Concrete Domains

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Abstract

This paper introduces the theory of a particular kind of computation domains called *concrete domains*. The purpose of this theory is to find a satisfactory framework for the notions of coroutine computation and sequentiality of evaluation.

> Diagrams are emphasized because I believe
> that an important part of learning lattice theory
> is the acquisition of skill in drawing diagrams.
> *George Grätzer*

1 Domains of computation

In general, we follow Scott's approach [Sco70]. To every syntactic object one associates a semantic object which is found in an appropriate semantic domain. For technical details, we follow [Mil73] and [Plo78] rather than Scott.

Definition 1.1 *A* partial order *is a pair* $< D; \leq >$ *where* D *is a non-empty set and* $\leq$ *is a binary relation satisfying:*

i) $\forall x \in D \qquad x \leq x$ *(reflexivity)*

ii) $\forall x, y \in D \qquad x \leq y,\ y \leq x \Rightarrow x = y$ *(antisymmetry)*

iii) $\forall x, y, z \in D \qquad x \leq y,\ y \leq z \Rightarrow x \leq z$ *(transitivity)*

One writes $x < y$ when $x \leq y$ and $x \neq y$. Two elements x and y are *comparable* when either $x \leq y$ or $y \leq x$. When this is not the case, the elements x and y are *incomparable* and this relation is written $x \parallel y$. A partial order in which any two elements are comparable is a *chain*.

Usual terms: In a partial order $< D; \leq >$, let H be a subset of D and x an element of H. The element x is an *upper bound* of H iff $\forall y \in H\ y \leq x$. It is a *lower bound* of H iff $\forall y \in H\ x \leq y$. It is a *least upper bound (lub)* of H iff it is an upper bound of H and

$$\forall z \text{ upper bound of } H \qquad x \leq z$$

It is a *greatest lower bound (glb)* of H iff it is a lower bound of H and

$$\forall z \text{ lower bound of } H \qquad z \leq x$$

When x is a lub (resp. glb) of H, one writes $x = \bigcup H$ (resp. $x = \bigcap H$). If $H = \{a, b\}$, these notations are shortened to $x = a \vee b$ and $x = a \wedge b$ respectively.

Two elements x and y in D are *compatible* if $\{x, y\}$ has an upper bound. This relation is noted $x \uparrow y$, and its complement, the *incompatibility* relation, is written $x \# y$.

An element x in H is a *maximum* iff $x = \bigcup H$. It is a *minimum* iff $x = \bigcap H$.

Definition 1.2 *In a partial order* $< D; \leq >$ *a subset* X *of* D *is* directed *iff* X *is non-empty and*

$$\forall x_1, x_2 \in X \quad \exists x_3 \in X : \ x_1 \leq x_3,\ x_2 \leq x_3$$

Remark: By definition the set which is the support of a chain is *a fortiori* directed.

Definition 1.3 *A partial order* $< D; \leq >$ *is* complete *iff*

i) D has a minimum element $\perp$

ii) Any directed subset X of D has a least upper bound

Definition 1.4 *A partial order* $< D; \leq >$ *is* conditionally complete *iff any subset X of D that has an upper bound has a least upper bound.*

Remarks:

i) Since D is non-empty, the empty set $\emptyset$ has an upper bound. Hence if $< D; \leq >$ is conditionally complete, D must have a minimum element $\perp = \bigcup \emptyset$

ii) The terminology used here, although standard, may not be ideal since a partial order may be complete without being conditionally complete.

Proposition 1.1 *A complete partial order* $< D; \leq >$ *is conditionally complete iff every* pair *of compatible elements* $< x, y >$ *has a least upper bound* $x \vee y$.

Proof: Consider a complete partial order $< D; \leq >$ in which every pair of compatible elements has a least upper bound and let X be a bounded subset of D. If $X = \emptyset$ then $\bigcup X = \perp$. If X is reduced to a single element x, this x is the least upper bound of X. If X contains exactly two elements x and y, and has an upper bound, then x and y are compatible and $\bigcup X = x \vee y$.

Consider now a finite subset X of D that has an upper bound, with $|X| \geq 2$ and $X = X' \dot{\cup} \{x\}$. Since X has an upper bound, so does X' which has, by induction hypothesis, a least upper bound $\bigcup X'$. As any upper bound of X must dominate both $\bigcup X'$ and x, these elements must be compatible and hence $\bigcup X = \bigcup X' \vee x$. Now if X is infinite, let Y be the set of least upper bounds of its finite subsets. The set Y is directed, so it has a least upper bound $\bigcup Y$. For any x in X, $x \leq \bigcup Y$ since $\{x\}$ is a finite subset for which $\bigcup Y$ is an upper bound. Since any upper bound of X must at least dominate $\bigcup Y$ we obtain

$$\bigcup X = \bigcup Y$$

The converse is trivial. $\Box$

Proposition 1.2 *In a conditionally complete partial order $< D; \leq >$, any non-empty subset X of D has a greatest lower bound $\bigcap X$.*

Proof: Let Y be the set of elements in D dominated by X. Since X is non-empty, some x in X dominates Y. Thus Y has a lub $\bigcup Y$. For any x in X it is the case that $\forall y \in Y \ y \leq x$ hence also $\bigcup Y \leq x$. So $\bigcup Y$ is a lower bound of X, and $\bigcup Y = \bigcap X$. □

Definition 1.5 *In a partial order $< D; \leq >$ a subset X of D is* consistent *iff any two elements in X are compatible.*

Definition 1.6 *A partial order $< D; \leq >$ is* coherent *iff any consistent subset X of D has least upper bound.*

Remarks:

1. A subset that has an upper bound is consistent. Hence if a partial order is coherent it is *a fortiori* conditionally complete.

2. The empty set $\emptyset$ is consistent. Hence it has a least upper bound $\perp$. A directed set is consistent. Hence if a partial order is coherent it is *a fortiori* complete.

Proposition 1.3 *A complete partial order $< D; \leq >$ is coherent iff any consistent* triple $< x, y, z >$ *has a least upper bound.*

Proof: Any consistent X that has at most 3 elements obviously has a least upper bound. Now consider a consistent finite subset $X = \{x_1, x_2, \ldots, x_n\}$ of D such that $|X| = n \geq 3$. Assume, by induction hypothesis, that any consistent subset Y such that $1 \leq |Y| < n$ has a lub. Now the set $\{x_1 \vee x_2, x_2 \vee x_3, \ldots, x_{n-2} \vee x_{n-1}, x_n\}$ contains at most $n-1$ elements. Any two elements in it are compatible, because

i) if both are of the form $x_i \vee x_{i+1}$, they are dominated by $\bigcup\{x_1, x_2, \ldots, x_{n-1}\}$, which exists by induction hypothesis.

ii) $x_i \vee x_{i+1}$ and x_n are compatible since the triple $\{x_i, x_{i+1}, x_n\}$ is consistent and thus admits a lub.

Consequently, using again the induction hypothesis, the set X has a lub. If now X is infinite, the set Y of the lubs of the finite subsets of X is a directed set and we have $\bigcup X = \bigcup Y$. □

Definition 1.7 *In a partial order* $< D; \leq >$*, an element* x *is* isolated *(or* compact*) iff in any directed set with a lub that dominates* x *one can find an element* y *that dominates* x*. In symbols:*

$$\forall X \subset D,\ X \text{ directed} \quad x \leq \bigcup X \Rightarrow \exists y \in X\ x \leq y$$

Notation: The set of isolated elements less than x is noted $\mathcal{A}(x)$. An element in $\mathcal{A}(x)$ is called an approximant of x. The set of all isolated elements in $< D; \leq >$ is written $\mathcal{A}(D)$.
Remark: An element x is isolated iff $x \in \mathcal{A}(x)$. Hence $\mathcal{A}(D) = \bigcup_{x \in D} \mathcal{A}(x)$

Proposition 1.4 *In a conditionally complete partial order* $< D; \leq >$

i) If two isolated elements a *and* b *are compatible then* $a \vee b$ *is isolated.*

ii) For any x*, the set* $\mathcal{A}(x)$ *is directed.*

Proof:

i) Since a and b are compatible, their lub $a \vee b$ exists. Consider now a directed set S such that $a \vee b \leq \bigcup S$. Since a and b are isolated, from $a \leq \bigcup S$ and $b \leq \bigcup S$ we deduce that there are two elements a' and b' in S with $a \leq a'$ and $b \leq b'$. Since S is directed, there is a c in S with $a' \leq c$ and $b' \leq c$ hence $a \leq c$ and $b \leq c$ and thus $a \vee b \leq c$. Hence $a \vee b$ is isolated.

ii) If a and b are two approximants of x, the element $a \vee b$ is isolated by i) and dominated by x, thus it is also an approximant of x. Hence $\mathcal{A}(x)$ is directed. □

Definition 1.8 *A partial order* $< D; \leq >$ *is* algebraic *iff for any* x *in* D *the set* $\mathcal{A}(x)$ *is directed and*

$$x = \bigcup \mathcal{A}(x)$$

If additionally $\mathcal{A}(D)$ *is denumerable,* $< D; \leq >$ *is* ω–algebraic*.*

Definition 1.9 *We will call* computation domain *a coherent and* ω*–algebraic partial order.*

Notation From now on we abandon the precise notation $< D; \leq >$. We merely use the same letter for the set and the partial order, unless more precision becomes necessary.

Lemma 1.1 *In a computation domain* $x \leq y \Leftrightarrow \mathcal{A}(x) \subset \mathcal{A}(y)$*.*

Proof: From left to right the implication is immediate. Conversely, since $\mathcal{A}(x)$ and $\mathcal{A}(y)$ are directed they have lubs that verify $\bigcup \mathcal{A}(x) \leq \bigcup \mathcal{A}(y)$ and by algebraicity we deduce $\bigcup \mathcal{A}(x) = x \leq y = \bigcup \mathcal{A}(y)$. □

Corollary 1.1 *In a computation domain, if x is isolated and $x < y$ then there is an approximant z of y with $x < z \leq y$.*

Proof: Let t be an element of the necessarily non empty set $\mathcal{A}(y) \backslash \mathcal{A}(x)$. Since x and t are both approximants of y, so is $x \vee t$. Taking $z = x \vee t$, we have $x < z \leq y$. □

Corollary 1.2 *If an element y in a computation domain is not isolated, then one can find an infinite strictly increasing chain of isolated elements $\{\bot, x_1, x_2, \ldots, x_n, \ldots\}$ approximating y, i.e. with*

$$\bot < x_1 < x_2 < \cdots < x_n < \cdots < y$$

Proof: The minimum element $\bot$ is isolated and we have $\bot < y$. Now assume that we have a chain $\{\bot, x_1, x_2, \ldots, x_{n-1}\}$ of n isolated elements such that

$$\bot < x_1 < x_2 < \cdots < x_{n-1} < y$$

Since x_{n-1} is isolated, one can find by the previous Corollary an isolated element x_n with $x_{n-1} < x_n \leq y$. But since y is not isolated, certainly $x_n < y$ and the chain has been extended to contain $n + 1$ elements. □

Proposition 1.5 *The cartesian product of a countable number of computation domains is a computation domain.*

Proof: Let α be an ordinal, $1 \leq \alpha \leq \omega$ and $\{< D_i; \leq_i >\}_{i<\alpha}$ a family of computation domains. An element x in $D = \prod_{i<\alpha} D_i$ is a vector $< x_0, x_1, \ldots, x_i, \ldots >$. The set D inherits the relation $\leq$ defined componentwise:

$$\forall x, y \in D \quad x \leq y \iff \forall i < \alpha \ x_i \leq y_i$$

Two elements in D are compatible iff they are compatible componentwise. Indeed, if x and y are compatible, there exists z with $x \leq z$ and $y \leq z$ hence $\forall i \ x_i \leq_i z_i$ and $\forall i \ y_i \leq_i z_i$, so x and y are compatible componentwise. Conversely, if $\forall i \, \exists z_i \ x_i \leq_i z_i, y_i \leq_i z_i$, the vector $z =< z_0, z_1, \ldots, z_i, \ldots >$ dominates x and y which are thus compatible. Similarly, if $x \uparrow y$ we have $x \vee y =< x_0 \vee y_0, \ldots, x_i \vee y_i, \ldots >$. A subset X of D is consistent iff it is

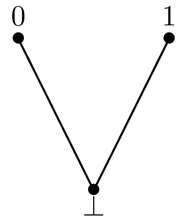


Figure 1: The domain T

consistent componentwise. Hence if each of the partial orders $< D_i; \leq_i >$ is coherent, so is $< D; \leq >$.

Let us prove now that $< D; \leq >$ is ω-algebraic. Consider the subset of D defined by

$$I = \bigcup_{i<\alpha} \{x | x_i \in \mathcal{A}(D_i) \text{ and } \forall j < \alpha, j \neq i, x_j = \perp_{D_j}\}$$

The elements of I are vectors all components of which are the minimum element in the relevant domain, except possibly for the i-th component which is an isolated element in D_i. Any element in I is isolated in D. Indeed, let X be a directed subset of D with $x \leq \bigcup X$. Since the i-th component of X is a directed set and x_i is isolated in D_i, there exists z_i in X_i with $x_i \leq z_i$. As well for any j with $j < \alpha, j \neq i$ we have $x_j = \perp_{D_j} \leq_j z_j$ so we obtain $x \leq z$.

Consider now an arbitrary element x in D. The set Y_x defined by $Y_x = \{y \mid y \in I, y \leq x\}$ has a least upper bound $\bigcup Y_x$ since it is consistent. Of course $\bigcup Y_x \leq x$. But since each of the $< D_i; \leq_i >$ is ω-algebraic we have also

$$(\bigcup Y_x)_i = \bigcup (y_i | y \in Y_x) \geq \bigcup \mathcal{A}(x_i) = x_i$$

thus $\bigcup Y_x = x$. Let Z_x be the directed set obtain by adding to Y_x the least upper bounds of its finite subsets. We still have $\bigcup Z_x = x$. Hence if x is isolated, there exists an element z in Z_x with $x \leq z$. But z must be less than x, so $z = x$. *An element in D is isolated iff it is the least upper bound of finitely many elements of I.* Hence D contains at most denumerably many isolated elements. Futhermore, Z_x is directed and $x = \bigcup Z_x$, so that the domain is ω-algebraic. We have shown that D is *coherent* and *ω-algebraic*, so it is a computation domain.□

Example: Let $T =< \{\perp, 0, 1\}; \leq >$ be the three element computation domain where $0 \parallel 1$. The cartesian product of denumerably many copies of T is the computation domain T^ω. This domain is discussed in detail by Plotkin [Plo78] who shows that it is a *universal* domain in a precise mathematical sense.

Definition 1.10 *Let $< D; \leq >$ and $< D'; \leq' >$ be two complete partial orders. A function f from D to D' is* continuous *iff*

$$(1) \qquad \forall X \subset D,\ X\, directed \quad f(\bigcup X) = \bigcup{}' \{f(x) | x \in X\}$$

This definition is not very convenient to use. In a computation domain, we will use the following characterization:

Lemma 1.2 *Consider two computation domains $< D; \leq >$ and $< D'; \leq' >$. A function f from D to D' is continuous iff*

$$(2) \qquad \begin{cases} \text{i) } f \text{ is monotonic, i.e.} \forall x, y \in D\ x \leq y \Rightarrow f(x) \leq f(y) \\ \text{ii) } \forall e \in \mathcal{A}(f(x))\, \exists d \in \mathcal{A}(x) \text{ such that } e \leq' f(d) \end{cases}$$

Proof:

a) We show first that (1) implies (2). Consider a function f verifying (1) and two elements x and y in D with $x \leq y$. The set $\{x, y\}$ is directed since $y = x \vee y$. Therefore $f(y) = f(x) \vee' f(y)$. Hence $f(x)$ and $f(y)$ are comparable and $f(x) \leq' f(y)$. Thus f is monotonic. The image of a directed set by a monotonic function is a directed set $f(X)$ and in particular, since for any x the set $\mathcal{A}(x)$ is directed, the set $f(\mathcal{A}(x))$ is directed. Let e be an arbitrary approximant of $f(x)$. We have

$$e \leq' f(x) = f(\bigcup \mathcal{A}(x)) = \bigcup{}' f(\mathcal{A}(x))$$

Since e is isolated and $f(\mathcal{A}(x))$ is directed, there exists an element d in $\mathcal{A}(x)$ with $e \leq f(d)$.

b) We show now that (2) implies (1). Let X be a directed subset of D and f a function from D to D' verifying (2). Since f is monotonic, the set $f(X)$ is directed and $\bigcup' f(X) \leq' f(\bigcup X)$. To prove the converse inequality $f(\bigcup X) \leq' \bigcup' f(X)$ consider an arbitrary approximant e of $f(\bigcup X)$. By (2) one can find d in $\mathcal{A}(\bigcup X)$ with $e \leq' f(d)$. Since d is isolated and X is directed, from $d \leq \bigcup X$ one deduces that there is an element x in X such that $d \leq x$. We have $f(x) \leq' \bigcup' f(X)$ and, since f is monotonic, $f(d) \leq' f(x)$ so

$$\forall e \in \mathcal{A}(f(\bigcup X)) \quad e \leq' \bigcup{}' f(X)$$

and consequently $\mathcal{A}(f(\bigcup X)) \subset \mathcal{A}(\bigcup' f(X))$. By Lemma 1.1 $f(\bigcup X) \leq' \bigcup' f(X)$ and finally $f(\bigcup X) = \bigcup' f(X)$.□

Proposition 1.6 *Consider the computation domains D_1, D_2, and D. A function f from $D_1 \times D_2$ to D is continuous iff the functions $f_1 = \lambda y.f(x_1, y)$ and $f_2 = \lambda y.f(y, x_2)$ are continuous for any x_1 in D_1 and any x_2 in D_2.*

Proof: First, if f is continuous, so are the functions in the familyf_1 and f_2. Let us show this for family f_1. Consider a directed subset S_1 of D_2, and the subset S of $D_1 \times D_2$ defined by $S = \{< x_1, y > | y \in S_1\}$. Now

$$f_1(\bigcup_2 S_1) = f(x_1, \bigcup_2 S_1) = f(\bigcup S) = \bigcup f(S) = \bigcup f(x_1, S_1) = \bigcup_2 f_1(S1)$$

Assume now conversely that the families of functions f_1 and f_2 are continuous. Then f is monotonic. Indeed, if $< x_1, y_1 > \leq < x_2, y_2 >$ then $f(x_1, y_1) \leq f(x_2, y_1) \leq f(x_2, y_2)$. Consider now a directed subset S of $D_1 \times D_2$, and let S_1 and S_2 be its projections on D_1 and D_2. Take $T = \{< x, y > | x \in S_1, y \in S_2\}$. Because the families f_1 and f_2 are continuous we can write:

$$f(\bigcup X) = f(\bigcup S_1, \bigcup S_2) = \bigcup f(S_1, \bigcup S_2) = \bigcup f(S_1, S_2) = \bigcup f(T)$$

Since S is directed and f is monotonic, we now that $f(S)$ is directed. Furthermore, S is included in T, so $\bigcup f(S) \leq \bigcup f(T)$. Take now an arbitrary element $< x, y >$ in T. There are certainly two elements $< x, y_1 >$ and $< x_1, y >$ in S because S_1 and S_2 are projections of S. Since S is directed, there is $< x_2, y_2 >$ in S that dominates both, thus $< x, y > \leq < x_2, y_2 >$. As f is monotonic, we obtain $\bigcup f(T) \leq \bigcup f(S)$. We conclude $f(\bigcup S) = \bigcup f(T) = \bigcup f(S)$, thus f is continuous.□

The result above generalizes trivially to functions with more than two arguments.

In a computation domain D, two elements x and y always have a greatest lower bound $x \wedge y$ (Proposition 1.2) and one can define a function $\wedge$ from D^2 to D by $\wedge = \lambda xy.\, x \wedge y$.

Proposition 1.7 *If D is a computation domain $\wedge$ is a continuous function from D^2 to D.*

Proof: By the previous result, it is sufficient to prove that the functions $\wedge_1 = \lambda y.x \wedge y$ and $\wedge_2 = \lambda y.y \wedge x$ are continuous. Since $\wedge$ is commutative, it is in fact sufficient to prove that $\wedge_1$ is continuous. We use the characterization of Lemma 1.2.

i) $\wedge_1$ is monotonic: $y_1 \leq y_2 \Rightarrow x \wedge y_1 \leq x \wedge y_2$

ii) Le e be an approximant of $x \wedge y$. The element e is an approximant of x and y. So, taking this e in $\mathcal{A}(y)$ we have $e \leq x \wedge e = \wedge_1(e)$. □

Theorem 1.1 (Knaster-Tarski) *If D is a computation domain, any continous function f from D to D has a least fixed point Yf and*

$$Yf = \bigcup\{f^n(\perp)|n \geq 0\}$$

Proof: Take $S = \{f^n(\perp)|n \geq 0\}$. The set S is not empty because it contains $\perp = f^0(\perp)$. Since f is monotonic, it is trivial to show by induction that

$$\forall n \geq 0 \quad f^n(\perp) \leq f^{n+1}(\perp)$$

hence S is a chain. Thus S has a least upper bound $\bigcup S$. Consider $Yf = \bigcup S$. Since f is continuous and S is directed:

$$f(Yf) = f(\bigcup S) = \bigcup f(S) = \bigcup\{f^n(\perp)|n \geq 1\}$$

But since $\perp$ is the minimum element of D

$$\bigcup\{f^n(\perp)|n \geq 1\} = \bigcup\left[\{f^n(\perp)|n \geq 1\}\bigcup\{\perp\}\right] = \bigcup S = Yf$$

Thus $Yf = f(Yf)$ which shows that Yf is a fixed point of f. Consider now any fixed point x of f. We have $f^0(\perp) = \perp \leq x$ and if $f^n(\perp) \leq x$, because f is monotonic $f^{n+1}(\perp) = f(f^n(\perp)) \leq f(x) = x$. Therefore S is dominated by x, and so is its lub Yf. Hence Yf is the least fixed point of f.□
Notation: If D and E are computation domains, we will note $[D \to E]$ the set of continuous functions from D to E. This space inherits an ordering relation defined by extensionality:

$$\forall f, g \in [D \to E] \qquad f \leq g \iff \forall x \in D \ f(x) \leq_E g(x)$$

The constant function $\lambda x.\perp_E$ is the minimum element in $[D \to E]$. The following result is fundamental.

Theorem 1.2 *If D and E are computation domains, the set $[D \to E]$ together with its natural ordering is a computation domain.*

Proof:

a) Let F be a consistent subset of $[D \to E]$. For any x in D the set $\{f(x)|f \in F\}$ is consistent and thus admits a lub g_x. Let us show that the function $\lambda x.g_x$ is continuous. Let X be a directed subset of D with lub z

$$g_z = \bigcup_E \{f(z)|f \in F\}$$

Since all functions in F are continuous,

$$\begin{aligned} g_z &= \bigcup_E \{f(x)|x \in X, f \in F\} \\ &= \bigcup_E \{g_x|x \in X\} \end{aligned}$$

hence $\lambda x.g_x$ is the least upper bound of F in $[D \to E]$. Thus $[D \to E]$ is coherent.

b) We must show now that $[D \to E]$ is ω-algebraic. Consider the family of functions indexed over $\mathcal{A}(D) \times \mathcal{A}(E)$ defined by:

$$\varphi_{d,e}(x) = \begin{cases} e & \text{if } d \leq x \\ \bot_E & \text{otherwise} \end{cases} \qquad (d \in \mathcal{A}(D),\ e \in \mathcal{A}(E))$$

1. *The functions in this family, called step functions, are continuous* Indeed:

   i) $\varphi_{d,e}$ is monotonic (obvious)

   ii) Let a be an approximant of $\varphi_{d,e}(x)$. If $\varphi_{d,e}(x) = \bot_E$, then

   $$a = \bot_E \leq \varphi_{d,e}(\bot_D) \quad \text{with} \quad \bot_D \in \mathcal{A}(x)$$

   If $\varphi_{d,e}(x) = e$, then $d \leq x$ thus $d \in \mathcal{A}(x)$ since d is isolated. But then $a \leq \varphi_{d,e}(d) = e$ with $d \in \mathcal{A}(x)$.

2. *The step functions are isolated elements of* $[D \to E]$. Let F be a directed subset of $[D \to E]$ such that $\varphi_{d,e} \leq \bigcup F$. The result obtained in part a) allows one to write:

   $$e = \varphi_{d,e}(d) \leq (\bigcup F)(d) = \bigcup \{f(d)|f \in F\}$$

   but e is isolated and $\{f(d)|f \in F\}$ is a directed set. Thus there exists a function g in F with $e = \varphi_{d,e}(d) \leq g(d)$. But now if $x \geq d$ then $\varphi_{d,e}(x) = e \leq g(d) \leq g(x)$, and otherwise $\varphi_{d,e}(x) = \bot_E \leq g(x)$ so that $\varphi_{d,e} \leq g$.

3. *Any continuous function in* $[D \to E]$ *is the least upper bound of the step functions under it.* Define $S(f) = \{\varphi_{d,e} | \varphi_{d,e} \leq f\}$. Remark that $\varphi_{d,e} \in S(f) \iff e \in \mathcal{A}(f(d))$. This obvious from left to right because $\varphi_{d,e}(d) = e$ and from right to left by monotonicity of f. Using now the continuity of f

$$\begin{aligned} \forall x f(x) = f(\bigcup \mathcal{A}(x)) &= \textstyle\bigcup_{d \in \mathcal{A}(x)} f(d) \\ &= \textstyle\bigcup_{d \in \mathcal{A}(x), e \in \mathcal{A}(f(d))} e \\ &= \textstyle\bigcup_{d \in \mathcal{A}(x), e \in \mathcal{A}(f(d))} \varphi_{d,e}(x) \\ &= \textstyle\bigcup_{e \in \mathcal{A}(f(d))} \varphi_{d,e}(x) \end{aligned}$$

   So $\forall x\ f(x) = (\bigcup S(f))(x)$, thus $f = \bigcup S(f)$

4. *The isolated elements of* $[D \to E]$ *are exactly the finite unions of step functions.* Consider an isolated element f in $[D \to E]$, and the set $S'(f)$ obtained in closing $S(f)$ by finite unions. The set $S'(f)$ is directed and we have $f = \bigcup S(f) = \bigcup S'(f)$. Since f is isolated, there exists in $S'(f)$ an element g such that $f \leq g$. But since g is a finite union of elements of $S(f)$ we also have $g \leq f$. Thus $f = g$ showing that f is a finite union of step functions.

5. $[D \to E]$ *is* ω*-algebraic.* For all f we have $f = \bigcup S(f) = \bigcup S'(f)$. Thus $[D \to E]$ is algebraic. As D and E have at most denumerably many isolated elements, there exists only denumerably many step functions, hence only denumerably many isolated elements in $[D \to E]$.

We have proved that when D and E are computation domains, $[D \to E]$ is coherent and ω-algebraic, hence also a computation domain.□

The theorem above allows one, starting from computation domains, to construct a hierarchy of computation domains such as $[D \to E]$, $[D \to [D \to E]]$, $[[D \to E] \to [D \to E]]$ etc.

2 Concrete domains of computation

In this section, we try to translate into mathematical form a number of ideas that come from earlier research. It is difficult to figure out what is critical to the well-functioning of a complex operational mechanism. In contrast, we have more experience in finding the general conditions under which a mathematical result is valid [1].

[1] A similar approach is followed by J-J. Lévy in his Ph. D. Thesis [Lev78]

The central result of this work is the Representation Theorem that, in a sense indicates that we have been successful in our endeavor. Starting from the general idea of a computation domain, we justify progressively the need to restrict this notion until we reach the definition of a *concrete* computation domain and study its properties.

2.1 Initial motivations

In the model theory of programming languages as developed starting with the work of Scott [Sco70, Sco76], there is no distinction between *data* and *functions*. A single mathematical structure, the *computation domain* is defined and all objects with which one computes are found in appropriate computation domains. This is not surprising because the main objective of this theory was, at least initially, to develop a functional model of the λ-calculus of Church, language where these distinctions don't exist. Indeed certain programming languages such as ISWIM [Lan76], GEDANKEN [Rey72], ML [GRW78], etc. exhibit similar characteristics. However, most programming languages make a very clear distinction between *data* and *procedures*. Is it possible to rediscover this distinction in the models of programming languages, i.e. through the study of their denotational semantics? Is it possible to analyze more precisely the structure of computation domains so as to separate, for example, the domains whose structure is sufficiently simple that they don't need to be understood as function spaces?

Examples:

We call $\perp$ the single element computation domain, 0 the computation domain with two elements, $T =< \{\perp, 0, 1\}; \leq >$ the three element domain in which 0 and 1 are incomparable. These three spaces, as well as their cartesian products in a finite number of copies are clearly data spaces rather than functional spaces.

The examples above might lead one to partition computation domains into two classes, according to their being finite or infinite. Such a categorization is much too rough for two reasons:

i) We will be unable to give a representation as a data structure for certain *finite* domains.

ii) On the other hand, certain infinite domains must clearly be categorized as data spaces. For example, this will be the case for $N_\perp$ and $\overline{N}$, defined from the set N of natural numbers by:

$$\begin{cases} N_\perp & =< \{\perp\} \cup N; \leq > \quad \text{with } \forall x, y \in N \; x \neq y \Longrightarrow x \parallel y \\ \overline{N} & =< N \cup \{\infty\}; \leq > \quad \text{where } \leq \text{ is the natural order on } N \\ & \qquad\qquad\qquad\quad \text{completed by } \forall x \in N, x < \infty \end{cases}$$

We are going to characterize axiomatically a certain class of computation domains. In this endeavour, we shall follow two fundamental principles:

1. (M. Smyth) All axioms that we postulate specify a property of the *isolated elements* in a computation domain. Other elements are *constructed* from the stock of isolated elements by a limit mechanism; their properties will therefore be *deduced* from the properties of isolated elements.

2. The class of computation domains that we are trying to define must be closed by certain elementary constructions, such as finite or infinite cartesian products, or taking *upper sections* (cf. section 1.2). However, it doesn't need to be closed by *exponentiation*, i.e. when constructing function spaces.

2.2 The isolated elements axiom

Isolated elements in a computation domain are meant to stand for *finite* amounts of information. When dealing with data, we would like to be able to reason by induction on these elements. This implies that the set of isolated elements should be well founded with respect to the relation $\leq$, i.e. that there should be no infinite chain $\{x_1, x_2, \ldots, x_n, \ldots\}$ with

$$\{x_1 > x_2 > \cdots > x_n > \cdots\}$$

In this way, an isolated element cannot be *decomposed* indefinitely. We want also to express the intuitive idea that an isolated element can be built using only a finite number of components. This leads to considering property I:

Property I
Between any two distinct comparable isolated elements, any chain of isolated elements is finite.

Proposition 2.1 *Let $< D; \leq >$ be a computation domain satisfying property I. Consider an arbitrary element x in D and an isolated element y. If x is dominated by y, then x is isolated.*

Proof: If x is not isolated, then by Corollary 1.2 there is an infinite chain of isolated elements $\{\bot, x_1, x_2, \ldots, x_n, \ldots\}$ with

$$\bot < x_1 < x_2 < \ldots < x_n < \ldots < x$$

If y is isolated and $x \leq y$, then necessarily $x < y$. Hence the chain

$$\{\bot, x_1, x_2, \ldots, x_n, \ldots, y\}$$

is an infinite increasing chain of isolated elements between $\bot$ and y. The existence of this chain contradicts property I, so x is isolated.□

Corollary 2.1 *In a computation domain, Property I is equivalent to* I_1*:*

Property I_1
Between any two distinct comparable isolated elements, any chain is finite.

Proof: Property I_1 implies obviously Property I. Conversely, if x and y are isolated and $x \leq y$, then by the previous result, any element z such that $x \leq z \leq y$ is isolated. Since any chain between x and y contains only isolated elements, it is finite. □

Definition 2.1 *In a conditionally complete partial order* $< D; \leq >$*, an* ideal *is a non empty subset* J *of* D *such that:*

i) $\forall x \in J,\ \forall y \in D \qquad y \leq x \Longrightarrow y \in J$ *(i.e.* J *is downward closed)*

ii) $\forall x, y \in J \qquad x \uparrow y \Longrightarrow x \vee y \in J$

Corollary 2.2 *In a computation domain, property I is equivalent to property* I_2*:*

Property I_2	The set of isolated elements is a well founded ideal.

Proof: If a computation domain D verifies property I, then the set of its isolated elements is an ideal by Proposition 1.4 and Proposition 2.1. Since I implies I_1, there is no infinite decreasing chain in $\mathcal{A}(D)$. Hence property I implies property I_2.

Conversely, assume D has property I_2. Consider an arbitray x less than some isolated element y in D. There is no infinite decreasing chain between x and y since $\mathcal{A}(D)$ is well-founded. If there were an infinite increasing chain

$$\{x, z_1, z_2, \ldots, z_n, \ldots, y\} \text{ with } x < z_1 < z_2 < \cdots < z_n < \cdots < y$$

one would have $\bigcup z_i = z \leq y$. Now z is not isolated and $z < y$, which contradicts the hypothesis that $\mathcal{A}(D)$ is an ideal.

Consider now any chain C between x and y. Since C does not contain infinite decreasing chains, C is an *ordinal.* If C is infinite, then it contains the smallest limit ordinal ω. But ω contains an infinite increasing chain, which cannot be the case for C. Hence C is a finite chain, and we conclude that property I_2 implies property I.$\Box$

Examples: Domain $D_1 =< N \cup \{\infty, \top\}; \leq >$ with the natural ordering on N and $\forall x \in N\ x < \infty$ and $\infty < \top$ does not satisfy property I_2 because $\mathcal{A}(D_1)$ is not an ideal ($\top$ is isolated, but ∞ is not). Domain $D_2 =< Z \cup \{-\infty, +\infty\}; \leq >$ with the natural ordering on Z and $\forall x \in Z\ -\infty < x < +\infty$ does not verify I_2 because $\mathcal{A}(D_2)$ is not well founded. However, all finite domains, as well as $N_\perp$ and $\overline{N}$ have property I.

Definition 2.2 *Consider a partial order $< D; \leq >$ and two elements x and y in D. We say that y* covers *x iff:*

i) $x < y$

ii) $\forall z \quad x \leq z \leq y \Longrightarrow x = z$ *or* $y = z$

One may also say that y is just above x. This relation is noted $x \prec y$. Its reflexive closure is written $x \preceq y$

Proposition 2.2 *Consider a computation domain $< D; \leq >$ with property I. If x and y are isolated elements in D, then we have $x \leq y$ iff:*

- *Either $x = y$*
- *Or there exists a finite sequence $\{z_0, z_1, \ldots, z_n\}$ of elements in $\mathcal{A}(D)$ with $z_0 = x$, $z_n = y$ and $z_i \prec z_{i+1}$ for $0 \leq i < n$.*

Proof: First, if such a sequence exists, then by transitivity $x \leq y$. Conversely, assume $x < y$. Let H be the set of chains with elements in $\mathcal{A}(D)$ with minimum x and maximum y. The set H is not empty because it contains in particular the chain $\{x, y\}$, and we can order it by inclusion. In the partial order $< H; \subset >$ there cannot be an infinite increasing chain because $< D; \leq >$ has property I. Let $C = \{z_0, z_1, \ldots, z_n\}$ be a maximal element in $< H; \subset >$; we will call such a chain a maximal chain between x and y. Without loss of generality we may assume $z_0 < z_1 < \cdots < z_n$.

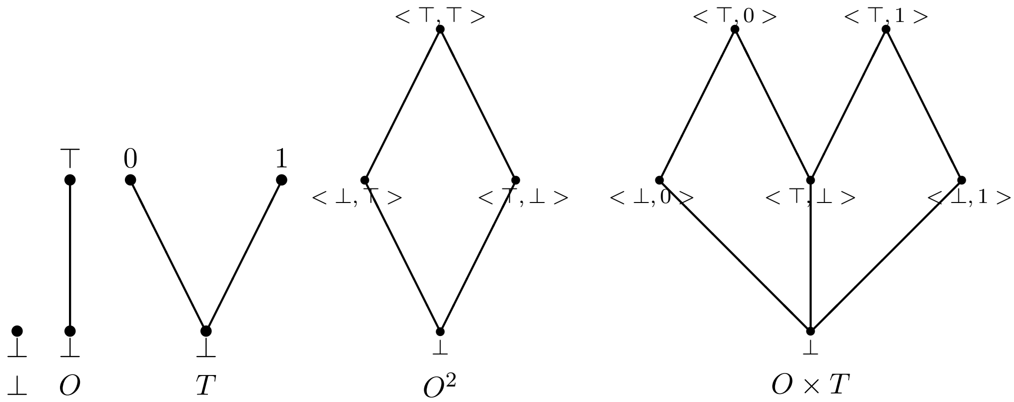


Figure 2: Sample finite domains

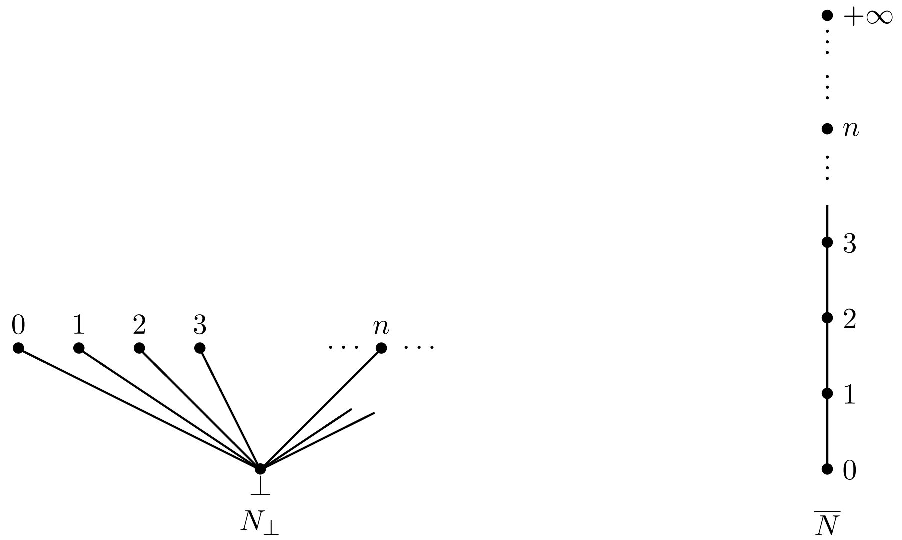


Figure 3: Sample infinite domains

Now we must have $z_i \prec z_{i+1}$ $(0 \leq i < n)$, because otherwise one could extend C with an isolated element z such that $z_i < z < z_{i+1}$ contradicting maximality of C in $< H; \subset >$. Similarly, it must be the case that $z_0 = x$ and $z_n = y$.□

From now on we will find it useful to represent configurations of elements belonging to a partial order, or partial orders themselves, by graphs called *Hasse diagrams*. The nodes in a diagram associated to $< D; \leq >$ denote elements in D and two nodes a and b are connected by an edge going upwards iff $a \prec b$ in D. Simple conventions will be used to represent infinite domains. As an example, Figures 2 and 3 show a number of partial orders that we have already mentioned.

Before proceeding with the study of computation domains that satisfy property I, we notice that only trivial function spaces have this property.

Lemma 2.1 *If D and E are computation domains, if D is infinite and E has at least two elements, then $[D \to E]$ does not satisfy property I.*

Proof: Observe first that if D has infinitely many elements, then it has infinitely many isolated elements by Corollary 1.2. As well, if E has at least two elements, then there is an isolated element e in E with $\perp_E \neq e$. Consider now the infinite partial order $< \mathcal{A}(D); \leq_D >$. By Koenig's lemma:

a) Either there exists an infinite increasing chain of elements in $\mathcal{A}(D)$,

b) Or there is an element d in $\mathcal{A}(D)$, and an infinite set $\{d_i\}_{i \in N}$ of elements in $\mathcal{A}(D)$ with

$$\begin{cases} \forall i \in N & d < d_i \\ \forall i, j \in N & d_i \parallel d_j \text{ if } i \neq j \end{cases}$$

<u>Case a.</u> Consider an infinite increasing chain $\{d_1, d_2, \ldots, d_n, \ldots\}$ in $\mathcal{A}(D)$, i.e. such that $d_1 < d_2 < \cdots < d_n < \cdots$. and the sequence of step functions $\varphi_{d_i,e}$. This infinite sequence of isolated elements in $[D \to E]$ is *decreasing*

$$\varphi_{d_1,e} > \varphi_{d_2,e} > \varphi_{d_3,e} > \cdots > \varphi_{d_n,e} > \cdots$$

thus $\mathcal{A}([D \to E])$ is not well-founded and $[D \to E]$ does not have property I_2.
<u>Case b.</u> In that case we have $\forall i \in N\ \varphi_{d_i,e} < \varphi_{d,e}$ since $d < d_i$. The set Φ of functions $\{\varphi_{d_i,e}\}_{i \in N}$ has an upper bound. Since $[D \to E]$ is a computation domain, it has a least upper bound ϕ. Naturally we have $\phi \leq \varphi_{d,e}$. But since $\forall i \in N\ \varphi_{d_i,e}(d) = \perp_E$ necessarily $\phi(d) = \perp_e$. But $\varphi_{d,e}(d) = e \neq \perp_E$, so $\phi < \varphi_{d,e}(d)$.

Let us show now that ϕ is not isolated in $[D \to E]$. If ϕ were isolated, there would exist a finite subset J of N with $\phi = \bigcup_{j \in J} \varphi_{d_j,e}$. Take an integer k not in J. Since $\varphi_{d_k,e}(d_k) = e$ and $\varphi_{d_k,e} \leq \phi$ we have $e \leq \phi(d_k)$. But by hypothesis

$$\forall j \in J\ d_j \parallel d_k$$

so that $\varphi_{d_j,e}(d_k) = \perp_E$ and also $\phi(d_k) = \perp_E$. Since e is different of $\perp_E$, we have a contradiction. So ϕ is not isolated in $[D \to E]$. Then $\mathcal{A}([\mathcal{D} \to \mathcal{E}])$ is not an ideal.

We have shown in both cases that $[D \to E]$ does not satisfy I.□

Remark: This lemma distinguishes sharply between domains that appear to be very similar. For example, the domain $[N_\perp \to O]$ does not have property

I. In contrast O^ω, the cartesian product of denumerably many copies of O, satisfies property I. This is because O^ω is only isomorphic to the set of *strict* functions in $[N_\perp \to O]$, i.e. the functions f such that $f(\perp) = \perp_O$. To be very precise, the non-strict function $\lambda x.\top_O$ in $[N_\perp \to O]$ is isolated and it does not correspond to any element in O^ω. But this function dominates the non-strict function ψ defined by:

$$\psi(x) = \begin{cases} \top_O & \text{if } x \neq \perp_{N_\perp} \\ \perp_O & \text{if } x = \perp_{N_\perp} \end{cases}$$

which is not isolated in $[N_\perp \to O]$.

Definition 2.3 *Consider a partial order $< D; \leq >$ with a minimum element $\perp$. An* atom *is an element of D that covers $\perp$, and we say that D is* atomic *iff any element distinct from $\perp$ dominates an atom.*

In symbols:

$$\forall x \neq \perp \exists y \quad \perp \prec y \leq x$$

Proposition 2.3 *A computation domain that verifies property I is atomic.*

Proof: Consider first an isolated element x with $x \neq \perp$. By Proposition 2.2 there exists a finite sequence $\{z_0, z_1, \ldots, z_n\}$ of elements in $\mathcal{A}(D)$ with $\perp = z_0 \prec z_1 \prec \cdots \prec z_n = x$. Hence z_1 is an atom and $\perp \prec z_1 \leq x$.

If now x is not isolated, let e be an element in $\mathcal{A}(D)$ which is distinct from $\perp$. Such an element must exist, otherwise $\mathcal{A}(x) = \{\perp\} = \mathcal{A}(\perp)$ and thus, by Lemma 1.1, $x = \perp$. Now we have just shown that there exists an element y with $\perp \prec y \leq e$. By transitivity, we obtain $\perp \prec y \leq x$.□

Property I and its Corollary, atomicity, are interesting properties for a computation domain, and they seem to capture a certain intuition about data domains. We will see now that these properties are not preserved under a fundamental operation on computation domains.

Definition 2.4 *Consider a partial order $< D; \leq >$ and two elements x and y in D with $x \leq y$. The* interval *$[x, y]$ is the set $\{z | x \leq z \leq y\}$ and the* upper section of *x, noted $[x)$ is the set $\{z | x \leq z\}$. Of course, intervals and upper sections inherit the partial order $\leq$.* [2]

Proposition 2.4 *Intervals and upper sections of a computation domain are computation domains.*

[2]We also call $[x, y]$ and $[x)$ the partial orders thus defined

Proof: As reasoning proceeds identically in both cases, we will only prove the result for upper sections. Consider an arbitrary upper section $[x)$ in a computation domain $< D; \leq >$. Any non empty consistent subset of $[x)$ is a consistent subset of D and therefore has a least upper bound in D. This least upper bound is necessarily in $[x)$.

Furthermore, the empty set also admits a least upper bound in $[x)$. So $< [x); \leq >$ is a coherent partial order. Let us show now that is is also ω-algebraic. Let $\{d_i\}_{i \in I}$ be an enumeration of $\mathcal{A}(D)$. For any i in I define

$$c_i = \begin{cases} x \vee d_i & \text{if } x \uparrow d_i \\ x & \text{otherwise} \end{cases}$$

Of course, each element c_i defined in this way belongs to $[x)$ and we will show that $\{c_i\}_{i \in I} = \mathcal{A}([x))$.

First, the element x is minimum in $[x)$, so it is isolated in $< [x); \leq >$. Consider now an element c_i different of x, and a directed subset X of $[x)$ such that $c_i \leq \bigcup X$. Since $c_i = x \vee d_i$, we have also $d_i \leq \bigcup X$. Since d_i is isolated in D, and X is directed, we have $d_i \leq y$ for some y in X. Since y is in $[x)$, thus larger than x we have $c_i = x \vee d_i \leq x \vee y = y$ which proves that c_i is isolated in $[x)$. Thus $\{c_i\}_{i \in I} \subset \mathcal{A}([x))$.

Consider now an arbitrary element of $[x)$. Since D is algebraic

$$y = \bigcup \{d_i | i \in I, d_i \leq y\}$$

Since y dominates x, we have also $y \vee x = y = \bigcup \{d_i \vee x | i \in I, d_i \leq y\}$. But $d_i \leq y$ iff $d_i \vee x \leq y$

$$y = \bigcup \{c_i | i \in I, c_i \leq y\}$$

The equality above proves that $< [x); \leq >$ is algebraic. Furthermore, the set $\{c_i | i \in I, c_i \leq y\}$ is directed, so if y is isolated in $< [x); \leq >$, for some j in I $y = c_j$. It follows that $\mathcal{A}([x)) = \{c_i\}_{i \in I}$ so $\mathcal{A}([x))$ is denumerable.

The partial order $< [x); \leq >$ is coherent and ω-algebraic, so it is a computation domain. □

The counterexample on Figure 4(a) shows that if a computation domain has property I, it is not necessarily the case for its upper sections. In that domain, we have a chain $\{\bot, x_1, x_2, \ldots, x_n, \ldots\}$ where

$$\bot \prec x_1 \prec x_2 \prec x_3 \cdots \prec x_n \prec \cdots$$

with limit x. Additionally atom a_1 is assumed to be compatible with x, and incomparable with each of the x_i (thus x). Let us now assume also:

$$\forall j \geq 1, \forall k \geq j \ x_k \parallel a_j \text{ and } \exists a_{j+1} \text{ with } x_j \prec a_{j+1} < x_j \vee a_j$$

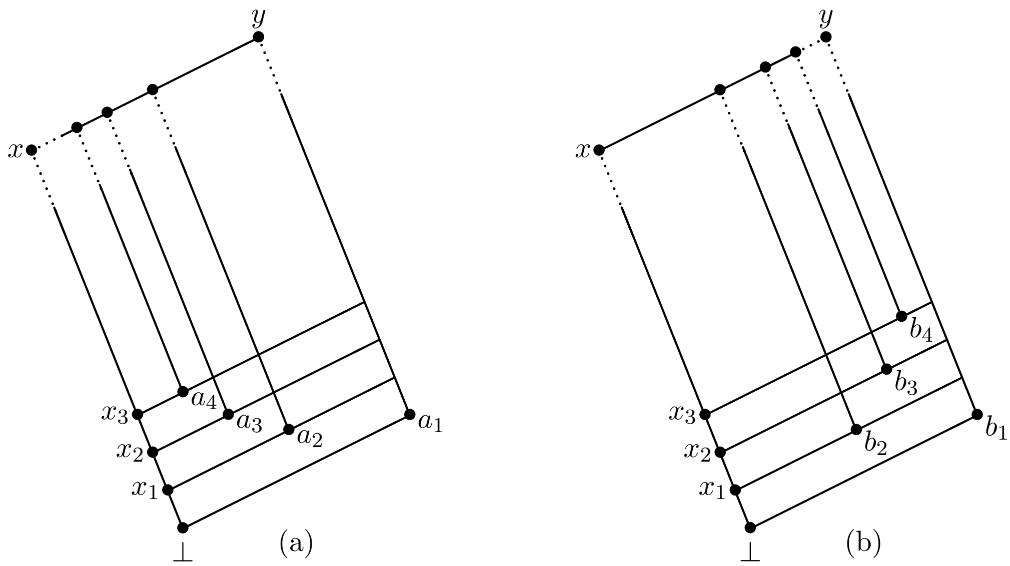


Figure 4: Property I is not valid in upper sections

and $\forall j \geq 1$ $x \vee a_j > x \vee a_{j+1}$. The partial order defined in this fashion is a computation domain satisfying property I. In $[x)$ the sequence $\{x \vee a_j\}_{j \geq 1}$ is an infinitely decreasing chain of isolated elements of $< [x); \leq >$ between x and y. (Similarly, one can construct an example exhibiting an infinite increasing chain of isolated elements of $< [x); \leq >$ between x and y, see Figure 4(b)).

As we indicated in the introduction to this section, we consider it desirable for the notion of data domain to be preserved under upper sections and intervals. This means that we have to consider a stronger property than property I.

3 The covering relation

We have seen that the isolated elements of $< [x); \leq >$ are, but for x itself, of the form $x \vee d$ with d isolated, compatible and incomparable with x. The following property postulates a similar characterization of the *atoms* in an upper section.

Property C

If x and y are two compatible isolated elements

$$x \wedge y \prec x \implies y \prec x \vee y$$

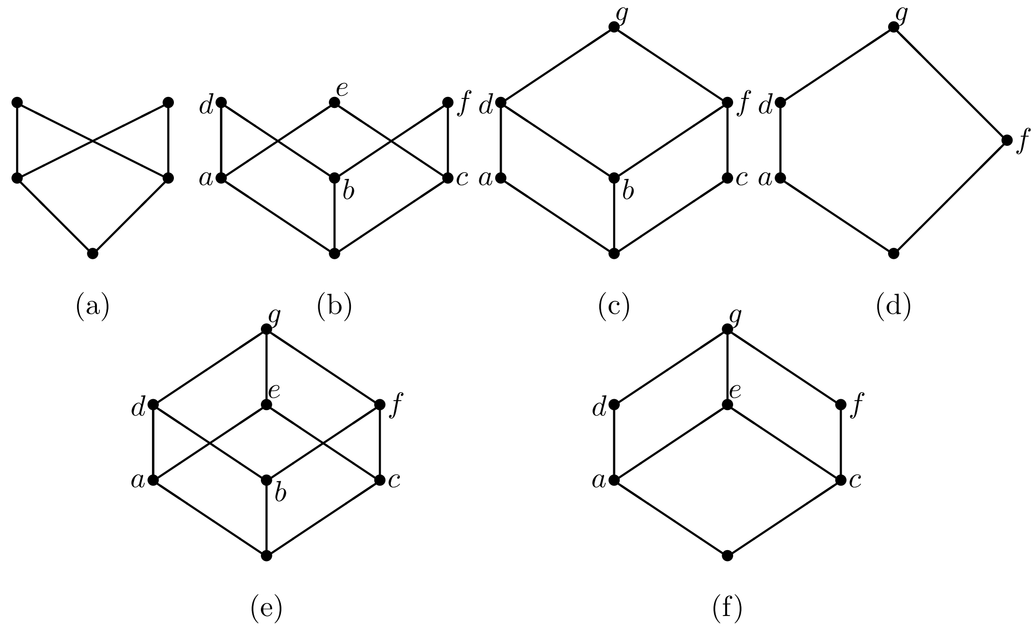


Figure 5: Investigating Property C

Remarks:

i) If x and y are *comparable* and verify $x \wedge y \lessdot x$, one cannot have $x \leq y$ otherwise $x \wedge y = x \lessdot x$ which is impossible. Hence $y \leq x$ and $x \wedge y = y \lessdot x$. In that case, property C holds trivially.

ii) While property I did not exclude any finite domain, this is not the case for property C. This is not too surprising, as it already happens for some axioms of computation domains. For example, the partial order on Figure 5(a) is not conditionally complete, the partial order on Figure 5(b) is not consistent. The partial orders on Figure 5(c) and 5(d) do not satisfy Property C.

In the diagram on Figure 5(b), coherence forces one to add a maximum element g, yielding the domain of Figure 5(e).

In the domain of Figure 5(c), elements a and c are compatible and $\bot = a \wedge c \lessdot a$ and $\bot \lessdot c$ as well. So by property C, one should have $a \lessdot a \vee c$ and $c \lessdot a \vee c$. If we add an element $e = a \vee c$ that covers a and c and is covered by g, we obtain again the domain of Figure 5(e) that satisfies C. Finally, in the domain of Figure 5(d), we have $\bot = a \wedge f \lessdot f$ but $a \vee f = g$ does not cover a. If we add an element c

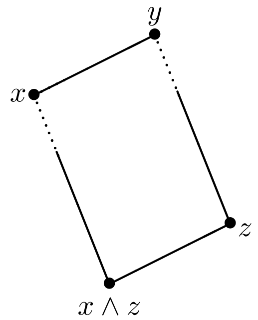
so that $\bot \prec c \prec f$ and $b = a \vee c$ with $b \prec g$, we obtain the domain of Figure 5(f) that has property C.

iii) Property C concerns only pairs of compatible elements. This property can only constrain the structure of sub-lattices in a computation domain. In lattice theory, this property is known as the *lower covering condition*[Bir67]. Although a computation domain is not a lattice, the forthcoming developments are largely inspired by the study of this condition in lattice theory.

We begin by showing, in several steps, that if the set of isolated elements in a computation domain has property I and C, then the whole domain has property C.

Proposition 3.1 *Let $D =< D; \leq >$ be a computation domain with properties I and C. We have $\forall x, y \in D\ x \prec y\ \Rightarrow \exists z \in \mathcal{A}(y)\ x \wedge z \prec z$ and $y = x \vee z$.*

Proof:



If $x \prec y$, *a fortiori* $x < y$. Consider an element d of $\mathcal{A}(y) \setminus \mathcal{A}(x)$, which must exist by Lemma 1.1. Since d is not an approximant of x we have $x \wedge d \neq d$. As D has property I, we deduce:

i) $x \wedge d \in \mathcal{A}(D)$ because $d \in \mathcal{A}(D)$.

ii) $\exists z \in \mathcal{A}(y)\ x \wedge d \prec z \leq d$ by Proposition 2.2.

This element z is not dominated by x, otherwise it would also be dominated by $x \wedge d$. Hence $x \wedge z = x \wedge d$. Since x and d are compatible, so are x and z and by Property C, $x \prec x \vee z$.

Since x and z are both less than y, we obtain: $x \prec x \vee z \leq y$. But $x \prec y$ so $y = x \vee z$, which proves the result. □

Proposition 3.2 *In a computation domain having Property I, Property C is equivalent ot Property* $\widehat{\mathrm{C}}$*:*

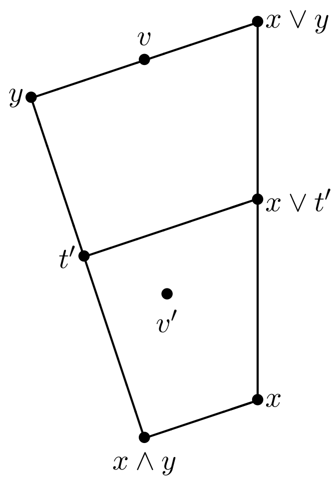
Property $\widehat{\mathbf{C}}$

If x and y are any two compatible elements

$$x \wedge y \prec x \Longrightarrow y \prec x \vee y$$

Proof: Property $\widehat{\mathrm{C}}$ trivially implies property C. The converse is shown in two steps.

1. Assume first that x is isolated and y is arbitrary, with $x \uparrow y$ and $x \wedge y \prec x$. As we have already observed, only the case where $x \parallel y$ is interesting. By Property I_3, if x is isolated, so is $x \wedge y$. Assume now that there exists a v such that $y < v < x \vee y$. Property C excludes this possibility when y is isolated. Since $y < v$, there exists an approximant v_1 of v which is not an approximant of y. Since x covers $x \wedge y$, x cannot dominate v, because we would then have $y = x \wedge y \prec x = x \vee y$.



Therefore, there is an approximant v_2 of v which is not an approximant of x. Since $x \vee y$ is isolated by Property I_3, we can construct the isolated element $v' = v_1 \vee v_2 \vee (x \wedge y)$. This element verifies:

$$v' \in \mathcal{A}(v) \quad v' \notin \mathcal{A}(y) \quad v' \notin \mathcal{A}(x) \quad x \wedge y \leq v'$$

Note also that v' doesn't dominate x, otherwise v would, which would contradict $v < x \vee y$. Since v' is dominated by $x \vee y$ we have now

$$v' \leq x \vee y = x \vee (\bigcup \mathcal{A}(y)) = \bigcup_{z \in \mathcal{A}(y)} (x \vee z)$$

Since v' is isolated and the set $\{x \vee z | z \in \mathcal{A}(y)\}$ is directed, there exists an approximant t of y such that $v' \leq x \vee t$. Now take $t' = t \vee (x \wedge y)$:

$$v' \vee (x \wedge y) = v' \leq x \vee (t \vee (x \wedge y)) = x \vee t'$$

The element t' cannot dominate x, otherwise we would have $x \vee t' = t'$ thus $v' \leq t'$, which is impossible because v' is not an approximant of y. So $t' \wedge x = x \wedge y$ and by Property C $t' \prec x \vee t'$. Take then $w = v' \wedge t'$. We have $t' \leq w \leq x \vee t'$ so that either $w = t'$ or $w = x \vee t'$. The first case, $w = t'$ is impossible because it implies $v' \leq t'$, hence $v' \in \mathcal{A}(y)$. The case $w = x \vee t'$ is also impossible, because $w = v' \vee t'$ is an approximant of v that cannot dominate x without contradicting $v < x \vee v$. The existence of v leads to a a contradiction in all cases. So necessarily $y \prec x \vee y$.

2. Assume now x to be an arbitrary element in the domain. By Proposition 3.1, if $x \wedge y \prec x$, one can find an approximant z of x with $(x \wedge y) \wedge z \prec z$ and $x = (x \wedge y) \vee z$. From the first inequality we deduce $y > z$. But $x \wedge y \leq y$ implies also $(x \wedge y) \wedge z \leq y \wedge z$. Thus $(x \wedge y) \wedge z \leq y \wedge z < z$ and $(x \wedge y) \wedge z = y \wedge z$. Since y and z are compatible because y and x are, we can apply the result of part 1 and deduce $y \prec y \vee z$. Since $x = (x \wedge y) \vee z$ we have now

$$x \vee y = (x \wedge y) \vee z \vee y = y \vee z$$

and thus also $y \prec x \vee y$. □

Corollary 3.1 *In a computation domain D satisfying I and C, any upper section (and any interval) is atomic.*

Proof: Here again, we give only the proof for an upper section $[x)$. Let y be an element such that $x < y$. By Lemma 1.1, we can find an approximant z of y which is not an approximant of x and therefore $x \wedge z < z$. Since z and $x \wedge z$ are isolated, there is a t in $\mathcal{A}(D)$ with $x \wedge z = x \wedge t \prec t \leq z$. Since $x \uparrow z$ implies $x \uparrow t$, we obtain using property $\widehat{\mathrm{C}}$ $x \prec x \vee t \leq x \vee z \leq y$. □

Proposition 3.3 *In a computation domain satisfying I, Property C is equivalent to Property* C_1*:*

Property C_1

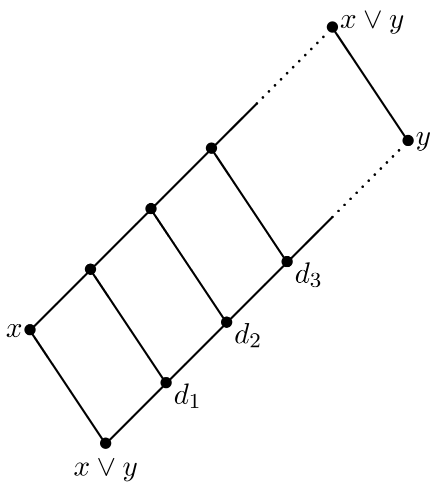
If x and y are two distinct compatible elements

$$\exists z \quad z -\!\!< x,\, z -\!\!< y \implies x -\!\!< x \vee y,\, y -\!\!< x \vee y$$

Proof:

1. <u>$\widehat{\mathrm{C}}$ implies C_1</u> Indeed, if x and y are distinct, element z is their glb and Property $\widehat{\mathrm{C}}$ implies immediately $x -\!\!< x \vee y$ and $y -\!\!< x \vee y$.

2. <u>C_1 implies $\widehat{\mathrm{C}}$</u> Consider two compatible isolated elements x and y such that $x \wedge y -\!\!< y$. We will prove by induction that $y -\!\!< x \vee y$ using Proposition 2.2.



   - Base cases. If $y = x \wedge y$ then immediately $y -\!\!< x \vee y = x$. If y covers $x \wedge y$ then C_1 gives $y -\!\!< x \vee y$.
   - Inductive step. Assume that C is valid when there exists a maximal chain with at most n element between $x \wedge y$ and y and consider two isolated elements x and y such that there is a sequence of $n+1$ elements $\{d_0, d_1, \ldots, d_n\}$ with $x \wedge y = d_0 -\!\!< d_1 -\!\!< d_2 -\!\!< \cdots -\!\!< d_n = y$. By property C_1 we have $d_1 -\!\!< d_1 \vee x$. Since $x < y$, $d_1 \vee x$ is not less than y, so $d_1 = (d_1 \vee x) \wedge y$. Using the induction hypothesis, we obtain $y -\!\!< (d_1 \vee x) \vee y$. Since d_1 is less than y, we deduce $y -\!\!< x \vee y$. $\square$

Definition 3.1 *A partial order satisfies the* Jordan-Dedekind condition *if, between any two comparable elements, all maximal chains are finite and have the same length.*

Theorem 3.1 *If D is a computation domain satisfying* I *and* C*, then $\mathcal{A}(D)$ satisfies the Jordan-Dedekind condition.*

Proof: The proof follows closely the proof of Theorem 14, in chapter 2 of [Bir67]. We show by induction that if between any two comparable elements a and b of $\mathcal{A}(D)$ there is a maximal chain of length n, then all maximal chains have length n. Assume $a \leq b$. If $a = b$ then all maximal chains between a and b have length 0. If $a \prec b$, there doesn't exist a c with $a < c < b$, so $\{a, b\}$ is the only maximal chain between a and b.

Assume now the property valid when there exists, between two comparable elements, a chain with length less than $n + 1 (n \geq 1)$ and take two isolated elements a and b with a maximal chain of length $n + 1$ between them:

$$a = x_0 \prec x_1 \prec x_2 \prec x_3 \cdots \prec x_n \prec x_{n+1} = b$$

Since D has property I, all maximal chains between a and b are finite and built up with elements of $\mathcal{A}(D)$. Take any maximal chain $\{y_0, y_1, \ldots, y_l\}$ between a and b. Two cases are possible:

- Case 1. $x_1 = y_1$. By induction hypothesis, all maximal chains between x_1 and b have length n, so $l = n + 1$.

- Case 2. $x_1 \neq y_1$. Since x_1 and y_1 are dominated by b, we have $x_1 \uparrow y_1$ and, by C_1: $x_1 \prec x_1 \vee y_1$ and $y_1 \prec x_1 \vee y_1$. By induction hypothesis, all maximal chains between x_1 and b have length n, so in particular those that have $x_1 \vee y_1$ as their first element. Hence all maximal chains between $x_1 \vee y_1$ and b have length n_1. Take such a chain $\{z_0 = x_1 \vee y_1, z_1, \ldots, z_{n-1} = b\}$. The chain $\{y_1, z_0, \ldots, z_{n-1}\}$ is a maximal chain between y_1 and b. Using again the induction hypothesis, we obtain that all maximal chains between y_1 and b have length n so in particular $\{y_1, y_2, \ldots, y_l\}$. Again $l = n + 1$. □

The Theorem above allows one to define an absolute notion of *height* for isolated elements.

Definition 3.2 *In a partial order $< D; \leq >$ with a minimum element $\perp$, a* height function *is a function h from D to N such that:*

i) $h(\bot) = 0$

ii) $x \prec y \iff x \leq y$ *and* $h(y) = 1 + h(x)$

Corollary 3.2 *In a computation domain satisfying I and C, the function h from $\mathcal{A}(D)$ to N that associates to any isolated x the common length of all maximal chains between $\bot$ and x is a height function.*

Proof: By definition $h(\bot) = 0$. Assume now $x \prec y$. Any maximal chain $\{\bot, x_1, \ldots, x_{h(x)}\}$ from $\bot$ to x can be extended to a maximal chain $\{\bot, x_1, \ldots, x_{h(x)}, y\}$ hence $h(y) = 1 + h(x)$. Conversely, assume $x \leq y$ and $h(y) = 1 + h(x)$. All maximal chains from x to y must have length 1, hence $x \prec y$. □

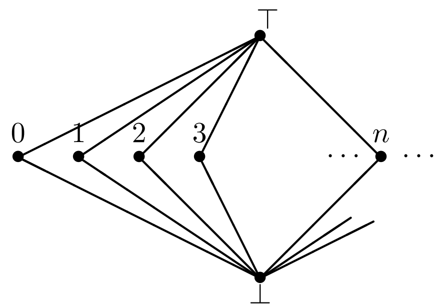
Recall the computation domain $\overline{N} < N \cup \{\infty\}; \leq >$ where $\leq$ is the natural ordering on N and ∞ is a maximum element. The height function h from $\mathcal{A}(D)$ to N may be extended to an element of $[D \rightarrow \overline{N}]$ because it is monotonic. Then we will have $h(x) = \infty$ iff x is not isolated, by Corollary 1.2. This property legitimates calling *finite* the elements of $\mathcal{A}(D)$ and *infinite* the elements of D that are not isolated.

Remark: Properties C and I do not exclude the possibility that a finite element might dominate an infinite number of finite elements, as illustrated by the counter example of Figure 6.

To prove the fundamental inequality of the next Theorem 3.2, we need the following technical result:

Lemma 3.1 *In a partial order with Property* C_1 *we have*

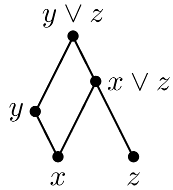
$$\forall x, y, z \qquad x \prec y,\ z \uparrow y \implies x \vee z \preceq y \vee z$$



$N_\bot$

Figure 6: $\top$ dominates infinitely many elements

Proof:



Since y and z are compatible, *a fortiori* x and z are. Let us examine the possibilities for $x \vee z$.

1. $x \vee z = x$ i.e $z \leq x \leq y$. Then $x \vee z = x \prec y = y \vee z$
2. $x \vee z = y$ i.e. $z \leq y$ so $x \vee z = y = y \vee z$
3. $x \vee z \parallel y$. Then by property C $x \vee z \prec (x \vee z) \vee y = (x \vee y) \vee z = y \vee z$.
4. $x \vee z \geq y$. Then $x \vee z \geq y \vee z$. But from $x \leq y$ we also deduce $x \vee z \leq y \vee z$, so $x \vee z = y \vee z$.□

Theorem 3.2 *Let D be a computation domain with properties I and C, and consider two compatible finite elements a and b in D. The following inequality holds:*

$$h(a) + h(b) \geq h(a \wedge b) + h(a \vee b)$$

Proof: If a and b are comparable, assume for example $a \leq b$. Since $a \wedge b = a$ and $a \vee b = b$, we have trivially $h(a) + h(b) = h(a \wedge b) + h(a \vee b)$. Suppose now that $a \parallel b$ and consider a maximal chain $\{x_0, x_1, \ldots, x_n\}$ with

$$a \wedge b = x_0 \prec x_1 \prec x_2 \cdots \prec x_n = b$$

All elements in this chain are compatible with a and by the previous Lemma:

$$\forall j \quad 0 \leq j \leq n-1 \quad x_j \vee a \preceq x_{j+1} \vee a$$

Hence, since h is a height function

$$\forall j \quad 0 \leq j \leq n-1 \quad h(x_{j+1} \vee a) - h(x_j \vee a) \leq 1$$

Summing these inequalities

$$\sum_{0 \leq j \leq n-1} [h(x_{j+1} \vee a) - h(x_j \vee a)] \ \leq n = h(b) - h(a \wedge b)$$

So reducing the left hand side we obtain $h(b \vee a) - h(a) \leq h(b) - h(a \wedge b)$ and hence $h(a) + h(b) \geq h(a \wedge b) + h(a \vee b)$. □

Lemma 3.2 *Let D be a computation domain with properties I and C, and consider two arbitrary elements a and b in D with $a \leq b$. If there exists a maximal chain with finite length n between a and b, then all chains in $[a, b]$ are finite and have a length less than n.*

Proof: As in the proof of Theorem 3.1, we reason by induction on n. If $n = 0$ or $n = 1$ we have respectively $a = b$ or $a \prec b$, and the result is immediate. Assume now that the result is true provided there exists a maximal chain between two elements with length less than $n+1$. Consider two elements a and b for which there exists a maximal chain of length $n+1$:

$$a = x_0 \prec x_1 \prec x_2 \cdots \prec x_n \prec x_{n+1}$$

Take $Y = \{y_i\}_{i \in I}$ to be an arbitrary chain in $[a, b]$. Choose in Y an arbitrary element y distinct of a. Two cases may occur:

1. $\underline{x_1 \leq y}$ All chains from y to b are finite and include at most n elements by induction hypothesis, thus the set $Z = \{y_i | i \in I, y \leq y_i\}$ has at most $n+1$ elements.

2. $\underline{x_1 \parallel y}$ Then $y \prec x_1 \vee y$ by Property C_1 and $x_1 \neq x_1 \vee y$. By induction hypothesis, all chains between $x_1 \vee y$ and b are finite and include at most n elements. Thus, there exists a chain with at most $n+1$ elements between y and b, and by induction hypothesis the set Z defined above has at most $n+1$ elements. Since y was arbitrary different of a, the set $\{y_i \neq a\}_{i \in I}$ has at most $n+1$ elements, so Y has at most $n+2$ elements, and the chain Y has at most length $n+1$. $\Box$

We are now ready to prove the final result of this section.

Theorem 3.3 *Any upper section $[x)$ and any interval $[x, y]$ in a computation domain satisfying I and C is a computation domain satisfying these properties.*

Proof: We prove the result only for an upper section $[x)$. We have seen that $[x)$ is a computation domain in Proposition 3.3. Its isolated elements are of the form $x \vee d$ with $d \in \mathcal{A}(D)$ and $x \uparrow d$. Take an element d in $\mathcal{A}(D)$ which is not less than x. Since $x \wedge d$ and d are isolated, there exists a maximal chain

$$x \wedge d = z_0 \prec z_1 \prec \cdots \prec z_n = d$$

By Lemma 3.1, we have $z_j \vee x \preceq z_{j+1} \vee x \quad (0 \leq j \leq n-1)$. So

$$x \preceq z_1 \vee x \preceq z_2 \vee x \preceq \cdots \preceq z_n \vee x = d \vee x$$

Hence there exists a finite maximal chain from x to $x \vee d$ and, by the previous lemma, all chains from x to $x \vee d$ are finite. Hence $[x)$ has property I. Since D has property C_1, the upper section $[x)$ has property C. □

Definition 3.3 *We say that y is* finite relative to *x if y is isolated in $[x)$. This relation is written $x \prec y$.*

Corollary 3.3 *In a computation domain satisfying I and C, if y is finite relative to x then all maximal chains from x to y are finite and have the same length.*

Proof: Simply use Theorem 3.1 in $[x)$.□
Remarks: Standard texts about lattice theory provide alternate equivalents to property C, which is frequently called the *lower covering condition*. In [Bir67], a lattice that satisfies this condition and in which all chains are finite is called *semi-modular*. In [Mae72] the term *symmetric lattice* is used. Elements that cover the minimum element are also called *points* and the interest in semi-modular lattices comes from geometry. A lattice is called *geometric* if first it is semi-modular and second any element is the least upper bound of a set of *points*. The computation domains that we consider do not have this property which is replaced by *algebraicity*.

4 The incompatibility relation

Properties C and I concern only the structure of the sublattices in a computation domain. We must now examine more carefully the incompatibility relation. This study will lead us to postulate a new property concerning this relation.

Proposition 4.1 *If S is a consistent subset in a computation domain and all elements in S are compatible with a given element x, then $\bigcup S$ and x are compatible.*

Proof: The set $T = S \cup \{x\}$ is consistent and admits a least upper bound $\bigcup T$. Since S is consistent and included in T, $\bigcup S \leq \bigcup T$. Hence $\bigcup S$ and x are both less than $\bigcup T$, thus they are compatible.□

Corollary 4.1 *If a and x are two arbitrary elements in a computation domain, there exists a maximum element x/a less or equal to x and compatible with a. The element $a \vee (x/a)$ is called the* pseudo least upper bound *of a and x, and noted $a \underline{\vee} x$.*

Proof: Let S be the set of elements less than x compatible with a. By the previous proposition, $\bigcup S$ is compatible with a and the result is proved using $x/a = \bigcup S$. $\Box$

Proposition 4.2 *For any element a in a computation domain, the functions $\lambda x.x/a$ and $\lambda x.a\underline{\vee}x$ are continuous.*

Proof: We use the characterization of Lemma 1.2. First both functions are monotonic:

$$\begin{cases} x \leq x' \implies x/a \leq x'/a \\ x \leq x' \implies a\underline{\vee}x = a \vee x/a \;\leq\; a \vee x'/a = a\underline{\vee}x' \end{cases}$$

Consider now an approximant e of x/a. Since e is compatible with a we have $e = e/a$ so the function $\lambda x.x/a$ is continuous. Consider now an approximant e of $a\underline{\vee}x$. Since e is isolated and $e \leq a \vee x/a = \bigcup_{z \in \mathcal{A}(x/a)}(a \vee z)$, there exists an approximant d of x/a such that $e \leq a \vee d$. But when a and d are compatible, $a\underline{\vee}d = a \vee d$, hence we obtain $e \leq a\underline{\vee}d$. Therefore the function $\lambda x.a\underline{\vee}x$ is continuous. $\Box$

Remark: The function $\lambda x \lambda y.x\underline{\vee}y$ is not monotonic in its first argument. For example in domain T we have $\bot\underline{\vee}1 = 1$ and $0\underline{\vee}1 = 0$.

In a computation domain satisfying I and C, we can give a more precise characterization of the incompatibility relation.

Definition 4.1 *An interval [a,b] is called* prime *when $a \prec b$.*

Proposition 4.3 *In a partial order D, the intervals are ordered by the relation $\leq$ defined by:*

$$[a,b] \leq [c,d] \iff a = b \wedge c \text{ and } d = b \vee c$$

The resulting partial order is noted $I(D)$.

Proof:

- Reflexivity If $[a,b]$ is an interval, then $a \leq b$ so $a = b \wedge a$ and $b = b \vee a$. So $[a,b] \leq [a,b]$.

- Antisymmetry If $[a,b] \leq [c,d]$ then also $a \leq c$ and $b \leq d$. So from $[a,b] \leq [c,d] \leq [a,b]$ we deduce $a \leq c \leq a$ and $b \leq d \leq b$. By antisymmetry in D we obtain $a = c$ and $b = d$.

- Transitivity Consider three intervals $[a,b]$, $[c,d]$, $[e,f]$ and assume $[a,b] \leq [c,d] \leq [e,f]$. Using the definition we write

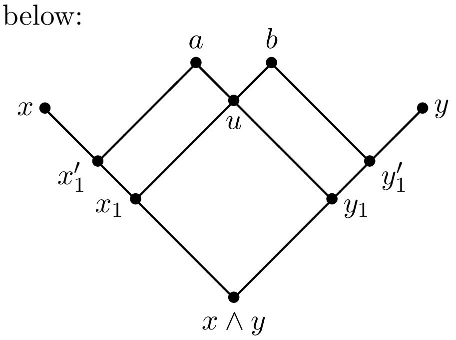
$$\begin{cases} a = b \wedge c, c = d \wedge e & \text{hence} \quad a = b \wedge d \wedge e \\ d = b \vee c, f = d \vee e & \text{hence} \quad f = b \vee c \vee e \end{cases}$$

  Now $b \leq d$ and $c \leq e$ yield $a = b \wedge e$ and $f = b \vee e$, i.e. $[a,b] \leq [e,f]$. □

Proposition 4.4 *Let D be a computation domain satisfying I and C. Two elements x and y in D are incompatible iff there are two prime intervals $[x_1, x_1']$ and $[y_1, y_1']$ included respectively in $[x \wedge y, x]$ and $[x \wedge y, y]$, and two prime intervals $[u,a]$ and $[u,b]$ with:*

$$[x_1, x_1'] \leq [u,a] \quad [y_1, y_1'] \leq [u,b] \quad x \wedge y \prec u \quad a \# b$$

Proof: The situation described in the statement of the proposition is summarized in the figure below:



Consider two incompatible elements x and y and let us reason in the computation domain $[x \wedge y)$. Since $\mathcal{A}(y)$ is a directed set, hence consistent, there exists necessarily an element y_0 in $\mathcal{A}(y)$ that is incompatible with x. Take $y_1 = y_0/x$. Since y_1 is less than y_0 which is isolated in $[x \wedge y)$, it is also isolated by Property I. Take for y_1' any element such that $y_1 \prec y_1' \leq y$. Such an element must exist because y_1 is compatible with x thus different of y, which is not, by hypothesis. By definition of y_1 we must have $y_1' \# x$. We notice then that $x \wedge y_1' = x \wedge y$ and perform the construction again, finding x_1 and x_1' isolated such that:

$$x_1 \uparrow y_1' \quad x_1 \prec x_1' \# y_1'$$

Now we take $u = x_1 \vee y_1$, $a = x_1' \vee y_1$, and $b = x_1 \vee y_1'$. Since x_1 and y_1 are isolated in $[x \wedge y)$, so is u. Since x_1' and y_1' dominate respectively x_1 and y_1, we can write:

$$a = x_1' \vee (x_1 \vee y_1) = x_1' \vee u$$

and

$$b = (x_1 \vee y_1) \vee y_1' = u \vee y_1'$$

Finally u dominates neither x_1' nor y_1' because $x_1' \# y_1'$. Thus $u \wedge x_1' = x_1$ and $u \wedge y_1' = y_1$. Using Property C, we conclude $u \prec x_1' \vee u = a$ and $u \prec y_1' \vee u = b$ and, since x_1' and y_1' are incompatible, $a \# b$.

The proposition is proved from left to right. Conversely, assume that we have two prime intervals $[x_1, x_1']$ and $[y_1, y_1']$ included respectively in $[x \wedge y, x]$ and $[x \wedge y, y]$, and two prime intervals $[u, a]$ and $[u, b]$ with:

$$[x_1, x_1'] \leq [u, a] \quad [y_1, y_1'] \leq [u, b] \quad a \# b$$

Elements a and b are incompatible and $b = u \vee y_1'$. Since a and u are compatible, then a and y_1' must be incompatible. But $a = x_1' \vee u$ and $u \uparrow y_1'$. So finally $x_1' \# y_1'$, and consequently $x \# y$. □

We introduce now a new property, Property Q, that restricts the way in which incompatibilities may appear.

Property Q

If x and y are two incompatible isolated elements

$$x \wedge y \prec x \implies \exists! t \quad t \# x,\ x \wedge y \prec t \leq y$$

Very simple finite computation domains fail to have Property Q.

For example the domains whose diagrams are represented on Figure 7 do not satisfy Q. For the first one, we observe that a and b are incompatible, with $a \wedge b = \perp$ and $\perp \prec b$. But c is the only element in $[\perp, a]$ that covers $\perp$, and it is compatible with d. So there exists *no element* t such that $a \wedge b \prec t \leq a$ and $t \# b$. In the second case, the domain of Figure 7 (b), it is unicity that is

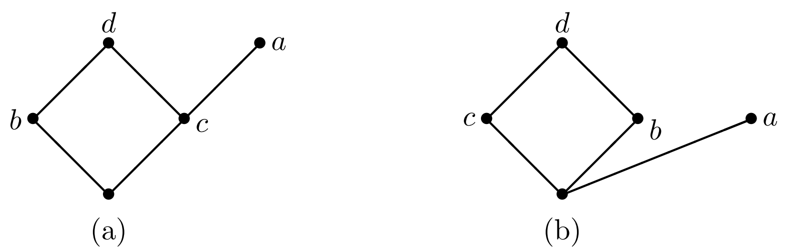


Figure 7: (a) and (b) fail to have Property Q

not satisfied. Indeed, elements a and d are incompatible, and $\bot = a \wedge d \prec a$. But both b and c cover $\bot$, are less than d and are incompatible with a.

These examples suggest that Property Q may be considered as the conjunction of two simpler properties.

Notation: Let x be an arbitrary element in a computation domain D. We will note P_x the set $\{z | x \prec z\}$ of atoms of $[x)$. On P_x we can define the relation $\mathcal{R}_x$ by

$$a\mathcal{R}_x b \iff a\#b \text{ or } a = b$$

Relation $\mathcal{R}_x$ is of course reflexive and symmetric.

Proposition 4.5 *In a computation domain D, Property* Q *is equivalent to the conjunction of the following properties* $\mathrm{Q_E}$ *and* $\mathrm{Q_U}$*:*

Property $\mathrm{Q_E}$ (Existence of a minimal incompatible element)

$$\forall x, y \in \mathcal{A}(D) \quad x\#y,\ x \wedge y \prec x \implies \exists t\#x,\ x \wedge y \prec t \leq y$$

Property $\mathrm{Q_U}$ (Uniqueness)

$$\forall x \in \mathcal{A}(D) \quad \mathcal{R}_x \text{ is an equivalence relation on } P_x$$

Proof:

i) Q implies $\mathrm{Q_E}$ and $\mathrm{Q_U}$. It is immediate that Q implies $\mathrm{Q_E}$, which is weaker. But we already know that $\mathcal{R}_x$ is reflexive and symmetric, so we need only to show that Q implies that $\mathcal{R}_x$ is transitive. Consider three elements a, b, and c of P_x with $a\mathcal{R}_x b$ and $b\mathcal{R}_x c$. If $a = b$ or $b = c$ we have immediately $a\mathcal{R}_x c$. Suppose now $a\#b$ and $b\#c$. We need to show that either $a = c$ or $a\#c$. Assume we had $a \uparrow c$. From $b\#a$ and $b\#c$ we deduce $b\#a \vee c$. There can be only one element t such that $b\#t \leq a \vee c$ by Property Q. But both a and c satisfy this condition. Hence $a = c$.

ii) Assume now $\mathrm{Q_E}$ and $\mathrm{Q_U}$. Consider two isolated elements x and y with $x\#y$ and $x \wedge y \prec x$. By $\mathrm{Q_E}$ there exists an element t with $x\#t$ and $x \wedge y \prec t \leq y$. Let now t' be an arbitrary element such that $x\#t'$ and $x \wedge y \prec t' \leq y$. In $P_{x\wedge y}$ we have $x\mathcal{R}_{x\wedge y}t$ and $x\mathcal{R}_{x\wedge y}t'$. Thus, since $\mathcal{R}_{x\wedge y}$ is an equivalence relation $t\mathcal{R}_{x\wedge y}t'$. But t and t' are compatible, because both are less than y. So $t = t'$. Hence $\mathrm{Q_E}$ and $\mathrm{Q_U}$ imply Q. □

Definition 4.2 *Two prime intervals $[x, x']$ and $[y, y']$ are* equipollent *when $x = y$ and $x'\mathcal{R}_x y'$.*

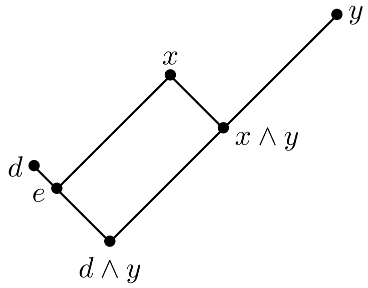
We call $IP(X)$ the set of prime intervals in a partial order X. The previous result shows that if D has Property Q, *equipollence is an equivalence relation* on $IP(\mathcal{A}(D))$.

Following what we did for Property C, we will show that it is sufficient to postulate property Q on the isolated elements in a computation domain for it to be valid in the whole domain.

Proposition 4.6 *In a computation domain satisfying I and C, consider two arbitrary elements x and y such that $x\#y$ and $x \wedge y \prec x$. There exists an approximant e of x with*

$$e\#y,\ e \wedge y \prec e \quad \textit{and} \quad e \vee (x \wedge y) = x$$

Proof:



If x is incompatible with y, there exists an approximant d of x incompatible with y since $\mathcal{A}(x)$ is a consistent subset, using Proposition 4.1. Since d is therefore not comparable with y, we have necessarily $d \wedge y < d$. We can then find, by Corollary 3.1 an element e with $d \wedge y \prec e \leq d$. Since e covers $d \wedge y$ and is not less than y we have also $e \wedge y = d \wedge y$. By Property $\widehat{\mathrm{C}}$ we obtain $x \wedge y \prec (x \wedge y) \vee e$. Since e is an approximant of x, the element $(x \wedge y) \vee e$ is less than x. As x covers $x \wedge y$ by hypothesis, we obtain $(x \wedge y) \vee e = x$. Finally, elements e and y are incompatible, otherwise we would have $x = e \vee (x \wedge y) \leq e \vee y$ so x and y would be incompatible, which contradicts the hypothesis. □

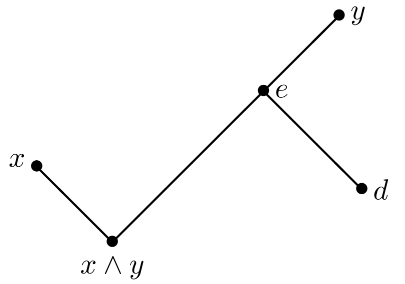
Lemma 4.1 *In a computation domain satisfying properties I and C, Property* $\mathrm{Q_E}$ *is equivalent to Property* $\widehat{\mathrm{Q_E}}$*:*

Property $\widehat{\mathrm{Q_E}}$

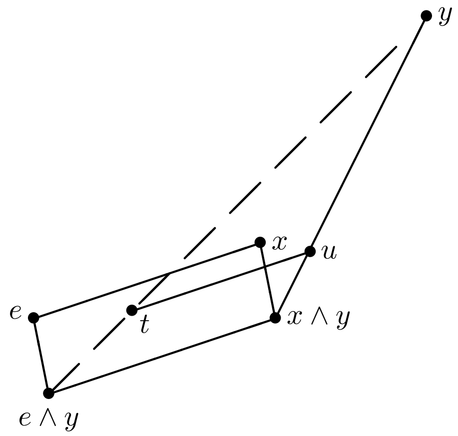
$$\forall x, y \quad x\#y,\ x \wedge y \prec x \Longrightarrow \exists t\#x,\ x \wedge y \prec t \leq y$$

Proof: Property $\widehat{\mathrm{Q_E}}$ trivially implies Property $\mathrm{Q_E}$. The converse is proved in two steps.

1. Assume first that x is *isolated* and y is an arbitrary element with $x\#y$ and $x \wedge y \prec x$. As we have remarked before, there exists an approximant d of y which is incompatible with x. Since both d and $x \wedge y$ are less than y, define e by $e = d \vee (x \wedge y)$. The element e is isolated because both d and $x \wedge y$ are, and incompatible with x because d is. Hence $x \wedge y = x \wedge e$ and we can use property $\mathrm{Q_E}$. There exists t with $x\#t$ and $x \wedge e \prec t \leq e$, and we deduce immediately $x\#t$ and $x \wedge y \prec t \leq y$.



2. Consider now an arbitrary x. By Proposition 4.6, there exists an isolated element e with $e\#y$, $e \wedge y \prec e$, and $e \vee (x \wedge y) = x$.



So we can use the result of the first case and find an element t with $e\#t$ and $e \wedge y \prec t \leq y$. We notice now first that t and $x \wedge y$ are compatible (both are less than y) and second that t is not less than $x \wedge y$ (because t is incompatible with e); so we deduce $t \wedge (x \wedge y) = e \wedge y$. Using property

$\widehat{\mathrm{C}}$:

$$x \wedge y \prec (x \wedge y) \vee t = u$$

The element u is incompatible with e thus with x and we have as requested $x \wedge y \prec u \leq y$. □

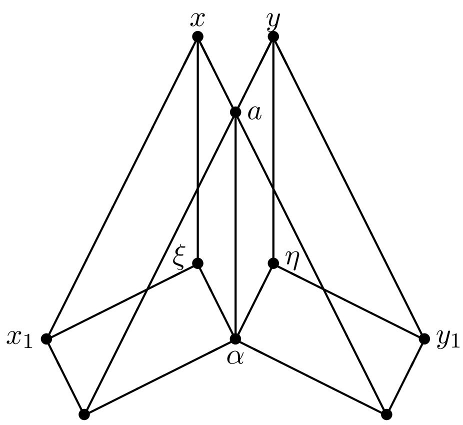
Proposition 4.7 *In a computation domain with properties I and C, let a, x, and y be three elements satisfying*

$$(G_1) \qquad a \prec x,\, a \prec y,\, x\#y$$

Then there are three elements α, ξ, and η approximants (resp.) of a, x, and y in the configuration correponding to (G_1), as well as:

$$x = \xi \vee a \quad \textit{and} \quad y = \eta \vee a$$

Proof:



Applying twice Proposition 4.6, we can find x_1 and y_1, approximants of x and y (resp.) with

$$\begin{cases} x_1\#y, x_1 \wedge y \prec x_1, x_1 \vee a = x \\ y_1\#x, y_1 \wedge x \prec y_1, y_1 \vee a = y \end{cases}$$

Now take $\alpha = (x_1 \wedge y) \vee (y_1 \wedge x)$. The element α is an approximant of a and it dominates neither x_1 nor y_1. So:

$$\begin{cases} \alpha \wedge x_1 = x_1 \wedge y \prec x_1 \\ \alpha \wedge y_1 = x \wedge y_1 \prec y_1 \end{cases}$$

By property C we obtain

$$\begin{cases} \alpha \prec\!\!\!\prec \alpha \vee x_1 = \xi \\ \alpha \prec\!\!\!\prec \alpha \vee y_1 = \eta \end{cases}$$

and since ξ and η are necessarily incomparable with a:

$$\begin{cases} x = \xi \vee a \\ y = \eta \vee a \end{cases}$$

If ξ and η were compatible, the set $\{\xi, \eta, a\}$ would be consistent, admitting thus a lub that would dominate $\xi \vee a$ and $\eta \vee a$. But this is impossible because x and y are incompatible by hypothesis. So we have:

$$\alpha \prec\!\!\!\prec \xi,\ \alpha \prec\!\!\!\prec \eta,\ \xi \# \eta$$

□

Proposition 4.8 *In a computation domain with properties I and C, let a, x, y, and z be four elements satisfying*

$$(G_2) \qquad a \prec\!\!\!\prec x,\ a \prec\!\!\!\prec y,\ x\#y,\ y\#z,\ x \neq z$$

Then there are four elements α, ξ, η, and ζ approximants (resp.) of a, x, y, and z satisfying (G_2) as well as:

$$x = \xi \vee a \quad y = \eta \vee a \quad z = \zeta \vee a$$

Proof: First we apply the previous result to the three elements a, x, and y. We can find α_1, ξ_1, and η_1 approximants of a, x, and y with:

$$\begin{cases} \alpha_1 \prec\!\!\!\prec \xi_1,\ \alpha_1 \prec\!\!\!\prec \eta_1,\ \xi_1 \# \eta_1 \\ x = \xi_1 \vee a,\ y = \eta_1 \vee a \end{cases}$$

Consider now $[\alpha_1)$. By Proposition 3.1, we can find an element ζ such that $\alpha_1 \prec \zeta$ with

$$\begin{cases} \zeta \wedge a = \alpha \prec\!\!\!\prec \zeta \\ z = \zeta \vee a \end{cases}$$

Since α_1 is isolated, so is ζ as well as the elements ξ and η defined by

$$\begin{cases} \xi = \xi_1 \vee \alpha \\ \eta = \eta_1 \vee \alpha \end{cases}$$

(Since ξ_1 and η_1 are compatible with a, they are *a fortiori* compatible with α). Since ξ_1 and η_1 cannot be less than a hence than α

$$\begin{cases} \xi_1 \wedge \alpha = \alpha_1 \prec \xi_1 \\ \eta_1 \wedge \alpha = \alpha_1 \prec \eta_1 \end{cases}$$

and by property C:

$$\begin{cases} \alpha \prec \xi \\ \alpha \prec \eta \end{cases}$$

We also have $\alpha \prec \zeta$. Let us show the remaining properties. First, $\xi \# \eta$ since $\xi_1 \# \eta_1$. Next we have:

$$\begin{cases} x = a \vee \xi_1 = a \vee \xi_1 \vee \alpha = a \vee \xi \\ y = a \vee \eta_1 = a \vee \eta_1 \vee \alpha = a \vee \eta \\ z = a \vee \zeta \end{cases}$$

If η and ζ were compatible the set $\{a, \eta, \zeta\}$ would be consistent, which contradicts the fact that x and y are incompatible. So we have also $\eta \# \zeta$. Last, since $x \neq z$, we have trivially $\xi \neq \zeta$. □

Remark: In the previous propositions, as well as in several propositions in this section, we use freely coherence, which sometimes leads to shorter proofs. However this property is not necessary for the results to hold.

Lemma 4.2 *In a computation domain satisfying properties I and C, Property* $\mathrm{Q_U}$ *is equivalent to Property* $\widehat{\mathrm{Q_U}}$*:*

Property $\widehat{\mathrm{Q_U}}$

In $IP(D)$, equipollence is an equivalence relation.

Proof: Property $\widehat{\mathrm{Q_U}}$ implies trivially property $\mathrm{Q_U}$ which is weaker. The converse is a corollary of the previous result. Let $[a, x]$, $[a, y]$, and $[a, z]$ be three intervals with $[a, x]\mathcal{R}[a, y]$ and $[a, y]\mathcal{R}[a, z]$. As in Proposition 4.5, the only non-trivial case is when $x \uparrow z$ with $x \# y$, $y \# z$, and $x \neq z$. By Proposition 4.6, we can then find approximants α,ξ,η,ζ for a,x,y,z with:

$$\alpha \prec \xi \quad \alpha \prec \eta \quad \alpha \prec \zeta \quad \xi \# \eta \quad \eta \# \zeta \quad \xi \neq \zeta$$

as well as $x = a \vee \xi$ and $z = a \vee \zeta$. So if x and z are compatible, so are ξ and ζ. But property $\mathrm{Q_U}$ excludes this possibility. So x and z must be incompatible and the equipollence relation is an equivalence on prime intervals. □

Corollary 4.2 *In a domain satisfying I and C, property Q is equivalent to property* $\widehat{\mathrm{Q}}$*:*

Property $\widehat{\mathrm{Q}}$

If x *and* y *are two incompatible elements*

$$x \wedge y \prec x \implies \exists! t \quad t \# x,\ x \wedge y \prec t \leq y$$

Proof: It is easy to show, as in Proposition 4.5, that $\widehat{\mathrm{Q}}$ is equivalent to the conjunction of $\widehat{\mathrm{Q_E}}$ and $\widehat{\mathrm{Q_U}}$. □

Corollary 4.3 *In a domain D satisfying properties I, C, and Q, an upper section also satisfies these properties.*

Proof: Consider an arbitrary upper section $[a)$. As a computation domain, $[a)$ has properties I and C. If x and y are two elements of $[a)$, then $x \wedge y$ also belongs to $[a)$. So if D satisfies property $\widehat{\mathrm{Q}}$, so does $[a)$. □
Notation: If $[a, b]$ and $[c, d]$ are equipollent prime intervals, we write now $[a, b] \simeq [c, d]$.

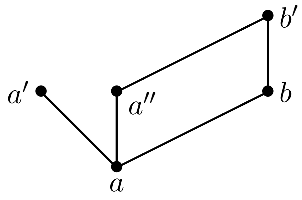
Definition 4.3 *In a partial order D, two intervals are* transposed *iff they are* comparable *as elements of* $I(D)$*.*

We call $\mathcal{T}$ the transposition relation. This relation is obviously reflexive and symmetric.

Lemma 4.3 *In a computation domain satisfying I, C, and Q, equipollence and transposition* commute *on* $IP(D)$*, i.e.* $\simeq \circ \mathcal{T} = \mathcal{T} \circ \simeq$.

Proof: Consider prime intervals $[a, a']$, $[a, a'']$, and $[b, b']$ such that $[a, a'] \simeq [a, a'']$ and $[a, a'']\mathcal{T}[b, b']$. We must show that there exists a prime interval $[b, b'']$ such that $[a, a']\mathcal{T}[b, b'']$ and $[b, b''] \simeq [b, b']$. If $a' = a''$ then $[a, a']\mathcal{T}[b, b']$ and we can take $[b, b''] = [b, b']$. Thus, assume $a' \# a''$. If $[a, a''] = [b, b']$, we can take $[b, b''] = [a, a']$. Two cases are still possible:

<u>Case 1: $[a, a''] < b, b']$</u>

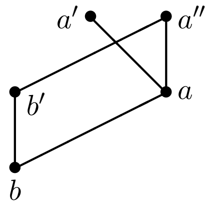


In this case, a' is necessarily compatible with b. Assume indeed $a'\#b$. By property Q, there exists an element t with $a'\#t$ and $a \prec t \leq b$. Therefore $[a, a''] \simeq [a, a'] \simeq [a, t]$. By Q again $[a, a''] = [a, t]$. Now

- either $a''\#t$, but this is impossible because both a' and t are less than b'
- or $a'' = t$, but this is also impossible because $a'' \wedge b = a \neq a''$ so a'' is not less than b while t is less than b.

So we can take $b'' = a' \vee b$. Since $a' \wedge b = a \prec a'$, by property C $b \prec b''$. Finally, elements b'' and b cannot be compatible, because otherwise a' and a'' would be compatible, which contradicts the hypothesis. We have $[a, a']\mathcal{T}[b, b'']$ and $[b, b''] \simeq [b, b']$, which concludes this case.

Case 2: $[a, a''] > [b, b']$



In this case, a' and b' are necessarily incompatible. Indeed, if a' and b' were compatible, the element $a' \vee b' = a' \vee a \vee b' = a \vee a''$ would exist, which contradicts $a'\#a''$. From $a'\#b'$ we deduce by Q, since $a' \wedge b' = b \prec b'$, that there exists an element b'' with $b''\#b'$ and $b \prec b' \leq a'$. This element b'' is not less than a, otherwise a'' would dominate b' and b'', so $b'' \vee a = a'$ and $b'' \wedge a = b$. So $[a, a']\mathcal{T}[b, b'']$ and $[b, b''] \simeq [b, b']$, which concludes this case and the proof of the Lemma. □

Definition 4.4 *The* projectivity *relation is the transitive closure of transposition.*

This relation is an equivalence relation written $\sim$. If intervals $[a, b]$ and $[c, d]$ satisfy $[a, b] \sim [c, d]$, they are called *projective* intervals. We will only consider this relation for prime intervals.

Theorem 4.1 *On the prime intervals of a partial order satisfying I, C, and Q, equipollence and projectivity are commuting equivalence relations, i.e:*

$$\simeq \circ \sim = \sim \circ \simeq$$

Proof: By the previous lemma we know that $\simeq \circ \mathcal{T} = \mathcal{T} \circ \simeq$. Let us show by induction that for any n, n positive, we have:

$$\simeq \circ \mathcal{T}^n = \mathcal{T}^n \circ \simeq$$

The case were $n = 1$ is immediate and

$$\begin{aligned} \simeq \circ \mathcal{T}^{n+1} &= (\simeq \circ \mathcal{T}^n) \circ \mathcal{T} \\ &= (\mathcal{T}^n \circ \simeq) \circ \mathcal{T} && \text{by induction hypothesis} \\ &= \mathcal{T}^n \circ (\simeq \circ \mathcal{T}) && \text{by associativity} \\ &= \mathcal{T}^n \circ (\mathcal{T} \circ \simeq) \\ &= \mathcal{T}^{n+1} \circ \simeq && \text{by associativity again} \end{aligned}$$

As $[a, b] \simeq \circ \sim [c, d]$ iff there is an integer n such that $[a, b] \simeq \circ \mathcal{T}^n [c, d]$, we have then also $[a, b] \mathcal{T}^n \circ \simeq [c, d]$ hence $[a, b] \sim \circ \simeq [c, d]$. □

The product of the equivalence relations $\simeq$ and $\sim$ is again an equivalence relation that we will write $\approx$. Since the relation $\approx$ extends $\simeq$, we will say *from now on* that the prime intervals $[a, b]$ and $[c, d]$ are *equipollent* iff $[a, b] \approx [c, d]$.

Before studying further equipollence and projectivity, we try to give an intuitive feeling for the meaning of these relations.

Example 1: Consider the domain O^3 whose diagram is shown on Figure 8. Since this domain is a lattice, it cannot be used to illustrate equipollence. However, there are three equivalence classes for the projectivity relation $\sim$.

1. $[(\bot, \bot, \bot), (\top, \bot, \bot)] \sim [(\bot, \top, \bot), (\top, \top, \bot)] \sim [(\bot, \top, \top), (\top, \top, \top)] \sim [(\bot, \bot, \top), (\top, \bot, \top)]$
2. $[(\bot, \bot, \bot), (\bot, \top, \bot)] \sim [(\top, \bot, \bot), (\top, \top, \bot)] \sim [(\top, \bot, \top), (\top, \top, \top)] \sim [(\bot, \bot, \top), (\bot, \top, \top)]$
3. $[(\bot, \bot, \bot), (\bot, \bot, \top)] \sim [(\top, \bot, \bot), (\top, \bot, \top)] \sim [(\top, \top, \bot), (\top, \top, \top)] \sim [(\bot, \top, \bot), (\bot, \top, \top)]$

Example 2: Consider the domain $O \times T$ whose diagram is shown on Figure 9. Here, there are three equivalence classes for the projectivity relation $\sim$.

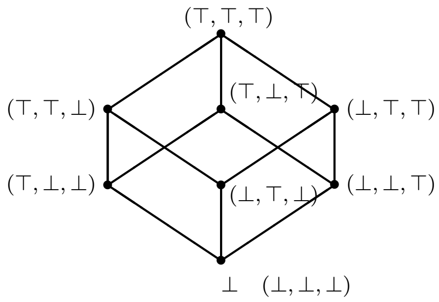


Figure 8: Domain O^3

1. $[(\bot,\bot),(0,\bot)] \sim [(\bot,\top),(0,\top)]$
2. $[(\bot,\bot),(1,\bot)] \sim [(\bot,\top),(1,\top)]$
3. $[(0,\bot),(0,\top)] \sim [(\bot,\bot),(\bot,\top)] \sim [(1,\bot),(1,\top)]$

The union of classes 1 and 2 is an equivalence class for the equipollence relation, while class 3 is a second one. The fact the O contains two incompatible atoms results in the first equipollence class containing exactly two projectivity classes. The fact that we have a cartesian product of two domains can be seen in the presence of two equipollence classes. With the help of these two equivalence relations, we are able to analyze the structure of a computation domain. Naturally, the Representation Theorem will be based on these relations, that we study now in greater depth.

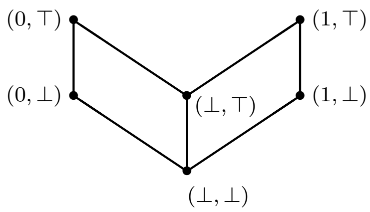


Figure 9: Domain $O \times T$

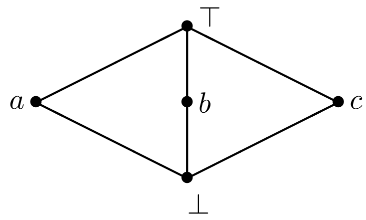


Figure 10: The diamond domain

5 The projectivity relation

If two prime intervals are projective, we would like them to represent *the same elementary information increment*, possibly taking place in two distinct global states. We shall call now an *elementary decision*, or more briefly a *decision*, an equivalence class of projective prime intervals. However, such an interpretation of projectivity faces an inconsistency that can only be eliminated by postulating an additional property.

Consider the partial order on Figure 10. It is trivial to verify that this partial order is a computation domain satisfying I,C, and Q. Since we have also

$$[\bot, a] \leq [b, \top] \geq [\bot, c] \leq [a, \top] \geq [\bot, b] \leq [c, \top]$$

all prime intervals in this lattice belong to one and the same projectivity class. It is difficult to accept that a single elementary decision may allow the construction of four different elements. More specifically, two precise facts run counter to our interpretation:

i) All prime intervals of the form $[\bot, x]$ are projective, and should constitute the same elementary decision,

ii) To go from $\bot$ to b, for example, the "decision" is the same one as to go from b to $\top$.

The lattice of Figure 10 plays an important role in lattice theory so one might try simply to exclude such a configuration with five elements from a computation domain. We will see that if a computation domain is a lattice, this idea is valid. But as there are incompatible elements, the situation is more intricate. Consider for example the domain of Figure 11, which is represented by a Hasse diagram "seen from above".

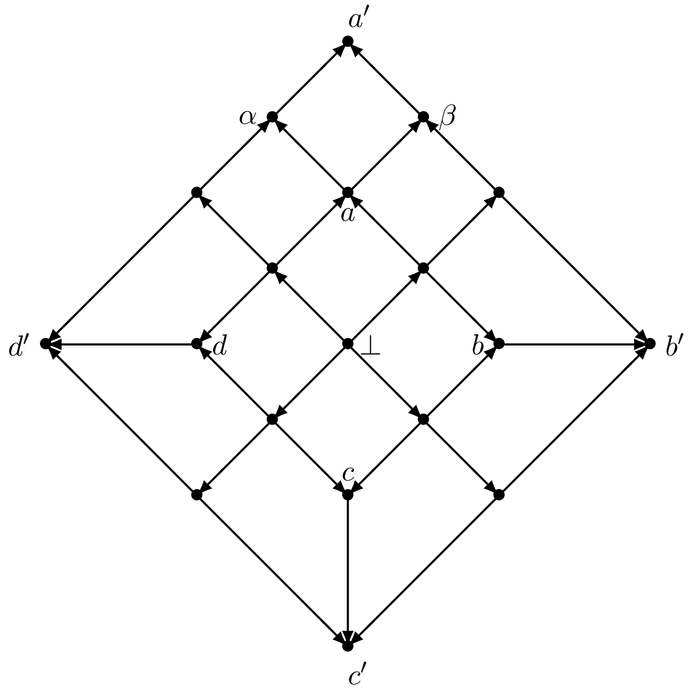


Figure 11: Diamond is not a sublattice

Arrows point upwards in the partial order. A sublattice of this domain must be a sublattice of one of the intervals $[\bot, a']$, $[\bot, b']$, $[\bot, c']$, or $[\bot, d']$ because elements a', b', c', and d' are maximal and incompatible. But it is clear that none of these intervals contains a sublattice that is isomomorphic to the five element lattice of Figure 10. However, phenomena that we have considered above as inconsistent with our intuition still occur: in the interval $[a, a']$ all prime intervals are projective. In a similar fashion, the 25 element domain of Figure 12 shows that two distinct prime intervals may be simultaneously projective and equipollent: $[a, a_1] \sim [a, a_2]$ and $a_1 \# a_2$. But in our understanding, two distinct equipollent prime intervals should correspond to two contradictory elementary information increases.

The examples above, due to Gordon Plotkin, point to a new property, that we call property R.

Property R

If $[a, x]$ and $[a, y]$ are two *projective* prime intervals with isolated endpoints, then $x = y$

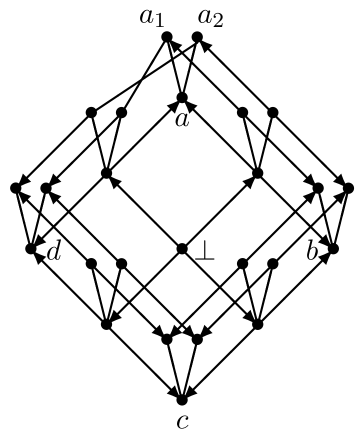


Figure 12: Another counterexample

This property can be stated in the following way: if a is an isolated element, then two distinct elementary increases from a are two distinct decisions.

Before examining the many consequences of property R, we show as is now customary that the property is valid for two arbitrary prime intervals.

Proposition 5.1 *Consider two prime intervals with isolated endpoints* $[a, a']$ *and* $[b, b']$*, in a computation domain satisfying I and C. If there exists a prime interval* $[c, c']$ *such that*

$$[a, a'] \le [c, c'] \ge [b, b']$$

then there exists a prime interval $[d, d']$ *with* isolated endpoints *such that*

$$[a, a'] \le [d, d'] \ge [b, b']$$

Proof: By hypothesis, $c' = a \vee c$ hence $c' = a \vee (\bigcup \mathcal{A}(c))$ and $c' = \bigcup_{z \in \mathcal{A}(c)} (a' \vee z)$. The set $\{a' \vee z | z \in \mathcal{A}(c)\}$ is directed and it dominates b'. So there exists an isolated element e with $b' \le a' \vee e,\ e \in \mathcal{A}(c)$. Take $d = e \vee a \vee b$. The element d is an approximant of c that dominates a and b. So d dominates neither a' nor b' and, by property C, $d \prec d \vee a'$ and $d \prec d \vee b'$. So since $b' \le a' \vee e \le a' \vee e \vee a \vee b = a' \vee d$ we have $d \prec d \vee b' \le d \vee a'$. Elements $d \vee b'$ and $d \vee a'$ are thus equal to the same element d' and

$$[a, a'] \le [d, d'] \ge [b, b']$$

□

Proposition 5.2 *In a computation domain satisfying I and C, property R is equivalent to property* $\widehat{\mathrm{R}}$*:*

Property $\widehat{\mathrm{R}}$

If $[a,x]$ and $[a,y]$ are two projective prime intervals then $x = y$

Proof: Property $\widehat{\mathrm{R}}$ implies trivially property R. Conversely, consider two arbitrary projective prime intervals $[a,x]$ and $[a,y]$. There exists a sequence $\{[x_i,x_i']\}_{0\leq i\leq n}$ with $[x_0,x_0'] = [a,x]$ and $[x_n,x_n'] = [a,y]$ such that

$$[x_0,x_0']\mathcal{T}[x_1,x_1']\cdots\mathcal{T}[x_n,x_n']$$

By Lemma 3.1, we can find intervals with isolated endpoints $[z_i,z_i'] \leq [x_i,x_i']$ $(0 \leq i \leq n)$. If we take now $[t_i,t_i'] = [x_i \vee x_{i+1}, x_i' \vee x_{i+1}']$ $(0 \leq i \leq n-1)$ we have

$$[z_i,z_i'] \leq [t_i,t_i'] \geq [z_{i+1},z_{i+1}'] \quad (0 \leq i \leq n-1)$$

By the previous proposition, there are prime intervals with isolated endpoints $[u_i,u_i']$ $(0 \leq i \leq n-1)$ such that

$$[z_i,z_i'] \leq [u_i,u_i'] \geq [z_{i+1},z_{i+1}'] \quad (0 \leq i \leq n-1)$$

As a consequence, $[z_0,z_0']$ and $[z_n,z_n']$ are projective in $\mathcal{A}(D)$. From $[z_0,z_0'] \leq [a,x]$ and $[z_n,z_n'] \leq [a,y]$ we deduce that z_0 and z_n are both less than a and we can take $z = z_0 \vee z_n$. This element z cannot dominate z_0' nor z_n' since it is an approximant of a that does not dominate them. Hence

$$\begin{cases} z \wedge z_0' = z_0 \\ z \wedge z_n' = z_n \end{cases}$$

therefore

$$\begin{cases} z \prec z \vee z_0' = z' \\ z \prec z \vee z_n' = z'' \end{cases}$$

which shows that $[z_0,z_0'] \leq [z,z']$ and $[z_n,z_n'] \leq [z,z'']$. Since z is isolated and $[z,z'] \sim [z,z'']$, we can use property R and deduce $z' = z''$. Since we have also $[z,z'] \leq [a,x]$ and $[z,z''] \leq [a,y]$ we conclude $x = y$. $\square$

Corollary 5.1 *In a domain satisfying I,C, Q , and R, any upper section (and any interval) satisfyies these properties.*

Proof: Consider an upper section $[b)$. If the prime intervals $[a,x]$ and $[a,y]$ are projective in $[b)$, they are also projective in the whole domain. Hence $x = y$, so the upper section $[b)$ has property R. We know from before that it has properties I,C, and Q. $\square$

Proposition 5.3 *In a computation domain D satisfying I,C, and R consider two compatible elements x and y. If $[a, a']$ is a prime interval such that*

$$a \leq x \wedge y \quad \textit{and} \quad a' \leq x \vee y$$

Then either $a' \leq x$ or $a' \leq y$.

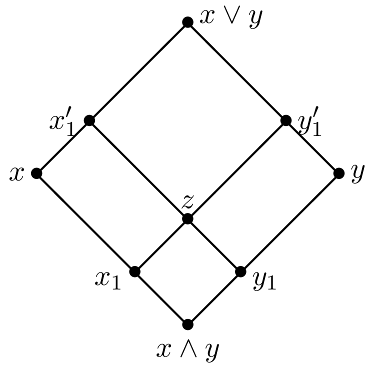
Proof: Notice first that in the case where x and y are comparable, say $x \leq y$, we have immediately $a' \leq y = x \vee y$ so that the proposition holds trivially. Suppose now $x \parallel y$. We can also assume $a' \not\leq x \wedge y$ otherwise the proposition is again immediate. Consider first the case where x and y are finite relative to $x \wedge y$.

Case 1. $x \wedge y \prec x, x \wedge y \prec y$ The proof is by induction on the sum $\delta(x, y)$ of the lengths of the maximal chains from $x \wedge y$ to x and from $x \wedge y$ to y.

i) Base case Since $x \parallel y$ the first case to consider is when $\delta(x, y) = 2$, i.e. $x \wedge y -\!\!< x$ and $x \wedge y -\!\!< y$. From $a' \parallel x \wedge y$ we deduce $a = a' \wedge x \wedge y$, and by property C, which we can use because $a' \uparrow x \wedge y$ we obtain: $x \wedge y -\!\!< a'' \leq x \vee y$ with $a'' = a' \vee (x \wedge y)$. Now either $a'' = x$ and then $a' \leq x$, or $a'' \neq x$ and then, by Property C, we have $x -\!\!< a'' \vee x \leq x \vee y$. But we have also $x -\!\!< x \vee y$ so $a'' \vee x = x \vee y$. From $[x \wedge y, a''] \leq [x, x \vee y] \geq x \wedge y, y]$ we deduce by property R that $a'' = y$.

As a result, we have indeed when $\delta(x, y) = 2$ either $a' \leq x$ or $a' \leq y$.

ii) Induction step Assume now $\delta(x, y) = n$, $n \geq 2$. Since x and y are incomparable we have $x \wedge y < x$ and $x \wedge y < y$. By atomicity, there are two elements x_1 and y_1 with $x \wedge y -\!\!< x_1 \leq x$ and $x \wedge y -\!\!< y_1 \leq y$. Take now $z = x_1 \vee y_1$, $x'_1 = x \vee y_1 = x \vee z$, $y'_1 = y \vee x_1 = y \vee z$. Elements x'_1, y'_1, and z do exist because x and y are compatible.



Two cases are now possible:

Case 1.1 $a' \leq z$: Then the result of the base case may be used to deduce that either $a' \leq x_1$ or $a' \leq y_1$, thus either $a' \leq x$ or $a' \leq y$.

Case 1.2 $a' \not\leq z$: Then $a = a' \wedge z$. Since a' and z are both less than $x \vee y$ they are comaptible and we can use property C. With $a'' = z \vee a'$ we have $z \prec\!\!\!\cdot\; a'' \leq x \vee y$. But $x_1' \vee y_1' = x \vee z \vee y \vee z = x \vee y$ and since z is less than x_1' and y_1' we have also $z \leq x_1' \wedge y_1'$. To be in a position to apply the induction hypothesis to the interval $[z, a'']$ and elements x_1' and y_1', we need only verify that $\delta(x_1', y_1') < \delta(x, y)$. Now $\delta(x_1', y_1')$ is less than the sum of the lengths of maximal chains from z to x_1' and from z to y_1'. So $\delta(x_1', y_1') \leq n - 2$. Applying the induction hypothesis yields that either $a'' \leq x_1'$ or $a'' \leq y_1'$. Assume without loss of generality that $a'' \leq x_1'$. Since $a' \leq a''$ we have also $a' \leq x_1'$. But $\delta(x, z) \leq n - 1$. We can use the induction hypothesis again for the interval $[a, a']$ and the elements x and z, to conclude that either $a' \leq z$ or $a' \leq x$. We have assumed that $a' \not\leq z$. So $a' \leq x$.

Case 2. Assume now x and y are arbitrary and take again $a'' = a' \vee (x \wedge y)$. Since the upper section $[x \wedge y)$ is a computation domain, there are approximants x' and y' of x and y in this domain such that the atom a'' is dominated by $x' \vee y'$. Then $a \leq x \wedge y \leq x' \wedge y'$ and $a' \leq x' \vee y'$ with $x' \wedge y' \prec x'$ and $x' \wedge y' \prec y'$. Using the result of the first case, we deduce that either $a' \leq x'$ or $a' \leq y'$, so that again $a' \leq x$ or $a' \leq y$.□

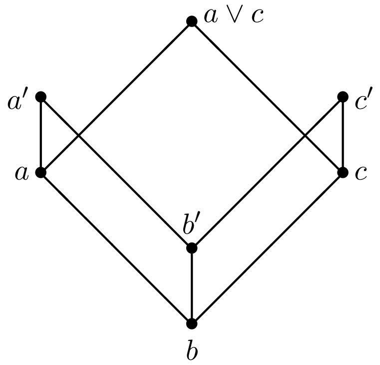
Corollary 5.2 *In a computation domain satisfying properties I, C and R, no sublattice is isomorphic to the sublattice of figure 10.*

Proof: Let x and y be two arbitrary compatible, incomparable elements. Take any z such that $x \wedge y < z < x \vee y$. By atomicity, there is an element t with $x \wedge y \prec\!\!\!\cdot\; t \leq z$. By the previous result, either $t \leq x$ or $t \leq y$. In the first case, $x \wedge y < x \wedge z$ and in the second case $x \wedge y < y \wedge z$. □

To prove the converse, we need a very useful result that limits the cases that we need to consider when two intervals are projective. This result is obtained in two steps.

Proposition 5.4 *In a computation domain satisfying I, C, and R, consider three prime intervals* $[a, a']$, $[b, b']$, *and* $[c, c']$ *such that* $[a, a'] \geq [b, b'] \leq [c, c']$. *If* a *and* c *are compatible, then we have also* $[a, a'] \leq [a \vee c, a' \vee c'] \geq [c, c']$.

Proof:



By definition of the relation $\leq$ for intervals we have $a' = a \vee b'$ and $c' = b' \vee c$. Since a and c are compatible, the triple $\{a, b', c\}$ is consistent. It has a lub $d' = a \vee b' \vee c$. But

$$\begin{cases} d' &= a \vee c \vee b' &= (a \vee c) \vee (c \vee b') &= (a \vee c) \vee c' \\ d' &= a \vee c \vee b' &= (a \vee c) \vee (a \vee b') &= (a \vee c) \vee a' \end{cases}$$

Take $d = a \vee c$. Since b' is not less than a nor c, by the previous proposition b' is not less than $a \vee c$. Thus d' is different from d, so $d \prec d'$ by property C. Since d' dominates a' and c', $d' = a' \vee c'$. Since we have $[a, a'] \leq [d, d'] \geq [c, c']$ the result follows. □

Definition 5.1 *We call* concrete domain *a domain of computation satisfying properties I, C, Q, and R.*

Lemma 5.1 *In a concrete domain, two distinct prime intervals* $[a, a']$ *and* $[b, b']$ *are projective iff there exists an* alternating *sequence of prime intervals* $\{[x_0, x'_0], [x_1, x'_1], \ldots, [x_n, x'_n]\}$ *i.e.* $[a, a'] = [x_0, x'_0]$, $[b, b'] = [x_n, x'_n]$,

$$\text{and either} \quad \begin{cases} [a, a'] < [x_1, x'_1] > [x_2, x'_2] < [x_3, x'_3] \cdots [x_n, x'_n] \\ [a, a'] > [x_1, x'_1] < [x_2, x'_2] > [x_3, x'_3] \cdots [x_n, x'_n] \end{cases}$$

satisfying additionally condition Z:

$$\forall i \in [0, n-2] \quad [x_i, x'_i] > [x_{i+1}, x'_{i+1}] < [x_{i+2}, x'_{i+2}] \;\Rightarrow\; x_i \# x_{i+2}$$

Proof: The proof proceeds by induction on the length of the sequence of transpositions that are needed to go from $[a, a']$ to $[b, b']$. If $[a, a']\mathcal{T}[b, b']$ the result is immediate. Assume now the property to be true for two projective prime intervals for which there is a sequence of transpositions of length at most $n-1$, and suppose $[a, a']\mathcal{T}[x_1, x_1'] \cdots [x_{n-1}, x_{n-1}']\mathcal{T}[b, b']$. By induction hypothesis there is an alternating sequence $\{[y_1, y_1'], \ldots, [y_e, y_e']\}$ between $[x_1, x_1']$ and $[b, b']$. Thus two cases are possible:

Case 1: $[x_1, x_1'] < [y_1, y_1'] > [y_2, y_2'] < \cdots [b, b']$

Case 1.1:$[a, a'] \leq [x_1, x_1']$. Then we have also $[a, a'] \leq [y_1, y_1']$ by transitivity and so $[a, a'] < [y_1, y_1'] > [y_2, y_2'] < \cdots [b, b']$

Case 1.2: $[a, a'] > [x_1, x_1']$. Then if $a \# y_1$ the sequence

$$\{[a, a'], [x_1, x_1'], [y_1, y_1'], \ldots, [b, b']\}$$

satisfies condition Z. Otherwise, by the previous result, we have:

$$[a, a'] < [a \vee y_1, a' \vee y_1'] > [y_1, y_1']$$

and the sequence $\{[a, a'], [a \vee y_1, a' \vee y_1'], [y_2, y_2'] < \ldots [b, b']\}$ is an alternating sequence. If y_3 exists, we know that $y_1 \# y_3$ so *a fortiori* $a \vee y_1 \# y_3$ and the sequence satisfies Z.

Case 2: $[x_1, x_1'] > [y_1, y_1'] < [y_2, y_2'] > \cdots [b, b']$

Case 2.1: $[a, a'] < [x_1, x_1']$. Then $\{[a, a'], [x_1, x_1'], [y_1, y_1'], \ldots, [b, b']\}$ is an acceptable alternating sequence.

Case 2.2: $[a, a'] \geq [x_1, x_1']$. Then by transitivity $[a, a'] > [y_1, y_1']$ and the sequence $\{[a, a'], [y_1, y_1'], \ldots, [b, b']\}$ is an alternating sequence. Since we had $x_1 \# y_2$, certainly $a \# y_2$ and the sequence satisfies Z. □

Corollary 5.3 *If a concrete domain is a* lattice*, two prime intervals* $[a, a']$ *and* $[b, b']$ *are projective iff there exists a prime interval* $[c, c']$ *such that*

$$[a, a'] \leq [c, c'] \geq [b, b']$$

Proof: Since two elements cannot be incompatible, the only alternating sequences of prime intervals between two distinct prime intervals $[a, a']$ and $[b, b']$ are of the form:

1. $[a, a'] < [b, b']$
2. $[a, a'] > [b, b']$
3. $[a, a'] < [c, c'] > [b, b']$

Collecting these three cases with the case where $[a, a']$ and $[b, b']$ are identical, we obtain $[a, a'] \leq [c, c'] \geq [b, b']$. The converse is immediate. □

Theorem 5.1 *If a computation domain is a lattice satisfying I and C, the property R is equivalent to property* $\mathrm{R}_{\mathcal{T}}$*:*

Property $\mathrm{R}_{\mathcal{T}}$

No sublattice is isomorphic to the lattice of Figure 10.

Proof: We already know by Corollary 5.2 that R implies $\mathrm{R}_{\mathcal{T}}$. Assume now that $\mathrm{R}_{\mathcal{T}}$ holds and consider two projective prime intervals $[a, x]$ and $[a, y]$. By Corollary 5.2, there exists a prime interval $[c, c']$ such that

$$[a, x] \leq [c, c'] \geq [a, y]$$

We will reason by induction on $\delta(a, c)$, the length of the maximal chains from a to c to prove that such a configuration implies $x = y$ when $a \prec c$, and then by continuity to prove the result in general.

Case $\delta(a, c) = 0$. Then $a = c$ and $c' = x \vee c = x \vee a = x$ as well as $c' = y \vee c = y \vee a = y$ so $x = y$.

Case $\delta(a, c) = 1$. Then $\delta(a, c') = 2$. Since $c \vee x \neq c$ and $c \vee y \neq c$, necessarily $c \neq x$ and $c \neq y$. It is not possible to have $c' = x \vee y$ because the sublattice including $a, x, c, y, c \vee y$ would be isomorphic to the lattice of figure foo. Hence $x \vee y < c'$, which implies $\delta(a, x \vee y) \leq 1$. Consequently x and y are comparable. As both cover a they must be equal.

Case $\delta(a, c) = n > 1$. Then there exists an element d with $a \prec d \leq c$ so $\delta(d, c) = n - 1$. Since $a = x \wedge c = x \wedge d$ and $a = y \wedge c = y \wedge d$, using property C we deduce $d \prec d \vee x$ and $d \prec d \vee y$. We have immediately

$$[d, d \vee x] \leq [c, c'] \geq [d, d \vee y]$$

By induction hypothesis the $d \vee x = d \vee y$. But then, if x and y were distinct, the lattice including $a, x, y, d, c \vee x$ would be isomorphic to the lattice of Figure 10. So we must have $x = y$.

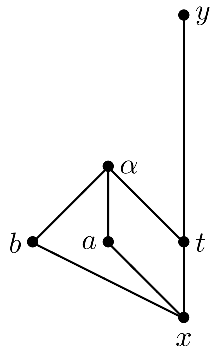
We conclude the proof using Proposition 5.1. If $[a, x] \leq [c, c'] \geq [a, y]$ there exists a prime interval $[\gamma, \gamma']$ with $a \prec \gamma$ and $[a, x] \leq [\gamma, \gamma'] \geq [a, y]$. Hence here again $x = y$. □

An interesting consequence of property $\mathrm{R}_{\mathcal{T}}$ is that it excludes a domain like the one on Figure 6. More precisely:

Proposition 5.5 *In a concrete domain, an interval* $[x, y]$ *of height* n *contains at most* n *elements covering* x*.*

Proof: We reason again by induction on the height of the interval $[x, y]$. The result is immediate when $x = y$ and $x \prec y$. If all maximal chains from x to y have length 2, then consider two elements a and b covering x and less than y. If they are distinct, we have $y = a \vee b$ by property C. Property $\mathrm{R}_{\mathcal{T}}$ excludes the possibility of a third element c less than y covering x.

Now in the general case, assume all maximal chains from x to y have length n, with $n > 2$. Consider an arbitrary element t such that $x \prec t \leq y$. The interval $[t, y]$ is of height $n - 1$ and by induction hypothesis there are at most $n - 1$ elements covering t in that interval.



By property $\mathrm{R}_{\mathcal{T}}$, the mapping that associates to any element of $[x, y]$ covering x the element $x \vee t$ is an *injection.* So there are at most $n-1$ elements of $[x, y]$ covering x and distinct from t. If we now count t, the result is established. □

Corollary 5.4 *In a concrete domain, if* $x \prec y$ *the interval* $[x, y]$ *contains only finitely many elements.*

Proof: We reason again by induction on the height $\delta(x, y)$ of the interval $[x, y]$. If $\delta(x, y) = 0$ or $\delta(x, y) = 1$ the result is immediate. Suppose now $\delta(x, y) = n > 1$. Then for any a covering x in $[x, y]$ there are, by induction hypothesis, finitely many elements in $[a, y]$. Since the number of elements covering x in $[x, y]$ is finite, there are finitely many elements in $[x, y]$. □

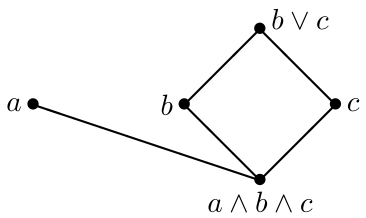
Corollary 5.5 *In a concrete domain, a finite element dominates only finitely many elements.*

Remark: We are not too concerned with the independence of the various axioms that we postulate for computation domains, nor of the properties that we have studied so far. But one may notice here that properties C and $R_{\mathcal{T}}$ imply respectively conditional completeness and coherence, which in a way is another argument in favor of these axioms. Since coherence has been studied relatively little in the literature, we prove that it is not independent of I,C, Q and $R_{\mathcal{T}}$.

Proposition 5.6 *If an algebraic partial order is conditionally complete, and it satisfies properties I,C,Q, and* $R_{\mathcal{T}}$*, then it is coherent.*

Proof: By proposition 1.2, we need only to show that any pairwise consistent triple a, b, c has a least upper bound. We reason by induction on $\delta(a \wedge b \wedge c, a)$.

a) Base cases: If $\delta(a \wedge b \wedge c, a) = 0$, then a,b, and c are less than $b \vee c$. If $\delta(a \wedge b \wedge c, a) = 1$, then suppose a were incompatible with $b \vee c$.



By property Q, since $a \wedge b \wedge c = a \wedge (b \vee c)$ there exists a t such that

$$a \wedge b \wedge c \prec t \leq b \vee c \quad \text{and} \quad a \# t$$

But by proposition 5.2 (whose proof doesn't rely on coherence!) that can be applied since $a \wedge b \wedge c \leq b \wedge c$, either $t \leq b$ or $t \leq c$. But then, in either case the set $\{a, b, c\}$ cannot be pairwise consistent. If for example t is less than b, then b cannot be compatible with a. So $a \uparrow (b \vee c)$ and by conditional completeness $a \vee (b \vee c)$ exists.

b) Induction step: Assume the property holds when $\delta(a \wedge b \wedge c, a) < n \geq 1$ and assume $\delta(a \wedge b \wedge c, a) = n$. consider a maximal chain

$$a \wedge b \wedge c = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n = a$$

from $a \wedge b \wedge c$ to a. Since the triple $\{a, b, c\}$ is pairwise consistent, so is the triple $\{x_{n-1}, b, c\}$. By induction hypothesis, it admits a least upper bound $x_{n-1} \vee b \vee c$. We can use the argument of the base case to the triple $\{a, x_{n-1} \vee b, x_{n-1} \vee c\}$. Finally, $a \vee x_{n-1} \vee b \vee c = a \vee b \vee c$.

c) Continuity argument: If a is not finite relatively to $a \wedge b \wedge c$, consider an arbitrary α approximant of a. The triple $\{a, b, c\}$ is pairwise consistent, so is the triple $\{\alpha, b, c\}$, thus $\alpha \vee (b \vee c)$ exists. In the upper section $[a \wedge b \wedge c)$ we have:

$$\bigcup_{\alpha \in \mathcal{A}(a)} (\alpha \vee (b \vee c)) = \left(\bigcup_{\alpha \in \mathcal{A}(a)} \alpha \right) \vee (b \vee c)$$

By algebraicity we have $\bigcup_{\alpha \in \mathcal{A}(a)} \alpha = a$ and consequently $a \vee b \vee c$ exists.□

We return to our central concern, the study of the consequences of property R.

Lemma 5.2 *Consider two compatible elements x and y in a concrete domain. If $[x, x']$ is a prime interval included in $[x, x \vee y]$, then there exists a prime interval $[u, u']$ included in $[x \wedge y, y]$ which is projective with it.*

Proof: Remark first that y cannot be less than x because then we would have $x \vee y = x$ and the prime interval $[x, x']$ could not be included in $[x, x \vee y]$. Now we reason by induction on the length $\delta(x \wedge y, y)$ of the maximal chains from $x \wedge y$ to y.

a) Base case: $\delta(x \wedge y, y) = 1$, i.e. $x \wedge y \prec y$. By property C we have $x \prec x \vee y$. Since we have also $x \prec x' \leq x \vee y$ we deduce $x' = x \vee y$. The intervals $[x \wedge y, y]$ and $[x, x']$ are transposed.

b) Induction step: Assume $\delta(x \wedge y, y) = n > 1$. Consider an arbitrary element v covered by y. By Lemma 3.1 we have $v \vee x \preceq v \vee x'$. We examine both cases in turn:

   Case 1: $v \vee x = v \vee x'$. We can apply the induction hypothesis because $x \wedge v = x \wedge y$ so $\delta(x \wedge v, v) = \delta(x \wedge y, v) = n - 1$. Thus there exists an interval $[u, u']$ included in $[x \wedge y, v]$ – thus *a fortiori* in $[x \wedge y, y]$ – projective with $[x, x']$.

   Case 2: $v \vee x \prec v \vee x'$. Note that this case implies that y is not less than $v \vee x$: we would then have $v \vee x \vee y = x \vee y = v \vee x$ and

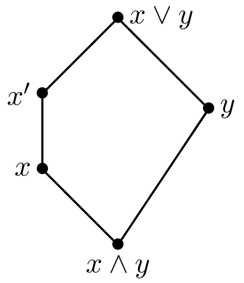
$$v \vee x \prec v \vee x' \leq x \vee y = v \vee x$$

   which is impossible. Thus $(v \vee x) \wedge y = v$ and we can use property C and deduce $v \vee x \prec (v \vee x) \vee y = x \vee y$. But $v \vee x \prec v \vee x' \leq x \vee y$

hence $v \vee x' = x \vee y = (v \vee x) \vee y$ which means that the following holds:

$$[x, x'] \leq [v \vee x, v \vee x'] \geq [v, y]$$

This concludes the proof when $\delta(x \wedge y, y)$ is finite.



c) Continuity argument: If now y is not finite relative to $x \wedge y$, there exists nevertheless an element $d \in \mathcal{A}(y)$ with $x \wedge y \prec d$ and $x \mathrel{-\!\!<} x' \leq x \vee d$ and we can apply the previous results to the elements x,x', and d. □

Remark:

1. This proof doesn't use property R. It is included in this Section because we need the Lemma here.
2. In fact, we can prove with a minor adjustment of the induction argument that there exists a prime interval $[t, t']$ and $[x, x'] \leq [t, t'] \geq [u, u']$.

Corollary 5.6 *In a concrete domain, if* $[x, x']$ *is a prime interval included in in the interval* $[\bot, a \vee b]$*, there exists a prime interval projective with it either in* $[\bot, a]$ *or in* $[\bot, b]$.

Proof: Using Lemma 3.1 we obtain $a \vee x \mathrel{=\!\!<} a \vee x'$ and $b \vee x \mathrel{=\!\!<} b \vee x'$.

Case 1: $a \vee x \mathrel{-\!\!<} a \vee x'$ and $b \vee x \mathrel{-\!\!<} b \vee x'$. Then we have

$$[a \vee x, a \vee x'] \geq [x, x'] \leq [b \vee x, b \vee x']$$

thus by Proposition 5.3, since $(a \vee x) \uparrow (b \vee x)$

$$[a \vee x, a \vee x'] \leq [a \vee b \vee x, a \vee b \vee x'] \geq [b \vee x, b \vee x']$$

But there is a contradiction since $a \vee b = a \vee b \vee x = a \vee b \vee x'$, making it impossible for the interval $[a \vee b \vee x, a \vee b \vee x']$ to be prime. This case cannot happen.

Case 2: $a \vee x = a \vee x'$ (the case $b \vee x = b \vee x'$ is handled symmetrically). Then the prime interval $[x, x']$ is included in $[x, x \vee a]$. By the previous Lemma, there exists a prime interval $[u, u']$ included in $[x \wedge a, a]$ (hence *a fortiori* in $[\bot, a]$) with $[x, x'] \sim [u, u']$. □

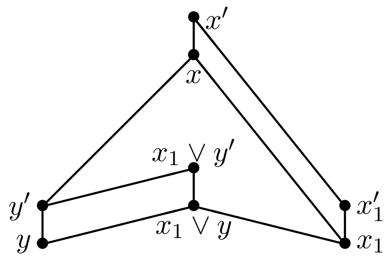
Lemma 5.3 *Consider two projective prime intervals $[x, x']$ and $[z, z']$ in a concrete domain. If there exists a prime interval $[y, y']$ projective with $[x, x']$ in $[\bot, x]$, then there exists a prime interval projective with $[z, z']$ in $[\bot, z]$.*

Proof: The proof proceeds by induction on the length Z of the alternating sequence of transposed prime intervals between $[x, x']$ and $[z, z']$. If $Z = 0$ the intervals $[x, x']$ and $[z, z']$ are identical, so the result is immediate. Assume now $Z = n, n > 0$. Two cases are possible, depending on the form of the alternating sequence.

Case 1: $[x, x'] < [x_1, x_1'] > \cdots [z, z']$. In that case the interval $[y, y']$ is also included in $[\bot, x_1]$. By induction hypothesis, there exists a prime interval $[t, t']$ in $[\bot, z]$ with $[t, t'] \sim [y, y']$ because the alternating sequence from $[x_1, x_1']$ to $[z, z']$ is of length $n - 1$.

Case 2: $[x, x'] > [x_1, x_1'] < \cdots [z, z']$. Since x_1 and y are compatible, we deduce by Lemma 3.1 $x_1 \vee y \mathrel{-\!\!\!\!\prec\!\!\!=} x_1 \vee y'$, thus two cases are possible.

Case 2.1: $x_1 \vee y \prec x_1 \vee y'$



Since x_1 and y are both less than x, so is $x_1 \vee y$. Therefore x_1' is not less than $x_1 \vee y$, because otherwise x_1' would be less than x and $x \prec x' = x_1' \vee x$ would be impossible. So $x_1 = (x_1 \vee y) \wedge x_1'$ and by property C:

$$x_1 \vee y \prec (x_1 \vee y) \vee x_1' = x_1' \vee y$$

Hence we have $[x_1 \vee y, x_1 \vee y'] \sim [y, y'] \sim [x, x'] \sim [x_1, x_1'] \sim [x_1 \vee y, x_1' \vee y]$. By property $\widehat{\mathrm{R}}$ we conclude $x_1 \vee y' = x_1' \vee y$. But then $x_1' \leq x_1 \vee y' \leq x$, which we have seen is impossible. There is a contradiction, so this case cannot happen.

Case 2.2: $x_1 \vee y = x_1 \vee y'$. Then we can use Lemma 5.1. There is a prime interval $[u, u']$ projective with $[y, y']$ in $[x_1 \wedge y, x_1]$. By transitivity $[u, u'] \sim [x_1, x_1']$. Using the induction hypothesis, we deduce that there exists a prime interval $[t, t']$ in $[\bot, z]$ with $[u, u'] \sim [t, t']$ and thus $[y, y'] \sim [t, t']$. □

Theorem 5.2 *In a concrete domain, if $[x, x']$ is a prime interval, then the interval $[\bot, x]$ contains no equipollent prime interval.*

Proof:

A. We prove first that there cannot be a prime interval $[y, y']$ in $[\bot, x]$ with $[x, x'] \sim [y, y']$. The proof is by induction on $h(x)$ the height of x. If $h(x) = 0$ the result is immediate. If $h(x) = n > 0$, assume some $[y, y']$ included $[\bot, x]$ verified $[y, y'] \sim [x, x']$. By the previous lemma, there exists $[t, t'] \sim [y, y']$ with $[t, t']$ included in $[\bot, y]$. But $h(y) < h(x)$ so by induction hypothesis this is impossible. Hence the property is proved for any finite x. If now $h(x)$ is infinite, there exists by Proposition 3.1 a prime interval with finite endpoints $[\eta, \eta']$ with $[\eta, \eta'] \leq [y, y']$. In the upper section $[\eta')$ there exists a finite $[\xi, \xi']$ with $[\xi, \xi'] \leq [x, x']$. The prime intervals $[\xi, \xi']$ and $[\eta, \eta']$ are now projective intervals with finite endpoints and the reasoning above applies.

B. We prove now that there can't be a prime interval equipollent to $[x, x']$ in $[\bot, x]$. Assume such an interval $[y, y']$ would exist,i.e. $[y, y'] \approx [x, x']$. By definition $\approx = \simeq \circ \sim = \sim \circ \simeq$. Hence $[y, y'] \sim \circ \simeq [x, x']$, which means that there is a prime interval $[x, x'']$ with $[y, y'] \sim [x, x'']$. This is impossible by the result of part A. □

In the five sections above, we have defined the essential properties that a computation domain should satisfy to be considered plausibly a data domain rather than a functional domain. The mathematical consequences of these properties are consistent with our intuition. But it remains to show that these properties are *sufficient* to characterize truly a notion of concrete computation domain. This is the role of the forthcoming sections that develop a representation theory for concrete domains.

6 The information matrix

To start with, we expose the essential facts on which the representation of concrete domains will be based.

Definition 6.1 *An equivalence class of* equipollent *prime intervals will be called a* cell*.*

Notation: Let $[x, x']$ be a prime interval. We denote $d[x, x']$ the equivalence class of $[x, x']$ under projectivity (the *decision* associated to $[x, x']$) and $c[x, x']$ the cell associated with $[x, x']$, i.e. its equivalence class under equipollence.

Definition 6.2 *If $[x, x']$ is a prime interval and a dominates x', we say that a* occupies cell $c[x, x']$ *and* contains decision $d[x, x']$*. We note:*

$$\begin{array}{ll} \Gamma(a) & = \{c[x, x'] \,|\, x \prec x' \quad \textit{and} \quad x' \leq a\} \\ \Delta(a) & = \{d[x, x'] \,|\, x \prec x' \quad \textit{and} \quad x' \leq a\} \end{array}$$

Proposition 6.1 *For any a:*

$$\begin{array}{ll} \Gamma(a) & = \{c[x, x'] \,|\, x \prec x' \quad \textit{and} \quad x' \in \mathcal{A}(a)\} \\ \Delta(a) & = \{d[x, x'] \,|\, x \prec x' \quad \textit{and} \quad x' \in \mathcal{A}(a)\} \end{array}$$

Proof: This result is a simple application of Proposition 3.1. For any prime interval $[y, y']$ with $y' \leq a$, there is a prime interval $[x, x']$ with finite endpoints such that $[x, x'] \leq [y, y']$, hence

$$\begin{array}{ll} c[x, x'] & = c[y, y'] \\ d[x, x'] & = d[y, y'] \end{array}$$

Since $y' \leq a$, *a fortiori* $x' \leq a$. As x' is finite, it is an approximant of a. □

Proposition 6.2 *Consider a consistent subset X in a concrete domain. We have the following equalities:*

$$\left\{ \begin{array}{ll} \Gamma(\bigcup X) & = \bigcup_{x \in X} \Gamma(x) \\ \Delta(\bigcup X) & = \bigcup_{x \in X} \Delta(x) \end{array} \right.$$

Proof: First, by coherence, if X is consistent it has a least upper bound $\bigcup X$. Now by definition of Γ and Δ:

$$\left\{ \begin{array}{ll} x \leq y & \Rightarrow \Gamma(x) \subset \Gamma(y) \\ x \leq y & \Rightarrow \Delta(x) \subset \Delta(y) \end{array} \right.$$

So immediately:

$$\left\{ \begin{array}{l} \bigcup_{x \in X} \Gamma(x) \subset \Gamma(\bigcup X)) \\ \bigcup_{x \in X} \Delta(x) \subset \Delta(\bigcup X)) \end{array} \right.$$

We prove now the converse inequalities by induction on the cardinal of X when X is finite and then by continuity.

a) Base Cases: If $|X| = 0$ then $\bigcup X = \bot$ and $\Gamma(\bot) = \Delta(\bot) = \emptyset$. If $|X| = 1$ then $X = \{x\}$ and $\bigcup X = x$. So obviously $\Gamma(x) \supset \Gamma(\bigcup X)$ and $\Delta(x) \supset \Delta(\bigcup X)$.

b) Induction step: let $X = \{x_1, x_2, \ldots, x_{n-1}, x_n\}(n > 1)$. If X is consistent, so is $X' = \{x_1, x_2, \ldots, x_{n-1}\}$. By induction hypothesis:

$$\begin{cases} \bigcup_{x \in X'} \Gamma(x) \supset \Gamma(\bigcup X')) \\ \bigcup_{x \in X'} \Delta(x) \supset \Delta(\bigcup X')) \end{cases}$$

Since $\bigcup X = (\bigcup X') \vee x_n$, so by Corollary 5.6, any prime interval $[x, x']$ included in $[\bot, \bigcup X]$ is projective with a prime interval included either in $[\bot, \bigcup X']$ or in $[\bot, x_n]$. Hence

$$\begin{cases} \Gamma(\bigcup X) \subset \Gamma(\bigcup X') \cup \Gamma(x_n) \\ \Delta(\bigcup X) \subset \Delta(\bigcup X') \cup \Delta(x_n) \end{cases}$$

Using the induction hypothesis we obtain:

$$\begin{cases} \Gamma(\bigcup X) \subset \bigcup_{x \in X} \Gamma(x) \\ \Delta(\bigcup X) \subset \bigcup_{x \in X} \Delta(x) \end{cases}$$

c) Continuity argument: consider an arbitrary prime interval $[x, x']$ with finite endpoints included in $[\bot, \bigcup X]$. Since x' is finite less than $\bigcup X$ and the set obtained by adding to X the least upper bounds of its finite subsets is directed, we can find a finite subset Y of X whose least upper bound dominates x'. Thus by the previous result:

$$\begin{cases} c[x, x'] \in \bigcup_{y \in Y} \Gamma(y) \\ d[x, x'] \in \bigcup_{y \in Y} \Delta(y) \end{cases}$$

so we deduce

$$\begin{cases} \Gamma(\bigcup X) \subset \bigcup_{x \in X} \Gamma(x) \\ \Delta(\bigcup X) \subset \bigcup_{x \in X} \Delta(x) \end{cases}$$

□

In a concrete domain, we have a property that is far stronger than the Jordan-Dedekind condition.

Lemma 6.1 *Consider an arbitrary element x in a concrete domain and a maximal chain $\{\bot = x_0, x_1, \ldots, x_n, \ldots\}$ between $\bot$ and x. We have the equalities:*

$$\begin{aligned} \Gamma(x) &= \{c[x_i, x_{i+1}] | i \geq 0\} \\ \Delta(x) &= \{d[x_i, x_{i+1}] | i \geq 0\} \end{aligned}$$

Proof: the equalities are proved by induction on $h(x)$.

a) Base Cases: if $h(x) = 0$ then $x = \perp$ and $\Gamma(\perp) = \Delta(\perp) = \emptyset$. If h(x)=1, then x is an atom and the property is obvious again.

b) Induction step: assume now $h(x) = n > 1$. Take an arbitrary prime interval $[y, y']$ in $[\perp, x]$. Since y' and x_{n-1} are compatible, by Lemma 3.1 we have $x_{n-1} \vee y \preceq\!\!\!\!< x_{n-1} \vee y'$ and two cases have to be considered:

Case 1: $x_{n-1} \vee y = x_{n-1} \vee y'$. In that case, by Lemma 5.1 there exists a prime interval $[z, z']$ in $[\perp, x_{n-1}]$ projective with $[y, y']$. Since x_{n-1} is of height $n-1$, we can use the induction hypothesis. Hence there exists an interval $[x_k, x_{k+1}]$ with $k \leq n-2$ and $[z, z'] \sim [x_k, x_{k+1}]$ i.e. $d[z, z'] = d[x_k, x_{k+1}]$ and therefore

$$\begin{array}{ll} d[y, y'] & \in \{d[x_i, x_{i+1}] | i \geq 0\} \\ c[y, y'] & \in \{c[x_i, x_{i+1}] | i \geq 0\} \end{array}$$

Case 2: $x_{n-1} \vee y \prec x_{n-1} \vee y'$. In that case the prime interval $[x_{n-1} \vee y, x_{n-1} \vee y']$ is included in the prime interval $[x_{n-1}, x]$ which implies

$$\begin{array}{ll} x_{n-1} & = x_{n-1} \vee y \\ x & = x_{n-1} \vee y' \end{array}$$

so $[y, y'] \leq [x_{n-1}, x]$ and here again

$$\begin{array}{ll} d[y, y'] & \in \{d[x_i, x_{i+1}] | i \geq 0\} \\ c[y, y'] & \in \{c[x_i, x_{i+1}] | i \geq 0\} \end{array}$$

c) Continuity argument: If x is not finite, we know nevertheless by proposition 6.1 that

$$\begin{array}{llll} \Gamma(x) & = \{c[y, y'] | [y, y'] & \text{prime and} & y, y' \in \mathcal{A}(x)\} \\ \Delta(x) & = \{d[y, y'] | [y, y'] & \text{prime and} & y, y' \in \mathcal{A}(x)\} \end{array}$$

Consider then a prime interval $[y, y']$ with finite endpoints. The maximal chain from $\perp$ to x is a directed set so there is a finite element x_n in the chain such that $y' \leq x_n$. Using the result of the finite case, we can find an interval $[x_i, x_{i+1}](i \leq n-1)$ projective with $[y, y']$. □

Corollary 6.1 *For any x in a concrete domain $h(x) = |\Gamma(x)| = |\Delta(x)|$.*

Proof: Assume first x is finite. By the previous lemma, we know that $|\Gamma(x)| \leq h(x)$ and $|\Delta(x)| \leq h(x)$. But by Theorem 5.1 a maximal chain cannot contain two equipollent prime intervals. So $h(x) \leq |\Gamma(x)|$ and $h(x) \leq |\Delta(x)|$. Now if x is infinite, using Theorem 5.1 we have $|\Gamma(x)| = \infty$ and $|\Delta(x)| = \infty$. □

We prove now a technical result that is much stronger than Proposition 5.4.

Proposition 6.3 *Consider two projective prime intervals $[a, a']$ and $[b, b']$ in a concrete domain. If a and b are compatible we have also:*

$$[a, a'] \leq [c, c'] \geq [b, b']$$

with $c = a \vee b$ and $c' = a' \vee b = a \vee b'$

Proof: First a' and $a \vee b$ are compatible. Indeed if we had $a' \# a \vee b$, there would exist an element t such that $a \prec t \leq a \vee b$ and $t \# a'$. By Lemma 5.1 there would exist an interval $[u, u']$ in $[\bot, b]$ with $[u, u'] \sim [a, t]$ thus $[u, u'] \approx [a, a']$. But since $[a, a'] \sim [b, b']$ we deduce $[u, u'] \approx [b, b']$, which is impossible by Theorem 5.1. Symmetrically we can show $b' \uparrow c = a \vee b$. The same reasoning also shows that a' and b' are not less than c. By Property C we deduce

$$\begin{aligned} c \prec (a \vee b) \vee a' &= a' \vee b \\ c \prec (a \vee b) \vee b' &= a \vee b' \end{aligned}$$

But the prime intervals $[a \vee b, a' \vee b]$ and $[a \vee b, a \vee b']$ are projective. So by property R we obtain $a' \vee b = a \vee b'$. □

Corollary 6.2 *If $[x, x']$ and $[y, y']$ are two equipollent prime intervals included in the same interval $[\bot, z]$ then they are projective.*

Proof: From $[x, x'] \approx [y, y']$ we deduce that there exists a a prime interval $[y, y'']$ such that $[x, x'] \sim [y, y''] \approx [y, y']$. But x and y are compatible, so by the previous result $[x, x'] \leq [x \vee y, x' \vee y] \geq [y, y'']$. As $x' \vee y$ is less than z, so is y''. Since y' is also dominated by z we must have $y' = y''$ and therefore $[x, x'] \sim [y, y']$. □

Theorem 6.1 *In a concrete domain*

$$x \leq y \iff \Delta(x) \subset \Delta(y)$$

Proof: By definition of Δ we have $x \leq y \Rightarrow \Delta(x) \subset \Delta(y)$, so we need only to prove the converse implication. We reason by induction on the height of x.

a) Base Case: If $h(x) = 0$ then $x = \bot$ and for any y we have $x \leq y$.

b) Induction step: Assume we have $\Delta(x) \subset \Delta(y) \;\Rightarrow\; x \leq y$ when the height of x is less than n, and assume $h(x) = n$. Consider an arbitrary maximal chain $\bot = x_0 \prec x_1 \prec \cdots \prec x_{n-1} \prec x_n = x$ from $\bot$ to x, and assume $\Delta(x) \subset \Delta(y)$. Since $x_{n-1} \leq x$, we have $\Delta(x_{n-1}) \subset \Delta(x) \subset \Delta(y)$. As $h(x_{n-1}) = n-1$ we can use the induction hypothesis to deduce $x_{n-1} \leq y$. Now $d[x_{n-1}, x_n]$ belongs to $\Delta(x)$ thus to $\Delta(y)$ so there exists a prime interval $[z, z']$ in $[\bot, y]$ with $[x_{n-1}, x_n] \sim [z, z']$. Both elements x_{n-1} and z are less than y so we can use Proposition 6.3:

$$[x_{n-1}, x_n] \leq [x_{n-1} \vee z, t] \geq [z, z']$$

$$t = x_{n-1} \vee z' = x_n \vee z$$

But since both x_{n-1} and z' are less than y so is t, therefore x_n is less than y. As $x_n = x$ we obtain $x \leq y$.

c) Continuity argument: From $\Delta(x) \subset \Delta(y)$ we deduce

$$\forall a \in \mathcal{A}(x)\ \Delta(a) \subset \Delta(y)$$

thus by the result of the finite case $\forall a \in \mathcal{A}(x)\ a \leq y$. By algebraicity $x = \bigcup_{a \in \mathcal{A}(x)} a$ and therefore $x \leq y$. □

Definition 6.3 *A prime interval is called* minimal *if it is minimal for the relation $\leq$ between intervals.*

Definition 6.4 *An element x is* join–irreducible *iff*

i) $x \neq \bot$

ii) $x = a \vee b \Rightarrow x = a \quad$ *or* $\quad x = b$

Proposition 6.4 *In a concrete domain, for any prime interval $[x, x']$ there exists a prime interval $[y, y']$ less than $[x, x']$ where y' is join–irreducible.*

Proof: By Proposition 3.1 it is sufficient to examine the case where $[x, x']$ has finite endpoints. We reason by induction on $h(x')$.

a) Base Case: $h(x') = 1$. The element x' is an atom thus necessarily join–irreducible. The result is immediate.

b) Induction step: Assume $h(x') = n, n > 1$. If x' is join–irreducible, the property is proved immediately. Otherwise $x' = a \vee b$ together with $a < x'$ and $b < x'$. By Corollary 5.6 there exists a prime interval $[u, u']$ included either in $[\bot, a]$ or in $[\bot, b]$ such that $[u, u'] \sim [x, x']$. Since both u and x are both less than x', by Proposition 6.3:

$$[u, u'] \leq [x \vee u, x \vee u'] \geq [x, x']$$

Since $x \vee u' \leq x'$ necessarily $x \vee u' = x'$ and thus $x \vee u = x$ so $[u, u'] \leq [x, x']$. But since u' is either less than a or less than b we have in fact $[u, u'] < [x, x']$, which implies $h(u) < h(x')$ and we can apply the induction hypothesis to the prime interval $[u, u']$. There exists a prime interval $[y, y']$ with y' join-irreducible and $[y, y'] \leq [u, u']$ and *a fortiori* $[y, y'] \leq [x, x']$. □

Corollary 6.3 *In a concrete domain, a prime interval $[x, x']$ is minimal iff x' is join–irreducible.*

Proof: Assume first that x' is join–irreducible and consider a prime interval $[y, y']$ such that $[y, y'] \leq [x, x']$. By definition of $\leq$ we have $x' = x \vee y'$. Since x' is join–irreducible and $x \neq x'$ we must have $y' = x'$. Thus $y = y' \wedge x = x' \wedge x = x$, and $[y, y'] = [x, x']$. So $[x, x']$ is minimal.

Conversely, assume that $[x, x']$ is minimal. By the previous propostion there exists $[y, y']$ with y' join–irreducible and $[y, y'] \leq [x, x']$. By minimality $[y, y'] = [x, x']$ so $y' = x'$ which proves that x' is join–irreducible. □

Proposition 6.5 *In a concrete domain, if the prime interval $[x, x']$ is minimal, then any prime interval $[x, x'']$ such that $[x, x''] \simeq [x, x']$ is also minimal.*

Proof: Consider an arbitrary prime interval $[y, y'']$ such that

$$[y, y''] \leq [x, x''] \simeq [x, x']$$

Since $\leq \circ \simeq = \simeq \circ \leq$ there exists a y' such that

$$[y, y''] \simeq [y, y'] \leq [x, x']$$

Since $[x, x']$ is minimal $[y, y'] = [x, x']$ so $y = x$. Hence $x \wedge y'' = x$ which implies $x \leq y''$. Since $x'' = x \vee y''$ we have $x'' = y''$ and therefore $[y, y''] = [x, x'']$, which proves that $[x, x'']$ is minimal. □

Definition 6.5 *In a concrete domain, consider a decision δ and a set of decisions Δ. We say that Δ* enables δ *iff there is a minimal prime interval $[x, x']$ such that:*

$$\begin{cases} d[x, x'] = \delta \\ \Delta(x) = \Delta \end{cases}$$

By the previous proposition, if Δ enables δ it also enables all decisions equipollent to δ so we can say that Δ *enables cell* γ iff there exists a minimal prime interval $[x, x']$ such that:

$$\begin{cases} c[x, x'] = \gamma \\ \Delta(x) = \Delta \end{cases}$$

Remarks:

1. If the interval $[x, x']$ is minimal, elements x and x' are finite. Therefore, since $|\Delta(x)| = h(x)$, a cell is always enabled by a finite number of decisions.

2. In general, within a given equivalence class of projective prime intervals, there are several distinct minimal intervals. Therefore, several distinct sets of decisions may enable a given cell. The case where any cell γ is enabled by a single set of decisions Δ is a very important special case that we will consider in section 10.

We are now ready to build a whole class of concrete domains, using the notions introduced in this section.

Definition 6.6 *An* information matrix *is a quadruple $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ where*

1. *Γ is a* countable set*. Its elements will be called cells.*

2. *V is a* countable set*.*

3. *$\mathcal{V}$ is a function from Γ to $\mathcal{P}(V)$ that maps any cell c in Γ to the subset $\mathcal{V}(c)$ of* possible values *at c. We simply say that $\mathcal{V}(c)$ is the type of c. We call* decision *a pair $< c, v >$ where c is a cell and v is a possible value at c, i.e. $c \in \Gamma$ and $v \in \mathcal{V}(c)$. We note Δ_M the set of decisions defined by Γ,V, and $\mathcal{V}$, and $\mathcal{F}(\Delta_M)$ the set of finite subsets of Δ_M.*

4. *the* enabling function $\mathcal{E}$ *maps* Γ *to* $\mathcal{P}(\mathcal{F}(\Delta_M)) - \emptyset$*. If a finite set of decisions* $\{d_1, d_2, \ldots, d_n\}$ *belongs to* $\mathcal{E}(c)$ *we say that* $\{d_1, d_2, \ldots, d_n\}$ *enables cell c.*

Notations: Let $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ be an information matrix with set of decision Δ_M. If $d =< c, v >$ ($c \in \Gamma, v \in \mathcal{V}(c)$) is a decision, we say that this decision *concerns* cell c; if $\{d_1, d_2, \ldots, d_n\}$ is a set of decisions in $\mathcal{E}(c)$, we say that this set *enables cell c and decision d*. This relation is written:

$$d_1, d_2, \ldots, d_n \vdash d$$

If the empty set enables a cell (resp. a decision) we say that this cell (resp. this decision) is *initial*.

Definition 6.7 *Consider an information matrix M and a decision d in M. A finite sequence of decisions* $d_0, d_1, d_2, \ldots, d_{n-1}, d_n = d$ *is a* proof of d *iff for any j with* $0 \leq j \leq n$ *there is a subset* $\{d_{j_1}, d_{j_2}, \ldots, d_{j_k}\}$ *of* $\{d_0, \ldots, d_{j-1}\}$ *that enables* d_j*, i.e.* $d_{j_1}, d_{j_2}, \ldots, d_{j_k} \vdash d_j$.

Definition 6.8 *In an information matrix, a subset of decisions X is* connected *by another subset Y iff any decision in X has a proof included in Y. A subset X that is connected by itself is called* connected.

Remarks: If X is connected by Y we have $X \subset Y$. If X is connected by Y, em a fortiori X is connected by any superset of Y. If two sets of decisions are connected, so is their set union. A proof is of course connected. From these last two remarks, we deduce that any finite subset X of a connected set may be included in a finite connected subset: simply include a proof of each element of X.

Definition 6.9 *In an information matrix M a* configuration *is a connected set of decisions in which* no two distinct decisions concern the same cell.

Let Σ_M be the set of configurations of an information matrix M. Any configuration σ is a subset of Δ_M by definition, so Σ_M is naturally ordered by inclusion.

Example: Consider the matrix $M_1 =< \Gamma_1, V_1, \mathcal{V}_1, \mathcal{E}_1 >$ defined by

1. $\Gamma_1 = \{c_1, c_2, c_3\}$
2. $V_1 = \{\top\}$

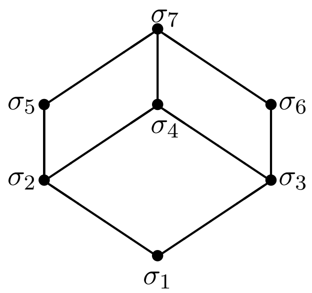


Figure 13: Σ_{M_1}

3. $\mathcal{V}_1 = \lambda c.\{\top\}$
4. $\mathcal{E}_1(c_1) = \mathcal{E}_1(c_2) = \{\emptyset\} \quad \mathcal{E}_1(c_3) = \{\{c_1\}, \{c_2\}\}$

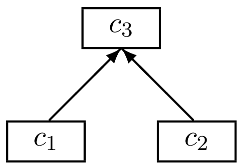
Since V_1 contains a single value, the set of decisions is isomorphic to Γ_1 and the set of configurations Σ_{M_1} comprises the following seven configurations:

$$\sigma_1 = \emptyset \quad \sigma_2 = \{c_1\} \quad \sigma_3 = \{c_2\} \quad \sigma_4 = \{c_1, c_2\}$$

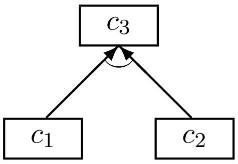
$$\sigma_5 = \{c_1, c_3\} \quad \sigma_6 = \{c_2, c_3\} \quad \sigma_7 = \{c_1, c_2, c_3\}$$

The diagram of the partial order $< \Sigma_{M_1}; \subset >$ is shown on Figure 13.

We have used extensively Hasse diagrams to represent partial orders; in the same manner it is useful to represent in a synthetic manner an information matrix. Such a graphical representation is only feasible when all cells may only contain a single value $\top$ (i.e. $\mathcal{V} = \lambda c.\{\top\}$). In that case Γ and Δ_M are isomorphic and $\mathcal{E}$ maps Γ to $\mathcal{P}(\mathcal{F}(\Gamma))$ so that we can use a representation by "and-or" graphs that is familiar in computer science. Each cell in M is represented by a node in the graph and if we have $c_1, c_2, \ldots, c_{n-1} \vdash c_n$ the graph of M has $n-1$ edges $c_i \rightarrow c_n$ and they are drawn connected by an arc (for "and"). For example matrix M_1 that we have just seen is represented here:



Matrix $M_2 =< \Gamma_1, V_1, \mathcal{V}_1, \mathcal{E}_2 >$ where $\mathcal{E}_2(c_1) = \mathcal{E}_2(c_2) = \{\emptyset\}$ and $\mathcal{E}_2(c_3) = \{\{c_1, c_2\}\}$ is represented by



Simple conventions allow representing infinite matrices in this manner (cf. Figure 14).

Lemma 6.2 *In the partial order* $< \Sigma_M; \subset >$ *of the configurations of an information matrix* M *ordered by set inclusion, two configurations* σ_1 *and* σ_2 *are compatible iff the set* $\sigma_1 \cup \sigma_2$ *is a configuration. Furthermore* $\sigma_1 \vee \sigma_2 = \sigma_1 \cup \sigma_2$.

Proof: First if $\sigma_1 \cup \sigma_2$ is a configuration, since $\sigma_1 \subset \sigma_1 \cup \sigma_2$ and $\sigma_2 \subset \sigma_1 \cup \sigma_2$, we have $\sigma_1 \uparrow \sigma_2$. Assume conversely $\sigma_1 \uparrow \sigma_2$, i.e. that there is a configuration σ with $\sigma_1 \subset \sigma$ and $\sigma_2 \subset \sigma$ and consider the set of decisions $\sigma_1 \cup \sigma_2$. We remarked earlier that since σ_1 and σ_2 are connected, so is their union. If in $\sigma_1 \cup \sigma_2$ two distinct decisions concerned the same cell, then this would also be the case in σ that includes $\sigma_1 \cup \sigma_2$. But this is impossible because σ is a configuration. Thus $\sigma_1 \cup \sigma_2$ is a configuration.

Since any configuration dominating σ_1 and σ_2 must contain (hence dominate) $\sigma_1 \cup \sigma_2$ we have $\sigma_1 \vee \sigma_2 = \sigma_1 \cup \sigma_2$. $\Box$

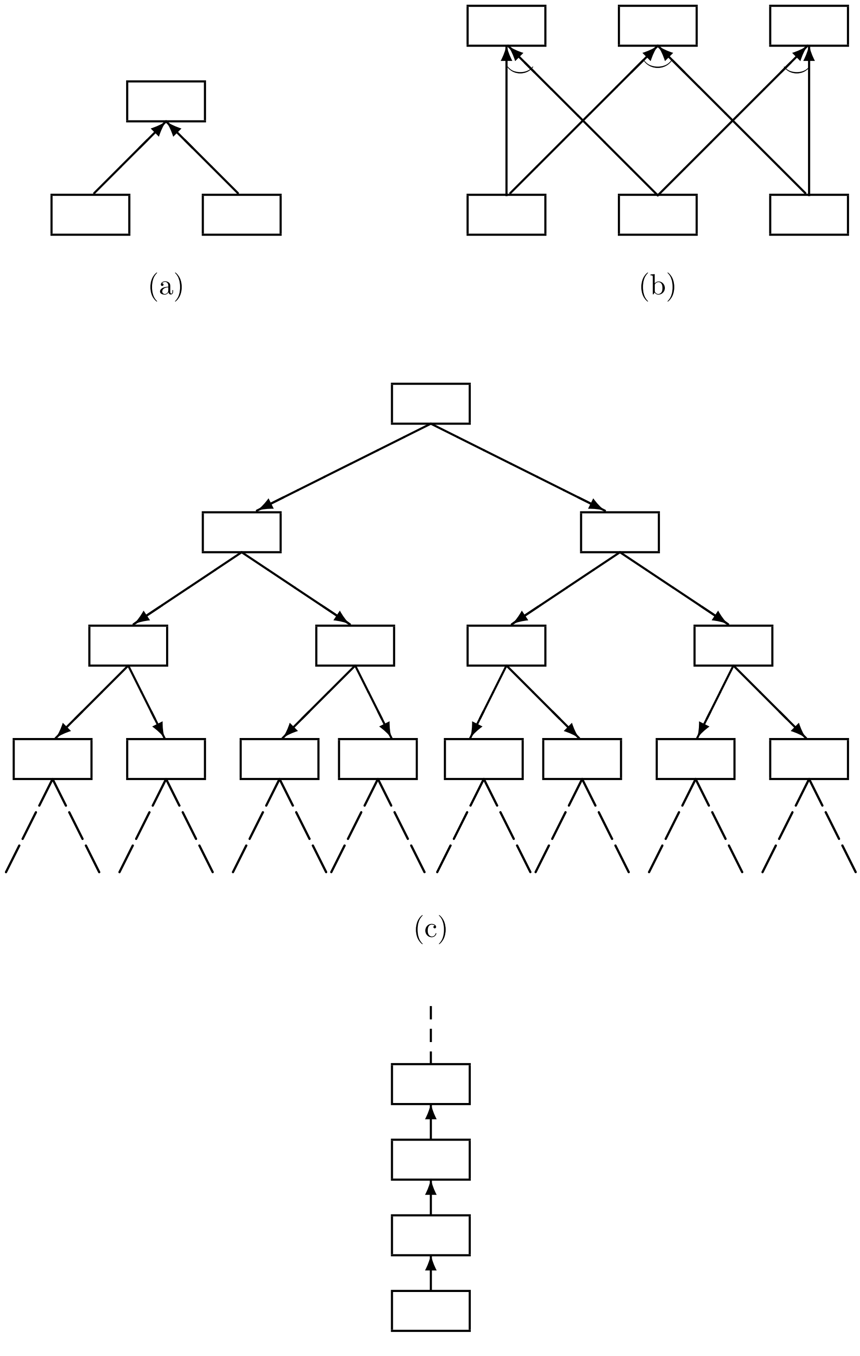
Remark: However, the set intersection of two configurations is not necessarily a configuration because it may not be connected. For example in the matrix M_1 considered earlier, we have $\sigma_5 \cap \sigma_6 = \{c_3\}$ and $\{c_3\}$ is not connected. In fact $\sigma_5 \wedge \sigma_6 = \sigma_1 = \emptyset \neq \sigma_5 \cap \sigma_6$.

Lemma 6.3 *In the partial order* $< \Sigma_M; \subset >$ *configuration* σ_2 *covers configuration* σ_1 *iff there exists a decision* d *such that* $\sigma_2 = \sigma_1 \dot{\cup} d$.

Proof: Assume first that σ_1 and σ_2 are two configurations such that $\sigma_2 = \sigma_1 \dot{\cup} d$. Then $\sigma_1 \subset \sigma_2$ and $\sigma_1 \neq \sigma_2$. Let σ be an arbitrary configuration in $[\sigma_1, \sigma_2]$, i.e $\sigma_1 \subset \sigma \subset \sigma_2$. Since σ_1 and σ_2 differ only by the element d, either σ doesn't contain d and $\sigma_1 = \sigma$ or σ contains d and $\sigma = \sigma_2$. Thus we have indeed $\sigma_1 \prec \sigma_2$.

Conversely assume $\sigma_1 \prec \sigma_2$. Let d be an arbitrary decision in σ_2 not in σ_1. Such a decision exists since σ_1 and σ_2 are distinct. Since σ_2 is connected, there is a proof of d in σ_2:

$$d_0, d_1, d_2, \ldots, d_{n-1}, d_n = d$$



(a) (b) (c) (d)

Figure 14: Example information matrices

Consider the first decision d_j in this proof that does not belong to σ_1. The set $\sigma_1 \dot{\cup} \{d_j\}$ is connected since d_j has a proof entirely contained in it. Now $\sigma_{\subset}\sigma_1 \dot{\cup} \{d_j\} \subset \sigma_2$ and $\sigma_1 \neq \sigma_1 \dot{\cup} \{d_j\}$. So since σ_2 covers σ_1 we must have $\sigma_1 \dot{\cup} \{d_j\} = \sigma_2$. □

Theorem 6.2 *For any information matrix M the partial order $< \Sigma_M; \subset >$ is a concrete domain.*

Proof:

Part 1: $< \Sigma_M; \subset >$ is a computation domain.

1. $< \Sigma_M; \subset >$ is coherent. Let X be a consistent set of configurations and consider the set of decisions obtained in taking all decisions of all elements of X. This set σ is connected because it is a union of connected sets. Suppose two decisions in σ would concern the same cell. These two decisions could not be included in the same element of X, because X contains only configurations. But they cannot come from two distinct elements x_1 and x_2 of X, otherwise $x_1 \cup x_2$ would not be a configuration, contradicting the hypothesis $x_1 \uparrow x_2$ by Lemma 6.2. Thus σ is a configuration. It is the smallest configuration that dominates all elements of X, so $\sigma = \bigcup X$.
2. $< \Sigma_M; \subset >$ is ω–algebraic. Let us show that the finite configurations are exactly the isolated elements in $< \Sigma_M; \subset >$.

   First we show that finite configurations are isolated. Let X be a directed set of configurations and τ a finite set of decisions such that $\tau \subset \bigcup X$. We reason by induction on the size (cardinal) of τ. In the base case, if $|\tau| = 0$ then $\tau = \emptyset$ and for any x in X, $\tau \subset x$. If now $|\tau| = n (n > 0)$ then choose an arbitrary decision d in τ and take $\tau = \tau' \dot{\cup} \{d\}$. Since $|\tau'| < n$ by the induction hypothesis there exists x_1 in X such that $\tau' \subset x_1$. Now there must exists a configuration x_2 in X that contains decision d, otherwise it wouldn't be a decision of $\bigcup X$, which would contradict $\tau \subset \bigcup X$. Since X is directed, there is x in X with $x_1 \subset x$ and $x_2 \subset x$, so $\tau \subset x$.

   Consider now an arbitrary configuration x. If a is a finite subset of x, we have seen that a may be included in a finite connected subset $\overline{a}$ of x, which is then a configuration. As X is the union of all its finite parts, we have $x = \cup\{\overline{a} | a \in \mathcal{F}(x)\}$. On the right hand side of this equation is a directed set of configurations, so we have also:

$$x = \bigcup \{\overline{a} | a \in \mathcal{F}(x)\}$$

So if x is isolated, there exists a finite subset a of x with $x \subset \overline{a}$ and therefore, since $\overline{a} \subset x$, $\overline{a} = x$, proving that x is a finite configuration.

We have proved that the finite elements of $< \Sigma_M; \subset >$ are exactly the finite configurations. As there are only denumerably many finite subsets in a denumerable set, we conclude that $< \Sigma_M; \subset >$ is ω–algebraic. This terminates the first part.

Part 2: $< \Sigma_M; \subset >$ is a concrete domain. We check in turn that $< \Sigma_M; \subset >$ has properties I,C, Q, and R.

1. Property I. The set of finite configurations is trivially an ideal of $< \Sigma_M; \subset >$. As there are only finitely many subsets of a finite set, *a fortiori* there are only finitely many configurations included in a finite configuration. So the ideal is well founded.
2. Property C. Let σ_1 and σ_2 be two compatible finite configurations such that $\sigma_1 \wedge \sigma_2 \prec \sigma_1$. By Lemma 6.3 we have $\sigma_1 = \sigma_1 \wedge \sigma_2 \dot{\cup} \{d\}$. By Lemma 6.2, if $\sigma_1 \uparrow \sigma_2$ then $\sigma_1 \vee \sigma_2 = \sigma_1 \cup \sigma_2$, so:

   $$\sigma_1 \vee \sigma_2 = \sigma_1 \wedge \sigma_2 \cup \{d\} \cup \sigma_2 = \sigma_2 \cup \{d\}$$

   If element d belonged to σ_2, we would have $\sigma_1 \vee \sigma_2 = \sigma_2$ thus $\sigma_1 \subset \sigma_2$ and $\sigma_1 \wedge \sigma_2 = \sigma_1$ which contradicts the hypothesis. Therefore:

   $$\sigma_1 \vee \sigma_2 = \sigma_2 \dot{\cup} \{d\}$$

   and by Lemma 3.2 again $\sigma_2 \prec \sigma_1 \vee \sigma_2$.
3. Property Q. If two configurations σ_1 and σ_2 are incompatible, the set $\sigma_1 \cup \sigma_2$ is not a configuration by Lemma 6.2. Since $\sigma_1 \cup \sigma_2$ is connected, there must exist two distinct decisions d_1 and d_2, with $d_1 \in \sigma_1$ and $d_2 \in \sigma_2$ concerning the same cell. Consider two incompatible and finite configurations σ_1 and σ_2 with $\sigma_1 \wedge \sigma_2 \prec \sigma_1$. Let $d_1 =< c, x >$ and $d_2 =< c, y >$ $(x \neq y)$. Since $\sigma_1 \wedge \sigma_2$ is less than σ_1 and σ_2, it cannot contain a decision concerning cell c. Thus $\sigma_1 = \sigma_1 \wedge \sigma_2 \dot{\cup} \{d_1\}$. The decision d_1 has a proof $\{d'_0, d'_1, \ldots, d'_n = d_1\}$. Without loss of generality we can assume this proof has no earlier occurrence of d_1, i.e. the elements d'_i $(0 \leq i \leq n-1)$ are all in $\sigma_1 \wedge \sigma_2$. Since d_1 and d_2 concern the same cell, we have:

   $$d'_0, d'_1, \ldots, d'_{n-1} \vdash d_2$$

hence the set $\tau = \sigma_1 \wedge \sigma_2 \dot{\cup} \{d_2\}$ is connected, and since d_2 is the only decision concerning c, it is a configuration. We have now $\sigma_1 \wedge \sigma_2 \prec \tau \subset \sigma_2$ and $\sigma_1 \# \tau$, so Property $\mathrm{Q_E}$ is satisfied. Consider now three configurations σ_1, σ_2, and σ_3 covering σ i.e.

$$\sigma_1 = \sigma \dot{\cup} \{d_1\} \quad \sigma_2 = \sigma \dot{\cup} \{d_2\} \quad \sigma_3 = \sigma \dot{\cup} \{d_3\}$$

If $\sigma_1 \mathcal{R}_\sigma \sigma_2$ and $\sigma_2 \mathcal{R}_\sigma \sigma_3$ we must have $d_1 =< c, v_1 >$, $d_2 =< c, v_2 >$, and $d_3 =< c, v_3 >$. If $v_3 = v_1$ then $\sigma_1 = \sigma_3$ and if $v_3 \neq v_1$ then $\sigma_1 \# \sigma_3$. So property $\mathrm{Q_U}$ is satisfied as well.

4. Property R. We will prove that if two prime intervals $[\sigma_1, \sigma_1']$ and $[\sigma_2, \sigma_2']$ there exists a decision d with $\sigma_1' = \sigma_1 \dot{\cup} \{d\}$ and $\sigma_2' = \sigma_2 \dot{\cup} \{d\}$. In fact, since projectivity is the transitive closure of transposition, it is sufficient to prove this property when $[\sigma_1, \sigma_1'] \mathcal{T} [\sigma_2, \sigma_2']$. If $[\sigma_1, \sigma_1'] \leq [\sigma_2, \sigma_2']$ we have seen in part 2 of this proof that $\sigma_2' = \sigma_1' \vee \sigma_2 = \sigma_2 \dot{\cup} \{d\}$. If $[\sigma_1, \sigma_1'] \geq [\sigma_2, \sigma_2']$ assume $\sigma_2' = \sigma_2 \dot{\cup} \{d'\}$ and $\sigma_1' = \sigma_1 \dot{\cup} \{d\}$. By definition, $\sigma_1' = \sigma_2' \vee \sigma_1 = (\sigma_2 \dot{\cup} \{d'\}) \cup \sigma_1$. But we know that $\sigma_2 \subset \sigma_1$, so $\sigma_1' = \sigma_1 \dot{\cup} \{d'\} = \sigma_1 \dot{\cup} \{d\}$. Hence $d = d'$ and $\sigma_2' = \sigma_2 \dot{\cup} \{d\}$.

   Now if $[\sigma, \sigma']$ and $[\sigma, \sigma'']$ are projective, we must have $\sigma' = \sigma \dot{\cup} \{d\}$ and $\sigma'' = \sigma \dot{\cup} \{d\}$, hence $\sigma = \sigma'$ which proves property R. □

Remark: In $< \Sigma_M; \subset >$, the height $h(\sigma)$ of a configuration σ is simply $|\sigma|$ if σ is finite, and infinite otherwise. From the set theoretic equality:

$$|A| + |B| = |A \cap B| + |A \cup B|$$

we deduce, since $\sigma_1 \wedge \sigma_2 \subset \sigma_1 \cap \sigma_2$:

$$h(\sigma_1) + h(\sigma_2) \geq h(\sigma_1 \wedge \sigma_2) + h(\sigma_1 \vee \sigma_2)$$

an inequality that we have already proved. It is clear here that there will be a strict inequality whenever $\sigma_1 \wedge \sigma_2 \neq \sigma_1 \cap \sigma_2$.

7 The representation Theorem

The theorem that we are going to prove now is a representation theorem that plays a role similar to the two classical representation theorems of Lattice Theory([Bir67]):

1. every boolean lattice is isomorphic to a field of sets
2. every distributive lattice is isomorphic to a ring of sets

Here, given an arbitrary concrete domain, we will construct an *information matrix* whose space of configurations, which is a concrete domain by the result of the previous section, is isomorphic to the concrete domain that we started with.

Theorem 7.1 *Every concrete domain is isomorphic to the set of configurations of an information matrix.*

Proof: Consider an arbitrary concrete domain D.

Part 1: Construction of the information matrix.

We build an information matrix $M = < \Gamma, V, \mathcal{V}, \mathcal{E} >$ in the manner that is implicit in our terminology.

i) Γ is the set of cells (equivalence classes under equipollence) of D (cf. Definition 6.1). Since the cardinality of this set is less than the cardinality of the set of isolated elements in D, the set Γ is countable.

ii) V is the set of decisions of D (equivalence classes under projectivity), which is countable for the same reason.

iii) If c is a cell in D, it is the union of equivalence classes under projectivity, so we take $\mathcal{V}(c)$ to be the set of projectivity classes in c. Thus if c_1 and c_2 are two distinct cells in D, the sets $\mathcal{V}(c_1)$ and $\mathcal{V}(c_2)$ are disjoint sets. Therefore *the set Δ_M of decisions of M is isomorphic to V*. In other words, all cells in M have a distinct type.

iv) Function $\mathcal{E}$ is the function that maps any cell c to the set of finite parts of Δ_M (i.e. of V) that enable c (cf. Definition 6.5).

The set of configurations of the matrix M built in this manner is a concrete domain by Theorem 6.2.

Part 2: The injection ϕ from D to $< \Sigma_M; \subset >$.

Any element x in D defines the set $\Delta(x)$ of the decisions that it contains (cf. Definition 6.2). The set $\Delta(x)$ is a subset of V in one-one correspondence with a subset $\phi(x)$ of Δ_M. We prove by induction on $h(x)$ that $\phi(x)$ is a configuration of M.

a) Base case: If $h(x) = 0$ then $x = \bot$ and $\Delta(x) = \phi(x) = \emptyset$. The empty set is a configuration.

b) Induction step: Assume $h(x) = n\,(n > 0)$. Two cases are to be considered:

Case 1: x is not join–irreducible. Then $x = a \vee b$ with $a < x$ and $b < x$, thus $h(a) < n$ and $h(b) < n$. By induction hypothesis $\phi(a)$ and $\phi(b)$ are configurations. Since $\Delta(x) = \Delta(a) \cup \Delta(b)$ by Proposition 6.2, we have also $\phi(x) = \phi(a) \cup \phi(b)$. Thus $\phi(x)$ is a connected set of decisions. By Corollary 5.2, if two prime intervals dominated respectively by a and b are equipollent they are projective, therefore $\phi(a) \cup \phi(b)$ does not contain two distinct decisions in Δ_M concerning the same cell. Hence $\phi(x)$ is a configuration of M.

Case 2: x is join–irreducible. If the element x is join–irreducible it has a (unique) predecessor $\bar{x}$ and $h(\bar{x}) = n - 1$. By induction hypothesis $\phi(\bar{x})$ is a configuration. By definition, in D the set $\Delta(\bar{x})$ enables cell $[\bar{x}, x]$, so the set $\phi(\bar{x}) \cup d[\bar{x}, x]$ is connected in M. Furthermore it is a configuration by Theorem 5.1. Since $\bar{x}$ is a predecessor of x, we have $\phi(x) = \phi(\bar{x}) \cup d[\bar{x}, x]$ so $\phi(x)$ is a configuration.

c) Continuity argument: If x is infinite $\Delta(x) = \bigcup_{\xi \in \mathcal{A}(x)} \Delta(\xi)$ by Proposition 6.2. Thus $\phi(x) = \bigcup_{\xi \in \mathcal{A}(x)} \phi(\xi)$. Since for any finite ξ the set $\phi(\xi)$ is a configuration, the set $\phi(x)$ is connected. By Corollary 6.2 we obtain that $\phi(x)$ is a configuration.

Now $x \leq y$ implies $\Delta(x) \subset \Delta(y)$, i.e. $\phi(x) \subset \phi(y)$. Function ϕ is monotonic. By Theorem 6.1, if $\phi(x) = \phi(y)$ we have $x = y$. Hence ϕ is a monotonic *injection*.

Part 3: Function ϕ is onto.

Since Σ_M is a concrete domain, we reason naturally by induction on the size of an element σ in Σ_M, i.e. on $|\sigma|$.

a) Base case: If $|\sigma| = 0$ then σ is the empty configuration. It is the case that $\phi(\perp_D)$ is the empty configuration.

b) Induction step: Assume that any configuration in Σ_M of cardinality less than $n(n > 0)$ is the image by ϕ of some element in D anc consider a configuration σ with $|\sigma| = n$. Two cases are to be considered:

Case 1: σ is not join–irreducible in Σ_M. Then $\sigma = \sigma_1 \vee \sigma_2$, with $|\sigma_1| < n$ and $|\sigma_2| < n$. By induction hypothesis, there are two elements x_1 and x_2 in D with $\sigma_1 = \phi(x_1)$ and $\sigma_2 = \phi(x_2)$. The elements x_1 and x_2 are compatible, because otherwise, by Proposition 4.4 we could find two equipollent non projective prime intervals $[\xi_1, \xi_1']$ and $[\xi_2, xi_2']$ in $[\bot, x_1]$ and $[\bot, x_2]$ respectively. But then σ would contain two distinct decisions $d[\xi_1, \xi_1']$ and $d[\xi_2, xi_2']$ concerning the same cell, which is impossible. So the element $x_1 \vee x_2$ exists in D and $\phi(x_1 \vee x_2) = \phi(x_1) \cup \phi(x_2) = \phi(x_1) \vee \phi(x_2) = \sigma$.

Case 2: σ is join–irreducible in Σ_M. Let $\bar{\sigma}$ be the unique predecessor of σ. Since $|\bar{\sigma}| = |\sigma| - 1$ there exists an element $\bar{x}$ in D such that $\phi(\bar{x}) = \bar{\sigma}$ by induction hypothesis. Since σ covers $\bar{\sigma}$, there is a decision d with $\sigma = \bar{\sigma} \dot{\cup} d$ and d has a proof $\pi \dot{\cup} d$ with $\pi \subset \bar{\sigma}$. Given the way we have constructed $\mathcal{E}$, there exists therefore in D a *minimal* prime interval $[\xi, \xi']$ with $d[\xi, \xi'] = d$ and $\Delta(\xi) \subset \pi$.

Since $\Delta(\xi) \subset \bar{\sigma} = \Delta(\bar{x})$ we conclude $\xi \leq \bar{x}$ by Theorem 6.1. Since $\bar{\sigma} \dot{\cup} d$ is a configuration, there is no prime interval in $\Delta(\bar{x})$ in the equipollence class of $[\xi, \xi']$. Hence ξ' is compatible with $\bar{x}$ and is not less than $\bar{x}$. Take now $x = \bar{x} \vee \xi'$. Then $\Delta(x) = \Delta(\bar{x}) \cup \Delta(\xi')$ and

$$\Delta(x) = \Delta(\bar{x}) \cup \pi \cup d = \Delta(\bar{x}) \dot{\cup} d = \bar{\sigma} \dot{\cup} d = \sigma$$

and consequently $\phi(x) = \sigma$.

c) Continuity argument: Assume now that σ is an infinite configuration. Since Σ_M is algebraic, we have $\sigma = \bigcup\{\tau | \tau \in \mathcal{A}(\sigma)\}$. Any configuration in $\mathcal{A}(\sigma)$ is finite, so it is the image of some ξ in D. The inverse image of $\mathcal{A}(\sigma)$ by ϕ is a directed set. Let now x be defined by $x = \bigcup\{\xi | \phi(\xi) \in \mathcal{A}(\sigma)\}$. By Proposition 6.2 we obtain $\Delta(x) = \bigcup\{\tau | \tau \in \mathcal{A}(\sigma)\}$ and therefore $\phi(x) = \sigma$.

Theorem 6.1 can now be rewritten in the following manner:

$$x \leq y \iff \phi(x) \subset \phi(y)$$

which concludes the proof of the isomorphism between D and $< \Sigma_M; \subset >$.□

Examples: We show now on a few simple examples how one obtains an information matrix that represents a concrete domain.

Example 1: The diagram of Figure 15 (a) has three equivalence classes of prime intervals for equipollence, so we build three cells. The join–irreducible elements are underlined: a, a', c, c'. Since $\Delta(\bot) = \{\emptyset\}$, cells A and B (corresponding to equipollence classes $\{[\bot, a], [c, b], [c', b']\}$ and $\{[\bot, c], [a, b], [a', b']\}$ respectively) are initial. The domain is a lattice, so each cell can only have one possible value (no incompatibility may arise). Finally cell C, which represents

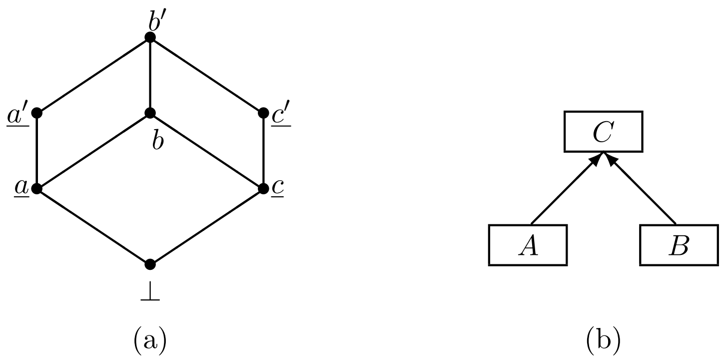


Figure 15: Example 1

equipollence class $\{[a, a'], [b, b'], [c, c']\}$ is enabled either by $\Delta(a)$ or by $\Delta(c)$. In other words, C is enabled by any decision on A *or* on B.

It is easy to verify that the set of configurations of the information matrix on Figure 15 (b) is isomorphic to the partial order on Figure 15 (a) with for example the following correspondence:

domain element	*Configuration*
$\bot$	$\emptyset$
a	$\{< A, \top >\}$
c	$\{< B, \top >\}$
b	$\{< A, \top >, < B, \top >\}$
a'	$\{< A, \top >, < C, \top >\}$
c'	$\{< B, \top >, < C, \top >\}$
b'	$\{< A, \top >, < B, \top >, < C, \top >\}$

Example 2:

The diagram of Figure 16 (a) has two equipollence classes, so we build two cells A and B ($A = \{[\bot, a], [b', a'], [\bot, c], [b', c']\}$ and $B = \{[a, a'], [\bot, b'], [c, c']\}$). As the three join–irreducible elements are atoms, both cells are initial. Finally, cell A contains two equivalence classes of projective prime intervals, and so it may take two distinct values. To double–check, we fill out the correspondence table:

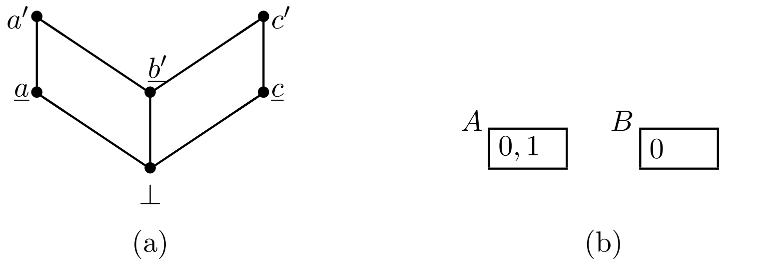


Figure 16: Example 2

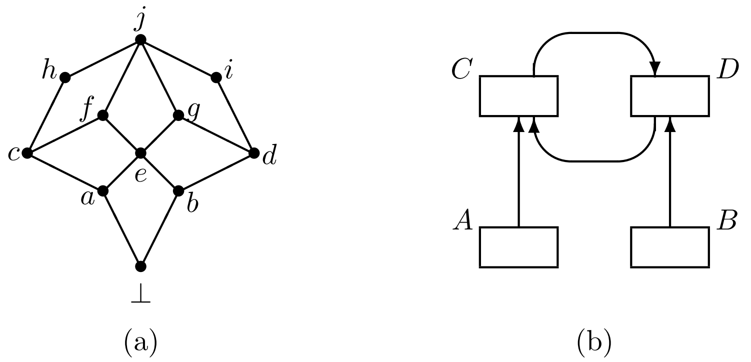


Figure 17: Example 3

domain element	*Configuration*
$\perp$	$\emptyset$
a	$\{< A, 0 >\}$
c	$\{< A, 1 >\}$
b'	$\{< B, 0 >\}$
a'	$\{< A, 0 >, < B, 0 >\}$
c'	$\{< A, 1 >, < B, 0 >\}$

Remark: The domain on Figure 16 (a) is the cartesian product $T \times O$. Note that O is represented by a single cell that may take only a single value, and T is represented by a single cell that may take two values. We will see in the next section that the cartesian product of two concrete domains is represented by the juxtaposition of their representations.

Example 3:

Here again, the diagram of Figure 17 (a) is a lattice, thus all cells in its representation as an information matrix may take only one value.

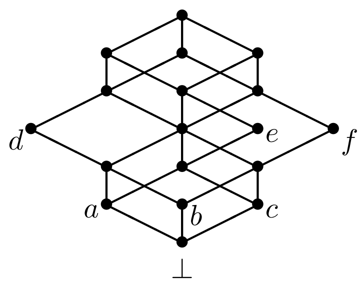


Figure 18: Example 4

There are four cells:

$$
\begin{array}{ll}
A & = \{[\bot, a], [b, e], [d, g], [i, j]\} \\
B & = \{[\bot, b], [a, e], [c, f], [h, j]\} \\
C & = \{[a, c], [e, f], [g, j], [d, i]\} \\
D & = \{[b, d], [e, g], [f, j], [c, h]\}
\end{array}
$$

and six join–irreducible elements: a, b, c, d, h, i. Hence cells A and B are initial, and sets $\{A, C\}$ and $\{B\}$ enable cell D; as well sets $\{B, D\}$ and $\{A\}$ enable cell C. We notice here that the representation theorem doesn't yield a "minimal" representation since the matrix on Fig. 17 (b) is equivalent, i.e. gives rise to the same configurations, but includes less constraints than the one we have built. In view of the symmetry, we give only half of the correspondence between the domain and the configurations of the information matrix.

domain element	*Configuration*
$\bot$	$\emptyset$
a	$\{< A, \top >\}$
e	$\{< A, \top >, < B, \top >\}$
c	$\{< A, \top >, < C, \top >\}$
f	$\{< A, \top >, < B, \top >, < C, \top >\}$
h	$\{< A, \top >, < C, \top >, < D, \top >\}$
j	$\{< A, \top >, < B, \top >, < C, \top >, < D, \top >\}$

Remark: Cell C enables cell D and conversely. This "loop" cannot be eliminated.
Example 4:

The lattice on Figure 18 is the free distributive lattice with three generators. Any *finite* distributive lattice has property $R_{\mathcal{T}}$ and therefore is automatically a concrete domain. The lattice has six equivalence classes of projective prime intervals and each class contains a *single* minimal interval. We will see later that this fact is general in presence of distributivity. The diagram of the representation is on Figure 14 (b).

For the moment, we do not give examples of infinite domains, beyond the well–known domain of infinite sequences. We must first examine a number of basic operations that allow one to construct concrete domains.

8 Basic Operations

In this section and in the next one, we study certain operations that allow one to construct complex concrete domains starting from simpler ones. For example, we have seen that the cartesian product of two computation domains is a computation domain. Similarly:

Proposition 8.1 *The cartesian product of two concrete domains is a concrete domain*

Proof: If D and E are two concrete domains, their cartesian product is ordered componentwise:

$$< x, y > \leq_{D\times E} < x', y' > \Leftrightarrow \ x \leq_D x' \quad \text{and} \quad y \leq_E y'$$

The isolated points in $D \times E$ are pairs of the form $< d, e >$ where $d \in \mathcal{A}(D)$ and $e \in \mathcal{A}(E)$. One checks immediately that the covering and incompatiblity relations are given by

$$\begin{cases} < d, e > \prec < d', e' > & \Leftrightarrow \ (d \prec_D d' \text{ and } e = e') \text{ or } (d = d' \text{ and } e \prec_E e') \\ < d, e > \# < d', e' > & \Leftrightarrow \ (d \#_D d') \text{ or } (e \#_E e') \end{cases}$$

We can now verify that $D \times E$ has all the properties of a concrete domain.

1. Property I: Consider two isolated elements $< d, e >$ and $< d', e' >$ in $D \times E$. Any element $< x, y >$ in the interval $[< d, e >, < d', e' >]$ satisfies:

   $$\begin{cases} d \leq x \leq d' \\ e \leq y \leq e' \end{cases}$$

   There are only finitely many such pairs by Property I in D and E, and *a fortiori* all chains in this interval are finite.

2. Property C: Upper and lower bounds in $D \times E$ are taken componentwise. Assume then we have $< x, x' > \uparrow < y, y' >$ and $< x \wedge y, x' \wedge y' > \prec < x, x' >$. Two cases are to be considered:

   Case 1. $x = x \wedge y$ and $x' \wedge y' \prec x'$. Then by C in E, $y' \prec x' \vee y'$ and of course $x \vee y = y$. Hence $< y, y' > \prec < x \vee y, x' \vee y' > = < x, x' > \vee < y, y' >$.

   Case 2. $x' = x' \wedge y'$ and $x \wedge y \prec x$. Property C in D yields similarly $< y, y' > \prec < x, x' > \vee < y, y' >$.

   So $D \times E$ has property C.

3. Property Q: Let $< x, x' >$ and $< y, y' >$ be two incompatible elements in $D \times E$ such that $< x \wedge y, x' \wedge y' > \prec < x, x' >$. We have either $x \# y$ or $x' \# y'$ and these conditions are not mutually exclusive. Two (symmetric) cases are possible:

   Case 1. $x \wedge y = x$. Then x and y are comparable and therefore $x' \# y'$; since $x' \wedge y' \prec x'$, by Property Q in E there exists an element t' such that $x' \wedge y' \prec t' \leq y'$ and $x' \# t'$. Thus

$$\begin{cases} < x \wedge y, x' \wedge y' > \prec < x, t' > \leq < y, y' > \\ < x, x' > \# < x, t' > \end{cases}$$

   so Property $\mathrm{Q_E}$ is established in this case. Since Property $\mathrm{Q_U}$ is valid in E, there cannot exist an element t'' distinct from t' with

$$\begin{cases} < x \wedge y, x' \wedge y' > \prec < x, t'' > \leq < y, y' > \\ < x, x' > \# < x, t'' > \end{cases}$$

   Furthermore, any element of the form $< u, x' \wedge y' >$ with $x \wedge y = x \prec u$ is compatible with $< x, x' >$. Thus Property $\mathrm{Q_U}$ is valid in this case.

   Case 2. $x' \wedge y' = x'$. This case is treated symmetrically.

   Property Q is therefore established in $D \times E$.

4. Property R: To establish Property R, we must have closer look at the prime intervals in $D \times E$ and the transposition relation. First, the interval $[< d, e >, < d', e' >]$ is prime iff

$$\text{Either } [d, d'] \text{ is prime and } e = e'$$
$$\text{Or } [e, e'] \text{ is prime and } d = d'$$

Take two intervals $[< d_1, e_1 >, < d'_1, e'_1 >]$ and $[< d_2, e_2 >, < d'_2, e'_2 >]$. If $[< d_1, e_1 >, < d'_1, e'_1 >] \leq [< d_2, e_2 >, < d'_2, e'_2 >]$ then

$$\begin{cases} d_1 = d'_1 \wedge d_2 & \text{and} \quad e_1 = e'_1 \wedge e_2 \\ d'_2 = d'_1 \vee d_2 & \text{and} \quad e'_2 = e'_1 \vee e_2 \end{cases}$$

If $[d_1, d'_1]$ is prime and $e_1 = e'_1$ then

$$\begin{cases} [d_1, d'_1] \leq [d_2, d'_2] \\ e_1 = e'_1 = e_2 = e'_2 \end{cases}$$

If $[e_1, e'_1]$ is prime and $d_1 = d'_1$ then

$$\begin{cases} [e_1, e'_1] \leq [e_2, e'_2] \\ d_1 = d'_1 = d_2 = d'_2 \end{cases}$$

By symmetry and transitivity we obtain that if

$$[< d_1, e_1 >, < d'_1, e'_1 >] \sim [< d_2, e_2 >, < d'_2, e'_2 >]$$

$$\begin{cases} \text{Either} & [d_1, d'_1] \sim [d_2, d'_2] \quad \text{and} \quad e_1 = e'_1 = e_2 = e'_2 \\ \text{Or} & [e_1, e'_1] \sim [e_2, e'_2] \quad \text{and} \quad d_1 = d'_1 = d_2 = d'_2 \end{cases}$$

where both cases are mutually exclusive.

Assume now that we have $[< d, e >, < d', e' >] \sim [< d, e >, < d'', e'' >]$.

1. either $[d, d'] \sim_D [d, d'']$, and by Property R, $d' = d''$. Since $e = e' = e''$ we have indeed $< d', e' >=< d'', e'' >$
2. or $[e, e'] \sim_E [e, e'']$ and by Property R, $e' = e''$. Since $d = d' = d''$ we have also $< d', e' >=< d'', e'' >$.

Property R is therefore valid in $D \times E$. $\Box$

Remark: To prove that a computation domain is concrete we have two strategies. Either we examine in turn, as we just did, the properties that must be verified. Or we make use of the representation theorem, i.e. we produce an information matrix whose set of configurations is isomorphic to the domain in question. These two strategies have their own advantages and we will illustrate this in the sequel.

Definition 8.1 *Consider two information matrices* $M' =< \Gamma', V', \mathcal{V}', \mathcal{E}' >$ *and* $M'' =< \Gamma'', V'', \mathcal{V}'', \mathcal{E}'' >$ *whose sets of cells are* disjoint. *The* juxtaposition *of* M' *and* M'' *is the information matrix* $< \Gamma, V, \mathcal{V}, \mathcal{E} >$ *defined as follows:*

$$\begin{cases} \Gamma = \Gamma' \dot{\cup} \Gamma'' \\ V = V' \cup V'' \\ \forall c \in \Gamma' \ \mathcal{V}(c) & = \mathcal{V}'(c) \\ \forall c \in \Gamma'' \ \mathcal{V}(c) & = \mathcal{V}''(c) \\ \forall c \in \Gamma' \ \mathcal{E}(c) & = \mathcal{E}'(c) \\ \forall c \in \Gamma'' \ \mathcal{E}(c) & = \mathcal{E}''(c) \end{cases}$$

Proposition 8.2 *If* M' *and* M'' *are two information matrices and* M *is their juxtaposition, then* $< \Sigma_M; \subset >=< \Sigma_{M'}; \subset > \times < \Sigma_{M''}; \subset >$.

Proof: Consider an arbitrary configuration σ of M. Since the set of cells of M is the disjoint union of the sets of cells of M' and M'', configuration σ is the disjoint union of two sets of decisions σ' and σ'' concerning respectively cells in M' and in M''. The sets σ' and σ'' are connected by definition of the accessibility relation in M. As connected subsets of a configuration σ' and σ'' are configurations of M in in trivial correspondence with configurations of M' and M''. So to any element in Σ_M we can associate an element in $\Sigma_{M'} \times \Sigma_{M''}$. Conversely, by definition of the juxtaposition of two matrices, to any element in $\Sigma_{M'} \times \Sigma_{M''}$ we can associate a configuration in Σ_M. Finally:

$$\sigma_1 \subset_M \sigma_2 \quad \Leftrightarrow \quad (\sigma_1' \subset_{M'} \sigma_2') \text{ and } (\sigma_1'' \subset_{M''} \sigma_2'')$$

hence the one-one mapping between Σ_M and $\Sigma_{M'} \times \Sigma_{M''}$ is order preserving. Thus the domains Σ_M and $\Sigma_{M'} \times \Sigma_{M''}$ are isomorphic. □

From the proposition above, we deduce a quick proof that the the cartesian product of two concrete domains is concrete. If D' and D'' are two concrete domains, represented respectively by matrices M' and M'', the set of configurations of the juxtaposition of M' and M'' is isomorphic to $D' \times D''$. Hence $D' \times D''$ is a concrete domain. The reasoning can be extended to a countable number of information matrices, so we obtain as well:

Corollary 8.1 *The cartesian product of a countable domain of concrete domains is concrete.*

Example: Domain T on Figure 19 (a) is associated to the matrix represented on Figure 19 (b), and T^ω, the universal computation domain of Plotkin ([Plo78]) is associated to the matrix of Figure 19 (c). Hence T^ω is a concrete domain. Similarly $N_\perp^\omega$, the domain underlying the language LUCID ([AW77]) is a concrete domain.

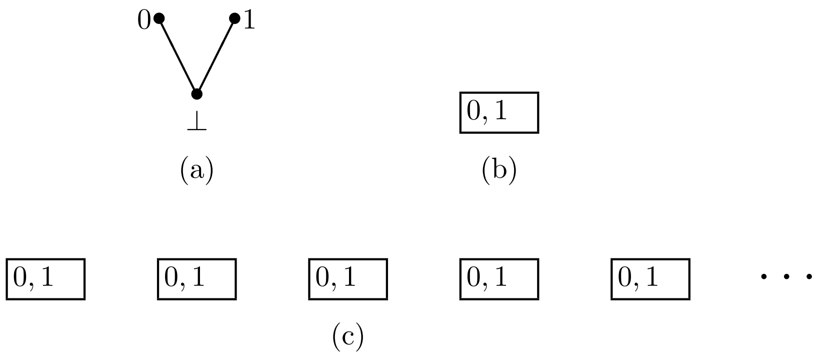


Figure 19: T and T^ω

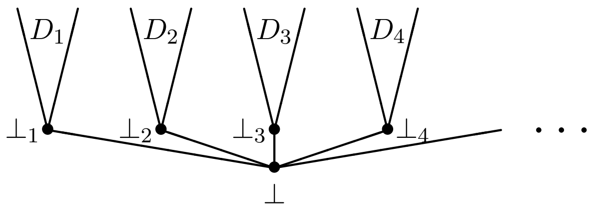


Figure 20: Separated sum

Definition 8.2 *Consider* $\{< D_i; \leq_i >\}_{i \in I}$ *a countable family of partial orders whose domains are disjoint. The* separated sum *of this family is the partial order defined by*

i) $D = \{\bot\} \dot\cup \bigcup_{i \in D_i} D_i$

ii) $x \leq y \quad \Leftrightarrow \quad x = \bot$ *or* $\exists i \in I \; x \leq_i y$

(The element $\bot$ *is not in any of the sets* D_i*).*

Proposition 8.3 *The separated sum of countably many concrete domains is concrete.*

Proof: It is immediate that the separated sum of a countable number of computation domains is a computation domain whose isolated elements are those of the component domains plus the new element $\bot$. Property I is valid as soon as it is valid in the component domains. Property C carries because no new pair of compatible and incomparable elements has been created. The only pairs $< x, y >$ with $x \# y$ and $x \wedge y \prec x$ that have appeared in the separated sum are of the form $< \perp_i, d_j >$ with $i \neq j$ and $d_j \in D_j$, since in that case $\perp_i \wedge d_j = \perp$. But then $\perp_j$ is the unique element such that $\perp_i \# \perp_j, \perp \prec \perp_j \leq d_j$. Hence the separated sum D has property Q. Property R remains valid because the only prime intervals that have appeared in D are of the form $[\perp, \perp_j]$ and they are alone in their projectivity class. □

The separated sum of a family of concrete domains $\{< D_i; \leq_i >\}_{i \in I}$ contains only one new cell that is the equipollence class of the prime intervals of the form $[\perp, \perp_i](i \in I)$. This cell is enabled by the empty set. This remark leads into the following definition.

Definition 8.3 *Consider a finite or countable set of information matrices with disjoint sets of cells $\{M_i\}_{i \in I}$. The* sum *of this family of matrices is the matrix M defined by:*

i) $\Gamma = (\dot{\bigcup}_{i \in I} \Gamma_i) \dot{\cup} \{\gamma\}$

ii) $V = (\bigcup_{i \in I} V_i) \cup \{I\}$

iii) $\mathcal{V}(\gamma) = I$ *and* $\forall c \in \Gamma_i \; \mathcal{V}(c) = \mathcal{V}_i(c)$

iv) $\mathcal{E}(\gamma) = \{\emptyset\}$ *and* $\forall c \in \Gamma_i \; \mathcal{E}(c) = \{e \dot{\cup} \{< \gamma, i >\} | e \in \mathcal{E}_i(c)\}$

Proposition 8.4 *The set of configurations of the sum of a countable family of information matrices is isomorphic to the separated sum of the sets of configurations of this family. concrete.*

Proof: Consider a countable set of information matrices with disjoint sets of cells $\{M_i\}_{i \in I}$ and their sum M. A non empty configuration σ of M contains necessarily one and only one decisions of the form $< \gamma, i >$. Thus all other decisions in σ are decisions in M_i and they form a configuration in Σ_{M_i}. Thus there is an injection of M in the separated sum $(\Sigma_{M_i})_{i \in I}$. Conversely it is trivial to associate a configuration of M to any element in the separated sum. Thus there is a one-one mapping that preserves order, so it is an isomorphism. □

Remark: The choice of a *separated* sum of concrete domains is not arbitrary. Indeed, the *coalesced* sum of two concrete domains is not necessarily a concrete domains; nor is the *skew* sum where one of the minimal elements is taken to be the minimal element of the result. The figure below illustrates the fact that property Q may fail in both cases. Domain Q is either the coalesced sum of O^2 and O, or the skew sum of O^2 and $\perp$. But Q doesn't have property Q.

O^2 O $\perp$ Q

Remark: Domain $\perp$ may be represented by the information matrix with no cells. Domain $N_\perp$ is the separated sum of a countable number of copies of $\perp$. Hence $N_\perp$ may be represented by a unique cell that can take an arbitrary integer as value.

Definition 8.4 *In a coherent partial order* $< D; \leq >$ *a* coherent ideal *is a non-empty subset* J *of* D *such that:*

i) $\forall x \in J, \forall y \in D \quad y \leq x \Longrightarrow y \in J$

ii) $\forall X \subset J \quad X$ *consistent* $\Longrightarrow \bigcup X \in J$

Remark: Since two compatible elements form a consistent set, this definition is a generalization of Definition 2.1.

Proposition 8.5 *In a concrete domain* $< D; \leq >$*, any coherent ideal* J *is a concrete sub–domain.*

Proof: By definition $< J; \leq >$ is coherent. If d is an isolated element in D belonging to J, then d is certainly isolated in J. Conversely, by algebraicity of D, for any d in J we have $d = \bigcup \mathcal{A}(d)$. But all elements in $\mathcal{A}(d)$ belong to J since they are less than d. Hence if d is isolated in J it is also isolated in D. Thus the isolated elements in J are exactly the isolated elements of D belonging to J. So $< J; \leq >$ is a sub–domain of $< D; \leq >$. Lets us show now that J is concrete.

Property I: Since $\mathcal{A}(J) = \mathcal{A}(D) \cap J$ it is immediate that $\mathcal{A}(J)$ is a well-founded ideal of J.

Property C: If x and y are compatible elements in J, then $x \wedge y \in J$ and $x \vee y \in J$. Since Property C holds in D it is valid in J.

Property Q: If x and y are incompatible elements in J, the whole interval $[x \wedge y, y]$ is contained in J. Thus the validity of Q in D implies its validity in J.

Property R: If R were not valid in J, it would not be valid in D. Hence R is satisfied. □

Before exhibiting the representation of coherent ideals, we note an interesting result whose validity relies on the entire property R.

Lemma 8.1 *In a concrete domain, the coherent ideal generated by a finite set of finite elements is finite.*

Proof: Let X be a finite set of finite elements in a concrete domain D. Take $\Delta = \cup\{\Delta(x)|x \in X\}$. The set Δ is finite. Let J be the coherent ideal generated by X, i.e. the intersection of all coherent ideals containing X. Consider the set $K = \{z|\Delta(z) \subset \Delta\}$. This set K contains X and it is a coherent ideal:

1. If $x \in X$ then $\Delta(x) \subset \Delta$, thus $x \in K$
2. If $x \leq y$ and $y \in K$, we have $\Delta(x) \subset \Delta(y) \subset \Delta$, thus $x \in K$
3. If Y is a consistent subset of D such that $\forall y \in Y \ \Delta(y) \subset \Delta$, then by proposition 6.2
$$\Delta(\bigcup Y) = \bigcup_{y \in Y} \Delta(y) \quad \subset \Delta$$
thus K is coherent.

Therefore $J \subset K$ and $\forall z \in J \ \Delta(z) \subset \Delta$. By Theorem 6.1 $z_1 \neq z_2 \implies \Delta(z_1) \neq \Delta(z_2)$ thus $|J| \leq |\mathcal{P}(\Delta)|$. Since Δ is finite, so is $\mathcal{P}(\Delta)$. Hence J is finite. □

Remark: It is easy to generalize the example of Figure 12 to show that the property above is not a consequence of $R_{\mathcal{T}}$ alone.

Definition 8.5 *Let* $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ *be an information matrix and* X *be an arbitrary subset of* Σ_M*. Take* $\Delta_X = \bigcup X$*. The* restriction M_X *of* M *to* X *is the information matrix* $< \Gamma', V', \mathcal{V}', \mathcal{E}' >$ *defined as follows:*

i) $\Gamma' = \{c| < c, v > \in \Delta_X\}$

ii) $V' = \{v| < c, v > \in \Delta_X\}$

iii) $v \in \mathcal{V}'(c)$ *iff* $< c, v > \in \Delta_X$

iv) A set of decisions Δ *in* M_X *enables* c *iff* $\Delta \in \mathcal{E}(c)$

Remark that two restrictions M_X and M_Y are distinct iff Δ_X and Δ_Y are distinct. The restrictions of a given information matrix are naturally ordered by inclusion and we have:

Lemma 8.2 *Let M be an information matrix. The set of restrictions of M ordered by inclusion is isomorphic to the set of coherent ideals of Σ_M.*

Proof:

1. Consider an arbitrary subset X of Σ_M and the restriction M_X of M to X. Let ϕ be the function that, for any X, maps M_X to Σ_{M_X}. We show first that Σ_{M_X} is a coherent ideal of Σ_M.

   i) A configuration σ of M_X is also a configuration of M. If σ' is an arbitrary configuration of M such that $\sigma' \subset \sigma$, then σ' is certainly a configuration of M_X.

   ii) Let S be a consistent set of configurations of M_X. The set $\cup_{\sigma \in S}\sigma$ is also a configuration of M_X. But by Lemma 6.2, in $\Sigma_M \bigcup S = \cup_{\sigma \in S}\sigma$. Therefore $\bigcup S \in \Sigma_{M_X}$, which proves that Σ_{M_X} is a coherent ideal of Σ_M.

   Function ϕ is trivially monotonic. We show that it is an injection. COnsider two distinct restriction M_X and M_Y of M. By the remark above we have $\Delta_X \neq \Delta_Y$. Hence there exists a configuration σ in Y such that not all of its decisions are in Δ_X. This configuration σ is an element of Σ_{M_Y} that is not in Σ_{M_X}.

2. Conversely let J be a coherent ideal of Σ_M, and consider the restriction M_J. By Part 1, the set Σ_{M_J} is a coherent ideal of M that contains J. If we had $J \neq \Sigma_{M_J}$, there would be a decision in Σ_{M_J} that is not in J. But by Definition 8.5 this is impossible. So $J = \Sigma_{M_J}$ and ϕ is onto. □

In a computation domain, the dual concept of an ideal is that of an upper section. Recall that any upper section in a concrete domain is a concrete domain. Upper sections have naturally the dual interpretation of that of ideals.

Definition 8.6 *Let $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ be an information matrix and σ be an arbitrary configuration of M. Take $\mathcal{O}_\sigma = \{c \,|\, < c, v > \in \sigma\}$. The* extension *$M^\sigma$ of σ in M is the information matrix $< \Gamma', V', \mathcal{V}', \mathcal{E}' >$ defined as follows:*

i) $\Gamma' = \Gamma \backslash \mathcal{O}_\sigma$

ii) $V' = V$

iii) $\mathcal{V}'$ *is the restriction of* $\mathcal{V}$ *to* Γ'

iv) If a set of decisions Δ *in* M *enables* c *in* Γ' *then* $\Delta\backslash\sigma$ *enables* c *in* M^σ*; conversely if* Δ' *enables* c *in* M^σ *then it must be the case that* $\Delta' \cup \sigma$ *enables* c *in* M *.*

Lemma 8.3 *Let* M *be an information matrix. The set of extensions* M^σ *of the configurations* σ *of* M *is isomorphic to the set of upper sections* Σ_M*.*

Proof: A set of decisions τ in M^σ is a configuration of M^σ iff $\sigma \cup \tau$ is a configuration of M. □

Definition 8.7 *In a partial order* $< D; \leq >$*, a subset* X *of* D *is* convex *iff whenever it contains* x *and* y *with* $x \leq y$*, it contains all elements in the interval* $[x, y]$*.*

In a computation domain D, a sub-domain H has a minimum element $\perp_H$. If H is convex, then H is a coherent ideal of $[\perp_H)$. Hence any convex sub–domain of a concrete domain is concrete. A convex sub–domain is naturally interpreted as the restriction of the extension of some configuration.

Definition 8.8 *In a computation domain* D*, an* open set *is an arbitrary union of upper sections of finite elements.*

Remarks:

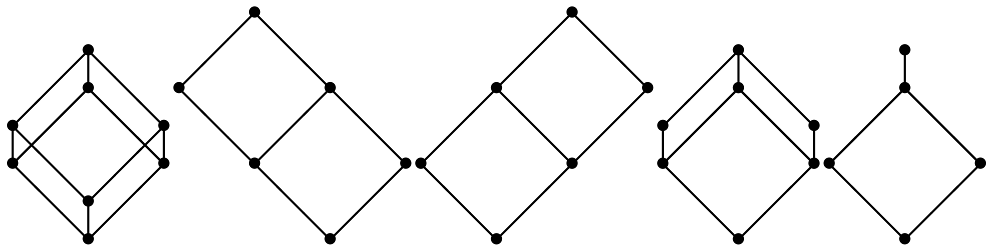
1. The family $\mathcal{F}$ of subsets of D defined in this way has the following properties:

   (O1) $D \in \mathcal{F}$ since $D = [\perp)$

   (O2) Arbitrary union of elements of $\mathcal{F}$ are also elements of $\mathcal{F}$

   (O3) Finite intersections of elements of $\mathcal{F}$ are also elements of $\mathcal{F}$ by Proposition 1.4.

   Therefore the family $\mathcal{F}$ constitues a family of open sets in the usual sense, which justifies our terminology. Note that the upper sections of finite elements form a basis for this topology, and the the upper sections of the join–irreducible elements are a sub–basis, i.e. that any element of the basis is obtained by finite intersection of the elements of the sub–basis (using Corollary 5.5).



F_1 F_2 F_3 F_4 F_5

Figure 21: $F_i = O^2 \overset{\mathcal{O}_i}{\frown} O$

2. A subset of D is an open set iff it is the inverse image of $\top$ by a continuous function from D to O. Indeed, first if f is a continuous function from D to O it is the lub of a family of step functions $\phi_{d,\top}$ with d isolated in D. But $\phi^{-1}_{d,\top}(\top) = [d)$, hence $f^{-1}(\top)$ is an open set. Conversely if $\mathcal{O}$ is an open set, the function f defined by

$$\begin{cases} f(x) = \top & \text{if } x \in \mathcal{O} \\ f(x) = \bot & \text{otherwise} \end{cases}$$

is monotonic and continuous.

Definition 8.9 *Consider two computation domains* $< D; \leq_D >$ *and* $< E; \leq_E >$*, and an open set* $\mathcal{O}$ *in* D*. The* graft of E on D at $\mathcal{O}$*, noted* $D \overset{\mathcal{O}}{\frown} E$*, is the partial order* $< F; \leq >$ *defined as follows:*

i) $F = \{< d, e > \mid d \in D, e \in E \quad \text{and} \quad d \in \mathcal{O} \text{ or } e = \bot\}$

ii) $\leq$ *is the partial order induced by* $D \times E$ *on* F*.*

Example: Take $D = O^2$ and $E = O$. The open sets in O^2 are the sets $\mathcal{O}_i (0 \leq i \leq 5)$ defined by:

$$\mathcal{O}_1 = [\bot) \quad \mathcal{O}_2 = [0) \quad \mathcal{O}_3 = [1) \quad \mathcal{O}_4 = [0) \cup [1) \quad \mathcal{O}_5 = [\top)$$

and the grafts of E on D at $\mathcal{O}_i$ are the F_i whose diagram is shown on Figure 21.

Proposition 8.6 *If* D *and* E *are concrete domains, any graft* F *of* E *on* D *is a concrete domain, and* D *is isomorphic to a coherent ideal of* F*.*

Proof: Consider an arbitrary open set $\mathcal{O}$ in D and take $F = D \overset{\mathcal{O}}{\frown} E$. The set F is a subset of $D \times E$. If two elements in F are compatible, they are compatible in $D \times E$. Conversely, if two elements $< d_1, e_1 >$ and $< d_2, e_2 >$ of F are compatible in $D \times E$, they have a lub $< d_1 \vee d_2, e_1 \vee e_2 >$. Two cases may occur:

i) Either d_1 or d_2 is in $\mathcal{O}$. Then $d_1 \vee d_2 \in \mathcal{O}$ and $< d_1 \vee d_2, e_1 \vee_2 > \in F$.

ii) Or neither d_1 nor d_2 are in $\mathcal{O}$. Then $e_1 = e_2 = \bot_E$ so $e_1 \vee e_2 = \bot_E$ and $< d_1 \vee d_2, e_1 \vee_2 > \in F$.

Therefore two elements in F are compatible iff they are compatible in $D \times E$, and the least upper bounds in F are those in $D \times E$. It follows immediately that F is *coherent*. We show now that F is ω-algebraic. If x is an isolated element in $D \times E$ belonging to F, it is obviously isolated in F. Furthermore, any element $< x, y >$ in F is the lub of its approximants in $D \times E$ by algebraicity of $D \times E$. Consider an approximant $< d, e >$ of $< x, y >$ that is in $D \times E$ but not in F. Then $d \notin \mathcal{O}$ and $e \neq \bot_E$. Hence $y \neq \bot_E$ and therefore $x \in \mathcal{O}$ Since the characteristic function of $\mathcal{O}$ is continuous, there exists c in $\mathcal{A}(x) \cap \mathcal{O}$ such that $d \leq c \leq x$. Now $< d, e >$ is less than $< c, e >$ which is an isolated element in F. Thus $< x, y > = \bigcup \{ < d, e > \mid < d, e > \in \mathcal{A}(D \times E) \cap F \}$. It follows that F is ω-algebraic.

Property I is trivially inherited from $D \times E$. Before checking further properties, remark that $< d, e > \prec_F < d', e' >$ implies $< d, e > \prec_{D \times E} < d', e' >$. Indeed two cases may occur:

<u>Case 1</u>: $d \in \mathcal{O}$. Then $< d, e > \prec_F < d', e' > \Longleftrightarrow < d, e > \prec_{D \times E} < d', e' >$.

<u>Case 2</u>: $d \notin \mathcal{O}$. Then $e = \bot$ and $< d, e > \prec_F < d', e' >$ implies $d \prec_D d'$ and $e' = e = \bot$.

Now if we have $< d, e > \uparrow < d', e' >$ and $< d, e > \wedge_F < d', e' > \prec_F < d, e >$ we must have $< d, e > \wedge_F < d', e' > \prec_{D \times E} < d, e >$. By Property C in $D \times E$ we have

$$< d', e' > \prec_{D \times E} < d \vee e, d' \vee e' >$$

and therefore $< d', e' > \prec_F < d \vee e, d' \vee e' >$ which proves property C.

Similarly if $< d, e > \# < d', e' >$ and $< d, e > \wedge_F < d', e' > \prec_F < d, e >$ then $< d, e > \wedge_F < d', e' > = < d \wedge d', e \wedge e' >$ and by Property Q in $D \times E$ there exists a unique $< t, t' >$ such that $< d \wedge d', e \wedge e' > \prec_{D \times E} < t, t' > \leq < d', e' >$ and $< d, e > \# t, t' >$. Two cases may occur:

<u>Case 1</u>: $d \wedge d' \in \mathcal{O}$. Then $< t, t' > \in F$.

<u>Case 2</u>: $d \wedge d' \notin \mathcal{O}$. Then if $e \neq \perp$ then $d \wedge d' = d$ but in that case $< d, e > \notin F$. Therefore $e = \perp$ and $d \wedge d' \multimap< d$. If $t' \neq \perp$ then $t = d \wedge d'$ aand $< d, d > \uparrow < t, t' >$. So $t' = \perp$ and $< t, t' > \in F$. Hence Property Q holds in F.

Finally, if two intervals of F are transposed, they are also transposed in $D \times E$ thus Property R must be valid in F.

Domain D is isomorphic to the partial order of the pairs of the form $< d, \perp >$ in F which is a coherent ideal of F. □

Remarks:

1. The domains D and $D \overset{D}{\frown} E$ are isomorphic, so that we can consider a cartesian product as a particular kind of graft.
2. If D is finite, the set of maximal points in D is an open set $\mathcal{M}$. The construction $D \overset{\mathcal{M}}{\frown} E$ is particularly useful, so we write it simply $D \frown E$.

Proposition 8.7 *Let* $M_1 =< \Gamma_1, V_1, \mathcal{V}_1, \mathcal{E}_1 >$ *and* $M_2 =< \Gamma_2, V_2, \mathcal{V}_2, \mathcal{E}_2 >$ *be two information matrices, and* X *be an arbitrary set of finite configurations of* M_1. *Define* $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ *as follows:*

i) $\Gamma = \Gamma_1 \dot\cup \Gamma_2$ *(One may assume* Γ_1 *and* Γ_2 *disjoint w.l.o.g.)*

ii) $V = V_1 \cup V_2$

iii) $\mathcal{V}(c) = \begin{cases} \mathcal{V}_1(c) & \text{if } x \in \Gamma_1 \\ \mathcal{V}_2(c) & \text{if } x \in \Gamma_2 \end{cases}$

iv) The function $\mathcal{E}$ *is defined by cases:*

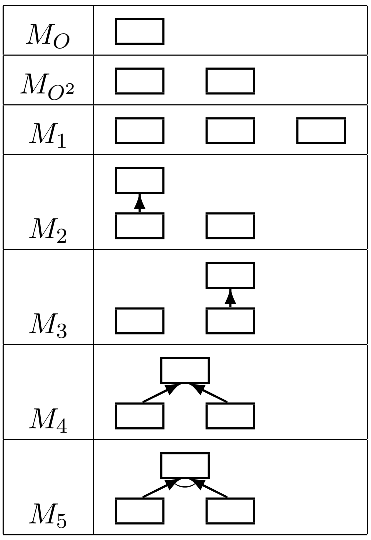
1. *If* $\gamma \in \Gamma_1$ *then* $\mathcal{E}(\gamma) = \mathcal{E}_1(\gamma)$
2. *If* $\gamma \in \Gamma_2$ *and* $\Delta \in \mathcal{E}_2(\gamma)$ *then* $\forall \sigma \in X \quad \{\sigma\} \cup \Delta \in \mathcal{E}(\gamma)$

Then if we take $\mathcal{O} = \{\sigma' | \sigma' \supset \sigma \in X\}$ *we have:*

$$\Sigma_M = \Sigma_{M_1} \overset{\mathcal{O}}{\frown} \Sigma_{M_2}$$

Proof: It is immediate by definition that any configuration in Σ_M is a configuration of the juxtaposition of M_1 and M_2, hence that Σ_M is included in $\Sigma_{M_1} \times \Sigma_{M_2}$. Furthermore, the ordering on Σ_M is inherited from $\Sigma_{M_1} \times \Sigma_{M_2}$.

If σ is a configuration of Σ_M, let σ_1 and σ_2 be the restrictions of σ to Γ_1 and Γ_2 respectively. By definition of $\mathcal{E}$, either $\sigma_2 = \emptyset$ and σ_1 is a configuration of M_1, or $\sigma_2 \neq \emptyset$ and then σ_1 must contain at least one element of X. Hence there is an injection between Σ_M and $\Sigma_{M_1} \overset{\mathcal{O}}{\frown} \Sigma_{M_2}$. Conversely, any element of $\Sigma_{M_1} \overset{\mathcal{O}}{\frown} \Sigma_{M_2}$ is a compatible set of decisions in $\Sigma_{M_1} \times \Sigma_{M_2}$, and by definition of $\mathcal{E}$ it is connected in M, which concludes the proof of the isomorphism. □
Example: Matrices M_{O^2} and M_O represent respectively O^2 and O. Matrices M_i in the table represent each one of the grafts F_i of O on O^2.



9 Inverse limit constructions

We investigate now the possibility of constructing concrete domains by a limiting process. Of course, since the property of being concrete is not in general preserved by exponentiation, it is impossible to preserve it by arbitrary inverse limits. However, it is also clear that certain restricted limit constructions will preserve this property.

Definition 9.1 *If D and E are two computation domains, a* projection *is a pair of continuous functions $< \phi, \psi >$ with $\phi \in [D \to E]$ and $\psi \in [E \to D]$ such that*

i) $\forall x \in D \quad \psi(\phi(x)) = x$

ii) $\forall x \in E \quad \phi(\psi(x)) \leq x$

Definition 9.2 *A projection $<\phi, \psi>$ between D and E is* rigid *iff*

$$\forall d \in \mathcal{A}(D), e \in \mathcal{A}(E) \quad e \leq \phi(d) \Rightarrow e = \phi(\psi(e))$$

Proposition 9.1 *A projection $<\phi, \psi>$ between D and E is rigid iff*

$$\forall x \in D, y \in E \quad y \leq \phi(x) \Rightarrow y = \phi(\psi(y))$$

Proof: Consider an arbitrary approximant e of y in E. If e is less than $\phi(x)$, since ϕ is continuous, there exists an approximant d of x with $e \leq \phi(d)$. But $<\phi, \psi>$ is a rigid projection so $e = \phi(\psi(e))$. As ϕ and ψ are continuous, so is $\phi \circ \psi$ and thus

$$y = \bigcup_{e \in \mathcal{A}(y)} e = \bigcup_{e \in \mathcal{A}(y)} \phi(\psi(e)) = \phi(\psi(\bigcup_{e \in \mathcal{A}(y)} e))$$

and therefore $y = \phi(\psi(y))$. □

Proposition 9.2 *Between two computation domains D and E, there exists a rigid projection iff D is isomorphic to a coherent ideal of E.*

Proof:

Part 1: Consider a coherent ideal J of E and let ϕ be the restriction to J of the identity function on E. Map any x in E to $\psi(x)$ defined by $\psi(x) = \bigcup\{z | z \in \mathcal{A}(x) \cap J\}$. Since E is coherent, the element $\psi(x)$ exists; since J is coherent, the element is in J. We show that $\psi(x)$ is continuous using the characterization of Lemma 1.2. First ψ is trivially monotonic. Consider now an arbitrary approximant e of $\psi(x)$. Since e is isolated and the set $\{z | z \in \mathcal{A}(x) \cap J\}$ is directed, there exists some z with $e \leq z$ and $z \in \mathcal{A}(x) \cap J$. Since for any z in J we have $\psi(z) = z$:

$$\forall e \in \mathcal{A}(\psi(x)) \, \exists z \in \mathcal{A}(x) \quad e \leq \psi(z)$$

which proves that ψ is a continuous function. The pair $<\phi, \psi>$ is a projection between J and E as:

i) $\forall x \in J \quad \psi(\phi(x)) = \psi(x) = x$

ii) $\forall x \in E \quad \psi(x) \leq x$ thus $\phi(\psi(x)) = \psi(x) \leq x$

Consider now two elements x and y with x in J and y in E. If $y \leq \phi(x) = x$, since J is an ideal, element y is in J and therefore $\psi(y) = y$ and also $y = \phi(\psi(y))$. Hence the projection $<\phi, \psi>$ is rigid.

Part 2: Assume that there is a rigid projection $< \phi, \psi >$ between D and E. Take $J = \phi(D)$. We show first that J is a coherent ideal of E.

i) J is downward closed. Consider an arbitrary element y less than $\phi(x)$, for some x in D. Since $< \phi, \psi >$ is rigid, we have $y = \phi(\psi(y))$ by Proposition 9.1. Hence y belongs to $\phi(D)$.

ii) J is coherent. Consider a consistent subset X of $\phi(D)$ and let Y be the inverse image of X by ϕ. The set Y is consistent: consider two arbitrary elements a and b in Y. Since X is consistent, elements $\phi(a)$ and $\phi(b)$ are compatible and we have:

$$\begin{cases} a = \psi(\phi(a)) \leq \psi(\phi(a) \vee \phi(b)) \\ b = \psi(\phi(b)) \leq \psi(\phi(a) \vee \phi(b)) \end{cases}$$

hence a and b are compatible. Since Y is consistent, it has a l.u.b η. Since ϕ is monotonic $\forall x \in X \; x \leq \phi(\eta)$ and therefore, since X is consistent $\bigcup X \leq \phi(\eta)$ and $\bigcup X = \phi(\psi(\bigcup X))$ since $< \phi, \psi >$ is rigid. Thus $\bigcup X$ belongs to $\phi(D)$ and $\phi(D)$ is a coherent ideal.

Finally, if $< \phi, \psi >$ is a projection between D and E, the partial orders D and $\phi(D)$ are isomorphic. We conclude that D is isomorphic to a coherent ideal of E when $< \phi, \psi >$ is rigid. □

Notation: If D and E are concrete domains, we write $D \leq E$ when D is isomorphic to a coherent ideal of E or, equivalently when there is a rigid projection from D to E.

Proposition 9.3 *Among concrete domains, relation $\leq$ is a preorder.*

Proof:

i) If D is an arbitrary concrete domain, D is a coherent ideal of itself.

ii) Assume $D \leq E \leq F$ i.e. that there are two rigid projections $< \phi_1, \psi_1 >$ and $< \phi_2, \psi_2 >$ with:

$$\begin{cases} \forall x \in D \quad \psi_1\psi_2(\phi_2\phi_1(x)) = \psi_1(\psi_2\phi_2(\phi_1(x))) = \psi_1\phi_1(x) = x \\ \forall x \in E \quad \phi_2\phi_1(\psi_1\psi_2(X)) \leq \phi_2(\psi_2(x)) \leq x \end{cases}$$

Assume now that, for some x in D and for some y in F we have $y \leq \phi_2\phi_1(x)$. Since $< \phi_2, \psi_2 >$ is rigid $y = \phi_2(\psi_2(y))$. But $\psi_2(y) \leq \psi_2\phi_2\phi_1(x) = \phi_1(x)$. Hence since $< \phi_1, \psi_1 >$ is rigid, $\psi_2(y) = \phi_1\psi_1\psi_2(y)$. So finally $y = \phi_2\phi_1\psi_1\psi_2(y)$ which proves that $< \phi_2 \circ \phi_1, \psi_1 \circ \psi_2 >$ is rigid. Therefore D is isomorphic to an ideal of F, i.e. $D \leq F$.□

Definition 9.3 *A sequence* $\{D_1, D_2, \cdots, D_n, \cdots\}$ *of computation domains is a* directed sequence *iff for all* $i (i \geq 1)$ *there exists a projection* $< \phi_{i,i+1}, \psi_{i+1,i} >$ *between* D_i *and* D_{i+1}.

Between two domains D_i and D_j of a directed sequence ($i < j$), there exists then a projection noted $< \phi_{i,j}, \psi_{j,i} >$. By convention we note $< \phi_{i,i}, \psi_{i,i} >$ the pair $< I_i, I_i >$ where I_i is the identity function on D_i. If all projections $< \phi_{i,i+1}, \psi_{i+1,i} >$ are rigid, we say that the sequence is *rigid*, which we note

$$D_1 \leq D_2 \leq \cdots \leq D_n \leq \cdots$$

By Proposition 9.3, all projections $< \phi_{i,j}, \psi_{j,i} >$ are also rigid.

Definition 9.4 *Consider a directed sequence* $\{D_1, D_2, \cdots, D_n, \cdots\}$ *of computation domains. The* inverse limit *of this sequence is the partial order* $< D; \leq >$ *where*

i) D *is the set of sequences* $< x_1, x_2, \ldots, x_n, \ldots >$ *with*

$$\begin{cases} \forall i \geq 1 & x_i \in D_i \\ \forall j \geq i & x_i = \psi_{j,i}(x_j) \end{cases}$$

ii) $\leq$ *is the partial order defined componentwise:*

$$x \leq_D y \Leftrightarrow \forall i \geq 1 \; x_i \leq_{D_i} y_i$$

Theorem 9.1 *The inverse limit of a rigid sequence of concrete domains is a concrete domain.*

Proof: Let D be the inverse limit of the rigid sequence

$$D_1 \leq D_2 \leq \cdots \leq D_n \leq \cdots$$

1. The partial order D is coherent. Let X be a consistent subset of D and for all $i (i \geq 1)$ X_i be the set of i-th coordinates of the elements of X. Each of the X_i is consistent in D_i and therefore has a lub $\bigcup X_i$. We show that the sequence $< \bigcup X_1, \bigcup X_2, \ldots, \bigcup X_i, \ldots >$ is in D. Since X is a subset of D:

   $$\forall x \in X \; x_i = \psi_{j,i}(x_j) \quad (i \leq j)$$

   hence

   $$\bigcup X_i = \bigcup_{x_j \in X_j} \psi_{j,i}(x_j)$$

Let X'_j be the directed set obtained from X_j by adding all lubs of its finite subsets. By continuity:

$$\bigcup_{x_j \in X_j} \psi_{j,i}(x_j) = \bigcup_{x_j \in X'_j} \psi_{j,i}(x_j) = \psi(\bigcup X'_j) = \psi(\bigcup X_j)$$

and therefore

$$(\bigcup X)_i = \bigcup X_i = \psi_{j,i}(\bigcup X_j) = \psi_{j,i}(\bigcup X)_j$$

2. The partial order D is ω-algebraic. We must identify the isolated elements in D. To this end, define two collections of functions $\{\phi_{i,\infty}\}$ and $\{\psi_{\infty,i}\}$ from D_i to D and from D to D_i respectively in the following fashion:

$$\begin{cases} \forall e \in D_i \quad (\phi_{i,\infty}(e))_j = \phi_{i,j}(e) \quad (j \geq i) \\ \forall e \in D_i \quad (\phi_{i,\infty}(e))_j = \psi_{i,j}(e) \quad (j < i) \\ \forall x \in D \quad \psi_{\infty,i}(x) = x_i \end{cases}$$

This definition makes sense provided $\forall i \geq 1, \forall e \in D_i, \quad \phi_{i,\infty}(e) \in D$. Take $x = \phi_{i,\infty}(e)$. For any k, it is immediate that x_k belongs to D_k. We must check now the second condition, i.e. $\forall n \geq m \; x_m = \psi_{n,m}(x_n)$. There are three cases:

<u>Case 1. $m \geq i$.</u> Then $x_m = \phi_{i,m}(e)$ and $x_n = \phi_{i,n}(e)$. We compute:

$$\begin{aligned} \psi_{n,m}(x_n) = \psi_{n,m}(\phi_{i,n}(e)) \quad &= \psi_{n,m}(\phi_{m,n}(\phi_{i,m}(e))) \\ &= \psi_{n,m}(\phi_{m,n}(x_n)) \\ &= x_m \end{aligned}$$

<u>Case 2. $n \geq i$.</u> Then $x_m = \psi_{i,m}(e)$ and $x_n = \psi_{i,n}(e)$. We compute:

$$\psi_{n,m}(x_n) = \psi_{n,m}(\psi_{i,n}(e)) = \psi_{i,m}(e) = x_m$$

<u>Case 3. $n \geq i > m$.</u> Then $x_m = \psi_{i,m}(e)$ and $x_n = \phi_{i,n}(e)$. Therefore:

$$\psi_{n,m}(x_n) = \psi_{n,m}(\phi_{i,n}(e)) = \psi_{i,m}(\psi_{n,i}(\phi_{i,n}(e))) = \psi_{i,m}(e)$$

and here again $x_m = \psi_{n,m}(x_n)$.

It is immediate that, for any i, the functions $\phi_{i,\infty}$ and $\psi_{\infty,i}$ are continuous. We show now that *the pairs $< \phi_{i,\infty}, \psi_{\infty,i} >$ are projections from D_i to D*. First,

$$\forall i \geq 1, \forall e \in D_i \quad \psi_{\infty,i}(\phi_{i,\infty}(e)) = (\phi_{i,\infty}(e))_i = \phi_{i,i}(e) = e$$

To prove the second condition, namely

$$\forall i \geq 1, \forall d \in D \quad \phi_{i,\infty}(\psi_{\infty,i}(d)) \leq d$$

we examine the j-th coordinate and distinguish two cases:

Case 1. $j < i$. Then $(\phi_{i,\infty}(\psi_{\infty,i}(d)))_j = (\phi_{i,\infty}(d_j))_j = \psi_{i,j}(d_i)$. But d belongs to D thus, if $j < i$ then $\psi_{i,j}(d_i) = d_j$. We have the required inequality for all coordinates with rank less than i.

Case 2. $j \geq i$. Then $(\phi_{i,\infty}(\psi_{\infty,i}(d)))_j = (\phi_{i,\infty}(d_i))_j = \phi_{i,j}(d_i)$. But d belongs to D thus, if $j \geq i$ then $d_i = \psi_{j,i}(d_j)$. Therefore

$$(\phi_{i,\infty}(\psi_{\infty,i}(d)))_j = \phi_{i,j}(\psi_{j,i}(d_j)) \leq d_j$$

since the pair $< \phi_{i,j}, \psi_{j,i} >$ is a projection. The inequality is established in this case as well.

To conclude, we show now that *the isolated elements of* D *are exactly the* $\phi_{i,\infty}(e)$ for any $i (i \geq 1)$ and e isolated in D_i. Consider first an element d with $d = \phi_{i,\infty}(e)$ and e isolated in D_i. Let X be an arbitrary directed subset of D such that $d \leq \bigcup X$. On the i-th coordinate, we have:

$$d_i = (\phi_{i,\infty}(e))_i = \phi_{i,i}(e) = e \leq (\bigcup X)_i = \bigcup X_i$$

As e is isolated and X_i is directed, there exists x in X with $e \leq x_i$. By monotonicity of $\phi_{i,\infty}$ we conclude $\phi_{i,\infty}(e) = d \leq \phi_{i,\infty}(x_i)$. We are left to prove that $\phi_{i,\infty}(x_i) \leq x$.

i) $j < i$: $(\phi_{i,\infty}(x_i))_j = \psi_{i,j}(x_i) = x_j$

ii) $j \geq i$: $(\phi_{i,\infty}(x_i))_j = \phi_{i,j}(x_i) = \phi_{i,j}(\psi_{j,i}(x_j)) \leq x_j$.

We conclude that $d \leq x$ with $x \in X$ hence d is isolated in D. Similarly, one shows that $\forall i, k \; i \leq k \; \phi_{i,k}(e) \in \mathcal{A}(D_k)$. Thus the set

$$\{z | z \leq x \text{ and } z = \phi_{i,\infty}(e)\}$$

is directed and its lub is x. Thus $\mathcal{A}(D) = \{\phi_{i,\infty}(e) | i \geq 1 \text{ and } e \in D_i\}$ and D is ω-algebraic.

3. The pairs $< \phi_{i,\infty}, \psi_{\infty,i} >$ are rigid. Assume that we have $y \leq \phi_{i,\infty}(x)$ for some y in D and x in D_i. We have to show that $y = \phi_{i,\infty}(\psi_{\infty,i}(y))$.

i) $j < i$: Then $y_j = \psi_{i,j}(y_i)$ hence $y_j = (\phi_{i,\infty}(y_i))_j = (\phi_{i,\infty}(\psi_{\infty,i}(y)))_j$.

ii) $\underline{j \geq i}$: Then $(\phi_{i,\infty}(x))_j = \phi_{i,j}(x)$. Since the pairs $< \phi_{i,j}, \psi_{j,i} >$ are rigid, from $y_j \leq \phi_{i,j}(x)$ we deduce $y_j = \phi_{i,j}(\psi_{j,i}(y_j))$. But $\psi_{j,i}(y_j) = y_i$ so that we obtain:

$$y_i = \phi_{i,j}(y_i) = (\phi_{i,\infty}(y_i))_j = (\phi_{i,\infty}(\psi_{\infty,i}(y)))_j$$

In both cases we have the desired inequality, The pairs $< \phi_{i,\infty}, \psi_{\infty,i} >$ are therefore rigid, and all domains D_i are isomorphic to coherent ideals of D.

4. The domain D is concrete. We check first Property I. If $\phi_{i,\infty}(e)$ and $\phi_{j,\infty}(f)$ are two isolated elements in D with $\phi_{i,\infty}(e) \leq \phi_{j,\infty}(f)$, then $\phi_{i,\infty}(e)$ belongs to $\phi_{j,\infty}(D_j)$ since $\phi_{j,\infty}(D_j)$ is an ideal of D. Since $\phi_{j,\infty}(D_j)$ is isomorphic to D_j that has Property I, there cannot be an infinite chain between $\phi_{i,\infty}(e)$ and $\phi_{j,\infty}(f)$. The remaining properties C,Q,and R are expressed in terms of a finite number of finite elements in D. There exists always a coherent ideal $\phi_{k,\infty}(D_k)$ that contains all these elements, and therefore the properties are valid in D because they are valid in D_k. □

Proposition 9.4 *Any concrete domain is the inverse limit of a rigid sequence of some of its finite coherent ideals.*

Proof: Consider an enumeration $\{c_1, c_2, \ldots, c_n, \ldots\}$ of the finite elements in a concrete domain D. This enumeration exists since D is ω-algebraic. Let us build a sequence $\{J_1, J_2, \ldots, J_n, \ldots\}$ of ideals where J_i is the coherent ideal generated by $\{c_1, c_2, \ldots, c_i\}$. By Lemma 8.1, each one of these ideals is finite, and by Proposition 8.6, each one of them is a concrete domain. Since for any i domain J_i is a coherent ideal of J_{i+1}, the sequence $\{J_i\}$ is a rigid sequence of concrete domains, and its inverse limit J is a concrete domain. We have to show that J is isomorphic to D.

By Proposition 9.2, if J_i is a coherent ideal of J_j the pair $< \phi_{i,j}, \psi_{j,i} >$ with $i \leq j$ and

$$\begin{cases} \forall x \in J_i \quad \phi_{i,j}(x) = x \\ \forall x \in J_j \quad \psi_{i,j}(x) = \bigcup\{z | z \in \mathcal{A}(x) \cap J_i\} \end{cases}$$

is a rigid projection between J_i and J_j. Take $x =< x_1, x_2, \ldots, x_n, \ldots >$ an element of J. From $x_i = \psi_{j,i}(x_j)$ we deduce $\forall i, j \geq i \quad x_i \leq x_j$. The sequence $\{x_1, x_2, \ldots, x_n, \ldots\}$ is increasing and has a lub $\phi(x)$. It is immediate that function ϕ is a monotonic function from J to D.

1. $\underline{\phi \text{ is onto}}$. Consider an arbitrary element d in D and the sequence $\delta =< d_1, d_2, \ldots, d_n, \ldots >$ where $d_i = \bigcup\{z | z \in \mathcal{A}(d) \cap J_i\}$. The sequence δ belongs to J because if $i \leq j$ then $J_i \subset J_j$ and therefore

$$\begin{aligned} d_i &= \bigcup\{z|z \in \mathcal{A}(d) \cap J_i\} = \bigcup\{z|z \in \mathcal{A}(d) \cap J_j \cap J_i\} \quad (i \leq j) \\ &= \bigcup\{z|z \in \mathcal{A}(d_j) \cap J_i\} = \psi_{j,i}(d_j) \end{aligned}$$

   Finally $\phi(\delta) = \bigcup_{i \geq 1} d_i = d$ since the family $\{J_i\}_{i \geq 1}$ covers $\mathcal{A}(D)$.

2. $\underline{\phi \text{ is one-one}}$. Consider two distinct elements $x =< x_1, \ldots, x_n, \ldots >$ and $x' =< x'_1, \ldots, x'_n, \ldots >$ of J and let k be the smallest integer such that $x_k \neq x'_k$. We must have $x_k = x'_k \vee c_k$ or the symmetric equality. From $\forall l \geq k \ x_k = \bigcup\{z | z \in \mathcal{A}(D_l) \cap J_k\}$ we deduce $\forall l \geq k \ x_l \not\geq c_k$ and therefore $\phi(x) = \bigcup_{i \geq 1} x_i \not\geq c_k$. But $\phi(x') \geq x'_k \geq c_k$ so that necessarily $\phi(x) \neq \phi(x')$.$\square$

We give now a result that justifies our expressing all properties in terms of isolated elements.

Theorem 9.2 (Ideal Completion) *Let $< L; \leq >$ be a partial order where L is denumerable and*

i) Any consistent finite subset of L has a lub.

ii) Between any two elements of L, all chains are finite.

iii) L has properties C, Q, and R.

Consider then the partial order $\widehat{L}$ of the directed ideals *of L* ordered by inclusion. *Then $\widehat{L}$ is a concrete domain and L is isomorphic to $\mathcal{A}(\widehat{L})$.*

Proof:

1. $\underline{\widehat{L} \text{ is coherent}}$. Let X be a consistent family of directed ideals. Consider two compatible elements J_1 and J_2 of J. They are compatible, so there exists a directed ideal J_3 with $J_1 \subset J_3$ and $J_2 \subset J_3$. For any $a \in J_1$ and $b \in J_2$ we have also $a \in J_3$ and $b \in J_3$ so a and b are compatible. Let X' be the union of all ideals in X and J the set obtained from X' in adding the lubs of all of the finite subsets of X' (they exist by hypothesis i)) and the elements dominated by these lubs. It is immediate that J is a directed ideal. Since any directed ideal containing the elements of X must include J we deduce $J = \bigcup_{\widehat{L}} X$ and therefore $\widehat{L}$ is coherent.

2. $\widehat{L}$ is ω-algebraic. We show that the principal ideal of L, i.e. the sets of the form J_a with

$$J_a = \{z | z \leq a\} \quad (a \in L)$$

are exactly the isolated elements in $\widehat{L}$. Consider a directed subset X of $\widehat{L}$ such that $J_a \subset \bigcup_{\widehat{L}} X$. We have $a \leq \bigcup_L (\bigcup_{\widehat{L}} X)$. But in L, all elements a are isolated because all chains from $\perp$ to a are finite by hypothesis ii). Thus there exists an element x in the directed ideal $\bigcup_{\widehat{L}} X$ with $a \leq x$, and therefore an ideal Ξ in X that contains x. We obtain $J_a \subset \Xi$ which proves that J_a is isolated.

Consider now an arbitrary element J in $\widehat{L}$. Trivially we have $J = \cup_{a \in J} J_a$. But $\cup_{a \in J} J_a \subset \bigcup_{a \in J} J_a \subset \cup_{a \in J} J_a$ hence $\bigcup_{a \in J} J_a = \cup_{a \in J} J_a$. Finally $J = \bigcup_{a \in J} J_a$, which proves that $\widehat{L}$ is algebraic, and that the principal ideal of L are the isolated elements of $\widehat{L}$. Since L is denumerable, $\widehat{L}$ is ω-algebraic.

Finally we note that $\mathcal{A}(\widehat{L})$ is isomorphic to L. Consequently, properties C, Q, and R are valid in $\mathcal{A}(\widehat{L})$ hence in $\widehat{L}$. This concludes the proof that $\widehat{L}$ is a concrete domain. □

10 Distributive concrete domains

We are going to study now a special case of importance in applications, that of concrete domains in which there is *a unique minimal prime interval* in each equivalence class of projective prime intervals (by Proposition 6.4, there exists at least one minimal interval in each projectivity class). We call this unicity property Property U. It is defined as follows:

Property U

If $[a, a']$ and $[b, b']$ are two minimal projective prime intervals, then $[a, a'] = [b, b']$.

Proposition 10.1 *Property* U *is equivalent to Property* U'*:*
If $[a, a']$ *and* $[b, b']$ *are two minimal projective prime intervals and there exists a prime interval* o *with* $[a, a'] \leq o \geq [b, b']$ *then* $[a, a'] = [b, b']$*.*

Proof: It is immediate that Property U implies Property U'. Assume now that U' holds, and consider an alternating sequence of transposed prime intervals between two minimal intervals $[a, a']$ and $[b, b']$

$$\{[a, a'], [x_1, x_1'], \ldots, [x_{n-1}, x_{n-1}'], [b, b']\}$$

Since $[a, a']$ and $[b, b']$ are minimal, we have necessarily $[a, a'] \leq [x_1, x_1']$ and $[b, b'] \leq [x_{n-1}, x_{n-1}']$ hence n is an even number. Take $n = 2p$ and reason by induction on p. If $p = 1$, we are in the configuration of Property U', so $[a, a'] = [b, b']$. If p is larger than 1, two cases are possible:

Case 1: x_2' is join–irreducible. By U' we have $[a, a'] = [x_2, x_2']$. There exists now an alternating chain of length $2(p - 1)$ of prime intervals between $[a, a']$ and $[b, b']$. By induction hypothesis we conclude $[a, a'] = [b, b']$.

Case 2: x_2' is not join–irreducible. Then there exists a minimal prime interval $[\overline{x_2}, \overline{x_2'}]$ with $[\overline{x_2}, \overline{x_2'}] \leq [x_2, x_2']$. But then $[a, a'] \leq [x_1, x_1'] \geq [x_2, x_2'] \geq [\overline{x_2}, \overline{x_2'}]$ and by Property U' we obtain $[a, a'] = [\overline{x_2}, \overline{x_2'}]$. The sequence $\{[\overline{x_2}, \overline{x_2'}], [x_3, x_3'], \ldots, [b, b']$ is an alternating sequence of length $2(p - 1)$, and $[\overline{x_2}, \overline{x_2'}] = [b, b']$ by induction hypothesis. We conclude $[a, a'] = [\overline{x_2}, \overline{x_2'}] = [b, b']$. $\square$

Lemma 10.1 *In a concrete domain D, the following properties are* equivalent*:*

1. *Property U*
2. *Conditional distributivity:*

$$\forall a, b, c \in D \quad b \uparrow c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

3. *Conditional modularity:*

$$\forall a, b, c \in D \quad a \uparrow b, a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

4. $\forall x, y \in D \quad \Delta(x \wedge y) = \Delta(x) \cap \Delta(y)$
5. *The height function is a valuation, i.e.*

$$\forall x, y \in \mathcal{A}(D) \quad x \uparrow y \Rightarrow h(x) + h(y) = h(x \vee y) + h(x \wedge y)$$

Proof:

a) 1 implies 4. We know already, by Proposition 6.2, that $\Delta(x \wedge y) \subset \Delta(x)$ and $\Delta(x \wedge y) \subset \Delta(y)$ and therefore $\Delta(x \wedge y) \subset \Delta(x) \cap \Delta(y)$. Consider now a decision d belonging to $\Delta(x)$ and $\Delta(y)$, and two prime intervals $[u, u']$ and $[v, v']$ included respectively in $[\bot, x]$ and $[\bot, y]$ and in the projectivity class of d. By Proposition 6.4, we can find two minimal intervals $[\overline{u}, \overline{u'}]$ and $[\overline{v}, \overline{v'}]$ such that $[\overline{u}, \overline{u'}] \leq [u, u']$ and $[\overline{v}, \overline{v'}] \leq [u, u']$. Since $[u, u'] \sim [v, v']$ Property U allows one to deduce $[\overline{u}, \overline{u'}] = [\overline{v}, \overline{v'}]$. Since $\overline{u'}$ and $\overline{v'}$ are dominated respectively by x and y we have $u' = v' \leq x \wedge y$. Thus decision d belongs to $\Delta(x \wedge y)$. We have shown the inequality $\Delta(x) \cap \Delta(y) \subset \Delta(x \wedge y)$ and we conclude $\Delta(x \wedge y) = \Delta(x) \cap \Delta(y)$.

b) 4 implies 5. In the lattice of finite subsets of an arbitrary set, we have the equation $|A \cup B| = |A| + |B| - |A \cap B|$. Consider two arbitrary compatible elements x and y in D. By Proposition 6.2 we have $\Delta(x \vee y) = \Delta(x) \cup \Delta(y)$. Therefore

$$\begin{aligned} |\Delta(x \vee y)| &= \Delta(x) \cup \Delta(y)| \\ &= |\Delta(x)| + |\Delta(y)| - |\Delta(x) \cap \Delta(y)| \\ &= |\Delta(x)| + |\Delta(y)| - |\Delta(x \wedge y)| \quad \text{by 4} \end{aligned}$$

Using the result of Proposition 6.4, we obtain

$$x \uparrow y \Rightarrow \ h(x) + h(y) = h(x \vee y) + h(x \wedge y)$$

c) 5 implies 1. We show that 5 implies Property U', which is sufficient by Proposition 12.1. Assume we have $[a, a'] \leq [z, z'] \geq [b, b']$ with $[a, a']$ and $[b, b']$ minimal. Let us show that either $[a, a'] = [b, b']$ or $a \wedge b = a' \wedge b'$. Suppose we had $a \wedge b < a' \wedge b'$. By relative atomicity, there would exist an element t such that $a \wedge b \prec t \leq a' \wedge b'$. Thus either $t \not\leq a$ or $t \not\leq b$. Assume w.l.o.g. that $t \not\leq a$. Then $t \wedge a = a \wedge b$ and by Property C $a \prec a \vee t \leq a'$. Since we have also $a \prec a'$ we must have $a \vee t = a'$ and $[a \wedge b, t] \leq [a, a']$. Since $[a, a']$ is minimal $a \wedge b = a$ and $t = a'$. Since $[a, a']$ and $[b, b']$ are projective, by Theorem 5.1 $a' \leq b$ is not possible. Hence $[a, a'] \leq [b, b']$. But $[b, b']$ is also minimal, so $[a, a'] = [b, b']$. We have proved by contradiction that if $[a, a']$ and $[b, b']$ are distinct $a' \wedge b' = a \wedge b$. But Proposition 6.5 allows one to write:

$$[a, a'] \leq [a \vee b, a' \vee b'] \geq [b, b']$$

By hypothesis, function h is a valuation and we have

$$h(a \wedge b) = h(a) + h(b) - h(a \vee b)$$

thus $1 + h(a \wedge b) = h(a' \wedge b')$, which contradicts $a \wedge b = a' \wedge b'$. We conclude that $[a, a'] = [b, b']$ thereby proving Property U'.

d) 4 implies 2. Consider three elements a, b, c in D with $b \uparrow c$.

$$
\begin{aligned}
\Delta(a \wedge (b \vee c)) &= \Delta(a) \cap \Delta(b \vee c) && \text{by 4} \\
&= \Delta(a) \cap (\Delta(b) \cup \Delta(c)) && \text{(Proposition 6.2)} \\
&= (\Delta(a) \cap \Delta(b)) \cup (\Delta(a) \cap \Delta(c)) && \text{(set theory)} \\
&= \Delta(a \wedge b) \cup \Delta(a \wedge c) && \text{by 4 again} \\
&= \Delta((a \wedge b) \vee (a \wedge c)) && \text{(Proposition 6.2)}
\end{aligned}
$$

And by Theorem 6.1 we conclude $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

e) 2 implies 3. This is a standard proof in lattice theory. Assume $a \uparrow b$ and $a \leq c$. By distributivity:

$$
\begin{aligned}
(a \vee b) \wedge (a \vee c)) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) \\
&= a \vee ((a \vee b) \wedge c) \\
&= a \vee ((a \wedge c) \vee (b \wedge c)) && \text{by distributivity} \\
&= a \vee a \vee (b \wedge c) && \text{since } a \leq c
\end{aligned}
$$

We obtain the required modularity law $(a \vee b) \wedge c = a \vee (b \wedge c)$.

f) 3 implies 1. Assume we have the modularity law and consider two minimal prime intervals $[a, a']$ and $[b, b']$ such that $[a, a'] \leq [a \vee b, a' \vee b'] \geq [b, b']$. Since $a \vee b' = a' \vee b = a' \vee b'$ and $b \leq b'$ we obtain by modularity:

$$b' = (a' \vee b) \wedge b' = b \vee (a' \wedge b')$$

But if $[a, a']$ and $b, b']$ are distinct, we have seen that $a' \wedge b' = a \wedge b$ thus $b' = a \vee (a \wedge b) = b$ which is a contradiction. Since $[a, a'] = [b, b']$ Property U' holds. □

The result above justifies calling a domain satisfying Property U either *modular* or *distributive* or even *metric.*

Proposition 10.2 *A concrete domain D is distributive iff it is isomorphic to the partial order of configurations of a matrix $< \Gamma, V, \mathcal{V}, \mathcal{E} >$ with*

$$\forall \gamma \in \Gamma,\ |\mathcal{E}(\gamma)| = 1$$

In other words D is represented by a matrix without disjunctions.

Proof: From left to right, the result is a direct consequence of the construction used in the Representation Theorem and Property U. Conversely, consider a matrix $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ verifying the condition $\forall \gamma \in \Gamma \ |\mathcal{E}(\gamma)| = 1$. For any decision d, let $p(d)$ the unique set of decisions that enables d. We show that, in such an information matrix, if a decision d has a proof, then it has a unique irredundant proof. The proof is by induction on the length $l(d)$ of the proof of d.

Base Case: $l(d) = 1$, i.e. d is initial and $p(d) = \emptyset$. The proof $\{d\}$ is irredundant and any other proof of d includes it, hence it is unique.

Inductive step: $l(d) = n(n > 1)$. Then d has a proof $d_1, d_2, \ldots, d_{n-1}, d$. Since only $p(d)$ enables d, we must have $p(d) \subset \{d_1, d_2, \ldots, d_{n-1}\}$. Thus all decisions in $p(d)$ have proof of length less than n, therefore a unique irredundant proof by induction hypothesis. Let now $\Delta(d)$ be the union of all unique irredundant proofs of all elements of $p(d)$. The set $\Delta(d) \cup \{d\}$ is a proof of d. Any proof of d contains d and the irredundant proofs of the elements of $p(d)$. Therefore $\Delta(d) \cup \{d\}$ is the unique irredundant proof of d.

Consider now σ_1 and σ_2 two finite compatible configurations of M. Since σ_1 and σ_2 are compatible, the set of decisions $\sigma_1 \cap \sigma_2$ doesn't contain two distinct decisions concerning the same cell because it is included in $\sigma_1 \cup \sigma_2$. If d is an arbitrary decision in $\sigma_1 \cap \sigma_2$ it has a unique irredundant proof π. Since σ_1 and σ_2 are connected $\pi \subset \sigma_1$ and $\pi \subset \sigma_2$ thus $\pi \subset \sigma_1 \cap \sigma_2$ and the set $\sigma_1 \cap \sigma_2$ is connected. Hence it is a configuration and $\sigma_1 \wedge \sigma_2 = \sigma_1 \cap \sigma_2$. Then $|\sigma_1| + |\sigma_2| = |\sigma_1 \wedge \sigma_2| + |\sigma_1 \vee \sigma_2|$ and the height of the elements of Σ_M is a valuation. By Lemma 10.1 the concrete domain $< \Sigma_M; \subset >$ is distributive. □

Remark: The previous results states that if $< \Sigma_M; \leq >$ is distributive, then there exists a matrix M' with $< \Sigma_M; \subset >=< \Sigma'_M; \subset >$. But it is perfectly possible for M to contain disjunctions, as shown in the example of Figure 22.

The following proposition characterizes a frequent case, where distributivity can be proved quickly.

Proposition 10.3 *A concrete domain is distributive iff the domain is the partial order of configurations of some information matrix $M =< \Gamma, V, \mathcal{V}, \mathcal{E} >$ where any cell is enabled by sets of decisions that concern a single set of cells.*

Proof: The proof follows the pattern of the proof of the previous result. The property is immediate from left to right. For any d let $q(d)$ be the common set of cells occupied by all sets of decisions that enable the cell of d. We

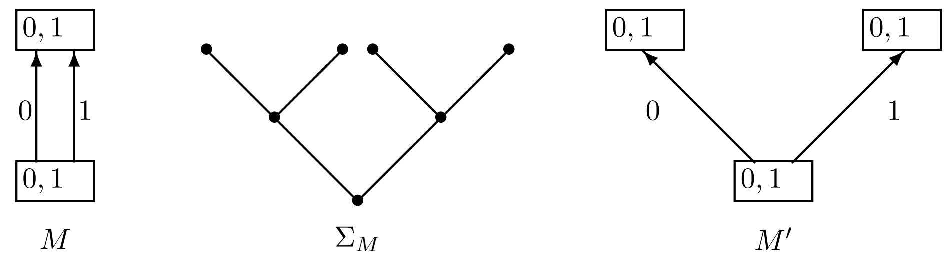


Figure 22: M and M' have the same configuration space Σ_M

show that in such an information matrix, if a decision d has a proof, then all irredundant proofs of d occupy the same set of cells. We proceed by induction on the length $l(d)$ of the length of d.

Base Case: $l(d) = 1$. The empty set is the only one that enables d. Hence the cell of d is occupied by any proof of d.

Inductive step: $l(d) = n (n > 1)$. Then d has a proof $d_1, d_2, \ldots, d_{n-1} \vdash d$. Let $\mathcal{O}(\{d_1, d_2, \ldots, d_{n-1}\})$ be the set of cells occupied by the decisions in $\{d_1, d_2, \ldots, d_{n-1}\}$. Any set of decisions enabling d occupies $q(d)$ so $q(d) \subset \mathcal{O}(\{d_1, d_2, \ldots, d_{n-1}\})$. Consider an element Δ in $\mathcal{E}(d)$ included in $\{d_1, d_2, \ldots, d_{n-1}\}$. By induction hypothesis, all irredundant proofs of the elements of Δ occupy the same set of cells. Let γ be the cell occupied by d. Taking the union of all these cells with γ we obtain a set of cells $\Gamma(d)$ and any irredundant proof of d contains $\Gamma(d)$.

Consider now two finite and compatible configurations σ_1 and σ_2 of M and take an arbitrary decision d in $\sigma_1 \cap \sigma_2$. Any irredundant proof of d occupies $\Gamma(d)$. Hence σ_1 and σ_2 occupy $\Gamma(d)$. Therefore d has a proof in $\sigma_1 \cap \sigma_2$ and thius set of decisions is connected. Hence $\sigma_1 \wedge \sigma_2 = \sigma_1 \cap \sigma_2$ and Σ_M is a distributive concrete domain. □

Proposition 10.4 *The separated sum of a finite or denumerable number of distributive concrete domains, the cartesian product of a finite or denumerable number of distributive concrete domains, the inverse limit of any rigid sequence of distributive concrete domains are distributive concrete domains.*

Proof: It is immediate that the sum and the juxtaposition of an arbitrary number of information matrices in which all cells are enabled by a unique set of decisions is of this kind as well. Let D be the inverse limit of a rigid sequence of distributive concrete domains $D_1, \leq D_2 \leq \cdots \leq D_n \leq \cdots$. If $[x, x']$ and $[y, y']$ are two minimal prime intervals with $[x, x'] \leq [x \vee y, x' \vee y'] \geq [y, y]$, consider the coherent ideal generated by the isolated elements x' and y'. The ideal J is finite and thus there exists an integer k such that $J \leq D_k$. Since D_k is distributive, by Property U' we obtain $[x, x'] = [y, y']$ which proves Property U' in D. □

Proposition 10.5 *If D and E are two distributive concrete domains, and if $\mathcal{O}$ is an open set such that*

$$\forall d, e \in \mathcal{O}\, minimal \quad \Gamma(d) = \Gamma(e)$$

then $D \overset{\mathcal{O}}{\frown} E$ is a distributive concrete domain.

Proof: By construction of the matrix associated to $D \overset{\mathcal{O}}{\frown} E$, it is immediate that it satisfies the condition of Proposition 10.3. □
Example: It is easy to check on Figure 21 that only F_4 is not distributive.

Historical Note(1978): The essential part of the research reported here was carried out in Autumn 1975 at the University of Edinburgh. Preliminary versions of this text have been distributed privately during seminars on Semantics in Sophia-Antipolis in Autumn 1977 and on the Theory of Continuous Lattices in Darmstadt, July 1978.

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