LOGARITHMETICS, INDEX POLYNOMIALS AND BIFURCATING ROOT-TREES

Thesis for the degree of Doctor of Philosophy

by

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1959

CONTENTS

INTRODUCTION

Logarithmetics and index polynomials originated in Etherington's study of train algebras of rank 3 (cf. (12), (16)). In (16) Etherington showed that if λ, μ are the right and the left principal train roots and X is a non-idempotent element of the algebra then $X^2 = X + u$, $uX = \lambda u$, $Xu = \mu u$, $u^2 = 0$, where $u = X^2 - X$. It follows that $X^P = X + \theta_P u$ where θ_P denotes a polynomial in λ and μ, the index θ-polynomial of the index P or the power X^P. ψ-polynomials were introduced in the same paper: $\psi_P = 1 + (\lambda + \mu - 1)\theta_P$ $(\lambda + \mu \neq 1)$, i.e. if $v = (\lambda + \mu - 1)^{-1}u$ and $I = X - v$ then $X^P = I + \psi_P v$.

These concepts were further developed in a research course given by Etherington in 1954-55 at the University of Edinburgh in which he discussed his previous work and some new results which he later published in (18) and (19). During this course, which the author attended, Etherington suggested representations of the free logarithmetic by index polynomials in non-commuting indeterminates and by bifurcating root-trees. The intimate connection between these was pointed out by the author and developed in (25). This paper is incorporated in Chapter I . Index polynomials are defined there as sums of terms which can be interpreted as coordinates of certain knots in the corresponding tree. Apart from the ψ- and the θ-polynomials two further types are defined, the χ- and the ω-polynomials. The latter are used in Chapter IV to define Ω-trees, a generalization of bifurcating root-trees to trees of infinite altitudes.

In Chapter I we also prove two inequalities relating the potency and the altitude of a bifurcating root-tree to its mutability; these results were obtained in (30).

Etherington (15) gave a most elegant interpretation of indices of the free logarithmetic and of some of its homomorphs as partitions. This was pursued further by Hourston (24). His approach however was purely formalist and is not acceptable to the author (see the introduction in Chapter IV). A constructivist theory of bifurcating root-trees of infinite altitudes is developed in Chapter IV (cf. (31)).

Etherington (15), Robinson (36) and Evans (22) gave systems of postulates for the free logarithmetic . Evans also obtained some results in the non-associative number theory. This was followed by the author in (26); the contents of this paper constitute Chapter II.

Chapter III contains mainly results obtained in (27). The only addition is a section on index ψ- and θ-polynomials in commuting indeterminates λ, μ. This is the only part of the thesis which has not been previously published or submitted for publication by the author.

The thesis concludes with Chapter V on enumeration of indices (cf. (27), (28), (29)).

The author is indebted to Dr. I.M.H. Etherington for his supervision, advice and constructive criticism in the writing of this work.

CHAPTER I. INDEX POLYNOMIALS AND BIFURCATING ROOT-TREES.

1. THE FREE LOGARITHMETIC

A groupoid is a set closed with respect to a binary operation. It is cyclic if it is generated by one element.

Let x be the generator of the free cyclic (multiplicative) groupoid $\mathfrak{A}$. Then any element of $\mathfrak{A}$ can be written in the form x^P where

$$x^1 = x \quad \text{and} \quad x^{Q+R} = x^Q x^R .$$

Thus, e.g., $x^{((1+1)+(1+(1+1)))+1} = ((xx)(x(xx)))x$.

Call P the index of x^P and the above defined binary operation on indices - addition of indices. We say that two indices Q and R are equal and write $Q = R$ if and only if $x^Q = x^R$.

For brevity we can use the symbols of natural numbers for the right principal powers:

$$x^1 = x, \; x^2 = x^1 x, \; x^3 = x^2 x, \; \dots , \; x^n = x^{n-1}x, \dots,$$

and similar symbols with dots to denote the left principal powers:

$$x^{\dot{1}} = x, \; x^{\dot{2}} = xx^{\dot{1}}, \; x^{\dot{3}} = xx^{\dot{2}}, \; \dots , \; x^{\dot{n}} = xx^{\dot{n-1}}, \dots .$$

The set of all indices $\mathcal{J}$ and their addition form a cyclic groupoid generated by 1. Call it the additive free logarithmetic* and denote it by $\mathcal{L}^+$. $\mathcal{L}^+$ is obviously isomorphic with $\mathfrak{A}$.

We define for $\mathfrak{I}$ another binary operation of <u>multiplication</u>: PQ is defined by

$$x^{PQ} = (x^P)^Q.$$

We easily prove that the multiplication is associative and right distributive with respect to addition:

$$x^{PQ.R} = (x^{PQ})^R = ((x^P)^Q)^R = (x^P)^{QR} = x^{P.QR};$$

and $x^{P(Q+R)} = (x^P)^{Q+R} = (x^P)^Q.(x^P)^R = x^{PQ}x^{PR} = x^{PQ+PR}$.

The semi-group formed by the set of all indices $\mathfrak{I}$ and their multiplication is called the multiplicative free logarithmetic and is denoted by $\mathcal{L}^\times$. 1 is the identity element of $\mathcal{L}^\times$.

In this chapter we investigate the properties and representations of the algebra of indices with the two operations, addition and multiplication, which we call the <u>free logarithmetic</u> and denote by $\mathcal{L}$.

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* This is Etherington's free logarithmetic B (cf. (17)).

2. BIFURCATING ROOT-TREES

Indices can be represented graphically by bifurcating root-trees which may be defined as follows:

Definition. (a) $\cdot$ and $\vee$ are bifurcating root-trees;

(b) If (P) and (Q) are two bifurcating root-trees $^{(P)}\vee^{(Q)}$ is also a bifurcating root-tree.

Trees will mean bifurcating root-trees unless otherwise stated.

The points of a tree where lines meet or terminate are called *knots*. We call the diagram $\vee$ a *fork*, its upper two knots its *left* and *right ends*, its bottom knot the *node* of the fork and the two lines joining its knots the *left* and the *right arm* of the fork. A knot is said to *cover* another knot if the latter is the node of a fork of which the former is an end.

In each tree there is exactly one knot which does not cover any other knot; it is called the *root* of the tree. The knots of a tree which are not covered by other knots are the *free ends* of the tree. In the tree $\cdot$ the only free end coincides with the root of the tree.

The tree $^{(P)}\vee^{(Q)}$ is called the *sum* of trees (P) and (Q) which are respectively the left and the right <u>subtrees of the first order</u> of 

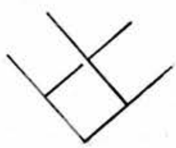
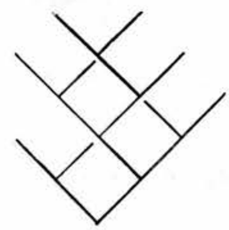
. We write

(P) + (Q) = 

.

If (S) is a subtree of the first order of a subtree of (n-1)th order of (R) then (S) is called a subtree of (R) of <u>n^{th} order</u>.

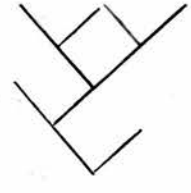
<u>Example</u>.

 is a subtree of 2^{nd} order of 

The other subtrees of 2^{nd} order are · , · and  .

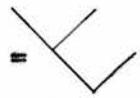
Since  = · + · , it follows from the inductive definition of trees that any tree can be generated by repeated addition of · . Hence the set of all trees and their addition form a free cyclic groupoid, obviously isomorphic with $\mathscr{L}^+$.

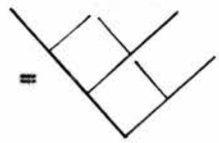
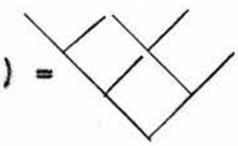
The fundamental correspondence is $1 \longleftrightarrow \cdot$. For example

(1 + (2 + 2)) + 1 $\longleftrightarrow$ 

We can now define multiplication of trees so that $\mathcal{T}$, the algebra of trees with the operations of addition and multiplication, is isomorphic with $\mathscr{L}$.

Definition. The product (P)(Q) of two trees is the tree obtained by joining to each free end of (Q) the root of a tree identical with (P).

Examples. If (P) =  and (Q) = 

then (P)(Q) =  while (Q)(P) =  .

Evidently multiplication of trees is right-distributive.

It can be easily proved by non-associative induction* that if indices P, Q correspond to trees (P), (Q) then their product PQ corresponds to tree (P)(Q). Thus $\mathcal{L}$ is isomorphic with $\mathcal{T}$. Because of this isomorphism we can use the terminology and the notation for trees and for indices indiscriminately. We can also write = without ambiguity in place of $\longleftrightarrow$, or even speak of the tree 1, a tree P + Q, etc.

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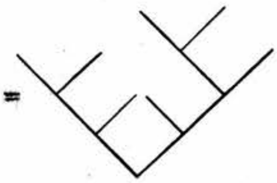
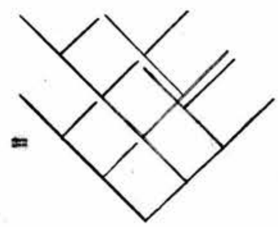
* Principle of non-associative induction: A proposition Z_X involving index X is true for all $X \varepsilon \mathcal{Y}$ if Z_1 is true and $(Z_P \text{ and } Z_Q) \Longrightarrow Z_{P+Q}$. (Cf. (15), p. 446.)

This can be proved by appeal to the principle of mathematical induction which in turn must be regarded as the fundamental intuitive principle of mathematics.

Examples.

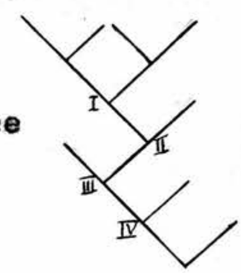
$3 + (1 + 3) =$  ; $3(1 + 3) =$  .

3. KNOTS AND TERMS

Given any knot other than the root there is one and only one knot in the tree which is covered by the given knot, viz. the node of the fork of which the given knot is an end. Hence the chain of knots of which the root of the tree is the first and the given knot the last element (v. (6), p. 10) is uniquely determined. Thus we can determine uniquely the position of any knot in a tree by specifying for each knot in this chain, other than the root, whether it is a left (λ) or a right (μ) end.

Example.

In the tree



knot I is a left end,
knot II is a right end,
knot III is a left end,
knot IV is a left end

and knot I can be specified (reading from knot to root) by the ordered set $(\lambda, \mu, \lambda, \lambda)$.

In general: the root can be represented by the empty set and any other knot by the ordered set $(\lambda, \nu_1, \nu_2, \ldots)$

or by $(\mu, \nu_1, \nu_2, \ldots)$, where $(\nu_1, \nu_2, \ldots)$ is the set representing the knot covered by the given knot, according as the latter is a left or a right end.

It will be convenient to write these ordered sets as products, so that a knot is represented by a monomial in λ, μ (e.g. $\lambda\mu\lambda^2$) which will be called its <u>term</u>, and to add terms to form <u>polynomials</u>. These polynomials will be subject to the ordinary laws of algebra, excepting the commutative law of multiplication. Thus we shall be operating in the ring $\mathfrak{M}[\lambda,\mu]$ obtained by adjoining two non-commuting indeterminates λ, μ to the domain $\mathfrak{M}$ of integers. $\mathfrak{M}[\lambda,\mu]$ consists of all polynomials in λ, μ with integer coefficients*.

<u>Definition</u>. (i) The term of the root of a tree is 1.

(ii) The term of a knot, which covers a knot whose term is ν, is $\lambda\nu$ or $\mu\nu$ according as the former is a left or a right end.

It follows immediately from the definition that if a tree contains a knot whose term is $\nu_1\nu_2\nu_3\ldots\nu_n$ (where $\nu_i = \lambda$ or μ as the case may be) then it contains <u>all</u> the knots whose terms are right divisors of $\nu_1\nu_2\nu_3\ldots\nu_n$.

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* Although the index polynomials to be introduced have only <u>positive</u> integer coefficients, minus signs are needed for expressing relations connecting different index polynomials.

4. ALTITUDE, POTENCY AND MUTABILITY

We define the altitude of

an index P

as the ordinal number α_P such that

$$\alpha_1 = 0 \text{ and}$$

$$\alpha_{Q+R} = 1 + \max(\alpha_Q, \alpha_R).$$

a knot

as the degree in λ, μ of the corresponding term.

(Alternatively we define the altitude of the root as 0, and that of any other knot as equal to 1 + (altitude of the knot covered by it).)

a tree

as the maximum of the altitudes of all its knots.

Further, we define the potency* of

an index P

as the cardinal number δ_P such that

$$\delta_1 = 1$$

$$\delta_{Q+R} = \delta_Q + \delta_R.$$

(Alternatively: δ_P is the number which P denotes if all symbols in P are interpreted as numbers and operations in ordinary arithmetic.)

a tree P

as the (cardinal) number δ_P of all free ends of P.

(Alternatively:

δ_P = (no. of forks in P) + 1.)

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* Called "degree of an index" by Etherington (11), (15) and Popova (34) and "length of a non-associative number" by Evans (22). The term "potency" is to be preferred here in order to avoid confusion with degree, in the ordinary sense, of the corresponding index polynomials.

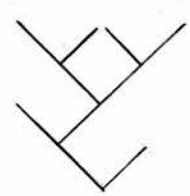
It is easily seen that these definitions are consistent with the isomorphism of $\mathcal{L}$ and $\mathcal{T}$.

Two trees are said to be <u>conformal</u> if the corresponding indices become equal when addition of indices is commutative. A knot of a tree P is called <u>unbalanced</u> if the subtree of P of which the knot is the root has non-conformal subtrees of the first order; otherwise it is <u>balanced</u>. The <u>mutability</u> μ_P of P is equal to the number of unbalanced knots in P. Thus the number of trees conformal to a given tree P (or indices conformal to a given index P) is 2^{μ_P}. We have (cf. (11)):

$$\mu_1 = 0, \quad \mu_{P+Q} = \begin{cases} 2\mu_P, & \text{if } P, Q \text{ are conformal,} \\ \mu_P + \mu_Q + 1, & \text{if } P, Q \text{ are not conformal.} \end{cases}$$

<u>Example</u>.

$$P = (1 + 2.2) + 1 =$$



The terms of the knots are

$$\lambda^2\mu\lambda,\ \mu\lambda\mu\lambda,\ \lambda\mu^2\lambda,\ \mu^3\lambda,\ \lambda\mu\lambda,\ \mu^2\lambda,\ \lambda^2,\ \mu\lambda,\ \lambda,\ \mu,\ 1.$$

Their altitudes are

4, 4, 4, 4, 3, 3, 2, 2, 1, 1, 0,

respectively; and we have

$$\alpha_P = 4, \quad \delta_P = 6, \quad \mu_P = 2.$$

From the definition of the product of two trees we deduce the formulae

$$\delta_{PQ} = \delta_P\delta_Q, \quad \alpha_{PQ} = \alpha_P + \alpha_Q, \quad \mu_{PQ} = \delta_Q\mu_P + \mu_Q .$$

These are easily provable by non-associative induction applied to Q.

Etherington has shown (11) that α , δ , μ must satisfy the following conditions:

(1) $2^\alpha \geq \delta \geq \alpha + 1$;

(2) $\delta \geq \mu + 2$ $(\delta \neq 1)$; the equality holding only when $\delta = \alpha + 1$, i.e. when the tree is primary (v. infra, § 5);

(3) $\mu \leq 3.2^{\alpha-3} - 1$ $(\alpha \geq 3)$.

For a given δ or a given α conditions (2), (3) prescribe the maximal value for μ. In this section we find two minimal conditions for μ and show constructively that for any given non-negative integers α, δ satisfying condition (1) the least number satisfying these minimal conditions is in fact the mutability of a tree of altitude α and potency δ.

Potency δ_P can be expressed uniquely as a sum of h_P distinct powers of 2 : $\delta = 2^{i_1} + 2^{i_2} + \dots + 2^{i_h}$ $(i_1 > i_2 > \dots > i_h \geq 0)$. Obviously $i_1 = [\log_2 \delta]$ and h is equal to the sum of digits in δ written in the binary scale of notation.

LEMMA. $h_P + h_Q \geq h_{P+Q}$.

For clearly in any scale of notation the total sum of digits in several natural numbers cannot be exceeded by that of the sum of these numbers.

THEOREM 1.1 . Let δ and μ be the potency and the mutability of a tree P and let $\delta = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_h}$ $(i_1 > i_2 > \ldots > i_h)$. Then $\mu \geq h - 1$.

Proof. Use non-associative induction.
If $P = 1$: $\mu = 0$, $\delta = 2^0$, hence $h = 1$ and $\mu = h - 1$.
Let $P = Q + R$ and assume that the theorem holds for Q and R. Then (i) if Q and R are not conformal

$$\begin{aligned}
\mu_{Q+R} &= \mu_Q + \mu_R + 1 \\
&\geq (h_Q - 1) + (h_R - 1) + 1, \text{ by the induction hypothesis}, \\
&= (h_Q + h_R) - 1 \\
&\geq h_{Q+R} - 1, \text{ by preceding lemma;}
\end{aligned}$$

(ii) if Q and R are conformal

$$\begin{aligned}
\mu_{Q+R} &= 2\mu_Q \\
&\geq 2h_Q - 2, \text{ by the induction hypothesis}, \\
&= 2h_{Q+R} - 2 \\
&\geq h_{Q+R} - 1 \text{ since } h_{Q+R} \geq 1 .
\end{aligned}$$

<u>THEOREM 1.2</u> . If α, δ, μ are the altitude, potency and mutability of tree P then $\alpha - \mu \leq \log_2(\delta - \mu)$.

<u>Proof</u>. If $P = 1$ then $\alpha - \mu = 0 - 0 = \log_2(1 - 0) = \log_2(\delta - \mu)$. Let $P = Q + R$ and assume that $\alpha_Q - \mu_Q \leq \log_2(\delta_Q - \mu_Q)$ and $\alpha_R - \mu_R \leq \log_2(\delta_R - \mu_R)$. Without loss of generality assume that $\alpha_Q \geq \alpha_R$. Then

(i) if Q, R are not conformal

$$\begin{aligned}
\alpha_{Q+R} - \mu_{Q+R} &= (\alpha_Q + 1) - (\mu_Q + \mu_R + 1) \\
&= (\alpha_Q - \mu_Q) + (\alpha_R - \mu_R) - \alpha_R \\
&\leq \log_2(\delta_Q - \mu_Q) + \log_2(\delta_R - \mu_R) - \log_2\delta_R, \\
&\qquad \text{using the induction hypothesis and } \alpha \geq \log_2\delta, \\
&= \log_2\{\delta_Q - \mu_Q + \mu_R\delta_R^{-1}(\mu_Q - \delta_Q)\} \\
&\leq \log_2(\delta_Q - \mu_Q + \delta_R - \mu_R - 1), \\
&\qquad \text{since } \mu_R\delta_R^{-1}(\mu_Q - \delta_Q) \leq 0 \leq \delta_R - \mu_R - 1, \\
&= \log_2(\delta_{Q+R} - \mu_{Q+R}) .
\end{aligned}$$

(ii) if Q, R are conformal then $\delta_Q = \delta_R$, $\mu_Q = \mu_R$ and

$$\begin{aligned}
\alpha_{Q+R} - \mu_{Q+R} &= (\alpha_Q + 1) - 2\mu_Q \\
&= (\alpha_Q - \mu_Q) + \log_2 2 - \mu_Q \\
&\leq /
\end{aligned}$$

$$\leq \log_2(2\delta_Q - 2\mu_Q) - \mu_Q,$$

by the induction hypothesis,

$$\leq \log_2(2\delta_Q - 2\mu_Q)$$

$$= \log_2(\delta_{Q+R} - \mu_{Q+R}) .$$

<u>THEOREM 1.3</u> . Let α, δ be non-negative integers such that $\alpha + 1 \leq \delta \leq 2^{\alpha}$ and let $\delta = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_h}$ $(i_1 > i_2 > \ldots > i_h \geq 0)$.
If μ is the least integer such that

$$\alpha - \mu \leq \log_2(\delta - \mu) \qquad \text{(I)}$$

and $\qquad \mu \geq h - 1 , \qquad$ (II)

then there exists a tree (in general not unique) of altitude α, potency δ and mutability μ .

<u>Proof</u>. (i) If $\alpha \leq i_1 + h - 1$ then the tree

$$((((2^{i_1} + 2^{i_2}) + 2^{i_3}) + \ldots) + 2^{i_p}) + (2^{i_{p+1}} + (2^{i_{p+2}} + (\ldots + (2^{i_{h-1}} + 2^{i_h})))),$$

where $p = \alpha - i_1 < h$, has altitude α, potency δ and mutability $h - 1$. Moreover, since any number less than $h - 1$ contravenes condition (II), $\mu = h - 1$.

(ii) If $\alpha \geq i_1 + h$ then δ can be written in the form $\delta = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_k}$ where $k = \alpha - j_1 + 1$ and

either (a) $j_1 > j_2 > \ldots > j_{k-1} = j_k$,

or (b) $j_1 > j_2 > \ldots > j_{r-1} \geq j_r > j_{r+1} = j_{r+2} = \ldots = j_k = 0 \quad (r \geq 3)$,

or (c) $j_1 > j_2 = j_3 = \ldots = j_k = 0$.

This can be done as follows. We first partition 2^{i_h} into powers of 2, seeking to make k, the number of terms in an expression for δ, equal to $\alpha - i_1 + 1$ so that $i_1 + k - 1 = \alpha$. If $\alpha < i_1 + h - 2 + 2^{i_h}$ we can achieve that by partitioning 2^{i_h} and thereby obtain an expression of the form (a), (b) or (c). If $\alpha = i_1 + h - 2 + 2^{i_h}$ we can just do it by partitioning 2^{i_h} entirely into 1's (then $k = h - 1 + 2^{i_h} = \alpha - i_1 + 1$). If α exceeds this value we partition $2^{i_{h-1}}$ also; and so on. If in the course of this process we obtain an expression $2^n + 1 + \ldots + 1$ $(n \geq 2)$ and the number of terms is still insufficient the next step in partitioning is $2^{n-1} + 2^{n-2} + 2^{n-2} + 1 + \ldots + 1$ because the values of "$j_1 + k - 1$" for $2^n + 1 + \ldots + 1$ and for $2^{n-1} + 2^{n-1} + 1 + \ldots + 1$ are equal and in what follows both partitions would yield the same tree. Thus the first two terms in an expression for δ are never equal and $k \geq 3$ unless $k = h$. Finally if α has its maximum value $\delta - 1$ we reach $\delta = 2 + 1 + \ldots + 1$, where $j_1 = 1$, $k = \delta - 1 = \alpha - j_1 + 1$.

Now consider the tree $(((2^{j_1} + 2^{j_2}) + 2^{j_3}) + \ldots) + 2^{j_k}$. Its altitude is α, its potency δ and its mutability $k - 1$. In order to prove that $\mu = k - 1$ we have to show that no tree can have altitude α, potency δ and mutability $k - t$ where $t > 1$. We prove that

$\alpha - (k - t) > \log_2\{\delta - (k - t)\}$.

If (a), then $\delta \leq 2^{j_1} + 2^{j_1-1} + \dots + 2^{j_1-(k-2)} + 2^{j_1-(k-2)}$

$$= 2^{j_1+1} \leq 2^{j_1+1} + k - 3 \quad \text{since } k \geq 3 .$$

If (b), then $\delta \leq 2^{j_1} + 2^{j_1-1} + \dots + 2^{j_1-(r-2)} + 2^{j_1-(r-2)} + (k - r)$

$$= 2^{j_1+1} + k - r \leq 2^{j_1+1} + k - 3 .$$

If (c), then $\delta = 2^{j_1} + (k - 1) \leq 2^{j_1+1} + k - 3$.

Thus in all cases $\delta \leq 2^{j_1+1} + k - 3$.

Hence
$$\begin{aligned}
\alpha - (k - t) &= j_1 + t - 1 \\
&= \log_2 2^{j_1+t-1} \\
&= \log_2(2^{j_1+1} . 2^{t-2}) \\
&\geq \log_2\{(\delta - k + 3) . 2^{t-2}\} \\
&\geq \log_2\{(\delta - k + 3) + (t - 2)\} \\
&> \log_2\{\delta - (k - t)\} .
\end{aligned}$$

5. SUBORDINATES

A fork of a tree is said to be free if both its ends are free ends.

A fork in a tree P is called the leading fork if either (1) it coincides with P (if $P = 2$),

or (2), if $P = P' + P''$ ($P \neq 2$) and $\alpha_{P'} \geq \alpha_{P''}$, it is the leading fork of P',

or (3), if $P = P' + P''$ $(P \neq 2)$ and $\alpha_{P'} < \alpha_{P''}$,

it is the leading fork of P''.

The leading fork of a tree P is necessarily a free fork and the altitude of its ends is α_P.

A tree P is called <u>primary</u> if $P \neq 1$ and each of its forks has at least one free end .

A tree Q is called a first <u>subordinate</u> of a tree P, if Q can be obtained from P by removing a single free fork. It is called an n^{th} subordinate of P if it is a first subordinate of an $(n-1)^{th}$ subordinate of P. It is convenient to regard P as its own subordinate (of order 0). If Q is an n^{th} subordinate of P we call P an n^{th} <u>superior</u> of Q.

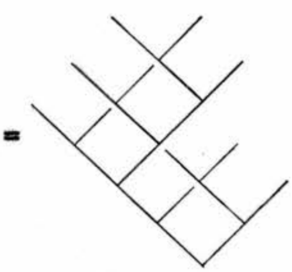
<u>Examples</u>. The $(\delta_P - 2)^{th}$ subordinate of P $(\delta_P \geq 2)$ is necessarily the tree  .

$(P + Q) + 1$ and $(P + 1) + 1$ are respectively a $(\delta_R - 1)^{th}$ and a $(\delta_R + \delta_Q - 2)^{th}$ subordinate of $(P + Q) + R$.

We say that Q, a first subordinate of a tree P, is the first <u>principal</u> subordinate of P if Q does not contain the leading fork of P. The n^{th} principal subordinate of P is the first principal subordinate of the $(n - 1)^{th}$ principal subordinate of P.

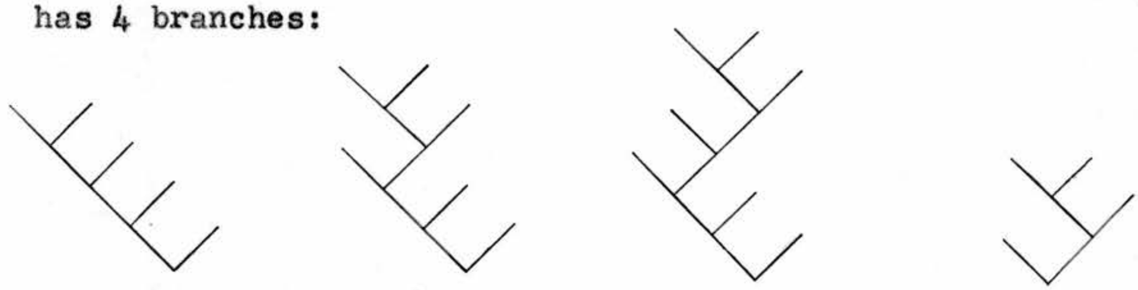
If <u>all</u> free forks of Q ($Q \neq 1$), a subordinate of P, are free forks of P, then Q is said to be a <u>component</u> of P. If a component of P is primary it is called a <u>branch</u> of P.

<u>Example</u>.

The tree P = 

$((2 + (2 + 3)) + 1) + 3$

has 4 branches:



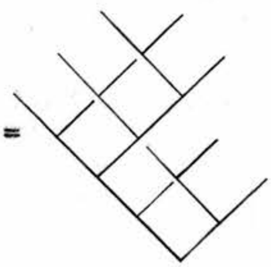
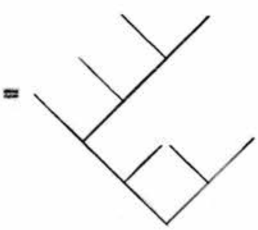
and altogether $15 = {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 2^4 - 1$ components, obtainable by superimposing any combination of the branches.

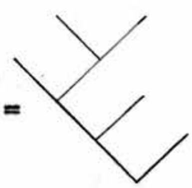
Call the number of free forks in a tree P ($P \neq 1$) the <u>lineage</u> of P and denote it by γ_P. γ_P is the number of distinct branches of P. Call the trees of lineage 2, 3, ..., n binary, ternary, ... , n-ary.

Further, we say that Q is a first <u>total</u> subordinate of P if Q is obtained from P by removing a single free fork from <u>each</u> branch of P (i.e. by removing all free forks of P). R is the n^{th} total subordinate of P if it is the first total subordinate of the $(n - 1)^{th}$ total subordinate of P. Write $R = d^nP$.

We have $$\delta_{dP} = \delta_P - \gamma_P.$$

<u>Example</u>.

P =  , dP =  ,

d^2P =  , $d^6P = \cdot$.

The index dP is obtained from the index P (written in its shortest additive form) by replacing each

$$2,\ 3,\ \dot{3},\ 4,\ \dot{4},\ 5,\ \dot{5},\ 6,\ \ldots$$

by $$1,\ 2,\ \dot{2},\ 3,\ \dot{3},\ 4,\ \dot{4},\ 5,\ \ldots\ .$$

If $P = QR$ and $Q \neq 1$ then $dP = (dQ)R$.

<u>Examples</u>. (i) $P = ((2 + (2 + 3)) + 1) + 3,$

$dP = ((1 + (1 + 2)) + 1) + 2 = (\dot{4} + 1) + 2,$

$d^2P = (\dot{3} + 1) + 1, \quad d^3P = (\dot{2} + 1) + 1 = 4,$

$d^4P = 3, \quad d^5P = 2, \quad d^6P = 1.$

(ii) $P = 3.2, \qquad dP = (d3).2 = 2.2$,

$d^2P = (d2).2 = 2, \qquad d^3P = d2 = 1$.

6. INDEX POLYNOMIALS

Consider the ring $\mathfrak{M}[\lambda,\mu]$ of non-commutative polynomials defined in §3. Suppose that π is a function mapping the set of all trees on to Π, a subset of $\mathfrak{M}[\lambda,\mu]$: $P \to \pi_P$, with the property that whenever $\pi_P = \pi_{P'}$ and $\pi_Q = \pi_{Q'}$ then $\pi_{P+Q} = \pi_{P'+Q'}$. It can then be proved by non-associative induction that also $\pi_{PQ} = \pi_{P'Q'}$. We can then define for Π the operations

$$\pi_P \oplus \pi_Q = \pi_{P+Q}, \quad \pi_P \otimes \pi_Q = \pi_{PQ}.$$

The algebra Π so determined is said to form a representation of $\mathcal{T}$, the algebra of all trees, by means of <u>index π-polynomials</u>.

<u>Examples</u>. (i) The function mapping all trees on to a fixed polynomial (e.g. $\lambda^2 + \mu\lambda$) gives a representation (though a trivial one) of $\mathcal{T}$.

(ii) The polynomials defined inductively by

$$\pi_1 = 1, \qquad \pi_{P+Q} = \lambda(\pi_P + \pi_Q)$$ cf. p. 74

give a representation of $\mathcal{T}$. It will be shown in Chapter III that these polynomials give a faithful (v. infra) representation of the free commutative entropic logarithmetic, i.e. of the homomorph of $\mathcal{L}$ determined by the congruence relations

$$\left.\begin{array}{c} P + Q \sim Q + P \\ (P + Q) + (R + S) \sim (P + R) + (Q + S) \end{array}\right\} \text{all } P,Q,R,S.$$

If the correspondence $P \leftrightarrow \pi_P$ is one-one the index π-polynomials are said to represent $\mathcal{Y}$ (and thus also $\mathcal{L}$) faithfully. Faithful representations by index polynomials are of special interest, particularly in the case of index polynomials whose terms actually represent knots in the corresponding trees.

The most obvious polynomials to possess this property are the χ-polynomials.

Definition. The index χ-polynomial of a tree P is the polynomial whose terms correspond to all the knots of P.

Obviously $\chi_1 = 1$ and from the definition of χ-polynomials and that of a term of a knot it follows that

$$\chi_{P+Q} = \chi_P\lambda + \chi_Q\mu + 1$$

(which is an alternative definition of index χ-polynomials).

Evidently the χ-polynomials provide a faithful representation of trees.

The χ-polynomials, however, are very unwieldy because of the inclusion of the terms corresponding to all the knots of a tree.

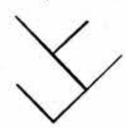
7. ψ-POLYNOMIALS

It appears that a tree could be completely determined by the terms corresponding to all its free ends or equally well by the terms corresponding to all the other knots. We now introduce two important types of index polynomials which contain precisely these terms and eventually we prove that either of these representations is a faithful~~l~~ one.

<u>Definition</u>. The index ψ-polynomial of a tree P is the polynomial whose terms correspond to all free ends of P.

<u>Examples</u>.

P =  $\psi_P = \lambda + \mu$.

Q =  $\psi_Q = \lambda^2\mu + \mu\lambda\mu + \mu^2 + \lambda$.

R = · $\psi_R = 1$.

There are δ_P terms in ψ_P $(P \neq 1)$.

<u>THEOREM 1.4</u> . (i)
$$\psi_1 = 1 ,$$
$$\psi_{P+Q} = \psi_P\lambda + \psi_Q\mu ;$$

(ii)
$$\psi_{PQ} = \psi_P\psi_Q .$$

(N.B. (i) provides an alternative definition of ψ-polynomials.)

<u>Proof</u>. (i) The terms corresponding to the free ends of $\overset{(P)\ (Q)}{\vee}$ are the terms corresponding to the free ends of P post-multiplied by λ (i.e. those of $\psi_P\lambda$) and the terms corresponding to the free ends of Q post-multiplied by μ (i.e. those of $\psi_Q\mu$). This follows from the definition of the term of a knot (v. §3).

(ii) is provable by non-associative induction applied to Q. The theorem is true for $Q = 1$:

$$\psi_{P1} = \psi_P = \psi_P\psi_1.$$

Assume $\psi_{PR} = \psi_P\psi_R$ and $\psi_{PS} = \psi_P\psi_S$; then

$$\begin{aligned}\psi_{P(R+S)} = \psi_{PR+PS} &= \psi_{PR}\lambda + \psi_{PS}\mu \\ &= \psi_P\psi_R\lambda + \psi_P\psi_S\mu, \text{ by induction hypothesis,} \\ &= \psi_P(\psi_R\lambda + \psi_S\mu) \\ &= \psi_P\psi_{R+S}, \text{ by (i)} .\end{aligned}$$

This direct representation of multiplication of trees (or of indices) by multiplication of the corresponding ψ-polynomials is very convenient, particularly in the study of factorization of trees and indices.

The ψ-polynomials obviously provide a representation of $\mathcal{T}$, since to each tree corresponds a uniquely determined index ψ-polynomial. We prove that this representation is faithful.

<u>LEMMA</u>. If ψ_P is an index ψ-polynomial and

$$\psi_P = \varphi_1\lambda + \varphi_2\mu, \qquad \varphi_1, \varphi_2 \in \mathfrak{M}[\lambda,\mu]$$

then φ_1, φ_2 are both index ψ-polynomials and are uniquely determined.

For $P \neq 1$; therefore $P = Q + R$ and $\psi_P = \psi_Q\lambda + \psi_R\mu$. Each term of ψ_P has a well-determined last (right) factor, either λ or μ. The terms having λ for last factor are those of $\psi_Q\lambda$. Hence ψ_Q is uniquely determined. But also $\varphi_1\lambda$ contains all the terms of ψ_P ending in λ. Therefore $\varphi_1\lambda = \psi_Q\lambda$, i.e. $\varphi_1 = \psi_Q$. Similarly $\varphi_2 = \psi_R$.

<u>THEOREM 1.5</u> . Index ψ-polynomials represent faithfully the algebra of trees $\mathcal{T}$ (and thus also the logarithmetic $\mathcal{L}$).

<u>Proof</u>. It suffices to prove that

$$(\psi_P = \psi_Q) \Longrightarrow (P = Q) \tag{A}$$

Use induction on α_P, the altitude of P.
If $\alpha_P = 0$, $P = 1$, $\psi_P = \psi_Q = 1$ and $Q = 1$.
Suppose (A) is true for $\alpha_P < a$. Let $\alpha_P = \alpha_Q = a$ (>0) and let $P = P_1 + P_2$ and $Q = Q_1 + Q_2$. Then

$\psi_P = \psi_{P_1}\lambda + \psi_{P_2}\mu$, $\psi_Q = \psi_{Q_1}\lambda + \psi_{Q_2}\mu$ and, by the Lemma,

ψ_{P_1}, ψ_{P_2}, ψ_{Q_1}, ψ_{Q_2} are uniquely determined index ψ-polynomials. Thus $\psi_P = \psi_Q$ implies $\psi_{P_1} = \psi_{Q_1}$, $\psi_{P_2} = \psi_{Q_2}$ and, since altitudes of P_1, Q_1, P_2, Q_2 are all less than a, by the induction hypothesis $P_1 = Q_1$, $P_2 = Q_2$, i.e. $P = Q$.

8. Θ-POLYNOMIALS

<u>Definition</u>. The index Θ-polynomial of a tree P is the polynomial whose terms correspond to all nodes of P (i.e. to all knots of P which are not free ends). $\Theta_1 = 0$.

<u>THEOREM 1.6</u>. (i) $\Theta_1 = 0$,

$$\Theta_{P+Q} = \Theta_P\lambda + \Theta_Q\mu + 1 .$$

(This is an alternative definition of Θ-polynomials.)

(ii) $\Theta_{PQ} = \Theta_P + \Theta_Q + \Theta_P(\lambda + \mu - 1)\,\Theta_Q$.

(iii) $\psi_P = 1 + (\lambda + \mu - 1)\,\Theta_P$.

(iv) $\chi_P = \Theta_{2P} = 1 + (\lambda + \mu)\,\Theta_P$.

<u>Proof</u>. (i) The terms corresponding to nodes of $(P)\vee(Q)$ are those corresponding to nodes of P post-multiplied by λ, those corresponding to nodes of Q post-multiplied by μ and the term corresponding to the root of $(P)\vee(Q)$. Hence the result.

(ii) The formula is true for $Q = 1$:

$\Theta_{P1} = \Theta_P = \Theta_P + \Theta_1 + \Theta_P(\lambda + \mu - 1)\,\Theta_1$ since $\Theta_1 = 0$.

Assume that it is true for $Q = R$ and $Q = S$,

i.e. $$\theta_{PR} = \theta_P + \theta_R + \theta_P(\lambda + \mu - 1)\theta_R \, ,$$

$$\theta_{PS} = \theta_P + \theta_S + \theta_P(\lambda + \mu - 1)\theta_S \, .$$

Then

$$\begin{aligned}
\theta_{P(R+S)} &= \theta_{PR+PS} \\
&= \theta_{PR}\lambda + \theta_{PS}\mu + 1 \\
&= (\theta_P + \theta_R + \theta_P(\lambda+\mu-1)\theta_R)\lambda + (\theta_P + \theta_S + \theta_P(\lambda+\mu-1)\theta_S)\mu + 1 \\
&= \theta_P + (\theta_R\lambda + \theta_S\mu + 1) + \theta_P(\lambda+\mu-1)(\theta_R\lambda + \theta_S\mu + 1) \\
&= \theta_P + \theta_{R+S} + \theta_P(\lambda + \mu - 1)\theta_{R+S} \, .
\end{aligned}$$

Alternatively, we can prove (iii) first and then

$$\begin{aligned}
(\lambda+\mu-1)\theta_{PQ} &= \psi_{PQ} - 1 \\
&= \psi_P\psi_Q - 1 \\
&= (1 + (\lambda+\mu-1)\theta_P)(1 + (\lambda+\mu-1)\theta_Q) - 1 \\
&= (\lambda+\mu-1)(\theta_P + \theta_Q + \theta_P(\lambda+\mu-1)\theta_Q) \, .
\end{aligned}$$

(iii) $\psi_1 = 1 = 1 + (\lambda + \mu - 1)\theta_1$, since $\theta_1 = 0$.

To prove the formula by induction assume

$$\psi_Q = 1 + (\lambda + \mu - 1)\theta_Q \quad \text{and} \quad \psi_R = 1 + (\lambda + \mu - 1)\theta_R \, .$$

Then
$$\begin{aligned}
\psi_{Q+R} &= \psi_Q\lambda + \psi_R\mu \\
&= (1 + (\lambda + \mu - 1)\theta_Q)\lambda + (1 + (\lambda + \mu - 1)\theta_R)\mu \\
&= 1 + (\lambda + \mu - 1)(\theta_Q\lambda + \theta_R\mu + 1) \\
&= 1 + (\lambda + \mu - 1)\theta_{Q+R} \, .
\end{aligned}$$

(iv) $\quad \chi_P = \psi_P + \theta_P$, by definition of these index polynomials,

$$= 1 + (\lambda + \mu - 1)\theta_P + \theta_P, \text{ by (iii)},$$

$$= 1 + (\lambda + \mu)\theta_P$$

$$= \theta_{2P}, \text{ by (ii)} .$$

From the formula $\psi_P = 1 + (\lambda + \mu - 1)\theta_P$ and Theorem 1.5 (or equally well from the formula $\chi_P = 1 + (\lambda + \mu)\theta_P$) follows

<u>THEOREM 1.7</u> . Index θ-polynomials represent faithfully the algebra of trees $\mathcal{Y}$ (and thus also the logarithmetic $\mathcal{L}$).

A necessary and sufficient condition that a polynomial $\varphi \in \mathfrak{M}[\lambda,\mu]$ should be an index θ-polynomial is that

either $\qquad \varphi = 0$

or $\qquad \varphi = \varphi_1\lambda + \varphi_2\mu + 1$

and both φ_1 and φ_2 are θ-polynomials.

We can have, however, a much more direct criterion:

<u>THEOREM 1.8</u> . A polynomial $\varphi \in \mathfrak{M}[\lambda,\mu]$ is an index θ-polynomial if and only if: either $\varphi = 0$

or (i) all coefficients of φ are equal to 1

and (ii) with each of its terms φ contains also <u>all</u> the (positive) right divisors of that term.

To prove the necessity, i.e. that

$$(\varphi = \theta_P) \Longrightarrow (\varphi = 0 \text{ or (i) and (ii)})$$

use induction on the altitude of P. The conditions are evidently necessary for $\alpha = 0$ and 1. Assume necessity for $\alpha < a$ $(a > 1)$ and let $\alpha_P = a$. Then $\theta_P = \theta_{P_1}\lambda + \theta_{P_2}\mu + 1$ where P_1 and P_2 are the subtrees of P of the first order and their altitudes are less than a. (i) obviously follows from the induction hypothesis.

If φ contains a term $\nu_1\nu_2\nu_3\ldots\nu_k\lambda$ (where each $\nu_i = \lambda$ or μ) then θ_{P_1} must contain the term $\nu_1\nu_2\nu_3\ldots\nu_k$ and, by the induction hypothesis, all its other right divisors: $\nu_2\nu_3\ldots\nu_k$, $\nu_3\ldots\nu_k$, ... , ν_k, 1. Hence φ contains the terms $\nu_2\nu_3\ldots\nu_k\lambda$, $\nu_3\ldots\nu_k\lambda$, ... , $\nu_k\lambda$, λ and since it also contains the term 1 it contains all the right divisors of $\nu_1\nu_2\nu_3\ldots\nu_k\lambda$. Similarly for a term of the form $\nu'_1\nu'_2\nu'_3\ldots\nu'_h\mu$.

We prove the sufficiency of conditions (i) and (ii) by induction on n, the degree of φ in λ, μ.
If $n = 0$, $\varphi = 0$ or 1 and $\varphi = \theta_1$ or θ_2.
Assume the conditions are sufficient for polynomials of degree less than m $(m > 0)$ and let φ be of degree m. Then (ii) implies that φ has a term 1, i.e. we can write $\varphi = \varphi_1\lambda + \varphi_2\mu + 1$ and if φ satisfies (i) and (ii) then so do φ_1 and φ_2. Hence by the induction hypothesis φ_1, φ_2 are θ-polynomials and so is φ.

An immediate consequence of the foregoing is that the Θ-polynomial of a primary tree is uniquely determined by its leading term (i.e. the term of highest degree in λ, μ).

We say that a tree P is the <u>union</u> of trees $P_1, P_2, \ldots, P_r$, and write

$$P = P_1 \cup P_2 \cup \ldots \cup P_r = \bigcup_{i=1}^{r} P_i$$

if $$\{\Theta_P\} = \{\Theta_{P_1}\} \cup \{\Theta_{P_2}\} \cup \cdots \cup \{\Theta_{P_r}\} = \bigcup_{i=1}^{r} \{\Theta_{P_i}\}$$

where $\{\Theta_Q\}$ denotes the set of all the terms of Θ_Q.
$\bigcup_{i=1}^{r} P_i$ is therefore the least common superior of the P_i's.

It is easily seen that Q is a subordinate of P if and only if $\{\Theta_Q\} \subset \{\Theta_P\}$.

<u>THEOREM 1.9</u>. Any tree $P \neq 1$ is the union of its branches P_i, i.e. $\{\Theta_P\} = \bigcup_i \{\Theta_{P_i}\}$.

<u>Proof</u>. Use induction on α_P, the altitude of P. The theorem is true for $\alpha = 1$ as the only branch of $\vee$ is $\vee$ itself. Suppose it is true for $\alpha < a$ $(a > 1)$. Let $\alpha_P = a$. Then $\Theta_P = \Theta_Q\lambda + \Theta_R\mu + 1$ where $\alpha_Q, \alpha_R < a$ and at least one of $\alpha_Q, \alpha_R \neq 0$. Now, free forks of Q, R are free forks of P, i.e. if Q_s and R_t are branches of Q and R respectively $\Theta_{Q_s}\lambda + 1$ and $\Theta_{R_t}\mu + 1$ are Θ-polynomials of branches of P. Moreover, if Q_s and R_t run through all branches of Q and R, the Θ-polynomials of all branches of P are thus obtained.

Hence, $\{\Theta_P\} = \{\Theta_Q\lambda + \Theta_R\mu + 1\}$

$$= (\bigcup_s \{\Theta_{Q_s}\lambda + 1\}) \cup (\bigcup_t \{\Theta_{R_t}\mu + 1\})$$

$$= \bigcup_i \{\Theta_{P_i}\} .$$

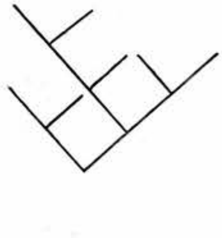
COROLLARY. The index Θ-polynomial of any tree $P \neq 1$ is uniquely determined by the leading terms of the Θ-polynomials of all the branches of P .

9. ω-POLYNOMIALS

The corollary to Theorem 1.9 suggests that the algebra of trees can be represented faithfully by index polynomials containing fewer terms than either ψ- or Θ-polynomials.

Definition. The index ω-polynomial of a tree P is the polynomial whose terms correspond to all the nodes of free forks of P. $\omega_1 = 0$.

Example.

P =  $\omega_P = \lambda^2\mu + \mu^2 + \lambda.$

The definition of ω-polynomials amounts to this: ω_P ($P \neq 1$) consists of all those terms of Θ_P which are not right-divisors of any other terms of Θ_P.

Whence, the necessary and sufficient condition for a polynomial $\varphi \in \mathfrak{M}[\lambda,\mu]$ to be an index ω-polynomial is that either $\varphi = 0$

or (i) all coefficients of φ are equal to 1

and (ii) if φ_1 is a term of φ, φ does <u>not</u> contain any other right-divisors of φ_1.

The proof is similar to that in Section 8.

<u>THEOREM 1.10</u> . (i) $\omega_1 = 0, \quad \omega_2 = 1,$

$$\omega_{P+Q} = \omega_P\lambda + \omega_Q\mu \quad (P + Q \neq 2).$$

(This is an alternative definition of ω-polynomials.)

$$\text{(ii)} \quad \omega_{PQ} = \omega_Q \quad \text{if} \quad P = 1,$$
$$\omega_{PQ} = \omega_P\psi_Q \quad \text{if} \quad P \neq 1.$$

$$\text{(iii)} \quad \psi_Q = \omega_{2Q}.$$

$$\text{(iv)} \quad \omega_{PQ} = \omega_Q \quad \text{if} \quad P = 1,$$
$$\omega_{PQ} = \omega_P\omega_{2Q} \quad \text{if} \quad P \neq 1.$$

The proof of (i) is almost identical with the proof of Theorem 1.4 (i). Proposition (ii) is obviously true if either P = 1 or Q = 1. To complete the proof assume that it is true for two trees A and B, i.e. that

$$\omega_{PA} = \omega_P\psi_A, \quad \omega_{PB} = \omega_P\psi_B.$$

Then
$$\begin{aligned}\omega_{P(A+B)} = \omega_{PA+PB} &= \omega_{PA}\lambda + \omega_{PB}\mu \\ &= \omega_P\psi_A\lambda + \omega_P\psi_B\mu \\ &= \omega_P(\psi_A\lambda + \psi_B\mu) \\ &= \omega_P\psi_{A+B}.\end{aligned}$$

(iii) is a particular case of (ii) when $P = 2$;

(iv) follows from (ii) and (iii).

We have seen (Corollary to Theorem 1.9) that there is a one-one correspondence $\theta_P \leftrightarrow \omega_P$. Hence

<u>THEOREM 1.11</u> . The algebra of trees $\mathcal{Y}$ (and thus also the logarithmetic $\mathcal{L}$) is faithfully represented by index ω-polynomials.

For each branch of a tree P ($P \neq 1$) the index polynomial ω_P contains one term. There are therefore γ_P terms in ω_P, i.e.

$$\gamma_P = \omega_P(1,1).$$

Also since the potency of a tree P can be defined as the number of free ends in P, the number of terms in ψ_P is δ_P, i.e.

$$\delta_P = \psi_P(1,1).$$

But $\psi_P = \omega_{2P}$ and thus

$$\delta_P = \gamma_{2P}.$$

Now, $\gamma_P \leq \frac{1}{2}\gamma_{2P}$, i.e. $\gamma_P \leq \frac{1}{2}\delta_P$, the equality sign holding only if P itself is of the form 2Q.

Thus an ω-polynomial of a tree ($\neq 1$) contains at most only half as many terms as the ψ-polynomial of the same tree. Moreover, since the number of terms in the θ-polynomial of P

$$\theta_P(1,1) = \delta_P - 1 = \psi_P(1,1) - 1,$$

the number of terms in ω_P is at most $\frac{1}{2}$(1 + number of terms in θ_P).

Thus the ω-polynomials have the great advantage of brevity. θ-polynomials again give a more direct representation of trees showing the position of each node (cf. the definition of the union of trees).

The principal advantage of the ψ-polynomials is that they represent the multiplication of trees by multiplication of polynomials. This will be found particularly useful in the study of factorization of trees and indices.

10. FACTORIZATION

A tree or an index P is said to be prime if

$$(P \neq 1) \text{ and } ((P = QR) \Longrightarrow (Q = 1 \text{ or } R = 1)).$$

Since $\delta_{PQ} = \delta_P \delta_Q$ and $\delta_1 = 1$, a tree is prime if its potency is a prime number.

For similar reasons the index ψ-polynomial of a prime tree must be prime in Ψ (the algebra of all ψ-polynomials), but it is by no means obvious that it is also prime in $\mathfrak{M}[\lambda,\mu]$. Indeed an ω-polynomial of a prime tree need not be prime in $\mathfrak{M}[\lambda,\mu]$,

e.g. $$\omega_{(3+2)+1} = \lambda^3 + \mu\lambda = (\lambda^2 + \mu)\lambda$$

although $(3 + 2) + 1$ is prime.

Thus even if the unique factorization law holds in $\mathfrak{M}[\lambda,\mu]$ it may not be so in Ψ. It is not a priori impossible that, e.g., a ψ-polynomial

$$\psi_P = \varphi_1 \cdot \varphi_2 \cdot \varphi_3 \cdot \varphi_4 \cdot \varphi_5 \cdot \varphi_6,$$

where $\varphi_i \varepsilon \mathfrak{M}[\lambda,\mu]$ but are not ψ-polynomials, and that

$$\varphi_1\varphi_2 = \psi_Q, \qquad \varphi_3\varphi_4 = \psi_R, \qquad \varphi_5\varphi_6 = \psi_S,$$

$$\varphi_1\varphi_2\varphi_3 = \psi_T, \qquad \varphi_4\varphi_5\varphi_6 = \psi_V$$

where ψ_Q, ψ_R, ψ_S, ψ_T, ψ_V are ψ-polynomials of prime trees. Then $\psi_P = \psi_Q\psi_R\psi_S = \psi_T\psi_V$ and ψ_P though perhaps uniquely factorizable in $\mathfrak{M}[\lambda,\mu]$ would not have unique prime factors in Ψ.

We prove, however, that this is in fact impossible and that the ψ-polynomials and therefore also the indices and trees are uniquely factorizable.

<u>LEMMA</u>. If $\psi_P\psi_Q = \psi_R\psi_S$ and ψ_P, ψ_R are both prime, then $\psi_P = \psi_R$ and $\psi_Q = \psi_S$.

The lemma is true if ψ_Q is of degree 0; for $\psi_Q = 1$ implies that $\psi_P = \psi_R\psi_S$ and since ψ_P, ψ_R are prime we have $\psi_S = 1$ and $\psi_P = \psi_R$. Assume that the lemma holds if the degree of ψ_Q is less than n. Let ψ_Q be a ψ-polynomial of degree n $(n > 0)$.

Then $\psi_Q = \psi_{Q_1}\lambda + \psi_{Q_2}\mu$

and $\psi_S = \psi_{S_1}\lambda + \psi_{S_2}\mu$ $\quad (\psi_S \neq 1$, since $(\psi_S = 1) \Longrightarrow (\psi_Q = 1))$.

$$\psi_{PQ} = \psi_P\psi_Q = \psi_P\psi_{Q_1}\lambda + \psi_P\psi_{Q_2}\mu = \psi_R\psi_S = \psi_R\psi_{S_1}\lambda + \psi_R\psi_{S_2}\mu.$$

Hence by the lemma to Theorem 1.5

$$\psi_P\psi_{Q_1} = \psi_R\psi_{S_1} \quad \text{and} \quad \psi_P\psi_{Q_2} = \psi_R\psi_{S_2}$$

and since $\psi_{Q_1}, \psi_{S_1}, \psi_{Q_2}, \psi_{S_2}$ are of degree less than n and ψ_P, ψ_R are prime, $\psi_P = \psi_R$, $\psi_{Q_1} = \psi_{S_1}$, $\psi_{Q_2} = \psi_{S_2}$, by the induction hypothesis. Therefore $\psi_Q = \psi_S$.

<u>THEOREM 1.12</u> . ("Unique factorization into primes.")

(i) If $\psi_P = \psi_{P_1}\psi_{P_2}\cdots\psi_{P_{r-1}}\psi_{P_r} = \psi_{Q_1}\psi_{Q_2}\cdots\psi_{Q_{s-1}}\psi_{Q_s}$,

where r, s are finite ordinals and ψ_{P_i}, ψ_{Q_i} are prime ψ-polynomials then $r = s$ and $\psi_{P_i} = \psi_{Q_i}$ $(1 \leq i \leq r)$.

(ii) If $P = P_1 P_2 \cdots P_{r-1} P_r = Q_1 Q_2 \cdots Q_{s-1} Q_s$

where r, s are finite ordinals and P_i, Q_i are prime trees or indices then $r = s$ and $P_i = Q_i$ $(1 \leq i \leq r)$.*

<u>Proof</u>. (i) Multiplication of ψ-polynomials is associative, therefore $\psi_{P_1}(\psi_{P_2}\cdots\psi_{P_r}) = \psi_{Q_1}(\psi_{Q_2}\cdots\psi_{Q_s})$.
ψ_{P_1}, ψ_{Q_1} are prime, hence by the Lemma

$$\psi_{P_1} = \psi_{Q_1} \quad \text{and} \quad \psi_{P_2}\cdots\psi_{P_r} = \psi_{Q_2}\cdots\psi_{Q_s} .$$

Now r and s are finite. The result follows (formally by induction on r).

(ii) follows from (i) and Theorem 1.5.

<u>COROLLARY</u>. ("Unique division".) If P, Q are two trees there is <u>at most</u> one tree X and one tree Y satisfying

$$PX = Q, \qquad YP = Q.$$

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* This is also a direct corollary to Etherington's Theorem on unique factorization of partitioned serials ((15), p.448).

One of the most remarkable properties of index ψ-polynomials is that ψ-polynomials of prime trees are prime in $\mathfrak{M}[\lambda,\mu]$, i.e. $\varphi'\varphi''$ cannot be a ψ-polynomial unless both φ' and φ'' are ($\pm$) ψ-polynomials. The following quite obvious lemma will be required. Denote by $n(\varphi)$ the degree of φ in λ, μ.

LEMMA. If ψ_P is a ψ-polynomial of degree k it can be expressed in the form $\psi_P = (\lambda + \mu)\varphi_1 + \varphi_2$, where $n(\varphi_1) = k - 1$, $n(\varphi_2) \leq k - 1$, and $\varphi_1 + \varphi_2$ is a ψ-polynomial.

For if P has r forks with ends at the maximal altitude k (i.e. if φ_1 has r terms of degree k-1) $\varphi_1 + \varphi_2$ is the ψ-polynomial of the rth principal subordinate of P or of a subordinate of that tree.

THEOREM 1.13 . The product of two polynomials φ', $\varphi'' \in \mathfrak{M}[\lambda,\mu]$, with non-negative coefficients, is an index ψ-polynomial if and only if both φ' and φ'' are ψ-polynomials.

Proof. Sufficiency. If φ' and φ'' are ψ-polynomials, $\varphi' = \psi_P$, $\varphi'' = \psi_Q$ say, then $\varphi'\varphi'' = \psi_P\psi_Q = \psi_{PQ}$.

Necessity. Use induction on $n(\varphi'\varphi'')$. The condition is necessary if the degree of the product is 0, because if $\varphi'\varphi''$ is a ψ-polynomial of degree 0 then

$$\varphi'\varphi'' = 1 \quad \text{and} \quad \varphi' = \varphi'' = 1 = \psi_1 .$$

Assume that it is necessary for products whose degree is less than m $(m>0)$. Let $n(\varphi'\varphi'') = m$. If either φ' or φ'' is 1 there is nothing to prove.

Suppose φ', $\varphi'' \neq 1$. Since $\varphi'\varphi''$ is an index ψ-polynomial of degree $m>0$, $\varphi'\varphi''$ cannot contain 1 as a term and all coefficients of $\varphi'\varphi''$ are 1. Thus at most one of φ', φ'' contains 1 as a term.

<u>Firstly</u>, if φ'' does not contain 1 as a term it must contain terms ending in λ as well as terms ending in μ (since $\varphi'\varphi''$ being a ψ-polynomial must contain such terms). Let $\varphi'' = \varphi''_1\lambda + \varphi''_2\mu$.

Then $$\varphi'\varphi'' = \varphi'\varphi''_1\lambda + \varphi'\varphi''_2\mu$$

and, by the lemma to Theorem 1.5, $\varphi'\varphi''_1$ and $\varphi'\varphi''_2$ are ψ-polynomials.

Hence by the induction hypothesis φ', φ''_1, φ''_2 are ψ-polynomials, i.e. φ' and $\varphi'' = \varphi''_1\lambda + \varphi''_2\mu$ are index ψ-polynomials.

<u>Secondly</u> we prove that φ'' cannot contain 1 as a term. We do this by showing that » $\varphi'' \neq 1$ and φ'' contains 1 as a term" is incongruous with the premiss » $\varphi'\varphi''$ is an index ψ-polynomial", viz. that from these premises it can be concluded that φ'' is an index ψ-polynomial which is impossible if φ'' contains 1 as a term and $\varphi'' \neq 1$.

Suppose then that $\varphi' = \varphi_1'\lambda + \varphi_2'\mu$

and $\varphi'' = \varphi_1''\lambda + \varphi_2''\mu + 1$

where at least one of φ_1'', φ_2'' is non-zero.

$$\varphi'\varphi'' = (\varphi_1'\lambda + \varphi_2'\mu)(\varphi_1''\lambda + \varphi_2''\mu + 1)$$

$$= (\varphi_1'(\lambda\varphi_1'' + 1) + \varphi_2'\mu\varphi_1'')\lambda + (\varphi_2'(\mu\varphi_2'' + 1) + \varphi_1'\lambda\varphi_2'')\mu$$

and by the lemma to Theorem 1.5

both $\varphi_1'(\lambda\varphi_1'' + 1) + \varphi_2'\mu\varphi_1''$ and $\varphi_2'(\mu\varphi_2'' + 1) + \varphi_1'\lambda\varphi_2''$ are ψ-polynomials.

(i) If $n(\varphi_1') > n(\varphi_2')$ then, by the lemma applied to the above two ψ-polynomials,

$$\varphi_1' = (\lambda + \mu)\varphi_{11}' + \varphi_{12}' \quad \text{where} \quad n(\varphi_{11}' + \varphi_{12}') = n(\varphi_1') - 1$$

and $(\varphi_{11}' + \varphi_{12}')(\lambda\varphi_1'' + 1) + \varphi_2'\mu\varphi_1''$, $\varphi_2'(\mu\varphi_2'' + 1) + (\varphi_{11}' + \varphi_{12}')\lambda\varphi_2''$

are ψ-polynomials. Thus

$$((\varphi_{11}' + \varphi_{12}')(\lambda\varphi_1'' + 1) + \varphi_2'\mu\varphi_1'')\lambda + (\varphi_2'(\mu\varphi_2'' + 1) + (\varphi_{11}' + \varphi_{12}')\lambda\varphi_2'')\mu$$

$$= ((\varphi_{11}' + \varphi_{12}')\lambda + \varphi_2'\mu)(\varphi_1''\lambda + \varphi_2''\mu + 1)$$

$$= ((\varphi_{11}' + \varphi_{12}')\lambda + \varphi_2'\mu).\varphi'' \text{ is a } \psi\text{-polynomial.}$$

But $n(((\varphi_{11}' + \varphi_{12}')\lambda + \varphi_2'\mu).\varphi'') = m - 1.$

Hence by the induction hypothesis φ'' is an index ψ-polynomial, $\neq 1$, with 1 as one of its terms. Contradiction.

(ii) If $n(\varphi_1') \leq n(\varphi_2')$ the proof is similar. Note that although $n(\varphi_1') = n(\varphi_2')$ implies $n(\varphi_1'\lambda\varphi_1'') = n(\varphi_2'\mu\varphi_1'')$ the terms of the two free ends of a fork at maximal altitude in the tree corresponding to the ψ-polynomial $\varphi_1'(\lambda\varphi_1'' + 1) + \varphi_2'\mu\varphi_1''$ cannot be contained one in $\varphi_1'\lambda\varphi_1''$ and the other in $\varphi_2'\mu\varphi_1''$, unless $\varphi_1' = \varphi_2' = 1$. In this case, however, $\varphi' = \lambda + \mu$ and $\varphi'\varphi''$, a ψ-polynomial, would contain λ and μ as terms as well as other terms; this is impossible.

<u>COROLLARY</u>. A ψ-polynomial of a prime tree is prime in $\mathfrak{M}[\lambda,\mu]$.

If P is a left (right) divisor of Q and Q is a left (right) divisor of R then P is a left (right) divisor of R.

If a tree or an index P is a <u>left</u> divisor of Q and of R then it is also a left divisor of Q+R. For if $Q = PX$ and $R = PY$ then $Q + R = PX + PY = P(X + Y)$. This property does not hold, in general, for right divisors.

A very useful criterion of primeness of many trees is the following:

The sum of two unequal trees which have no common proper left divisor is a prime tree.

For $P + Q = RS$ where $S \neq 1$, i.e. $S = S_1 + S_2$, would imply $P = RS_1$ and $Q = RS_2$ and, since P and Q have no common proper left divisor and $P \neq Q$, R is equal to 1.

On the other hand if two trees or indices Q and R have a common left divisor P then every index expressible in the form $QX + RY$ has P as left divisor.

All indices having P as left divisor form a subgroupoid of $\mathcal{L}^+$. This cyclic subgroupoid (which is a proper subgroupoid if $P \neq 1$) is additively isomorphic with $\mathcal{L}^+$ itself. Obviously the product of any such index postmultiplied by any other index has P as left divisor. Thus all indices having P as left divisor form a right ideal of $\mathcal{L}$.

11. THE LATTICE OF ALL TREES

The union of two trees $P \cup Q$ has been defined by

$$\{\theta_{P \cup Q}\} = \{\theta_P\} \cup \{\theta_Q\}.$$

We can similarly define the intersection $P \cap Q$ of P and Q by

$$\{\theta_{P \cap Q}\} = \{\theta_P\} \cap \{\theta_Q\}$$

and consider the lattice of all trees L. L is partially ordered by the relation of subordination, i.e. $P \leq Q$ if P is a subordinate of Q. It is obviously a distributive lattice.

Since
$$\delta_P + \delta_Q = \delta_{P \cup Q} + \delta_{P \cap Q}$$

and $P < Q$ (i.e. $P \leq Q$ and $P \neq Q$) implies $\delta_P < \delta_Q$, potency is a positive valuation on L and L is a metric lattice (v. (6), pp. 74-76).

If we now define the <u>distance</u> from P to Q

$$\partial(P, Q) = \delta_{P \cup Q} - \delta_{P \cap Q}$$

i.e. as the number of forks belonging either to P or to Q but not to both we have

$$\partial(P, P) = 0 \quad \text{while} \quad \partial(P, Q) > 0 \quad \text{if} \quad P \neq Q,$$
$$\partial(P, Q) = \partial(Q, P),$$
$$\partial(P, Q) + \partial(Q, R) \geq \partial(P, R),$$

and thus the set of all trees, with distance so defined, forms a metric space.

If P is a subordinate of Q then all trees X such that $P \leq X \leq Q$ form a sublattice of $\mathcal{L}$. Such a sublattice is called a <u>closed interval</u> of $\mathcal{L}$ and is denoted by $[P, Q]$. Q is its greatest and P its least element. In particular the closed interval $[1, P]$ is the sublattice of all subordinates of P. Similarly the sublattice of all superiors of P may be denoted by $[P, \infty]$.

The concept of the lattice of all trees should prove particularly useful in the study of those collapsed logarithmetics (v. (15), p. 452) which can be regarded as closed intervals or as lattice homomorphs of $\mathcal{L}$.

CHAPTER II. NON-ASSOCIATIVE NUMBER THEORY

Etherington (15) has obtained some basic results in the non-associative number theory. Notably he proved the unique factorization in the free logarithmetic $\mathcal{L}$ (see also (22) and (25) or Theorem 1.12), both cancellation laws and the associative law for multiplication in $\mathcal{L}$. Evans (22) deduced these and other simple properties of "non-associative numbers" (i.e. indices of $\mathcal{L}$) from Peano-like postulates. In particular he proved that if S is a proper factor of P, not a right-factor, and $P = Q + R$ then S is a proper factor of Q and R. A similar result was obtained independently by the author in (25) (v. Chap. I, § 10).

Addition and multiplication in $\mathcal{L}$ are both non-commutative. Two indices commute additively if and only if they are equal. In Theorem 2.1 we give a necessary and sufficient condition that two indices of $\mathcal{L}$ should commute multiplicatively. In Theorems 2.2 and 2.3 we solve Diophantine-like equations. In (22) Evans has proved "Fermat's Last Theorem" for non-associative numbers. Theorem 2.3 generalizes this result which is then deduced as a corollary.

It is convenient to extend the definition of exponentiation of an index (cf. (15), p. 449): We define $P^0 = 1$ for all indices P of $\mathcal{L}$.

<u>THEOREM 2.1</u> . P and Q, two indices of $\mathscr{L}$, commute with respect to multiplication, i.e. $PQ = QP$, if and only if they are powers of the same index.

<u>Proof</u>. The condition is obviously sufficient. To prove necessity denote the number of prime factors in an index X by pr(X) and use induction on pr(PQ).
If pr(PQ) = 0 or 1 then either P or Q is equal to 1 and either $P = Q^{o}$ or $Q = P^{o}$.
Now let pr(PQ) = a $(a > 1)$ and assume that the condition is necessary for all pairs of commuting indices whose products contain less than <u>a</u> prime factors. Without loss of generality we can suppose that $pr(P) \leq pr(Q)$.
If $P = 1$ then $P = Q^{o}$. Otherwise $PQ = QP$ gives $Q = PQ'$, where Q' is a proper left-divisor of Q or is equal to 1. It follows that $P^2Q' = PQ'P$ and therefore $PQ' = Q'P$. Now, $pr(PQ') < a$ and, by the induction hypothesis, $P = R^s$ and $Q' = R^t$ for some index R. Hence $P = R^s$ and $Q = R^{s+t}$.

<u>THEOREM 2.2</u> . If $X^mP = Y^n$ or $PX^m = Y^n$, where X, P, Y are indices of $\mathscr{L}$, P is prime and m, n are integers greater than 1, then X and Y are both powers of P.

<u>Proof</u>. $X^mP = Y^n$ implies either (i) $X = Y^sY'$ or (ii) $Y = X^tX'$, where s and t are maximal in the sense

that Y', X' are proper left-divisors of Y, X respectively or are equal to 1.

If (i) $X = Y^sY'$ then $XP = Y^sY'P$ and, since XP is a proper right-divisor of Y^n and $pr(Y^s) < pr(XP) \leq pr(Y^{s+1})$, $XP = Y''Y^s$ where Y'' is a proper right-divisor of Y or is equal to Y. But $pr(Y'') = pr(Y'P) = pr(XP) - pr(Y^s)$ and since Y'' is a left-divisor of XP it is a left-divisor of X and thus of Y. Hence $Y'' = Y'Q$ where Q is a prime index. Now, Y'' is also a right-divisor of Y so that Q is the prime right-divisor of Y. Therefore $Q = P$ and $Y^sY'P = Y'PY^s$, i.e. Y^s and $Y'P$ commute. Hence, by Theorem 2.1, $Y^s = R^a$ and $Y'P = R^b$ for some index R. Note that P is a right-divisor of R and R is a left-divisor of Y and let $R = Y_1Y_2\ldots Y_{k-1}P$ where the Y_i are prime. Then

$$XP = (Y_1\ldots Y_{k-1}P)^{a+b}, \qquad X = (Y_1\ldots Y_{k-1}P)^{a+b-1}Y_1\ldots Y_{k-1}$$

and, as $X^mP = Y^n$ and $m > 1$,

$$(Y_1\ldots Y_{k-1}P)^{a+b-1}Y_1\ldots Y_{k-1}(Y_1\ldots Y_{k-1}P)^{a+b-1}\ldots P$$
$$= (Y_1\ldots Y_{k-1}P)^{g}\ldots ,$$

where $g \geq 2a + 2b - 1$. Since factorization into primes is unique in $\mathscr{L}$ the $(k(a+b))$th, $(k(a+b)+1)$th, ... , $(k(a+b)+k-1)$th prime factors on both sides are equal, i.e. $Y_1 = P$, $Y_2 = Y_1$, ..., $P = Y_{k-1}$. Hence $Y_1 = \ldots = Y_{k-1} = P$ and X and Y are both powers of P.

If (ii) $Y = X^tX'$ consider first the case when $X' = 1$. Then $X^mP = X^{nt}$. Hence $P = X^{nt-m}$ and since P is prime $X = P$ and $Y = P^t$.

If $X' \neq 1$ it is a proper left-divisor of X. Since P is a right-divisor of Y it is a right-divisor of X'. Let $X' = X''P$. Now, $pr(X^t) < pr(Y) < pr(X^{t+1})$ and Y is a right-divisor of X^mP; therefore $Y = X'''X^tP$, where X''' is a proper right-divisor of X or is equal to 1. But X''' is a left-divisor of Y and therefore of X. Also X'' is a left-divisor of X and $pr(X'') = pr(X''')$. It follows that $X'' = X'''$ and $Y = X^tX''P = X''X^tP$. Hence $X^tX'' = X''X^t$, i.e. X^t and X'' commute and, by Theorem 2.1,

$$X^t = (X_1 \ldots X_h)^c \qquad \text{and} \qquad X'' = (X_1 \ldots X_h)^d$$

where the X_i are prime. Remembering that $n > 1$ we have

$$(X_1 \ldots X_h)^f \ldots P = (X_1 \ldots X_h)^{c+d} P (X_1 \ldots X_h)^{c+d} \ldots ,$$

where $f \geq 2c + 2d$. Comparing the $(h(c+d)+1)$th, $(h(c+d)+2)$th, ..., $(h(c+d)+h)$th prime factors on both sides we have $X_1 = P$, $X_2 = X_1$, $X_3 = X_2$, ... , $X_h = X_{h-1}$. Hence $X_1 = \ldots = X_h = P$ and both X and Y are powers of P.

If $PX^m = Y^n$ the proof is similar.

<u>Note</u>. In the statements and proofs of Theorems 2.1 and 2.2 no direct use is made of the operation of addition. It follws that these theorems are really about the free **multiplicative** logarithmetic $\mathcal{L}^{\times}$, and they actually apply

to free semigroups. Indeed the free semigroup with ρ generators is isomorphic to any subsemigroup of $\mathscr{L}^{\times}$ generated by ρ prime indices.

<u>THEOREM 2.3</u> . If $X^p + Y^q = Z^r$ where X, Y, Z are indices of $\mathscr{L}$ and p, q, r are integers greater than 1 then $X = 2^k$, $Y = 2^m$, $Z = 2^n$ and $kp = mq = nr - 1$.

<u>Proof</u>. Let $Z = Z_1 \ldots Z_n$, where the Z_i are prime. $X^p + Y^q = Z^r$ implies $X^p = (Z_1 \ldots Z_n)^{r-1} Z_1 \ldots Z_{n-1} Z'_n$ and $Y^q = (Z_1 \ldots Z_n)^{r-1} Z_1 \ldots Z_{n-1} Z''_n$ where $Z'_n + Z''_n = Z_n$ (Cf. (22), p. 302 or Chap. I, §10). Since Z_n is prime, Z'_n and Z''_n must be mutually left-prime.

Suppose $Z'_n, Z''_n \neq 1$. Then $X = UX'$ and $Y = UY'$ where X', Y' are mutually left-prime and neither is 1. Therefore $U(X'X^{p-1} + Y'Y^{q-1}) = Z^r$ and $X'X^{p-1} + Y'Y^{q-1}$ is prime. Since $p,q > 1$ the potency of $X'X^{p-1} + Y'Y^{q-1}$ is greater than that of U and hence $X'X^{p-1} + Y'Y^{q-1}$ is not a factor of U. Thus a prime occurs only once as a factor of $X^p + Y^q$. But every prime factor of Z^r occurs at least r times, i.e. more than once and $X^p + Y^q = Z^r$ which is a contradiction. Therefore either Z'_n or Z''_n is equal to 1.

If Z'_n is equal to 1, $X^p Z_n = Z^r$, where Z_n is prime and $p,r > 1$. Therefore, by Theorem 2.2, $X = Z_n^k$, $Z = Z_n^n$. Thus $Y^q = Z_n^{nr-1} Z''_n$. Since $q > 1$ the potency of Y is not greater than half of that of $Z_n^{nr-1}Z_n''$. Now, $nr - 1 > 0$ and the potency of Z_n'' is less than that of Z_n, which is prime. Hence all prime factors of Y are equal to Z_n and $Z_n'' = 1$. Therefore $Z_n = Z_n' + Z_n'' = 2$ and $X = 2^k$, $Y = 2^m$, $Z = 2^n$ where $kp = mq = nr - 1$.

If $Z_n'' = 1$ the proof is similar.

<u>COROLLARY</u> ("Fermat's Last Theorem").

$$X^r + Y^r = Z^r \text{ implies } r = 1 .$$

CHAPTER III. HOMOMORPHS OF THE FREE LOGARITHMETIC

1. INTRODUCTION

Addition in the free logarithmetic $\mathcal{L}$ is non-associative and non-commutative. Multiplication in $\mathcal{L}$ is associative and right-distributive but not commutative nor left-distributive. The following congruence relations determine therefore homomorphisms on $\mathcal{L}$:

commutative: $$P + Q \sim Q + P, \tag{c}$$

palintropic: $$PQ \sim QP, \tag{p}$$

left-distributive: $$(P + Q)R \sim PR + QR, \tag{d}$$

and entropic: $$(P + Q) + (R + S) \sim (P + R) + (Q + S) \tag{e}$$

(cf. (15)).

We denote the homomorph of $\mathcal{L}$ determined by congruence relation (r) by $\mathcal{L}_r$. It is known that $\mathcal{L}_e$ is a homomorph of $\mathcal{L}_p$ and that $\mathcal{L}_p$ is isomorphic to $\mathcal{L}_d$. The above four relations determine therefore only five distinct homomorphs: the free commutative logarithmetic $\mathcal{L}_c$, the free palintropic logarithmetic $\mathcal{L}_p$, the free entropic logarithmetic $\mathcal{L}_e$, the free commutative palintropic logarithmetic $\mathcal{L}_{cp}$ and the free commutative entropic logarithmetic $\mathcal{L}_{ce}$.

2. CONGRUENCE RELATIONS ON $\mathcal{L}$

$P = Q$ means that P and Q represent the same index in $\mathcal{L}$ or the same tree in $\mathcal{T}$. We shall say that P is congruent to Q modulo (r) and write $P \sim Q \bmod (r)$ if (r) is an equivalence relation on $\mathcal{L}$ and

either (i) $P = Q$;

or (ii) $P \sim Q \bmod (r)$ by direct application of (r) (e.g. $2 + 3 \sim 3 + 2 \bmod (c)$; $2.3 \sim 3.2 \bmod (p)$; etc.);

or (iii) $P = P' + P''$, $Q = Q' + Q''$ and $P' \sim Q'$, $P'' \sim Q'' \bmod (r)$;

or (iii') $P = RP'$ and $Q = RQ'$ and $P' \sim Q' \bmod (r)$;

or (iv) $P = R_1 \sim R_2 \sim \ldots \sim R_k = Q$ where $R_i \sim R_{i+1} \bmod (r)$ $(1 \leq i \leq k - 1)$ by virtue of (i) or (ii) or (iii) or (iii').

We prove that "congruence" on $\mathcal{L}$ thus defined is a congruence relation in the usual sense (cf. (6), p. vii). It is obviously a congruence relation for addition. It suffices to prove

THEOREM 3.1 . If $P \sim P'$ and $Q \sim Q' \bmod (r)$ then $PQ \sim P'Q' \bmod (r)$.

Proof. $PQ \sim PQ'$, by (iii'). We prove that the premises of the theorem imply $PQ' \sim P'Q' \bmod (r)$. Use non-associative induction on Q'.

If $Q' = 1$ there is nothing to prove. Otherwise let $Q' = Q_1' + Q_2'$. Assume that $PQ_1' \sim P'Q_1'$ and $PQ_2' \sim P'Q_2'$. Then, by (iii), $PQ_1' + PQ_2' \sim P'Q_1' + P'Q_2'$, i.e. $PQ' \sim P'Q'$, since multiplication in $\mathcal{L}$ is right-distributive.

For all congruence relations considered in the preceding section case (iii') of the definition follows from the other four cases.

<u>THEOREM 3.2</u> . Let ρ_1, ρ_2, ρ_3 be equivalence relations on $\mathcal{L}$ defined as follows:

(1) $P\rho_1 Q$ if (I) $P = S + T$ and $Q = T + S$;

(2) $P\rho_2 Q$ if (I) $P = ST$ and $Q = TS$;

(3) $P\rho_3 Q$ if (I) $P = (S + T) + (U + V)$ and

$$Q = (S + U) + (T + V);$$

also $P\rho_i Q$ $(i = 1, 2, 3)$ if

either (II) $P = Q$;

or (III) $P = P' + P''$, $Q = Q' + Q''$ where $P'\rho_i Q'$ and $P''\rho_i Q''$;

or (IV) $P = R_1 \rho_i R_2 \rho_i R_3 \rho_i \ldots \rho_i R_k = Q$ where $R_s \rho_i R_{s+1}$ by virtue of (I) or (II) or (III).

Then $P\rho_1 Q$ is equivalent to $P \sim Q \bmod (c)$,

$P\rho_2 Q$ is equivalent to $P \sim Q \bmod (p)$

and $P\rho_3 Q$ is equivalent to $P \sim Q \bmod (e)$.

<u>Proof</u>. ρ_1, ρ_2, ρ_3 are obviously congruence relations for addition. It remains to prove that $P\rho_i Q$ implies

$RP\rho_i RQ$ $(i = 1, 2, 3)$. We consider the four cases in which $P\rho_i Q$.

(1) If (I) $P = S + T$ and $Q = T + S$ then

$$\begin{aligned} R(S + T) &= RS + RT, \text{ since multiplication in } \mathcal{L} \text{ is right-distributive,} \\ &\rho_1 RT + RS, \text{ by } (\rho_1), \\ &= R(T + S). \end{aligned}$$

Hence, by (IV), $RP\rho_1 RQ$.

(2) If (I) $P = ST$ and $Q = TS$ the proof is by non-associative induction.

When $T = 1$, $P = Q$ and thus $RP\rho_2 RQ$, by (II).

Let $T = T' + T''$ and assume that $RST'\rho_2 RT'S$ and $RST''\rho_2 RT''S$. Then

$$\begin{aligned} RST &= RST' + RST'' \\ &\rho_2 RT'S + RT''S, \text{ by the induction hypothesis and (III),} \\ &\rho_2 SRT' + SRT'', \text{ since } (RT')S\rho_2 S(RT') \text{ and } (RT'')S\rho_2 S(RT''), \\ &= SR(T' + T'') \\ &= SRT \\ &= RTS, \text{ by } (\rho_2). \end{aligned}$$

Hence $RP\rho_2 RQ$.

(3) If $P = (S + T) + (U + V)$, $Q = (S + U) + (T + V)$ then

$$\begin{aligned} RP &= (RS + RT) + (RU + RV), \text{ since multiplication in } \mathcal{L} \text{ is right-distributive,} \\ &\rho_3 (RS + RU) + (RT + RV), \text{ by } (\rho_3), \end{aligned}$$

$= R((S + U) + (T + V))$

$= RQ$.

Hence $RP\rho_3 RQ$.

Further, for all three relations:

if (II) $P = Q$, we have $RP = RQ$ and thus $RP\rho_i RQ$;

if (III) $P = P' + P''$, $Q = Q' + Q''$, where $P'\rho_i Q'$, $P''\rho_i Q''$, the result is easily provable by induction on altitude (or potency) of P;

if (IV) $P = R_1 \rho_i R_2 \rho_i R_3 \rho_i \ldots \rho_i R_k = Q$, where $R_s \rho_i R_{s+1}$ by virtue of (I), (II) or (III), the proof is by induction on k.

It follows from the above theorem that congruence relations mod (p), (c), (e) are completely defined by cases (i), (ii), (iii) and (iv) of the definition and in all subsequent proofs in which it is premised that indices or trees are congruent it will suffice to consider these four cases only.

3. FREE COMMUTATIVE LOGARITHMETIC. FREE ENTROPIC LOGARITHMETIC

Commutative logarithmetics have been studied by Etherington ((11), (12) and (15)). In this section we only add a theorem on faithful representations of the free commutative logarithmetic $\mathscr{L}_c$.

Denote the homomorphs of Ψ, Θ, Ω (the algebras of all ψ-, θ-, ω-polynomials) determined by congruence relations

$$\psi(\lambda,\mu) \sim \psi(\mu,\lambda), \quad \theta(\lambda,\mu) \sim \theta(\mu,\lambda), \quad \omega(\lambda,\mu) \sim \omega(\mu,\lambda)$$

by Ψ_ι, Θ_ι, Ω_ι respectively.

THEOREM 3.3 . $\mathscr{L}_c$ is faithfully represented by Ψ_ι, also by Θ_ι and by Ω_ι.

Proof. We prove first that

$$\psi(\lambda,\mu) \sim \psi(\mu,\lambda) \qquad (\iota)$$

implies $\psi_{P+Q} \sim \psi_{Q+P}$.

$$\begin{aligned}\psi_{P+Q}(\lambda,\mu) &= \psi_P(\lambda,\mu).\lambda + \psi_Q(\lambda,\mu).\mu \\ &\sim \psi_{P+Q}(\mu,\lambda), \quad \text{by } (\iota), \\ &= \psi_P(\mu,\lambda).\mu + \psi_Q(\mu,\lambda).\lambda \\ &\sim \psi_P(\lambda,\mu).\mu + \psi_Q(\lambda,\mu).\lambda, \quad \text{by } (\iota), \\ &= \psi_{Q+P}(\lambda,\mu).\end{aligned}$$

To prove the converse, i.e. that $\psi_{P+Q} \sim \psi_{Q+P}$ implies (ι), note that $\psi_1 \sim \psi_1 \bmod (\iota)$ and use induction on the altitude of $P + Q$.
When $\alpha_{P+Q} = 1$, $\psi_{P+Q}(\lambda,\mu) = \lambda + \mu$ while $\psi_{P+Q}(\mu,\lambda) = \mu + \lambda$. Suppose the theorem holds for altitudes less than a $(a > 1)$ and let $\alpha_{P+Q} = a$. Then

$$
\begin{aligned}
\psi_{P+Q}(\lambda,\mu) &\sim \psi_{Q+P}(\lambda,\mu) \\
&= \psi_P(\lambda,\mu).\mu + \psi_Q(\lambda,\mu).\lambda \\
&\sim \psi_P(\mu,\lambda).\mu + \psi_Q(\mu,\lambda).\lambda, \quad \text{by the induction hypothesis since } \alpha_P, \alpha_Q < a, \\
&= \psi_{P+Q}(\mu,\lambda) .
\end{aligned}
$$

The proof for θ- and ω-polynomials is almost identical.

The free entropic logarithmetic $\mathscr{L}_e$ is a homomorph of the free palintropic logarithmetic, that is to say PQ and QP are congruent modulo (e) for all P and Q. This result was essentially obtained by Murdoch ((32), Corollary to Theorem 10) and in a more general form by Etherington ((15), Theorem 4). Etherington has also proposed the question ((16), p. 249) whether the free entropic logarithmetic is represented faithfully by index θ-polynomials in <u>commuting</u> indeterminates λ, μ. Call index polynomials in commuting indeterminates palindromic. It is known that palindromic Θ-polynomials represent faithfully the logarithmetic of the general train algebra of rank 3 (v. (16), p. 249). Etherington's question amounted therefore to this: are the free entropic logarithmetic and the logarithmetic of the general train algebra of rank 3 isomorphic? In 1954 I communicated to Dr. Etherington the following example which answers the question in the negative.

Example. The indices $(\dot{4} + 1) + (1 + 3)$ and $(\dot{3} + 1) + (1 + 4)$ are not congruent mod (e) although their palindromic Θ-polynomials are both equal to $\lambda^2\mu^2 + \lambda^2\mu + \lambda\mu^2 + \lambda^2 + \mu^2 + \lambda + \mu + 1$.

4. PALINDROMIC ψ- AND Θ-POLYNOMIALS

Palindromic index ψ- and Θ-polynomials are polynomials in two commuting indeterminates λ, μ over the domain of integers and are defined as follows:

$$\psi_1 = 1, \qquad \psi_{P+Q} = \lambda\psi_P + \mu\psi_Q \ ;$$

$$\Theta_1 = 0, \qquad \Theta_{P+Q} = \lambda\Theta_P + \mu\Theta_Q + 1 \ .$$

The algebras of palindromic index polynomials are homomorphs of the algebras of the corresponding index polynomials (in non-commuting indeterminates) determined by the congruence relation

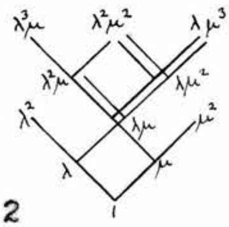
$$\lambda\mu = \mu\lambda \qquad (t)\,.$$

This homomorphism induces a homomorphism on $\mathcal{L}$. The resulting homomorph is called the <u>free palindromic logarithmetic</u> and is denoted by $\mathcal{L}_t$. $\mathcal{L}_t$ is a homomorph of $\mathcal{L}_e$ (cf. §3). It is not known if $\mathcal{L}_t$ is equationally definable on $\mathcal{L}$. Note that even if $\mathcal{L}_t$ is not equationally definable on $\mathcal{L}$ it may be so on L, the lattice of all trees (or indices).

The terms of a palindromic index polynomial of a tree P still represent knots of P: those of ψ_P represent the free ends of P and those of Θ_P the nodes of forks in P. Each term of an index polynomial is determined by one or more knots of the corresponding tree; a term $\nu\lambda^r\mu^s$ is the sum of ν monomials $\lambda^r\mu^s$ which are determined by ν knots in the tree. The palindromic ψ- or Θ-polynomial can be written down by inspection of the tree, exactly as the corresponding general index polynomials, except that the distinction between $\lambda\mu$ and $\mu\lambda$ is here ignored.

<u>Example</u>.

$$P = \dot{4} + (2.2 + 1)$$



$$\psi_P = \lambda^3\mu + 3\lambda^2\mu^2 + 2\lambda\mu^3 + \lambda^2\mu + \lambda^2 + \mu^2,$$

$$\Theta_P = \lambda^2\mu + 2\lambda\mu^2 + 2\lambda\mu + \lambda + \mu + 1\,.$$

Terms of palindromic index polynomials can be represented by weighted lattice points on a "tree pattern" in the following way: Let Oi, Oj be two semi-axes making 45°, or any other convenient acute angle, with the upward vertical; the term $\nu\lambda^i\mu^j$ corresponds to the point (i, j) and its weight is ν. This representation is suggested by the fact that if a tree is drawn so that the arms of all its forks are of equal length and make 45° with the upward vertical then knots coincide if and only if their terms are equal (when λ, μ commute) (cf. fig. on p. 60). Palindromic index polynomials are represented by certain sets of weighted lattice points on this pattern; terms not appearing in the polynomial can be regarded as having weight 0.

Etherington (16) gave necessary and sufficient conditions that a given polynomial $\sum \pi_{ij}\lambda^i\mu^j$ should be a palindromic index Θ-polynomial. They are: (i) all π_{ij} are non-negative integers, (ii) if i,j are not both zero, $\pi_{ij} \leq \pi_{i-1,j} + \pi_{i,j-1}$, (iii) $\pi_{00} = 0$ or 1. It follows that in any representation of a palindromic Θ-polynomial on a tree pattern the weights of the points (i, 0), (0, j) cannot exceed 1 and that of the point (i+1, j+1) cannot exceed the sum of the weights of points (i+1,j) and (i, j+1).

<u>Example</u>. $P = \dot{4} + (2.2 + 1)$,

$$\psi_P = \lambda^3\mu + 3\lambda^2\mu^2 + 2\lambda\mu^3 + \lambda^2\mu + \lambda^2 + \mu^2,$$

$$\theta_P = \lambda^2\mu + 2\lambda\mu^2 + 2\lambda\mu + \lambda + \mu + 1 .$$

(In the remainder of this section "index polynomials" will mean "palindromic index polynomials" and ψ_P, θ_P will denote palindromic ψ-, θ-polynomials.)

We now obtain a necessary and sufficient condition that $\sum \nu_{ij}\lambda^i\mu^j$ be an index ψ-polynomial (Theorem 3.6).

Call a polynomial $\sum \nu_{ij}\lambda^i\mu^j$ <u>ordered with respect to</u> λ if the term $\nu_{pq}\lambda^p\mu^q$ precedes the term $\nu_{rs}\lambda^r\mu^s$ when $p + q > r + s$ or when $p + q = r + s$ and $p > r$. The first term of a polynomial φ, when φ has been ordered with respect to λ, is called the <u>leading</u> term of φ.

<u>THEOREM 3.4</u> . If a ψ-polynomial ψ can be written in the form $\psi = \lambda^i\mu^j + \varphi$, where φ is a polynomial in λ, μ with non-negative coefficients, and ψ' is any ψ-polynomial then $\lambda^i\mu^j\psi' + \varphi$ is a ψ-polynomial.

<u>Proof</u>. Use induction on $n(\psi)$, the degree of ψ. If $n(\psi) = 1$, $\psi = \lambda + \mu$ and the theorem holds since both $\lambda + \mu\psi'$ and $\lambda\psi' + \mu$ are ψ-polynomials.

Suppose that the theorem is true for ψ-polynomials of degree less than N $(N > 1)$ and let $n(\psi) = N$. Then $\psi = \lambda\psi_A + \mu\psi_B$ and either (i) ψ_A contains a term $\lambda^{i-1}\mu^j$, or (ii) ψ_B contains a term $\lambda^i\mu^{j-1}$, or both.

If (i): let $\psi_A = \lambda^{i-1}\mu^j + \varphi_A$, where φ_A is a polynomial in λ, μ with positive integer coefficients.

$\psi = \lambda^i\mu^j + \varphi$ and $\psi = \lambda\psi_A + \mu\psi_B$ imply $\varphi = \lambda\varphi_A + \mu\psi_B$.

Now, $n(\psi_A) < N$ and therefore, by the induction hypothesis, $\lambda^{i-1}\mu^j\psi' + \varphi_A$ is an index ψ-polynomial. Hence

$$\lambda(\lambda^{i-1}\mu^j\psi' + \varphi_A) + \mu\psi_B = \lambda^i\mu^j\psi' + (\lambda\varphi_A + \mu\psi_B) = \lambda^i\mu^j\psi' + \varphi$$

is also one. Similarly if (ii) is the case.

Note that if ψ, ψ' are the ψ-polynomials of P and Q respectively then $\lambda^i\mu^j\psi' + \varphi$ is the ψ-polynomial of a tree obtained by joining the root of Q to any free end of P whose term is $\lambda^i\mu^j$.

<u>COROLLARY 1</u>. If $\varphi_1, \varphi_2, \dots, \varphi_r$ are polynomials in λ, μ with positive integer coefficients such that $\sum \varphi_i$ is a ψ-polynomial and $\psi_1, \psi_2, \dots, \psi_r$ are any ψ-polynomials then $\sum_i \varphi_i\psi_i$ is a ψ-polynomial.

<u>COROLLARY 2</u>. If ψ is a ψ-polynomial containing the term $\nu_{ij}\lambda^i\mu^j$ then $\psi + (\lambda + \mu - 1)\lambda^i\mu^j$ is also a ψ-polynomial.

<u>COROLLARY 3</u>. If ψ is a ψ-polynomial containing the term $\nu_{ij}\lambda^i\mu^j$ then $\psi + (\lambda + \mu - 1)\nu\lambda^i\mu^j$, where ν is a positive integer not greater than ν_{ij}, is also a ψ-polynomial.

<u>Proof</u>. Apply Corollary 2 ν times.

Note that the converse of Theorem 3.4 is not true, i.e. the fact that $\lambda^i\mu^j\psi' + \varphi$ and ψ' are ψ-polynomials does not imply that $\lambda^i\mu^j + \varphi$ is one. For example $\psi_{3+3} = \lambda\mu\psi_2 + \lambda^3 + \mu^3 + 2\lambda\mu$ but $\lambda\mu + \lambda^3 + \mu^3 + 2\lambda\mu$ is <u>not</u> a ψ-polynomial. We can, however, prove a somewhat weakened form of the converse of Corollary 2:

<u>THEOREM 3.5</u> . Let ψ be a ψ-polynomial of degree n in λ,μ $(n \neq 0)$ and let $\nu\lambda^r\mu^s$ be its leading term. Then $\psi - (\lambda + \mu - 1)\lambda^{r-1}\mu^s$ is a ψ-polynomial.

<u>Proof</u>. If ψ is the ψ-polynomial of P then $\psi - (\lambda + \mu - 1)\lambda^{r-1}\mu^s$ is the ψ-polynomial of the first principal subordinate of P. To prove the theorem formally use induction on n.
If $n = 1$, $\psi = \lambda + \mu$ and $\lambda^{r-1}\mu^s = 1$. Therefore $\psi - (\lambda + \mu - 1)\lambda^{r-1}\mu^s = 1 = \psi_1$.
Assume that the theorem holds for all ψ-polynomials of degree less than N $(N > 1)$. Let ψ be of degree N.

Then $\psi = \lambda\psi' + \mu\psi''$ where ψ', ψ'' are ψ-polynomials of degree less than N. If $\nu'\lambda^{r'}\mu^{s'}$, $\nu''\lambda^{r''}\mu^{s''}$ are the leading terms of ψ', ψ'' then either (i) $r = r' + 1$ and $s = s'$ or (ii) $r = r''$ and $s = s'' + 1$ (or both). Also, by the induction hypothesis, $\psi' - (\lambda + \mu - 1)\lambda^{r'-1}\mu^{s'}$ and $\psi'' - (\lambda + \mu - 1)\lambda^{r''-1}\mu^{s''}$ are ψ-polynomials. Therefore

if (i):
$$\begin{aligned} &\lambda(\psi' - (\lambda + \mu - 1)\lambda^{r'-1}\mu^{s'}) + \mu\psi'' \\ &= \lambda\psi' + \mu\psi'' - (\lambda + \mu - 1)\lambda^{r'}\mu^{s'} \\ &= \psi - (\lambda + \mu - 1)\lambda^{r-1}\mu^{s} \end{aligned}$$

is a ψ-polynomial;

and if (ii):
$$\begin{aligned} &\lambda\psi' + \mu(\psi'' - (\lambda + \mu - 1)\lambda^{r''-1}\mu^{s''}) \\ &= \lambda\psi' + \mu\psi'' - (\lambda + \mu - 1)\lambda^{r''-1}\mu^{s''+1} \\ &= \psi - (\lambda + \mu - 1)\lambda^{r-1}\mu^{s} \end{aligned}$$

is a ψ-polynomial.

<u>COROLLARY</u>. The premises of Theorem 3.5 imply also that $\psi - (\lambda + \mu - 1)\nu\lambda^{r-1}\mu^{s}$ is a ψ-polynomial.

<u>Proof</u>. If $\nu = 1$ the corollary reduces to Theorem 3.5. If $\nu > 1$, the leading term of $\psi - (\lambda + \mu - 1)\lambda^{r-1}\mu^{s}$ is $(\nu - 1)\lambda^{r}\mu^{s}$. Hence $\psi - (\lambda + \mu - 1)2\lambda^{r-1}\mu^{s}$ is a ψ-polynomial; and so on.

<u>THEOREM 3.6</u> . A polynomial $\varphi = \sum \nu_{ij}\lambda^{i}\mu^{j}$ ($\varphi \neq 1$) with positive integer coefficients and leading term $\nu_{rs}\lambda^{r}\mu^{s}$ is a ψ-polynomial if and only if $\varphi - (\lambda + \mu - 1)\nu_{rs}\lambda^{r-1}\mu^{s}$ is a ψ-polynomial.

Proof. Necessity follows from the Corollary to Theorem 3.5 .

Sufficiency: Since all coeffients of φ are positive integers, the ψ-polynomial $\varphi - (\lambda + \mu - 1)\nu_{rs}\lambda^{r-1}\mu^s$ contains the term $\nu\lambda^{r-1}\mu^s$ $(\nu \geq \nu_{rs})$. Then, by Corollary 3 to Theorem 3.4,
$(\varphi - (\lambda + \mu - 1)\nu_{rs}\lambda^{r-1}\mu^s) + (\lambda + \mu - 1)\nu_{rs}\lambda^{r-1}\mu^s = \varphi$
is a ψ-polynomial.

Theorem 3.6 provides a useful algorithm for ascertaining if a given polynomial φ is a ψ-polynomial. In fact δ, the sum of coefficients in φ, is greater than that of $\varphi - (\lambda + \mu - 1)\nu_{rs}\lambda^{r-1}\mu^s$ and the process will determine whether φ is a ψ-polynomial or not in less than δ steps.

Obviously a polynomial $\varphi = \sum \pi_{ij}\lambda^i\mu^j$ is a Θ-polynomial if and only if $(\lambda + \mu - 1)\varphi + 1$ is a ψ-polynomial. We may expect that if all π_{ij} are positive integers and the leading term of φ is $\pi_{r-1,s}\lambda^{r-1}\mu^s$ then φ is a Θ-polynomial if and only if $\varphi - \pi_{r-1,s}\lambda^{r-1}\mu^s$ is one. The condition is in fact necessary, viz.

THEOREM 3.7 . If Θ is a Θ-polynomial and $\pi_{r-1,s}\lambda^{r-1}\mu^s$ its leading term then $\Theta - \pi_{r-1,s}\lambda^{r-1}\mu^s$ is a Θ-polynomial.

Proof. The premises imply that $(\lambda + \mu - 1)\Theta + 1$ is a ψ-polynomial with leading term $\pi_{r-1,s}\lambda^r\mu^s$.

Therefore, by Theorem 3.6,

$$((\lambda + \mu - 1)\Theta + 1) - (\lambda + \mu - 1)\pi_{r-1,s}\lambda^{r-1}\mu^{s}$$
$$= (\lambda + \mu - 1)(\Theta - \pi_{r-1,s}\lambda^{r-1}\mu^{s}) + 1$$

is a ψ-polynomial, say ψ_P. Hence $\Theta - \pi_{r-1,s}\lambda^{r-1}\mu^{s} = \Theta_P$.

The condition, however, is not sufficient, i.e. the mere fact that $\varphi - \pi_{r-1,s}\lambda^{r-1}\mu^{s}$ is a Θ-polynomial does not imply that φ is one. For example, $(5\lambda^2 + \lambda + 1) - 5\lambda^2$ is a Θ-polynomial but $5\lambda^2 + \lambda + 1$ is not a Θ-polynomial. To see why it should be so, let us try to use the method of proof in Theorem 3.7 to prove the sufficiency of the condition. Let therefore $\varphi - \pi_{r-1,s}\lambda^{r-1}\mu^{s}$ be a Θ-polynomial, Θ_P say. Then

$$\psi_P = (\lambda + \mu - 1)(\varphi - \pi_{r-1,s}\lambda^{r-1}\mu^{s})$$
$$= ((\lambda + \mu - 1)\varphi + 1) - (\lambda + \mu - 1)\pi_{r-1,s}\lambda^{r-1}\mu^{s}$$

and, by Theorem 3.6, $(\lambda + \mu - 1)\varphi + 1$ is a ψ-polynomial (and thus φ is a Θ-polynomial) provided that all coefficients of $(\lambda + \mu - 1)\varphi + 1$ are positive integers. Since φ itself is a polynomial with positive integer coefficients, this additional condition amounts to: $\pi_{ij} \leq \pi_{i-1,j} + \pi_{i,j-1}$ (i, j not both 0), which is Etherington's condition (ii).

5. FREE COMMUTATIVE ENTROPIC LOGARITHMETIC

This logarithmetic has particularly interesting faithful representations by index polynomials. We introduce for it a special nomenclature and notation. If two indices or trees, P and Q, are congruent mod (c)(e) we call them <u>concordant</u> and we write $P \sim Q$.

Observe that $(P + Q) + (R + S) \sim (A + B) + (C + D)$ where (A, B, C, D) is any of the 4! permutations of (P, Q, R, S). Indeed this fact together with the relation $1 + P \sim P + 1$ are equivalent to (c)(e). This suggests

<u>THEOREM 3.8</u> . If two subtrees of the same order of a tree P be transposed the resulting tree Q is concordant to P.

<u>Proof</u>. The theorem holds trivially when P is of altitude 1. We use induction on altitude and assume that the theorem is true for trees of altitudes less than a. Let P be of altitude a.

(1) If $P = P' + 1$ or $1 + P'$, both subtrees must belong to P', a tree of altitude $a-1$; the result follows by the induction hypothesis.

(2) If $P = (P_1 + P_2) + (P_3 + P_4)$ then:

(a) If both subtrees belong to $P_1 + P_2$, a tree of altitude a-1, the theorem again follows by the induction hypothesis.

Similarly if the two subtrees belong to $P_3 + P_4$.

(b) If one subtree belongs to P_1 and the other to P_3 (or one to P_2 and the other to P_4) the result follows from (a) since

$$(P_1 + P_2) + (P_3 + P_4) \sim (P_1 + P_3) + (P_2 + P_4).$$

(c) If one subtree belongs to P_1 and the other to P_4 (or one to P_2 and the other to P_3) the result follows from (a) since

$$(P_1 + P_2) + (P_3 + P_4) \sim (P_1 + P_2) + (P_4 + P_3)$$
$$\sim (P_1 + P_4) + (P_2 + P_3).$$

(Note. The proof of the equivalent proposition (which is false; v. Example in §3) for the non-commutative entropic logarithmetic fails in case (2)(c).)

We shall require a more general form of this result.

LEMMA. If a tree P has a free end at altitude a then any tree Q concordant to it has also a free end at the same altitude.

Proof. The lemma is quite obvious if (i) $P = Q$, or if (ii)(1) $P = (R+S) + (T+U)$ and $Q = (R+T) + (S+U)$, or if (ii)(2) $P = R + S$ and $Q = S + R$.
It is easily provable by induction on altitude if (iii) $P = R + S$, $Q = R' + S'$ and $R \sim R'$, $S \sim S'$ and by induction on k if (iv) $P = R_1 \sim R_2 \sim \ldots \sim R_k = Q$.

P is an nth superior of R if R is an nth subordinate of P. We shall use the following notation: Let a first superior of R be denoted by $\bar{R}$ or by $\bar{R}_{(\alpha)}$ if the node of the additional fork in the superior is at altitude α. $\bar{R}$ (or $\bar{R}_{(\alpha)}$) denotes a definite though unspecified tree, not the class of all first superiors of R.

<u>THEOREM 3.9</u> . If P and Q are two concordant trees, each with a free end at altitude a, then $\bar{P}_{(a)} \doteq \bar{Q}_{(a)}$.

<u>Proof</u>. Consider in turn the four cases defining $P \doteq Q$.

(i) If P = Q the result follows from Theorem 3.8 .

(ii)(1) $P = (R+S) + (T+U)$ and $Q = (R+T) + (S+U)$.

One at least of R, S, T, U has a free end at altitude a-2; let it be R. Then

$$\begin{aligned} \bar{P}_{(a)} &\doteq (\bar{R}_{(a-2)} + S) + (T + U), \text{ by Theorem 3.8,} \\ &\doteq (\bar{R}_{(a-2)} + T) + (S + U), \text{ by (e),} \\ &\doteq \bar{Q}_{(a)}, \text{ by Theorem 3.8 .} \end{aligned}$$

(ii)(2) $P = R + S$ and $Q = S + R$. First suppose that R has a free end at altitude a-1. Then

$$\begin{aligned} \bar{P}_{(a)} &\doteq \bar{R}_{(a-1)} + S, \text{ by Theorem 3.8,} \\ &\doteq S + \bar{R}_{(a-1)}, \text{ by (c),} \\ &\doteq \bar{Q}_{(a)}, \text{ by Theorem 3.8 .} \end{aligned}$$

If R has no free ends at altitude a-1, S must have one; the proof is then similar.

(iii) $P = R + S$, $Q = R' + S'$ and $R \sim R'$, $S \sim S'$. Suppose that R has a free end at altitude a-1. Then, by the lemma, R' has a free end at the same altitude and $\bar{P}_{(a)} \sim \bar{R}_{(a-1)} + S$, $\bar{Q}_{(a)} \sim \bar{R}'_{(a-1)} + S'$. These are concordant if $\bar{R}_{(a-1)} \sim \bar{R}'_{(a-1)}$. Use therefore induction on a. Again, if R has no free ends at altitude a-1 then S must have one and the proof is similar.

(iv) If $P = R_1 \sim R_2 \sim \ldots \sim R_k = Q$ then by the lemma each R_i has a free end at altitude a. The proof is by induction on k.

We are now in a position to prove the principal theorem on the structure of concordant trees.

<u>THEOREM 3.10</u>. Two trees are concordant if and only if they have the same number of free ends at each altitude.

<u>Proof</u>. Let the two trees be P and Q.

<u>Necessity</u>. Let p_i, q_i, r_i, s_i, t_i, u_i denote the numbers of free ends at altitude i in the trees P, Q, R, S, T, U respectively.

(i) If $P = Q$ there is nothing to prove.

(ii) If $P = (R + S) + (T + U)$ and $Q = (R + T) + (S + U)$ then $p_i = q_i = r_{i-2} + s_{i-2} + t_{i-2} + u_{i-2}$ $(2 \leq i \leq \alpha_P)$ and $p_0 = p_1 = q_0 = q_1 = 0$.

(ii)(2) If $P = R + S$ and $Q = S + R$ then

$p_i = q_i = r_{i-1} + s_{i-1}$ $(1 \leq i \leq \alpha_P)$

and $p_0 = q_0 = 0$.

(iii) $P = R + S$, $Q = T + U$ and $R \sim T$, $S \sim U$. The condition is obviously necessary if $\alpha_P = 1$. Use induction on altitude of P. The lemma to Theorem 3.9 implies that altitudes of concordant trees are equal. The altitudes of P and Q are therefore equal and those of R, S, T, U are all less than α_P. Thus, by the induction hypothesis, $r_i = t_i$ and $s_i = u_i$ for all i. But $p_i = r_{i-1} + s_{i-1}$, $q_i = t_{i-1} + u_{i-1}$. Hence $p_i = q_i$.

(iv) If $P = R_1 \sim R_2 \sim \ldots \sim R_k = Q$ the necessity is proved by induction on k.

<u>Sufficiency</u>. Note that the potency of any tree T is $\delta_T = \sum_i t_i$ and use induction on the potency of P.

If $\delta_P = 1$, $P = Q = 1$ and the condition is obviously sufficient. Assume that it is sufficient for trees of potency less than d. Let $\delta_P = \delta_Q = d$, $\alpha_P = \alpha_Q = a$ and let R, S be the first principal subordinates of P, Q respectively. Then, since $p_i = q_i$ $(1 \leq i \leq a)$, $r_i = s_i = p_i$ $(1 \leq i \leq a-2)$, $r_{a-1} = s_{a-1} = p_{a-1} + 1$ and $r_a = s_a = p_a - 2$. But the potencies of R and S are equal to d-1. Thus, by the induction hypothesis, R and S are concordant and , by Theorem 3.9, $\bar{R}_{(a-1)} \sim \bar{S}_{(a-1)}$. Now, by Theorem 3.8, $P \sim \bar{R}_{(a-1)}$ and $Q \sim \bar{S}_{(a-1)}$. Hence the result.

A similar necessary and sufficient condition can be obtained for numbers of nodes (or of all knots) at each altitude.

6. ψ- AND θ-POLYNOMIALS IN ONE INDETERMINATE

The altitude of a knot is equal to the degree in λ, μ of its term. This and Theorem 3.10 suggest that concordant trees (or indices) can be represented by polynomials in one indeterminate in which the degree of each term corresponds to the altitude and the coefficient to the number of free ends at this altitude. We now introduce such index polynomials, study their properties and prove that $\mathscr{L}_{ce}$ is faithfully represented by them. It turns out that these are Etherington's original index polynomials (cf. (12)).

The algebras of the two types of index polynomials defined below are homomorphs of Ψ and Θ determined by the congruence relations:

$$\psi(\lambda,\mu) \sim \psi'(\lambda,\mu) \quad \text{if} \quad \psi(\lambda,\lambda) = \psi'(\lambda,\lambda) \quad \text{in } \mathfrak{M}[\lambda,\mu];$$

$$\theta(\lambda,\mu) \sim \theta'(\lambda,\mu) \quad \text{if} \quad \theta(\lambda,\lambda) = \theta'(\lambda,\lambda) \quad \text{in } \mathfrak{M}[\lambda,\mu].$$

It is convenient therefore to call them <u>index ψ- and θ-polynomials in one indeterminate</u> and to denote them by $\psi_P(\lambda)$ and $\theta_P(\lambda)$ or, where no confusion is likely to arise (as in this section), simply ψ- and θ-polynomials and to write ψ_P and θ_P.

<u>Definitions</u>.

(i) Index ψ-polynomials in one indeterminate:

$$\psi_1(\lambda) = 1, \qquad \psi_{P+Q}(\lambda) = \lambda(\psi_P(\lambda) + \psi_Q(\lambda)).$$ cf p. 24

(ii) Index θ-polynomials in one indeterminate:

$$\theta_1(\lambda) = 0, \qquad \theta_{P+Q}(\lambda) = \lambda(\theta_P(\lambda) + \theta_Q(\lambda)) + 1.$$

We have $\psi_P = (2\lambda - 1)\theta_P + 1$. This is easily proved by non-associative induction. For, since $\theta_1 = 0$, $\psi_1 = (2\lambda - 1)\theta_1 + 1$ and if we assume that $\psi_Q = (2\lambda - 1)\theta_Q + 1$ and $\psi_R = (2\lambda - 1)\theta_R + 1$ then

$$\begin{aligned}\psi_{Q+R} &= \lambda(\psi_Q + \psi_R) \\ &= \lambda((2\lambda - 1)\theta_Q + 1 + (2\lambda - 1)\theta_R + 1) \\ &= (2\lambda - 1)(\lambda(\theta_Q + \theta_R) + 1) + 1 \\ &= (2\lambda - 1)\theta_{Q+R} + 1 .\end{aligned}$$

Call the term of maximal degree in λ in a polynomial $\varphi(\lambda)$ the <u>leading</u> term of $\varphi(\lambda)$. It is easily seen that all coefficients in index polynomials defined above are non-negative integers and that the coefficient of the leading term of ψ_P $(P \neq 1)$ is even.

THEOREM 3.11 . The polynomial $\varphi = 2\lambda^i + \varphi'$, where φ' is a polynomial in λ with non-negative integer coefficients, is an index ψ-polynomial if and only if $\lambda^{i-1} + \varphi'$ is one.

Proof. If $2\lambda^i + \varphi'$ is an index ψ-polynomial, ψ_P say, then $(2\lambda^i + \varphi') - 2\lambda^i + \lambda^{i-1} = \lambda^{i-1} + \varphi'$ is the ψ-polynomial of a first subordinate of either P or of a tree concordant to P. Again, if $\lambda^{i-1} + \varphi'$ is a ψ-polynomial, ψ_Q say, then $(\lambda^{i-1} + \varphi') - \lambda^{i-1} + 2\lambda^i = 2\lambda^i + \varphi'$ is the ψ-polynomial of $\bar{Q}_{(i-1)}$, a first superior of Q.
To these somewhat loose remarks we add a formal proof.

Necessity. Use induction on $n(\varphi)$, the degree of φ. If $n(\varphi) = 1$, $\varphi = 2\lambda$, i.e. $i = 1$, $\varphi' = 0$ and therefore $\lambda^{i-1} + \varphi' = 1 = \psi_1$. Assume that the condition is necessary for ψ-polynomials of degree less than m. Let $n(\varphi) = m$. Then $\varphi = 2\lambda^i + \varphi' = \lambda\psi_A + \lambda\psi_B$ and
either (a) ψ_A or ψ_B contains a term $\nu\lambda^{i-1}$ with $\nu \geq 2$;
or (b) ψ_A and ψ_B each contains a term λ^{i-1}.
If (a): suppose that ψ_B contains a term $\nu\lambda^{i-1}$ and let $\psi_B = 2\lambda^{i-1} + \varphi_B$ where φ_B is a polynomial with non-negative coefficients. Then, since $n(\psi_B) \leq m - 1$, $\lambda^{i-2} + \varphi_B$ is a ψ-polynomial and

$$\begin{aligned} \lambda^{i-1} + \varphi' &= \varphi - 2\lambda^i + \lambda^{i-1} \\ &= \lambda\psi_A + \lambda\psi_B - 2\lambda^i + \lambda^{i-1} \\ &= \lambda\psi_A + (2\lambda^i + \lambda\varphi_B) - 2\lambda^i + \lambda^{i-1} \\ &= \lambda\psi_A + \lambda(\lambda^{i-2} + \varphi_B) \end{aligned}$$

is also a ψ-polynomial.

If (b): let $\psi_A = \lambda\psi_{A_1} + \lambda\psi_{A_2}$, $\psi_B = \lambda\psi_{B_1} + \lambda\psi_{B_2}$ and suppose that ψ_{A_1} and ψ_{B_1} each contains a term λ^{i-2}. Then $\varphi = \lambda(\lambda\psi_{A_1} + \lambda\psi_{B_1}) + \lambda(\lambda\psi_{A_2} + \lambda\psi_{B_2}) = \lambda\psi_C + \lambda\psi_D$, say, where ψ_C contains the term $2\lambda^{i-2}$ and the proof proceeds as in case (a).

Sufficiency. Let $\psi_P = \lambda^{i-1} + \varphi'$. Then $\varphi = \psi_P + (2\lambda - 1)\lambda^{i-1}$. Use induction on $n(\psi_P)$. If $n(\psi_P) = 0$, $\psi_P = 1$, $i = 1$ and $\varphi = 1 + (2\lambda - 1) = 2\lambda = \psi_2$. Suppose that the condition is sufficient for polynomials of degree less than m $(m > 0)$ and let $n(\psi_P) = m$. Then $\psi_P = \lambda\psi_Q + \lambda\psi_R$ and either ψ_Q or ψ_R contains a term $\nu\lambda^{i-2}$ $(\nu > 0)$; let it be ψ_R. Hence ψ_R can be written in the form $\lambda^{i-2} + \varphi_R$, where φ_R is a polynomial with non-negative coefficients, and, since $n(\psi_R) < m$, $2\lambda^{i-1} + \varphi_R$ is an index ψ-polynomial. But

$$\begin{aligned}\varphi &= \psi_P + (2\lambda - 1)\lambda^{i-1}\\ &= \lambda\psi_Q + \lambda(\lambda^{i-2} + \varphi_R) + (2\lambda - 1)\lambda^{i-1}\\ &= \lambda\psi_Q + \lambda(\varphi_R + 2\lambda^{i-1})\end{aligned}$$

and is therefore also a ψ-polynomial.

COROLLARY 1. A necessary and sufficient condition for $\varphi(\lambda)$, a polynomial of degree n $(n \geq 1)$ with positive integer coefficients, to be an index ψ-polynomial is that $\varphi - (2\lambda - 1)\lambda^{n-1}$ should be one.

<u>COROLLARY 2</u>. $\varphi(\lambda)$, a polynomial with positive coefficients and leading term $\nu\lambda^n$, is an index ψ-polynomial if and only if $\varphi - (2\lambda - 1)\frac{\nu}{2}\lambda^{n-1}$ is one.

Necessary and sufficient conditions that $\sum_{i=0}^{n} \nu_i\lambda^i$ should be an index Θ-polynomial have been given by Etherington ((16), p.251). They are: (i) all ν_i are non-negative integers; (ii) if $i \neq 0$, $\nu_i \leq 2\nu_{i-1}$; (iii) $\nu_0 = 0$ or 1. The necessity of these is quite obvious. The sufficiency can be proved by above Corollary 2. For $\sum_{i=0}^{n} \nu_i\lambda^i$ is a Θ-polynomial if $\varphi = (2\lambda - 1)(\sum_{i=0}^{n} \nu_i\lambda^i) + 1$ is a ψ-polynomial and this is so if

$$\begin{aligned}\varphi' &= (2\lambda - 1)(\sum_{i=0}^{n}\nu_i\lambda^i) + 1 - (2\lambda - 1)\nu_n\lambda^n \\ &= (2\lambda - 1)(\sum_{i=0}^{n-1}\nu_i\lambda^i) + 1\end{aligned}$$

is one. Now the degree of φ' is less than that of φ. The proof is by induction on degree.

It is worth noting that if $\sum_{i=0}^{n}\nu_i\lambda^i$ is an index Θ-polynomial then so is $\sum_{i=0}^{r}\nu_i\lambda^i$ $(0 \leq r \leq n)$.

<u>THEOREM 3.12</u> . The free commutative entropic logarithmetic is faithfully represented by index ψ-polynomials in one indeterminate.

<u>Proof</u>. I. To prove that concordant indices have the same ψ-polynomial, i.e. that $(P \sim Q) \Rightarrow (\psi_P(\lambda) = \psi_Q(\lambda))$:

(i) If $P = Q$ then obviously $\psi_P = \psi_Q$

(ii) (1) If $P = (R+S)+(T+U)$ and $Q = (R+T)+(S+U)$ then $\psi_P = \psi_Q = \lambda^2(\psi_R + \psi_S + \psi_T + \psi_U)$.

(ii)(2) If $P = R+S$ and $Q = S+R$ then $\psi_P = \psi_Q = \lambda(\psi_R + \psi_S)$.

(iii) If $P = R+S$, $Q = T+U$ and $R \sim T$, $S \sim U$ then $\psi_P = \lambda(\psi_R + \psi_S)$ and $\psi_Q = \lambda(\psi_T + \psi_U)$ which are equal if $\psi_R = \psi_T$ and $\psi_S = \psi_U$. Use induction on altitude.

(iv) If $P = R_1 \sim R_2 \sim \ldots \sim R_k = Q$, use induction on k.

II. To prove that $(\psi_P = \psi_Q) \Rightarrow (P \sim Q)$: $\psi_P = \psi_Q$ implies that $\delta_P = \delta_Q$ and $\alpha_P = \alpha_Q$. Use induction on δ_P. If $\delta_P = 1$, $P = Q = 1$. Assume that the theorem holds for indices of potency less than d. Let $\delta_P = d$. Consider the trees P and Q. Their first principal subordinates have both the ψ-polynomial $\psi_P - (2\lambda - 1)\lambda^{\alpha_P - 1}$. The potency of these subordinates is d-1 so that, by the induction hypothesis, they are concordant. Now, P and Q are their superiors satisfying the premises of Theorem 3.9. The result follows.

COROLLARY. The free commutative entropic logarithmetic is faithfully represented by index Θ-polynomials in one indeterminate.

For $\quad (P \dot{\sim} Q) \Longleftrightarrow (\psi_P = \psi_Q)$

$$\Longleftrightarrow ((2\lambda - 1)\Theta_P + 1 = (2\lambda - 1)\Theta_Q + 1)$$

$$\Longleftrightarrow (\Theta_P = \Theta_Q).$$

CHAPTER IV. BIFURCATING ROOT-TREES OF INFINITE ALTITUDE

1. INTRODUCTION

The development of a mathematical theory depends both in form and in meaning on its author's conception of the nature and the purpose of mathematics. This is not so evident in theories dealing with finite quantities only, as any theory which concerns a finite number of given mathematical entities is per se constructive (unless one adopts the extraordinary course of defining the finite by means of the infinite). When, however, a theory deals with non-terminating processes, its development is entirely conditioned by its author's philosophical point of view.

Root-trees of infinite altitude have been studied, essentially from a formalist point of view, by Hourston (24). In the present chapter we develop a constructivist theory of bifurcating root-trees of infinite altitude.

The sequence N of natural numbers is a basal intuition of mathematics. This sequence is non-terminating, i.e. every natural number has a successor. This is precisely what we mean when we say that the sequence of natural numbers is <u>infinite</u>. What is definitely not meant is that there is an actual completed infinite aggregate of all natural numbers. Such an assertion would be tantamount to saying that an unfinishable process can be completed which is contradictory. We shall use the terms: ordered species, similar ordered species, order type, segment and sum of an ordered species of ordered species, in their usual meaning as, e.g. defined in (8) (where, however, order type is called Ordinalzahl).

A sequence S of mathematical entities, called elements of S, is a law which gives the first element of S and a method of constructing the nth element when its predecessors are known. A sequence is said to be finite if it is similar to a segment of N and infinite if it is similar to N itself.

A well-ordered species is defined as follows:

(1) A sequence is a well-ordered species;

(2) The sum of a sequence of well-ordered species is a well-ordered species.

The order type of a well-ordered species is called its ordinal number.

A finite bifurcating root-tree can be defined (cf. (15)) as a law which at the first stage partitions a given ordered species, called the basis of the tree, into a left and a right subspecies, and at each subsequent stage partitions all subspecies which do not consist of a single element into a left and a right subspecies. After the final stage all subspecies consist of single elements. The ordinal number of stages is the altitude of the tree.

This definition of finite trees can be extended to trees of transfinite altitude (cf.(24)). As a transfinite well-ordered species of partitions may not have a final stage and as some stages may not be immediately preceded by another stage, we can require only that every element of the basis should be ultimately separated from any other. In order that this definition be constructive, the basis and the law of partition must be given initially in such a way that there is an a priori certainty that the conditions of the definition are complied with. Such a definition would be unwieldy, moreover it would be difficult to define trees of large transfinite altitude. This can be avoided if we abandon partitions as the basis of our definition. It is clear that transfinite trees can be defined as infinite sequences of finite trees where each tree of the sequence is a subordinate of its successor. This approach gives a very satisfactory definition which enables us to define even trees which cannot be defined constructively by means of partitions (i.e. which would require a "non-denumerable" basis). It is not possible, however, to define constructively in this manner any trees of altitude greater than ω. In order to define a tree of any transfinite altitude we generalize the concept of index ω-polynomials and represent trees by sums of transfinite products.

2. SEQUENCES OF TREES AND LIMIT TREES

An infinite sequence of trees $\{P_i\}$ is called <u>increasing</u> if $P_i \leq P_{i+1}$, i.e. if P_i is a subordinate of P_{i+1}, for all i. In what follows "sequence" will mean "infinite increasing sequence".

A sequence $\{P_i\}$ is said to be an <u>echelon</u> sequence if α_{P_i}, the altitude of P_i, is equal to i and $P_i = 2^i \cap P_{i+1}$ for all i. For instance the sequences $2, 2^2, 2^3, 2^4, \ldots$ and $2, 2+2, 3+3, 4+4, \ldots$ are echelon sequences while the sequences $4, 5, 6, 7, \ldots$ and $2, 3, 3^2, 3^3, 3^4, \ldots$ are not.

Sequence $\{Q_i\}$ is called a <u>subordinate</u> of sequence $\{P_i\}$ if for each i a number $j = j(i)$ can be found such that $Q_i \leq P_j$. We write $\{Q_i\} \leq \{P_i\}$.

If all trees in a sequence are equal to a fixed tree the sequence is said to be <u>constant</u>. Any sequence subordinate to a constant sequence is called <u>bounded</u>. Obviously a sequence is bounded if the corresponding sequence of altitudes is bounded. If the sequence of altitudes corresponding to the sequence $\{P_i\}$ is unbounded, i.e. if, given any number K, a number $I = I(K)$ can be found such that $\alpha_{P_I} > K$, the sequence $\{P_i\}$ is said to be unbounded. A sequence need not be either bounded or unbounded. Consider the sequence $\{P_i\}$, defined as follows: $P_i = 2$ if among the first i digits in the decimal expansion of π no sequence 0123456789 occurs, and $P_i = i$ (where i denotes the right-principal tree of altitude i-1) if it does, is well defined but is neither bounded nor unbounded.

THEOREM 4.1 . If $\{P_i\} \leq \{P_i'\}$ and $\{Q_i\} \leq \{Q_i'\}$ then

(i) $\{P_i + Q_i\} \leq \{P_i' + Q_i'\}$,

(ii) $\{P_i \cup Q_i\} \leq \{P_i' \cup Q_i'\}$,

(iii) $\{P_i \cap Q_i\} \leq \{P_i' \cap Q_i'\}$.

Proof. Let i be any suffix. Since $\{P_i\} \leq \{P_i'\}$ and $\{Q_i\} \leq \{Q_i'\}$ numbers h and k can be found such that $P_i \leq P_h'$ and $Q_i \leq Q_k'$. Let n be any number greater than both h and k. Then since the sequences $\{P_i\}$ and $\{Q_i\}$ are increasing $P_i \leq P_n'$ and $Q_i \leq Q_n'$, i.e. $\theta_{P_n'} = \theta_{P_i} + \varphi_1$ and $\theta_{Q_n'} = \theta_{Q_i} + \varphi_2$, where φ_1, φ_2 are polynomials with non-negative coefficients.

(i)
$$\begin{aligned}\theta_{P_n'+Q_n'} &= \theta_{P_n'}\lambda + \theta_{Q_n'}\mu + 1\\ &= (\theta_{P_i} + \varphi_1)\lambda + (\theta_{Q_i} + \varphi_2)\mu + 1\\ &= \theta_{P_i}\lambda + \theta_{Q_i}\mu + 1 + \text{(terms with non-negative coefficients)}\\ &= \theta_{P_i+Q_i} + \text{(terms with non-negative coefficients)}.\end{aligned}$$

Therefore

$P_i + Q_i \leq P'_n + Q'_n$, i.e. $\{P_i + Q_i\} \leq \{P'_i + Q'_i\}$.

(ii) Since $\{\theta_{P_i}\} \subset \{\theta_{P'_n}\}$ and $\{\theta_{Q_i}\} \subset \{\theta_{Q'_n}\}$ then obviously $\{\theta_{P_i}\} \cup \{\theta_{Q_i}\} \subset \{\theta_{P'_n}\} \cup \{\theta_{Q'_n}\}$, i.e. $P_i \cup Q_i \leq P'_n \cup Q'_n$.

(iii) Similarly $\{\theta_{P_i}\} \cap \{\theta_{Q_i}\} \subset \{\theta_{P'_n}\} \cap \{\theta_{Q'_n}\}$, i.e. $P_i \cap Q_i \leq P'_n \cap Q'_n$.

Note that we cannot state a similar theorem about $\{P_i Q_i\}$ since the fact that $\{P_i\}$ and $\{Q_i\}$ are sequences does not imply that $\{P_i Q_i\}$ is an (increasing) sequence. For example, the sequences $\{P_i\} = \dot{3}+1, \dot{3}+1, \dot{3}+1, \ldots$ and $\{Q_i\} = 2, 3, 4, \ldots$ are both increasing but $(\dot{3}+1)2$ is not a subordinate of $(\dot{3}+1)3$, for

$\theta_{(\dot{3}+1)3} = \mu\lambda^3 + \mu\lambda\mu\lambda + \lambda^3 + \lambda\mu\lambda + \mu\lambda\mu + \lambda^2 + \lambda\mu + \mu\lambda + \lambda + \mu + 1$

while $\theta_{(\dot{3}+1)2} = \mu\lambda^2 + \mu\lambda\mu + \lambda^2 + \lambda\mu + \lambda + \mu + 1$ and $\{\theta_{(\dot{3}+1)2}\}$ is not a subspecies of $\{\theta_{(\dot{3}+1)3}\}$.

If $\{P_i\} \leq \{Q_i\}$ and $\{Q_i\} \leq \{P_i\}$ we say that the two sequences are <u>equivalent</u> and write $\{P_i\} \sim \{Q_i\}$. It is easily seen that this relation is reflexive, symmetric and transitive.

We define the <u>sum</u>, the <u>union</u> (l.u.b.) and the <u>intersection</u> (g.l.b.) of two sequences $\{P_i\}$ and $\{Q_i\}$:

$$\{P_i\} + \{Q_i\} = \{P_i + Q_i\},$$
$$\{P_i\} \cup \{Q_i\} = \{P_i \cup Q_i\},$$
$$\{P_i\} \cap \{Q_i\} = \{P_i \cap Q_i\}.$$

<u>THEOREM 4.2</u> . $\{P_i\} + \{Q_i\} \sim \{P_i'\} + \{Q_i'\}$ if and only if $\{P_i\} \sim \{P_i'\}$ and $\{Q_i\} \sim \{Q_i'\}$.

<u>Proof</u>. <u>Sufficiency</u> is obvious.

<u>Necessity</u>: We have $\{P_i + Q_i\} \sim \{P_i' + Q_i'\}$. Therefore there exist numbers h, k such that $P_i + Q_i \leq P_h' + Q_h'$ and $P_i' + Q_i' \leq P_k + Q_k$. But this implies

$$P_i \leq P_h', \quad Q_i \leq Q_h', \quad P_i' \leq P_k, \quad Q_i' \leq Q_k.$$

For $\{\theta_{P_i}\lambda + \theta_{Q_i}\mu + 1\} \subset \{\theta_{P_h'}\lambda + \theta_{Q_h'}\mu + 1\}$ and, since the only terms on the right-hand side ending in λ are $\{\theta_{P_h'}\lambda\}$, $\{\theta_{P_i}\lambda\} \subset \{\theta_{P_h'}\lambda\}$, i.e. $\{\theta_{P_i}\} \subset \{\theta_{P_h'}\}$. A similar argument holds for the other subordinates.

Given any two increasing sequences $\{P_i\}$, $\{Q_i\}$,

either (i) $\{P_i\} \leq \{Q_i\}$,

or (ii) $\{Q_i\} \leq \{P_i\}$,

or (iii) $\{P_i\} \not\leq \{Q_i\}$, i.e. we can prove that $\{P_i\} \leq \{Q_i\}$ is impossible,

or (iv) $\{Q_i\} \not\leq \{P_i\}$,

or (v) we may not be able to determine whether either of these sequences is subordinate of the other or not.

If $\{P_i\} \leq \{Q_i\}$ and $\{Q_i\} \not\leq \{P_i\}$ we write $\{P_i\} < \{Q_i\}$. Note that $\{P_i\} \leq \{Q_i\}$ is not equivalent to: $\{P_i\} < \{Q_i\}$ or $\{P_i\} \sim \{Q_i\}$. The latter means

$\{P_i\} \leq \{Q_i\}$ <u>and</u> $(\{Q_i\} \not\leq \{P_i\}$ or $\{Q_i\} \leq \{P_i\})$

and is therefore stronger than $\{P_i\} \leq \{Q_i\}$.

<u>Definition</u>. A sequence is called <u>effectively increasing</u> if we can find an echelon or a constant sequence equivalent to it.

<u>Example</u>. The sequence P_i where $P_i = 1.2.3...i$ (i.e. P_i is the product of the first i right-principal trees) is effectively increasing because

(i) $\alpha_{P_i} = 0 + 1 + 2 + \ldots + (i - 1) = \frac{1}{2}i(i - 1)$

and thus $P_i \leq 2^{\frac{1}{2}i(i-1)}$,

(ii) we can prove by induction on i that 2^i is a subordinate of P_{i+1}. If $i = 1$ then $2^i = 2$ and $P_{i+1} = 2$. Assume $2^i \leq P_{i+1}$ for $i < a$ $(a > 1)$. We want to prove that $\Theta_{P_{a+1}}$ contains all possible terms of degree a, i.e. that all such terms are right-divisors of terms of $\omega_{P_{a+1}}$. Now, $P_{a+1} = P_a(a + 1)$. Therefore

$$\begin{aligned}\omega_{P_{a+1}} &= \omega_{P_a} \cdot \psi_{a+1} \\ &= \omega_{P_a}(\lambda^a + \mu\lambda^{a-1} + \mu\lambda^{a-2} + \ldots + \mu\lambda^2 + \mu\lambda + \mu)\end{aligned}$$

and, since all possible terms of degree a-1 (and thus all terms of degree less than a) are by the induction hypothesis right-divisors of terms of ω_{P_a}, all possible terms of degree a are right-divisors of terms of $\omega_{P_{a+1}}$. Hence $2^i \leq P_{i+1}$ and $\{P_i\} \sim \{2^i\}$. Since $\{2^i\}$ is an echelon sequence, $\{P_i\}$ is effectively increasing.

Call the species of sequences equivalent to the sequence $\{P_i\}$ the <u>limit tree</u> of $\{P_i\}$ and denote it by $\lim P_i$. $\mathrm{Lim}\, P_i$ is said to be infinite or finite according as $\{P_i\}$ is unbounded or bounded.

<u>THEOREM 4.3</u> . Two effectively increasing sequences have the same limit tree if and only if they are equivalent to the same echelon sequence or to the same constant sequence.

<u>Proof</u>. The sufficiency follows directly from the definition of limit trees. To prove the necessity we have to show that if two echelon or constant sequences are equivalent they are identical. This is quite obvious for constant sequences. Let $\{P_i\}$, $\{Q_i\}$ be two equivalent echelon sequences. Then $P_i \leq Q_{i+k}$ for some $k \geq 0$. Since $\alpha_{P_i} = i$ we have $P_i \leq 2^i \frown Q_{i+k}$. But $\{Q_i\}$ is an echelon sequence. Therefore

$$Q_i = 2^i \cap Q_{i+1}$$
$$= 2^i \cap (2^{i+1} \cap Q_{i+2}) = 2^i \cap Q_{i+2}$$
$$= \dots \dots$$
$$= 2^i \cap Q_{i+k}.$$

P_i is therefore a subordinate of Q_i. Similarly $Q_i \leq P_i$. Hence $P_i = Q_i$ for all i.

The preceding theorem should not be interpreted as an assertion that we can establish a one-one correspondence between finite limit trees and constant sequences and between infinite limit trees and echelon sequences, for we may not be able even to determine whether a given sequence is constant or not (cf. example preceding Theorem 4.1) or indeed whether two echelon sequences are equivalent or not. Let, e.g., k be the least number such that the (k-9)th, (k-8)th, ... ,(k-1)th, kth digits in the decimal expansion of π are all 9 and define $P_i = 2^i$ if $i \leq k$, $P_i = 2^k \cap (i+1)$ if $i > k$; then we cannot say whether $\{P_i\}$ and $\{2^i\}$ are equivalent or not. but...

The limit tree $\lim P_i$ is said to be a <u>subordinate</u> of the limit tree $\lim Q_i$, written $\lim P_i \leq \lim Q_i$, if $\{P_i\} \leq \{Q_i\}$.

<u>THEOREM 4.4</u> . (i) The relation $\lim P_i \leq \lim Q_i$ is well defined.
(ii) If $\lim P_i \leq \lim Q_i$ and $\lim Q_i \leq \lim P_i$ then the two limit trees are identical.

<u>Proof</u>. (i) We prove that $\{P_i\} \sim \{P_i'\}$, $\{Q_i\} \sim \{Q_i'\}$ and $\{P_i\} \leq \{Q_i\}$ imply $\{P_i'\} \leq \{Q_i'\}$. If $\{P_i\}$ and $\{P_i'\}$ are equivalent then, given any number i, we can find a number $j = j(i)$ such that $P_i' \leq P_j$. $\{P_i\} \leq \{Q_i\}$ implies that $P_j \leq Q_k$ for some k. Finally $\{Q_i\} \sim \{Q_i'\}$ implies $Q_k \leq Q_h'$ for some h. It follows that $P_i' \leq Q_h'$ and thus $\{P_i'\} \leq \{Q_i'\}$.

(ii) If $\lim P_i \leq \lim Q_i$ and $\lim Q_i \leq \lim P_i$ then, by part (i) of the theorem and the definition of a subordinate of a limit tree, $\{P_i\} \leq \{Q_i\}$ and $\{Q_i\} \leq \{P_i\}$ and therefore $\{P_i\} \sim \{Q_i\}$. This, by the definition of a limit tree means that $\lim P_i = \lim Q_i$.

We now define the sum, union and intersection of two limit trees having regard to the following considerations:

The sum, the union and the intersection of two limit trees should be limit trees. Thus we cannot define the union of two limit trees as the logical sum of the two species.

The union and the intersection of two limit trees should be their l.u.b. and g.l.b. respectively, i.e. whenever $\lim P_i \leq \lim R_i$ and $\lim Q_i \leq \lim R_i$ then $\lim P_i \cup \lim Q_i \leq \lim R_i$ and dually for intersection.

The following definitions satisfy these requirements.

<u>Definitions</u>. $\lim P_i + \lim Q_i = \lim(P_i + Q_i)$,

$$\lim P_i \cup \lim Q_i = \lim(P_i \cup Q_i),$$

$$\lim P_i \cap \lim Q_i = \lim(P_i \cap Q_i).$$

Clearly these definitions do not depend specifically on sequences $\{P_i\}$ and $\{Q_i\}$. For if $\{P_i\} \sim \{P_i'\}$ and $\{Q_i\} \sim \{Q_i'\}$ then $\{P_i \cup Q_i\} \sim \{P_i' \cup Q_i'\}$ (apply Theorem 4.1 twice), i.e. if $\lim P_i = \lim P_i'$ and $\lim Q_i = \lim Q_i'$ then $\lim(P_i \cup Q_i) = \lim(P_i' \cup Q_i')$. Similarly for the intersection. Also, by Theorem 4.2,
$\lim P_i + \lim Q_i = \lim P_i' + \lim Q_i'$ if and only if $\lim P_i = \lim P_i'$ and $\lim Q_i = \lim Q_i'$.

Lim $P_i \cup \lim Q_i$ and $\lim P_i \cap \lim Q_i$ are the l.u.b. and the g.l.b. of limit trees $\lim P_i$ and $\lim Q_i$. Thus limit trees form a lattice $\mathcal{T}$ which obviously is distributive. The sublattice of all finite limit trees defined by means of effectively increasing sequences and $\mathcal{L}$, the lattice of all trees (cf. Chap. I, § 11), are lattice isomorphic.

Lim 2^i is the greatest element of $\top$. For if lim P_i is any limit tree and $\alpha(i)$ is the altitude of P_i then $P_i \leq 2^{\alpha(i)}$. Evidently lim 1 is the least element of $\top$.

3. Ω -TREES

Trees are faithfully represented by index ω-polynomials. ω-polynomials of primary trees are monic monomials in non-commuting indeterminates λ and μ. To a sequence of primary trees corresponds a sequence of these monomials. Each term in this sequence is a right-divisor of every term following it in the sequence. A primary limit tree therefore can be represented uniquely by a finite (in the case of finite limit trees) or an infinite product of the form $\nu_n \ldots \nu_3 \nu_2 \nu_1$ or $\ldots \nu_3 \nu_2 \nu_1$ (ν_i = λ or μ) such that if $\{P_i\} \in \lim P_i$ then ω_{P_i} is a right-divisor of this product for all i.

Unions* of primary limit trees and the limit tree lim 1 are said to be <u>convergent</u>. Limit trees for which it can be shown that they cannot be constructed as a union of primary limit trees are called <u>divergent</u>. Evidently all finite limit trees are convergent. Lim 2^i is divergent. Primary limit trees bear the same relation to lim 2^i as real numbers to the continuum.

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* i.e. constructible unions.

Indeed there is an obvious one-one correspondence between primary limit trees and binary fractions between 0 and 1 (fractions such as 0.11, 0.110, 0.1100, 0.10111... being considered as distinct symbols) and $\lim 2^i$ can be interpreted as a spread (cf. (23)) defining the linear continuum.

To a tree P ($P \neq 1$) of lineage γ, i.e. a union of γ distinct trees, corresponds an ω-polynomial of γ terms, each of which represents a branch of P. Convergent limit trees can be represented by a species of infinite products of the form $\nu_n \ldots \nu_3 \nu_2 \nu_1$ or $\ldots \nu_3 \nu_2 \nu_1$. It is convenient to write these species as sums. For example the limit tree $\lim P_i$ with $P_1 = 2$, $P_2 = 2 + 2$, $P_{i+2} = 2 + (Q_i + R_i)$, where Q_i and R_i are the left and right principal trees of altitude i, is represented by $\lambda + \ldots\lambda\lambda\lambda\mu + \ldots\mu\mu\mu\mu$. The altitude of a limit tree is equal to $1 + \max \alpha_i$ where α_i are the duals of the order types of the terms (these being well-ordered from right to left) in the corresponding polynomial.

Although it is not possible to construct limit trees of altitudes greater than ω† we can define polynomials

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† ω here obviously denotes the ordinal number ω which is quite unrelated to the symbol "ω" in "ω-polynomial".

involving terms whose order types are duals of any (denumerable) ordinal number. Since for finite altitudes these polynomials are simply ω-polynomials which represent trees faithfully and for all altitudes not exceeding ω they represent convergent limit trees, they provide a natural generalization of the concept of trees. We shall call these polynomials <u>Ω-trees</u>. We define first finitary Ω-trees.

<u>Definition</u>. (i) 0 and 1 are finitary Ω-trees. The latter is a primary Ω-tree.

(ii) The dual species of a well-ordered species of symbols λ and μ, written as a formal product, is a primary (and therefore finitary) Ω-tree.

(iii) A polynomial of γ terms each of which is a primary Ω-tree is a finitary Ω-tree provided that no term of the polynomial is a right-divisor of any other term.

γ is called the <u>lineage</u> of the Ω-tree. Each term in a finitary Ω-tree is called a <u>branch</u> of the tree. Two finitary Ω-trees are equal if and only if they contain the same branches.

<u>Addition</u> of two Ω-trees ρ and σ is defined by analogy with the formation of the ω-polynomial of a sum of two trees (cf. Chap. I, § 9):

$$\rho \oplus \sigma = \begin{cases} 1 & \text{if } \rho = \sigma = 0, \\ \rho\lambda + \sigma\mu & \text{otherwise,} \end{cases}$$

where $\oplus$ denotes addition of Ω-trees and $+$ the formal summation of terms in an Ω-tree.

We define the <u>potency</u> δ_ρ and the <u>altitude</u> α_ρ of a finitary Ω-tree ρ in such a way that whenever ρ can be interpreted as a well-ordered species of partitions the potency of ρ is the order type of its basis and the altitude of ρ is the ordinal number of stages of partitions (v. §1):

(i) $\delta_0 = 1,\ \delta_1 = 2; \quad \alpha_0 = 0,\ \alpha_1 = 1$.

(ii) If ρ is a primary Ω-tree:

$$\delta_\rho = \overline{\rho(\mu)}^* + 2 + \overline{\rho(\lambda)} , \qquad \alpha_\rho = \bar{\rho}^* + 1 ,$$

where $\bar{\rho}^*$ is the dual of the order type of ρ and $\overline{\rho(\mu)}$, $\overline{\rho(\lambda)}$ denote the order types of the ordered subspecies of ρ composed entirely of μ's and λ's respectively.

(iii) If ρ is of lineage γ ($\gamma > 1$), then ρ can be expressed in the form $\rho = (\sigma\lambda + \tau\mu)\pi = (\sigma \oplus \tau)\pi$, where π is the primary Ω-tree which is the greatest common right-divisor of all branches ρ_i of ρ and σ, τ are Ω-trees of lineage less than γ; and we define

$$\delta_\rho = \overline{\pi(\mu)} + \delta_\sigma + \delta_\tau + \overline{\pi(\lambda)} , \qquad \alpha_\rho = \max(\alpha_{\rho_i}) .$$

We have, as in the case of ordinary finite trees, $\delta_{\rho\oplus\sigma} = \delta_\rho + \delta_\sigma$ and $\alpha_{\rho\oplus\sigma} = 1 + \max(\alpha_\rho, \alpha_\sigma)$.

Note that the Ω-tree 0 is not an identity element with respect to the operation $\oplus$, e.g. $\ldots\lambda\mu\lambda\mu\lambda\mu \oplus 0 = \ldots\lambda\mu\lambda\mu\lambda\mu\lambda$. In fact $\rho \oplus 0 = \rho$ if and only if $\ldots\lambda\lambda\lambda\lambda$ is a right-divisor of ρ and $0 \oplus \sigma = \sigma$ if and only if $\ldots\mu\mu\mu\mu$ is a right-divisor of σ.

Two finitary Ω-trees commute with respect to the operation $\oplus$ if and only if they are equal. For $\rho \oplus \sigma = \sigma \oplus \rho$ means $\rho\lambda + \sigma\mu = \sigma\lambda + \rho\mu$ and, since Ω-trees are equal only if their branches are identical, $\rho = \sigma$.

$\rho \oplus \sigma = \rho + \sigma$ if either $\rho\lambda = \rho$ and $\sigma\mu = \sigma$ i.e. if $\ldots\lambda\lambda\lambda\lambda$ is a right-divisor of ρ and $\ldots\mu\mu\mu\mu$ is a right-divisor of σ (ρ, σ not both 0) or $\rho\lambda = \sigma$ and $\sigma\mu = \rho$ i.e. $\ldots\lambda\mu\lambda\mu\lambda\mu$ is a right-divisor of ρ and $\ldots\mu\lambda\mu\lambda\mu\lambda$ is a right-divisor of σ.

No finitary Ω-tree is idempotent with respect to addition $\oplus$, since the lineage of $\rho \oplus \rho$ is twice that of ρ if $\rho \neq 0$ and if $\rho = 0$ then $\rho \oplus \rho = 0 \oplus 0 = 1 \neq \rho$. Note that there exists an idempotent limit tree, viz. $\lim 2^i = \lim 2^i + \lim 2^i$. Moreover if $\lim P_i$ is idempotent $P_r + P_r = P_r . 2 \in \{P_i\}$ and thus $P_r . 2^n \in \{P_i\}$. But $2^n \leq P_r . 2^n$ and therefore $\{2^i\} \leq \{P_i\}$. Now $\{P_i\} \leq \{2^i\}$ and therefore $\{P_i\} \sim \{2^i\}$. Hence the only idempotent limit tree is $\lim 2^i$. This is obvious from the graphical representation which we introduce in the next section.

It is possible to define non-finitary Ω-trees by replacing part (iii) of the definition of finitary Ω-trees by:

(iii') A species, written as a formal series, of primary Ω-trees is an Ω-tree provided that no term of this series is a right-divisor of any other term.

It follows at once that if ρ and σ are (non-finitary) Ω-trees, not both 0, then $\rho \oplus \sigma = \rho\lambda + \sigma\mu$ is an Ω-tree.

Although in many particular cases such trees can be constructively defined and from this definition one could define their lineage, potency and altitude, it seems that in the general case constructive definitions of altitude and potency are not possible. We could define altitude as the "l.u.b." of the altitudes of all branches of the Ω-tree and potency as the order type of the basis, the construction of which can be achieved in particular cases (e.g. by using the graphical representation of the next section). These definitions, however, in the general case would be essentially non-constructive and the uniqueness of potency probably would not be constructively provable.

We can define index ψ- and Θ-polynomials representing finitary Ω-trees. Again, in the case of non-finitary Ω-trees, due to the enormous chaos of possibilities it is doubtful if these definitions can be regarded as constructive.

<u>Definitions</u>. The <u>Θ-polynomial</u> of a finitary Ω-tree is the polynomial containing all the terms of the Ω-tree and all their proper right-divisors (including 1) each term appearing only once.

For example, the Θ-polynomial of the finitary Ω-tree $\rho = \lambda + \ldots\lambda\lambda\lambda\mu + \ldots\mu\mu\mu\mu$ is $\Theta(\rho) = 1 + \lambda + \mu + \lambda\mu + \lambda^2\mu + \lambda^3\mu + \ldots + \ldots\lambda\lambda\lambda\mu + \mu^2 + \mu^3 + \ldots + \ldots\mu\mu\mu\mu$.

The <u>ψ-polynomial</u> of a finitary Ω-tree σ is defined as follows:

$$\psi(\sigma) = 1 + (\lambda + \mu - 1)\,\Theta(\sigma) .$$

Thus the ψ-polynomial of ρ is

$$\psi(\rho) = \lambda^2 + \mu\lambda + \mu\lambda\mu + \mu\lambda^2\mu + \mu\lambda^3\mu + \ldots + \ldots\lambda\lambda\lambda\mu + \lambda\mu^2 + \lambda\mu^3 + \ldots + \ldots\mu\mu\mu\mu .$$

<u>THEOREM 4.5</u> . If ρ and σ are finitary Ω-trees then

$$\Theta(\rho \oplus \sigma) = \Theta(\rho)\lambda + \Theta(\sigma)\mu + 1,$$
$$\psi(\rho \oplus \sigma) = \psi(\rho)\lambda + \psi(\sigma)\mu .$$

<u>Proof</u>. If $\rho = \sigma = 1$ the theorem is obvious. Otherwise $\rho \oplus \sigma = \rho\lambda + \sigma\mu$. $\Theta(\rho)\lambda$ and $\Theta(\sigma)\mu$ contain all terms of $\rho\lambda$ and $\sigma\mu$ and all their proper right-divisors except 1. Therefore $\Theta(\rho)\lambda + \Theta(\sigma)\mu + 1 = \Theta(\rho\oplus\sigma)$.

$$\begin{aligned}\psi(\rho\oplus\sigma) &= 1 + (\lambda + \mu - 1)\,\Theta(\rho\oplus\sigma)\\ &= 1 + (\lambda + \mu - 1)\{\Theta(\rho)\lambda + \Theta(\sigma)\mu + 1\}\\ &= \{1 + (\lambda+\mu-1)\Theta(\rho)\}\lambda + \{1 + (\lambda+\mu-1)\Theta(\sigma)\}\mu\\ &= \psi(\rho)\lambda + \psi(\sigma)\mu .\end{aligned}$$

We now define multiplication of Ω-trees by analogy with the formation of the ω-polynomial of a product of two trees.

Definition. $\rho \otimes 0 = \rho$,

$\rho \otimes \sigma = \rho\, \psi(\sigma) \quad (\sigma \neq 0).$

An immediate consequence of the definition is that the Ω-tree 0 is a multiplicative identity and $\psi(\sigma) = 1 \otimes \sigma$. Note that we have defined Ω-trees so that finite Ω-trees are the ω-polynomials of the corresponding bifurcating root-trees and therefore the Ω-trees 0 and 1 correspond to trees 1 and 2 respectively.

We now prove that the above-defined multiplication of finitary Ω-trees has the same properties as multiplication of trees.

LEMMA 1. If ρ and σ are finitary Ω-trees then

(i) $\theta(\rho\sigma) = (\theta(\rho) - 1)\sigma + \theta(\sigma)$,

(ii) $\psi(\rho\sigma) = \psi(\rho)\sigma + \psi(\sigma) - (\lambda + \mu)\sigma$.

Proof.

(i) $\theta(\rho\sigma) = \rho\sigma + \sum$(all proper right-divisors of terms of $\rho\sigma$)

$= \rho\sigma + \sum$(all proper right-divisors of terms of ρ)σ +

$+ \sum$(all proper right-divisors of terms of σ)

$= \rho\sigma + (\theta(\rho) - \rho)\sigma + (\theta(\sigma) - \sigma)$

$= (\theta(\rho) - 1)\sigma + \theta(\sigma)$.

(ii) $\psi(\rho\sigma) = 1 + (\lambda + \mu - 1)((\theta(\rho) - 1)\sigma + \theta(\sigma))$

$$= 1 + (\psi(\rho) - 1)\sigma - (\lambda + \mu - 1)\sigma + \psi(\sigma) - 1$$

$$= \psi(\rho)\sigma + \psi(\sigma) - (\lambda + \mu)\sigma \ .$$

LEMMA 2. The polynomial whose terms are all distinct proper right-divisors of terms of $\psi(\sigma)$ is $\theta(\sigma)$.

Proof. $\psi(\sigma) = 1 + (\lambda + \mu - 1)\ \theta(\sigma)$. Therefore proper right-divisors of terms of $\psi(\sigma)$ are proper right-divisors of terms of $\lambda\theta(\sigma)$ and those of terms of $\mu\theta(\sigma)$ less those of terms of $\theta(\sigma)$. Hence the terms of $\theta(\sigma)$ are precisely the proper right-divisors of terms of $\psi(\sigma)$. Some of them of course may be proper right-divisors of several terms of $\psi(\sigma)$.

The ψ-polynomial of an Ω-tree σ, $\psi(\sigma)$, is itself an Ω-tree, viz. $1 \otimes \sigma$. The ψ- and θ-polynomial of $\psi(\sigma)$ are therefore well defined.

LEMMA 3 . If σ is a finitary Ω-tree then

(i) $\theta(\psi(\sigma)) = \psi(\sigma) + \theta(\sigma)$,

(ii) $\psi(\psi(\sigma)) = (\lambda + \mu)\ \psi(\sigma)$.

Proof.

(i) $\theta(\psi(\sigma)) = \psi(\sigma) + \sum$(all proper right-divisors of terms of $\psi(\sigma)$)

$= \psi(\sigma) + \theta(\sigma)$, by Lemma 2.

(ii) $\psi(\psi(\sigma)) = 1 + (\lambda + \mu - 1)\ \theta(\psi(\sigma))$

$$= 1 + (\lambda + \mu - 1)(\psi(\sigma) + \theta(\sigma))$$

$$= 1 + (\lambda + \mu - 1)\,\psi(\sigma) + \psi(\sigma) - 1$$
$$= (\lambda + \mu)\,\psi(\sigma)\ .$$

<u>THEOREM 4.6</u> . If ρ and σ are finitary Ω-trees then
$$\psi(\rho \otimes \sigma) = \psi(\rho)\,\psi(\sigma)\ .$$

<u>Proof</u>.

$$\begin{aligned}\psi(\rho \otimes \sigma) &= \psi(\rho\,\psi(\sigma)), \text{ by the definition of the } \otimes \text{ product,}\\ &= \psi(\rho)\,\psi(\sigma) + \psi(\psi(\sigma)) - (\lambda + \mu)\,\psi(\sigma), \text{ by Lemma 1,}\\ &= \psi(\rho)\,\psi(\sigma) + (\lambda + \mu)\,\psi(\sigma) - (\lambda + \mu)\,\psi(\sigma), \text{ by Lemma 3,}\\ &= \psi(\rho)\,\psi(\sigma)\ .\end{aligned}$$

<u>THEOREM 4.7</u> . Multiplication $\otimes$ of finitary Ω-trees is (i) associative,

(ii) right-distributive with respect to addition $\oplus$.

<u>Proof</u>. Let ρ, σ, π be any finitary Ω-trees.

(i)
$$\begin{aligned}(\rho \otimes \sigma) \otimes \pi &= \rho\,\psi(\sigma) \otimes \pi\\ &= \rho\,\psi(\sigma)\,\psi(\pi)\ ;\\ \rho \otimes (\sigma \otimes \pi) &= \rho\,\psi(\sigma \otimes \pi)\\ &= \rho\,\psi(\sigma)\,\psi(\pi), \text{ by Theorem 4.6 .}\end{aligned}$$

(ii)
$$\begin{aligned}\rho \otimes (\sigma \oplus \pi) &= \rho\,\psi(\sigma \oplus \pi)\\ &= \rho(\psi(\sigma)\lambda + \psi(\pi)\mu)\\ &= \rho\,\psi(\sigma)\lambda + \rho\,\psi(\pi)\mu\\ &= (\rho \otimes \sigma)\lambda + (\rho \otimes \pi)\mu\\ &= \rho \otimes \sigma \oplus \rho \otimes \pi\ .\end{aligned}$$

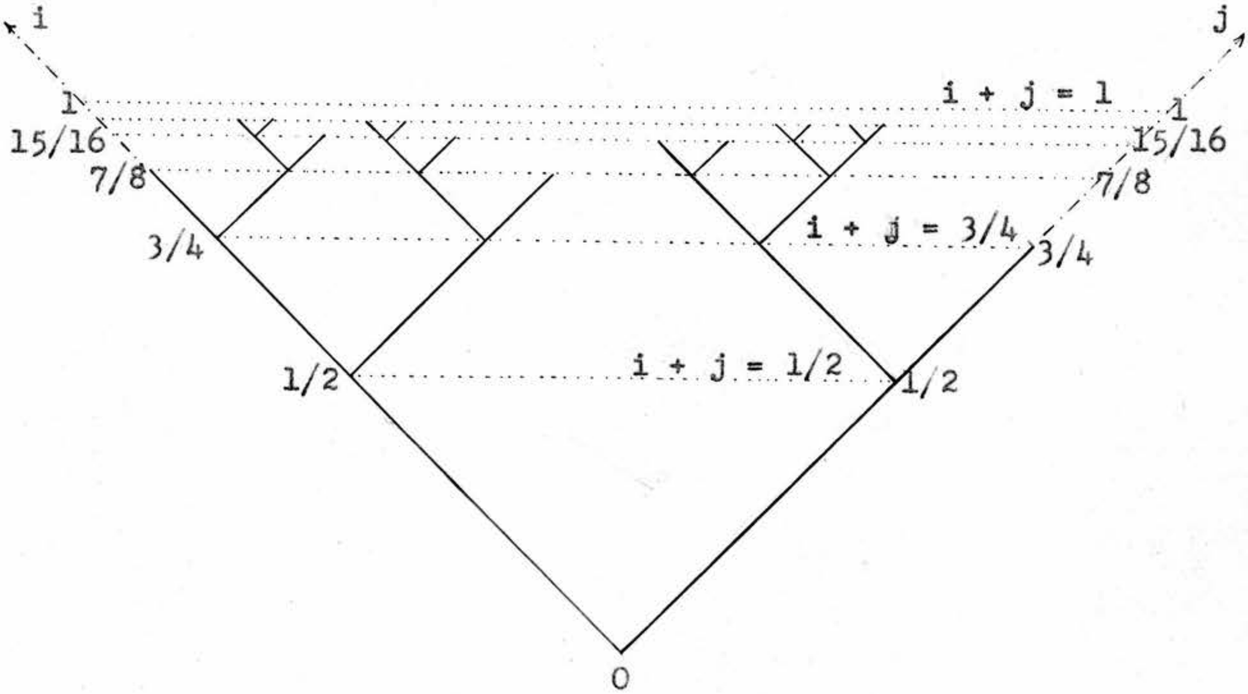
4. GRAPHICAL REPRESENTATION OF LIMIT TREES AND Ω-TREES

In the study of bifurcating root-trees it is convenient to adopt a graphical representation in which the arms of all forks of a tree are of equal length and are at 45°, or any other convenient acute angle, to the upward vertical. This graphical representation cannot be used for sequences of trees of ever increasing altitudes. We shall make the length of the arms decrease in geometric progression with altitude.

Refer knots of a tree to two semi-axes originating from the root and making an angle of 45° with the upward vertical. Call the axis on the left the i-axis and the axis on the right the j-axis. The coordinates of the root are (0, 0) (the corresponding term is 1). The coordinates of knots whose terms are λ and μ are $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. In general, if (i_o, j_o) are the coordinates of the knot whose term is $\nu_{k-1}\nu_{k-2}\cdots\nu_2\nu_1$ ($\nu_i = \lambda$ or μ) then the coordinates of the knots corresponding to the terms $\lambda\nu_{k-1}\nu_{k-2}\cdots\nu_2\nu_1$ and $\mu\nu_{k-1}\nu_{k-2}\cdots\nu_2\nu_1$ are $(i_o + 2^{-k}, j_o)$ and $(i_o, j_o + 2^{-k})$ respectively. Clearly all arms are parallel to the axes and all knots of altitude α lie on the line $i + j = \sum_{r=1}^{\alpha} 2^{-r}$ (see example).

<u>Example</u>. $P = ((1 + 3) + 4) + (2.\dot{3} + 1)$,

$\omega_P = \lambda\mu\lambda^2 + \lambda^2\mu\lambda + \lambda^2\mu + \lambda\mu\lambda\mu + \mu^2\lambda\mu$.



Terms	Coords.	Terms	Coords.	Terms	Coords.
1	$(0, 0)$	$\mu^2\lambda$	$(\frac{1}{2}, \frac{3}{8})$	$\mu^2\lambda\mu$	$(\frac{1}{4}, \frac{11}{16})$
λ	$(\frac{1}{2}, 0)$	$\lambda^2\mu$	$(\frac{3}{8}, \frac{1}{2})$	$\lambda^2\mu\lambda^2$	$(\frac{27}{32}, \frac{1}{8})$
μ	$(0, \frac{1}{2})$	$\mu\lambda\mu$	$(\frac{1}{4}, \frac{5}{8})$	$\mu\lambda\mu\lambda^2$	$(\frac{13}{16}, \frac{5}{32})$
λ^2	$(\frac{3}{4}, 0)$	$\lambda\mu\lambda^2$	$(\frac{13}{16}, \frac{1}{8})$	$\lambda^3\mu\lambda$	$(\frac{23}{32}, \frac{1}{4})$
$\mu\lambda$	$(\frac{1}{2}, \frac{1}{4})$	$\mu^2\lambda^2$	$(\frac{3}{4}, \frac{3}{16})$	$\mu\lambda^2\mu\lambda$	$(\frac{11}{16}, \frac{9}{32})$
$\lambda\mu$	$(\frac{1}{4}, \frac{1}{2})$	$\lambda^2\mu\lambda$	$(\frac{11}{16}, \frac{1}{4})$	$\lambda^2\mu\lambda\mu$	$(\frac{11}{32}, \frac{5}{8})$
μ^2	$(0, \frac{3}{4})$	$\mu\lambda\mu\lambda$	$(\frac{5}{8}, \frac{5}{16})$	$\mu\lambda\mu\lambda\mu$	$(\frac{5}{16}, \frac{21}{32})$
λ^3	$(\frac{7}{8}, 0)$	$\lambda^3\mu$	$(\frac{7}{16}, \frac{1}{2})$	$\lambda\mu^2\lambda\mu$	$(\frac{9}{32}, \frac{11}{16})$
$\mu\lambda^2$	$(\frac{3}{4}, \frac{1}{8})$	$\mu\lambda^2\mu$	$(\frac{3}{8}, \frac{9}{16})$	$\mu^3\lambda\mu$	$(\frac{1}{4}, \frac{23}{32})$
$\lambda\mu\lambda$	$(\frac{5}{8}, \frac{1}{4})$	$\lambda\mu\lambda\mu$	$(\frac{5}{16}, \frac{5}{8})$		

There is a one-one correspondence between ω-polynomials of primary trees and points (i, j) such that i and j are proper non-negative fractions with denominators dividing $2^{\alpha-1}$, where α is the altitude of the corresponding tree, and $i + j = \sum_{r=1}^{\alpha-1} 2^{-r}$ ($i + j = 0$ if $\alpha = 1$). There is therefore a one-one correspondence between these points and primary trees.

An unbounded sequence of primary trees is represented by a sequence of points converging to a definite point on the line $i + j = 1$. This point is called the <u>limit point</u> of the limit tree to which the sequence belongs. The i-coordinate of this point can be obtained by interpreting the dual of the primary Ω-tree corresponding to the limit tree as a binary fraction where λ, μ represent the digits 1, 0. The j-coordinate can be obtained similarly by interpreting λ, μ as the digits 0, 1, or simply by subtracting the i-coordinate from 1. For example, the limit tree corresponding to the Ω-tree $\ldots\mu\lambda\lambda\mu\lambda\lambda\mu\lambda\lambda$ has the limit point whose i-coordinate is $0.110110110\ldots$, i.e. the point $(\frac{6}{7}, \frac{1}{7})$. Each point on the line $i + j = 1$ corresponds to a unique infinite limit tree except a denumerable species of double points whose coordinates are of the form $(k/2^n, 1 - k/2^n)$, where n is a positive integer and k is a non-negative integer not exceeding 2^n. Each of these double points is a limit point of two distinct infinite limit trees corresponding to primary Ω-trees of the form

$\ldots\mu\mu\mu\mu\lambda\nu_h\nu_{h-1}\ldots\nu_2\nu_1$ and $\ldots\lambda\lambda\lambda\lambda\mu\nu_h\nu_{h-1}\ldots\nu_2\nu_1$

($\nu_i = \lambda$ or μ). For example, $(\frac{3}{4}, \frac{1}{4})$ is the limit point of limit trees corresponding to Ω-trees $\ldots\mu\mu\mu\mu\lambda\lambda$ and $\ldots\lambda\lambda\lambda\lambda\mu\lambda$. The analogy between these double points and the two alternative notations for binary fractions such as $0.11 = 0.10111\ldots$ is obvious.

A convergent infinite limit tree has a well-defined, finite or denumerably infinite, species of limit points on the line $i + j = 1$. For a divergent limit tree any point on certain segments of this line is a limit point.

<u>Example</u>. The sequence

$\{P_i\} = \{2,\ 2+2,\ 3+(2+2),\ 4+(3+(2+2),\ \ldots\}$,

where $P_i = i + P_{i-1}$, is the union of the sequences of primary trees $\{P_i^{(1)}\}$, $\{P_i^{(2)}\}$, $\{P_i^{(3)}\}$, $\{P_i^{(4)}\}$, ... :

$\{P_i^{(1)}\} = \{2, 3, 4, 5, \ldots\}$,

$\{P_i^{(2)}\} = \{2,\ 1+2,\ 1+3,\ 1+4,\ \ldots\}$,

$\{P_i^{(3)}\} = \{2,\ 1+2,\ 1+(1+2),\ 1+(1+3),\ 1+(1+4),\ \ldots\}$,

$\{P_i^{(4)}\} = \{2,\ 1+2,\ 1+(1+2),\ 1+(1+(1+2)),\ 1+(1+(1+3)),\ \ldots\}$,

....

$\{P_i^{(\omega+1)}\} = \{\dot{2}, \dot{3}, \dot{4}, \dot{5}, \ldots\}$.

The corresponding primary Ω-trees are:
$\ldots\lambda\lambda\lambda\lambda$, $\ldots\lambda\lambda\lambda\mu$, $\ldots\lambda\lambda\lambda\mu^2$, $\ldots\mu^3$, ... , $\ldots\mu\mu\mu\mu$.
Therefore $\lim P_i$ is convergent and its limit points are:
$(1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{4}, \frac{3}{4})$, $(\frac{1}{8}, \frac{7}{8})$, $(\frac{1}{16}, \frac{15}{16})$, ... , $(0, 1)$.

The limit tree $\lim(((1+2^i)+1)+(2^{i+1}+1))$ is divergent. Any point (i, j) such that $\frac{1}{4} \leq i \leq \frac{1}{2}$ or $\frac{3}{4} \leq i \leq \frac{7}{8}$ and $j = 1 - i$ is a limit point of the limit tree.

We turn now to the graphical representation of Ω-trees. Finite Ω-trees are represented in the same way as the corresponding trees, viz. by graphs consisting of forks whose nodes are points (i, j) given by the terms of the Θ-polynomial of the Ω-tree. The arms of a fork with its node at altitude α are of length $2^{-\alpha-1}$. Similarly the coordinates of nodes of an Ω-tree ρ of a transfinite altitude are given by the terms of $\Theta(\rho)$. If the term of a node is of the form $\sigma\pi$ $(1 \leq \bar{\sigma}^* \leq \omega,\ \bar{\pi}^* = \omega.k)$ then the coordinates of the point are $(2^{-k}i_1 + i_0, 2^{-k}j_1 + j_0)$ where (i_1, j_1), (i_0, j_0) are the coordinates of the points corresponding to σ, π respectively. In other words if the term is of the form $\pi_{k+1}\pi_k\ldots\pi_2\pi_1$ where $\pi_1, \pi_2, \ldots, \pi_k$ are of order type ω^* and the order type of π_{k+1} is either finite or ω^* then the coordinates of the point are found as follows: Let $p(\pi_i)$ be the value of the dual of π_i interpreted as a binary fraction in which λ, μ represent the digits 1, 0 and $q(\pi_i)$ the value when λ, μ are the digits 0, 1. Then the coordinates of the point are

$$\left(\sum_{r=1}^{k+1} p(\pi_r)/r, \sum_{r=1}^{k+1} q(\pi_r)/r\right).$$

For example, the point corresponding to the term $\lambda\mu^2\lambda^2 \ldots\lambda\lambda\lambda\lambda \ldots\lambda\mu\lambda\mu\lambda\mu$ has coordinates

$$\left(\frac{1}{4}\cdot\frac{25}{32} + \frac{1}{2}\cdot 1 + \frac{1}{3}, \frac{1}{4}\cdot\frac{3}{16} + \frac{1}{2}\cdot 0 + \frac{2}{3}\right) = \left(\frac{395}{384}, \frac{137}{192}\right).$$

In the preceding discussion $\omega.k$ represents a finite multiple of ω. A straightforward extension of the above procedure would represent terms of order types $(\omega^2)^*$, $(\omega^3)^*$, $(\omega^4)^*$, ... by points on the lines $i + j = 2$, $i + j = 3$, $i + j = 4$, ... Conceivably one might represent terms of higher order types on a finite graph by more drastic contractions of the scale.

CHAPTER V. ENUMERATION OF INDICES

1. INDICES OF $\mathscr{L}$ AND $\mathscr{L}_c$

The numbers a_δ, p_α of possible indices in $\mathscr{L}$ of given potency δ and of given altitude α respectively are given (cf. (11)) by the recurrence formulae

$$a_\delta = a_1 a_{\delta-1} + a_2 a_{\delta-2} + a_3 a_{\delta-3} + \dots + a_{\delta-1} a_1, \quad a_1 = 1;$$

$$p_{\alpha+1} = 2p_\alpha(p_0 + p_1 + p_2 + \dots + p_{\alpha-1}) + p_\alpha^2, \quad p_0 = 1.$$

The formulae for b_δ, q_α, the corresponding numbers of possible non-congruent indices in $\mathscr{L}_c$, are (ibid.):

$$b_1 = b_2 = q_0 = 1;$$

$$b_{2\delta-1} = b_1 b_{2\delta-2} + b_2 b_{2\delta-3} + \dots + b_{\delta-1} b_\delta,$$

$$b_{2\delta} = b_1 b_{2\delta-1} + b_2 b_{2\delta-2} + \dots + b_{\delta-1} b_{\delta+1} + \tfrac{1}{2} b_\delta (b_\delta + 1);$$

$$q_{\alpha+1} = q_\alpha(q_0 + q_1 + q_2 + \dots + q_{\alpha-1}) + \tfrac{1}{2} q_\alpha (q_\alpha + 1).$$

If we denote the number of indices in $\mathscr{L}$ of altitude not greater than α by s_α, i.e. $s_\alpha = \sum_{i=0}^{\alpha} p_i$, the formula for $p_{\alpha+1}$ becomes

$$\begin{aligned} p_{\alpha+1} &= 2p_\alpha s_{\alpha-1} + p_\alpha^2 \\ &= (s_{\alpha-1} + p_\alpha)^2 - s_{\alpha-1}^2 \\ &= s_\alpha^2 - s_{\alpha-1}^2. \end{aligned}$$

Alternatively,
$$\begin{aligned} p_{\alpha+1} &= p_\alpha(s_\alpha + s_{\alpha-1}) \\ &= \prod_{i=1}^{\alpha} (s_i + s_{i-1}). \end{aligned}$$

The first of these formulae gives

$$s_{\alpha+1} = 1 + s_\alpha^2.$$

Similarly if $t_\alpha = \sum_{i=0}^{\alpha} q_i$ we obtain the corresponding formulae

$$2q_{\alpha+1} = t_\alpha^2 - t_{\alpha-1}^2 + q_\alpha = q_\alpha(t_\alpha + t_{\alpha-1} + 1),$$

$$q_{\alpha+1} = 2^{-\alpha}\prod_{i=1}^{\alpha}(t_i + t_{i-1} + 1),$$

$$2t_{\alpha+1} = t_\alpha^2 + t_\alpha + 2.$$

The problem of enumeration of indices of $\mathscr{L}$ of given potency δ ($\delta > 1$) and given altitude α ($\alpha + 1 \leq \delta \leq 2^\alpha$, cf. Chap. I, §4) is essentially one of finding the number of partitions of a sequence of δ objects according to the following rules (cf. (15); also Chap. 4, §1):

(1) At the first stage the sequence of δ objects is partitioned so that the first σ objects are in the left subsequence and the remaining $\delta - \sigma$ objects in the right subsequence.

(2) At stage ν all subsequences which do not consist of single elements are again partitioned into a left subsequence and a right subsequence.

(3) There are α stages. After stage α all subsequences consist of single elements.

The corresponding problem for indices of $\mathcal{L}_c$ is equivalent to the enumeration of partitions of an unordered set of δ identical objects according to similar rules.

As there is an index of potency 1 and altitude 0 we may say that a set of a single element can be partitioned at stage 0.

Let $p(\alpha, \delta)$ denote the number of indices of altitude α and potency δ in $\mathcal{L}$. Obviously $p(0, 1) = 1$. If $\alpha > 1$, any index X of altitude α and potency δ is the sum of its left subindex X' and its right subindex X", i.e. $X = X' + X''$. We can obtain all required indices by:

(1) Letting subtree X' run through all indices of altitude $\alpha - 1$ and X" through all indices of altitude less than $\alpha - 1$ and potency $\delta - \delta_{X'}$ (where $\delta_{X'}$ denotes the potency of X'). There are

$$\sum_{d=\alpha}^{\delta-1}\left\{p(\alpha-1, d)\sum_{a=0}^{\alpha-2} p(a, \delta-d)\right\}$$ such indices;

(2) as in (1) but interchanging the roles of X' and X";

and (3) if $\delta - \alpha \geq \alpha$, letting X' run through all indices of altitude $\alpha - 1$ and potency d ($d = \alpha, \alpha + 1, \ldots, \delta - \alpha$), and X" through all indices of altitude $\alpha - 1$ and potency $\delta - d$. There are

$$\sum_{d=\alpha}^{\delta-\alpha} p(\alpha-1, d)\, p(\alpha-1, \delta-d)$$ of these.

Hence

<u>THEOREM 5.1</u> .

$$p(\alpha,\ \delta) = \sum_{d=\alpha}^{\delta-1} \left\{ p(\alpha-1,\ d) \sum_{a=0}^{\alpha-2} 2p(a,\ \delta-d) + p(\alpha-1,\ \delta-d) \right\},$$

where $p(x,\ y) = 0$ whenever $x + 1 > y$ or $y > 2^x$.

Denote the number of non-congruent commutative indices of $\mathcal{L}_c$ of altitude α and potency δ by $q(\alpha,\ \delta)$. Then $q(0,\ 1) = 1$. If $\alpha \geq 1$ and $X = X' + X''$ is an index of altitude α and potency δ, we obtain all such non-congruent indices by:

(1) letting X' run through all indices of $\mathcal{L}_c$ of altitude $\alpha - 1$ and X'' through all indices of altitude less than $\alpha - 1$ and of potency $\delta - \delta_{X'}$. There are
$$\sum_{d=\alpha}^{\delta-1} \left\{ q(\alpha-1,\ d) \sum_{a=0}^{\alpha-2} q(a,\ \delta-d) \right\}$$ such indices;

and (2)(a) if δ is odd and $\frac{1}{2}(\delta - 1) \geq \alpha$, letting X' run through all indices of $\mathcal{L}_c$ of altitude $\alpha - 1$ and potency d $(d = \alpha,\ \alpha+1,\ \ldots,\ \frac{1}{2}(\delta-1))$ and X'' through all indices of altitude $\alpha - 1$ and potency $\delta - d$. There are
$$\sum_{d=\alpha}^{\frac{1}{2}(\delta-1)} q(\alpha-1,\ d)\ q(\alpha-1,\ \delta-d)$$ of these.

(b) if δ is even and $\frac{1}{2}\delta - 1 \geq \alpha$

(i) letting X' run through all indices of $\mathcal{L}_c$

of altitude $\alpha-1$ and potency d $(d = \alpha, \alpha+1, \ldots$ $\ldots, \frac{1}{2}\delta-1)$ and X" through all indices of altitude $\alpha - 1$ and potency $\delta - d$. There are

$\sum_{d=\alpha}^{\frac{1}{2}\delta-1} q(\alpha-1, d)\, q(\alpha-1, \delta-d)$ of these; and

(ii) letting both X' and X" run through all indices of $\mathcal{L}_c$ of altitude $\alpha - 1$ and potency $\frac{1}{2}\delta$ but taking only one index from each thus obtained pair of congruent indices except when $X' \sim X''$. There are $\frac{1}{2}q(\alpha-1, \frac{1}{2}\delta)\{q(\alpha-1, \frac{1}{2}\delta) + 1\}$ of these.

Thus

<u>THEOREM 5.2</u> .

$$q(\alpha, \delta) = \sum_{d=\alpha}^{\delta-1}\Big\{q(\alpha-1, d) \sum_{a=0}^{\alpha-2} q(a, \delta-d)\Big\} + Q(\alpha, \delta), \quad \text{where}$$

$$Q(\alpha, \delta) = \begin{cases} \displaystyle\sum_{d=\alpha}^{\frac{1}{2}(\delta-1)} q(\alpha-1, d)\, q(\alpha-1, \delta-d), & \text{if } \delta \text{ is odd,} \\ \displaystyle\sum_{d=\alpha}^{\frac{1}{2}\delta-1} q(\alpha-1, d)\, q(\alpha-1, \delta-d) + \\ \qquad + \frac{1}{2}q(\alpha-1, \frac{1}{2}\delta)\{q(\alpha-1, \frac{1}{2}\delta) + 1\}, & \text{if } \delta \text{ is even.} \end{cases}$$

$Q(x, y) = q(x, y) = 0$ whenever $x + 1 > y$ or $y > 2^x$.

We calculate

α	0	1	2	2	3	3	3	3	3	4	4	4	4	4	4	4	4	4	4	4	
δ	1	2	3	4	4	5	6	7	8	5	6	7	8	9	10	11	12	13	14	14	16
$p(\alpha,\delta)$	1	1	2	1	4	6	6	4	1	8	20	40	68	94	114	116	94	60	28	8	1
$q(\alpha,\delta)$	1	1	1	1	1	2	2	1	1	1	3	5	7	8	9	7	7	4	3	1	1

2. INDICES OF $\mathcal{L}_t$ AND OF $\mathcal{L}_{ce}$

Let ι_P be the number of trees (or indices) which have the same palindromic ψ- and Θ-polynomial as a given tree P and let $\mathcal{U}_P$ be the number of trees (or indices) concordant to P.

<u>THEOREM 5.3</u> . If P is a given tree and $\cancel{\prod_{i,j}} \nu_{ij}\lambda^i\mu^j$, Σ

Σ $\cancel{\prod_{i,j}} \pi_{ij}\lambda^i\mu^j$ are its palindromic index ψ-polynomial and Θ-polynomial respectively then $\iota_P = \prod_{i,j} \binom{\nu_{ij}+\pi_{ij}}{\pi_{ij}}$.

(This result was conjectured by Etherington.)

<u>Proof</u>. Use induction on the potency of P. The formula holds trivially for potency 1. Assume that it holds for

trees of potencies less than d. If P is of potency d and $\nu_{rs}\lambda^r\mu^s$ is the leading term of ψ_P then $\psi_Q = \psi_P - (\lambda + \mu - 1)\nu_{rs}\lambda^{r-1}\mu^s$ is the ψ-polynomial of the (ν_{rs})th principal subordinate of P (cf. Theorem 3.5). The potency of Q is $d - \nu_{rs}$ and thus, by the induction hypothesis, $\iota_Q = \prod_{\substack{i,j \neq \\ r-1,s}} \binom{\nu_{ij} + \pi_{ij}}{\pi_{ij}}$.

Now, the coefficient of $\lambda^{r-1}\mu^s$ in ψ_Q is $\nu_{r-1,s} + \nu_{rs}$ and therefore any tree congruent to Q mod (t) has $\nu_{r-1,s} + \nu_{rs}$ free ends corresponding to this term (i.e. corresponding to the point $(r-1,\ s)$ on the tree pattern). To obtain all trees congruent to P mod (t) we join the nodes of ν_{rs} forks to these free ends in all possible manners. For each tree congruent to Q mod (t) this can be done in $\binom{\nu_{r-1,s} + \nu_{rs}}{\nu_{rs}}$ distinct ways. Therefore

$$\iota_P = \binom{\nu_{r-1,s} + \nu_{rs}}{\nu_{rs}} \iota_Q = \binom{\nu_{r-1,s} + \pi_{r-1,s}}{\pi_{r-1,s}} \iota_Q, \text{ since } \nu_{rs} = \pi_{r-1,s},$$

$$= \prod_{i,j} \binom{\nu_{ij} + \pi_{ij}}{\pi_{ij}}.$$

<u>Example</u>. To find the number of trees congruent to 3.3.4 mod (t).

The palindromic index polynomials of 3.3.4 are

$$\psi_{3.3.4} = \lambda^7 + 3\lambda^6\mu + 3\lambda^5\mu^2 + \lambda^4\mu^3 + 3\lambda^5\mu + 6\lambda^4\mu^2 + 3\lambda^3\mu^3 + \\ + \lambda^4\mu + 5\lambda^3\mu^2 + 4\lambda^2\mu^3 + 2\lambda^2\mu^2 + 3\lambda\mu^3 + \mu^3 ,$$

$$\theta_{3.3.4} = \lambda^6 + 2\lambda^5\mu + \lambda^4\mu^2 + \lambda^5 + 4\lambda^4\mu + 3\lambda^3\mu^2 + \lambda^4 + 4\lambda^3\mu + \\ + 4\lambda^2\mu^2 + \lambda^3 + 3\lambda^2\mu + 3\lambda\mu^2 + \lambda^2 + 2\lambda\mu + \mu^2 + \lambda + \mu + 1 .$$

$\lambda^i\mu^j$:	$\lambda^5\mu$,	$\lambda^4\mu^2$,	$\lambda^4\mu$,	$\lambda^3\mu^2$,	$\lambda^2\mu^2$
ν_{ij} :	3	6	1	5	2
π_{ij} :	2	1	4	3	4

For all other terms either ν_{ij} or π_{ij} is 0 and thus $\binom{\nu_{ij}+\pi_{ij}}{\pi_{ij}} = 1$.

$$\mathscr{V}_{3.3.4} = \binom{5}{2}\binom{7}{1}\binom{5}{4}\binom{8}{3}\binom{6}{4} = 294\ 000 .$$

<u>THEOREM 5.4</u> . If $\psi_P = \sum_i \nu_i\lambda^i$ and $\theta_P = \sum_i \pi_i\lambda^i$ are index polynomials in one indeterminate of a given tree P then $\mathscr{V}_P$, the number of trees concordant to P, is equal to $\prod_i \binom{\nu_i + \pi_i}{\pi_i}$.

The proof is similar to that of Theorem 5.3 (v. (27), p. 191).

<u>Example</u>. To find the number of trees concordant to 3.3.4.

$$\psi_{3.3.4} = 8\lambda^7 + 12\lambda^6 + 10\lambda^5 + 5\lambda^4 + \lambda^3 ,$$

$$\theta_{3.3.4} = 4\lambda^6 + 8\lambda^5 + 9\lambda^4 + 7\lambda^3 + 4\lambda^2 + 2\lambda + 1 .$$

$$\left.\begin{array}{lllll} \lambda^i: & \lambda^6, & \lambda^5, & \lambda^4, & \lambda^3; \\ \nu_i: & 12, & 10, & 5, & 1; \\ \pi_i: & 4, & 8, & 9, & 7. \end{array}\right\} \begin{array}{l} \text{For all other terms either} \\ \nu_i \text{ or } \pi_i \text{ is 0 and thus} \\ \binom{\nu_i + \pi_i}{\pi_i} = 1. \end{array}$$

$$\mathcal{N}_{3.3.4} = \binom{16}{4}\binom{18}{8}\binom{14}{9}\binom{8}{7} = 1\ 275\ 507\ 192\ 960.$$

Finally we give two formulae: one for the number of non-concordant indices of altitude α and the other for the number of non-concordant indices of potency δ.

Let ν be a non-negative integer, λ an indeterminate and i any non-negative integer such that $2^i \geq \nu$. Denote by Λ the operator defined as follows:

$$\Lambda(\nu\lambda^i) = \nu\lambda^i \quad \text{if} \quad \nu = 0 \text{ or } 1$$

and
$$\Lambda(\nu\lambda^i) = (\nu - 2)\lambda^i + \lambda^{i-1} \quad \text{if} \quad \nu \geq 2.$$

Define the Λ-value of ν, denoted Λ_ν, as the number of all possible (different) polynomials in λ obtained by operating with Λ in all possible manners on $\nu\lambda^i$ and on terms of thus derived polynomials.

Example. To find the Λ-value of 7. We have

$$7\lambda^i,$$
$$\Lambda(7\lambda^i) = 5\lambda^i + \lambda^{i-1},$$
$$(\Lambda(5\lambda^i)) + \lambda^{i-1} = 3\lambda^i + 2\lambda^{i-1},$$
$$(\Lambda(3\lambda^i)) + 2\lambda^{i-1} = \lambda^i + 3\lambda^{i-1},$$
$$3\lambda^i + \Lambda(2\lambda^{i-1}) = 3\lambda^i + \lambda^{i-2},$$
$$\lambda^i + \Lambda(3\lambda^{i-1}) = \lambda^i + \lambda^{i-1} + \lambda^{i-2},$$

i.e. 6 distinct polynomials and it is impossible to obtain more than 6. Hence the Λ-value of 7 is 6.

It is easily seen that

$$\begin{array}{llll}
\Lambda_0 = 1, & \Lambda_4 = 4, & \Lambda_8 = 10, & \Lambda_{12} = 20, \\
\Lambda_1 = 1, & \Lambda_5 = 4, & \Lambda_9 = 10, & \Lambda_{13} = 20, \\
\Lambda_2 = 2, & \Lambda_6 = 6, & \Lambda_{10} = 14, & \Lambda_{14} = 26, \\
\Lambda_3 = 2, & \Lambda_7 = 6, & \Lambda_{11} = 14, & \Lambda_{15} = 26, \text{ etc.}
\end{array}$$

In fact we have

LEMMA.

$$\Lambda_{2\nu+1} = \Lambda_{2\nu} = \sum_{r=0}^{\nu} \Lambda_r.$$

Proof. Use induction on ν. The formula gives correct Λ-value for $\nu = 1$. Assume that the formula holds for integers less than ν. Consider $\Lambda_{2\nu}$, the number of all distinct polynomials which can be obtained from $2\nu\lambda^i$ by the process described above. All such polynomials with a term in λ^i are obtained from $2\nu\lambda^i = 2\lambda^i + 2(\nu - 1)\lambda^i$ by operating with Λ in all possible manners on the term $2(\nu - 1)\lambda^i$ and on terms derived from it. There are $\Lambda_{2(\nu-1)}$ such polynomials. All the derived polynomials of degree less than i are obtained by operating in the same way on $\nu\lambda^{i-1}$. There are Λ_ν of these.
Hence $\Lambda_{2\nu} = \Lambda_{2(\nu-1)} + \Lambda_\nu$. But, by the induction hypothesis, $\Lambda_{2(\nu-1)} = \sum_{r=0}^{\nu-1} \Lambda_\nu$. Thus $\Lambda_{2\nu} = \sum_{r=0}^{\nu-1} \Lambda_r + \Lambda_\nu = \sum_{r=0}^{\nu} \Lambda_r$.

$\Lambda_{2\nu+1}$ is the number of possible polynomials obtained from $(2\nu + 1)\lambda^i$ by the same process. Now, $(2\nu + 1)\lambda^i = \lambda^i + 2\nu\lambda^i$ and since $\Lambda(\lambda^i) = \lambda^i$ all the required polynomials are obtained by operating with Λ on the term $2\nu\lambda^i$ and on terms derived from it. Thus $\Lambda_{2\nu+1} = \Lambda_{2\nu}$.

Denote by r_α the number of all non-concordant indices of altitude α, i.e. the number of all index ψ-polynomials, $\psi(\lambda)$, of degree α.

<u>THEOREM 5.5</u> .

$$r_{\alpha+1} = \sum_{i=0}^{2^\alpha-1} \Lambda_i, \quad \text{i.e.} \quad r_\alpha = \Lambda_{2^\alpha-2}.$$

<u>Proof</u>. All possible trees of altitude $\alpha + 1$ are subordinates of the plenary tree $2^{\alpha+1}$. Moreover, Theorem 3.11 implies that if we operate with Λ on a term of a ψ-polynomial ψ_P and the resulting polynomial φ differs from ψ_P then φ is the ψ-polynomial of a first subordinate of P. Thus all index ψ-polynomials of degree $\alpha + 1$ can be obtained by operating with Λ on $2^{\alpha+1}\lambda^{\alpha+1}$, the index ψ-polynomial of the plenary tree $2^{\alpha+1}$, and on terms of the derived polynomials in such a way as to leave in each resulting polynomial a term in $\lambda^{\alpha+1}$. We can obtain all these polynomials in the following way: first operate with Λ on the leading terms only and obtain the sequence of ψ-

polynomials of the first, second, ... , $(2^{\alpha} - 1)$th principal subordinates of $2^{\alpha+1}$:

$$2^{\alpha+1}\lambda^{\alpha+1}; \quad (2^{\alpha+1} - 2)\lambda^{\alpha+1} + \lambda^{\alpha}, \quad (2^{\alpha+1} - 4)\lambda^{\alpha+1} + 2\lambda^{\alpha},$$
$$\ldots, \quad (2^{\alpha+1} - 2i)\lambda^{\alpha+1} + i\lambda^{\alpha}, \quad \ldots,$$
$$4\lambda^{\alpha+1} + (2^{\alpha} - 2)\lambda^{\alpha}, \quad 2\lambda^{\alpha+1} + (2^{\alpha} - 1)\lambda^{\alpha}.$$

Now, from each $(2^{\alpha+1} - 2i)\lambda^{\alpha+1} + i\lambda^{\alpha}$ we can obtain all ψ-polynomials of degree $\alpha + 1$ with leading term $(2^{\alpha+1} - 2i)\lambda^{\alpha+1}$ by leaving the term in $\lambda^{\alpha+1}$ alone and operating with Λ on $i\lambda^{\alpha}$ and on other resulting terms. But, by the definition of Λ-value, we can obtain in this manner exactly Λ_i polynomials. Hence $r_{\alpha+1} = \sum_{i=0}^{2^{\alpha}-1} \Lambda_i$. Now, by the Lemma $\sum_{i=0}^{2^{\alpha-1}-1} \Lambda_i = \Lambda_{2^{\alpha}-2}$ and so $r_{\alpha} = \Lambda_{2^{\alpha}-2}$.

For $\alpha = 0, 1, 2, 3, 4, 5, 6, 7, \ldots$

$r_{\alpha} = 1, 1, 2, 6, 26, 166, 1626, 25510, \ldots$.

Let $f(x) = \Lambda_0 + \Lambda_1 x + \Lambda_2 x^2 + \ldots$. Then

$$\frac{f(x)}{1 - x} = (\Lambda_0 + \Lambda_1 x + \Lambda_2 x^2 + \ldots)(1 + x + x^2 + \ldots)$$
$$= \Lambda_0 + (\Lambda_0 + \Lambda_1)x + (\Lambda_0 + \Lambda_1 + \Lambda_2)x^2 + \ldots$$
$$= \Lambda_0 + \Lambda_2 x + \Lambda_4 x^2 + \ldots = \Lambda_1 + \Lambda_3 x + \Lambda_5 x^2 + \ldots .$$

Hence $$f(x) = \frac{f(x^2)}{1 - x^2} + \frac{xf(x^2)}{1 - x^2},$$

i. e. $$f(x) = \frac{f(x^2)}{1 - x} .$$

This functional equation is easily solved by iteration:

$$f(x) = \frac{1}{1-x} f(x^2) = \frac{1}{1-x}\cdot\frac{1}{1-x^2} f(x^4)$$

$$= \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^4} f(x^8) = \ldots = \prod_{i=0}^{\infty}(1 - x^{2^i})^{-1}$$

$$= (1+x+x^2+x^3+x^4+\ldots)(1+x^2+x^4+\ldots)(1+x^4+x^8+\ldots)$$

$$\cdot(1+x^8+\ldots)\ldots .$$

Thus r_α $(\alpha > 0)$ is the coefficient of $x^{2\alpha-2}$ in the Maclaurin expansion of this function. Alternatively, r_α is the coefficient of $x^{2\alpha}$ in $1 + x^2 f(x)$ (all α).

(The preceding paragraph on the generating function $f(x)$ was communicated to me by Dr Etherington.)

A non-zero polynomial $\sum_{i=0}^{n} c_i\lambda^i$ where the c_i are positive integers is a θ-polynomial in one indeterminate if and only if $c_0 = 1$ and $c_{i+1} \leq 2c_i$ $(i = 0, 1, 2, \ldots , n-1)$ (cf. Chap. III, § 6). Also if P has $\sum_{i=0}^{n} c_i\lambda^i$ for its θ-polynomial the potency of P is equal to $1 + \sum_{i=0}^{n} c_i$. Hence the problem of finding the number of non-concordant indices of potency $d+1$, i.e. the number of distinct θ-polynomials $\sum_{i=0}^{n} c_i\lambda^i$ such that $\sum_{i=0}^{n} c_i = d$, is equivalent to the problem of finding the number of partitions of d such that

$d = 1 + c_1 + c_2 + \ldots + c_n$ where $c_1 = 1$ or 2 and $c_{i+1} \leq 2c_i$ To solve it consider the more general problem: given two positive integers c and d find the number of partitions of d such that $d = c + c_1 + c_2 + \ldots + c_n$ where $c_1 \leq 2c$ and $c_{i+1} \leq 2c_i$. Denote this number by $v(c, d)$. Since c_1 can take any value between 1 and $\min(2c, d-c)$ we have: $v(c, d) = \sum_{i=1}^{2c} v(i, d-c)$, where $v(x,y) = 0$ whenever $x > y$. The formula expresses $v(c, d)$ in terms of values of the function for smaller values of the second argument. Since $v(x, x) = 1$ for all positive x we can calculate $v(c, d)$ for any given c and d by repeated use of the formula. Thus

c \ d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
							v(c, d)							
1	1	1	2	3	5	9	16	28	50	89	159	285	510	914
2	0	1	1	2	4	7	12	22	39	70	126	225	404	725
3	0	0	1	1	2	4	7	13	24	42	76	137	245	441
4	0	0	0	1	1	2	4	7	13	24	43	78	140	251
5	0	0	0	0	1	1	2	4	7	13	24	43	78	
6	0	0	0	0	0	1	1	2	4	7	13	24	43	78
7	0	0	0	0	0	0	1	1	2	4	7	13	24	43
8	0	0	0	0	0	0	0	1	1	2	4	7	13	24

The first row ($c = 1$) in the above table gives the number of non-concordant indices of potency $d + 1$.

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