Topological Domain Theory

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Abstract

This thesis presents Topological Domain Theory as a powerful and flexible framework for denotational semantics. Topological Domain Theory models a wide range of type constructions and can interpret many computational features. Furthermore, it has close connections to established frameworks for denotational semantics, as well as to well-studied mathematical theories, such as topology and computable analysis.

We begin by describing the categories of Topological Domain Theory, and their categorical structure. In particular, we recover the basic constructions of domain theory, such as products, function spaces, fixed points and recursive types, in the context of Topological Domain Theory.

As a central contribution, we give a detailed account of how computational effects can be modelled in Topological Domain Theory. Following recent work of Plotkin and Power, who proposed to construct effect monads via free algebra functors, this is done by showing that free algebras for a large class of parametrised equational theories exist in Topological Domain Theory. These parametrised equational theories are expressive enough to generate most of the standard examples of effect monads. Moreover, the free algebras in Topological Domain Theory are obtained by an explicit inductive construction, using only basic topological and set-theoretical principles.

We also give a comparison of Topological and Classical Domain Theory. The category of ω-continuous dcpos embeds into Topological Domain Theory, and we prove that this embedding preserves the basic domain-theoretic constructions in most cases. We show that the classical powerdomain constructions on ω-continuous dcpos, including the probabilistic powerdomain, can be recovered in Topological Domain Theory.

Finally, we give a synthetic account of Topological Domain Theory. We show that Topological Domain Theory is a specific model of Synthetic Domain Theory in the realizability topos over Scott's graph model $\mathcal{P}\omega$. We give internal characterisations of the categories of Topological Domain Theory in this realizability topos, and prove the corresponding categories to be internally complete and weakly small. This enables us to show that Topological Domain Theory can model the polymorphic λ-calculus, and to obtain a richer collection of free algebras than those constructed earlier.

In summary, this thesis shows that Topological Domain Theory supports a wide range of semantic constructions, including the standard domain-theoretic constructions, computational effects and polymorphism, all within a single setting.

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Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of the results of chapter 3 have appeared in [9], parts of the results of chapters 4 and 5 in [8, 7], and parts of the results of chapter 6 in [6].

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List of Categories

Acronym	Objects and Morphisms	Chapters
Top	topological spaces and continuous maps	1,2,4
ω**Top**	countably-based topological spaces and continuous maps	2
kTop	compactly-generated spaces and continuous maps	2,4
kP	compactly-generated predomains and continuous maps	3,5
Seq	sequential spaces and continuous maps	2,4
SeqP	sequential predomains and continuous maps	3,5
Mon	monotone convergence spaces and continuous maps	3
ω**Mon**	ω-monotone convergence spaces and continuous maps	3
QCB	topological quotients of countably-based spaces and continuous maps	1,2,4,5
QCB$_0$	T_0-quotients of countably-based spaces and continuous maps	2,6
TP	topological predomains and continuous maps	3,5,6
TD	topological domains and continuous maps	3,5
TD$_\perp$	topological domains and strict maps	3,5
Equ	equilogical spaces and equilogical morphisms	2
ω**Equ**	countably-based equilogical spaces and equilogical morphisms	2,6
PQ	ω-projecting equilogical spaces and equilogical morphisms	2
AdmRep	spaces with admissible representation and continuous maps	2
DCPO	dcpos and Scott-continuous maps	1,3,5
DCPPO	pointed dcpos and Scott-continuous maps	1,3
DCPPO$_\perp$	pointed dcpos and strict maps	1,3
CONT	continuous dcpos and Scott-continuous maps	1
CONT$_\perp$	pointed continuous dcpos and Scott-continuous maps	1

Contents

Acronym	Objects and Morphisms	Chapters
$\omega\mathbf{CONT}$	countably-based continuous dcpos and Scott-continuous maps	1,3,5
$\omega\mathbf{CONT}_\perp$	countably-based pointed continuous dcpos and Scott-continuous maps	1,3
$\omega\mathbf{FS}$	countably-based FS-domains and Scott-continuous maps	3
$\omega\mathbf{FS}_\perp$	countably-based pointed FS-domains and Scott-continuous maps	3
$\omega\mathbf{CPO}$	chain complete posets and ω-Scott-continuous maps	3
$\omega\mathbf{CPPO}_\perp$	chain complete pointed posets and strict ω-Scott-continuous maps	3
$\mathbf{Asm}(\mathcal{P}\omega)$	assemblies over $\mathcal{P}\omega$ and realizable maps	6
$\mathbf{Mod}(\mathcal{P}\omega)$	modest sets over $\mathcal{P}\omega$ and realizable maps	6
$\mathbf{Ext}(\mathcal{P}\omega)$	extensional assemblies and realizable maps	6
$\mathbf{CE}(\mathcal{P}\omega)$	complete extensional assemblies and realizable maps	6
$\mathbf{RT}(\mathcal{P}\omega)$	realizability topos over $\mathcal{P}\omega$ (also $\mathbf{RT}$)	6
$\mathbf{\Sigma Pos}$	Σ-posets and $\mathbf{RT}$-morphisms	6
$\mathbf{\Sigma Cpo}$	Σ-cpos and $\mathbf{RT}$-morphisms	6
$\mathbf{Sep}$	$\neg\neg$-separated objects and $\mathbf{RT}$-morphisms	6
$\mathbf{Orth}$	orthogonal objects and $\mathbf{RT}$-morphisms	6
$\mathbf{Orth}_{\neg\neg}$	$\neg\neg$-separated orthogonal objects and $\mathbf{RT}$-morphisms	6
$\mathbb{C}_\Sigma$	Σ-algebras in $\mathbb{C}$ and Σ-homomorphisms (for Σ being a signature for a parameterised algebraic theory for any category $\mathbb{C}$)	4,5
$\mathbb{C}_{(\Sigma,\mathcal{E})}$	$(\Sigma,\mathcal{E})$-algebras in $\mathbb{C}$ and Σ-homomorphisms (for $(\Sigma,\mathcal{E})$ being a parameterised equational theory for any category $\mathbb{C}$)	4,5
$\mathbb{C}_\mathcal{T}$	$\mathcal{T}$-algebras in $\mathbb{C}$ and $\mathcal{T}$-homomorphisms (for $\mathcal{T}$ being an equational algebraic theory for $\mathbf{RT}$ and $\mathbb{C}$ any definable subcategory of $\mathbf{RT}$)	6

1 Introduction

1.1 Motivation

Denotational Semantics aims at giving mathematical models to computational systems, in order to provide abstract methods for reasoning about the concrete systems. Amongst other tasks, one of its principal goals is to provide a flexible framework for interpreting programming languages. Such a framework should be able to interpret a variety of type constructions, it should be able to model a wide range of computational phenomena, and it should have a well-studied underlying mathematical theory. More specifically, one is interested in a framework supporting the following aspects.

Type Constructions: Functional programming languages [106, 157] are in general based on some form of the λ-calculus, and so *function types* are explicitly needed in an accurate model [156]. Moreover, *product* and *sum types* are supported in common functional programming languages like ML [106, 102, 92] or Haskell [157, 61]. Other useful constructions are *polymorphic* [119] and *recursive types* [106]. A good framework should be able to support these type constructions.

Computational phenomena: Arguably one the most important programming mechanism is *recursion*, allowing the programmer to write simple algorithms which can be applied to infinitely many cases. Furthermore, programs often interact with their underlying computational system, invoking certain forms of non-functional behaviour. These non-functional behaviours are called *computational effects* [97], and examples are nontermination, nondeterminism, probabilistic behaviour, exceptions, input/output and side-effects. A good framework has to be able to deal with recursion and a wide range of computational effects.

Underlying mathematics: In addition to the mentioned features, it is desirable for a good framework for denotational semantics to have a well-developed and well-established underlying mathematical theory. The reason for this is that an established underlying theory provides many ways to reason about the programming constructs. Furthermore, such a connection shows that programming languages are subject to similar mathematical concepts as other disciplines. The ultimate aim here would be to have an underlying mathematical theory, subject to logical principles, strong enough that they enable automated reasoning about programs.

This thesis shows that Topological Domain Theory [138, 10] gives a mathematical

toolkit for denotational semantics, which satisfies all these demands. More specifically, after recalling the definitions of the categories of Topological Domain Theory and the results of their closure properties from the literature [90, 129, 138, 28, 9], we show how computational effects can be modelled in this framework via a free algebra approach [8, 7], following recent work of Plotkin and Power [112]. Furthermore, we give a detailed comparison of the categorical structure and the free algebra construction in Topological and Classical Domain Theory [9, 7, 11]. Finally, we strengthen the categorical closure properties by embedding the categories of Topological Domain Theory into a model of intuitionistic set-theory, showing that the resulting categories are internally complete and weakly small. This shows that polymorphic types can be modelled in Topological Domain Theory, and that free algebras exist for a very general class of algebraic theories, extending the results we obtain in the earlier parts of the thesis.

The starting point of Topological Domain Theory is the fact that the category **QCB** of topological quotients of countably-based spaces (qcb-spaces) can be viewed as a full reflective exponential ideal of Scott's category of ω-equilogical spaces [14]. Moreover, **QCB** has also been discovered as a category in computable analysis. Weihrauch's Type Two Theory of Effectivity (TTE) [165] uses Type-2 machines for modelling computations on sets with cardinality up to the continuum, and employs Baire space representations of topological spaces to develop a theory for computable analysis. It turns out that the T_0-qcb-spaces form the largest class of topological spaces on which computability can be modelled accurately by this approach [129]. The choice of topology as underlying mathematical theory for denotational semantics is justified by Smyth's paradigm [143, 144, 27], which draws an analogy between topological and computational entities.

The category of those qcb-spaces that are also monotone convergence spaces [36] (equivalently d-spaces in the sense of Wyler [166]), is a full reflective exponential ideal of **QCB** [9]. We call this category **TP**, and its objects *topological predomains* [138]. **TP** inherits the rich categorical structure of **QCB**, and has a full cartesian-closed subcategory **TD** of objects whose specialization order has a least element, which are called *topological domains* [10]. Continuous endomaps of topological domains have least fixed points, and thus recursion can be modelled in **TD**. A continuous map between topological domains which preserves the least element is called *strict*, and these maps form the category **TD**$_\perp$, which is not cartesian-closed but symmetric monoidal closed. **TD**$_\perp$ turns out to be algebraically compact, and thus has solutions for recursive domain equations [33, 29]. It follows that recursive types can be modelled here.

The first major contribution of this thesis is to show how computational effects can be modelled in Topological Domain Theory. Computational effects capture non-functional behaviour of computational systems, such as e.g. nontermination, nondeterminism, probabilistic behaviour, input/output, exceptions, side-effects. Eugenio Moggi [95, 97] suggested modelling such computational effects by considering a type construc-

tion which assigns to any result type X a computational type TX, which models the corresponding non-functional behaviour on the result type X. Semantically he used monads to model the construction of computational types and developed the computational λ-calculus. Recently, Gordon Plotkin and John Power [110, 111, 112] have refined this monadic approach and have shown that many computationally interesting effect monads arise as free algebra functors for algebraic theories. This approach has a distinct advantage over Moggi's theory: one can easily combine effects, by combining the underlying algebraic theories appropriately. However, a limitation of the free algebra approach is that it does not capture all computational effects; a notable counterexample is given by continuations which cannot be described algebraically, see [50].

We show that **QCB**, **TP** and **TD**$_\perp$ have free algebras for parameterised equational theories, a sufficiently rich class of algebraic theories to model the effects captured by Plotkin and Power's refinement [8]. For example, all the effects mentioned above can thus be modelled in Topological Domain Theory. Moreover, the classical powerdomain constructions for nondeterminism can be recovered in Topological Domain Theory with this approach [7], as they are constructed as free algebras for inequational algebraic theories [43, 109, 2], and these are subsumed by parameterised equational theories. We also obtain a probabilistic powerdomain construction using the parameterised equational theory for a convex space [58, 40]. Finally, our free algebra construction follows from basic set-theoretic and topological principles, giving us insights to the structure of the computational types.

The second main contribution of the thesis is to explicitly describe the interrelations of Topological Domain Theory with established theories, putting particular emphasis on the comparison with Classical Domain Theory. Classical Domain Theory, which was started by Scott in the early 1970s [135], is arguably the most popular and successful framework for denotational semantics to date. Its idea is to interpret datatypes by complete partially ordered sets, and it has attracted numerous researchers from mathematics and computer-science, see [30] for an overview. As a framework for denotational semantics, it satisfies all the demands above. However, it has its limitations, as inside Classical Domain Theory some of these features seem to be incompatible with each other. For instance, it is not known whether it is possible to combine function types with probabilistic features in a category of ω-continuous dcpos [69], which is arguably the largest class of classical domains, in which a notion of computability is available. Similarly, it is not known whether polymorphism can be combined with computational effects in Classical Domain Theory [64].

We show that Topological Domain Theory not only subsumes Classical Domain Theory as a framework for denotational semantics, by providing a single set of closely interrelated categories which meet all our demands, but we also argue that Topological

Domain Theory is a genuine extension of Classical Domain Theory. In particular, we show that the category ω**CONT**, of ω-continuous dcpos and Scott-continuous maps between them, is a full subcategory of the category of topological predomains, and this inclusion preserves the construction of finite products and countable coproducts [9]. Moreover, the restriction of the embedding to ω**FS** [63], the largest cartesian-closed subcategory of pointed ω-continuous dcpos, preserves the function space construction [9]. Also the construction of free algebras in both frameworks coincides in many cases on ω-continuous dcpos, for instance, for the classical powerdomain constructions for nondeterminism via inequational algebraic theories [7]. Finally, the probabilistic powerdomain construction in Topological Domain Theory, as the free convex topological predomain, coincides with the traditional construction in Classical Domain Theory, due to Claire Jones and Gordon Plotkin [59, 58], on all pointed ω-continuous dcpos [11].

The final contribution of the thesis is to give a synthetic account of Topological Domain Theory. It is well-known from [14, 12] that Scott's category of ω-equilogical spaces can be embedded into the realizability topos **RT**$(\mathcal{P}\omega)$ over his graph model $\mathcal{P}\omega$ [136]. As **QCB** is a full reflective exponential ideal of the category of ω-equilogical spaces, it follows that it can also be embedded into **RT**$(\mathcal{P}\omega)$, and it turns out that Topological Domain Theory forms a concrete model of Synthetic Domain Theory [48, 153, 103, 86, 101, 118]. Synthetic Domain Theory has succeeded in providing categories with very strong categorical closure properties, so that they are able to model a wide range of type constructions, for instance polymorphic types. However, in changing to an intuitionistic setting, some of the classical reasoning principles are lost, and perhaps for this reason, Synthetic Domain Theory has not attracted the general audience in the way Classical Domain Theory did.

We show that, when embedded into **RT**$(\mathcal{P}\omega)$, the categories **QCB** and **TP** become equivalent to the categories of regular Σ-posets, respectively Σ-cpos [117, 116], which are well-studied notions in Synthetic Domain Theory, see [116, 101]. We then prove these categories to be internally complete and weakly small, which enables us to show that they are able to interpret the polymorphic λ-calculus [119]. Finally, we apply an internal Adjoint Functor Theorem, in order to obtain an abstract proof of the existence of free algebra functors for a wide class of algebraic theories, generalising the parameterised equational theories discussed above. This establishes Topological Domain Theory as a bridge between Classical and Synthetic Domain Theory, inheriting the powerful synthetic constructions, while at the same time having an easily accessible classical characterisation.

4

1.2 History and related work

In the early 1970s, Dana Scott [136] proposed the use of dcpos (<u>d</u>irected <u>c</u>omplete <u>p</u>artially <u>o</u>rdered sets) for giving denotational semantics to programming languages. His key insight was that the order on a dcpo can be used to model the amount of information that is obtained from a result; larger elements carry more information than smaller ones. Thus one gets a theory of approximation, which is necessary when giving an expressive mathematical interpretation to programs on a concrete computational system. A specific application of this thinking is the fixed point theorem for continuous endofunctions on dcppos (pointed dcpos), which allows one to model recursion [135]. Subsequently, Scott showed that a categorical fixed point principle can be applied to the category **DCPO** itself, to obtain recursive types. For instance, he constructed a nontrivial dcpo D for which the continuous endofunction space $[D \to D]$ is isomorphic to D itself [136], and thus obtained an abstract model for the untyped λ-calculus, see also [145]. Following Scott's results, the study of dcpos found wide interest in both the computer science and mathematical communities, and has blossomed into a rich research field, see [30] for an overview. Nowadays, this research is known as (Classical) Domain Theory.

Continuous dcpos [36] can be characterised as those dcpos which are well-behaved with respect to the concept of approximation, and the category ω**CONT** of countably-based continuous dcpos is widely regarded as the largest category of dcpos, which may support a reasonable notion of computability (although to the best of the author's knowledge a detailed account has been given only for subcategories of ω**CONT**, see e.g. [26, 141]). A further benefit of (countably-based) continuous dcpos is that they provide a framework which is well-suited for exact real number computation, see [25] for an overview. However, unlike the category of all dcpos, ω**CONT** is not cartesian-closed, so it does not allow higher type constructions. Jung [63] has characterised the category of pointed ωFS-domains as the largest cartesian-closed subcategory of pointed ω-continuous dcpos. In order to model nondeterminism and probabilistic behaviour, powerdomain constructions have been defined [108, 142, 127, 59]. But, although ω**CONT** is closed under the classical powerdomain constructions, in case of the probabilistic powerdomain it is still an open question whether this holds for any of its cartesian-closed subcategories [69]. In another direction, Coquand, Gunter and Winskel [23] have shown that it is possible to give a model for the polymorphic λ-calculus in Classical Domain Theory. However, their model does not obey Reynolds' relational parametricity condition [121], which is necessary for proving important program equivalences. Moreover, it is known that this polymorphic model does not support some of the classical powerdomains comstructions [64].

In the late 1970s, Scott suggested changing the underlying setting for domain theory

from classical to intuitionistic set-theory. The original motivation behind this was to simplify the domain-theoretic approach; a domain was supposed to be just a set with special properties, and every function was supposed to be continuous. This approach is known today as Synthetic Domain Theory. Based on works in realizability semantics [47, 89, 103], the first specific categories for Synthetic Domain Theory were defined in Rosolini's thesis [123]. In subsequent research [48, 153, 34, 104, 51, 85, 86, 118, 101] further categories have been found, and it has been shown that they have very strong closure properties, making them powerful tools for modelling type constructions, such as polymorphic types. Furthermore, using an internal adjoint functor theorem, one can show that they are closed under suitable powerdomain constructions. However, the change to an intuitionistic setting involves abstract and nontrivial constructions, and some of the classical reasoning principles are lost.

A very general and classical approach to semantics was proposed by Michael Smyth [143, 144]. He compared computational concepts with topological ones. A datatype τ has a set of semi-decidable (or observable [1]) properties, which is closed under unions and finite intersections, resembling the properties of a topology. Moreover, a program from τ to τ' transfers semi-decidable properties of τ' to semi-decidable properties of τ, resembling the property of a continuous map. Thus Smyth proposed the following dictionary: a datatype τ translates to a topological space X_τ, a semi-decidable (or observable) property P of τ translates to an open subset U_P of X_τ, and a program accepting values of type τ and returning values of type τ' translates to a continuous map $X_\tau \to X_{\tau'}$. One quickly notices small mismatches in this analogy, in particular there is no convincing argument why every continuous function should correspond to a computable program, but nevertheless, the simplicity of this approach is appealing. Furthermore, with the interpretation of open sets as observations, the specialization order becomes indeed an order of information, as advocated by Scott. One of the challenges of the topological approach is to find a category of topological objects suitable as a basis for denotational semantics. A major obstacle is that the category **Top** of all topological spaces and continuous maps is not cartesian-closed, hence cannot provide a function type construction.

Let us remark that Classical Domain Theory is a specific instance of Smyth's ideas, which is able to resolve the problem of constructing function types, but it has other disadvantages, as we have remarked above. Another topological framework for denotational semantics is given by stably compact spaces, studied by Achim Jung, Matthias Kegelmann and Andrew Moshier [66]. The benefits of the stably compact framework are that it contains important classes of topological spaces, such as FS-domains or compact Hausdorff spaces [66], and it is well-suited for modelling nondeterministic and probabilistic computations [65, 99]. Perhaps its most significant benefit is that it has a logical counterpart along the lines of Abramsky's Domain Theory in Logical

Form [1, 68, 70, 99]. However, the category of stably compact spaces is not cartesian-closed, hence the stably compact framework does not present an intrinsic function type construction. Further topological frameworks for denotational semantics are Martin Escardó's Synthetic Topology [27], where Smyth's ideas are extended to give computational meaning to concepts like compactness, and Paul Taylor's Abstract Stone Duality [15, 154, 155], which aims at providing an axiomatic framework for general mathematics, based, amongst other ideas, on Smyth's dictionary.

Several cartesian-closed categories of topological spaces have been applied in algebraic topology, for instance the category of compactly-generated spaces [20, 150] or the category of sequential spaces [32, 46, 56]. In the late 1990s, however, Scott [137, 14] proposed to use a cartesian-closed supercategory of **Top** for denotational semantics, namely the category of equilogical spaces, **Equ**, which are topological spaces equipped with an equivalence relation. The motivation for this proposal was that **Equ** also has a countably-based counterpart ω**Equ** which is cartesian-closed and closely related to realizability semantics, see [12]. In 1998, Matías Menni and Alex Simpson [90] began to study subcategories of so-called topological objects in ω**Equ**. They developed the concept of an ω-projecting quotient map, which is a special kind of topological quotient map, and it turned out that **PQ**, the category of equilogical spaces whose quotient map is ω-projecting, is the largest subcategory of ω**Equ** having a purely topological counterpart. Remarkably, **PQ** turns out to be a full reflective exponential ideal of ω**Equ**, see *op.cit.*

At around the same time, Matthias Schröder [129] studied topological spaces in Weihrauch's TTE framework [165]. Schröder found the category **AdmRep**, of topological spaces with admissible quotient representation as the maximal category of topological spaces to which the TTE approach can be applied [130]. Andrej Bauer, a PhD-student of Scott at that time, was aware of the works of Menni/Simpson and Schröder, and showed that the categories **PQ** and **AdmRep** are in fact equivalent [13] (modulo the T_0-property). Shortly after that, Schröder succeeded in showing that any quotient of a countably-based space can be obtained as an ω-projecting quotient of a countably-based space, thus the topological counterpart of **PQ** is in fact the category **QCB**, see [90, 129].

Motivated by these discoveries, Simpson [138] started an extensive study of **QCB**. He quickly found out that **QCB** is equivalent to the category of extensional objects in the realizability model over Scott's graph model $\mathcal{P}\omega$, giving it a very rich categorical structure. Moreover, he identified a category of dcpo-like objects in **QCB** with an intrinsic topological characterisation, and called them topological predomains [138, 10]. The category **TP** of these topological predomains turns out to be cartesian-closed as well and to be equivalent to the category of complete extensional objects over $\mathcal{P}\omega$, a category that has already arisen in Synthetic Domain Theory [117].

Together with Martín Escardó and Jimmie Lawson, Simpson [28] examined topological properties of **QCB**. One main result of their research was that **QCB** inherits its rich structure from a wide range of cartesian-closed subcategories of **Top**, amongst them the categories of compactly generated spaces and sequential spaces, where the latter case was already shown in [90]. This is remarkable, as product and function space constructions in general differ in these categories. More recently, Schröder [132] has shown that there is also an embedding of **QCB** into the category of filter spaces [46], which preserves the cartesian-closed structure. These results show empirically that **QCB** lies at the core of any approach towards a topological framework with function types.

In a subsequent research programme on "Topological Domain Theory", Simpson was first joined by Schröder and then the author, to investigate the application of **QCB** and its subcategory **TP** to computational phenomena. In this thesis we recall the categorical properties of the categories of Topological Domain Theory, and outline their embeddings into the categories of equilogical spaces, compactly-generated spaces and sequential spaces, as well as the connection to TTE. We then show how computational effects can be modelled in **QCB** and its subcategories, and finally give a detailed account of how Topological Domain Theory fits into the realizability model over $\mathcal{P}\omega$. We also give a thorough comparison of the constructions in Topological Domain Theory and their counterparts in Classical Domain Theory. In addition to presenting our own contributions to this programme, the more general aim of this thesis is to give a complete account of the developments in Topological Domain Theory. Thus, in addition to the technical contributions we make, this thesis is intended to serve as a comprehensive reference for Topological Domain Theory. All results and proofs to which we did not explicitly contribute are explicitly flagged as such.

1.3 Overview

Next, we give a chapter by chapter overview of the thesis.

Chapter 2: This chapter surveys published work on the categorical structure of the category of topological quotients of countably-based spaces. It begins by following the steps of Menni and Simpson [90], and recalling how **QCB** is characterised as the largest full topological subcategory of Scott's ω-equilogical spaces [14], and that it is a full reflective exponential ideal in ω**Equ**. After that we mention some results of Escardó, Lawson and Simpson [28], showing that **QCB** inherits its structure from the important cartesian-closed categories of compactly-generated and sequential spaces, and that it can be characterised via countable pseudobases in each case. The sequential part of this follows from Schröder's thesis [129]. Finally, we outline how Schröder was led to **QCB** in *op.cit.* as an important class of topological spaces in Weihrauch's TTE framework

[165].

Chapter 3: In chapter 3, we present joint work with Schröder and Simpson [9] on the categorical structure of the categories of topological predomains and topological domains. In particular, we show how recursion and recursive types can be modelled in Topological Domain Theory, by introducing the categories **TP** and **TD** of topological predomains and topological domains [138, 9]. Topological predomains are those qcb-spaces which are monotone convergence spaces [36] (or d-spaces in [166]). We prove that **TP** is a full reflective exponential ideal in **QCB**, and that the reflection **QCB** → **TP** is simply given by the restriction of the monotone convergence reflection of topological spaces [166] to qcb-spaces. This is done by considering a corresponding reflection in the supercategories of compactly-generated and sequential spaces. A result of Schröder and Simpson [134] then shows that the reflection functor **QCB** → **TP** preserves countable products. After that, we recall the definitions of the categories **TD** of topological domains and **TD**$_\perp$ of strict maps between topological domains [9, 10]. We prove **TD** to be a full exponential ideal of **TP** and **TD**$_\perp$ to be a symmetric monoidal closed subcategory of **TP** with a reflection functor **TP** → **TD**$_\perp$. Moreover, we prove a fixed point theorem for **TD**, showing that it can model recursion, and that the category **TD**$_\perp$ is algebraically compact [33] and hence has solutions for recursive domain equations along the lines of [29]. Finally we compare the categorical structure of these categories with the structure of categories in Classical Domain Theory. The category ω**CONT** of ω-continuous dcpos embeds into **TP**, and we show that the restriction of this embedding to ω**FS**$_\perp$ [63], the largest cartesian-closed category of pointed ω-continuous dcpos, preserves the cartesian closed structure. Furthermore, we give an example of two continuous dcpos for which the **TP**-function space between them does not carry the Scott-topology. In such cases we argue that the function space construction in Topological Domain Theory is preferable to the the one in Classical Domain Theory.

Chapter 4: In chapter 4 we begin to study computational effects in Topological Domain Theory, following our paper [8], which was presented at MFPS 2006 in Genoa. We start by recalling Moggi's monadic approach [97] and its refinement by Plotkin and Power [112], using free algebras. Subsequently, we define parameterised equational theories as an appropriate class of algebraic theories to model computational effects, following the Plotkin/Power approach. We show that free algebras for such theories exist in the categories of compactly-generated and sequential spaces and describe their construction. More specifically, we show that the elements of absolutely free algebras can be constructed in a transfinite inductive process, that the topology of the absolutely free algebra can be constructed in an ordinary ω-inductive process and that the equations can be built in by factorising the space terms by a suitable congruence. Then the main result of this chapter is Theorem 4.4.3, which shows that **QCB** is closed under the construction of finitary qcb-parameterised equational theories in the category

of compactly-generated spaces, and that it is closed under the construction of ω-ary qcb-parameterised equational theories in the category of sequential spaces. This shows that a wide range of computational effects can be modelled in **QCB** via free algebra constructions. Moreover, our approach to showing the existence of free algebras in **QCB** yields a direct and inductive construction of the free algebras, using only basic set-theoretic and topological principles. We conclude this chapter by showing that the monads induced by the free algebra constructions are in fact strong.

Chapter 5: Chapter 5 extends the results of the previous chapter to the categories of topological predomains and topological domains with strict maps. This follows our article [7], which has been submitted to a special journal issue for selected articles from MFPS 2006. In particular, we show that the reflection functor **QCB** $\to$ **TP** lifts to a functor on the corresponding categories of algebras and use this to prove the existence of free algebra functor for ω-ary tp-parameterised equational theories in the category **TP**. Furthermore we present a free algebra construction for the category **TD**$_\perp$, which yields that a wide range of computational effects can be modelled along the lines of Plotkin and Power in the categories of Topological Domain Theory. We then show that parameterised equational theories are a genuine generalisation of ordinary inequational algebraic theories, which have been considered by Abramsky and Jung [2] in Classical Domain Theory. Subsequently we compare the constructions of free algebras for parameterised equational theories in Topological and Classical Domain Theory, showing that for a wide class of theories the constructions coincide in both frameworks on those topological (pre)domains which carry the Scott-topology [7]. In particular, this yields that the constructions of the classical powerdomains for nondeterminism, i.e. the upper, lower and convex powerdomain constructions, can be extended to all topological predomains. A more subtle investigation shows that such a generalisation result also holds for the Jones/Plotkin probabilistic powerdomain [59]. Finally, we give a counterexample, showing that a topological probabilistic powerspace construction given by Heckmann [41] does not preserve qcb-spaces in general, and hence is not an alternative construction for modelling probabilistic computations in Topological Domain Theory. The counterexample is obtained using the space of Gruenhage and Streicher [38], who showed that **QCB** is not closed under the topological construction of sobrification.

Chapter 6: In chapter 6, we analyse the connections of **QCB** and its subcategories to the realizability model over Scott's graph model $\mathcal{P}\omega$ [136]. The first section of this chapter was already done in the author's Diplomthesis [6]; the second part is joint with Simpson and has not appeared before. It is known [90, 12] that the category of ω-equilogical spaces is equivalent to **Asm**($\mathcal{P}\omega$), the category of assemblies over $\mathcal{P}\omega$. Thus, the results of chapters 2 and 3 yield that the categories of Topological Domain Theory appear as subcategories of **Asm**($\mathcal{P}\omega$). We start the chapter by characterising the corresponding subcategories in terms of the realizability model. From work of Hyland

[47], it is known that $\mathbf{Asm}(\mathcal{P}\omega)$ is a full reflective subcategory of the realizability topos over $\mathcal{P}\omega$, which is a model of intuitionistic set-theory. It follows that Topological Domain Theory has a counterpart in Synthetic Domain Theory. In the second part of the chapter, we give internal characterisations to the synthetic counterparts of the categories of Topological Domain Theory. We then show that these categories are internally complete and weakly small, and argue that this structure is powerful enough to model polymorphic types [119]. Finally, we apply an internal adjoint functor theorem to the synthetic version of Topological Domain Theory, and show that these categories have free algebras for a wide class of algebraic theories, generalising the parameterised equational theories of chapters 4 and 5. However, this time the existence follows from abstract principles, hence we do not gain the same direct construction of the free algebras as before.

Chapter 7: We conclude the thesis by giving a summary of the obtained results, and an overview over open problems and future work in this project.

1.4 Prerequisites

The reader of the thesis is expected to be familiar with the following subjects.

Topology: The reader is expected to know the definitions of topological spaces, open and closed subsets, bases and neighbourhoods, continuous maps, subspaces and subspace inclusions, topological quotients and topological quotient maps, discrete and indiscrete spaces, and the separation axioms T_0–T_2 (Hausdorff). As an introduction to this subject we recommend [144].

For simplicity, we identify a topological space $(X, \mathcal{O}_X)$ usually just with its underlying set X. In cases where ambiguity may occur, we use sub- and superscripts to avoid these, for instance $\mathbb{N}$ denotes the natural numbers (including 0) with the discrete topology, and $\mathbb{N}^\uparrow$ the natural numbers with the topology whose open sets are of the form $\{n, n+1, n+2, \dots\}$; similarly $\mathbb{R}$ denotes the real numbers with the Euclidean topology, and $\mathbb{R}^\uparrow$ the real numbers with the topology of lower semi-continuity, i.e. generated by the positively unbounded open intervals (a, ∞). By $\mathbf{1}$ we denote a chosen one-point space, and by $\mathbb{S}$ Sierpinski space, the space of two points $\{\bot, \top\}$ where the singleton $\{\top\}$ is open and $\{\bot\}$ is not.

We often talk about *countably-based spaces*, by which we mean spaces for which the topology has a countable base.

A *compact* space is a space satisfying the Heine-Borel property, i.e. a space for which every open cover has a finite subcover. Accordingly, a subset of a topological space X is compact if it satisfies the Heine-Borel property when equipped with the subspace topology. Notice that we do not assume a compact space to be Hausdorff here; in the literature our notion of compactness is sometimes referred to as *quasi-compact*.

The *specialization order* $\sqsubseteq$ on a topological space X is defined as $x \sqsubseteq y$ if every open $U \subseteq X$ containing x also contains y. It is a straightforward observation that the specialization order is in general a pre-order on X; it is a partial order if and only if X is a T_0-space, and it is discrete if and only if X is a T_1-space.

For an element $x \in X$, the *closure* of x is the smallest closed subset of X containing x. It can be described with the specialization order as $\downarrow x := \{y \in X \mid y \sqsubseteq x\}$. The (open) *neighbourhood filter* of x is the set $\mathcal{U}_x := \{U \in \mathcal{O}_X \mid x \in U\}$.

Recall that a non-empty set $\mathcal{F}$ of open subsets of X is called a *filter*, if it does not contain the empty set $\emptyset$ and, if $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$. A filter $\mathcal{F}$ is called *completely prime*, if additionally $\bigcup_{i \in I} U_i \in \mathcal{F}$ implies the existence of some $i_0 \in I$ with $U_{i_0} \in \mathcal{F}$. Any neighbourhood filter is a completely prime filter, and if every complete prime filter is the neighbourhood filter of a unique element of X, then X is called *sober*. For any space X, the *sobrification $Sob(X)$* is the space of complete prime filters of X equipped with the topology given by sets of the form

$$\Box U := \{\mathcal{F} \in Sob(X) \mid U \in \mathcal{F}\},$$

for $U \in \mathcal{O}_X$, see e.g. [36].

Category Theory: A basic category-theoretic background is expected, and can found for instance in [87]. In particular we use the following concepts: category, object and morphism, isomorphism, (regular) monomorphism and (regular) epimorphism, functor, product and coproduct, equaliser and coequaliser, initial and terminal object, adjoint functors including Freyd's Adjoint Functor Theorem, Kleisli-category and Eilenberg-Moore algebras, cartesian closedness and the λ-calculus. In the final parts of Chapter 6, we also use some topos theory, in particular the internal language of a topos. As references for this, we recommend [80, 105].

Category theory has turned out to be very useful for semantics, as the usual type constructions, like product, sum or function types have category-theoretic counterparts, and this can furthermore be generalised to recursive and polymorphic types. Consequently, we make use of these abstract constructions, and present the framework of Topological Domain Theory in the language of category theory.

Classical Domain Theory: By Classical Domain Theory we refer to the study of the category **DCPO**, of dcpos and continuous maps, and its subcategories. It is based on the work of Dana Scott [135, 136], and has been studied extensively over the last decades by many computer scientists and mathematicians, see [30] for an overview. The basic definitions we need is that of a dcpo, the Scott topology, continuous maps between dcpos, way-below order $\ll$, bases and continuous dcpos, and powerdomain constructions.

A dcpo with a least element is called a *dcppo* (for <u>d</u>irected <u>c</u>omplete <u>p</u>ointed <u>p</u>artially

ordered set), and **DCPPO** denotes the category of dcppos and continuous maps, whereas **DCPPO**$_\perp$ stands for the category of dcppos and strict continuous maps, i.e. continuous maps which additionally preserve the least element. It is known that the categories **DCPO** and **DCPPO** are cartesian-closed, but **DCPPO**$_\perp$ is not; it is symmetric monoidal closed.

The category of continuous dcpos, respectively continuous dcppos, and continuous maps between them is denoted by **CONT**, respectively **CONT**$_\perp$. Continuous dcpos with a countable basis are called ω-continuous dcpos, and they form the category ω**CONT**, their pointed counterpart the category ω**CONT**$_\perp$.

1 Introduction

2 The framework QCB

In this chapter we establish the category **QCB**, of topological quotients of countably-based spaces and continuous maps between them, as an ideal framework for Smyth's dictionary of datatypes as topological spaces [144]. The well-suitedness of **QCB** has three reasons: firstly, it appears as a full cartesian-closed subcategory of Scott's category of ω-equilogical spaces, see [90], and thus a notion of computability may be developed here, along the lines of [12]; secondly, it includes all computationally interesting topological spaces and the categorical constructions in **QCB** are universal, in the sense that they are inherited from a wide range of cartesian-closed subcategories and supercategories of **Top**, see [28]; thirdly, it arises as the largest class of topological spaces which have admissible representations in Weihrauch's Type Two Theory of Effectivity (TTE) [165], hence it is the appropriate notion of a topological space for computable mathematics, see [129].

For an overview, consider the following categorical roadmap, where all arrows are functors, upwards pointing arrows are inclusion functors, and $\cong$ denotes equivalence of categories.
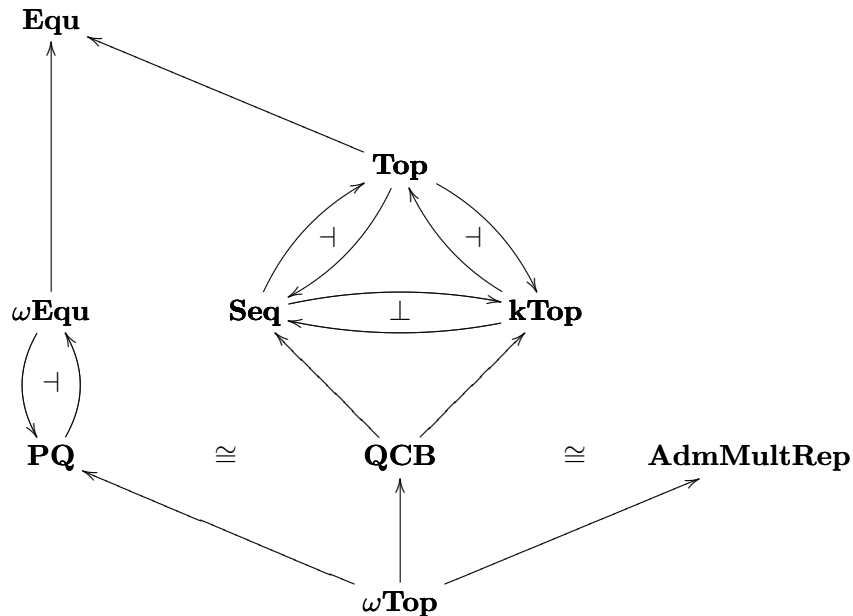
(In fact, the functor **Seq** $\rightarrow$ **kTop** in the above diagram is also an inclusion functor, but we give the diagram as it is, since we want to distinguish the inclusions of **QCB**

into **Seq** and **kTop**.)

In the first section we obtain the left-hand side of this diagram, along the lines of Menni and Simpson's [90]. In particular, we identify the largest topological subcategory **PQ** [90] of Scott's category of ω-equilogical spaces [137], which contains all countably-based spaces. Moreover, we outline that this category is a full reflective exponential ideal in ω**Equ** and that it is equivalent to **QCB**.

We then establish the middle part of the diagram, by mentioning results of Escardó, Lawson and Simpson's [28], which shows that **QCB** embeds into **kTop**, the category of compactly-generated spaces, and **Seq**, of sequential spaces. Both, **kTop** and **Seq**, are cartesian-closed categories of topological spaces, and in both cases the cartesian-closed structure is inherited by **QCB**. This is a remarkable result, since in general the construction of function spaces differs in **kTop** and **Seq**. Moreover, we mention that in both cases, **QCB** can be characterised as the class of objects which have a countable pseudobase. The results for the connection to sequential spaces also follow from Schröder's thesis [129].

Finally, we outline how Schröder [129] identified **QCB** as the largest class of topological spaces which have admissible representations in Weihrauch's TTE [165]. This shows that **QCB** fits nicely into an established framework for computable analysis, and hence is also an attractive category for semantics from a purely mathematical point of view.

2.1 Equilogical spaces

Equilogical spaces were introduced to denotational semantics by Dana Scott [137, 14] with the purpose of providing a framework with strong connections to Smyth's dictionary [144], while also allowing the construction of function spaces. The underlying idea of the equilogical framework is that topological spaces serve as representations for datatypes, providing names for the data elements. The formal definition is given as follows, where we drop Scott's original T_0-restriction.

Definition 2.1.1. An *equilogical space* is a tuple $(X, \sim_X)$, where X is a topological space and $\sim_X$ an equivalence relation on X. An *(equilogical) morphism* between equilogical spaces $(X, \sim_X)$ and $(Y, \sim_Y)$ is a function between the quotient sets $f : (X/\sim_X) \to (Y/\sim_Y)$ for which there exists a (not necessarily unique) continuous

map $\widehat{f} : X \to Y$ making the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \widehat{f}\ } & Y \\
\downarrow & & \downarrow \\
X/\!\sim_X & \xrightarrow[f]{} & Y/\!\sim_Y
\end{array}
$$

The relation preserving map $\widehat{f}$ is called an *equivariant map* for f.

Equilogical spaces and morphisms between them form the category **Equ**.

Notice that equilogical morphisms are continuous when the quotient sets are equipped with the quotient topologies. Thus we may consider the quotients as topological spaces and the morphisms as special continuous maps. We call the space X, underlying the equilogical space $(X, \sim_X)$, its *representing space*. In contrast to the category of topological spaces, **Equ** does provide function spaces, as the following Theorem, due to Bauer, Birkedal and Scott [14], shows.

Theorem 2.1.2. *The category* **Equ** *is cartesian-closed and has countable limits and colimits.*

Proof. The original proof of cartesian closure for equilogical spaces such that the representing space satisfies the T_0-axiom is Theorem 3.13 of [14]. The more general case here follows from section 1 of Rosolini's [125]. For the countable limits and colimits see e.g. Bauer's thesis [12]. □

A fact that makes **Equ** particularly interesting for computer science is that it has a countably-based counterpart, which is also cartesian-closed, see section 2 of Bauer's [13].

Definition 2.1.3. An ω-*equilogical space* is an equilogical space $(X, \sim_X)$ for which X is countably-based. The ω-equilogical spaces form the full subcategory ω**Equ** of **Equ**.

Theorem 2.1.4. *The category* ω**Equ** *is cartesian-closed with countable limits and colimits, and this structure is inherited from* **Equ**.

As a framework for denotational semantics, ω**Equ** works as follows. Datatypes are interpreted by the quotients $X/\!\sim_X$, where each element $a \in X/\!\sim_X$ has a set of names $[a] \subseteq X$. Programs are modelled by continuous functions on the names, with soundness being ensured by the equivariant condition.

There is an obvious functor $Q : $ **Equ** $\to$ **Top**, associating to an equilogical space $(X, \sim_X)$ the topological quotient $X/\!\sim_X$, and to a morphism $(X, \sim_X) \to (Y, \sim_Y)$ the corresponding continuous map between the quotients. This functor is not full, as not

all topological maps between the quotients can be lifted to equivariant maps between the representing spaces. This observation motivates the following definition, taken from [90].

Definition 2.1.5. A *topological subcategory* of **Equ** is a subcategory $\mathbb{C}$ of **Equ** for which the restriction $Q : \mathbb{C} \to$ **Top** is full.

With Smyth's dictionary in mind, we are particularly interested in topological subcategories of ω**Equ**, as here a notion of computability may be developed along the lines of [12]. An obvious such category is the full subcategory consisting of objects of the form $(X, =)$, i.e. objects for which the equivalence relation is the identity. Under the functor Q this category becomes equivalent to ω**Top**, the category of countably-based spaces and continuous maps. Unfortunately, it is well-known that this category is not cartesian-closed. However, it turns out that there are larger topological subcategories of ω**Equ** which are cartesian-closed.

Observe that, by definition, a topological subcategory of ω**Equ** has the property that for any two objects $(X, \sim_X)$ and $(Y, \sim_Y)$, every continuous map between the quotients $X/\sim_X \to Y/\sim_Y$ can be lifted to an equivariant map $X \to Y$. This leads to the following definition, again from [90].

Definition 2.1.6. Let X, Y be topological spaces. An ω-*projecting map* $q : X \twoheadrightarrow Y$ is a continuous map, such that for every countably-based space A, and every continuous map $f : A \to Y$, there exists a (not necessarily unique) continuous $\widehat{f} : A \to X$, as in:

$$
\begin{array}{ccc}
A & \dashrightarrow^{\widehat{f}} & X \\
& {\scriptstyle f} \searrow & \downarrow {\scriptstyle q} \\
& & Y
\end{array}
$$

It is called an ω-*projecting quotient map*, if in addition it is a topological quotient map.

By **PQ** we denote the full subcategory of ω**Equ** consisting of the equilogical spaces $(X, \sim_X)$, for which the topological quotient $X \twoheadrightarrow X/\sim_X$ is ω-projecting.

Menni and Simpson have shown, see Theorem 1 of [90], that **PQ** forms a maximal topological subcategory of ω**Equ**. We remark that there exist other maximal topological subcategories of ω**Equ**.

Proposition 2.1.7. PQ *is the largest topological subcategory of ω**Equ** containing all objects of the form $(X, =)$.*

As a rather surprising fact, it turns out that **PQ** inherits a rich structure from ω**Equ**, as the next result shows, which follows from Theorem 2 of *op.cit.* and section 4 of [13].

Theorem 2.1.8. PQ *is closed in* ω**Equ** *under isomorphisms, countable products, countable coproducts, regular subobjects and function spaces.*

We see that **PQ** is a topological subcategory of equilogical spaces, which still has a very rich categorical structure. Moreover, one can address computability issues in **PQ**, by the following considerations. The category ω**Equ** is equivalent to the category of assemblies over Scott's graph model $\mathcal{P}\omega$ [14]. This means that all ω-equilogical spaces can be obtained as subquotients of $\mathcal{P}\omega$, and equilogical morphisms as equivalence classes of continuous endofunctions of $\mathcal{P}\omega$. On $\mathcal{P}\omega$ itself, which is a model of the untyped λ-calculus [136], a notion of computability has been defined e.g. in *op.cit.* From this one can derive a notion of computability on ω**Equ**, hence also on **PQ**. For more details on this subject, we refer the reader to Bauer's thesis [12].

Now the question arises how the topological counterpart of **PQ** can be characterised. Clearly, the corresponding spaces are exactly those topological spaces which arise as ω-projecting topological quotients of countably-based spaces. The following pivotal result, due to Schröder [129], shows that in fact every topological quotient of a countably-based space arises in this fashion.

Theorem 2.1.9 (Schröder)**.** *Every quotient of a countably-based topological space can be obtained as an ω-projecting quotient of a countably-based topological space.*

Proof. In Theorem 3.2.4 of *op.cit.*, Schröder shows that a quotient of a countably-based topological space is a sequential space that has an admissible multirepresentation, and in Theorem 3.4.7 of *op.cit.*, he shows that such spaces can be obtained as ω-projecting quotients of countably-based topological spaces. A slightly less general result for T_0-quotients of countably-based topological spaces is due to Bauer [13]. $\qquad\square$

This result motivates one to consider the category of quotients of countably-based topological spaces and continuous maps as a framework for denotational semantics. In the following definition we also distinguish those objects satisfying the T_0-separation axiom, which play a prominent role in Chapter 6 below.

Definition 2.1.10. The category **QCB** is the category of topological quotients of countably-based topological spaces and continuous maps. Its objects are simply called *qcb-spaces*. The full subcategory of qcb-spaces satisfying the T_0-separation axiom is denoted by **QCB**$_0$, and its objects are called *qcb$_0$-spaces*.

Observe that **QCB**$_0$ is a full reflective subcategory of **QCB**, with the reflection being the well-known T_0-reflection of topological spaces. Now, Theorem 2.1.9 yields the main result of this section.

Theorem 2.1.11. QCB *is equivalent to* **PQ**, *which is a full reflective exponential ideal of* ω**Equ**.

Proof. Observe that by the definition of ω-projecting maps, ω-projecting quotients are unique (up to isomorphism) in ω**Equ**, in the sense that whenever $X/\sim_X$ and $Y/\sim_Y$ are topologically isomorphic and the respective quotient maps ω-projecting, then the isomorphisms $X/\sim_X \leftrightarrow Y/\sim_Y$ can be lifted to equivariant maps $X \leftrightarrow Y$. Thus, Theorem 2.1.9 yields that **QCB** and **PQ** are equivalent categories. The reflection functor ω**Equ** $\to$ **PQ** is obtained by associating to an ω-equilogical space $(X, \sim_X)$ the ω-equilogical space $(Y, \sim_Y)$, where $X/\sim_X \cong Y/\sim_Y$ as topological spaces, and $Y \twoheadrightarrow Y/\sim_Y$ is ω-projecting. $\square$

Corollary 2.1.12. QCB *and* $\mathbf{QCB}_0$ *are cartesian-closed categories which have countable limits and countable colimits.*

Proof. Clear, by Theorems 2.1.8 and 2.1.11 and the straightforward observation that $\mathbf{QCB}_0$ is closed under all the relevant constructions in **QCB**. $\square$

Throughout the thesis, we denote the exponential of X and Y in **QCB** by Y^X; other function spaces have a different notation.

The following result has an interesting application to the interpretation of Smyth's dictionary in **QCB**.

Proposition 2.1.13. *A qcb-space X is hereditarily Lindelöf, i.e. for every family $\{U_i\}_{i \in I}$ there exists a countable subfamily $J \subseteq I$ such that $\bigcup_{j \in J} U_j = \bigcup_{i \in I} U_i$.*

Proof. Immediate from the fact that countably-based spaces are hereditarily Lindelöf, and this property is preserved under quotients. $\square$

It follows that for the open subsets of a qcb-space, closure under countable unions is sufficient to form a topology. This gives a computational justification for identifying open subsets with semi-decidable properties of a datatype; it appears to be physically plausible for semi-decidable properties to be closed under countable unions, as it should be possible to run the countably many semi-decision processes in parallel (or sequentially by interleaving), to form another semi-decision process.

Those observations, together with the fact that **QCB** supports function type constructions and has access to a notion of computability, indicate that **QCB** is indeed a most suitable framework for denotational semantics in Smyth's sense.

2.2 Compactly-generated spaces

So far, we have seen that **QCB** is a category of topological spaces with rich categorical structure inherited from **Equ**, a cartesian-closed extension of the category of topological spaces. In this and the next section, we outline how **QCB** also lives in and inherits its structure from well-studied cartesian-closed subcategories of **Top**. Moreover, in these

frameworks it arises intrinsically as the full subcategory of objects having a countable *pseudobase*.

The first category we consider is the category of compactly-generated spaces. It goes back to Brown [20] and Steenrod [150], and has since found many applications, for instance in algebraic topology [150, 24] and topological group theory [78, 113]. The results we present are taken from [28], and we do not give the proofs but rather provide references to the corresponding theorems in *op.cit.*

Definition 2.2.1. A subset U of a topological space X, is open in the *compactly-generated topology* on X if for every compact Hausdorff space K and continuous map $p : K \rightarrow X$, the preimage $p^{-1}(U)$ is open in K. We write $k(X)$ for X equipped with the compactly generated topology, and call X *compactly-generated* if $X = k(X)$.

By **kTop** we denote the full subcategory of **Top** whose objects are the compactly-generated spaces.

It is well-known that **kTop** is a full coreflective subcategory of **Top**, the coreflection mapping a space X to $k(X)$. The category **kTop** is complete and cocomplete, with colimits calculated as in **Top**, and limits obtained by coreflecting the limits in **Top**. Moreover, **kTop** is cartesian-closed, with function spaces obtained as follows.

Definition 2.2.2. For topological spaces X, Y, the *compact-open topology* on the set of continuous functions $C(X, Y)$ is generated by subbasic open sets of the form

$$\langle K, U \rangle := \{ f \in C(X, Y) | f(K) \subseteq U \},$$

where $K \subseteq X$ is compact and $U \subseteq Y$ open. We write $[X \rightarrow_{co} Y]$ for $C(X, Y)$ equipped with the compact-open topology.

The following is a standard result, which can be derived from the discussion at the beginning of Section 5, Remark 5.20 and Proposition 5.21 in [28].

Proposition 2.2.3. kTop *is cartesian-closed with the exponential* $X \Rightarrow_k Y$ *given by* $k([X \rightarrow_{co} Y])$.

Let us remark that the coreflection in this construction is necessary, as $[X \rightarrow_{co} Y]$ need not be compactly-generated. For instance the function space $[2 \rightarrow_{co} Y]$ is isomorphic to the topological product Y^2, and it is well-known that there exist compactly-generated spaces X for which the compactly-generated product $X \times X$ carries a strictly finer topology than the topological product.

The following Theorem due to Jimmie Lawson can be found as Theorem 4.7 in [28], and is of great importance in the subsequent chapters.

Theorem 2.2.4 (Lawson). *Every dcpo equipped with the Scott-topology is a compactly-generated space.*

Proof. [Lawson] Suppose X is a dcpo carrying the Scott-topology. As the Sierpinski space $\mathbb{S}$ is compactly generated, any continuous map $\mathbb{S} \to X$ is also continuous when considered as a map $\mathbb{S} \to k(X)$. It follows that X and $k(X)$ have the same specialization order, and so any open subset of $k(X)$ is upper in the specialization order of X. Thus we only need to show that an upper set in X which is open in $k(X)$ is Scott-open, or equivalently that a lower set in X which is closed in $k(X)$ is Scott-closed, i.e. closed under suprema of directed subsets. Let L be such a lower set and $D \subseteq L$ be directed. If $\bigvee^{\uparrow} D \notin L$, we might assume D to be a counterexample of least cardinality. By the Theorem of Iwomura, see Theorem 1 of [88], one can write D as a well-ordered (by inclusion) family of directed subsets of lower cardinality. Thus, the supremum of each of these subsets must belong to L, and these suprema form a well-ordered subset C_0 of L. Close up this well-ordered chain C_0 under suprema to obtain a chain C closed under suprema in X which is also well-ordered. (Let A be a nonempty subset of C. Pick $a_0 \in A$, then $a_1 \in A$ with $a_1 < a_0$ and continue this process. Either it terminates at the least element of A or one obtains a strictly decreasing sequence in A. In the latter case we choose for each a_i an element $b_i \in C_0$ with $a_{i+1} < b_i \leq a_i$. But then the sequence (b_i) is infinite and decreasing in C_0, violating the fact that it is well-ordered.)

Now the chain C is complete in its order, and hence it is a compact Hausdorff space in its order topology (see page 217 in [36]). Since it is closed under suprema in X, the embedding $C \hookrightarrow X$ into X under the Scott-topology is continuous, and hence a probe. Since the inverse image of L contains C_0, a dense subset of C, we conclude $C \subseteq L$. But C contains the supremum of C_0 which is the supremum of D, by construction. $\qquad\square$

An interesting application of the above results is the following, which appears as Corollary 5.16 in [28].

Lemma 2.2.5. *For any compactly-generated space X, the function space $X \Rightarrow_k \mathbb{S}$ is isomorphic to $\mathcal{O}_X$, the lattice of open subsets of X, equipped with the Scott-topology for the inclusion order.*

An easy observation, see Proposition 7.1 in [28], shows that every qcb-space is compactly-generated. More remarkable is that **QCB** inherits its structure from **kTop**, which follows from Corollary 7.3 and Remark 7.4 of [28].

Theorem 2.2.6. **QCB** *is a full subcategory of* **kTop**, *and the inclusion functor preserves countable limits, countable colimits and the cartesian-closed structure. Moreover, for an equalizer diagram in* **kTop**:

$$X \longrightarrow Y \rightrightarrows Z$$

it holds that X is in **QCB** *whenever Y is.*

It turns out that **QCB** can be characterised intrinsically in **kTop** using the following definition of a pseudobase, due to E. Michael [91].

Definition 2.2.7. A *Michael pseudobase* for a topological space X is a family $\mathcal{B}$ of (not necessarily open) subsets of X, such that whenever $K \subseteq U$ with K compact and U open, then there exist finitely many $B_1, \ldots, B_k \in \mathcal{B}$ with $K \subseteq \bigcup_{i=1}^{k} B_i \subseteq U$.

Notice that, although a base for the topology of X is always a Michael pseudobase for X, a Michael pseudobase, in general, may give very little information about the topology. For instance, the powerset $\mathcal{P}(X)$ is always a Michael pseudobase for X, in fact the largest pseudobase for X.

It is a fairly straightforward observation that a space has a finite Michael pseudobase, if and only if it has a finite topology, since every open is the union of the elements of the pseudobase it contains. Thus, the first interesting class of topological spaces, in terms of minimal Michael pseudobases, is given by spaces which can be equipped with a countable Michael pseudobase. For this we have the following result, which was proved in Theorem 6.10 of [28], where Proposition 3.1.14 and Theorem 3.2.4 of [129] were used.

Theorem 2.2.8. *A compactly-generated space has a countable Michael pseudobase if and only if it is a qcb-space.*

It follows that the category of compactly-generated spaces with countable Michael pseudobase is **QCB**, giving it an intrinsic characterisation inside **kTop**. Furthermore, this characterisation via the size of a Michael pseudobase yields the following useful result.

Proposition 2.2.9. *A qcb-space X is hereditarily separable, i.e. for any subset $A \subseteq X$ there exists a countable subset $C \subseteq A$ such that C is dense in the subspace topology on A.*

Proof. Let $\mathcal{B}$ be a countable Michael pseudobase for X. Then, as every singleton $\{x\} \subseteq X$ is compact, we get that $\bigcup \mathcal{B}$ covers X. Moreover, we get that $\mathcal{B}' := \{B \in \mathcal{B} \mid B \cap A \neq \emptyset\}$ is countable. For $B \in \mathcal{B}'$ choose $x_B \in B \cap A$. We claim that $\{x_B\}_{B \in \mathcal{B}'}$ is dense in A. To see this let $U \subseteq A$ be non-empty, say $x \in U$, and open in the subspace topology of X. Then there exists $U' \in X$ with $U' \cap A = U$. Moreover, there exist finitely many $B_1, \ldots B_k \in \mathcal{B}$ with $x \in \bigcup_{i=1}^{k} B_i \subseteq U'$, and so we have

$$x \in \bigcup_{i=1}^{k} (A \cap B_i) = A \cap \bigcup_{i=1}^{k} B_i \subseteq U.$$

Surely there exists some $1 \leq i_0 \leq k$ with $B_{i_0} \cap A \neq \emptyset$ and we get $x_{B_{i_0}} \in U$, as required. $\qquad\square$

We finish this section by giving a condition under which the **QCB**-function space construction preserves countably-based spaces. The central role is played by the locally compact spaces.

Definition 2.2.10. A topological space X is *locally compact* if for every $x \in X$ and open U containing x, there exists a compact neighbourhood K of x, such that $K \subseteq U$.

In Corollary 6.11 of [28] it was shown that a locally compact space is a qcb-space if and only if it is countably-based.

Proposition 2.2.11. *Let X, Y be countably-based spaces and X be locally-compact. Then the compactly-generated function space $X \Rightarrow_k Y$ is countably-based.*

Proof. In [81], Lambrinos and Papadopoulos show that in this case the compact-open topology on $[X \to_{co} Y]$ is countably-based. Thus the result follows with Proposition 2.2.3 and Theorem 2.2.6, and the fact that countably-based spaces are compactly-generated. $\square$

2.3 Sequential spaces

In this section, we study how **QCB** lives inside the category **Seq** of sequential spaces. Sequential spaces were introduced formally by Franklin [32], who classified them as those topological spaces, in which all properties are determined by sequence convergence. The sequential spaces form a full reflective cartesian-closed subcategory of the category of Kuratowski limit spaces [77], and their categorical structure has been studied, for instance, by Hyland [46] and Johnstone [56].

Like **kTop**, **Seq** includes **QCB** as a full subcategory, as can be easily seen from Franklin's characterisation theorem (Corollary 1.14 in [32] and recalled below as Theorem 2.3.2), and again **QCB** inherits its categorical structure under this inclusion. Moreover, the view of qcb-spaces as sequential spaces has turned out to be very useful, as the sequential constructions seem to be easier to handle than their compactly-generated counterparts, see e.g. [129, 134]. This fact allows us to generalise some of the obtained results about qcb-spaces, and it helps us in Chapters 4 and 5 below, when we study free algebra constructions in **QCB** and its subcategories. Let us start by recalling the definition of sequential spaces.

Definition 2.3.1. A subset V of a topological space (or more generally an L-space in the sense of [46]) X is *sequentially open* if, given $x \in V$ and a sequence $(x_i)_{i \in \mathbb{N}}$ converging to x, the sequence $(x_i)_{i \in \mathbb{N}}$ is eventually in V, i.e. there exists $i_0 \in \mathbb{N}$ such that for all $i > i_0$, it holds that $x_i \in V$. The sequentially open subsets of X form the *sequential topology* of X. By $Seq(X)$ we denote X with the sequential topology and call X a *sequential space* if $X = Seq(X)$.

By **Seq**, we denote the full subcategory of **Top** consisting of the sequential spaces.

Clearly, every open subset is sequentially open, hence the sequential topology on X is always finer than the original one. Indeed, like **kTop**, **Seq** is a full coreflective subcategory of **Top**. It is complete and cocomplete with colimits computed as in **Top**, and limits are obtained by coreflecting the limits in **Top**. Moreover, it is not hard to see that the sequential topology of a space is finer than the compactly-generated one, see for instance [28], hence the **Seq** is a subcategory of **kTop** and the coreflection **Top** → **Seq** cuts down to a coreflection functor **kTop** → **Seq**.

The following characterisation theorem of sequential spaces is due to Franklin, and appears as Corollary 1.14 in [32].

Theorem 2.3.2. *For a topological space X the following are equivalent:*

(i) X is a sequential space,

(ii) X is a topological quotient of a first-countable space, i.e. a space in which every point $x \in X$ has a countable neighbourhood base,

(iii) X is a topological quotient of a metric space.

Considering characterisation (ii), it seems that **Seq** is a very natural topological supercategory of **QCB**. As mentioned above, **Seq** is also cartesian-closed, with a very intuitive function space construction, which is given by the topology of continuous convergence, which is defined in §20.VI of [77]. In *op.cit.*, it is also shown that this function space satisfies the properties of an exponential in **Seq**. A formal statement of the following result can be found in [46].

Proposition 2.3.3. *Let X, Y be sequential spaces, and $C(X, Y)$ the set of continuous functions. Define a convergence relation on $C(X, Y)$ as follows. $(f_i)_{i \in \mathbb{N}}$ converges to f, if for all $(x_i)_{i \in \mathbb{N}}$ converging to x in X, the sequence $(f_i(x_i))_{i \in \mathbb{N}}$ converges to $f(x)$ in Y. Then the sequential function space $X \Rightarrow_{seq} Y$ is given by $C(X, Y)$ equipped with the sequential topology induced by this convergence relation.*

Schröder has shown that the sequential function space construction can also be given similarly as in **kTop**. The proof of the following Proposition can be found as Lemma 4.2.2 in [129].

Proposition 2.3.4. Seq *is cartesian-closed with exponential $X \Rightarrow_{seq} Y$ given by $Seq([X \to_{co} Y])$.*

It follows that for sequential spaces X, Y, the sequential function space $X \Rightarrow_{seq} Y$ has in general a finer topology than the compactly-generated function space $X \Rightarrow_k Y$, and there are examples in which it is strictly finer, see [28].

In any cartesian-closed category, the functor $X \times (-)$ preserves regular epis, and so, since topological quotient maps are closed under composition, it follows that in **kTop**

and **Seq**, finite products of topological quotient maps are again topological quotient maps. In **Seq** this result can be extended to countable products, as the following result shows, which is due to Schröder and Simpson, see Theorem 4.2 in [134].

Theorem 2.3.5. *If for all $n \in \mathbb{N}$, the map $q_n : X_n \to Q_n$ is a topological quotient map between sequential spaces, then the map $\prod_{n \in \mathbb{N}} q_n : \prod_{n \in \mathbb{N}} X_n \to \prod_{n \in \mathbb{N}} Q_n$ is a topological quotient map if the products are those in* **Seq**.

We do not know if a similar result holds for $\mathbb{N}$-indexed products in **kTop**, which is an indication that sequential constructions are simpler than their compactly-generated counterparts. Indeed, infinite product and subspace constructions differ in **Seq** and **kTop**, as examples in [28] show. The construction of finite products however coincides, see Theorem 5.4 in *op.cit.*

The following two results are analogous to Theorem 2.2.6, and follow again from Corollary 7.3 and Remark 7.4 of [28].

Theorem 2.3.6. **QCB** *is a full subcategory of* **Seq**, *and the inclusion functor preserves countable limits, countable colimits and the cartesian-closed structure. Moreover, for an equalizer diagram in* **Seq**:

$$ X \longrightarrow Y \rightrightarrows Z $$

it holds that X is in **QCB**, *whenever Y is.*

It follows that the constructions of function spaces and countable limits in **Seq** and **kTop** coincide on the common subcategory **QCB**. In fact, this result can be transferred to a wide range of cartesian-closed subcategories of **Top**. For a more thorough treatment on this, we refer again to [28].

In fact, we can strengthen the result on equalizers as follows.

Proposition 2.3.7. *A sequentially continuous map $m : X \to Y$ between qcb-spaces, which is a regular mono in* **Seq**, *arises as an equalizer in* **QCB**.

Proof. Let $\nabla 2$ denote the two-element indiscrete space $\{0, 1\}$, and $\chi_X : Y \to \nabla 2$ the characteristic map of X, i.e. $y \mapsto 1$ if and only if $y \in X$. Then the following is an equalizer diagram in **QCB**.

$$ X \xrightarrow{\ m\ } Y \underset{\chi_X}{\overset{1}{\rightrightarrows}} \nabla 2 $$

$\square$

We remark that this does not mean that regular monos in **QCB** are necessarily topological embeddings. The problem is that the subspace topology might not be

sequential and there exist counterexamples, for instance the sobrification of the Grunhage/Streicher counterexample X [38] is a topological subspace of $\mathbb{S}^{\mathbb{S}^X}$ (equipped with the Scott-topology) but it is not sequential.

In the previous section we characterised **QCB** as the full subcategory of **kTop** consisting of those compactly-generated spaces having a countable Michael pseudobase. An analogous notion of pseudobase exists in **Seq**, and again it turns out that **QCB** is exactly the full subcategory of sequential spaces consisting of the objects that have a countable pseudobase. The appropriate definition is due to Schröder [129].

Definition 2.3.8. A *sequential pseudobase* for a topological space X is a family $\mathcal{B}$ of (not necessarily open) subsets of X, such that whenever $(x_i)_{i \in \mathbb{N}}$ is a sequence converging to $x \in U$, then there exist finitely many $B_1, \ldots, B_k \in \mathcal{B}$, containing x, such that $x_i \in \bigcup_{j=1}^k B_j$ for all but finitely many $i \in \mathbb{N}$.

The proof of the following result can be found as Proposition 3.1.14 and Theorem 3.2.4 in [129].

Theorem 2.3.9. *A sequential space has a countable sequential pseudobase if and only if it is a qcb-space.*

We finish this section by mentioning that in **QCB** several notions of compactness agree. The proof of this is quite technical, and was given by Nyikos [16].

Proposition 2.3.10. *For a subset K of a qcb-space X, the following properties are equivalent.*

(i) *K is compact;*

(ii) *K is countably compact, i.e. for any countable family of opens $\{U_i\}_{i \in I}$ with $K \subseteq \bigcup_{i \in I} U_i$, there exists a finite $F \subseteq I$ such that $K \subseteq \bigcup_{i \in F} U_i$;*

(iii) *K is sequentially compact, i.e. every sequence $(x_i)_{i \in \mathbb{N}}$ of elements in K has a subsequence $(x_j)_{j \in \mathbb{N}}$ which converges to some $x \in K$.*

Proof. That (i) and (ii) are equivalent follows from Proposition 2.1.13. The implication (iii) $\Rightarrow$ (ii) is well-known. The implication (ii) $\Rightarrow$ (iii) is the nontrivial part, and was proved by Nyikos in *op.cit.* $\qquad\qquad\square$

In fact, the technique of Nyikos' proof has been adapted by Schröder to show that a topological space has a countable Michael pseudobase if and only if it has a countable sequential pseudobase [131].

2.4 Type Two Theory of Effectivity

We finish this chapter by summarising how Schröder [129] discovered quotients of countably-based spaces as an interesting class of topological spaces in Weihrauch's Type Two Theory of Effectivity (TTE) [165], a widely accepted approach to capturing computability in analysis. TTE provides a theory for modelling computations on sets of uncountable cardinality, using Type-2 Turing machines.

A Type-2 machine is a special kind of Turing machine which can be described as follows. Its underlying alphabet is given by a finite or countably-infinite set with at least 2 elements. It has an input tape, on which it is fed by an infinite sequence of symbols over the given alphabet, and an output tape, on which it produces an infinite sequence of symbols, subject to a given (finite) transition table. The output head can only move forward, and cannot delete already written symbols. If at some point the output head stops writing, the computation is intepreted as having failed. Hence a Type-2 machine gives rise to a physically feasible computation and can be implemented on a conventional computer.

Formally, if Σ denotes the underlying alphabet of a Type-2 machine, then, in a successful computation, the machine transduces an infinite input sequence $x \in \Sigma^\omega$ into an infinite output sequence $f(x) \in \Sigma^\omega$. Thus, the machine gives rise to a partial function $f : \Sigma^w \rightharpoonup \Sigma^\omega$. The behaviour of the output head implies that every finite prefix of the output is computed using only a finite prefix of the input. Thus, a function computed by a Type-2 machine is continuous, if its domain is equipped with the relative Cantor-topology, which is defined as follows, see [165].

Definition 2.4.1. Let Σ be a set, Σ^* and Σ^ω denote the sets of finite, respectively infinite sequences over Σ. Let furthermore $p^{<k} : \Sigma^\omega \to \Sigma^*$ denote the map assigning to an infinite sequence $x \in \Sigma^\omega$, its first k elements, i.e. $x = (x_n)_{n \in \mathbb{N}} \mapsto (x_1, \ldots, x_k)$. The *Cantor-topology* on Σ^ω is generated by sets of the form

$$\{x \in \Sigma^\omega \mid p^{<k}(x) = (y_1, \ldots, y_k)\},$$

for $(y_1, \ldots, y_k)$ ranging over Σ^*.

For a subset $A \subseteq \Sigma^\omega$, the *relative Cantor-topology* on A is the topology of A as a subspace of Σ^w with the Cantor-topology.

A partial map $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$ is called *partial continuous* if it is continuous with respect to the relative Cantor-topology on its domain.

We remark that not every partial continuous function $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$ can be given by a corresponding Type-2 machine. A simple argument for this is that by definition there can only exist countably many different Type-2 machines for a given alphabet Σ, because the machine is uniquely determined by its finite transition table. However,

there are uncountably-many partial continuous maps $\Sigma^\omega \rightharpoonup \Sigma^\omega$.

In TTE, computations on sets of cardinality $\leq |\Sigma^\omega|$ are modelled by representing elements of such sets by names in Σ^ω, and giving a corresponding computable function on the names. Formally, the definition is the following, see also [165, 129].

Definition 2.4.2. A *representation* of a set X is a surjective partial map $\delta : \Sigma^\omega \rightharpoonup X$. A *name* of $x \in X$ is an element $n \in \Sigma^\omega$ with $\delta(n) = x$. We denote the set of all names of x by $N(x)$. The tuple (X, δ) is called a *represented space*.

A partial function $f : \Sigma^\omega \rightharpoonup \Sigma^\omega$, which arises from a Type-2 machine, is called *computable*. A map $f : X \to Y$, between represented spaces (X, δ_X) and (Y, δ_Y), is called *relatively computable* (with respect to δ_X and δ_Y) if there exists a computable map $g : \Sigma^\omega \rightharpoonup \Sigma^\omega$, such that for all names $n \in N(x)$, it holds that $g(n) \in N(f(x))$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
dom(\delta_X) & \xrightarrow{\ g\ } & dom(\delta_Y) \\
\delta_X \downarrow & & \downarrow \delta_Y \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

The map $f : X \to Y$ is called *relatively continuous*, if there exists a corresponding partial continuous map $g : \Sigma^\omega \rightharpoonup \Sigma^\omega$.

Although the notion of relative computability is strictly stronger than the one of relative continuity, there are practical reasons for considering the latter, especially when it comes to investigating if a given representation is computationally appropriate. As an enlightening example, Schröder [129] gives the decimal representation of real numbers, used in ordinary floating point arithmetic, which for general arithmetic purposes is not computationally appropriate; for instance addition $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not relatively continuous with respect to the decimal representation, since $0.99\ldots = 1.00\ldots$, and consequently $0.33\ldots + 0.66\ldots = 1.00\ldots$ which violates the condition that any digit of the output is determined by a finite prefix of the input.

One can generalise the above definition of representations to topological spaces instead of sets.

Definition 2.4.3. A representation of a topological space X is a surjective partial continuous map $\delta : \Sigma^\omega \rightharpoonup X$.

Then the question arises for which representations of topological spaces X and Y, the notions of continuity and relative continuity with respect to a representation coincide. Notice that in general neither implies the other. For instance if X is the indiscrete space consisting of n elements, and Y the discrete space consisting of n elements, then

these spaces can be equipped with the same representation so that the identity map $X \to Y$ becomes relatively continuous, although it is not continuous. Conversely, if X is Σ^ω itself, equipped with the indiscrete topology, then every set-theoretic map $X \to X$ is continuous, but certainly they are not all relatively continuous with respect to the representation given by the identity map.

In order to relate continuity and relative continuity for represented topological spaces, Kreutz and Weihrauch [76] introduced the notion of an admissible representation, which has been generalised by Schröder in [130], to the following:

Definition 2.4.4. Let $\delta : \Sigma^\omega \rightharpoonup X$ and $\delta' : \Sigma^\omega \rightharpoonup X$ be representations of a set X. Then δ is *continuously translatable* to δ', written $\delta \preccurlyeq \delta'$, if the identity map $\mathrm{id}_X : X \to X$ is relatively continuous with respect to δ and δ', i.e. if there exists a partial continuous map $g : \Sigma^\omega \to \Sigma^\omega$ making the following diagram commute:

$$
\begin{array}{ccc}
dom(\delta) & \xrightarrow{\ g\ } & dom(\delta') \\
\delta \downarrow & & \downarrow \delta' \\
X & \xrightarrow[\mathrm{id}_X]{} & X
\end{array}
$$

Let X be a topological space. A representation $\delta : \Sigma^\omega \rightharpoonup X$ of X is called an *admissible representation* of X, if it is continuous with respect to the relative Cantor-topology on $dom(\delta)$, and any other representation $\delta' : \Sigma^\omega \rightharpoonup X$ which is continuous with respect to the relative Cantor-topology on $dom(\delta')$, is continuously translatable to δ.

One of Schröder's main results is Theorem 4 of [130], showing the following.

Theorem 2.4.5. *Let X and Y be topological spaces and $\delta_X : \Sigma^\omega \rightharpoonup X$ and $\delta_Y : \Sigma^\omega \rightharpoonup Y$ be admissible representations. Then a function $f : X \to Y$ is sequentially continuous if and only if it is relatively continuous with respect to δ_X and δ_Y.*

It follows that for sequential spaces X and Y with admissible representations δ_X and δ_Y, the continuous maps $X \to Y$ coincide with the relative continuous maps with respect to δ_X and δ_Y. Thus, the class of sequential spaces which have an admissible representation arises as an important class of topological spaces in TTE. An interesting result for the objects of this class is the following, which is Theorem 7 in [130].

Theorem 2.4.6. *If X is a sequential space and $\delta : \Sigma^\omega \rightharpoonup X$ an admissible representation of X, then $\delta : dom(\delta) \to X$ is a topological quotient map.*

Hence, a sequential space with an admissible representation is a qcb-space. One of the major achievements of Schröder's thesis [129] was to show that conversely every

qcb$_0$-space has in fact an admissible quotient representation. This is shown by using the fact that qcb$_0$-spaces are the sequential T_0-spaces with countable sequential pseudobase, see Theorem 2.3.9, and using the following result, which appeared as Theorem 13 in [130].

Theorem 2.4.7. *A topological space X has an admissible representation if and only if it is a T_0-space with a countable sequential pseudobase.*

Thus, if **AdmRep** denotes the category of admissible representations of sequential spaces and relatively continuous maps between them, we get the following theorem.

Theorem 2.4.8. *The categories* **AdmRep** *and* **QCB**$_0$ *are equivalent.*

It follows that **QCB**$_0$ appears as an important category in TTE; it forms the largest class of topological spaces on which continuity can be modelled faithfully by relative continuity with respect of the Cantor-topology on Σ^ω. In fact, it turns out that the T_0-axiom can be dropped if one extends the above definition of a representation to so-called multirepresentations (the difference being that in a multirepresentation a name can represent more than one element of the represented set X, hence it is given by a partial map $\delta : \Sigma^\omega \rightharpoonup \mathcal{P}(X)$ into the power set of X). Then one gets that the category **AdmMultRep** of sequential spaces with admissible multi-representations coincides with **QCB**. In any case, **QCB** appears as an important class of spaces in the TTE framework, making **QCB** a most appealing framework for computations from a mathematical viewpoint. Moreover, the connection to TTE also yields another framework of computability for **QCB**.

3 Topological domains

In the previous chapter we have argued that **QCB** provides an ideal framework for instantiating Smyth's dictionary in denotational semantics. In particular, it was shown that **QCB** models a wide range of type constructions, including function types, which moreover are inherited from important cartesian-closed categories of spaces, and that it forms an important class of topological spaces, living faithfully in Weihrauch's TTE framework.

One aspect that **QCB** fails to provide is a model for recursion. Recursion lies at the centre of Classical Domain Theory, and is modelled using Scott's fixed point theorem [36] which states that any continuous endofunction of a dcppo has a least fixed point. However, it is obvious that endofunctions of qcb-spaces need not have fixed points in general. Thus, our approach to modelling recursion in Topological Domain Theory is to follow the ideas of Classical Domain Theory and identify an appropriate class of dcppo-like objects inside **QCB** for which a fixed point theorem holds. We divide this characterisation into two steps. The first step is to identify dcpo-like spaces, i.e. spaces that have least upper bounds for directed subsets with respect to the specialization order. Following [138], we call such objects topological predomains. In the second step, we add least elements to obtain topological domains [10]. Observe that this conforms with the development in Classical Domain Theory, where the mathematical research has been developed in such a way that many results are first obtained for the category of dcpos, and then restricted to dcppos [36]. However, since the topology is the principal entity in Topological Domain Theory, and the order derived, our framework is richer than that of Classical Domain Theory.

The first section of this chapter introduces the right notion of directed completeness for topological spaces, which is that of a monotone convergence space, introduced by Wyler as d-space in [166]. These monotone convergence spaces form a full reflective subcategory **Mon** of **Top** and the reflection functor restricts to **kTop** and, with minor amendments, to **Seq**. The resulting categories **kP** of compactly-generated predomains, respectively **SeqP** of sequential predomains, turn out to inherit the cartesian-closed structure from **kTop**, respectively **Seq**.

In the following section, we show that the class of compactly-generated predomains in **QCB** coincides with the class of sequential predomains in **QCB**, and that **QCB** is closed under the corresponding reflection functors. The resulting objects are called
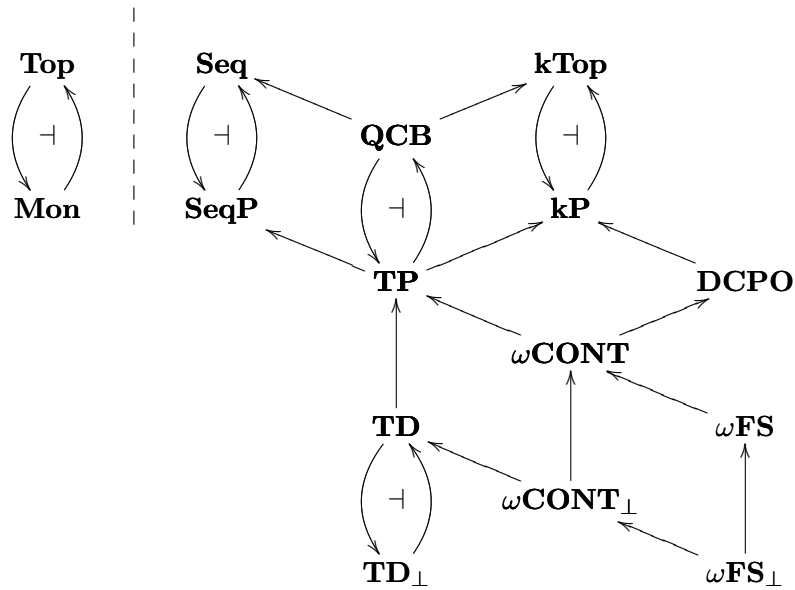
Figure 3.1: Categorical connections

topological predomains, and they form the category **TP**. This category is a full reflective exponential ideal of **QCB**, and inherits its categorical structure from it. Following the development in Classical Domain Theory, we then consider least elements and, subsequently, strict maps in our framework, i.e. maps which preserve the least elements. This leads to the introduction of the category **TD** of continuous maps between topological domains, which are topological predomains with a least element, and **TD**$_\perp$ of strict maps between topological domains. The category **TD** is again cartesian-closed, and we can obtain the expected fixed-point theorem. The category **TD**$_\perp$, on the other hand, is not cartesian-closed. We show that it has a symmetric monoidal closed structure, and that it is algebraically compact in the sense of [33]. Hence, one can solve recursive domain equations in it, following the treatment in Fiore's thesis [29]. Thus, our framework is able to model recursive types.

In the final part of this chapter, we give a detailed comparison between the type constructions in Topological and Classical Domain Theory. The first two sections of the Chapter establish the categorical connections given in Figure 3.1. As a result the category ω**CONT**, of ω-continuous dcpos equipped with the Scott-topology, forms a full subcategory of **TP**. We investigate in how far this subcategory inclusion preserves the categorical structure known from Classical Domain Theory. This structure is not preserved in general, and we give specific counterexamples in which the construction of (countable) products, respectively function spaces, does not preserve the Scott-topology. However, we also show that the inclusion of the largest cartesian-closed subcategory of ω**CONT**$_\perp$, Jung's category ω**FS**$_\perp$ of pointed ωFS-domains [63], into **TP** does preserve the cartesian-closed structure.

The chapter closely follows the development of the paper [9], which was jointly written by Schröder, Simpson and the author. Following our policy, results to which the author did not contribute are explicitly attributed.

3.1 Monotone convergence spaces

Our first task is to classify dcpo-like topological spaces. Since an order on a topological space is already given by the specialization order, our task is reduced to finding a suitable notion of directed complete space with respect the specialization order. For this we use the following definition going back to Wyler's notion of d-space [166].

Definition 3.1.1. A topological space X is a *monotone convergence space* if the specialization order on X is directed-complete, and every open subset $U \subseteq X$ is Scott-open with respect to this order.

By **Mon** we denote the category of monotone convergence spaces and continuous maps between them.

Observe that a monotone convergence space is necessarily a T_0-space. The condition that every open subset needs to be Scott-open can be justified by the following reasoning. If $D \subseteq X$ is directed with respect to the specialization order, and $x = \bigvee^{\uparrow} D$, then we expect to be able to approximate observations about x by observations on elements of D. Using Smyth's dictionary, this means that if U is an open neighbourhood of x, then $U \cap D \neq \emptyset$.

Let us recall the following well-known Lemma.

Lemma 3.1.2. *Any continuous map $f : X \to Y$ between monotone convergence spaces is Scott-continuous with respect to the specialization orders on X and Y.*

Proof. If $D \subseteq X$ is directed and $V \subseteq Y$ an open neighbourhood of $f(\bigvee^{\uparrow} D)$, then $f^{-1}(V) \cap D$ is nonempty, as $f^{-1}(V)$ is Scott open. Thus we get $f(\bigvee^{\uparrow} D) \sqsubseteq \bigvee^{\uparrow} f(D)$ and the converse is trivial, showing that f is indeed Scott-continuous. $\square$

Proposition 3.1.3. Mon *is a full reflective subcategory of* **Top**.

Proof. [Wyler] The original proof is Theorem 2.7 in [166], we sketch the argument: First observe that any subdcpo A of a sober space X is a monotone convergence space under the subspace topology, since whenever $\bigvee^{\uparrow} \mathcal{F}_i = \mathcal{F}$ in A (as the poset of completely prime filters) and $\mathcal{F} \in \square U$ for some open $U \subseteq X$, then $U \in \mathcal{F} = \bigcup \mathcal{F}_i$, and so there exists i_0 with $U \in \mathcal{F}_{i_0}$, as required. Now let X be any topological space and $\iota : X \to Sob(X)$ denote the continuous map of X into its sobrification $Sob(X)$, see e.g. section V-4 in [36]. By $\mathcal{M}(X)$ denote the smallest subdcpo of $Sob(X)$ containing $\iota(X)$, equipped with the subspace topology. Then by the observation we just made $\mathcal{M}(X)$ is

a monotone convergence space. Notice that we also get that $\iota : X \to Sob(X)$ restricts to a continuous map $\eta : X \to \mathcal{M}(X)$. We next show that $\mathcal{M}$ can be extended to a reflection functor.

In Proposition V-4.7 of [36] it is shown that the sobrification Sob is functorial and that it is a reflection functor. Let $f : X \to Y$ be a continuous map. Then it can be uniquely lifted to a continuous map $Sob(f) : Sob(X) \to Sob(Y)$ such that $Sob(f) \circ \iota_X = \iota_Y \circ f$. Moreover, $Sob(f)$ is Scott-continuous on the specialization orders of $Sob(X)$ and $Sob(Y)$ by Lemma 3.1.2 and the fact that a sober space is clearly a subdcpo of itself. We show that $Sob(f)^{-1}(\mathcal{M}(Y))$ is a subdcpo of $Sob(X)$. For this let $D \subseteq Sob(f)^{-1}(\mathcal{M}(Y))$ be directed. Then we have that $Sob(f)(\bigvee^{\uparrow} D) = \bigvee^{\uparrow} Sob(f)(D) \in \mathcal{M}(Y)$, as this is a dcpo in the specialization order, and so $\bigvee^{\uparrow} D \in Sob(f)^{-1}(\mathcal{M}(Y))$. But clearly $\iota(X) \subseteq Sob(f)^{-1}(\mathcal{M}(Y))$, and hence $\mathcal{M}(X) \subseteq Sob(f)^{-1}(\mathcal{M}(Y))$. This shows that $Sob(f)$ can be restricted to a continuous map $\mathcal{M}(f) : \mathcal{M}(X) \to \mathcal{M}(Y)$ with $\mathcal{M}(f) \circ \eta_X = \eta_Y \circ f$. That $\mathcal{M}$ is indeed a reflection functor can again be concluded from the fact that Sob is a reflection and will be left to the reader. □

Thus, we obtain the reflection functor $\mathcal{M} : \mathbf{Top} \to \mathbf{Mon}$, which plays an important role throughout this thesis. Observe that like $\mathbf{Top}$, the category $\mathbf{Mon}$ is not cartesian-closed. However, it turns out that the restriction of $\mathbf{Mon}$ to $\mathbf{kTop}$ inherits the cartesian-closed structure. We start with the definition of spaces which belong to this restriction.

Definition 3.1.4. A *compactly-generated predomain* is a compactly-generated space that also is a monotone convergence space. Compactly-generated predomains form the full subcategory $\mathbf{kP}$ of $\mathbf{kTop}$.

With Lawson's Theorem 2.2.4, we immediately get the following two results.

Proposition 3.1.5. $\mathbf{DCPO}$ *is a full coreflective subcategory of* $\mathbf{kP}$.

Proof. By Lemma 3.1.2 any continuous function $f : X \to Y$ between compactly-generated predomains is Scott-continuous. Thus, on objects the coreflection $\mathcal{S} : \mathbf{kP} \to \mathbf{DCPO}$ is obtained by assigning to a compactly-generated predomain X the underlying set of X equipped with the Scott-topology for the specialization order of X, which is a compactly-generated predomain by Lawson's Theorem 2.2.4. □

Throughout the thesis, we denote the coreflection functor by $\mathcal{S} : \mathbf{kP} \to \mathbf{DCPO}$.

Lemma 3.1.6. *If X is a monotone convergence space, so is $k(X)$.*

Proof. [Simpson] As observed in the proof of Theorem 2.2.4 X and $k(X)$ have the same specialization order, hence the specialization order on $k(X)$ is a dcpo. Furthermore, by Lawson's Theorem 2.2.4, the space $\mathcal{S}(X)$ is a compactly-generated predomain. Hence it follows that the topology on $k(X)$ must be coarser than the one on $\mathcal{S}(X)$, which is the Scott-topology. □

Next we show that the reflection $\mathcal{M}$ cuts down to a reflection $\mathcal{M} : \mathbf{kTop} \to \mathbf{kP}$. For this let $\iota : X \hookrightarrow \mathcal{M}(X)$ denote the continuous map of a space X into its monotone convergence reflection.

Lemma 3.1.7. *The following are equivalent for a subset $V \subseteq \mathcal{M}(X)$.*

(i) V *is open;*

(ii) $\iota^{-1}(V)$ *is open in X, and $V = \uparrow (V \cap \iota(X))$ in the specialization order of $\mathcal{M}(X)$;*

(iii) $\iota^{-1}(V)$ *is open in X, and V is Scott-open in the specialization order of $\mathcal{M}(X)$.*

Proof. [Simpson] The part (i) $\Rightarrow$ (iii) is trivial, so we show (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii).

For (ii) $\Rightarrow$ (i), suppose $V = \uparrow(V \cap \iota(X))$, and $\iota^{-1}(V)$ is open in X. By definition of the topology on $\mathcal{M}(X)$, the set $\{\mathcal{F} \in \mathcal{M}(X) \mid \iota^{-1}(V) \in \mathcal{F}\}$ is open in $\mathcal{M}(X)$. We show that $V = \{\mathcal{F} \in \mathcal{M}(X) \mid \iota^{-1}(V) \in \mathcal{F}\}$. It holds that $\{\mathcal{F} \in \mathcal{M}(X) \mid \iota^{-1}(V) \in \mathcal{F}\} \subseteq V$, for if $\iota^{-1}(V) \in \mathcal{F}$, then there exists some $x \in \iota^{-1}(V)$ for which $\mathcal{U}_x \subseteq \mathcal{F}$ (otherwise every $x \in \iota^{-1}(V)$ had an open neighbourhood U_x not in $\mathcal{F}$ contradicting that $\mathcal{F}$ is completely prime). We conclude that $\mathcal{F} \in \uparrow(V \cap \iota(X)) = V$. For the converse suppose $\iota^{-1}(V) \notin \mathcal{F}$, then for all $x \in \iota^{-1}(V)$, we have $\mathcal{U}_x \nsubseteq \mathcal{F}$, and so $\mathcal{F} \notin \uparrow(V \cap \iota(X)) = V$, showing the claim.

For (iii) $\Rightarrow$ (ii), suppose $\iota^{-1}(V)$ is open in X, and V Scott open in $\mathcal{M}(X)$. Obviously, we have $\uparrow(V \cap \iota(X)) \subseteq V$, as V is an upper set. For the converse, suppose that $\mathcal{F} \notin \uparrow (V \cap \iota(X))$. Let S denote the Scott closure of $\iota(X \setminus \iota^{-1}(V))$ in $\mathcal{M}(X)$, for which we get $S \cap V = \emptyset$, because $\mathcal{M}(X) \setminus V$ is Scott-closed. Observe that $S \cup \uparrow(V \cap \iota(X))$ is closed under directed suprema and includes $\iota(X)$ as a subset. Hence $S \cup \uparrow(V \cap \iota(X)) = \mathcal{M}(X)$, and we can conclude $\mathcal{F} \in S$ and $\mathcal{F} \notin V$, as required. $\square$

We remark that the above statement does not hold for the sobrification replacing the monotone convergence reflection. In particular, (iii) does not imply (i) in that case. To see this let X be the space of the natural numbers equipped with the cofinite topology, then the filter $\mathcal{F}$ of all nonempty open subsets of X is completely prime, and the singleton $\{\mathcal{F}\}$ satisfies condition (iii) of the Lemma but is not open in $Sob(X)$.

Proposition 3.1.8. *If X is a compactly-generated space, so is $\mathcal{M}(X)$.*

Proof. [Simpson] We have to show that $\mathcal{M}(X)$ carries the compactly-generated topology, that is $\mathcal{M}(X) \cong k(\mathcal{M}(X))$. So suppose $U \subseteq k(\mathcal{M}(X))$ is open. Then by Lawson's Theorem 2.2.4, U is Scott open with respect to the specialization order on $\mathcal{M}(X)$. Moreover, as X is compactly-generated, the pre-embedding ι is continuous as a map $X \to k(\mathcal{M}(X))$, showing $\iota^{-1}(U)$ is open in X. But then the previous lemma proves that U must already be open in $\mathcal{M}(X)$, hence $\mathcal{M}(X) \cong k(\mathcal{M}(X))$. $\square$

An interesting question in its own right is whether Lemma 3.1.6 and Proposition 3.1.8 can be unified by showing that for every topological space X, one has $k(\mathcal{M}(X)) \cong \mathcal{M}(k(X))$. This problem seems to be nontrivial, and at the time of writing, we do not know how to solve it.

The above results yield that **kP** is a full reflective subcategory of **kTop**. In fact we can do better, as the following shows.

Theorem 3.1.9. kP *is a full reflective exponential ideal of* **kTop**.

Proof. [Simpson] Let X be a compactly-generated space and Y be a compactly-generated predomain. We have to show that $X \Rightarrow_k Y$ is a compactly-generated predomain. It is an easy observation that the specialization order on $X \Rightarrow_k Y$ is a dcpo, since it is the pointwise order on functions. So it remains to show that the topology on $X \Rightarrow_k Y$ is coarser than the Scott-topology. By Lawson's Theorem 2.2.4, it suffices to show that every compact-open subset $\langle K, U \rangle \subseteq X \Rightarrow_k Y$ is Scott-open. So assume $\{f_i\}_{i \in I}$ is a directed set of continuous maps with $\bigvee^{\uparrow}_{i \in I} f_i \in \langle K, U \rangle$. Then we have $K \subseteq (\bigvee^{\uparrow}_{i \in I} f_i)^{-1}(U) = \bigcup^{\uparrow}_{i \in I}(f_i^{-1}(U))$, and by compactness of K, this yields the existence of some $i_0 \in I$ with $K \subseteq f_{i_0}^{-1}(U)$, as required. $\qquad\square$

Similarly to predomains in **kTop**, one can also define predomains in **Seq**. However, just to take these to be the intersection of **Seq** and **Mon** does not yield the desired result, because **Seq** is not closed under the reflection $\mathcal{M} : \mathbf{Top} \to \mathbf{Mon}$, as the following example shows.

Let X be the space of all countable ordinals ordered by the usual order and equipped with the Scott-topology. Then $\mathcal{M}(X)$ has as underlying set $X \cup \{\omega_1\}$, where ω_1 denotes the first uncountable ordinal, which is the directed sup of all countable ordinals. In particular, the singleton $\{\omega_1\}$ is not open in $\mathcal{M}(X)$. However, ω_1 cannot be obtained as limit of any sequence of elements in X, since X is closed under countable suprema, and so $\{w_1\}$ is sequentially open in $\mathcal{M}(X)$, showing $\mathcal{M}(X)$ is not sequential.

We see that there is a serious incompatibility between sequentiality and the Scott-topology, as directed suprema are in general not accessible merely by limits of a sequence. To fix this, we have to consider ω-monotone convergence spaces and the ω-Scott-topology instead.

Definition 3.1.10. A subset U of a poset X is open in the ω-*Scott-topology* if it is upper with respect to the specialization order and whenever $x \in U$, and $\{x_n\}_{n \in \mathbb{N}}$ is an ascending sequence with supremum x, then there exists $n_0 \in \mathbb{N}$ with $x_{n_0} \in U$.

A space X is an ω-*monotone convergence space* if its specialization order has suprema of ascending sequences (i.e. is an ω-cpo), and every open $U \subseteq X$ is open in the ω-Scott-topology. The ω-monotone convergence spaces and continuous maps form the category ω**Mon**.

A *sequential predomain* is a sequential space that also is an ω-monotone convergence space. The sequential predomains form the full subcategory **SeqP** of **Seq**.

Notice that the ω-Scott-topology on a space X is in general finer than the Scott-topology, hence every monotone convergence space is an ω-monotone convergence space. Analogous to Lawson's Theorem, we get the following result, whose proof, in contrast, is trivial.

Lemma 3.1.11. *Every ω-cpo X equipped with the ω-Scott-topology is a sequential predomain.*

Proof. In an ωcpo X equipped with the ω-Scott-topology, sequentially open subsets $U \subseteq X$ are open. For if U is sequentially open, and $x \in U$, then for any ascending sequence $\{x_n\}_{n \in \mathbb{N}}$ with supremum x, it holds that $(x_n)_{n \in \mathbb{N}}$ converges to x. But this implies the existence of an $n_0 \in \mathbb{N}$ with $x_{n_0} \in U$, showing U is ω-Scott-open. $\qquad\square$

Like **Mon**, the category ω**Mon** is a full reflective subcategory of **Top**, with a reflection functor $\mathcal{M}^{\omega} : \textbf{Top} \to \omega\textbf{Mon}$, which can be described as follows. Let again $\iota : X \to Sob(X)$ denote the pre-embedding into the sobrification. Then $\mathcal{M}^{\omega}(X)$ is given by the smallest subspace of $Sob(X)$, containing $\iota(X)$ and being closed under suprema of ascending sequences.

Proposition 3.1.12. *For a sequential space X, $\mathcal{M}^{\omega}(X)$ is sequential.*

Proof. [Schröder/Simpson] The original proof by Schröder and Simpson can be found as Proposition 5.2 in [134], and is by transfinite induction. However, we conjecture that one can also adapt the proof for Lemma 3.1.7 and Proposition 3.1.8 to the situation of ω-monotone convergence and sequential spaces. $\qquad\square$

Thus it follows, as above for **kTop** and **kP**, that **SeqP** is a full reflective exponential ideal of **Seq**. Moreover, Schröder and Simpson [134] have shown that, in this case, the reflection preserves countable products, a fact that turns out to be useful when we construct free algebras to model computational effects in Chapter 6.

Theorem 3.1.13. SeqP *is a full reflective exponential ideal of* **Seq***, and the reflection functor preserves countable products.*

Proof. [Schröder/Simpson] The preservation of countable products is nontrivial, and a proof is given as Theorem 5.6 in [134]. That **SeqP** is a full exponential ideal then follows from Theorem 1.857 of [35]. $\qquad\square$

3.2 Topological domains

In the previous section, we have identified categories of dcpo-like objects in **kTop**, respectively **Seq**, and seen that these have good closure properties. We now show that

these results can be transferred to **QCB**, and moreover that here the two reflection functors, $\mathcal{M}$ and $\mathcal{M}^{\omega}$, coincide. The resulting objects are called *topological predomains* and they lie at the core of Topological Domain Theory.

Subsequently, we consider topological predomains with least elements, so-called *topological domains*. These topological domains form another cartesian-closed category, in which one can model recursion like in Classical Domain Theory. Furthermore, we consider *strict maps* between topological domains, and show that the resulting category is symmetric monoidal closed and algebraically compact [33], hence has solutions for recursive domain equations.

Naturally, we start by studying the behaviour and closure properties of monotone convergence spaces in **QCB**.

Lemma 3.2.1. *A qcb-space X is an ω-monotone convergence space if and only if it is a monotone convergence space.*

Proof. [Simpson] As every monotone convergence space X is an ω-monotone convergence space, we only have to show that the converse is true for qcb-spaces. So assume X is a qcb-space and an ω-monotone convergence space. Let $D \subseteq X$ be a directed subset of X. By Proposition 2.2.9, D has a countable dense subset $\{y_n\}_{n\in\mathbb{N}}$, and by directedness of D, we can define an increasing sequence as follows. Let $x_0 = y_0$, and $x_{n+1} \in D$ with $x_{n+1} \sqsupseteq x_n, y_n$.

We observe that the sequence $(x_n)_{n\in\mathbb{N}}$ is also dense in D, as for any open $U \subseteq X$ which has a nonempty intersection with D, there exists $n_0 \in \mathbb{N}$ with $y_{n_0} \in U$ and hence $x_{n_0} \in U$. Moreover the upper bounds of the ascending sequence $(x_n)_{n\in\mathbb{N}}$ coincide with the upper bounds of D which can be seen as follows: if $z \sqsupseteq x_n$ for all $n \in \mathbb{N}$, but as the sequence $(x_n)_{n\in\mathbb{N}}$ is dense in D this means that for all $y \in D$ and open $U \subseteq X$ containing y, there exists $n_0 \in \mathbb{N}$ with $x_{n_0} \in U$ and consequently $z \in U$, showing that $z \sqsupseteq y$.

As X is an ω-monotone convergence space, this yields that the least upper bound of D exists in X. Now assume $\bigvee^{\uparrow} D = \bigvee^{\uparrow}_{n\in\mathbb{N}} x_n \in U$, and $U \subseteq X$ open. Then we find $n_0 \in \mathbb{N}$ with $x_{n_0} \in U$, and hence $D \cap U \neq \emptyset$, showing that U is Scott-open, and hence X a monotone convergence space. $\qquad\square$

Definition 3.2.2. A *topological predomain* is a qcb-space that also is a monotone convergence space, or equivalently an ω-monotone convergence space. By **TP** we denote the full subcategory of **QCB** consisting of the topological predomains.

Notice that it is clear that a topological predomain is in fact a qcb$_0$-space. For our next theorem, which shows that **QCB** is closed under the monotone convergence reflection, we need the following lemma.

Lemma 3.2.3. *For a compactly-generated T_0-space X, the mapping*

$$\eta_X : X \to ((X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}),$$

obtained as the double exponential transpose of the identity map $\mathbb{S}^X \to \mathbb{S}^X$ is a regular mono in **kTop**.

Proof. [Schröder/Simpson] We show that η_X is a topological embedding; then it is easily obtained as the equalizer of maps $((X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}) \to \nabla \mathbf{2}$ into the two-element indiscrete space, as in the proof of Proposition 2.3.7. For the proof, we identify a map $f : Y \to \mathbb{S}$ with the corresponding open subset $U_f \subseteq Y$, following Lemma 2.2.5.

So let X be a compactly-generated T_0-space. The T_0-axiom ensures that the η_X is injective, and by cartesian-closure it is continuous, as is $\eta_{(X \Rightarrow_k \mathbb{S})} : (X \Rightarrow_k \mathbb{S}) \to (((X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S})$. Moreover, the maps are given by the λ-terms:

$$\lambda x : X. \, \lambda f : (X \Rightarrow_k \mathbb{S}). \, f(x),$$

respectively

$$\lambda f : (X \Rightarrow_k \mathbb{S}). \, \lambda F : ((X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}). \, F(f).$$

It follows that $f(x) = \top$ if and only if $\eta_X(x)(f) = \top$, which in turn is the case if and only if $\eta_{(X \Rightarrow_k \mathbb{S})}(f)(\eta_X(x)) = \top$. Consequently $x \in U_f$ if and only if $\eta_X(x) \in U_{\eta_{(X \Rightarrow_k \mathbb{S})}(f)}$. It follows that every open $U \subseteq X$ is the preimage of an open $V \subseteq ((X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S})$ and the claim follows. $\square$

Theorem 3.2.4. *If X is a qcb-space, then $\mathcal{M}(X)$ and $\mathcal{M}^\omega(X)$ are a qcb-spaces, as well.*

Proof. [Schröder/Simpson] We show the claim for $\mathcal{M}(X)$. The compactly-generated function spaces $\mathcal{M}(X) \Rightarrow_k \mathbb{S}$ and $X \Rightarrow_k \mathbb{S}$ are isomorphic by Lemma 2.2.5 and Lemma 3.1.7. It follows that also $(\mathcal{M}(X) \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}$ and $(X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}$ are isomorphic, the latter being the **QCB**-exponential $\mathbb{S}^{\mathbb{S}^X}$, as **QCB** inherits function spaces from **kTop**. Since $\mathcal{M}(X)$ is a T_0-space, the map

$$\eta_{\mathcal{M}(X)} : \mathcal{M}(X) \to ((\mathcal{M}(X) \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S})$$

is a regular mono in **kTop**, by the previous Lemma. Thus, there exists a regular mono $\mathcal{M}(X) \to \mathbb{S}^{\mathbb{S}^X}$ in **kTop**, and the result follows from Theorem 2.2.6. $\square$

This leads to the following pivotal theorem.

Theorem 3.2.5. **TP** *is a full reflective exponential ideal of* **QCB**, *which has countable products and coproducts. The reflection* **QCB** $\to$ **TP** *is obtained by restricting the*

reflection $\mathcal{M} : \mathbf{kTop} \to \mathbf{kP}$ to qcb-spaces, or equivalently by restricting the reflection $\mathcal{M}^\omega : \mathbf{Seq} \to \mathbf{SeqP}$ to qcb-spaces. Furthermore, the reflection preserves countable products and the inclusion preserves countable coproducts.

Proof. The previous lemma shows that we obtain a reflection functor $\mathcal{M} : \mathbf{QCB} \to \mathbf{TP}$ as the restriction of the reflection $\mathbf{kTop} \to \mathbf{kP}$ to qcb-spaces. Moreover, by Lemma 3.2.1, we know that for a qcb-space X, $\mathcal{M}^\omega(X)$ is in fact a monotone convergence space, and so it must equal $\mathcal{M}(X)$ by the definition of these functors. Thus, the reflection is also given by the restriction of $\mathcal{M}^\omega : \mathbf{Seq} \to \mathbf{SeqP}$ to qcb-spaces. That the reflection preserves countable products follows now from Theorem 3.1.13, and hence Theorem 1.857 of [35] yields again that $\mathbf{TP}$ is an exponential ideal in $\mathbf{QCB}$. The existence and preservation of countable coproducts follows easily from the fact that all involved coproducts are just the usual topological coproducts. $\square$

For convenience, we denote the reflection functor by $\mathcal{M} : \mathbf{QCB} \to \mathbf{TP}$.

Corollary 3.2.6. $\mathbf{TP}$ *is a full cartesian-closed subcategory of* $\mathbf{kP}$ *and* $\mathbf{SeqP}$*, and inherits the cartesian-closed structure, as well as countable products and coproducts.*

Having obtained good closure properties for topological predomains, we now turn our attention towards least elements and recursion.

Definition 3.2.7. A topological predomain X whose specialization order has a least element (usually denoted by $\perp_X$) is called a *topological domain*. Topological domains and continuous maps between them form the category $\mathbf{TD}$.

A *strict map* between topological domains X, Y is a continuous map $f : X \to Y$ that preserves the least element, i.e. $f(\perp_X) = \perp_Y$. Topological domains and strict maps form the category $\mathbf{TD}_\perp$.

The category $\mathbf{TD}$ inherits a lot of categorical structure from $\mathbf{TP}$ as the following result shows.

Theorem 3.2.8. *The category* $\mathbf{TD}$ *is an exponential ideal of* $\mathbf{TP}$ *and closed under the formation of countable products in* $\mathbf{TP}$*.*

Proof. Straightforward. $\square$

Furthermore, in $\mathbf{TD}$ the expected fixed point property holds, and the fixed points are uniform, in the sense of Exercise 2.30 of Plotkin's Pisa notes [109].

Proposition 3.2.9. *For a topological domain X every continuous map $f : X \to X$ has a least fixed point, that is, there exists an element $Fix_f \in X$ such that $f(Fix_f) = Fix_f$ and Fix_f is the smallest element having this property. Moreover, given continuous functions $f : X \to X$, $g : Y \to Y$ and a strict map $h : X \to Y$, such that $h \circ f = g \circ h$, one gets $h(Fix_f) = Fix_g$.*

Proof. The existence proof is the same as Scott's original proof for dcppos. The map f is Scott-continuous by Proposition 3.1.5, and it clearly holds that $\perp_X \sqsubseteq f(\perp_X)$. So if we define inductively $f^0(x) = x$, and $f^{n+1}(x) = f(f^n(x))$, then we get for all $n \in \mathbb{N}$, that $f^n(\perp_X) \sqsubseteq f^{n+1}(\perp_X)$. As X is a topological domain $Fix_f := \bigvee_{n \in \mathbb{N}}^{\uparrow} f^n(\perp_X)$ exists in X, and by Scott continuity of f, we get

$$f(Fix_f) = f(\bigvee_{n \in \mathbb{N}} f^n(\perp_X)) = \bigvee_{n \in \mathbb{N}} f^{n+1}(\perp_X) = Fix_f.$$

Furthermore for any other fixpoint y of f, we have $\perp_X \sqsubseteq y$, hence inductively for all $n \in \mathbb{N}$, $f^{n+1}(\perp_X) \sqsubseteq f^{n+1}(y) = y$, and thus y is an upper bound of $\{f^n(\perp_X) | n \in \mathbb{N}\}$, showing $Fix_f \sqsubseteq y$, as required.

The uniformity follows from the straightforward calculation that for all $n \in \mathbb{N}$, $h(f^n(\perp_X)) = g^n(h(\perp_X)) = g^n(\perp_Y)$. $\qquad\qquad\square$

$\mathbf{TD}_\perp$ on the other hand is not cartesian-closed for well-known reasons: the one point domain $\mathbf{1}$ is a zero object in $\mathbf{TD}_\perp$, i.e. it is initial and terminal object, and any cartesian-closed category with zero object is trivial. Nevertheless, $\mathbf{TD}_\perp$ has countable limits, which are inherited from $\mathbf{QCB}$.

Lemma 3.2.10. *The inclusion functor* $\mathbf{TD}_\perp \to \mathbf{QCB}$ *creates countable limits.*

Proof. Clearly, countable $\mathbf{QCB}$-products of topological domains are again topological domains, and they satisfy the properties of a categorical product. Moreover, if

$$X \xrightarrow{\ e\ } Y \overset{f}{\underset{g}{\rightrightarrows}} Z$$

is an equalizer diagram in $\mathbf{QCB}$, such that Y and Z are topological domains and f, g strict maps, then the specialization order of X has a least element and $e(\perp_X) = \perp_Y$. Since f and g are Scott-continuous by Proposition 3.1.5, the specialization order on X is directed complete. Finally, the topology on X is the sequentialisation of the subspace topology inherited from Y, and by Lawson's Theorem 2.2.4, coarser than the Scott-topology. It follows that X is a topological domain, and the above diagram is an equalizer diagram in $\mathbf{TD}_\perp$. Thus, the inclusion functor $\mathbf{TD}_\perp \hookrightarrow \mathbf{QCB}$ creates countable products and equalizers, and the claim follows. $\qquad\qquad\square$

In the following, we show that $\mathbf{TD}_\perp$ has a symmetric monoidal closed structure and that there exists a reflection functor $\mathbf{TP} \to \mathbf{TD}_\perp$, which extends the lifting functor from Classical Domain Theory. Let us start by defining this lifting functor in its greatest generality.

Definition 3.2.11. The *lifting functor* $(\cdot)_\perp$ on topological spaces is defined as follows. For a topological space X, the space $X_\perp$, has as underlying set the disjoint union

$X \cup \{\perp_{X_\perp}\}$ and the topology given by the open subsets of X together with $X \cup \{\perp_{X_\perp}\}$. For a continuous map $f : X \to Y$, the obvious extension $f_\perp : X_\perp \to Y_\perp$ maps $\perp_{X_\perp}$ to $\perp_{Y_\perp}$.

Observe that $\perp_X$ becomes the least element with respect to the specialization order on $X_\perp$. Moreover, it is easy to see that $(\cdot)_\perp$ carries continuous maps between topological predomains to strict maps between topological domains.

Lemma 3.2.12. *The lifting functor* $(\cdot)_\perp : \mathbf{TP} \to \mathbf{TD}_\perp$ *is left adjoint to the inclusion functors* $\mathbf{TD}_\perp \hookrightarrow \mathbf{TP}$ *and* $\mathbf{TD}_\perp \hookrightarrow \mathbf{TD}$.

Proof. Straightforward. $\qquad\square$

We immediately get the following unique extension theorem for the lifting functor.

Corollary 3.2.13. *For every topological predomain X, topological domain Y and continuous map $f : X \to Y$, there exists a unique strict map $\widehat{f} : X_\perp \to Y$, making the following diagram commute:*

$$
\begin{array}{ccc}
X_\perp & \overset{\widehat{f}}{-\,-\,-\!\!\!\rightarrow} & Y \\[2pt]
i \Big\uparrow & \nearrow{\scriptstyle f} & \\[2pt]
X & &
\end{array}
$$

The lifting functor $(\cdot)_\perp$ has an inverse on objects, which is employed repeatedly below, due to its well-behavedness.

Definition 3.2.14. For a topological domain X, we write $X^\dagger$ for the open subspace of X with underlying set $X \setminus \{\perp_X\}$.

We remark that for a topological domain X, the space $X^\dagger$ is always a topological predomain, because open subspaces of sequential spaces are again sequential, and a countable sequential pseudobase for $X^\dagger$ is easily obtained by restricting the pseudobasis sets of X. However, $(\cdot)^\dagger$ cannot be extended to a functor in our categories of total maps, as a continuous map $f : X \to Y$ between topological domains might map elements $x \neq \perp_X$ to $\perp_Y$. In fact, lifting and $(\cdot)^\dagger$ establish an equivalence between $\mathbf{TD}_\perp$ and the category of topological predomains and partial continuous maps. However, we prefer to work with $\mathbf{TD}_\perp$, and do not consider partial maps in this thesis.

We make the following straightforward observation about the operation $(\cdot)^\dagger$.

Proposition 3.2.15. *The Scott-topology on $X^\dagger$ is the relative Scott-topology inherited from X.*

Now we show the categorical closure properties of $\mathbf{TD}_\perp$.

Theorem 3.2.16. $\mathbf{TD}_\perp$ *is a symmetric monoidal closed category.*

Proof. The tensor product $X \otimes Y$ on $\mathbf{TD}_\perp$ is given by the subspace of $X \times Y$ consisting of $\{(x,y) \mid x \neq \perp_X \text{ and } y \neq \perp_Y\} \cup \{(\perp_X, \perp_Y)\}$. This is indeed a qcb-space, as it can be obtained as $(X^\dagger \times Y^\dagger)_\perp$. The tensor product $\otimes$ obviously is commutative and associative, and it induces a symmetric monoidal structure on $\mathbf{TD}_\perp$, with the unit being Sierpinski space $\mathbb{S}$.

The closed structure $X \Rightarrow_\perp Y$ is given by the closed subspace of the $\mathbf{QCB}$-exponential Y^X consisting of the strict maps. The natural isomorphism

$$\mathbf{TD}_\perp(A \otimes X, Y) \cong \mathbf{TD}_\perp(A, X \Rightarrow_\perp Y)$$

between the Hom-sets is given as follows. For $f \in \mathbf{TD}_\perp(A \otimes X, Y)$ the corresponding element $f' \in \mathbf{TD}_\perp(A, X \Rightarrow_\perp Y)$ is the map, assigning to $\perp_A$ the constant $\perp_Y$-map, and to any other $a \in A$, the strict map

$$x \mapsto \begin{cases} \perp_Y & \text{if } x = \perp_X \\ f(a,x) & \text{otherwise.} \end{cases}$$

We show that f' is sequentially continuous. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in A, converging to a, and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X, converging to x. Since $X \Rightarrow_\perp Y$ is a sequential subspace of the sequential function space $X \Rightarrow_{seq} Y$, by Theorem 2.3.6, we have to show that $(f'(a_n)(x_n))_{n \in \mathbb{N}}$ converges to $f'(a)(x)$ in Y. If $a = \perp_A$ or $x = \perp_X$, then $f'(a)(x) = \perp_Y$, and the claim follows trivially. So let $a \neq \perp_A$ and $x \neq \perp_X$. Then we may assume without loss of generality that $a_n \neq \perp_A$ and $x_n \neq \perp_X$ for all $n \in \mathbb{N}$. But then for all $n \in \mathbb{N}$, it holds that $f'(a_n)(x_n) = f(a_n, x_n)$, and $(f(a_n, x_n))_{n \in \mathbb{N}}$ converges to $f(a,x) = f'(a)(x)$ in Y, showing the sequential continuity of f'.

Conversely, for $g \in \mathbf{TD}_\perp(A, X \Rightarrow_\perp Y)$, the corresponding element $g' \in \mathbf{TD}_\perp(A \otimes X, Y)$ is given by $(a,x) \mapsto g(a)(x)$ for any $(a,x) \in A \otimes X$. Sequential continuity of g' is shown as follows. A sequence $((a_n, x_n))_{n \in \mathbb{N}}$ converges to (a,x) in $A \otimes X$ if and only if $(a_n)_{n \in \mathbb{N}}$ converges to a in A and $(x_n)_{n \in \mathbb{N}}$ converges to x in X, because $A \otimes X$ is a sequential subspace of the cartesian product $A \times X$. Moreover, for all $n \in \mathbb{N}$, we have $g'(a_n, x_n) = g(a_n)(x_n)$, and $(g(a_n)(x_n))_{n \in \mathbb{N}}$ converges to $g(a)(x) = g'(a,x)$ in Y, as required.

Naturality in A and Y is obvious. $\qquad\qquad\square$

In the literature on Classical Domain Theory the counterpart of our tensor product is usually called the *smash product*.

In the presence of least elements, colimits turn out to be problematic, see [45]. From the general reasons in *op.cit.*, it follows that $\mathbf{TD}$ does neither have binary coproducts nor an initial object. However, the situation can be amended as the following result shows.

Proposition 3.2.17. TD$_\perp$ *has countable coproducts.*

Proof. Let $\{X_i\}_{i \in I}$ be a countable family of topological domains. We claim the **TD**$_\perp$-coproduct $\bigoplus_{i \in I} X_i$ is given by $(\coprod_{i \in I} X_i^\dagger)_\perp$, the lifting of the **QCB**-coproduct of the $X_i^\dagger$. To see this, let $j_i : X_i \to (\coprod_{i \in I} X_i^\dagger)_\perp$ denote the unique strict extensions of the inclusions $X_i^\dagger \hookrightarrow \coprod_{i \in I} X_i^\dagger \hookrightarrow (\coprod_{i \in I} X_i^\dagger)_\perp$. Suppose Y is a topological domain, and for each $i \in I$, $f_i : X_i \to Y$ is a strict map. Then the restrictions $f_i^* : X_i^\dagger \to Y$ are continuous, hence we get a unique continuous map $\coprod_{i \in I} f_i^* : \coprod_{i \in I} X_i^\dagger \to Y$, which by Corollary 3.2.13 has a unique strict extension $\bigoplus_{i \in I} f_i : (\coprod_{i \in I} X_i^\dagger)_\perp \to Y$. Moreover, the strictness of all involved maps ensures that $\bigoplus_{i \in I} f_i \circ j_i = f_i$ for all $i \in I$. □

Finally, we turn our attention towards categorical fixed points, in order to obtain solutions for recursive domain equations. Just as in Classical Domain Theory this can be used to model recursive types. For our development in Topological Domain Theory, we follow the treatment in Fiore's thesis [29], where, based on the work of Freyd [33], he investigates categorical fixed points in ω**CPO**-enriched categories.

Let us start by recalling that an ω**CPO**-enriched category is a locally small category $\mathbb{C}$ in which for every pair of objects X, Y, the Hom-set $\mathbb{C}(X, Y)$ carries an ωcpo-structure, and composition $\circ : \mathbb{C}(X, Y) \times \mathbb{C}(Y, Z) \to \mathbb{C}(X, Z)$ is an ω**CPO**-morphism, i.e. ω-Scott-continuous. Similarly, an ω**CPPO**$_\perp$-enriched category is a locally small category $\mathbb{C}$ in which for every pair of objects X, Y, the Hom-set $\mathbb{C}(X, Y)$ carries an $\omega cppo$-structure, and composition $\circ : \mathbb{C}(X, Y) \times \mathbb{C}(Y, Z) \to \mathbb{C}(X, Z)$ is a strict ω-Scott-continuous map. We have the following straightforward lemma.

Lemma 3.2.18. TD$_\perp$ *is* ω**CPPO**$_\perp$*-enriched, hence also* ω**CPO**-*enriched.*

Furthermore, let us recall that an ω**CPO**-enriched functor $F : \mathbb{C} \to \mathbb{D}$ is a functor between ω**CPO**-enriched categories such that for all $\mathbb{C}$-objects X, Y, the assignment $F_{X,Y} : \mathbb{C}(X, Y) \to \mathbb{D}(FX, FY)$ is ω-Scott-continuous.

In Fiore's thesis [29], the central notion for having categorical fixed points is that of ω**CPO**-algebraic compactness, which goes back to Freyd [33].

Definition 3.2.19. An ω**CPO**-algebraically compact category is an ω**CPO**-enriched category $\mathbb{C}$ in which every ω**CPO**-enriched endofunctor $F : \mathbb{C} \to \mathbb{C}$ has an initial F-algebra whose inverse is a final F-coalgebra.

Fiore shows that in an ω**CPO**-algebraically compact category $\mathbb{C}$, for every ω**CPO**-enriched bifunctor $f : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ there exists a categorical fixed point, i.e. a $\mathbb{C}$-object $FixF$, such that $F(FixF, FixF) \cong FixF$. A rough sketch of Fiore's argument can be given as follows. Using Lambek's Lemma [79], one gets that in an ω**CPO**-algebraically compact category, every ω**CPO**-enriched endofunctor $F : \mathbb{C} \to \mathbb{C}$ has a fixed point, given by its initial algebra. This, in turn, is used to show that every ω**CPO**-enriched

contravariant endofunctor $G : \mathbb{C}^{op} \to \mathbb{C}$ has a fixed point, given by $Fix(G \circ G^{op})$. Having this, one can prove that every $\omega\mathbf{CPO}$-enriched bifunctor $F : \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$ induces an $\omega\mathbf{CPO}$-enriched functor $F^* : \mathbb{C}^{op} \to \mathbb{C}$, by the assignment $X \mapsto FixF(X, -)$. Now for $FixF^*$, one gets:

$$F(Fix\, F^*, Fix\, F^*) \cong F(Fix\, F^*, F^*(Fix\, F^*)) \cong F^*(Fix\, F^*) \cong Fix\, F^*.$$

Hence one has obtained a categorical fixed point for F.

Thus, in order to model recursive types in Topological Domain Theory, we want to show that $\mathbf{TD}_\perp$ is $\omega\mathbf{CPO}$-algebraically compact. We do this again with the help of Fiore's results [29].

Theorem 3.2.20. $\mathbf{TD}_\perp$ *is $\omega\mathbf{CPO}$-algebraically compact.*

Proof. By Theorem 7.1.3 and Corollary 7.2.5 of [29], we have to show that:

(i) $\mathbf{TD}_\perp$ has an e-initial object, i.e. an object $\mathbf{0}$ such that for all topological domains X, there exists a strict map $X \to \mathbf{0}$ with $(X \to \mathbf{0} \to X) \leq \mathrm{id}_X$,

(ii) $\mathbf{TD}_\perp$ has colimits for ω-chains of embeddings.

Clearly, (i) is fulfilled, as the one-element space $\mathbf{1}$ is e-initial in $\mathbf{TD}_\perp$. For (ii), observe that colimits of embeddings coincide with limits of projections in $\mathbf{TD}_\perp$. Thus we have to show that limits of ω-chains of projections exist in $\mathbf{TD}_\perp$, which follows from Lemma 3.2.10. $\square$

Corollary 3.2.21. *Every $\omega\mathbf{CPO}$-enriched bifunctor $F : \mathbf{TD}_\perp^{op} \times \mathbf{TD}_\perp \to \mathbf{TD}_\perp$ has a categorical fixed point.*

Thus we have recursive types for type constructions corresponding to $\omega\mathbf{CPO}$-enriched functors in the category of topological domains and strict maps. Furthermore, it immediately follows that these type constructions transfer to the $\omega\mathbf{CPO}$-enriched super-categories $\mathbf{TD}$ and $\mathbf{TP}$. Hence recursive types, for type constructors corresponding to $\omega\mathbf{CPO}$-enriched functors on $\mathbf{TD}_\perp$, can be modelled in Topological Domain Theory.

In fact, Fiore's treatment of categorical fixed points is more thorough, as he also considers parameterised bifunctors $F : \mathbb{A} \times \mathbb{C}^{op} \times \mathbb{C} \to \mathbb{C}$, where A also has the appropriate enrichment. For details we refer the reader to Chapters 6 and 7 in [29]. We just remark that by Theorem 7.1.12 in *op.cit.*, $\mathbf{TD}_\perp$ is parameterised $\omega\mathbf{CPO}$-algebraically compact, hence also has solutions for parameterised and mutually recursive domain equations.

3.3 Comparison with Classical Domain Theory

In the previous section, we have presented a category of topological predomains, consisting of topological spaces with a dcpo-like structure, and a category of topological domains, in which recursion can be modelled. A great influence in this interplay between topological and order-theoretic concepts has been Classical Domain Theory. In fact, we have shown that the most general category in Classical Domain Theory, **DCPO**, occurs as a full coreflective subcategory of **kP**, the category of compactly-generated predomains, which also includes **TP** faithfully. Moreover, it is clear that any dcpo whose Scott-topology has a countable base, is a qcb-space. In particular, we get the following.

Proposition 3.3.1. *The category* ω**CONT***, of Scott-continuous maps between* ω-*continuous dcpos, is a full subcategory of* **TP***.*

We now study in how far the categorical constructions, such as products, coproducts and function spaces, coincide in Classical and Topological Domain Theory. We recall that in Classical Domain Theory all these constructions employ the pointwise order, and then the Scott-topology on it. It is an easy observation that the specialization order on the corresponding constructions in Topological Domain Theory is also the pointwise order. So the problem reduces to the question whether the categorical constructions of Topological Domain Theory preserve the Scott-topology.

We start by investigating coproducts, which is the easiest case. Subsequently we consider products, and finally function spaces.

Lemma 3.3.2. *For any family* $\{X_i\}_{i \in I}$ *of dcpos with Scott-topology, the compactly-generated coproduct* $\coprod_{i \in I} X_i$ *carries the Scott-topology.*

Proof. Trivial, because **DCPO** is a coreflective subcategory of **kP**, by Proposition 3.1.5, and thus the inclusion functor **DCPO** $\hookrightarrow$ **kP** preserves colimits. Moreover, coproducts in **kP** coincide with coproducts in **kTop** (in fact, they are just topological coproducts). $\square$

Corollary 3.3.3. *For a countable family* $\{X_i\}_{i \in I}$ *of topological predomains carrying the Scott-topology, the* **QCB***-coproduct* $\coprod_{i \in I} X_i$ *carries the Scott-topology. Furthermore, if* $\{X_i\}_{i \in I}$ *is a countable family of topological domains with Scott-topology, then the* **TD**$_\perp$-*coproduct* $\bigoplus_{i \in I} X_i$ *carries the Scott-topology.*

Proof. The first part follows immediately from the above lemma and Corollary 3.2.6. For the second part, observe that the $X_i^\dagger$ are topological predomains and carry the Scott-topology. Moreover, for the **QCB**-coproduct $\coprod_{i \in I} X_i^\dagger$, which carries the Scott-topology, we have that $(\coprod_{i \in I} X_i^\dagger)_\perp \cong \bigoplus_{i \in I} X_i$. But clearly, $(\coprod_{i \in I} X_i^\dagger)_\perp$ carries the Scott-topology, as this is trivially preserved by lifting. $\square$

That the construction of finite products in **kTop** preserves the Scott-topology, follows (almost) immediately from the cartesian-closedness.

Lemma 3.3.4. *If X, Y are dcpos with Scott-topology, then the **kTop**-product $X \times Y$ carries the Scott-topology.*

Proof. [Simpson] The topology of the compactly-generated product $X \times Y$ is coarser than the Scott-topology, so it is sufficient to show that the two topologies are set-isomorphic. The Scott-topology on $X \times Y$ is set-isomorphic to $\mathbf{DCPO}(\mathcal{S}(X \times Y), \mathbb{S})$, which in turn is isomorphic to $\mathbf{DCPO}(X, Y \Rightarrow_{dcpo} \mathbb{S})$. However, $Y \Rightarrow_{dcpo} \mathbb{S} \cong Y \Rightarrow_k \mathbb{S}$, by Lemma 2.2.5, and thus $\mathbf{DCPO}(X, Y \Rightarrow_{dcpo} \mathbb{S})$ and $\mathbf{kTop}(X, Y \Rightarrow_k \mathbb{S})$ are set-isomorphic (via an obvious isomorphism identifying the respective map with itself). But the latter is isomorphic to $\mathbf{kTop}(X \times Y, \mathbb{S})$, which is the compactly-generated topology on $X \times Y$. $\qquad\square$

However, infinite products in **kTop** do not preserve the Scott-topology in general, as the following counterexample shows. Let **2** be the two-element discrete space, and consider the countable product $\mathbf{2}^{\mathbb{N}}$, which is a Hausdorff space, hence the Scott-topology on it is discrete. However, the **kTop**-product $\mathbf{2}^{\mathbb{N}}$ is a qcb-space, as **QCB** inherits countable products from **kTop**. But the discrete topology on $\mathbf{2}^{\mathbb{N}}$ does not have a countable Michael pseudobase, hence is not a qcb-space, showing that the **kTop**-product $\mathbf{2}^{\mathbb{N}}$ does not carry the Scott-topology.

Surprisingly, the situation gets resolved in the presence of least element as the following result shows, which is adapted from Heckmann's Theorem 7.8 in [42].

Lemma 3.3.5. *Let $\{X_i\}_{i \in I}$ be a family of dcppos with Scott topology. Then the **kTop**-product $\prod_{i \in I} X_i$ carries the Scott-topology.*

Proof. [Heckmann] The Scott-topology on the product refines the compactly-generated one, hence it is sufficient to show that the identity map $\prod_{i \in I} X_i \to \mathcal{S}(\prod_{i \in I} X_i)$ is continuous. For $J \subseteq I$, let $\pi_J : \prod_{i \in I} X_i \to \prod_{i \in J} X_i$ be the continuous projection onto the J-components. Furthermore, let $\rho_J : \prod_{i \in J} X_i \to \prod_{i \in I} X_i$ be the continuous map assigning to $(x_i)_{i \in J}$ the tuple $((x_i)_{i \in J}, (\bot_{X_i})_{i \in I \setminus J})$. For a finite $F \subseteq I$, we have $\prod_{i \in F} X_i \cong \mathcal{S}(\prod_{i \in F} X_i)$, by Lemma 3.3.4, hence the composite

$$\prod_{i \in I} X_i \xrightarrow{\ \pi_F\ } \prod_{i \in F} X_i \xrightarrow{\ \rho_F\ } \mathcal{S}(\prod_{i \in I} X_i)$$

is continuous. But the supremum $\bigvee^{\uparrow} \{\rho_F \circ \pi_F \,|\, F \subseteq I \text{ finite}\}$ in the compactly-generated function space $\prod_{i \in I} X_i \Rightarrow_k \mathcal{S}(\prod_{i \in I} X_i)$ is the set-theoretic identity map, which thus is continuous. $\qquad\square$

Corollary 3.3.6. *Let $\{X_i\}_{i \in I}$ be a countable family of topological predomains carrying the Scott-topology, such that all but a finite subfamily have a least element. Then the **QCB**-product $\prod_{i \in I} X_i$ carries the Scott-topology.*
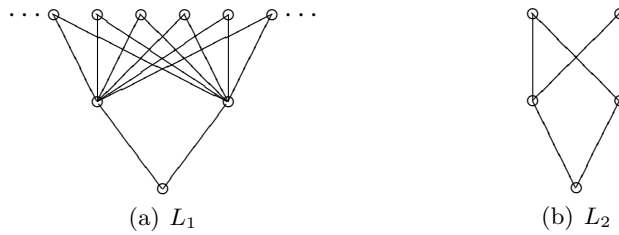
(a) L_1 (b) L_2

Figure 3.2: Two L-domains

Proof. This follows immediately from Lemmas 3.3.4 and 3.3.5, and Corollary 3.2.6. □

We finally turn our attention towards function spaces. Here the general situation is simple; neither does the compactly-generated function space construction (or the one in **QCB**) preserve the Scott-topology, nor does pointedness resolve this issue. The former can be easily seen by the above example for countable products, since the countable product $2^{\mathbb{N}}$ is isomorphic to the function space $\mathbb{N} \Rightarrow_k 2$. For the latter consider the following example.

Let L_1 and L_2 be the dcppos presented in Figure 3.2. Both are easily seen to be ω-continuous dcppos, hence locally compact countably-based spaces. Thus the compactly-generated function space $L_1 \Rightarrow_k L_2$ is countably-based, by Proposition 2.2.11. However, the dcpo function space $L_1 \Rightarrow_{dcpo} L_2$ is an algebraic dcpo, but not countably-based (see Jung's dissertation [62] where it is shown that $L_1 \Rightarrow_{dcpo} L_1$ has $2^{\mathbb{N}}$ compact elements; the same follows easily for $L_1 \Rightarrow_{dcpo} L_2$). Thus $L_1 \Rightarrow_k L_2$ and $L_1 \Rightarrow_{dcpo} L_2$ disagree.

In fact the domains of the previous example are both L-domains, hence they belong to one of the two maximal cartesian-closed subcategories of **CONT**$_{\perp}$, the category of pointed continuous dcpos and continuous maps between them, see [63]. (Notice that the subscript $\perp$ in **CONT**$_{\perp}$ has a different meaning from the one in **TD**$_{\perp}$. We use this notation to conform with the literature.) Regarding the above example, one might say that, considering the function space construction, Classical Domain Theory does not embed faithfully into Topological Domain Theory. However, we remark that the above example shows that the **DCPO**-construction of function spaces does not preserve the property of having a countable base (which from a computational viewpoint would be desirable). This phenomenon is reflected by the fact that although the category of L-domains forms a cartesian-closed category of continuous dcppos, its countably-based counterpart is not cartesian-closed.

In contrast, the other maximal cartesian-closed category of **CONT**$_{\perp}$ is given by the category of pointed FS-domains, in which the function space construction does preserve the countable base, and we now show that the compactly-generated function space of two FS-domains does carry the Scott-topology. We start by defining FS-domains, and remark that our definition differs slightly from Jung's original one of [63], but is

equivalent as Lemma 2 of *op.cit.* shows.

Definition 3.3.7. An *FS-domain* is a continuous dcppo X for which there exists a directed family $\{f_i\}_{i \in I}$ of continuous endofunctions, each *strongly finitely separated* from id_X, i.e. for each f_i there exists a finite separating set $M_{f_i} \subseteq X$ such that for all $x \in X$, there exists some $m \in M_{f_i}$ with $f_i(x) \sqsubseteq m \ll x$, and $\bigvee^{\uparrow}_{i \in I} f_i = \mathrm{id}_X$. An ωFS-*domain* is an FS-domain with a countable basis.

FS-domains equipped with Scott-topology and continuous maps form the category **FS**, the ωFS-domains its full subcategory ω**FS**.

Next, we give a refinement of Proposition 2.2.11 for function spaces between continuous dcpos in the category **TP**.

Definition 3.3.8. For topological spaces X, Y, the *point-open topology* (sometimes called topology of pointwise convergence) on the set of continuous maps $C(X, Y)$ is generated by subbasic open sets of the form $\langle x, U \rangle$, defined as

$$\langle x, U \rangle := \{f \in C(X, Y) \mid f(x) \in U\},$$

for $x \in X$ and $U \subseteq Y$ open. The point-open function space is denoted by $[X \to_{pto} Y]$.

Notice that the point-open topology is the subspace topology of $C(X, Y)$ in the product space $Y^{|X|}$.

Lemma 3.3.9. *For an ω-continuous dcpo X and a countably-based space Y, the compactly-generated function space $X \Rightarrow_k Y$ carries the point-open topology.*

Proof. We know from the proof of Propositions 2.2.3 and 2.2.11 that $X \Rightarrow_k Y$ carries the compact-open topology, which obviously refines the point-open topology. Thus it suffices to show that every subbasic compact open set $\langle K, U \rangle$ is open in the point-open topology. But for $f \in \langle K, U \rangle$, we have $K \subseteq f^{-1}(U)$, hence by X being a continuous dcpo, $K \subseteq \bigcup_{x \in f^{-1}(U)} \mathop{\uparrow}\!\!\!\uparrow x$. By compactness of K, this yields a finite $F \subseteq f^{-1}(U)$ with $K \subseteq \bigcup_{x \in F} \mathop{\uparrow}\!\!\!\uparrow x$. But now we have that $\bigcap_{x \in F} \langle x, U \rangle$ is a point-open neighbourhood of f, and contained in $\langle K, U \rangle$, showing that $\langle K, U \rangle$ is open in the point-open topology, as required. $\qquad \square$

Finally, we need the following result which is a mild generalisation of Corollary 1.36 of [62], but has essentially the same proof.

Proposition 3.3.10. *If D is a dcpo and E a continuous dcpo such that $[D \to E]$ with the pointwise order is a continuous dcpo, then $f \ll g$ implies $f(x) \ll g(x)$ for all $x \in D$.*

Proof. Suppose $\{e_i\}_{i \in I}$ is a directed family of elements in E with $\bigvee^{\uparrow} e_i = g(x)$. We have to show that there exists $i_0 \in I$ such that $e_{i_0} \sqsupseteq f(x)$. Observe that $[\downarrow x \to \downarrow g(x)]$ is a retract of $[D \to E]$, and therefore a continuous dcpo. For each $h : D \to E$, let $h' : \downarrow x \to \downarrow g(x)$ denote the image of h under the retraction.

Then $f' \ll g'$, for if $\{\psi_j\}_{j \in J}$ is a directed family of functions in $[\downarrow x \to \downarrow g(x)]$ with $\bigvee^{\uparrow} \psi_j = g'$, then $\{\Psi_j\}_{j \in J}$, defined as

$$\Psi_j(y) = \begin{cases} \psi_j(y) & \text{if } y \sqsubseteq x \\ g(y) & \text{otherwise} \end{cases}$$

is a directed family of functions in $[D \to E]$ with $\bigvee^{\uparrow} \Psi_j = g$. Thus there exists $j_0 \in J$ such that $\Psi_{j_0} \sqsupseteq f$, and hence $\psi_{j_0} \sqsupseteq f'$.

Now for each $i \in I$, let $c_{e_i} : \downarrow x \to \downarrow g(x)$ denote the constant function with value e_i. Then $\bigvee^{\uparrow} c_{e_i} \sqsupseteq g'$, hence there exists $i_0 \in I$ such that $c_{e_{i_0}} \sqsupseteq f'$, giving

$$e_{i_0} = c_{e_{i_0}}(x) \sqsupseteq f'(x) = f(x),$$

as desired. $\qquad \square$

Now we can show the central result about function spaces between FS-domains in Topological Domain Theory.

Theorem 3.3.11. *If D is an FS-domain and E a continuous dcpo such that $[D \to E]$ is a continuous dcpo, then the compactly-generated function space $X \Rightarrow_k Y$ carries the Scott-topology.*

Proof. We have to show that the Scott-topology and the point-open topology on $[X \to Y]$ coincide. Notice that this does not directly follow from Lemma 3.3.9, as we have dropped the countability conditions. However, once we have shown $[X \to_{pto} Y] \cong [X \to_{Scott} Y]$, Lawson's Theorem 2.2.4 yields that $[X \to_{pto} Y]$ is compactly-generated and hence

$$[X \to_{Scott} Y] \cong [X \to_{pto} Y] \cong [X \to_{co} Y] \cong X \Rightarrow_k Y.$$

So let $h \in [D \to E]$ and $\mathcal{U}$ be a Scott-open neighbourhood of h. We show that there exists a point-open neighbourhood $\mathcal{V}$ of h such that $\mathcal{V} \subseteq \mathcal{U}$.

Since D is an FS-domain, there exists a directed set $\{f_i\}_{i \in I}$ of endofunctions, each strongly finitely seperated from id_D, with finite seperating sets M_{f_i}, and $\bigvee^{\uparrow} f_i = \mathrm{id}_D$. Furthermore, since $[D \to E]$ is a continuous dcpo, there exists a directed set $\{\psi_j\}_{j \in J}$ of endofunctions such that each $\psi_j \ll h$ and $\bigvee^{\uparrow} \psi_j = h$.

Thus $\{\psi_j \circ f_i\}_{i \in I, j \in J}$ also is a directed set such that each $\psi_j \circ f_i \ll h$ and $\bigvee^{\uparrow} \psi_j \circ f_i = h$, hence a Scott-open neighbourhood basis for $h \in [D \to E]$ is given by $\{\Uparrow (\psi_j \circ f_i)\}_{i \in I, j \in J}$. Thus, there exist $i_0 \in I$ and $j_0 \in J$ such that $h \in \Uparrow (\psi_{j_0} \circ f_{i_0}) \subseteq \mathcal{U}$.

Set $\mathcal{V} = \bigcap_{m \in M_{f_{i_0}}} (m, \uparrow \psi_{j_0}(m))$, then $h \in \mathcal{V}$, by Proposition 3.3.10. We claim that $\mathcal{V} \subseteq \mathcal{U}$.

For that let $x \in D$ and $h' \in \mathcal{V}$. Then there exists $m \in M_{f_{i_0}}$ with $f_{i_0}(x) \sqsubseteq m \ll x$, and so

$$(\psi_{j_0} \circ f_{i_0})(x) \sqsubseteq \psi_{j_0}(m) \ll h'(m) \sqsubseteq h'(x).$$

Thus $\mathcal{V} \subseteq \uparrow (\psi_{j_0} \circ f_{i_0}) \subseteq \mathcal{U}$, showing the claim. $\qquad\qquad\square$

We remark that it is possible to derive the above result directly from the literature, namely from Corollary III-5.18 of [36]. However, the proof here is more elementary and so we give it for our self-contained treatment of Topological Domain Theory. With the above theorem, we immediately get the desired preservation property.

Corollary 3.3.12. *For ωFS-domains X, Y, the* **QCB***-exponential Y^X carries the Scott-topology.*

Thus we find the largest cartesian-closed category of Classical Domain Theory on which a notion of computability may be developed, as a faithful subcategory in Topological Domain Theory. Moreover, we claim that in the more general case the function space construction of Topological Domain Theory is preferable to the classical Scott-function space, because Proposition 2.2.11 shows that if X and Y are ω-continuous dcpos then the **QCB**-function space Y^X is again countably-based, carrying the compact-open topology.

We finish the comparison of categorical structures in Classical and Topological Domain Theory by identifying another class of compactly-generated function spaces, which carry the Scott-topology. More specifically, we generalise Lemma 2.2.5. The starting point for this is the following result which follows from Lemma 7.14 of [128].

Lemma 3.3.13. *If a continuous dcpo has binary infima, then it has infima for all nonempty compact subsets.*

Theorem 3.3.14. *If Y is a continuous dcpo with binary infima, then for all compactly-generated spaces X, the function space $X \Rightarrow_k Y$ carries the Scott-topology.*

Proof. We need to show that every Scott-open subset $W \subseteq X \Rightarrow_k Y$ is indeed open in $X \Rightarrow_k Y$. Consider any continuous map $p : K \to X \Rightarrow_k Y$ for a compact Hausdorff space K. We must show that for all $k \in p^{-1}(W)$, there exists an open neighbourhood of k included in $p^{-1}(W)$.

Let D be the set of compact neighbourhoods of k, ordered by reverse inclusion. Then D is directed, because as a compact Hausdorff space, K is locally compact. For $L \in D$, define $h_L : X \to Y$ by

$$h_L(x) = \bigwedge_{z \in L} p(z)(x),$$

which is continuous, since it arises as the composite of continuous maps

$$X \xrightarrow{\tilde{p}} (K \Rightarrow_k Y) \longrightarrow (L \Rightarrow_k Y) \xrightarrow{M_L} Y$$

the components of which we describe next.

The map $\tilde{p}$ is obtained applying the exponential transpose twice to p:

$$\frac{\dfrac{K \to (X \Rightarrow_k Y)}{K \times X \to Y}}{X \to (K \Rightarrow_k Y)}$$

The map $(K \Rightarrow_k Y) \to (L \Rightarrow_k Y)$ is just the restriction function, and M_L given by $M_L(f) = \bigwedge_{z \in L} f(z)$. This is continuous, because if $y \ll M_L(f)$, then there exists a y' with $y \ll y' \ll M_L(f)$, and so $\langle L, \uparrow y' \rangle$ is a neighbourhood of f in $L \Rightarrow_k Y$ satisfying $y \ll M_L(g)$ for all $g \in \langle L, \uparrow y' \rangle$.

Clearly if $L' \leq L$ in D, then $h_{L'} \sqsubseteq h_L$, as D is ordered by reverse inclusion. But then $H = \{h_L \mid L \in D\}$ is directed in $X \Rightarrow_k Y$, and we claim $\bigvee^\uparrow H = p(k)$.

Trivially, it holds that $\bigvee^\uparrow H \sqsubseteq p(k)$, since for $L \in D$, we have $k \in L$, and thus we get $h_L(x) = \bigwedge_{z \in L} (p(z))(x) \sqsubseteq (p(k))(x)$. For the converse, suppose $x \in X$ and $y \ll (p(k))(x)$. By continuity of p, $U = \{z \in K \mid y \ll (p(z))(x)\}$ is open in K with $k \in U$. By local compactness, we find a compact neighbourhood L of k with $L \subseteq U$, and for this L, we have $h_L(x) = \bigwedge_{z \in L} (p(z))(x) \sqsupseteq y$, hence $y \sqsubseteq (\bigvee^\uparrow H)(x)$.

Summarising, we have a directed $H \subseteq X \Rightarrow_k Y$ with $\bigvee^\uparrow H = p(k) \in W$. As W is Scott-open, there exists $L \in D$ such that $h_L \in W$. But then for any $z \in L$, it holds that $p(z) \sqsupseteq h_L$, hence $p(z) \in W$, and so $L \subseteq p^{-1}(W)$, showing $p^{-1}(W)$ is an open neighbourhood of k, as required. $\qquad\square$

We do not know whether the condition for Y having binary meets is necessary to obtain the above result. However, for a restricted class of spaces we can improve it, using Proposition 2.2.3, Lawson's Theorem 2.2.4 and Proposition II-4.6 of [36].

Proposition 3.3.15. *Let X be locally compact, and Y be a bounded-complete continuous dcppo. Then $X \Rightarrow_k Y$ carries the Scott-topology and is itself a bounded-complete continuous dcppo.*

Proof. Proposition II-4.6 of [36] says that the compact-open and the Scott-topology on the function space $[X \to Y]$ agree, hence with Proposition 2.2.3 and Lawson's Theorem 2.2.4, the result follows. $\qquad\square$

Corollary 3.3.16. *If Y is an ω-continuous dcpo with binary infima, then for all qcb-spaces X, Y^X carries the Scott-topology. If moreover, X is locally compact, and Y a bounded-complete ω-continuous dcppo, then Y^X is itself a bounded-complete ω-continuous dcppo.*
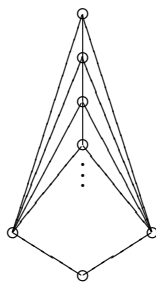
Figure 3.3: U_ω

Notice however, that in general, if Y is a continuous dcppo with binary infima, the function space $X \Rightarrow_k Y$ need not be a continuous dcpo again, although it carries the Scott-topology by Theorem 3.3.14. Consider the ω-algebraic dcppo U_ω given in Figure 3.3. Then it is well-known from Classical Domain Theory that the Scott-function space $[U_\omega \rightarrow_{dcpo} U_\omega]$ ($\cong U_\omega^{U_\omega}$) is not continuous. However, despite this lack of continuity, our results show that it is countably-based, and has the point-open topology, by Lemma 3.3.13. Thus from the viewpoint of Topological Domain Theory, the function space $U_\omega^{U_\omega}$ is well-behaved.

To summarise, we have shown that Classical Domain Theory embeds into Topological Domain Theory, since ω**CONT** is a full subcategory of **TP**. Moreover, this embedding is faithful in the sense that countable coproducts and finite products are preserved in general, and countable products in the presence of least elements. Furthermore, the inclusion of the unique largest cartesian-closed subcategory of ω**CONT**$_\perp$, the category of pointed ωFS-domains, into **TP** preserves the cartesian-closed structure. In other cases, the Scott-function space construction leads out of the category of ω**CONT** for various reasons: the countable basis might be lost, continuity might not be preserved, or both. In Topological Domain Theory however, the function space construction between two ω-continuous dcpos is always countably-based, and although it might not carry the Scott-topology, the point-open topology on it gives an intuitive and straightforward characterisation of the function space topology.

4 Computational effects for QCB

In the previous chapters we have investigated the categorical structure of **QCB** and its subcategories of topological (pre)domains. The results were that this structure is very rich and enables us to model the usual type constructions of a programming language based on the simply-typed λ-calculus, as well as recursive types. However, in real-world computing there are features which are not captured by such a pure calculus. We think of them as non-functional behaviour or *computational effects*. Common examples of computational effects are nontermination, nondeterminism, probabilistic computations, interactive input/output, exceptions, global state or continuations.

The initial approach to model these computational effects semantically was taken by Moggi [95, 97, 96], who suggested to distinguish between types of values and types of computations. For each type of values X in the language, there is a corresponding type of computations, TX, on values of X, which involves a given effect. In Moggi's theory, T is the functor of a monad, and a semantic model for a language with effects (his computational λ-calculus [95]) is given by the Kleisli-category $\mathbb{C}_T$ over some cartesian-closed category $\mathbb{C}$. Recently, Plotkin and Power [111, 112, 110] have shown that a surprising number of the computationally interesting monads are generated as free algebras for equational theories in the sense of universal algebra. What is striking about their approach is that the associated algebraic operations are exactly the natural computational primitives for generating the relevant effects. The first section of this chapter summarizes this work.

In view of Plotkin and Power's approach, one would like to show that a general framework for denotational semantics supports free algebra constructions for a sufficiently rich class of equational theories, in order to model a wide class of computational effects. An appropriate class of theories is given by *parameterised equational theories* and is defined below. In the second and third section of this chapter, we give an explicit construction of free algebras for such parameterised equational theories in the categories of compactly-generated and sequential spaces. More specifically, we show that the elements of these free algebras are given by terms constructed from the signature and the underlying space factorised by a congruence relation induced by the set of equations. The procedure here is exactly as in universal algebra. The topology on the free compactly-generated and sequential algebras is then constructed explicitly as a quotient topology of an inductively defined topology on the set of terms. This

inductive definition of the topology, given in Theorem 4.2.4, is surprisingly obtained by an ordinary ω-limit process, independent of the size of the arity sets of the operations.

In the fourth section of this chapter, these results are used to show that **QCB** is closed under the free algebra construction for ω-ary parameterised equational theories in **Seq**, and for finitary parameterised equational theories in **kTop**. This yields the desired free algebra construction for **QCB**. In the final section of the chapter, we show that the computational monads induced by parameterised equational theories are strong. In particular, this shows that the classical computational effects can be modelled in **QCB** using strong monads, and so **QCB** provides a model for the computational λ-calculus.

The results of the later sections of this chapter have been presented at MFPS 2005 in Genoa, and appeared in the conference proceedings [8]. A revised version [7], also including important results of the following chapter, has been submitted to a journal for selected papers from this conference. Unfortunately, in the process of the examination of this dissertation a mistake in these articles has been found. In the articles an incorrect generalisation of Theorem 2.3.5 has been used to show certain topological quotient constructions. Here we fix this mistake, by adapting the notion of congruences from universal algebra to parameterised equational theories and obtain our Main Theorem 4.4.3 on this alternative way.

4.1 Effects via algebras

The principles of Moggi's theory of modelling computational effects semantically can be described with help of the following example. Suppose we want to give a set-theoretical denotational semantics for a programming language based on the simply-typed call-by-value λ-calculus. Furthermore, assume the execution of a program has some nondeterministic behaviour. Clearly we cannot model such an effect on every set. However, there might exist a class of special sets which model this nondeterministic behaviour. Now we would like to have an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$, assigning to any ordinary set X a corresponding special set TX of computations returning values over X. For instance T can be taken to be the finite nonempty powerset functor. Then a given finite subset $F \subseteq X$ denotes a nondeterministic computation whose possible results are the values of F.

If T is the functorial part of a monad, as is the case for the finite nonempty powerset functor, then a model for our programming language involving nondeterministic behaviour is obtained by the Kleisli-category $\mathbf{Set}_T$, which has arrows of the form $X \to TY$. Hence the return type of a program is a computational type over the values we expect it to take.

In this vein, Moggi [97] identified computational effects denotationally with monads, so-called *effect monads*. He introduced the computational λ-calculus [95], and used

Kleisli categories for giving denotational semantics to it. Particular examples of effect monads for a set-theoretic model of the computational λ-calculus are the following:

Nontermination: The underlying functor for the nontermination monad on **Set** assigns to a set X the set $X \cup \{\perp_X\}$, where $\perp_X$ denotes nontermination.

Nondeterminism: Here the corresponding monad is the finite nonempty powerset monad $\mathcal{P}_f$. The idea is that a nondeterministic computation might result in several different values, hence it is modelled by the finite subset consisting of the corresponding elements.

Probabilistic computations: One possible monad for modelling probabilistic computations on sets is given by the finite convex combinations monad, which assigns to X the set $FConv(X)$ of finite convex combinations $\lambda_0 x_0 + \ldots \lambda_k x_k$ over X, where $\lambda_0, \ldots, \lambda_k \in \mathbb{R}^+$, with $\lambda_0 + \cdots + \lambda_k = 1$, and $x_0, \ldots, x_k \in X$. Here one interprets $\lambda_0 x_0 + \cdots + \lambda_k x_k$ as a computation, returning x_0 with probability λ_0, x_1 with probability λ_1, and so on.

Exceptions: Let E be the set of possible exceptions, then the exception monad assigns to a set X the set $X + E$, since a computation can either result in a value $x \in X$ or in an exception $e \in E$.

Input/Output: The monad for interactive I/O assigns to a set X the smallest set Y such that $Y \cong X + X \times O + Y^I$, where I and O are sets of possible inputs and outputs, respectively. The idea behind this is that a computation either simply returns a value of X, or it returns a value together with some output in O, or it expects an input from I to determine how the computation continues.

Global State: For sets L of memory locations and V of values, define a set S of states by $S = V^L$, i.e. a state is an assignment of values to the set of memory locations. Then the global state monad is given by $X \mapsto (X \times S)^S$, because a program depends on the current state and returns the result of a computation together with a new state.

Continuations: For a monad for continuations, Moggi suggested to use the functor $X \mapsto R^{R^X}$, where R is a set of possible results, see [95].

For non set-theoretic models the above monads can be adjusted. For instance for a domain-theoretic semantics, nontermination is modelled by the lifting monad, nondeterminism by the upper, lower or convex powerdomain monad [2], probabilistic computations by the probabilistic powerdomain [58], and the last four effects by similar categorical constructions to the ones above.

In practice, monads have been implemented into the programming language Haskell, where they are used to invoke certain effects, enabling the programmer to communicate with the program environment, see e.g. [61, 60, 163, 164].

One major drawback of Moggi's approach is compositionality; given two effect monads T_1 and T_2, it is not clear how to combine them suitably into a monad T_3 capturing both effects. Moggi suggested to use so-called monad-transformers [96], which are spe-

cific to each monad and enable one to build one effect on top of another. However, this treatment appears to be rather ad-hoc, and a more intrinsic approach is desired.

This problem has been addressed by Plotkin and Power [111, 112, 110, 52], who investigated how effect monads can be constructed from elementary principles, and successively refined Moggi's approach. Notice that the above definitions of effect monads do not explicitly describe the nonfunctional behaviour which triggers the effects. For instance nondeterminism is caused by nondeterministic choices of the underlying system. Such a nondeterministic choice typically corresponds to a binary operation on the underlying type, which is commutative, associative and idempotent. If we consider the nondeterminism monad, the finite nonempty powerset monad $\mathcal{P}_f$, then we see that for all sets X, there exists a binary operation $\cup : (\mathcal{P}_f X)^2 \to \mathcal{P}_f X$ (given by set-union), which satisfies these laws. In fact, $\mathcal{P}_f X$ is the free algebra over X for an equational theory given by such a binary operation. Hence, the nondeterminism monad is induced by a free algebra construction for an equational theory describing a nondeterministic choice operation.

This way of considering computational effects is strongly influenced by Plotkin's work [108, 43, 109] on using algebraic principles in operational and denotational semantics, which culminated in the powerdomain constructions of Classical Domain Theory; the classical powerdomain constructions for nondeterminism are induced by free algebra functors for inequational algebraic theories, see [2].

It has been shown by Plotkin and Power [112] that, with exception of the continuations monad, all of the classical computational effect monads are induced by free algebra functors for equational algebraic theories. In the case of the standard examples, we get the following:

Nontermination: The algebraic theory for nontermination is given by a 0-ary operation (or constant), $\bot$, and no equations. The free algebra functor for this theory is exactly the one given above.

Nondeterminism: As described, here we have a binary operation *choose* $: X^2 \to X$ subject to commutativity, associativity and idempotence, for which the free algebra functor induces the finite nonempty powerset monad.

Probabilistic choice: Similar to the nondeterminism example, probabilistic computations are caused by a probabilistic choice operation $+ : \mathbb{I} \times X^2 \to X$, which is binary on X and parameterised by the unit interval $\mathbb{I}$. Then $+(\lambda, x, y)$, or simply $x +_\lambda y$, denotes a computation returning x with probability λ and y with probability $1 - \lambda$. The suitable axioms for this probabilistic choice operation are:

- $x +_1 y = x$,

- $x +_\lambda x = x$,

- $x +_\lambda y = y +_{1-\lambda} x$,

- $(x +_\lambda y) +_{\lambda'} z = x +_{\lambda\lambda'} (y +_{1 - \frac{\lambda'(1-\lambda)}{1 - \lambda\lambda'}} z)$, for $\lambda\lambda' \neq 1$.

The free algebra functor for this equational theory is the functor $FConv$ of finite convex combinations, assigning to a set X, the free convex set $FConvX$ over X.

Exceptions: For each exception $e \in E$, let $raise_e$ be a constant of an equational theory with no equations. Then the free algebra functor on **Set** is $X \mapsto X + E$. (Notice here that there is an issue with exception handling, as $handle_e$ is not an operation in the usual sense. There have been attempts to define and characterise $handle$ as a *deconstructor* by Plotkin and Pretnar [private communication]. Also Levy [83] has incorporated $handle$ into his Call-by-Push-Value calculus.)

Interactive I/O: Interactive I/O is caused by a read operation and a write operation. Concretely, they are given by $read : X^I \to X$, where I is a set of input symbols, and $write : O \times X \to X$, for O being a set of output symbols. There are no equations. The idea is that $read$ accepts inputs $i \in I$ and continuously evaluates values $x \in X$ dependent on i, whereas $write$ takes an output $o \in O$ and a computation $x \in X$ and disposes the output by printing it. The corresponding free algebra functor yields the monad for I/O given above.

Global State: Let, as above, L be a set of memory locations and V be a set of values. Then the Global State monad is induced by the following equational algebraic theory. We have two operations: $lookup : X^V \to X^L$, such that $lookup(f)(l)$ evaluates to $f(v)$, where v is the value stored in memory location l, and $update : X \to X^{L \times V}$, such that $update(x)(l, v)$ denotes that the value v is stored at memory location l and then the execution of the computation x continues. There are 7 suitable axioms for this equational theory, expressed by demanding the following diagrams to commute:

1. The first diagram expresses that updating a memory location by the value just read out of it does no change:

$$
\begin{array}{ccccc}
X & \xrightarrow{\;update\;} & X^{L \times V} & \xrightarrow{\;\cong\;} & (X^V)^L \\
\Big\downarrow{\scriptstyle X^!} & & & & \Big\downarrow{\scriptstyle lookup^L} \\
X^L & \xleftarrow{\;X^{\delta_L}\;} & X^{L \times L} & \xleftarrow{\;\cong\;} & (X^L)^L
\end{array}
$$

2. This expresses that if a computation is dependent on two values, which are to be

determined by reading out a single memory location, then one lookup is sufficient:

$$
\begin{array}{ccccc}
(X^V)^V & \xrightarrow{\ lookup^V\ } & (X^L)^V & \xrightarrow{\ \cong\ } & (X^V)^L \\
\cong \downarrow & & & & \downarrow lookup^L \\
X^{V \times V} & & & & (X^L)^L \\
X^{\delta_V} \downarrow & & & & \downarrow \cong \\
X^V & \xrightarrow{\ lookup\ } & X^L & \xleftarrow{\ X^{\delta_L}\ } & X^{L \times L}
\end{array}
$$

3. This diagram says that if we update the same memory location twice in succession, the first update is redundant:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ update\ } & X^{L \times V} & \xrightarrow{\ update^{L \times V}\ } & (X^{L \times V})^{L \times V} \\
update \downarrow & & & & \downarrow \cong \\
X^{L \times V} & \xrightarrow{\ X^{L \times \pi_1}\ } & X^{L \times V \times V} & \xleftarrow{\ X^{\delta_L \times V \times V}\ } & X^{L \times L \times V \times V}
\end{array}
$$

4. The following expresses that if we lookup a memory location we have just updated, then the obtained value is the one we put into it:

$$
\begin{array}{ccccc}
X^V & \xrightarrow{\ lookup\ } & X^L & \xrightarrow{\ update^L\ } & (X^{L \times V})^L \\
update^V \downarrow & & & & \downarrow X^{\delta_L \times V} \\
(X^{L \times V})^V & & \xrightarrow{\ X^{L \times \delta_V}\ } & & X^{L \times V}
\end{array}
$$

5. This diagram says that if we lookup two distinct locations to determine a computation, the order is irrelevant (L_2 denotes the set of pairs of distinct memory locations, which is evidently a subset of $L \times L$):

$$
\begin{array}{ccccccc}
(X^V)^V & \xrightarrow{\ lookup^V\ } & (X^L)^V & \xrightarrow{\ \cong\ } & (X^V)^L & \xrightarrow{\ lookup^L\ } & (X^L)^L \\
swap \downarrow & & & & & & \downarrow swap \\
(X^V)^V & & & & & & (X^L)^L \\
lookup^V \downarrow & & & & & & \downarrow X^{inc} \\
(X^L)^V & \xrightarrow{\ \cong\ } & (X^V)^L & \xrightarrow{\ lookup^L\ } & (X^L)^L & \xrightarrow{\quad} & X^{L_2}
\end{array}
$$

6. Similar to the previous diagram, the following expresses that when updating two

distinct locations, it does not matter which is updated first:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ update\ } & X^{L\times V} & \xrightarrow{\ update^{L\times V}\ } & (X^{L\times V})^{L\times V} & \xrightarrow{\ swap\ } & (X^{L\times V})^{L\times V} \\[4pt]
\downarrow{\scriptstyle update} & & & & & & \downarrow{\scriptstyle X^{inc\times V\times V}} \\[4pt]
X^{L\times V} & & \xrightarrow{\qquad update^{L\times V}\qquad} & & (X^{L\times V})^{L\times V} & \xrightarrow{\ X^{inc\times V\times V}\ } & X^{L_2\times V\times V}
\end{array}
$$

7. This final diagram expresses that a computation determined by looking up a memory location and updating another distinct location, is independent of the order of those operations:

$$
\begin{array}{ccccccc}
X^V & \xrightarrow{\ lookup\ } & X^L & \xrightarrow{\ update^L\ } & (X^{L\times V})^L & \xrightarrow{\ \cong\ } & (X^L)^{L\times V} \\[4pt]
\downarrow{\scriptstyle update^V} & & & & & & \downarrow{\scriptstyle X^{inc\times V}} \\[4pt]
(X^{L\times V})^V & \xrightarrow{\ \cong\ } & (X^V)^{L\times V} & \xrightarrow{\ lookup^{L\times V}\ } & (X^L)^{L\times V} & \xrightarrow{\ X^{inc\times V}\ } & X^{L_2\times V}
\end{array}
$$

We see that the underlying equational algebraic theories inducing these effect monads are very similar to the equational theories in universal algebra [22].

Let us recall that a signature in set-theoretic universal algebra is given by a set of operation symbols Σ such that each $\sigma \in \Sigma$ has a finite arity $|\sigma| \in \mathbb{N}$. An algebra for Σ is then a set A, together with an operation $\sigma_A : A^{|\sigma|} \to A$ for each $\sigma \in \Sigma$. The last example of Global State shows that this notion of operations of universal algebra is not general enough to capture the algebraic theories needed to model computational effects, the reason being that in this example we also have an arity in the codomain of an operation, and that, in general, the set of values V is not finite.

Hyland, Plotkin and Power [112, 114, 53] advertise the use of enriched Lawvere Theories as an extension of universal algebra for modelling computational effects in a general categorical setting. However, for Topological Domain Theory, the following definition of parameterised equational theories is more suitable. The reason for this is that, although Lawvere Theories are more general and the existence of free algebras for them follows from the general results presented in Chapter 6 below, we do not know how to give an elementary characterisation for the corresponding free algebra functors in Topological Domain Theory. The parameterised equational theories in turn, which we introduce next, are more elementary, hence easier to handle in our specific setting. We are grateful to Gordon Plotkin for suggesting the use of parameterised theories in Topological Domain Theory.

In the following definition we distinguish between parameterised operations of finite and countable arities for the following reason. As we have argued above, finite arities are

in general not sufficient enough to capture the operations which generate computational effects. For instance, the lookup operation generating the Global State monad has a non-finite arity and the same holds for the read operation generating the Input/Output monad. Thus we propose countable arities as an adequate generalisation to model computational effects algebraically in Topological Domain Theory. However in the next chapter we want to compare free algebra constructions in Topological Domain Theory with free algebra constructions in Classical Domain Theory. So far, we have only succeeded to compare them for algebraic theories with operations of finite arity. The reason for this is technical and explained below.

Definition 4.1.1. Let $\mathbb{C}$ be a category with finite (countable) products. A *signature* Σ for a *parameterised algebraic theory* for $\mathbb{C}$ is a set of operation symbols, such that:

- each symbol $\sigma \in \Sigma$, has an associated *parameter object* P_σ of $\mathbb{C}$,

- for each symbol $\sigma \in \Sigma$, there exists an associated set $|\sigma|$ of finite (countable) cardinality, called its *arity*.

A Σ-*algebra* in $\mathbb{C}$ (also $\mathbb{C}_\Sigma$-algebra) is a tuple $(A, \{\sigma_A\}_{\sigma \in \Sigma})$ (shortly $(A, \{\sigma_A\})$) such that A is a $\mathbb{C}$-object and for all $\sigma \in \Sigma$,

$$\sigma_A : P_\sigma \times A^{|\sigma|} \to A$$

is a $\mathbb{C}$-morphism. Here $A^{|\sigma|}$ denotes the $|\sigma|$-fold power of A.

A Σ-*homomorphism* between Σ-algebras $(A, \{\sigma_A\})$ and $(B, \{\sigma_B\})$ is a $\mathbb{C}$-morphism $h : A \to B$ such that for all $\sigma \in \Sigma$, the following diagram commutes:

$$
\begin{array}{ccc}
P_\sigma \times A^{|\sigma|} & \xrightarrow{P_\sigma \times h^{|\sigma|}} & P_\sigma \times B^{|\sigma|} \\
\sigma_A \downarrow & & \downarrow \sigma_B \\
A & \xrightarrow{\quad h \quad} & B
\end{array}
$$

If the arities of all operations in Σ have finite cardinality, the signature Σ is called *finitary*. If the arities are countable, Σ is called ω-*ary*.

The category $\mathbb{C}_\Sigma$ is given by the Σ-algebras and Σ-homomorphisms between them.

We remark that this definition is strictly weaker than the enriched Lawvere Theories of Plotkin and Power. The reason for this is that our arities are given by sets, instead of objects of a more general enriching category.

Let us next consider how we can express the equations of the above examples accordingly. In set-theoretic universal algebra, equations are given by pairs of Σ-terms s and t, which can be viewed as maps $s, t : A^n \to A$ for some $n \in \mathbb{N}$. Let us recall that these Σ-terms can be defined inductively as follows.

- For any $n \in \mathbb{N}$, the projection $\pi : A^n \to A$ is a Σ-term.

- For every operation symbol $\sigma \in \Sigma$, the map $\sigma_A : A^{|\sigma|} \to A$ is a Σ-term.

- Suppose $t : A^n \to A$ is a Σ-term, and for every $i \leq n$, we are given a Σ-term $t_i : A^m \to A^n$, then the composite:

$$A^m \xrightarrow{\langle t_i \rangle_{i \leq n}} A^n \xrightarrow{\quad t \quad} A$$

  is a Σ-term.

Notice that these Σ-terms are in fact natural transformations $U_n \Rightarrow U$, where $U : \mathbb{C}_\Sigma \to \mathbb{C}$ denotes the obvious forgetful functor and U_n is the composite:

$$\mathbf{Set}_\Sigma \xrightarrow{\quad U \quad} \mathbf{Set} \xrightarrow{\quad \Delta \quad} \mathbf{Set}^n$$

(Here Δ denotes the diagonal functor assigning to an object X the n-th power X^n.) The naturality means that whenever $t : U_n \Rightarrow U$ is a Σ-term and $h : (A, \{\sigma_A\}) \to (B, \{\sigma_B\})$ is a Σ-homomorphism, then the following diagram commutes:

$$
\begin{array}{ccc}
A^n & \xrightarrow{\ h^n\ } & B^n \\
{\scriptstyle t_A}\downarrow & & \downarrow{\scriptstyle t_B} \\
A & \xrightarrow{\ h\ } & B
\end{array}
$$

Thus the Σ-terms can be viewed as a generalised form of Σ-operations. In Plotkin and Power's [111] such generalised operations are called *algebraic operations.*

We now transfer this classical inductive definition of terms and equations to the parameterised setting. Following Plotkin and Power, we call the terms algebraic operations. For the formal definition, let U_α denote again the composite functor:

$$\mathbb{C}_\Sigma \xrightarrow{\quad U \quad} \mathbb{C} \xrightarrow{\quad \Delta \quad} \mathbb{C}^\alpha$$

where α is a set and $\mathbb{C}$ is sufficiently complete.

Definition 4.1.2. Let Σ be a finitary (ω-ary) signature for a parameterised algebraic theory for $\mathbb{C}$. A *(Σ)-algebraic operation* with parameter object $Z \in Ob(\mathbb{C})$ is a natural transformation $t : Z \times U_\alpha \Rightarrow U$, for some finite (countable) set α, built according to the following inductive definition:

- The projections $\pi : Z \times A^\alpha \to A$ are algebraic operations $t : Z \times U_\alpha \Rightarrow U$.

- An operation $\sigma : P_\sigma \times A^{|\sigma|} \to A$ is an algebraic operation $t : P_\sigma \times U_{|\sigma|} \Rightarrow U$.

- If $t : Z \times U_\alpha \Rightarrow U$ is an algebraic operation, $\{t_i\}_{i \in \alpha}$ a family of algebraic operations $Y \times U_\beta \Rightarrow U$, and $f : Z' \to Z$ a morphism in $\mathbb{C}$, then the composite:

$$Z' \times Y \times U_\beta \xrightarrow{f \times \langle t_i \rangle_{i \in \alpha}} Z \times U_\alpha \xrightarrow{t} U$$

  is an algebraic operation.

One calls α the arity of the algebraic operation $t : Z \times U_\alpha \to U$.

We remark that it might well be possible that this inductive definition does not capture all generalised operations (which commute with homomorphisms) on an arbitrary category $\mathbb{C}$. For this reason, we worked with a potentially more general definition of algebraic operations in the papers [8] and [7]. However, we conjecture that with the generalised Birkhoff's variety theorem of [115], it follows that for parameterised signatures in **Set**, **Top**, **kTop** and **Seq** all generalised operations can be obtained according the inductive construction above.

Now we can define equations for parameterised algebraic theories.

Definition 4.1.3. Let Σ be a signature for a parameterised algebraic theory for $\mathbb{C}$. An *equation* $(e : t = t')$, is given by a pair of Σ-algebraic operations $t, t' : P_e \times U_{|e|} \Rightarrow U$. Again we call P_e the parameter object of e and $|e|$ its arity.

A Σ-algebra $(A, \{\sigma_A\})$ is said to *satisfy* e, if $t_A = t'_A$ as $\mathbb{C}$-morphisms. If $\mathcal{E}$ is a set of equations for Σ, we call $(\Sigma, \mathcal{E})$ a *parameterised equational theory* for $\mathbb{C}$, and a Σ-algebra is a $(\Sigma, \mathcal{E})$-*algebra* (also $\mathbb{C}_{(\Sigma, \mathcal{E})}$-algebra) if it satisfies all equations $e \in \mathcal{E}$.

If Σ is a finitary (ω-ary) signature, and the arities of all equations in $\mathcal{E}$ are finite (countable) sets, we call the equational theory $(\Sigma, \mathcal{E})$ *finitary* (ω-*ary*).

The category $\mathbb{C}_{(\Sigma, \mathcal{E})}$ is given by the $(\Sigma, \mathcal{E})$-algebras and Σ-homomorphisms between them. In particular we get $\mathbb{C}_\Sigma = \mathbb{C}_{(\Sigma, \emptyset)}$.

Let us remark that in the commuting diagrams expressing the equations for Global State, all the compositions can be transformed into algebraic operations, fitting into the above definition. Let Σ is the parameterised signature for **Set**, consisting of operations $lookup : L \times X^V \to X$ and $update : L \times V \times X \to X$. Then for instance the algebraic operations of equation 5, are obtained as maps $L_2 \times A^{V \times V}$, given by

$$(l, l', (a_{(i,j)})_{(i,j) \in V \times V}) \mapsto lookup(l, (lookup(l', (a_{(i,j)})_{i \in V})_{j \in V})),$$

and

$$(l, l', (a_{(i,j)})_{(i,j) \in V \times V}) \mapsto lookup(l', (lookup(l, (a_{(i,j)})_{j \in V})_{i \in V})).$$

Our aim in this and the next chapter is to show that the categories of Topological Domain Theory have free algebras for ω-ary parameterised equational theories, and

thus they are capable of modelling computational effects, which are induced by such theories. Let us finish this section by recalling the definition of a free algebra.

Definition 4.1.4. Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for $\mathbb{C}$, and X be a $\mathbb{C}$-object. A $(\Sigma, \mathcal{E})$-algebra $(FX, \{\sigma_{FX}\})$ is called the *free $(\Sigma, \mathcal{E})$-algebra* over X, if:

- there exists a $\mathbb{C}$-morphism $\eta_X : X \to FX$, such that

- for any $(\Sigma, \mathcal{E})$-algebra $(B, \{\sigma_B\})$ each $\mathbb{C}$-morphism $f : X \to B$ can be uniquely extended to a Σ-homomorphism along η_X, as in:

$$
\begin{array}{ccc}
FX & \xrightarrow{\;\widehat{f}\;} & B \\
{\scriptstyle \eta_X}\big\uparrow & \nearrow{\scriptstyle f} & \\
X & &
\end{array}
$$

For $\mathcal{E} = \emptyset$, the free $\mathbb{C}_{(\Sigma, \mathcal{E})}$-algebra (or simply the free $\mathbb{C}_\Sigma$-algebra) over X is called the *absolutely free Σ-algebra*.

Recall that the universal property expresses that the free algebra construction is functorial. That is, for each $\mathbb{C}$-object X the free $\mathbb{C}_{(\Sigma, \mathcal{E})}$-algebra $(FX, \{\sigma_{FX}\})$ over X exists if and only if the forgetful functor $U : \mathbb{C}_{(\Sigma, \mathcal{E})} \to \mathbb{C}$ has a left adjoint F, which is the *free algebra functor*. In this case the adjunction $F \dashv U$ gives rise to a monad $T = U \circ F$. The Plotkin/Power approach to modelling computational effects, presented above, yields that for a wide range of computational effects, a corresponding effect monad can be obtained using such an adjunction for a free algebra construction for a parameterised equational theory.

4.2 Free algebras in categories of topological spaces

One standard method to obtain the existence of a free algebra functor is to apply Freyd's Adjoint Functor Theorem (FAFT) [87] for showing that the forgetful functor from the category of algebras to its underlying category has a left adjoint. However, one of the necessary requirements for applying the FAFT is that the category of algebras is complete, which is not the case for categories of algebras over **QCB** or its subcategories, as **QCB** itself is only countably complete. Thus we have to use a different strategy to show the existence of free algebras for ω-ary parameterised equational theories in **QCB**. Let us outline our plan.

Our approach is to apply the FAFT in suitable complete supercategories of **QCB**, namely in **kTop** and **Seq**, and then to show that **QCB** is in fact closed under the free algebra construction in these supercategories. This is done by first showing that

the countably-based spaces are closed under the absolutely free algebra constructions for ω-ary theories in **kTop** and **Seq**. In a second step we show that set-theoretically the free algebras can be explicitly constructed by taking the quotient of the absolutely free algebras by a congruence relation, as in classical universal algebra. However, for this quotient to be a topological quotient map, one needs that topological quotients are preserved by the product constructions in the respective categories. As **kTop** is cartesian-closed, we get that for finitary parameterised equational theories, the free algebras are topological quotients of the absolutely free algebras. Moreover, in **Seq** we can use Schröder and Simpson's Theorem 2.3.5 to show that for ω-ary parameterised equational theories, the free algebras are topological quotients of the absolutely free algebras.

We consider both the compactly-generated and the sequential framework for the free algebra construction, as each has its own merits. The compactly-generated framework also subsumes Classical Domain Theory and is therefore useful in the comparison of free algebras in Classical and Topological Domain Theory in the next chapter. The sequential framework, on the other hand, gives us a more general result, since we can show that here ω-ary theories preserve qcb-spaces, whereas in the compactly-generated framework this result could only be obtained for finitary theories so far.

In this section we show that countably-based spaces are closed under the absolutely free algebra constructions in **Top**, **kTop** and **Seq**. In the next section we investigate congruences for parameterised equational theories in **kTop** and **Seq** and in the subsequent section we put our results together to show the existence of free algebras for ω-ary parameterised equational theories in **QCB**.

We start by using Freyd's Adjoint Functor Theorem to show that free algebras exist in **Set**, **Top**, **kTop** and **Seq**.

Theorem 4.2.1. *For any parameterised equational theory* $(\Sigma, \mathcal{E})$ *for* $\mathbb{C}$ *(where* $\mathbb{C}$ *ranges over* **Set***,* **Top***,* **kTop** *and* **Seq***), the forgetful functor* $U : \mathbb{C}_{(\Sigma, \mathcal{E})} \to \mathbb{C}$ *has a left adjoint, the free algebra functor* $F : \mathbb{C} \to \mathbb{C}_{(\Sigma, \mathcal{E})}$.

Proof. We roughly outline the proof considering the category **Top**, the other cases being very similar. For applying the FAFT, we need to show that $\mathbf{Top}_{(\Sigma, \mathcal{E})}$ is locally small and complete, that U preserves small limits, and that the Solution Set Condition is satisfied. Local smallness of $\mathbf{Top}_{(\Sigma, \mathcal{E})}$ is clear, and products and equalizers can be constructed as in **Top**, since the fact that all arrows are homomorphisms implies the existence of an algebra structure on the corresponding constructions. It immediately follows that U preserves limits. For the Solution Set Condition, observe that if X is a topological space, then one can construct TX, the set of all Σ-terms over X in an inductive process (see below). TX is now a set-theoretic Σ-algebra, where the operations can be defined in an obvious way. Next, consider all quotients $TX/\sim$ such

that in $TX/\sim$ all equations of $\mathcal{E}$ are satisfied. On each quotient, one gets a set of compatible topologies, where compatible means that all operations and the inclusion map of X into $TX/\sim$ are continuous. The solution set for X is given by all these quotients of TX with the compatible topologies. $\qquad\qquad\qquad\qquad\qquad\square$

Absolutely free algebras in **Set** can also be obtained using the following inductive construction, which is well-known from universal algebra:

Let Σ be a parameterised signature for **Set**, and $\mathcal{T}$ be the functor assigning to a set X, the set

$$\{\sigma(p, (x_i)_{i\in|\sigma|})|\ \sigma \in \Sigma,\ p \in P_\sigma,\ \forall i \in |\sigma|.\ x_i \in X\},$$

(here $\sigma(p, (x_i)_{i\in|\sigma|})$ is a formal expression). One gets that $\mathcal{T}(-) \cong \coprod_{\sigma\in\Sigma} P_\sigma \times (-)^{|\sigma|}$. Now define inductively for all ordinals α, $F_0(X) = X$, $F_{\alpha+1}(X) = X \cup \mathcal{T}(F_\alpha(X))$ and for limit ordinals $F_\alpha(X) = \bigcup_{\beta<\alpha} F_\beta(X)$. In other words, to obtain $F_{\alpha+1}(X)$, we add all terms obtained by applying the operations of Σ to terms of $F_\alpha(X)$ and for a limit ordinal α, $F_\alpha(X)$ is the union of all terms appearing in some $F_\beta(X)$ for $\beta < \alpha$. Clearly, $F_\alpha(X) \subseteq F_{\alpha+1}(X)$, and for all terms t constructed in this inductive process, we define $occ(t)$ to be the smallest ordinal α such that $t \in F_\alpha(X)$. Observe that $occ(t)$ is never a limit ordinal. Let κ be the smallest regular cardinal which is strictly larger than the cardinality of all $|\sigma|$. Then we get that $F_\kappa(X) \cong X \cup \mathcal{T}(F_\kappa(X))$, as one can show by induction on $occ(t)$ that whenever $t \in X \cup \mathcal{T}(F_\kappa(X))$, then t was already in $F_\kappa(X)$. It follows that $F_\kappa(X)$ is the free term algebra $\underline{F}X$ over X, where operations are defined by $\sigma_{\underline{F}X}(p, (t_i)_{i\in|\sigma|}) = \sigma(p, (t_i)_{i\in|\sigma|})$.

We now relate this set-theoretic construction to its topological counterparts. For this, let $\mathbb{C}$ be one of the categories **Top**, **kTop** or **Seq**, and $(\Sigma, \mathcal{E})$ be a parameterised equational theory for $\mathbb{C}$. Moreover, for a parameterised equational theory $(\Sigma, \mathcal{E})$ for $\mathbb{C}$, let $(\nabla\Sigma, \nabla\mathcal{E})$ denote the parameterised equational theory on **Set** obtained by forgetting all the topological structure present in $(\Sigma, \mathcal{E})$, i.e. the topological structure of the parameter objects. We use ∇ also to define the forgetful functor $\mathbb{C} \to$ **Set**.

Proposition 4.2.2. *Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for $\mathbb{C}$ (where $\mathbb{C}$ ranges over* **Top***,* **kTop** *and* **Seq***). Then for any space X in $\mathbb{C}$, the underlying set of the free $(\Sigma, \mathcal{E})$-algebra FX, is given by the free $(\nabla\Sigma, \nabla\mathcal{E})$-algebra over the underlying set ∇X of X, and the topology on FX is the finest compatible topology $\mathcal{O}$, i.e. the finest topology satisfying:*

(I) the inclusion map $\eta_X : X \to (\nabla FX, \mathcal{O})$ is continuous, and

(II) for all $\sigma \in \Sigma$,

$$\sigma_{\nabla FX} : P_\sigma \times (\nabla FX, \mathcal{O})^{|\sigma|} \to (\nabla FX, \mathcal{O})$$

is continuous (where the product carries the $\mathbb{C}$-product topology).

Proof. It is an easy observation that such a finest topology exists, since whenever $\mathcal{C}$ is a set of topologies satisfying (I) and (II), then so does the topology generated by $\bigcup \mathcal{C}$. In the case of $\mathbb{C} = \mathbf{kTop}, \mathbf{Seq}$, it is an easy observation that this topology is compactly-generated, respectively sequential, as (I) and (II) are invariant under the corresponding coreflections. Thus it remains to show that $(A, \{\sigma_A\}) := (\nabla FX, \mathcal{O}, \{\sigma_{\nabla FX}\})$ satisfies the universal property of a free algebra over X.

For this let $(B, \{\sigma_B\})$ be a $\mathbb{C}_{(\Sigma, \mathcal{E})}$-algebra, and $f : X \to B$ be a continuous map. Forgetting the topological structure, we get a $\mathbf{Set}_{(\nabla \Sigma, \nabla \mathcal{E})}$-algebra $(\nabla B, \{\sigma_{\nabla B}\})$, and thus a unique homomorphism lifting as in:

$$
\begin{array}{ccc}
\nabla A & \xdashrightarrow{\widehat{f}} & \nabla B \\
{\scriptstyle \eta_{\nabla X}} \uparrow & \nearrow {\scriptstyle f} & \\
\nabla X & &
\end{array}
$$

But now the topology on ∇A generated by $\{\widehat{f}^{-1}(U) \,|\, U \in \mathcal{O}(B)\}$ satisfies conditions (I) and (II), hence $\widehat{f} : A \to B$ must be continuous. That this lifting is unique follows again from the freeness of $(\nabla A, \{\sigma_{\nabla A}\})$ in **Set**. Thus $(A, \{\sigma_A\})$ is indeed the free $(\Sigma, \mathcal{E})$-algebra over X, as claimed. $\qquad\square$

An interesting question in its own right is, under which circumstances the inclusion map $\eta_X : X \to FX$ into the free algebra is a regular mono, and so really deserves to be called an inclusion map. For this we have the following simple answer.

Lemma 4.2.3. *Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for $\mathbb{C}$ (where $\mathbb{C}$ ranges over* **Top**, **kTop** *and* **Seq***). Then the inclusion map $\eta_X : X \to FX$ into the free algebra is a regular mono in $\mathbb{C}$ for all objects X if and only if $\eta_{\mathbb{S}} : \mathbb{S} \to F\mathbb{S}$ is a regular mono.*

Proof. First of all, observe that for all choices of $\mathbb{C}$, a map $\mathbb{S} \to Y$ is a regular mono if and only if it is a topological subspace embedding. Thus, we have to show that $\eta_X : X \to FX$ is a topological subspace embedding for all objects X of $\mathbb{C}$ if and only if $\eta_{\mathbb{S}} : \mathbb{S} \to F\mathbb{S}$ is a topological subspace embedding. So assume that $\eta_{\mathbb{S}} : \mathbb{S} \to F\mathbb{S}$ is a topological subspace embedding and X an arbitrary object of $\mathbb{C}$. By Proposition 4.2.2 and the inductive construction of set-theoretic free algebras, η_X is injective if and only if we cannot derive the equation $x = y$ from $\mathcal{E}$. But if $\eta_{\mathbb{S}}$ is a topological subspace embedding, than it is injective and hence $\eta(\bot) \neq \eta(\top)$, so $x = y$ cannot be derive from $\mathcal{E}$, showing that η_X must also be injective.

It remains to show that for every open subset $U \subseteq X$, there exists an open $\overline{U} \subseteq FX$

with $U = \eta_X^{-1}(\overline{U})$. For this consider the following diagram:

$$
\begin{array}{ccc}
FX & \xrightarrow{\quad\widehat{\eta_{\mathbb{S}} \circ \chi_U}\quad} & F\mathbb{S} \\
\uparrow{\scriptstyle \eta_X} & \quad\mathbb{S}\nearrow{\scriptstyle \eta_{\mathbb{S}}} & \\
X & \nearrow{\scriptstyle \chi_U} &
\end{array}
$$

Let now $W \subseteq F\mathbb{S}$ be an open subset with $\eta_{\mathbb{S}}^{-1}(W) = \{\top\}$. Then $\overline{U} = \widehat{\eta_{\mathbb{S}} \circ \chi_U}^{-1}(W)$ is open in FX, and satisfies $\eta_X^{-1}(\overline{U}) = U$, as required. Thus, η_X is indeed a topological subspace embedding. $\qquad\square$

We now show a surprising result: In contrast to the construction of terms, which in general is a transfinite inductive process, the topology on the absolutely-free $\mathbf{Top}_\Sigma$-algebra can always be constructed in an ordinary ω-inductive limit process. This construction is crucial for showing that for appropriate ω-ary theories the absolutely-free algebra functor $\underline{F} : \mathbf{Top} \to \mathbf{Top}_\Sigma$ preserves countably-based spaces, which does not seem to be deducible from the above characterisation of the free algebra topology. So suppose we want to construct the absolutely-free algebra over a space X, and we have constructed the absolutely free set-theoretic algebra $(\nabla\underline{F}X)$ together with the inclusion map $\eta : X \hookrightarrow \nabla\underline{F}X$. Let Ω_0 be the topology generated by $\{\eta(U)| U \in \mathcal{O}(X)\}$, and for a given topology Ω_n, let Ω_{n+1} be the topology generated by

$$\Omega_n \cup \{\sigma_{\nabla\underline{F}X}(V \times U)| \sigma \in \Sigma,\ V \in \mathcal{O}(P_\sigma),\ U \in \Omega_n(\underline{F}X^{|\sigma|})\},$$

where $\Omega_n(\underline{F}X^{|\sigma|})$ is the product topology with respect to Ω_n. We obviously get that Ω_{n+1} is finer than Ω_n, and so we obtain the diagram:

$$(\nabla\underline{F}X, \Omega_0) \longleftarrow (\nabla\underline{F}X, \Omega_1) \longleftarrow (\nabla\underline{F}X, \Omega_2) \longleftarrow \cdots$$

Let Ω_∞ be the topology generated by $\bigcup_{n\in\mathbb{N}} \Omega_n$. Then we get:

Theorem 4.2.4. *For all topological spaces X, the topology on the absolutely-free $\mathbf{Top}_\Sigma$-algebra $(\underline{F}X, \{\sigma_{\underline{F}X}\})$ is given by Ω_∞.*

Proof. We have to show that $\Omega_\infty = \mathcal{O}(\underline{F}X)$, where $\mathcal{O}(\underline{F}X)$ denotes the topology of the absolutely free algebra. Notice that by construction all the Ω_n are compatible in the sense of Lemma 4.2.2, and so as Ω_∞ is generated by $\bigcup_{n\in\mathbb{N}} \Omega_n$, we get $\Omega_\infty \subseteq \mathcal{O}(\underline{F}X)$.

Thus, it remains to show that $\mathcal{O}(\underline{F}X) \subseteq \Omega_\infty$, which we do by showing that for all terms $t \in \underline{F}X$, and $U \subseteq \mathcal{O}(\underline{F}X)$ containing t, there exists $V \in \Omega_\infty$ such that $t \in V \subseteq U$. For this we use transfinite induction on $occ(t)$. If $occ(t) = 0$, then $t = x \in X$, and as $\eta_X : X \to \underline{F}X$ is continuous, we get for $V = \eta^{-1}(U)$ that $\eta(V) \in \Omega_0$ and $t \in \eta(V) \subseteq U$. So let $occ(t) = \alpha + 1$ and for all terms t' with $occ(t') \le \alpha$ and opens $U' \in \mathcal{O}(\underline{F}X)$

containing t', there exists $V' \in \Omega_\infty$ such that $t' \in V' \subseteq U'$. Suppose $t = \sigma(p, (t_i)_{i \in |\sigma|})$. As all operations are continuous on $\underline{F}X$, we have that $(p, (t_i)_{i \in |\sigma|}) \in \sigma_{\underline{F}X}^{-1}(U)$, and find a basic open $W \times \prod_{i \in |\sigma|} U_i \subseteq \sigma_{\underline{F}X}^{-1}(U)$, containing $(p, (t_i)_{i \in |\sigma|})$, such that $U_i = \underline{F}X$ for $i \in |\sigma| \setminus F$, where $F \subseteq |\sigma|$ is finite. For all $i \in F$, $occ(t_i) \leq \alpha$ and so we can apply the induction hypothesis, i.e. for all $i \in F$, there exists $V_i \in \Omega_\infty$ such that $t_i \in V_i \subseteq U_i$. Now, for each $i \in F$, there exists $n_i \in \mathbb{N}$ such that $V_i \in \Omega_{n_i}$, and so, as F is finite, $m = max_{i \in F}\, n_i$ exists and we get for

$$V = \{\sigma(p, (s_i)_{i \in |\sigma|})|\; p \in W, \forall i \in F.\; s_i \in V_i, \forall i \in |\sigma| \setminus F.\; s_i \in \underline{F}X)\},$$

that $V \in \Omega_{m+1} \subseteq \Omega_\infty$ and $t \in V \subseteq U$, showing the claim. $\qquad\square$

From the inductive construction of the absolutely free algebra topology we can derive the following technical lemma, which enables us to show that the absolutely free algebra construction for sequential spaces is obtained by coreflecting the absolutely free topological algebras.

Lemma 4.2.5. *Let X be a topological space and $\underline{F}X$ be the absolutely free* **Top**$_\Sigma$-*algebra over X. Then the sequence convergence relation $\rightarrow$ on $\underline{F}X$ is given by the smallest relation $\rightsquigarrow \subseteq \underline{F}X^{\mathbb{N}} \times \underline{F}X$ for which the following two conditions hold:*

- *For any sequence $(x_n)_{n \in \mathbb{N}}$ which converges to x in X, it holds that $(\eta(x_n)) \rightsquigarrow \eta(x)$.*

- *Let $t \in \underline{F}X$ be a term and $(t_n)_{n \in \mathbb{N}}$ be a sequence of terms that have eventually the same structure as t, i.e. if $t = \sigma(p, (s_i)_{i \in |\sigma|})$, then there exists $K \in \mathbb{N}$ such that for all $k > K$, it holds that $t_k = \sigma(p^k, (s_i^k)_{i \in |\sigma|})$. Moreover assume that the sequence $(p^k)_{k > K}$ converges to p in P_σ, and, for all $i \in |\sigma|$, it holds that $(s_i^k)_{k > K} \rightsquigarrow s_i$. Then it holds that $(t_n)_{n \in \mathbb{N}} \rightsquigarrow t$.*

Proof. Using the fact that the inclusion map η and all operations are sequentially continuous, it is trivial to show that the sequence convergence relation $\rightarrow$ on $\underline{F}X$ satisfies the conditions above. Thus, we only have to show that $(t_n)_{n \in \mathbb{N}} \rightarrow t$ implies $(t_n)_{n \in \mathbb{N}} \rightsquigarrow t$, for $\rightsquigarrow$ being the smallest relation satisfying the conditions above. We use the inductive construction of the topology of $\underline{F}X$ for this, and show the claim again by induction on $occ(t)$.

For $occ(t) = 0$, we have $t = \eta(x)$ for some $x \in X$. Let $V \subseteq X$ be any open neighbourhood of x. Then $\eta(V)$ is open in $\underline{F}X$, by the inductive construction of the topology, and so $(t_n)_{n \in \mathbb{N}} \rightarrow t$ implies the existence of $K \in \mathbb{N}$ with $t_k \in \eta(V)$ for $k > K$. But then $t_k = \eta(x_k)$ for some $x_k \in V$, and it follows that $(x_k)_{k > K}$ converges to x in X, and hence $(t_n)_{n \in \mathbb{N}} \rightsquigarrow t$.

Now let $occ(t) = \alpha + 1$, say $t = \sigma(p, (s_i)_{i \in |\sigma|})$, and suppose for all $z \in \underline{F}X$ with $occ(z) \leq \alpha$, it holds that $(z_n)_{n \in \mathbb{N}} \rightarrow z$ implies $(z_n)_{n \in \mathbb{N}} \rightsquigarrow z$. Observe that the inductive

construction of the topology yields that the set

$$\{\sigma(a, (b_i)_{i \in |\sigma|})| \ a \in P_\sigma \text{ and } \forall i \in |\sigma|. \ b_i \in \underline{F}X\}$$

is open in $\underline{F}X$, hence $(t_n)_{n \in \mathbb{N}} \to t$ implies the existence of some $K \in \mathbb{N}$ such that for all $k > K$, it holds that $t_k = \sigma(p^k, (s_i^k)_{i \in |\sigma|})$. Moreover, for any open neighbourhood $V \subseteq P_\sigma$ of p, the set

$$\{\sigma(a, (b_i)_{i \in |\sigma|})| \ a \in V \text{ and } \forall i \in |\sigma|. \ b_i \in \underline{F}X\}$$

is open in $\underline{F}X$, hence $(t_k)_{k > K} \to t$ implies that $(p^k)_{k > K}$ converges to p in P_σ. Finally, for every $i_0 \in |\sigma|$ and open neighbourhood $U \subseteq \underline{F}X$ of s_{i_0}, the inductive construction of the topology shows that the set

$$\{\sigma(a, (b_i)_{i \in |\sigma|})| \ a \in P_\sigma \text{ and } \forall i \in |\sigma| \setminus \{i_0\}. \ b_i \in \underline{F}X \text{ and } b_{i_0} \in U\}$$

is open in $\underline{F}X$, hence $(t_k)_{k > K} \to t$ implies that $(s_{i_0}^k)_{k > K} \to s_{i_0}$. Thus, as $occ(s_{i_0}) \leq \alpha$, the induction hypothesis yields $(s_{i_0}^k)_{k > K} \rightsquigarrow s_{i_0}$. Now the properties of $\rightsquigarrow$ show that $(t_k)_{k > K} \rightsquigarrow t$, as required. $\qquad\qquad\square$

Proposition 4.2.6. *Let Σ be a signature for a parameterised algebraic theory for* **Top** *such that all parameter spaces are sequential. Then for a topological space X, the absolutely free* **Seq**$_\Sigma$*-algebra over $Seq(X)$ is given by the coreflection (into* **Seq**) *of the absolutely free* **Top**$_\Sigma$*-algebra over X.*

Proof. Just for the proof, let $\underline{F}_{\mathbf{Seq}} : \mathbf{Seq} \to \mathbf{Seq}_\Sigma$ and $\underline{F}_{\mathbf{Top}} : \mathbf{Top} \to \mathbf{Top}_\Sigma$ denote the respective absolutely free algebra functors. Lemma 4.2.5 shows that $\underline{F}_{\mathbf{Top}}(Seq(X))$ and $\underline{F}_{\mathbf{Top}}(X)$ have the same converging sequences. Hence, without loss of generality, we may assume X to be a sequential space, and have to show that the identity maps $\underline{F}_{\mathbf{Seq}}X \leftrightarrow Seq(\underline{F}_{\mathbf{Top}}X)$ are both sequentially continuous.

As coreflection functors preserves limits, it is easy to see that $Seq(\underline{F}_{\mathbf{Top}}X)$ is a **Seq**$_\Sigma$-algebra, and therefore the universal property of $\underline{F}_{\mathbf{Seq}}X$ yields that the identity $\underline{F}_{\mathbf{Seq}}X \to Seq(\underline{F}_{\mathbf{Top}}X)$ is continuous. For the converse, suppose that $(t_n)_{n \in \mathbb{N}}$ is a sequence of terms in $\underline{F}_{\mathbf{Top}}X$, converging to t. By Lemma 4.2.5, we get that the $(t_n)_{n \in \mathbb{N}} \rightsquigarrow t$, where $\rightsquigarrow$ denotes again the smallest relation, satisfying the properties in the claim of Lemma 4.2.5. Again, by the fact that the inclusion map $\eta : X \to \underline{F}_{\mathbf{Seq}}X$ and all operations on $\underline{F}_{\mathbf{Seq}}X$ are sequentially continuous, it follows that the sequence convergence relation of $\underline{F}_{\mathbf{Seq}}X$ satisfies the conditions of Lemma 4.2.5, showing that the identity map $Seq(\underline{F}_{\mathbf{Top}}X) \to \underline{F}_{\mathbf{Seq}}X$ is sequentially continuous. $\qquad\square$

We remark that such a result does not hold once we introduce equations. In [78], Lamartin gives an example of a compactly-generated space X (which in fact is se-

quential), for which the free compactly-generated group over X is not the compactly-generated coreflection of the free topological group over X. The same holds for the sequential coreflection, and we see that in both cases the free algebra construction is not given by merely coreflecting the free construction for **Top**.

We can now show the following result, which is an important part of the proof of Theorem 4.4.3, which is the main result of this chapter.

Theorem 4.2.7. *Let Σ be an ω-ary signature for a parameterised equational theory for* **Top** *such that all parameter spaces are countably-based. Then the absolutely free topological algebra construction preserves countably-based spaces, and so does the absolutely free compactly-generated algebra construction and the absolutely free sequential algebra construction.*

Proof. Recall the inductive construction of the topology of the free topological algebra from Theorem 4.2.4. By induction, it is easily shown that for countably-based space X, the spaces $(\nabla FX, \Omega_n)$ are again countably-based, given that the set Σ is countable, the arity sets of all operations are countable and the parameter spaces are countably-based. It follows that the absolutely free topological algebra over X, which is the limit of the $(\nabla FX, \Omega_n)$ is also a countably-based space, and hence sequential. Thus, by Proposition 4.2.6, it follows that the absolutely free topological and the absolutely free sequential algebra constructions for a countably-based space coincide. As, by Proposition 4.2.2, the topology of the absolutely free compactly-generated algebra is finer than the topology of the absolutely free topological algebra and coarser than the topology of the absolutely free sequential algebra, it follows that it coincides with those two. □

4.3 Congruences for parameterised theories

In this section we generalise the classical notion of congruences from universal set-theoretical algebras to parameterised equational theories on the categories **Seq** and **kTop**, in order to show that free equational algebras can be constructed explicitly as topological quotients of the respective absolutely-free algebras. Subsequently, we show that the free algebra functors preserve topological quotient maps, in order to obtain the necessary results for proving that **QCB** is closed under the free algebra construction for finitary parameterised equational theories in **kTop** and ω-ary parameterised equational theories in **Seq**.

Most of the arguments in this section are purely set-theoretic. Only for Lemmas 4.3.2 and 4.3.3 we use a topological argument to show that the quotient topology for a congruence relation is compatible for certain cases in the categories **Seq** and **kTop**.

Definition 4.3.1. Let Σ be a signature for a parameterised algebraic theory for $\mathbb{C}$ $(= \mathbf{Set}, \mathbf{Top}, \mathbf{Seq}, \mathbf{kTop})$. A Σ-*congruence* on a $\mathbb{C}_\Sigma$-algebra $(A, \{\sigma_A\})$ is an equivalence relation $\sim$ on A such that for all $\sigma \in \Sigma$ and $p \in P_\sigma$, it holds that:

$$\forall i \in |\sigma|.\, a_i \sim a_i' \;\Rightarrow\; \sigma_A(p, (a_i)_{i \in |\sigma|}) \sim \sigma_A(p, (a_i')_{i \in |\sigma|}).$$

Lemma 4.3.2. *Let Σ be an ω-ary signature for a parameterised algebraic theory for* **Seq***, $(A, \{\sigma_A\})$ be a* **Seq**$_\Sigma$*-algebra and $\sim$ be a Σ-congruence on $(A, \{\sigma_A\})$. Then the topological quotient $A/\sim$ can be equipped with a* **Seq**$_\Sigma$*-algebra structure such that the topological quotient map $q : A \twoheadrightarrow A/\sim$ becomes a* **Seq**$_\Sigma$*-homomorphism.*

Proof. We define $\sigma_{A/\sim}(p, (q(a_i))_{i \in |\sigma|}) := q(\sigma_A(p, (a_i)_{i \in |\sigma|}))$. This map is well-defined since $\sim$ is a Σ-congruence. That $\sigma_{A/\sim} : P_\sigma \times (A/\sim)^{|\sigma|} \to A/\sim$ is continuous follows from the fact that in **Seq** countable products preserve topological quotient maps, see Theorem 2.3.5. Thus, we get that in the following commuting diagram:

$$
\begin{array}{ccc}
P_\sigma \times A^{|\sigma|} & \xrightarrow{\;\;\sigma_A\;\;} & A \\
{\scriptstyle P_\sigma \times q^{|\sigma|}}\downarrow & & \downarrow{\scriptstyle q} \\
P_\sigma \times (A/\sim)^{|\sigma|} & \xrightarrow[\;\;\sigma_{A/\sim}\;\;]{} & A/\sim
\end{array}
$$

$P_\sigma \times q^{|\sigma|}$ is a topological quotient map. Hence for any open $U \subseteq A/\sim$, we have that $V := (q \circ \sigma_A)^{-1}(U) = (P_\sigma \times q^{|\sigma|})^{-1}(\sigma_{A/\sim}^{-1}(U))$ is open in $P_\sigma \times A^{|\sigma|}$ and $P_\sigma \times q^{|\sigma|}$-saturated, and so $\sigma_{A/\sim}^{-1}(U) = P_\sigma \times q^{|\sigma|}(V)$ is open in $P_\sigma \times (A/\sim)^{|\sigma|}$, as required.

Moreover, by the definition of the algebra structure on $A/\sim$ it follows directly that the quotient map $q : A \to A/\sim$ is a **Seq**$_\Sigma$-homomorphism. $\qquad\square$

The crucial fact used in this proof is that in the category of sequential spaces topological quotient maps are preserved by countable products. This is not the case in **Top**, hence an analogous result cannot be obtained here. In the compactly-generated framework we get the following.

Lemma 4.3.3. *Let Σ be a finitary signature for a parameterised algebraic theory for* **kTop***, $(A, \{\sigma_A\})$ be a* **kTop**$_\Sigma$*-algebra and $\sim$ be a Σ-congruence on $(A, \{\sigma_A\})$. Then the topological quotient $A/\sim$ can be equipped with a* **kTop**$_\Sigma$*-algebra structure such that the topological quotient map $q : A \twoheadrightarrow A/\sim$ becomes a* **kTop**$_\Sigma$*-homomorphism.*

Proof. As a simple consequence of the facts that the topological quotient maps are exactly the regular epis in **kTop** and that **kTop** is cartesian-closed, we get that finite products preserve topological quotient maps here. Thus we can apply the same reasoning as in Lemma 4.3.2. $\qquad\square$

Lemma 4.3.4. *Let Σ be an ω-ary signature for a parameterised algebraic theory for* **Seq***, respectively a finitary signature for a parameterised algebraic theory for* **kTop***. Let furthermore $h : (A, \{\sigma_A\}) \to (B, \{\sigma_B\})$ be a Σ-homomorphism. Then the kernel relation $\sim_h$ of h is a Σ-congruence on $(A, \{\sigma_A\})$ and h factors as:*

$$(A, \{\sigma_A\}) \xrightarrow{q} (A/\sim_h, \{\sigma_{A/\sim_h}\}) \xrightarrow{h^*} (B, \{\sigma_B\}) \,,$$

where the algebra structure on $(A/\sim_h, \{\sigma_{A/\sim_h}\})$ is the canonical one obtained in Lemma 4.3.2 and h^* is a Σ-homomorphism. Moreover, h^* is the unique Σ-homomorphism for which this factorisation holds.

Proof. We prove this and the following results only for ω-ary theories for **Seq**, as the argument for finitary theories for **kTop** is the same.

Clearly, the kernel relation of h is an equivalence relation on A. We show that it also satisfies the additional requirement of a Σ-congruence. For this, suppose we are given $\sigma \in \Sigma$, $p \in P_\sigma$ and for all $i \in |\sigma|$, $a_i \sim_h a_i'$. Then we get that:

$$\begin{aligned}
h(\sigma_A(p, (a_i)_{i \in |\sigma|})) &= \sigma_B(p, (h(a_i))_{i \in |\sigma|}) \\
&= \sigma_B(p, (h(a_i'))_{i \in |\sigma|}) \\
&= h(\sigma_A(p, (a_i')_{i \in |\sigma|})),
\end{aligned}$$

and so $\sigma_A(p, (a_i)_{i \in |\sigma|}) \sim_h \sigma_A(p, (a_i')_{i \in |\sigma|})$, as required.

Now let $q : (A, \{\sigma_A\}) \to (A/\sim_h, \{\sigma_{A/\sim_h}\})$ denote the topological quotient map, which by Lemma 4.3.2 is a Σ-homomorphism. As the kernel relations of h and q coincide, it is clear that $h^* : q(a) \mapsto h(a)$, is well-defined and that $h = h^* \circ q$. The fact that q is a topological quotient map makes h^* continuous, and a simple calculation yields:

$$\begin{aligned}
h^*(\sigma_{A/\sim_h}(p, q(a_i)_{i \in |\sigma|})) &= h^*(q(\sigma_A(p, (a_i)_{i \in |\sigma|}))) \\
&= h(\sigma_A(p, (a_i)_{i \in |\sigma|})) \\
&= \sigma_B(p, (h(a_i))_{i \in |\sigma|}) \\
&= \sigma_B(p, (h^*(q(a_i')))_{i \in |\sigma|}),
\end{aligned}$$

showing that h^* is indeed a Σ-homomorphism. As the kernel relations for q and h coincide, it is clear that h^* is the unique map for which $h = h^* \circ q$ holds. $\qquad\square$

Corollary 4.3.5. *Let Σ be an ω-ary signature for a parameterised algebraic theory for* **Seq***, respectively a finitary signature for a parameterised algebraic theory for* **kTop***. Then the Σ-congruences on an algebra $(A, \{\sigma_A\})$ are exactly the kernel relations for Σ-homomorphisms with domain $(A, \{\sigma_A\})$.*

Moreover, if $\sim$ and $\approx$ are Σ-congruences on a Σ-algebra $(A, \{\sigma_A\})$ with $\sim \subseteq \approx$, then the factorisation $(A, \{\sigma_A\}) \twoheadrightarrow (A/\sim, \{\sigma_{A/\sim}\}) \twoheadrightarrow (A/\approx, \{\sigma_{A/\approx}\})$ consists of Σ-homomorphisms (where the algebra structures on the quotients are the canonical ones

obtained in Lemma 4.3.2).

Definition 4.3.6. Let Σ be a signature for a parameterised algebraic theory for **Seq**, respectively **kTop**, and $\mathcal{E}$ be a set of Σ-equations. Then the *congruence generated by* $\mathcal{E}$ on a Σ-algebra $(A, \{\sigma_A\})$ is the smallest Σ-congruence $\sim_{\mathcal{E}}$ on $(A, \{\sigma_A\})$ such that:

$$\forall (e : t = t') \in \mathcal{E}. \ \forall p \in P_e. \ \forall (a_i)_{i \in |e|} \in A^{|e|}. \ t_A(p, (a_i)_{i \in |e|}) \sim_{\mathcal{E}} t'_A(p, (a_i)_{i \in |e|}).$$

Lemma 4.3.7. *Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for **Seq**, respectively **kTop**. Furthermore let $(A, \{\sigma_A\})$ be a Σ-algebra and $\sim_{\mathcal{E}}$ the congruence on $(A, \{\sigma_A\})$ generated by $\mathcal{E}$. Then the following hold:*

 (i) *The algebra $(A/\!\sim_{\mathcal{E}}, \{\sigma_{A/\sim_{\mathcal{E}}}\})$ satisfies all equations in $\mathcal{E}$.*

 (ii) *If $h : (A, \{\sigma_A\}) \to (B, \{\sigma_B\})$ is a Σ-homomorphism, and all equations in $\mathcal{E}$ are satisfied by $(B, \{\sigma_B\})$, then for the kernel relation $\sim_h$ of h, it holds that $\sim_{\mathcal{E}} \subseteq \sim_h$.*

Proof. (i) Let $q : (A, \{\sigma_A\}) \to (A/\!\sim_{\mathcal{E}}, \{\sigma_{A/\sim_{\mathcal{E}}}\})$ denote the topological quotient map, and suppose $(e : t = t')$ is an equation in $\mathcal{E}$. Then, by definition of the congruence $\sim_{\mathcal{E}}$, we get for $p \in P_e$ and $(a_i)_{i \in |e|} \in A^{|e|}$ the following:

$$
\begin{aligned}
t_{A/\sim_{\mathcal{E}}}(p, (q(a_i))_{i \in |e|}) &= q(t_A(p, (a_i)_{i \in |e|})) \\
&= q(t'_A(p, (a_i)_{i \in |e|})) \\
&= t'_{A/\sim_{\mathcal{E}}}(p, (q(a_i))_{i \in |e|}),
\end{aligned}
$$

as required.

 (ii) Let again $(e : t = t')$ be an equation in $\mathcal{E}$, $p \in P_e$ and $(a_i)_{i \in |e|} \in A^{|e|}$. Then we get:

$$
\begin{aligned}
h(t_A(p, (a_i)_{i \in |e|})) &= t_B(p, (h(a_i))_{i \in |e|}) \\
&= t'_B(p, (h(a_i))_{i \in |e|}) \\
&= h(t'_A(p, (a_i)_{i \in |e|})),
\end{aligned}
$$

showing that $t_A(p, (a_i)_{i \in |e|}) \sim_h t'_A(p, (a_i)_{i \in |e|})$. As $\sim_h$ is a congruence, by Lemma 4.3.4, it follows that $\sim_{\mathcal{E}} \subseteq \sim_h$, showing the claim. $\square$

Now we are ready to construct the explicit congruences for obtaining free algebras for ω-ary parameterised equational theories in **Seq** as topological quotients of the respective absolutely-free algebras.

Theorem 4.3.8. (i) *Let $(\Sigma, \mathcal{E})$ be an ω-ary parameterised equational theory for **Seq** and $\underline{F} : \mathbf{Seq} \to \mathbf{Seq}_\Sigma$ denote the absolutely-free algebra functor. Let furthermore for a sequential space X, $\sim_{\mathcal{E}}$ denote the Σ-congruence on $(\underline{F}X, \{\sigma_{\underline{F}X}\})$ by $\mathcal{E}$. Then the free $\mathbf{Seq}_{(\Sigma, \mathcal{E})}$-algebra over X is given by $(\underline{F}X/\!\sim_{\mathcal{E}}, \{\sigma_{\underline{F}X/\sim_{\mathcal{E}}}\})$.*

(ii) Let $(\Sigma, \mathcal{E})$ be a finitary parameterised equational theory for **kTop** *and $\underline{F} :$* **kTop** $\rightarrow$ **kTop**$_\Sigma$ *denote the absolutely-free algebra functor. Let furthermore for a sequential space X, $\sim_\mathcal{E}$ denote the Σ-congruence on $(\underline{F}X, \{\sigma_{\underline{F}X}\})$ by $\mathcal{E}$. Then the free* **kTop**$_{(\Sigma, \mathcal{E})}$*-algebra over X is given by $(\underline{F}X/\!\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\})$.*

Proof. By Lemma 4.3.7 (i), $(\underline{F}X/\!\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\})$ is a **Seq**$_{(\Sigma, \mathcal{E})}$-algebra, and the inclusion map $\delta_X : X \rightarrow \underline{F}X/\!\sim_\mathcal{E}$ is given by the composition:

$$X \xrightarrow{\;\eta_X\;} \widehat{F}X \xrightarrow{\;q\;} \widehat{F}X/\!\sim_\mathcal{E}$$

Thus we only need to show the universal property for $(\underline{F}X/\!\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\})$. So let $(A, \{\sigma_A\})$ be a **Seq**$_{(\Sigma, \mathcal{E})}$-algebra and $f : X \rightarrow A$ be a continuous map. Then there exists a unique **Seq**$_\Sigma$-homomorphism extension $\widehat{f} : (\underline{F}X, \{\sigma_{\underline{F}X}\}) \rightarrow (A, \{\sigma_A\})$. By Lemma 4.3.4, the kernel relation $\sim_{\widehat{f}}$ is a congruence, and by Lemma 4.3.7 (ii), we have $\sim_\mathcal{E} \subseteq \sim_{\widehat{f}}$. Now we can apply again Lemma 4.3.4 and Corollary 4.3.5 and get the following factorisation of $\widehat{f}$:

$$(\underline{F}X, \{\sigma_{\underline{F}X}\}) \xrightarrow{\;q\;} (\underline{F}X/\!\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\}) \xrightarrow{\;r\;} (\underline{F}X/\!\sim_{\widehat{f}}, \{\sigma_{\underline{F}X/\sim_{\widehat{f}}}\}) \xrightarrow{\;s\;} (A, \{\sigma_A\})$$

where all maps are **Seq**$_\Sigma$-homomorphisms, and so is $\overline{f} := s \circ r$. It follows that $f = \overline{f} \circ \delta_X$, and it remains to show that $\overline{f}$ is the unique such extension.

Assume $h : (\underline{F}X/\!\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\}) \rightarrow (A, \{\sigma_A\})$ is another such extension. Then the universal property of $(\underline{F}X, \{\sigma_{\underline{F}X}\})$ gives $h \circ q = \widehat{f} = \overline{f} \circ q$, and the fact that q is a regular epi in **Seq** yields $h = \overline{f}$, as required.

The argument for the compactly-generated case is the same. $\qquad\square$

Now we use this result to show that the free algebra functors preserve topological quotient maps for the respective parameterised equational theories in **Seq** and **kTop**.

Theorem 4.3.9. *Let Σ be a signature for an ω-ary parameterised equational theory in* **Seq** *then the free algebra functor $F :$* **Seq** $\rightarrow$ **Seq**$_{(\Sigma, \mathcal{E})}$ *preserves topological quotient maps. Similarly, if Σ is a signature for a finitary parameterised equational theory in* **kTop** *then the free algebra functor $F :$* **kTop** $\rightarrow$ **kTop**$_{(\Sigma, \mathcal{E})}$ *preserves topological quotient maps.*

Proof. Again, we only show the claim for sequential spaces. If $q : X \rightarrow Q$ is a topological quotient map, then it can be obtained as the coequalizer of a pair of maps $f, g : Y \rightarrow X$ in **Seq**. As the free algebra functor $F :$ **Seq** $\rightarrow$ **Seq**$_{(\Sigma, \mathcal{E})}$ is a left adjoint, it preserves coequalizer diagrams, and so $Fq : FX \rightarrow FQ$ is the coequalizer of $Ff, Fg : FY \rightarrow FX$ in **Seq**$_{(\Sigma, \mathcal{E})}$. Now let $\sim_{Fq}$ denote the kernel relation of Fq, which by Corollary 4.3.5 is a congruence on FX. By Lemmas 4.3.2 and 4.3.7, the algebra $(FX/\!\sim_{Fq}, \{\sigma_{FX/\sim_{Fq}}\})$ satisfies all equations in $\mathcal{E}$ and the topological quotient map

$p : FX \to FX/\sim_{Fq}$ becomes a Σ-homomorphism. Moreover, p coequalises Ff and Fg, since by definition of the kernel relation for Fq, we have that $Fq \circ Ff(t) = Fq \circ Fg(t)$ if and only if $p \circ Ff(t) = p \circ Fg(t)$. We thus get the following commuting diagram:

$$FY \underset{Fg}{\overset{Ff}{\rightrightarrows}} FX \xrightarrow{Fq} FQ$$

with $r : FX \to FX/\sim_{Fq}$

where the map $FX/\sim_{Fq} \to FQ$ is obtained from Lemma 4.3.4 and $FQ \to FX/\sim_{Fq}$ from the universal property of the coequalizer. Again the universal property of the coequalizer yields immediately that $FQ \to FX/\sim_{Fq} \to FQ$ is the identity map, and the definition of $\sim_{Fq}$ yields that $FX/\sim_{Fq} \to FQ \to FX/\sim_{Fq}$ is also the identity map. Thus $FQ \cong FX/\sim_{Fq}$ and the claim follows. $\qquad\square$

Finally, we get the following Corollary which is the second main ingredient for the proof of Theorem 4.4.3.

Corollary 4.3.10. *(i) Let $(\Sigma, \mathcal{E})$ be an ω-ary parameterised equational theory for* **Seq**. *Then the free algebra functor $F :$ **Seq** $\to$ **Seq**$_{(\Sigma, \mathcal{E})}$ preserves topological quotient maps. Moreover if $\underline{F} :$ **Seq** $\to$ **Seq**$_\Sigma$ denotes the absolutely free algebra functor, then for any sequential space X, the free algebra FX over X is given by $(\underline{F}X/\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\})$, where $\sim_\mathcal{E}$ is the congruence generated by $\mathcal{E}$.*

(ii) Let $(\Sigma, \mathcal{E})$ be a finitary parameterised equational theory for **kTop**. *Then the free algebra functor $F :$ **kTop** $\to$ **kTop**$_{(\Sigma, \mathcal{E})}$ preserves topological quotient maps. Moreover if $\underline{F} :$ **kTop** $\to$ **kTop**$_\Sigma$ denotes the absolutely free algebra functor, then for any compactly-generated space X, the free algebra FX over X is given by $(\underline{F}X/\sim_\mathcal{E}, \{\sigma_{\underline{F}X/\sim_\mathcal{E}}\})$, where $\sim_\mathcal{E}$ is the congruence generated by $\mathcal{E}$.*

4.4 Free algebras in QCB

In this section we show that **QCB** is closed under the free algebra construction in **kTop** and **Seq** for wide classes of parameterised equational theories. Before turning towards the Main Theorem 4.4.3, we first show another technical result.

Theorem 4.4.1. *Let Σ be a signature for a parameterised algebraic theory for* **kTop**, *respectively* **Seq**, *such that all parameter spaces are qcb-spaces. Then there exists a parameterised equational theory $(\tilde{\Sigma}, \mathcal{E})$ such that all parameter spaces are countably-based, and* **kTop**$_\Sigma \cong$ **kTop**$_{(\tilde{\Sigma}, \mathcal{E})}$, *respectively* **Seq**$_\Sigma \cong$ **Seq**$_{(\tilde{\Sigma}, \mathcal{E})}$. *Moreover, the sets Σ and $\tilde{\Sigma}$ are canonically isomorphic and the isomorphism preserves the arities of the operations.*

Proof. We only prove the statement for the category **kTop**, the proof for **Seq** is analogous. Let $\tilde{\Sigma}$ be set-isomorphic to Σ via an isomorphism $\sigma \mapsto \tilde{\sigma}$, with $|\tilde{\sigma}| = |\sigma|$ and $P_{\tilde{\sigma}}$ being a countably-based space such that there exists a topological quotient map $q_\sigma : P_{\tilde{\sigma}} \to P_\sigma$. For every $\tilde{\sigma} \in \tilde{\Sigma}$ and $z \in P_{\tilde{\sigma}}$, let t^z be the algebraic operation $\mathbf{1} \times U_{|\tilde{\sigma}|} \Rightarrow U$ induced by the composite:

$$\mathbf{1} \times A^{|\tilde{\sigma}|} \xrightarrow{z \times A^{|\tilde{\sigma}|}} P_{\tilde{\sigma}} \times A^{|\tilde{\sigma}|} \xrightarrow{\tilde{\sigma}_A} A$$

Now, for every $\tilde{\sigma} \in \tilde{\Sigma}$, we denote by $\mathcal{E}_{\tilde{\sigma}}$, the set of equations given by:

$$\bigcup_{p \in P_\sigma} \{(t^z = t^{z'}) \mid z, z' \in q_\sigma^{-1}(p)\}.$$

Finally, let $\mathcal{E}$ be the set of equations, given by $\mathcal{E} := \bigcup_{\tilde{\sigma} \in \tilde{\Sigma}} \mathcal{E}_{\tilde{\sigma}}$. We show that $\mathbf{kTop}_\Sigma \cong \mathbf{kTop}_{(\tilde{\Sigma}, \mathcal{E})}$.

Suppose $(A, \{\sigma_A\})$ is a $\mathbf{kTop}_\Sigma$-algebra. For each $\sigma \in \Sigma$, we define $\tilde{\sigma}_A$ as the composite:

$$P_{\tilde{\sigma}} \times A^{|\tilde{\sigma}|} \xrightarrow{q_\sigma \times A^{|\sigma|}} P_\sigma \times A^{|\sigma|} \xrightarrow{\sigma_A} A$$

which makes $(A, \{\tilde{\sigma}_A\})$ into a $\mathbf{kTop}_{\tilde{\Sigma}}$-algebra. Let $e \in E$, say $(e : t^z = t^{z'}) \in \mathcal{E}_{\tilde{\sigma}}$, then by definition of $\tilde{\sigma}_A$, we have for all $(a_i)_{i \in |\tilde{\sigma}|} \in A^{|\tilde{\sigma}|}$ that:

$$
\begin{aligned}
\tilde{\sigma}_A(z, (a_i)_{i \in |\tilde{\sigma}|}) &= \sigma_A(q_\sigma(z), (a_i)_{i \in |\sigma|}) \\
&= \sigma_A(q_\sigma(z'), (a_i)_{i \in |\sigma|}) \\
&= \tilde{\sigma}_A(z', (a_i)_{i \in |\tilde{\sigma}|}),
\end{aligned}
$$

which shows that $(A, \{\tilde{\sigma}_A\})$ satisfies e, and thus is a $\mathbf{kTop}_{(\tilde{\Sigma}, \mathcal{E})}$-algebra.

Moreover, for every Σ-homomorphism $h : (A, \{\sigma_A\}) \to (B, \{\sigma_B\})$, the following diagram commutes:

$$
\begin{array}{ccc}
P_{\tilde{\sigma}} \times A^{|\tilde{\sigma}|} & \xrightarrow{P_{\tilde{\sigma}} \times h^{|\tilde{\sigma}|}} & P_{\tilde{\sigma}} \times B^{|\tilde{\sigma}|} \\
\downarrow{\scriptstyle q_\sigma \times A^{|\sigma|}} & & \downarrow{\scriptstyle q_\sigma \times B^{|\sigma|}} \\
P_\sigma \times A^{|\sigma|} & \xrightarrow{P_\sigma \times h^{|\sigma|}} & P_\sigma \times B^{|\sigma|} \\
\downarrow{\scriptstyle \sigma_A} & & \downarrow{\scriptstyle \sigma_B} \\
A & \xrightarrow{h} & B
\end{array}
$$

Hence h is a $\tilde{\Sigma}$-homomorphism.

Conversely, let $(A, \{\tilde{\sigma}_A\})$ be a $\mathbf{kTop}_{(\tilde{\Sigma},\mathcal{E})}$-algebra. For every $\sigma \in \Sigma$ and $z \in P_\sigma$, pick $y \in q_\sigma^{-1}(z)$, and define $\sigma : P_\sigma \times A^{|\sigma|} \to A$, as:

$$(z, (a_i)_{i \in |\sigma|}) \mapsto \tilde{\sigma}_A(y, (a_i)_{i \in |\sigma|}).$$

This map is well-defined, because $(A, \{\tilde{\sigma}_A\})$ satisfies all equations in $\mathcal{E}$, and so for any other $y' \in q_\sigma^{-1}(z)$, we get that $\tilde{\sigma}_A(y, (a_i)_{i \in |\sigma|}) = \tilde{\sigma}_A(y', (a_i)_{i \in |\sigma|})$. The operation σ_A is shown to be continuous as follows. It holds that $\tilde{\sigma}_A = \sigma_A \circ (q \times A^{|\sigma|})$, and $(q \times A^{|\sigma|})$ is a topological quotient map by the cartesian-closure of $\mathbf{kTop}$. Hence for every open $U \subseteq A$, the set $V := (\sigma_A \circ (q \times A^{|\sigma|}))^{-1}(U)$ is open and $(q \times A^{|\sigma|})$-saturated in $P_{\tilde{\sigma}} \times A^{|\tilde{\sigma}|}$. Thus, $(q \times A^{|\sigma|})(V) = \sigma_A^{-1}(U)$ is open in $P_\sigma \times A^{|\sigma|}$. It follows that $(A, \{\sigma_A\})$ is a $\mathbf{kTop}_\Sigma$-algebra.

As a consequence of the reasoning above, any $\tilde{\Sigma}$-homomorphism between $\mathbf{kTop}_{(\tilde{\Sigma},\mathcal{E})}$-algebras $(A, \{\tilde{\sigma}_A\})$ and $(B, \{\tilde{\sigma}_B\})$ is in fact a Σ-homomorphism between the corresponding algebras $(A, \{\sigma_A\})$ and $(B, \{\sigma_B\})$, concluding the proof that $\mathbf{kTop}_\Sigma \cong \mathbf{kTop}_{(\tilde{\Sigma},\mathcal{E})}$.

$\square$

Now we define the class of parameterised equational theories for which we show the existence of free algebras in **QCB**.

Definition 4.4.2. Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for **Top**, **kTop** or **Seq**. We call it

(i) *finitary qcb-parameterised* if all parameter spaces are qcb-spaces, the arity sets of all operations and equations are finite and the set Σ is countable,

(ii) *ω-ary qcb-parameterised* if all parameter spaces are qcb-spaces, and the arity sets of all operations, equations and the set Σ are countable.

Let us remark that all examples of algebraic theories for computational effects, presented in the previous section, satisfy the countability conditions. A possible counterexample would be a theory describing probabilistic choices with an $\mathbb{R}$-indexed family of operations $\{+_r\}_{r \in \mathbb{R}}$. Here the set of operations is too large to fit the definition above, which is to be expected, as such a theory captures continuity with respect to the discrete topology on the parameter space $\mathbb{R}$, which, of course, is not an object of **QCB**.

We are now ready to prove our Main Theorem.

Theorem 4.4.3. *For an ω-ary qcb-parameterised equational theory $(\Sigma, \mathcal{E})$, the forgetful functor $U : \mathbf{QCB}_{(\Sigma,\mathcal{E})} \to \mathbf{QCB}$ has a left adjoint, the free algebra functor $F : \mathbf{QCB} \to \mathbf{QCB}_{(\Sigma,\mathcal{E})}$. This free algebra construction is inherited from* **Seq**. *If moreover $(\Sigma, \mathcal{E})$ is finitary qcb-parameterised, then the free algebra construction is also inherited from* **kTop**.

Proof. It is sufficient to show that for an ω-ary qcb-parameterised equational theory $(\Sigma, \mathcal{E})$, the free algebra construction in **Seq** preserves qcb-spaces, as the universal property follows immediately. So let X be a qcb-space, say a topological quotient of the countably-based space Z. Construct an ω-ary parameterised equational theory $(\tilde{\Sigma}, \tilde{\mathcal{E}})$ with countably-based parameter spaces, such that $\mathbf{Seq}_\Sigma \cong \mathbf{Seq}_{(\tilde{\Sigma}, \tilde{\mathcal{E}})}$, along the lines of Theorem 4.4.1. Let now $F : \mathbf{Seq} \to \mathbf{Seq}_{(\Sigma, \mathcal{E})}$ be the free $(\Sigma, \mathcal{E})$-algebra functor in **Seq**, and $\tilde{F} : \mathbf{Seq} \to \mathbf{Seq}_{(\tilde{\Sigma}, \tilde{\mathcal{E}})}$ be the free $(\tilde{\Sigma}, \tilde{\mathcal{E}})$-algebra functor, which is isomorphic to the absolutely free Σ-algebra functor by Theorem 4.4.1. Finally, let $\underline{F} : \mathbf{Seq} \to \mathbf{Seq}_{\tilde{\Sigma}}$ be the absolutely free $\tilde{\Sigma}$-algebra functor. Then we get the following commuting diagram:

$$
\begin{array}{ccc}
\underline{F}Z & \longrightarrow\!\!\!\!\!\rightarrow & \underline{F}X \\
\downarrow & & \downarrow \\
\tilde{F}Z & \longrightarrow\!\!\!\!\!\rightarrow & \tilde{F}X \\
\downarrow & & \downarrow \\
FZ & \longrightarrow\!\!\!\!\!\rightarrow & FX
\end{array}
$$

where by Corollary 4.3.10 all maps are topological quotient maps. As topological quotient maps are closed under composition, we thus obtain FX as a topological quotient of $\underline{F}Z$, which is a countably-based space by Theorem 4.2.7.

If $(\Sigma, \mathcal{E})$ is finitary qcb-parameterised, we can mimic this proof with **kTop** replacing **Seq**, as then again Corollary 4.3.10 ensures that the diagram consists only of topological quotient maps. $\qquad\square$

We conjecture that the free algebra construction for **QCB** is inherited from **kTop** also in the case of ω-ary qcb-parameterised equational theories. In order to show this one needs to obtain an analogous result to Theorem 2.3.5 for the compactly-generated framework.

It follows that we can model a wide range of computational effects and combinations of them in **QCB**. For instance, we can model all the effects considered by Plotkin and Power [112]. In fact, nontermination, nondeterminism, probabilistic computations and exceptions are induced by finitary qcb-parameterised equational theories, and Input/Output and global state by ω-ary qcb-parameterised equational theories, provided the space I of input symbols for the I/O case and the space V of values in the global state case are discrete. We remark that for I discreteness is a reasonable choice, whereas for V one would like to generalise to other types of spaces. However, the above proof method relies heavily on having an underlying set-theoretic construction, which is not possible anymore, once we go beyond arities given as sets in the parameterised theo-

ries. The theory would get more complicated then, and the only way to get a more general result, we know so far, is to use the connections to Synthetic Domain Theory and employ an internal Adjoint Functor Theorem, as in Chapter 6.5 below.

We remark that also continuations can be modelled in **QCB**, as by cartesian-closure, the functor $(-) \mapsto R^{R^{(-)}}$ is well-defined for any qcb-space R. Suggestions on how to combine continuations with other effects have been given by Hyland, Levy, Plotkin and Power [50].

We furthermore remark that the above results give us an intuitive description of the free **QCB**-algebras. The underlying set is just the free set-theoretic algebra, which is the set of Σ-terms factorised by the congruence induced by the equations. Also, the free algebra topology is the quotient topology obtained by factorising the set of terms, equipped with the inductively defined topology from Theorem 4.2.4, by this congruence.

Finally, we remark that parameterised equational theories might also be of interest beyond the theory of computational effects. For instance real topological vector spaces are algebras in **Top** for the parameterised equational theory $(\Sigma, \mathcal{E})$, with operations given by a constant $0 \in X$, and continuous maps $+ : X^2 \to X$ and $\cdot : \mathbb{R} \times X \to X$, subject to the usual vector space axioms. Similarly, one can describe topological modules over a fixed topological ring, and other similar structures by parameterised equational theories.

4.5 A note on strength

The expert reader will have noticed that we have not shown our computational monads, which are constructed via free algebra functors, to be strong. However, strength was an essential demand on a computational monad in Moggi's original work [97]. The strength of a monad allows one to work with multi-variable expressions in the computational λ-calculus. The reason we neglected strength is that we can show that free algebra monads for parameterised equational theories are automatically strong if the underlying category is well-pointed and cartesian-closed.

The crucial observation for this is the following result:

Lemma 4.5.1. *Let Σ be a signature for a parameterised algebraic theory for a cartesian-closed category $\mathbb{C}$. Then $\mathbb{C}_\Sigma$-algebras are closed under exponentiation in $\mathbb{C}$, i.e. for every $\mathbb{C}$-object X and $\mathbb{C}_\Sigma$-algebra $(A, \{\sigma_A\})$, the exponential A^X can be equipped with a Σ-algebra structure.*

Proof. We define σ_{A^X} to be the exponential transpose of the composite:

$$P_\sigma \times (A^X)^{|\sigma|} \times X \cong P_\sigma \times (A^{|\sigma|})^X \times X \xrightarrow{eval_{A^{|\sigma|},X}} P_\sigma \times A^{|\sigma|} \xrightarrow{\sigma_A} A$$

In terms of the λ-calculus, σ_{A^X} is given by $\lambda p.\lambda(f_i).\lambda x.\sigma_A(p,(f_i(x)))$. It follows that $(A^X, \{\sigma_{A^X}\})$ is a $\mathbb{C}_\Sigma$-algebra. $\qquad\square$

Provided $\mathbb{C}$ has equalizers, this result makes the category of algebras, $\mathbb{C}_\Sigma$, into a $\mathbb{C}$-enriched category. Moreover, it motivates the following definition of left (and right) homomorphisms.

Definition 4.5.2. Let Σ be a signature for a parameterised algebraic theory for a cartesian-closed category $\mathbb{C}$. Let furthermore $(A, \{\sigma_A\}), (B, \{\sigma_B\})$ be $\mathbb{C}_\Sigma$-algebras and X be an arbitrary $\mathbb{C}$-object. A $\mathbb{C}$-morphism $f : A \times X \to B$ is a *left homomorphism* if its exponential transpose $\overline{f} : A \to B^X$ is a homomorphism $(A, \{\sigma_A\}) \to (B^X, \{\sigma_{B^X}\})$.

Similarly we define a *right homomorphism* $f : X \times A \to B$.

Now we show that, whenever $\mathbb{C}$ is well-pointed, then Lemma 4.5.1 carries over to the equational setting, i.e. $(A^X, \{\sigma_{A^X}\})$ satisfies all equations satisfied by $(A, \{\sigma_A\})$. For the following two lemmas we assume that Σ is a signature for a parameterised algebraic theory for a well-pointed cartesian-closed category $\mathbb{C}$.

Lemma 4.5.3. *Let $t : Z \times U_\alpha \Rightarrow U$ be a Σ-algebraic operation. Then for all $\mathbb{C}$-objects X, and $\mathbb{C}_\Sigma$-algebras $(A, \{\sigma_A\})$, the following diagram commutes:*

$$
\begin{array}{ccc}
Z \times (A^X)^\alpha & \xrightarrow{\quad t_{A^X} \quad} & A^X \\
& \searrow^{const \times (A^X)^\alpha} \quad \nearrow_{t_A^X} & \\
& Z^X \times (A^X)^\alpha \cong (Z \times A^\alpha)^X &
\end{array}
$$

where const is the exponential transpose of the obvious projection $Z \times X \to Z$.

Proof. For a global element $x : \mathbf{1} \to X$, let $\pi_x : A^X \to A^{\mathbf{1}} \cong A$ denote the evident projection map, which, by the definition of operations on A^X, is a Σ-homomorphism. Then, by well-pointedness of $\mathbb{C}$, it suffices to show that for all global elements $x : \mathbf{1} \to X$,

$$\pi_x \circ t_{A^X} = \pi_x \circ t_A^X \circ (const \times (A^X)^\alpha).$$

Naturality of t ensures that the following diagram commutes:

$$
\begin{array}{ccc}
Z \times (A^X)^\alpha & \xrightarrow{t_{A^X}} & A^X \\
\downarrow^{Z \times \pi_x^\alpha} & & \downarrow^{\pi_x} \\
Z \times A^\alpha & \xrightarrow{\quad t_A \quad} & A
\end{array}
$$

Hence, $\pi_x \circ t_{A^X} = t_A \circ (Z \times \pi_x^\alpha)$.

On the other hand, a simple application of the λ-calculus yields that:

$$\lambda z.\lambda(f_i).t_A^X(const(z),(f_i))(x) = \lambda z.\lambda(f_i).t_A(z,(f_i(x))),$$

hence $t_A \circ (Z \times \pi_x^\alpha) = \pi_x \circ t_A^X \circ (const \times (A^X)^\alpha)$, and so the claim follows. $\qquad\square$

Lemma 4.5.4. *Suppose X is a $\mathbb{C}$-object, $(A, \{\sigma_A\})$ and $(B, \{\sigma_B\})$ are $\mathbb{C}_\Sigma$-algebras, and $f : A \times X \to B$ is a left homomorphism. Then the following diagram commutes for all algebraic operations $t : Z \times U_\alpha \Rightarrow U$:*

$$
\begin{array}{ccccc}
Z \times A^\alpha \times X & \xrightarrow{\ \ \delta\ \ } & Z \times (A \times X)^\alpha & \xrightarrow{\ \ Z \times f^\alpha\ \ } & Z \times B^\alpha \\
{\scriptstyle t_A \times X}\downarrow & & & & \downarrow{\scriptstyle t_B} \\
A \times X & & \xrightarrow{\hspace{4cm} f \hspace{4cm}} & & B
\end{array}
$$

where $\delta : Z \times A^\alpha \times X \to Z \times A^\alpha \times X^\alpha \cong Z \times (A \times X)^\alpha$ is the evident diagonal morphism.

Proof. Again we use the fact that a cartesian-closed category is a model of the λ-calculus. Thus we have to show that:

$$\lambda z.\lambda(a_i).\lambda x.f(t_A(z,(a_i)),x) = \lambda z.\lambda(a_i).\lambda x.t_B(z,(f(a_i,x))).$$

But the exponential transpose of the left-hand side of this equation is given by

$$\lambda z.\lambda(a_i).\overline{f}(t_A(z,(a_i))),$$

whereas the exponential transpose of the right-hand side evaluates to

$$\lambda z.\lambda(a_i).t_B^X(const(z),(\overline{f}(a_i)))$$

which by the previous Lemma is equal to:

$$\lambda z.\lambda(a_i).t_{B^X}(z,(\overline{f}(a_i))).$$

Now the naturality of t yields that these terms coincide, hence the above equation does hold. $\qquad\square$

This enables us to show the desired Theorem:

Theorem 4.5.5. *Let Σ be a signature for a parameterised algebraic theory for a well-pointed cartesian-closed category $\mathbb{C}$, $(A, \{\sigma_A\})$ be a $\mathbb{C}_\Sigma$-algebra and X be a $\mathbb{C}$-object. If $(A, \{\sigma_A\})$ satisfies an equation $(e : t = t')$, then so does $(A^X, \{\sigma_{A^X}\})$.*

Proof. We have to show that the maps $t_{A^X}, t'_{A^X} : P_e \times (A^X)^{|e|} \to A^X$ coincide. The previous Lemma shows that the following diagram commutes:

$$
\begin{array}{ccccc}
P_e \times (A^X)^{|e|} \times X & \xrightarrow{\ \delta\ } & P_e \times (A^X \times X)^{|e|} & \xrightarrow{P_e \times eval^\alpha_{A,X}} & P_e \times A^{|e|} \\
{\scriptstyle t_{A^X} \times X} \downarrow & & & & \downarrow {\scriptstyle t_A} \\
A^X \times X & & \xrightarrow{\hspace{4cm} eval_{A,X} \hspace{4cm}} & & A
\end{array}
$$

since the evaluation map $eval_{A,X} : A^X \times X \to A$ is evidently a left homomorphism. But then we get that $eval_{A,X} \circ (t_{A^X} \times X) = eval_{A,X} \circ (t'_{A^X} \times X)$, because the diagram commutes if we replace t by t'. Now the universal property of the exponential yields $t_{A^X} = t'_{A^X}$, as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

As a consequence of this result, we get the following.

Corollary 4.5.6. *Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for a well-pointed cartesian-closed category $\mathbb{C}$. Then free algebras are parametrically free, i.e. for every map $f : X \times Y \to A$ into a $\mathbb{C}_{(\Sigma, \mathcal{E})}$-algebra $(A, \{\sigma_A\})$, there exists a unique left homomorphism $\widehat{f} : FX \times Y \to A$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
FX \times Y & \dashrightarrow^{\ \widehat{f}\ } & A \\
{\scriptstyle \eta_X \times Y} \uparrow & \nearrow {\scriptstyle f} & \\
X \times Y & &
\end{array}
$$

Similarly, there exists a unique right homomorphism $\widetilde{f} : X \times FY \to A$ extending f along $X \times \eta_Y$.

We finally show that parametrically free algebras guarantee the associated monad to be strong.

Let us first recall that a monad (T, λ, μ) on a category $\mathbb{C}$ with finite products is strong, if there exists a natural transformation $t_{X,Y} : X \times TY \to T(X \times Y)$ which

satisfies the following three commuting diagrams:

$$(A) \quad \begin{array}{ccc} X \times TY & \xrightarrow{\;t_{X,Y}\;} & T(X \times Y) \\ {\scriptstyle X \times \eta_Y} \uparrow & \nearrow{\scriptstyle \eta_{X \times Y}} & \\ X \times Y & & \end{array} \qquad (B) \quad \begin{array}{ccc} X \times T(Y \times Z) & \xrightarrow{\;t_{X,Y \times Z}\;} & T(X \times Y \times Z) \\ {\scriptstyle X \times t_{Y,Z}} \uparrow & \nearrow{\scriptstyle t_{X \times Y,Z}} & \\ X \times Y \times TZ & & \end{array}$$

$$(C) \quad \begin{array}{ccccc} X \times T^2 Y & \xrightarrow{\;t_{X,TY}\;} & T(X \times TY) & \xrightarrow{\;T(t_{X,Y})\;} & T^2(X \times Y) \\ {\scriptstyle X \times \mu_Y} \downarrow & & & & \downarrow {\scriptstyle \mu_{X \times Y}} \\ X \times TY & & \xrightarrow{\hspace{4cm} t_{X,Y} \hspace{4cm}} & & T(X \times Y) \end{array}$$

Lemma 4.5.7. *Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for a cartesian-closed category $\mathbb{C}$ such that there exists a free algebra functor $F : \mathbb{C} \to \mathbb{C}_{(\Sigma, \mathcal{E})}$. If for every $\mathbb{C}$-object X, FX is parametrically free, then the induced monad is strong.*

Proof. We define the strength $t_{X,Y} : X \times FY \to F(X \times Y)$ as in diagram (A) above, using the fact that FY is parametrically free:

$$\begin{array}{ccc} X \times FY & \dashrightarrow{\;\widetilde{\eta}_{X \times Y}\;} & F(X \times Y) \\ {\scriptstyle X \times \eta_Y} \uparrow & \nearrow{\scriptstyle \eta_{X \times Y}} & \\ X \times Y & & \end{array}$$

The universal property of a parametrically free algebra ensures that this is in fact natural in X and Y, and that diagram (B) does commute, as well. That diagram (C) commutes is a consequence of the following 4 facts:

(i) For all $\mathbb{C}$-objects X, $\mu_X : F^2 X \to FX$ is a homomorphism, which is clear.

(ii) If $f : (A, \{\sigma_A\}) \to (B, \{\sigma_B\})$ is a homomorphism, then the definition of the algebra structures on A^X and B^X yields that $f^X : (A^X, \{\sigma_{A^X}\}) \to (B^X, \{\sigma_{B^X}\})$ is a homomorphism, as well.

(iii) If $f : X \times A \to B$ is a right homomorphism, and $g : B \to C$ a homomorphism, then $g \circ f : X \times A \to C$ is a right homomorphism, which follows from (ii) and the fact that the exponential transpose of $g \circ f$ is given by the composite:

$$A \xrightarrow{\;\overline{f}\;} B^X \xrightarrow{\;g^X\;} C^X$$

(iv) If $f : A \to B$ is a homomorphism and $g : X \times B \to C$ a right homomorphism, then $g \circ (X \times f) : X \times A \to C$ is a right homomorphism, which follows from the fact that the exponential transpose of $g \circ (X \times f)$ is obtained as the composite:

$$A \xrightarrow{\quad f \quad} B \xrightarrow{\quad \bar{g} \quad} C^X$$

Now it is easily shown that both compositions $X \times F^2Y \to F(X \times Y)$ in diagram (C) are right homomorphisms and extend $X \times FY \to F(X \times Y)$ along $X \times \eta_Y$, hence commutativity follows by the uniqueness of such an extension in a parametrically free algebra. $\qquad \square$

The main result of this section follows now immediately.

Theorem 4.5.8. *Let $(\Sigma, \mathcal{E})$ be a parameterised equational theory for a well-pointed cartesian-closed category $\mathbb{C}$. Then, if the free algebra functor $F : \mathbb{C} \to \mathbb{C}_{(\Sigma, \mathcal{E})}$ exists, the monad induced by F is strong.*

As **QCB** is well-pointed and cartesian-closed, it follows that the computational monads we constructed here, as free algebra functors for ω-ary qcb-parameterised equational theories, are indeed strong, and so we can interpret Moggi's computational λ-calculus in **QCB**.

We remark that Plotkin and Power obtain strength for their computational monads in [111] by demanding their algebraic operations to be $\mathbb{C}$-enriched natural transformations $t : U_\alpha \Rightarrow U$; in their more general framework the underlying categories $\mathbb{C}$ need not be cartesian-closed, but $\mathbb{V}$-closed and have $\mathbb{V}$-cotensors, where $\mathbb{V}$ is an appropriate enriching category. Then, if $\mathbb{C}$ has equalizers, the category of algebras $\mathbb{C}_\Sigma$ becomes $\mathbb{V}$-closed, as in Lemma 4.5.1. It can then be shown that $\mathbb{V}$-enrichment of an algebraic operation implies that a diagram similar to the one in Lemma 4.5.4 is satisfied. The rest of the proof is analogous to our situation. Of course, cartesian-closure implies $\mathbb{C}$ to be enriched over itself. The presence of equalizers implies the category of algebras to be $\mathbb{C}$-closed, and well-pointedness implies that all algebraic operations are $\mathbb{C}$-enriched. Thus, the strength of our monads follows from Plotkin and Power's work.

There are also connections to the work of Kock [72, 73, 74], who examined under which conditions monads on symmetric monoidal categories carry a strength. He essentially obtains a result similar to our Lemma 4.5.7. We remark that his Theorem 2.6 of [73], which is specific to cartesian-closed categories, cannot be applied to our case. The reason for this is that he assumes the monadic functor to commute with finite products which is generally not the case for free algebra functors. We believe that in order to apply his more general results, Corollary 4.5.6 above needs to be shown.

In any case, we think it is worth presenting a self-contained approach for our specific

situation of free algebra monads for parameterised equational theories in cartesian-closed categories.

5 Computational effects for topological domains

We have seen how computational effects can be modelled in **QCB** via free algebras for parameterised equational theories. In this chapter, we investigate the extent to which such an approach can be used to model effects in the categories of topological predomains and topological domains. A technical investigation enables us to transfer the results of the previous chapter to predomains. However, we give an example showing that free algebras, even for simple equational theories, need not exist in the category of topological domains and continuous maps. Nevertheless, we show that a free algebra construction does exist in the category of topological domains and strict maps.

Furthermore, we examine in how far these constructions compare to the treatment of computational effects in Classical Domain Theory. Abramsky and Jung [2] have shown how to construct free algebras for ordinary non-parameterised inequational algebraic theories in Classical Domain Theory, and how the upper, lower and convex powerdomains are obtained as such free algebras. Strüder [152], a student of Jung, has considered parameterised inequational theories for the category of continuous dcpos. We show that in the category of topological predomains parameterised equational theories have the same expressive power as parameterised inequational theories, and that whenever all operations are finitary, then the free algebra constructions in Classical Domain Theory and Topological Domain Theory coincide. This immediately yields that the classical upper, lower and convex powerdomain constructions can be generalised to all topological predomains. Furthermore, a more detailed investigation shows, that also the classical probabilistic powerdomain construction of Jones and Plotkin[1] [58, 59] can be recovered for ω-continuous dcppos as a free algebra construction in Topological Domain Theory.

Finally, we investigate how the parameterised equational theory for probabilistic computations compares to the classical probabilistic powerspace construction [65, 41], which is an extension of the probabilistic powerdomain to all topological spaces. The result is that we cannot identify a larger class of spaces on which the two constructions

---

[1] In fact, Jones and Plotkin consider a combination of probabilistic computations and nontermination, whereas we distinguish these effects. Their probabilistic powerdomain in [58, 59] consists of the subprobability valuations, and consequently we call it the *classical subprobabilistic powerdomain*. By the *classical probabilistic powerdomain*, we refer to the dcpo of probability valuations equipped with the Scott-topology. The details are given in chapter 5.4.

coincide beyond ω-continuous dcppos. In fact, using a space given by Grunhage and Streicher [38] we show that the classical probabilistic powerspace construction does not preserve qcb-spaces, in general.

5.1 Free algebras for topological predomains

We start by giving some technical results showing under which circumstances free algebra constructions can be transferred to reflective and coreflective subcategories. The result will be that free algebras for parameterised equational theories exist in the category of topological predomains and that they are obtained by reflecting the free **QCB**-algebras. Moreover we will also be able to apply our results in one of the later sections of this chapter to show that free algebras for parameterised equational theories exist in Classical Domain Theory and that for a wide range of theories these free algebra constructions in Classical and Topological Domain Theory coincide.

In [2] non-parameterised inequational theories have been considered for the categories of Classical Domain Theory, and the classical powerdomains for nondeterminism have been recovered as free algebras for such inequational theories. Therefore, inequational theories are a naturally interesting class of algebraic theories in any poset-enriched category. Towards the end of this section we show that parameterised equational theories subsume inequational theories, since the inequations can be recovered by adding auxiliary operations with appropriate parameter objects. In particular this will help us to generalise the classical powerdomain constructions to the categories of Topological Domain Theory below.

For the beginning, let us observe that any product preserving functor $F : \mathbb{C} \to \mathbb{D}$ lifts to a functor of algebras on these categories: if Σ is a signature for a parameterised algebraic theory for $\mathbb{C}$, then we can define the signature $F\Sigma$ on $\mathbb{D}$ to have the same underlying set as Σ, say $\{\sigma^F | \sigma \in \Sigma\}$, such that $|\sigma^F| = |\sigma|$ and $P_{\sigma^F} = FP_\sigma$. In a similar fashion, one can also apply the functor to algebraic operations, as given in Definition 4.1.2, by extending the functor along the inductive construction of algebraic operations. Then we get:

Proposition 5.1.1. *If $(\Sigma, \mathcal{E})$ is a finitary, respectively ω-ary parameterised equational theory and $F : \mathbb{C} \to \mathbb{D}$ a functor preserving finite, respectively countable products, then it lifts to a functor $\mathbb{C}_{(\Sigma, \mathcal{E})} \to \mathbb{D}_{(F\Sigma, F\mathcal{E})}$.*

Proof. We assume that $(\Sigma, \mathcal{E})$ is an ω-ary parameterised equational theory for $\mathbb{C}$ and F preserves countable products. The finitary case is analogous. Let $(A, \{\sigma_A\})$ be a $\mathbb{C}_\Sigma$-algebra. Then for $\sigma \in \Sigma$, we have

$$F\sigma_A : FP_\sigma \times FA^{|\sigma|} \cong P_{\sigma^F} \times FA^{|\sigma^F|} \to FA.$$

Thus setting $\sigma_{FA}^F := F\sigma_A$ makes (FA, σ_{FA}^F) into a $D_{F\Sigma}$-algebra. With this construction, it is clear that F carries Σ-homomorphisms to $F\Sigma$-homomorphisms.

As for algebraic operations, it is clear that a product preserving functor preserves projections, and thus we can extend F to an algebraic operation $t : U_\alpha \Rightarrow U$ along the inductive construction of Definition 4.1.2 to obtain an algebraic operation $t^F : U'_\alpha \Rightarrow U'$, where $U' : \mathbb{D}_{F\Sigma} \to \mathbb{D}$ denotes the corresponding forgetful functor. Thus we obtain an ω-ary parameterised equational theory $(F\Sigma, F\mathcal{E})$ on $\mathbb{D}$, and clearly if $(e : t = t')$ is an equation in $\mathcal{E}$, then by definition of the corresponding equation $(e^F : t^F = t'^F)$, we have that $t_A = t'_A$ implies $t_{FA}^F = t'^F_{FA}$, showing the claim. $\qquad\square$

In the special case of a reflection or a coreflection functor, we get the following.

Lemma 5.1.2. *Let $\mathbb{C}$ and $\mathbb{D}$ be categories with finite (countable) products, such that $\mathbb{D}$ is a full subcategory of $\mathbb{C}$. If $(\Sigma, \mathcal{E})$ is a finitary (ω-ary) parameterised equational theory for $\mathbb{C}$ such that for all $\sigma \in \Sigma$ and $e \in \mathcal{E}$, P_σ and P_e are objects of $\mathbb{D}$, then:*

(i) *If $\mathbb{D}$ is a reflective subcategory of $\mathbb{C}$, then $\mathbb{D}_{(\Sigma, \mathcal{E})}$ is a subcategory of $\mathbb{C}_{(\Sigma, \mathcal{E})}$, and if the reflection functor $R : \mathbb{C} \to \mathbb{D}$ preserves finite (countable) products, it lifts to a functor $R : \mathbb{C}_{(\Sigma, \mathcal{E})} \to \mathbb{D}_{(\Sigma, \mathcal{E})}$. Moreover, the unit of the reflection $\eta_A : A \to RA$ is a Σ-homomorphism for all $(\Sigma, \mathcal{E})$-algebras $(A, \{\sigma_A\})$ in $\mathbb{C}$.*

(ii) *If $\mathbb{D}$ is a coreflective subcategory of $\mathbb{C}$, and the inclusion functor $\mathbb{D} \hookrightarrow \mathbb{C}$ preserves finite (countable) products, then $\mathbb{D}_{(\Sigma, \mathcal{E})}$ is a subcategory of $\mathbb{C}_{(\Sigma, \mathcal{E})}$ and the coreflection functor lifts to a functor $C : \mathbb{C}_{(\Sigma, \mathcal{E})} \to \mathbb{D}_{(\Sigma, \mathcal{E})}$. Moreover, the unit of the coreflection $\mu_A : CA \to A$ is a Σ-homomorphism for all $(\Sigma, \mathcal{E})$-algebras $(A, \{\sigma_A\})$ in $\mathbb{C}$.*

Proof. Clearly, in both cases the category $\mathbb{D}_{(\Sigma, \mathcal{E})}$ is well-defined and a subcategory of $\mathbb{C}_{(\Sigma, \mathcal{E})}$, since the inclusion functor $\mathbb{D} \hookrightarrow \mathbb{D}$ preserves products in either case. It only remains to show that in (i) the unit of the reflection $\eta_A : A \to RA$ and in (ii) the unit of the coreflection $\mu_A : CA \to A$ are Σ-homomorphisms. We only show the case (i), as (ii) is analogous.

By definition of the algebra structure on $(RA, \{\sigma_{RA}\})$, we have that $\sigma_{RA} = R\sigma_A$. Thus, the naturality of η and the fact that the reflection preserves countable products yield the following commuting diagram:

$$
\begin{array}{ccc}
A^{|\sigma|} & \xrightarrow{\ \sigma_A\ } & A \\[2pt]
{\scriptstyle \eta_{A^{|\sigma|}} \cong \eta_A^{|\sigma|}}\Big\downarrow & & \Big\downarrow{\scriptstyle \eta_A} \\[2pt]
RA^{|\sigma|} & \xrightarrow[R\sigma_A \cong \sigma_{RA}]{} & RA
\end{array}
$$

which shows the claim. $\qquad\square$

Next we turn our attention towards the construction of free algebras in reflective and coreflective subcategories and get the following result.

Theorem 5.1.3. *Let $\mathbb{C}$ be a category with finite (countable) products, $\mathbb{D}$ a full subcategory of $\mathbb{C}$ with finite (countable) products, and $(\Sigma, \mathcal{E})$ a finitary (ω-ary) parameterised equational theory for $\mathbb{C}$ such that for all $\sigma \in \Sigma$ and $e \in \mathcal{E}$, P_σ and P_e are objects of $\mathbb{D}$. Let furthermore $F : \mathbb{C} \to \mathbb{C}_{(\Sigma, \mathcal{E})}$ be the free algebra functor. Then:*

(i) If $\mathbb{D}$ is a reflective subcategory of $\mathbb{C}$ and the reflection functor $R : \mathbb{C} \to \mathbb{D}$ preserves finite (countable) products, then the free $(\Sigma, \mathcal{E})$-algebra $(A, \{\sigma_A\})$ in $\mathbb{D}$ over X is given by $(RFX, \{\sigma_{RFX}\})$,

(ii) if $\mathbb{D}$ is a coreflective subcategory of $\mathbb{C}$, and the inclusion functor $\mathbb{D} \hookrightarrow \mathbb{C}$ preserves finite (countable) products, then F restricts to a free algebra functor on $\mathbb{D}$.

Proof. We first show (i). By Lemma 5.1.2, $(RFX, \{\sigma_{RFX}\})$ is a $(\Sigma, \mathcal{E})$-algebra in $\mathbb{D}$, and we get a morphism $X \to RFX$ as the composite:

$$X \xrightarrow{\iota_X} FX \xrightarrow{\eta_{FX}} RFX$$

where $\iota_X : X \to FX$ is the inclusion map into the free algebra.

We only have to show that the morphism $\eta_{FX} \circ \iota_X : X \to RFX$ satisfies the unique extension property of a free algebra. For this, let $(B, \{\sigma_B\})$ be any $(\Sigma, \mathcal{E})$-algebra in $\mathbb{D}$, and $f : X \to B$ be a morphism. As $(FX, \{\sigma_{FX}\})$ is the free algebra over X in $\mathbb{C}$, and by Corollary 5.1.2, $(B, \{\sigma_B\})$ a $(\Sigma, \mathcal{E})$-algebra in $\mathbb{C}$, we have a unique homomorphism lifting $\widehat{f} : FX \to B$ for f. Thus $R\widehat{f} : RFX \to RB \cong B$ is a homomorphism lifting for f in $\mathbb{D}$, and uniqueness follows from the universal property of a reflection functor.

For showing (ii), recall that by Lemma 5.1.2 $(CFX, \{\sigma_{CFX}\})$ is a $(\Sigma, \mathcal{E})$-algebra in $\mathbb{C}$ and the unit of the coreflection $\mu_{FX} : CFX \to FX$ is a Σ-homomorphism. Thus, if $\iota_X : X \to FX$ is the inclusion morphism into the free algebra, we have the following commuting diagram:

and it follows that $\mu_{FX} \circ \widehat{C\iota_X} \cong \mathrm{id}_{FX}$, by the freeness of FX over X. But then it also follows that $C(\mu_{FX} \circ \widehat{C\iota_X}) = C(\mathrm{id}_{FX}) \cong \mathrm{id}_{CFX} \cong C\mu_{FX}$, and so $C(\widehat{C\iota_X}) \cong \mathrm{id}_{CFX}$. As C is a coreflection, this yields $C(\widehat{C\iota_X} \circ \mu_{FX}) \cong \mathrm{id}_{CFX} \cong C(\mathrm{id}_{CFX})$, and we can

conclude $\widehat{C\iota_X} \circ \mu_{FX} \cong \mathrm{id}_{CFX}$. Thus we get the following two commuting diagrams:

$$
\begin{array}{ccc}
FX = \!\!=\!\!=\!\!= FX & \quad & CFX = \!\!=\!\!=\!\!= CFX \\
\searrow_{\widehat{C\iota_X}} \quad \nearrow_{\mu_{FX}} & & \searrow_{\mu_{FX}} \quad \nearrow_{\widehat{C\iota_X}} \\
CFX & & FX
\end{array}
$$

And it follows that $FX \cong CFX$, hence it is indeed a $\mathbb{D}$-object, and trivially free as a $(\Sigma, \mathcal{E})$-algebra over X in $\mathbb{D}$. $\qquad \square$

We remark that part (i) of the above theorem can also be shown using Butler's Theorem (see Section 3.7 in [5]).

For applying the above results to the category of topological predomains, recall Schröder and Simpson's Theorem 3.1.13, which says that the reflection functor $\mathcal{M} :$ $\mathbf{QCB} \to \mathbf{TP}$ preserves countable products. It follows with Proposition 5.1.1 that for appropriate equational theories $(\Sigma, \mathcal{E})$, the reflection $\mathcal{M}$ cuts down to a reflection functor $\mathbf{QCB}_{(\Sigma,\mathcal{E})} \to \mathbf{TP}_{(\mathcal{M}\Sigma, \mathcal{M}\mathcal{E})}$. Furthermore, Theorem 5.1.3 shows that a free algebra functor on $\mathbf{TP}$ can be obtained using this reflection. The following definition gives us an appropriate class of parameterised equational theories for topological predomains.

Definition 5.1.4. A parameterised equational theory $(\Sigma, \mathcal{E})$ for $\mathbf{Top}$ is called *ω-ary tp-parameterised* if all parameter spaces are topological predomains, and the arity sets of all operations and equations, and the set Σ are countable. A *finitary tp-parameterised* equational theory is one such that the arity sets of all operations and equations are finite, and the set Σ is countable.

Theorem 5.1.5. *For an ω-ary tp-parameterised equational theory $(\Sigma, \mathcal{E})$, the forgetful functor $U : \mathbf{TP}_{(\Sigma,\mathcal{E})} \to \mathbf{TP}$ has a left adjoint, the free algebra functor $F : \mathbf{TP} \to$ $\mathbf{TP}_{(\Sigma,\mathcal{E})}$. It is obtained as $\mathcal{M}F_{\mathbf{QCB}}$, where $F_{\mathbf{QCB}} : \mathbf{QCB} \to \mathbf{QCB}_{(\Sigma,\mathcal{E})}$ is the free algebra functor on $\mathbf{QCB}$ and $\mathcal{M} : \mathbf{QCB}_{(\Sigma,\mathcal{E})} \to \mathbf{TP}_{(\Sigma,\mathcal{E})}$ the lifting of the reflection functor to the categories of algebras. Moreover, the free algebra functor F induces a strong monad on $\mathbf{TP}$.*

Proof. Immediate with Theorems 3.1.13, 5.1.3 and 4.5.8. $\qquad \square$

In fact, we can apply Theorem 5.1.3 also to $\mathbf{kTop}$, if all operations are finitary. Then we obtain that there is a free algebra functor for $\mathbf{kP}$ (which can also be shown in greater generality with the FAFT), and that the free algebra constructions in $\mathbf{TP}$ and $\mathbf{kP}$ coincide. This result is useful for comparing the treatment of computational effects in Classical and Topological Domain Theory in the following sections.

Corollary 5.1.6. *For a finitary tp-parameterised equational theory $(\Sigma, \mathcal{E})$, the free algebra construction on $\mathbf{TP}$ is the restriction of the free algebra functor $F : \mathbf{kP} \to$ $\mathbf{kP}_{(\Sigma,\mathcal{E})}$ to topological predomains.*

As **TP** and **kP** are poset-enriched categories, one is naturally interested in algebraic theories with inequations. For instance, using Scott's idea of an information order, one would like to have an inequation $\bot \le x$ for the algebraic theory describing nontermination, expressing that a nonterminating program gives no information. For parameterised algebraic theories in a poset-enriched category, the natural definition of inequational theories is as follows.

Definition 5.1.7. Let Σ be a signature for a parameterised algebraic theory for a poset-enriched category $\mathbb{C}$. An *inequation* $(i : t \le t')$, is given by a pair of Σ-algebraic operations $t, t' : P_i \times U_{|i|} \Rightarrow U$. Again we call P_i the parameter object and $|i|$ the arity of i.

A Σ-algebra $(A, \{\sigma_A\})$ is said to *satisfy* i, if $t_A \le t'_A$ in the Hom-poset $\mathbb{C}(P_i \times A^{|i|}, A)$. If $\mathcal{I}$ is a set of inequations for Σ, we call $(\Sigma, \mathcal{I})$ a *parameterised inequational theory* for $\mathbb{C}$, and a Σ-algebra is a $(\Sigma, \mathcal{I})$-*algebra* (also $\mathbb{C}_{(\Sigma,\mathcal{I})}$-algebra) if it satisfies all inequations $i \in \mathcal{I}$.

This yields a well-defined category $\mathbb{C}_{(\Sigma,\mathcal{I})}$ of $(\Sigma, \mathcal{I})$-algebras and Σ-homomorphisms in $\mathbb{C}$.

The following result shows that in the presence of parameter spaces, we do not gain more expressivity by allowing inequations instead of equations. Recall that by $\mathbb{S}$, we denote the Sierpinski space.

Lemma 5.1.8. *Let $\mathbb{C}$ be a full subcategory of* **Top** *consisting of T_0-spaces and containing Sierpinski space $\mathbb{C}$. Then for any parameterised inequational theory $(\Sigma, \mathcal{I})$ for $\mathbb{C}$, there exists a parameterised equational theory $(\Sigma', \mathcal{E})$, such that $\mathbb{C}_{(\Sigma,\mathcal{I})} \cong \mathbb{C}_{(\Sigma',\mathcal{E})}$. The same holds for* **TP**.

Proof. For each inequation $(i : t \le t')$ in $\mathcal{I}$, we introduce an auxiliary operation σ^i with $P_{\sigma^i} = \mathbb{S} \times P_i$, and $|\sigma^i| = |i|$. Thus we construct

$$\Sigma' = \Sigma \cup \{\sigma^i | \, i \in \mathcal{I}\}.$$

Now we introduce for $(i : t \le t')$ two equations $(e^i : \sigma^i(\bot, p, (x_j)_{j\in|i|}) = t(p, (x_j)_{j\in|i|}))$ and $(e'^i : \sigma^i(\top, p, (x_j)_{j\in|i|}) = t'(p, (x_j)_{j\in|i|}))$, and set

$$\mathcal{E} = \{e^i | \, i \in \mathcal{I}\} \cup \{e'^i | \, i \in \mathcal{I}\}.$$

If $(A, \{\sigma_A\})$ is a $(\Sigma, \mathcal{I})$-algebra in $\mathbb{C}$, then it becomes a Σ'-algebra, if we define for $i \in \mathcal{I}$,

$$\sigma^i_A(z, p, (a_j)_{j\in|i|}) = \begin{cases} t_A(p, (a_j)_{j\in|i|}) & \text{if } z = \bot \\ t'_A(p, (a_j)_{j\in|i|}) & \text{if } z = \top \end{cases}$$

Notice that $\sigma^i_A : \mathbb{S} \times P_i \times A^{|i|} \to A$ is continuous, because if $U \subseteq A$ is open, then

$(\sigma_A^i)^{-1}(U) = \{\bot\} \times t_A^{-1}(U) \cup \{\top\} \times t'^{-1}_A(U)$, and, as $t_A^{-1}(U) \subseteq t'^{-1}_A(U)$, this is equal to $\mathbb{S} \times t_A^{-1}(U) \cup \{\top\} \times t'^{-1}_A(U)$, which is open by the continuity of t_A and t'_A. It immediately follows that all equations in $\mathcal{E}$ are satisfied by A.

Conversely, if $(A, \{\sigma_A\})$ is a $(\Sigma', \mathcal{E})$-algebra in $\mathbb{C}$, then as $\Sigma \subseteq \Sigma'$, it also is a Σ-algebra. If $(i : t \leq t')$ is an inequation in $\mathcal{I}$, then A satisfies i, as we have $\sigma^i(\bot, p, (x_j)_{j \in |i|}) = t(p, (x_j)_{j \in |i|})$ and $\sigma^i(\top, p, (x_j)_{j \in |i|}) = t'(p, (x_j)_{j \in |i|})$, hence

$$
\begin{aligned}
(p, (a_j)_{j \in |i|}) \in t_A^{-1}(U) &\Rightarrow (\bot, p, (a_j)_{j \in |i|}) \in (\sigma_A^i)^{-1}(U) \\
&\Rightarrow (\top, p, (a_j)_{j \in |i|}) \in (\sigma_A^i)^{-1}(U) \; . \\
&\Rightarrow (p, (a_j)_{j \in |i|}) \in t'^{-1}_A(U)
\end{aligned}
$$

Thus $\mathbb{C}_{(\Sigma, \mathcal{I})}$-algebras and $\mathbb{C}_{(\Sigma', \mathcal{E})}$-algebras coincide, and as the extra operations of Σ' are defined in terms of Σ-algebraic operations, Σ-homomorphisms are Σ'-homomorphisms, showing that indeed $\mathbb{C}_{(\Sigma, \mathcal{I})} \cong \mathbb{C}_{(\Sigma', \mathcal{E})}$. □

It follows that parameterised equational theories have the same expressive power as parameterised inequational theories in the categories like **kP**, **TP**, **SeqP** or **DCPO**. Thus, in the following we freely use inequations, when considering parameterised equational theories, knowing that with the above lemma, we can always translate the inequations into equations using auxiliary $\mathbb{S}$-parameterised operations.

5.2 Free algebras for topological domains

The results of the previous section raise the question whether we also have free algebras in presence of least elements, i.e. for topological domains. However, it turns out that even for the simplest non-parameterised equational theories, this is not the case, as the following example shows.

Let Σ be given by a single nullary operation c, i.e. a constant. Then a $\mathbf{TD}_\Sigma$-algebra is a topological domain A together with a distinguished element $c_A \in A$, written as (A, c_A). We show by contradiction that there exists no free $\mathbf{TD}_\Sigma$-algebra over the one-element domain $\mathbf{1} = \{*\}$. Assume such a free algebra (A, c_A) exists, with inclusion map $\eta_{\mathbf{1}} : \mathbf{1} \to A$. Clearly, $(\mathbb{S}, \top)$ and $(\mathbb{S}, \bot)$ are both $\mathbf{TD}_\Sigma$-algebras, and so we obtain the following commuting diagrams

where the upper maps are homomorphism extensions, i.e. $h(c_A) = \top$ in the left and $h'(c_A) = \bot$ in the right diagram. It follows immediately from the left diagram that

$c_A \not\sqsubseteq \eta(*)$, and from the right that $\eta(*) \not\sqsubseteq c_A$. Thus, as A is a topological domain, it has a least element $\perp_A$, which is distinct from c_A and $\eta(*)$. But now the map $\top : \mathbf{1} \to (\mathbb{S}, \top)$ has at least two homomorphism extensions making the diagram below commute:

$$(A, c_A) \dashrightarrow (\mathbb{S}, \top)$$

$$\eta_X \uparrow \qquad \nearrow \top$$

$$\mathbf{1}$$

The first one being the constant map $\top : A \to \mathbb{S}$, the second being the strict map

$$h : x \mapsto \begin{cases} \perp & \text{if } x = \perp_A \\ \top & \text{otherwise} \end{cases}$$

This contradicts the unique extension property of a free algebra.

Nevertheless, there do exist parameterised equational theories for which free algebras exist in **TD**. One particular such case is when the construction of free **TP**-algebras preserves the existence least elements, which is the case for parameterised equational theories, for which all operations are idempotent in a suitable sense.

Definition 5.2.1. A parameterised equational theory $(\Sigma, \mathcal{E})$ for **TP** is called *idempotent* if for each operation $\sigma \in \Sigma$, there exists an equation $e \in \mathcal{E}$ of the form $x = \sigma(p, (x)_{i \in |\sigma|})$, i.e. the following diagram commutes:

$$P_\sigma \times A \xrightarrow{\quad \pi_2 \quad} A$$

$$P \times \delta_A^{|\sigma|} \searrow \qquad \nearrow \sigma$$

$$P_\sigma \times A^{|\sigma|}$$

If $(\Sigma, \mathcal{E})$ is such an idempotent parameterised equational theory, D a topological domain and FD the free $TP_{(\Sigma, \mathcal{E})}$-algebra over D, then we can deduce for all terms $t \in FD$, that $\perp_D \sqsubseteq t$, using the inductive absolutely free algebra construction, and the fact that the reflection $\mathcal{M} : \mathbf{QCB} \to \mathbf{TP}$ only adds elements which arise as suprema of ascending chains. Examples for idempotent theories are the algebraic theories for the upper, lower and convex powerdomain, or the free convex space construction for probabilistic computations.

Observe that for an idempotent parameterised equational theory, the inclusion map $D \to FD$ is strict. Moreover, it is clear that in this case the unique homomorphism extension of a strict map is again strict, and thus the free algebra construction readily carries over to the category $\mathbf{TD}_\perp$ of topological domains and strict maps. It turns out that strictness is a crucial property for constructing free algebras over topological

domains. In fact we can show that there exists a free algebra construction in $\mathbf{TD}_\perp$, for arbitrary ω-ary tp-parameterised equational theories. The crucial observation for this is that $\mathbf{TD}_\perp$ itself is the category of $\mathbf{TP}$-algebras for the lifting monad, i.e. the equational theory describing nontermination. Thus least elements of objects and strictness of maps can be described purely algebraically. The details are given in the following Theorem.

Theorem 5.2.2. *Let $(\Sigma, \mathcal{E})$ be an ω-ary tp-parameterised equational theory, and X be a topological domain. Then there exists a $\mathbf{TD}_{(\Sigma,\mathcal{E})}$-algebra $(FX, \{\sigma_{FX}\})$ and a strict map $\eta_X : X \to FX$ with the following universal property. For any strict map $f : X \to A$ into a $\mathbf{TD}_{(\Sigma,\mathcal{E})}$-algebra $(A, \{\sigma_A\})$, there exists a unique Σ-homomorphism extension $\widehat{f} : FX \to A$, as in:*

$$
\begin{array}{ccc}
FX & \dashrightarrow^{\widehat{f}} & A \\
\eta_X \uparrow & \nearrow_{f} & \\
X & &
\end{array}
$$

Moreover, the algebra extension $\widehat{f}$ is strict.

Proof. Let $(\Sigma_\perp, \mathcal{E}_\perp)$ denote the ω-ary tp-parameterised equational theory, given by $\Sigma_\perp = \Sigma \cup \{\perp\}$, where $\perp$ is a constant, and $\mathcal{E}_\perp = \mathcal{E} \cup \{(e : \perp \leq x)\}$. In other words, $(\Sigma_\perp, \mathcal{E}_\perp)$ extends $(\Sigma, \mathcal{E})$ by lifting. Assume X is a topological domain. We claim that the universal property is satisfied by $F_{\mathbf{TP}} X^\dagger$, where $F_{\mathbf{TP}} : \mathbf{TP} \to \mathbf{TP}_{(\Sigma_\perp, \mathcal{E}_\perp)}$ denotes the free $(\Sigma_\perp, \mathcal{E}_\perp)$-algebra functor for topological predomains, and $(-)^\dagger$ is the mapping given in Definition 3.2.14.

To verify this claim, observe that the categories $(\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp$ and $\mathbf{TP}_{(\Sigma_\perp, \mathcal{E}_\perp)}$ are isomorphic. Let now $f : X \to A$ be a strict map into a $\mathbf{TD}_{(\Sigma,\mathcal{E})}$-algebra $(A, \{\sigma_A\})$. This map restricts to a continuous map $g : X^\dagger \to A$, such that $g = f \circ \iota_{X^\dagger}$ for the canonical embedding $\iota_{X^\dagger} : X^\dagger \to X \cong X^\dagger_\perp$. Thus we obtain the following commuting diagram:

$$
\begin{array}{ccc}
F_{\mathbf{TP}} X^\dagger & \dashrightarrow^{\widehat{g}} & A \\
\gamma_{X^\dagger} \uparrow & & \\
& X & \nearrow_{f} \\
& \nearrow_{\iota_{X^\dagger}} & \\
X^\dagger & &
\end{array}
$$

where $\widehat{g}$ is the unique $\Sigma_\perp$-homomorphism extension of $g = f \circ \iota_{X^\dagger}$. As $F_{\mathbf{TP}} X^\dagger$ is a $(\Sigma_\perp, \mathcal{E}_\perp)$-algebra, it has a least element, the constant $\perp_{F_{\mathbf{TP}} X^\dagger}$, hence it is a topological domain. Thus there exists a unique strict map $\eta_X : X \to F_{\mathbf{TP}} X^\dagger$, with $\eta_{X^\dagger} = \gamma_X \circ \iota_{X^\dagger}$,

completing the above diagram to:

$$
\begin{array}{ccc}
F_{\mathbf{TP}}X^\dagger & \overset{\widehat{g}}{\dashrightarrow} & A \\
\uparrow \gamma_{X\dagger} & \eta_X \quad f & \nearrow \\
& X & \\
\nearrow \iota_{X\dagger} & & \\
X^\dagger & &
\end{array}
$$

As $\widehat{g}$ is a strict Σ-homomorphism, it only remains to show that it is unique amongst such extensions for f. But this follows immediately from the fact that it was the unique $\Sigma_\perp$-homomorphism extension for of $g = f \circ \iota_{X\dagger}$. □

This theorem yields the following result.

Corollary 5.2.3. *For any ω-ary tp-parameterised equational theory $(\Sigma, \mathcal{E})$, the forgetful functor $U : (\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp \to \mathbf{TD}_\perp$ has a left adjoint $\mathbf{TD}_\perp \to (\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp$. Moreover, the corresponding monad on $\mathbf{TD}_\perp$ is strong (with respect to the cartesian structure on $\mathbf{TD}_\perp$).*

Proof. The existence of a left adjoint follows immediately from the previous theorem. That the corresponding monad is strong follows from the fact that $(\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp$ is isomorphic to $\mathbf{TP}_{(\Sigma_\perp,\mathcal{E}_\perp)}$. Moreover, the free algebra functor $F : \mathbf{TP} \to \mathbf{TP}_{(\Sigma_\perp,\mathcal{E}_\perp)}$ is strong by Theorem 4.5.8. It is a straightforward verification that, for topological domains X, Y, the strength $t_{X,Y} : X \times FY \to F(X \times Y)$ of this monad is a strict map. Thus, by construction of the left adjoint $\mathbf{TD}_\perp \to (\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp$, it follows that the strength t transfers to a strength for the monad generated by the adjunction

$$
\mathbf{TD}_\perp \to (\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp \dashv (\mathbf{TD}_{(\Sigma,\mathcal{E})})_\perp \to \mathbf{TD}_\perp.
$$

□

We remark that although we have the unique extension property with respect to strict maps, the operations of the signature need not be strict in general. For instance consider the inequational theory $(\Sigma, \mathcal{E})$ with just one unary operation symbol, σ, and no equations. Then the free $\mathbf{TD}_{\perp(\Sigma,\mathcal{E})}$-algebra $F\mathbf{1}$ over the one point domain $\mathbf{1}$ is isomorphic to the free $\mathbf{TP}_{(\Sigma_\perp,\mathcal{E}_\perp)}$-algebra over the empty space $\mathbf{0}$. This free $\mathbf{TP}$-algebra in turn is the monotone convergence reflection of the free $\mathbf{QCB}_{(\Sigma_\perp,\mathcal{E}_\perp)}$-algebra over $\mathbf{0}$. But the elements of the free $\mathbf{QCB}$-algebra are of the form $\sigma^n(\perp)$, for $n \in \mathbb{N}$. Furthermore, we have the inequation $\perp \sqsubseteq \sigma(\perp)$, and consequently $\sigma^n(\perp) \sqsubseteq \sigma^m(\perp)$ for all $n \leq m$, as σ is continuous. It follows that the free $\mathbf{QCB}_{(\Sigma_\perp,\mathcal{E}_\perp)}$-algebra over $\mathbf{0}$ is

given by $(\mathbb{N}^{\uparrow}, 0, s)$, where $\mathbb{N}^{\uparrow}$ is the generic ascending chain and $s : \mathbb{N}^{\uparrow} \to \mathbb{N}^{\uparrow}$ is the successor operation $n \mapsto n + 1$. Hence, the free $\mathbf{TD}_{\perp(\Sigma, \mathcal{E})}$-algebra $F\mathbf{1}$ is isomorphic to $(\mathcal{M}(N^{\uparrow}), 0, s)$, where $\mathcal{M}(\mathbb{N}^{\uparrow})$ is the monotone convergence reflection of the generic chain, and s again the successor operation. But, of course, $s(0) \neq 0$, hence the operation s is not strict. It follows that the left adjoint of Corollary 5.2.3 is not a free algebra functor in the sense of Definition 4.1.4. The reason for this is that the categories $(\mathbf{TD}_{(\Sigma, \mathcal{E})})_{\perp}$ and $(\mathbf{TD}_{\perp})_{(\Sigma, \mathcal{E})}$ are not isomorphic in general. For these categories to be isomorphic, the operations need to be strict.

Let us also remark that one can always force the operations to satisfy strictness conditions, by expressing these directly within the equational theory $(\Sigma, \mathcal{E})$. For instance a bistrict version of nondeterminism, is given by the ordinary inequational theory with signature $\{\perp, +\}$, where $\perp$ is a constant and $+$ binary, and the following set of inequations:

- $\perp \leq x$,

- $x + x = x$,

- $x + y = y + x$,

- $(x + y) + z = x + (y + z)$,

- $x + \perp = \perp$.

Strictness of $+$ follows already from the second equation, whereas bistrictness is obtained by the third and last equations. As $(\Sigma, \mathcal{E})$ already has an intrinsic least element, one gets that the theories $(\Sigma, \mathcal{E})$ and $(\Sigma_{\perp}, \mathcal{E}_{\perp})$ have the same algebras over the category of topological (pre)domains. Consequently, the categories $(\mathbf{TD}_{(\Sigma, \mathcal{E})})_{\perp}$ and $(\mathbf{TD}_{\perp})_{(\Sigma, \mathcal{E})}$ are isomorphic, and thus the above construction yields a free algebra construction in $\mathbf{TD}_{\perp}$.

We furthermore remark, that, by Definition 4.1.1, the operations use the cartesian products of $\mathbf{TD}$, and not the monoidal structure of $\mathbf{TD}_{\perp}$. It might be interesting to investigate how far one can recover the above results if one allows operations to use the monoidal structure instead. Certainly, we would have to recover the technical results of the previous section for the adjunction $(-)_{\perp} \dashv I : \mathbf{TD}_{\perp} \hookrightarrow \mathbf{TP}$, with respect to the monoidal structure on $\mathbf{TD}_{\perp}$, in order to show the existence of a free algebra construction in $\mathbf{TD}_{\perp}$ for corresponding algebraic theories. We conjecture that free algebras for the monoidal structure correspond to imposing multistrictness on the operations.

Summarising, we have shown that the category of topological domains and strict maps has a construction of computational types for a wide class of parameterised equational theories. On the other hand, free algebras, in the sense of Definition 4.1.4, exist

only for certain classes of algebraic theories, such as the idempotent theories of Definition 5.2.1 in **TD**, and for theories with strict operations in **TD**$_\perp$. Examples of idempotent theories are the inequational theories for the classical powerdomain constructions, see Chapter 6 of [2], and the parameterised equational theory for probabilistic computations, given in the previous chapter. Furthermore, strictness can always be imposed onto the algebraic theories directly.

5.3 Comparison with Classical Domain Theory

Having seen that a good collection of free algebras exist in the categories **TP** and **TD**$_\perp$, we now investigate how our results compare to the construction of free algebras in Classical Domain Theory. Abramsky and Jung [2] have investigated free algebra constructions for ordinary inequational theories in Classical Domain Theory. Specific instances of such free algebra constructions are the classical powerdomains for nondeterminism by Plotkin [108] and Smyth [142]. We show that for a wide range of ordinary inequational theories, the free algebra constructions of Classical and Topological Domain Theory coincide on common spaces. This shows in particular that the classical powerdomain constructions for nondeterminism can be generalised to all topological predomains.

Recall that by Proposition 3.1.5, the category of dcpos and continuous maps is a full coreflective subcategory of **kP**, and the inclusion functor **DCPO** $\hookrightarrow$ **kP** preserves finite products by Lemma 3.3.4. Using Theorem 5.1.3, we thus obtain the following result.

Proposition 5.3.1. *Let* $(\Sigma, \mathcal{E})$ *be a finitary parameterised equational theory for* **kP** *such that all parameter spaces carry the Scott-topology. Then the restriction of the free algebra functor* $F : \mathbf{kP} \to \mathbf{kP}_{(\Sigma, \mathcal{E})}$ *to dcpos with Scott-topology, yields a free algebra functor for* **DCPO***.*

Proof. Follows immediately from Proposition 3.1.5 and Theorem 5.1.3. $\qquad\square$

This gives the following result on comparing free algebras in Classical and Topological Domain Theory.

Theorem 5.3.2. *Let* $(\Sigma, \mathcal{E})$ *be a finitary tp-parameterised equational theory such that all parameter spaces carry the Scott-topology. Then the free algebra constructions in* **TP** *and* **DCPO** *coincide on topological predomains which carry the Scott-topology.*

Proof. This follows from the previous proposition and Corollary 5.1.6. $\qquad\square$

Observe that all finitary non-parameterised inequational theories fall into the scope of this result by Lemma 5.1.8. Thus we get that for all such theories the free algebra

constructions in Classical and Topological Domain Theory coincide. In particular, we get that the upper, lower and convex powerdomain constructions can be generalised to all topological predomains. As an example, we pick the upper powerdomain.

Traditionally, the upper powerdomain $P^u X$ for a dcpo X is given as the free **DCPO**-algebra over X for the inequational theory given by a binary operation $\wedge : A^2 \to A$, subject to the inequations:

- $x = x \wedge x$,

- $x \wedge y = y \wedge x$,

- $x \wedge (y \wedge z) = (x \wedge y) \wedge z$,

- $x \wedge y \leq x$.

(The lower powerdomain is obtained by changing left and right-hand side of the last inequation, and the convex powerdomain by omitting this inequation completely, i.e. by the equational theory for nondeterminism, given in the previous chapter.) Using Lemma 5.1.8, we can formulate a parameterised equational theory $(\Sigma, \mathcal{E})$ for Topological Domain Theory, where we again have a binary operation $\wedge : A^2 \to A$, and an auxiliary operation $\sigma : \mathbb{S} \times A^2 \to A$, subject to the first three equations above, and the following two:

- $\sigma(\bot, x, y) = x \wedge y$,

- $\sigma(\top, x, y) = x$.

If $F : \mathbf{TP} \to \mathbf{TP}_{(\Sigma, \mathcal{E})}$ is the free algebra functor for this theory, which exists by Theorem 5.1.5, then this construction coincides with the upper powerdomain construction in the sense that, for any topological predomain X, which carries the Scott-topology, it holds that $FX \cong P^u X$.

Observe that Theorem 5.3.2 in particular applies to all ω-continuous dcpos. It is known, see e.g. [2], that the upper powerdomain over an ω-continuous dcpo X is given by the set KX of nonempty compact-saturated subsets equipped with the Scott-topology for the reverse inclusion order, hence so is FX, the free **TP**-algebra for the parameterised equational theory above.

An interesting question in its own right is, for which algebraic theories the free algebra construction in **TP** and **DCPO** preserves ω-continuous dcpos. Abramsky and Jung [2] have shown that free algebras for non-parameterised inequational theories exist for continuous domains, and Koslowski [75] has shown that, in fact, **CONT** is closed under the free algebra construction for such theories in **DCPO**. Hence for finitary non-parameterised inequational theories, ω**CONT** is closed under the free algebra construction in **TP**. However, this cannot be generalised to the parameterised setting,
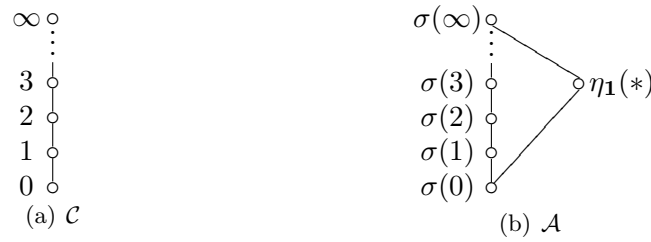
Figure 5.1: Counterexample

as Strüder [152], a student of Jung, has shown. The following counterexample is taken from her thesis, and slightly simplified.

Let $\mathcal{C}$ be the dcpo given in Figure 5.1(a). Let furthermore $(\Sigma, \mathcal{E})$ be the parameterised inequational theory for **DCPO**, given by one operation σ with $P_\sigma = \mathcal{C}$ and $|\sigma| = 0$, subject to the inequations $\sigma(0) \leq x$ and $x \leq \sigma(\infty)$. In other words, a $\mathbf{DCPO}_{(\Sigma,\mathcal{E})}$-algebra has a least element $\sigma(0)$ and a greatest element $\sigma(\infty)$. Then the free $\mathbf{DCPO}_{(\Sigma,\mathcal{E})}$-algebra over the one-element dcpo $\mathbf{1}$, which also is the free $\mathbf{TP}_{(\Sigma,\mathcal{E})}$-algebra, is $\mathcal{A}$, from Figure 5.1(b), which is a standard example of a non-continuous dcpo. Moreover, considering that $\mathcal{C}$ itself is a $\mathbf{CONT}_{(\Sigma,\mathcal{E})}$-algebra, one can prove that there is no free $\mathbf{CONT}_{(\Sigma,\mathcal{E})}$-algebra over $\mathbf{1}$, by showing that $\eta(*)$ must be incomparable with $\sigma(n)$ for all $n \notin \{0, \infty\}$.

In her thesis, Strüder gives a condition for the equations under which free $\mathbf{CONT}_{(\Sigma,\mathcal{E})}$-algebras can be constructed, but she also shows that this condition is not necessary for the existence of a free algebra functor in **CONT**. Also following Koslowski's approach in a parameterised setting yields a complicated condition for **CONT** to be closed under the free algebra construction in **DCPO**, and again this condition is not necessary. We leave it for future research to investigate in how far free algebras for parameterised equational theories do exist for continuous dcpos.

Nevertheless, we get the following obvious consequence of Theorem 5.3.2.

Corollary 5.3.3. *Let $(\Sigma, \mathcal{E})$ be a finitary tp-parameterised equational theory for* **kP** *such that all parameter spaces carry the Scott-topology. Let furthermore the free algebra functor* $\mathbf{DCPO} \rightarrow \mathbf{DCPO}_{(\Sigma,\mathcal{E})}$ *restrict to a free algebra functor* $\mathbf{CONT} \rightarrow \mathbf{CONT}_{(\Sigma,\mathcal{E})}$. *Then the inclusion functor* $\omega\mathbf{CONT} \hookrightarrow \mathbf{TP}$ *preserves the constructions of free $(\Sigma, \mathcal{E})$-algebras.*

As we have seen, this Corollary applies to the upper, lower and convex powerdomains, but also to the equational theories describing nontermination and exceptions. It does not, however, apply to the equational theory for probabilistic computations, as the parameter space $\mathbb{I}$ does not carry the Scott-topology. Thus, we devote the next section to a more subtle comparison between the free algebra construction for the equational theory for probabilistic computations, as convex spaces in **TP**, and the Jones/Plotkin

probabilistic powerdomain [59, 58].

Before investigating the probabilistic powerdomain, let us first mention the following two questions.

1. Can Theorem 5.3.2 be generalised to ω-ary tp-parameterised equational theories?

2. Does $\omega\mathbf{CONT}$ have free algebras for non-parameterised ω-ary equational theories, and if so are they inherited from $\mathbf{DCPO}$?

Notice that the FAFT can be applied to $\mathbf{DCPO}$, in order to show the existence of free algebras for ω-ary parameterised equational theories. However, our results do not tell us whether this construction coincides with the one for topological predomains, although we expect this not to hold, since the construction of countable products differs in general. Furthermore, the results of Abramsky and Jung [2] and Koslowski [75] about free algebras for continuous domains cannot be applied beyond finitary theories.

5.4 The probabilistic powerdomain

The first approach to modelling probabilistic behaviour in Classical Domain Theory via a probabilistic powerdomain construction was suggested by Saheb-Djahromi [127]. This construction was later generalised by Jones and Plotkin [59, 58], and we show that for ω-continuous dcppos the classical probabilistic powerdomain construction coincides with the free algebra construction for convex spaces in $\mathbf{TP}$ which has been suggested for modelling probabilistic computations by Graham [37] and, more recently, Plotkin and Power [112]. Thus, Topological Domain Theory gives a solution to an open problem in Classical Domain Theory [69]: it provides a framework for denotational semantics, which has connections to notions of computability, allows higher type constructions and is closed under a probabilistic powerdomain construction.

Traditionally, the classical probabilistic powerdomain is not given as the free algebra construction for an algebraic theory, but as the dcpo of continuous probability measures. However, Jones [58] and Heckmann [40] have shown that for continuous dcppos, it can be recovered as the free algebra in $\mathbf{DCPO}$ for the equational theory for pointed convex spaces. That free algebras for this theory exist in $\mathbf{DCPO}$, although the parameter space $\mathbb{I}$, the unit interval with the Euclidean topology, is not an object of $\mathbf{DCPO}$ itself, was already shown by Graham [37]. However, since $\mathbb{I}$ does not carry the Scott-topology, we cannot simply apply Theorem 5.3.2 for comparing the free algebra constructions for this algebraic theory in $\mathbf{DCPO}$ and $\mathbf{TP}$. Instead, we use an auxiliary algebraic theory, proposed by Heckmann in [40], to show that the free algebra construction in $\mathbf{TP}$ for the equational theory for probabilistic computations extends the classical probabilistic powerdomain construction.

Let us start by recalling the construction of the (sub)probabilistic powerdomain as given by Jones and Plotkin [58, 59].

By $\mathbb{I}^{\uparrow}$, we denote the unit interval $[0,1]$ equipped with the topology given by open intervals of the form $(x,1]$ for $x \in \mathbb{I}$, which is the Scott topology with respect to the usual order. A *continuous subprobability valuation* over a topological space X is a continuous map $\nu : \mathcal{O}(X) \to \mathbb{I}^{\uparrow}$, where $\mathcal{O}(X)$ carries the Scott-topology, such that $\nu(\emptyset) = 0$, and $\nu(U \cup V) + \nu(U \cap V) = \nu(U) + \nu(V)$. If additionally $\nu(X) = 1$, we call ν a *continuous probability valuation* over X. We denote the set of continuous subprobability valuations, respectively probability valuations, over X by $\mathcal{V}_{\leq 1}(X)$, respectively $\mathcal{V}_1(X)$. These sets have an intrinsic ordering given by $\nu \leq \nu'$ if for all open subsets $U \in \mathcal{O}(X)$, it holds that $\nu(U) \leq \nu'(U)$. With this ordering, $\mathcal{V}_{\leq 1}(X)$ and $\mathcal{V}_1(X)$ become dcpos, and so we can equip them with the Scott-topology. We denote the resulting topological spaces by $\mathcal{V}_{\leq 1}^{\uparrow}(X)$, respectively $\mathcal{V}_1^{\uparrow}(X)$.

As mentioned before, in [58, 59], Plotkin and Jones call $\mathcal{V}_{\leq 1}^{\uparrow}(X)$ the probabilistic powerdomain, as it can be used to model a combination of nontermination and probabilistic computations. However, we think it is useful to distinguish these effects, and so we call $\mathcal{V}_{\leq 1}^{\uparrow}(X)$ the *classical subprobabilistic powerdomain*, and $\mathcal{V}_1^{\uparrow}(X)$ the *classical probabilistic powerdomain*. The two powerdomains can be related using the following result.

Lemma 5.4.1. *Let X be a topological space, and $X_{\perp}$ be its lifting. Then it holds that $\mathcal{V}_{\leq 1}^{\uparrow}(X) \cong \mathcal{V}_1^{\uparrow}(X_{\perp})$.*

Proof. Straightforward. $\qquad\square$

Graham [37] and Plotkin and Power [110] have suggested the standard equational theory for a convex space for modelling probabilistic computations. This theory is given by a binary $\mathbb{I}$-parameterised operation $+ : \mathbb{I} \times X^2 \to X$, subject to the equations:

- $x +_1 y = x$,

- $x +_\lambda x = x$,

- $x +_\lambda y = y +_{1-\lambda} x$,

- $(x +_\lambda y) +_{\lambda'} z = x +_{\lambda\lambda'} (y +_{1 - \frac{\lambda'(1-\lambda)}{1-\lambda\lambda'}} z)$, for $\lambda\lambda' \neq 1$.

We call the **TP**-algebras for this equational theory *convex topological predomains*, and denote the free algebra functor by $F_{conv} : \mathbf{TP} \to \mathbf{TP}_{conv}$. The homomorphisms between convex topological predomains are called *affine maps*.

This algebraic theory for probabilistic computations can be easily extended to model nontermination, by adding a constant $\perp : \mathbf{1} \to X$ together with an inequation $\perp \leq x$. We call the **TP**-algebras for this extended theory *convex topological domains*, the

homomorphisms *strict affine maps*, and denote the free algebra functor by $F_{con\perp}$: $\mathbf{TP} \to \mathbf{TP}_{con\perp}$.

Observe that the operation $+$ is idempotent, hence F_{conv} preserves topological domains, and the inclusion $\eta_X : X \hookrightarrow F_{conv}X$ preserves least elements. The proof of Theorem 5.2.2 then shows:

Lemma 5.4.2. *For a topological predomain* X, $F_{con\perp}X \cong F_{conv}X_\perp$.

Notice the analogy between Lemmas 5.4.2 and 5.4.1.

Our aim is to show that for an ω-continuous dcpo X, $F_{con\perp}X$ is isomorphic to the classical subprobabilistic powerdomain $\mathcal{V}_{\leq 1}^\uparrow(X)$, and for an ω-continuous dcppo X, $F_{conv}X$ is isomorphic to the classical probabilistic powerdomain $\mathcal{V}_1^\uparrow(X)$. As we cannot apply Theorem 5.3.2, because the parameter space $\mathbb{I}$ does not carry the Scott-topology, we use the following auxiliary parameterised equational theory, introduced by Heckmann in [40].

Definition 5.4.3. A $\mathbf{TP}$-*interval cone* is a topological predomain A with a constant $\perp$, and continuous operations $\oplus : A^2 \to A$ and $\cdot : \mathbb{I}^\uparrow \times A \to A$, subject to the following axioms:

- $0 \cdot x = \perp$,

- $1 \cdot x = x$,

- $\lambda \cdot \perp = \perp$,

- $\lambda \cdot (\mu \cdot x) = \lambda\mu \cdot x$,

- $x \oplus x = x$,

- $x \oplus y = y \oplus x$,

- $(x \oplus y) \oplus (u \oplus v) = (x \oplus u) \oplus (y \oplus v)$,

- $(\frac{\lambda+\mu}{2}) \cdot x = \lambda \cdot x \oplus \mu \cdot x$,

- $\lambda \cdot (x \oplus y) = \lambda \cdot x \oplus \lambda \cdot y,$.

A continuous map $h : A \to B$ between $\mathbf{TP}$-interval cones is called an *IC-homomorphism*, if $h(\perp) = \perp$, for all $x, y \in A$, $h(x \oplus y) = h(x) \oplus h(y)$, and for all $\lambda \in \mathbb{I}^\uparrow$, $h(\lambda \cdot x) = \lambda \cdot h(x)$.

With Theorem 5.1.5, we get that for any topological predomain X, the free $\mathbf{TP}$-interval cone $F_{IC}X$ over X exists. Moreover, the parameter space $\mathbb{I}^\uparrow$ carries the Scott-topology, and thus we can apply Theorem 5.3.2.

Proposition 5.4.4. *For an ω-continuous dcpo X, $F_{IC}X$ is isomorphic to the classical subprobabilistic powerdomain* $\mathcal{V}_{\leq 1}^\uparrow(X)$.

Proof. Heckmann has shown in [40] that **CONT** is closed under the free interval cone construction in **DCPO** which yields the subprobabilistic powerdomain in this case. Thus, with 5.3.3, if X is an ω-continuous dcpo with Scott-topology then the free **TP**-interval cone over X coincides with the free **DCPO**-interval cone over X, which is $\mathcal{V}^{\uparrow}_{\leq 1}(X)$. $\qquad\qquad\square$

We now show that for a continuous dcpo X, the free **TP**-interval cone over X is isomorphic to the free convex topological domain over X. For this, we need to show three lemmas, the first of which is a generalisation of a well-known domain-theoretic fact.

Lemma 5.4.5. *Let X, Y be arbitrary topological spaces and D be a continuous dcpo. Then a map $f : X \times D \to Y$, where $X \times D$ carries the product topology, is (jointly) continuous if and only if it is continuous in each argument separately.*

Proof. Trivially, a jointly continuous map is continuous in each argument. For the converse, let $V \subseteq Y$ be open, and $f(x_0, d_0) \in V$. We show the existence of an open neighbourhood $W \subseteq f^{-1}(V)$ in the product topology of $X \times D$, which contains (x_0, d_0). By continuity in the second argument and the fact that D is a continuous dcpo, there exists $d_1 \ll d_0$ such that $f(x_0, d_1) \in V$. Next, by continuity in the first argument, we find an open neighbourhood $U \subseteq X$ of x_0 such that for all $x \in U$, $f(x, d_1) \in V$. But by separate continuity, the map f is monotone with respect to the product of the specialization orders on X and D, which is the specialization order on the topological product. Thus, for all $x \in U$ and $d \gg d_1$, $f(x, d) \in V$, hence setting $W = U \times {\uparrow} d_1$ yields the required open subset $W \subseteq f^{-1}(V)$. $\qquad\square$

Lemma 5.4.6. *Every **TP**-interval cone $(A, \{\bot_A, \oplus_A, \cdot_A\})$, that is a continuous dcpo, can be equipped with a convex topological domain structure. Moreover, every IC-homomorphism $h : (A, \{\bot_A, \oplus_A, \cdot_A\}) \to (B, \{\bot_B, \oplus_B, \cdot_B\})$ into a **TP**-interval cone, that also carries a convex topological domain structure, becomes a strict affine map between the corresponding convex topological domains.*

Proof. A proof of the first part can be found in Sections 5-7 of [40]; we sketch the argument, dropping the subscripts of the operations for readability.

Using the equations of interval cones, each term can be rewritten as a finite tree, such that all occurrences of $\cdot$ are at the leaves, and we assign the weight $\frac{\lambda}{2^k}$ to each leaf, where λ is the first argument of the $(\cdot)$-operation at the leaf, and k is the number of branches from the root to the leaf, e.g. for $(\lambda_1 \cdot x \oplus \lambda_2 \cdot y) \oplus \lambda_3 \cdot z$, we assign weight $\frac{\lambda_1}{4}$ to x, $\frac{\lambda_2}{4}$ to y, and $\frac{\lambda_3}{2}$ to z. Using the equations one can show that terms resulting in the same weights are indeed equivalent in A. Thus, we get elements of the form $\langle \frac{\lambda_1}{2^{k_1}}, x_1; \ldots; \frac{\lambda_n}{2^{k_n}}, x_n \rangle$, where $\sum_{i=1}^{n} \frac{1}{2^{k_i}} = 1$. Now we use the fact that topological predomains have least upper bounds and define $x +_\lambda y$ as the least upper bound of terms

resulting in $\langle \frac{\lambda_i}{2^{k_i}}, x; \frac{\mu_i}{2^{l_i}}, y \rangle$ such that $\{\frac{\lambda_i}{2^{k_i}}\}_i$ is increasing with $\bigvee^{\uparrow} \frac{\lambda_i}{2^{k_i}} = \lambda$, and $\{\frac{\mu_i}{2^{l_i}}\}_i$ is increasing with $\bigvee^{\uparrow} \frac{\mu_i}{2^{l_i}} = 1 - \lambda$; the existence of such sequences follows from elementary combinatorial principles. The directedness of $\langle \frac{\lambda_i}{2^{k_i}}, x; \frac{\mu_i}{2^{l_i}}, y \rangle$ can be shown using the equations for interval cones. This construction results in a well-defined operation $+ : \mathbb{I} \times A^2 \to A$, which is continuous in each argument separately. But Lemma 5.4.5 and the fact that A is a continuous dcpo, guarantee that $+$ is jointly continuous, and hence A a convex topological predomain.

For the second part, strictness of h is immediate, as $\perp$ is the least element of the specialization order of a **TP**-interval cone. Furthermore, for all terms of the form $\langle \frac{\lambda}{2^k}, x; \frac{\mu}{2^l}, y \rangle$, it holds that $h(\langle \frac{\lambda}{2^k}, x; \frac{\mu}{2^l}, y \rangle) = \langle \frac{\lambda}{2^k}, h(x); \frac{\mu_i}{2^l}, h(y) \rangle$, since h is an IC-homomorphism. Now let, as above, $\{\frac{\lambda_i}{2^{k_i}}\}_i$ be increasing with $\bigvee^{\uparrow} \frac{\lambda_i}{2^{k_i}} = \lambda$, and $\{\frac{\mu_i}{2^{l_i}}\}_i$ be increasing with $\bigvee^{\uparrow} \frac{\mu_i}{2^{l_i}} = 1 - \lambda$. Then, using the fact that any map between topological predomains is Scott-continuous, we get that

$$
\begin{aligned}
h(x +_\lambda y) &= h(\bigvee^{\uparrow} \langle \tfrac{\lambda_i}{2^{k_i}}, x; \tfrac{\mu_i}{2^{l_i}}, y \rangle) \\
&= \bigvee^{\uparrow} h(\langle \tfrac{\lambda_i}{2^{k_i}}, x; \tfrac{\mu_i}{2^{l_i}}, y \rangle) \\
&= \bigvee^{\uparrow} \langle \tfrac{\lambda_i}{2^{k_i}}, h(x); \tfrac{\mu_i}{2^{l_i}}, h(y) \rangle \quad, \\
&= h(x) +_\lambda h(y)
\end{aligned}
$$

showing that h is indeed affine. $\qquad\square$

Finally, we show a converse to the above Lemma.

Lemma 5.4.7. *Any convex topological domain $(A, \{\perp_A, +_A\})$ can be equipped with a* **TP**-*interval cone structure. Moreover, any strict affine map $h : (A, \{\perp_A, +_A\}) \to (B, \{\perp_B, +_B\})$ into a convex topological domain is an IC-homomorphism between the corresponding interval cones.*

Proof. Again we drop the subscripts of the operations for readability. Define $x \oplus y$ as $x +_{\frac{1}{2}} y$ and $\lambda \cdot x$ as $x +_\lambda \perp$. Then all operations for an interval cone are well-defined, and $\oplus$ is obviously continuous. For the continuity of $(\cdot)$, observe that it is clearly continuous as a map $\mathbb{I} \times A \to A$, so we only have to show that for all open $V \subseteq A$, the first component of $\cdot^{-1}(V)$ is upper closed with respect to the usual order of $\mathbb{I}$. So let $\lambda \cdot x \in V$, and $\lambda < \mu$. Then $x +_\lambda \perp \in V$, hence $x +_\lambda (x +_{\frac{\mu - \lambda}{1 - \lambda}} \perp) \in V$, since $\perp$ is the least element of A, and $+$ continuous. But the last term evaluates to $x +_\mu \perp$, showing that the first component of $\cdot^{-1}(V)$ is indeed upper closed. To prove that all the IC-axioms hold in A is a straightforward task and left to the reader.

For the second part, observe that all IC-operations are defined directly from the convex operation $+$. Thus a strict affine map is an IC-homomorphism, as required. $\quad\square$

Proposition 5.4.8. *For an ω-continuous dcpo X, $F_{IC}X$ is isomorphic to $F_{con\perp}X$.*

Proof. Let X be a continuous dcpo, $\eta_X : X \hookrightarrow F_{IC}X$ and $\iota_X : X \hookrightarrow F_{con\perp}X$ be the inclusion maps into the free algebras. Then by Proposition 5.4.4, $F_{IC}X$ is a continuous dcpo, hence by Lemma 5.4.6, it carries a convex structure. Conversely, $F_{con\perp}X$ is a convex topological domain, and so, by Lemma 5.4.7, carries an interval cone structure. Now the universal properties of free algebras yield the following two commuting diagrams:

$$\begin{array}{ccc}
F_{IC}X & \overset{h}{-\!\!\!-\!\!\!\succ} & F_{con\perp}X \\
\eta_X \uparrow & \nearrow \iota_X & \\
X & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
F_{con\perp}X & \overset{h^*}{-\!\!\!-\!\!\!\succ} & F_{IC}X \\
\iota_X \uparrow & \nearrow \eta_X & \\
X & &
\end{array}$$

where h is the unique IC-homomorphism extending ι_X, and h^* the unique strict affine map extending η_X. By Lemma 5.4.6, h is a strict affine map, and by Lemma 5.4.7, h^* is an IC-homomorphism. Thus, the universal property of free algebras yields that $h \circ h^* \cong \mathrm{id}_{F_{con\perp}X}$ and $h^* \circ h \cong \mathrm{id}_{F_{IC}X}$, as required. $\qquad\square$

Thus we finally can show our desired result.

Theorem 5.4.9. *Let X be an ω-continuous dcpo. Then the classical subprobabilistic powerdomain $\mathcal{V}^{\uparrow}_{\leq 1}(X)$ is isomorphic to the free convex topological domain $F_{con\perp}X$ over X. If furthermore X has a least element, then the classical probabilistic powerdomain $\mathcal{V}^{\uparrow}_1(X)$ is isomorphic to the free convex topological predomain $F_{conv}X$ over X.*

Proof. The first part follows by Propositions 5.4.4 and 5.4.8. For the second part, assume that X is an ω-continuous dcppo. Then, using Lemma 5.4.1, we get that $\mathcal{V}^{\uparrow}_1(X) \cong \mathcal{V}^{\uparrow}_{\leq 1}(X^{\dagger}) \cong F_{con\perp}X^{\dagger}$. But $F_{con\perp}X^{\dagger} \cong F_{conv}X$, as we have observed in Lemma 5.4.2, which shows the claim. $\qquad\square$

Let us remark that our results show that one can equally use the algebraic theory of interval cones for modelling probabilistic computations in **TP**. Such interval cones appear in some recent work of Keimel, Plotkin and Tix [159] on combining powerdomains. Nevertheless, the convex space construction is the more natural one when it comes to modelling probability on non-dcpo-like qcb-spaces, hence we consider Theorem 5.4.9 as the pivotal result of this section.

5.5 The probabilistic powerspace construction

Having seen that the free convex topological predomain construction yields the classical probabilistic powerdomain for ω-continuous dcppos, the question naturally arises, whether it yields a classical construction for a more general class of spaces.

If we drop the pointedness condition, then in general the free convex topological pre-domain does not coincide with the classical probabilistic powerdomain, as the following example shows.

Counterexample 1: Let $\mathbf{2} = \{a, b\}$ be the two-element discrete space, then the underlying set of $\mathcal{V}_1^{\uparrow}(\mathbf{2})$ is the unit interval, and the same holds for $F_{conv}(\mathbf{2})$. However, the intrinsic order on $\mathcal{V}_1^{\uparrow}(\mathbf{2})$ is discrete, hence so is the Scott-topology. Thus, $\mathcal{V}_1^{\uparrow}(\mathbf{2})$ is not a qcb-space, as it has too many open sets. It follows that it cannot be isomorphic to the free convex topological predomain over $\mathbf{2}$.

Let us consider $F_{conv}(\mathbf{2})$ a bit more closely. Every term $t \in F_{conv}(\mathbf{2})$ can be uniquely written as $a +_{\lambda} b$ for some $\lambda \in \mathbb{I}$. We show this by induction on the term construction, using Proposition 4.2.2 and Theorems 4.4.3 and 5.1.5. We have:

- $a = a +_1 b$,

- $b = a +_0 b$,

- $(a +_{\lambda} b) +_{\mu} (a +_{\zeta} b) = a +_{\lambda\mu+\zeta-\mu\zeta} b$.

Conversely, every $\lambda \in \mathbb{I}$ induces a term $a +_{\lambda} b \in F_{conv}(\mathbf{2})$, and for $\lambda \neq \mu$, the terms $a +_{\lambda} b$ and $a +_{\mu} b$ are distinct. It follows that the underlying set of $F_{conv}(\mathbf{2})$ is isomorphic to the unit interval $\mathbb{I}$. Moreover, the Euclidean unit interval $\mathbb{I}$ itself is a convex space with the weighted sum operation. The isomorphism $F_{conv}(\mathbf{2}) \to \mathbb{I}$ is the unique homomorphism extension of the map $\mathbf{2} \to \mathbb{I}$, with $a \mapsto 1$ and $b \mapsto 0$. It follows that the free topology on $F_{conv}(\mathbf{2})$ is at least as fine as the Euclidean topology. Conversely, for any strictly finer topology than the Euclidean topology on $F_{conv}(\mathbf{2})$, the map $+ : \mathbb{I} \times F_{conv}(\mathbf{2})^2 \to F_{conv}(\mathbf{2})$ is not continuous, and it follows that $F_{conv}(\mathbf{2}) \cong \mathbb{I}$.

This is an interesting result, as there exists a well-known generalisation of the probabilistic powerdomain construction to all topological spaces, the so-called probabilistic powerspace construction [71, 41], which applied to $\mathbf{2}$ also yields the Euclidean unit interval. It has the following definition.

Definition 5.5.1. The *(classical) probabilistic powerspace* $\mathcal{V}_1^w$ over a topological space X is given by the set of continuous probability valuations equipped with the topology induced by the point-open function space $[\mathcal{O}(X) \to_p \mathbb{I}^{\uparrow}]$, i.e. a subbasis for $\mathcal{V}_1^w$ is given by sets of the form:
$$\langle U, r \rangle := \{\nu \in \mathcal{V}_1(X) \mid \nu(U) > r\}.$$

In the literature, the topology of the probabilistic powerspace is often referred to as the *weak topology*, because it is possible to extend any continuous function $f : X \to \mathbb{I}^{\uparrow}$ to a map $f^* : \mathcal{V}_1(X) \to \mathbb{I}^{\uparrow}$ by introducing a notion of integration, and it turns out that the topology of the probabilistic powerspace is the weakest topology making this extension continuous for every f, see [71, 39, 41, 158, 159]. Moreover, for a continuous

dcppo, the weak topology on $\mathcal{V}_1(X)$ coincides with the Scott-topology [39], and so in a sense, this probabilistic powerspace construction extends the classical probabilistic powerdomain beyond the world of Classical Domain Theory. For instance Jung [65] has studied this probabilistic powerspace construction for stably compact spaces, and shown that it preserves this class of spaces.

The above example raises the question whether $F_{conv}X \cong \mathcal{V}_1^w(X)$ holds for all topological predomains. However, that this is not the case can be seen as follows.

Counterexample 2: We show that for the unit interval $\mathbb{I}$ with the Euclidean topology, the free convex topological predomain $F_{conv}\mathbb{I}$ is not isomorphic to $\mathcal{V}_1^w(\mathbb{I})$.

From Proposition 4.2.2, Theorems 4.4.3 and 5.1.5, and the fact that the free algebra functor $F_{conv} : \mathbf{QCB} \to \mathbf{QCB}_{conv}$ preserves Hausdorff-spaces, one can conclude that an element $F_{conv}\mathbb{I}$ is given by a unique finite convex combination of point valuations. However, the classical Lebesgue measure restricts to a valuation $\lambda : \mathcal{O}(\mathbb{I}) \to \mathbb{I}$ on $\mathbb{I}$, which is not merely given by a finite convex combination of point measures. But λ clearly is continuous, when $\mathcal{O}(\mathbb{I})$ is equipped with the Scott-topology, hence $\lambda \in \mathcal{V}_1^w(\mathbb{I})$, showing $F_{conv}\mathbb{I}$ and $\mathcal{V}_1^w(\mathbb{I})$ are not isomorphic.

Nevertheless it would be a pleasant coincidence if the topology on $F_{conv}X$ was, in general, given by the induced point-open topology. That this does not hold shows the following, rather technical, counterexample.

Counterexample 3: We show that the topology of the free convex topological predomain $F_{conv}\mathbb{N}$ over the natural numbers is strictly finer than the induced point-open topology.

First, observe that the induced point-open topology on the finite convex combinations over $\mathbb{N}$ is compatible and Hausdorff. It follows by the characterisation theorems of free predomain algebras, 4.2.2, 4.4.3 and 5.1.5, that the free algebra topology is at least as fine as it. Hence the underlying set of $F_{conv}\mathbb{N}$ is simply given by the finite convex combinations of natural numbers, i.e. of the form $\sum_{n \in F} \lambda_n \cdot n$, such that $F \subseteq \mathbb{N}$ is finite and $\forall n \in F.\ \lambda_n > 0$. The elements of $\mathcal{V}_1^w(\mathbb{N})$ on the other hand are all countable convex combinations of natural numbers, hence $F_{conv}\mathbb{N}$ is a subset of $\mathcal{V}_1^w(\mathbb{N})$.

Now consider the following topology on $F_{conv}\mathbb{N}$. Let $(\xi_k)_{k \in \mathbb{N}}$ be a sequence of positive real numbers, and set

$$(\xi_k)_{k \in \mathbb{N}}^{\circ} := \{\sum_{n \in F} \lambda_n \cdot n \in F_{conv}\mathbb{N} |\ \forall n \in F.\ \lambda_n < \xi_n\}.$$

These open sets form the basis for a topology Ω on $F_{conv}\mathbb{N}$, which obviously includes all point-open subsets as we have:

$$\langle U, r \rangle = \bigcup \{(\xi_k)_{k \in \mathbb{N}}^{\circ} |\ \sum_{k \notin U} \xi_k < 1 - r\}.$$

Lemma 5.5.2. Ω *is a compatible topology on* $F_{conv}\mathbb{N}$.

Proof. Let $(\sum_{n \in F} \alpha_n \cdot n) +_\lambda (\sum_{m \in F'} \beta_m \cdot m) \in (\xi_k)^\circ_{k \in \mathbb{N}}$, and set for $n \notin F$, respectively $m \notin F'$, $\alpha_n = 0$, respectively $\beta_m = 0$. Then for all $k \in \mathbb{N}$, $\lambda\alpha_k + (1 - \lambda)\beta_k < \xi_k$. Thus, there exist $\kappa, \epsilon > 0$ such that for all $k \in F \cup F'$, $\lambda\alpha_k + (1-\lambda)\beta_k + \kappa(\alpha_k + \beta_k) + 2\epsilon < \xi_k$. Set now for $k \in F \cup F'$, $\xi^0_k = \alpha_k + \frac{1}{\lambda+\kappa}\epsilon$ and $\xi^1_k = \beta_k + \frac{1}{1-\lambda+\kappa}\epsilon$. Then for $\alpha'_k < \xi^0_k$, $\beta'_k < \xi^1_k$ and $\lambda - \kappa < \lambda' < \lambda + \kappa$,

$$
\begin{aligned}
\lambda'\alpha'_k + (1 - \lambda')\beta'_k &< (\lambda + \kappa)(\alpha_k + \tfrac{1}{\lambda+\kappa}\epsilon) + (1 - \lambda + \kappa)(\beta_k + \tfrac{1}{1-\lambda+\kappa}\epsilon) \\
&= (\lambda + \kappa)\alpha_k + (1 - \lambda + \kappa)\beta_k + 2\epsilon \\
&< \xi_k.
\end{aligned}
$$

For $k \notin F \cup F'$, set $\xi^0_k = \xi^1_k = \xi_k$. Then for $\alpha'_k < \xi^0_k$, $\beta'_k \le \xi^1_k$ and $0 \le \lambda' \le 1$,

$$
\begin{aligned}
\lambda'\alpha'_k + (1 - \lambda')\beta'_k &< \lambda'\xi_k + (1 - \lambda')\xi_k \\
&= \xi_k.
\end{aligned}
$$

Thus,

$$
(\lambda - \kappa, \lambda + \kappa) \times (\xi^0_k)^\circ_{k \in \mathbb{N}} \times (\xi^1_k)^\circ_{k \in \mathbb{N}} \subseteq +^{-1}(\xi_k)^\circ_{k \in \mathbb{N}},
$$

and $\sum_{n \in F} \alpha_n \cdot n \in (\xi^0_k)^\circ_{k \in \mathbb{N}}$, and $\sum_{m \in F'} \beta_m \cdot m \in (\xi^1_k)^\circ_{k \in \mathbb{N}}$, showing that Ω is indeed compatible. $\qquad\square$

We actually have shown that Ω is compatible in **Top**, hence sequential compatibility follows, and so the free algebra topology on $F_{conv}\mathbb{N}$ refines Ω.

Lemma 5.5.3. *The set* $(2^{-k})^\circ_{k \in \mathbb{N}}$ *is not open in the point-open topology.*

Proof. First observe that $(2^{-k})^\circ_{k \in \mathbb{N}}$ is nonempty, as it includes the valuation $\frac{3}{4} \cdot \delta_0 + \frac{1}{4} \cdot \delta_1$, where δ_0 and δ_1 are the respective point measures. Now let $\bigcap_{j=0}^K \langle U_j, r_j \rangle$ be a point-open open set such that $\sum_{n \in F} \lambda_n \cdot n \in \bigcap_{j=0}^K \langle U_j, r_j \rangle$. Then for all $j \le K$, $\sum_{n \in U_j \cap F} \lambda_n > r_j$, hence there exist some $\epsilon_j > 0$ such that $\sum_{n \in U_j \cap F} \lambda_n - r_j > \epsilon_j$. Let now $\epsilon = min\{\epsilon_j \mid j \le K\} \cup \{\lambda_n \mid n \in F\}$. Then $\epsilon > 0$, hence there exists $N_0 \in \mathbb{N} \setminus F$ with $2^{-N_0} < \epsilon$. Now pick $N_1 \in F$ and set $\alpha_{N_1} = \lambda_{N_1} - \epsilon$, $\alpha_{N_0} = \epsilon$, and $\alpha_n = \lambda_n$ for all $n \in F \setminus \{N_1\}$. Then $\sum_{n \in F \cup \{N_0\}} \alpha_n \cdot n \in \bigcap_{j=0}^K \langle U_j, r_j \rangle$, but $\alpha_{N_0} = \epsilon > 2^{-N_0}$, hence $\sum_{n \in F \cup \{N_0\}} \alpha_n \notin (2^{-k})^\circ_{k \in \mathbb{N}}$, showing that $(2^{-k})^\circ_{k \in \mathbb{N}}$ cannot be open in the point-open topology. $\qquad\square$

It follows that the free topology on $F_{conv}\mathbb{N}$, which is at least as fine as Ω, is strictly finer than the induced point-open topology, and we get the following result.

Theorem 5.5.4. *In general the free topological predomain* $F_{conv}X$ *does not carry the induced point-open topology.*

So it seems that, in general, the free convex topological predomain construction yields quite unnatural spaces, and cannot be described in terms of the classical probabilistic powerspace construction. However, this probabilistic powerspace construction always yields sober spaces, and it preserves countably-based spaces, as the following Lemma shows.

Lemma 5.5.5. *For all topological spaces X, the probabilistic powerspace $\mathcal{V}_1^w(X)$ is sober. Moreover, if X is a countably-based space, so is $\mathcal{V}_1^w(X)$, and hence it is a topological predomain.*

Proof. The sobriety of $\mathcal{V}_1^w(X)$ has been shown by Heckmann as Proposition 5.1 in [41]. We observe that it preserves countably-based spaces.

Let $\mathcal{B}$ be a basis for the topology on X, and without loss of generality, assume $\mathcal{B}$ is closed under finite unions. We claim that a countable subbasis for $\mathcal{V}_1^w(X)$ is given by sets of the form $\langle V, q \rangle$, where $V \in \mathcal{B}$ and q rational. To verify this claim, observe that for every open $U \subseteq X$, we have that $U = \bigcup \{V \in \mathcal{B}| \ V \subseteq U\}$, and this union is directed by the closure of $\mathcal{B}$ under finite unions. Moreover by continuity, we have that for every $\nu \in \mathcal{V}_1^w(X)$, $\nu(U) > r$ if and only if there exists a $V \in \mathcal{B}$, with $V \subseteq U$ and $\nu(V) > r$, and so there also exists a rational q such that $\nu(V) > q > r$. It follows that $\langle U, r \rangle = \bigcup \{\langle V, q \rangle| \ V \in \mathcal{B}, \ V \subseteq U \text{ and } q > r\}$, which verifies our claim. $\qquad\square$

Now the question arises whether $\mathcal{V}_1^w$ preserves qcb-spaces. In other words: Can we apply the classical probabilistic powerspace construction for modelling probabilistic computations in Topological Domain Theory?

The answer is again negative, as the next counterexample shows. The space used in this counterexample was given by Grunhage and Streicher in [38], to show that **QCB** is not closed under the sobrification.

Counterexample 4: Let us recall the definition of the Grunhage/Streicher space G. The underlying set of G is $\mathbb{N} \times \mathbb{N}$, and a basis for the topology is given by sets of the form

$$[(n, m), f] := \{(n, m)\} \cup \{(k, l) \in \mathbb{N} \times \mathbb{N}| \ k > n \text{ and } l \geq f(k - n - 1)\},$$

for $(n, m) \in \mathbb{N} \times \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ is any set-theoretic function. It is easily verified that these sets form indeed the basis for a topology, which makes G into a non-sober, non-countably based qcb-space, which satisfies the T_1-axiom, see [38].

In *op.cit.*, it was shown that the elements of the sobrification of G are the neighbourhood filters $\mathcal{U}_x$, for $x \in G$, and the filter $\mathcal{U}_\infty := \mathcal{O}(G) \setminus \{\emptyset\}$. For $\lambda \in \mathbb{I}$ and $\mathcal{F} \in Sob(G)$, we denote by $\lambda \chi_{\mathcal{F}} : \mathcal{O}(G) \to \mathbb{I}$ the continuous map, assigning λ to U, if $U \in \mathcal{F}$, and 0 otherwise. Similarly, we define for $\sum_{x \in G} \lambda_x \leq 1$, $\sum_{x \in G} \lambda_x \chi_{\mathcal{U}_x}$ to be the continuous map assigning to U the value $\sum_{x \in U} \lambda_x$.

Theorem 5.5.6. *Every valuation $\nu \in \mathcal{V}_1^w(G)$ is of the form $\lambda_\infty \chi_{\mathcal{U}_\infty} + \sum_{x \in G} \lambda_x \chi_{\mathcal{U}_x}$, such that $\lambda_\infty + \sum_{x \in G} \lambda_x = 1$.*

Proof. For the valuation $\nu : \mathcal{O}(G) \to \mathbb{I}$, we define for $x \in G$, $\lambda_x = 1 - \nu(G \setminus \{x\})$, and $\lambda_\infty = 1 - \sum_{x \in G} \lambda_x$. We claim $\nu = \nu'$ for $\nu' := \lambda_\infty \chi_{\mathcal{U}_\infty} + \sum_{x \in G} \lambda_x \chi_{\mathcal{U}_x}$.

First, observe that ν' is indeed a continuous valuation. Clearly, for all finite $F \subseteq G$, $\sum_{x \in F} \lambda_x = 1 - \nu(G \setminus F)$, hence $1 - \sum_{x \in G} \lambda_x$ is a nonnegative real number. For all finite $F \subseteq G$, define $\nu_F := \lambda_\infty \chi_{\mathcal{U}_\infty} + \sum_{x \in F} \lambda_x \chi_{\mathcal{U}_x}$. Then we have that $\nu' = \bigvee^\uparrow \nu_F$, and since for all finite F, it holds that $\nu_F(\emptyset) = 0$, $\nu_F(U) \leq \nu_F(V)$ whenever $U \subseteq V$, and $\nu_F(U \cup V) + \nu_F(U \cap V) = \nu_F(U) + \nu_F(V)$, all these hold for ν', as well. Moreover, $\nu'(G) = 1$, and ν' is continuous by its definition, making it a continuous probability valuation.

Next we show by induction that $\nu(V_n) = \nu'(V_n)$ for all open sets which are of the form $V_n := \{(i, j) \in G | \ i \geq n\}$. For $V_0 = G$, this is clear. So suppose $\nu(V_n) = \lambda_\infty + \sum_{x \in V_n} \lambda_x$. For each $x \in V_n \setminus V_{n+1}$, $V_{n+1} \cup \{x\}$ is open, and we have

$$\nu(V_{n+1} \cup \{x\}) = \nu(G) + \nu(V_{n+1}) - \nu(G \setminus \{x\}) = \nu(V_{n+1}) + \lambda_x.$$

But then by continuity of ν,

$$\nu(V_{n+1}) + \sum_{x \in V_n \setminus V_{n+1}} \lambda_x = \bigvee_{F \subseteq_{fin} V_n \setminus V_{n+1}} \left(\nu(V_{n+1}) + \sum_{x \in F} \lambda_x \right) = \bigvee_{F \subseteq_{fin} V_n \setminus V_{n+1}} \nu(V_{n+1} \cup F) = \nu(V_n),$$

hence

$$\nu(V_{n+1}) = \nu(V_n) - \sum_{x \in V_n \setminus V_{n+1}} \lambda_x = \lambda_\infty + \sum_{x \in V_{n+1}} \lambda_x,$$

by induction hypothesis.

We show the general case $\nu(U) = \nu'(U)$, similarly. Let $k = \min\{i \in \mathbb{N} | \ (i, j) \in U\}$, then for all $x \in V_k$, $U \cup \{x\}$ is open. Hence we can show as above,

$$\nu(U) = \nu(V_k) - \sum_{x \in V_n \setminus U} \lambda_x = \lambda_\infty + \sum_{x \in U} \lambda_x,$$

showing $\nu = \nu' = \lambda_\infty \chi_{\mathcal{U}_\infty} + \sum_{x \in G} \lambda_x \chi_{\mathcal{U}_x}$. $\qquad\square$

Lemma 5.5.7. *The set*

$$W := \{\nu \in \mathcal{V}_1(G) | \ \nu = \lambda_\infty \chi_{\mathcal{U}_\infty} + \sum_{x \in G} \lambda_x \chi_{\mathcal{U}_x} \text{ and } \lambda_\infty > 0\}$$

is not open in $\mathcal{V}_1^w(G)$.

Proof. By definition, the topology on $\mathcal{V}_1^w(G)$ has a subbasis given by sets of the form $\langle U, r \rangle := \{\nu \in \mathcal{V}_1(G) | \ \nu(U) > r\}$. We show that W does not contain any finite inter-

section of such subbasic open sets. Let $\bigcap_{i\in F}\langle U_i, r_i\rangle$ be such a finite intersection such that all the U_i are nonempty. Then, by the structure of G, $\bigcap_{i\in F} U_i$ is nonempty. So let $x \in \bigcap_{i\in F} U_i$, and $\delta_x : \mathcal{O}(G) \to \mathbb{I}$ be the Dirac measure for x, i.e. assigning 1 to U if U contains x, and 0 otherwise. Then δ_x is a continuous valuation and $\delta_x \in \bigcap_{i\in F}\langle U_i, r_i\rangle$. However, $\delta_x = \chi_{\mathcal{U}_x}$, and thus $\lambda_\infty = 0$ for δ_x. It follows that $\bigcap_{i\in F}\langle U_i, r_i\rangle$ is not a subset of W. $\qquad\square$

Lemma 5.5.8. *Let $(\nu_i)_{i\in\mathbb{N}}$ is a sequence of valuations over G converging to ν_∞ in $\mathcal{V}_1^w(G)$, such that $\nu_i = \lambda_\infty^i \chi_{\mathcal{U}_\infty} + \sum_{x\in G} \lambda_x^i \chi_{\mathcal{U}_x}$ and $\nu_\infty = \lambda_\infty^\infty \chi_{\mathcal{U}_\infty} + \sum_{x\in G} \lambda_x^\infty \chi_{\mathcal{U}_x}$. Then for all finite $F \subseteq G$, $\{1 - \sum_{x\in F} \lambda_x^i\}_{i\in\mathbb{N}}$ converges to $1 - \sum_{x\in F} \lambda_x^\infty$ in the Scott topology. Equivalently, for all finite $F \subseteq G$ and $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $i \geq N$, $\sum_{x\in F} \lambda_x^i \leq \sum_{x\in F} \lambda_x^\infty + \epsilon$.*

Proof. As $\nu_i \to \nu_\infty$, we have that for all finite $F \subseteq G$, $\nu_i(G \setminus F) \to \nu_\infty(G \setminus F)$ in the Scott topology of $\mathbb{I}$. Thus $\{1 - \sum_{x\in F} \lambda_x^i\}_{i\in\mathbb{N}}$ converges to $1 - \sum_{x\in F} \lambda_x^\infty$, as claimed. $\qquad\square$

Lemma 5.5.9. *For every $r \in \mathbb{I}^\uparrow$, the set*

$$B := \{\nu \in \mathcal{V}_1(G)| \nu = \lambda_\infty \chi_{\mathcal{U}_\infty} + \sum_{x\in G} \lambda_x \chi_{\mathcal{U}_x} \text{and } \lambda_\infty \leq r\}$$

is sequentially closed in $\mathcal{V}_1^w(G)$.

Proof. We show that if $(\nu_i)_{i\in\mathbb{N}}$ is a sequence in B converging to some ν_∞ in $\mathcal{V}_1^w(G)$, then $\nu_\infty \in B$. Suppose that $\nu_i = \lambda_\infty^i \chi_{\mathcal{U}_\infty} + \sum_{x\in G} \lambda_x^i \chi_{\mathcal{U}_x}$ with $\lambda_\infty^i \leq r$. Then for $\nu_\infty = \lambda_\infty^\infty \chi_{\mathcal{U}_\infty} + \sum_{x\in G} \lambda_x^\infty \chi_{\mathcal{U}_x}$, we have to show that $\lambda_\infty^\infty \leq r$. Clearly, this holds for $r = 1$, so we only have to show it for $r < 1$.

Assume for a contradiction $\lambda_\infty^\infty = r + \rho$ for some $\rho > 0$. Let now $\frac{\rho}{3} > \epsilon > 0$, and choose $F \subseteq G$ finite such that $\sum_{x\in G\setminus F} \lambda_x^\infty \leq \epsilon$. Define $V_n := \{(k,l) \in G \setminus F| k \geq n\}$. As $(\nu_i)_{i\in\mathbb{N}}$ converges to ν_∞, we can, without loss of generality, assume that for all $i \in \mathbb{N}$, $\nu_i(V_i) \geq \nu_\infty(V_i) - \epsilon > r + \frac{2\rho}{3}$; otherwise we construct an appropriate subsequence of $(\nu_i)_{i\in\mathbb{N}}$ inductively. Then for each $i \in \mathbb{N}$, we find a finite $F_i \subseteq V_i$, for which it holds that $\sum_{x\in F_i} \lambda_x^i > \frac{\rho}{2}$. Let $K(F_i) := min\{k| (k,l) \in F_i\}$, then $\{K(F_i)\}_{i\in\mathbb{N}}$ is an unbounded sequence, and, again without loss of generality, we can assume it is strictly increasing. For each $i \in \mathbb{N}$, let now $L(F_i) := max\{l| (k,l) \in F_h \text{ for some } h \leq i\}$. Furthermore we define $A_0 := \{(k,l) \in G| k \leq K(F_1) \text{ and } l \leq L(F_1)\}$, and inductively

$$A_n := \{(k,l) \in G| K(F_n) < k \leq K(F_{n+1}) \text{ and } l \leq L(F_{n+1})\}.$$

Then $\bigcup_{i\in\mathbb{N}} F_i \subseteq \bigcup_{n\in\mathbb{N}} A_n$, and $\bigcup_{n\in\mathbb{N}} A_n$ is closed in G; the reason is that for each fixed $k_0 \in \mathbb{N}$, $\{l| (k_0,l) \in \bigcup_{n\in\mathbb{N}} A_n)\}$ is bounded. Thus, the set $V := (G \setminus \bigcup_{n\in\mathbb{N}} A_n) \cup F$ is open and disjoint with all the F_i, as F is disjoint with all the F_i. Now we have

$\nu_i(V) < 1 - \frac{\rho}{2}$, as $\sum_{x \in F_i} \lambda_x^i > \frac{\rho}{2}$, but we also have

$$\nu_\infty(V) = \lambda_\infty^\infty + \sum_{x \in V} \lambda_x^\infty \geq \lambda_\infty^\infty + \sum_{x \in F} \lambda_x^\infty = 1 - \sum_{x \in G \setminus F} \lambda_x^\infty > 1 - \frac{\rho}{3}.$$

This clearly contradicts that $(\nu_i)_{i \in \mathbb{N}}$ converges to ν_∞, proving the claim. $\square$

Recall however that by Lemma 5.5.7, the set W is not open in $\mathcal{V}_1^w(G)$, hence $\mathcal{V}_1^w(G)$ is not sequential.

Theorem 5.5.10. QCB *is not closed under the probabilistic powerspace construction.*

As a last question one may ask whether there exists a better probabilistic powerdomain construction for Topological Domain Theory, in the sense that it coincides with the classical powerspace construction on all countably-based spaces. And indeed the answer for this question is positive. There exists an observational approach to model computational effects proposed by Simpson [133], and one can apply this approach to obtain an observationally-induced probabilistic powerdomain in Topological Domain Theory which for countably-based spaces yields the classical probabilistic powerspace. For more details on this, we refer the reader to [11].

6 A synthetic account of Topological Domain Theory

In the previous chapters we have investigated Topological Domain Theory from a purely classical mathematical point of view. We have proved that it supports many useful constructions, and arguably surpasses Classical Domain Theory as a framework for denotational semantics. In fact, we have shown that it can be viewed as a genuine extension of Classical Domain Theory, in which topology, instead of a partial order, is the principal entity.

However, this classical viewpoint of Topological Domain Theory makes it difficult to obtain a model for polymorphic type theories, because **QCB** and its subcategories are only countably complete. A more subtle restriction is that our construction of free algebras from simple classical principles, in Chapters 4 and 5, only allows operations whose arity is a countable set, and not an object of **QCB**, respectively **TP**, itself. The reason is again that **QCB** is not complete, and hence an adjoint functor theorem is not available. Furthermore, for more general algebraic theories we cannot obtain Theorem 4.4.3, using our existing methods, as the free algebra construction in **Seq** for such nonstandard arities cannot easily be compared with the set-theoretic construction, as in Proposition 4.2.2.

It is well-known that Synthetic Domain Theory [48, 153, 117, 85, 116, 86, 118, 101] is able to provide categories of domains which have strong enough closure properties to interpret polymorphic type theories, and in which an adjoint functor theorem is applicable. The trick of Synthetic Domain Theory is to change from the classical to an intuitionistic (or constructive) setting. More specifically, one identifies domains as special sets in a realizability topos over a partial combinatory algebra. Then one uses the internal logic of the topos to show that the full subcategory formed by these special sets supports constructions known from Classical Domain Theory, is internally complete, and weakly equivalent to an internal (and thus small) category of the topos.

From Theorem 2.1.11 we can conclude that **QCB** and its subcategories can be embedded into the realizability topos $\mathbf{RT}(\mathcal{P}\omega)$ over Scott's graph model $\mathcal{P}\omega$ [136], because $\omega\mathbf{Equ}$ has been shown to be equivalent to the category $\mathbf{Asm}(\mathcal{P}\omega)$ of assemblies over $\mathcal{P}\omega$, see [14, 90]. We show that Topological Domain Theory is equivalent to a specific model of Synthetic Domain Theory, and investigate this model from a synthetic view-

point. Our aim is to find subcategories of $\mathbf{RT}(\mathcal{P}\omega)$ which are equivalent to $\mathbf{QCB}_0$ and $\mathbf{TP}$, respectively, and to show that these subcategories have sufficient internal closure properties to form a model for the polymorphic λ-calculus [119], and to support an internal adjoint functor theorem.

The characterisation of the respective subcategories of $\mathbf{RT}(\mathcal{P}\omega)$ is done in two steps in the first two sections of this chapter. First, we give a direct external characterisation of the images of $\mathbf{QCB}_0$ and $\mathbf{TP}$ under the equivalence $\omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$. This yields that $\mathbf{QCB}_0$ is equivalent to the category $\mathbf{Ext}(\mathcal{P}\omega)$ of extensional assemblies [34, 117, 124], and $\mathbf{TP}$ is equivalent to $\mathbf{CE}(\mathcal{P}\omega)$, the category of complete extensional assemblies [117]. Then, in the second section, we internalise the categories $\mathbf{Ext}(\mathcal{P}\omega)$ and $\mathbf{CE}(\mathcal{P}\omega)$ using the internal logic of $\mathbf{RT}(\mathcal{P}\omega)$. This yields the categories $\boldsymbol{\Sigma}\mathbf{Pos}$ of Σ-posets and $\boldsymbol{\Sigma}\mathbf{Cpo}$ of Σ-cpos. In the subsequent section, we investigate these categories for their closure conditions, showing that they are internally complete and weakly small. We then use these closure properties to show how the polymorphic λ-calculus can be interpreted in $\mathbf{QCB}$. In the final section of the chapter, we apply an internal adjoint functor theorem to $\boldsymbol{\Sigma}\mathbf{Pos}$ and $\boldsymbol{\Sigma}\mathbf{Cpo}$ to show that these categories have free algebras for a very general class of algebraic theories, thus generalising the existence results of Chapters 4 and 5.

This main contribution of this chapter is to give a complete account of the synthetic properties of Topological Domain Theory, in particular, showing that Topological Domain Theory is able to model the polymorphic λ-calculus, and that we have free algebras for a more general class of algebraic theories. Within this chapter we recall many proofs from the Synthetic Domain Theory literature [160, 44, 54, 118, 18, 19], and apply them to our model. The synthetic characterisations of the categories of Topological Domain Theory were stated by Simpson [138], but here we give for the first time a detailed presentation of the required translations into the internal language of the realizability topos.

6.1 Characterisation of Topological Domain Theory in $\mathbf{Asm}(\mathcal{P}\omega)$

We start our synthetic investigation of Topological Domain Theory by giving an external characterisation of $\mathbf{QCB}_0$ and $\mathbf{TP}$ in the realizability model over Scott's graph model $\mathcal{P}\omega$ [136]. More specifically, we show that under the equivalence of $\omega\mathbf{Equ}$ and $\mathbf{Asm}(\mathcal{P}\omega)$, the category $\mathbf{QCB}_0$ becomes equivalent to the full reflective subcategory of extensional objects [34, 124] in $\mathbf{Asm}(\mathcal{P}\omega)$, and $\mathbf{TP}$ to the full reflective subcategory of complete extensional objects [34, 124]. These results were claimed in a paper by Simpson [138], and were shown in the author's Diplomarbeit [6]. We recall the proofs in order to give a complete picture of the synthetic characterisations of Topological

Domain Theory.

Let us first recall the basic property that makes $\mathcal{P}\omega$, the power set of the natural numbers, into a model for the untyped λ-calculus, and thus a total combinatory algebra. First, observe that $\mathcal{P}\omega$ is an ω-algebraic lattice with respect to the inclusion order. If we talk about $\mathcal{P}\omega$ as a topological space, we refer to this lattice equipped with the Scott-topology. Scott [135] has shown that every T_0-space can be embedded into a continuous lattice, and that the continuous lattices are the injective objects in the category of T_0-topological spaces. It follows that continuous maps between T_0-spaces lift to continuous maps between the lattices into which they are embedded. As has been pointed out by Escardó (see section 6 of [90]), this can be generalised beyond the T_0-spaces, if one uses the following definition of a topological pre-embedding, see for instance [90].

Definition 6.1.1. A continuous map $f : X \to Y$ between topological spaces is a *topological pre-embedding* if for every open subset $U \subseteq X$, there exists an open subset $V \subseteq Y$ with $U = f^{-1}(V)$.

Then $\mathcal{P}\omega$ is injective in ω**Top** with respect to topological pre-embeddings, as the following Theorem expresses.

Theorem 6.1.2 (Scott, Escardó). *Let X and Y be topological spaces. Then:*

(i) *There exists a topological pre-embedding $X \to \mathcal{P}\omega$ if and only if X is a countably-based space.*

(ii) *If $i_X : X \to \mathcal{P}\omega$ and $i_Y : Y \to \mathcal{P}\omega$ are topological pre-embeddings, then every continuous map $f : X \to Y$ can be lifted to a (not necessarily unique) continuous endofunction $\widehat{f} : \mathcal{P}\omega \to \mathcal{P}\omega$ such that the following diagram commutes.*

$$
\begin{array}{ccc}
\mathcal{P}\omega & \xrightarrow{\ \widehat{f}\ } & \mathcal{P}\omega \\
\ \ \uparrow{\scriptstyle i_X} & & \ \ \uparrow{\scriptstyle i_Y} \\
X & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

Proof. The claims are consequences of Scott's original proofs of Theorems 1.5 and 1.6 in [136], and the fact that every topological space can be pre-embedded into its T_0-reflection. $\qquad\square$

This property of $\mathcal{P}\omega$ can now be used to show that it is universal amongst ω-continuous lattices in the following sense.

Theorem 6.1.3 (Scott). *The (Scott-)continuous retracts of $\mathcal{P}\omega$ are exactly the ω-continuous lattices. In particular, $\mathcal{P}\omega$ is reflexive, i.e. the lattice of continuous endofunctions $[\mathcal{P}\omega \to \mathcal{P}\omega]$ is a Scott-continuous retract of $\mathcal{P}\omega$.*

Proof. This follows from Theorem 6.1.2 above, Theorem 2.12 in [135], and the remarks after Proposition III-4.12 in [36]. □

It follows that $\mathcal{P}\omega$ is a model for the untyped λ-calculus, see [136] for details. Thus, it is a total combinatory algebra, in the sense of [4], and we can identify an element $a \in \mathcal{P}\omega$ with a continuous endomorphism via the retraction $\Phi : \mathcal{P}\omega \to [\mathcal{P}\omega \to \mathcal{P}\omega]$. We write $a \cdot b$ for $(\Phi(a))(b)$.

Having seen that $\mathcal{P}\omega$ serves as an untyped computational model, a standard way of adding types to such a model is as follows. A type is simply a set, the elements of which are realized by elements of $\mathcal{P}\omega$. A morphism between types is a function between the corresponding sets, which is tracked by a program which acts accordingly on the realizers. This leads to the well-studied category of assemblies over $\mathcal{P}\omega$.

Definition 6.1.4. An *assembly* over $\mathcal{P}\omega$ is a tuple $X = (|X|, \|\cdot\|_X)$ such that $|X|$ is a set and $\|\cdot\|_X : |X| \to \mathcal{P}(\mathcal{P}\omega) \setminus \{\emptyset\}$ a map, assigning to each $x \in |X|$ a nonempty set of *realizers* $\|x\|_X \subseteq \mathcal{P}\omega$. A morphism between assemblies X and Y is a set-theoretic map $f : |X| \to |Y|$ for which there exists some $a \in \mathcal{P}\omega$, which *tracks* (or *realizes*) f, in the sense that for all $x \in |X|$ and $b \in \mathcal{P}\omega$, it holds that:

$$b \in \|x\|_X \Rightarrow a \cdot b \in \|f(x)\|_Y.$$

The category of assemblies over $\mathcal{P}\omega$ and morphisms between them is denoted by $\mathbf{Asm}(\mathcal{P}\omega)$.

We remark that assemblies give a notion of abstract types over $\mathcal{P}\omega$. They are too crude to serve as datatypes. For instance, **Set** appears as a full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, if one chooses the realizers accordingly, see Section 1.2.3 of [85]. However, $\mathbf{Asm}(\mathcal{P}\omega)$ provides a rich class of abstract types which turns out to be useful when it comes, for instance, to modelling polymorphic types, see [54, 122]. It is well-known that $\mathbf{Asm}(\mathcal{P}\omega)$ carries a very rich structure. In particular, it is a quasi-topos, see Theorem 5.1 of [47] together with Theorem 6.2.3 below.

A better notion of datatype over $\mathcal{P}\omega$ is given by the following definition of modest sets in $\mathbf{Asm}(\mathcal{P}\omega)$.

Definition 6.1.5. A *modest set* over $\mathcal{P}\omega$ is an assembly $X = (|X|, \|\cdot\|_X)$ such that the elements of $|X|$ do not share realizers, i.e. for $x, y \in |X|$,

$$\|x\|_X \cap \|y\|_X \neq \emptyset \Rightarrow x = y.$$

By $\mathbf{Mod}(\mathcal{P}\omega)$ we denote the full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ consisting of the modest sets.

In contrast to the assemblies, in a modest set each realizer carries the full information about which element it represents.

We can now establish the basis of the connection between Topological Domain Theory and the realizability model over $\mathcal{P}\omega$.

Theorem 6.1.6. *The categories* $\omega\mathbf{Equ}$ *and* $\mathbf{Asm}(\mathcal{P}\omega)$ *are equivalent.*

Proof. The proof is essentially due to Bauer, Birkedal and Scott, see Theorem 4.3 of [14]. However, as already mentioned in Chapter 2.1, their equilogical spaces are subject to the representing spaces satisfying the T_0-axiom and thus their category becomes equivalent to $\mathbf{Mod}(\mathcal{P}\omega)$. For our case see Proposition 8.1 of [90]. The countably-based version of that proof is easily obtained with Theorems 6.1.2 and 6.1.3, see e.g. Theorem 1.3.4 of [6]. For the subsequent development, we introduce the equivalence functors $I : \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$ and $J : \mathbf{Asm}(\mathcal{P}\omega) \to \omega\mathbf{Equ}$:

- Let $(X, \sim)$ be an ω-equilogical space, and $i_X : X \to \mathcal{P}\omega$ a topological pre-embedding. Then $I(X) = (X/\sim, \|\cdot\|)$ with $\|[x]_\sim\| = \{i_X(y)|\ y \sim x\}$. The existence of a realizer for an ω-equilogical morphism $f : (X, \sim_X) \to (Y, \sim_Y)$ is guaranteed by Theorem 6.1.2. (Notice that this functor makes use of the Axiom of Choice.)

- Let $X = (|X|, \|\cdot\|_X)$ be an assembly, and A be the topological space with underlying set $\{(x, a) \in |X| \times \mathcal{P}\omega|\ a \in \|x\|_X\}$ and the topology being the coarsest topology making the projection $\pi : A \to \mathcal{P}\omega$ continuous. Then $J(X)$ is given by $(A, \sim)$ with $(x, a) \sim (y, b)$ if and only if $x = y$. The definition of morphisms in the respective categories ensures that J becomes functorial. $\qquad\square$

It follows, with Theorems 2.1.11 and 3.2.5, that there are full reflective exponential ideals of $\mathbf{Asm}(\mathcal{P}\omega)$, which are equivalent to $\mathbf{QCB}_0$ and $\mathbf{TP}$, respectively. In order to obtain a characterisation of these subcategories of $\mathbf{Asm}(\mathcal{P}\omega)$, we next identify some important assemblies. In doing this, we always have the connections to topological and equilogical spaces in mind. In fact all the objects we identify below are induced by countably-based topological spaces, i.e. they are in the image of the composite functor $\omega\mathbf{Top} \hookrightarrow \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$.

- The terminal object of $\mathbf{Asm}(\mathcal{P}\omega)$ is given by the assembly $\mathbf{1} = (\{*\}, \|\cdot\|_{\mathbf{1}})$ with $\|*\|_{\mathbf{1}} = \mathcal{P}\omega$.

- The assembly $\nabla\mathbf{2} = (\{0, 1\}, \|\cdot\|_{\nabla\mathbf{2}})$ has $\|0\|_{\nabla\mathbf{2}} = \|1\|_{\nabla\mathbf{2}} = \mathcal{P}\omega$. It is easily shown, see Theorem 5.1 (c) of [47], that $1 : \mathbf{1} \to \nabla\mathbf{2}$ is a regular subobject classifier

in $\mathbf{Asm}(\mathcal{P}\omega)$, in the sense that $m : X \to Y$ is a regular mono in $\mathbf{Asm}(\mathcal{P}\omega)$, if and only if there exists a map $f : Y \to \nabla 2$ such that the following diagram is a pullback in $\mathbf{Asm}(\mathcal{P}\omega)$:

$$
\begin{array}{ccc}
X & \longrightarrow & \mathbf{1} \\
\Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle 1} \\
Y & \xrightarrow{\ f\ } & \nabla\mathbf{2}
\end{array}
$$

The assembly $\nabla\mathbf{2}$ is isomorphic to the image of the indiscrete 2-element space in $\mathbf{Asm}(\mathcal{P}\omega)$ under the inclusion $\omega\mathbf{Top} \hookrightarrow \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$.

- The assembly $\mathcal{N} = (\mathbb{N}, \|\cdot\|_{\mathcal{N}})$, with $\|n\|_{\mathcal{N}} = \{\{n\}\}$, is a natural numbers object in $\mathbf{Asm}(\mathcal{P}\omega)$, see Proposition 1.2.8 in [85] (the object given there has different realizers, but is easily seen to be isomorphic to ours). It is isomorphic to the image of the space of natural numbers with the usual discrete topology under the inclusion $\omega\mathbf{Top} \hookrightarrow \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$.

- The assembly $\Sigma = (\{\bot, \top\}, \|\cdot\|_{\Sigma})$, with $\|\bot\|_{\Sigma} = \{\emptyset\}$ and $\|\top\|_{\Sigma} = \{\{0\}\}$, is isomorphic to the image of Sierpinski space in $\mathbf{Asm}(\mathcal{P}\omega)$ under the inclusion $\omega\mathbf{Top} \hookrightarrow \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$. This assembly is semantically of paramount interest. It can be shown to be a dominance in the sense of [123], and thus induces a notion of partiality for $\mathbf{Asm}(\mathcal{P}\omega)$, and a lifting functor $L : \mathbf{Asm}(\mathcal{P}\omega) \to \mathbf{Asm}(\mathcal{P}\omega)$. For details, we refer to Chapter 4 of [85], in which this construction has been given a nice external description. For the purposes of this section, it is helpful to think of Σ as an object in $\mathbf{Asm}(\mathcal{P}\omega)$, characterising observable properties.

- Let ω^+ be the assembly $(\mathbb{N} \cup \{\infty\}, \|\cdot\|_{\omega^+})$ with

$$\|n\|_{\omega^+} = \,\downarrow n = \{k \in \mathbb{N} \mid k < n\}.$$

Observe that ω^+ is a subobject of $\Sigma^{\mathcal{N}}$ in $\mathbf{Asm}(\mathcal{P}\omega)$, corresponding to a generic ascending chain with least upper bound. Thus, it is isomorphic to the image of $\mathcal{M}(\mathbb{N}^{\uparrow})$ under the inclusion $\omega\mathbf{Top} \hookrightarrow \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$.

- The assembly $\omega = (\mathbb{N}, \|\cdot\|_{\omega})$, with $\|n\|_{\omega} = \|n\|_{\omega^+}$, is the regular subobject of ω^+ in $\mathbf{Asm}(\mathcal{P}\omega)$, serving as a generic ascending chain without least upper bound. It is isomorphic to the image of $\mathbb{N}^{\uparrow}$ under the inclusion $\omega\mathbf{Top} \hookrightarrow \omega\mathbf{Equ} \to \mathbf{Asm}(\mathcal{P}\omega)$.

As mentioned, Σ is used to give a notion of observable properties in $\mathbf{Asm}(\mathcal{P}\omega)$; an observable property of an assembly X is given by morphism $X \to \Sigma$, hence an element of Σ^X. One can formulate an observable (or extensional) equality on X, by saying x

and y are *extensionally equal* if for all $f \in |\Sigma^X|$, it holds that $f(x) = f(y)$. In this vein, the intrinsic equality on an assembly X is extensional if for all extensionally equal $x, y \in |X|$, it holds that $x = y$ as elements of $|X|$. In categorical terms this means that the map $\eta_X : X \to \Sigma^{\Sigma^X}$, which is given by the exponential transpose of the evaluation map $\Sigma^X \times X \to \Sigma$, and tracked by $\lambda xy.yx$, is a mono.

On the other hand, we can view the realizers of elements of Σ^{Σ^X} as programs on observable properties of X. It seems natural to expect the existence of a uniform method which, given a program realizing some $\eta_X(x) \in \Sigma^{\Sigma^X}$, produces a realizer of x, i.e. elements of X are realized in a uniform way with respect to the observable properties induced by the computational model $\mathcal{P}\omega$. In case of $\eta_X : X \to \Sigma^{\Sigma^X}$ being a mono, this amounts to demanding this mono to be regular in $\mathbf{Asm}(\mathcal{P}\omega)$.

This discussion motivates the following definition, originating in [48, 34, 124].

Definition 6.1.7. An assembly X is a *(Σ-)extensional object* if the map $\eta_X : X \to \Sigma^{\Sigma^X}$ is a regular mono in $\mathbf{Asm}(\mathcal{P}\omega)$. The full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ of extensional objects is denoted by $\mathbf{Ext}(\mathcal{P}\omega)$.

Notice that an extensional object is necessarily a modest set, because Σ is modest and $\mathbf{Mod}(\mathcal{P}\omega)$ is an exponential ideal and closed under subobjects in $\mathbf{Asm}(\mathcal{P}\omega)$, see Theorem 3.5 in [86].

Corollary 6.1.8. *The category* $\mathbf{Ext}(\mathcal{P}\omega)$ *is a full subcategory of* $\mathbf{Mod}(\mathcal{P}\omega)$.

One can argue that $\mathbf{Ext}(\mathcal{P}\omega)$ provides an even better notion of datatype over $\mathcal{P}\omega$ than $\mathbf{Mod}(\mathcal{P}\omega)$, since for an extensional object $(|X|, \|\cdot\|_X)$, elements can be distinguished by observable properties, as outlined above. An example for a modest set which is not extensional is $X = (\{0, 1\}, \|\cdot\|_X)$ with $\|0\|_X = \{\{0\}, \{0, 1, 2\}\}$ and $\|1\|_X = \{\{0, 1\}, \{0, 1, 2, 3\}\}$. Then by continuity, any morphism $f : X \to \Sigma$ gives $f(0) = f(1)$, and in this sense we cannot distinguish the elements of X by procedures in the computational model $\mathcal{P}\omega$.

The following Theorem establishes the main connection between Topological Domain Theory and realizability semantics. It was claimed by Simpson in [138] and proved as Theorem 4.1.1 in the author's Diplomthesis [6].

Theorem 6.1.9 (Simpson)**.** *The categories* $\mathbf{QCB}_0$ *and* $\mathbf{Ext}(\mathcal{P}\omega)$ *are equivalent.*

Proof. For the proof, we use the following facts:

(I) By Theorem 6.1.6, $\mathbf{Asm}(\mathcal{P}\omega)$ is equivalent to $\omega\mathbf{Equ}$ with equivalence functors $I : \omega\mathbf{Equ} \leftrightarrow \mathbf{Asm}(\mathcal{P}\omega) : J$.

(II) From Theorems 2.1.8, 2.1.11 and Corollary 2.1.12, one can conclude that $\mathbf{PQ}_0$, the full subcategory of $\mathbf{PQ}$ which consists of those objects $(X, \sim_X)$, for which the

topological quotient $X/\sim_X$ is a T_0-space, is a full reflective exponential ideal of $\omega\mathbf{Equ}$ and closed under the formation of regular subobjects in $\omega\mathbf{Equ}$.

(III) By Theorems 2.1.9 and 2.1.11, $\mathbf{QCB}$ is equivalent to $\mathbf{PQ}$, and $\mathbf{QCB}_0$ is equivalent to $\mathbf{PQ}_0$. By $E : \mathbf{QCB} \to \mathbf{PQ}$ we denote the equivalence functor, which cuts down to the equivalence $\mathbf{QCB}_0 \to \mathbf{PQ}_0$.

(IV) By Lemma 3.2.3, and Theorem 2.3.6 and Proposition 2.3.7 above, it holds that for any qcb$_0$-space X, the map $\eta_X : X \to \mathbb{S}^{\mathbb{S}^X}$ is a regular mono in $\mathbf{QCB}$.

(V) Let $\mathcal{S}$ be the equilogical space $(\mathbb{S}, =)$. Then $\mathcal{S}$ is an object of $\mathbf{PQ}_0$, and it holds that $J(\Sigma) \cong \mathcal{S} \cong E(\mathbb{S})$.

By (I) and (V) it suffices to show that the class of ω-equilogical spaces $(X, \sim_X)$ for which the canonical map $\eta_{(X,\sim_X)} : (X, \sim_X) \to \mathcal{S}^{\mathcal{S}^{(X,\sim_X)}}$ is a regular mono in $\omega\mathbf{Equ}$ is precisely $\mathbf{PQ}_0$.

By (II) and (V), it holds that for every ω-equilogical space $(X, \sim_X)$, the object $\mathcal{S}^{\mathcal{S}^{(X,\sim_X)}}$ is in $\mathbf{PQ}_0$, and again by (II) that any regular $\omega\mathbf{Equ}$-subobject of $\mathcal{S}^{\mathcal{S}^{(X,\sim_X)}}$ is in $\mathbf{PQ}_0$, establishing one direction of the proof.

Conversely, observe that by (III) and (IV), for every object of $\mathbf{PQ}_0$, the morphism $\eta_{(X,\sim_X)} : (X, \sim_X) \to \mathcal{S}^{\mathcal{S}^{(X,\sim_X)}}$ is a regular mono in $\mathbf{PQ}$. But as $\mathbf{PQ}$ is a reflective subcategory of $\omega\mathbf{Equ}$, the inclusion functor $\mathbf{PQ} \hookrightarrow \omega\mathbf{Equ}$ preserves regular monos, hence $\eta_{(X,\sim_X)}$ is indeed a regular mono in $\omega\mathbf{Equ}$, as required. $\square$

The identification of domain-like datatypes over $\mathcal{P}\omega$ is again split into two steps. The first step is already done, by using the fact that Σ is a dominance, and thus induces a lifting functor $L : \mathbf{Asm}(\mathcal{P}\omega) \to \mathbf{Asm}(\mathcal{P}\omega)$, which in turn is used to obtain a notion of pointed objects. More specifically, a pointed object is an algebra for the lifting functor induced by Σ. For the details, we again refer the reader to Chapter 4 of [85].

The second step is to identify objects in which, in a suitable sense, ascending chains have least upper bounds. For this, we observe that a notion of ascending chain in an assembly X is given by a morphism $\omega \to X$, and a notion for a least upper bound of an ascending chain by a morphism $\omega^+ \to X$. Thus we obtain the assemblies of ascending chains X^ω, respectively ascending chains with least upper bounds X^{ω^+}, over X. There is an obvious embedding $\iota : \omega \to \omega^+$, tracked by the identity $\lambda x.x$, which induces a morphism $X^\iota : X^{\omega^+} \to X^\omega$. This leads to the following definition of completeness, due to Longley and Simpson [85, 86].

Definition 6.1.10. An assembly X is called *complete* if the map $X^\iota : X^{\omega^+} \to X^\omega$ is an isomorphism in $\mathbf{Asm}(\mathcal{P}\omega)$.

We call an assembly a *complete extensional object* if it is complete and extensional. The full subcategory of complete extensional objects in $\mathbf{Asm}(\mathcal{P}\omega)$ is denoted by $\mathbf{CE}(\mathcal{P}\omega)$.

The next technical lemma will show that under the equivalence $\mathbf{QCB}_0 \simeq \mathbf{Ext}(\mathcal{P}\omega)$, the category of topological predomains is equivalent to the category of complete extensional objects. It has been claimed by Simpson [138], and proved in a restricted version as Theorem 4.2.5 in the author's Diplomthesis [6].

Lemma 6.1.11 (Simpson). *Let* $i : \mathbb{N}^{\uparrow} \to \mathcal{M}(\mathbb{N}^{\uparrow})$ *be the embedding of* $\mathbb{N}^{\uparrow}$ *into its monotone convergence reflection (i.e.* $\mathcal{M}(\mathbb{N}^{\uparrow}) = \mathbb{N} \cup \infty$ *equipped with the Scott-topology with respect to the usual order).* *Then for a topological space* X, *the following are equivalent:*

(i) X *is an* ω-*monotone convergence space,*

(ii) *Every continuous* $f : \mathbb{N}^{\uparrow} \to X$ *extends uniquely along* i, *as in:*

$$
\begin{array}{ccc}
\mathcal{M}(\mathbb{N}^{\uparrow}) & \stackrel{\widehat{f}}{\dashrightarrow} & X \\
i \uparrow & \nearrow{\scriptstyle f} & \\
\mathbb{N}^{\uparrow} & &
\end{array}
$$

(iii) *For every topological space* Z, *every continuous* $f : Z \times \mathbb{N}^{\uparrow} \to X$ *extends uniquely along* $Z \times i$, *as in:*

$$
\begin{array}{ccc}
Z \times \mathcal{M}(\mathbb{N}^{\uparrow}) & \stackrel{\widehat{f}}{\dashrightarrow} & X \\
Z \times i \uparrow & \nearrow{\scriptstyle f} & \\
Z \times \mathbb{N}^{\uparrow} & &
\end{array}
$$

Proof. It is clear that $(iii) \Rightarrow (ii)$ holds. So we show $(ii) \Rightarrow (i)$ and $(i) \Rightarrow (iii)$.

For $(ii) \Rightarrow (i)$, suppose X satisfies the unique extension property. First, observe that this implies X to be a T_0-space, since if this is not the case, say x and y have the same open neighbourhoods, then the constant map $x : \mathbb{N}^{\uparrow} \to X$ has two extensions along i, contradicting the assumption. We show that X is an ω-monotone convergence space. Assume $(x_n)_{n \in \mathbb{N}}$ is an ascending chain in the specialization order of X. Then the map $f : \mathbb{N}^{\uparrow} \to X$, defined as $f(n) = x_n$ is continuous, and by assumption extends uniquely to a map $\widehat{f} : \mathcal{M}(\mathbb{N}^{\uparrow}) \to X$. Hence there exists an element $x_{\infty} := \widehat{f}(\infty) \in X$. We show that in the specialization order of X, it holds that $x_{\infty} = \bigvee^{\uparrow}_{n \in \mathbb{N}} x_n$. Clearly, it holds for all $n \in \mathbb{N}$ that $x_n \sqsubseteq x_{\infty}$, since $\widehat{f}$ is continuous. Now suppose $x_n \sqsubseteq y$ for all $n \in \mathbb{N}$, and $x_{\infty} \in U$ for some open $U \subseteq X$. Then, as $\widehat{f}^{-1}(U) \subseteq \mathcal{M}(\mathbb{N}^{\uparrow})$ is open, there exists $n_0 \in \mathbb{N}$ with $n_0 \in \widehat{f}^{-1}(U)$, and so $x_{n_0} \in U$. But then it follows that $y \in U$, and so $x_{\infty} \sqsubseteq y$, showing that x_{∞} is indeed the least upper bound of $(x_n)_{n \in \mathbb{N}}$. It also follows that every open $U \subseteq X$ is ω-Scott-open, and thus X is an ω-monotone convergence space.

For $(i) \Rightarrow (iii)$, let X be an ω-monotone convergence space. We have to show that every continuous map $f : Z \times \mathbb{N}^{\uparrow} \to X$ extends uniquely along $Z \times i$. For this, observe that $Z \times i$ is a topological subspace embedding. Define the map $\widehat{f} : Z \times \mathcal{M}(\mathbb{N}^{\uparrow}) \to X$, as $\widehat{f}(z, n) = f(z, n)$, for $n \in \mathbb{N}$, and $\widehat{f}(z, \infty) = \bigvee^{\uparrow}_{n \in \mathbb{N}} f(z, n)$. Obviously $\widehat{f}$ is well-defined, because for every $z \in Z$, the sequence $(f(z, n))_{n \in \mathbb{N}}$ is ascending in the specialization order of X. It follows that $\widehat{f}$ extends f along $Z \times i$. We show that $\widehat{f}$ is continuous. For this, let $\widehat{f}(z, \infty) \in U$ for some open $U \subseteq X$. As U is ω-Scott-open, there exists some $n_0 \in \mathbb{N}$ with $\widehat{f}(z, n_0) \in U$, and so $f(z, n_0) \in U$. Thus, $f^{-1}(U) \subseteq Z \times \mathbb{N}^{\uparrow}$ is open and nonempty, and, since $Z \times i$ is a topological embedding, there exists an open $V \subseteq Z \times \mathcal{M}(\mathbb{N}^{\uparrow})$ with $f^{-1}(U) = (Z \times i)^{-1}(V)$. By definition of $\widehat{f}$, it follows immediately that $V = \widehat{f}^{-1}(U)$, showing that $\widehat{f}$ is indeed continuous. This continuity also ensures that $\widehat{f}$ is the unique morphism extending the above diagram, as for any other such $g : Z \times \mathcal{M}(\mathbb{N}^{\uparrow})$, it must hold that $g(z, \infty) = \bigvee^{\uparrow}_{n \in \mathbb{N}} f(z, n) = \widehat{f}(z, \infty)$. This completes the proof. $\qquad \square$

Thus we get the following result which again has been claimed by Simpson [138], and shown as Theorem 4.2.5 in the author's Diplomthesis [6].

Theorem 6.1.12 (Simpson)**.** *The categories* **TP** *and* **CE**$(\mathcal{P}\omega)$ *are equivalent.*

Proof. Let $E : \mathbf{QCB}_0 \to \mathbf{Ext}(\mathcal{P}\omega)$ be the equivalence functor, obtained in Theorem 6.1.9 above. Then $(\iota : \omega \to \omega^+) \cong E(i : \mathbb{N}^{\uparrow} \to \mathcal{M}(\mathbb{N}^{\uparrow}))$, where $i : \mathbb{N}^{\uparrow} \to \mathcal{M}(\mathbb{N}^{\uparrow})$ is the topological embedding of $\mathbb{N}^{\uparrow}$ into its ω-monotone convergence reflection. By Theorem 6.1.9, it suffices to show that for a qcb$_0$-space X, the morphism $X^i : X^{\mathcal{M}(\mathbb{N}^{\uparrow})} \to X^{\mathbb{N}^{\uparrow}}$ is an isomorphism in $\mathbf{QCB}_0$ if and only if X is a topological predomain. By cartesian-closedness of $\mathbf{QCB}_0$ this is equivalent to showing that every continuous map $f : Z \times \mathbb{N}^{\uparrow} \to X$ extends uniquely along $Z \times i$ as in (iii) of the previous Lemma (but with the compactly-generated product topology). We recall Theorem 5.4 of [28] shows that if one of the factors is locally compact, then the compactly generated product $A \times B$ carries the product topology. As both, $\mathbb{N}^{\uparrow}$ and $\mathcal{M}(\mathbb{N}^{\uparrow})$ are easily seen to be locally compact, we can thus apply Lemma 6.1.11 and the claim follows. $\qquad \square$

Let us remark that in the literature on Synthetic Domain Theory one finds several notions of completeness for an assembly X. One, given by Phoa [103], is induced by maps $\mathcal{N} \to X$ together with an order on X, defined in the internal logic of the associated realizability topos. Although this notion of completeness might in general disagree with the one given above, see [86], Theorem 3.3.3 of [117] shows that they do agree in the realizability model over $\mathcal{P}\omega$. Another notion of completeness is given by Longley and Simpson's *well-completeness* [85, 86]. An object X of $\mathbf{Asm}(\mathcal{P}\omega)$ is well-complete if its lifting LX is complete, where L is the lifting functor induced by the dominance Σ. Surprisingly, it turns out to be quite hard, in a general setting, to show

that lifting preserves completeness. That this is the case in our situation follows from Theorem 1.22 of [101]. Thus in $\mathbf{Ext}(\mathcal{P}\omega)$ the notions of completeness, in the sense of Phoa and as above, and the notion of well-completeness all coincide.

6.2 Characterisation of Topological Domain Theory in $\mathbf{RT}(\mathcal{P}\omega)$

We have now obtained external characterisations of the categories of Topological Domain Theory in the realizability model over $\mathcal{P}\omega$. However, the full power of Synthetic Domain Theory lies in the internal constructions obtained from the realizability topos associated to the underlying computational model. In our case this is $\mathbf{RT}(\mathcal{P}\omega)$ (or simply $\mathbf{RT}$ below), the realizability topos over the graph model. Its structure is well-known, see [49, 47, 107], and is given as follows.

Let $\langle -, - \rangle : \mathcal{P}\omega \times \mathcal{P}\omega \to \mathcal{P}\omega$ be a continuous pairing function, and $p_1, p_2 \in \mathcal{P}\omega$ be continuous projections such that $p_1 \cdot \langle x, y \rangle = x$ and $p_2 \cdot \langle x, y \rangle = y$ for all $x, y \in \mathcal{P}\omega$. The existence of such functions is evident from the fact that $\mathcal{P}\omega$ is a model of the untyped λ-calculus. The lattice $\mathcal{H} = \mathcal{P}(\mathcal{P}\omega)$, the power set of $\mathcal{P}\omega$, carries a Heyting-algebra structure, given as follows:

- $A \wedge B := \{\langle x, y \rangle | \ x \in A \text{ and } y \in B\}$,

- $A \vee B := \{\langle \{0\}, x \rangle | \ x \in A\} \cup \{\langle \{1\}, x \rangle | \ x \in B\}$,

- $A \to B := \{a \in \mathcal{P}\omega | \text{ if } x \in A \text{ then } a \cdot x \in B\}$,

- $\perp := \emptyset$,

- $\top := \mathcal{P}\omega$.

As usual in intuitionistic logic, one defines $\neg A$ as $A \to \perp$.

For every set X, the set of $\mathcal{H}$-*predicates* on X is given by the set of functions $\mathcal{H}^X$, and carries a pointwise Heyting-algebra structure, i.e. for all $x \in X$ and $\phi, \psi \in \mathcal{H}^X$, the elements $(\phi \wedge \psi)(x), (\phi \vee \psi)(x), (\phi \to \psi)(x) \in \mathcal{H}$ are defined using the respective combinators pointwise in $\mathcal{H}$. Now we can derive an entailment relation $\vdash_X$ on $\mathcal{H}^X$ by:

$$\phi \vdash_X \psi \text{ if } \bigcap \{(\phi \to \psi)(x) | \ x \in X\} \neq \emptyset.$$

Then, as shown in Proposition 1.1 of [47], the preorder $(\mathcal{H}^X, \vdash_X)$ forms itself a Heyting pre-algebra. An element ϕ of this Heyting-prealgebra $\mathcal{H}^X$ is called *valid*, if $\top \vdash_X \phi$.

Furthermore, any set-theoretic function $f : X \to Y$ induces a Heyting-prealgebra homomorphism $f^* : (\mathcal{H}^Y, \vdash_Y) \to (\mathcal{H}^X, \vdash_X)$, defined as $f^*(\phi) = \phi \circ f$. This homomorphism can be shown to have left and right adjoints, $\exists_f, \forall_f : (\mathcal{H}^X, \vdash_X) \to (\mathcal{H}^Y, \vdash_Y)$.

Explicitely, one gets:

$$(\forall_f(\phi))(y) := \bigcap\{|f(x) = y| \to \phi(x)| \ x \in X\}$$

and

$$(\exists_f(\phi))(y) := \bigcup\{|f(x) = y| \wedge \phi(x)| \ x \in X\},$$

where

$$|f(x) = y| := \begin{cases} \mathcal{P}\omega & \text{if } f(x) = y \\ \emptyset & \text{otherwise.} \end{cases}$$

In general, one writes $\exists x.\phi(x, y) \in \mathcal{H}$, respectively $\forall x.\phi(x, y) \in \mathcal{H}$, for the respective adjoints induced by the obvious projections, i.e. $\exists x.\phi(x, y) = (\exists_\pi(\phi))(y)$ and $\forall x.\phi(x, y) = (\forall_\pi(\phi))(y)$ for $\phi \in \mathcal{H}^{X \times Y}$ and $\pi : X \times Y \to Y$.

Now we can describe the objects and morphisms of the realizability topos **RT**:

- An object of **RT** is a tuple $(|X|, =)$, where $|X|$ is a set and $=: |X| \times |X| \to \mathcal{H}$ an $\mathcal{H}$-predicate on $|X| \times |X|$ such that

(symmetry)	$x = y \to y = x,$
(transitivity)	$x = y \wedge y = z \to x = z$

  are valid.

- a morphism $[G] : (|X|, =) \to (|Y|, =)$ in **RT** is given by an equivalence class of *functional relations*, where $G \in \mathcal{H}^{|X| \times |Y|}$ is a functional relation if the following are all valid:

(relationality)	$G(x, y) \wedge x = x' \wedge y = y' \to G(x', y'),$
(strictness)	$G(x, y) \to (x = x \wedge y = y),$
(single-valuedness)	$G(x, y) \wedge G(x, y') \to y = y',$
(totality)	$x = x \to \exists y.G(x, y).$

  The functional relations $G, H \in \mathcal{H}^{|X| \times |Y|}$ are called equivalent if

  $$G(x, y) \leftrightarrow H(x, y)$$

  is valid.

  The identity morphism $(|X|, =) \to (|X|, =)$ is given by the equivalence class of the equality relation, and if $[G] : (|X|, =) \to (|Y|, =)$ and $[H] : (|Y|, =) \to (|Z|, =)$ are morphisms, then the composite $[H \circ G] : (|X|, =) \to (|Z|, =)$ is given by the equivalence class of the functional relation $(x, z) \mapsto \exists y. \, G(x, y) \wedge H(y, z)$.

As the name suggests, we get the following result for **RT**.

Theorem 6.2.1. *The category* **RT** *is a topos.*

Proof. This is Theorem 2.13 in [49]. □

It follows that **RT** is a model for intuitionistic type theory, see e.g. [80, 105]. As excellent references on how the internal logic works in realizability toposes, we recommend [47, 105].

For the subsequent development, we adopt the convention of arguing in the internal language of **RT**, using the rules of intuitionistic logic and intuitionistic type theory. Although at times this appears to be informal, our results are justified by the general theory of [80, 105]. Nevertheless, we emphasise which parts of our proofs and constructions are done externally, and which make use of the internal structure of the topos. As usual, we write **RT** $\models \phi$ to express that the formula ϕ is valid in **RT**.

We next give an internal characterisation for the category **Asm**$(\mathcal{P}\omega)$ in **RT**. For this we recall some general definitions of intuitionistic type theory.

Definition 6.2.2. An object $X = (|X|, =)$ in a topos $\mathcal{E}$ is called $\neg\neg$-separated if

$$\mathcal{E} \models \forall x, y \in X.\ \neg\neg x = y \to x = y.$$

We write **Sep** for the full subcategory of **RT** given by the $\neg\neg$-separated objects.

An mono $m : X \to Y$ in a topos $\mathcal{E}$ is called $\neg\neg$-*closed* if

$$\mathcal{E} \models \forall y \in Y.\ (\neg\neg(\exists x \in X.\ m(x) = y)) \to (\exists x \in X.\ m(x) = y).$$

In this case, X is called a $\neg\neg$-*closed subobject* of Y.

Next, we recall some results about realizability toposes from [47].

Theorem 6.2.3. *In* **RT** *the following hold:*

(i) **Sep** *is a full reflective subcategory of* **RT** *and it is equivalent to* **Asm**$(\mathcal{P}\omega)$.

(ii) *For a given $\neg\neg$-separated object Y, there is a one-to-one correspondence between $\neg\neg$-closed subobjects $X \to Y$ and subobjects $X \to Y$ arising as equalizers of pairs of maps $Y \rightrightarrows Z$ into a $\neg\neg$-separated object.*

Proof. For (i), the reflectivity of **Sep** and the equivalence to **Asm**$(\mathcal{P}\omega)$ follows essentially from Propositions 5.1, 6.1 and 6.3 of [47]. The correspondence for (ii) follows from the Definition of $\neg\neg$-closed monos in section 5 of *op.cit.* □

It is a fact that under the equivalence **Asm**$(\mathcal{P}\omega) \simeq$ **Sep**, the assembly Σ of the previous section becomes the $\neg\neg$-separated object $(\{\bot, \top\}, =)$ with $|\bot = \bot| = \{\emptyset\}$ and $|\top = \top| = \{\{0\}\}$. We denote this object again by Σ. It is a subobject of the subobject

classifier Ω, and thus we can again interpret morphisms $f \in X \to \Sigma$ as observable predicates of X.

The following definition gives an internal characterisation of $\mathbf{Ext}(\mathcal{P}\omega)$, and hence $\mathbf{QCB}_0$, in $\mathbf{RT}$.

Definition 6.2.4. An object X of $\mathbf{RT}$ is called a Σ-*poset*, if the map $\eta_X : X \to \Sigma^{\Sigma^X}$, given by $\lambda x : X.\lambda f : \Sigma^X. fx$ is a mono. It is called a *regular* Σ-*poset*, if η_X is a $\neg\neg$-closed mono.

The regular Σ-posets form the full subcategory $\boldsymbol{\Sigma}\mathbf{Pos}$ of $\mathbf{RT}$.

The notion of (regular) Σ-posets can be tracked back to several sources, e.g. [117, 116, 118, 101]. The name is justified by the fact that one can define a relation $\leq$ on objects X of $\mathbf{RT}$ internally, by

$$x \leq y := \forall f \in \Sigma^X. f(x) \to f(y).$$

This relation satisfies the axioms of a partial order, if X is a Σ-poset. For details on this, we refer the reader to Chapter 2 of Reus' thesis [116].

We get the following result.

Theorem 6.2.5. *The category* $\boldsymbol{\Sigma}\mathbf{Pos}$ *is a full reflective subcategory of* $\mathbf{RT}$ *and equivalent to* $\mathbf{Ext}(\mathcal{P}\omega)$.

Proof: By Theorem 6.2.3 (i), the equivalence of $\mathbf{Asm}(\mathcal{P}\omega)$ and $\mathbf{Sep}$ carries extensional assemblies to objects for which η_X arises as an equalizer in $\mathbf{Sep}$, and by Theorem 6.2.3 (ii), these are exactly the objects for which η_X is a $\neg\neg$-closed mono, i.e. the regular Σ-posets. This yields the equivalence of $\boldsymbol{\Sigma}\mathbf{Pos}$ and $\mathbf{Ext}(\mathcal{P}\omega)$. As $\mathbf{Ext}(\mathcal{P}\omega)$ is a full reflective subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$, by Theorems 2.1.11, 6.1.6 and 6.1.9, and $\mathbf{Sep}$ reflective in $\mathbf{RT}$, the claim follows. $\qquad\square$

Corollary 6.2.6. *The categories* $\boldsymbol{\Sigma}\mathbf{Pos}$ *and* $\mathbf{QCB}_0$ *are equivalent.*

In fact the definition of regular Σ-posets above is equivalent to a more general one, as the following standard result shows.

Theorem 6.2.7. *For an object X of $\mathbf{RT}$ the following are equivalent.*

(i) for the evident morphism $\eta_X : X \to \Sigma^{\Sigma^X}$, it holds that:

$$\mathbf{RT} \models \text{``}\eta_X \text{ is a } \neg\neg - \text{closed mono''},$$

(ii) there exists an object Y in $\mathbf{RT}$ and a morphism $m : X \to \Sigma^Y$ such that:

$$\mathbf{RT} \models \text{``}m \text{ is a } \neg\neg - \text{closed mono''},$$

Proof. See e.g. Theorem 2.6.6 of [116]. □

We can also give an internal characterisation of completeness. For this, let $\mathcal{N}$ be the natural numbers object in **RT**, which is isomorphic to the equally denoted assembly under the equivalence $\mathbf{Asm}(\mathcal{P}\omega) \simeq \mathbf{Sep}$. Let furthermore ω^+ be the subobject of $\Sigma^{\mathcal{N}}$ in **RT** given by $\{f \in \Sigma^{\mathcal{N}} |\ \forall n \in \mathcal{N}.f(n+1) \to f(n)\}$, and ω the subobject of ω^+, given by $\{f \in \omega^+ |\ \neg\neg(\exists n \in \mathcal{N}.f(n) = \bot)\}$. Then ω^+, ω and the inclusion $\iota : \omega \to \omega^+$ become isomorphic to the image of the equally denoted entities of the previous section under the equivalence. This allows us to generalise the definition of completeness of the previous section to all objects to **RT**. In particular, we get:

Definition 6.2.8. An object X of **RT** is *complete* if

$$\mathbf{RT} \models \text{``}X^{\iota} \text{ is an isomorphism''}.$$

The category Σ**Cpo** is the full subcategory of **RT** given by *complete regular Σ-posets* (or simply Σ-*cpos*), i.e. objects X satisfying:

$$\mathbf{RT} \models \text{``}\eta_X \text{ is a } \neg\neg - \text{closed mono''} \wedge \text{``}X^{\iota} \text{ is an isomorphism''}.$$

The notion Σ-cpo can be found in [116], and is derived from [34, 48]. Now, Definition 6.1.10 and Theorem 6.2.5 give that under the equivalence $\mathbf{Asm}(\mathcal{P}\omega) \simeq \mathbf{Sep}$, the category of complete extensional assemblies, $\mathbf{CE}(\mathcal{P}\omega)$, becomes equivalent to Σ**Cpo**. Hence, we get the following.

Corollary 6.2.9. *The category Σ***Cpo** *is a full reflective subcategory of* **RT** *and equivalent to* $\mathbf{CE}(\mathcal{P}\omega)$ *and* **TP**.

Proof. The equivalence is clear by the definition of Σ-cpos and Theorem 6.1.12. Reflectivity follows from the fact that **TP** is reflective in $\mathbf{QCB}_0$, by Theorem 3.2.5. □

We have thus obtained internal characterisations in **RT** of $\mathbf{QCB}_0$ and **TP**, two categories at the center of Topological Domain Theory.

6.3 Internal closure properties of ΣPos and ΣCpo

Our next aim is to investigate the internal closure properties of regular Σ-posets, respectively Σ-cpos. In particular we want to show that the corresponding categories are complete and small in a suitable sense. We start by recalling some general definitions which relate the intuitionistic type-theoretic interpretation of a topos with the theory of fibrations, see also [80, 105]. Our aim is to use the fibrational theory as little as

possible, and work in intuitionistic type theory instead. However, mentioning the main connections of these two approaches cannot be avoided, as we crucially use results from [54], which works in the fibrational setting.

For the definition of fibrations, we refer the reader to Chapter 2 of [105]. The base category of the fibrations we use, will always be a topos $\mathcal{E}$. Recall Definition 2.3.5 of *op.cit.*, which defines completeness of such a fibration $p : \mathbb{C} \to \mathcal{E}$. It says that $p : \mathbb{C} \to \mathcal{E}$ is *complete* if each slice $\mathbb{C}_I$ has finite limits and each reindexing functor $f^* : \mathbb{C}_J \to \mathbb{C}_I$ has a right adjoint $\prod_f : \mathbb{C}_I \to \mathbb{C}_J$, corresponding to an internal product construction, which satisfies the Beck-Chevalley condition.

The interpretation of intuitionistic type theory in a topos $\mathcal{E}$ heavily uses the codomain fibration $cod : \mathcal{E}^2 \to \mathcal{E}$, see e.g. [55, 105]. In particular, it uses the fundamental theorem of topos theory, which states that for a topos $\mathcal{E}$ and an object I of $\mathcal{E}$, the slice category $\mathcal{E}/I$ is again a topos, and for any morphism $f : I \to J$ in $\mathcal{E}$, the reindexing functor $f^* : \mathcal{E}/J \to \mathcal{E}/I$ is logical, i.e. preserves the topos structure, see e.g. Theorem 3.4.3 and Corollary 3.4.7 of [105]. The following describes the intuitionistic type-theoretic interpretation of families of objects in a topos, see e.g. Definition 3.8.1 in *op.cit.*

Definition 6.3.1. Let $\mathcal{E}$ be a topos and I be an object of $\mathcal{E}$. An *I-indexed family* of objects $\{X_i\}_{i \in I}$ of $\mathcal{E}$ is given by an object $\begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix}$ of the slice topos $\mathcal{E}/I$. Similarly, an I-indexed family of morphisms $\{f_i : X_i \to Y_i\}_{i \in I}$ of $\mathcal{E}$ is given by a morphism $f : \begin{pmatrix} X \\ \downarrow \\ I \end{pmatrix} \to \begin{pmatrix} Y \\ \downarrow \\ I \end{pmatrix}$ in the slice topos $\mathcal{E}/I$.

Now we can describe indexed limits and internal products in $\mathcal{E}$ via families of objects, respectively morphisms. If $\{X_i\}_{i \in I}$ and $\{Y_i\}_{i \in I}$ are I-indexed families of objects of $\mathcal{E}$, then the product $\{X_i\}_{i \in I} \times \{Y_i\}_{i \in I}$ in the slice $\mathcal{E}/I$ is constructed pointwise, i.e. given by $\{X_i \times Y_i\}_{i \in I}$. (In fact, it is given by the pullback of the corresponding maps $X \to I$ and $Y \to I$ in $\mathcal{E}$.) Similarly, for a pair $\{f_i, g_i : X_i \to Y_i\}$ of I-indexed families of morphisms of $\mathcal{E}$, the equalizer in $\mathcal{E}/I$ is given pointwise by $\{e_i : A_i \to X_i\}_{i \in I}$, where $e_i : A_i \to X_i$ is the equalizer of $f_i, g_i : X_i \to Y_i$. Finally, the terminal object of $\mathcal{E}/I$ is given by the family $\{\mathbf{1}\}_{i \in I}$.

For a every morphism $f : I \to J$ in a topos $\mathcal{E}$. the reindexing functor $f^* : \mathcal{E}/J \to \mathcal{E}/I$ has left and right adjoints, denoted by $\coprod_f, \prod_f : \mathcal{E}/I \to \mathcal{E}/J$, see e.g. Corollary 3.4.7 of [105]. These adjoints satisfy the Beck-Chevalley condition and form the internal coproduct and internal product functors in the intuitionistic type theory given by $\mathcal{E}$. In particular, if $f : I \to J$ is a morphism in $\mathcal{E}$ and $\{X_i\}_{i \in I}$ an I-indexed family of objects of $\mathcal{E}$, then the internal product of $\{X_i\}_{i \in I}$ along f is given by the J-indexed family $\{\prod_{f(i)=j} X_i\}$. This construction makes the codomain fibration $\mathcal{E}^2 \to \mathcal{E}$ complete.

Now let us give the notion of definable full subcategories of a topos $\mathcal{E}$. For this,

observe that the internal language of a topos $\mathcal{E}$ (see [80, 105]) can be extended with free type and term variables. This is done in the expected way, by constructing types, terms and formulae with free variables, according to the rules of intuitionistic type theory and intuitionistic logic (see *op.cit.*) together with the type and term constructors given by the structure of $\mathcal{E}$.

Definition 6.3.2. A *definable full subcategory* $\mathbb{C}$ of a topos $\mathcal{E}$ is a subcategory for which there exists a formula $\phi(\underline{X})$ in the internal logic of $\mathcal{E}$ extended with a free type variable, such that an object X of $\mathcal{E}$ belongs to $\mathbb{C}$ if and only if:

$$\mathcal{E} \models \phi(X),$$

The notion of a definable full subcategory of a topos $\mathcal{E}$ should not be confused with that of an *internal subcategory* of $\mathcal{E}$, which is given explicitly by a pair of objects of $\mathcal{E}$ see section 3 of [47]. For instance, the categories $\mathbf{Sep}$, $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are all definable in $\mathbf{RT}$, with the respective formulae given in Definitions 6.2.2, 6.2.4 and 6.2.8, but none of these is an internal subcategory, because none is small. (There is no object of objects).

A definable full subcategory $\mathbb{C}$ always gives rise to a full subfibration of $cod : \mathcal{E}^2 \to \mathcal{E}$, the fibration of families of objects and morphisms of $\mathbb{C}$. In particular, if $\phi(\underline{X})$ is the formula characterising $\mathbb{C}$. Then an I-indexed family of objects of $\mathbb{C}$ is an I-indexed family $\{X_i\}_{i\in I}$ of $\mathcal{E}$ for which it holds that:

$$\mathcal{E} \models \forall i \in I.\, \phi(X_i).$$

Definition 6.3.3. A definable full subcategory $\mathbb{C}$ of a topos $\mathcal{E}$ is *internally complete* if the corresponding fibration of families of $\mathbb{C}$ is complete.

We get the following trivial condition for internal completeness of definable full subcategories.

Proposition 6.3.4. *Let $\mathbb{C}$ be a definable full subcategory of a topos $\mathcal{E}$, such that families of $\mathbb{C}$ are closed under the construction of indexed finite limits of families of $\mathcal{E}$, and $\mathbb{C}$ is closed under internal products in $\mathcal{E}$. Then $\mathbb{C}$ is internally complete.*

Now we turn our attention back to the realizability topos $\mathbf{RT}$. For the next definition, observe that the assembly $\nabla\mathbf{2}$, of Section 6.1, becomes the object $(\{0, 1\}, =)$, with

$$|x = y| := \begin{cases} \mathcal{P}\omega & \text{if } x = y \\ \emptyset & \text{otherwise,} \end{cases}$$

under the equivalence of $\mathbf{Asm}(\mathcal{P}\omega)$ and $\mathbf{Sep}$. We denote this object again by $\nabla\mathbf{2}$.

Definition 6.3.5. An object X of **RT** is called *orthogonal* (to $\nabla \mathbf{2}$) if for the terminal map $! : \nabla \mathbf{2} \to \mathbf{1}$, it holds that:

$$\mathbf{RT} \models \text{``}X^! : X \to X^{\nabla \mathbf{2}} \text{ is an isomorphism''}.$$

By **Orth**, we denote the full subcategory of **RT** on the orthogonal objects, and by **Orth**$_{\neg\neg}$ the full subcategory on the $\neg\neg$-separated orthogonal objects.

Clearly, **Orth** and **Orth**$_{\neg\neg}$ are definable full subcategories of **RT**. Moreover, we get the following result.

Theorem 6.3.6. *The categories* **Sep**, $\mathbf{\Sigma Pos}$, $\mathbf{\Sigma Cpo}$, **Orth** *and* **Orth**$_{\neg\neg}$ *are all internally complete in* **RT**.

Proof. We argue in the internal language of **RT**. For the existence of indexed limits in these categories, we have to show that all the formulae $\phi(\underline{X})$, defining objects of our categories, are invariant under the corresponding constructions. As indexed limits are calculated pointwise, the result follows from the fact that closure under finite limits is provable in the internal logic using the formulae defining the categories.

As an example we show that $\mathbf{\Sigma Pos}$ is closed under indexed equalizers in $\mathcal{E}$, all other cases are easily shown analogously. So let $\{f_i, g_i : X_i \to Y_i\}_{i \in I}$ be families of morphisms between families of Σ-posets, and let $\{e_i : Z_i \to X_i\}_{i \in I}$ be the equalizer of $\{f_i, g_i\}_{i \in I}$. By Theorem 6.2.7, we have to show that there exists an I-indexed family $\{A_i\}_{i \in I}$ of objects of $\mathcal{E}$ with an I-indexed family of monos $\{m_i : Z_i \to \Sigma^{A_i}\}_{i \in I}$ such that:

$$\mathbf{RT} \models \forall i \in I. \text{``}m_i : Z_i \to \Sigma^{A_i} \text{ is a } \neg\neg - \text{closed mono''}.$$

As $\{X_i\}_{i \in I}$ is a family of Σ-posets, we have that:

$$\mathbf{RT} \models \forall i \in I. \text{``}\eta_{X_i} : X_i \to \Sigma^{\Sigma^{X_i}} \text{ is a } \neg\neg - \text{closed mono''}.$$

Moreover, by Theorem 6.2.3 (ii), we have that:

$$\mathbf{RT} \models \forall i \in I. \text{``}e_i : X_i \to Y_i \text{ is a } \neg\neg - \text{closed mono''},$$

and as $\neg\neg$-closed monos obviously compose, we get our result by:

$$\mathbf{RT} \models \forall i \in I. \text{``}\eta_{X_i} \circ e_i : Z_i \to \Sigma^{\Sigma^{X_i}} \text{ is a } \neg\neg - \text{closed mono''}.$$

Next, we consider internal products. By Proposition 2.1 in [54], **Orth** is closed under the formation of internal products in $\mathcal{E}$, and so are the complete objects, according to Definition 6.2.8, which are also characterised by an orthogonality condition. Also it is

a standard result that $\neg\neg$-separated objects and $\neg\neg$-separated monos are closed under internal products in any topos.

We show explicitely that regular Σ-posets are closed under the formation of internal products in $\mathcal{E}$. Suppose $\{X_i\}_{i\in I}$ is an I-indexed family of regular Σ-posets. Then the I-indexed family $\{\eta_{X_i} : X_i \to \Sigma^{\Sigma^{X_i}}\}_{i\in I}$ is a family of $\neg\neg$-closed monos. As $\neg\neg$-closed monos are closed under internal products, we get that

$$\{ \prod_{f(i)=j} m_i : \prod_{f(i)=j} X_i \to \prod_{f(i)=j} \Sigma^{\Sigma^{X_i}} \}_{j\in J}$$

is a J-indexed family of $\neg\neg$-closed monos. Furthermore, in intuitionistic type theory, it holds that $\prod_{f(i)=j} \Sigma^{\Sigma^{X_i}}$ is isomorphic to $\Sigma^{\coprod_{f(i)=j} \Sigma^{X_i}}$. Hence, we get, for any given $j \in J$, a J-indexed family of $\neg\neg$-closed monos $\prod_{f(i)=j} X_i \to \Sigma^{\coprod_{f(i)=j} \Sigma^{X_i}}$. With Theorem 6.2.7, it follows that $\{\prod_{f(i)=j} X_i\}_{j\in J}$ is a J-indexed family of regular Σ-posets.

As $\mathbf{Orth}_{\neg\neg}$ is given by objects satisfying the formulae of $\mathbf{Orth}$ and $\mathbf{Sep}$, and $\Sigma\mathbf{Cpo}$ by the formulae for complete objects and $\Sigma\mathbf{Pos}$, and all these formulae are invariant under the formation of internal products, the claim follows. $\qquad\square$

Let us next investigate smallness of the respective categories. Here we use the following notion of weak smallness, which can be found e.g. in [126, 140].

Definition 6.3.7. A definable full subcategory $\mathbb{C}$ of a topos $\mathcal{E}$ is called *weakly small* if there exists a family $\{C_j\}_{j\in J}$ of objects of $\mathbb{C}$ such that for every family $\{X_i\}_{i\in I}$ of objects of $\mathbb{C}$, it holds that:

$$\mathcal{E} \models \forall i \in I. \exists j \in J. X_i \cong C_j,$$

where $X_i \cong C_j$ ought to be understood as $\exists f : X_i \to C_j$. "f is an isomorphism".

This definition is indeed a weaker notion than genuine (strong) smallness; the validity of the above formula in the topos does not give a functor which assigns an object of $\{C_j\}_{j\in J}$ to every object of $\{X_i\}_{i\in I}$. In fact, the definition amounts to the fibration of families of $\mathbb{C}$ to be weakly equivalent to a fibration with generic object. For more details about the difference of weak and strong notions in a topos, we refer the reader to [54].

In fact, the following result of *op.cit.* is the starting point for our proof to show that $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are weakly small.

Theorem 6.3.8. *The category* $\mathbf{Orth}$ *is weakly small.*

Proof. We argue in the internal logic of $\mathbf{RT}$. In Theorem 7.1 of [54], it is shown that, in the effective topos [47], there is a fibred weak equivalence between the fibration of families of orthogonal objects and the fibration of global subquotients of $\mathbb{N}$, where $\mathbb{N}$ is

the carrier set of the partial combinatory algebra, generating the effective topos. The same reasoning as in [54] shows that, in **RT** the fibration of families of objects of **Orth** is weakly equivalent to the fibration of global subquotients of $\mathcal{P}\omega$. This yields that for every family $\{X_i\}_{i \in I}$, there exists a family $\{Y_j\}_{j \in J}$ of global subquotients of $\mathcal{P}\omega$ such that:

$$\mathbf{RT} \models \forall i \in I. \exists j \in J. X_i \cong Y_j.$$

But the families of global subquotients of $\mathcal{P}\omega$ are exactly the families obtained as pullbacks of the family $\{C_q\}_{q \in Q}$, which generates the internal category of subquotients of $\mathcal{P}\omega$ in **RT**. This family $\{C_q\}_{q \in Q}$ is constructed as in section 3 of [54] (after Lemma 3.2) with the object Q being given by:

$$Q = \{R \in \Omega^{\mathcal{P}\omega \times \mathcal{P}\omega} | \text{ ``}R \text{ is symmetric and transitive''}\},$$

with Ω being the subobject classifier in **RT**. Then we get that:

$$\mathbf{RT} \models \forall j \in J. \exists q \in Q. Y_j \cong C_q,$$

and hence:

$$\mathbf{RT} \models \forall i \in I. \exists q \in Q. X_i \cong C_q,$$

By Corollary 6.2 and Lemma 6.3 of *op.cit.*, we get that the family $\{C_q\}_{q \in Q}$ is indeed a family of orthogonal objects, and thus **Orth** is weakly small. $\qquad\square$

Next we show that the property of being weakly small transfers to definable full subcategories.

Lemma 6.3.9. *Let $\mathbb{C}$ be a definable full subcategory of a topos $\mathcal{E}$ and $\mathbb{D}$ be a definable full subcategory of $\mathbb{C}$. Then $\mathbb{D}$ is weakly small if $\mathbb{C}$ is.*

Proof. We argue again in the internal logic of **RT**. Let $\{C_j\}_{j \in J}$ be the family of objects of $\mathbb{C}$, making it weakly small. Let furthermore $\phi(\underline{X})$ be the formula defining the objects of $\mathbb{D}$. Now form the subobject $J' := \{j \in J | \phi(C_j)\}$ of J. We claim that the family $\{C_j\}_{j \in J'}$ makes $\mathbb{D}$ weakly small. First, observe that $\{C_j\}_{j \in J'}$ is indeed a family of objects of $\mathbb{D}$ since, by construction, it holds that:

$$\mathcal{E} \models \forall j \in J'. \phi(C_j).$$

Furthermore, since $\mathbb{D}$ is a subcategory of $\mathbb{C}$, we have that for every family $\{X_i\}_{i \in I}$ of objects in $\mathbb{D}$, it holds that:

$$\mathcal{E} \models \forall i \in I. \exists j \in J. X_i \cong C_j \wedge \phi(X_i).$$

But then it holds that:

$$\mathcal{E} \models \forall i \in I.\, \exists j \in J.\, X_i \cong C_j \wedge \phi(X_i) \wedge \phi(C_j),$$

since any intuitionistic type-theoretic formula ϕ is invariant under isomorphisms. Thus, by the construction of J', we conclude that:

$$\mathcal{E} \models \forall i \in I.\, \exists j \in J'.\, X_i \cong C_j,$$

showing the claim. $\hspace{1cm}\square$

Finally, we show that $\mathbf{\Sigma Pos}$ and $\mathbf{\Sigma Cpo}$ are full subcategories of $\mathbf{Orth}$. The result implying this is the following, which can also be found in [54].

Lemma 6.3.10. *The categories* $\mathbf{Orth}_{\neg\neg}$ *and* $\mathbf{Mod}(\mathcal{P}\omega)$ *are equivalent.*

Proof. By the equivalence $\mathbf{Asm}(\mathcal{P}\omega) \simeq \mathbf{Sep}$, it suffices to show that $\mathbf{Mod}(\mathcal{P}\omega)$ is the full subcategory of $\mathbf{Asm}(\mathcal{P}\omega)$ on the objects which are orthogonal to the terminal map $!: \nabla \mathbf{2} \to \mathbf{1}$. As $\mathbf{Asm}(\mathcal{P}\omega)$ is cartesian-closed, an assembly $X = (|X|, \|\cdot\|_X)$ is orthogonal to $!$, if every morphism $f: Z \times \nabla \mathbf{2} \to X$ in $\mathbf{Asm}(\mathcal{P}\omega)$ factors uniquely through the projection $Z \times \nabla \mathbf{2} \to Z$, as in:

$$
\begin{array}{ccc}
Z & \dashrightarrow^{\widehat{f}} & X \\
\uparrow & \nearrow{\scriptstyle f} & \\
Z \times \nabla \mathbf{2} & &
\end{array}
$$

So assume this is the case for X, and $x, y \in |X|$ with $a \in \|x\|_X \cap \|y\|_X$. Then the constant map $\lambda x.a$ tracks the map $f: Z \times \nabla \mathbf{2} \to X$, given by:

$$(z, s) \mapsto \begin{cases} x & \text{if } s = 0 \\ y & \text{if } s = 1. \end{cases}$$

But then the factorisation property yields

$$x = f(z, 0) = \widehat{f}(z) = f(z, 1) = y,$$

showing that X is a modest set.

Conversely, suppose $X = (|X|, \|\cdot\|_X)$ is a modest set, and $f: Z \times \nabla \mathbf{2} \to X$ a morphism in $\mathbf{Asm}(\mathcal{P}\omega)$ tracked by $a \in \mathcal{P}\omega$. As X is modest, and $(z, 0)$ and $(z, 1)$ have the same sets of realizers, it follows that $f(z, 0) = f(z, 1)$. Thus, the map $\widehat{f}: Z \to X$, defined as $z \mapsto f(z, 0)$ is well-defined, and moreover tracked by $\lambda x.a \cdot \langle x, \{0\} \rangle$. Clearly,

$\hat{f}$ extends f along the projection, and it is unique with this property, showing that X is orthogonal to $! : \nabla\mathbf{2} \to \mathbf{1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 6.3.11. *The categories $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are full subcategories of $\mathbf{Orth}$.*

Proof. We know that under the equivalence $\mathbf{Asm}(\mathcal{P}\omega) \simeq \mathbf{Sep}$, $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are equivalent to $\mathbf{Ext}(\mathcal{P}\omega)$ and $\mathbf{CE}(\mathcal{P}\omega)$, from Theorem 6.2.5 and Corollary 6.2.9. But $\mathbf{Ext}(\mathcal{P}\omega)$ and $\mathbf{CE}(\mathcal{P}\omega)$ are full subcategories of $\mathbf{Mod}(\mathcal{P}\omega)$, by Corollary 6.1.8, which is equivalent to $\mathbf{Orth}_{\neg\neg}$ under the equivalence $\mathbf{Asm}(\mathcal{P}\omega) \simeq \mathbf{Sep}$, by Lemma 6.3.10. It follows that $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are full subcategories of $\mathbf{Orth}_{\neg\neg}$, and hence of $\mathbf{Orth}$. $\quad\square$

Now Theorem 6.3.8, Lemma 6.3.9 and Corollary 6.3.11, yield the following.

Theorem 6.3.12. *The categories $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are weakly small in $\mathbf{RT}$.*

Thus, we obtain the main result about internal closure properties of the internal version of Topological Domain Theory, with Theorems 6.3.6 and 6.3.12.

Theorem 6.3.13. *The categories $\Sigma\mathbf{Pos}$ and $\Sigma\mathbf{Cpo}$ are internally complete and weakly small.*

6.4 Polymorphism in Topological Domain Theory

We now show how the polymorphic λ-calculus can be modelled in $\Sigma\mathbf{Pos}$, and hence in $\mathbf{QCB}_0$. A model in $\Sigma\mathbf{Cpo}$, respectively $\mathbf{TP}$, can be obtained in the same way. Let us start by fixing a family $\{C_j\}_{j \in J}$ of Σ-posets, making $\Sigma\mathbf{Pos}$ weakly small, which exists by Theorem 6.3.12.

Next, we recall the basic type construction rules for the polymorphic λ-calculus [119]. We denote type variables by $X, X_0, X_1, \ldots$ and types by $\sigma, \tau, \tau_0, \tau_1, \ldots$. Then the type constructions are given by

$$\tau ::= X \mid \sigma \to \tau \mid \forall X.\tau.$$

A *type environment* for a type τ is given by an assignment ρ, assigning to each free type variable $X \in FTV(\tau)$, a Σ-poset $\rho(X)$ (equivalently, a type environment for τ is given by a $FTV(\tau)$-indexed family $\{\rho(X_i)\}_{i \in FTV(\tau)}$ of Σ-posets). If ρ is an assignment of Σ-posets to the free type variables of $FTV(\tau) \setminus \{X\}$, then $\rho_{(A/X)}$ denotes the type environment for τ which extends ρ by the assignment $X \mapsto A$. We give the following inductive definition of a type interpretation $[\![\cdot]\!]_\rho$ for a type τ in a type environment ρ:

- $[\![X]\!]_\rho := \rho(X)$,

- $[\![\sigma \to \tau]\!]_\rho := [\![\tau]\!]_\rho^{[\![\sigma]\!]_\rho}$, the latter being the exponential in $\Sigma\mathbf{Pos}$,

- Let $\mathbf{Gpd}(\mathbf{\Sigma Pos})$ denote the subcategory of $\mathbf{\Sigma Pos}$ given by the isomorphisms. In order to give an interpretation for $\forall X.\tau$, we associate to the open type τ the functor $F_\tau : \mathbf{Gpd}(\mathbf{\Sigma Pos}) \to \mathbf{\Sigma Pos}$, given by the assignment $A \mapsto [\![\tau]\!]_{\rho_{(A/X)}}$. Thus, we then define, following Simpson [140]:

$$[\![\forall X.\tau]\!]_\rho := \{a \in \prod_{j \in J} F_\tau(C_j) \mid \forall j, j' \in J. \, \forall i : C_j \cong C_{j'}. \, a_{C_j} = F_\tau(i)(a_{C_{j'}})\},$$

which is a regular Σ-poset by internal completeness.

We remark, that the condition demanding the elements of the components of polymorphic types to be invariant under isomorphisms can be viewed as a weak form of Reynold's relational parametricity [121]. In fact, we need this invariance condition on $[\![\forall X.\tau]\!]_\rho$ for getting unique interpretations of type applications, see below.

Let us now turn our attention towards terms. We denote term variables by $x, x_0, x_1, \ldots$ and terms by $s, t, t_0, t_1, \ldots$. Then the term constructions are given by

$$t ::= x \mid \lambda x : \sigma.t \mid ts \mid \Lambda X.t \mid t(\sigma).$$

The constructions of well-formed terms in the polymorphic λ-calculus are subject to the usual well-typedness rules, see [119].

Assume $t : \tau$ is a term, and ρ a type environment for τ. Then a *term environment* for $t : \tau$ is a map μ, assigning to each free variable $x : \sigma$ of t an element $\mu(x) \in [\![\sigma]\!]_\rho$. As above, if μ is an assignment of variables to elements of matching regular Σ-posets, such that $dom(\mu) = FV(t) \setminus \{x : \sigma\}$, then we denote by $\mu_{(a/x)}$ the term environment for t extending μ by $x \mapsto a \in [\![\sigma]\!]_\rho$.

Now suppose we are given a type environment ρ and a term environment μ for $t : \tau$. Then we can give the following inductive definition of a term interpretation $[\![\cdot]\!]_{\rho,\mu}$. We start with the abstraction rules:

- $[\![x]\!]_{\rho,\mu} := \mu(x)$,

- $[\![\lambda x : \sigma.t]\!]_{\rho,\mu} := \lambda a : [\![\sigma]\!]_\rho.[\![t]\!]_{\rho,\mu_{(a/x)}}$, which can be defined using the cartesian-closed structure of $\mathbf{\Sigma Pos}$,

- $[\![\Lambda X.t]\!]_{\rho,\mu} := \langle [\![t]\!]_{\rho_{(C_j/X)},\mu} \rangle_{j \in J}$.

Obviously these interpretations are sound with respect to the type interpretations, i.e. we get $[\![t]\!]_{\rho,\mu} \in [\![\tau]\!]_\rho$. It remains to give sound interpretations for the application rules. They are given as follows:

- Assume ts is well-formed with $t : \tau_0$, $s : \tau_1$ and $ts : \tau_2$. Then the definition of type interpretations ensures that $[\![\tau_0]\!]_\rho = [\![\tau_2]\!]_\rho^{[\![\tau_1]\!]_\rho}$. Thus, we can define:

$$[\![ts]\!]_{\rho,\mu} := [\![t]\!]_{\rho,\mu}([\![s]\!]_{\rho,\mu}).$$

- Assume $t(\sigma)$ is a well-formed term, then the well-typedness conditions ensure that $t : \forall X.\tau$, hence $[\![t]\!]_{\rho,\mu}$ is an element of a J-indexed product, i.e. of the form

$$\langle [\![s]\!]_{\rho(C_j/X),\mu}\rangle_{j\in J} \in \prod_{j\in J} F_\tau(C_j),$$

for $F_\tau : \mathbf{Gpd}(\mathbf{\Sigma Pos}) \to \mathbf{\Sigma Pos}$ being the functor corresponding to the type τ, as defined above. Moreover, by the weak smallness of $\mathbf{\Sigma Pos}$, there exists some $j_0 \in J$ with an isomorphism $i : C_{j_0} \cong [\![\sigma]\!]_\rho$. Now one can show straightforwardly, using induction on the type constructors, that

$$F_\tau(i) : F_\tau(C_{j_0}) \to F_\tau([\![\sigma]\!]_\rho)$$

is an isomorphism. Thus we define:

$$[\![t(\sigma)]\!]_{\rho,\mu} := F_\tau(i)(\pi_{j_0}([\![t]\!]_{\rho,\mu})).$$

The parametricity condition on the elements of $[\![\forall X.\tau]\!]_\rho$ ensures that the result is independent of the choice of the $j_0 \in J$, as long as $C_{j_0} \cong [\![\sigma]\!]_\rho$. Hence this definition is well-defined.

It follows that for all types $t : \tau$ in a type environment ρ and term environment μ, we have that $[\![t]\!]_{\rho,\mu} \in [\![\tau]\!]_\rho$. Moreover, it is now straightforward to show that our interpretations are sound with respect to the rules of the polymorphic λ-calculus. Furthermore, we can apply the same approach to $\mathbf{\Sigma Cpo}$, if we exchange the family making the model category weakly small.

Corollary 6.4.1. *The polymorphic λ-calculus can be interpreted in $\mathbf{\Sigma Pos}$ and $\mathbf{\Sigma Cpo}$.*

We remark that although we have some weak form of parametricity for polymorphic types, the model given here is not relationally parametric in the sense of Reynolds [121]. Thus, we cannot prove some important program equivalences using the semantics, as done e.g. in [162]. One way to amend this situation is to give an internal characterisation of parametricity, and to define the polymorphic types to only contain parametric elements. Approaches in this direction have been made by Rosolini and Simpson [126], and in a fibrational setting by Birkedal, Møgelberg and Pedersen [18, 19].

From Corollaries 6.2.6, 6.2.9 and 6.4.1, it follows that the polymorphic λ-calculus can be modelled in Topological Domain Theory, if we use the synthetic version of it. However the open question remains whether one can give a topological description of the polymorphic types for $\mathbf{QCB}_0$, see also the discussion in Chapter 7.2.

6.5 Free algebras synthetically

In this section we use the result that $\mathbf{\Sigma Pos}$ and $\mathbf{\Sigma Cpo}$ are internally complete and weakly small categories to apply an internal version of the Adjoint Functor Theorem. This enables us to show the existence of free algebra functors for a wide range of internally definable algebraic theories over Σ-posets and Σ-cpos. In particular, we obtain free algebra functors for a strictly more general class of parameterised algebraic theories than in Chapters 4 and 5.

Let us start by giving the definition of the algebraic theories for the realizability topos $\mathbf{RT}$, for which we show the existence of free algebras.

Definition 6.5.1. An *equational algebraic theory* $\mathcal{T}$ for $\mathbf{RT}$ is given by the following data:

- an object $\mathcal{O}$ of operations for $\mathcal{T}$,

- an $\mathcal{O}$-indexed family $\{A_o\}_{o \in \mathcal{O}}$ of objects, giving arities to the operations $o \in \mathcal{O}$,

- an object $\mathcal{E}$ of $\mathbf{RT}$, giving an indexing set of equations for $\mathcal{T}$,

- an $\mathcal{E}$-indexed family $\{A_e\}_{e \in \mathcal{E}}$ of objects, giving arities to the equations $e \in \mathcal{E}$,

- for each $e \in \mathcal{E}$, there is a pair (t_e, t'_e) of algebraic operations $t_e, t'_e : X^{A_e} \to X$ following Definition 4.1.2 above.

A $\mathcal{T}$-*algebra* is a tuple $(X, \{o_X\}_{o \in \mathcal{O}})$, where $\{o_X : X^{A_o} \to X\}_{o \in \mathcal{O}}$ is an $\mathcal{O}$-indexed family of morphisms in $\mathbf{RT}$, and it holds that:

$$\mathbf{RT} \models \forall e \in \mathcal{E}. \, \forall (x_i)_{i \in A_e} \in X^{A_e}. \, t_e((x_i)_{i \in A_e}) = t'_e((x_i)_{i \in A_e}).$$

A $\mathcal{T}$-*homomorphism* $(X, \{o_X\}_{o \in \mathcal{O}}) \to (Y, \{o_Y\}_{o \in \mathcal{O}})$ is given by a map $h : X \to Y$, such that:

$$\mathbf{RT} \models \forall o \in \mathcal{O}. \, \forall (x_i)_{i \in A_o} \in X^{A_o}. \, h(o_X((x_i)_{i \in A_o})) = o_Y((h(x_i))_{i \in A_o}).$$

The category of $\mathcal{T}$-algebras and $\mathcal{T}$-homomorphisms between them is denoted by $\mathbf{RT}_{\mathcal{T}}$.

A $\mathbf{\Sigma Pos}_{\mathcal{T}}$-algebra is a $\mathcal{T}$-algebra $(X, \{o_X\}_{o \in \mathcal{O}})$ such that X is a regular Σ-poset, and a $\mathbf{\Sigma Cpo}_{\mathcal{T}}$-algebra is a $\mathcal{T}$-algebra $(X, \{o_X\}_{o \in \mathcal{O}})$ such that X is a Σ-cpo. The $\mathbf{\Sigma Pos}_{\mathcal{T}}$-algebras and $\mathcal{T}$-homomorphisms between them form the category $\mathbf{\Sigma Pos}_{\mathcal{T}}$, and the $\mathbf{\Sigma Cpo}_{\mathcal{T}}$-algebras and $\mathcal{T}$-homomorphisms between them form the category $\mathbf{\Sigma Cpo}_{\mathcal{T}}$.

We remark that the above definition of algebraic theories subsumes the parameterised (in)equational theories of Chapters 4 and 5. This can be seen as follows. Suppose, we are given a parameterised operation $\sigma : P_\sigma \times X^A \to X$. Then its exponential transpose is a map $X^A \to X^{P_\sigma}$. But in $\mathbf{RT}$, giving a map $X^A \to X^{P_\sigma}$ is equivalent to giving a

P_σ-indexed family of maps $\{\sigma_p : X^A \to X\}_{p \in P_\sigma}$. Thus, if Σ is a set of parameterised algebraic operations, then in **RT** we can give an equivalent algebraic theory with a $\coprod_{\sigma \in \Sigma} P_\sigma$-indexed family of operations. We remark that a similar approach to parameterised operations was taken by Strüder in her Diplomthesis [152]. Definition 6.5.1 is strictly more general than the algebraic theories of Chapters 4 and 5, as it allows arbitrary objects of **RT** to be arities of operations.

We furthermore remark that the categories of algebras defined above are instances of a general notion of definable subcategories generalising our definable full subcategories (Definition 6.3.2). More specifically, $\mathbf{RT}_\mathcal{T}$ is not a subcategory of **RT**, but its objects and morphisms of are defined by logical formulae in the internal language of **RT** extended by free type and term variables; the formula for objects is given by:

$$\phi(\underline{X}, \langle \underline{o}_X : \underline{X}^{X^{A_o}} \rangle_{o \in \mathcal{O}}) \equiv \forall e \in \mathcal{E}. \, \forall (x_i)_{i \in A_e} \in \underline{X}^{A_e}. \, t_e((x_i)_{i \in A_e}) = t'_e((x_i)_{i \in A_e}),$$

where the terms t_e, t'_e possibly depend on the free variable $\langle \underline{o}_X \rangle$, and the formula for morphisms is given by:

$$\psi(\underline{X}, \underline{Y}, \langle \underline{o}_X \rangle, \langle \underline{o}_Y \rangle, \underline{h} : \underline{Y}^{\underline{X}}) \equiv \forall o \in \mathcal{O}. \, \forall (x_i) \in \underline{X}^{A_o}. \, h(\underline{o}_X((x_i))) = \underline{o}_Y((h(x_i))).$$

Similarly, for every definable full subcategory (following Definition 6.3.2) $\mathbb{C}$ of **RT**, the categories of $\mathbb{C}_\mathcal{T}$-algebras is an instance of a generalised definability notion, by modifying the formula for algebras appropriately. In particular, this holds for the categories $\mathbf{\Sigma Pos}_\mathcal{T}$ and $\mathbf{\Sigma Cpo}_\mathcal{T}$.

We do not give a formal definition of generalised definable categories here, as the only case we are interested in is that of $\mathcal{T}$-algebras, as defined above. Also, a more general account on definable categories would involve a detailed and careful syntactic construction, contrasting with the topological flavour of the thesis, since in a deeper study of definability one would prefer to work in a fibrational setting. The important observation of this generalised definability of categories of algebras is, that the formulae ϕ, ψ can be transferred to the slice categories $\mathbf{RT}/I$, by reindexing, and thus we get the following definition of families of algebras.

Definition 6.5.2. Let $\phi(\underline{X}, \langle \underline{o}_X : \underline{X}^{X^{A_o}} \rangle_{o \in \mathcal{O}}), \psi(\underline{X}, \underline{Y}, \langle \underline{o}_X \rangle, \langle \underline{o}_Y \rangle, \underline{h} : \underline{Y}^{\underline{X}})$ be formulae defining an algebraic theory $\mathcal{T}$ in **RT**. Then an I-indexed family of algebras is a tuple of I-indexed families $(\{X_i\}_{i \in I}, \{\{o_{X_i}\}_{o \in \mathcal{O}}\}_{i \in I})$ such that:

$$\mathbf{RT} \models \forall i \in I. \, \phi(X_i, \langle o_{X_i} \rangle_{o \in \mathcal{O}}),$$

so for each $i \in I$, the tuple $(X_i, \{o_{X_i}\}_{o \in \mathcal{O}})$ is a $\mathcal{T}$-algebra. Similarly, an I-indexed family of homomorphisms $(\{X_i\}_{i \in I}, \{\{o_{X_i}\}_{o \in \mathcal{O}}\}_{i \in I}) \to (\{Y_i\}_{i \in I}, \{\{o_{Y_i}\}_{o \in \mathcal{O}}\}_{i \in I})$ is given by an

I-indexed family of morphisms $\{h_i : X_i \to Y_i\}$ such that:

$$\mathbf{RT} \models \forall i \in I.\, \psi(X_i, Y_i, \langle o_{X_i} \rangle, \langle o_{Y_i} \rangle, h_i),$$

i.e. for each $i \in I$, the map $h_i : X_i \to Y_i$ is a homomorphism $(X_i, \{o_{X_i}\}_{o \in \mathcal{O}}) \to (Y_i, \{o_{Y_i}\}_{o \in \mathcal{O}})$.

Analogously, we define families of $\mathbf{\Sigma Pos}_{\mathcal{T}}$-algebras and families of $\mathbf{\Sigma Cpo}_{\mathcal{T}}$-algebras, as those families that in addition satisfy the formulae for families of Σ-posets, respectively Σ-cpos.

The next lemma gives the completeness properties of categories.

Lemma 6.5.3. *The forgetful functor* $U : \mathbf{\Sigma Pos}_{\mathcal{T}} \to \mathbf{\Sigma Pos}$*, mapping a* $\mathbf{\Sigma Pos}_{\mathcal{T}}$*-algebra* $(X, \{o_X\}_{o \in \mathcal{O}})$ *to its underlying* Σ*-poset* X *and a homomorphism* $(X, \{o_X\}_{o \in \mathcal{O}}) \to (Y, \{o_Y\}_{o \in \mathcal{O}})$ *to itself, created limits (that is finite limits and limits indexed by objects of* $\mathbf{RT}$*).*

The same holds for $\mathbf{\Sigma Cpo}_{\mathcal{T}}$*.*

Proof. We construct the equalizer of $f, g : (X, \{o_X\}_{o \in \mathcal{O}}) \to (Y, \{o_Y\}_{o \in \mathcal{O}})$ in $\mathbf{\Sigma Pos}_{\mathcal{T}}$. It is given by $e : (E, \{o_E\}_{o \in \mathcal{O}}) \to (X, \{o_X\}_{o \in \mathcal{O}})$, where $e : E \to X$ is the equalizer of $f, g : X \to Y$ in $\mathbf{\Sigma Pos}$, and the actions of the operations $o_E \in \mathcal{O}$ are inherited from the operations $o_X \in \mathcal{O}$ in the usual way, i.e.

$$o_E((x_i)_{i \in A_o}) := \mathbf{the}\ y \in E\ \mathbf{such\ that}\ o_X((e(x_i))_{i \in A_o}) = e(y)$$

following the notation of [105].

Similarly, we construct binary products pointwise, and the terminal object always carries a trivial algebra structure $(\mathbf{1}, \{!\}_{o \in \mathcal{O}})$, showing that $\mathbf{\Sigma Pos}_{\mathcal{T}}$ has indeed finite limits. (Similar to the proof of Theorem 6.3.6, one can show that this structure carries over to the fibres, giving the families of $\mathbf{\Sigma Pos}_{\mathcal{T}}$ finite indexed limits).

Finally, we construct the product of an I-indexed family $(\{X_i\}_{i \in I}, \{o_{X_i}\}_{o \in \mathcal{O}}\}_{i \in I})$ of $\mathbf{\Sigma Pos}_{\mathcal{T}}$-algebras. It is given by $(\prod_{i \in I} X_i, \{\prod_{i \in I} o_{X_i}\}_{o \in \mathcal{O}})$, which is a $\mathcal{T}$-algebra, because it holds that $\prod_{i \in I}(X_i^{A_o}) \cong (\prod_{i \in I} X_i)^{A_o}$. The projections are inherited from $\mathbf{\Sigma Pos}$, and easily shown to be homomorphisms. It follows that $\mathbf{\Sigma Pos}_{\mathcal{T}}$ inherits $\mathbf{RT}$-completeness from $\mathbf{\Sigma Pos}$.

For $\mathbf{\Sigma Pos}_{\mathcal{T}}$ being closed under isomorphisms, let $(X, \{o_X\}_{o \in \mathcal{O}})$ be a $\mathbf{\Sigma Pos}_{\mathcal{T}}$-algebra and $i : X \to Y$ be an isomorphism in $\mathbf{\Sigma Pos}$. Then the operations o_Y are obtained as the composites:

$$Y^{A_o} \xrightarrow{(i^{-1})^{A_o}} X^{A_o} \xrightarrow{o_X} X \xrightarrow{i} Y$$

giving the required $\mathcal{T}$-algebra $(Y, \{o_Y\}_{o \in \mathcal{O}})$. Clearly, $i : (X, \{o_X\}_{o \in \mathcal{O}}) \to (Y, \{o_Y\}_{o \in \mathcal{O}})$ becomes an isomorphism in $\mathbf{\Sigma Pos}_{\mathcal{T}}$. $\qquad \square$

This allows us to show our main result about free algebras.

Theorem 6.5.4. *Let $\mathcal{T}$ be an equational algebraic theory for* **RT**. *Then the following hold:*

(i) There exists a left adjoint for the forgetful functor $U : \Sigma\mathbf{Pos}_{\mathcal{T}} \to \Sigma\mathbf{Pos}$.

(ii) There exists a left adjoint for the forgetful functor $U : \Sigma\mathbf{Cpo}_{\mathcal{T}} \to \Sigma\mathbf{Cpo}$.

Proof. Again we show (i), and then (ii) follows essentially from the same argument. We prove the statement in the internal logic of **RT**, our argument follows along the lines of Freyd's Adjoint Functor Theorem. Let $\phi(\underline{X}, \langle \underline{o}_X \rangle), \psi(\underline{h})$ be the formulae defining $\mathcal{T}$-algebras and $\mathcal{T}$-homomorphisms (leaving out some of the variables and subscripts for readability). Also, recall that by Lemma 6.5.3 above, $\Sigma\mathbf{Pos}_{\mathcal{T}}$ has finite limits and **RT**-products, and the forgetful functor $U : \Sigma\mathbf{Pos}_{\mathcal{T}} \to \Sigma\mathbf{Pos}$ preserves them.

Next, we construct the solution set for U at a Σ-poset X. For this let $\{C_j\}_{j \in J}$ be a family of Σ-posets, making $\Sigma\mathbf{Pos}$ weakly small, along the lines of Theorem 6.3.12. We define the set K as:

$$K := \{(C_j, \{o_{C_j}\}_{o \in \mathcal{O}}) \mid j \in J \wedge \phi(C_j, \langle o_{C_j} \rangle)\},$$

By local smallness of $\Sigma\mathbf{Pos}$, for every $j \in J$, there is only a set of possible families of operations $\{o_{C_j}\}_{o \in \mathcal{O}}$, making $(C_j, \{o_{C_j}\}_{o \in \mathcal{O}})$ into a $\mathcal{T}$-algebra. Thus, K can indeed be defined as an object in **RT**, and we get K-indexed families $\{C_{j_k}\}_{k \in K}, \{\{o_{C_{j_k}}\}_{o \in \mathcal{O}}\}_{k \in K}$ which form a K-indexed family of algebras. Since $\Sigma\mathbf{Pos}_{\mathcal{T}}$ is closed under isomorphisms of $\Sigma\mathbf{Pos}$, by Lemma 6.5.3, we now get that for any $\Sigma\mathbf{Pos}_{\mathcal{T}}$-algebra $(Y, \{o_Y\}_{o \in \mathcal{O}})$, it holds that:

$$\mathbf{RT} \models \exists k \in K. \, \exists i : (C_{j_k}, \{o_{C_{j_k}}\}_{o \in \mathcal{O}}) \cong (Y, \{o_Y\}_{o \in \mathcal{O}}).$$

(In particular, i is an isomorphism and a homomorphism; both of these properties can be defined in the internal language.)

Thus, the solution set of U at X is given by:

$$Q := \{f : X \to U((C_{j_k}, \{o_{C_{j_k}}\}_{o \in \mathcal{O}})) \mid k \in K\};$$

whenever $(Y, \{o_Y\}_{o \in \mathcal{O}})$ is a $\Sigma\mathbf{Pos}_{\mathcal{T}}$-algebra and $g : X \to Y$ is a morphism in $\Sigma\mathbf{Pos}$, then:

$$\mathbf{RT} \models \exists q \in Q. \, \exists i : (C_q, \{o_{C_q}\}_{o \in \mathcal{O}}) \cong (Y, \{o_Y\}_{o \in \mathcal{O}}) \wedge i \circ f_q = g.$$

Now the argument of the standard Adjoint Functor Theorem can be applied for showing that U has a left adjoint F. In particular, FX is given by the subobject of $\prod_{f \in Q}(C_{j_k}, \{o_{C_{j_k}}\}_{o \in \mathcal{O}})$ on those elements of the product which are invariant under all

endomorphisms

$$h : \prod_{f \in Q} (C_q, \{o_{C_q}\}_{o \in \mathcal{O}}) \to \prod_{f \in Q} (C_q, \{o_{C_q}\}_{o \in \mathcal{O}}),$$

for which it holds that $h \circ \langle f \rangle_{f \in Q} = \langle f \rangle_{f \in Q}$. That this product and its subobject can be defined as algebras follows from the completeness result of Lemma 6.5.3. Internally, it then holds that there exists a natural isomorphism:

$$\mathbf{\Sigma Pos}_{\mathcal{T}}(FX, (Y, \{o_Y\}_{o \in \mathcal{O}})) \cong \mathbf{\Sigma Pos}(X, Y),$$

which can be expressed appropriately in the internal language as both $\mathbf{\Sigma Pos}$ and $\mathbf{\Sigma Pos}_{\mathcal{T}}$ are $\mathbf{RT}$-enriched. This completes the proof. $\qquad\square$

We remark that in this proof we have glossed over one important fact, namely that the definition of families of algebras (Definition 6.5.2) and, consequently, the construction of $\mathbf{RT}$-products in $\mathbf{\Sigma Pos}_{\mathcal{T}}$ (Lemma 6.5.3) are the "correct ones". This crucially depends on constructing the "correct" fibration $Fam(\mathbf{\Sigma Pos}_{\mathcal{T}}) \to \mathbf{RT}$ of families of algebras, which follows from a suitable generalisation of a notion of definable categories (Definition 6.3.2). However, we know that the constructions of Lemma 6.5.3 are the "correct" ones for Theorem 6.5.4, since they are inherited from $\mathbf{\Sigma Pos}$, and this is what makes the proof work.

Of course, it would be better to have a fully formal treatment of definable categories here, but this would take us too far from the main topological core of the thesis. Our arguments demonstrate the power of the synthetic approach, and we leave it to future work to provide a fully self-contained treatment of definable categories and free algebra constructions.

If we use the equivalence of $\mathbf{QCB}_0$ and $\mathbf{\Sigma Pos}$, respectively $\mathbf{TP}$ and $\mathbf{\Sigma Cpo}$, then Theorem 6.5.4 shows the existence of free algebra functors in Topological Domain Theory for a much wider class of algebraic theories than Theorems 4.4.3 and 5.1.5 did. Here again the question arises whether there is an elementary topological characterisation of free algebras for a wider class of algebraic theories than parameterised equational ones, see again the discussion in Chapter 7.2.

Finally, let us remark that we could have also embedded $\mathbf{QCB}_0$ and $\mathbf{TP}$ into the realizability topos over Kleene's second algebra $\mathbf{K}_2$, and obtained very similar results. For more details we refer again to Bauer's thesis [12].

7 Conclusions

We conclude the thesis by summing up the achievements of Topological Domain Theory, emphasising our contributions to it, and giving an overview of the open problems and future work in this research programme.

7.1 Achievements

We have shown that Topological Domain Theory is a flexible toolkit for denotational semantics, satisfying all the demands given in the introduction. It provides a family of closely related categories which are able to interpret all the usual type constructions of (functional) programming languages. Furthermore, it can model recursion and a wide variety of computational effects, and has close connections to established mathematical frameworks.

In particular, the categories of Topological Domain Theory are **QCB**, **QCB**$_0$, **TP**, **TD** and **TD**$_\perp$, related as in Figure 7.1 below:

- **QCB** is the category of topological quotients of countably-based topological spaces and continuous maps between them (Definition 2.1.10). It is equivalent to **PQ**, the largest full topological subcategory of Scott's category of ω-equilogical spaces which includes all countably-based spaces (Definition 2.1.6, Proposition 2.1.7 and Theorem 2.1.11).

- **QCB**$_0$ is the full subcategory of **QCB** of the T_0-quotients of countably-based spaces (Definition 2.1.10), and there is a reflection functor **QCB** $\to$ **QCB**$_0$, given by the well-known T_0-reflection of topological spaces.

- **TP** is the category of continuous maps between topological predomains, which are qcb-spaces that are also monotone convergence spaces (Definition 3.2.2). It is a full reflective subcategory of **QCB** and **QCB**$_0$, with the reflection being given by the monotone convergence reflection of topological spaces (Theorem 3.2.5).

- **TD** is the category of continuous maps between topological domains, which are topological predomains whose specialization order has a least element (Definition 3.2.7). It is a full subcategory of **TP**.
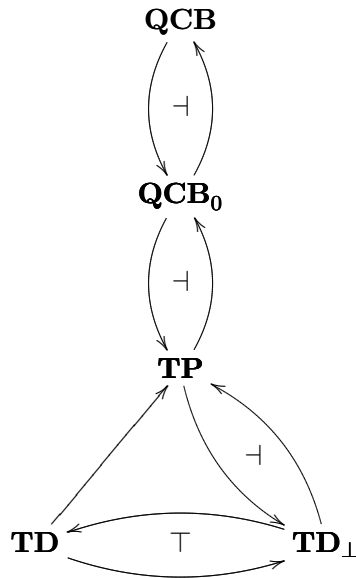
$$\begin{array}{c} \textbf{QCB} \\ \dashv \\ \textbf{QCB}_0 \\ \dashv \\ \textbf{TP} \\ \top \\ \textbf{TD} \qquad \textbf{TD}_\perp \end{array}$$

Figure 7.1: The categories of Topological Domain Theory

- $\textbf{TD}_\perp$ is the category of topological domains and strict maps between them, i.e. maps which preserve the least element. There exists a functor $\textbf{TP} \to \textbf{TD}_\perp$, analogous to the lifting functor which is well-known from Classical Domain Theory, and this functor is left adjoint to the inclusion $\textbf{TD}_\perp \to \textbf{TP}$ (Lemma 3.2.12). This adjunction restricts to an adjunction $(\textbf{TD} \to \textbf{TD}_\perp) \dashv (\textbf{TD}_\perp \to \textbf{TD})$ (Lemma 3.2.12).

The demands on a framework for denotational semantics, given in the introduction, are satisfied by Topological Domain Theory as follows.

Type Constructions: The type constructions of the simply-typed λ-calculus with product types can be interpreted in $\textbf{QCB}$, $\textbf{QCB}_0$, $\textbf{TP}$ and $\textbf{TD}$ in the usual way, as all these categories are cartesian-closed and have countable limits (Theorems 2.1.12, 3.2.5 and 3.2.8). Moreover, a strict λ-calculus with tensor products can be modelled in $\textbf{TD}_\perp$, which is symmetric monoidal closed (Theorem 3.2.16). Sum types can be modelled in $\textbf{QCB}$, $\textbf{QCB}_0$, $\textbf{TP}$ and $\textbf{TD}_\perp$, using the fact that they have countable coproducts (Theorems 2.1.12, 3.2.5 and Proposition 3.2.17). In $\textbf{TD}$ coalesced sums exist, given by the coproduct of $\textbf{TD}_\perp$, analogous to the case in Classical Domain Theory.

For modelling recursive types in Topological Domain Theory, Theorem 3.2.20 and its Corollary 3.2.21 can be used, which show that $\textbf{TD}_\perp$ is $\omega\textbf{CPO}$-algebraically compact, and thus has solutions for recursive domain equations given by $\omega\textbf{CPO}$-enriched bifunctors.

In order to interpret polymorphic types in Topological Domain Theory, we have embedded our categories into the realizability topos over Scott's graph model $\mathcal{P}\omega$. The result is that $\textbf{QCB}_0$ is equivalent to the category $\boldsymbol{\Sigma}\textbf{Pos}$, of Σ-posets (Theorem 6.2.5 and

Corollary 6.2.6), and **TP** is equivalent to the category $\mathbf{\Sigma Cpo}$, of Σ-cpos (Definition 6.2.8 and Corollary 6.2.9). Both these categories, $\mathbf{\Sigma Pos}$ and $\mathbf{\Sigma Cpo}$, are internally complete and weakly small in the realizability topos (Corollary 6.3.13), and they are models for the polymorphic λ-calculus (Corollary 6.4.1). This shows that polymorphic type constructors exist in the categories $\mathbf{QCB}_0$ and **TP**.

Computational phenomena: The fixed-point theorem of Classical Domain Theory can be applied in the category **TD** (Proposition 3.2.9), and hence recursion can be interpreted in Topological Domain Theory. Naturally, we cannot model recursion in the categories **QCB**, $\mathbf{QCB}_0$ and **TP**, because all these categories include the two-point discrete space **2**, which has an obvious endofunction without fixed-point. Thus, concerning recursion, the situation in Topological Domain Theory is as in Classical Domain Theory.

Computational effects have been modelled using a free algebra approach, based on the work of Plotkin and Power [112]. In particular, we have introduced parameterised equational theories as an appropriate class of algebraic theories for modelling effects (Definitions 4.1.2 and 4.1.3). We have then shown that **QCB** is closed under the construction of free algebras for ω-ary qcb-parameterised equational theories in the category of sequential spaces, where a free algebra functor was obtained using Freyd's Adjoint Functor Theorem (Theorems 4.2.1 and 4.4.3). As the T_0-reflection $\mathbf{QCB} \to \mathbf{QCB}_0$ preserves countable products, a free algebra construction in the category $\mathbf{QCB}_0$ can be obtained along the lines of Theorem 5.1.3. Thus, we can model a wide range of computational effects, including nontermination, nondeterminism, exceptions, input/output and side-effects, in **QCB** and $\mathbf{QCB}_0$.

Using a result by Schröder and Simpson, who have proved that also the reflection $\mathbf{QCB} \to \mathbf{TP}$ preserves countable products, we have shown that free algebras for ω-ary tp-parameterised equational theories exist in the category of topological predomains (Theorem 5.1.5). A simple equational theory was used to show that the category of topological domains and continuous maps does not have a free algebra functor in general. However, we proved that the category of topological domains and strict maps supports a free algebra construction for ω-ary tp-parameterised equational theories (Corollary 5.2.3). It follows that the categories **TP** and $\mathbf{TD}_\perp$ provide models for functional programming languages with computational effects.

By embedding the categories $\mathbf{QCB}_0$ and **TP** into the realizability topos, we could give an abstract proof of the existence of free algebra functors for a very general class of algebraic theories, subsuming the parameterised equational theories of Chapters 4 and 5 (Theorem 6.5.4). In particular, in the topological approach, we demanded the operations of parameterised equational theories to have arities given by (classical) sets. However, the synthetic algebraic theories of Definition 6.5.1 allow arbitrary objects of **QCB** as arities; for instance we can allow the Euclidean real numbers to be the arity

of values in the equational theory describing side-effects. Moreover, the countability conditions on the parameterised equational theories could be dropped in the synthetic case.

However, the main advantage of the free algebra construction in Chapters 4 and 5 was the intuitive characterisation of free algebras, using an inductive construction (Proposition 4.2.2 and Theorem 4.2.4). This intuition is lost in the abstract proof of Theorem 6.5.4, where an internal adjoint functor theorem was used instead. Furthermore, the topological characterisation in Chapters 4 and 5 allowed us to prove the coincidence of the free algebra constructions in Topological and Classical Domain Theory for a wide range of algebraic theories (Theorem 5.3.2).

Underlying mathematics: Naturally, a well-established underlying mathematical theory for Topological Domain Theory is given by topology. The categories **QCB**, **QCB**$_0$, **TP** and **TD** are all categories of continuous maps between special classes of topological spaces, hence they are full subcategories of **Top**, whereas **TD**$_\perp$ is a category of special continuous maps between a special class of topological spaces. Moreover, one can argue that, from a topological viewpoint, the categorical constructions of Topological Domain Theory are canonical. The cartesian-closed structure of **QCB** and its subcategories is inherited from wide range of well-studied cartesian-closed subcategories and supercategories of **Top**, such as compactly-generated spaces (Escardó, Lawson and Simpson, see Theorem 2.2.6), sequential spaces (Menni, Simpson and Schröder, see Theorem 2.3.5), Scott's equilogical spaces (Menni and Simpson, see Theorem 2.1.11), and, as recently shown by Schröder [132], the category of filter spaces. The approach of using topological spaces for denotational semantics can be motivated by Smyth's dictionary. The canonicity of the constructions in Topological Domain Theory show that **QCB** and its subcategories lie at the core of any approach towards denotational semantics with function types, following Smyth's ideas.

Furthermore, we have shown that Topological Domain Theory not only subsumes, but genuinely extends Classical Domain Theory as a framework for denotational semantics. In particular, with Lawson's Theorem 2.2.4, it follows that the category of ω-continuous dcpos, which from a computational viewpoint is arguably the largest interesting category in Classical Domain Theory, is a full subcategory of the category of topological predomains. Moreover, the inclusion functor ω**CONT** $\hookrightarrow$ **TP** preserves finite products and countable coproducts, and its restriction to the largest cartesian-closed subcategory of pointed ω-continuous dcpos, Jung's category ω**FS**, preserves the cartesian-closed structure (Lemma 3.3.3, Corollaries 3.3.6 and 3.3.12). We have also shown that the classical powerdomain constructions, including the Jones/Plotkin (sub)probabilistic powerdomain, can be generalised to all topological predomains (Corollary 5.3.3 and Theorem 5.4.9). In this sense, we can say that Classical Domain Theory embeds faithfully into Topological Domain Theory, and there is no loss, when one shifts from Classical Domain

Theory to the framework we propose.

In Chapter 6, we have established Topological Domain Theory as a specific instance of Synthetic Domain Theory. This was done by embedding the categories $\mathbf{QCB}_0$ and $\mathbf{TP}$ into the realizability topos over Scott's graph model $\mathcal{P}\omega$ (Corollaries 6.2.6 and 6.2.9). This made available the internal language of the topos for reasoning about the constructions in Topological Domain Theory. The research in Synthetic Domain Theory has shown that such an internal approach yields powerful applications for reasoning about programming constructs, see [116, 139].

Finally, we mention that Schröder has discovered $\mathbf{QCB}_0$ as an important category in Weihrauch's TTE framework (Theorem 2.4.8). This establishes a connection between Topological Domain Theory and computable analysis, and should make Topological Domain Theory an appealing framework for denotational semantics from a purely mathematical point of view.

7.2 Open Problems and Future Work

Although we have argued that Topological Domain Theory provides a very flexible, and nevertheless accessible, framework for denotational semantics, there are still some open problems in this programme and there are issues that have not been treated yet. In this final section, we give an overview over these unsolved problems and open issues, with hints on how solutions might be found.

Explicit descriptions of synthetic constructions: One of the principal aims of the research programme of Topological Domain Theory has been to give accessible constructions for this powerful toolkit, using very basic principles. Examples of such accessible constructions are the function spaces in Chapter 2 or the construction of free algebras in Chapter 4. However, our most general result, namely that $\mathbf{QCB}_0$ and $\mathbf{TP}$ are internally complete and weakly small, is obtained by changing the underlying mathematical universe to intuitionistic type theory. It follows that our synthetic constructions for polymorphic types and free algebras are very abstract and do not give a direct characterisation of the resulting spaces. It would be desirable to give purely topological constructions corresponding to the synthetic ones.

The main problem in this is that a straightforward externalisation of the internal construction, using fibrations along the lines of [105, 55], is unlikely to yield the desired results. The reason we believe this is that the first step of this externalisation is to investigate the fibration of uniform families of (complete) extensional assemblies over the category of assemblies itself. However, the uniformity of these families does not seem to have a topological counterpart, hence it appears that the fibrational approach is a dead end towards a purely topological characterisation.

One alternative is to start with a topological characterisation, then translate this into

the intuitionistic setting, and show that it satisfies the required properties. For instance, one would expect the topological characterisation of a polymorphic type $\forall X.F(X)$, to be a subspace of an appropriate product construction $\prod_{X \in \mathbf{Q}} F(X)$ in a supercategory of **QCB**, where **Q** denotes a set of isomorphism classes of objects for **QCB**. Furthermore, the corresponding subspace should be characterised by a modification of Reynold's relational parametricity [121]. So far, however, we have not succeeded in finding the right characterisation.

The most interesting generalisation of the synthetic free algebra construction of Theorem 6.5.4, compared to the classical one of Theorem 4.4.3, is that it allows the arities of operations to be arbitrary qcb-spaces instead of just classical sets. We do not know whether one can, analogously to the development in Chapter 4, give an inductive construction of free algebras for such theories in the category of sequential spaces, and then show that this construction preserves qcb-spaces. The main problem is that, in this case, Proposition 4.2.2 cannot be applied anymore. As a result, the categorical properties of **Seq** have to be examined more closely. In particular, the question arises in how far sequential colimits (of certain sizes) commute with the sequential function space construction. We conjecture that if an inductive construction of terms can be obtained, then it is straightforward to generalise the inductive construction of the topology in Theorem 4.2.4 to this case. In particular, if the arity spaces are locally compact, then with Theorem 2.2.11 it should be possible to show that an inductive construction of free sequential algebras preserves qcb-spaces.

Local and Dynamic Programming Features: Modern computational infrastructure, such as multi-processor systems or (large) networks, allows concurrent and distributed computations. Also, modern programming languages often support dynamic programming features. These concepts challenge the classical frameworks for denotational semantics, since a notion for the scope of a program has to be developed. The research in this area, going back to Reynolds [120] and his student Oles [100], suggests that functor categories surpass ordinary categories when it comes to modelling such local or dynamic programming features. More recently, Stark has used presheaf categories for giving denotational semantics for the ν-calculus [147, 148] and, with Cattani and Winskel [21, 149], the π-calculus, see also Fiore, Moggi and Sangiori's [31]. Plotkin and Power have obtained denotational semantics for languages supporting local state with a functor category in [110]. Furthermore, Stark [149] has shown how computational effects can be modelled in the dynamic metalanguage of the π-calculus, using free algebras following then Plotkin/Power approach.

It is an interesting task to adapt Topological Domain Theory to such approaches. Specifically, one has to investigate functor categories over the categories of Topological Domain Theory, to find out in how far these can interpret local and dynamic features. Certainly, the embedding of Topological Domain Theory into the realizability topos

over $\mathcal{P}\omega$ should be helpful for this task, as it gives a wide choice of possible functor categories going beyond ordinary presheaves. It would be an interesting result if there existed a (synthetic) functor category in $\mathbf{RT}(\mathcal{P}\omega)$, which models local and dynamic features, and which also carries (parts of) the structure present in Topological Domain Theory.

Computability: Perhaps the most pressing open issue in Topological Domain Theory is to give a detailed account of computability in this framework. There are two obvious "external" approaches to model computability in **QCB** and its subcategories. The first comes from Weihrauch's TTE, the second from the realizability model over the graph model $\mathcal{P}\omega$.

In Schröder's thesis [129], the first approach was taken, and it was shown that the category $\mathbf{QCB}_{eff}^{TTE}$, of effectively-admissibly represented qcb-spaces and computable maps between them, has good categorical closure properties. In particular, it is cartesian-closed and fits faithfully into Weihrauch's theory of computable analysis [165].

The modern approach to computability in the realizability model over $\mathcal{P}\omega$ is to consider the relative realizability topos $\mathbf{RT}(\mathcal{P}\omega, \mathcal{P}\omega_{r.e.})$, whose objects are arbitrary objects of $\mathbf{RT}(\mathcal{P}\omega)$, but whose morphisms are computable maps, see [12]. With this approach one obtains a category $\mathbf{QCB}_{eff}^{\mathcal{P}\omega}$ of effective qcb-spaces and computable maps, with the computability notion inherited from $\mathbf{RT}(\mathcal{P}\omega, \mathcal{P}\omega_{r.e.})$. The results of [12] show that also this category $\mathbf{QCB}_{eff}^{\mathcal{P}\omega}$ is cartesian-closed.

Weihrauch's TTE is closely related to the realizability model over Baire space $\mathbb{B}$ (also know as the second Kleene-algebra K_2). In particular, it is well-known that computability in TTE is equivalent to computability in $\mathbf{RT}(\mathbb{B}, \mathbb{B}_{r.e.})$, see [84, 12]. Moreover, it is known that $\mathbf{RT}(\mathbb{B}, \mathbb{B}_{r.e.})$ is an (applicative) retract of $\mathbf{RT}(\mathcal{P}\omega, \mathcal{P}\omega_{r.e.})$, see [12]. In particular, one can show that the categories $\mathbf{QCB}_{eff}^{TTE}$ and $\mathbf{QCB}_{eff}^{\mathcal{P}\omega}$ are equivalent.

However, rather than applying the notion of computability of a wider framework to ours, it would be more desirable to have a simple and natural account of computability specifically for Topological Domain Theory. For instance one might hope that a computability structure for qcb-spaces can be derived directly from the countable pseudobases of Theorems 2.2.8 and 2.3.9, similar to Spreen's [146] computability account for countably-based topological spaces. However, so far investigations of this approach have turned out to be fruitless.

Repleteness and a localic version of Topological Domain Theory: In the early 1990s, Hyland [48] and Taylor [153] independently introduced the so-called *replete objects* in Synthetic Domain Theory. These replete objects form a full reflective subcategory **Rep** in a realizability topos; in fact they form the smallest full reflective subcategory, containing the dominance Σ. From the synthetic viewpoint, **Rep** is a very well-behaved and well-studied category allowing numerous type constructions. However, the construction of the reflection into **Rep**, the so-called repletion, is very

complicated. Maybe the best description of this construction to date is Streicher's (transfinite) inductive characterisation [151].

The connection of Topological Domain Theory to Synthetic Domain Theory over Scott's graph model raised hopes that at least for this specific realizability model a more intuitive construction of the repletion could be obtained. More specifically, the characterisation of repleteness in a realizability topos is very similar to that of sobriety of a topological space. Thus, the hope was that for realizability over $\mathcal{P}\omega$ repletion for an extensional assembly is simply given by the sobrification of the corresponding qcb-space. However, Gruenhage and Streicher [38] have given a counterexample for this, showing that **QCB** is not closed under the topological sobrification. Hence, giving an explicit characterisation of repletion remains an open problem.

An open question related to the repleteness problem is the following. When reasoning about the behaviour of a computer program, the observations that are performed on a program run are the principal subjects of investigation, and not the raw data. Employing Smyth's dictionary this means that the open subsets of a topological space are the most important entity when it comes to semantic reasoning about operational properties. This leads to the mathematical field of locale theory [161], which consequently has found some interest in the theoretical computer science community [1, 68, 67, 70, 99]. The underlying observation of locale theory is the Stone-type duality between the category of sober topological spaces and the category of spatial locales (a locale being a special kind of lattice), see [57]. The lattice structures of the locales yield a framework for developing program logics for the interpreted programming language.

In this regard, an open problem for Topological Domain Theory is whether there exists a localic version of **QCB** and its subcategories, for example whether one can give a characterisation of the class of locales which arise as lattices of open subsets of qcb-spaces. Should that be the case, one might hope for a Stone-type duality between the category of replete qcb-spaces and the corresponding category of locales, which can be used to give means for logical reasoning about program behaviour. For Classical Domain Theory such a logical framework has been given by Abramsky [1], and one would hope to generalise his approach for a localic version of Topological Domain Theory.

We remark that there might be other ways to obtain a logical framework for reasoning about program behaviour in Topological Domain Theory. The connection to the realizability topos **RT**$(\mathcal{P}\omega)$, as given in Chapter 6, might serve as another starting point.

A metalanguage for Topological Domain Theory: After we have seen that Topological Domain Theory provides a very flexible framework for denotational semantics, which may serve as a basis for giving denotational semantics for a wide range of existing programming (meta)languages, it appears to be an interesting task to con-

struct a specific metalanguage for this framework, matching the categorical constructions shown above. The starting point of such a programme is to choose which of the categories of Topological Domain Theory should serve as a model for the metalanguage. Naturally, the categories **QCB**, **QCB**$_0$ and **TP** provide rich classes of basic objects, but they are not able to model recursion. On the other hand, the category **TD** is not suitable for modelling computational effects, as Chapter 5.2 has shown, and the treatment of recursion in **TD**$_\perp$ is rather uninteresting, as every strict map has a trivial fixed point.

Thus, we propose to use the adjunction between **TD** and **TD**$_\perp$ for a dually typed metalanguage for Topological Domain Theory. Examples of existing dually typed languages are Barber's type theory for dual intuitionistic linear logic [3], or Levy's Call-by-Push-Value [82]. An extension for Barber's calculus to include parametric polymorphism and recursion has been given Bierman, Pitts and Russo [17]. Semantic models for this extension have been obtained by Birkedal, Møgelberg and Pedersen [18, 19] and Rosolini and Simpson [126]. Levy [82] on the other hand uses a type theory which distinguishes between value types and computational types, and his Call-by-Push-Value is able to model computational effects. A modification of his type theory, to include parametric polymorphism, has recently been suggested by Møgelberg and Simpson [94, 93].

Our results indicate that the categories **TD** and **TD**$_\perp$ are able to model effects and polymorphism, and the strictness of the function spaces in **TD**$_\perp$ is a special form of linearity; after all **TD**$_\perp$ is a category of algebras and homomorphisms over **TP**. Thus we propose to adapt the dually typed metalanguages above to our setting, including polymorphism and effects. Possible further inputs for the strict setting might be Momigliano's strict λ-calculus [98], and Rosolini and Simpson's polymorphic calculus [126], based on strictness.

An interesting possibility in providing a metalanguage for Topological Domain Theory, is to incorporate the construction of effects as free algebras for operationally given algebraic theories. This should make the metalanguage very flexible, when it comes to incorporating computational effects. The difficulty in this task is to find an elegant and useful metalanguage, combining all the features modelled in Topological Domain Theory.

7 Conclusions

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