HOMOMORPHISMS AND DERIVATIONS ON BANACH ALGEBRAS

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ABSTRACT

This thesis is concerned with some problems in three areas of Banach algebra theory. These are dealt with separately in Chapters 2, 3 and 4.

Chapter 2 is concerned with certain automatic continuity problems for homomorphisms and derivations on Banach algebras. The main result is that if there exists a discontinuous homomorphism from a Banach algebra onto a semi-prime Banach algebra, or a discontinuous derivation on a semi-prime Banach algebra, then there exists a topologically simple radical Banach algebra.

The main result of Chapter 3 is that there are no Jordan derivations which are not also associative derivations on any semi-prime algebra over a field not of characteristic 2. It follows from this that every Jordan derivation on a semi-simple Banach algebra is a derivation, and therefore continuous.

The background to Chapter 4 is a theorem which states that if A is a C*-algebra with identity, acted on by a group G of isometric automorphisms in such a way that A is G-abelian, then the set of G-invariant states of A is a simplex. This was proved by Lanford and Ruelle in connection with the C*-algebra approach to statistical mechanics. Methods are developed to provide an alternative proof of this result and to investigate the possibility of similar results holding in special cases when A is not a C*-algebra.

PREFACE

The material presented in this thesis is claimed as original, with the exception of those sections and parts of sections where specific mention is made to the contrary.

CONTENTS

INTRODUCTION

This thesis is a presentation of some problems and results in three distinct areas of Banach algebra theory, each of which is concerned in some way with homomorphisms, with derivations, or with both. There are four chapters. Chapter 1 consists almost entirely of standard algebra and Banach algebra theory and Chapters 2,3 and 4 each deal with one of the three sets of problems. Chapters 2,3 and 4 are independent of each other.

Chapter 2 is concerned with what are often called 'automatic continuity problems'. Let A and B be algebras over the same field. Then a linear mapping h from A into B is a <u>homomorphism</u> if

$$h(ab) = h(a)h(b) \qquad (a,b \in A).$$

A typical automatic continuity result for homomorphisms states sufficient conditions on Banach algebras A and B for all homomorphisms from A into B, or from A onto B, to be continuous. The conditions are ideally of a purely algebraic nature, but they may be partly algebraic and partly topological. Alternatively, A or B may be a specified Banach algebra.

By a <u>derivation</u> on an algebra A, we shall always mean a linear mapping D of A into itself such that

$$D(ab) = aDb + (Da)b \qquad (a,b \in A).$$

As in the case of homomorphisms, an automatic continuity result for derivations typically states the continuity of <u>all</u> derivations on any Banach algebra satisfying certain conditions.

The history of automatic continuity problems in Banach algebra theory is well illustrated by the 'uniqueness of norm problem'. A Banach algebra A is said to have a unique complete norm topology if all the complete algebra norms on A are equivalent. If p and q are complete algebra norms on A, then the identity mapping on A is continuous, as a homomorphism from the Banach algebra A with the norm p onto the Banach algebra A with the norm q, if and only if p and q are equivalent. Thus the problem of whether or not a Banach algebra A has a unique complete norm topology - the uniqueness of norm problem for A - is a special case of the problem of whether or not every homomorphism from an arbitrary Banach algebra onto A is continuous.

One of the earliest uniqueness of norm results is that if X is a Banach space, then the Banach algebra $BL(X)$ of all bounded linear operators on X has a unique complete norm topology. This was proved by Eidelheit in 1940 [23]. At about the same time, Gelfand proved that every homomorphism from a commutative Banach algebra into a commutative semi-simple Banach algebra is continuous and that every commutative semi-simple Banach algebra therefore has a unique complete norm topology. About 1948, Rickart raised the problem of whether or not every non-commutative semi-simple Banach algebra has a unique complete norm topology. Although unable to solve this problem, he did show, for example, that every primitive Banach algebra with minimal idempotents has a unique complete norm topology, and that a homomorphism from a Banach algebra onto a semi-simple Banach algebra with a unique complete norm topology is automatically continuous [67, 68]. The uniqueness of the complete norm topology for non-commutative semi-simple Banach algebras was eventually proved by Johnson in 1967 [46], and this is still the most important result of this type. Between 1965 and 1974, the uniqueness of the complete norm topology was proved for Arens-Hoffman extensions of commutative semi-simple Banach algebras [10, 57], Banach algebras of formal power series [61], and the radical convolution algebra $L^1(0,1)$ [45].

The study of automatic continuity problems for derivations began somewhat later. In 1960, Sakai proved that a derivation on a C*-algebra is automatically continuous and, in 1967, Johnson and Sinclair extended this result to all semi-simple Banach algebras [71, 51]. Since then, similar results have been obtained for Banach algebras of formal power series [60], Arens-Hoffman extensions of certain Banach algebras [61] and $L^1(0,1)$ [45]. More recently, there has been some interest in automatic continuity problems for derivations into modules [5, 6, 44].

The most obvious example of a Banach algebra with non-equivalent complete algebra norms and discontinuous derivations is an infinite-dimensional Banach space with the zero product [68]. Such an algebra is nilpotent and therefore equal to its prime radical (see Section 3). An example due to Feldman shows that a Banach algebra with a 1-dimensional prime radical can also have a non-unique complete norm topology [4, 62]. One way to exclude both these examples is to consider only semi-prime Banach algebras [16, 49], and one of the main results of Chapter 2 is that if a semi-prime Banach algebra has two non-equivalent complete algebra norms, or a discontinuous derivation, then it also has a closed 2-sided ideal which is a topologically simple radical Banach algebra (see Section 10). As observed in [9], it is not known whether or not there are any such algebras. Thus, although this result does not prove that every semi-prime Banach algebra has a unique complete norm topology and automatically continuous derivations, it does indicate the kind of difficulties that would be involved in constructing a counter-example.

A long standing conjecture concerning derivations is that if D is a derivation on a commutative Banach algebra A, then the range of D is contained in the Jacobson radical of A. Singer and Wermer proved that this is true if A is complex and D is continuous [80]. This result also follows from a theorem in [75], which states that if D is a continuous derivation on a Banach algebra A, and P is a primitive ideal of A, then $D(P)$ is contained in P. In section 12 of Chapter 2, we show that if there is a (discontinuous) derivation D on a Banach algebra A, and a primitive ideal P of A such that $D(P)$ is not contained in P, then we may again conclude that there must be a topologically simple radical Banach algebra associated with A in a certain way. It follows from this that if the conjecture concerning the range of a derivation on a commutative Banach algebra is false, then there is a commutative topologically simple radical Banach algebra.

Most of the interest in automatic continuity problems for homomorphisms which are not onto has centred on the case when the domain algebra is a C*-algebra. This is a natural priority in view of the fact that, of all classes of Banach algebras, the C*-algebras are in many ways the best understood. For many years the major problem in this field was to determine whether or not there exists a discontinuous homomorphism from any C*-algebra into a Banach algebra. Dales [13] and Esterle [24] have recently proved that, assuming the continuum hypothesis, there is a discontinuous homomorphism from any infinite-dimensional commutative C*-algebra into a Banach algebra. On the other hand, it was shown in [47], that if A is a (non-commutative) unital C*-algebra, with no proper closed 2-sided ideals of finite codimension, then every homomorphism from A into a Banach algebra is continuous. Two of the remaining problems are to determine whether or not there exists a discontinuous homomorphism from a non-commutative C*-algebra onto a dense subalgebra of a semi-simple Banach algebra [77], and to find further necessary and sufficient conditions for a non-commutative C*-algebra to have a discontinuous homomorphism into any Banach algebra. Results in [78], show that the existence of discontinuous homomorphisms from a unital commutative C*-algebra A is equivalent to the existence of discontinuous homomorphisms with prime kernels from maximal ideals of A. In Section 9 of Chapter 2, we show that similar results hold in the non-commutative case. For example, we show that if a non-commutative C*-algebra A has a discontinuous homomorphism into a Banach algebra, then it has a closed 2-sided ideal M such that

(1) M has finite codimension

(2) There exists a discontinuous homomorphism k from M into a Banach algebra such that the kernel of k is a prime ideal of M.

It would be totally misleading to give the impression that there are large numbers of examples of homomorphisms and derivations which are not either obviously continuous or obviously discontinuous, but for which this may be decided by applying automatic continuity results. In fact, most naturally occurring homomorphisms and derivations are continuous, and can be proved to be so by elementary arguments. The role of automatic continuity results is not, therefore, to establish the continuity of specific homomorphisms or derivations, but rather to explain why it is so hard, and in many cases impossible, to construct discontinuous homomorphisms and derivations. It may be that the value of this admittedly rather esoteric exercise lies more in the stimulus it gives to the general development of Banach algebra theory, than in the results themselves.

To introduce the subject of Chapter 3, let A be any algebra or ring, and consider the operation $\circ$ defined on A by

$$a \circ b = ab + ba \qquad (a,b \in A).$$

This operation is called the Jordan product on A and is commutative, but not in general associative. Chapter 3 is concerned with one aspect of the relationship between the Jordan product on A and the associative product from which it is derived. This relationship has been studied by several authors, but most extensively by Herstein in a series of papers [31, 32, 33, 34, 35], most of the results of which are reproduced in [36]. Two of the subjects considered by Herstein in this connection are Jordan homomorphisms and Jordan derivations.

A Jordan homomorphism on a ring (or algebra) A is an additive (or linear) mapping of A into another ring (or algebra) such that

$$h(a \circ b) = h(a) \circ h(b) \qquad (a, b \in A).$$

Similarly, a Jordan derivation on A is an additive (or linear) mapping D of A into itself such that

$$D(a \circ b) = a \circ Db + (Da) \circ b \qquad (a, b \in A).$$

It is obvious that a homomorphism is a Jordan homomorphism and that a derivation is a Jordan derivation. It is also clear that an anti-homomorphism ($h(ab) = h(b)h(a)$) is a Jordan homomorphism, and that there is therefore a natural class of Jordan homomorphisms which are not homomorphisms.

There are, however, very few examples of Jordan derivations which are not derivations, and the main problem in this area has always been to explain this scarcity by showing that, on large classes of rings and algebras, Jordan derivations which are not derivations definitely do not exist. Probably the first result of this kind was proved by Jacobson and Rickart in [43]. They showed that if A is a ring with identity such that every Jordan homomorphism on A is the sum of a homomorphism and an anti-homomorphism, then every Jordan derivation on A is a derivation. In [34], Herstein showed that every Jordan derivation on a prime ring in which $2x = 0$ implies $x = 0$, is a derivation. In [76], Sinclair used this result to prove that every continuous Jordan derivation on a semi-simple Banach algebra is a derivation and also asked the following question :

'Is every Jordan derivation on a semi-simple Banach algebra continuous?'

The main purpose of Chapter 3 is to answer this question. Since it is known that every derivation on a semi-simple Banach algebra is continuous, one way of doing this is to prove that every Jordan derivation on a semi-simple Banach algebra is a derivation. This is the approach adopted in Chapter 3. In fact, the only properties of semi-simple Banach algebras used in the proof are shared by any semi-prime algebra over a field of characteristic different from 2. We therefore conclude that there are no Jordan derivations which are not derivations on any algebra of this kind.

Essentially the results of Chapter 3, but in the terminology of rings rather than algebras, have been published in [12]. This paper also contains some simple examples of Jordan derivations which are not derivations.

Banach algebra theory cannot be expected to survive indefinitely unless it produces solutions to at least some problems which are not, like the problems considered in Chapters 2 and 3, generated entirely within the theory itself. One of the most successful parts of Banach algebra theory, from this point of view, is the theory of C*-algebras. Chapter 4 is concerned, although only rather indirectly, with one area in which C*-algebras have been used in the last decade (1966 - 1976). This is in the so-called 'algebraic approach' to statistical mechanics.

The use of C*-algebras in physics was pioneered by Segal [73, 74] and, later, by Haag and Kastler [30], mainly in connection with quantum field theory. The developments with which Chapter 4 is concerned are contained in a series of papers [17, 18, 19, 20, 53, 55], which appeared in the years 1966 to 1969. These papers describe properties of triples of the form (A,G,T), where A is a unital C*-algebra, G is a group and T is a (group) homomorphism of G into the group of star-automorphisms of A. The basic idea of the algebraic approach to statistical mechanics is to impose extra conditions and structure on triples of this form so as to produce useful mathematical models of physical systems such as gases and magnetic materials, which are traditionally studied by the methods of classical or quantum statistical physics. In such a model, the 'observables' of the system are represented by certain elements of the algebra A, the group G represents certain physical symmetries of the system, and the equilibrium states of the system are represented by the G-invariant states of A. A <u>state</u> on A is a continuous linear functional f on A such that $\|f\| = 1$ and

$$f(a^*a) \geqslant 0 \qquad (a \in A),$$

and f is <u>G-invariant</u> if

$$f(T(g)a) = f(a) \qquad (a \in A,\ g \in G).$$

In the simplest cases, the group G is an abelian group of space translations, for example, $\mathbb{R}^\nu$ for 'continuous models' of gases, or $\mathbb{Z}^\nu$ for 'lattice models' of magnetic systems ($\nu = 1,2,3$). In these cases A typically has a 'quasi-local structure' consisting of a set $\{A_S : S \in K\}$ of closed star-subalgebras such that the following conditions are satisfied :

(1) If $G = \mathbb{R}^\nu$, then K is the set of all bounded Lebesgue measurable subsets of $\mathbb{R}^\nu$, and if $G = \mathbb{Z}^\nu$, then K is the set of all finite subsets of $\mathbb{Z}^\nu$,

(2) If S is contained in S', then A_S is contained in $A_{S'}$,

(3) $T(g)A_S = A_{S+g} \qquad (S \in K,\ g \in G)$,

(4) If S and S' are disjoint, then $ab = ba$ for all a in A_S and b in $A_{S'}$,

(5) The union of the subalgebras A_S is a dense star-subalgebra of A.

In models of this kind, the observables of that part of the system in the region S are identified with all or some of the self-adjoint elements of the subalgebra A_S. Conditions (2) and (3) may be interpreted as meaning that, if a is an observable in the region S, then a is also an observable in any region containing S, and for each g in G, $T(g)a$ is the corresponding observable in the region $S + g$. In classical models, the algebra A is chosen to be commutative, and in this case condition (4) is redundant. In quantum models the algebras A_S are not commutative. This reflects the fact that the order in which two observations are made may affect the results obtained. On the other hand, observations made far enough apart from each other may be expected not to interact, and this is reflected in condition (4). It may easily be shown that a triple (A,G,T) with a quasi-local structure satisfying conditions (1) to (5) is <u>asymptotically abelian</u> in the sense that, for all a and b in A,

$$\|a(T(g)b) - (T(g)b)a\| \to 0 \quad \text{as} \quad g \to \infty.$$

The relationship between the C*-algebra approach to statistical mechanics and more obviously statistical methods can be established by a detailed description of the G-invariant states of triples (A,G,T) associated with specific models of physical systems [69]. On the other hand, some results which may be of physical significance can be more easily studied in a more abstract setting. The problems considered in Chapter 4 were suggested by a result of this kind due to Lanford and Ruelle, which states that if (A,G,T) is G-abelian, then the set of all G-invariant states is a simplex [55].

The term 'G-abelian' refers to a condition which is satisfied by all asymptotically abelian triples (A,G,T), and is related to the 'non-interaction of observables at a distance'. The precise definition is given in Section 17.

Let $D(A,G)$ denote the set of all G-invariant states of a triple (A,G,T). Then $D(A,G)$ is a weak* compact convex subset of the dual space A' of A, and an element f of $D(A,G)$ is said to be a G-ergodic state if it is an extreme point of $D(A,G)$, that is, if it cannot be expressed in the form

$$f = tf_1 + (1 - t)f_2,$$

with $0 < t < 1$, f_1 and f_2 in $D(A,G)$ and f_1 not equal to f. It follows from the general theory of compact convex sets that each G-invariant state f has an integral representation of the form

$$f(a) = \int x(a)\,d\mu(x) \qquad (a \in A),$$

where μ is a positive boundary measure on $D(A,G)$.

$D(A,G)$ is said to be a simplex if, for each f, this representation is unique (see Section 19). If A is separable, then $D(A,G)$ is metrizable, and a positive Baire measure on $D(A,G)$ is a boundary measure if and only if it is supported on the set of G-ergodic states. Claims for the physical significance of the result that, if (A,G,T) is G-abelian, then $D(A,G)$ is a simplex, are based on this fact, and on the following two arguments, which we illustrate in terms of a system consisting entirely of water molecules [69].

(1) A G-invariant state representing an equilibrium state of the system, represents a pure thermodynamic phase (i.e. ice, water or steam) if and only if it is G-ergodic.

(2) On physical grounds, a G-invariant state f representing

a mixture (of ice and water, for example) should therefore have a unique decomposition into G-ergodic states. This 'unique ergodic decomposition' is provided by the unique boundary measure representing f on the simplex $D(A,G)$.

The preceeding remarks do not do justice to the subtlety and complexity of the algebraic approach to equilibrium statistical mechanics, and we have mentioned only one of many interesting results in C*-algebra theory which have been proved in this connection. However, there are clearly grounds for doubt that such an abstract approach can be of much use in physics. That these doubts exist is confirmed by the following statement made by a leading exponent of the algebraic approach, 'It is a fact that people who know something about statistical mechanics do not usually know much about C*-algebras and vice versa; this situation has led to a certain amount of skepticism on one side and to unjustified claims of the solution of big problems on the other side' (Ruelle [70]).

The purpose of Chapter 4 is not, however, to make any contribution to statistical mechanics, but rather to investigate the extent to which C*-algebras can be replaced by Banach algebras which are not C*-algebras, while still retaining some form of 'asymptotic abelianness' and the possibility that the set of G-invariant states (suitably redefined) is a simplex.

The first stage is to extend the terminology to cover triples of the form (A,G,T), where A is a complex unital Banach algebra, and T is a homomorphism of G into the group of isometric automorphisms of A. By a <u>state</u> on A, we shall mean, as in numerical range theory, a continuous linear functional f on A such that

$$\|f\| = f(1) = 1.$$

As before, a state f is G-invariant if

$$f(T(g)a) = f(a) \qquad (a \in A,\ g \in G),$$

and the set $D(A,G)$ of G-invariant states is a convex weak* compact subset of the dual space A' of A. It therefore makes sense to ask whether or not $D(A,G)$ is a simplex.

Now assume that the group G is amenable, and let M be a 2-sided and inversion invariant mean on $l^{\infty}(G)$ (see Section 16). The triple (A,G,T) is M-asymptotically abelian if, for all f in $D(A,G)$ and a and b in A,

$$M(w) = 0, \text{ where } w(g) = f(aT(g)b - (T(g)b)a) \qquad (g \in G).$$

If A is a C*-algebra, then (A,G,T) is M-asymptotically abelian if and only if it is G-abelian (see Section 18).

If A is commutative and $T(g) = 1$ for all g in G, then (A,G,T) is clearly M-asymptotically abelian, and the G-invariant state space $D(A,G)$ coincides with the set $D(A)$ of all the states of A. Since there do exist commutative Banach algebras A, such that $D(A)$ is not a simplex, we conclude immediately that the theorem of Lanford and Ruelle does not extend to the case when A is not a C*-algebra. When A is a unital C*-algebra, $D(A)$ is a simplex if and only if A is commutative. However, when A is not a C*-algbera very little is known about what kind of algebraic properties A satisfies if $D(A)$ is a simplex. There is therefore little hope of finding necessary conditions for $D(A,G)$ to be a simplex, and we therefore concentrate on sufficient conditions. The main result is that there is a unital Banach algebra B

associated with (A,G,T) and M such that

(1) If B is commutative, then (A,G,T) is M-asymptotically abelian

(2) If the state space $D(B)$ of B is a simplex, then the G-invariant state space $D(A,G)$ is a simplex

(3) If A is a C*-algebra, then B is a C*-algebra, and B is commutative if and only if (A,G,T) is G-abelian.

These results imply that if B is a commutative C*-algebra, then (A,G,T) is M-asymptotically abelian and $D(A,G)$ is a simplex. They also provide an alternative proof of the known result that if A is a C*-algebra, G is amenable, and (A,G,T) is G-abelian, then $D(A,G)$ is a simplex.

The final part of Section 19 of Chapter 4 is an attempt to establish how far the scope of these results extends beyond the case when A is a C*-algebra.

Let $H(A) = \{a \in A : f(a) \text{ is real for all } f \text{ in } D(A)\}$. Then, by the Vidav-Palmer theorem (see Section 19), A is a C*-algebra if and only if $A = H(A) + iH(A)$.

Now let $H(A,G) = \{a \in A : f(a) \text{ is real for all } f \text{ in } D(A,G)\}$. We show that, at least when A is Arens regular, a necessary and sufficient condition on A for B to be a C*-algebra is $A = H(A,G) + iH(A,G)$.

Since $H(A)$ is contained in $H(A,G)$, this condition is satisfied when A is a C*-algebra, but may also be satisfied, as may be confirmed by trivial examples, when A is not a C*-algebra. However, the main conclusion to be drawn from these results is that if (A,G,T) is M-asymptotically abelian, but A is not a C*-algebra, then it is in general difficult to determine whether or not $D(A,G)$ is a simplex, but unlikely that it is so. It is to be hoped that the methods used to reach this rather negative conclusion may at least be of some interest.

Most of the new results are in Sections 8 to 13 of Chapter 2, Section 15 of Chapter 3 and Sections 18 and 19 of Chapter 4. No new results are proved in Sections 1, 2, 3, 5, 6, 14, 16 and 17.

Knowledge of basic functional analysis is assumed throughout and, in Section 9 and in Chapter 4, some more specialised results concerning C*-algebras are quoted without proof. Lomonosov's theorem concerning the existence of invariant subspaces for compact operators is used in Section 10 and, in Section 19, parts of the theory of compact convex sets and boundary measures play a major role. Section 19 also uses a certain amount of numerical range theory, including the Vidav-Palmer theorem.

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Chapter One

THE JACOBSON AND PRIME RADICALS OF A BANACH ALGEBRA

1. Introduction.

The purpose of Chapter 1 is to provide an introduction to that part of the ideal theory of non-commutative algebras and Banach algebras which will be used in Chapters 2 and 3. Section 2 is concerned principally with primitive ideals and the Jacobson radical, and Section 3 with prime ideals and the prime radical. Some examples of Banach algebras, and of continuous derivations and homomorphisms on Banach algebras, are described in Section 4.

Familiarity with the basic definitions of Banach algebra theory is assumed and, in particular, free use is made of concepts related to the Gelfand representation theory for commutative Banach algebras, and of the Gelfand-Mazur theorem on complex normed division algebras, and its real analogue (lemma 2.23).

With the possible exceptions of Example 4.14 and Example 4.17, nothing in Chapter 1 is original, but specific references are given only for some of the more recent results. The principal sources for Sections 2 and 3 are the books, 'Banach algebras', by C.E. Rickart, [68], 'Complete normed algebras', by F.F. Bonsall and J. Duncan, [9], and 'The theory of rings', by N.H. McCoy, [63].

The main purpose of the rest of this section is to summarise some basic terminology and notation.

1.1 Algebras and algebra norms.

An algebra over a field F is a linear space A over F, with a specified associative bilinear product. If F is the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers, then F is called the scalar field of A.

A is a real algebra if $F = \mathbb{R}$, and a complex algebra if $F = \mathbb{C}$.

An algebra norm is a norm $\|\cdot\|$ on a real or complex algebra A such that

$$\|ab\| \leqslant \|a\|\|b\| \qquad (a,b \in A).$$

A real or complex algebra A with a specified algebra norm $\|\cdot\|$ is called a normed algebra, and may be written $(A,\|\cdot\|)$ when more than one algebra norm on A is under consideration. A normed algebra is a Banach algebra if it is complete, in the sense that every Cauchy sequence converges. A complete algebra norm is an algebra norm $\|\cdot\|$ on a real or complex algebra A, such that $(A,\|\cdot\|)$ is a Banach algebra.

A normed algebra A is unital if it has an identity element 1, such that $\|1\| = 1$.

The following notation will be used for sums and products of subsets X and Y of an algebra A:

$$X + Y = \{a + b : a \in X,\ b \in Y\}$$

$$XY = \{ab : a \in X,\ b \in Y\}.$$

For any subset X of A, or of any linear space, span(X) will denote the linear span of X.

1.2 Homomorphisms and derivations.

Let A and B be algebras over a field F. A linear mapping h from A into B is a homomorphism if,

$$h(ab) = h(a)h(b) \qquad (a,b \in A).$$

A homomorphism from A into B is a monomorphism if it is 1:1, and an epimorphism if it is onto.

An automorphism of A is a 1:1 homomorphism of A onto itself.

A derivation on A is a linear mapping D of A into itself such that

$$D(ab) = aDb + (Da)b \qquad (a,b \in A).$$

This condition is an abstraction from the product rule for the differentiation of the product of two differentiable functions. A derivation D on A also satisfies the Leibnitz identity :

$$D^n(ab) = \sum_{i=0}^{n} \binom{n}{i} (D^i a)(D^{n-i} b) \qquad (a,b \in A,\ n = 1,2,\ldots\ldots).$$

This is easily proved by induction on n.

1.3 Ideals and quotient algebras.

Let J be a linear subspace of an algebra A. Then, J is a left ideal if $AJ \subseteq J$, a right ideal if $JA \subseteq J$, and a 2-sided ideal if $AJ + JA \subseteq J$.

Let A/J denote the difference space of A modulo J, and let Q denote the natural mapping

$$a \to a + J \qquad (a \in A),$$

of A onto A/J. If I is any subspace of A containing J, then I/J will denote the subspace QI of A/J.

Now suppose that J is a 2-sided ideal of A. Then A/J is an algebra, the quotient algebra of A modulo J, with respect to the product defined by

$$(a + J)(b + J) = (ab + J) \qquad (a,b \in A).$$

If I is a left (right) ideal of A, containing J, then I/J

is a left (right) ideal of A/J.

If A is a Banach algebra, and J is a closed 2-sided ideal of A, then A/J is a Banach algebra, with respect to the <u>quotient norm</u> defined by

$$\|a + J\| = \inf\{\|b\| : b - a \in J\} \qquad (a \in A).$$

In this case, if I is a closed subspace of A containing J, then I/J is a closed subspace of A/J.

1.4 Modules.

Let A be an algebra over a field F.

A <u>left A-module</u> is a linear space X over F, with a specified bilinear mapping $(a,x) \longrightarrow a.x : A \times X \longrightarrow X$ such that

$$a.(b.x) = (ab).x \qquad (a, b \in A, x \in X).$$

Similarly, a <u>right A-module</u> is a linear space X over F, with a specified bilinear mapping $(x,a) \longrightarrow x.a : X \times A \longrightarrow X$ such that

$$(x.b).a = x.(ba) \qquad (a, b \in A, x \in X).$$

If A is a Banach algebra, then a <u>Banach left A-module</u> is a left A-module X, which is also a Banach space, and which satisfies the condition

$$\|a.x\| \leqslant M\|a\|\|x\| \qquad (a \in A, x \in X),$$

for some constant $M > 0$.

A linear mapping T from a left A-module X into a left A-module Y is an <u>A-module homomorphism</u> if

$$T(a.x) = a.Tx \qquad (a \in A, x \in X).$$

X and Y are <u>algebraically equivalent</u> if there exists a 1:1 A-module homomorphism from X onto Y.

If J is a left ideal of A, then A/J is a left

A-module, with respect to the module operation defined by

$$a.(b + J) = ab + J \qquad (a,b \in A).$$

If A is a Banach algebra, and J is a closed left ideal, then A/J is a Banach left A-module and

$$\|a.(b + J)\| \leq \|a\|\|b + J\| \qquad (a,b \in A).$$

An A-submodule of a left A-module X is a linear subspace Y of X such that

$$a.y \in Y \qquad (a \in A,\ y \in Y).$$

Similarly, if X is a right A-module, then a subspace Y of X is an A-submodule if

$$y.a \in Y \qquad (a \in A,\ y \in Y).$$

2. The Jacobson radical.

In this Section we describe those properties of the irreducible modules, primitive ideals and Jacobson radical of a Banach algebra which will be used in Sections 9 to 13 of Chapter 2. Most of the definitions and results are independent of any topology and were originally developed in the more general setting of ring theory [42], although their application to Banach algebras is well known [9, 68]. Of the results specific to Banach algebras, one of the most fundamental is the fact that every primitive ideal in a Banach algebra is closed. This makes the primitive ideals and Jacobson radical of a Banach algebra considerably easier to deal with than the prime ideals and prime radical considered in Section 3.

There are several possible approaches to the definition of the Jacobson radical and related concepts. Wherever possible, the concept of an irreducible left A-module is treated as basic, and results concerning irreducible right A-modules, modular ideals, quasi-invertible elements and quasi-nilpotent elements are included only as necessary or convenient.

Throughout this Section, A will denote an algebra over a field F.

2.1 Definition. A left (right) A-module X is irreducible if $A.X \neq \{0\}$ $(X.A \neq \{0\})$, and $\{0\}$ and X are the only A-submodules of X.

A 2-sided ideal P of A is left primitive if there exists an irreducible left A-module X such that

$$P = \{a \in A : a.X = \{0\}\},$$

and right primitive if there exists an irreducible right

A-module X such that

$$P = \{a \in A : X.a = \{0\}\}.$$

P is <u>primitive</u> if it is either left primitive or right primitive.

As an example of an irreducible left A-module, let X be a Banach space, let A be the algebra $BL(X)$ of all bounded linear operators on X, and define the module operation on X by

$$T.x = T(x) \qquad (T \in BL(X), \ x \in X).$$

In this case, the corresponding primitive ideal is the zero ideal (0) of A.

An algebra (such as $BL(X)$) in which the zero ideal is a primitive ideal is called a <u>primitive algebra</u>.

If X is an irreducible left A-module, then the set $Z = \{x \in X : A.x = \{0\}\}$ is an A-submodule. Thus $Z = \{0\}$ and so $A.x = X$ for all non-zero x in X.

<u>2.2 Definition</u>. A left ideal I of A is a <u>modular left ideal</u> if there exists an element e of A such that $ae - a$ is in I for all a in A. Any such element e is called a <u>right modular unit</u> for I.

A left ideal I of A is a <u>maximal left ideal</u> if I is not equal to A, and A and I are the only left ideals of A which contain I. A <u>maximal modular left ideal</u> is a modular left ideal which is also a maximal left ideal.

<u>2.3 Lemma</u>. Every proper modular left ideal is contained in a maximal modular left ideal.

<u>Proof</u>. This is a straightforward application of Zorn's lemma (see, for example, [9, Proposition 9.2]).

2.4 Lemma.

(1) Let M be a maximal modular left ideal of A. Then the left A-module A/M is irreducible.

(2) If X is an irreducible left A-module, then there exists a maximal modular left ideal M of A such that A/M is algebraically equivalent to X.

Proof. To prove (1), let e be a right modular unit for the maximal modular left ideal M and let a be an element of A not in M. Then $ae - a \in M$ implies $ae \notin M$, and therefore $A.(A/M) \neq \{0\}$. Since the A-submodules of A/M are clearly in 1:1 correspondence with the left ideals of A containing M, it follows that A/M is an irreducible left A-module.

Now let X be an irreducible left A-module, let x be any non-zero element of X, and let $M = \{a \in A : a.x = 0\}$. Then M is a left ideal and the mapping T of A/M into X defined by

$$T(a + M) = a.x \qquad (a \in A),$$

is a well-defined 1:1 module homomorphism of A/M onto X. Let e be any element of A such that $e.x = x$. Then $ae - a$ is in M for all a in A, and M is therefore a modular left ideal. If I is any left ideal of A containing M, then $T(I)$ is an A-submodule of X. Thus $I = M$ or $I = A$, and M is therefore a maximal left ideal. This completes the proof of (2).

2.5 Definition. The Jacobson radical R of A is the intersection of all the primitive ideals of A. If $R = A$ (i.e. if A has no primitive ideals), then A is a radical algebra.

A is <u>semi-simple</u> if $R = (0)$. Every primitive algebra is therefore semi-simple.

Although the definition of the Jacobson radical in terms of primitive ideals is the most suitable definition for the purposes of Chapter 2, the concept of quasi-invertibility is a useful tool for the proof of some of the results of this section. A complete characterisation of the Jacobson radical in terms of quasi-invertibility is given in [9, p.124].

An element a of A is <u>quasi-invertible</u> if there is an element b of A such that $a + b - ab = 0 = a + b - ba$.

<u>2.6 Lemma</u>. Let J be the intersection of all the left primitive ideals of A. Then every element of J is quasi-invertible.

<u>Proof</u>. Let e be any element of J and let $I = \{be - b : b \in A\}$. Suppose there is no element b of A such that $e + b - be = 0$. Then e is not an element of I, and I is therefore a proper modular left ideal. By lemma 2.3, I is contained in some maximal modular left ideal M.

Let $P = \{a \in A : aA$ is contained in $M\}$. Then, by lemma 2.4 (1), P is a left primitive ideal. Thus e is in P, and $e = (e - e^2) + e^2$ is therefore in M. Since e is a right modular unit for M, this is impossible. This contradiction proves that there must be an element b of A, such that $e + b - be = 0$. To prove that e is quasi-invertible, it is sufficient to show that $e + b - eb = 0$. To do this, note that $b = be - e$ is in J. Thus, by repeating the argument above with b instead of e, we obtain an element c of A such that $b + c - cb = 0$. But then, $c = cb - b =$

$cbe - ce - be + e = (cb - c - b)e + e = e$, and this completes the proof.

<u>2.7 Corollary</u>. The Jacobson radical R of A is equal to the intersection of all the left primitive ideals of A.

<u>Proof</u>. As in lemma 2.6, let J be the intersection of all the left primitive ideals of A. Then R is contained in J. Suppose R is not equal to J and let a be any element of J which is not in R. Then there exists an irreducible right A-module X such that $X.a \neq \{0\}$. Let x be any element of X such that $x.a \neq 0$. Then $X = x.aA$, and so there is an element d of A such that $x = x.ad$. Let $e = ad$. Then e is in J, and, by lemma 2.6, there is an element b of A such that $e + b - eb = 0$. But then, $x = x.e = x.(eb - b) = 0$, which is a contradiction proving that $J = R$ as required.

<u>2.8 Corollary</u>. Let R be the Jacobson radical of A, and let a in A and b in R satisfy $ab = a$. Then $a = 0$.

<u>Proof</u>. By lemma 2.6, there is an element c of A such that $b + c - bc = 0$. Therefore, $a = a + ac - ac = ab + ac - abc = 0$.

<u>2.9 Lemma</u>. Let I and P be 2-sided ideals of A such that I is contained in P. Then P/I is a left primitive ideal of A/I if and only if P is a left primitive ideal of A.

<u>Proof</u>. Suppose that P/I is a left primitive ideal of A/I, and let X be an irreducible left (A/I)-module such that $P/I = \{b \in A/I : b.X = \{0\}\}$. Then X may be regarded as an irreducible left A-module by means of the definition

$$a.x = (a + I).x \qquad (a \in A,\ x \in X),$$

and so $P = \{a \in A : a.X = \{0\}\}$ is a left primitive ideal of A. Conversely, if X is an irreducible left A-module such that $P = \{a \in A : a.X = \{0\}\}$, then X may be regarded as an irreducible left A/I-module by means of the definition

$$(a + I).x = a.x \qquad (a \in A,\ x \in X),$$

and P/I is therefore a left primitive ideal of A/I.

<u>2.10 Corollary</u>. If I is a 2-sided ideal of A contained in the Jacobson radical R of A, then the Jacobson radical of A/I is R/I. In particular, the Jacobson radical of A/R is (0) and A/R is therefore semi-simple.

<u>Proof</u>. This follows immediately from Corollary 2.7 and lemma 2.9.

<u>2.11 Corollary</u>. Every right primitive ideal of A is an intersection of left primitive ideals.

<u>Proof</u>. Let P be a right primitive ideal of A. Then the Jacobson radical of A/P is (0). By Corollary 2.7, P is equal to an intersection of ideals J of A such that J contains P and J/P is a left primitive ideals of A/P. By lemma 2.9, the ideals J are all left primitive ideals of A.

<u>2.12 Lemma</u>. Let I be a 2-sided ideal of A and let R be the Jacobson radical of A. Then $I \cap R$ is the Jacobson radical of I.

<u>Proof</u>. Let $R(I)$ be the Jacobson radical of I, let P be a left primitive ideal of A, and let X be an irreducible left A-module such that $P = \{a \in A : a.X = \{0\}\}$. Then either I is contained in P, or X is an irreducible left I-module. In

either case, $R(I)$ is contained in $P \cap I$. This proves that $R(I)$ is contained in $R \cap I$. Now let a be any element of $R \cap I$ and let Y be an irreducible left I-module. Suppose $a.Y \neq \{0\}$. Then there exist elements c of I and y of Y such that $ca.y = y \neq 0$. Let $e = ca$. Then e is in R, and so, by lemma 2.6, there is an element b of A such that $e + b - be = 0$. But then, $y = e.y = (be - b).y = 0$, which is a contradiction. This proves that $a.Y = \{0\}$ and that a is therefore in $R(I)$ as required.

The following lemma indicates how the theory of this Section applies when A is commutative.

<u>2.13 Lemma</u>. If A is commutative and P is any ideal of A, then the following are equivalent :

(1) P is primitive

(2) P is maximal modular

(3) A/P is a field.

<u>Proof</u>. If P is primitive, then, by lemma 2.4, there is a maximal modular ideal M such that $P = \{a \in A : aA \subseteq M\}$. But then $M \subseteq P \subsetneq A$ and so $P = M$. The implications (2) implies (3) and (3) implies (1) are obvious.

<u>2.14 Definition</u>. Let X be an irreducible left A-module, let $L(X)$ denote the algebra of all linear operators on X, and let

$$D = \{T \in L(X) : T(a.x) = a.Tx \quad (x \in X,\ a \in A)\}.$$

Then D is called the <u>centralizer of A on X</u> and is clearly a subalgebra of $L(X)$.

Let T be a non-zero element of D. Then the range and kernel of T are A-submodules of X, and T is therefore invertible. Let S be the inverse of T. Then

$$a.Sx = ST(a.Sx) = S(a.(TSx)) = S(a.x) \qquad (a \in A,\ x \in X),$$

and S is therefore in D. This proves that D is a division algebra.

Let $x_0, \ldots\ldots, x_n$ be any $n + 1$ elements of X. Then x_0 is said to be a <u>linear combination over D</u> of $x_1, \ldots\ldots, x_n$, if there exist $T_1, \ldots\ldots, T_n$ in D such that

$$x_0 = T_1 x_1 + \ldots\ldots + T_n x_n.$$

A non-empty set E of X is <u>linearly independent over D</u> if for all $x_1, \ldots\ldots, x_n$ in E and $T_1, \ldots\ldots, T_n$ in D, $T_1 x_1 + \ldots\ldots + T_n x_n = 0$ implies $T_1 = T_2 = \ldots\ldots = T_n = 0$.

The terms '<u>n-dimensional over D</u>', '<u>finite-dimensional over D</u>' and '<u>infinite-dimensional over D</u>', refering to subspaces Y of X such that DY is contained in Y, should be interpreted exactly as they would be if D were a field and X a linear space over D.

<u>2.15 Lemma</u>. Let X be an irreducible left A-module, let D be the centralizer of A on X, and let x and y be elements of X linearly independent over D. Then there exists a in A such that $a.x = 0$ and $a.y \neq 0$.

<u>Proof</u>. Suppose that $ax = 0$ implies $a.y = 0$. Then we may define a linear mapping T from $X = A.x$ onto $X = A.y$ by

$$T(a.x) = a.y \qquad (a \in A).$$

Simple calculations show that T is in D and that $Tx = y$. This contradicts the linear independence of x and y over D. There must therefore be an element a of A with the required property.

2.16 Lemma. Let X be an irreducible left A-module and let D be the centralizer of A on X. Then, for all n greater than or equal to 2 and elements $x_1, \ldots\ldots, x_n$ of X linearly independent over D, there is an element a of A such that
$a.x_1 = \ldots\ldots = a.x_{n-1} = 0$ and $a.x_n \neq 0$.

Proof. The proof is by induction on n, and the case $n = 2$ is lemma 2.15. Assume that the result holds for some $n \geqslant 2$, and let $x_1, \ldots\ldots, x_{n+1}$ be linearly independent over D.

Let $I = \{a \in A : a.x_1 = \ldots\ldots = a.x_{n-1} = 0\}$. Then, by the inductive hypothesis, there exists b in I such that $b.x_n \neq 0$. Since I is a left ideal, we have $X = I.x_n$. Now suppose that $a.x_1 = \ldots\ldots\ a.x_n = 0$ implies $a.x_{n+1} = 0$. Then we may define an element T of D by

$$T(a.x_n) = a.x \qquad (a \in I),$$

and for all a in I, we then have $a.(Tx_n - x_{n+1}) = 0$. Suppose $x_1, \ldots\ldots, x_{n-1}, Tx_n - x_{n+1}$ are linearly independent over D. Then, by the inductive hypothesis, there is an element a of I such that $a.(Tx_n - x_{n+1}) \neq 0$. This shows that $x_1, \ldots\ldots, x_{n-1}, Tx_n - x_{n+1}$ are not linearly independent over D, and therefore contradicts the linear independence over D of $x_1, \ldots\ldots, x_{n+1}$. This proves that the statement holds for $n + 1$, and so completes the proof of the lemma.

2.17 Theorem (Jacobson's density theorem).

Let X be an irreducible left A-module, let D be the centralizer of A on X, and let $x_1, \ldots\ldots, x_n$ and $y_1, \ldots\ldots, y_n$ be elements of X. Then, if $x_1, \ldots\ldots, x_n$ are linearly independent over D, there is an element a of A such that

$$a.x_i = y_i \qquad (i = 1, \ldots\ldots, n).$$

Proof. By lemma 2.16, there are elements $b_1, \ldots\ldots, b_n$ of A such that $b_i . x_i \neq 0$ and

$$b_i . x_j = 0 \qquad (j \neq i).$$

For each i, $Ab_i . x_i = X$, and there is therefore an element c_i of A such that $c_i b_i . x_i = y_i$. The element $a = \sum_{i=1}^{n} c_i b_i$ of A has the required property.

The following Corollary is an immediate consequence of the Wedderburn structure theorem for finite-dimensional semi-simple algebras, which follows easily from Jacobson's density theorem (see, for example, [9, p.134]).

2.18 Corollary. Let J be a finite-dimensional semi-simple subalgebra of A. Then J has an identity element e. If J is a 2-sided ideal, then $J = Ae$ and e commutes with every element of A.

We now specialise to the case when A is a Banach algebra.

2.19 Lemma. If A is a Banach algebra and M is a maximal modular left ideal of A, then M is closed.

Proof. Let e be a right modular unit for M and let

$$U = \{a \in A : \|e - a\| < 1\}.$$

Suppose $M \cap U$ is not empty, let a be any element of $M \cap U$, and let $b = \sum_{n=1}^{\infty} (e - a)^n$. Then $b - b(e - a) = e - a$ and so $e = a + ba + b - be$ is in M. This is impossible, since $M \neq A$ implies $e \notin M$. Thus $M \cap U$ must be empty, and, since U is an open set of A, it follows that e is not in the closure $\bar{M}$

of M. But $\bar{M}$ is then a proper left ideal of A containing M and is therefore equal to M. Thus M is closed.

2.20 Corollary. Let P be a left primitive ideal of A. Then there exists an irreducible Banach left A-module X such that $P = \{a \in A : a.X = \{0\}\}$ and $\|a.x\| \leqslant \|a\|\|x\| \quad (a \in A, x \in X)$.

Proof. By lemma 2.4, there is a maximal modular left ideal M such that $P = \{a \in A : aA \subseteq M\}$. Since M is closed, A/M is a Banach left A-module with the required property.

2.21 Corollary. Every primitive ideal of a Banach algebra is closed.

Proof. By Corollary 2.11, it is sufficient to show that every left primitive ideal is closed. This follows immediately from Corollary 2.20.

2.22 Corollary. The Jacobson radical of a Banach algebra is closed.

The proof of the following result is based on certain elementary properties of the spectrum of an element of a normed algebra. Since this spectral theory is not used (explicitly) anywhere else, the proof of the lemma is omitted. It may be found in [9, p. 71 - 74].

2.23 Lemma. Let D be a real or complex normed division algebra. Then D is isomorphic to $\mathbb{C}$, if it is complex, and to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, if it is real, where $\mathbb{H}$ is the real quaternion algebra. In particular, D is finite-dimensional over its scalar field.

2.24 Lemma. Let A be a Banach algebra, let X be an irreducible left A-module and let D be the centralizer of A on X. Then D is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Proof. By lemma 2.4, we can assume that $X = A/M$, for some maximal modular left ideal M of A. By lemma 2.23, it is sufficient to prove that D is contained in the algebra $BL(A/M)$ of bounded linear operators on A/M, since it is then a normed division algbera. To do this, let T be in D, let e be a right modular unit for M and let a be any element of A. If b is in A and $a - b$ is in M, then $a - be = b - be + a - b$ is in M, and therefore $\|T(a + M)\| = \|T(be + M)\| = \|T(b.(e + M))\| = \|b.T(e + M)\| \leqslant \|b\|\|T(e + M)\|$. Thus $\|T(a + M)\| \leqslant \|a + M\|\|T(e + M)\|$, and T is therefore continuous.

The final lemma of this section is used in Section 4 to aid the recognition of radical Banach algebras.

An element a of a Banach algebra is quasi-nilpotent if $\|a^n\|^{\frac{1}{n}} \to 0$ as $n \to \infty$.

2.25 Lemma. Let A be a Banach algebra and let J be a left ideal of A such that every element of J is quasi-nilpotent. Then J is contained in the Jacobson radical R of A.

Proof. Suppose J is not contained in R. Then, by Corollary 2.20, there is an irreducible Banach left A-module X such that $J.X \neq \{0\}$ and $\|a.x\| \leq \|a\|\|x\|$, for all a in A and x in X. Let x be any element of X such that $J.x \neq \{0\}$. Then, since J is a left ideal, $J.x = X$ and there is therefore an element e of J such that $e.x = x \neq 0$. But then $e^n.x = x$, for all n, and so $\|x\|^{\frac{1}{n}} \leq \|e^n\|^{\frac{1}{n}}\|x\|^{\frac{1}{n}} \to 0$, which is a contradiction. Thus J must be contained in R.

It is also true that every element of the Jacobson radical of a Banach algebra is quasi-nilpotent. A proof of this well-known result (using spectral theory) may be found in [9, p.126].

3. The prime radical.

This Section contains the definitions and elementary properties of the prime radical of an algebra and related concepts, and has applications in Chapter 2 and Chapter 3. As in the previous Section, most of the theory is purely algebraic. However, this Section differs from the last in that most of the results specific to Banach algebras were first proved fairly recently. Of these, the most important is the theorem of Grabiner which states that a nil Banach algebra is nilpotent.

As in Section 2, A will denote an algebra over a field F.

3.1 Definition. Let P be a 2-sided ideal of A. Then P is a prime ideal if

$$aAb \subseteq P \text{ implies } a \in P \text{ or } b \in P \qquad (a,b \in A),$$

and a semi-prime ideal if

$$aAa \subseteq P \text{ implies } a \in P \qquad (a \in A).$$

Note that a prime ideal is also semi-prime.

A is a prime algebra if (0) is a prime ideal, and a semi-prime algebra if (0) is a semi-prime ideal.

If A is commutative then an ideal P of A is prime if and only if

$$ab \in P \text{ implies } a \in P \text{ or } b \in P \qquad (a,b \in A),$$

and semi-prime if and only if

$$a^2 \in P \text{ implies } a \in P \qquad (a \in A).$$

A commutative prime algebra is usually referred to as an integral domain.

3.2 Lemma. Let h be a homomorphism of A onto an algebra B.

If P is a prime (semi-prime) ideal of B, then $h^{-1}(P)$ is a prime (semi-prime) ideal of A. Conversely, if N is a prime (semi-prime) ideal of A containing the kernel of h, then $h(N)$ is a prime (semi-prime) ideal of B.

<u>Proof</u>. This follows immediately from the definitions.

<u>3.3 Corollary</u>. Let P be a 2-sided ideal of A. Then P is a prime (semi-prime) ideal if and only if A/P is a prime (semi-prime) algebra.

<u>3.4 Definition</u>. An element a of A is <u>nilpotent</u> if $a^n = 0$ for some n.

A subset E of A is <u>nil</u> if all its elements are nilpotent, and <u>nilpotent</u> if there is a natural number N such that $E^N = (0)$, where $E^N = \{a_1 \ldots\ldots a_N : a_1, \ldots\ldots, a_N \in E\}$.

Note that a nilpotent ideal (for example) is a nil ideal, but that in general a nil ideal is not nilpotent.

<u>3.5 Lemma</u>. The following are equivalent :

(1) A is semi-prime

(2) A has no non-zero nilpotent left ideals

(3) A has no non-zero nilpotent right ideals

(4) A has no non-zero nilpotent 2-sided ideals.

<u>Proof</u>. Suppose A is semi-prime and let I be a nilpotent left ideal or a nilpotent right ideal. Suppose I is not the zero ideal, let n be the least natural number such that $I^n = (0)$, and let a be any non-zero element of I^{n-1}. Then $aAa \subseteq I^{n-1}AI \subseteq I^n = (0)$, and therefore $a = 0$, which contradicts the choice of a. Thus I must be the zero ideal.

Since the implications (2) implies (4) and (3) implies (4) are obvious, it remains to prove that (4) implies (1).

Let $N = \{a \in A : aA = (0)\}$. Then N is a 2-sided ideal, and $N^2 \subseteq NA = (0)$. Suppose A has no non-zero nilpotent 2-sided ideals. Then $N = (0)$, and a similar argument shows that if $Aa = (0)$, then $a = 0$. To show that A is semi-prime, we must prove that if $aAa = (0)$, then $a = 0$. Let a be any element of A and let I be the 2-sided ideal of A spanned by the set AaA. If $aAa = (0)$, then $I = (0)$ and so $baA = (0)$ for all b in A. But then, $Aa = (0)$ and therefore $a = 0$. This completes the proof.

<u>3.6 Definition</u>. The <u>prime radical</u> L of A is the intersection of all the prime ideals of A.

<u>3.7 Lemma</u>. Let L be the prime radical of A. Then A/L is semi-prime.

<u>Proof</u>. Let a be any element of A such that aAa is contained in L. Then a is in each prime ideal of A, and therefore in L. Thus L is a semi-prime ideal, and by Corollary 3.3, A/L is therefore a semi-prime algebra.

In Proposition 3.9, we prove that A is semi-prime if and only if its prime radical is the zero ideal. This requires the following lemma, which is also used in the proofs of lemmas 3.11 and 3.12.

<u>3.8 Lemma</u>. Let M be a non-empty subset of A such that

(1) 0 is not in M

(2) For all a and b in M, there is an element x of A such that axb is in M.

Then there is a prime ideal P of A such that the intersection of P with M is empty.

Proof. Let X be the set of all 2-sided ideals P of A such that $P \cap M$ is empty. By condition (1), X is non-empty, and, if X is partially ordered by inclusion, then it clearly satisfies the conditions of Zorn's lemma and so has a maximal element P. We show that P is prime. Suppose not, and let a and b be elements of $A \setminus P$ such that aAb is contained in P. Let $(a) = Fa + Aa + aA + \mathrm{span}(AaA)$ and $(b) = Fb + Ab + bA + \mathrm{span}(AbA)$. Then $P + (a)$ and $P + (b)$ are 2-sided ideals, and , by the maximality of P, there exist elements c in $(P + (a)) \cap M$ and d in $(P + (b)) \cap M$. Let x be an element of A such that cxd is in M. Then $cxd \in (P + (a))A(P + (b)) \subseteq P$, and so $M \cap P$ is not empty. This is a contradiction, proving that P must be a prime ideal.

3.9 Proposition. Let L be the prime radical of A. Then A is semi-prime if and only if $L = (0)$.

Proof. If $L = (0)$, then A is semi-prime, by lemma 3.7.

To prove the converse, suppose that A is semi-prime, and let a be any non-zero element of A. To show that a is not in L it is sufficient to prove that there is a prime ideal P of A such that a is not in P.

Let $a_1 = a$, and choose $a_2, a_3, \ldots..$ inductively so that $a_{n+1} \in a_nAa_n \setminus (0)$. This can be done, since $a_n \neq 0$ implies $a_nAa_n \neq (0)$. Let $M = \{a_n : n = 1,2, \ldots.\}$. A simple inductive argument shows that if n is less than or equal to m, then

$a_{n+1} \in a_n A a_n \subseteq a_m A a_n \cap a_n A a_m$. Thus M satisfies the conditions of lemma 3.8, and there is therefore a prime ideal P such that $M \cap P$ is empty. Since a is in M, this completes the proof.

3.10 Corollary. A 2-sided ideal of A is a semi-prime ideal if and only if it is an intersection of prime ideals.

3.11 Lemma. The prime radical of A is a nil ideal of A containing all the nilpotent ideals of A.

Proof. Let a be a non-nilpotent element of A and let

$$M = \{a^n : n = 1, 2, \ldots\}.$$

Then M satisfies the conditions of lemma 3.8, and there is therefore a prime ideal P of A such that a is not in P. Thus every element of the prime radical L is nilpotent and L is therefore a nil ideal.

Let N be a left or right nilpotent ideal of A. Then $(N + L)/L$ is a nilpotent ideal of A/L and, by lemmas 3.7 and 3.5, N is therefore contained in L.

If A is commutative, and a is a nilpotent element of A, then the ideal aA is nilpotent. Thus a is in the prime radical L. It follows that L is simply the set of all nilpotent elements of A.

When A is not commutative, L may not even contain all the nil ideals of A, and is certainly not in general equal to the set of all nilpotent elements.

3.12 Lemma. Let I be a 2-sided ideal of A and let L be the prime radical of A. Then $I \cap L$ is the prime radical of I.

Proof. Let P be a prime ideal of A and let a and b be elements of I such that $aIb \subseteq P \cap I$. Then $aAIb \subseteq P$, and so either $a \in P$ or $Ib \subseteq P$. If $Ib \subseteq P$, then $IAb \subseteq P$, and so $I \subseteq P$ or $b \in P$. In either case we have $a \in P \cap I$ or $b \in P \cap I$ and $P \cap I$ is therefore a prime ideal of I. It follows that the prime radical $L(I)$ of I is contained in $I \cap L$. Now let a be any element of I not in $L(I)$, and let N be a prime ideal of I such that a is not in N. Then $M = I \setminus N$ satisfies the conditions of lemma 3.8, and there is therefore a prime ideal P of A such that $M \cap P$ is empty. In particular, a is not in $L \cap I$. This completes the proof.

3.13 Definition. Let I be a 2-sided ideal of A and let P be a prime ideal of A containing I. Then P is a *minimal prime ideal over I* if, whenever N is a prime ideal of A such that $I \subseteq N \subseteq P$, then $N = I$ or $N = P$. If $I = (0)$, then P is referred to simply as a *minimal prime ideal*. Note that a prime ideal P containing I is minimal over I if and only if P/I is a minimal prime ideal of A/I.

The existence of minimal prime ideals is guaranteed by the following lemma, the proof of which is a simple application of Zorn's lemma.

3.14 Lemma. Let I be a 2-sided ideal of A and let P be a prime ideal containing I. Then there exists a prime ideal N of A such that N is contained in P and N is minimal over I.

3.15 Corollary. The prime radical of A is equal to the intersection of all the minimal prime ideals of A.

We now briefly consider the relationship between the prime radical and the Jacobson radical.

<u>3.16 Lemma</u>. Let P be a primitive ideal of A. Then P is a prime ideal.

<u>Proof</u>. Suppose that P is a left primitive ideal, and let X be an irreducible left A-module such that $P = \{a \in A : a.X = \{0\}\}$. Let a and b be elements of A such that aAb is contained in P and b is not in P. Then, $Ab.X = X$, and therefore $a.X = aAb.X = \{0\}$, which proves that a is in P. Thus P is a prime ideal. A similar argument shows that a right primitive ideal is a prime ideal.

<u>3.17 Corollary</u>. The prime radical of A is contained in the Jacobson radical of A.

<u>3.18 Lemma</u>. If A is finite-dimensional, then every proper prime ideal of A is primitive.

<u>Proof</u>. We may assume without loss of generality that $P = (0)$ and that $A \neq (0)$. Since A cannot have a strictly descending sequence of subspaces, it must have a minimal left ideal, that is, a non-zero left ideal X which does not properly contain any non-zero left ideal. X may be regarded as a left A-module in the obvious way, and is clearly irreducible. Since (0) is a prime ideal, $a.X = (0)$ if and only if $a = 0$. Thus (0) is a left primitive ideal.

<u>3.19 Corollary</u>. The Jacobson radical of a finite-dimensional algebra is equal to its prime radical and is nilpotent.

The following lemma was first proved by Nagata and Higman [64, 38], and is the essential tool for investigating the special algebraic properties of the prime radical of a Banach algebra. Before stating it, we define the <u>algebra obtained from A by adjoining an identity element</u>. This is the algebra B of all ordered pairs (a,s) such that a is in A and s is in F, with the operations defined by

$$(a,s) + (b,t) = (a + b,s + t)$$

$$t(a,s) = (ta,ts)$$

$$(a,s)(b,t) = (ab + sb + ta,st),$$

for all a and b in A and s and t in F. The element $(0,1)$ of B is written 1 and is an identity element for B, The map $a \rightarrow (a,0)$ is a monomorphism of A into B, and A may therefore be identified with the subalgebra $\{(a,0) : a \in A\}$ of B. We then have $(a,s) = a + s1 = a + s$, for all a in A and s in F.

<u>3.20 Lemma (Nagata-Higman)</u>. If $a^n = 0$, for all a in A, and the characteristic of F is 0, then A is nilpotent, and $A^{2^n - 1} = (0)$.

<u>Proof</u>. The following proof appears in [42], where it is attributed to P.J. Higgins.

The result is clearly true for $n = 1$. Suppose that it holds for some n greater than or equal to 1, and that $a^{n+1} = 0$, for all a in A. Let B be the algebra obtained from A by adjoining an identity element, and for all b in B, let $b^0 = 1$. Let $t_0, \ldots\ldots, t_{n-1}$ be distinct non-zero elements of F. Then the Vandermonde matrix V, in which the (i,j)th entry is t_{i-1}^{j-1}, is invertible (see, for example

41, p.115).

For any a and b in A, and non-zero t in F, we have $0 = t^{-1}(a + tb)^{n+1} = P_1(a,b) + tP_2(a,b) + \ldots\ldots + t^{n-1} P_n(a,b)$, where $P_1(a,b), \ldots\ldots, P_n(a,b)$ are elements of A independent of t. By the invertibility of the matrix V, it follows that $P_1(a,b) = \ldots\ldots = P_n(a,b) = 0$, and that, in particular,

$$P_1(a,b) = \sum_{i=0}^{n} a^i ba^{n-i} = 0 \qquad (a,\ b \in A).$$

Let a, b and c be any elements of A. Then,

$$(n+1)a^n cb^n = a^n c \sum_{j=0}^{n} b^j b^{n-j} = \sum_{i=0}^{n} a^i c \sum_{j=0}^{n} b^j a^{n-i} b^{n-j} =$$

$$\sum_{j=0}^{n}\left(\sum_{i=0}^{n} a^i cb^j a^{n-i}\right)b^{n-j} = 0, \text{ and therefore } a^n cb^n = 0.$$

Let I be the 2-sided ideal of A generated by the set $\{a^n : a \in A\}$. Then $IAI = (0)$, and for all x in A/I, $x^n = 0$. By the inductive hypothesis applied to A/I, we have $A^M \subseteq I$, where $M = 2^n - 1$. Thus $A^{2^{n+1}-1} = (0)$, and A is nilpotent as required.

<u>3.21 Theorem</u> (Grabiner [27]). A nil Banach algebra is nilpotent.

<u>Proof</u>. Let A be a nil Banach algebra over F (= $\mathbb{R}$ or $\mathbb{C}$). For each natural number n, let $X(n) = \{a \in A : a^n = 0\}$. Then each $X(n)$ is closed, and $A = \bigcup\{X(n) : n = 1, 2, \ldots\ldots\}$. By the Baire Category Theorem, there is a natural number n such that $X(n)$ has an interior point. Let b be any interior point of $X(n)$, and let a be any element of A. Then there exists a positive real number t, such that, if s is in F and $|s| < t$, then $(b + sa)^n = 0$. But then $a^n = 0$, and the result therefore follows from lemma 3.20.

The following lemma is used by Grabiner in [29].

3.22 Lemma. Let A be a Banach algebra, and let a be any element of A. Then the ideals Aa and aA are nil if and only if they are nilpotent.

Proof. We may assume without loss of generality that the norm of a is less than 1. We show that if Aa is nil, then it is nilpotent. A similar argument works for the right ideal aA, and the converse is obvious.

Assume that Aa is nil, and define a norm $\|\cdot\|'$ on Aa by $\|x\|' = \inf\{\|b\| : ba = x\}$. To see that $\|\cdot\|'$ is a complete norm on Aa, note that if T is the bounded linear operator on A defined by $Tb = ba$, then $A/\mathrm{Ker}(T)$ is isometrically isomorphic to $(Aa, \|\cdot\|')$. Now let x and y be elements of Aa, with $x = ba$ and $y = ca$. Then $\|xy\|' \leq \|bac\| \leq \|b\|\|c\|$, and $\|\cdot\|'$ is therefore an algebra norm. Thus $(Aa, \|\cdot\|')$ is a nil Banach algebra and, by Theorem 3.21, Aa is therefore nilpotent.

3.23 Theorem (Grabiner [29], Dixon [16]). Every left or right nil ideal of a Banach algebra is contained in a sum of nilpotent 2-sided ideals.

Proof. The following proof is due to Grabiner.

Let A be a Banach algebra, and let B be the algebra obtained from A by adjoining an identity element. Define a norm on B by

$$\|a + s\| = \|a\| + |s| \qquad (a \in A,\ s \in F).$$

Then B is a Banach algebra. Let I be a left nil ideal of A and let a be in I. Then Ba is contained in I, and is therefore nil. By lemma 3.22, applied with $B = A$, Ba must be nilpotent. Suppose $(Ba)^N = (0)$. Then, $(BaB)^N \subseteq (Ba)^N B = (0)$. Thus span$(BaB)$ is a 2-sided nilpotent ideal of A. It follows that I is equal to the sum (and the union) of the set $\{\text{span}(BaB) : a \in I\}$ of 2-sided nilpotent ideals. A similar argument shows that any right nil ideal is a sum of nilpotent 2-sided ideals.

3.24 Corollary (Dixon [16]). The prime radical of a Banach algebra is the sum of its 2-sided nilpotent ideals and contains all the 2-sided nil ideals.

If the prime radical L of a Banach algebra is closed, then, by Theorem 3.21, it is nilpotent. Conversely, if L is nilpotent, then there is a natural number n such that L is contained in $X(n) = \{a \in A : a^n = 0\}$. Since $X(n)$ is closed, the closure of L is contained in $X(n)$ and, by lemma 3.20, is therefore nilpotent. It follows that L is closed if and only if it is nilpotent.

4. Examples.

This Section contains a few examples of semi-prime Banach algebras, and of continuous derivations and automorphisms. An extensive list of examples of Banach algebras is given in the Appendix of [68], and a large proportion of these are semi-simple, and therefore also semi-prime. However, there are rather few published examples of semi-prime Banach algebras which are not semi-simple. By lemma 3.12 and Corollary 2.22, the Jacobson radical of a semi-prime Banach algebra is also a semi-prime Banach algebra. There is therefore no serious loss of generality involved in restricting attention to radical semi-prime Banach algebras. Most, if not all, of the known examples of Banach algebras of this type are convolution algebras of some kind, but it is not at all clear whether this reflects some theoretical restriction on the structure of such algebras, or merely a lack of ingenuity in constructing examples. However, convolution Banach algebras usually have large numbers of closed ideals, and it is not obvious that this is a necessary feature of semi-prime radical Banach algebras in general (see Section 10).

Let A be a Banach algebra, and let b be an element of A which does not commute with every element of A. Then the map $a \rightarrow ab - ba$ on A is a non-zero derivation, usually referred to as an <u>inner derivation</u>. Every non-commutative Banach algebra therefore has continuous non-zero derivations. However, this is certainly not true for commutative Banach algebras. In fact, a commutative semi-simple Banach algebra has no non-zero derivations (see Section 12), and an example of Newman shows that radical Banach algebras may also have no non-zero derivations [65] (see also [9, p.97]).

Examples of continuous homomorphisms are much more abundant. In this section, we concentrate on isometric automorphisms. In particular, we show how semi-groups of isometric automorphisms may be used to construct a class of convolution Banach algebras which includes some non-commutative radical semi-prime Banach algebras.

The section ends with an example of a completely regular semi-simple commutative Banach algebra with non-maximal closed prime ideals. This is of interest, because in a large class of such algebras, all the proper closed prime ideals are maximal. This last example, and Example 4.14, are the only examples in this section which may be original.

We begin by defining what is usually meant by a Banach algebra of formal power series.

<u>4.1 Definition</u>. Let $\mathbb{C}[[t]]$ denote the algebra of all formal power series in the indeterminate t, with complex coefficients, and with the algebra operations defined by

$$\sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} (a_n + b_n) t^n$$

$$z \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} (z a_n) t^n \qquad (z \in \mathbb{C})$$

$$\left(\sum_{n=0}^{\infty} a_n t^n\right)\left(\sum_{n=0}^{\infty} b_n t^n\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} a_m b_{n-m}\right) t^n,$$

where, for all n, a_n and b_n are complex numbers.

$\mathbb{C}[[t]]$ is a commutative algebra with identity and is an integral domain.

Let A be a subalgebra of $\mathbb{C}[[t]]$, and let $\|\cdot\|$ be a complete algebra norm on A. Then $(A, \|\cdot\|)$ is a Banach algebra of formal power series if

(1) t is in A, and A is generated by t, or by t and 1

(2) For each m, the co-ordinate functional

$\sum_{n=0}^{\infty} a_n t^n \longrightarrow a_m : (A, \|\cdot\|) \longrightarrow \mathbb{C}$, is continuous.

We now describe one of the earliest examples of a radical semi-prime Banach algebra. It is in fact isometrically isomorphic to a Banach algebra of formal power series (see, for example, [28]), and is therefore an integral domain.

4.2 Example [39, 68, 28].

Let $\Delta = \{z \in \mathbb{C} : |z| \leqslant 1\}$, and let $A(\Delta)$ be the Banach space of all continuous functions on Δ which are analytic on the interior of Δ, and with the norm

$$\|f\|_{\infty} = \sup\{|f(z)| : z \in \Delta\} \qquad (f \in A(\Delta)).$$

With the ordinary pointwise product, $A(\Delta)$ is a semi-simple commutative Banach algebra, usually referred to as the disc algebra.

A different product $*$ can be defined on $A(\Delta)$ by

$$(f * g)(z) = \int_{[0,z]} f(w)g(z - w)dw \qquad (z \in \Delta,\ f,g \in A(\Delta)),$$

where $[0,z]$ is the line segment from 0 to z.

The Banach algebra so obtained is radical, and is called the disc algebra with convolution. Now define D on $A(\Delta)$ by

$$Df(z) = zf(z) \qquad (z \in \Delta,\ f \in A(\Delta)).$$

Then D is clearly a bounded linear operator of norm 1 and, with respect to the convolution product, is a derivation, since for all f and g in $A(\Delta)$ and z in Δ,

$$D(f * g)(z) = z(f * g)(z) = \int_{[0,z]} zf(w)g(z - w)dw$$
$$= \int_{[0,z]} wf(w)g(z - w)dw + \int_{[0,z]} f(w)(z - w)g(z - w)dw$$
$$= (Df * g)(z) + (f * Dg)(z).$$

A full account of Banach algebras of formal power series, including a description of their derivations and automorphisms, is given in [28].

A well-known example of a commutative radical convolution algebra which is not semi-prime, is the Banach algebra $L^1(0,1)$ of all (equivalence classes of) complex-valued Lebesgue integrable functions on $(0,1)$, with the norm

$$\|f\| = \int_{[0,1]} |f(x)|\, dx \qquad (f \in L^1(0,1))$$

and the product

$$(f * g)(x) = \int_0^x f(y)g(x - y)dy \qquad (f,g \in L^1(0,1),\ x \in (0,1)).$$

A full description of the derivations and automorphisms of $L^1(0,1)$ is given in [52].

Let A be an algebra with identity, and let a be an invertible element of A. Then the map

$$b \longrightarrow a^{-1}ba \qquad (b \in A)$$

is an automorphism of A, called an <u>inner automorphism</u>. A non-commutative Banach algebra always has non-trivial inner automorphisms (see [9, p.87]).

Semi-simple commutative Banach algebras generally have many automorphisms induced by homeomorphisms of their carrier spaces, as in the following example.

4.3 Example. Let $A(\Delta)$ be the disc algebra with the pointwise product and let t be any real number. Then the operator h_t on $A(\Delta)$ defined by

$$h_t(f)(z) = f(e^{it}z) \qquad (f \in A(\Delta),\ z \in \Delta),$$

is an isometric automorphism.

Examples of non-trivial automorphisms on commutative radical Banach algebras are harder to find. The following well known result indicates one source of such examples, and is one of the reasons for the interest in derivations.

4.4 Lemma. Let D be a continuous derivation on a Banach algebra A. Then $\exp(D)$ is a continuous automorphism on A.

Proof. See, for example, [9, p.87].

4.5 Example. Let D be the derivation on the disc algebra with convolution described in Example 4.2. Then

$$\exp(D)(f)(z) = \exp(z)f(z) \qquad (z \in \Delta,\ f \in A(\Delta)).$$

4.6 Definition. Let A be a Banach algebra and let S be a semi-group. Then, by a representation of S on A by isometric automorphisms, we shall mean a semi-group homomorphism T of S into the group of isometric automorphisms of A.

When there is no risk of confusion, the notation

$$t.a = T(t)(a) \qquad (a \in A,\ t \in S)$$

will be used. In terms of this notation, the definition requires that the following conditions be satisfied :

(1) $t.ab = (t.a)(t.b) \qquad (t \in S,\ a,\ b \in A)$

(2) $t.(a + zb) = t.a + z(t.b)$ $\quad (t \in S,\ a,b \in A,\ z \in F)$

(3) $ts.a = t.(s.a)$ $\quad (t,s \in S,\ a \in A)$

(4) $\|t.a\| = \|a\|$ $\quad (t \in S,\ a \in A)$. (5) $a \to t.a$ is onto $(t \in S)$

If S is written additively, then condition (3) should be written $(t + s).a = t.(s.a)$.

4.7 Example. Let A be the disc algebra (with pointwise product), let S be any sub-semi-group of the additive group of the real numbers, and define T by

$$T(t) = h_t \qquad (t \in S),$$

where h_t is as defined in Example 4.3.

4.8 Example. Let A be any Banach algebra and let h be any isometric automorphism of A. Then, if S is either $\mathbb{N}$ (the natural numbers) or $\mathbb{Z}^+$ (the non-negative integers), and

$$T(n) = h^n \qquad (n \in S),$$

then T is a representation of S on A by isometric automorphisms.

Further examples, with S a group, are given in Chapter 4.

We now describe a class of convolution algebras, which might be called 'weighted cross products'. The aim is to construct examples of non-commutative radical semi-prime Banach algebras.

4.9 Definition. Let A be a Banach algebra, let S be a semi-group, and let w be a real-valued function on S such that the following conditions are satisfied :

(1) $w(s) > 0 \quad (s \in S)$

(2) $w(st) \leqslant w(s)w(t) \quad (s, t \in S)$.

We will denote by $l^1(S,A,w)$ the Banach space of all functions f from S into A such that

$$\|f\| = \sum_{s \in S} \|f(s)\| w(s) < \infty .$$

It is well known that $l^1(S,A,w)$ is a Banach algebra with respect to the convolution product $*$ defined by

$$(f * g)(s) = \sum_{tu=s} f(t)g(u) \qquad (s \in S,\ f,g \in l^1(S,A,w)).$$

Now let T be a representation of S on A by isometric automorphisms (see Definition 4.6), and for all f and g in $l^1(S,A,w)$, let

$$(f *_T g)(s) = \sum_{tu=s} f(t)t.g(u) \qquad (s \in S),$$

where, as before, $t.a = T(t)(a)$. Then,

$$\|f *_T g\| \leqslant \sum_{s \in S} \sum_{tu=s} \|f(t)\| \|g(u)\| w(t)w(u) \leqslant \|f\|\|g\|,$$

and $*_T$ is clearly bilinear.

To show that $*_T$ is associative, let f, g and h be any elements of $l^1(S,A,w)$. Then, for all z in S,

$$(f *_T g) *_T h(z) = \sum_{xy=z} \Big(\sum_{tu=x} f(t)t.g(u) \Big) x.h(y) =$$

$$\sum_{tuy=z} f(t)(t.g(u))(tu.h(y)) = \sum_{tx=z} f(t) \sum_{uy=x} (t.g(u))(tu.h(y)) =$$

$$\sum_{tx=z} f(t)t.\Big(\sum_{uy=x} g(u)u.h(y)\Big) = f *_T (g *_T h)(z).$$

Thus $l^1(S,A,w)$ with the product $*_T$ is also a Banach algebra, which we will denote by $l^1(S,A,w,T)$.

We will write $w = 1$, if $w(s) = 1$ for all s in S, and $T = 1$, if $T(s) = 1$ for all s in S. For all a in A and

s in S, $k(s,a)$ will denote the element of $l^1(S,A,w)$ defined by $k(s,a)(t) = \begin{cases} a, & s = t \\ 0, & s \neq t. \end{cases}$

Note that

$$k(s,a) *_T k(u,b) = k(su, a(s.b)) \qquad (a,b \in A,\ s,u \in S).$$

We will denote the nth power of an element f of $l^1(S,A,w,T)$ by f^{*^n}, to avoid any confusion with pointwise products.

Hirschfeld and Rolewicz used the crossed product $*_T$ (with $w = 1$, $S = \mathbb{Z}^+$ and T as in Example 4.8) to construct an example of a non-commutative Banach algebra with no non-zero zero divisors [40]. The construction has also been used extensively (with $w = 1$) in the case when A is a C*-algebra and S is a group [82]. Note that $l^1(S,\mathbb{C},w,1)$ is the ordinary weighted semi-group algebra (see, for example, [9, p.8]). The proof of the following lemma is straightforward and is therefore omitted.

<u>4.10 Lemma</u>. The linear span of the set $\{k(s,a) : a \in A,\ s \in S\}$ is a dense subalgebra of $l^1(S,A,w,T)$.

The following Proposition shows that restrictions on w (and by implication on S) are sufficient to ensure that $l^1(S,A,w,T)$ is a radical algebra. In particular, there is no reason why A should be chosen to be non-unital.

<u>4.11 Proposition</u>. Let w satisfy the following two conditions (in addition to conditions (1) and (2) of Definition 4.10) :

~~(3) w(st) = w(ts) (t, s ∈ S)~~ $\omega(s_1 \cdots s_n) = \omega(\sigma(s_1 \cdots s_n)) \quad (\sigma \in S_n)$.

(4) $w(s^n)^{\frac{1}{n}} \longrightarrow 0$ as $n \longrightarrow \infty$ $(s \in S)$.

Then $l^1(S,A,w,T)$ is a radical algebra.

Proof. Let R denote the Jacobson radical of $l^1(S,A,w,T)$. By Corollary 2.22 and lemma 4.10, it is sufficient to prove that $k(u,a)$ is in R, for all u in S and a in A.

A simple inductive argument shows that

$$k(u,a)^{*^n} = k(u^n, a(u.a)(u^2.a).....(u^{n-1}.a)) \qquad (n \geqslant 2).$$

Thus $\|k(u,a)^{*^n}\| \leqslant \|a\|^n w(u^n)$, and so $k(u,a)$ is quasi-nilpotent by condition (4). However, since $l^1(S,A,w,T)$ may be non-commutative, this is not enough to prove that $k(u,a)$ is in R. By lemma 2.25, it is sufficient to prove that $f*_T k(u,a)$ is quasi-nilpotent for all f in $l^1(S,A,w,T)$. To do this, let $g = f*_T k(u,a)$. Then

$$g^{*^n}(s) = \sum_{x_1.....x_n = s} g(x_1)(x_1.g(x_2)).....(x_1.....x_{n-1}.g(x_n)) \quad (s \in S, n \geqslant 2)$$

and
$$g(x) = \sum_{yu = x} f(y)(y.a) \qquad (x \in S),$$

and therefore,
$$\|g^{*^n}\| = \sum_{s \in S} \|g^{*^n}(s)\| w(s) \leqslant$$

$$\sum_{s \in S}\left(\sum_{x_1 \cdots x_n = s} \left(\sum_{y_1 u = x_1} \|f(y_1)\| \|a\| \right) ..... \left(\sum_{y_n u = x_n} \|f(y_n)\| \|a\| \right) w(s) \right).$$

By conditions (2) and (3), we have $w(s) \leqslant w(u^n)w(y_1)...w(y_n)$, when $y_1 u.....y_n u = s$. Thus,

$$\|g^{*^n}\| \leqslant \|a\|^n w(u^n) \sum_{s \in S}\left(\sum_{y_1 u \cdots y_n u = s} \|f(y_1)\| w(y_1)..... \|f(y_n)\| w(y_n) \right)$$

$\leqslant \|a\|^n \|f\|^n w(u^n)$. Since $w(u^n)^{\frac{1}{n}} \longrightarrow 0$, this shows that g is quasi-nilpotent and so completes the proof.

It is easy to check that, when A is unital, $l^1(S,A,w,T)$

is non-commutative if S is non-commutative or $T \neq 1$.

4.12 Example [21].

Let $FS(2)$ be the free semi-group on 2 generators u,v, and for each 'word' s of $FS(2)$, let $n(s)$ denote the number of letters in s (for example, $n(uv^3u^{234}vuv) = 241$).

Let $w(s) = (n(s)!)^{-1}$ $\quad (s \in FS(2))$.

Then w satisfies conditions (1) and (2) of Definition 4.9 and (3) and (4) of Proposition 4.11. Thus $l^1(FS(2),\mathbb{C},w,1)$ is a non-commutative radical Banach algebra.

It is easy to check that $l^1(FS(2),\mathbb{C},w,1)$ has no non-zero zero divisors. In particular, it is prime, and therefore semi-prime.

In both of the next two examples, we take $S = \mathbb{N}$, and

$$w(n) = (n!)^{-1} \qquad (n \in S).$$

They are both non-commutative radical Banach algebras.

4.13 Example. This is essentially the example of Hirschfeld and Rolewicz [40], except that, in their example, $w = 1$ and $S = \mathbb{Z}^+$.

Let A be any commutative unital Banach algebra which is also an integral domain, and let h be a non-trivial isometric automorphism on A (for example, let $A = A(\Delta)$ and $h = h_t$, as in Example 4.3). As in Example 4.8, let $T(n) = h^n$. Let f and g be any non-zero elements of $l^1(\mathbb{N},A,w,T)$ and let n and m be the least natural numbers such that $f(n) \neq 0$ and $g(m) \neq 0$ respectively. Then,

$$(f *_T g)(n + m) = f(n)n.g(m) \neq 0.$$

Thus $l^1(\mathbb{N},A,w,T)$ has no non-zero zero divisors.

The final example of this type exhibits more typical behaviour for non-commutative semi-prime algebras, since it does have non-zero nilpotent elements.

4.14 Example. Let A be the algebra $C[-1,1]$ of all continuous complex-valued functions on the interval $[-1,1]$, and define T on $\mathbb{N}$ by

$$T(n)a(x) = a((-1)^n x) \qquad (n \in \mathbb{N},\ a \in C[-1,1],\ x \in [-1,1]).$$

Let a be a non-zero element of $C[-1,1]$ such that

$$a(x)a(-x) = 0 \qquad (x \in [-1,1]).$$

Then, $k(1,a)^{*2} = k(2,a(1.a)) = 0$, and $k(1,a)$ is therefore a non-zero nilpotent element.

Let f be in $l^1(\mathbb{N},C[-1,1],w,T)$ and suppose

$$f *_T g *_T f = 0 \qquad (g \in l^1(\mathbb{N},C[-1,1],w,T)).$$

If $f \neq 0$, then there is a least natural number n such that $f(n) \neq 0$. But then,

$$(f *_T k(m,1) *_T f)(2n + m) = f(n)((n + m).f(n)) = 0,$$

and therefore, whether n is odd or even, $f(n)^2 = 0$ and so $f(n) = 0$. This contradiction proves that $l^1(\mathbb{N},C[-1,1],w,T)$ is semi-prime.

The section ends with an example of a completely regular unital commutative semi-simple Banach algebra with non-maximal proper closed prime ideals.

Let $C^\infty[0,1]$ denote the algebra of all infinitely differentiable complex-valued functions on $[0,1]$, and for all continuous complex-valued functions f on $[0,1]$ let

$$\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}.$$

Let $\{M_n : n = 0, 1, 2, \ldots\ldots\}$ be any sequence of strictly positive real numbers such that $M_0 = 1$ and

$$M_n \geq \binom{n}{r} M_r M_{n-r} \qquad (0 \leq r \leq n).$$

Then $D(M_n)$ will denote the Banach algebra of all f in $C^\infty[0,1]$ such that

$$\|f\| = \sum_{n=0}^{\infty} \frac{1}{M_n} \|f^{(n)}\|_\infty < \infty.$$

These Banach algebras were described by Lorch in 1944, and are semi-simple [59].

Let $P = \{f \in D(M_n) : f^{(n)}(0) = 0,\ n = 0, 1, 2, \ldots\ldots\}$. Then P is a closed prime ideal of infinite co-dimension and is therefore non-maximal. In fact, P is the kernel of the homomorphism

$$f \rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) t^n$$

of $D(M_n)$ into the integral domain $\mathbb{C}[[t]]$ (see Definition 4.1). The aim is to show that $\{M_n\}$ can be chosen so that $D(M_n)$ is completely regular. This then provides the required example.

$D(M_n)$ is <u>natural</u> if the carrier space of $D(M_n)$ is $[0,1]$. Dales and Davie, in [14], state, but do not prove, that a sufficient condition for $D(M_n)$ to be natural is

$$\sum_{r=1}^{n-1} \binom{n}{r} \frac{M_r M_{n-r}}{M_n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.$$

For the sake of completeness, we include a proof of a weaker version of this result (lemma 4.16).

4.15 Lemma. The following are equivalent :

(1) $D(M_n)$ is natural

(2) If f is in $D(M_n)$, and $|f(x)| \geqslant 1$, for all x in $[0,1]$, then f^{-1} is in $D(M_n)$.

Proof. This is easily proved by routine arguments (see [14]).

4.16 Lemma. If $\sum_{n=2}^{\infty}\left(\sum_{r=1}^{n-1}\binom{n}{r}\frac{M_r M_{n-r}}{M_n}\right)<\infty$, then $D(M_n)$ is natural.

Proof. Let f in $D(M_n)$ satisfy $|f(x)| \geqslant 1$, for all x in $[0,1]$, and let $g = f^{-1}$. Then, for n greater than 1,

$$0 = (fg)^{(n)}(x) = \sum_{r=0}^{n}\binom{n}{r} g^{(r)}(x)f^{(n-r)}(x) \qquad (x \in [0,1]),$$

and therefore,

$$|g^{(n)}(x)| \leqslant |-g^{(n)}(x)f^{(0)}(x)| \leqslant \sum_{r=0}^{n-1}\binom{n}{r}\|g^{(r)}\|_\infty\|f^{(n-r)}\|_\infty \qquad (x \in [0,1]).$$

Dividing by M_n, and using the fact that $\|g\|_\infty \leqslant 1$, we get

$$\frac{\|g^{(n)}\|_\infty}{M_n} \leqslant \frac{\|f^{(n)}\|_\infty}{M_n} + \|f\|\sum_{r=1}^{n-1}\binom{n}{r}\frac{M_r M_{n-r}}{M_n}\frac{\|g^{(r)}\|_\infty}{M_r} \qquad (n \geqslant 2).$$

Now choose N such that

$$\sum_{r=1}^{n-1}\binom{n}{r}\frac{M_r M_{n-r}}{M_n} \leqslant (2\|f\|)^{-1} \qquad (n \geqslant N),$$

and a real number M such that $\|f\| \leqslant 2^{-1}M$ and

$$M_r^{-1}\|g^{(r)}\|_\infty \leqslant M \qquad (0 \leqslant r \leqslant N).$$

Suppose that $M_r^{-1}\|g^{(r)}\|_\infty \leqslant M$, for $0 \leqslant r \leqslant n$, where $n \geqslant N$. Then,

$$M_{n+1}^{-1}\|g^{(n+1)}\|_\infty \leqslant \|f\| + M(2\|f\|)^{-1}\|f\| \leqslant M,$$

and so, by induction, $M_n^{-1}\|g^{(n)}\|_\infty$ is less than or equal to M for all n. But then,

$$\sum_{n=2}^{\infty}\frac{\|g^{(n)}\|_\infty}{M_n} \leqslant \|f\| + \|f\| M\sum_{n=2}^{\infty}\left(\sum_{r=1}^{n-1}\binom{n}{r}\frac{M_r M_{n-r}}{M_n}\right) < \infty,$$

and therefore g is in $D(M_n)$. By lemma 4.15, this proves that $D(M_n)$ is natural.

4.17 Example. For all real numbers a such that $0 < a \leq 1$, define the functions f_a and g_a on $[0,1]$ by

$$f_a(x) = \begin{cases} \exp(-(a - x)^{-1}) & (0 \leq x < a) \\ 0 & (a \leq x \leq 1), \end{cases}$$

and $$g_a(x) = f_a(1 - x) \qquad (0 \leq x \leq 1).$$

Let $F = \{f_a, g_a : 0 < a \leq 1\}$. Then F is contained in $C^\infty[0,1]$, and for any closed set E of $[0,1]$ and x in $[0,1] \setminus E$, there is an element f of F such that

$$f(E) = \{0\} \quad \text{and} \quad f(x) \neq 0.$$

We prove the following statements.

(1) $k_n = \sup\{\|f^{(n)}\|_\infty : f \text{ is in } F\} < \infty \qquad (n = 0, 1, 2, \ldots.)$

(2) For any sequence $\{k_n\}$ of positive real numbers, $\{M_n\}$ can be chosen so that $D(M_n)$ is natural and

$$\sum_{n=0}^{\infty} \frac{K_n}{M_n} < \infty .$$

It then follows that $\{M_n\}$ can be chosen so that $D(M_n)$ is natural and contains F. By [9, Corollary 23.9, p.118], $D(M_n)$ is then completely regular as required.

Proof of (1). Differentiating f_a, n times $(n \geq 1)$, we get

$$f_a^{(n)}(x) = \begin{cases} \exp(-(a - x)^{-1}) \sum_{i=n+1}^{2n} d(n,i)(a - x)^{-i} & (0 \leq x < a) \\ 0 & (a \leq x \leq 1), \end{cases}$$

(where the numbers $d(n,i)$ do not depend on a), and therefore,

$$|f_a^{(n)}(x)| \leq \exp(-(a - x)^{-1})(a - x)^{-2n} n c_n \qquad (0 \leq x < a \leq 1),$$

where $c_n = \max\{d(n,i) : i = n + 1, \ldots\ldots, 2n\}$. But $\exp(-t)t^{2n}$ is less than or equal to $(2n)^{2n} \exp(-2n)$ for all t greater than 1, and therefore,

$$\|f_a^{(n)}\|_\infty \leqslant (2n)^{2n}\exp(-2n)nc_n \qquad (0 < a \leqslant 1, \quad n = 1, 2, \ldots\ldots).$$

This proves (1), since

$$\|f_a^{(0)}\|_\infty \leqslant 1 \text{ and } \|g_a^{(n)}\|_\infty = \|f_a^{(n)}\|_\infty \qquad (0 < a \leqslant 1, \quad n \geqslant 0).$$

Proof of (2). Without loss of generality, we can assume that k_n is greater than 2^n, for all n. Let $\{M_n\}$ be the sequence defined by $M_0 = M_1 = 1$ and

$$M_n = 2^n \sum_{r=1}^{n-1} \binom{n}{r} M_r M_{n-r} + k_n^2 \qquad (n \geqslant 2).$$

Then,

$$M_n \geqslant \binom{n}{r} M_r M_{n-r} \qquad (0 \leqslant r \leqslant n),$$

and $$\sum_{n=2}^{\infty}\left(\sum_{r=1}^{n-1}\binom{n}{r}\frac{M_r M_{n-r}}{M_n}\right) \leqslant \sum_{n=2}^{\infty} 2^{-n} < \infty .$$

$D(M_n)$ is therefore natural, by lemma 4.16. Finally,

$$\sum_{n=2}^{\infty}\frac{k_n}{M_n} \leqslant \sum_{n=2}^{\infty}\frac{k_n}{k_n^2} \leqslant \sum_{n=2}^{\infty} 2^{-n} < \infty .$$

This completes the proof.

Chapter Two

AUTOMATIC CONTINUITY OF HOMOMORPHISMS AND DERIVATIONS

5. Introduction.

One of the properties of Banach algebras which makes automatic continuity results possible is their completeness, and the usual way in which completeness is brought to bear is through the closed graph theorem.

Let S be a linear mapping from a Banach space X into a Banach space Y. Then the separating space of S is the set $\mathfrak{S}(S) = \{y \in Y : \text{there exist } x_n \to 0 \text{ in } X \text{ with } Sx_n \to y \text{ in } Y\}$, and the closed graph theorem is equivalent to the statement that S is continuous if and only if $\mathfrak{S}(S) = \{0\}$.

Almost all the results of Chapter 2 are based on lemma 6.4, which states that if $\{T_n\}$ and $\{R_n\}$ are sequences in $BL(X)$ and $BL(Y)$ respectively, such that $ST_n - R_nS$ is continuous for all n, then there exists a natural number N such that

$$(R_1 \ldots\ldots R_n \mathfrak{S}(S))^- = (R_1 \ldots\ldots R_N \mathfrak{S}(S))^- \qquad (n \geqslant N).$$

This lemma has already been applied successfully to several automatic continuity problems, and has especially strong implications for the automatic continuity of epimorphisms and derivations. This was first demonstrated by Sinclair and Jewell in the following result which appears as Theorem 2 in [45].

<u>5.1 Theorem [45].</u> Let A be a Banach algebra with the

property that for each infinite dimensional closed (two-sided) ideal J in A there is a sequence $b_1, b_2, \dots$ in A such that $(Jb_n \dots b_1)^- \supsetneq (Jb_{n+1} \dots b_1)^-$ for all positive integers n. If A contains no non-zero finite-dimensional nilpotent ideal, then a homomorphism from a Banach algebra onto A, and a derivation on A are continuous.

The essential point in the proof of this theorem is that the separating space of a derivation on A, or of an epimorphism from a Banach algebra onto A, is a closed 2-sided ideal J which, by lemma 6.4, has the property that, for every sequence $\{b_n\}$ in A, there exists a natural number N such that $(Jb_n \dots b_1)^- = (Jb_N \dots b_1)^-$, for all $n \geq N$.

To avoid tedious repetition, any closed 2-sided ideal J with this property will be referred to as a separating ideal, whether or not it is the separating space of an epimorphism or a derivation.

Theorem 5.1 is still the best starting point from which to prove the automatic continuity of epimorphisms and derivations on specific Banach algebras. It applies to a wide range of Banach algebras, including semi-simple Banach algebras, $L^1(0,1)$ and Banach algebras of formal power series (see Definition 4.1).

Chapter 2 is concerned mainly with the structure of separating ideals in general, and with the automatic continuity results which can be deduced from knowledge of this structure. For the most part, we consider only problems concerning non-nilpotent separating ideals, although with the underlying objective of showing that many (possibly all) Banach algebras do not have non-zero radical separating ideals which are not nilpotent. In fact, Theorem 5.1, and its application to non semi-prime Banach algebras such as $L^1(0,1)$, suggest that it may be the absence of non-zero finite-dimensional nilpotent ideals which is crucial for automatic continuity results, rather than the absence of non-zero nilpotent ideals in general. This possibility suggests the problem of determining whether or not a non-zero nilpotent separating ideal necessarily contains a non-zero finite-dimensional ideal, but we do not consider this problem further.

We now describe briefly the contents of each of the sections 6 to 13 of Chapter 2.

Section 6 is concerned with properties of separating spaces of linear mappings in general, and includes a proof of the fundamental lemma 6.4.

In Section 7, the term 'separating ideal' is generalised to 'B-separating ideal', so as to include the separating space of any homomorphism from a Banach algebra onto a dense subalgebra B of a Banach algebra A (Definition 7.1).

The main technical results are proved in Section 8. Of these, one of the most important is that if J is a B-separating ideal, and L is the prime radical of $J \cap B$, then $L = \bar{L} \cap B$. In particular, if $A = B$ (so that J is a separating ideal in the original sense), then the prime radical of J is a closed ideal, and therefore nilpotent.

In Section 9, the results of Section 8 are used to prove results concerning discontinuous homomorphisms from Banach algebras onto dense subalgebras of Banach algebras, with the emphasis on the case when the domain algebra is a non-commutative C*-algebra.

Sections 10, 11, 12 and 13 are concerned with derivations, epimorphisms and the uniqueness of norm problem. We show that if there is a separating ideal with non-nilpotent Jacobson radical, then there is a topologically simple radical Banach algebra (see Definition 10.4).From this it follows that there is a topologically simple radical Banach algebra if any one of the following propositions fails to be true :

(1) Every epimorphism from a Banach algebra onto a Banach algebra has a nilpotent separating space.

(2) Every epimorphism from a Banach algebra onto a semi-prime Banach algebra is continuous.

(3) Every semi-prime Banach algebra has a unique complete norm topology.

(4) Every derivation on a Banach algebra has a ~~non~~-nilpotent separating space.

(5) Every derivation on a semi-prime Banach algebra is continuous.

(6) Every derivation on a commutative Banach algebra maps the algebra into its Jacobson radical.

(7) Every derivation on a Banach algebra A maps each primitive ideal of A into itself.

We also show that (1) and (2) are equivalent, and that, if (4) is true, then so are (5), (6) and (7).

6. The separating space.

Let X and Y be Banach spaces, and let S be a linear mapping from X into Y. Then the separating space of S is the set

$$\mathfrak{S}(S) = \{y \in Y : \text{there exist } x_n \to 0 \text{ in } X \text{ with } Sx_n \to y \text{ in } Y\}.$$

This section is concerned with some elementary properties of separating spaces which are used in the rest of the Chapter. Proofs of lemmas 6.1 to 6.3 are given in [79, Section 1].

6.1 Lemma. $\mathfrak{S}(S)$ is a closed linear subspace of Y, and S is continuous if and only if $\mathfrak{S}(S) = \{0\}$.

6.2 Lemma. Let X,Y and Z be Banach spaces, let S be a linear mapping from X into Y, and let R be a continuous linear mapping from Y into Z. Then

(1) RS is continuous if and only if $R\mathfrak{S}(S) = \{0\}$.

(2) $(R\mathfrak{S}(S))^{-} = \mathfrak{S}(RS)$.

6.3 Lemma. Let S be a linear mapping from a Banach space X into a Banach space Y, and let X_0 and Y_0 be closed linear subspaces of X and Y respectively, such that SX_0 is contained in Y_0. If $S_0 : X/X_0 \to Y/Y_0$ is defined by

$$S_0(x + X_0) = Sx + Y_0 \qquad (x \in X),$$

then S_0 is continuous if and only if $\mathfrak{S}(S)$ is contained in Y_0.

6.4 Lemma. Let X and Y be Banach spaces, let S be a linear mapping from X into Y, and let $\{T_n\}$ and $\{R_n\}$ be sequences of bounded linear operators on X and Y respectively, such that $ST_n - R_nS$ is continuous for all n. Then there is a natural number N such that

$$(R_1 \cdot\cdots\cdot R_n \mathfrak{S}(S))^- = (R_1 \cdot\cdots\cdot R_N \mathfrak{S}(S))^- \qquad (n \geqslant N).$$

Proof. Let $V_n = R_1 \cdot\cdots\cdot R_nS - ST_1 \cdot\cdots\cdot T_n$. Then V_1 is continuous, and $V_{n+1} = V_nT_{n+1} - R_1 \cdot\cdots\cdot R_n(ST_{n+1} - R_{n+1}S)$, so that, by induction, V_n is continuous for all n. It follows that $\mathfrak{S}(R_1 \cdot\cdots\cdot R_nS) = \mathfrak{S}(ST_1 \cdot\cdots\cdot T_n)$, and therefore, by lemma 6.2, that $(R_1 \cdot\cdots\cdot R_n \mathfrak{S}(S))^- = \mathfrak{S}(ST_1 \cdot\cdots\cdot T_n)$, for all n.

Let y be in $\mathfrak{S}(ST_1 \cdot\cdots\cdot T_{n+1})$, and let $\{x_m\}$ be a sequence in X such that $x_m \to 0$ and $ST_1 \cdot\cdots\cdot T_{n+1}x_m \to y$. Then $T_{n+1}x_m \to 0$, and y is therefore in $\mathfrak{S}(ST_1 \cdot\cdots\cdot T_n)$. This proves that $\mathfrak{S}(ST_1 \cdot\cdots\cdot T_{n+1})$ is contained in $\mathfrak{S}(ST_1 \cdot\cdots\cdot T_n)$, for all n. Suppose there is no number N with the required property. Then we may suppose without loss of generality that

$$\mathfrak{S}(ST_1 \cdot\cdots\cdot T_{n+1}) \subsetneqq \mathfrak{S}(ST_1 \cdot\cdots\cdot T_n) \qquad (n = 1, 2, \ldots).$$

We may also assume that the norm of T_n is less than or equal to 1, for all n. We obtain a contradiction by constructing an element z of X such that, for all n, $\|Sz\| \geqslant n$.

Let Q_n denote the natural mapping of Y onto the Banach space $Y/\mathfrak{S}(ST_1 \cdot\cdots\cdot T_n)$. Then $Q_n\mathfrak{S}(ST_1 \cdot\cdots\cdot T_n) = \{0\}$ and therefore, by lemma 6.2, $Q_nST_1 \cdot\cdots\cdot T_n$ is continuous. On the other hand, $\mathfrak{S}(Q_{n+1}ST_1 \cdot\cdots\cdot T_n) = (Q_{n+1}\mathfrak{S}(ST_1 \cdot\cdots\cdot T_n))^- \neq \{0\}$ and so $Q_{n+1}ST_1 \cdot\cdots\cdot T_n$ is discontinuous. Using this information, we may inductively choose a sequence $\{x_n : n = 1, 2, \ldots\}$ in X such that, for all n, $\|x_n\| \leqslant 2^{-n}$, and

$$\|Q_{n+1}ST_1 \cdots T_nx_n\| \geqslant n + \|Q_{n+1}ST_1 \cdots T_{n+1}\| + \|Q_{n+1}S\sum_{j=1}^{n-1} T_1 \cdots T_jx_j\|.$$

Let $z = \sum_{j=1}^{\infty} T_1 \cdots T_jx_j$.

Then $\|Sz\| \geqslant \|Q_{n+1} Sz\|$

$$\geqslant \|Q_{n+1} ST_1 \ldots T_n x_n\| - \|Q_{n+1} S \sum_{j=1}^{n-1} T_1 \ldots T_j x_j\| -$$

$$\|Q_{n+1} ST_1 \ldots T_{n+1} (x_{n+1} + \sum_{j=n+2}^{\infty} T_{n+2} \ldots T_j x_j)\|$$

$$\geqslant n.$$

This completes the proof.

A weaker version of lemma 6.4, in which $ST_n - R_n S$ is required to be zero, is given as lemma 1.6 in [79]. The version given here appears in [45] and a stronger version in [56].

7. Separating ideals.

This section starts with a formal definition of the term 'B-separating ideal'.

7.1 Definition. Let B be a dense subalgebra of a Banach algebra A. Then a subset J of A is a B-separating ideal of A, if it is a closed 2-sided ideal of A with the property that, for every sequence $\{b_n\}$ in B, there exists a natural number N (depending on $\{b_n\}$) such that

$$(Jb_n \ldots\ldots b_1)^- = (Jb_N \ldots\ldots b_1)^- \qquad (n \geqslant N).$$

When $B = A$, we shall refer to J simply as a separating ideal.

Note that any finite-dimensional ideal of a Banach algebra is a separating ideal, and that any closed 2-sided ideal in a nilpotent Banach algebra is a separating ideal.

7.2 Proposition. Let S be a linear mapping from a Banach space X into a Banach algebra A and let B be a dense subalgebra of A. Suppose that there exist continuous linear operators T_b and U_b on X, for all b in B, such that the maps

$$x \to ST_b x - (Sx)b \quad \text{and} \quad x \to SU_b x - b(Sx)$$

from X into A are continuous. Then the separating space $\mathfrak{S}(S)$ of S is a B-separating ideal of A.

Proof. By lemma 6.1, $\mathfrak{S}(S)$ is a closed linear subspace of A. Let a be any element of $\mathfrak{S}(S)$, and let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ and $Sx_n \to a$. Then, for all b in B,

$$ST_b x_n - (Sx_n)b \to 0 \quad \text{and} \quad SU_b x_n - bSx_n \to 0.$$

Thus, $ab = \lim ST_b x_n$ and $ba = \lim SU_b x_n$ and therefore, ba and ab are in $\mathfrak{S}(S)$. This proves that $\mathfrak{S}(S)$ is a 2-sided ideal, since B is dense in A.

Now let $\{b_n\}$ be any sequence in B, and let

$$R_n a = ab_n \quad \text{and} \quad T_n = T_{b_n} \qquad (a \in A,\ n = 1, 2, \ldots\ldots).$$

By lemma 6.4, there exists a natural number N such that

$$(R_1 \ldots\ldots R_n \mathfrak{S}(S))^- = (R_1 \ldots\ldots R_N \mathfrak{S}(S))^- \qquad (n \geqslant N).$$

Since $R_1 \ldots\ldots R_n \mathfrak{S}(S) = \mathfrak{S}(S) b_n \ldots\ldots b_1$, this completes the proof.

7.3 Corollary. Let h be a homomorphism from a Banach algebra C onto a dense subalgebra B of a Banach algebra A. Then the separating space $\mathfrak{S}(h)$ of h is a B-separating ideal of A.

Proof. For b in B, let

$$T_b x = xc \quad \text{and} \quad U_b x = cx \qquad (x \in C),$$

where c is some element of C such that $h(c) = b$. Now apply Proposition 7.2, with $X = C$ and $h = S$.

7.4 Corollary. Let D be a derivation on a Banach algebra A. Then the separating space $\mathfrak{S}(D)$ of D is a separating ideal of A.

Proof. In this case, we have $X = A = B$, $D = S$, and for all b and x in A, $T_b x = xb$ and $U_b x = bx$.

Thus, by the defining condition for a derivation,

$$D(T_b x) - (Dx)b = T_{Db} x \quad \text{and} \quad D(U_b x) - bDx = U_{Db} x \qquad (x, b \in A).$$

Since T_{Db} and U_{Db} are continuous, the conditions of Proposition 7.2 are satisfied. $\mathfrak{S}(D)$ is therefore a separating ideal of A.

As observed in Section 5, Corollary 7.3 (with $A = B$) and Corollary 7.4 are contained in [45] (although not using this terminology). Considerable use is made of lemma 6.4 in the study of discontinuous homomorphisms with dense range (Corollary 7.3 with $A \neq B$) in [78, 79].

<u>7.5 Proposition</u>. Let B be a dense subalgebra of a Banach algebra A, and let J be a closed 2-sided ideal of A. Then the following are equivalent :

(1) J is a B-separating ideal of A

(2) For every non-empty subset E of B such that $JE \neq (0)$, there exists an element z of E such that $Jz \neq (0)$, and for each element b of B with bz in E, either $Jbz = (0)$ or $(Jbz)^- = (Jz)^-$.

<u>Proof</u>. Suppose that J is a B-separating ideal and that E is a non-empty subset of B such that $JE \neq (0)$. Let b_1 be any element of E such that $Jb_1 \neq (0)$. If there is no element z of E with the required property, then there exists b_2 in B such that $b_2 b_1$ is in E and $(0) \subsetneq (Jb_2 b_1)^- \subsetneq (Jb_1)^-$. Continuing in this way, we obtain a sequence $\{b_n\}$ in B such that, for all n, $(Jb_{n+1} \ldots\ldots b_1)^- \subsetneq (Jb_n \ldots\ldots b_1)^-$. This proves that (1) implies (2).

Now let J satisfy condition (2) and let $\{b_n\}$ be any sequence in B. Let $E = \{b_n \ldots\ldots b_1 : n \in \mathbb{N}\}$. If $Jb_N \ldots\ldots b_1 = (0)$ for some N, then clearly $(Jb_n \ldots\ldots b_1)^- = (Jb_N \ldots\ldots b_1)^- = (0)$, for all $n \geqslant N$. Thus we may suppose that $Jz \neq (0)$ for all z in E. But then there exists $z = b_N \ldots\ldots b_1$ in E such that $(Jbb_N \ldots\ldots b_1)^- = (Jb_n \ldots\ldots b_1)^- = (Jz)^-$, whenever b is in B and $bb_N \ldots\ldots b_1$ is in E. In particular,

$$(Jb_n \cdots\cdots b_1)^- = (Jb_N \cdots\cdots b_1)^- \qquad (n \geqslant N),$$

and so J is a B-separating ideal of A.

Proposition 7.5 is related to Corollary 1.7 of [79]. If, in the following lemma, J is the separating space of a homomorphism from a Banach algebra onto B, then the conclusion of the lemma follows immediately from lemma 6.2 and Corollary 7.3.

7.6 Lemma. Let B be a dense subalgebra of a Banach algebra A, and let J be a B-separating ideal of A. If Q is the natùral mapping of A onto A/M, where M is a closed 2-sided ideal of A, then $\overline{QJ}$ is a QB-separating ideal of A/M.

Proof. Clearly, QB is a dense subalgebra of A/M, and $\overline{QJ}$ is a closed 2-sided ideal. Let $\{b_n\}$ be any sequence in B and choose N such that

$$(Jb_n \cdots\cdots b_1)^- = (Jb_N \cdots\cdots b_1)^- \qquad (n \geqslant N).$$

Then, for all $n \geqslant N$, $(\overline{QJ}Qb_n \cdots\cdots Qb_1)^- = (Q((Jb_n \cdots\cdots b_1)^-))^-$ $= (Q((Jb_N \cdots\cdots b_1)^-))^- = (\overline{QJ}Qb_N \cdots\cdots Qb_1)^-$. Since every sequence in QB is of the form $\{Qb_n\}$ it follows that $\overline{QJ}$ is a QB-separating ideal of A/M.

In the proof of lemma 7.6, we have used the fact that if S is a subspace of a Banach algebra A and a is in A, then $(Sa)^- = (\bar{S}a)^-$. Identities of this kind are used throughout the Chapter.

The term 'separating ideal' was first used by Rickart to describe the ideal

$\mathfrak{S} = \{a \in A$: there exist a_n in A with $\|a_n\|_1 \longrightarrow 0$ and $\|a_n - a\|_2 \longrightarrow 0\}$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ are complete algebra norms on A [67]. Since $\mathfrak{S}$ is then the separating space of the identity mapping on A, regarded as a homomorphism from $(A, \|\cdot\|_1)$ onto $(A, \|\cdot\|_2)$, it is also a separating ideal in the sense of Definition 7.1.

8. The prime radical of a separating ideal.

This section is concerned with the relationship between a B-separating ideal J of a Banach algebra A and the prime ideals of the dense subalgebra B. The starting point is the purely algebraic fact that if b is any non-nilpotent element of $J \cap B$, or more generally, any element of $J \cap B$ not in the prime radical of B, then there is a minimal prime ideal P of B such that b is not in P. If $\bar{P}$ is the closure of P in A, then P is clearly contained in $\bar{P} \cap B$. We show in Corollary 8.3, that P is in fact equal to $\bar{P} \cap B$. It follows that if L is the prime radical of $J \cap B$* then $\bar{L} \cap B = L$ (Theorem 8.5), and that, if Q is the natural mapping of A onto $A/\bar{L}$, then $QJ \cap QB$ is semi-prime*(Theorem 8.7). By lemma 7.6, QJ is a QB-separating ideal of $A/\bar{L}$. For many purposes, including Theorem 8.8, we can therefore assume without too much loss of generality that $J \cap B$ is already semi-prime. Note that if $J \cap B$ is not commutative and not closed, then it may conceivably be both semi-prime and nil.

Throughout this section, B will denote a dense subalgebra of a Banach algebra A, and for any subset E of B (or A), the closure of E in A will be denoted by $\bar{E}$. Most of the implications of the results of this section for the case $B = A$ are left until Section 10.

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* If I is a 2-sided ideal of an algebra, then by the prime radical L of I, we shall always mean the prime radical of I as an algebra in its own right. The statement 'I is semi-prime' will mean that I is a semi-prime algebra (i.e. $L = (0)$), <u>not</u> that I is a semi-prime ideal (see Definition 3.1).

8.1 Lemma. Let J be a B-separating ideal of A and let P be a prime ideal of B such that $J\cap B \not\subseteq P$. Then there exist elements z of $B\setminus P$ and y of B such that the following conditions are satisfied :

(1) If $b \in B$ and $bz \notin P$, then $(Jbz)^- = (Jz)^-$

(2) $Jzy \neq (0)$

(3) If $b \in B$, then $(Jzbzy)^- = (Jzy)^-$ or $Jzbzy = (0)$

(4) $((J\cap B)z)^- = (Jz)^-$.

Proof. Let $E = B\setminus P$. Then E contains $J\cap B\setminus P$ and is therefore non-empty. If $b \in B$ and $Jb = (0)$, then $(J\cap B)Bb \subseteq (0)$ $\subseteq P$, and b is therefore in P. Thus $Jb \neq (0)$ for every element b of E. By Proposition 7.5, there is therefore an element z of E such that, for all $b \in B$, $bz \in E$ implies $(Jbz)^- = (Jz)^-$. Let z be any such element. Then z satisfies condition (1).

Now let $F = zB$. Then F is non-empty, and $(J\cap B)zB \not\subseteq P$ implies $JF \neq (0)$. Applying Proposition 7.5 again, we obtain an element w of F such that $Jw \neq (0)$ and, for each element b of B with $bw \in F$, either $Jbw = (0)$ or $(Jbw)^- = (Jw)^-$.

Let y be any element of B such that $w = zy$. Then z and y satisfy condition (2). They also satisfy condition (3), since, if $b \in B$, then $zbw = zbzy \in F$, and therefore, $(Jzbzy)^- = (Jzy)^-$ or $Jzbzy = (0)$.

To prove that z also satisfies condition (4), let b be any element of $J\cap B$ such that $bz \notin P$. By condition (1), $(Jbz)^- = (Jz)^-$. But $(J\cap B)^-$ is a closed 2-sided ideal of A, and therefore, $(Jz)^- = (Jbz)^- \subseteq ((J\cap B)^- z)^- = ((J\cap B)z)^- \subseteq (Jz)^-$. Thus $(Jz)^- = ((J\cap B)z)^-$, and this completes the proof of the lemma.

$J \cap B$ is a 2-sided ideal of B. By lemma 3.12 and Definition 3.6, the prime radical L of $J \cap B$ is therefore equal to the intersection of $J \cap B$ with all the prime ideals of B. It follows that if $J \cap B$ is contained in every prime ideal of B, then $J \cap B$ is equal to L and, by lemma 3.11, therefore contains no non-nilpotent elements.

<u>8.2 Theorem</u>. Let J be a B-separating ideal of A and let P be a prime ideal of B such that $J \cap B \not\subseteq P$. Then there exists a closed 2-sided ideal N of A such that $N \cap B$ is a prime ideal of B contained in P.

<u>Proof</u>. Let z in $B \setminus P$ and y in B satisfy conditions (1) to (4) of lemma 8.1, and let $N = \{a \in A : JzAaJzy = (0)\}$. Then N is clearly a closed 2-sided ideal of A. Let b be any element of $B \setminus N$. Then, since B is dense in A and $(J \cap B)z$ is dense in Jz, there exist elements c of B and d of $J \cap B$ such that $Jzcbdzy \neq (0)$. But then, by condition (3) of lemma 8.1, $(Jzy)^- = (Jzcbdzy)^- \subseteq (JzAbJzy)^- \subseteq (Jzy)^-$. This proves that, if $b \in B$, then b is in $B \setminus N$ if and only if $(Jzy)^- = (JzAbJzy)^-$.

To prove that $N \cap B$ is a prime ideal of B, let a and b be elements of B such that $aBb \subseteq N \cap B$ and $b \in B \setminus N$. Then $(JzAaJzy)^- = (JzAa(JzAbJzy)^-)^- = (JzAa(Jz)^-\bar{B}bJzy)^-$. By condition (4) of lemma 8.1, we have $(Jz)^- = ((J \cap B)z)^-$. Thus $(JzAaJzy)^- = (JzAa((J \cap B)z)^-\bar{B}bJzy)^- = ((JzAa(J \cap B)zBbJzy)^- \subseteq (JzAaBbJzy)^- = (0)$, and a is therefore in $N \cap B$.

Now let b be any element of $N \cap B$. If b is not in P then $zBb(J \cap B)z \not\subseteq P$, and there are therefore elements c of B and d of $J \cap B$ such that $zcbdz \notin P$. But then, by condition (1) of lemma 8.1, $(Jzy)^- = (Jzcbdzy)^- \subseteq (JzAbJzy)^- = (0)$, contradicting condition (2) of lemma 8.1. This proves that $N \cap B$ is contained in P and so completes the proof of the Theorem.

Recall that P is a minimal prime ideal if it does not strictly contain any other prime ideal, and that, by lemma 3.14, every prime ideal contains a minimal prime ideal.

<u>8.3 Corollary</u>. Let P be a minimal prime ideal of B such that $J \cap B \not\subseteq P$. Then $P = \overline{P} \cap B$.

<u>Proof</u>. Let P be a minimal prime ideal of B such that $J \cap B \not\subseteq P$. Then, by Theorem 8.2, there exists a closed 2-sided ideal N of A such that $N \cap B$ is a prime ideal of B and $N \cap B \subseteq P$. But then, $N \cap B = P$, and therefore $P \subseteq \overline{P} \cap B \subseteq N \cap B = P$.

<u>8.4 Corollary</u>. If J is a separating ideal of A (i.e. $A = B$), and P is a minimal prime ideal of A such that $J \not\subseteq P$, then P is automatically closed.

<u>8.5 Theorem</u>. Let J be a B-separating ideal of A and let L be the prime radical of $J \cap B$. Then $\overline{L} \cap B = L$.

<u>Proof</u>. Let $K = \cap \{J \cap \overline{P} : P$ is a prime ideal of $B\}$. Then L is contained in $K \cap B$ and K is closed. If $b \in K \cap B \setminus L$, then there is a prime ideal N of B such that $b \notin N$, and therefore a minimal prime ideal P of B such that $b \in J \cap B \setminus P$. By Corollary 8.3, $K \cap B \subseteq J \cap \overline{P} \cap B = J \cap P$, and therefore b is in P. This contradiction proves that $K \cap B = L$. It follows that $\overline{L} \cap B \subseteq K \cap B = L \subseteq \overline{L} \cap B$, and that L is therefore equal to $\overline{L} \cap B$.

The commutative case of Theorem 8.5 is essentially contained in Corollary 11.6 of [79]. Note that since L is a 2-sided ideal of B, its closure $\bar{L}$ is a 2-sided ideal of A.

<u>8.6 Corollary</u>. The prime radical of a separating ideal is a closed 2-sided nilpotent ideal.

<u>Proof</u>. In this case, $B = A$ and so $L = \bar{L}$. By lemma 3.11, L is nil. By Theorem 3.21, it is therefore nilpotent.

<u>8.7 Theorem</u>. Let J be a B-separating ideal of A, let L be the prime radical of $B \cap J$, and let Q be the natural homomorphism of A onto $A/\bar{L}$. Then QJ is a QB-separating ideal of $A/\bar{L}$, and $QB \cap QJ$ is isomorphic to $B \cap J/L$ and is therefore semi-prime.

<u>Proof</u>. Since $\bar{L} \subseteq J$, QJ is closed, and, by lemma 7.6, is therefore a QB-separating ideal of $A/\bar{L}$. Define $T : B \cap J/L \longrightarrow QB \cap QJ$ by

$$T(b + L) = Qb \qquad (b \in B \cap J).$$

Then T is well-defined, since $L \subseteq \bar{L}$, and is a homomorphism. It is also onto since $QB \cap QJ = Q((B + L) \cap (J + L)) = Q((B + L) \cap J) = Q(B \cap J)$. If $b \in B \cap J$, and $T(b + L) = 0$, then $b \in B \cap \bar{L} = L$, by Theorem 8.5. Thus T is a monomorphism and $QB \cap QJ$ is therefore isomorphic to $B \cap J/L$. By lemma 3.7, $B \cap J/L$ is semi-prime, and $QB \cap QJ$ is therefore semi-prime.

<u>8.8 Theorem</u>. Let J be a B-separating ideal of A such that $J \cap B$ is semi-prime, and let P be a minimal prime ideal of B such that $J \cap B \not\subseteq P$. Then there exists a unique closed 2-sided ideal I of A satisfying the following conditions :

(1) I is contained in J

(2) $(I \cap B)^{-} = I \neq (0)$

(3) If K is any closed 2-sided ideal of A such that K is contained in I, then either $K \cap B = (0)$ or $K = I$.

(4) $P = \{b \in B : bI = (0)\}$.

Conversely, if I is any closed 2-sided ideal of A satisfying conditions (1) to (3), then the ideal P defined by condition (4) is a minimal prime ideal of B with $J \cap B \not\subseteq P$.

<u>Proof</u>. The main statement of the Theorem is that each minimal prime ideal P of B with $J \cap B \not\subseteq P$ is the intersection with B of the left annihilator of a closed 2-sided ideal of A satisfying conditions (1) to (3). The rest of the Theorem follows from this alone, by routine arguments similar to some of the arguments used in [81].

Let P be a minimal prime ideal of B such that $J \cap B \not\subseteq P$. Then it follows from the proofs of Theorem 8.2 and Corollary 8.3 (with $N \cap B = P$), that there exist elements z of $B \setminus P$ and y of B satisfying conditions (1) to (4) of lemma 8.1, and such that

$P = \{b \in B : JzAbJzy = (0)\} = \{b \in B : (JzAbJzy)^{-} \neq (Jzy)^{-}\}$.

Let $I = (\mathrm{span}(JzyJ))^{-}$. Then I is a closed 2-sided ideal of A contained in J.

To prove that I satisfies condition (2), note that $(I \cap B)^{-}$ is in any case a closed 2-sided ideal of A, and that $(J \cap B)zy$ is a non-zero, and therefore non-nilpotent, left ideal of $J \cap B$.

Thus $I \cap B \supseteq (J \cap B)zy(J \cap B)zy \neq (0)$, and therefore, $((J \cap B)zy(J \cap B)zy)^{-} = (Jzy(J \cap B)zy)^{-} = (Jzy)^{-}$, by (3) and (4) of lemma 8.1. But then, $(I \cap B)^{-} \supseteq ((I \cap B)^{-} J)^{-} \supseteq ((Jzy)^{-} J)^{-} =$

$(JzyJ)^{-}$, and therefore $I = (I \cap B)^{-}$ and $I \neq (0)$, as required.

To prove that $\{b \in B : bI = (0)\}$ is contained in P, let b be any element of B not in P. Then $(JzAbI)^{-} \supseteq (JzAbJzyJ)^{-} = (JzyJ)^{-} \neq (0)$, and therefore, $bI \neq (0)$. Now let b be in P. Then $(0) = JzAbJzy = JzAbJzyJ \supseteq (JzyJ)b(JzyJ)$, and therefore $IbI = (0)$. But then, $b(I \cap B)$ is a nilpotent right ideal of $J \cap B$, and therefore $bI = (b(I \cap B))^{-} = (0)$, by lemma 3.5. This proves that $P = \{b \in B : bI = (0)\}$.

To prove that I satisfies condition (3), let K be a closed 2-sided ideal of A such that $K \cap B \neq (0)$ and $K \subseteq I$. Then, since $K \cap B$ is not nilpotent, there exists an element b of $K \cap B$ such that $b(K \cap B) \neq (0)$. But then, $b \notin P$, and therefore, $K \supseteq ((JzAbJzy)^{-}J)^{-} = ((Jzy)^{-}J)^{-} = (JzyJ)^{-}$, and so $K = I$.

To prove the uniqueness of I, let I' be any other closed 2-sided ideal of A satisfying conditions (1) to (4). Suppose $(I' \cap B)(I \cap B) = (0)$. Then $I' \cap B \subseteq P$, and therefore $(I' \cap B)^2 = (0)$. But then, $I' \cap B$ is a non-zero nilpotent 2-sided ideal of $B \cap J$, which, by lemma 3.5, is impossible. We therefore have $(I'I) \cap B \supseteq (I' \cap B)(I \cap B) \neq (0)$, and therefore, by condition (3), $I = (\mathrm{span}(I'I))^{-} = I'$.

Finally, let I be a closed 2-sided ideal of A satisfying conditions (1) to (3), and let $P = \{b \in B : bI = (0)\}$. Then, since $I \cap B$ is not contained in P, $J \cap B$ is not contained in P. Let b be in $B \setminus P$. Then $(b(I \cap B))^2 \neq (0)$, and therefore, $Bb(I \cap B) \neq (0)$. By condition (3), I is then equal to $(\mathrm{span}(BbI))^{-}$, and if a is in B, and $aBb \subseteq P$, then $aBbI \subseteq PI = (0)$, and therefore $aI = (0)$ and a is in P. This proves that P is a prime ideal of B. If N

is any prime ideal of B contained in P, then $P(I \cap B) \subseteq PI = (0) \subseteq N$, and therefore $P \subseteq N$ or $I \subseteq B \cap N$. If $I \subseteq B \cap N$, then $(I \cap B)^2 \subseteq NI \subseteq PI = (0)$, which is impossible. Thus $P \subseteq N$ and therefore $P = N$. This proves that P is a minimal prime ideal of B, and so completes the proof of the Theorem.

Theorem 8.8 establishes a 1:1 correspondence between the set $X(J)$ of all minimal prime ideals P of B such that $J \cap B$ is not contained in P, and the set $Y(J)$ of all the closed 2-sided ideals I of A satisfying conditions (1) to (3). If (0) is a prime ideal of B, then $X(J)$ clearly contains only one element. In Section 10, we show that, if B is equal to A, then the sets $X(J)$ and $Y(J)$ are necessarily finite. It is not clear whether or not this is true in general, when B is not equal to A (see Theorem 9.9).

9. Applications to homomorphisms with dense range.

Let h be a homomorphism from a Banach algebra C into a Banach algebra A. By replacing A with the closure of the range of h, if necessary, we may assume that $B = h(C)$ is a dense subalgebra of A and, by Corollary 7.3, that the separating space $\mathfrak{S}(h)$ of h is therefore a B-separating ideal of A. The results of the previous section may therefore be applied to obtain information about discontinuous homomorphisms, although they are too weak, by themselves, to yield any new automatic continuity results, except for epimorphisms, which are considered in Section 11.

As stated in the Introduction, most of the interest in discontinuous homomorphisms which are not onto has centred on the case when the domain algebra C is a C*-algebra.

A C*-algebra is a complex Banach algebra C with an involution $*$ such that the following conditions are satisfied :

(1) $(a + b)^* = a^* + b^* \qquad (a, b \in C)$

(2) $(za)^* = \bar{z}a^* \qquad (z \in \mathbb{C}, a \in C)$

(3) $(ab)^* = b^*a^* \qquad (a, b \in C)$

(4) $(a^*)^* = a \qquad (a \in C)$

(5) $\|a^*a\| = \|a\|^2 \qquad (a \in C)$.

A commutative C*-algebra C is isometrically isomorphic to the Banach algebra $C_0(X)$ of all continuous complex-valued functions vanishing at infinity on some locally compact Hausdorff space X, with the norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\} \qquad (f \in C_0(X))$$

and the involution

$$f^*(x) = \overline{f(x)} \qquad (x \in X, f \in C_0(X)).$$

X is compact if and only if C has an identity element, in which case $C_0(X)$ is equal to the algebra $C(X)$ of all continuous complex-valued functions on X [9, p.189].

The main purpose of this section is to extend to the non-commutative case some results which are proved in the commutative case in [78] (see also [79]). We begin by stating, for purposes of comparison, some known results concerning discontinuous homomorphisms from commutative C*-algebras (Theorems 9.1 and 9.2).

<u>9.1 Theorem</u> [78, Theorem 4.2, p.172].

Let X be a compact Hausdorff space and suppose that there is a discontinuous homomorphism h from $C(X)$ onto a dense subalgebra of a Banach algebra A. Then there is a closed ideal M in A such that $k : C(X) \longrightarrow A/M$, defined by $k(f) = h(f) + M$, is a discontinuous homomorphism whose kernel is a prime ideal in $C(X)$.

<u>9.2 Theorem</u> [78, Theorem 4.3 (iii), p.173].

Let Y be a locally compact Hausdorff space. If h is a discontinuous homomorphism from $C_0(Y)$ onto a dense subalgebra of a radical Banach algebra A, then for each c in $C_c(Y)$ with $h(c) \neq 0$ there is a d in $C_0(Y)$ such that Qh is a discontinuous homomorphism whose kernel is a prime ideal in $C_0(Y)$, and $Qh(c) \neq 0$, where Q is the natural quotient map from A onto A/J and $J = \{a \in A : ah(d)A = (0)\}$.

Theorems 9.1 and 9.2 are proved in [78] by combining information obtained from lemma 6.4 about discontinuous homomorphisms from arbitrary Banach algebras with more precise information available from other sources about discontinuous homomorphisms from C*-algebras. This is also the procedure adopted here. The following four results have been extracted, for ease of reference, from fairly extensive studies of the properties of discontinuous homomorphisms from C*-algebras, made in the commutative case by Bade and Curtis [4], and in the non-commutative case by Johnson [47] and Sinclair [77] (see also [79]).

<u>9.3 Proposition</u> [77, Theorem 4.1, p.448].

Let h be a discontinuous homomorphism from a C*-algebra C onto a dense subalgebra B of a Banach algebra A. Then there exists a closed 2-sided ideal M of C, and a discontinuous homomorphism h_o from M into A such that

(1) M has finite co-dimension in C and does not contain the identity element, if any, of C

(2) $h_o(M)$ is a dense subalgebra of the separating space $\mathfrak{S}(h)$ of h, and $\mathfrak{S}(h) = \mathfrak{S}(h_o)$

(3) The Jacobson radical of A is contained in $\mathfrak{S}(h)$.

<u>9.4 Corollary</u> [79, Corollary 12.4, p.69].

If a unital C*-algebra C has no proper closed 2-sided ideals of finite co-dimension, then every homomorphism from C into a Banach algebra is continuous.

<u>9.5 Proposition</u> [78, Lemma 4.1, p.172].

Let h be a homomorphism from a C*-algebra C into a Banach algebra, and let c be any hermitian element of C.

Then either $h(c) = 0$ or $h(c)$ is non-nilpotent. (c is <u>hermitian</u> if and only if $c = c^*$).

<u>9.6 Corollary.</u> Let h be a homomorphism from a C^*-algebra C into a Banach algebra A. Then $h(C)$ is a semi-prime algebra, or equivalently, the kernel of h is a semi-prime ideal of C.

<u>Proof.</u> Let I be a nil left ideal of $h(C)$ and let c be any element of C such that $h(c)$ is in I. Then $h(c^*c) = h(c^*)h(c) \in I$, and so $h(c^*c)$ is nilpotent. Since c^*c is hermitian, Proposition 9.5 implies that $h(c^*c) = 0$. Let $|c| = (c^*c)^{\frac{1}{2}}$. By the general polar decomposition for C^*-algebras [11], there is an element u of C such that $c = u|c|^{\frac{1}{2}}$. Since $|c|^{\frac{1}{2}}$ is hermitian, and $h(|c|^{\frac{1}{2}})^4 = h(c^*c) = 0$, it follows that $h(|c|^{\frac{1}{2}}) = 0$, and therefore that $h(c) = h(u|c|^{\frac{1}{2}}) = 0$. Thus $I = (0)$ and, by lemma 3.5, $h(C)$ is semi-prime.

<u>9.7 Theorem.</u> Let h be a discontinuous homomorphism from a Banach algebra C onto a dense subalgebra B of a Banach algebra A, and let c be in C. If $h(c)$ is in the separating space $\mathfrak{S}(h)$ of h, but not in the prime radical of $\mathfrak{S}(h) \cap B$, then there exists a discontinuous homomorphism k from C into a Banach algebra such that the kernel of k is a prime ideal of C, and $k(c) \neq 0$.

<u>Proof.</u> Let P be a minimal prime ideal of B such that $h(c)$ is not in P. Then $\overline{P}$ is a closed 2-sided ideal of A and, by Corollary 8.3, $P = \overline{P} \cap B$. In particular, $h(c)$ is not in $\overline{P}$. Let Q be the natural quotient mapping of A onto $A/\overline{P}$, and let $k = Qh$. Then $k(c) = Qh(c) \neq 0$ and, by lemma 6.2, $\mathfrak{S}(k) = (Q\mathfrak{S}(h))^- \neq (0)$. By lemma 6.1, k is therefore discontinuous.

To complete the proof, note that $\mathrm{Ker}(k) = \{d \in C : h(d) \in \overline{P}\}$ $= h^{-1}(B \cap \overline{P}) = h^{-1}(P)$, which is a prime ideal of C, by lemma 3.2.

Theorem 9.7 applies in particular when $h(c)$ is a non-nilpotent element of $\mathfrak{S}(h)$, and in this case is the exact non-commutative analogue of [78, Theorem 3.3, p.170]. An alternative proof may be obtained by defining $k : C \longrightarrow A/N$ by $k(d) = h(d) + N$, where in the notation of Theorem 8.2, $N = \{a \in A : \mathfrak{S}(h)zAa\,\mathfrak{S}(h)zy = (0)\}$. The kernel $h^{-1}(N)$ of k is again a prime ideal of C not containing c.

We now apply Theorem 9.7 to the case when C is a C*-algebra. For the commutative case, see Theorem 9.1.

<u>9.8 Theorem</u>. Let C be a C*-algebra, and suppose there exists a discontinuous homomorphism from C into a Banach algebra. Then there exists a closed 2-sided ideal M of C and a prime ideal P of M, such that

(1) M is of finite co-dimension in C

(2) P is the kernel of a discontinuous homomorphism k from M into a Banach algebra.

<u>Proof</u>. By Proposition 9.3, there exists a closed 2-sided ideal M of finite co-dimension in C, and a discontinuous homomorphism h_o from M into a Banach algebra A such that $A = \mathfrak{S}(h_o) = \overline{B}$, where $B = h_o(C)$. By Corollary 9.6, B is semi-prime. Thus, by Proposition 3.9, the prime radical of $\mathfrak{S}(h_o) \cap B$, which is just the prime radical of B, is the zero ideal. Since h_o is discontinuous, there must be elements c of M such that $h_o(c) \neq 0$. By Theorem 9.7, there is therefore a

discontinuous homomorphism k from M into a Banach algebra such that the kernel of k is a prime ideal of M. This completes the proof of the Theorem.

The following Theorem is obtained directly from Theorem 8.8. Note that it applies in its entirety to the discontinuous homomorphism h_o of Proposition 9.3 and the proof of Proposition 9.8.

<u>9.9 Theorem</u>. Let h be a discontinuous homomorphism from a Banach algebra C onto a dense subalgebra B of a Banach algebra A, and suppose that B is semi-prime and that $\mathfrak{S}(h) = A$.

Let $X(h)$ be the set of all prime ideals P of C such that P is minimal over the kernel of h (see Definition 3.13), and let $Y(h)$ be the set of all closed 2-sided ideals I of A such that

(1) $(B \cap I)^{-} = I \neq (0)$

(2) If K is any closed 2-sided ideal of A such that K is contained in I, then either $K \cap B = (0)$ or $K = I$.

For each I in $Y(h)$, let

$$P(I) = \{c \in C : h(c)I = (0)\}.$$

Then $P(I)$ is in $X(h)$, and the map $I \rightarrow P(I)$ is a bijection of $Y(h)$ onto $X(h)$. For each I in $Y(h)$, let

$$J(I) = \{a \in A : aI = (0)\},$$

and let $Q(I)$ denote the natural mapping of A onto $A/J(I)$. Then $Q(I)h$ is a discontinuous homomorphism with kernel $P(I)$.

<u>Proof</u>. Let P be any element of $X(h)$. Then, since B is isomorphic to $C/\mathrm{Ker}(h)$ and is semi-prime and non-zero, $h(P)$

is a minimal prime ideal of B such that $\mathfrak{S}(h) \cap B = B \not\subseteq h(P)$. Conversely, if N is any minimal prime ideal of B, then there is a unique element P of $X(h)$ such that $h(P) = N$. All of the statements of the Theorem, other than the discontinuity of the homomorphisms $Q(I)h$, therefore follow directly from Theorem 8.8, on taking $J = \mathfrak{S}(h) = A$.

Now let I be in $Y(h)$. To prove that $Q(I)h$ is discontinuous, note that since B is semi-prime and $I \cap B \neq (0)$, we have $\mathfrak{S}(h)I \supseteq (I \cap B)^2 \neq (0)$. Thus, $\mathfrak{S}(h) \not\subseteq J(I)$, and, by lemma 6.2, $\mathfrak{S}(Q(I)h) = (Q(I)\mathfrak{S}(h))^{-} \neq (0)$, and $Q(I)h$ is therefore discontinuous.

Note that by Theorem 8.7, if B is not semi-prime and is not equal to its prime radical L, and if $\mathfrak{S}(h) = A$, then Theorem 9.9 may be applied to the homomorphism Qh, where Q is the natural mapping of A onto $A/\bar{L}$. In this case, Ker(Qh) is the intersection of all the prime ideals of C containing Ker(h).

It appears to be still an open question, even when C is a commutative C*-algebra (without identity) and A is radical, whether or not the sets $X(h)$ and $Y(h)$ are always finite (see [79, Theorem 11.7]).

The following Theorem is an immediate consequence of Theorem 9.9 and Proposition 9.3, and is a non-commutative version of Theorem 9.2.

9.10 Theorem. Let C be a C*-algebra without identity. If h is a discontinuous homomorphism from C onto a dense subalgebra B of a radical Banach algebra A, then for each c in C

with $h(c) \neq 0$, there exists a closed 2-sided ideal I of A such that Qh is a discontinuous homomorphism whose kernel is a prime ideal in C, and $Qh(c) \neq 0$, where Q is the natural quotient map from A onto A/J and $J = \{a \in A : aI = (0)\}$.

Proof. By Proposition 9.3 (3), $\mathfrak{S}(h)$ is equal to A. By Corollary 9.6, $h(C) = B$ is semi-prime. Thus, if c is in C and $h(c) \neq 0$, then there is a prime ideal P of C minimal over the kernel of h such that $c \notin P$. Apply Theorem 9.9.

It appears to be still an open question whether or not there exists a discontinuous homomorphism from a C*-algebra, or any other Banach algebra, onto a dense subalgebra of a semi-simple Banach algebra. The final Proposition of this section is only intended to show that in this case the construction of discontinuous homomorphisms with prime kernels is very much more straightforward, and probably therefore less significant.

9.11 Proposition. Let A be a semi-simple Banach algebra, and let h be a discontinuous homomorphism from a Banach algebra C onto a dense subalgebra B of A. Then there exists a primitive ideal P of A such that if Q is the natural map of A onto the Banach algebra A/P, then Qh is a discontinuous homomorphism with prime kernel.

Proof. Let P be any primitive ideal of A such that $\mathfrak{S}(h)$ is not contained in P, and let Q be the natural map of A onto A/P. By lemma 6.2, $\mathfrak{S}(Qh) \neq (0)$, and Qh is therefore discontinuous. By lemma 3.16 and Corollary 2.21, P is a closed prime ideal of A, and $B \cap P$ is therefore a prime ideal of B. But then, $\operatorname{Ker}(Qh) = h^{-1}(P)$ is a prime ideal of C.

10. The structure of non-nilpotent separating ideals.

The remaining sections of this chapter are concerned entirely with separating ideals of a Banach algebra A (i.e. with the case $A = B$ of Definition 7.1). This section begins with a lemma concerning sequences of mutually orthogonal elements of a separating ideal. This is used in Theorem 10.3 to strengthen the information available from Section 8. In the simplest case, when A is semi-prime, we show that if J is a non-zero separating ideal of A, then it contains minimal-closed 2-sided ideals $M_1, \ldots\ldots, M_n$ of A such that if $P_i = \{a \in A : aM_i = (0)\}$, for $i = 1, \ldots\ldots, n$, then $P_1, \ldots\ldots, P_n$ are minimal prime ideals of A such that $P_1 \cap \ldots\ldots \cap P_n \cap J = (0)$. For each i, M_i is either finite-dimensional, in which case P_i is primitive, or M_i is a topologically simple radical Banach algebra, in which case P_i is not primitive (Theorem 10.9). Using the fact that the prime radical of a separating ideal is closed, the structure of non-nilpotent separating ideals in general can be described in similar terms.

<u>10.1 Lemma</u>. Let J be a separating ideal of a Banach algebra A, and let $\{b_n\}$ be a sequence in J such that $b_n b_m = 0$ when $n \neq m$. Then there exists a natural number N such that $b_n^{N+2} = 0$ for all $n \geqslant N$.

<u>Proof</u>. By replacing b_n by $2^{-n}\|b_n\|^{-1} b_n$ when $b_n \neq 0$, we may assume that $\|b_n\| \leqslant 2^{-n}$, and may therefore define a_n in J by

$$a_n = \sum_{j=n}^{\infty} b_j \qquad (n = 1, 2, \ldots\ldots).$$

We show by induction that

$$a_n \cdots\cdots a_1 = \sum_{j=n}^{\infty} b_j^n \qquad (n = 1, 2, \ldots\ldots) \qquad (1).$$

This is clearly true for $n = 1$. If it is true for some $n \geqslant 1$, then

$$a_{n+1} a_n \cdots\cdots a_1 = \sum_{j=n}^{\infty} a_{n+1} b_j^n = \sum_{j=n}^{\infty}\left(\sum_{k=n+1}^{\infty} b_k b_j^n\right) = \sum_{j=n+1}^{\infty} b_j^{n+1},$$

since $b_k b_j^n$ is only non-zero when $k = j$. Thus (1) is true for all n. Since J is a separating ideal, there is a natural number N such that

$$(Ja_n \cdots\cdots a_1)^- = (Ja_N \cdots\cdots a_1)^- \qquad (n \geqslant N).$$

Let $n \geqslant N$. Then $b_n a_N \cdots\cdots a_1 b_n = b_n\left(\sum_{j=N}^{\infty} b_j^N\right) b_n = b_n^{N+2}$, and

$a_{n+1} \cdots\cdots a_1 b_n = \sum_{j=n+1}^{\infty} b_j^{n+1} b_n = 0$. But then, $b_n^{N+2} = b_n a_N \cdots\cdots a_1 b_n \in (Ja_N \cdots\cdots a_1)^- b_n = (Ja_{n+1} \cdots\cdots a_1)^- b_n = (0)$. This completes the proof of the lemma.

The important point in the conclusion of lemma 10.1 is that b_n is nilpotent for sufficiently large n.

<u>10.2 Definition</u>. Let I be a closed 2-sided ideal of a Banach algebra A. Then I is a <u>minimal-closed 2-sided ideal</u> of A if it does not properly contain any non-zero closed 2-sided ideal of A.

If, in Theorem 8.8, $A = B$, then the closed 2-sided ideals I of A satisfying conditions (1) to (3) are precisely the non-zero minimal-closed 2-sided ideals of A contained in J. This fact is used in the following theorem.

10.3 Theorem. Let J be a non-nilpotent separating ideal of a Banach algebra A, and let L be the prime radical of A. Then $L\cap J$ is the prime radical of J and is a closed nilpotent ideal. There exist closed prime ideals $P_1, \ldots\ldots, P_n$ of A and closed 2-sided ideals $M_1, \ldots\ldots, M_n$ of A such that the following conditions are satisfied :

(1) $L\cap J \subsetneq M_i \subseteq J \qquad (i = 1, \ldots\ldots, n)$

(2) $M_i / L\cap J$ is a non-nilpotent minimal-closed 2-sided ideal of $A/L\cap J \qquad (i = 1, \ldots\ldots, n)$

(3) $M_i M_j \subseteq L\cap J \qquad (i, j = 1, \ldots\ldots, n, i = j)$

(4) P_i is a minimal prime ideal of A and $J \not\subseteq P_i \qquad (i = 1, \ldots\ldots, n)$

(5) $P_i = \{a \in A : aM_i \subseteq L\cap J\}$ and $M_i \cap P_i = L\cap M_i = L\cap J$ $(i = 1, \ldots\ldots, n)$

(6) If P is any prime ideal of A, then either $J \subseteq P$ or $P_i \subseteq P$ for some i

(7) $J\cap P_1 \cap \ldots\ldots \cap P_n = L\cap J$

(8) If K is any closed 2-sided ideal of A such that $L\cap J \subsetneq K \subseteq J$, then $M_i \subseteq K$ for some i

(9) M_i is a separating ideal of $A \qquad (i = 1, \ldots\ldots, n)$.

Proof. By lemma 3.12, $L\cap J$ is the prime radical of J. By Corollary 8.6, $L\cap J$ is therefore a closed nilpotent ideal of A. Since J is not nilpotent, $L\cap J$ is not equal to J. By Theorem 8.7, $J/L\cap J$ is therefore a non-zero separating ideal of $A/L\cap J$ and is a semi-prime algebra. By Theorem 8.8, the set Y of all non-zero minimal-closed 2-sided ideals I of $A/L\cap J$ such that $I \subseteq J/L\cap J$ is not empty, and, by lemma 3.5, every element of Y is non-nilpotent. Suppose Y is not finite, and let $\{I_n\}$ be a sequence of distinct elements of Y.

If $I_n I_m \neq (0)$, then $I_n = (\mathrm{span}(I_n I_m))^- = I_m$, and therefore $n = m$. Thus if $\{b_n\}$ is any sequence in $J/L\cap J$ such that b_n is in I_n for all n, then $b_n b_m = 0$ for $n \neq m$. It follows from lemma 10.1 that I_n must be nil for sufficiently large n. But, for each n, I_n is not nilpotent and, by Theorem 3.21, is therefore not nil. Y must therefore be finite.

Let $I_1, \ldots\ldots, I_n$ be the distinct elements of Y and let

$$M_i = \{a \in A : a + L\cap J \in I_i\} \qquad (i = 1, \ldots\ldots, n).$$

Then each M_i is a closed 2-sided ideal of A and conditions (1) to (3) are clearly satisfied.

For each i, let $N_i = \{b \in A/L\cap J : bI_i = (0)\}$. By Theorem 8.8, each N_i is a minimal prime ideal of $A/L\cap J$ such that $J/L\cap J \not\subseteq N_i$. Let

$P_i = \{a \in A : a + L\cap J \in N_i\} = \{a \in A : aM_i \subseteq L\cap J\}$ $(i = 1, \ldots\ldots, n)$. Then, for each i, $J \not\subseteq P_i$ and, by lemma 3.2, P_i is a minimal prime ideal over $L\cap J$. But $L\cap J$ is contained in every prime ideal of A, and P_i is therefore a minimal prime ideal of A.

To prove that $M_i \cap P_i = L\cap M_i = L\cap J$, it is sufficient to prove that $M_i\cap P_i \subseteq L\cap J$, since the inclusions $L\cap J \subseteq L\cap M_i \subseteq M_i \cap P_i$ are obvious. If a is in $M_i \cap P_i$, then $aM_i \subseteq L\cap J$, and therefore a is in $L\cap J$. This proves that condition (5) is satisfied.

To prove that condition (6) is satisfied, let P be any prime ideal of A such that $J \not\subseteq P$, and let N be a minimal prime ideal of A such that $N \subseteq P$. Then $N/L\cap J$ is a minimal prime ideal of $A/L\cap J$ such that $J/L\cap J \not\subseteq N/L\cap J$, and, by Theorem 8.8, is therefore equal to N_i for some i. N is therefore equal to P_i for some i.

Condition (7) follows immediately from condition (6).

Let K be any closed 2-sided ideal of A such that $L \cap J \subsetneq K \subseteq J$. Then K is not contained in P_i for all i, and therefore $(K/L\cap J)I_i \neq (0)$ for some i. But then $I_i =$ $(\mathrm{span}((K/L\cap J)I_i))^- \subseteq K/L\cap J$, and M_i is therefore contained in K. This proves that condition (8) is satisfied.

To prove that M_i is a separating ideal, note that $M_i =$ $(\mathrm{span}(M_i J))^-$. Thus, if $\{a_n\}$ is any sequence in A, and

$$(Ja_n \cdots a_1)^- = (Ja_N \cdots a_1) \qquad (n \geqslant N),$$

then $(M_i a_N \cdots a_1)^- \subseteq (\mathrm{span}(M_i(Ja_N \cdots a_1)^-))^- =$ $(\mathrm{span}(M_i(Ja_n \cdots a_1)^-))^- \subseteq (M_i a_n \cdots a_1)^- \subseteq (M_i a_N \cdots a_1)^-$ for all $n \geqslant N$, and therefore

$$(M_i a_n \cdots a_1)^- = (M_i a_N \cdots a_1)^- \qquad (n \geqslant N).$$

This completes the proof of the Theorem.

10.4 Definition. A Banach algebra A is topologically simple if $A^2 \neq (0)$ and A is a minimal-closed 2-sided ideal of itself.

If A is a minimal-closed 2-sided ideal of itself and $A^2 = (0)$, then every closed linear subspace of A is a 2-sided ideal. Thus $A = (0)$ or A is 1-dimensional, and the condition $A^2 \neq (0)$ therefore only excludes these trivial cases.

10.5 Lemma. Let A be a topologically simple Banach algebra. Then A is a prime algebra, and is either primitive or radical.

Proof. Let $N = \{a \in A : Aa = (0)\}$. Then N is a closed

2-sided ideal of A and therefore $N = (0)$. Let a and b be any elements of A such that $aAb = (0)$, and let $K = \{c \in A : cAb = (0)\}$. Then K is a closed 2-sided ideal of A, and a is in K. If $a \neq 0$, then $K = A$ and therefore $Ab = (\mathrm{span}(A^2))^{-}b$ $= (\mathrm{span}(KAb))^{-} = (0)$, and so $b = 0$. This proves that A is a prime algebra.

Now suppose A has a primitive ideal P. Then P is a proper closed 2-sided ideal of A and so $P = (0)$ and A is therefore primitive. On the other hand, if A has no primitive ideals then it is equal to its Jacobson radical.

Let I be a non-nilpotent minimal-closed 2-sided ideal of a Banach algebra A and let $N = \{a \in I : IaI = (0)\}$. Then N is a closed 2-sided ideal of A, and therefore $N = (0)$. It follows that if K is any non-zero closed 2-sided ideal of I, then $IKI \neq (0)$, and therefore $I = (\mathrm{span}(IKI))^{-} = K$. This proves that I is a topologically simple Banach algebra. The ideals $M_1/L\cap J, \ldots\ldots, M_n/L\cap J$ in Theorem 10.3 are therefore topologically simple Banach algebras. In Theorem 10.9 we show that each $M_i/L\cap J$ is either finite-dimensional or radical. The proof of this result requires two lemmas, the first of which is taken from [51].

<u>10.6 Lemma</u> [51]. Let A be a Banach algebra and let X be an irreducible left Banach A-module. Let D be the centralizer of A on X and let $x_0, x_1, \ldots\ldots$ in X be linearly independent over D. Then there exists an element c of A such that $c.x_0 = 0$ and $c.x_1, c.x_2, \ldots\ldots$ are linearly independent over D.

Proof. Using lemma 2.15, choose b_1 in A such that $b_1.x_0 = 0$, $b_1.x_1 \neq 0$ and $\|b_1\| \leqslant \frac{1}{2}$. Then, using Theorem 2.16, choose $b_2, b_3, \ldots$ in A so that for all $i \geqslant 2$,

(1) $b_i.x_0 = b_i.x_1 = \ldots = b_i.x_{i-1} = 0$

(2) $b_i.x_i$ is not a linear combination over D of $c_i.x_1, \ldots, c_i.x_i$ where $c_i = b_1 + \ldots + b_{i-1}$.

(3) $\|b_i\| \leqslant 2^{-i}$.

Let $c = \sum_{n=1}^{\infty} b_n$. Then $c.x_0 = 0$,

$c.x_i = c_i.x_i + b_i.x_i \qquad (i = 1, 2, \ldots)$,

and, for $0 < j < i$, $c.x_j = c_i.x_j$. Suppose $c.x_1, c.x_2, \ldots$ are not linearly independent over D. Then there exists N and elements $T_1, \ldots, T_N$ of D such that $c.x_{N+1} = T_1(c.x_1) + \ldots + T_N(c.x_N) = T_1(c_{N+1}.x_1) + \ldots + T_N(c_{N+1}.x_N)$. But $b_{N+1}.x_{N+1}$ is then a linear combination over D of $c_{N+1}.x_1, \ldots, c_{N+1}.x_{N+1}$, contrary to the choice of b_{N+1}. Thus $c.x_1$, $c.x_2, \ldots$ are linearly independent over D, and the proof is complete.

10.7 Corollary. Let A be a Banach algebra and let X be an infinite-dimensional irreducible left Banach A-module. Then there exists an element a of A such that $a.X$ is infinite-dimensional over the centralizer D of A on X.

Proof. This follows immediately from lemma 10.6, and the fact that, by lemma 2.24, D is finite-dimensional over the scalar field of A, so that X is also infinite-dimensional over D.

10.8 Lemma. Let J be a separating ideal in a Banach algebra and let X be an irreducible Banach left J-module. Then X is finite-dimensional.

Proof. Suppose X is infinite-dimensional, and let E be the set of all a in J such that $a.X$ is infinite-dimensional over the centralizer D of $\overset{J}{\cancel{A}}$ on X. By Corollary 10.7, E is not empty, and for all a in E, $a.X \neq \{0\}$, and therefore $Ja \neq (0)$. By Proposition 7.5, there is therefore an element z of E such that for all a in J, $az \in E$ implies $(Jaz)^{-} = (Jz)^{-} \neq (0)$. Since $z.X$ is infinite-dimensional over D, there exist $x_0, x_1, \ldots\ldots$ in X such that $z.x_0, z.x_1, \ldots\ldots$ are linearly independent over D. By lemma 10.6, there is an element a of J such that $az.x_1, az.x_2, \ldots\ldots$ are linearly independent over D and $az.x_0 = 0$. But then az is in E, and therefore $J.(z.x_0) \subseteq (Jz)^{-}.x_0 = (Jaz)^{-}.x_0 = \{0\}$. Since X is irreducible, this implies that $z.x_0 = 0$, which contradicts the linear independence over D of $z.x_0, z.x_1, \ldots\ldots$. X must therefore be finite-dimensional.

The proof given above of lemma 10.8 differs only in detail from the first stage of the proof of Corollary 9 of [45], in which it is shown that if J is a closed 2-sided ideal in a semi-simple Banach algebra A and $J.X \neq \{0\}$ for some infinite-dimensional irreducible left A-module, then there exists a sequence $\{a_n\}$ in A such that $(Ja_{n+1} \ldots\ldots a_1)^{-} \subsetneq (Ja_n \ldots\ldots a_1)$ for all n.

In the statement and proof of the following Theorem and its Corollaries, we use the notation of Theorem 10.3.

10.9 Theorem. For each $i = 1, \ldots\ldots, n$, the following are equivalent :

(1) $M_i / L \cap J$ is a primitive Banach algebra

(2) $M_i / L \cap J$ is finite-dimensional

(3) A / P_i is finite-dimensional

(4) P_i is a primitive ideal of A.

If, for some i, these conditions are not satisfied, then $M_i / L \cap J$ is a topologically simple radical Banach algebra.

Proof. Only the proof of the implication (1) implies (2) uses the fact that J is a separating ideal. The rest of the proof consists of entirely routine arguments.

Suppose $M_i / L \cap J$ is a primitive algebra, and let X be an irreducible left $(M_i / L \cap J)$-module such that $(0) = \{b \in M_i / L \cap J : b.X = \{0\}\}$. By Corollary 2.20, we can assume that X is a Banach $(M_i / L \cap J)$-module. By Theorem 10.3 (9) and lemma 7.6, $M_i / L \cap J$ is a separating ideal of $A / L \cap J$. By lemma 10.8, X is therefore finite-dimensional. Since $M_i / L \cap J$ is isomorphic to a subalgebra of $BL(X)$, it follows that $M_i / L \cap J$ is finite-dimensional.

Now suppose that $M_i / L \cap J$ is finite-dimensional, and define $S : A \longrightarrow BL(M_i / L \cap J)$ by

$$S(a)(b + L \cap J) = ab + L \cap J \qquad (a \in A, b \in M).$$

Then $\mathrm{Ker}(S) = \{a \in A : aM_i \subseteq L \cap J\} = P_i$, and A / P_i is therefore isomorphic to a subspace of $BL(M_i / L \cap J)$ and so is finite-dimensional.

The implication (3) implies (4) follows immediately from lemma 3.18.

To prove the implication (4) implies (1), suppose P_i is a primitive ideal of A. Then $(M_i + P_i)/ P_i$ is a non-zero 2-sided ideal of the primitive algebra A / P_i, and is therefore a primitive algebra. Since $(M_i + P_i)/ P_i$ is isomorphic to

$M_i / P_i \cap M_i$ which, by Theorem 10.3 (5), is equal to $M_i / L \cap J$, this proves that $M_i / L \cap J$ is a primitive algebra.

The final statement of the Theorem follows from lemma 10.5.

<u>10.10 Corollary</u>. Let J be a separating ideal of a semi-simple Banach algebra. Then J is finite-dimensional.

<u>Proof</u>. In this case, we have $L = (0)$. If $J \neq (0)$, then J is non-nilpotent and, by lemma 2.12, each of the ideals $M_1, \ldots, M_n$ is a semi-simple algebra and, by lemma 10.5, is therefore primitive. By Theorem 10.9, the direct sum K of $M_1, \ldots, M_n$ is therefore finite-dimensional. By Corollary 2.18, K has an identity element e and e commutes with every element of A. Let a be any element of J. Then, for each i, $(a - ea)M_i = (a - ea)eM_i = (0)$. Thus $a - ea$ is in $P_1 \cap \ldots \cap P_n \cap J = L \cap J = (0)$, and therefore $J = eJ \subseteq K$. It follows that $J = K$ is finite-dimensional.

In the proof of Corollary 9 of [45], it is shown that if J is an infinite-dimensional closed 2-sided ideal in a semi-simple Banach algebra, then there is a sequence $\{a_n\}$ in A such that $(Ja_{n+1} \ldots a_1)^- \subsetneq (Ja_n \ldots a_1)^-$ for all n. This is equivalent to Corollary 10.10, which is not therefore a new result. The proof of Corollary 9 of [45] uses lemma 10.6, as described in the remark following lemma 10.8, and lemma 3.2 of [51] which states that if $X_0, X_1, X_2, \ldots$ are non-equivalent finite-dimensional irreducible left A-modules, then there is an element c of A such that $c.X_0 = \{0\}$ and $c.X_1 = X_1$, $c.X_2 = X_2, \ldots$. Use of this second lemma has been avoided in this section, by using lemma 10.1.

10.11 Corollary. Let R be the Jacobson radical of a Banach algebra A and let J be a separating ideal of A. Then $J \cap R$ is the Jacobson radical of J and $J / J \cap R$ is finite-dimensional.

Proof. By lemma 2.12, $J \cap R$ is the Jacobson radical of J, and by Corollary 2.10, A / R is semi-simple. By lemma 7.6, $((J + R)/ R)^-$ is a separating ideal of A / R. By Corollary 10.10, it is therefore finite-dimensional. But $J / J \cap R$ is isomorphic to $(J + R)/ R$, and $J / J \cap R$ is therefore finite-dimensional.

10.12 Corollary. Let A be a Banach algebra. Then the following statements are equivalent :

(1) A contains a separating ideal with non-nilpotent Jacobson radical

(2) A contains a non-nilpotent radical separating ideal M and a non-primitive closed prime ideal P such that $M \cap P$ is the prime radical of M and $M / M \cap P$ is a topologically simple radical Banach algebra.

The term 'radical separating ideal' should be interpreted as meaning a separating ideal contained in the Jacobson radical.

Proof. Let J be a separating ideal with non-nilpotent Jacobson radical. Then $L \cap J \subsetneq R \cap J$, where R is the Jacobson radical of A. If all the prime ideals $P_1, \ldots\ldots, P_n$ are primitive, then $R \cap J \subseteq P_1 \cap \ldots\ldots \cap P_n \cap J = L \cap J$. Thus for some i, P_i is not primitive. Let $M = M_i$ and $P = P_i$. Then, by

Theorem 10.9, $M/M\cap P = M/L\cap J$ is a topologically simple radical Banach algebra. By Corollary 2.10, the Jacobson radical of $A/L\cap J$ is $R/L\cap J$. Thus $M/L\cap J \subseteq R/L\cap J$, and therefore $M \subseteq R$. By Theorem 10.3 (9), M is therefore a radical separating ideal of A. By Theorem 10.3 (5), $M\cap P = M\cap L$, which is the prime radical of M, by lemma 3.12.

The implication (2) implies (1) is obvious.

<u>10.13 Corollary</u>. Let A be a Banach algebra in which every closed proper prime ideal is primitive. Then every separating ideal of A has nilpotent Jacobson radical.

<u>10.14 Corollary</u>. There exists a commutative Banach algebra with a non-nilpotent radical separating ideal if and only if there exists a commutative topologically simple radical Banach algebra.

<u>Proof</u>. If A is a commutative topologically simple radical Banach algebra, then $(Aa)^{-} = A$ for all non-zero elements a of A. Thus A is a separating ideal of itself.

Corollary 10.12 shows that there exists a separating ideal with non-nilpotent Jacobson radical if and only if there exists a topologically simple radical Banach algebra which is a separating ideal of itself. A non-commutative topologically simple radical Banach algebra may or may not be a separating ideal of itself. It is therefore possible that there exist no separating ideals with non-nilpotent Jacobson radicals even if there does exist a non-commutative topologically simple radical Banach algebra.

Let A be a commutative Banach algebra and let T be the <u>regular representation of A on itself</u>, which is the homomorphism of A into $BL(A)$ defined by

$$T(a)x = ax \qquad (a, x \in A).$$

A is <u>singly generated</u> if there is an element u of A such that A is the only closed subalgebra of A containing u. In this case, the closed ideals of A are precisely the closed invariant subspaces of the operator $T(u)$. It follows that if A is a singly generated topologically simple radical Banach algebra, then the operator $T(u)$ has no non-trivial closed invariant subspaces. It is an open question (the 'invariant subspace problem') whether or not there exists a bounded operator on any infinite-dimensional Banach space with no non-trivial closed subspaces [66].

If S is any bounded operator on a Banach space X, then a closed subspace Y of X is <u>hyperinvariant</u> for S if $UY \subseteq Y$ for all U in $BL(X)$ commuting with S.

If A is any topologically simple commutative radical Banach algebra, and u is any non-zero element of A, then $T(u)$ cannot have a non-trivial closed hyperinvariant subspace Y, since Y would then be a proper non-zero closed ideal of A.

In [58], Lomonosov proved that every non-zero compact operator on an infinite-dimensional Banach space has a non-trivial closed hyperinvariant subspace.

The following Definition was introduced by Alexander [1].

<u>10.15 Definition.</u> A <u>compact Banach algebra</u> is a Banach algebra A such that for each t in A, the mapping $a \longrightarrow tat$ is a compact linear operator on A.

The Banach algebra of all compact linear operators on a Banach space is a compact Banach algebra, and the disc algebra with convolution (see Example 4.2) is a commutative radical compact Banach algebra. Any closed subalgebra of a compact Banach algebra is a compact Banach algebra, and any quotient of a compact Banach algebra by a closed 2-sided ideal is a compact Banach algebra [1].

If A is a commutative topologically simple radical Banach algebra and t is any non-zero element of A, then the mapping $a \longrightarrow tat = t^2 a$ is non-zero. By Lomonosov's theorem, quoted above, it is therefore non-compact. We therefore have the following results.

<u>10.16 Lemma</u>. Let A be a commutative compact radical Banach algebra. Then A is not topologically simple.

<u>10.17 Theorem</u>. Let A be a commutative Banach algebra such that the Jacobson radical R of A is a compact Banach algebra. Then every separating ideal of A has nilpotent Jacobson radical.

<u>Proof</u>. Suppose A has a separating ideal J with non-nilpotent Jacobson radical. Then, by Corollary 10.12, there exists a closed prime ideal P of A and a closed 2-sided ideal M of A such that $M \subseteq R$ and $M / P \cap M$ is a topologically simple radical Banach algebra. Since $M / P \cap M$ is a compact Banach algebra, this contradicts lemma 10.16.

11. Applications to epimorphisms and the uniqueness of norm problem.

This section begins with a proof of the uniqueness of the complete norm topology of a semi-simple Banach algebra which shows that this result follows directly from lemma 6.4 without any need for lemma 10.1 or lemma 10.6.

11.1 Theorem (Johnson [46]).

Let A be a semi-simple Banach algebra. Then A has a unique complete norm topology.

Proof. We may assume without loss of generality that A is left primitive and that there is therefore an irreducible left Banach A-module X such that $a.X = \{0\}$ if and only if $a = 0$ (see [68, p.74]).

Suppose A does not have a unique complete norm topology, and let $\|\cdot\|_1$ be a complete algebra norm on A not equivalent to the given norm $\|\cdot\|$. Let $J = \{a \in A :$ there exist a_n in A with $\|a_n\|_1 \to 0$ and $\|a_n - a\| \to 0\}$. Then J is a non-zero separating ideal of A. By Proposition 7.5, or by a direct application of lemma 6.4, there is an element z of A such that $Jz \neq (0)$ and for all a in A, either $Jaz = (0)$ or $(Jaz)^- = (Jz)^-$. Suppose $z.X$ is not 1-dimensional over the centralizer D of A on X, and let x and y be elements of X such that $z.x$ and $z.y$ are linearly independent over D. By lemma 2.15, there is an element a of A such that $az.x = 0$ and $az.y \neq 0$. But $az.y \neq 0$ implies $Jaz \neq (0)$, and therefore, $J.(z.x) \subseteq (Jz)^-.x = (Jaz)^-.x = \{0\}$. It follows that $z.x = 0$, which contradicts the linear independence over D of $z.x$ and $z.y$.

$z.X$ must therefore be 1-dimensional over D. Let x be any element of X such that $z.x \neq 0$. Then $A.(z.x) = X$ and there is therefore an element a of A such that $az.x = x$. Let $e = az$. Then $e.x = x$ and there is a linear mapping T from X into D such that

$$e.y = T(y)x \qquad (y \in X).$$

For all y in X, $(e^2 - e).y = e.(T(y)x) - T(y)x = 0$. Thus e is a non-zero idempotent. Let h be the mapping from eAe into D defined by

$$h(eae) = T(a.x) \qquad (a \in A).$$

It is easy to check that h is a well-defined 1:1 anti-homomorphism of eAe onto D and that eAe is therefore a division algebra (i.e. e is a <u>minimal idempotent</u>). By lemma 2.23, eAe is therefore finite-dimensional. The restrictions of $\|\cdot\|_1$ and $\|\cdot\|$ to eAe are therefore equivalent. Let a be in J, and let $\{a_n\}$ be any sequence in A such that $\|a_n\|_1 \longrightarrow 0$ and $\|a_n - a\| \longrightarrow 0$. Then $ea_ne \longrightarrow 0$ in both norms, and therefore $eae = 0$. But J is a 2-sided ideal of A, and therefore $eAJAe = (0)$. Since A is a prime algebra, this implies that $e = 0$ or $J = (0)$. This is a contradiction, proving that A must have a unique complete norm topology.

The fact that a primitive Banach algebra with minimal idempotents has a unique complete norm topology was proved by Rickart [68, p.73].

The following lemma is well-known.

<u>11.2 Lemma</u>. Let h be an epimorphism from a Banach algebra C onto a semi-simple algebra A. Then the kernel of h is

closed.

Proof. The kernel of h is equal to the intersection of all the ideals $h^{-1}(P)$ such that P is a left primitive ideal of A. Let P be any left primitive ideal of A and let X be an irreducible left A-module such that $P = \{a \in A : a.X = \{0\}\}$. Then X may be regarded as a C-module by means of the definition

$$c.x = h(c).x \qquad (c \in C,\ x \in X).$$

X is then an irreducible C-module and $h^{-1}(P) = \{c \in C : c.X = \{0\}\}$ is therefore a primitive ideal of C. By Corollary 2.21, $h^{-1}(P)$ is closed. The kernel of h is therefore closed.

Alternative proofs of lemma 11.2 using different characterisations of semi-simplicity are given in [68, p.74] and [9, p.131]. The following Corollary of Theorem 11.1 and lemma 11.2 is also well-known.

11.3 Corollary. Let h be an epimorphism from a Banach algebra C onto a Banach algebra A. Then the separating space of h is contained in the Jacobson radical of A.

Proof. Let R be the Jacobson radical of A and let Q denote the natural mapping of A onto A/R. By lemma 11.2, the kernel of Qh is closed, and we may therefore define a complete algebra norm $\|\cdot\|_1$ on A/R by

$$\|Qhc\|_1 = \|c + \mathrm{Ker}(Qh)\| \qquad (c \in C).$$

By Theorem 11.1, there is a constant $M > 0$ such that

$$\|Qhc\| \leq M\|Qhc\|_1 \leq M\|c\|. \qquad (c \in C).$$

Qh is therefore continuous, and, by lemma 6.2, the separating space of h is therefore contained in R.

Let A be a Banach algebra and let h be an epimorphism from a Banach algebra onto A. By Corollary 7.3 (with $A = B$), the separating space $\mathfrak{S}(h)$ of h is a separating ideal of A. By Corollary 11.3, $\mathfrak{S}(h)$ is contained in the Jacobson radical of A. Suppose $\mathfrak{S}(h)$ is not nilpotent. By the results of the previous section, there is then a closed non-primitive prime ideal P of A and a closed 2-sided ideal M of A such that $\mathfrak{S}(h) \not\subseteq P$ and $M/P \cap M$ is a topologically simple radical Banach algebra. Let Q denote the natural homomorphism of A onto the prime Banach algebra A/P. Then Qh is discontinuous. It follows that there is an epimorphism h from a Banach algebra onto a Banach algebra such that $\mathfrak{S}(h)$ is not nilpotent if and only if there is a discontinuous epimorphism from a Banach algebra onto a prime Banach algebra.

The following Theorem follows immediately from Corollary 10.12.

<u>11.4 Theorem</u>. Let A be a semi-prime Banach algebra satisfying either of the following conditions :

(1) A has no non-zero radical minimal-closed 2-sided ideals

(2) Every proper closed prime ideal of A is primitive.

Then every epimorphism from a Banach algebra onto A is continuous and A has a unique complete norm topology.

The open questions suggested by the results so far may be formulated as follows :

(1) Is the separating space of an epimorphism from one Banach algebra onto another always a nilpotent ideal?

(2) Is every epimorphism from a Banach algebra onto a semi-

prime Banach algebra continuous?

(3) Is every epimorphism from a Banach algebra onto a prime Banach algebra continuous?

(4) Does every semi-prime Banach algebra have a unique complete norm topology?

(5) Is every epimorphism from a Banach algebra onto a semi-prime Banach algebra with a unique complete norm topology continuous?

(6) Is the kernel of an epimorphism from a Banach algebra onto a semi-prime Banach algebra always closed?

If the answer to any of these questions is 'no', then we may conclude that there exists a topologically simple radical Banach algebra. As has already been observed, questions (1), (2) and (3) are equivalent. If the answer to (6) is 'yes', then the argument used in the proof of Corollary 11.3 shows that the answer to (5) is then 'yes', and that (2) and (4) are then equivalent.

Another possibility suggested by these results is that there is a semi-prime Banach algebra with a non-unique complete norm topology if and only if there is a topologically simple radical Banach algebra with the same property. The following Theorem is a weaker result than this.

<u>11.5 Theorem</u>. The following statements are equivalent.

(1) There exists a semi-prime Banach algebra with two non-equivalent complete algebra norms

(2) There exists a prime algebra with two non-equivalent complete algebra norms and a non-zero 2-sided ideal I such that the

closures of I in both norms are topologically simple radical Banach algebras.

Proof. Let A be a semi-prime algebra with two non-equivalent complete algebra norms $\|\cdot\|_1$ and $\|\cdot\|_2$, and let J be the separating space of the identity mapping on A regerded as an epimorphism from $(A, \|\cdot\|_1)$ onto $(A, \|\cdot\|_2)$. By Theorem 10.3, with $L = (0)$, there exists a non-zero minimal-closed 2-sided ideal M of $(A, \|\cdot\|_2)$ and a prime ideal P of A such that $M \subseteq J$ and $P = \{a \in A : aM = (0)\}$. P is clearly closed in both norms, and since $J \not\subseteq P$, the quotient norms induced by $\|\cdot\|_1$ and $\|\cdot\|_2$ on the prime algebra A/P are non-equivalent complete algebra norms. We may therefore assume that A is already a prime algebra. By Corollary 11.3, M is a radical ideal. Repeating the above argument with the norms interchanged gives a non-zero radical minimal-closed 2-sided ideal N of $(A, \|\cdot\|_1)$. Let $I = N \cap M$. Then $(0) \neq NM \subseteq I$, and I is therefore $\|\cdot\|_1$-dense in N and $\|\cdot\|_2$-dense in M as required.

The implication (2) implies (1) is obvious, since a prime algebra is also semi-prime.

If, in the notation of the proof of Theorem 11.5, $N = M$, then I is a topologically simple Banach algebra with a non-unique complete norm topology.

12. Applications to derivations.

This section concentrates on the following questions :

(1) Is the separating space of a derivation on a Banach algebra always a nilpotent ideal?

(2) Does a derivation D on a Banach algebra A always satisfy the condition $D(P) \subseteq P$, for every primitive ideal P of A?

For the sake of completeness, proofs are included of several known results.

The first lemma applies to a derivation on an algebra over an arbitrary field.

12.1 Lemma. Let D be a derivation on an algebra A and let I be a 2-sided ideal of A. Then, for any n elements $a_1, \ldots, a_n$ of I, $D^n(a_1 \cdots a_n) - n!(Da_1) \cdots (Da_n)$ is in I.

Proof. We first show that if $n \geqslant 2$ and $0 \leqslant j < n$, then

$$D^j(a_1 \cdots a_n) \in I \qquad (a_1, \ldots, a_n \in I),$$

where $D^0 x = x$, for all x in A.

This is clearly true for $n = 2$, since

$$D(a_1 a_2) = a_1 Da_2 + (Da_1)a_2 \in I \qquad (a_1, a_2 \in I).$$

Suppose that it is true for some $n \geqslant 2$. Then, for any $n + 1$ elements $a_1, \ldots, a_{n+1}$ of I, and for $0 \leqslant j < n + 1$, Leibnitz's formula gives $D^j(a_1 \cdots a_{n+1}) = D^j(a_1(a_2 \cdots a_{n+1}))$ $= \sum_{i=0}^{j} \binom{j}{i} D^i(a_1) D^{j-i}(a_2 \cdots a_{n+1})$. By the inductive hypothesis, $D^{j-i}(a_2 \cdots a_{n+1})$ is in I for $1 \leqslant i \leqslant j$. This proves that $D^j(a_1 \cdots a_{n+1})$ is in I.

We now prove the statement of the lemma by induction on n. It is obviously true for $n = 1$. Suppose that it is true for some $n \geqslant 1$, and let $a_1, \ldots\ldots, a_{n+1}$ be any $n+1$ elements of I. Then

$$D^{n+1}(a_1 \ldots\ldots a_{n+1}) = \sum_{i=0}^{n+1} \binom{n+1}{i} D^i(a_1 \ldots\ldots a_n) D^{n+1-i}(a_{n+1}),$$

and the only term of the right hand side which is not in I is

$$\binom{n+1}{n} D^n(a_1 \ldots\ldots a_n) D(a_{n+1}) = (n+1) D^n(a_1 \ldots\ldots a_n) D(a_{n+1}).$$

By the inductive hypothesis, $D^n(a_1 \ldots\ldots a_n) - n!(Da_1)\ldots\ldots(Da_n)$ is in I, and so, $D^{n+1}(a_1 \ldots\ldots a_{n+1}) - (n+1)!(Da_1)\ldots\ldots(Da_{n+1})$
$= D^{n+1}(a_1 \ldots\ldots a_{n+1}) - (n+1)D^n(a_1 \ldots\ldots a_n)D(a_{n+1}) +$
$(n+1)(D^n(a_1 \ldots\ldots a_n) - n!(Da_1)\ldots\ldots(Da_n))Da_{n+1}$ is in I.
This completes the proof.

The special case of this lemma, in which $a_1 = \ldots\ldots = a_n$, is proved in [75], and the proof given above involves no new ideas.

12.2 Theorem (Sinclair [75]).

Let D be a continuous derivation on a Banach algebra and let P be a primitive ideal of A. Then $D(P)$ is contained in P.

Proof. By Corollary 2.11, every right primitive ideal of A is an intersection of left primitive ideals. We can therefore assume without loss of generality that P is left primitive and, by Corollary 2.20, that there is an irreducible Banach left A-module X such that $P = \{a \in A : a.X = \{0\}\}$ and $\|a.x\| \leqslant \|a\|\|x\|$, for all a in A and x in X.
Suppose $D(P)$ is not contained in P, and let a be any element of P such that Da is not in P. Then there is a

non-zero element y of X such that $Da.y \neq 0$. But then $ADa.y = X$, and there is therefore an element b of A such that $bDa.y = y$. Since $D(ba) = bDa + (Db)a$ and a is in P, it follows that $(D(ba))^n.y = y$, for all n. Since ba is in P, lemma 12.1 implies that $D((ba)^n) - n!(D(ba))^n$ is in P, for all n. But then,

$$1 = \|y\|^{-1} \|(D(ba))^n.y\| = \|y\|^{-1} \|\tfrac{1}{n!} D^n((ba)^n).y\| \leq \frac{1}{n!} \|D\|^n \|ba\|^n,$$

which is impossible, since $(n!)^{-1} \|D\|^n \|ba\|^n \to 0$ as $n \to \infty$.

<u>12.3 Corollary</u> (Singer and Wermer [80]).

Let A be a commutative Banach algebra and let D be a continuous derivation on A. Then $D(A)$ is contained in the Jacobson radical of A. If A is semi-simple, then $D = 0$.

<u>Proof</u>. This is the proof given in [75].

Let M be a maximal modular (= primitive) ideal of A. Then A/M is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$. By Theorem 12.2, $D(M)$ is contained in M. We may therefore define a (real linear) derivation D_o on A/M by

$$D_o(a + M) = Da + M \qquad (a \in A).$$

It is therefore sufficient to prove that there are no non-zero real linear derivations on $\mathbb{R}$ or on $\mathbb{C}$, and this is obvious.

<u>12.4 Lemma</u>. Let D be a derivation on an algebra A and let I be a nilpotent left ideal of A such that $I^n = (0)$. Then $n!D(I)$ is contained in the prime radical L of A.

<u>Proof</u>. Let a be in I, and let $r_1, \ldots\ldots, r_n$ be any n elements of A. By lemma 3.11, I is contained in L and, by

lemma 12.1, $D^n(r_1 a \cdots r_n a) - n!\,D(r_1 a)\cdots D(r_n a)$ is therefore in L. But, $r_1 a \cdots r_n a = 0$, and, for all i, $D(r_i a) - r_i Da$ is in L. $n!(r_1 Da)\cdots(r_n Da)$ is therefore in L. Let P be any prime ideal of A. Then $(An!\,Da)^n \subseteq P$, and therefore, $n!\,Da$ is in P. Since L is equal to the intersection of all the prime ideals of A, it follows that $n!\,Da$ is in L.

<u>12.5 Proposition</u>. Let D be a derivation on a Banach algebra A and let L be the prime radical of A. Then D(L) is contained in L.

<u>Proof</u>. Let I be any 2-sided nilpotent ideal. By lemma 12.4, D(I) is contained in L. By Corollary 3.24, L is equal to the sum of all the 2-sided nilpotent ideals. D(L) is therefore contained in L.

By Corollary 7.4, the separating space of a derivation on a Banach algebra is a separating ideal. We may therefore apply the results of Section 10.

<u>12.6 Lemma</u>. Let D be a derivation on a Banach algebra A. If the Jacobson radical of the separating space J of D is nilpotent, then J is nilpotent.

<u>Proof</u>. Let R be the Jacobson radical of A. By Corollary 10.11, $R \cap J$ is the Jacobson radical of J and $J/R \cap J$ is finite-dimensional. Let Q be the natural mapping of A onto $A/R \cap J$, and suppose J is not nilpotent. Then $J \neq R \cap J$, and QJ is therefore non-zero. By lemma 2.18, there is an element e of J such that Qe is an identity element for QJ. Let $e_1, \ldots, e_n$ be elements of J such that $Qe_1, \ldots, Qe_n$

is a basis of QJ. Then there exist continuous linear functionals $f_1, \ldots\ldots, f_n$ on QJ such that

$$Qa = \sum_{i=1}^{n} f_i(Qa)Qe_i \qquad (a \in J).$$

In particular,

$$ea - \sum_{i=1}^{n} f_i(Qea)e_i \in R \cap J \subseteq L \qquad (a \in A).$$

Let $\{a_m\}$ be a sequence in A such that $a_m \to 0$ and $Da_m \to e$. Then

$$D(ea_m) - \sum_{i=1}^{n} f_i(Qea_m)D(e_i) \in D(L) \subseteq L \qquad (m = 1, 2, \ldots),$$

by Proposition 12.5. For $i = 1, \ldots\ldots, n$, $f_i(Qea_m) \to 0$. Since $D(ea_m) = eDa_m + (De)a_m \to e^2$, it follows that e^2 is in the closure of L. By lemma 3.17, $\overline{L}$ is contained in R, and therefore $Qe^2 = (Qe)^2 = Qe = 0$, which is a contradiction.

<u>12.7 Corollary</u>. Let D be a derivation on a Banach algebra A such that the separating space J of D is not nilpotent. Then there is a closed non-primitive prime ideal P of A and a closed 2-sided ideal M of A such that $M / P \cap M$ is a topologically simple radical Banach algebra.

<u>Proof</u>. Suppose J is not nilpotent. Then, by lemma 12.6, the Jacobson radical of J is not nilpotent. Since J is a separating ideal of A, the required result therefore follows immediately from Corollary 10.12.

<u>12.8 Corollary</u>. Let A be a semi-prime Banach algebra satisfying either of the following conditions :

(1) A has no non-zero radical minimal-closed 2-sided ideals

(2) Every proper closed prime ideal of A is primitive.

Then every derivation on A is continuous.

12.9 Corollary (Johnson and Sinclair [51], see also [45]).

Every derivation on a semi-simple Banach algebra is continuous.

12.10 Theorem. Let D be a derivation on a Banach algebra A such that the separating space J of D is nilpotent. Then $D(P)$ is contained in P, for all primitive ideals P of A.

Proof. Let L be the prime radical of A and let Q be the natural mapping of A onto $A/\bar{L}$. Since J is nilpotent, it is contained in L. By lemma 6.2, QD is therefore continuous. By Proposi tion 12.5, $(QD)(L) = (0)$. By the continuity of QD, $QD(\bar{L}) = (0)$, and $D(\bar{L})$ is therefore contained in $\bar{L}$. We may therefore define a derivation D_o on $A/\bar{L}$ by

$$D_o(a + \bar{L}) = Da + \bar{L} \qquad (a \in A).$$

By lemma 6.3, D_o is continuous. Let P be a left primitive ideal of A. Then, by lemma 3.17, $\bar{L}$ is contained in P and, by lemma 2.9, $P/\bar{L}$ is therefore a left primitive ideal of $A/\bar{L}$. By Theorem 12.2, $D_o(P/\bar{L})$ is contained in $P/\bar{L}$, and $D(P)$ is therefore contained in P. By Corollary 2.11, every right primitive ideal is an intersection of left primitive ideals. It therefore follows that $D(P)$ is contained in P for all the primitive ideals P of A.

12.11 Corollary. If there is a derivation D on a Banach algebra A, and a primitive ideal P of A such that $D(P)$ is not contained in P, then there is a topologically simple radical Banach algebra.

<u>12.12 Corollary</u>. Let D be a derivation on a commutative Banach algebra A such that the separating space J of D is nilpotent. Then $D(A)$ is contained in the Jacobson radical R of A.

<u>Proof</u>. Let M be a maximal modular (= primitive) ideal of A. Then, by Theorem 12.10, $D(M)$ is contained in M. It follows that $D(R)$ is contained in R. We may therefore define a derivation D_o on A/R by

$$D_o(a + R) = Da + R \qquad (a \in A).$$

Since J is contained in R, D_o is continuous. By Corollary 12.3, $D_o = 0$, and $D(A)$ is therefore contained in R.

<u>12.13 Corollary</u>. If there is a derivation D on a commutative Banach algebra A such that $D(A)$ is not contained in the Jacobson radical of A, then there is a commutative topologically simple radical Banach algebra.

13. Arens-Hoffman extensions.

It is known that every Arens-Hoffman extension of a commutative semi-simple Banach algebra with identity has a unique complete norm topology (Lindberg [57]). The main purpose of this section is to give an easy proof of this result, using the theory of separating ideals, and to extend it to cover Arens-Hoffman extensions of a wider range of Banach algebras.

We begin by describing what is meant by an Arens-Hoffman extension of a Banach algebra.

Let A be a commutative algebra with identity, and let $A[t]$ denote the algebra of all polynomials in the indeterminate t with coefficients in A. Let $b(t) = t^n + b_{n-1} t^{n-1} + \ldots\ldots + b_1 t + b_0 \in A[t]$ be a monic polynomial of degree $n \geq 2$, and let $B = A[t]/(b(t))$, where $(b(t)) = A[t]b(t)$ is the principal ideal of $A[t]$ generated by $b(t)$. Then B is a commutative algebra with identity, and the map i from A into B defined by

$$i(a) = a + (b(t)) \qquad (a \in A)$$

is an algebra monomorphism. The important feature of B is that it contains a solution of the polynomial equation $b(t) = 0$. However, for the purposes of this section, it is only the A-bimodule structure of B which is important.

An <u>A-bimodule</u> is a left A-module X which is also a right A-module in such a way that the right and left module operations satisfy the consistency condition

$$a.(x.b) = (a.x).b \qquad (a, b \in A, x \in X).$$

A linear mapping T from an A-bimodule X into an A-bimodule Y is an <u>A-bimodule homomorphism</u> if

$$T(a.x) = a.Tx \quad \text{and} \quad T(x.a) = (Tx).a \qquad (a \in A, x \in X).$$

$B = A[t]/(b(t))$ is clearly an A-bimodule, with the module operations defined by

$$a.x = i(a)x = x.a \qquad (x \in B,\ a \in A).$$

For any algebra A, the notation $A(n)$ will be used for the linear space direct sum of n copies of A, regarded as an A-bimodule by means of the definitions

$$a.(a_1, \ldots\ldots, a_n) = (aa_1, \ldots\ldots, aa_n)$$
$$(a_1, \ldots\ldots, a_n).a = (a_1 a, \ldots\ldots, a_n a) \qquad (a,\ a_1,\ \ldots\ldots,\ a_n \in A).$$

13.1 Lemma. Define $T : A(n) \to B = A[t]/(b(t))$ by
$$T((a_1, \ldots\ldots, a_n)) = a_1 + a_2\underline{t} + \ldots\ldots + a_n\underline{t}^{n-1} \qquad (a_1, \ldots\ldots a_n \in A),$$
where $\underline{t} = t + (b(t))$. Then T is a 1:1 A-bimodule homomorphism from $A(n)$ onto B.

Proof. It is obvious from the definitions that T is an A-bimodule homomorphism, and from the identity $\underline{t}^n = -(b_0 + b_1\underline{t} + \ldots\ldots + b_{n-1}\underline{t}^{n-1})$, that T is onto. If $T((a_1, \ldots\ldots, a_n)) = 0$, then there exists an element $c(t)$ of $A[t]$ such that $a_1 + a_2 t + \ldots\ldots a_n t^{n-1} = c(t)b(t)$. Since $b(t)$ is monic and of degree n, it follows that $c(t) = 0$. But then, $a_1 = \ldots\ldots = a_n = 0$. T is therefore 1:1.

The module structure of B is the essential feature used in the construction of the Arens-Hoffman norms which are described in the following lemma. Proofs of the various statements made in the lemma can be found in [3].

13.2 Lemma. Let A be a commutative unital Banach algebra. Then there exist real numbers $s > 0$ such that $\|b_0\| + \|b_1\|s + \ldots\ldots + \|b_{n-1}\| s^{n-1} \leq s^n$. For any such s, define

$\|\cdot\|_s$ on $B = A[t]/(b(t))$ by

$$\|a_1 + a_2\underline{t} + \ldots\ldots + a_n\underline{t}^{n-1}\|_s = \sum_{i=0}^{n-1} \|a_{i+1}\| \, s^i \quad (a_1, \ldots, a_n \in A).$$

Then $\|\cdot\|_s$ is a complete algebra norm on B and the monomorphism $a \longrightarrow a + (b(t))$ from A into B is an isometry.

The Banach algebras $(B, \|\cdot\|_s)$ are called Arens-Hoffman extensions of A.

The module structure of B is used in [61] to prove that if every derivation on a Banach algebra A is continuous, then every derivation on any Arens-Hoffman extension of A is continuous.

In the following Theorem and its Corollaries, we consider Banach algebras satisfying the following two conditions :

(1) A has no non-zero finite-dimensional nilpotent ideals

(2) A has no infinite-dimensional separating ideals.

These are precisely the conditions of Theorem 2 of [45] (see Section 5, Theorem 5.1). As shown in [45], they imply that every separating ideal J of A contains an idempotent e such that $J = Ae$. To prove this, note that, by lemma 2.12 and Corollary 3.10, the Jacobson radical of J is a nilpotent ideal of A. By condition (1), J is therefore semi-simple. By Corollary 2.18, it therefore contains an identity element e.

Conditions (1) and (2) are satisfied by semi-simple Banach algebras, semi-prime Banach algebras with no radical minimal-closed ideals, semi-prime Banach algebras with no non-primitive proper closed prime ideals, $L^1(0,1)$, and Banach algebras of formal power series.

13.3 Theorem. Let A be a Banach algebra with no non-zero finite-dimensional nilpotent ideals and no infinite-dimensional separating ideals. Let $\|\cdot\|$ be any complete norm on $A(n)$ such that, for all a in A, the module operations

$$x \longrightarrow a.x \quad \text{and} \quad x \longrightarrow x.a \qquad (x \in A(n))$$

are continuous. Then $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_\infty$ defined on $A(n)$ by

$$\|(a_1, \ldots, a_n)\|_\infty = \max\{\|a_1\|, \ldots, \|a_n\|\} \qquad (a_1, \ldots, a_n \in A).$$

Proof. For each i, define $p_i : A(n) \longrightarrow A$ by

$$p_i((a_1, \ldots, a_n)) = a_i \qquad (a_1, \ldots, a_n \in A).$$

Then each p_i is a continuous A-bimodule homomorphism from $(A(n), \|\cdot\|_\infty)$ onto A. Let S denote the identity map from $(A(n), \|\cdot\|)$ to $(A(n), \|\cdot\|_\infty)$ and suppose that S is not continuous. Then $p_i \mathfrak{S}(S) \neq (0)$ for some i.

For each b in A, let T_b and U_b denote right and left module multiplication by b on $(A(n), \|\cdot\|)$. Then T_b and U_b are continuous and, for all b in A,

$$(p_i S)(T_b x) = (p_i Sx)b \quad \text{and} \quad (p_i S)(U_b x) = b(p_i Sx) \qquad (x \in A(n)).$$

By Proposition 7.2, the separating space $\mathfrak{S}(p_i S)$ of $p_i S$ is a separating ideal of A. Because of the conditions on A, $\mathfrak{S}(p_i S)$ is therefore finite-dimensional and contains an idempotent e_i such that $Ae_i = \mathfrak{S}(p_i S)$.

Let $X_i = \{(a_1, \ldots, a_n) \in A(n) : a_j \in e_i A,\ j = 1, \ldots, n\}$. Then X_i is finite-dimensional, and the restrictions of $\|\cdot\|$ and $\|\cdot\|_\infty$ to X_i are therefore equivalent. Let $\{x_m\}$ be a sequence in $A(n)$ such that $\|x_m\| \to 0$ and $p_i S x_m \longrightarrow e_i$ in A. Then $e_i.x_m$ is in X_i, for all m, and $\{e_i.x_m\}$ therefore converges to 0 in both norms. But then, $e_i = e_i^2 = \lim e_i((p_i S)(x_m)) = \lim (p_i S)(e_i.x_m) = 0$, and so $\mathfrak{S}(p_i S) =$

$\{0\}$. By lemma 6.2, $(p_i \mathfrak{S}(S))^- = \mathfrak{S}(p_i S) = (0)$, which is a contradiction. S must therefore be continuous and, by Banach's isomorphism theorem, $\|\cdot\|$ and $\|\cdot\|_\infty$ are therefore equivalent.

<u>13.4 Corollary</u>. Let A be a commutative unital Banach algebra with no non-zero finite-dimensional nilpotent ideals and no infinite-dimensional separating ideals. Then every Arens-Hoffman extension of A has a unique complete norm topology.

<u>Proof</u>. Let $\|\cdot\|'$ be any complete algebra norm on $B = A[t]/(b(t))$, where $b(t)$ is of degree n. Define $\|\cdot\|$ on $A(n)$ by

$$\|(a_1, \ldots\ldots, a_n)\| = \|a_1 + a_2\underline{t} + \ldots\ldots + a_n\underline{t}^{n-1}\|' \qquad (a_1, \ldots, a_n \in A).$$

By lemma 13.1 and Theorem 13.3, it is sufficient to prove that for each a in A, the map $a \longrightarrow a.x$ on $(A(n), \|\cdot\|)$ is continuous. To do this, let a and $a_1, \ldots\ldots, a_n$ be any elements of A. Then

$$\|a.(a_1, \ldots\ldots, a_n)\| = \|(a + (b(t)))(a_1 + \ldots\ldots + a_n\underline{t}^{n-1}\|'$$
$$\leqslant \|a + (b(t))\|' \|a_1 + \ldots\ldots + a_n\underline{t}^{n-1}\|' =$$
$= \|a + (b(t))\|' \|(a_1, \ldots\ldots, a_n)\|$. This completes the proof.

<u>13.5 Corollary</u> (Brown [10], Lindberg [57]).

Let A be a commutative semi-simple Banach algebra with identity. Then every Arens-Hoffman extension of A has a unique complete norm topology.

Chapter Three

JORDAN DERIVATIONS

14. Introduction.

Throughout this chapter, A will denote an algebra over a field F, such that the characteristic of F is not 2.

The Jordan product $\circ$ on A is defined by

$$a \circ b = ab + ba \qquad (a, b \in A),$$

and a linear mapping D of A into itself is a Jordan derivation if

$$D(a \circ b) = a \circ Db + (Da) \circ b \qquad (a, b \in A).$$

The main result of this chapter (Theorem 15.9) is that if L is the prime radical of A, and D is a Jordan derivation on A, then

$$D(ab) - aDb - (Da)b \in L \qquad (a, b \in A).$$

It follows immediately from this that if A is semi-prime (i.e. $L = (0)$), then D is a derivation. In particular, every Jordan derivation on a semi-simple Banach algebra is a derivation, and therefore continuous, by Corollary 12.9.

The restriction on the characteristic of F is necessary to avoid the case when A is commutative and satisfies the condition $2a = 0$ for all a in A. In this case the Jordan product $a \circ b = 2ab$ is zero, and all linear maps of A into itself are therefore Jordan derivations, but may not all be derivations.

15. Jordan derivations and the prime radical.

Throughout this section, D will denote a Jordan derivation on A and L will denote the prime radical of A. The following notation will be used in order to simplify the algebraic computations which make up most of the proofs.

$$d(a,b) = D(ab) - aDb - (Da)b \qquad (a,\ b \in A)$$

$$[a,b] = ab - ba \qquad (a,\ b \in A)$$

$$[a,b,c] = abc + cba \qquad (a,\ b,\ c \in A).$$

D is a derivation if and only if $d(a,b) = 0$ for all a and b in A. Recall that L is equal to the intersection of all the prime ideals of A. Let P be a prime ideal of A. We show that $d(a,b)$ is in P for all a and b in A. The proof divides into two cases depending on whether or not $[a,b]$ is in P. The first case $([a,b] \in P)$ includes the case when A/P is commutative.

The first two lemmas were proved (for rings) by Herstein, and are the starting point for his proof that every Jordan derivation on a prime ring in which $2x = 0$ implies $x = 0$ is a derivation.

15.1 Lemma (Herstein, [34]). For all a, b and c in A,

(1) $Da^2 = aDa + (Da)a$

(2) $D(aba) = (Da)ba + a(Db)a + abDa$

(3) $D([a,b,c]) = [Da,b,c] + [a,Db,c] + [a,b,Dc]$.

Proof. In the proof of (1) and (2) we use the fact that, because of the restriction on the characteristic of F, $2a = 0$ implies $a = 0$.

For all a in A,

$2Da^2 = D(a \circ a) = a \circ Da + (Da) \circ a = 2(aDa + (Da)a)$. This proves (1). To prove (2), first note the identity

$$x \circ (x \circ y) = x \circ (xy + yx) = x^2 \circ y + 2xyx \qquad (x, y \in A).$$

From this we obtain

$$\begin{aligned} 2D(aba) &= D(a \circ (a \circ b)) - D(a^2 \circ b) \\ &= a \circ D(a \circ b) + (Da) \circ (a \circ b) - a^2 \circ Db - Da^2 \circ b \\ &= a^2 \circ Db + 2a(Db)a + a(Da)b + abDa + (Da)ba + b(Da)a + \\ &\quad (Da)ab + (Da)ba + abDa + baDa - a^2 \circ Db - a(Da)b - \\ &\quad baDa - (Da)ab - b(Da)a \\ &= 2(Da)ba + 2a(Db)a + 2abDa. \end{aligned}$$

To prove (3), replace a by $a + c$ in (2). This gives $D((a + c)b(a + c)) = (D(a + c))b(a + c) + (a + c)(Db)(a + c) + (a + c)bD(a + c) = D(aba) + [Da,b,c] + [a,Db,c] + [a,b,Dc] + D(cbc)$. (3) follows on subtracting $D(aba + cbc)$ from each side.

Before stating the next lemma we introduce another notational device. This is the reversed product $\cdot$ on A defined by

$$a \cdot b = ba \qquad (a, b \in A).$$

The algebra obtained from A by reversing the product in this way will be denoted by rev(A) [9]. Since the Jordan products on A and rev(A) coincide, D is also a Jordan derivation on rev(A).

<u>15.2 Lemma</u> (Herstein). For all a and b in A,

(1) $d(a,b)[a,b] = 0$

(2) $[a,b]d(a,b) = 0$.

<u>Proof</u>. By (3) of lemma 15.1,

$D([a,b,ab]) = [Da,b,ab] + [a,Db,ab] + [a,b,D(ab)]$.

But $[a,b,ab] = (ab)^2 + ab^2 a$, and therefore, by (1) and (2) of lemma 15.1,

$D([a,b,ab]) = abD(ab) + D(ab)ab + (Da)b^2 a + a(Db^2)a + ab^2 Da$

$= abD(ab) + D(ab)ab + (Da)b^2 a + ab(Db)a + a(Db)ba + ab^2 Da$.

By comparing the two expressions for $D([a,b,ab])$, we get

$0 = (D(ab) - (Da)b - aDb)ab - (D(ab) - aDb - (Da)b)ba$

$= d(a,b)[a,b]$.

(2) now follows from (1) by reversing the product. Thus,

$0 = (D(b.a) - (Db).a - b.Da).(b.a - a.b) = [a,b]d(a,b)$.

<u>15.3 Corollary</u> (Herstein). For all a, b and c in A,

(1) $[c,b]d(a,b) + [a,b]d(c,b) = 0$

(2) $[c,a]d(a,b) + [a,b]d(c,a) = 0$.

<u>Proof</u>. By (2) of lemma 15.2,

$[a + c,b]d(a + c,b) = 0$. But $[a + c,b] = [a,b] + [c,b]$ and $d(a + c,b) = d(a,b) + d(c,b)$. Thus, since $[a,b]d(a,b) = 0 = [c,b]d(c,b)$, we have $[c,b]d(a,b) + [a,b]d(c,b) = 0$, as required.

Now interchange a and b in (1). This gives $[c,a]d(b,a) + [b,a]d(c,a) = 0$. But $d(a,b) + d(b,a) = D(a \circ b) - a \circ Db - Da \circ b = 0$, and therefore $d(b,a) = -d(a,b)$. Since $[b,a] = [-a,b]$, this proves (2).

<u>15.4 Lemma</u>. Let P be a prime ideal of A and let a and b be elements of A such that

$$[a,c]b \in P \qquad (c \in A).$$

Then b is in P, or $[a,c]$ is in P for all c in A.

<u>Proof</u>. Suppose $[a,c] \notin P$ for some element c of A, and let

x be any element of A. Then, $[a,c]xb = (acx - cxa)b + c(xa - ax)b = [a,cx]b + c[x,a]b \in P$. Thus $[a,c]Ab \subseteq P$ and therefore b is in P.

<u>15.5 Lemma</u>. Let P be a prime ideal of A. If $[a,b]$ is in P, then $d(a,b)$ is in P.

<u>Proof</u>. We first consider the case when A/P is non-commutative.

Suppose that $[a,b]$ is in P. Then, by Corollary 15.3,

$$[c,b]d(a,b) \in P \quad \text{and} \quad [c,a]d(a,b) \in P \qquad (c \in A).$$

It follows from lemma 15.4, that if there is an element c of A such that $[c,a] \notin P$ or $[c,b] \notin P$, then $d(a,b) \in P$.

Now suppose that $[c,a]$ and $[c,b]$ are in P for all c in A, and let c and e be any elements of A such that $[c,e] \notin P$. Then, $[e,b]$ and $[a + e,b]$ are in P and $[c,e]$ and $[c,a + e]$ are not in P. It follows that $d(e,b)$ and $d(a + e,b)$ are in P, and that $d(a,b) = d(a + e,b) - d(e,b)$ is therefore in P.

Now suppose that A/P is commutative. By lemma 15.2, we have $[a,b] \circ d(a,b) = 0$ and therefore, since D is a Jordan derivation, $0 = [a,b] \circ d(a,b) + D([a,b]) \circ d(a,b)$. Since A/P is a commutative integral domain, this implies that $D([a,b])d(a,b)$ is in P, and that, therefore, $d(a,b)$ is in P or $D([a,b])$ is in P. But,

$$\begin{aligned} 2d(a,b) &= D(ab + ba) + D(ab - ba) - 2(aDb + (Da)b) \\ &= D([a,b]) + (a \circ Db - 2(Da)b) + ((Da) \circ b - 2(Da)b) \in P, \end{aligned}$$

if $D([a,b])$ is in P. Thus in either case, $d(a,b) \in P$ and the proof is complete.

The final three lemmas are concerned with the case $[a,b] \notin P$. Lemma 15.6 is simply part of the calculation needed to prove lemma 15.7.

<u>15.6 Lemma</u>. For all a, b and x in A,

(1) $D([a,b]x) = (D(ab) - bDa - (Db)a)x + [a,b]Dx + xd(a,b)$

(2) $D(x[a,b]) = x(D(ab) - bDa - (Db)a) + Dx[a,b] + d(a,b)x$.

<u>Proof</u>. Note that $[a,b]x = [a,b,x] - x \circ ba$. Thus, by (3) of lemma 15.1, $D([a,b]x) = D([a,b,x]) - D(x \circ ba)$

$= [Da,b,x] + [a,Db,x] + [a,b,Dx] - xD(ba) - D(ba)x - (Dx)ba - baDx$

$= x(bDa + (Db)a - D(ba)) + ((Da)b + aDb - D(ba))x - (Dx)ba - baDx + abDx + (Dx)ba$.

Since D is a Jordan derivation, we have $(Da)b + aDb - D(ba) = D(ab) - (Db)a - bDa$ and $bDa + (Db)a - D(ba) = d(a,b)$, and therefore $D([a,b]x) = xd(a,b) + (D(ab) - bDa - (Db)a)x + [a,b]Dx$, as required.

(2) may be proved by a similar argument, or by reversing the product.

<u>15.7 Lemma</u>. For all a, b and c in A,

$$[[a,b],c,d(a,b)] = 0.$$

<u>Proof</u>. The idea of the proof is to evaluate $D([a,b]c[a,b])$ in two different ways and then obtain the required result from lemma 15.2 and the resulting identity.

Firstly, by replacing x by $[a,b]c$ in (2) of lemma 15.6 we get $D([a,b]c[a,b]) = d(a,b)[a,b]c + D([a,b]c)[a,b] + [a,b]c(D(ab) - bDa - (Db)a) = [a,b]Dc[a,b] + [D(ab)-bDa-(Db)a,c,[a,b]]$, by lemma 15.2 and lemma 15.6 (1).

We now observe that $D([a,b]) = (D(ab)-bDa-(Db)a) + d(a,b)$.
Thus, $D([a,b]c[a,b]) = [a,b]Dc[a,b] + D([a,b])c[a,b] + [a,b]cD([a,b]) - [d(a,b),c,[a,b]]$
$= D([a,b]c[a,b]) - [d(a,b),c,[a,b]]$, by lemma 15.1 (2).
The result now followson subtracting $D([a,b]c[a,b])$ from each side.

The final lemma is similar to [36, lemma 3,10].

15.8 Lemma. Let P be a prime ideal of A. Then $[a,c,b] = 0$ for all c in A implies $a \in P$ or $b \in P$.

Proof. Let h and k be arbitrary elements of A. Then

$$ahakb + bhaka = 0.$$

But $akb = -bka$ and $bha = -ahb$, and therefore $2ahbka = 0$. Thus $aAbAa \subseteq P$ and so $a \in P$ or $b \in P$.

15.9 Theorem. Let D be a Jordan derivation on an algebra A over a field F, such that the characteristic of F is not 2, and let L be the prime radical of A. Then $D(ab) - aDb - (Da)b$ is in L, for all a and b in A.

Proof. Let a and b be any elements of A and let P be any prime ideal of A. By lemma 15.7, $[[a,b],c,d(a,b)] = 0$, for all c in A. By lemma 15.8, either $[a,b] \in P$ or $d(a,b) \in P$. But, by lemma 15.5, $[a,b] \in P$ implies $d(a,b) \in P$. Thus, in either case, $d(a,b) \in P$. Since L is the intersection of all the prime ideals of A, this completes the proof.

15.10 Corollary. If A is semi-prime, then D is a derivation.

Proof. By Proposition 3.9, A is semi-prime if and only if L

$= (0)$. If $L = (0)$, then $D(ab) - aDb - (Da)b = 0$, for all a and b in A, and so D is a derivation.

<u>15.11 Theorem</u>. Let D be a Jordan derivation on a semi-simple Banach algebra. Then D is a continuous derivation.

<u>Proof</u>. This follows immediately from Corollary 15.10, Corollary 12.8, and the fact that any semi-simple algebra is semi-prime.

Chapter Four

INVARIANT STATES ON BANACH ALGEBRAS

16. Introduction.

Let A be a complex unital Banach algebra, let G be a group and let T be a representation of G on A by isometric automorphisms (see Definition 4.6). This chapter is concerned with the relationship between two conditions which the triple (A,G,T) may satisfy, both of which are expressed in terms of the G-invariant states of A.

Let A' denote the dual space of A. An element f of A' is a state if $\|f\| = 1 = f(1)$, and the state space of A is the set $D(A)$ of all states of A, with the weak* topology. $D(A)$ is non-empty, convex and compact (see [7, p.15]).

16.1 Definition. A linear functional f on A is G-invariant if
$$f(g.a) = f(a) \qquad (a \in A,\ g \in G),$$
where, as in Section 4,
$$g.a = T(g)(a) \qquad (a \in A,\ g \in G).$$

The set of G-invariant continuous linear functionals on A will be denoted by $A'(G)$, and the set of G-invariant states by $D(A,G)$.

It is clear that $A'(G)$ is a weak* closed subspace of A' and that $D(A,G)$ is a compact convex subset of $D(A)$, although it may be empty (see Example 16.3).

The two conditions on (A,G,T), with which this chapter is concerned are defined in Section 18 ((A,G,T) is M-asymptotically abelian) and Section 19 (D(A,G) is a simplex). The rest of this section consists of three examples, all of which, as will become apparent later, satisfy both of the conditions. The first example also serves to introduce the terminology of amenable groups.

<u>16.2 Definition</u>. Let G be a group and let $l^{\infty}(G)$ denote the C*-algebra of all bounded complex-valued functions on G. A state M on $l^{\infty}(G)$ is a <u>2-sided invariant mean</u> if, for all f in $l^{\infty}(G)$, $M({}_g f) = M(f_g) = M(f)$, where ${}_g f$ and f_g are the left and right translates of f, defined by

$${}_g f(x) = f(gx) \quad \text{and} \quad f_g(x) = f(xg) \qquad (x \in G).$$

If G is a finite group with n elements, then $l^{\infty}(G)$ clearly has a 2-sided invariant mean defined by

$$M(f) = \frac{1}{n}\sum_{g \in G} f(g) \qquad (f \in l^{\infty}(G)).$$

On the other hand, there do exist groups G such that $l^{\infty}(G)$ does not have a 2-sided invariant mean. One such example is the free group on 2 generators (see, for example, [37, p.236]).

A group G such that $l^{\infty}(G)$ does have a 2-sided invariant mean is said to be <u>amenable</u>. All abelian groups are amenable [37, p.231].

A state M on $l^{\infty}(G)$ is <u>inversion invariant</u> if

$$M(f) = M(\tilde{f}) \qquad (f \in l^{\infty}(G)),$$

where $\tilde{f}(g) = f(g^{-1})$ $(g \in G)$. If G is amenable, then it has a 2-sided and inversion invariant mean [37].

<u>16.3 Example</u>. Let H be any group and let $G = H \times H$. Define

$T: G \to BL(l^{\infty}(H))$ by

$T((h,k))(f)(x) = f(h^{-1}xk) \qquad (h, k, x \in H, f \in l^{\infty}(H))$.

Then T is a representation of G on $l^{\infty}(H)$ by isometric automorphisms and the G-invariant states of $l^{\infty}(H)$ are precisely the 2-sided invariant means on $l^{\infty}(H)$. If H is any non-amenable group, then the set $D(l^{\infty}(H),G)$ of G-invariant states is empty.

The next example is a simplified version of a mathematical structure used in the C*-algebra approach to statistical mechanics. For further details and more elaborate examples, see [69, Chapter 7].

Recall that an automorphism h on a unital C*-algebra is isometric if and only if it is a star-automorphism (i.e. $h(a^*) = (h(a))^*$ for all a in A) [15].

16.4 Example : Quantum lattice systems [69].

Let $G = \mathbb{Z}^{\nu}$, where $\nu = 1$, 2 or 3, and for each x in $\mathbb{Z}^{\nu}$, let H_x be a 2-dimensional Hilbert space. Let K denote the set of all finite subsets of $\mathbb{Z}^{\nu}$, and, for each non-empty S in K, let H_S be the Hilbert space tensor product of the Hilbert spaces $\{H_x: x \in S\}$. Thus,

$$H_S = \bigotimes_{x \in S} H_x.$$

Let $H_{\emptyset} = \{0\}$. Note that if S contains n points, then H_S is 2^n-dimensional.

For all S and S' in K, with $S \subseteq S'$, define $i(S,S'): BL(H_S) \to BL(H_{S'})$ by $i(S,S')(a)(x \otimes y) = a(x) \otimes y$, where $x \in H_S$ and $y \in H_{S'}$. This makes sense because of the natural isometric isomorphism between $H_{S'}$ and $H_S \otimes H_{S' \setminus S}$. The maps $i(S,S')$ satisfy the following conditions :

(1) $i(S,S')$ is an isometric star-monomorphism of $BL(H_S)$ into $BL(H_{S'})$ $\quad (S,\ S' \in K,\ S \subseteq S')$

(2) If $S \subseteq S' \subseteq S''$, then $i(S,S'') = i(S',S'')i(S,S')$

(3) If $S \cap S'$ is empty, then every element of $i(S,S \cup S')(BL(S))$ commutes with every element of $i(S',S \cup S')(BL(S')$.

Using these conditions it is easy to construct a unital C^*-algebra A, with a family of closed star-subalgebras $\{A_S : S \in K\}$ and maps $\{j_S : S \in K\}$, such that

(4) j_S is an isometric star-monomorphism of $BL(H_S)$ onto A_S

(5) If $S \subseteq S'$, then $A_S \subseteq A_{S'}$ and the following diagram commutes

$$\begin{tikzcd}
BL(H_S) \arrow[r, "i(S{,}S')"'] \arrow[d, "j_S"] & BL(H_{S'}) \arrow[d, "j_{S'}"] \\
A_S \arrow[r, "\text{inclusion}"'] & A_{S'}
\end{tikzcd}$$

(6) If $S \cap S'$ is empty, then every element of A_S commutes with every element of $A_{S'}$

(7) $\bigcup \{A_S : S \in K\}$ is a dense star-subalgebra of A.

We now construct a representation of $\mathbb{Z}^\nu$ on A by isometric automorphisms.

For each x in $\mathbb{Z}^\nu$, let $V_0(x)$ be a unitary mapping of H_0 onto H_x. The transformations $V_0(x)$ may be chosen arbitrarily subject only to the restriction $V_0(0) = 1$. Let

$$V_x(y) = V_0(x + y)V_0(x)^{-1} .$$

Then $V_x(y)$ is a unitary mapping of H_x onto H_{x+y}, and

(8) $\qquad V_x(y + z) = V_{x+y}(z)V_x(y) \qquad (x,\ y,\ z \in \mathbb{Z}^\nu)$.

For each S in K and y in $\mathbb{Z}^\nu$, let

$$V_S(y) = \bigotimes_{x \in S} V_x(y).$$

Then $V_S(y)$ is a unitary map of H_S onto H_{S+y}. Finally, for S in K, a in A_S, and y in $\mathbb{Z}^\nu$, let

$$T(y)a = j_{S+y}(V_S(y)(j_S^{-1}a)V_{S+y}(-y)).$$

Then $T(y)|A_S$ is an isometric star-monomorphism of A_S onto A_{S+y}. Using conditions (1) to (8), it is easy to check that $T(y)$ extends to an isometric automorphism of A (also denoted by $T(y)$) and that

$$T(y + z) = T(y)T(z) \qquad (y, z \in \mathbb{Z}^\nu).$$

Thus T is a representation of $\mathbb{Z}^\nu$ on A by isometric automorphisms. (A,G,T) has a quasi-local structure, as described on pages 9 and 10 of the Introduction, and is therefore asymptotically abelian. Triples (A,G,T) such that A is not a C^*-algebra can be obtained in a similar way by starting with finite-dimensional Banach spaces which are not Hilbert spaces.

<u>16.5 Example</u>. Let A be the disc algebra, let $G = \mathbb{R}$, and, as in Example 4.8, define T by

$$T(t)(f)(z) = f(e^{it} z) \qquad (z \in \Delta,\ f \in A(\Delta),\ t \in \mathbb{R}).$$

Then T is a representation of $\mathbb{R}$ on $A(\Delta)$ by isometric automorphisms and it is easy to check that the only $\mathbb{R}$-invariant state of $A(\Delta)$ is the character

$$f \longrightarrow f(0) \qquad (f \in A(\Delta)).$$

17. G-abelian C*-algebras.

None of the results in this section are original, and its main purpose is to demonstrate some of the basic techniques used in the study of triples (A,G,T) such that A is a C*-algebra.

Throughout this section A will denote a unital C*-algebra, G a group, and T a representation of G on A by isometric automorphisms. As in the previous section, let

$$D(A) = \{f \in A' : \|f\| = 1 = f(1)\}$$

and $$D(A,G) = \{f \in D(A) : f(g.a) = f(a) \quad (a \in A,\ g \in G)\}.$$

Recall that a linear functional f on A is <u>positive</u> if

$$f(a^*a) \geqslant 0 \qquad (a \in A).$$

It is well-known that if f is in A', then f is in $D(A)$ if and only if f is positive and of norm 1 [15].

We begin by describing the well-known Gelfand-Naimark-Segal construction. A <u>unitary representation</u> of G on a Hilbert space H is a (group) homomorphism of G into the group of unitary operators on H.

17.1 The Gelfand-Naimark-Segal construction (see, for example, [72]).

Let f be a G-invariant state on A and let

$$L_f = \{a \in A : f(a^*a) = 0\}.$$

Then L_f is a closed left ideal of A invariant under $T(g)$ for all g in G. For all a and b in A, let

$$(a + L_f, b + L_f)_f = f(b^*a).$$

Then $(\ ,\ )_f$ is an inner product on A/L_f. Let $(H_f, (\ ,\ )_f)$ be the Hilbert space completion of A/L_f and let $x_f = 1 + L_f \in H_f$. For all a in A, let

$$h_\varsigma(a)(b + L_\varsigma) = ab + L_\varsigma \qquad (b \in A).$$

Then $h_\varsigma(a)$ extends to a bounded linear operator on H_ς, also denoted by $h_\varsigma(a)$, such that

(1) h_ς is a star-homomorphism of A into $BL(H_\varsigma)$

(2) $(h_\varsigma(A)x_\varsigma)^- = H_\varsigma$

(3) $f(a) = (h_\varsigma(a)x_\varsigma, x_\varsigma)_\varsigma \qquad (a \in A)$.

For all g in G, let

$$U_\varsigma(g)(a + L_\varsigma) = g.a + L_\varsigma \qquad (a \in A).$$

Then $U_\varsigma(g)$ extends to a unitary operator on H_ς, also denoted by $U_\varsigma(g)$ such that

(4) U_ς is a unitary representation of G on H_ς

(5) $U_\varsigma(g)h_\varsigma(a) = h_\varsigma(g.a)U_\varsigma(g) \qquad (g \in G,\ a \in A)$

(6) $U_\varsigma(g)x_\varsigma = x_\varsigma \qquad (g \in G)$.

The notation of 17.1 will be used throughout this section. In addition, let $K_\varsigma = \{y \in H_\varsigma : U_\varsigma(g)y = y \quad (g \in G)\}$, and let P_ς be the orthogonal projection of H_ς onto K_ς. Note that x_ς is a unit vector in K_ς.

<u>17.2 Definition</u> (Lanford and Ruelle [55]).

(A,G,T) is <u>G-abelian</u> if $[P_\varsigma h_\varsigma(a)P_\varsigma, P_\varsigma h_\varsigma(b)P_\varsigma] = 0$ for all f in $D(A,G)$ and a and b in A.

For confirmation that the triples (A,G,T) used in the C*-algebra approach to statistical mechanics (and including Example 16.4) are G-abelian, see Chapters 7 and 8 of [69].

Definition 17.2 does not generalise very easily to the case when A is not a C*-algebra. In Theorem 17.5 we show that it is equivalent to a condition which, at least when G is an amenable group, can be generalised. The statement and proof of Theorem 17.5 require the following information concerning positive definite functions.

17.3 Definition. A positive definite function on G is an element f of $l^\infty(G)$ such that

$$\sum_{i,j=1}^{n} \overline{z_i} z_j f(g_i^{-1} g_j) \geqslant 0,$$

for all $n \geqslant 1$, $g_1, \ldots\ldots, g_n \in G$ and $z_1, \ldots\ldots, z_n \in \mathbb{C}$.
Equivalently, $f \in l^\infty(G)$ is positive definite if and only if there exists a unitary representation U of G on a Hilbert space H, and an element x of H such that

$$f(g) = (U(g)x,x) \qquad (g \in G).$$

Let $V(G)$ denote the linear span in $l^\infty(G)$ of the positive definite functions. Then $V(G)$ is equal to the set of all f in $l^\infty(G)$ such that

$$f(g) = (U(g)x,y) \qquad (g \in G),$$

for some unitary representation U of G on some Hilbert space H, and some x and y in H. It is also closed under translations (i.e. $f \in V(G)$ implies ${}_gf$ and $f_g \in V(G)$, for all g in G).

The following result is due to Godemont (see also [19]).

17.4 Theorem (Godemont [26]).

$\overline{V(G)}$ is a star-subalgebra of $l^\infty(G)$ containing the identity element 1 and closed under translations. There exists a unique state M on $\overline{V(G)}$ such that

(1) $M({}_gf) = M(f_g) = M(f) \qquad (f \in V(G),\ g \in G)$

(2) If U is a unitary representation of G on a Hilbert space H, x and y are in H, and $f(g) = (U(g)x,y)$, then $M(f) = (Px,y)$, where P is the projection of H onto $K = \{z$

$\in H : U(g)z = z\}$. Similarly, if $f(g) = (x, U(g)y)$, then $M(f) = (x, Py)$.

(3) If G is amenable, then the restriction to $\overline{V(G)}$ of every 2-sided invariant mean on $l^\infty(G)$ is equal to M.

M is determined uniquely by condition (1) and is referred to in [19] as the Godemont mean. For the proof of (3), see [19].

<u>17.5 Theorem</u> [20].

For all f in $D(A,G)$ and a and b in A, define $w(f,a,b) \in l^\infty(G)$ by

$$w(f,a,b)(g) = f(a(g.b) - (g.b)a) \qquad (g \in G).$$

Then $w(f,a,b) \in V(G)$, and the following statements are equivalent :

(1) (A,G,T) is G-abelian

(2) $M(w(f,a,b)) = 0 \qquad (f \in D(A,G),\ a,b \in A)$,

where M is the Godemont mean, as in Theorem 17.4.

<u>Proof</u>. For f in $D(A,G)$ and for each unit vector y of K_ζ, let

$$f_y(c) = (h_\zeta(c)y, y) \qquad (c \in A).$$

Then f_y is in $D(A,G)$ and, for all a and b in A,

$$w(f_y,a,b)(g) = (U_\zeta(g)h_\zeta(b)y, h_\zeta(a^*)y)_\zeta - (U_\zeta(g^{-1})h_\zeta(a)y, h_\zeta(b^*)y)_\zeta .$$

$w(f_y,a,b)$ is therefore in $V(G)$, and

$$M(w(f_y,a,b)) = (P_\zeta h_\zeta(b)y, h_\zeta(a^*)y)_\zeta - (P_\zeta h_\zeta(a)y, h_\zeta(b^*)y)_\zeta$$
$$= ([P_\zeta h_\zeta(a)P_\zeta, P_\zeta h_\zeta(b)P_\zeta]y, y)_\zeta .$$

In particular, taking $y = x_\zeta$, we have $w(f,a,b) \in V(G)$, and, if (A,G,T) is G-abelian, $M(w(f,a,b)) = 0$.

Conversely, if condition (2) is satisfied, then $M(w(f_y,a,b)) = 0$ for all y in K_ζ, and hence $[P_\zeta h_\zeta(a)P_\zeta, P_\zeta h_\zeta(b)P_\zeta] = 0$ for all $a, b \in A$ and $f \in D(A,G)$.

18. M-asymptotic abelianness.

The major obstacle to the generalisation of the term 'G-abelian' to cover triples (A,G,T) such that A is not a C*-algebra is the absence, in general, of the representations of A provided by the Gelfand-Naimark-Segal construction. There is also no reason, when A is not a C*-algebra, to expect that the functions $w(f,a,b)$ of Theorem 17.5 are in the domain of definition of the Godemont mean. Thus neither Definition 17.2 nor the equivalent definition given by Theorem 17.5 is meaning ful when A is not a C*-algebra. We avoid these problems by assuming that G is an amenable group, and replacing the Godemont mean by an invariant mean defined on the whole of $l^{\infty}(G)$.

Throughout this ~~this~~ section and the next, (A,G,T) will be a fixed triple consisting of a complex unital Banach algebra A, an amenable group G and a representation T of G on A by isometric automorphisms. M will denote a fixed 2-sided and invariant mean on $l^{\infty}(G)$.

The following notation will be used :

$A(G) = \{a \in A : g.a = a \quad (g \in G)\}$

$A'(G) = \{f \in A' : f(g.a) = f(a) \quad (a \in A,\ g \in G)\}$

$D(A) = \{f \in A' : \|f\| = 1 = f(1)\}$

$D(A,G) = A'(G) \cap D(A)$.

For reasons which will soon become apparent, it will be convenient to use the following integral type notation for M :

$$M(f) = ⨍ f(g)\,dM(g) \qquad (f \in l^{\infty}(G)).$$

The line through the integral sign indicates that, unless G is finite, M cannot be represented by a countably additive measure on G, so that M(f) is <u>not</u> a genuine integral.

In terms of this notation, the most important properties of M for the purposes of this section and the next may be expressed as follows :

18.1

(1) $$\int f(hg.a)dM(g) = \int f(g.a)dM(g) \qquad (f \in A', h \in G, a \in A)$$

(2) $$\int f(gh.a)dM(g) = \int f(g.a)dM(g) \qquad (f \in A', h \in G, a \in A)$$

(3) $$\left|\int f(g.a)dM(g)\right| \leqslant \|f\|\|a\| \qquad (f \in A'(G), a \in A)$$

(4) $$\int f((g.a)b)dM(g) = \int f(a(g.b))dM(g) \qquad (f \in A'(G), a, b \in A).$$

(4) is a consequence of the inversion invariance of M. Note that if f is a bounded complex-valued function on $G \times G$, then the 'double integrals'

$$\int\left(\int f(g,h)dM(g)\right)dM(h) \quad \text{and} \quad \int\left(\int f(g,h)dM(h)\right)dM(g)$$

make sense, but are not in general equal.

The following Proposition demonstrates that the restriction to amenable groups guarantees the existence of G-invariant states.

<u>18.2 Proposition</u>. For all f in A', let

$$(Ef)(a) = \int f(g.a)dM(g) \qquad (a \in A).$$

Then E is a continuous projection of A onto A'(G) such that $\|E\| = 1$ and $E(D(A)) = D(A,G)$.

Proof. The C*-algebra case is contained in lemma 2.3 of [18], and essentially the following argument is used in the proof of Proposition 6.2.13 of [69].

Let f be any element of A'. Then Ef is in A', $\|Ef\| \leq \|f\|$ and

$$(Ef)(h.a) = \int f(gh.a)dM(g) = \int f(g.a)dM(g) = Ef(a) \quad (a \in A,\ h \in G).$$

E is therefore a norm reducing linear mapping of A' into $A'(G)$. Now let f be any element of $A'(G)$. Then

$$(Ef)(a) = \int f(g.a)dM(g) = \int f(a)dM(g) = f(a)M(1) = f(a) \quad (a \in A).$$

This proves that $E(A') = A'(G)$ and that $E^2 = E$. Finally, if f is a state, then

$$(Ef)(1) = \int f(g.1)dM(g) = M(1) = 1,$$

and therefore $\|E\| = 1$ and $E(D(A)) = D(A,G)$.

It is well known that A' is equal to the linear span of the state space $D(A)$. In fact, given f in A', there exist $f_1, \ldots, f_4$ in $D(A)$ and ~~positive~~ non-negative real numbers $s_1, \ldots, s_4$ such that $f = s_1 f_1 - s_2 f_2 + i(s_3 f_3 - s_4 f_4)$ and $s_1 + s_2 + s_3 + s_4 \leq e\sqrt{2}\|f\|$ [8, p.100]. Now suppose that f is in $A'(G)$. Then $Ef = f = s_1 Ef_1 - s_2 Ef_2 + i(s_3 Ef_3 - s_4 Ef_4)$ and $Ef_1, \ldots, Ef_4$ are in $D(A,G)$. This proves the following result.

18.3 Corollary. Let f be any element of $A'(G)$. Then there exist $f_1, \ldots, f_4$ in $D(A,G)$ and positive real numbers s_1,

s_2, s_3, s_4 such that $f = s_1 f_1 - s_2 f_2 + i(s_3 f_3 - s_4 f_4)$ and $s_1 + s_2 + s_3 + s_4 \leqslant \sqrt{2}e\|f\|$.

It is not clear whether or not $A'(G)$ is always equal to the linear span of $D(A,G)$ when G is not amenable.

<u>18.4 Definition.</u> (A,G,T) is <u>M-asymptotically abelian</u> if

$$\int f(a(g.b) - (g.b)a)dM(g) = 0 \qquad (f \in D(A,G),\ a,\ b \in A).$$

Note that if A is commutative, then (A,G,T) is automatically M-asymptotically abelian.

In the case when A is a C*-algebra, this definition was used in [18]. In fact, when A is a C*-algebra, (A,G,T) is M-asymptotically abelian if and only if it is G-abelian. This follows immediately from the fact that the restriction of M to $\overline{V(G)}$ is the Godemont mean (see Theorem 17.4 (3) and Theorem 17.5).

The rest of this section is the result of an attempt to characterise M-asymptotically abelian triples in terms of a Banach algebra B associated with A in a certain way. The Banach algebra B is described in Theorem 18.5 and a simple calculation shows that if B is commutative, then (A,G,T) is M-asymptotically abelian. I do not know if B is necessarily commutative when (A,G,T) is M-asymptotically abelian. However, this is true when G is finite or A is Arens regular.

<u>Notation.</u> It will be convenient to regard A' as an A-bimodule, with the module operations defined by

$$(a.f)(x) = f(xa) \quad \text{and} \quad (f.a)(x) = f(ax) \qquad (a, x \in A, f \in A').$$

18.5 Theorem. Let E be the projection of A' onto $A'(G)$ defined, as in Proposition 18.2, by

$$(Ef)(a) = \int f(g.a)dM(g) \qquad (f \in A', a \in A).$$

For all a in A, define $Pa \in BL(A'(G))$ by

$$(Pa)(f) = E(a.f) \qquad (f \in A'(G)),$$

and let B be the closed subalgebra of $BL(A'(G))$ generated by the set $\{Pa : a \in A\}$. Then B is a unital Banach algebra and P is a continuous linear mapping of A into B. Define $Q : A'(G) \longrightarrow B'$ by

$$(Qf)(U) = (Uf)(1) \qquad (f \in A'(G), U \in B).$$

Then the following conditions are satisfied :

(1) $\|P\| = 1$ and $P(1) = 1$

(2) $P(h.a) = Pa \qquad (a \in A, h \in G)$

(3) $P(ab) = (Pa)(Pb)$ and $P(ba) = (Pb)(Pa) \qquad (a \in A(G), b \in A)$

(4) $P'Q = 1$ (where P' is the adjoint of P)

(5) Q is linear and $\|Qf\| = \|f\|$ for all f in $A'(G)$.

Proof. Note that

$$(Pa)(f)(b) = \int f((g.b)a)dM(g) \qquad (a, b \in A, f \in A'(G)).$$

Let f be any element of $A'(G)$. Then

$$\|Pa(f)\| \leqslant \|a.f\| \leqslant \|a\|\|f\| \qquad (a \in A)$$

and so $\|P\| \leqslant 1$. Also,

$$P(1)(f) = Ef = f \qquad (f \in A'(G)),$$

and, therefore, B is unital, $P(1) = 1$ and $\|P\| = 1$.

Now let a be in A and h be in G. Then,

$$P(h.a)(f)(b) = \int f((g.b)(h.a)dM(g) = \int f((h^{-1}g.b)a)dM(g) =$$

$$\int f((g.b)a)dM(g) = (Pa)(f)(b) \qquad (f \in A'(G),\ b \in A),$$

which proves (2). To prove (3), note that

$$a.f \in A'(G) \qquad (f \in A'(G),\ a \in A(G)).$$

Thus, if b is any element of A and a is in $A(G)$, then

$$P(ab)(f)(x) = \int f((g.x)ab)dM(g) = \int f((g.xa)b)dM(g) = (Pb(f))(xa)$$

$$= (a.Pb(f))(x) = E(a.Pb(f))(x) = (PaPb)(f)(x) \qquad (f \in A'(G),\ x \in A).$$

A similar argument shows that

$$P(ba) = PbPa \qquad (a \in A(G),\ b \in A).$$

To prove (4), let f be in $A'(G)$ and a be in A. Then,

$$(P'Q)(f)(a) = Qf(Pa) = Pa(f)(1) = \int f((g.1)a)dM(g) = f(a),$$

and therefore $P'Q = 1$.

Finally, $|Qf(U)| \leq \|U\|\|f\|$ for all f in $A'(G)$ and U in B, and therefore $\|Qf\| \leq \|f\|$, and $\|Qf\| \geq \sup\{|Qf(Pa)| : a \in A,\ \|a\| \leq 1\} = \sup\{|f(a)| : a \in A,\ \|a\| \leq 1\} = \|f\|$, so that Q is isometric, as required.

The notation of Theorem 18.5 will be used throughout the rest of the Chapter. Note that (4) is equivalent to

$$Pa(f)(1) = f(a) \qquad (a \in A,\ f \in A'(G)).$$

<u>18.6 Proposition.</u> If B is commutative, then (A,G,T) is M-asymptotically abelian.

<u>Proof.</u> Suppose that B is commutative. Then, for all f in $D(A,G)$ and a and b in A,

$$\int f(a(g.b) - (g.b)a)dM(g) = \int f((g.a)b)dM(g) - \int f((g.b)a)dM(g)$$

$$= (Pb(f))(a) - (Pa(f))(b) = [Pa,Pb](f)(\overset{1}{\cancel{a}}) = 0.$$

(A,G,T) is therefore M-asymptotically abelian.

For all f in $A'(G)$ and a, x and b in A, the function $(g,h) \longrightarrow f((g.a)x(h.b))$ is a bounded function on $G \times G$. The following lemma suggests that the converse of Proposition 18.6 may fail because of the non-reversibility of the order of 'integration' in the expression

$$\int\left(\int f((g.a)x(h.b)dM(h)\right)dM(g).$$

<u>18.7 Lemma</u>. If (A,G,T) is M-asymptotically abelian, then

$$\int\left(\int f((g.a)x(h.b))dM(h)\right)dM(g) - \int\left(\int f((g.a)x(h.b))dM(g)\right)dM(h)$$

$$= (PaPb - PbPa)(f)(x) \qquad (f \in A'(G),\ a,\ b,\ x \in A).$$

<u>Proof</u>. First note that, by Corollary 18.3 and Definition 18.4, we have

$$\int f(a(g.b) - (g.b)a)dM(g) = 0 \qquad (f \in A'(G),\ a,b \in A).$$

Let f be in $A'(G)$, and let a, b and x be any elements of A. Then, using 18.1 (3),

$$(PaPb)(f)(x) = \int Pb(f)((g.x)a)dM(g) = \int Pb(f)(a.(g.x))dM(g)$$

$$= \int Pb(f)((g.a)x)dM(g) = \int\left(\int f(h.((g.a)x)b)dM(h)\right)dM(g)$$

$$= \int\left(\int f((g.a)x(h.b))dM(h)\right)dM(g).$$

Similarly,

$$(PbPa)(f)(x) = \int Pa(f)((h.x)b)\,dM(h) = \int Pa(f)(x(h.b))\,dM(h) =$$

$$\int\left(\int f(g.(x(h.b))a)\,dM(g)\right)dM(h) = \int\left(\int f(ag.(x(h.b))\,dM(g)\right)dM(h)$$

$$= \int\left(\int f((g.a)x(h.b))\,dM(g)\right)dM(h).$$

We now describe the Arens products on the second dual A'' of A, and define the term 'Arens regular'. j will denote the natural embedding of A into A''.

<u>18.8 Definition</u> (see, for example, [9, p.50]).

For all S in A'' and f in A', define $S.f$ and $S_o f$ in A' by

$$(S.f)(a) = S(f.a) \quad \text{and} \quad S_o f(a) = S(a.f) \qquad (a \in A).$$

The <u>Arens products</u> $\cdot$ and $\circ$ on A'' are defined by

$$(R.S)(f) = R(S.f) \quad \text{and} \quad (R_o S)(f) = R(S_o f) \qquad (f \in A',\ R,S \in A'').$$

For both these products, A'' is a Banach algebra and j is a monomorphism of A into A''.

A is <u>Arens regular</u> if

$$R.S = S_o R \qquad (S,\ R \in A'').$$

<u>18.9 Theorem</u>. If A is Arens regular, then (A,G,T) is M-asymptotically abelian if and only if B is commutative.

<u>Proof</u>. By Proposition 18.6, it is sufficient to prove that if A is Arens regular and (A,G,T) is M-asymptotically abelian, then B is commutative.

Let f be in $A'(G)$, let a, b and x be any elements of A, and define I_1 and I_2 by

$$I_1 = \int\left(\int f((g.a)x(h.b))dM(h)\right)dM(g)$$

$$I_2 = \int\left(\int f((g.a)x(h.b))dM(g)\right)dM(h)$$

By lemma 18.7, it is sufficient to prove that if A is Arens regular, then $I_1 = I_2$. Let E' denote the adjoint of the operator E on A' defined in Proposition 18.2. We show that

$$I_1 = (E'ja).(jx).(E'jb)$$

$$I_2 = (E'jb)_o(jx)_o(E'ja),$$

from which the required result follows immediately, using Definition 18.8.

In the following calculations, we use the identities

$$((jy).S)(k) = S(k.y)$$

$$((jy)_o S)(k) = S(y.k)$$

$$(E'jy)(k) = \int k(g.y)dM(g),$$

where $k \in A'$, $y \in A$ and $S \in A''$.

Applying Definition 18.8, we get

$$(E'ja).(jx).(E'jb) = E'ja((jx.E'jb).f) =$$

$$\int((jx.E'jb).f)(g.a)dM(g) = \int(jx.E'jb)(f.(g.a))dM(g)$$

$$= \int E'jb((f.(g.a)).x)dM(g) = \int\left(\int(f.(g.a)x)(h.b)dM(h)\right)dM(g) = I_1.$$

Similarly, $(E'jb)_o(jx)_o(E'ja) = E'jb((jx_oE'ja)_of) =$

$$\int((jx_oE'ja)_of)(h.b)dM(h) = \int(jx_oE'ja)((h.b).f)dM(h)$$

$$= \int E'ja(x(h.b).f)dM(h) = \int\left(\int(x(h.b).f)(g.a)dM(g)\right. = I_2.$$

18.10 Corollary. If A is a C^*-algebra, then (A,G,T) is G-abelian if and only if B is commutative.

Proof. By, for example, [7, p.109-110], every C^*-algebra is Arens regular. The Corollary therefore follows immediately from the Theorem and the fact that (A,G,T) is G-abelian if and only if it is M-asymptotically abelian.

19. Uniqueness of ergodic decomposition.

One of the best known results concerning triples (A,G,T) such that A is a unital C*-algebra, is that if (A,G,T) is G-abelian, then the G-invariant state space $D(A,G)$ is a simplex (Lanford and Ruelle [55], see also [69] and [72]). The purpose of this section is to explore the rather limited possibilities for proving results of this kind when A is not a C*-algebra.

The section begins with a brief account of the theory of boundary measures on compact convex sets, leading to the definition of the term 'simplex', and based largely on the book 'Compact convex sets and boundary integrals', by E.M. Alfsen [2] (see also [69], p.206).

The natural setting for the theory of compact convex sets is an arbitrary locally convex real vector space. However, for ease of application, we consider only weak* compact convex sets in the dual space of a complex Banach space.

19.1 Boundary measures.

Let X be a complex Banach space, and let K be a weak* compact convex subset of the dual space X' of X.

A real-valued function a on K is *convex* if

$$a(tf + (1 - t)g) \leqslant ta(f) + (1 - t)a(g) \qquad (f, g \in K, 0 \leqslant t \leqslant 1),$$

and *affine* if

$$a(tf + (1 - t)g) = ta(f) + (1 - t)a(g) \qquad (f, g \in K, 0 \leqslant t \leqslant 1).$$

Let $C(K)$ denote the C*-algebra of all weak* continuous complex-valued functions on K, and let

$$C(K,\mathbb{R}) = \{f \in C(K) : f \text{ is real-valued}\}$$

$$\mathrm{Conv}(K) = \{ f \in C(K,\mathbb{R}) : f \text{ is convex}\}$$

$\mathrm{Aff}(K) = \{f \in C(K,\mathbb{R}) : f \text{ is affine}\}$.

Also, let $M(K)$ denote the dual space of $C(K)$, which can and will be identified with the space of all Baire measures on K (see [2, p.9]), and let

$M(K,\mathbb{R}) = \{m \in M(K) : m \text{ is real-valued}\}$

$M^+(K) = \{m \in M(K,\mathbb{R}) : m \geqslant 0\}$

$M_1^+(K) = \{m \in M^+(K) : \|m\| = 1\}$ $(= D(C(K)))$.

Define the binary relation $\prec$ on $M(K,\mathbb{R})$ by

$$m \prec n \iff m(a) \leqslant n(a) \qquad (a \in \mathrm{Conv}(K)).$$

Then $\prec$ is clearly reflexive and transitive. Suppose that $m \prec n \prec m$. Then $m(a) = n(a)$ $(a \in \mathrm{Conv}(K))$.

By [2, Proposition I.1.1], the linear span of $\mathrm{Conv}(K)$ is uniformly dense in $C(K)$. Thus, $m = n$ and $\prec$ is therefore a partial ordering on $M(K,\mathbb{R})$.

$m \in M(K,\mathbb{R})$ is a <u>boundary measure</u> if $|m|$ is maximal in $M^+(K)$ with respect to the ordering $\prec$.

The <u>extreme boundary</u> of K is the set $\mathrm{Ext}(K)$ of all extreme points of K. The following Proposition is relevant to the interpretation of the results of this section, although not to their proof.

<u>19.2 Proposition</u>. Let m be any element of $M(K,\mathbb{R})$. If m is a boundary measure, then $|m|(S) = 0$ for all Baire sets S of K such that $S \cap \mathrm{Ext}(K)$ is empty. If K is metrizable, then $\mathrm{Ext}(K)$ is a Baire set, and m is a boundary measure if and only if $|m|(K \setminus \mathrm{Ext}(K)) = 0$.

For a discussion and proof of the various parts of this Proposition, see [2, p. 31 - 44].

The basic result on the existence of boundary measures is the following. For the proof, which is a straightforward application of Zorn's lemma, see [2, p.36].

19.3 Lemma. For every m in $M^+(K)$, there exists a positive boundary measure n such that $m \prec n$.

19.4 Definition [2, p.22]. Let f be any element of K. Then m in $M^+(K)$ represents f if

$$a(f) = m(a) \qquad (a \in Aff(K)).$$

For all f in K, let $w_\varsigma(a) = a(f) \quad (a \in C(K))$. Then w_ς is in $M^+(K)$ and represents f.

19.5 Theorem (Choquet-Bishop-de Leeuw) [2, p.36].

Every point f of K can be represented by a positive boundary measure.

Proof. By lemma 19.3, there exists a positive boundary measure m such that $w_\varsigma \prec m$. If $a \in Aff(K)$, then $a \in Conv(K)$ and $-a \in Conv(K)$. Thus, $a(f) = w_\varsigma(a) \leqslant m(a) \leqslant w_\varsigma(a) = a(f)$. m therefore represents f.

19.6 Definition. K is a simplex if every point of K is represented by a unique positive boundary measure.

It follows easily from the Riesz representation theorem that the state space of a commutative C*-algebra with identity is a simplex. For an example of a commutative unital Banach algebra A such that the state space $D(A)$ of A is not a simplex, see [25].

There are many equivalent definitions of the term 'simplex'. The most useful of these for the purposes of this section is stated in Theorem 19.10. The proof of 19.10 requires the following three lemmas, all of which are well-known.

<u>19.7 Lemma</u> [2, Proposition I.2.3].

Let m in $M_1^+(K)$ represent the point f of K, let a be in $C(K,\mathbb{R})$, and let $\varepsilon > 0$. Then there exist $f_1, \ldots\ldots, f_n$ in K and positive real numbers $t_1, \ldots\ldots, t_n$ such that

$$f = \sum_{i=1}^{n} t_i f_i, \quad \sum_{i=1}^{n} t_i = 1 \quad \text{and} \quad \left| m(a) - \left(\sum_{i=1}^{n} t_i w_{f_i} \right)(a) \right| < \varepsilon .$$

<u>19.8 Lemma</u> [2, p.25].

Let f be in K and m in $M^+(K)$. Then $w_f \prec m$ if and only if m represents f.

For x in X, define $\hat{x}$ on X' by

$$\hat{x}(f) = f(x) \qquad (f \in X').$$

Note that, for each x in X, the real and imaginary parts $re(\hat{x}|K)$ and $im(\hat{x}|K)$ of $\hat{x}|K$ are in $Aff(K)$.

<u>19.9 Lemma</u>. Suppose that there is an element e in X such that $f(e) = 1$ $(f \in K)$. Then m in $M^+(K)$ represents f in K if and only if $f(x) = m(\hat{x}|K)$ $\quad (x \in X)$.

<u>Proof</u>. Suppose m represents f. Then $m(\hat{x}|K) = m(re(\hat{x}|K)) + im(im(\hat{x}|K)) = re(\hat{x}|K)(f) + i\,im(\hat{x}|K)(f) = f(x)$, for all x in X. Conversely, suppose $f(x) = m(\hat{x}|K)$, for all x in X. Then $m(re(\hat{x}|K)) = ref(x) = re(\hat{x}|K)(f)$, for all x in X. It is therefore sufficient to prove that the set $\{re(\hat{x}|K) : x \in X\}$ is uniformly dense in $Aff(K)$. This follows from [2, I.1.5].

A mapping S from K into a linear space is <u>affine</u> if $S(tf + (1 - t)g) = tS(f) + (1 - t)S(g) \quad (f,g \in K,\ 0 \leqslant t \leqslant 1)$.

<u>19.10 Theorem</u> [54, p.203, Ex.8].

Suppose, as in lemma 19.9, that there is an element e in X such that $f(e) = 1 \quad (f \in K)$. Then the following are equivalent :

(1) K is a simplex

(2) There is an affine mapping $f \longrightarrow m_f$ of K into $M_1^+(K)$ such that $m_f(\hat{x}|K) = f(x) \quad (f \in K,\ x \in X)$.

<u>Proof</u>. Let K be a simplex, and for all f in K, let m be the unique boundary measure representing f. Then, for all f and g in K and $0 \leqslant t \leqslant 1$, $tm_f + (1 - t)m_g$ is a boundary measure representing $tf + (1 - t)g$. The map $f \longrightarrow m_f$ is therefore affine.

Conversely, suppose condition (2) is satisfied and let n be any measure in $M^+(K)$ such that $m_f \prec n$. Then, by lemma 19.8, $w_f \prec m_f \prec n$, and so n represents f. To prove that m is the unique boundary measure representing f, it is therefore sufficient to prove that if n in $M^+(K)$ represents f, then $n \prec m_f$.

Let n represent f, let a be in Conv(K) and let $\varepsilon > 0$. By lemma 19.7, there exist $f_1, \ldots\ldots, f_n$ in K and positive real numbers $t_1, \ldots\ldots, t_n$ such that

$$f = \sum_{i=1}^{n} t_i f_i, \quad \sum_{i=1}^{n} t_i = 1 \text{ and } \left| n(a) - \left(\sum_{i=1}^{n} t_i w_{f_i}\right)(a) \right| < \varepsilon .$$

By lemma 19.8, $w_{f_i} \prec m_{f_i}$ for each i. Thus

$$n(a) - \varepsilon \leqslant \sum_{i=1}^{n} t_i w_{f_i}(a) \leqslant \sum_{i=1}^{n} t_i m_{f_i}(a) = m_f(a).$$

Since $\varepsilon > 0$ and

a in Conv(K) are arbitrary, this proves that $n \prec m_{\zeta}$, as required.

We now return to the study of the triple (A,G,T) as in Section 18. Recall that G is amenable and that M is a 2-sided and inversion invariant mean on $l^{\infty}(G)$. By Proposition 18.2, the G-invariant state space D(A,G) is non-empty.

A G-invariant state is <u>G-ergodic</u> if it is an extreme point of D(A,G). Let m be a positive boundary measure on D(A,G) representing a G-invariant state f. If A is separable, then D(A,G) is metrizable and, by Theorem 19.2, m is supported on the set Ext(D(A,G)) of G-ergodic states. Thus m may be regarded as an 'ergodic decomposition' of f, and D(A,G) is a simplex if and only if each f has a unique ergodic decomposition. If D(A,G) is not metrizable, then, because of the way in which the term 'boundary measure' has been defined, it still makes sense to ask whether or not D(A,G) is a simplex.

Let B be the subalgebra of BL(A'(G)) described in Theorem 18.5. We first establish sufficient conditions on B for D(A,G) to be a simplex. We use the notation of Section 18.

<u>19.11 Lemma</u>.

(1) $Q(D(A,G)) \subseteq D(B)$

(2) $P'(D(B)) = D(A,G)$.

<u>Proof</u>. Let f be in D(A,G). Then $\|Qf\| = \|f\| = 1 = f(1) = Qf(1)$, and Qf is therefore in D(B).

To prove (2), let f be in D(B). Then $\|P'f\| \leq \|f\| = 1 =$

$f(1) = f(P1) = (P'f)(1)$ and

$(P'f)(g.a) = f(P(g.a)) = f(Pa) = (P'f)(a) \qquad (a \in A,\ g \in G).$

$P'f$ is therefore a G-invariant state. Since $P'Q = 1$, we have $D(A,G) = P'Q(D(A,G)) \subseteq P'D(B) \subseteq D(A,G)$, and therefore $D(A,G) = P'D(B)$.

In applying Theorem 19.10 to $K = D(B)$ and $K = D(A,G)$ in the proof of the following result, we take $e = 1$.

<u>19.12 Theorem</u>. If the state space $D(B)$ of B is a simplex, then the G-invariant state space $D(A,G)$ of A is a simplex.

<u>Proof</u>. Suppose that $D(B)$ is a simplex. By Theorem 19.10, it is sufficient to construct an affine mapping $f \longrightarrow m_f$ of $D(A,G)$ into $M_1^+(D(A,G))$ such that

$$m_f(\hat{a}|D(A,G)) = f(a) \qquad (a \in A,\ f \in D(A,G)).$$

Let f be in $D(A,G)$. Then, by lemma 19.11, Qf is in $D(B)$. Let m_{Qf} be the unique positive boundary measure on $D(B)$ representing Qf. Then the mapping $f \longrightarrow m_{Qf}$ is affine and

$$m_{Qf}(\widehat{Pa}|D(B)) = Qf(Pa) = f(a) \qquad (a \in A,\ f \in D(A,G)).$$

Now define $h : C(D(A,G)) \longrightarrow C(D(B))$ by

$$h(c)(f) = c(P'f) \qquad (c \in C(D(A,G)),\ f \in D(B)).$$

This definition makes sense because P' is weak* continuous and, by lemma 19.11, $P'D(B) = D(A,G)$. Note that h is a linear isometry and that

$$h(\hat{a}|D(A,G)) = \widehat{Pa}|D(B) \qquad (a \in A).$$

Let h' denote the adjoint of h and let

$$m_f = h'(m_{Qf}) \qquad (f \in D(A,G)).$$

Then $f \longrightarrow m_f$ is an affine mapping of $D(A,G)$ into $M_1^+(D(A,G))$ and $m_f(\hat{a}|D(A,G)) = m_{Qf}(h(\hat{a}|D(A,G)) = m_{Qf}(\widehat{Pa}|D(B))$

$= f(a) \quad (a \in A,\ f \in D(A,G))$. This completes the proof.

19.13 Corollary. If B is a commutative C*-algebra, then $D(A,G)$ is a simplex and (A,G,T) is M-asymptotically abelian.

Proof. This follows immediately from the theorem and Proposition 18.6.

As would be expected, B is always a C*-algebra if A is a C*-algebra and (A,G,T) is G-abelian. (see Theorem 19.23). Thus Corollary 19.13 includes, for the case of amenable groups, the theorem of Lanford and Ruelle mentioned at the beginning of this section.

Now consider the triple $(A(\Delta),\mathbb{R},T)$ of Example 16.5. In this case, $A(\Delta)'(\mathbb{R})$, and therefore B, is 1-dimensional. Thus the conditions of Corollary 19.13 are satisfied. It is however quite obvious that $D(A(\Delta),\mathbb{R})$ is a simplex, since it has only one element.

The rest of this section is the result of an attempt, using numerical range theory, to find necessary and sufficient conditions on (A,G,T) for B to be a C*-algebra. The hope is that, in the absence of convincing examples, such conditions will give some idea of the scope of Corollary 19.13.

19.14 Definition [7,8].

Let A be any complex unital Banach algebra. Then the numerical range of the element a of A is the set

$$V(A,a) = \{f(a) : f \in D(A)\}.$$

An element a of A is hermitian if $V(A,a)$ is contained in $\mathbb{R}$. Let $H(A)$ denote the set of all hermitian elements of A.

If A is a C*-algebra, then $H(A) = \{a \in A : a = a^*\}$, and therefore $A = H(A) + iH(A)$. One of the most useful results in numerical range theory is that this condition in fact characterises unital C*-algebras. This is the Vidav-Palmer theorem which we now state. For the proof see [9] or [7].

<u>19.15 Theorem (Vidav-Palmer)</u>. Let A be a complex unital Banach algebra. Then A is a C*-algebra if and only if $A = H(A) + iH(A)$.

The statement 'A is a C*-algebra' should be interpreted as meaning A has an involution with respect to which it is a C*-algebra.

The following Corollary of Theorem 19.15 will be used in the proof of Theorem 19.23.

<u>19.16 Corollary</u>. Let B be a closed subalgebra of a complex unital Banach algebra A such that B contains the identity element of A and the following conditions are satisfied :

(1) $H(B)$ generates B as a Banach algebra

(2) B has a set of generators B_o such that for all $n \geqslant 1$ and $b_1, \ldots\ldots, b_n$ in B_o, $b_1 \ldots\ldots b_n$ is an element of $H(A) + iH(A)$.

Then B is a C*-algebra.

<u>Proof</u>. Let $J(B) = H(B) + iH(B)$ and $J(A) = H(A) + iH(A)$. Then, by [7, lemma 5.8], $J(B)$ and $J(A)$ are closed. It is sufficient to prove that $J(B)$ is an algebra, since in that case, $J(B) = B$ and, by Theorem 19.15, B is then a C*-algebra. We shall use the following two facts which are proved in

[7, p.59 - 60].

(a) If $h \in H(B)$ implies $h^2 \in H(B)$, then $J(B)$ is an algebra

(b) If $h \in H(A)$ and $h^2 \in J(A)$, then $h^2 \in H(A)$.

We first note that, by condition (2), B is contained in $J(A)$. Also, by lemma 5.2 of [7], $H(B) = H(A) \cap B$. Now let h be in $H(B)$. Then $h^2 \in J(A)$ and so by (b), $h^2 \in H(B)$. By (a), $J(B)$ is an algebra, and this completes the proof.

The following lemma will be used in the proof of lemma 19.21 to determine the numerical range of certain operators on $A'(G)$. For any subset S of the complex numbers, $\overline{co}(S)$ denotes the smallest convex closed set containing S.

<u>19.17 Lemma</u> [7, Theorem 9.5].

Let X be a complex Banach space and let U be any element of $BL(X')$. Then $V(BL(X'),U) = \overline{co}\{Uf(x) : f \in X',\ x \in X \text{ and } \|f\| = \|x\| = f(x) = 1\}$.

Let $H(A,G) = \{a \in A : f(a)$ is real for all f in $D(A,G)\}$. Then $H(A)$ is contained in $H(A,G)$ and, if A is a C^*-algebra, then $A = H(A,G) + iH(A,G)$. We first show that this condition is also satisfied when B is a C^*-algebra. We then show that if A is Arens regular, then $A = H(A,G) + iH(A,G)$ if and only if B is a C^*-algebra. As in section 18, the Arens regularity of A is needed only to justify reversing the order of 'integration' in expressions of the form

$$\int\left(\int f((g.a)x(h.b))dM(h)\right)dM(g).$$

19.18 Lemma. Let a be any element of A. Then a is in $H(A,G)$ if and only if Pa is in $H(B)$.

Proof. This follows immediately from (2) of lemma 19.11.

19.19 Corollary.

(1) If B is a C*-algebra and $P(A)$ is self-adjoint, then $A = H(A,G) + iH(A,G)$.

(2) If $A = H(A,G) + iH(A,G)$, then B is generated by $H(B)$.

Proof. Suppose that B is a C*-algebra such that $P(A) = P(A)^*$ and let a be in A. Then there exist elements b and c of $H(A,G)$ such that $Pa = Pb + iPc$. Let f be in $D(A,G)$. Then $f(a - (b + ic)) = Qf(Pa - (Pb + iPc)) = 0$, and therefore $a - (b + ic)$ is in $H(A,G)$. Since $H(A,G)$ is a real linear subspace of A, this proves that $A = H(A,G) + iH(A,G)$.

Now suppose that $A = H(A,G) + iH(A,G)$. Then B is generated by the subset $P(H(A))$ of $H(B)$. This proves (2).

Note that if P is onto, then it follows immediately that B is a C*-algebra if and only if $A = H(A,G) + iH(A,G)$.

19.20 Theorem. If B is a C*-algebra, then $A = H(A,G) + iH(A,G)$.

Proof. We first show that $P(A)$ is isometrically isomorphic to $A/\mathrm{Ker}(P)$ and therefore closed. Since P is norm decreasing, it is sufficient to prove that $\|Pa\| \geq \|a + \mathrm{Ker}(P)\|$ for all a in A. Let a be in A. Then, by the Hahn-Banach theorem, there exists a continuous linear functional f on $A/\mathrm{Ker}(P)$ such that $\|f\| = 1$ and $f(a + \mathrm{Ker}(P)) = \|a + \mathrm{Ker}(P)\|$. Define $\bar{f} \in A'$ by $\bar{f}(x) = f(x + \mathrm{Ker}(P))$. Then $\bar{f}$ is

G-invariant, since $g.x - x \in \mathrm{Ker}(P)$ for all x in A and g in G. Thus $\|Pa\| \geqslant |Q\bar{f}(Pa)| = |\bar{f}(a)| = \|a + \mathrm{Ker}(P)\|$.

Now suppose that B is a C*-algebra. By Corollary 19.19, it is sufficient to prove that $P(A)$ is self-adjoint. To do this we will use certain properties of the hermitian functionals on B. A continuous linear functional f on B is <u>hermitian</u> if $f = f^*$, where $f^*(x) = \overline{f(x^*)}$ $(x \in B)$. Let $H(B')$ denote the set of all hermitian functionals on B. Then $H(B') \cap iH(B') = \{0\}$ and $B' = H(B') + iH(B')$. Also, f in B' is hermitian if and only if $f = t_1 f_1 - t_2 f_2$ for some f_1 and f_2 in $D(B)$ and positive real numbers t_1 and t_2. Suppose that $P(A)$ is not self-adjoint, and let a be an element of A such that $(Pa)^*$ is not in $P(A)$. Since $P(A)$ is closed, there is an element f of B' such that $f(P(A)) = \{0\}$ and $f((Pa)^*) \neq 0$. Let f_1 and f_2 be hermitian functionals on B such that $f = f_1 + if_2$. Then $P'f = 0$ and therefore $P'f_1 = -iP'f_2$. Let $H(A',G) = \mathbb{R}^+D(A,G) - \mathbb{R}^+D(A,G)$. Then, by lemma 19.11 (2), and the characterisation of hermitian functionals quoted above, we have $P'f_1 \in H(A',G) \cap iH(A',G)$. By lemma 19.11(1), this implies that $Q(P'f\ ) \in H(B') \cap iH(B') = \{0\}$. Since Q is 1:1, this gives $P'f_1 = P'f_2 = 0$. But then $\overline{f((Pa)^*)} = f^*(Pa) = (f_1 - if_2)(Pa) = (P'f_1)(a) - i(P'f_2)(a) = 0$, which is a contradiction. This completes the proof of the Theorem.

To prove that the converse of Theorem 19.20 is true when A is Arens regular, we require the following two lemmas.

<u>19.21 Lemma.</u> Let U be a bounded linear operator on $A'(G)$

such that $Uf(x)$ is real for all f in $A'(G)$ and x in A such that $E(f.x)$ is in $D(A,G)$. Then U is hermitian.

Proof. Let L be the closed linear span of the set $\{a - g.a : a \in A, g \in G\}$. Then $A'(G) = \{f \in A' : f(L) = \{0\}\}$. Let q be the natural mapping of A onto A/L. Then the adjoint q' of q is an isometric isomorphism of $(A/L)'$ onto $A'(G)$. Let $S = (q')^{-1}Uq'$. Then $V(BL(A'(G),U) = V(BL((A/L)',S)$ and therefore, by lemma 19.17, $V(BL(A'(G)),U)$ $= \overline{co}\{(Sf)(qx) : f \in (A/L)', x \in A, \text{ and } \|qx\| = \|f\| = f(qx) = 1\}$ $= \overline{co}\{(Uf)(x) : f \in A'(G), x \in A \text{ and } \|f\| = \|qx\| = f(x) = 1\}$. Let f in $A'(G)$ and x in A satisfy $\|f\| = \|qx\| = f(x) = 1$. To complete the proof of the lemma, it is sufficient to prove that $E(f.x)$ is in $D(A,G)$. Let x' be any element of A such that $x - x'$ is in L. Then, for all a in A,

$$E(f.x')(a) = \int f(x'(g.a))dM(g) = \int f((g.x')a)dM(g) = E(a.f)(x') =$$

$E(a.f)(x) = E(f.x)(a)$. Thus $E(f.x) = E(f.x')$ and therefore $\|E(f.x)\| \leqslant \|x'\|$. This proves that $\|E(f.x)\| \leqslant \|qx\| = 1$.

$$\text{Finally, } E(f.x)(1) = \int f(x.(g.1))dM(g) = f(x) = 1, \text{ and}$$

therefore $E(f.x)$ is in $D(A,G)$.

19.22 Lemma. If A is Arens regular, then $(Pa_1 \ldots Pa_n)(f)(x) =$

$$\int\left(\ldots\left(\int E(f.x)(a_1(g_2.a_2)\ldots(g_n.a_n)dM(g_n)\right)\ldots\right)dM(g_2),$$

for all f in $A'(G)$, $n \geqslant 2$ and $a_1, \ldots, a_n, x$ in A.

Proof. Suppose that A is Arens regular. Then, as in the proof of Theorem 18.9, we have, for all f in $D(A,G)$ and a, x, b in A,

$$\int\left(\int f((g.a)x(h.b))dM(h)\right)dM(g) = \int\left(\int f((g.a)x(h.b))dM(g)\right)dM(h).$$

We prove the lemma by induction on n, beginning with the case $n = 2$, and making repeated use of (3) of 18.1. Let $a_1, a_2 \in A$.

Then $(Pa_1)(Pa_2)(f)(x) = \int((Pa_2)(f))((g.x)a_1)dM(g)$

$$= \int\left(\int f(g_2.((g.x)a_1)a_2)dM(g_2)\right)dM(g)$$

$$= \int\left(\int f((g.x)a_1(g_2.a_2))dM(g_2)\right)dM(g)$$

$$= \int\left(\int f((g.x)a_1(g_2.a_2))dM(g)\right)dM(g_2)$$

$$= \int\left(\int f(xg.(a_1(g_2.a_2)))dM(g)\right)dM(g_2)$$

$$= \int\left(\int (f.x)(g.(a_1(g_2.a_2)))dM(g)\right)dM(g_2)$$

$$= \int E(f.x)(a_1(g_2.a_2))dM(g_2).$$

Now suppose that the result is true for some $n \geq 2$ and let $a_1, \ldots, a_{n+1}$ and x be in A. Then

$(Pa_1 \ldots Pa_{n+1})(f)(x) = (Pa_1 \ldots Pa_n)(Pa_{n+1})(f)(x)$

$$= \int\left(\ldots\left(\int E(Pa_{n+1}f.x)(a_1(g_2.a\ )\ldots(g_n.a_n))dM(g_n)\right)\ldots\right)dM(g_2).$$

Let $g_1, \ldots, g_n$ be in G. Then, as in the proof of the case $n = 2$, $E((Pa_{n+1}f).x)(a_1(g_2.a\ )\ldots(g_n.a_n)) =$

$$\int(Pa_{n+1}f)((g.x)a_1(g_2.a_2)\ldots(g_n.a_n))dM(g)$$

$$= \int E(f.x)(a_1(g_2.a_2)\ldots(g_{n+1}.a_{n+1}))dM(g_{n+1}).$$

This completes the proof of the lemma.

19.23 Theorem. If A is Arens regular, then B is a C^*-algebra if and only if $A = H(A,G) + iH(A,G)$.

Proof. By Theorem 19.20, Corollary 19.16 and Corollary 19.19 (2), we have only to prove that if $A = H(A,G) + iH(A,G)$ and A is Arens regular, then $Pa_1 \cdots\cdots Pa_n$ is in $H(BL(A'(G)) + iH(BL(A'(G))$ for all $n \geqslant 1$ and $a_1, \ldots\ldots, a_n$ in A. The case $n = 1$ follows immediately from lemma 19.18.

Let $n \geqslant 2$, let $a_1, \ldots\ldots, a_n$ be in A, and suppose that $A = H(A,G) + iH(A,G)$. Then there exist functions x_1 and x_2 from G^{n-1} into $H(A,G)$ such that

$a_1(g_2 \cdot a_2)\cdots\cdots(g_n \cdot a_n) = x_1(g_2,\ldots\ldots,g_n) + ix_2(g_2,\ldots\ldots,g_n)$,

for all $g_2, \ldots\ldots, g_n$ in G. Let f be in $D(A,G)$. Then

$f(x_1(g_2,\ldots\ldots,g_n)) = re(f(a_1(g_2 \cdot a\ )\cdots\cdots(g_n \cdot a_n))$ and

$f(x_2(g_2,\ldots\ldots,g_n)) = im(f(a_1(g_2 \cdot a\ )\cdots\cdots(g_n \cdot a_n))$. Thus, for $i = 1$ and 2 and all $g_2,\ldots\ldots, g_n$ in G,

$|f(x_i(g_2,\ldots\ldots,g_n))| \leq \|a_1\|\cdots\cdots\|a_n\|$. Now let f be in $A'(G)$. By Corollary 18.3, there exist $f_1, \ldots, f_4$ in $D(A,G)$ and positive real numbers $t_1, \ldots, t_4$ such that $f = t_1 f_1 - t_2 f_2 + i(t_3 f_3 - t_4 f_4)$ and $t_1 + t_2 + t_3 + t_4 \leq \sqrt{2}e\|f\|$. Thus, for $i = 1$ and 2, $|f(x_i(g_2,\ldots\ldots,g_n))| \leq \sqrt{2}e\|f\|\|a_1\|\cdots\cdots\|a_n\|$, for all $g_2,\ldots\ldots, g_n$ in G. We may therefore define continuous linear functionals S_1 and S_2 on $A'(G)$ by

$$S_i f = \int\left(\cdots\left(\int f(x_i(g_2,\ldots\ldots,g_n))dM(g_n)\right)\cdots\right)dM(g_2) \quad (f \in A'(G)),$$

and bounded linear operators U_1 and U_2 on $A'(G)$ by

$(U_i f)(x) = S_i(E(f \cdot x)) \quad (f \in A'(G),\ x \in A)$. By lemma 19.22, $(Pa_1 \cdots\cdots Pa_n)(f)(x) =$

$$\int\left(\cdots\left(\int E(f.x)(a_1(g_2.a_2)\cdots(g_n.a_n)\,dM(g_n)\right)\cdots\right)dM(g_2)$$

$$= \int\left(\cdots\left(\int E(f.x)(x_1(g_2,\ldots,g_n) + ix_2(g_2,\ldots,g_n))\,dM(g_n)\right)\cdots\right)dM(g_2)$$

$$= (S_1 + iS_2)(E(f.x)) = (U_1 + iU_2)(f)(x).$$

Thus $Pa_1\cdots Pa_n = U_1 + iU_2$.

Now let f in $A'(G)$ and x in A satisfy the condition $E(f.x) \in D(A,G)$. Then, for $i = 1$ and 2,

$$(U_i f)(x) = \int\left(\cdots\left(\int E(f.x)(x_i(g_2,\ldots,g_n)\,dM(g_n)\right)\cdots\right)dM(g_2),$$

which is real, since $E(f.x)(x_i(g_2,\ldots,g_n))$ is real for all $g_2, \ldots, g_n$ in G. By lemma 19.21, U_1 and U_2 are therefore in $H(BL(A'(G))$. This completes the proof of the Theorem.

<u>19.24 Theorem</u>. If A is Arens regular, $A = A(A,G) + iH(A,G)$ and (A,G,T) is M-asymptotically abelian, then the G-invariant state space $D(A,G)$ is a simplex.

<u>Proof</u>. This follows immediately from Corollary 19.13, Theorem 19.23 and Theorem 18.9.

I do not know if Theorem 19.24 is true when A is not Arens regular. However, if A is a C*-algebra, then A is Arens regular and $H(A) \subseteq H(A,G)$. The following result therefore follows immediately from Theorem 19.24.

<u>19.25 Corollary</u> (Lanford and Ruelle). If A is a C*-algebra, G is amenable and (A,G,T) is G-abelian, then $D(A,G)$ is a simplex.

By replacing the invariant mean M by the Godemont mean and slightly modifying the definition of E, P, Q and B, Corollary 19.25 may be proved by these methods for non-amenable groups.

The conditions of Theorem 19.24 are satisfied by the triple (A,G,T) of Example 16.5, in which A is not a C^*-algebra. Further examples of this kind can be constructed using simple direct sum arguments. The fact that $H(A)$ is in general strictly contained in $H(A,G)$ suggests that less trivial examples probably do exist.

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