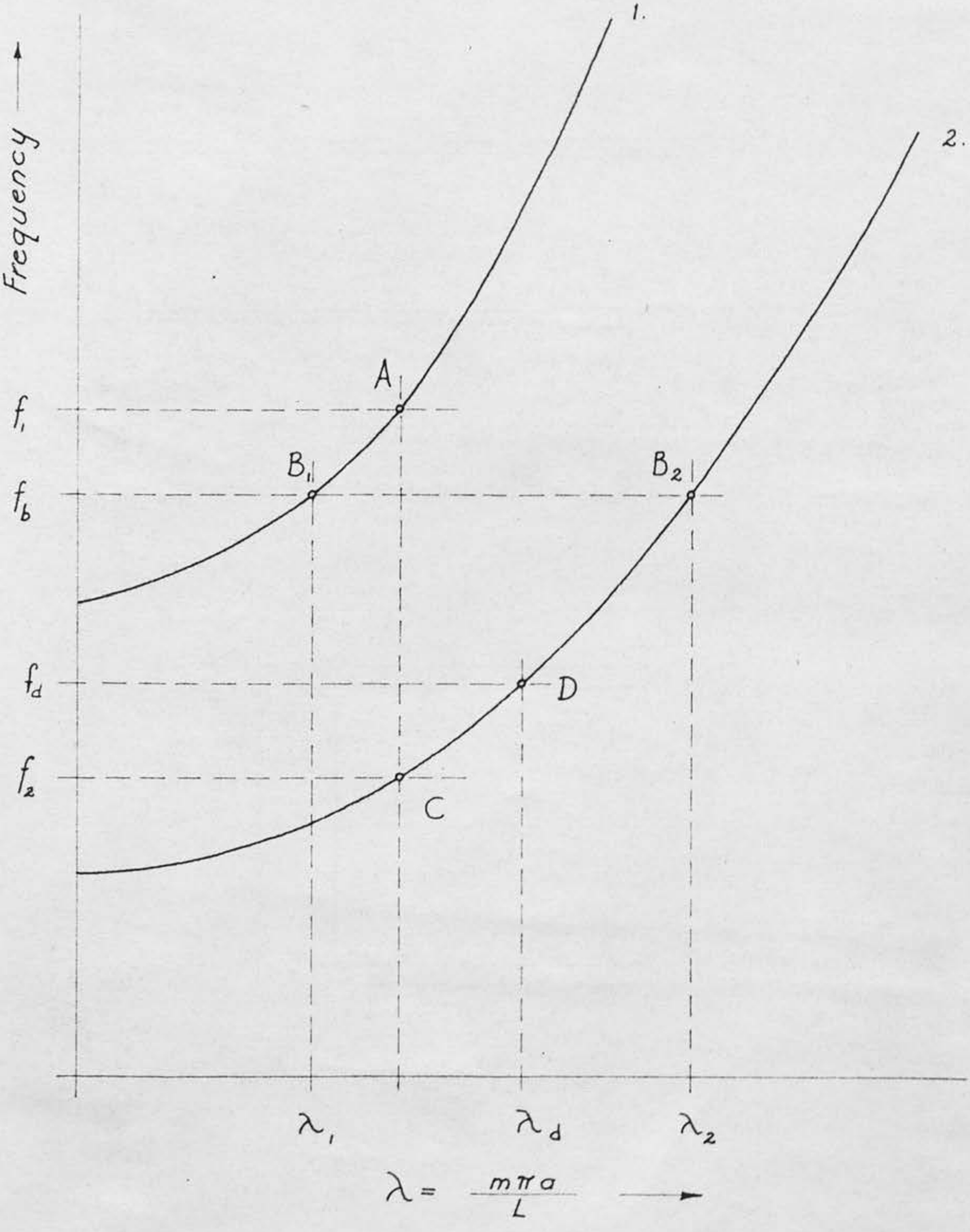
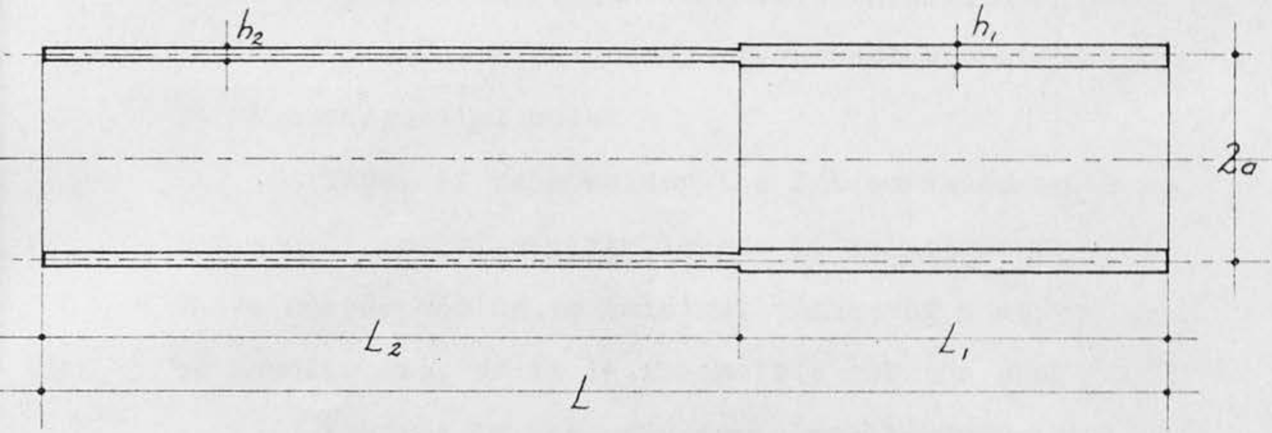
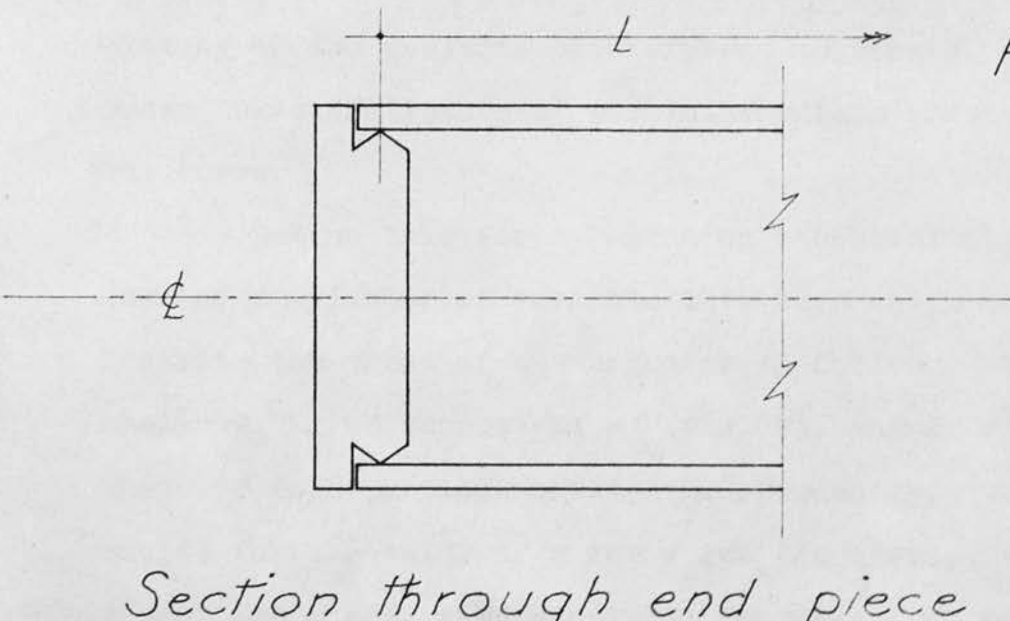

Frequency
1.
2.
A
B_1
B_2
C
D
f_1
f_b
f_d
f_2
λ_1
λ_d
λ_2
$\lambda = \frac{m\pi a}{L}$

Fig. 3



Section through experimental cylinder

Fig. 4



Section through end piece

Alternately, if we choose an arbitrary frequency on such a curve, we can find a value of L which will supply the necessary λ with an integral m value; a cylinder of that length will have the frequency chosen as its fundamental frequency of natural vibration with 2n circumferential nodes.

Thus, if we consider the f/λ curve to be drawn for a cylinder of a <u>given</u> length L, points other than those corresponding to integral values of m appear to be meaningless, as it is impossible for the end conditions to be satisfied if the vibration entails fractions of half-wavelengths. It is the examination of the behaviour of a cylindrical shell of constant mean radius, but consisting of two sections of different thickness, that shows the significance of the intermediate portions of the curve.

Let us take for a beginning a special simple case of a cylinder of variable thickness which will illustrate the trend of the argument to follow. In the shell of fig. 3 suppose $L_1 = \frac{\pi a}{\lambda_1}$(fig. 2), and $L_2 = \frac{\pi a}{\lambda_2}$. Then, if both portions existed independently, they would, for the value of n for which the curves are drawn, and m = 1, have a natural frequency f_b, as shown. Were they both of the same thickness (in which case $L_1 = L_2$), when joined together, they would form a cylinder for which f_b would represent the natural frequency for m = 2. Although they are not of the same thickness, they will still together form a shell whose natural frequency for m = 2 is f_b. It must be carefully noted that in such an event m supplies merely the number of axial wavelengths; since these are no longer equal, m itself does not provide any indication of the position of nodes.

It is legitimate - if not absolutely accurate - to assume that the mechanism of vibration of a non-uniform shell will be as described above, even when the node is not at the change of thickness; when there are more than one node apart from the two at the ends; and when the cylinder is vibrating in one of its fundamental modes.

Take, for instance, a cylinder of thickness h_1 and some given length L; then for particular values of n and m its natural frequency f_1 for that mode may be given by the point A on fig. 2; an exactly similar cylinder of thickness h_2 will have for the same mode a frequency f_2 represented by point C*. If we now consider a cylinder still of the same length L but of a section drawn in fig. 3, the natural frequency of its free vibrations in the mode under examination must

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* Here, and further below, unless otherwise stated, the mean radius is assumed to remain unchanged.

necessarily lie in the intermediate region; let that frequency be f_b such that $f_1 > f_b > f_2$ - it will correspond to B_1 on curve 1 and to B_2 on curve 2. The amplitude distribution in each portion will be sinusoidal - or very nearly so in practice - of the form $\sin\frac{\pi x}{l}$, where $2l$ is the wavelength and is given by:

$$l_1 = \frac{\pi a}{\lambda_1}, \qquad l_2 = \frac{\pi a}{\lambda_2}.$$

Thus complete half-wavelengths in either portion of the cylinder will be sinusoidal, while the curve which includes the change of section will be composed of two sine-curves which have the same amplitude and slope at the point where the thickness changes. This is merely a graphical representation of the form of vibration in keeping with the character of the proposed method of solution; more precisely, the stresses and bending moments existing at the change of thickness are of fundamental importance (they depend on the displacements and changes of curvature), and they may entail a slight deviation from sinusoidal form at that point; yet the assumption of purely sinusoidal deformation in each portion of the cylinder is a sufficiently close approximation for the practical purpose of calculating the natural frequencies of vibration: such an approach is not out of keeping with Rayleigh's principle for approximately calculating frequencies of free vibration.

We thus let the radial displacement in portion 1 vary as follows:-

$$w_1 = W_1 \sin\frac{\pi x}{l_1},$$

giving at the change of thickness a displacement

$$w = W_1 \sin\frac{\pi L_1}{l_1},$$

and a slope

$$\frac{dw_1}{dx} = W_1 \cos\frac{\pi L_1}{l_1}.$$

Let us consider for the other portion the origin of co-ordinates to be at its supported end. If we take for its displacement form

$$w_2 = W_2 \sin\frac{\pi x}{l_2},$$

we can equate the slopes and deflections at the change of thickness, and on eliminating W_1 and W_2 between the two equations, obtain

$$\tan\frac{\pi L_2}{l_2} = -\frac{l_1}{l_2}\tan\frac{\pi L_1}{l_1}. \quad (5)$$

[Appendix II, (1.14)]

Values of l_1 and l_2 are best chosen by trial and error, the criterion being that they must supply values of λ_1 and λ_2 corresponding to the same frequency; the fundamental mode of vibration has the frequency for which l_1 and l_2 satisfy the above equation so that the values of the tangents are in the first and second quadrants respectively. For further modes, with a higher number of axial nodes, these values will lie in successive quadrants compatible with the satisfaction of the equation.

In the trial or error method of determining l_1 and l_2 little difficulty or time is involved, and a solution can be obtained rapidly and without much computation. Specimen applications of the method are to be found in Appendix III.

There are cases, however, when the above equation cannot be solved for a particular mode. It will be found then that the required frequency is below the Rayleigh frequency f_R for portion 1 (i.e. that given by expression (4), corresponding to an infinite wavelength), and consequently a value of l is available for portion 2 (the thinner portion) only.

Mathematically, the significance of this becomes clear when we consider the general expressions for displacements (1) and recall what has been said about the values of the constant k_r on p. 9. At a certain value of frequency p^2, the value of k_r changes from a pure imaginary (which gave a simple sinusoidal waveform) to a complex or real number, introducing into the displacement function hyperbolic or, possibly, exponential terms.

In order to apply the "slope and deflection" method to this case, we must obtain an approximate expression for the waveform in the thicker portion.

It will by now be realized that the method is equivalent in its primary considerations to approximating the behaviour of a cylindrical shell to that of a beam; it is only after solving equation (5), that the values of l_1 and l_2 introduce the physical and dynamical characteristics of a shell. Thus now, to obtain an indication of the displacement function in the direction of the co-ordinate x, we consider a beam, consisting of two sections of flexural rigidity EI_1 and EI_2, of length L_1 and L_2 respectively, the total length of this beam being L; the ends of the beam are freely supported, i.e. displacements and moments vanish there. Such a beam is shown in Appendix II, fig. A1.

Let us assume that the displacement in portion 2 (the thinner portion) is still sinusoidal, of the form

$$w_2 = F \sin\frac{\pi x}{l_2},$$

but in portion 1 the displacement will be given by

$$w_1 = A \sin\frac{\pi x}{l_1} + C \sinh\frac{\pi x}{l_1},$$

which is the <u>exact</u> function for the beam, being obtained from the general solution by applying the two boundary conditions at the freely-supported end. l_1 is real, but unknown, and has to be determined together with A, C and F from the remaining four conditions at the point of change of section; these are supplied by considering at that point the expressions for the equality of displacement, slope, bending moment and shear force.

Having applied these conditions, as described in Appendix II, we find that l_1 is given by

$$\tan\frac{\pi L_1}{l_1} = \tanh\frac{\pi L_1}{l_1}, \qquad (6)$$

i.e. $$\frac{l_1}{L_1} = \frac{4}{5},\ \frac{4}{9},\ \frac{4}{13},\ \frac{4}{17},\ \text{etc.}, \qquad (7)$$

and $$\tan\frac{\pi L_1}{l_1} = 1. \qquad (8)$$

The frequency equation is obtained in the form

$$\tan\frac{\pi L_2}{l_2} = -\frac{l_1}{l_2}. \qquad (9)$$

It is seen immediately that it can be derived from equation (5) by applying condition (8).

Thus, for a fundamental mode, a solution for l_2 is obtained by taking $l_1 = \frac{4}{5}L_1$, and $\frac{\pi L_2}{l_2}$ in the second quadrant. For three axial nodes, $l_1 = \frac{4}{9}L_1$, and $\frac{\pi L_2}{l_2}$ is in the fourth quadrant; and so forth for higher modes.

Two very interesting features become apparent from the inspection of equation (9), viz.:- (i) since the determination of frequency is in this case reduced to solving the equation for l_2 only, values of frequency for different numbers of circumferential nodes will be obtained by simply applying to the relevant curves the <u>same</u> value of λ_2 (in fig. 2 - λ_d); e.g. if there is an instance of the insolubility of equation (5) in the case of, say, n = 3, m = 1, for some particular cylinder, then, having found l_2 (and hence λ_2) from equation (9), frequencies can be obtained for m = 1 and all values of n from 3 upwards; this is due to the fact that for n = 4, 5, etc. the two frequency/λ curves will be relatively further from each other, and λ_1 will always give a frequency below the Rayleigh frequency for portion 1; if we apply λ_1 to the curve for n = 2, we shall find that the frequency obtained lies above f_R for portion 1, indicating the applicability of equation (5); (ii) in the cases where equation (9) is to be applied, the natural frequency of a stepped cylinder is independent of the thickness of its thicker portion.

The latter point particularly was somewhat unexpected, and, in order to examine more closely these aspects of the problem, a comprehensive experimental investigation was designed and carried out. We shall proceed in the next chapter to the description of the experimental work, and the tabulation of its results.

3. EXPERIMENTAL INVESTIGATIONS

NOTATION FOR EXPERIMENTAL CYLINDERS.

All the experimental cylinders used were of the same mean radius and the same freely-supported length. Three different wall thicknesses were involved:- 0.25 in., 0.1875 in. and 0.125in., and to these the suffices 1, 2 and 3 respectively apply throughout this thesis.

The actual non-variable dimensions were:-

Freely-supported length (L)	17.5625 in.
Mean radius (a)	2.0725 in.

The numbers alone are employed to denote uniform cylinders of length L, mean radius a, and thickness 0.25 in., 0.1875 in. or 0.125 in., thus:- "cylinder 1", "cylinder 2", "cylinder 3".

In all cylinders which are not uniform, the ratio of the thinner portion to the freely-supported length L is always a multiple of $\frac{1}{8}$, and is denoted by successive letters of the alphabet A - G, thus:-

Ratio L_{thin}/L	Code letter
$\frac{1}{8}$	A
$\frac{1}{4}$	B
$\frac{3}{8}$	C
$\frac{1}{2}$	D
$\frac{5}{8}$	E
$\frac{3}{4}$	F
$\frac{7}{8}$	G

Hence "cylinder 1E3", for instance, refers to a cylinder with the following dimensions:-

Freely-supported length (L)	17.5625 in.
Mean radius (a)	2.0725 in.
Thickness (h_1) over $\frac{3}{8}L$ (= L_1)	0.250 in.
Thickness (h_3) over $\frac{5}{8}L$ (= L_3)	0.125 in.

Further, in general "cylinder 1 - 2", "2 - 3", "1 - 3" will denote cylinders consisting of sections of thickness to which the numbers refer (cf. above).

PART A. MEASUREMENT OF FREQUENCIES.

(i) General remarks:- An extensive experimental investigation formed part of the treatment of the present problem, and representative sets of results are included in this portion of the work.

It must be remembered that the theory developed is valid for what are termed <u>thin</u> shells, i.e. those whose thickness ratio does not exceed 0.1. This limitation, inherent in certain fundamental assumptions (in the expressions for stresses and strains), implies further in the case of shells of variable thickness that the relative ratios of the different thicknesses do not become disproportionately large or small; that, in fact, the vibration is perfectly continuous along the whole length of the shell.

Bearing this in mind, it was aimed in the preparation of experimental cylinders to obtain various ratios of lengths of sections rather than investigate specifically various relative ratios of the thicknesses themselves. Values of natural frequencies have, however, been calculated for cylinders whose thicknesses varied in different proportions, and experimental results for such cylinders have been obtained and are tabulated for purposes of comparison.

The condition of freely-supported ends was satisfied in a manner precisely similar to that adopted by Arnold and Warburton. Discs of the shape shown in fig. 4 were accurately machined and fitted in the ends of the cylinder. Such an arrangement effectively maintains the circularity of the shell at the section of support, yet does not otherwise constrain the motion. Tests were carried out designed to discover the effect of removing and replacing the end-pieces, but it was found that frequencies elicited in a series of experiments could always be repeated after the end-pieces had been disturbed. It must be pointed out, however, that after machining and first fitting the end-pieces into the cylinder a mark was scribed across the line of contact, and great care was taken to replace the end-pieces after each subsequent removal in exactly the same position.

The mean radius of the shell was kept constant throughout; hence in machining portions of the cylinder to a different thickness equal thicknesses of metal were taken off the outside and the inside of the wall. A sectional view of an experimental cylinder has already been given in fig. 3.

When investigation of the uniform cylinder was completed, it was machined to the new thickness over an eighth of its length, and one new end-piece made to fit the larger bore. Experiments were then carried out on the cylinder with $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, and $\frac{7}{8}$ of its length reduced in thickness, and finally the uniform cylinder of the reduced thickness was investigated*.

(ii) Experimental apparatus:- All cylinders were made of steel so that magnetic means of frequency excitation could be employed.

The general lay-out of the apparatus can be seen from plate I. The cylinder was set into vibratory motion through an earphone magnet, the source of frequency being a variable beat frequency oscillator. The output of the oscillator was first passed through a valve amplifier and a matching transformer; polarization of the exciting magnet was effected using a 50 volt D.C. supply in order to avoid frequency doubling. The gap between the magnet and the cylinder wall was kept as small as was possible without contact

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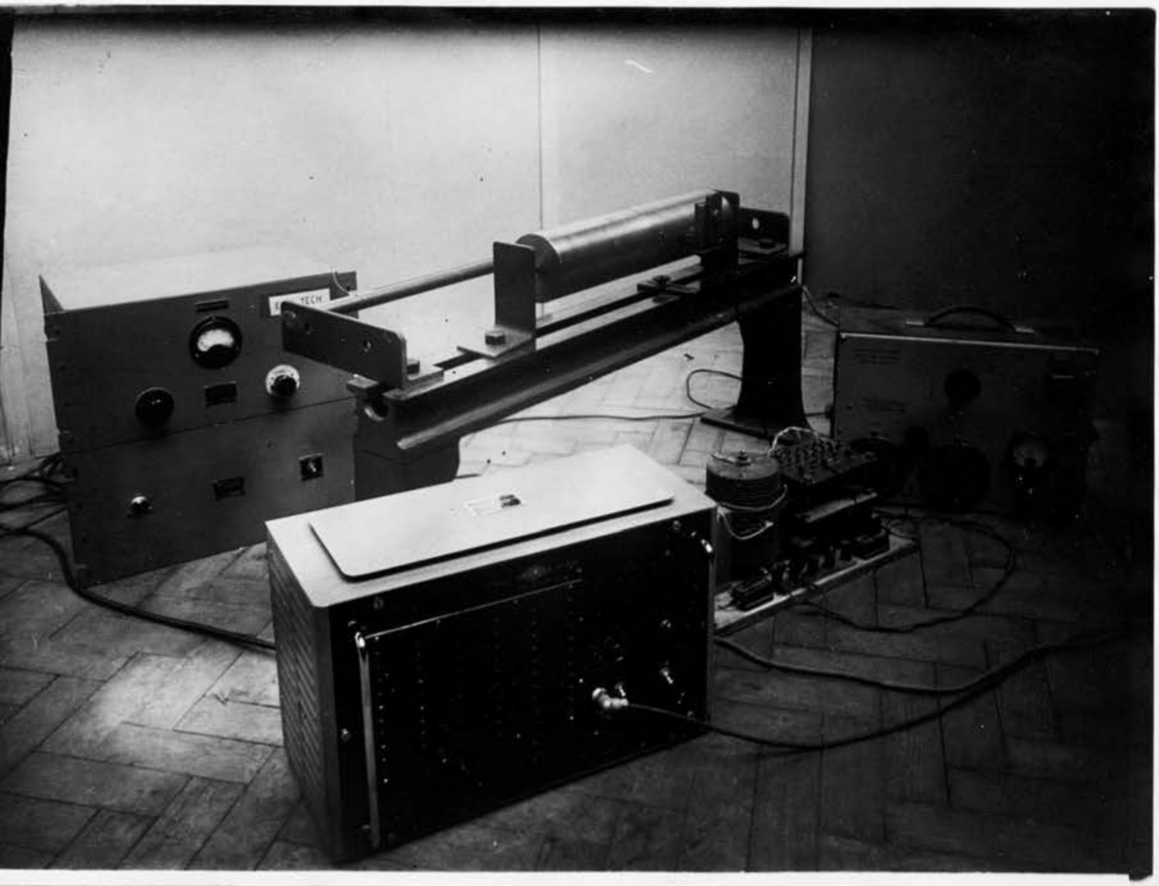
* The notation adopted to denote different experimental cylinders has already been described on p. 21.

PLATE I.



occurring during vibration.

Although the oscillator was equipped with two dials (high and low frequency range - differential reading) which were graduated to indicate the output frequency, this method of obtaining the values of natural frequencies was deemed inadequate; the dials were not calibrated in sufficiently small intervals to give results for the present type of investigation, where often small changes in frequency (due to the reduction in thickness of a portion of the cylinder) had to be measured.

Consequently an electronic counter (Airmec Type No. 704) was employed and the input frequency from the oscillator computed over an accurately measured time interval. This involved the use of an automatic switch mechanism which would not only maintain the counter in the oscillator circuit for a definite known period, but also effect both the "make" and the "break" instantaneously.

The arrangement adopted included a clock mechanism in conjunction with a remote contactor; the latter had an indicator arm rotating at one revolution per minute over a circular dial of which nearly one quadrant was coloured red. As the indicator arm passed out of that quadrant the circuit was "made" and the counter then registered beats from the oscillator for a period of 46 seconds, after which, as the indicator was entering the red quadrant, the circuit was "broken"; it remained so for the next 14 seconds, during which the reading on the counter could be noted and the counter reset in readiness for the next "make". It may be added that, were it so desired, it was a simple matter to utilize the 14 second period as the counting time.

Once wound, the motivating clock mechanism operated for some seven hours; the contactor itself could be switched off independently at any moment: for instance, when the "break" had been effected, the contactor could be stopped to enable any necessary adjustments to be made, and set in motion again (the "make" occurring at the usual place.) Finally, the counter had provision on its control panel for disconnecting it from the oscillator circuit (with this switch in the "off" position the instrument received no pulses whether the contactor had made or broken the circuit).

(iii) Experimental procedure:- The sound emitted by the cylinder at all frequencies which did not constitute its natural modes of vibration was of fairly small intensity. Consequently, as the oscillator was set at, or near, a frequency of the spectrum, the increase in amplitude was always noticeable, and by adjusting the value of the frequency so that the sound reached the peak of its intensity, it was possible to obtain the exact point of resonance. Although this method - i.e. a purely aural location of resonance frequencies - was for most cases perfectly satisfactory, constant use was made of the vibration pick-up system (described in detail in part B of this section) for that purpose. The point at which the amplitude of vibration rose to a maximum could be clearly seen on the screen of the cathode ray oscillograph; this was particularly useful at some of the more complex modes of vibration when the resonance response was not so violent, and the increase of amplitude of the sound emitted not so pronounced.

The position of the exciting magnet along the length of the cylinder did not in any way affect the point of resonance; it was occasionally changed, however, since, if it happened that the magnet was situated at a node corresponding to a certain natural mode of vibration, that mode could not then be elicited. Similarly, if two different modes had frequencies differing but slightly in magnitude, by setting the magnet at a node corresponding to one of them, the other could be obtained free from interference.

For the purpose of determining the nodal pattern a medical stethoscope was found convenient and quite satisfactory.

Once the required mode of vibration had been elicited, several 46 second counts were taken and their average noted. After the first two or three the oscillator was readjusted in an attempt to eliminate a possible error in obtaining the exact peak of resonance and consequently the accurate value of natural frequency; another few counts were then taken. It was found in general that the discrepancy between two such sets of results was extremely small, in fact, not exceeding the variation in readings which was always present due to the slight drift of the oscillator and possible minute changes in the contactor; this deviation was of the order of 1 in 14,000 over a period of 46 seconds.

To maintain a close watch on the operation of the contactor whose accuracy in timing the counting period was ever of paramount importance, a tuning fork was set up as a frequency standard. The fork used had a specified frequency of 523.3 c.p.s.; it was mounted rigidly in a vice arrangement and set into vibratory motion by a separate but exactly similar ear-phone magnet to that used on the cylinder. Every precaution was taken to guard the fork against possible disturbances and appreciable changes in temperature, and under these conditions its frequency could be said to remain constant. Before, during and after each series of experiments on the cylinder, the contactor was checked against the fork vibrating in its natural frequency; it was found on that basis to produce the interval of 46 seconds with an accuracy to within ± 0.2% this again includes the small possible drift of the oscillator for which no provision can be made.

(iv) Experimental results:- In the course of the investigation a very large number of natural frequencies had to be found, and corresponding theoretical values calculated. It was consequently decided to limit the range of modes considered to those with n = 2 to 5, and m = 1 to 4.

All the experimental results are tabulated in the following pages side by side with the calculated values; asterisks denote modes whose frequencies are below the Rayleigh frequency of the thicker portion, and were, therefore, calculated by means of the modified beam analogy method. There is also included a series of frequency curves for all the modes under consideration, drawn from calculated values with corresponding experimental points shown on them. They illustrate the variation of frequency from a uniform cylinder of thickness h_2 (see note on notation, p. 21) to one of thickness h_3 as the lengths of the two sections are progressively varied relative to each other in eight successive steps.

TABLE 1.

Frequencies for cylinder 1 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,700 1,626	4,360 4,353	8,130 8,297	12,725 13,367
3	Expt. Theory	2,430 2,390	4,615 4,611	8,335 8,541	12,930 13,583
4	Expt. Theory	3,810 3,833	5,170 5,167	 8,973	 13,957
5	Expt. Theory	5,430 5,545	6,040 6,072	9,280 9,612	13,685 14,495

TABLE 2.

Frequencies for cylinder 1A2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,635 1,630	4,300 4,346	7,920 7,850*	 11,310*
3	Expt. Theory	2,360 2,350	4,500 4,555	8,165 8,335	12,870 13,350
4	Expt. Theory	3,730 3,800	5,040 5,040	8,500 8,725	13,105 13,600
5	Expt. Theory	5,385 5,520	5,890 5,927	9,065 9,335	13,565 14,000

TABLE 3.

Frequencies for cylinder 1B2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,590	3,953		
	Theory	1,600	3,880*	6,655*	10,380*
3	Expt.	2,300	4,385	8,095	
	Theory	2,295	4,440	8,317	12,700*
4	Expt.	3,680	4,905	8,365	12,880
	Theory	3,755	4,920	8,635	13,450
5	Expt.	5,310	5,735	8,810	13,230
	Theory	5,470	5,840	9,105	13,700

TABLE 4.

Frequencies for cylinder 1C2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,525	3,595	6,410	
	Theory	1,550	3,510*	6,420*	10,185*
3	Expt.	2,255	4,360	7,540	
	Theory	2,275	4,430	7,720*	11,220*
4	Expt.		4,780	8,180	12,620
	Theory	3,740	4,840	8,340	13,000*
5	Expt.	5,240	5,593	8,625	12,930
	Theory	5,425	5,618	8,960	13,400

TABLE 5.

Frequencies for cylinder 1D2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,450 1,520	3,435 3,365*	6,270 6,330*	9,865 10,100*
3	Expt. Theory	2,240 2,255	4,205 4,310	6,955 7,070*	10,500 10,700*
4	Expt. Theory	3,650 3,693	4,690 4,670	7,965 8,510*	11,550 11,855*
5	Expt. Theory	5,275 5,395	5,515 5,555	8,380 8,595	12,750 13,200*

TABLE 6.

Frequencies for cylinder 1E2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,370 1,470*	3,343 3,320*	6,198 6,285*	9,808 10,070*
3	Expt. Theory	2,205 2,230	3,895 4,050*	6,615 6,660*	10,243 10,470*
4	Expt. Theory	3,593. 3,646	4,555 4,640	7,385 7,715*	10,915 11,220*
5	Expt. Theory	5,195 5,360	5,343 5,345	8,205 8,470	11,850 12,340*

TABLE 7.

Frequencies for cylinder 1F2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,300 1,350*	3,285 3,295*	6,153 6,258*	9,770 10,050*
3	Expt. Theory	2,150 2,185	3,665 3,800*	6,455 6,600*	10,080 10,335*
4	Expt. Theory	3,540 3,630	4,363 4,407	6,978 7,245*	10,533 10,855*
5	Expt. Theory	 5,325	5,218 5,288	7,725 8,210*	11,195 11,610*

TABLE 8.

Frequencies for cylinder 1G2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,215 1,280*	3,205 3,280*	6,100 6,260*	9,720 10,045*
3	Expt. Theory	2,050 2,128	3,493 3,625*	6,305 6,490*	9,923 10,250*
4	Expt. Theory	 3,600	4,088 4,455*	6,675 6,690*	10,285 10,620*
5	Expt. Theory	 5,300	5,000 5,050	7,273 7,680*	10,765 11,190*

TABLE 9.

Frequencies for cylinder 2 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,276	3,275	6,175	9,825
	Theory	1,259	3,271	6,229	10,027
3	Expt.	2,094	3,515	6,345	9,987
	Theory	2,104	3,507	6,419	10,196
4	Expt.	3,520	4,058	6,663	10,258
	Theory	3,593	4,058	6,772	10,484
5	Expt.	5,117	4,940	7,159	10,665
	Theory	5,293	4,941	7,314	10,907

TABLE 10.

Frequencies for cylinder 2A3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,271	3,253	5,638	
	Theory	1,267	3,260	5,525*	7,640*
3	Expt.	2,070	3,489	6,217	9,848
	Theory	2,100	3,447	6,262	9,985
4	Expt.	3,491	3,964	6,485	10,056
	Theory	3,545	3,955	6,549	10,253
5	Expt.	5,148	4,826	6,957	10,390
	Theory	5,205	4,819	7,055	10,620

TABLE 11.

Frequencies for cylinder 2B3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,243 1,250	2,960 2,810*	4,608 4,475*	7,146 6,950*
3	Expt. Theory	2,037 2,085	3,356 3,330	6,161 6,160	 8,715*
4	Expt. Theory	3,503 3,507	3,853 3,871	6,355 6,393	9,879 9,990
5	Expt. Theory	5,096 5,185	4,709 4,727	6,747 6,827	10,147 10,375

TABLE 12.

Frequencies for cylinder 2C3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,193 1,240	2,604 2,430*	 4,290*	 6,805*
3	Expt. Theory	2,016 2,045	3,326 3,300	5,539 5,425*	 7,565*
4	Expt. Theory	3,436 3,500	3,752 3,780	6,245 6,290	 9,205*
5	Expt. Theory	5,088 5,160	4,568 4,570	6,585 6,630	9,919 10,085

TABLE 13.

Frequencies for cylinder 2D3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,121	2,409	4,283	6,750
	Theory	1,190	2,280*	4,225*	6,750*
3	Expt.	1,993	3,169	4,900	7,271
	Theory	2,005	3,175	4,840*	7,185*
4	Expt.	3,421	3,624	5,909	8,145
	Theory	3,475	3,607	6,135*	8,075*
5	Expt.	5,041	4,453	6,372	9,292
	Theory	5,140	4,492	6,386	9,440*

TABLE 14.

Frequencies for cylinder 2E3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,046	2,307	4,214	6,673
	Theory	1,175	2,220*	4,200*	6,725*
3	Expt.	1,958	2,897	4,605	7,038
	Theory	1,980	2,990*	4,575*	7,010*
4	Expt.	3,309	3,423	5,314	7,586
	Theory	3,430	3,525	5,415*	7,560*
5	Expt.	4,981	4,332	6,164	8,397
	Theory	5,120	4,325	6,225	8,445*

TABLE 15.

Frequencies for cylinder 2F3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	983 1,065*	2,250 2,200*	4,173 4,180*	6,652 6,712*
3	Expt. Theory	1,925 1,925	2,666 2,720*	4,428 4,428*	6,830 6,908*
4	Expt. Theory	3,305 3,420	3,369 3,350	4,922 4,970*	7,264 7,280*
5	Expt. Theory	4,981 5,110	4,211 4,250	5,628 5,875*	7,821 7,880*

TABLE 16.

Frequencies for cylinder 2G3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	939 985*	2,199 2,187*	4,126 4,170*	6,593 6,700*
3	Expt. Theory	1,891 1,880	2,498 2,540*	4,301 4,340*	6,732 6,850*
4	Expt. Theory	3,303 3,410	3,138 3,140	4,657 4,740*	7,057 7,120*
5	Expt. Theory	4,911 5,100	4,051 4,045	5,204 5,395*	7,443 7,545*

TABLE 17.

Frequencies for cylinder 3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	900	2,190	4,104	6,550
	Theory	906	2,187	4,154	6,688
3	Expt.	1,865	2,426	4,244	6,704
	Theory	1,870	2,422	4,296	6,803
4	Expt.	3,290	3,000	4,571	6,982
	Theory	3,407	3,011	4,587	7,010
5	Expt.	4,898	3,951	5,012	7,280
	Theory	5,098	3,932	5,060	7,331

TABLE 18.

Frequencies for cylinder 1A3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,618	4,213	5,670	7,700
	Theory	1,625	4,300	5,525*	7,640*
3	Expt.	2,305	4,433	8,115	12,759
	Theory	2,350	4,455	8,315	13,370
4	Expt.	3,643	4,938	8,390	
	Theory	3,720	4,910	8,630	13,600
5	Expt.	5,307	5,800	8,867	
	Theory	5,480	5,800	9,140	13,900

TABLE 19.

Frequencies for cylinder 1B3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,575 1,580	3,153 2,810*	4,600 4,475*	7,200 6,950*
3	Expt. Theory	2,254 2,245	4,357 4,415	7,300 7,050*	8,850 8,715*
4	Expt. Theory	3,699 3,700	4,770 4,830	8,153 8,345	12,319 12,125*
5	Expt. Theory	5,255 5,415	5,518 5,630	8,513 8,725	12,865 13,415

TABLE 20.

Frequencies for cylinder 1C3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt. Theory	1,474 1,525	2,767 2,430*	4,486 4,290*	6,871 6,805*
3	Expt. Theory	2,211 2,195	4,173 4,295	5,764 5,425*	7,935 7,565*
4	Expt. Theory	3,565 3,670	4,554 4,555	7,581 7,665*	9,625 9,205*
5	Expt. Theory	5,204 5,350	5,316 5,348	8,215 8,405	11,670 11,600*

TABLE 21.

Frequencies for cylinder 1D3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,324	2,460	4,294	6,764
	Theory	1,410*	2,280*	4,225*	6,750*
3	Expt.	2,140	3,561	5,008	7,344
	Theory	2,160	3,505*	4,840*	7,185*
4	Expt.	3,451	4,441	6,291	8,300
	Theory	3,610	4,507	6,135*	8,075*
5	Expt.	5,124	5,085	7,705	9,657
	Theory	5,305	5,105	8,000*	9,440*

TABLE 22.

Frequencies for cylinder 1E3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,168	2,319	4,225	6,683
	Theory	1,200*	2,220*	4,200*	6,725*
3	Expt.	2,070	3,072	4,603	7,040
	Theory	2,100	2,990*	4,575*	7,010*
4	Expt.		4,132	5,419	7,614
	Theory	3,545	4,295	5,415*	7,560*
5	Expt.	5,060	4,867	6,594	8,493
	Theory	5,245	4,890	6,675*	8,445*

TABLE 23.

Frequencies for cylinder 1F3 - cycles per second.

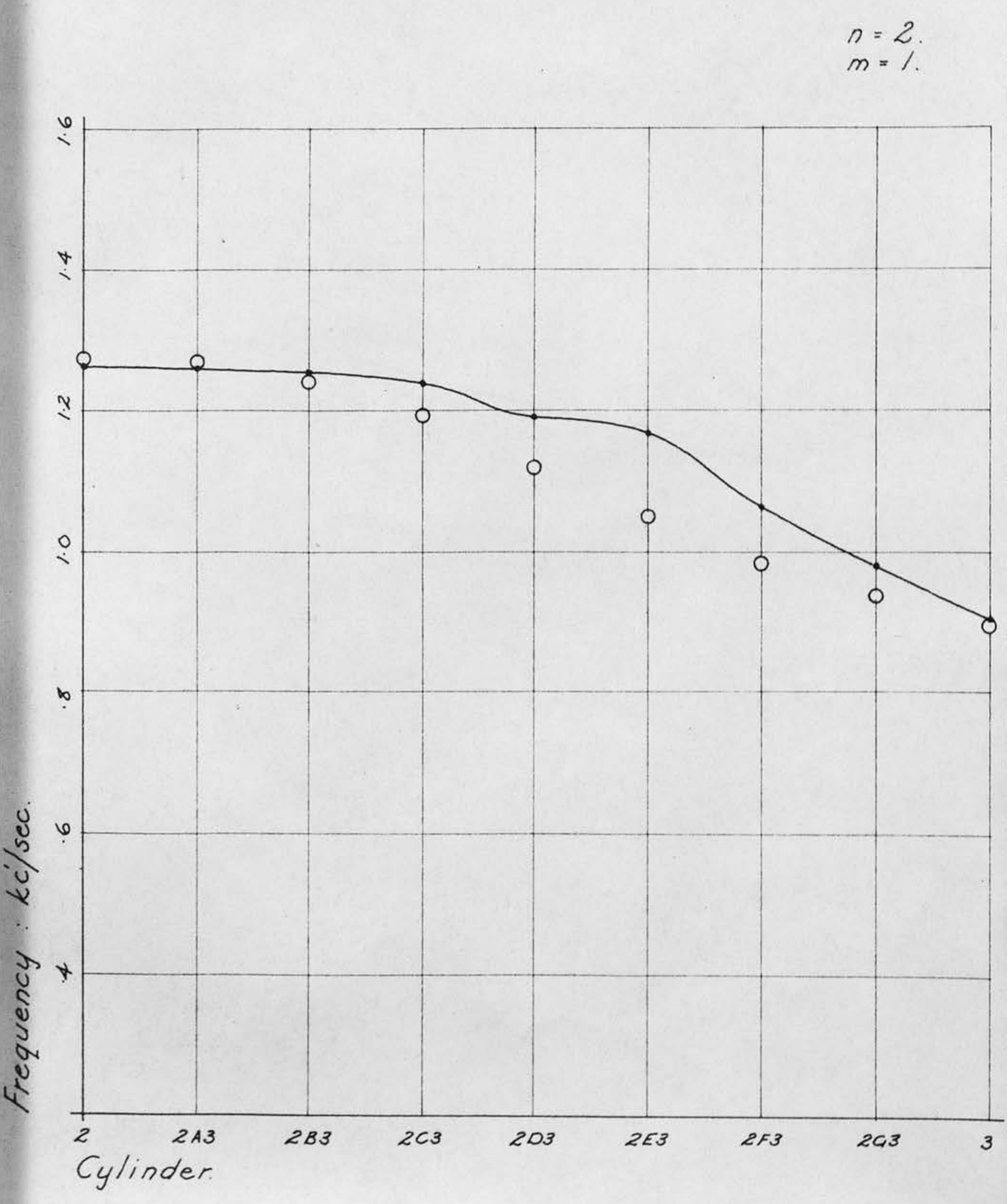
Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	1,048	2,259	4,167	6,638
	Theory	1,065*	2,200*	4,180*	6,712*
3	Expt.	2,003	2,758	4,437	6,836
	Theory	1,990	2,720*	4,428*	6,908*
4	Expt.		3,653	4,987	7,288
	Theory	3,520	3,800*	4,970*	7,280*
5	Expt.	5,089	4,593	5,797	7,879
	Theory	5,190	4,615	5,875*	7,880*

TABLE 24.

Frequencies for cylinder 1G3 - cycles per second.

Axial nodes (m + 1)		Circumferential nodes (2n)			
		4	6	8	10
2	Expt.	967	2,217	4,140	6,607
	Theory	985*	2,187*	4,170*	6,700*
3	Expt.	1,900	2,560	4,315	6,750
	Theory	1,895	2,540*	4,340*	6,850*
4	Expt.	3,390	3,257	4,724	7,100
	Theory	3,435	3,330*	4,740*	7,120*
5	Expt.	5,050	4,280	5,320	7,517
	Theory	5,130	4,295	5,395*	7,545*

Fig 6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1.2
1.0
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.6
.4
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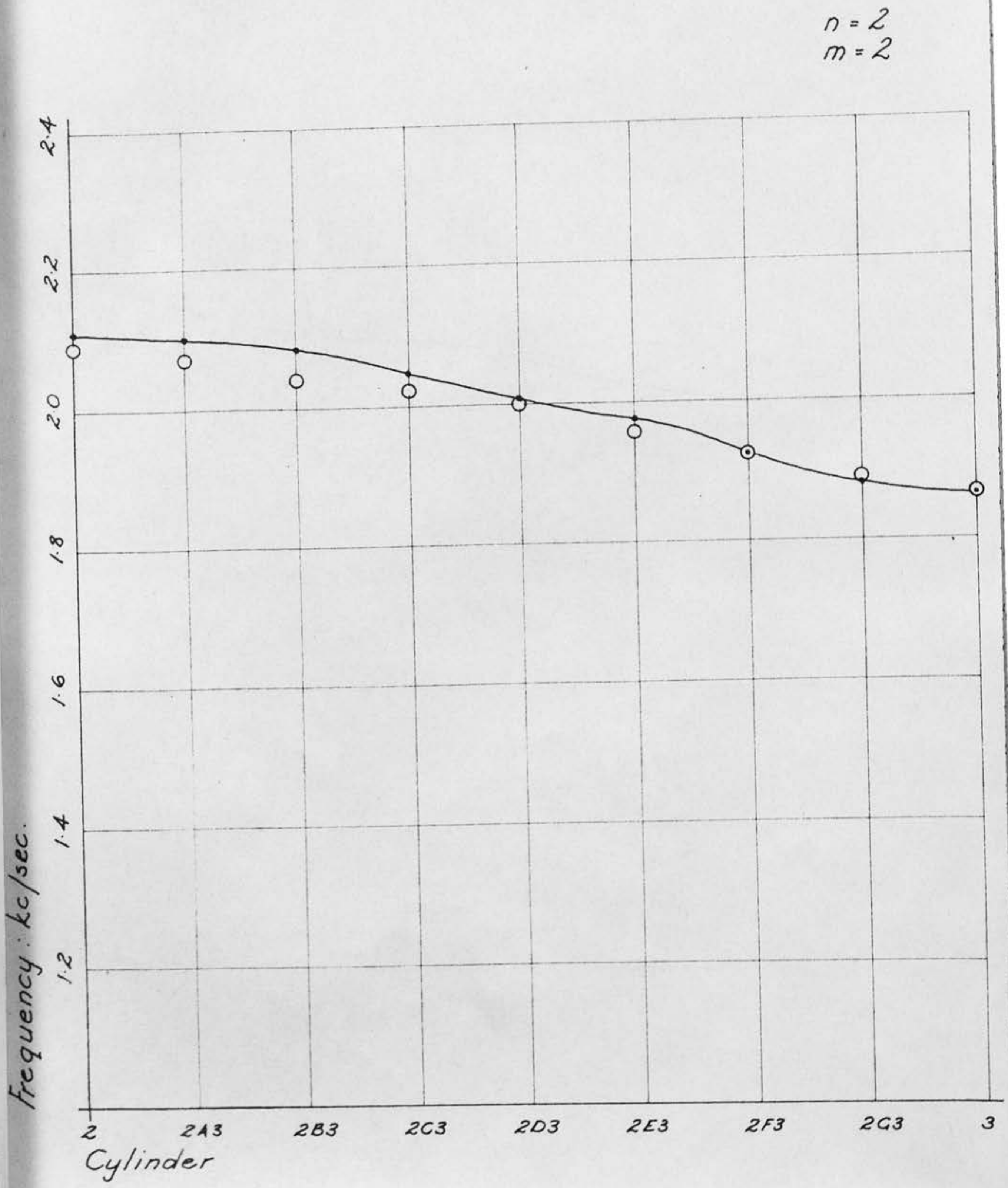
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder.

Fig 7.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m = 2
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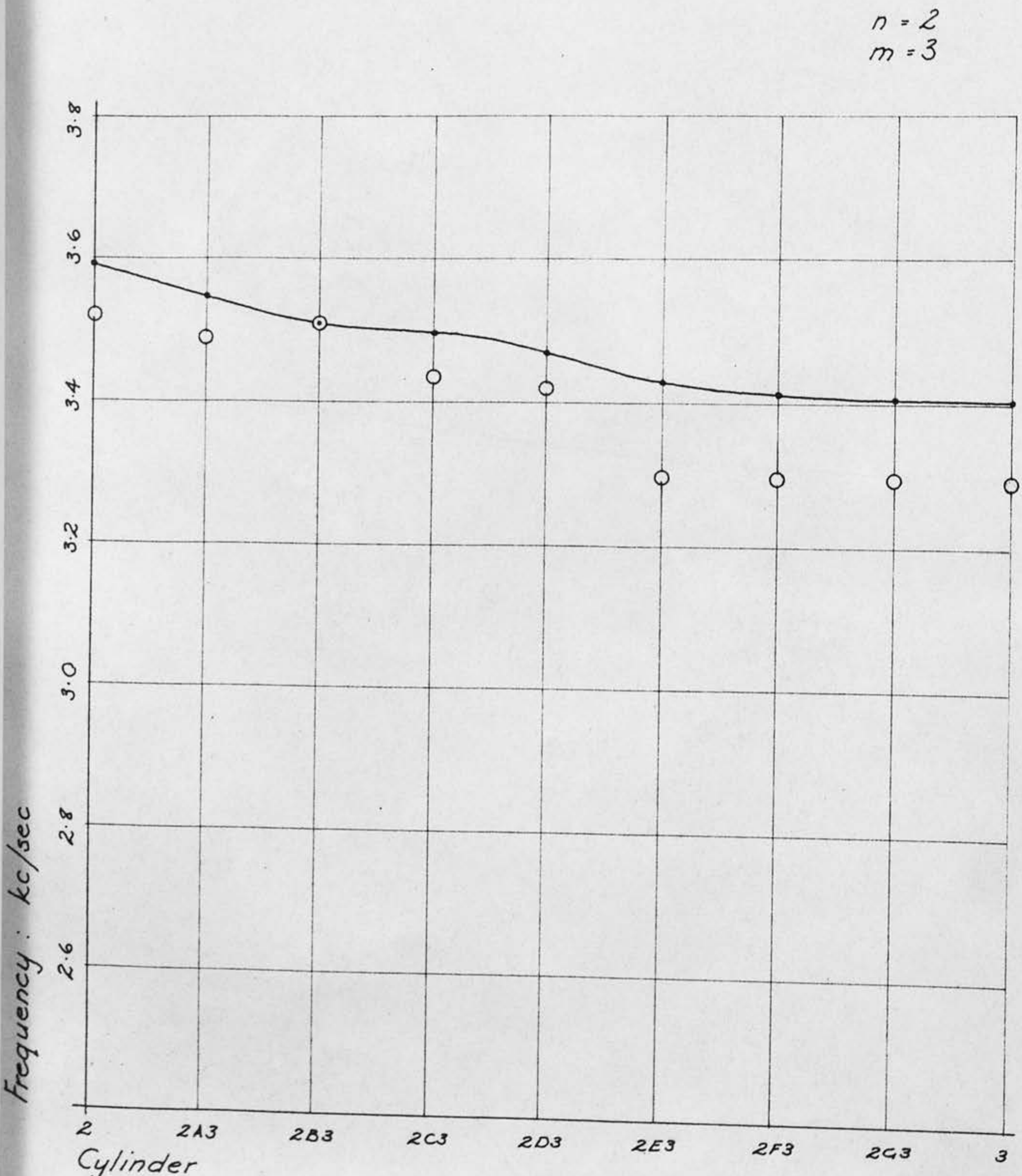
2.2
2.0
1.8
1.6
1.4
1.2
Frequency : kc/sec.
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 8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m = 3
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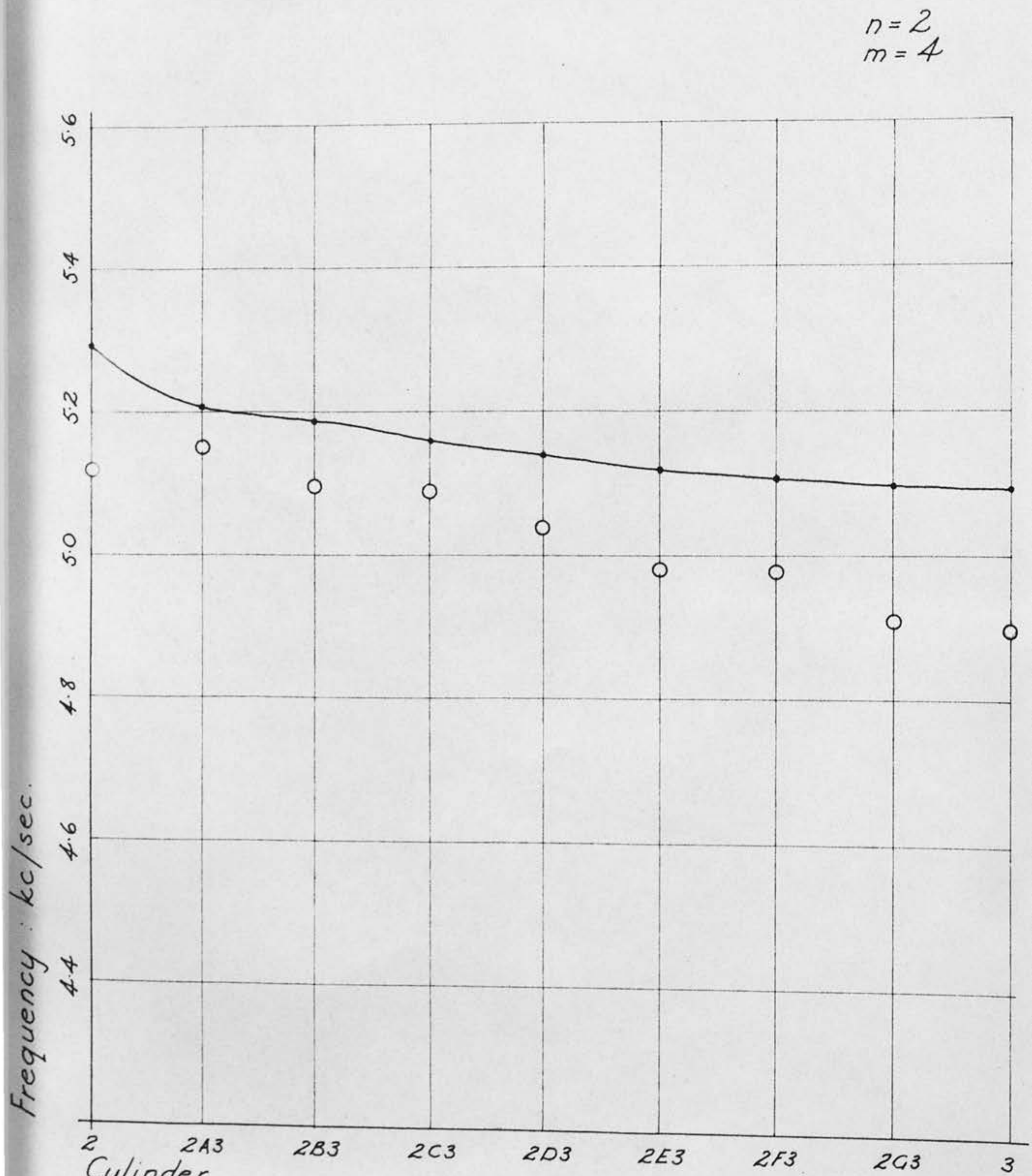
3·8
3·6
3·4
3·2
3·0
2·8
2·6
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 9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m = 4
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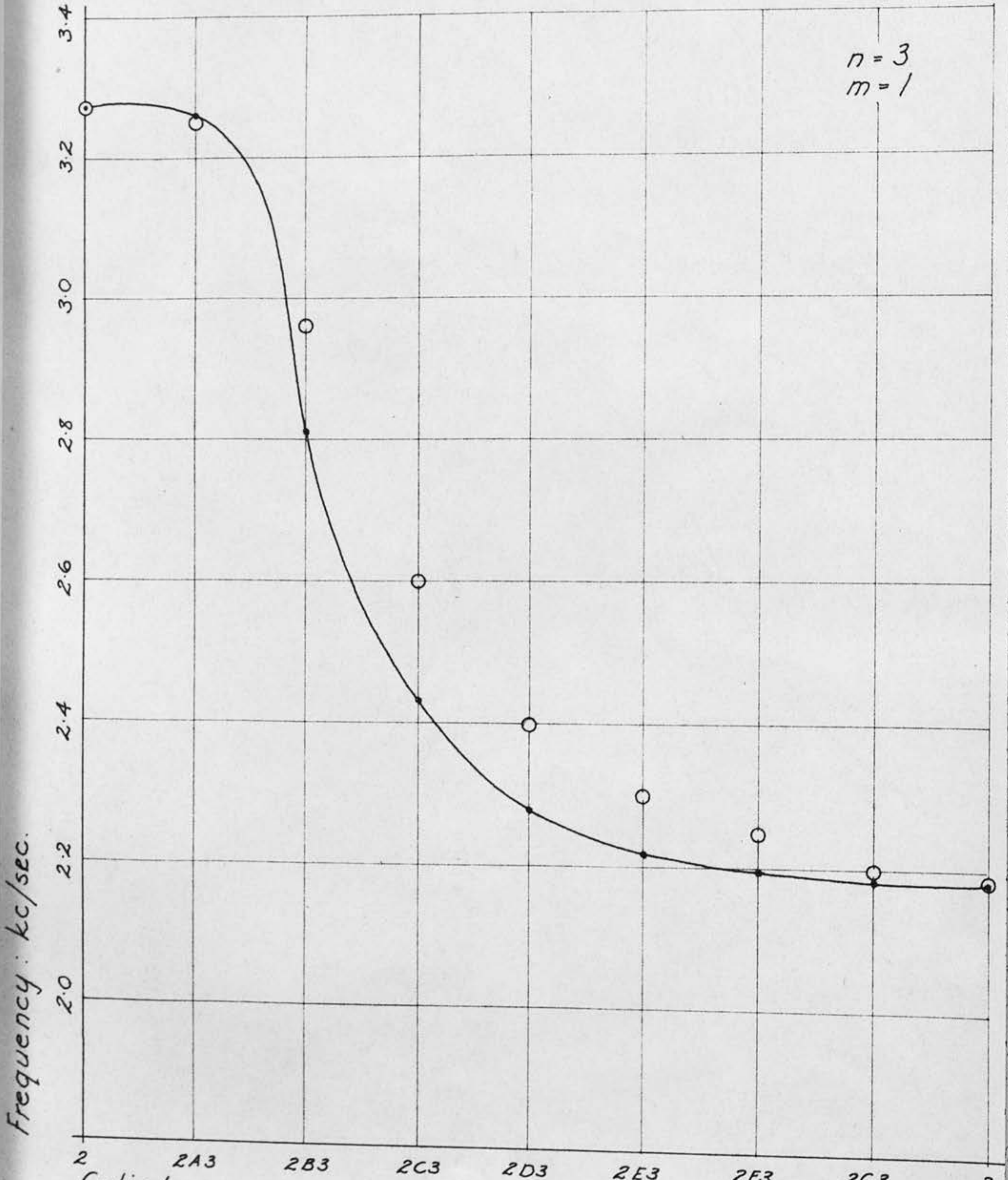
5·4
5·2
5·0
4·8
4·6
4·4
Frequency : kc/sec.
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 10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m = 1
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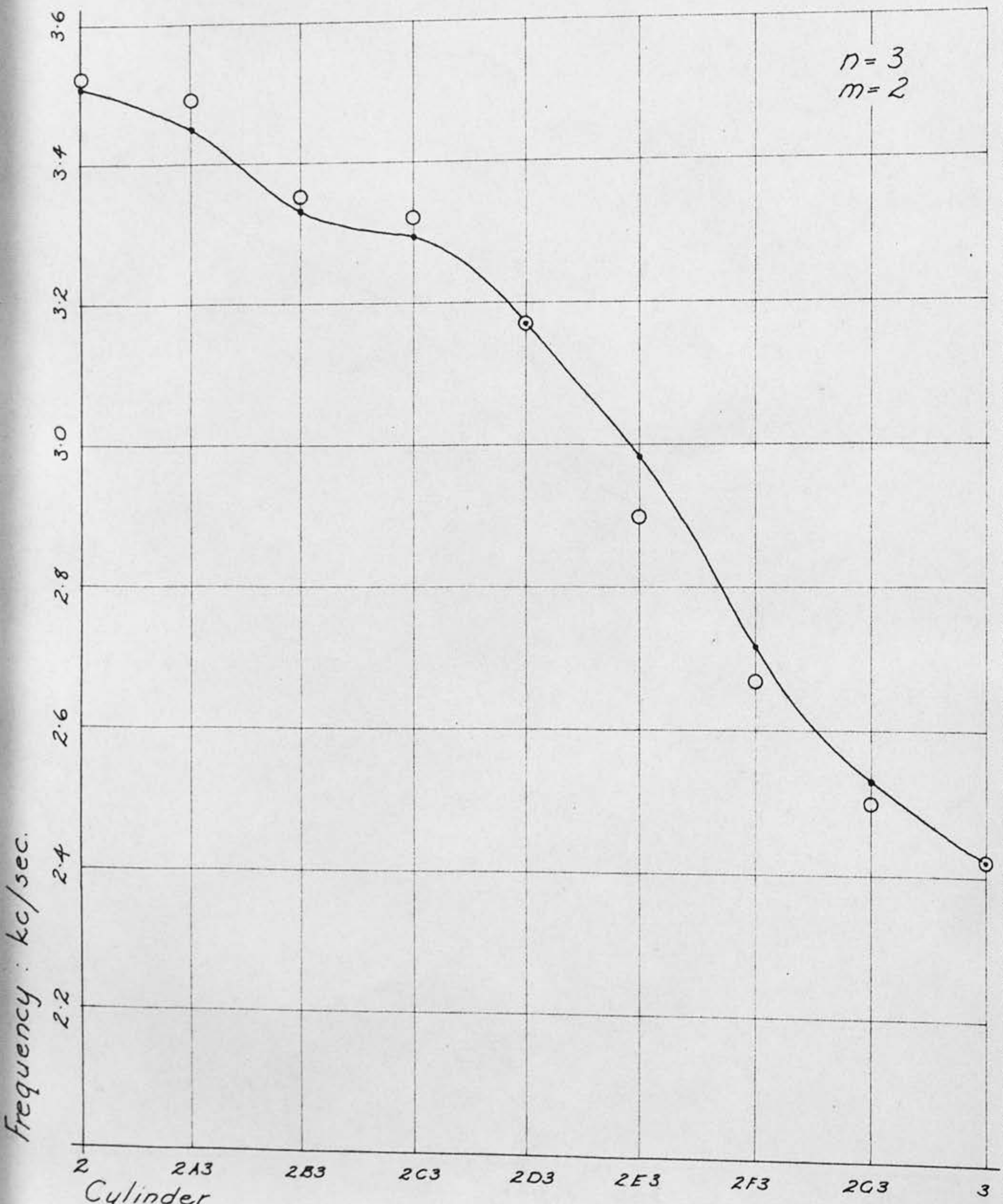
3·4
3·2
3·0
2·8
2·6
2·4
2·2
2·0
Cylinder
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3

Fig 11



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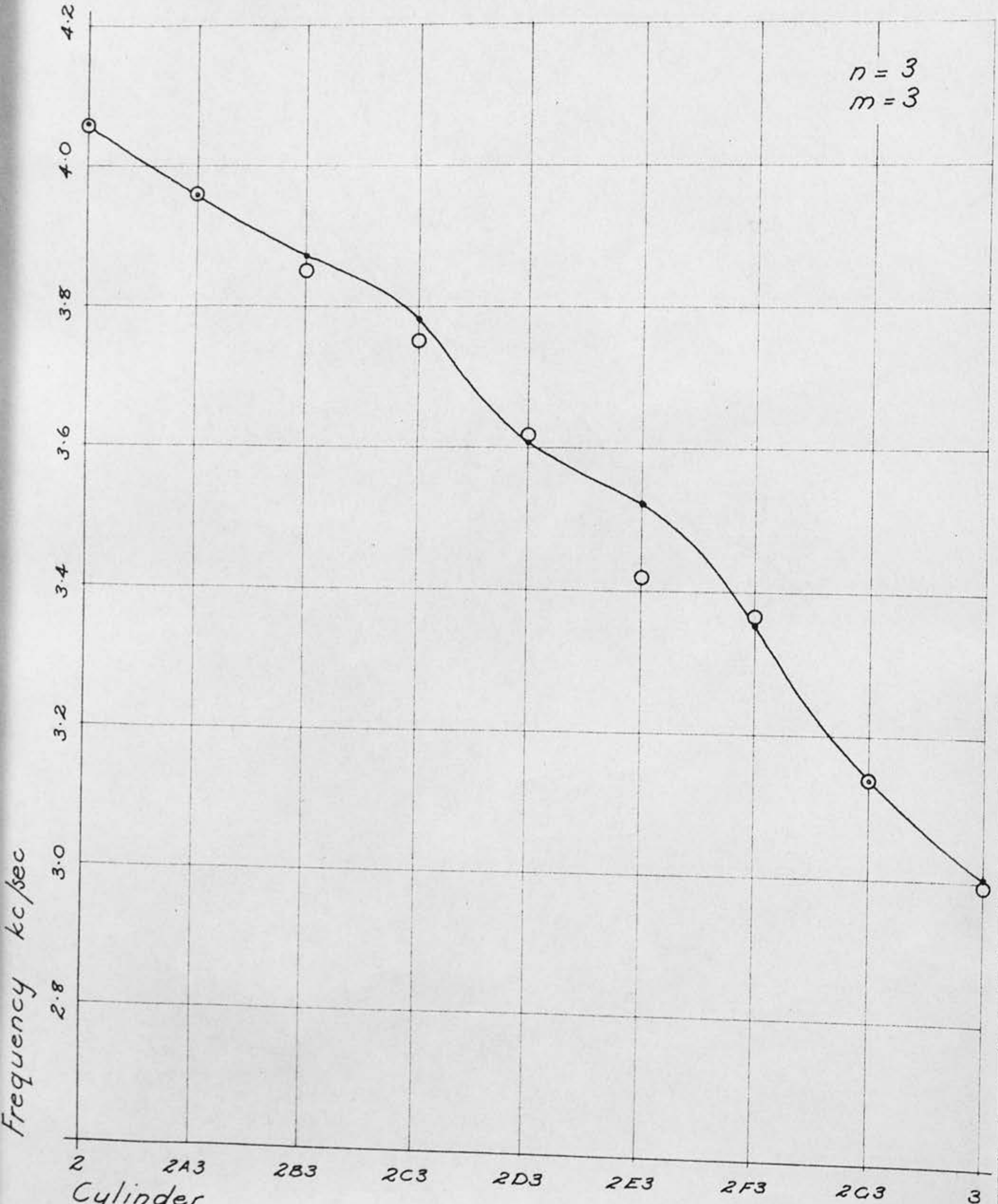
m= 2
3·6
3·4
3·2
3·0
2·8
2·6
2·4
2·2
Frequency : kc/sec.
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 12



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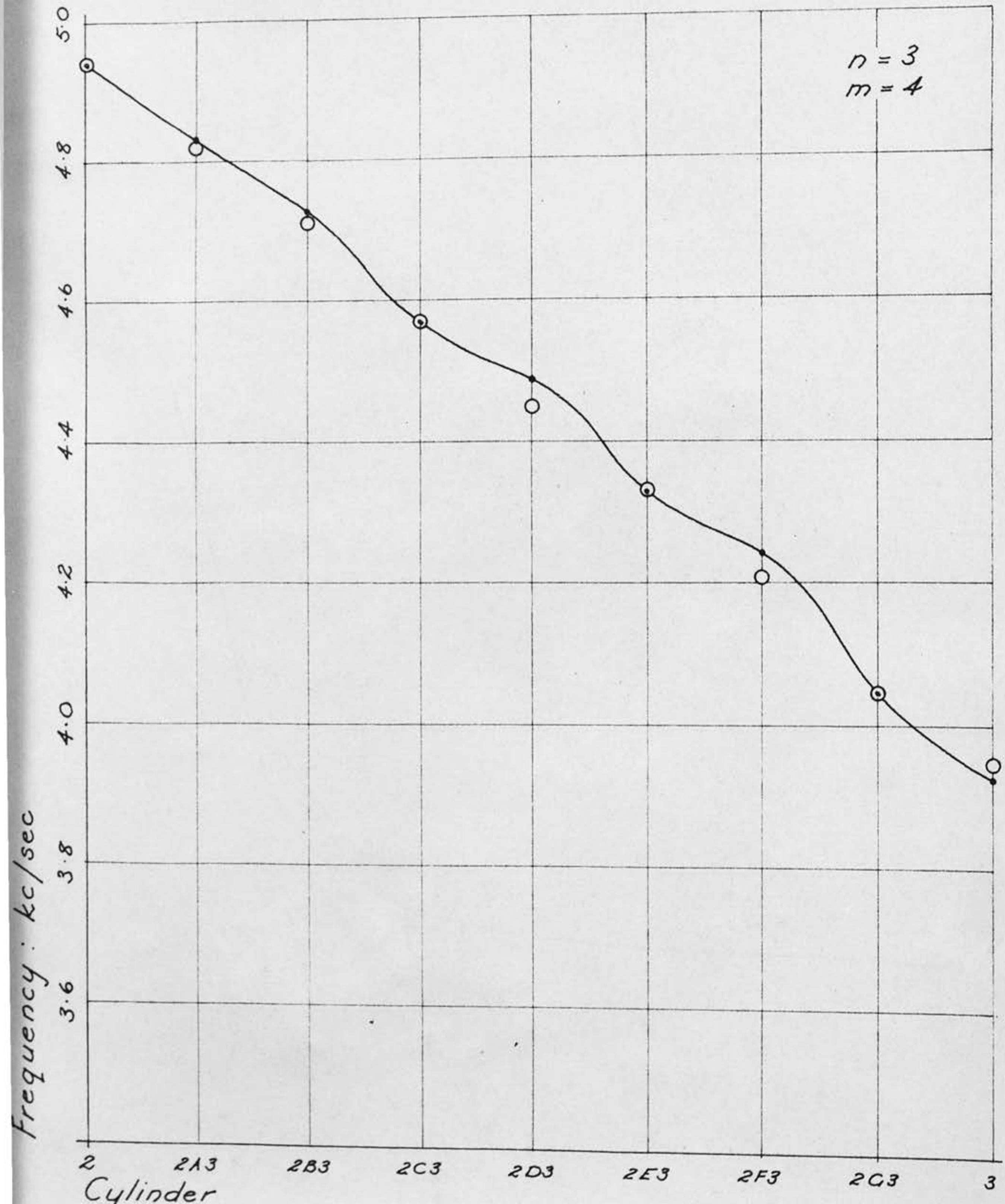
3.8
3.6
3.4
3.2
3.0
2.8
Frequency kc/sec
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 13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4·8
4·6
4·4
4·2
4·0
3·8
3·6
Frequency : kc/sec
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2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

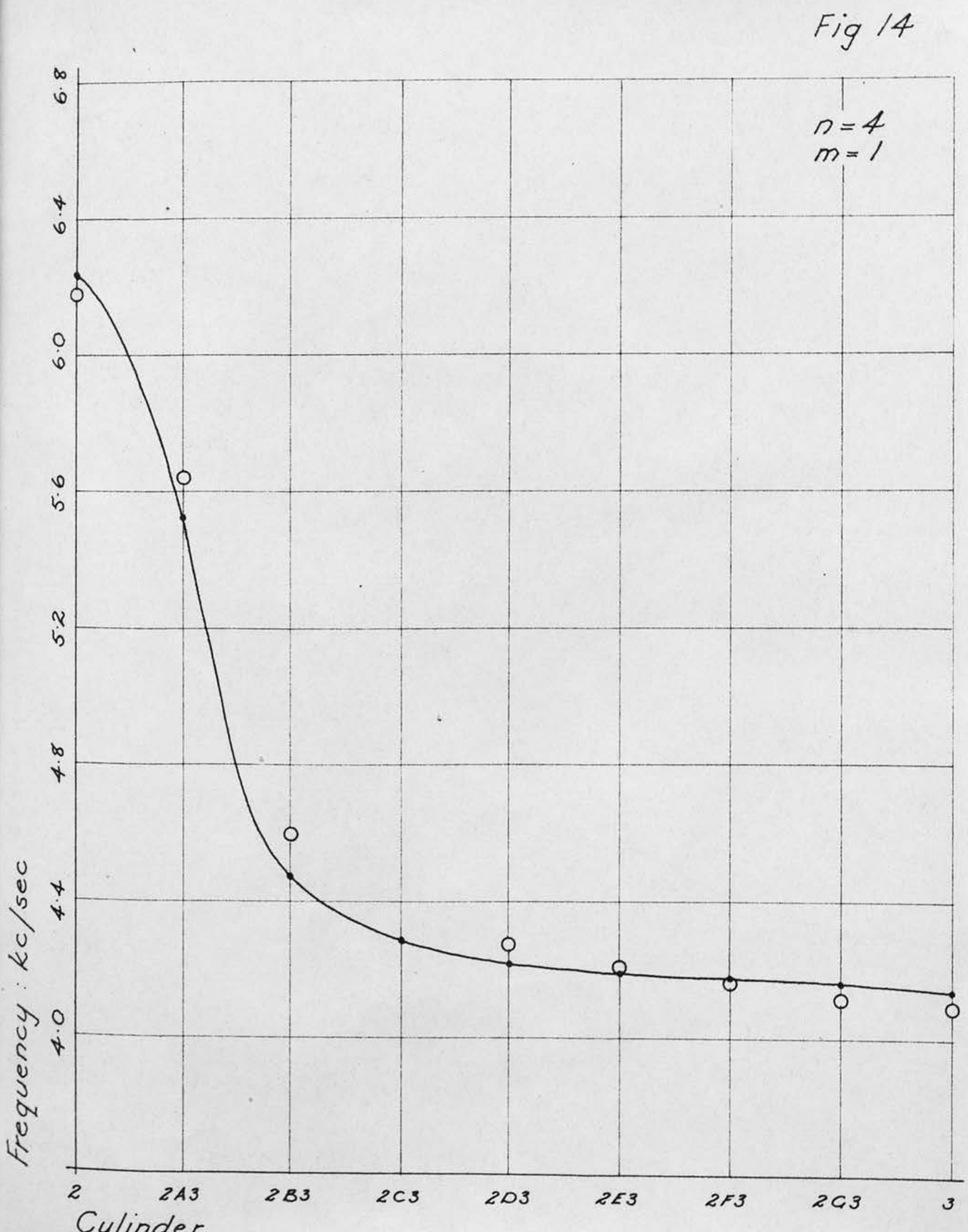
Fig 14
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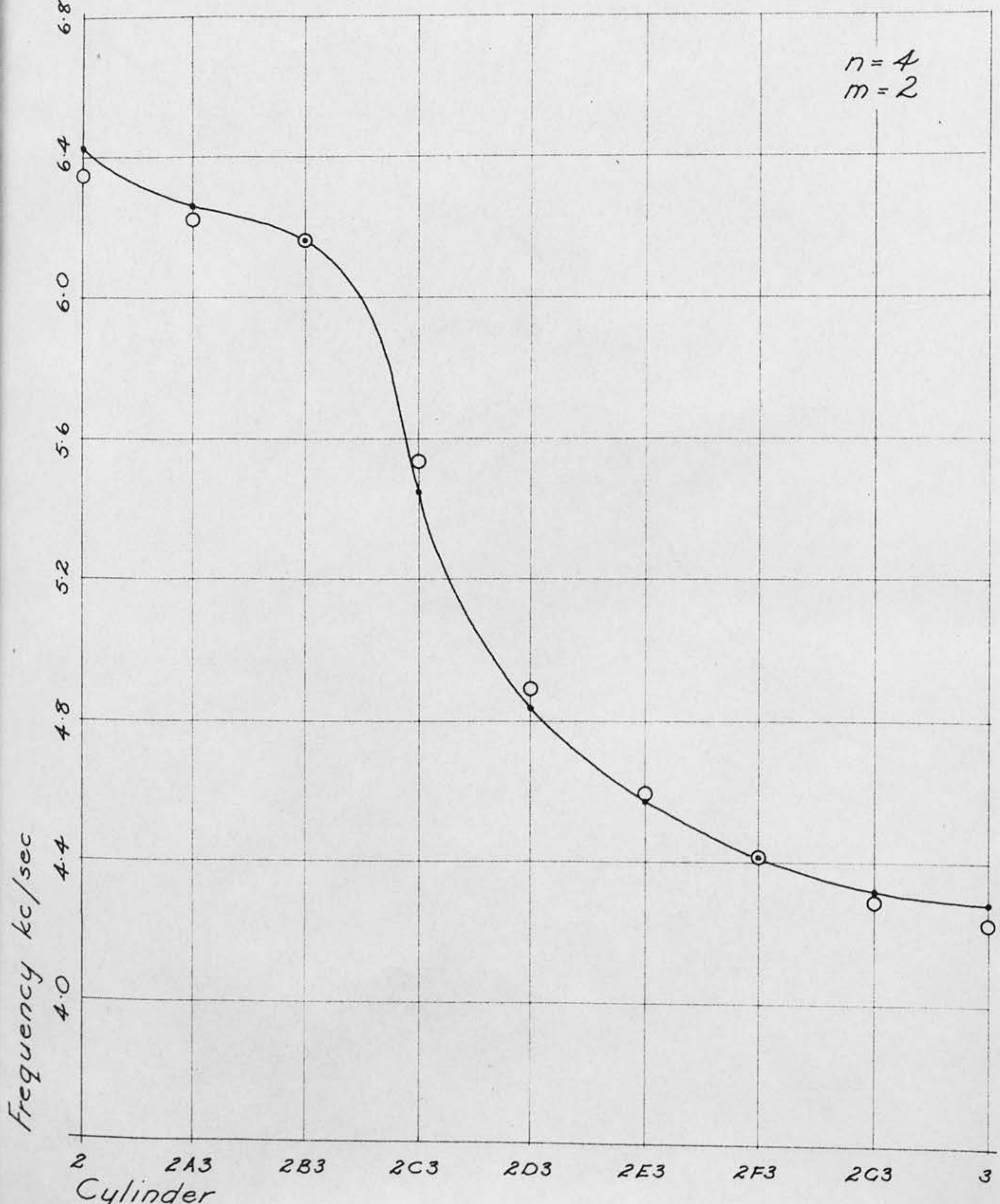
6.4
6.0
5.6
5.2
4.8
4.4
4.0
Frequency : kc/sec
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 15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m = 2
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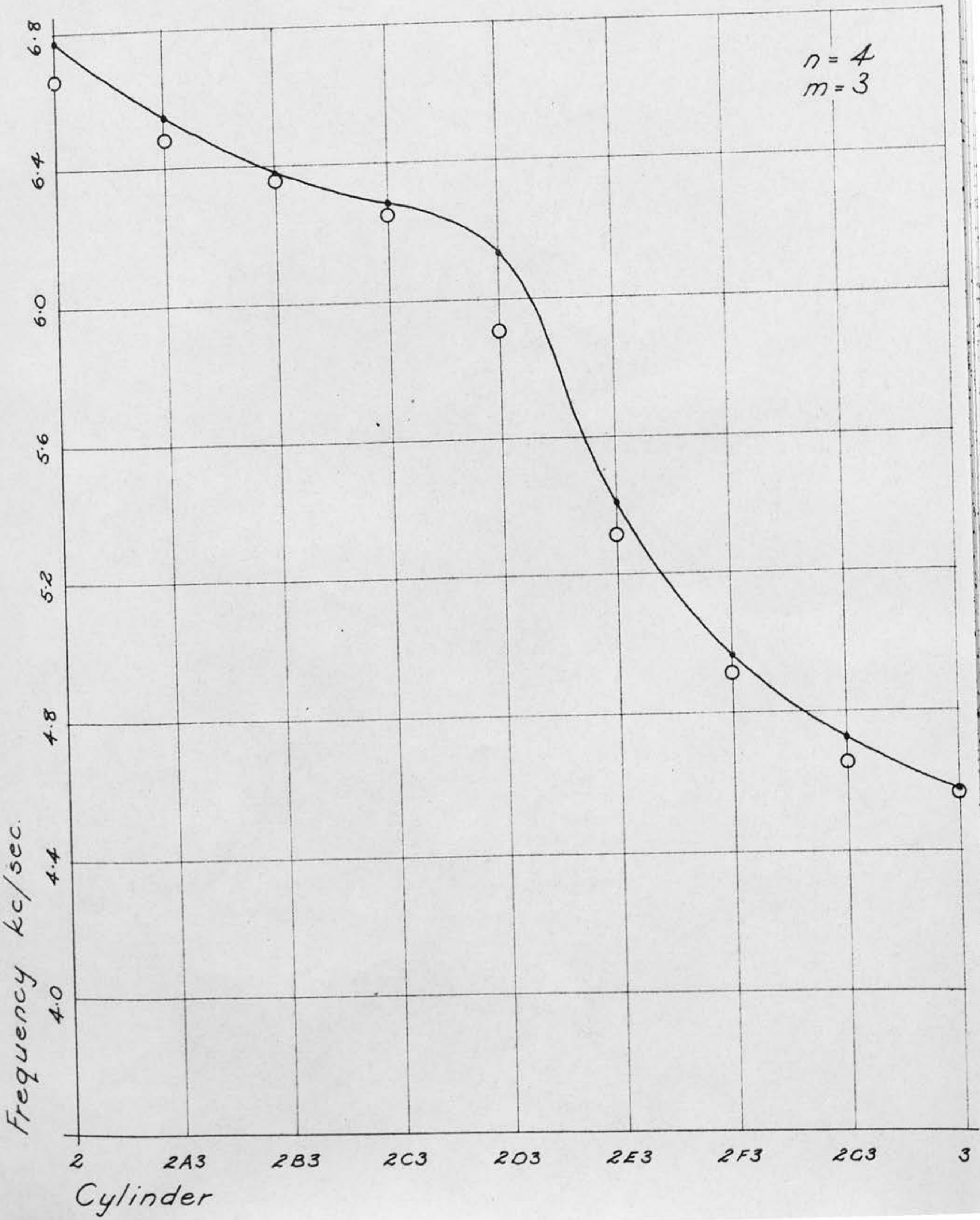
6·4
6·0
5·6
5·2
4·8
4·4
4·0
Frequency kc/sec
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 16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m = 3
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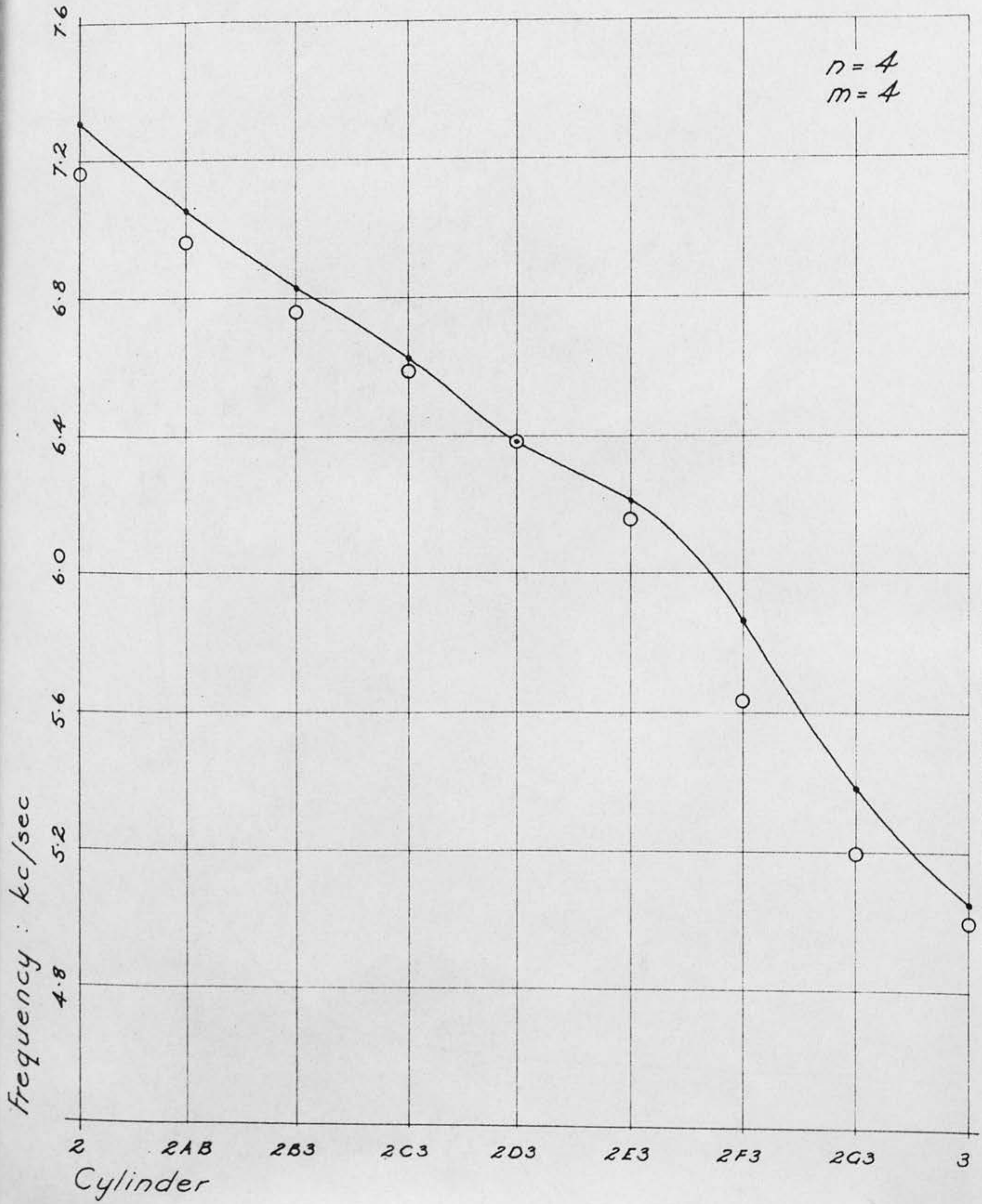
6.4
6.0
5.6
5.2
4.8
4.4
4.0
Frequency kc/sec.
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 17



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m = 4
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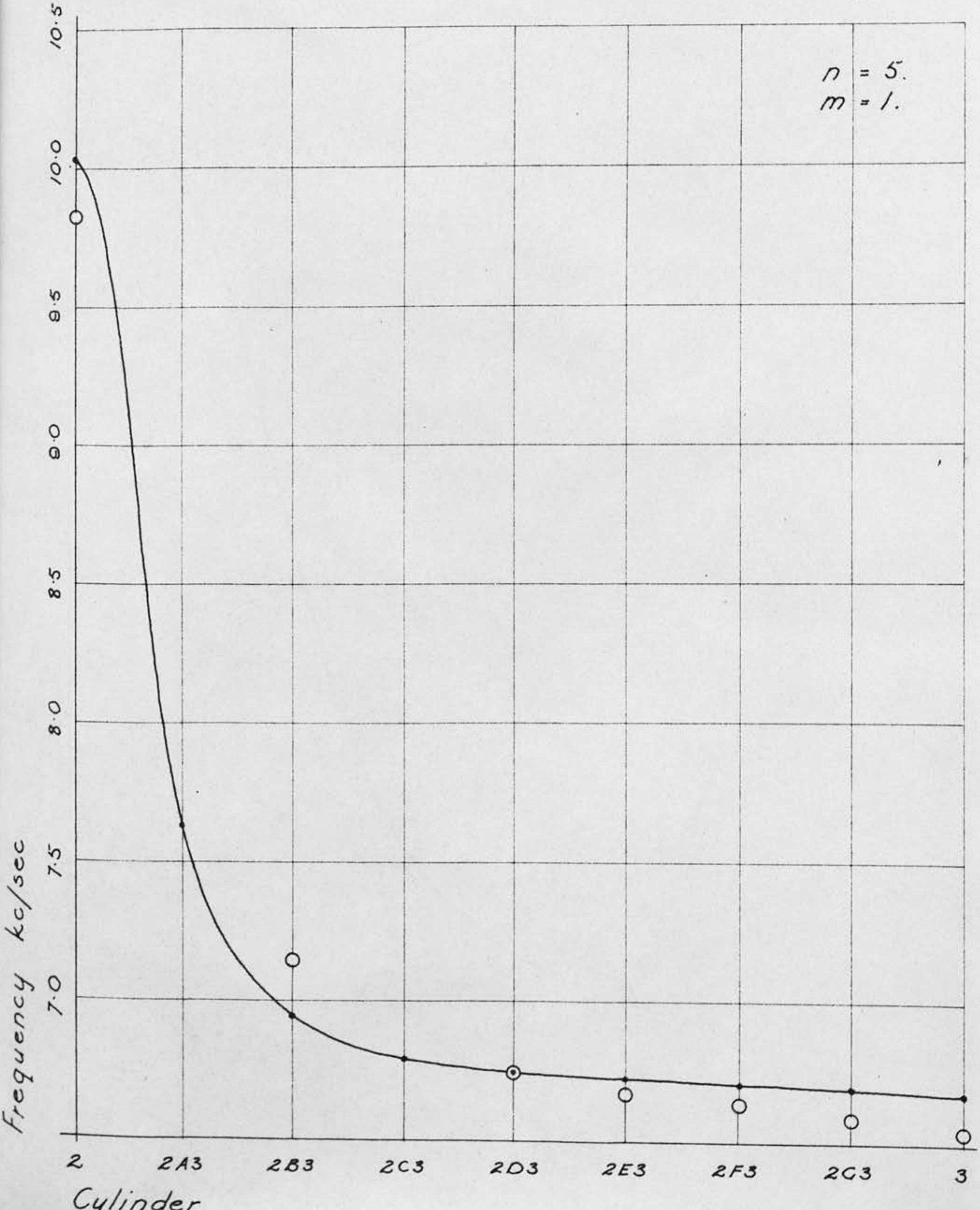
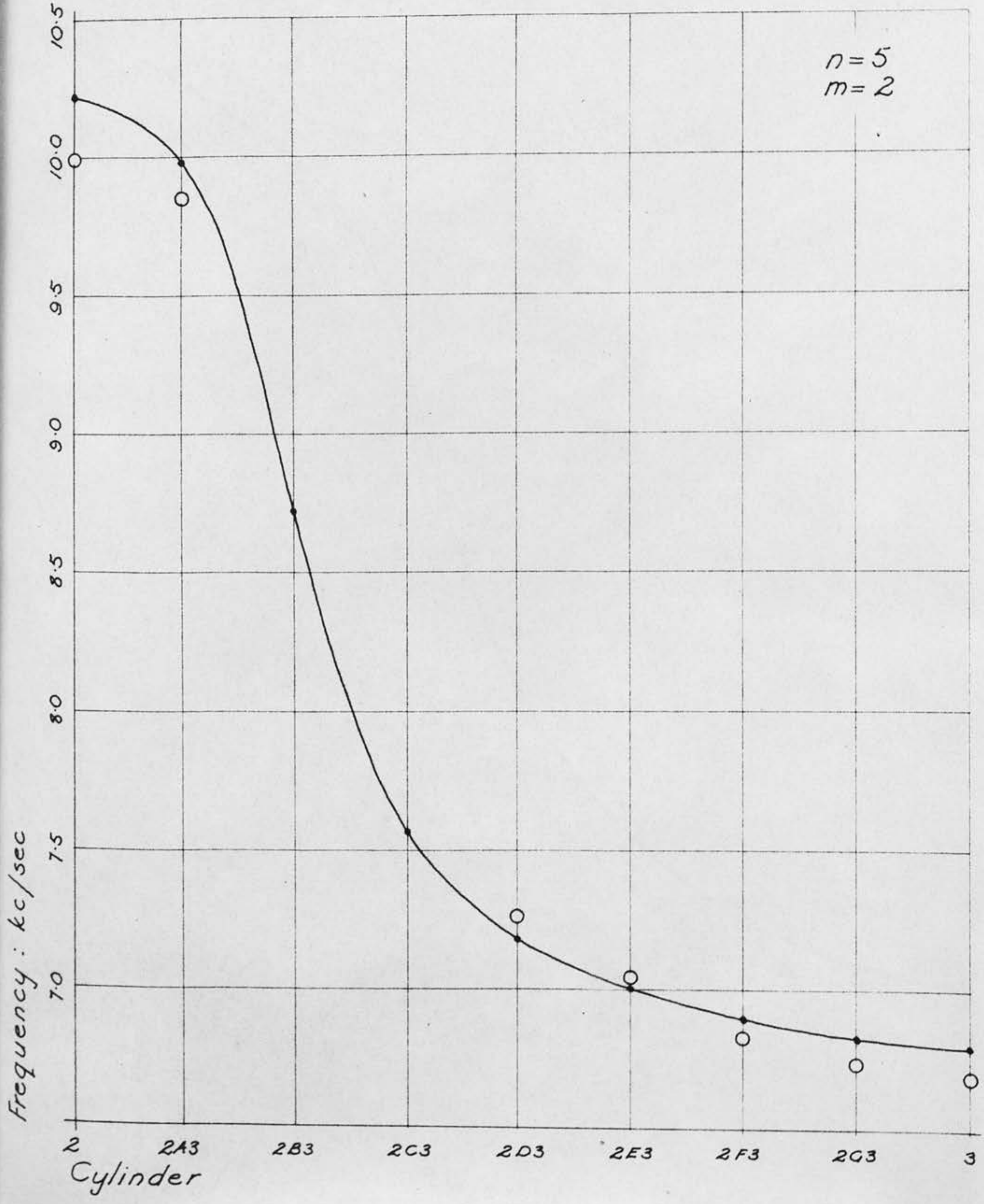
7.2
6.8
6.4
6.0
5.6
5.2
4.8
Frequency : kc/sec
2
2AB
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 18



n = 5.
m = 1.
Frequency kc/sec
10.5
10.0
9.5
9.0
8.5
8.0
7.5
7.0
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 19



n = 5
m = 2
10.5
10.0
9.5
9.0
8.5
8.0
7.5
7.0
Frequency : kc/sec
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

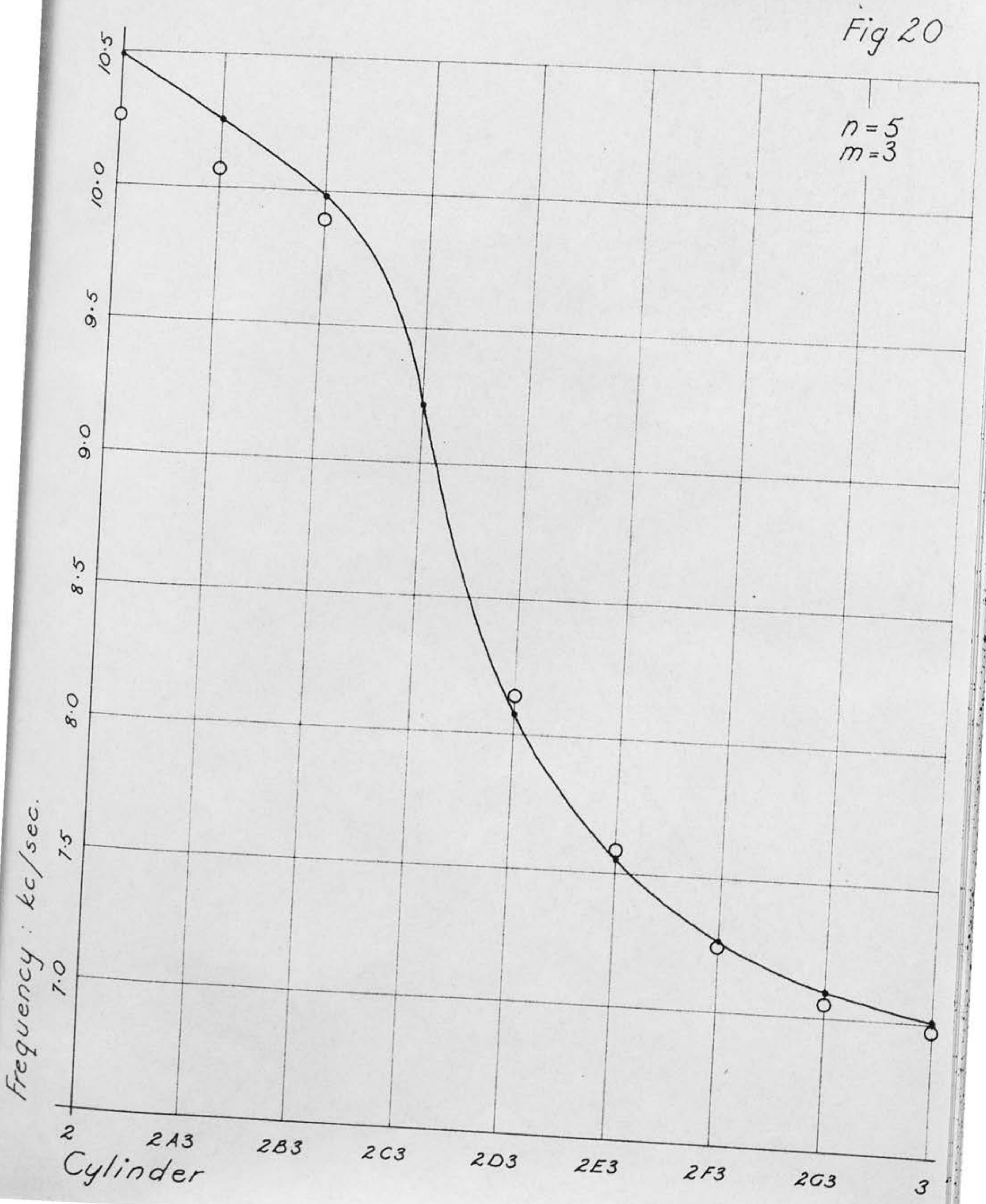
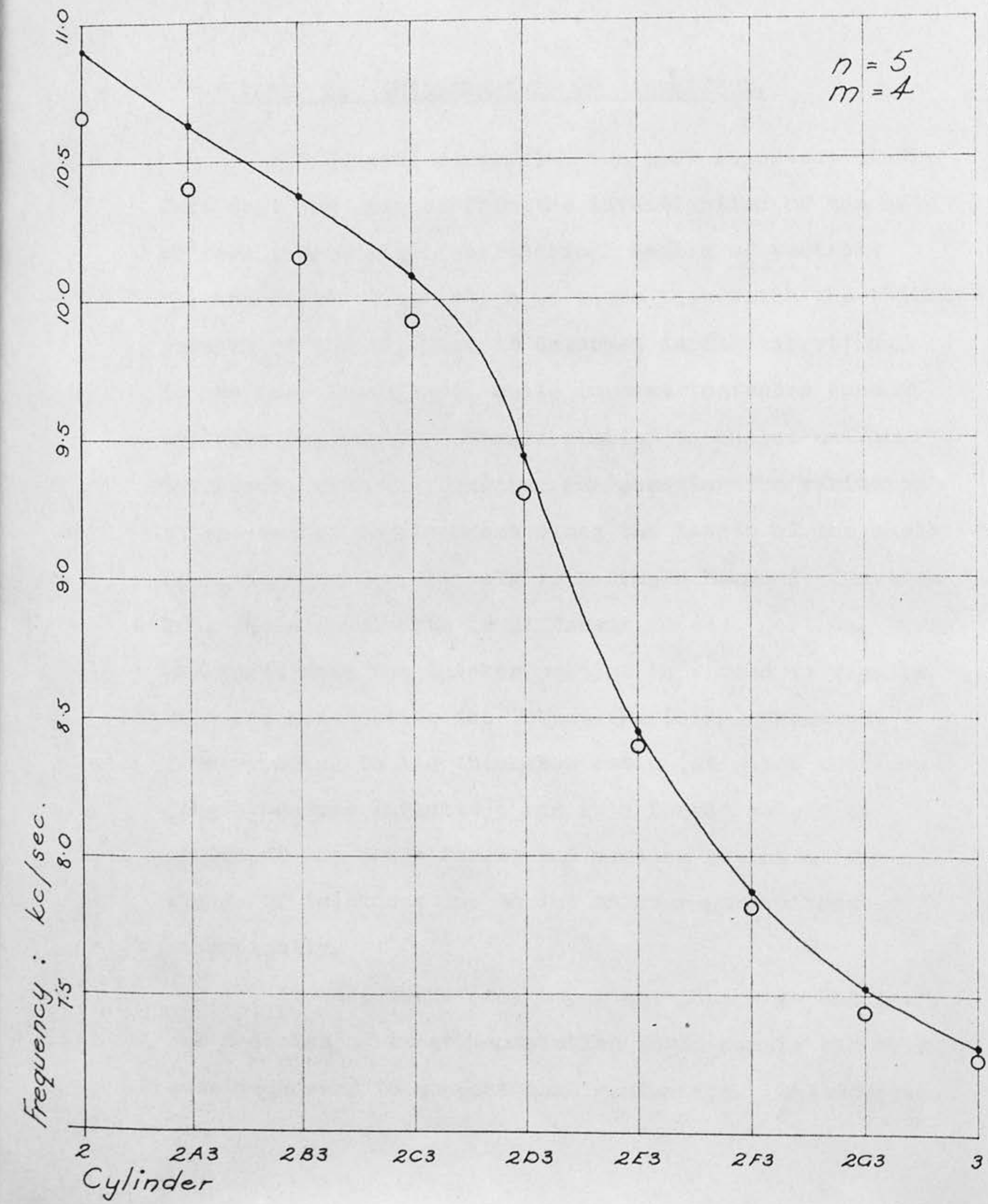


Fig 20
n = 5
m = 3
10·5
10·0
9·5
9·0
8·5
8·0
7·5
7·0
Frequency : kc/sec.
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

Fig 21.



n = 5
m = 4
Frequency : kc/sec.
11.0
10.5
10.0
9.5
9.0
8.5
8.0
7.5
2
2A3
2B3
2C3
2D3
2E3
2F3
2G3
3
Cylinder

PART B. MEASUREMENT OF AMPLITUDE.

(i) General remarks:- The most important single fact that has emerged from the investigation of the natural or free vibrations of cylindrical shells of variable thickness is the variation in shape into which the thicker portion of the cylinder is deformed during vibration. It has been shewn that, while in most instances such a cylinder behaves in a manner similar to one of uniform thickness, with the function representing the variation of the radial displacement along the length of the shell approximating very closely to a simple harmonic function of x whose wavelength is different in each portion, there are cases when the thicker portion is forced to vibrate at a frequency below the lowest (Rayleigh) frequency corresponding to its thickness ratio (at which the wave-length becomes infinite), and is deformed solely in virtue of the small forces and moments acting at the change of thickness due to the thinner part vibrating sinusoidally.

It was, therefore, important to design and carry out a series of experiments which would supply the evidence necessary to support this contention. The measurement of the amplitude of vibration presented certain difficulties which had to be surmounted - or circumvented - by choosing suitable experimental equipment and technique. Of these two have to be pointed out as the most important; (a) the small order of magnitude of the displacement to be measured, and (b) the difficulty of maintaining absolutely constant conditions of vibration (e.g. the input frequency) for any but a very short period of time.

(ii) Experimental apparatus:- The method employed in measuring displacements involved a capacitance pick-up in conjunction with a Southern Instruments Ltd. pre-amplifier and gauge oscillator unit, the output being fed simultaneously into Cossor D.C. and A.C. cathode ray oscillographs.

A sectional drawing of the pick-up head is shown in fig. 5, and a photograph of the complete set-up is included (plate II).

An amplitude distribution function such as had to be dealt with in the present case, - namely, involving point nodes along the length of the shell - would naturally have been best investigated with a point probe type pick-up; due, however, to the very small order of magnitude of the displacements a surface pick-up

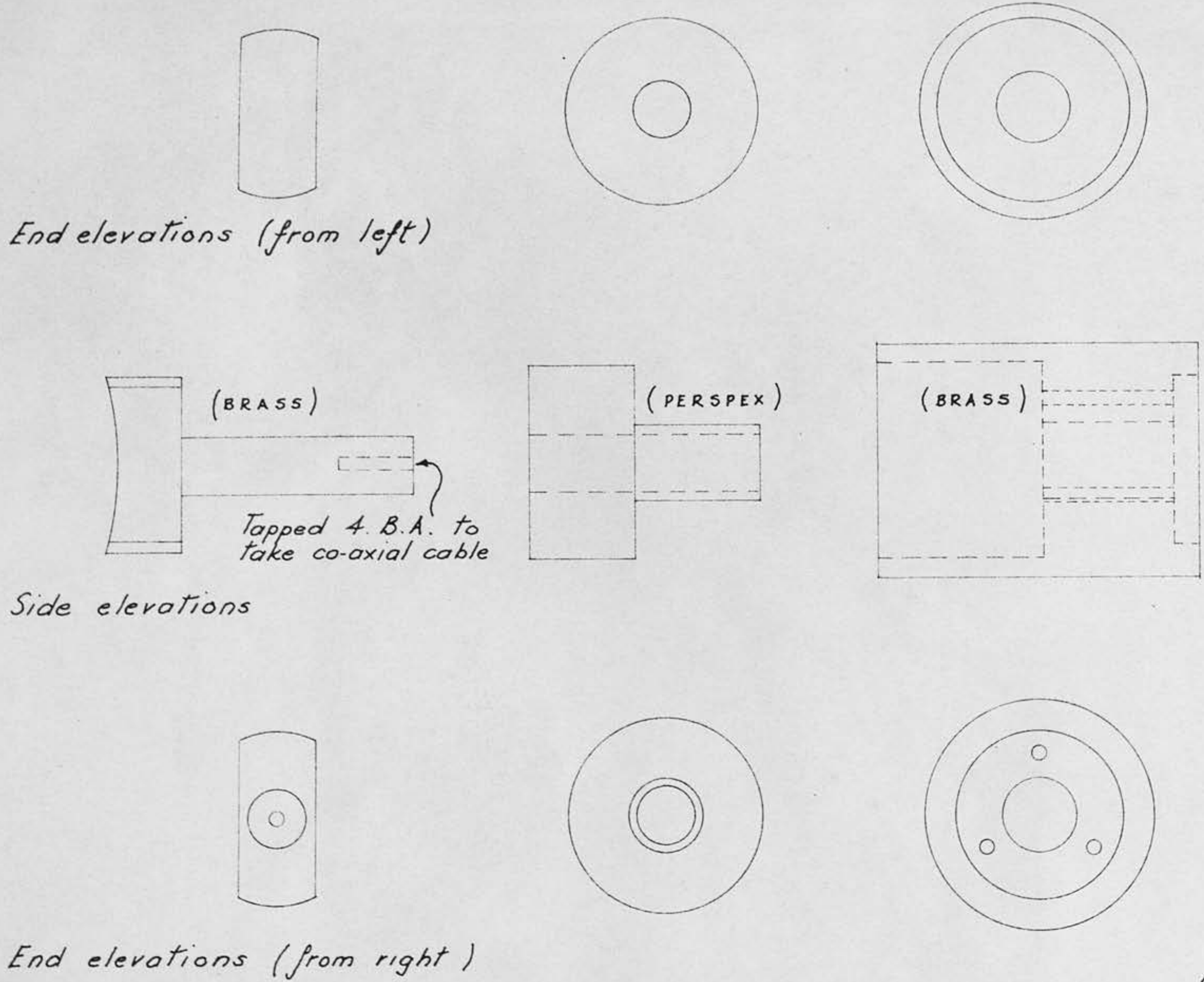


Fig. 5
(scale: full size)

PLATE II.

B

A C

was necessary to provide the requisite degree of sensitivity. The area and dimensions of the pick-up surface were designed to effect a most efficient compromise between these two consideration. The pick-up face was curved concavely with a radius of curvature equal to the mean radius of the experimental cylinders, so that at all times, for all thicknesses considered, two approximately parallel capacitance surfaces were obtained.

The oscillator and amplifier system employed are standard items of equipment developed by Southern Instruments Ltd. to measure rapidly varying pressure, force and vibration quantities using either variable capacity or variable inductance gauges. It consists of a radio frequency oscillator contained in a box (A)* located close to the pick-up head (B), and an amplifier (C) placed near the oscillograph recorders and the A.C. power supply; in this case two oscillographs were used: (a) the D.C. instrument to adjust the system initially, set the zero point and control the satisfactory operation of the system throughout, and (b) the A.C. instrument, which has a larger amplification, to indicate and measure the displacements recorded by the pick-up.

The pick-up and the oscillator were connected

---

* See plate II.

by a comparatively short coaxial cable, anf the oscillator and the amplifier were connected by a longer cable carrying H.T. and L.T. supplies to the former and the return H.F. signal to the latter. The amplifier was supplied from a valve stabilised power supply panel operated from A.C. mains and located in the amplifier box (C).

(iii) Experimental procedure:- The cylinder was set into vibratory motion as described in Part A of this chapter. After eliciting resonance in the mode to be investigated, the pick-up was set at an antinode, and the gap, exciting force and amplification of the A.C. oscillograph adjusted for best operation. It is desirable from the point of view of linearity of pick-up response and of elimination of any possible errors which might arise due to a small variation in the magnitude of the gap as the pick-up is moved along the cylinder, to make that gap as large as is consistent with the degree of sensitivity required.

Not all natural modes of vibration could be elicited with equal loudness, and the magnitude of the displacements which occurred in each varied within wide limits. In fact, only a proportion of modes which could be satisfactorily identified involved displacements large enough to be recorded by means of the equipment available. In many of those, to make investigation possible, the input power to the exciting magnet had to be considerably increased (higher power amplifier output - see part A), or the amplification of the cathode ray oscillograph made greater, or both these applied at once. The decrease in the size of the pick-up gap also, of course, made for higher sensitivity.

It must be borne in mind that excessive power input to the exciting magnet, if it did not actually affect the response of the magnet itself, could lead to the shell responding in more than one mode, and even to a certain amount of non-linear vibration. Too great an amplification in the oscillograph involved the presence of some considerable "hum" which rendered accurate measurement almost impossible. These considerations, together with the limitations imposed upon the size of pick-up gap - i.e., in fact, the intrinsic sensitivity of the apparatus, - all played a part in the initial setting-up.

As can be seen in plate II, the pick-up head was constructed to move along the cylinder on a key-way in a circular section bar running parallel to the axis of the shell; it could be moved in the radial direction - to vary the gap - by means of a fine thread adjusting screw bearing on the bar of spring steel which supported the head (a coarse adjustment for the gap was available by sliding the complete head in its holder).

When the initial setting-up was complete, the trace on the D.C. oscillograph screen was brought to the centre of the screen by operating the tuning adjustment of the discriminator circuit in the pre-amplifier unit. This position now fixed the gap size.

The procedure now was to take readings at regular, say $\frac{1}{2}$ inch or less, intervals along the shell by moving the pick-up along the shell. The intervals were sometimes reduced, as, for instance, in the vicinity of nodal or antinodal points, in order to locate these with greater accuracy.

Once the initial setting-up had been carried out as described, the gauge oscillator, pre-amplifier and cathode ray oscillograph controls remained unchanged. At each position of the pick-up, a small adjustment of the gap was usually required to bring the trace on the screen to its original position, thus ensuring that the gap was identical for each reading.

As there was always a slight risk of some "drift" in the audio-frequency oscillator, at each position of the pick-up it was ascertained that the input frequency did in fact produce maximum amplitude of vibration in the cylinder, i.e. that there was resonance; if necessary, the input frequency was slightly adjusted.

Rather than measure the height of the trace on the screen, it was found much more convenient and indeed more accurate to measure the voltage between the plates of the C.R. oscillograph on an AVO-meter or other external voltmeter. For purposes of illustration, however, photographs were taken of the C.R.O. record with the Cossor attachment camera. The technique was to switch off the time-base, move the beam the same small distance horizontally along the screen for each reading along the cylinder, and take a series of exposures on the same frame of the 35 mm. film. Three representative photographs are shown in figs. 34, 35 and 36; they were obtained for the following cases:- fig. 34 for mode n=2, m=1, of cylinder 3 (uniform); fig. 35 for mode n=3, m=1, of cylinder 1D3; fig. 36 for mode n=2, m=1, of cylinder 2E3.

Voltmeter readings are plotted in the form of displacement curves in figures 22 - 33. No attempt was made to measure absolute magnitudes of the displacements, and the vertical scales are therefore arbitrary, chosen to give in all cases a clear picture of (1) the characteristic features of the curves, and (2) the wavelength and nodal points.

By no means all the modes whose frequencies had been elicited could be investigated from the point of view of their amplitude distribution. In many, particularly the more complex, of them the displacement was too small to enable a significant record to be obtained. On the other hand, no useful purpose would be served by including here all the records which the author was able to obtain in the course of this part of his investigation. Figures 22 to 33 serve to illustrate in the fullest possible way the manner in which thin cylindrical shells of variable thickness deform during vibration; they are arranged as follows:-

Fig. 22 shows the mode n = 2, m = 1, for cylinder 2B3 ($L_2 = \frac{3}{4}L$, $L_3 = \frac{1}{4}L$); here the deflection form is composed of two sine-waves, whose wavelengths are measured and compared with the theoretical value. Fig. 23, for the same cylinder, shows the form of vibration for m = 1, and n = 3, 4 or 5; this form is the same for all three modes, and illustrates the case of the complete cylinder vibrating at a frequency below f_R of its thicker portion.

Figures 24 and 25 respectively refer in a similar fashion to cylinder 2D3 ($L_2 = L_3 = \frac{1}{2}L$), and fig. 26 and 27 to cylinder 2F3 ($L_2 = \frac{1}{4}L$, $L_3 = \frac{3}{4}L$).

Figures 28 and 29 illustrate some higher modes of vibration with sinusoidal wave-form.

Finally, fig. 30, 31, 32 and 33 show modes with

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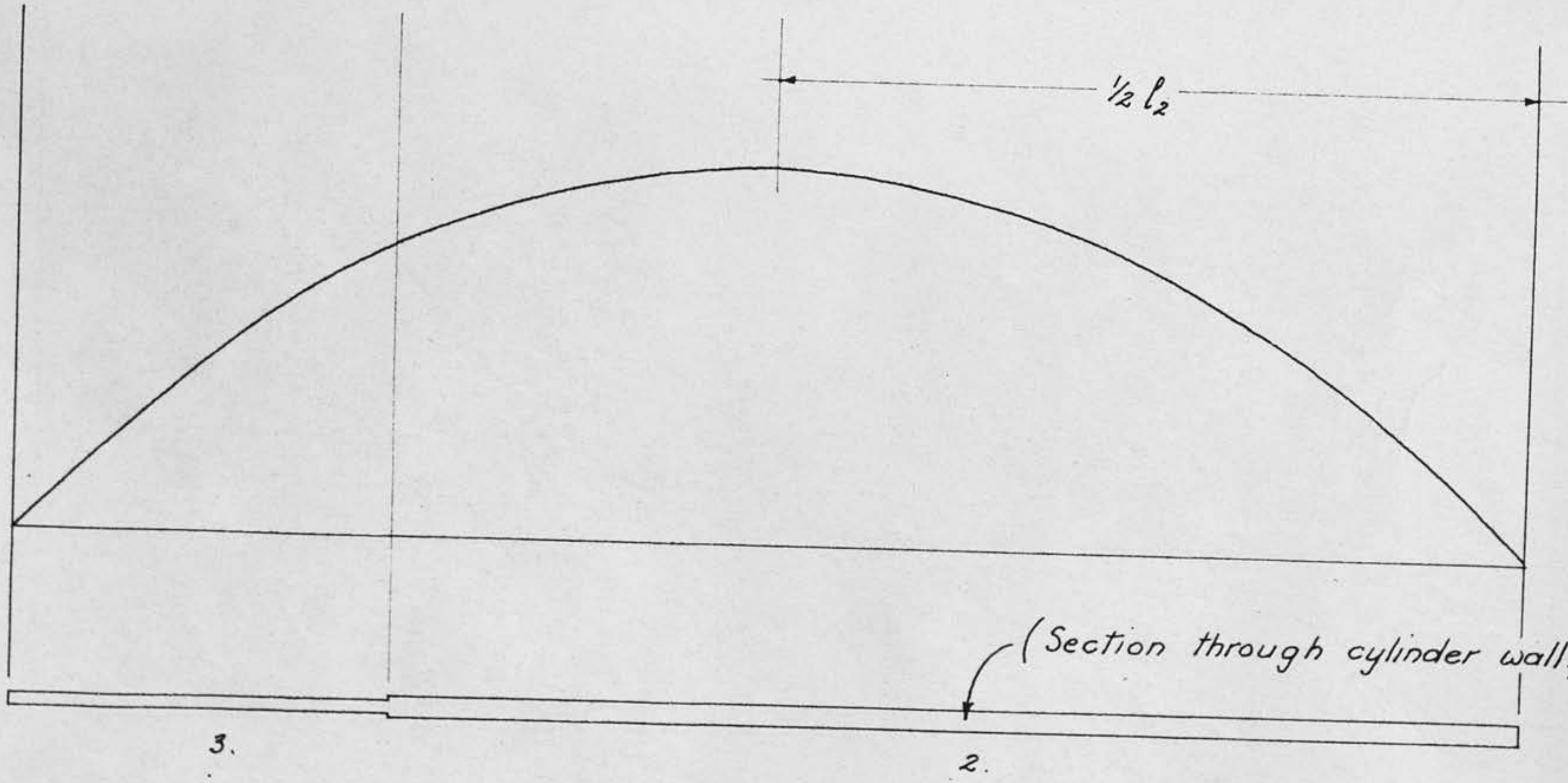
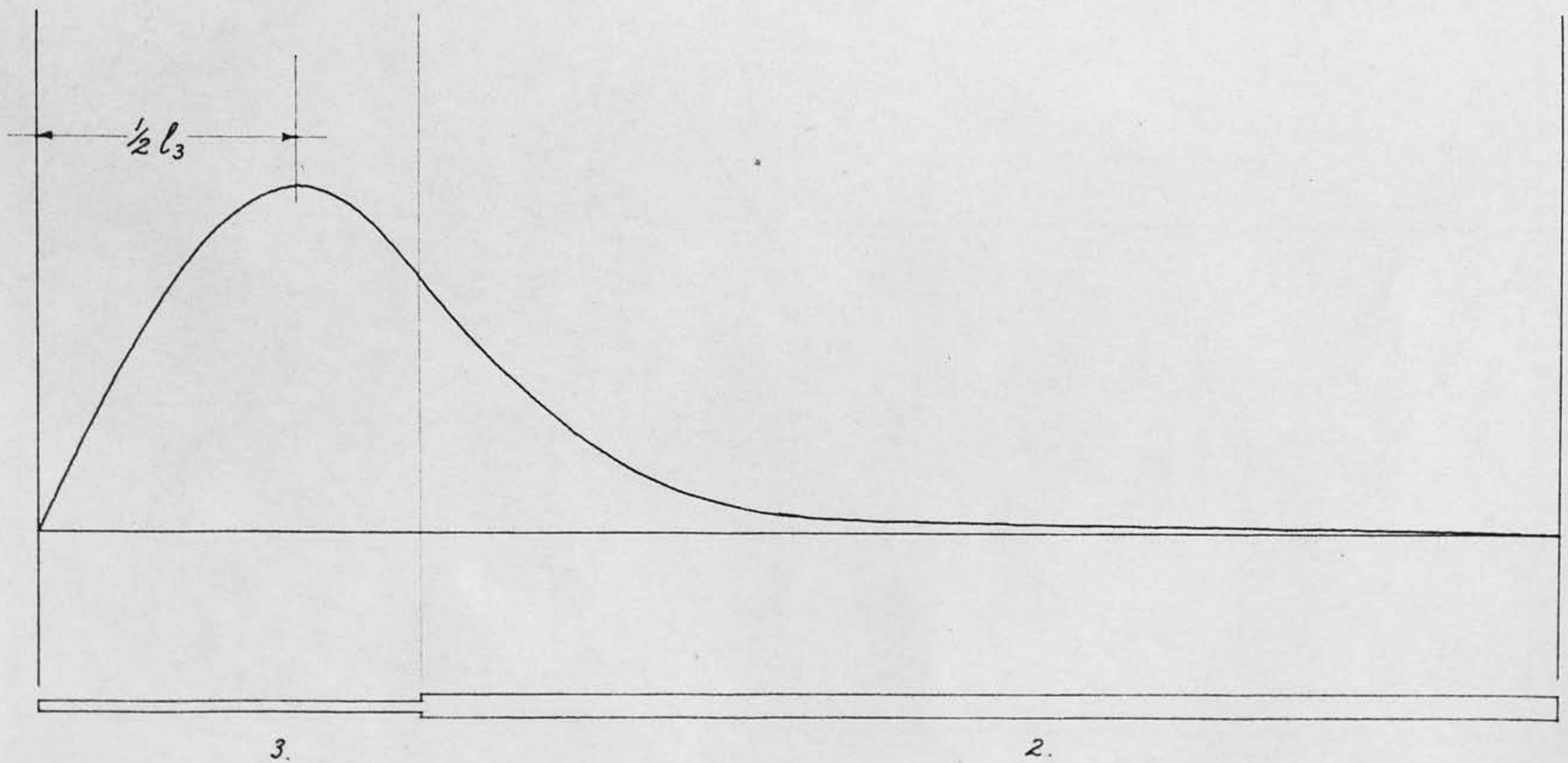
m = 1, 2, 3 and 4 respectively for cylinders 1D3 and 2D3, and various numbers of circumferential nodes.



l_2 expt. 17·60 ins
theory 17·70 ins.

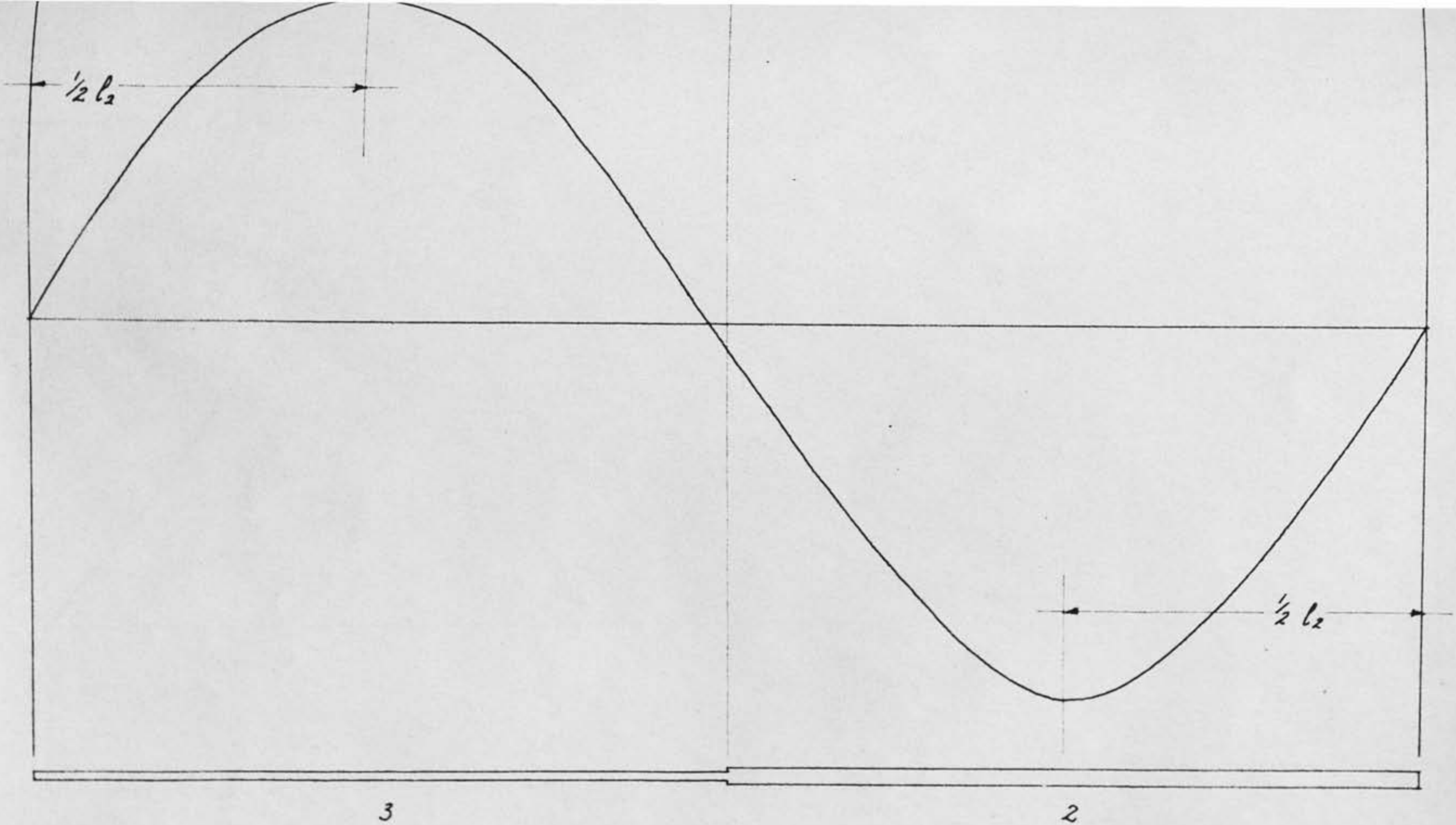
Cylinder 2B3 $n = 2$ $m = 1$.

Fig 22.



l_3 expt 6·0"
theory 6·5

Cylinder 2B3 $n = 3, 4, 5$, $m = 1$. Fig 23.

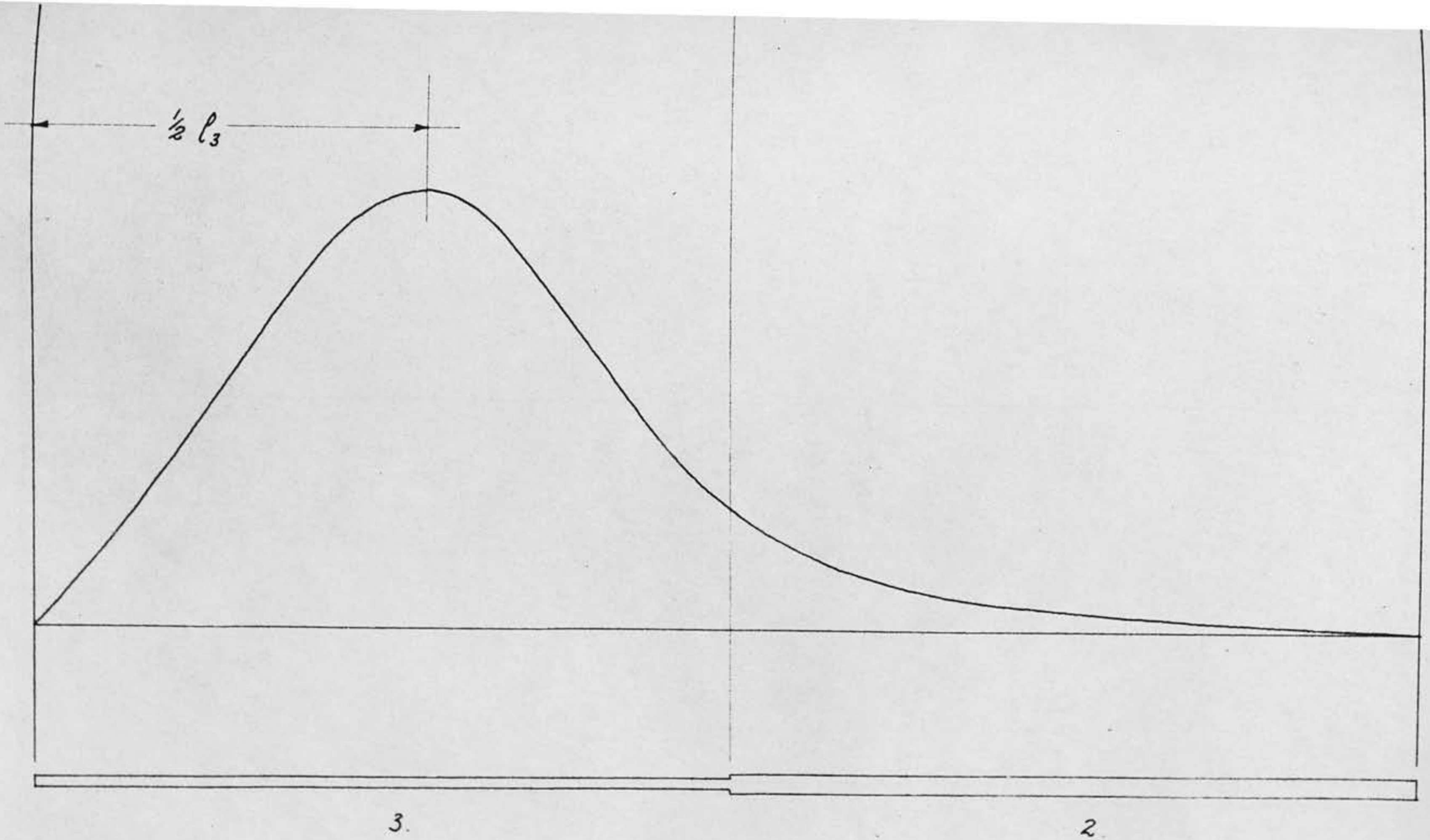


l_3 expt. 8.40"
theory. 8.30"

l_2 expt. 9.40"
theory 9.30"

Cylinder 2D3 $n=2$, $m=2$.

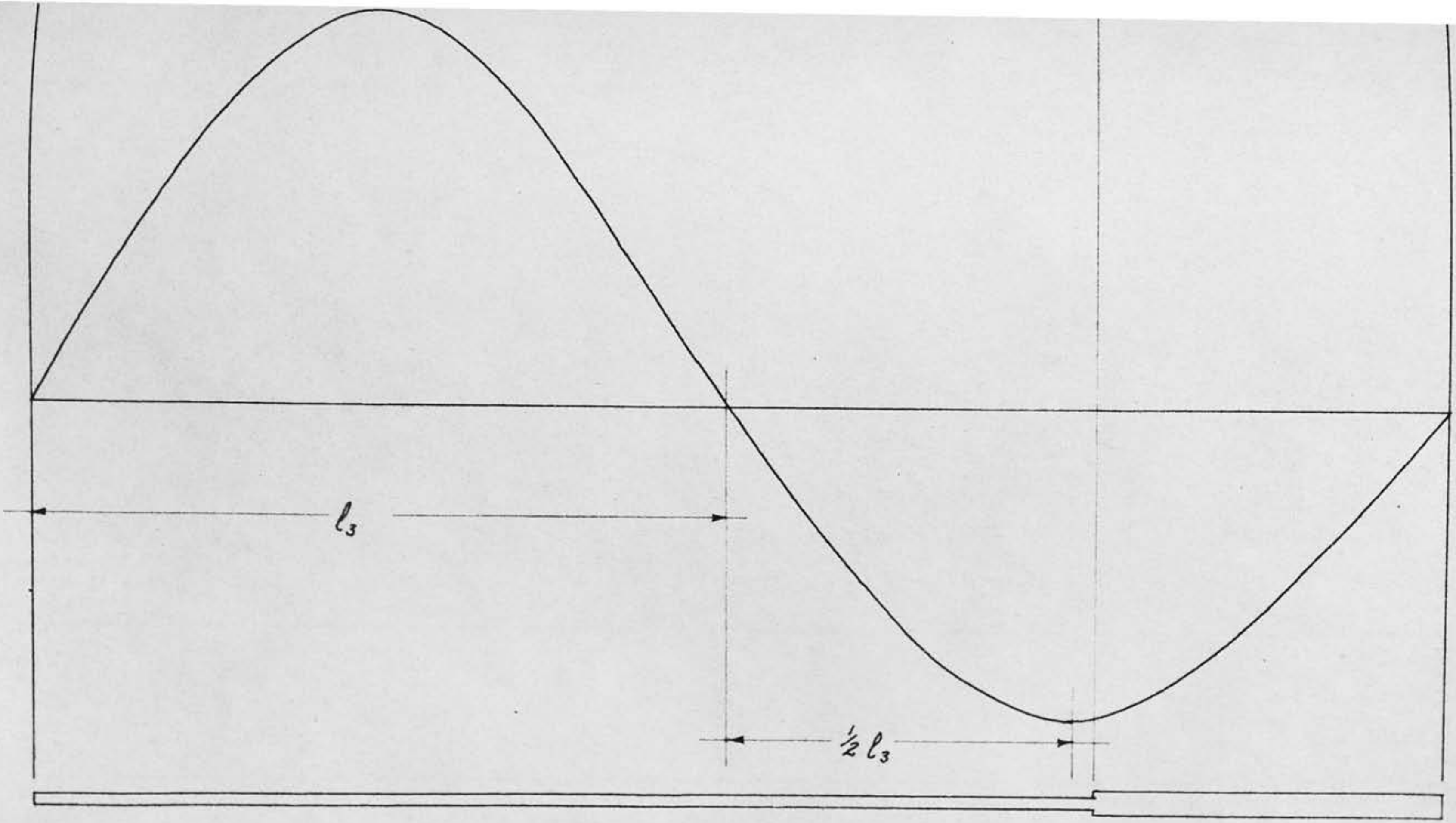
Fig 24.



l_3 expt. 10·0
theory 10·76

Cylinder 2D3 $n = 2, 3, 4, 5$ $m = 1$.

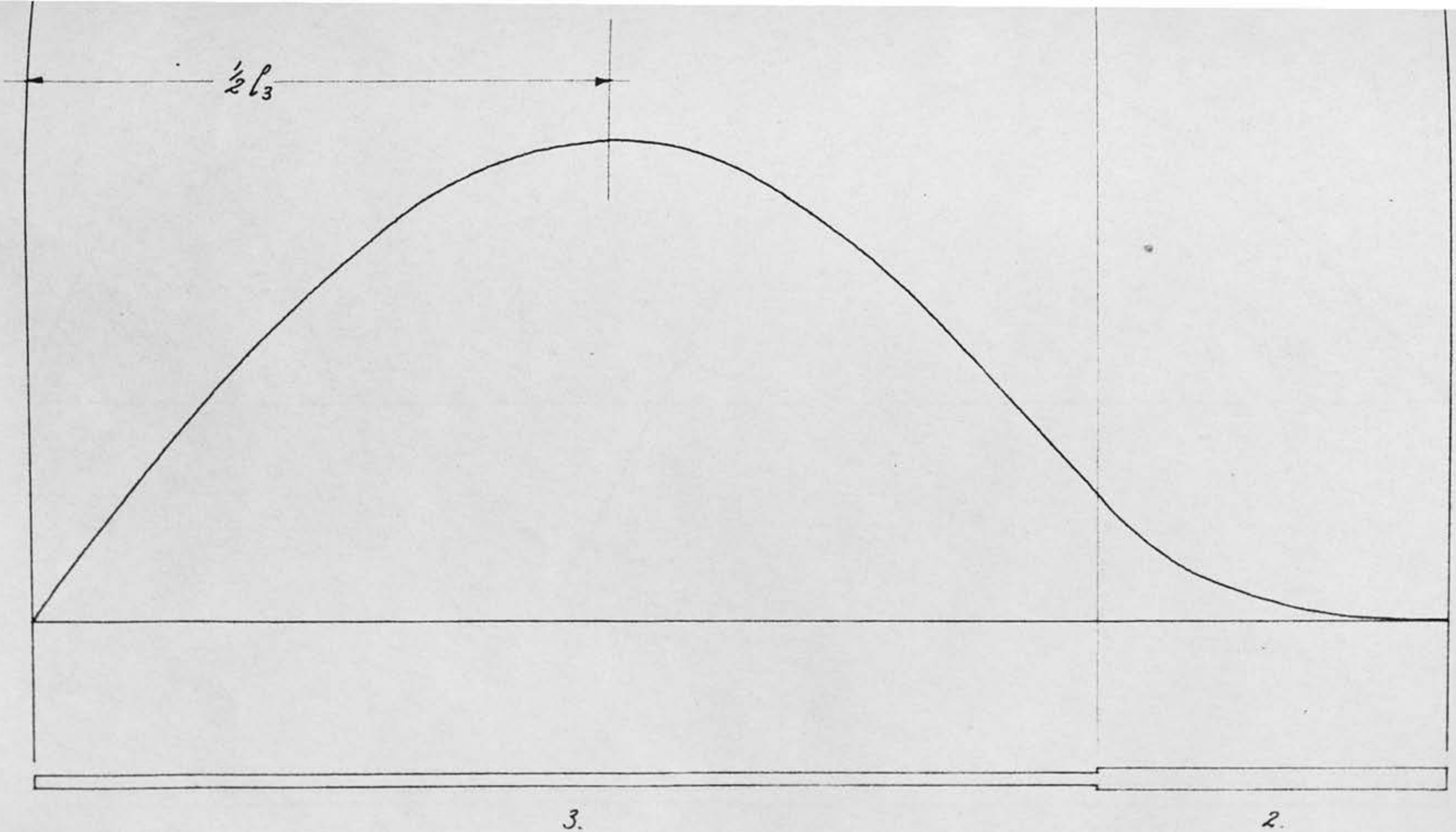
Fig 25



l_3 expt. 8·60"
theory 8·58"

Cylinder 2F3 $n = 2,\ m = 2.$

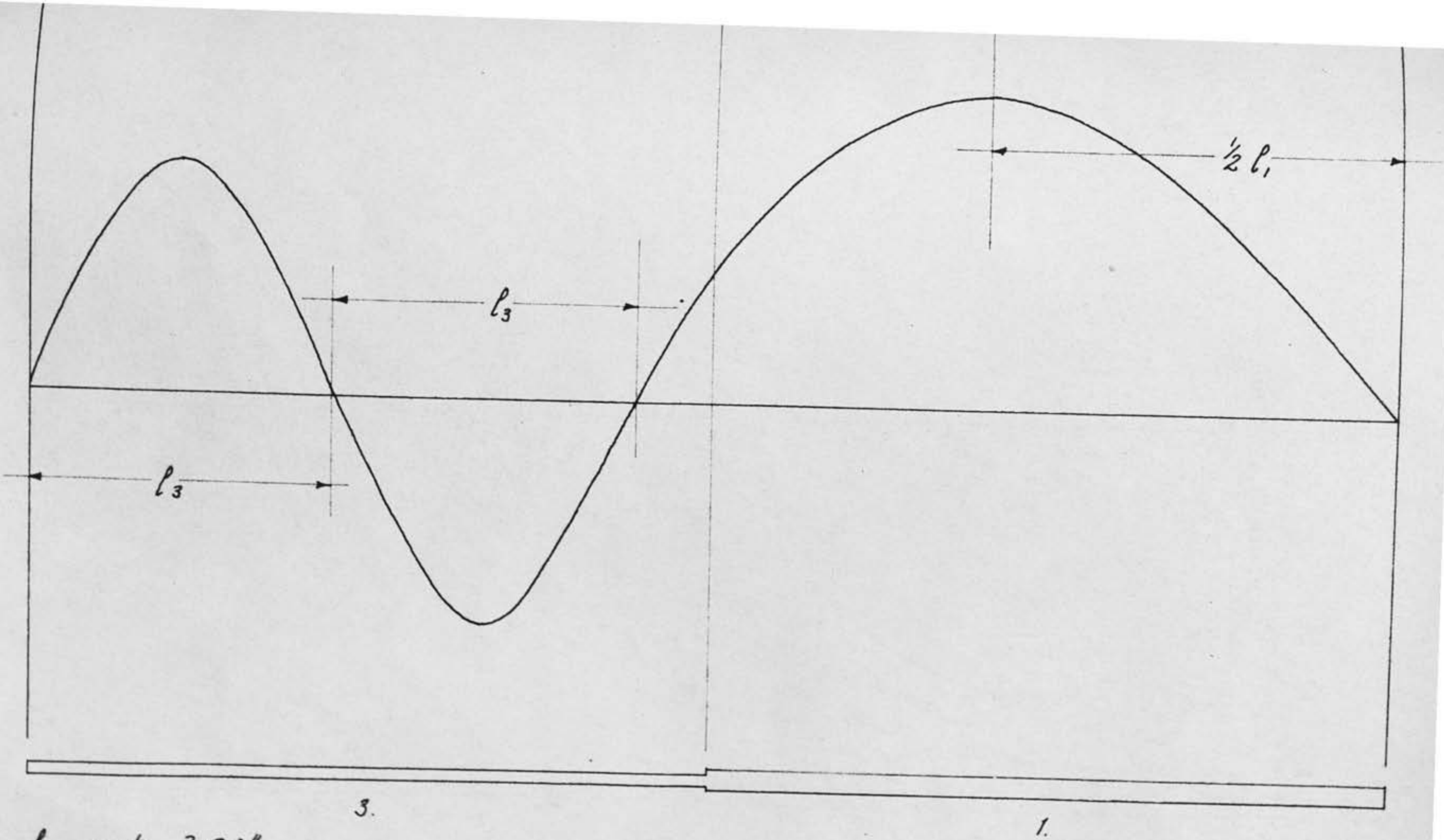
Fig 26



l_3 expt 14·4"
theory 14·27"

Cylinder 2F3 $n = 2, 3, 4, 5.$ $m = 1$

Fig 27.

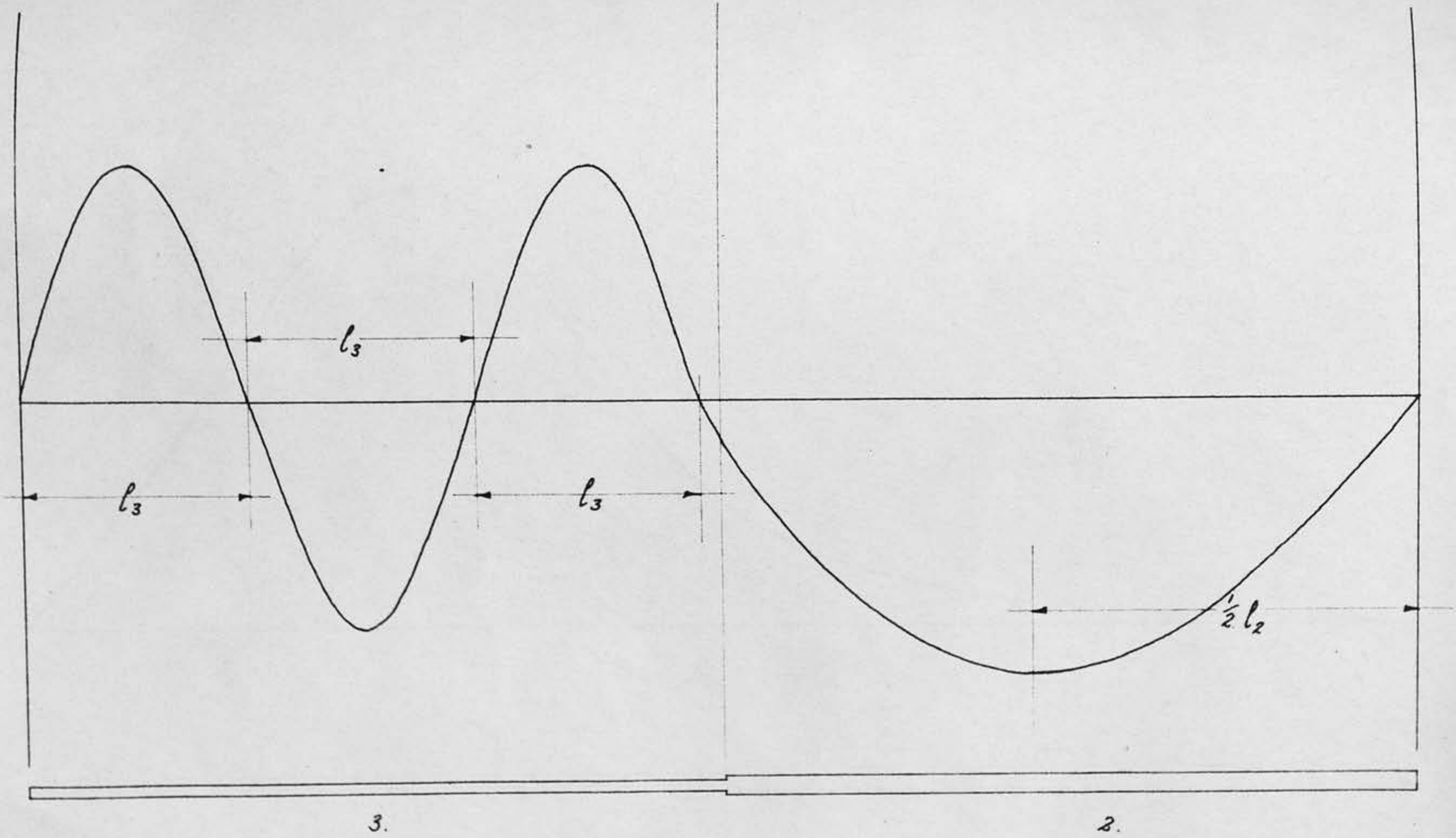


l_3 expt 3·90"
theory 3·86"

l_1 expt 10·50"
theory 10·32

Cylinder 1D3 $n = 3$ $m = 3$

Fig 28

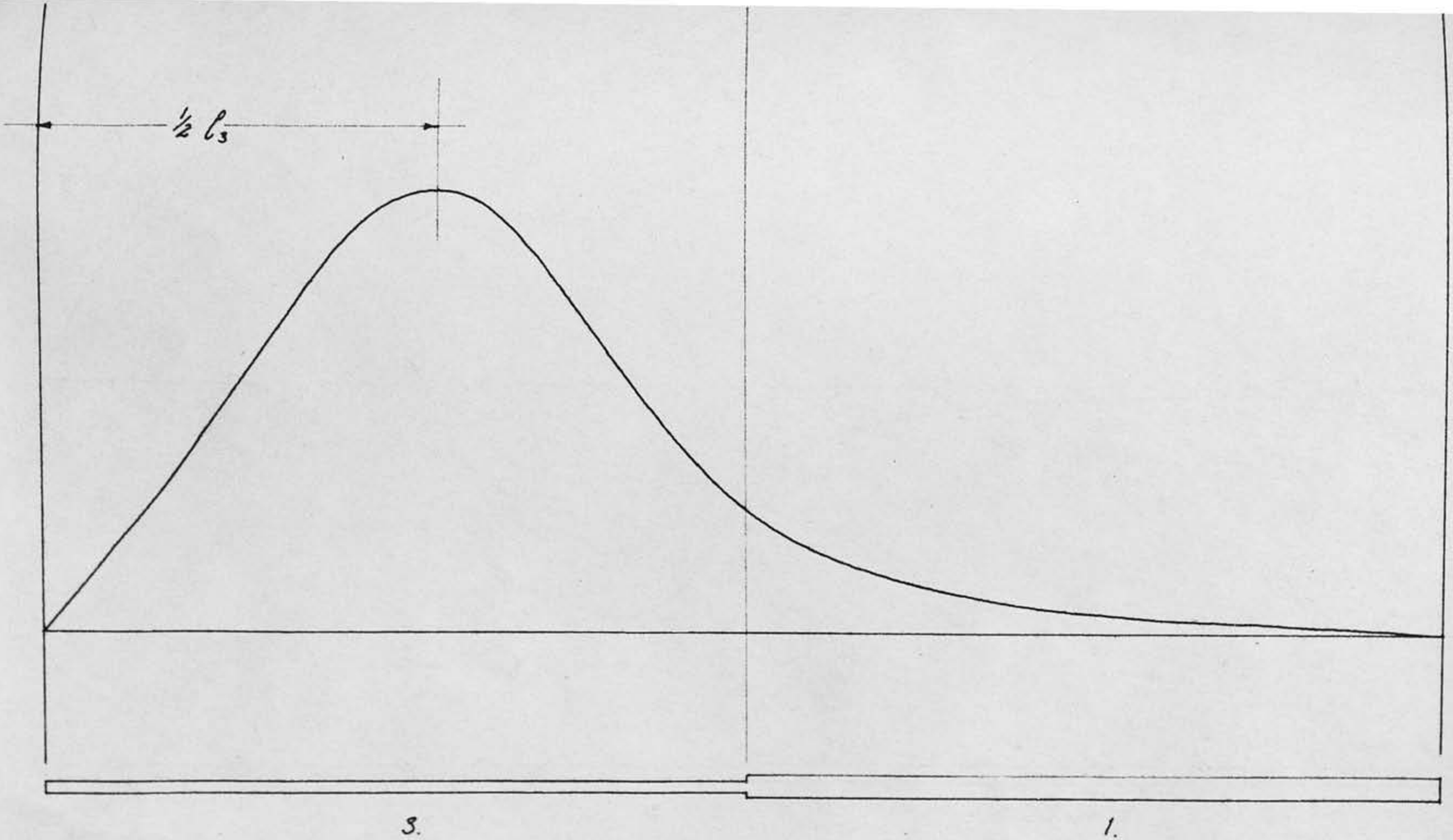


l_3 expt 2.86"
theory 2.80"

l_2 expt 9.63"
theory 9.41"

Cylinder 2D3 $n = 4$ $m = 4$

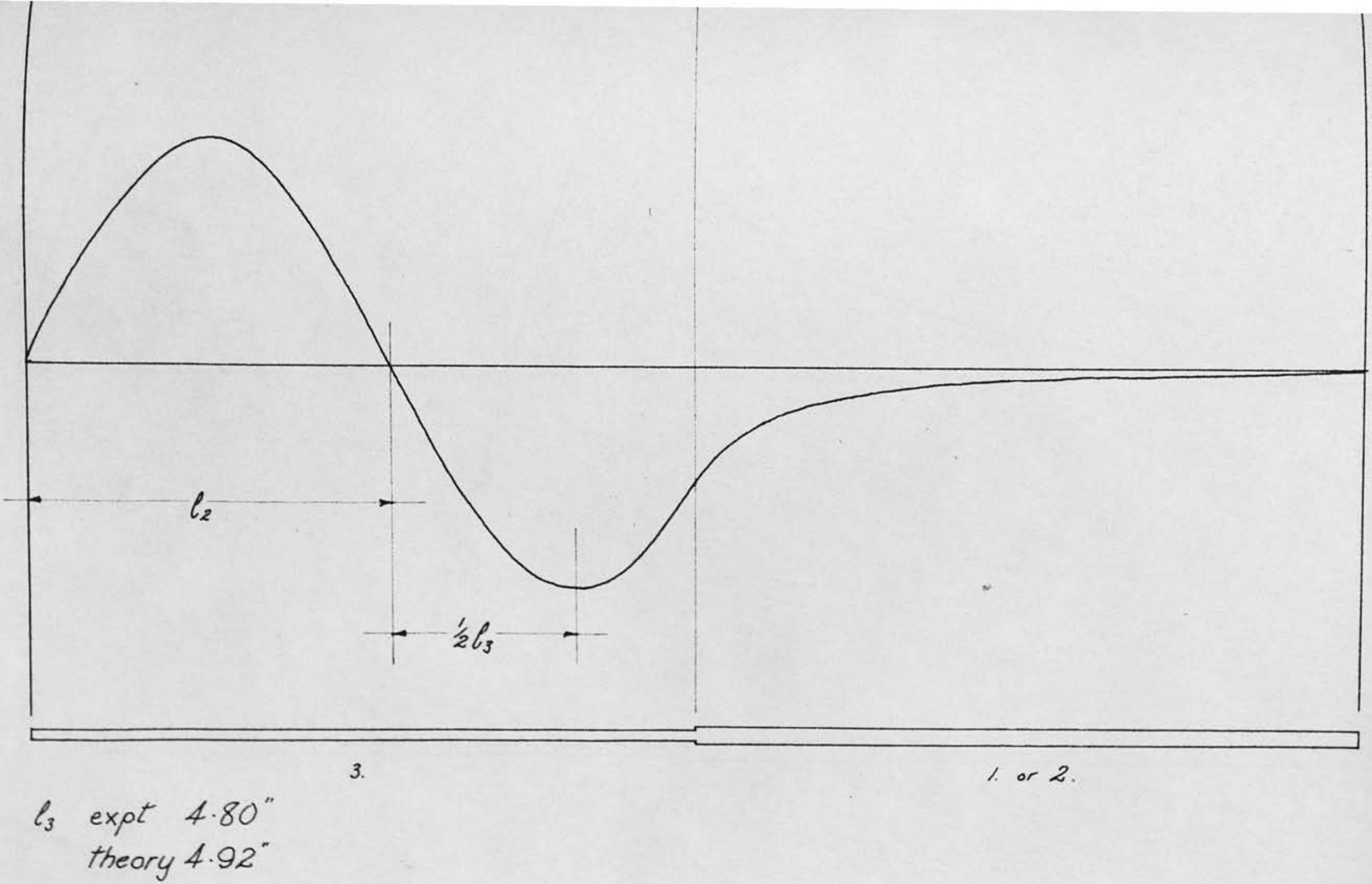
Fig 29



l_3 expt 10·0"
theory 10·76"

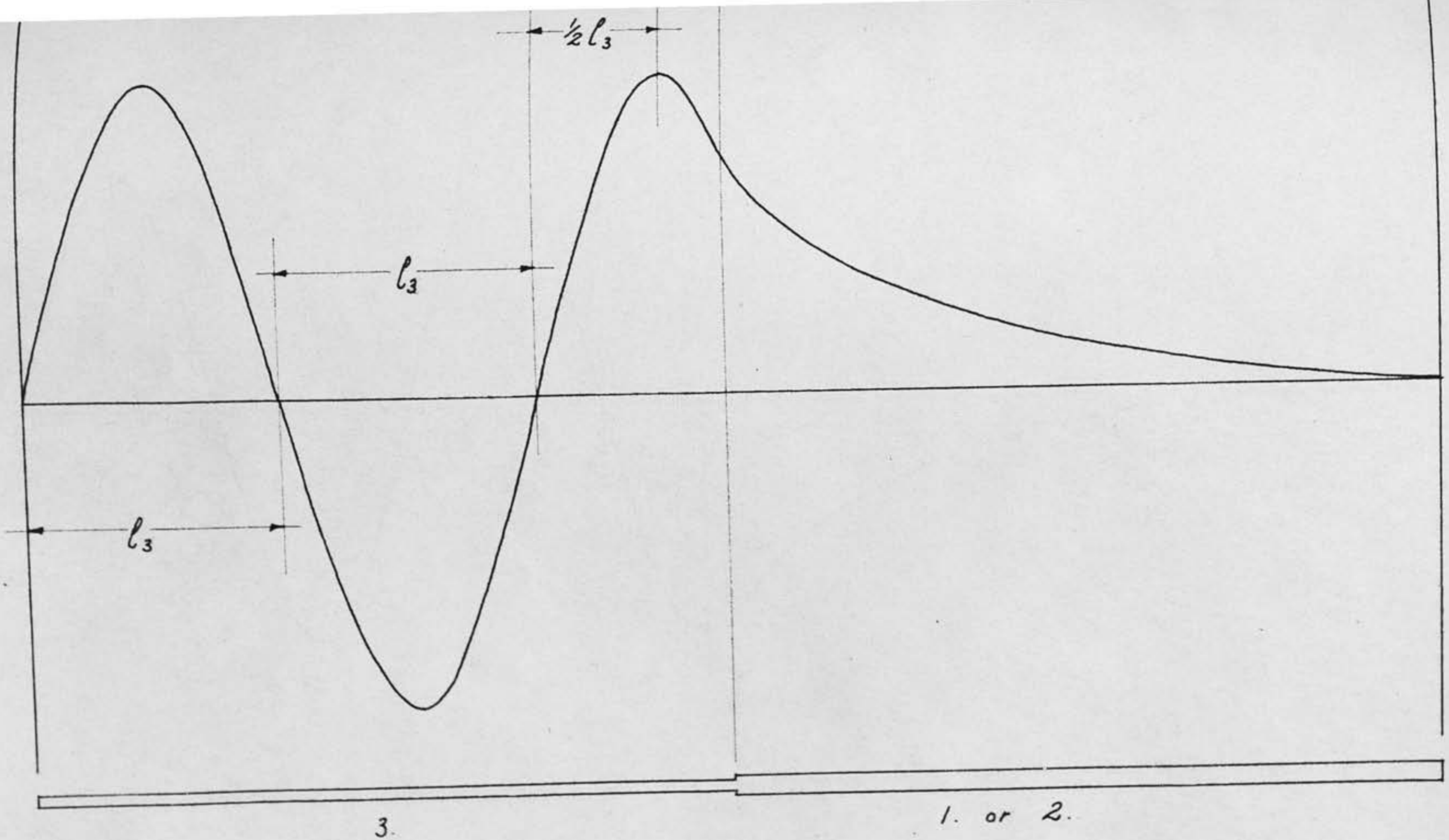
Cylinder 1D3 various n , m = 1.

Fig 30



Cylinder 1D3 or 2D3 various n, $m = 2$.

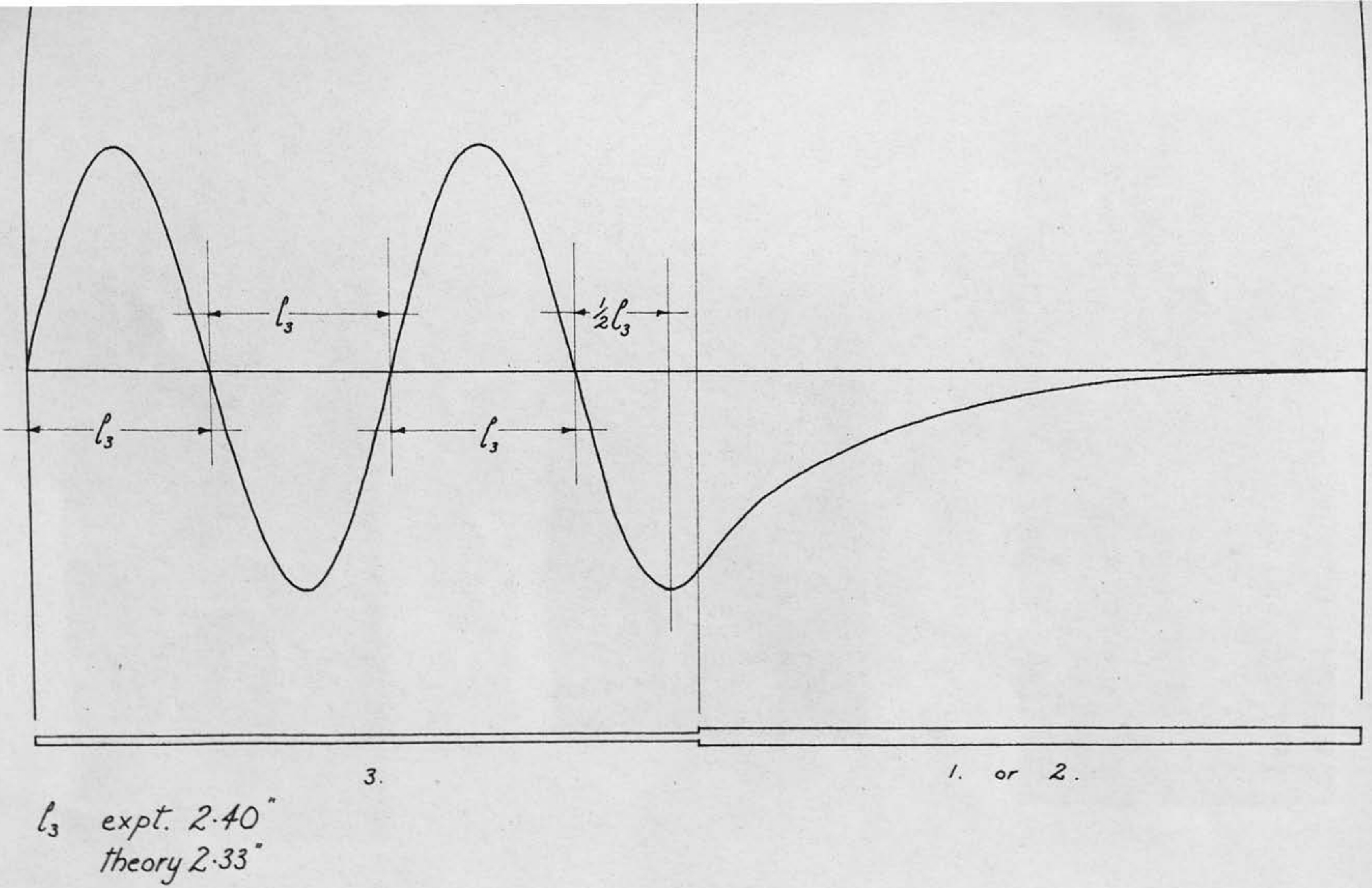
Fig 31.



l_3 expt 3·20"
theory 3·16

Cylinder 1D3 or 2D3 various n, $m = 3$

Fig 32



Cylinder 1D3 or 2D3 various n, $m = 4$

Fig 33

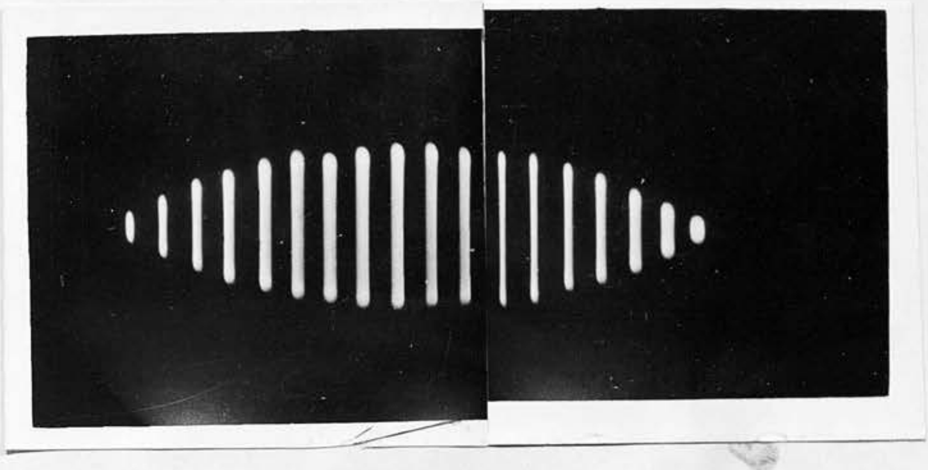


Fig. 34.

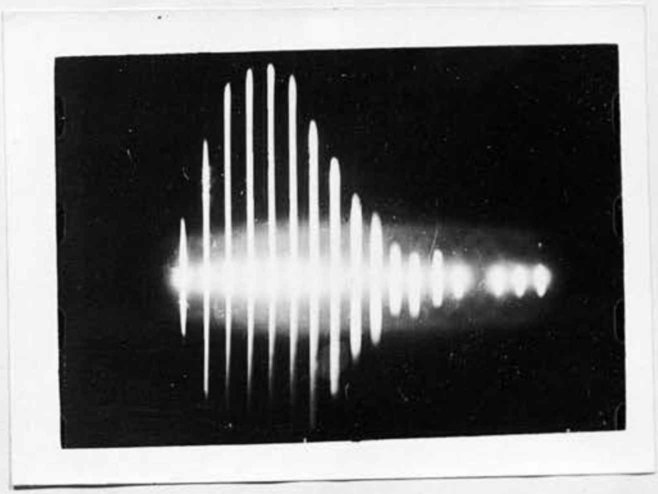


Fig. 35.

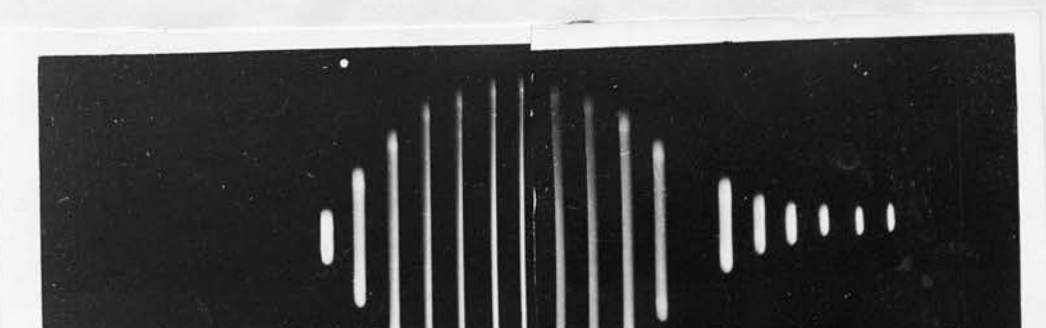


Fig. 36.

4. DISCUSSION.

At the end of parts A and B of Chapter 3 there are collected experimental values of natural frequencies and records of axial vibration forms respectively. Tables of natural frequencies were given priority of place, because the calculation of these was the object of this research; at this stage, however, it will be of benefit to the logical and orderly discussion of the results of this investigation to consider first the implications of the theory with regard to the manner of vibration of shells of variable thickness, and the extent to which the experimental findings agree with theoretical predictions.

In its simple fundamental aspect, the theory reduces, firstly, to the simplification of the complex three-dimensional problem of a shell to the analogous one-dimensional case of a beam (in the modes of vibration considered in this work radial displacements are strongly predominant (1) over those in the axial and tangential directions), and, secondly, to applying to the simplified problem Rayleigh's principle of assuming an approximate vibration form.

Thus, to begin with, a simple harmonic vibration form

$$w = W \sin \frac{\pi x}{l}$$

is assumed for each portion of the cylinder. It gives the frequency equation (5) of Chapter 2:-

$$\tan \frac{\pi L_2}{l_2} = -\frac{l_1}{l_2} \tan \frac{\pi L_1}{l_1} .$$

Although the assumed function is simple (remembering that the exact expression would be a series), very accurate values of natural frequencies are obtained; since, in solving the above equation, both l_1 and l_2 are determined, the physical and dynamic characteristics of the shell (as opposed to those of a beam) are allowed for in the fullest possible way. The manner in which the cylinder vibrates under these conditions is illustrated in figures 22, 24, 26, 28 and 29, where theoretical values of l (obtained from equation (5) of Chapter 2) are given for comparison.

When a mode of vibration cannot be calculated from this equation, due to the fact that only the value of l_2 (for the thinner portion) is available, - the frequency being below the Rayleigh frequency f_R for the thicker portion, - a more general vibration form must be assumed. As explained in Chapter 2 and Appendix II, this is achieved by taking for the thicker portion

$$w_1 = A \sin \frac{\pi x}{l_1} + C \sinh \frac{\pi x}{l_1} ,$$

where l_1 is real and unknown, and is determined together with the other unknown constants from the conditions at the point of change of thickness. This leads to the frequency equation (9) of Chapter 2:-

$$\tan \frac{\pi L_2}{l_2} = - \frac{l_1}{l_2} .$$

In this case no reference can be made to the f/λ curve for the thicker portion (portion 1), and the approximation to a beam is, therefore, even more complete. Consequently, since the vibration form in the thicker portion is now independent of the number of circumferential nodes, the sine-wave in the <u>thinner</u> portion will have the same wavelength (determined by l_2, as obtained from equation (9) of Chapter 2), irrespective of n. In fig. 23 such a vibration form is shown; only one curve is drawn for n = 3, 4 and 5, since experimental records for all these modes were congruent. Further illustrations are supplied by figures 25, 27, 30, 31, 32, and 33; the last two of these strikingly demonstrate the confinement of the axial nodal pattern to the thinner portion.

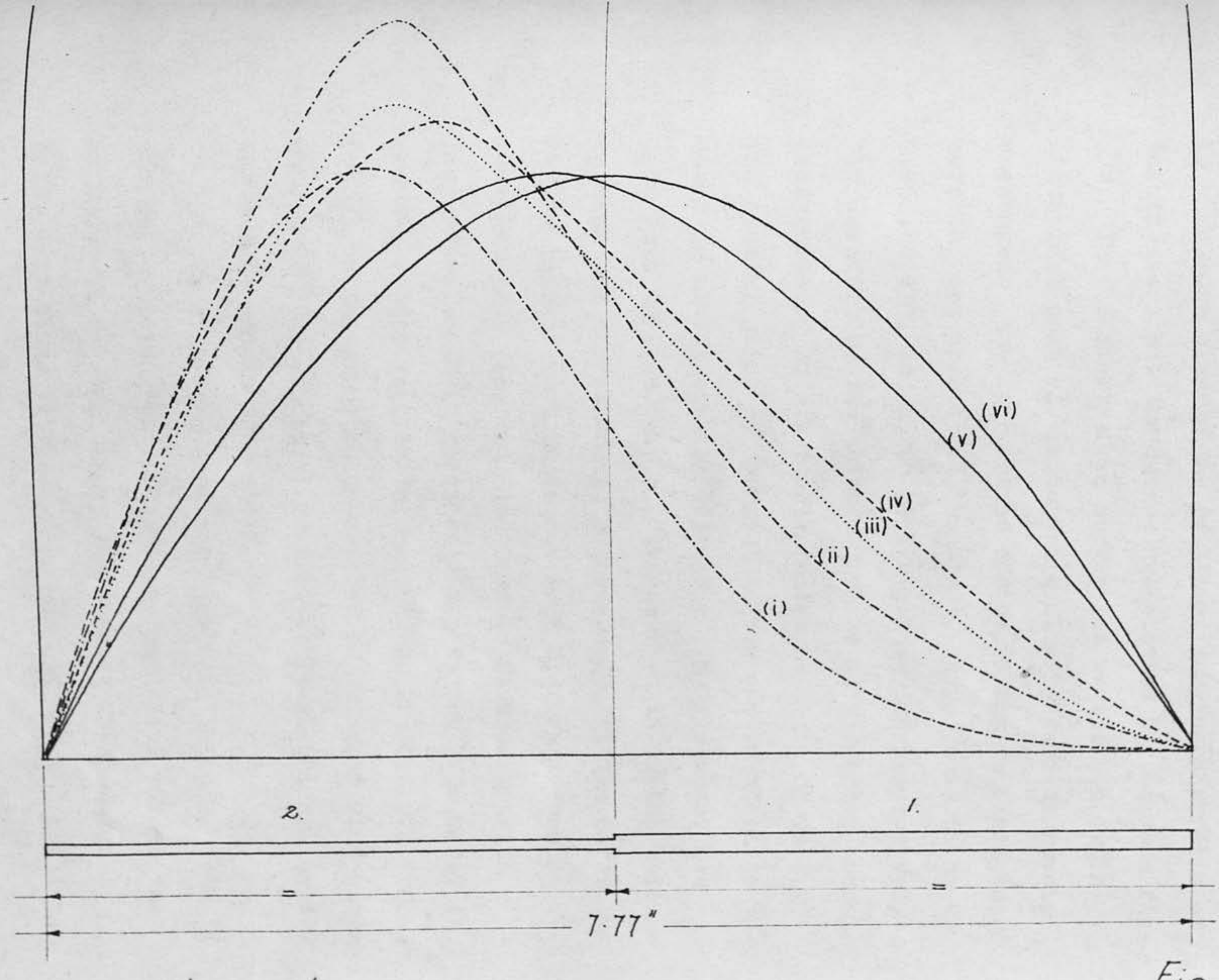
Figures 30 to 33 are not only drawn for more than one value of n; each of them represents in addition two curves - one for cylinder 1D3 and the other for cylinder 2D3 - which for the modes under consideration were found to be exactly similar. This confirms that, as stated in equation (9) of Chapter 2, the form and frequency of vibration of the shell are independent of the thickness of its thicker portion, provided, of course, the natural frequency falls below f_R for the thicker portion.

If we have a cylinder consisting of two portions of different thickness, and its vibration form for some value of n and m = 1 is, say, as shown in fig. 30, then by gradually decreasing the larger thickness while keeping the thinner portion unaltered, we should be able to observe the transition of the vibration form to that shown in fig. 22, say, where there is a sinusoidal amplitude distribution in each portion.

A series of experiments designed to show this progressive variation in the vibration form were carried out on a special cylinder of the following dimensions:-

$L = 7.770$ in.,	$h_1 = 0.1010$ in.,
$L_1 = L_2 = \frac{1}{2}L$,	$h_2 = 0.0505$ in.,
	$a = 1.9245$ in.

Keeping all other dimensions constant, thickness h_1 [(i)] was altered to (ii) 0.088 in., (iii) 0.076 in., (iv) 0.067 in., (v) 0.059 in., and finally (vi) 0.0505 in., - giving a uniform cylinder. Records of amplitude distribution obtained for these successive cylinders, for the mode n = 4, m = 1, are shown in fig. 37. In order



n = 4 , m = 1

Fig 37

to show more clearly the result of varying the ratio of thicknesses, all the curves are plotted in the same figure. In the first three curves the wavelength of the sinusoidal part of the curve is clearly seen to remain unchanged; the frequencies corresponding to these cases were in fact practically constant, being foumd to be 2,750, 2,730, and 2,700 cycles per second respectively; the waveform in the thicker portion is gradually becoming less curved. By a suitable choice of h_1 in curve (iv) the crucial point of transition from one waveform to the other is almost exactly hit upon; the frequency has come down to 2,560 c.p.s., the wavelength in the thinner portion has correspondingly increased, and the waveform in the thicker portion is not much different from a straight line (in fact, the form is sinusoidal with a large wavelength); on calculating f_R for $h_1 = 0.067$ in., and a = 1.9245 in., it is found to be 2,533 c.p.s. Curve (v) shows the waveform to consist of two near sine-waves, and curve (vi) was obtained for the final uniform cylinder and is purely sinusoidal.

It will be appreciated that the point of transition, viz. the ratio of thicknesses which allows the cylinder to have as one of its natural frequencies a value very close to that of f_R of its thicker portion, will not occur simultaneously for all values of n. This is due to the fact that the Rayleigh frequency for the thicker portion for different values of n will not correspond in all cases to the same value of λ on the f/λ curve for the thinner portion: this has already been mentioned in Chapter 2, p. 20.

Considering the physical interpretation of the behaviour of a shell of variable thickness in instances where its natural frequency lies below f_R for its thicker portion, the deformation to which that portion is subjected must be viewed as resulting from a forced vibration into which it is set by the small forces and moments acting at the change of thickness by virtue of the free vibrations of the thinner portion. We might be tempted to study such cases from the point of view of the thinner portion only, by considering it to have an end condition intermediate between free and fixed at the change of thickness. The danger in such reasoning lies in the extreme conditions which would logically have to be taken into account: by putting the thickness of the thicker portion equal to zero, a free end is obtained - this presents no difficulty; the other extreme, however, i.e. that of a large thickness, giving complete fixing to that end of the thinner portion, is incompatible with the assumed characteristics of a thin shell, as already stated in Chapter 3, p. 23.

The inspection of figs. 22 to 33 will show that the assumptions concerning the waveform, and the values of wavelength obtained from frequency equations are fully corroborated by the experimental findings.

Finally, as regards the values of natural frequencies themselves, the agreement between experimental and calculated results is complete and universal. The accuracy is not of the same high degree in all cases, and some comment must be made on the subject of sources of discrepancy.

In general it may be said that, although everywhere good, there is an improvement in the accuracy of calculated values when the ratio of the length of the thinner portion to the total length of the cylinder becomes at least $\frac{3}{8}$, and there is hardly any discrepancy between experimental and calculated frequencies when that ratio is $\frac{5}{8}$ or more. Further, frequencies obtained using the modified beam analogy method are on the whole slightly less accurate than those for which a value of could be determined in each portion; comment has already been made on this on p. 54.

More particularly, the greatest inaccuracies are to be observed in results for n = 5 of cylinders 1 - 2 and 1 - 3; for that, however, there is good reason. If we look first at Table 1, p. 32, in which natural frequencies of cylinder 1 are listed, discrepancies between experimental and theoretical results (which, in this case, are calculated from the <u>exact</u> frequency equation, as derived by Arnold and Warburton) for n = 5 are fairly large, of the order of 5.5 per cent. The method developed in this work, by its nature and basic assumptions, cannot at any time be in better agreement with experiment than the underlying exact theory. From the examination of Tables 2, 3 and 4 (p. 33 et seq.) especially, and, similarly, Tables 18 and 19 (p. 40 and 41), it will be seen that the corresponding discrepancies do not in fact exceed 5.5 per cent. in the modes considered, and follow exactly the pattern of the discrepancies in the uniform cylinder. Further, wherever for n = 5 in those cases the modified beam analogy method has to be used, it is then found to be nearer the experimental results than the full method: it will be remembered that those values are obtained without reference to the f/λ curve for thickness 1.

Cylinder 1 was made of thickness 0.250 in. (p. 21) in order to make available a suitable range of smaller thicknesses to be investigated; this gives a thickness ratio $\alpha = 0.120$, which is somewhat higher than 0.10, usually accepted as the upper limit of shells termed thin. This must account for the relative inaccuracy of experimental results for that cylinder. By comparison, a marked improvement is seen in results for cylinder 2, and for cylinder 3 a nearly perfect agreement of experiment with theory is reached.

Considering finally various numbers of circumferential nodes, $n = 3$ appears to provide experimental values closest to those obtained by calculation; even when the ratio of the length of the thinner portion to L is small, very good agreement exists - cf. here Tables 2, 10 and 18 (p. 32, 36 and 40 respectively); mention has already been made above of some instances of $n = 5$, and it remains now only to remark on some variation occasionally discernible in modes with $n = 2$: once more reference must be made to the paper of Arnold and Warburton, who encountered in their experimental work the fact of a certain amount of divergence for that particular value of n.

The overall average agreement between experimental and calculated values of natural frequencies is to within about 2 per cent., but in most cases it is even closer.

It is proposed in the next future to extend the present method of solution to cover the general case of a cylindrical shell consisting of any number of portions of different thickness; special consideration will have to be given to the modified beam analogy part of the method: already the natural frequencies of such a shell, which are higher than the Rayleigh frequency of its thickest portion, can be obtained by the process of "fitting together" successive sine-waves with wavelengths corresponding to the same frequency but different thickness ratios, starting from $w = 0$ at one end, in such a way as to obtain $w = 0$ at the other end. In addition to evolving the general method of solution, a comprehensive experimental investigation will have to be carried out.

APPENDIX I.

List of Frequently Employed Symbols.

a	mean radius of cylinder.
A,B,C,D	constants in beam analogy theory.
E	Young's modulus.
$f = p/2\pi$	frequency.
f_R	frequency of cylinder of infinite length (p. 12).
F,G,H,K	constants in beam analogy theory.
g	acceleration due to gravity.
h	thickness of cylinder.
K	coefficients in frequency equation for uniform thin shells.
l	half the axial wavelength, when $w = W \sin \frac{\pi x}{l}$.
L	length of cylinder.
L_1 etc.	lengths of portions of cylinder.
m	number of axial half-waves, or number of axial nodes minus one.
n	number of circumferential waves.
$p = 2\pi f$	circular frequency.
t	time.
u,v,w,	componenet displacements of a point on the middle surface of the shell.
U,V,W	functions of time, proportional to cos pt.

W_1, W_2	constants in "slope and deflection" method.
x	co-ordinate direction along the axis of the cylinder.
$\alpha = \frac{h}{a}$	thickness ratio of cylinder.
$\lambda = \frac{m\pi a}{L}$	axial wavelength factor.
ρ	density.
σ	Poisson's ratio.
ϕ	angular co-ordinate.

APPENDIX II.

Mathematics of Solution.

1. Cylinder "slope and deflection" method.

It will be profitable first of all to go over briefly the theory of free vibrations of thin cylindrical shells with freely-supported ends. The analysis has been carried out in detail in the paper by Arnold and Warburton (1), and here only the more important steps and results will be quoted.

The authors derive the frequency equation using the energy method. First, stresses acting on a small element of the shell are expressed in terms of the three componenet displacements u, v, and w, and their derivatives (Love, p. 529 et seq.). These displacements are taken as:-

$$u = U \cos n\phi \cos\frac{m\pi x}{L},$$
$$v = V \sin n\phi \sin\frac{m\pi x}{L}, \qquad (1.1)$$
$$w = W \cos n\phi \sin\frac{m\pi x}{L},$$

where ϕ defines the angular position of the point considered, n and $\frac{1}{2}$m are respectively the number of circumferential and axial wavelengths, U, V and W are functions of time only, and L is the length of the cylinder.

The above expressions satisfy the conditions at the ends of the shell and the general equations of vibration.

The strain energy S of the deformed cylinder and the kinetic energy T at any instant can now be expressed, and, since U, V and W are independent variables, the Lagrange equation is applicable. This gives

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{U}}\right] - \frac{\partial T}{\partial U} = -\frac{\partial S}{\partial U} \quad , \qquad (1.2)$$

and two similar equations in V and W.

We can write

$$\begin{aligned} U &= A \cos pt, \\ V &= B \cos pt, \\ W &= C \cos pt, \end{aligned} \qquad (1.3)$$

where A, B and C are constants, and $p/2\pi$ the natural frequency of vibration. Eliminating A, B and C from the three Lagrangian equations, the frequency equation is obtained. This is of the form

$$\Delta^3 - K_2\Delta^2 + K_1\Delta - K_0 = 0 , \qquad (1.4)$$

where

$$\Delta = \frac{\rho a^2(1-\sigma^2)p^2}{Eg} \qquad (1.5)$$

and the coefficients K_2, K_1 and K_0 are given by:-

$$K_0 = \tfrac{1}{2}(1-\sigma)^2(1+\sigma)\lambda^4 + \tfrac{1}{2}(1-\sigma) \times\beta\left[(\lambda^2+n^2)^4 - 2(4-\sigma^2)\lambda^4 n^2 - 8\lambda^2 n^4 - 2n^6 + 4(1-\sigma^2)\lambda^4 + 4\lambda^2 n^2 + n^4\right] , \qquad (1.6)$$

$$K_1 = \tfrac{1}{2}(1-\sigma)(\lambda^2+n^2)^2 + \tfrac{1}{2}(3-\sigma-2\sigma^2)\lambda^2 + \tfrac{1}{2}(1-\sigma)n^2 +\beta\left[\tfrac{1}{2}(3-\sigma)(\lambda^2+n^2)^3 + 2(1-\sigma)\lambda^4 - (2-\sigma^2)\lambda^2 n^2 - \tfrac{1}{2}(3+\sigma)n^4 + 2(1-\sigma)\lambda^2 + n^2\right] , \qquad (1.7)$$

$$K_2 = 1 + \tfrac{1}{2}(3-\sigma)(\lambda^2+n^2) +\beta\left[(\lambda^2+n^2)^2 + 2(1-\sigma)\lambda^2 + n^2\right] , \qquad (1.8)$$

in which

$$\beta = \frac{h^2}{12a^2} , \qquad \lambda = \frac{m\pi a}{L} . \qquad (1.9)$$

Equation (1.4) is a cubic in Δ, and gives three values of frequency for each nodal arrangement. In this thesis only the lowest of these was considered, the two remaining lying beyond the aural range.

Consider now a cylindrical shell such as is shown in fig. 3.

Referring to expressions (1.1), let the radial displacement in portion 1 be given, apart from factors depending on ϕ and t, by

$$w_1 = W_1 \sin\frac{\pi x}{l_1}, \qquad (1.10)$$

and in portion 2 by

$$w_2 = W_2 \sin\frac{\pi x}{l_2}, \qquad (1.11)$$

where l with the appropriate suffix is half the wave-length and gives, from expression (1.9) $\lambda = \frac{\pi a}{l}$.

Now, taking for each portion the origin of co-ordinates at its supported end, we have at the change of section

$$w_1 = w_2 ,$$

and

$$\frac{dw_1}{dx} = - \frac{dw_2}{dx} . \qquad (1.12)$$

Hence

$$W_1 \sin\frac{\pi L_1}{l_1} = W_2 \sin\frac{\pi L_2}{l_2},$$

and

$$\frac{\pi}{l_1} W_1 \cos\frac{\pi L_1}{l_1} = -\frac{\pi}{l_2} W_2 \cos\frac{\pi L_2}{l_2}, \qquad (1.13)$$

giving, after eliminating W_1 and W_2

$$\tan\frac{\pi L_2}{l_2} = -\frac{l_1}{l_2}\tan\frac{\pi L_1}{l_1}, \qquad (1.14)$$

which is our frequency equation. l_1 and l_2 must satisfy this equation by being chosen so that they give λ_1 and λ_2 corresponding to the same frequency (obtained from frequency/λ curves for the value of n required). For any one value of n frequencies for successive modes - with an increasing number of axial nodes - will be obtained by taking solutions with the values of the tangents located in successive quadrants.

For the limiting case, i.e. when, say, $L_1 = 0$, and consequently $L = L_2$, we have from equation (1.14)

$$\tan \frac{\pi L_2}{l_2} = 0,$$

i.e.

$$\frac{L_2}{l_2} = 1,\ 2,\ 3,\ 4 \ldots \text{ etc.},$$

which is the case of a uniform cylinder (similarly, of course, when $L_2 = 0$).

2. Beam analogy theory.

In the theory of lateral vibrations of bars or beams (5) we have the general differential equation of motion

$$\frac{\partial^2}{\partial x^2}\left[EI\,\frac{\partial^2 y}{\partial x^2}\right] = -\frac{\gamma A}{g}\,\frac{\partial^2 y}{\partial t^2}\ , \qquad (2.1)$$

where EI is the flexural rigidity of the beam,

A the cross-sectional area,

γ the weight of material per unit volume.

When the flexural rigidity does not change along the length of the beam, we may write the equation as:-

$$EI\,\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\,\frac{\partial^2 y}{\partial t^2} = 0,$$

or

$$a^2\,\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial x^2} = 0, \qquad (2.2)$$

where

$$a^2 = \frac{EIg}{A\gamma}\ .$$

All particular solutions of equation (2.2) will consist of a function of the co-ordinate x and a harmonic function of time cos pt or sin pt, where p determines the frequency of vibration.

Using the notation

$$\frac{p^2}{a^2} = \frac{p^2 A\gamma}{EIg} = k^4 ,$$

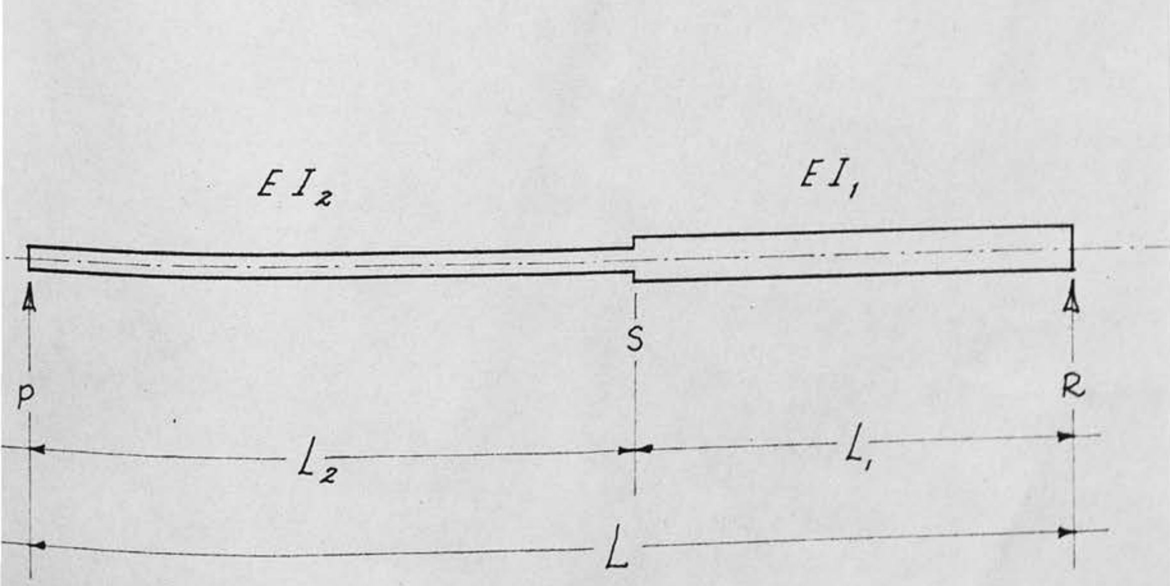
the general solution of equation (2.2) apart from the function of time multiplying it, is seen to be

$$X = A \sin kx + B \cos kx + C \sinh kx + D \cosh kx, \quad (2.3)$$

in which A, B, C and D are constants determined from boundary conditions for each particular case.

Let us now consider a beam as shown in fig. A1.

Fig A 1



It consists of two sections of different flexural rigidity, and is freely-supported at the ends P and R. The change of section occurs at S.

Let us omce more for the purpose of this analysis treat each portion from an origin of co-ordinates situated at its supported end.

Then we shall have for portion 1

$$(X)_1 = A \sin k_1x + B \cos k_1x + C \sinh k_1x + D \cosh k_1x,$$

and for portion 2

$$(X)_2 = F \sin k_2x + G \cos k_2x + H \sinh k_2x + K \cosh k_2x.$$

For the condition of freely-supported ends we have at P and R

$$\text{(i) } X = 0, \qquad (2.4)$$

$$\text{(ii) } \frac{\partial^2 X}{\partial x^2} = 0, \qquad (2.5)$$

and hence immediately

$$(X)_1 = A \sin k_1x + C \sinh k_1x,$$

$$(X)_2 = F \sin k_2x + H \sinh k_2x.$$

Now in applying this method to the case of cylindrical shells, let us for the sake of simplicity assume that the displacement function in the thinner portion is sinusoidal (as in the previous case), and we thus take

$$w_2 = F \sin \frac{\pi x}{l_2}, \qquad (2.6)$$

where l_2, as before, is half the wavelength and supplies the value of λ_2, since $\lambda_2 = \frac{\pi a}{l_2}$.

For the thicker portion we thus have

$$w_1 = A \sin \frac{\pi x}{l_1} + C \sinh \frac{\pi x}{l_1}, \qquad (2.7)$$

where l_1 is unknown and has to be determined together with A, C and F from the remaining four boundary conditions, i.e. those at S. They are concerned with the displacement, slope, bending moment and shear force at the change of section, and can be expressed as follows:-

$$\text{(i)} \quad (w_1) = (w_2) \qquad (2.8)$$

$$\text{(ii)} \quad \left[\frac{dw_1}{dx}\right]_{x=L_1} = -\left[\frac{dw_2}{dx}\right]_{x=L_2} \qquad (2.9)$$

$$\text{(iii)} \quad EI_1\left[\frac{d^2w_1}{dx^2}\right]_{x=L_1} = EI_2\left[\frac{d^2w_2}{dx^2}\right]_{x=L_2} \qquad (2.10)$$

$$\text{(iv)} \quad EI_1\left[\frac{d^3w_1}{dx^3}\right]_{x=L_1} = -EI_2\left[\frac{d^3w_2}{dx^3}\right]_{x=L_2} \qquad (2.11)$$

Hence we have:-

$$A \sin \frac{\pi L_1}{l_1} + C \sinh \frac{\pi L_1}{l_1} = F \sin \frac{\pi L_2}{l_2}, \qquad (2.12)$$

$$\frac{1}{l_1}\left[A \cos \frac{\pi L_1}{l_1} + C \cosh \frac{\pi L_1}{l_1}\right] = -\frac{F}{l_2} \cos \frac{\pi L_2}{l_2}, \qquad (2.13)$$

$$\frac{I_1}{l_1^2}\left[-A \sin \frac{\pi L_1}{l_1} + C \sinh \frac{\pi L_1}{l_1}\right] = -\frac{FI_2}{l_2^2} \sin \frac{\pi L_2}{l_2}, \qquad (2.14)$$

$$\frac{I_1}{l_1^3}\left[-A \cos \frac{\pi L_1}{l_1} + C \cosh \frac{\pi L_1}{l_1}\right] = \frac{FI_2}{l_2^3} \cos \frac{\pi L_2}{l_2}, \qquad (2.15)$$

From equations (2.12) and (2.14) we obtain

$$\frac{I_1 l_2^2}{I_2 l_1^2}\left[-A \sin\frac{\pi L_1}{l_1} + C \sinh\frac{\pi L_1}{l_1}\right] = -\left[A \sin\frac{\pi L_1}{l_1} + C \sinh\frac{\pi L_1}{l_1}\right],$$

or, putting $\frac{I_1 l_2^2}{I_2 l_1^2} = c,$

$$(c + 1)\, C \sinh\frac{\pi L_1}{l_1} = (c - 1)\, A \sin\frac{\pi L_1}{l_1}. \qquad (2.16)$$

From equations (2.13) and (2.15) we obtain

$$c\left[-A \cos\frac{\pi L_1}{l_1} + C \cosh\frac{\pi L_1}{l_1}\right] = -\left[A \cos\frac{\pi L_1}{l_1} + C \cosh\frac{\pi L_1}{l_1}\right],$$

$$(c + 1)\, C \cosh\frac{\pi L_1}{l_1} = (c - 1)\, A \cos\frac{\pi L_1}{l_1}. \qquad (2.17)$$

Hence

$$\frac{C}{A} = \frac{(c-1)\sin\frac{\pi L_1}{l_1}}{(c+1)\sinh\frac{\pi L_1}{l_1}} = \frac{(c-1)\cos\frac{\pi L_1}{l_1}}{(c+1)\cosh\frac{\pi L_1}{l_1}}, \qquad (2.18)$$

and l_1 is given at the same time by

$$\tan\frac{\pi L_1}{l_1} = \tanh\frac{\pi L_1}{l_1}, \qquad (2.19)$$

i.e.

$$\frac{l_1}{L_1} = \frac{4}{5},\ \frac{4}{9},\ \frac{4}{13}, \quad \text{etc.}, \qquad (2.20)$$

from which follows that $\tan\frac{\pi L_1}{l_1} = 1.$ (2.21)

Finally consider equations (2.12) and (2.13):-

$$A\left[\sin\frac{\pi L_1}{l_1} + \frac{C}{A}\sinh\frac{\pi L_1}{l_1}\right] = F \sin\frac{\pi L_2}{l_2},$$
$$\frac{A}{l_1}\left[\cos\frac{\pi L_1}{l_1} + \frac{C}{A}\cosh\frac{\pi L_1}{l_1}\right] = -\frac{F}{l_2}\cos\frac{\pi L_2}{l_2},$$

from which, by division:-

$$\tan\frac{\pi L_2}{l_2} = -\frac{l_1}{l_2}\left[\frac{\sin\frac{\pi L_1}{l_1} + \frac{C}{A}\sinh\frac{\pi L_1}{l_1}}{\cos\frac{\pi L_1}{l_1} + \frac{C}{A}\cosh\frac{\pi L_1}{l_1}}\right]$$

$$= -\frac{l_1}{l_2}\left[\frac{d\sin\frac{\pi L_1}{l_1}}{d\cos\frac{\pi L_1}{l_1}}\right] ,$$

where
$$d = 1 + \frac{c-1}{c+1} .$$

But from (2.20)

$$\sin\frac{\pi L_1}{l_1} = \cos\frac{\pi L_1}{l_1} ,$$

and, therefore,

$$\tan\frac{\pi L_2}{l_2} = -\frac{l_1}{l_2} . \qquad (2.22)$$

This is the frequency equation, since, when solved for l_2, the required frequency can be obtained from the appropriate f/λ curve, taking $\lambda_2 = \frac{\pi a}{l_2}$.

For the mode m = 1,

$$\pi > \frac{\pi L_2}{l_2} > \frac{\pi}{2}$$

and
$$l_1 = \frac{4}{5} L_1 .$$

For the mode m = 2,

$$2\pi > \frac{\pi L_2}{l_2} > \frac{3\pi}{2}$$

and
$$l_1 = \frac{4}{9} L_1 ,$$

and so forth for higher modes.

It will be observed that equation (2.22) is consistent with equation (1.14) for the case of sinusoidal amplitude distribution in the whole shell.

The latter states

$$\tan \frac{\pi L_2}{l_2} = -\frac{l_1}{l_2} \tan \frac{\pi L_1}{l_1},$$

and when the condition determining l_1 for the present case (where the wavelength in portion 1 is no longer real) is taken, i.e. $\tan \frac{\pi L_1}{l_1} = 1$ (2.21), we see that equation (2.22) is obtained.

As L_1, and hence l_1, decreases, the frequencies approach those for a uniform cylinder of thickness h_2, until, when $L_1 = 0$, and hence $l_1 = 0$, and $L = L_2$,

$$\tan \frac{\pi L_2}{l_2} = 0,$$

i.e. $$\frac{L_2}{l_2} = 1, 2, 3, 4 \ldots \text{ etc.},$$

which is the case of a uniform cylinder.

No significant answer is forthcoming from putting $L_2 = 0$, because any displacement in portion 1 is possible solely in virtue of the motion of portion 2.

APPENDIX III.

Specimen Calculations.

Although the method developed for the calculation of natural frequencies of vibration of the type of cylindrical shell considered in this thesis has been explained in detail in Chapter 2, it is desirable to illustrate the technique of its application by actually working out numerically some typical examples.

Let us, for instance, take cylinder 2A3, in which, according to our notation:-

$L = 17.5625$ in., $h_2 = 0.1875$ in.,

$L_2 = 15.3672$ in., $h_3 = 0.1250$ on.,

$L_3 = 2.1953$ in., $a = 2.0725$ in.,

and hence

$\alpha_2 = 0.0905$, $\alpha_3 = 0.0603$.

We must have previously calculated natural frequencies for each of the two uniform shells of thickness ratio α_2 and α_3 respectively, with, for convenience, $L =$ $= 17.5625$ in., from equation (1.4) of Appendix II. These are plotted against λ $(=\frac{m\pi a}{L})$ for $n = 2, 3, 4$ and 5, say, and for an adequate range of m, say $m = 1$ to 6. These curves are shown to a small scale in figs. A2, A3, A4 and A5.

n=2.

Fig A2

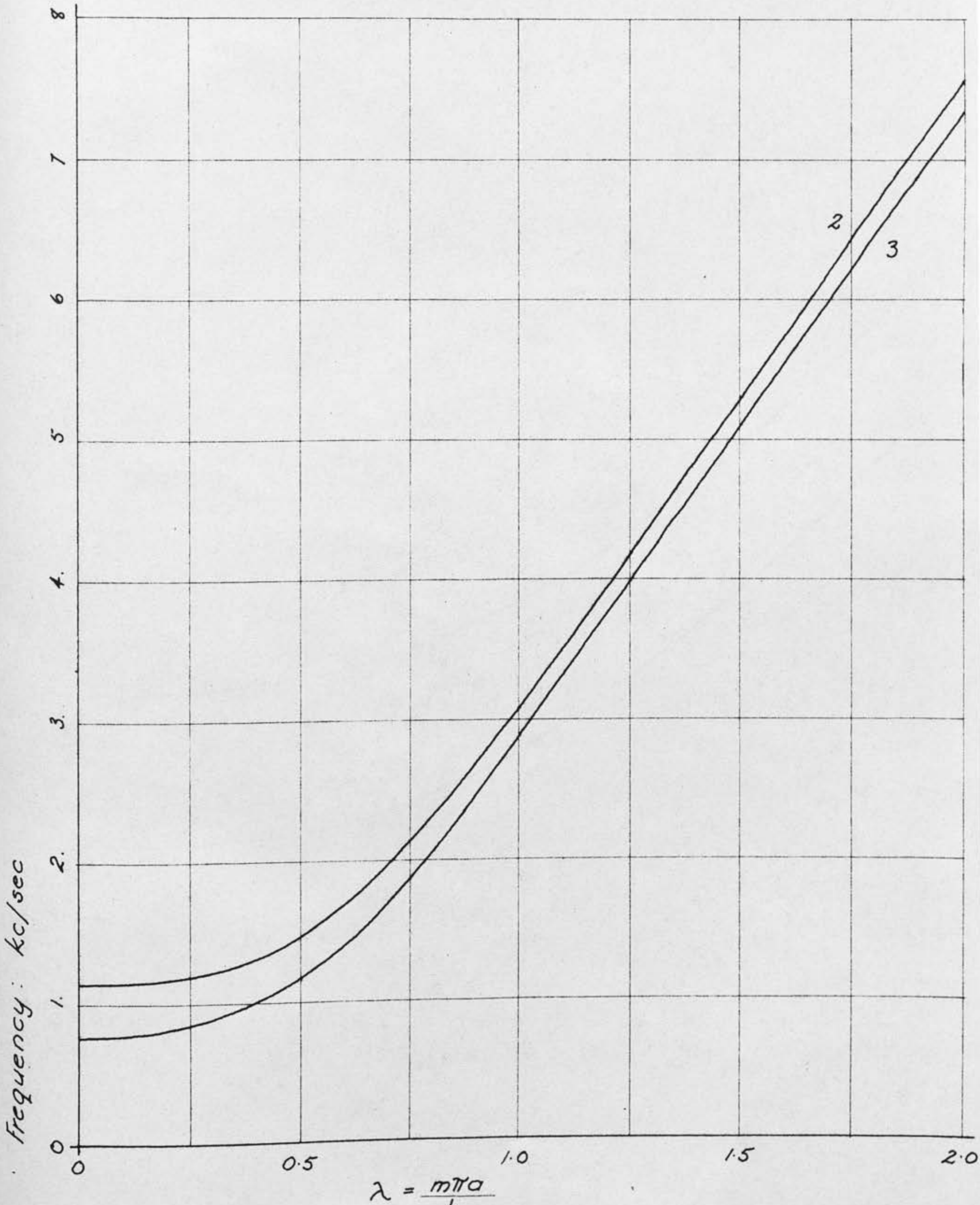
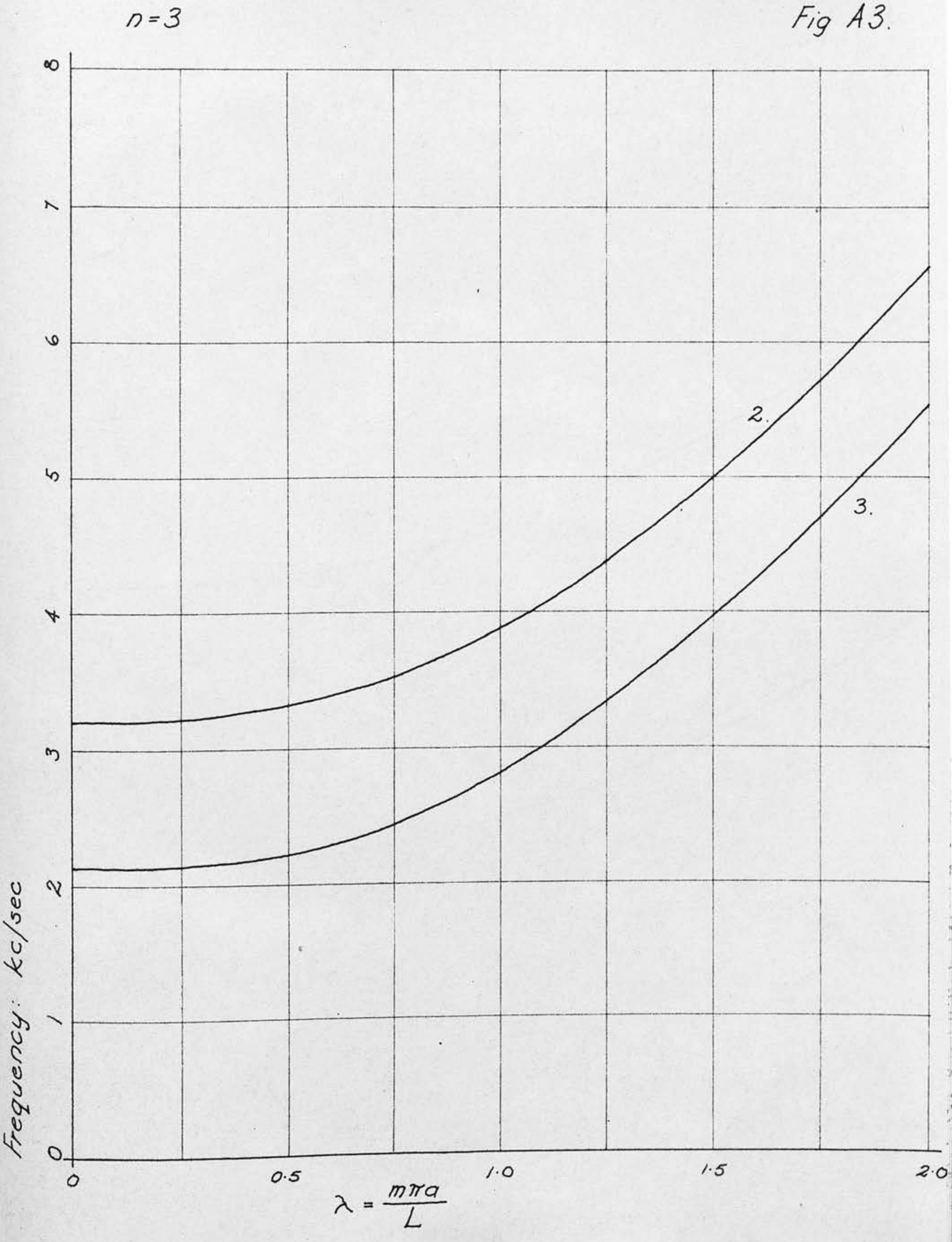


Fig A3.



n=3
Frequency: kc/sec
0
1
2
3
4
5
6
7
8
0
0·5
1·0
1·5
2·0
$\lambda = \frac{m\pi a}{L}$
2.
3.

$n = 4$. Fig A 4.

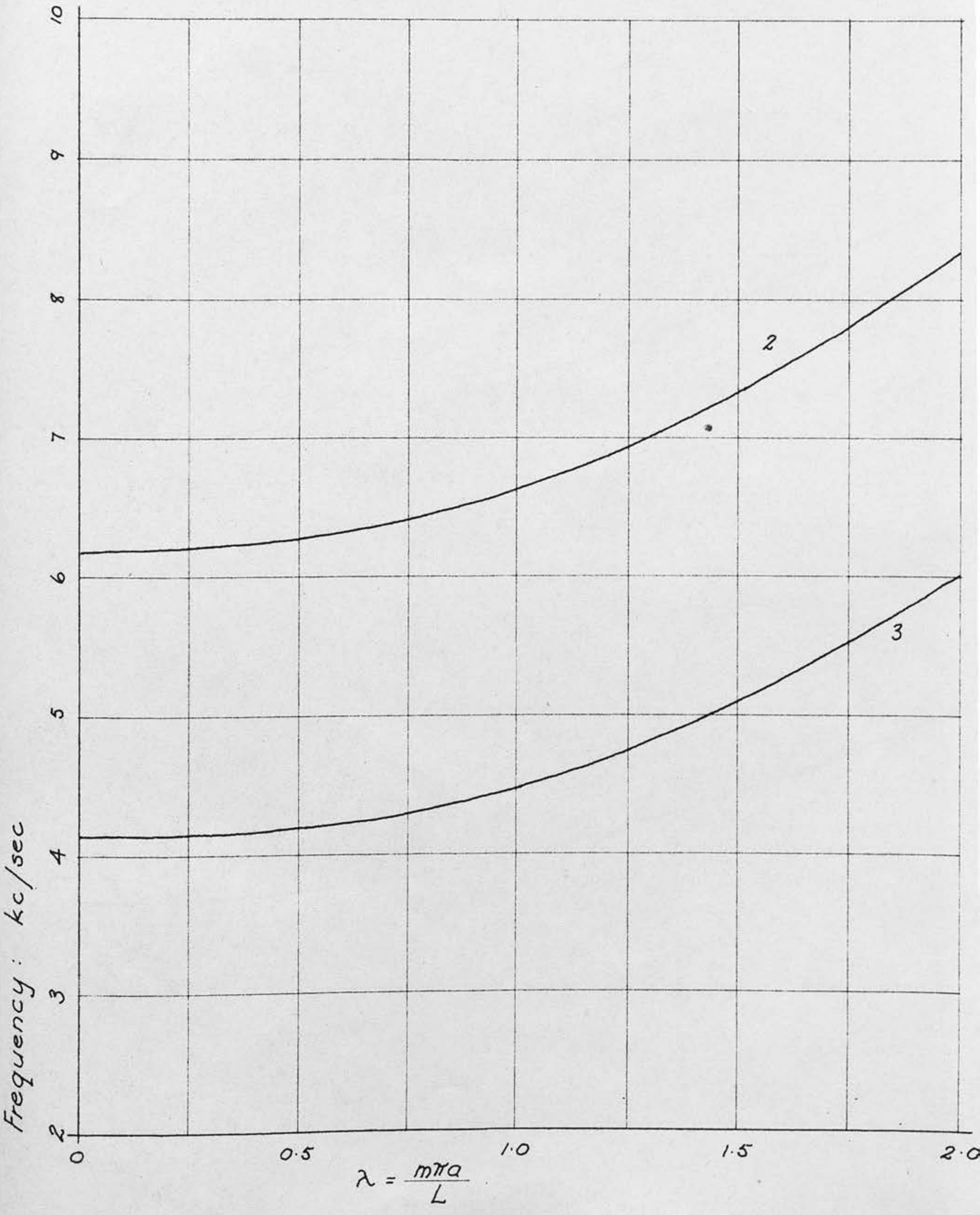
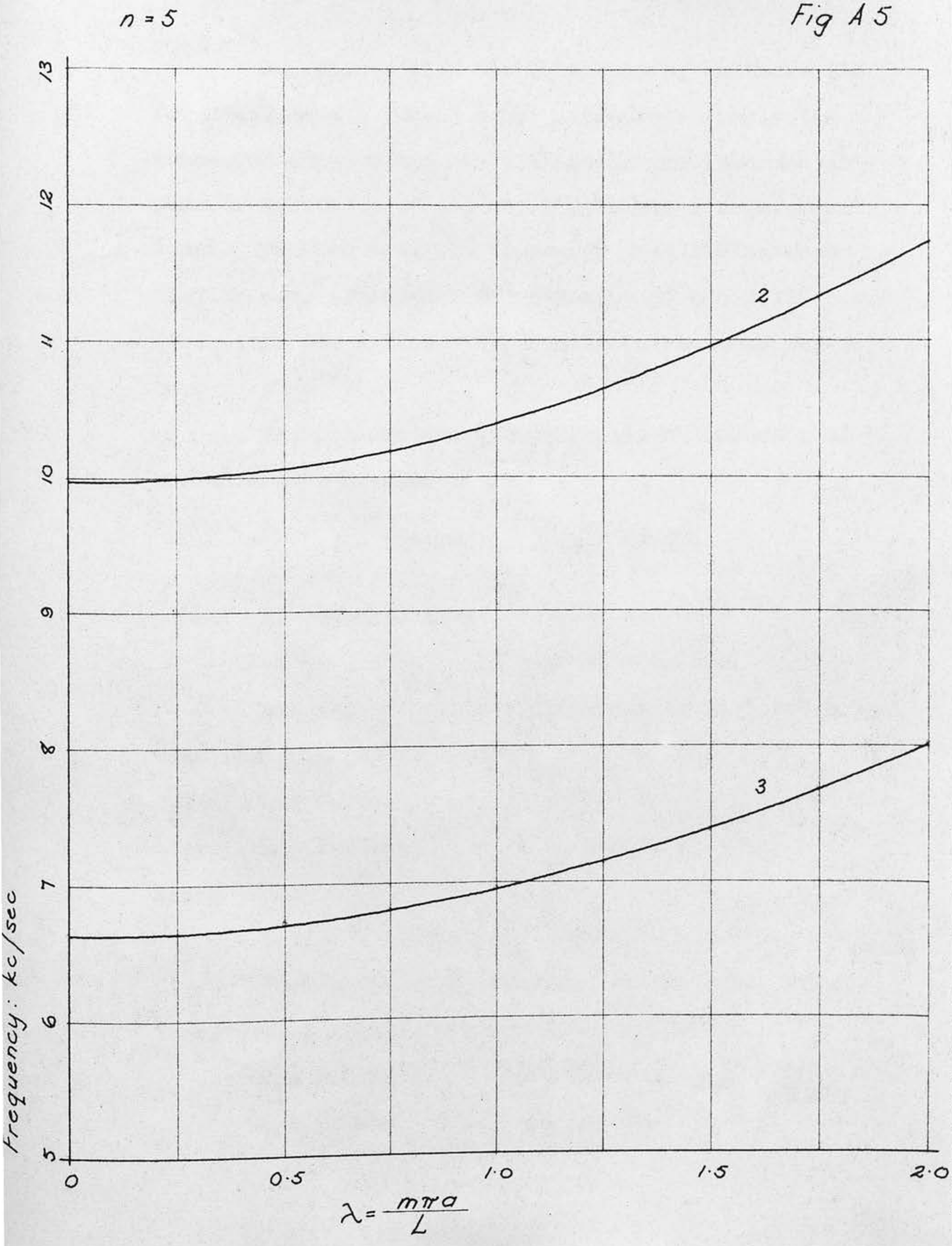


Fig A 5



Suppose we wish the frequency of cylinder 2A3 for n = 3, m = 1 (where m now represents simply the number of nodes minus one), which is the fundamental mode of vibration of the shell with six circumferential nodes. We know that the frequency must lie between 3,271 c.p.s. (frequency for cylinder 2) and 2,187 c.p.s. (frequency for cylinder 3), and be fairly near the former.

The frequency equation (1.14) of Appendix II for this case becomes

$$\tan \frac{395 \cdot 154^\circ}{l_3} = -\frac{l_2}{l_3} \tan \frac{2766 \cdot 078}{l_2}$$

We remember that

$$l = \frac{\pi a}{\lambda}, \qquad \text{and } \pi a = 6.5109.$$

Assume the required frequency to be 3,250 c.p.s. Then from fig. A3 we obtain:-

$$\lambda_2 = 0.275, \quad \lambda_3 = 1.2185, \quad \text{giving} \quad l_2 = 23.676, \quad l_3 = 5.343, \quad \text{and } \frac{l_2}{l_3} = 4.431.$$

Hence

L.H.S. = 3.487, R.H.S. = 8.770.

We see that the frequency is too low. Try, therefore, f = 3,260 c.p.s. Then, as before,

$$\lambda_2 = 0.325, \quad \lambda_3 = 1.223, \quad l_2 = 20.034, \quad l_3 = 5.324, \quad \frac{l_2}{l_3} = 3.763.$$

Hence

L.H.S. = 3.534, R.H.S. = 3.388.

By interpolation we obtain f = 3,259.8, or, practically, f = 3,260 c.p.s.

It will be seen that a guess of frequency which is considerably remote from the required value will be detected very readily. If the frequency is too high, the values of l_2 and l_3 obtained will indicate the presence of one or more axial nodes in the length of the cylinder; if it is too low, the frequency equation will not be satisfied.

The way in which nodal points are immediately indicated by the values of l_2 and l_3 will be illustrated by the following calculation, which will be carried out without further explanation.

Still with reference to cylinder 2A3 and n = 3, suppose the frequency of 3,450 c.p.s. is assumed; then we have

$\lambda_2 = 0.690$, $l_2 = 9.436$, $\frac{l_2}{l_3} = 1.884$.

$\lambda_3 = 1.300$, $l_3 = 5.008$,

L.H.S. = 5.097, R.H.S. = 4.417.

Assume then f = 3,425, which gives

$\lambda_2 = 0.670$, $l_2 = 9.718$, $\frac{l_2}{l_3} = 1.918$.

$\lambda_3 = 1.285$, $l_3 = 5.067$,

L.H.S. = 4.705, R.H.S. = 7.363.

By interpolation we have f = 3,447 c.p.s. as a natural frequency of vibration of the cylinder, and the values of l_2 and l_3 show that the mode is one with three axial nodes, i.e. m = 2.

Consider now cylinder 2E3, in which

L_2 = 6.5859 in., and L_3 = 10.9766 in.

Suppose, once again, the frequency required is for n = 3, m = 1. The frequency equation is for this case

$$\tan \frac{1975 \cdot 770^\circ}{l_3} = -\frac{l_2}{l_3} \tan \frac{1185 \cdot 462^\circ}{l_2}$$

We find, however, that the first solution gives f = 1.980 c.p.s., which corresponds to m = 2. This signifies that the modified beam analogy method must be used. Accordingly, we put $l_2 = \frac{4}{5} L_2$, and we have our frequency equation

$$\tan \frac{1975 \cdot 770^\circ}{l_3} = -\frac{5 \cdot 26872}{l_2}$$

which yields as the solution l_3 = 12.570. This gives λ_3 = 0.518, which then, from fig. A3, supplies the frequency for n = 3, m = 1, as 2,220 c.p.s. Applying this value of λ_3 to fig. A4 and A5, we can obtain at the same time frequencies of 4,200 c.p.s. for n = 4, and 6,725 c.p.s. for n = 5, respectively; applied to fig. A2, λ_3 = 0.518 gives f = 1,200, which is above f_R for cylinder 2: in fact, for $n = 2$, the original equation supplies a solution for $m = 1$, giving a frequency of 1,175 c.p.s.

We find that for $n = 3$, $m = 2$, as well, the modified beam analogy must be used. This time $l_2 = \frac{4}{9} L_2$, and the value of the tangent is in the fourth quadrant.

We have

$$\tan \frac{1975 \cdot 770^\circ}{l_3} = - \frac{2 \cdot 92706}{l_3} \quad ,$$

which gives for $n = 3$, $f = 2{,}990$ c.p.s.: for $n = 4$, $f = 4{,}575$ c.p.s.; and for $n = 5$, $f = 7{,}010$ c.p.s.

APPENDIX IV.

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