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Representations of Rational Cherednik Algebras: Koszulness and Localisation

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Declaration

I declare that I wrote this thesis myself alone and that the work inside is my own, except where explicitly stated otherwise in the text.

(*Rollo Jenkins*)

To Crozie and Joanna

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Lay Summary

An *algebra* is a typical object of study in pure mathematics. Take a collection of numbers (for example, all whole numbers or all decimal numbers). Inside, you can add and multiply, but with respect to these operations different collections can behave differently. Here is an example of what I mean by this. The collection of whole numbers is called $\mathbb{Z}$. Starting anywhere in $\mathbb{Z}$ you can get to anywhere else by adding other members of the collection: $9 + (-3) + (-6) = 0$. This is not true with multiplication; to get from 5 to 1 you would need to multiply by $\frac{1}{5}$ and $\frac{1}{5}$ doesn't exist in the restricted universe of $\mathbb{Z}$. Enter $\mathbb{R}$, the collection of all numbers that can be written as decimals. Now, if you start anywhere—apart from 0—you can get to anywhere else by multiplying by members of $\mathbb{R}$—if you start at zero you're stuck there.

By adjusting what you mean by 'add' and 'multiply', you can add and multiply other things too, like polynomials, transformations or even symmetries. Some of these collections look different, but behave in similar ways and some look the same but are subtly different. By defining an algebra to be *any collection of things with a rule to add and multiply in a sensible way*, all of these examples (and many more you can't imagine) can be treated in general. This is the power of abstraction: proving that an arbitrary algebra, A, has some property implies that every conceivable algebra (including $\mathbb{Z}$ and $\mathbb{R}$) has that property too.

In order to start navigating this universe of algebras it is useful to group them together by their behaviour or by how they are constructed. For example, $\mathbb{R}$ belongs to a class called simple algebras. There are mental laboratories full of machinery used to construct new and interesting algebras from old ones. One recipe, invented by Ivan Cherednik in 1993, produces *Cherednik algebras*.

Attached to each algebra is a collection of *modules* (also called *representations*). As shadows are to a sculpture, each module is a simplified version of the algebra, with a taste of its internal structure. They are not algebras in their own right: they have no sense of multiplication, only addition. Being individually simple, modules are often much easier to study than the algebra itself. However, everything that is interesting about an algebra is captured by the collective behaviour of its modules. The analogy fails here: for example, shadows encode no information about colour. Sometimes the interplay between its modules leads to subtle and unexpected insights about the algebra itself.

Nobody understands what the modules for Cherednik algebras look like. One first step is to simplify the problem by only considering modules which behave 'nicely'. This is what is referred to as category $\mathcal{O}$. Being *Koszul* is a rare property of an algebra that greatly helps to decribe its behaviour. Also, each Koszul algebra is mysteriously linked with another called its Koszul dual. One of the main results of the thesis is that, in some cases, the modules in category $\mathcal{O}$ behave as if they were the modules for some Koszul algebra. It is an interesting question to ask, what the Koszul dual might be and what this has to do with Cherednik's recipe.

Geometers study tangled, many-dimensional spaces with holes. In analogy with the algebraic world, just as algebraists use modules to study algebras, geometers use *sheaves* to study their spaces. Suppose one could construct sheaves on some space whose behaviour is precisely the same as Cherednik algebra modules. Then, for example, theorems from geometry about sheaves could be used to say something about Cherednik algebra modules. One way of setting up this analogy is called *localisation*. This doesn't always work in general. The last part of the thesis provides a rule for checking when it does.

Contents

Chapter 0

Introduction

The rational Cherednik algebra, $H(W)$, attached to a complex reflection group, W, is a degeneration of the *double-affine Hecke algebra*. It can be thought of as a flat family of non-commutative algebras depending on some parameters. Etingof and Ginzburg introduced them as algebras related to differential operators and Hilbert schemes. Since then, they have appeared in many areas of mathematics. This thesis focuses on their representation theory and some related geometry. Throughout, the term 'rational Cherednik algbera' is understood to mean 'rational Cherednik algebra with $t = 1$'. Unless explicitly stated otherwise, all the proofs in the thesis are original. The thesis is divided into three parts.

Algebra

Part I of the thesis builds to a proof of the following theorem.

Theorem 2.3.1. *Let W be a dihedral group and $H_{\mathbf{c}}(W)$ the associated rational Cherednik algebra for an arbitrary parameter* $\mathbf{c}$*. The category $\mathcal{O}$ associated to $H_{\mathbf{c}}(W)$ is Koszul.*

A *complex reflection group*, $(W, \mathfrak{h})$, is a finite group, W, together with a finite-dimensional complex representation, $\mathfrak{h}$, such that W is generated by elements, called *complex reflections*, which are non-trivial and fix some hyperplane in $\mathfrak{h}$ pointwise. They have been classified by Shephard–Todd in [ST54]: apart from thirty-four exceptions, they can all be written as $G(l, p, n)$ for some positive integers $l, p, n \in \mathbb{Z}$. Examples in the thesis will involve the following complex reflection groups:

(i) The cyclic group of order l, written $G(l, 1, 1)$ or μ_l.

(ii) The symmetric group on n letters, written $G(1, 1, n)$ or $\mathfrak{S}_n$.

(iii) The dihedral groups of order $2d$, written $G(d, d, 2)$ for $d > 1$.

The dihedral group of order 8, also known as the classical Lie group B_2, is also written $G(2, 1, 2)$.

The *rational Cherednik algebra*, $H_{\mathbf{c}}(W)$, is a quotient of the tensor algebra, $T_{\mathbb{C}}(\mathfrak{h} \oplus \mathfrak{h}^*)$, by commutation relations which involve a finite number of complex parameters, $\mathbf{c} = (c_1, \ldots, c_l)$; see Section 1.3.2. As a vector space, rational Cherednik algebras can be described by the PBW Theorem,

$$H_{\mathbf{c}}(W) \overset{\text{v.s.}}{\cong} \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W \otimes \mathbb{C}[\mathfrak{h}^*].$$

Studying the representation theory of $H_{\mathbf{c}}(W)$ by direct computation is difficult for all but the smallest examples. Instead, one considers *category $\mathcal{O}$*, a full subcategory, $\mathcal{O}_{\mathbf{c}}(W)$, of finitely generated left $H_{\mathbf{c}}(W)$-modules with a finiteness condition. Its definition is modelled on the celebrated *BGG-category $\mathcal{O}$* for a semisimple Lie algebra (see [Hum08]). The simple modules in $\mathcal{O}_{\mathbf{c}}(W)$ are in bijection with the irreducible complex representations of W. This category contains enough information about the rational Cherednik algebra to be useful—for example, it contains all the finite-dimensional modules—yet it is small enough to be wieldy. In fact, $\mathcal{O}_{\mathbf{c}}(W)$ is just the right size to have a property called *BGG-reciprocity*. This says that

(i) for each simple module, $L(\lambda)$, indexed by some irreducible representation, λ, of W, there is a corresponding *standard module*, $\Delta(\lambda)$, whose head is $L(\lambda)$,

(ii) the projective cover, $P(\lambda)$, of $L(\lambda)$ has a filtration by standard modules,

(iii) if $L(\mu)$ is another simple module then the multiplicity of $L(\mu)$ as a simple composition factor of $\Delta(\lambda)$ is the same as the multiplicity of $\Delta(\lambda)$ as a standard composition factor of $P(\mu)$.

In many cases this allows one to reconstruct the whole of category $\mathcal{O}$ from the Loewy series of the standard modules.

Although the definition of category $\mathcal{O}$ may seem artificial, it appears naturally in relation to other algebras. There is an exact and essentially surjective functor,

$$\mathsf{KZ}\colon \mathcal{O} \longrightarrow {}_{\mathcal{H}(W)}\mathbf{mod},$$

where $\mathcal{H}(W)$ is the *Iwahori–Hecke algebra* of W, a deformation of the group algebra $\mathbb{C}W$, also dependent on parameters. This is an example of a *highest weight cover* and, in the case $W = \mathfrak{S}_n$, the symmetric group, the category $\mathcal{O}$ is an *equivalent highest weight cover* to the *q-Schur algebra* over $\mathcal{H}(\mathfrak{S}_n)$; see [Rou08]. Being exact, the KZ-functor can be represented by some projective module, P_{KZ}. In this way $\mathcal{H}(W)$ is isomorphic to $\mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}})^{\mathrm{op}}$.

One of the key properties of the KZ-functor is the *double centraliser property*: if P is a projective generator for $\mathcal{O}$ then $\mathrm{End}_{\mathcal{O}}(P)^{\mathrm{op}}$ is Morita equivalent to $\mathrm{End}_{\mathcal{H}(W)}(\mathsf{KZ}(P))^{\mathrm{op}}$. More is known about the representation theory of the Hecke algebra; for example, Geck–Pfeiffer write down character formulae in [GP00]. The relationship between Hecke algebras and category $\mathcal{O}$ provided by the double centraliser property will be exploited in the proof of Theorem 2.3.1.

Suppose that A is a non-negatively graded left noetherian algebra with A_0 semisimple. The thesis is concerned with path algebras of quivers; such algebras satisfy these assumptions. One says that A is *Koszul* if there exists a projective resolution of A_0 in the category of graded A-modules,

$$\cdots \xrightarrow{d^2} P^1 \xrightarrow{d^1} P^0 \xrightarrow{d^0} A_0 \longrightarrow 0,$$

such that P^i is generated as an A-module in degree i.

Simple A-modules must be simple A_0-modules, so decomposing A_0 into simple summands gives all the simple modules up to shifting the grading. Define a graded ring by

$$E(A) := \mathrm{Ext}^\bullet(A_0, A_0) = \bigoplus_i \mathrm{Ext}^i(A_0, A_0),$$

whose multiplication is the cup product. Suppose that A is Koszul, $E(A)$ is right noetherian and that A is finitely generated as a module over A_0. Let $\mathrm{D^b}({}_A\mathbf{gmod})$ denote the bounded derived category of graded left A-modules. The right derived version of $\mathrm{gHom}_A(A_0, -)$, graded homomorphisms of A-modules, gives an equivalence of triangulated categories

$$\mathrm{RgHom}(A_0, -)\colon \mathrm{D^b}({}_A\mathbf{gmod}) \xrightarrow{\simeq} \mathrm{D^b}({}_{E(A)^{\mathrm{op}}}\mathbf{gmod}),$$

where simple modules are sent to indecomposable projectives and indecomposable injectives are sent to simples; see [BGS96, Theorem 1.2.6].

Koszul duality between graded rings is a deep phenomenon. The following is a simple example. Let $\Lambda(x, y)$ be the exterior algebra on $\mathbb{C}^2$: it has a basis $\{1, x, y, x \wedge y\}$. It can be written as a quotient of the algebra generated by x and y by the relation $a \wedge b + b \wedge a = 0$ for $a, b \in \{x, y\}$. Putting x and y in degree one gives a grading with respect to which it is Koszul. Koszul duality swaps the relation $x \otimes y + y \otimes x$ for $x \otimes y - y \otimes x$ and kills the relations $x \otimes x$ and $y \otimes y$, so that the Koszul dual is the polynomial ring $\mathbb{C}[x, y]$. This example is particularly striking because, whilst $\mathbb{C}[x, y]$ has global dimension two, $\Lambda(x, y)$ has infinite global dimension.

For the BGG category $\mathcal{O}$ of a semisimple Lie algebra, the blocks are Koszul and the Koszul duals are also blocks of similar categories (see [BGS96, Theorem 1.1.3]). There is no corresponding result for the category $\mathcal{O}$ of a rational Cherednik algebra. The following is an outline of the proof of Theorem 2.3.1. From the Jordan–Hölder series of the standard modules of $\mathcal{O}(W)$

given in [Chm06], use BGG reciprocity and known results about the corresponding Hecke algebras to calculate the projective covers. These give projective generators for the category so, by summing them, one gets a projective module P such that the Koszulness of $\mathcal{O}(W)$ is equivalent to the Koszulness of $\mathrm{End}_{\mathcal{O}}(P)$. Now there is a quiver with relations Q such that $\mathbb{C}Q$ is Morita equivalent to $\mathrm{End}_{\mathcal{O}}(P)$. These relations can be used to explicitly calculate projective resolutions of the simple modules.

This also provides a way to calculate the Koszul duals. The blocks of $\mathrm{End}_{\mathcal{O}}(P)$ are sometimes self-dual, but sometimes the duals are not Morita equivalent to blocks of $\mathcal{O}(W')$ for any dihedral group W'.

Combinatorics

Part II of the thesis is based on the following idea. Given a complex reflection group, $G(l,1,n) = \mu_l \wr \mathfrak{S}_n$ for some l and n, there is a way to construct a combinatorial object which is categorified by $\mathcal{O}_{\mathbf{c}}(G(l,1,n))$ for some specific values of $\mathbf{c}$. This object is called *Fock space*, $\Lambda^{s+\frac{\infty}{2}}$. It is an infinite-dimensional vector space with a basis of *charged l-multipartitions of n*; see Section 3.1 for the definition. Fock space has an action by the enveloping algebra of the quantum affine Lie algebra, $U_q(\hat{\mathfrak{sl}}_n)$, and it is the weight spaces with respect to this action that mimic the behaviour of the blocks of $\mathcal{O}_{\mathbf{c}}(G(l,1,n))$. In particular, each l-multipartition of n corresponds both to a basis vector of Fock space and an l-tuple of Young diagrams. These diagrams correspond to irreducible representations of $G(l,1,n)$, which, in turn, parametrise the simple modules in $\mathcal{O}_{\mathbf{c}}(G(l,1,n))$. If two multipartitions belong to the same weight space then the corresponding simples in $\mathcal{O}(G(l,1,n))$ belong to the same block.

In [CM], Chuang and Miyachi conjecture that, for these values of $\mathbf{c}$, the blocks of $\mathcal{O}_{\mathbf{c}}(G(l,1,n))$ are Koszul and the combinatorics which describe a block and its dual are related by a simple rule. This is called the *Level-Rank Duality Conjecture*. The conjecture is confirmed for some low rank cases when $W = \mathfrak{S}_2$, μ_2 and B_2; see Section 4.2.

Remark 0.0.1. *Recently, Rouquier–Shan–Varagnolo–Vasserot have proved this conjecture; see [RSVV13, Theorem 7.2]. See also [Los13] and [Web13].*

Geometry

Fix $W = G(l,1,n)$ and let $F = \mathbb{C}^*$. Part III of the thesis turns away from category $\mathcal{O}$ to look at a mechanism for constructing $H_{\mathbf{c}}(W)$ modules called *localisation*. This mechanism fits into a wider picture as follows.

Deformation Quantisation

A *conical symplectic resolution* is a complex smooth symplectic algebraic variety, Y, with an well-behaved action of F (see Section 5.7), such that the canonical map into its affinisation, $Y \longrightarrow Y_0 := \mathrm{Spec}(\mathbb{C}[Y])$, is a resolution of singularities. Deformation quantisation can be intepreted as a machine that produces localisation functors from conical symplectic resolutions together with a cohomology class,

Conical symplectic resolution and a character	$\Longrightarrow$	Localisation Functors
$Y \longrightarrow Y_0$, $\chi \in H^2(Y,\mathbb{C})$	$\longmapsto$	$\mathcal{G}_\chi \underset{\mathbb{S}}{\overset{\mathbb{T}}{\rightleftarrows}} \mathcal{A}_\chi$

where $\mathcal{G}_\chi$ is a category of sheaves on Y with a 'twist' by χ and $\mathcal{A}_\chi$ is a module category for an algebra of independent interest that is also dependent on χ.

Example 0.0.1. *The following varieties can be given a symplectic structure and an action of F so that they are examples of conical symplectic resolutions; see [BPW12] for more on this.*

(i) The cotangent bundle of a flag variety, G/B, where G is a simply connected, connected, complex Lie group such that $\mathfrak{g} = \mathrm{Lie}(G)$ is a semisimple Lie algebra and $B \subset G$ is a Borel. Here, Y_0 can be identified with the nilpotent cone of $\mathfrak{g}$.

(ii) The crepant resolution, $\widetilde{\mathbb{C}^2/\Gamma} \longrightarrow \mathbb{C}^2/\Gamma$, of the quotient singularity, where Γ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

(iii) The Hilbert Scheme of n points on the crepant resolution of the quotient singularity above, $\mathrm{Hilb}^n(\widetilde{\mathbb{C}^2/\Gamma}) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2/\Gamma)$. The resolution map is the Hilbert–Chow morphism to the singular space of n-tuples of points on $\mathbb{C}^2/\Gamma$.

(iv) Hypertoric varieties.

(v) Nakajima quiver varieties.

There is a considerable overlap between Examples (ii)—(v).

Deform quantising the first example yields the important Beilinson–Bernstein Localisation Theorem. Here $\mathcal{G}_\chi$ is the category of χ-twisted D-modules on the flag variety, G/B, and $\mathcal{A}_\chi$ is a subcategory of modules over $U(\mathfrak{g})$ which have central character χ; see Theorem 1.2.1. The localisation functor is an equivalence exactly when χ corresponds to a so-called 'anti-dominant' and 'regular' weight of $\mathfrak{g}$; these adjectives come from Lie Theory. The work in the thesis is effectively trying to find the corresponding adjectives in other cases.

If, instead of flag varieties, one chooses quiver varieties for a specific quiver—based on the fixed data, (l, n)—then (in good circumstances) the corresponding localisation functors produce modules for the rational Cherednik algebra of W. This construction depends on two choices instead of one. Let $G = \mathrm{GL}_n(\mathbb{C})^{\times l}$. The first choice is an algebraic group character, $\theta\colon G \longrightarrow \mathbb{C}^*$, which comes from Geometric Invariant Theory: The space Y_θ is constructed as the GIT quotient of an affine space, X, by G with respect to a linearisation determined by θ; see Section 5.4. Different choices of θ can produce different conical symplectic resolutions. Secondly, a character for $\mathfrak{g}$, $\chi \in (\mathfrak{g}^*)^G$, which determines the parameter, $\mathbf{c}$, for the corresponding Cherednik algebra $H_{\mathbf{c}}(W)$.

A method for deform quantising these varieties was developed by Rouquier and Kashiwara in [KR08]. They construct a W-algebra, $\mathcal{W}$, on the space Y_θ. Sections of $\mathcal{W}$ look like Laurent series in a formal parameter, $\hbar$, with coefficients in the sheaf of holomorphic functions. In order to compensate for the introduction of $\hbar$, they also put an action of $F = \mathbb{C}^*$ on $\mathcal{W}$, so that each power of $\hbar$ lies in a different weight space. For a conical symplectic resolution, $Y = T^*M$, that is a cotangent bundle on a smooth variety, M, taking global F-invariant sections produces the algebra of differential operators on M. They study the case, $l = 1$, which corresponds to the Hilbert Scheme of n points on the plane and show that there is a category of sheaves of $\mathcal{W}$-modules, $\mathcal{G}_\chi$ and corresponding localisation functors that produces Cherednik algebra modules for $W = \mathfrak{S}_n$. Bellamy and Kuwabara study the case $n = 1$, which correspond to hypertoric varieties in [BK12] and Gordon deals with the general case for $W = G(l, 1, n)$ in [Gor06].

For arbitrary choices of χ, the modules produced for this construction are actually modules for the spherical Cherednik algebra; see Section 1.4. The spherical Cherednik algebra is a subalgebra of $H_{\mathbf{c}}(W)$ which is Morita equivalent when $\mathbf{c}$ is a *spherical parameter*—these are dense in the parameter space. Thus, when the parameter, $\mathbf{c}$, corresponding to χ is spherical, $\mathcal{A}_\chi$ can be interpreted as being the category of $H_{\mathbf{c}}(W)$-modules. McGerty and Nevins prove that, when χ is chosen in this way, the localisation functors, $\mathbb{S}$ and $\mathbb{T}$, are derived equivalences. The question, 'for which χ does this construction produce actual equivalences?' is still open.

There is more to this story, though. The affine space, X, can also be used to construct $\mathcal{A}_\chi$. This is called *quantum hamiltonian reduction*.

Quantum Hamiltonian Reduction

The parameter space for θ can be divided into convex regions, called *GIT chambers*, inside which varying θ produces an isomorphic variety, Y_θ. These chambers are separated by hyperplanes called *GIT walls*. If θ is chosen so that it doesn't lie on a wall then there is a G-stable open subset, $X^{\mathrm{ss}} \subseteq X$, such that all the G-orbits in X^{ss} are closed and the action of G is free. This is a symplectic subvariety of X. Categories of χ-twisted sheaves of W-algebra modules can be constructed for X and X^{ss} and sheaves in the former category can be restricted to the latter; denote this functor Res. Quantum hamiltonian reduction is the functor $\mathbb{H} := \Gamma(X, -)^{G,F}$ that runs between χ-twisted sheaves on X and $H_{\mathbf{c}}(W)$-modules (when $\mathbf{c}$ is spherical). In [KR08] they prove that there is an equivalence, $\mathbb{E}$, that fits into Diagram 1.

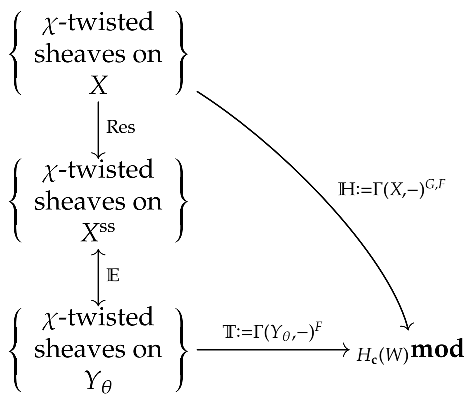


Figure 1: A (not necessary commuting) diagram of functors produced by deform quantising Nakajima quiver varieties corresponding to $W = G(l, 1, n)$.

One of the main results of the thesis is the following.

Theorem 6.1.2. *When* χ *is chosen so that* $\ker \mathrm{Res} \subseteq \ker \mathbb{H}$ *and the corresponding parameter* $\mathbf{c}$ *is spherical the following nice properties hold.*

(i) The diagram above commutes,

(ii) The functor $\mathbb{T}$ *is an equivalence,*

(iii) The kernel of the quantum hamiltonian reduction functor is precisely those sheaves supported on unstable points, $X \setminus X^{\mathrm{ss}}$*; that is,*
$$\ker \mathrm{Res} = \ker \mathbb{H}.$$

The Kirwan–Ness Stratification and Local Cohomology

Let X^{us} denote the unstable locus of X. One of the most important results in GIT is the Hilbert–Mumford Criterion, which says that the stability of points in X can be detected by the action of certain one-parameter subgroups of G. There is a rule for choosing, among these destabilising one-parameter subgroups, one which is maximally responsible for that point being unstable. This is called an optimal subgroup. The collection of all optimal subgroups produces the *Kirwan–Ness stratification* of X^{us}; each stratum is a locus of unstable points corresponding to some optimal subgroup in a conjugacy class; see Section 5.5.2. There is an ordering of these strata so that they can be written, $S_1, \ldots, S_q$ say, with each S_i corresponding to an optimal subgroup, λ_i. Let $X_i = X \setminus \bigcup_{j=1}^{i} S_i$. Each S_i is a closed subvariety of X_{i-1}.

Now the condition, $\ker \mathrm{Res} \subseteq \ker \mathbb{H}$, is equivalent to a condition on the vanishing of local cohomology of each X_{i-1} along S_i. The following theorem reduces the criteria further to checking if the parameter χ belongs to a set of hyperplanes inside the parameter space.

Theorem 6.3.1. *Let* $G = (\mathbb{C}^*)^l$ *acting on an affine symplectic variety* X *that satisfies Hypothesis 6.3.1. Then, for each* $i = 1, \ldots, q$ *there is a set of hyperplanes in the parameter space for* χ *such that, if* χ *avoids all of them, then the local cohomology vanishing condition is satisfied.*

These hyperplanes can be calculated by weights of each λ_i acting on the normal bundle to each stratum. McGerty–Nevins have proved a stronger version of this theorem, for the action of an arbitrary reductive group G. Their paper, [MN13], is not yet available. The main theorem of the thesis applies this criterion to some particular examples.

Theorem 7.3.1, 7.4.1, 7.5.1. *In each of the examples, $W = \mathfrak{S}_n, \mathbb{Z}/3\mathbb{Z}, B_2$, for any character $\chi \in (\mathfrak{g}^*)^G$ there is some θ, not lying on a GIT wall, such that* $\ker \mathrm{Res} \subseteq \ker \mathbb{H}$.

I conclude by conjecturing that the result extends to $G(l, 1, n)$.

Conjecture. *Let $W = G(l, 1, n)$. Then, for all χ there exists a θ, not lying on a GIT wall, such that* $\ker \mathrm{Res} \subseteq \ker \mathbb{H}$.

0.1 Notation

Every field in this thesis will be some extention of $\mathbb{Q}$, in particular, all the representation theory is done over $\mathbb{C}$. Bold-faced text will often refer to a category: given a ring A, let ${}_A\mathbf{Mod}$ and $\mathbf{Mod}_A$ denote the categories of left and right modules respectively and let ${}_A\mathbf{mod}$ and $\mathbf{mod}_A$ denote the respective subcategories of finitely generated left and right A-modules. If A is a graded ring then ${}_A\mathbf{gMod}$ will denote the category of graded left A-modules.

Given a set of vectors, S, in a k-vector space V, the span of S is denoted $\langle S \rangle_k$. Given a set of vectors, S, in a k-algebra A, the subalgebra generated by S is denoted $\langle S \rangle_{\mathrm{Alg}}$ and the left ideal generated by S is denoted AS. For emphasis that an isomorphism is a linear map and not necessarily a ring homomorphism the symbol $\overset{\text{v.s.}}{\cong}$ will be used.

By **variety** I mean a complex, quasi-compact, separated topological space, X, together with a sheaf, $\mathcal{O}_X$, and each point, $x \in X$, has an open neighbourhood, U, such that $(U, \mathcal{O}_X|U)$ is an affine algebraic variety. Given a smooth variety X, for any point x the corresponding affine open neighbourhood U, being an affine algebraic variety, has coordinate functions $x_1, \dots, x_n$ say. Call these **local coordinates** of x in X. For a variety, X, let $\mathbf{qCoh}_X$ and $\mathbf{coh}_X$ denote the categories of quasi-coherent and coherent sheaves on X respectively. Given an ideal I of a noetherian, commutative, reduced ring R, let $V(I)$ denote the subvariety $\mathrm{Spec}(R/I) \subseteq \mathrm{Spec}(R)$.

Part I

Algebra

Chapter 1

Algebraic Prerequisites

1.1 General Algebra

This section contains general prerequisite algebra for later chapters. The definitions are given for left modules; everything said in this section applies equally well for right modules.

1.1.1 Projective Covers and Blocks

Let R be a ring and M a left R-module. A **projective cover** of M is a projective left R-module, P, together with a surjective map, $p\colon P \longrightarrow M$, such that if Q is any proper submodule of P then $p(Q) \neq M$. By construction, projective covers, when they exist, are unique up to isomorphism, but this isomorphism need not be unique. A ring for which every left module has a projective cover is called left **perfect**. By [Bas60, Theorem P], left artinian rings are left perfect, so when R is a finite-dimensional $\mathbb{C}$-algebra projective left covers exist.

Let R be a ring and e a centrally primitive idempotent of R. That is, e is central, non-zero and cannot be written as a non-trivial sum of central idempotents. Then Re is said to be a **block** of R. A left R-module, M, is said to **belong** to a block, Re, of R if $eM = M$. Equivalently, a module belongs to a block if all of its simple composition factors belong to that block.

The decomposition of a ring, $R = \prod_{i\in I} B_i$, into blocks produces a decomposition of the category ${}_R\mathbf{Mod}$ into subcategories $\{{}_{B_i}\mathbf{Mod} \,|\, i \in I\}$ and similarly for ${}_R\mathbf{mod}$. Suppose A is an artinian algebra over $\mathbb{C}$ so that every finitely generated A-module has finite length. Then the simple modules are finite-dimensional. Let J be an indexing set for a complete list of isomorphism classes of simple modules, $\{S_j \,|\, j \in J\}$, and define $s_j := \dim S_j$ for all $j \in J$. The regular representation of A decomposes into a direct sum of projective indecomposables, $A = \oplus_{j\in J} P_j^{\oplus s_j}$, where each P_j is a projective cover of S_j.

1.1.2 Heads, Socles and Radicals

Let R be a ring and M an R-module. A submodule N is said to be **maximal** if M/N is simple. The **radical** of M, written $\mathrm{rad}(M)$, is the intersection of all maximal submodules. By successively taking radicals, one gets the descending **radical filtration** of M,

$$M \supseteq \mathrm{rad}(M) \supseteq \mathrm{rad}^2(M) \cdots .$$

The radical can also be described as the unique smallest submodule such that $M/\mathbf{rad}(M)$ is semisimple. This semisimple quotient is called the **head** of M.

Dually, the **socle** of a module, written $\mathrm{soc}(M)$, is the sum of all minimal non-zero submodules; it is also the unique maximal semisimple submodule. For $k > 1$, define $\mathrm{soc}^{k+1}(M)$ by $\mathrm{soc}^{k+1}(M)/\mathrm{soc}^{k}(M) = \mathrm{soc}(M/\mathrm{soc}^{k}(M))$, where $\mathrm{soc}^1(M) := \mathrm{soc}(M)$. This gives the **socle filtration** of M,

$$0 \subseteq \mathrm{soc}^1(M) \subseteq \mathrm{soc}^2(M) \subseteq \cdots .$$

Note that $\mathrm{rad}(\mathrm{soc}(M)) = 0$. When they are finite, the lengths of these two filtrations coincide and are called the **Loewy length** of M.

Lemma 1.1.1. *If $p\colon P \longrightarrow S$ is a projective cover of a simple left R-module then the head of P is isomorphic to S.*

Proof. Since the image of p is simple, $\ker p$ is a maximal submodule of P. The radical is the intersection of all maximal submodules, so p factors through the quotient of P by the radical; in other words, the head contains a summand isomorphic to S.

Suppose the head of P were $S \oplus M$ for some module M with

$$\pi\colon P \longrightarrow P/\mathrm{rad}(P) \cong S \oplus M,$$

the quotient map. Let Q be the submodule of P generated by all the preimages of S under π. Since P is a projective cover of S and $\pi(Q) = S$, $P = Q$ and so $M = 0$. □

From the definition of the radical of a module, one can see that the head of a direct sum of modules is a direct sum of the heads of the summands. This gives the following corollary.

Corollary 1.1.1. *Projective covers of simple modules are indecomposable.*

1.1.3 Extension Diagrams

Let L, M and N be left R-modules. A diagram with M stacked on top of N is used to refer to the following pair, a module L and a particular short exact sequence that describes L as a (possibly split) extension of M by N, that is,

$$\begin{matrix} M \\ N \end{matrix} := (L, i, p) \quad \Leftrightarrow \quad 0 \to N \xrightarrow{i} L \xrightarrow{p} M \to 0 \quad \text{is exact.}$$

The dependence of the diagram on the choice of short exact sequence is not obvious from the notation, but when the short exact sequence is split the diagram will be written $M \oplus N$, as usual.

This notation is ambiguous when three or more composition factors appear. For example, the diagram,

$$\begin{matrix} A \\ B, \\ C \end{matrix}$$

could mean an extension of A by $\begin{matrix} B \\ C \end{matrix}$ or an extension of $\begin{matrix} A \\ B \end{matrix}$ by C. The following proposition shows that these two interpetations are equivalent: they describe modules that are equal.

Proposition 1.1.1. *Let A, B and C be left R-modules. For each pair of short exact sequences,*

$$0 \longrightarrow B \xrightarrow{i_1} \begin{matrix} A \\ B \end{matrix} \xrightarrow{p_1} A \longrightarrow 0 \qquad 0 \longrightarrow C \xrightarrow{i_2} X \xrightarrow{p_2} \begin{matrix} A \\ B \end{matrix} \longrightarrow 0,$$

there exists a pair of short exact sequences,

$$0 \longrightarrow C \xrightarrow{i'_1} \begin{matrix} B \\ C \end{matrix} \xrightarrow{p'_1} B \longrightarrow 0, \qquad 0 \longrightarrow \begin{matrix} B \\ C \end{matrix} \xrightarrow{i'_2} Y \xrightarrow{p'_2} A \longrightarrow 0,$$

such that $X = Y$ and vice versa. The diagram refers to the equivalent data (X, i_1, i_2, p_1, p_2) and $(Y, i'_1, i'_2, p'_1, p'_2)$.

Proof. If X exists then there is a diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{i_1} & \begin{smallmatrix} A \\ B \end{smallmatrix} & \xrightarrow{p_1} & A & \longrightarrow & 0 \\
 & & & & \Big\uparrow{\scriptstyle p_2} & & & & \\
0 & \longrightarrow & C & \xrightarrow{i_2} & X & \xrightarrow{p_2} & \begin{smallmatrix} A \\ B \end{smallmatrix} & \longrightarrow & 0
\end{array}$$

Let $x \in X$. Then

$$p_1 p_2(x) = 0 \Leftrightarrow p_2(x) \in \ker p_1 \Leftrightarrow p_2(x) \in i_1(B) \Leftrightarrow x \in p_2^{-1}(i_1(B)),$$

so the kernel is precisely $K := p_2^{-1}(i_1(B))$. Now, p_2 is surjective and is the map which takes the quotient by $i_2(C)$ so

$$K/i_2(C) \cong p_2(K) = i_2(B) \cong B.$$

Define i_1' to be i_2 followed by the inclusion of $i_2(C)$ into K, define p_1' to be p_2 restricted to K, define i_2' to be the inclusion of K into X and p_2' to be the p. Then $X = Y$ as required. The converse argument is similar. □

By induction, this proposition makes extension diagrams of arbitrary length well-defined. These diagrams are especially useful for showing the composition factors of the radical or socle filtration of a module of finite length.

Lemma 1.1.2. *Suppose that an A-module M has a projective subquotient P. Then P is isomorphic to a submodule.*

Proof. Being a composition factor implies there are submodules N_1 and N_2 such that

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow P \longrightarrow 0$$

is a short exact sequence. Since P is projective the identity on P lifts to a homomorphism which splits the sequence. Hence P is isomorphic to a submodule of N_2 and so of M. □

1.1.4 Symmetric Algebras

A $\mathbb{C}$-algebra A is **symmetric** if it has an associative non-degenerate bilinear form,

$$\sigma\colon A \otimes A \longrightarrow \mathbb{C},$$

such that $\sigma(a, b) = \sigma(b, a)$ for all $a, b \in A$. If A is semisimple then, by the Artin–Wedderburn Theorem, it decomposes as a product of matrix rings over $\mathbb{C}$. One can then define $\sigma(a, b) = 0$ for any a and b belonging to different components of the decomposition and $\sigma(a, b) = \operatorname{tr}(ab)$ if a and b belong to the same component. In this way, all semisimple $\mathbb{C}$-algebras are symmetric.

Lemma 1.1.3. *Let A be an artinian symmetric algebra. Then the head of A is isomorphic to the socle of A.*

Proof. Since A is artinian, the regular representation decomposes into a sum of projective indecomposables. Now [Ben98, Theorem 1.6.3] shows that the simple head of any projective indecomposable is isomorphic to its socle. □

1.1.5 Morita Theory

This subsection closely follows the exposition in [Ben98, Section 2.2]. Two rings R and S are said to be **Morita equivalent** if there is an equivalence of abelian categories

$$F\colon {}_R\mathbf{Mod} \xrightarrow{\simeq} {}_S\mathbf{Mod}.$$

Definition 1.1.1. *Let R be a ring and P a left R-module. Then P is a* ***progenerator*** *for ${}_R$**Mod** if*

(i) P is finitely generated and projective

(ii) every left R-module is a homomorphic image of a direct sum of copies of P.

Theorem 1.1.1. *(Morita [Ben98, Theorem 2.2.6]) Two module categories ${}_R$**Mod** and ${}_S$**Mod** are equivalent if and only if ${}_R$**mod** and ${}_S$**mod** are equivalent. This happens if and only if $S \cong \mathrm{End}_R(P)^{\mathrm{op}}$ for some progenerator P for ${}_R$**Mod**.*

If R is an artinian ring with a complete list of projective indecomposables, $P_1, \ldots, P_n$, then $P = \oplus_{i=1}^n P_i^{\oplus m_i}$ is a progenerator if and only if each $m_i > 0$. The **basic algebra** of R is $\mathrm{End}(\oplus P_i)^{\mathrm{op}}$. When R is a $\mathbb{C}$-algebra the basic algebra is characterised by being the unique Morita equivalent algebra all of whose simple modules have dimension one.

1.2 D-Modules

The theory of D-modules originated in the 1960s as a tool in algebraic analysis. Since then, the subject has been developed, particularly by M. Sato, J. Bernstein, T. Kawai and M. Kashiwara, to the point where it has rich connections with other subjects. It has been used to resolve the Kazhdan–Lusztig conjecture by Brylinski–Kashiwara and Beilinson–Bernstein, to develop the Riemann–Hilbert correspondence and in the development of Hodge Theory. For an introduction see [HTT08].

1.2.1 The Sheaf of Differential Operators

Let X be a smooth algebraic variety over $\mathbb{C}$. Let Θ_X be the tangent sheaf or, equivalently, the sheaf of vector fields on X, and let D_X be the sheaf of differential operators. These are both locally free $\mathcal{O}_X$-modules with Θ_X coherent and D_X quasi-coherent. For example, if $X = \mathbb{A}^2$ with coordinates $\{x, y\}$ then

$$\Theta_X = \{f\partial_x + g\partial_y \,|\, f, g \in \mathcal{O}_X\} \subset D_X = \{f_{ab}\partial_x^a\partial_y^b \,|\, a, b \in \mathbb{Z}_{\geq 0},\ f_{ab} \in \mathcal{O}_X\},$$

where ∂_x and ∂_y denote the differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively.

The sections of D_X have a filtration called the **operator filtration**. It is defined on affine open subsets of X. Given an affine open subset U, choose local coordinates, $\{x_1, \ldots, x_n\}$ say. Then, for each $i = 1, \ldots, n$, $\partial_i := \frac{\partial}{\partial x_i}$ is a differential operator and $D_X(U)$ is generated by $\{x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\}$. The operator filtration is defined by setting $\deg x_i = 0$ and $\deg \partial_i = 1$ for each $i = 1, \ldots, n$; it is independent of the choice of coordinates.

Let $\pi \colon T^*X \longrightarrow X$ be the projection. Then, given an open set $U \subset X$ with local coordinates $x_1, \ldots, x_n$, there is an isomorphism

$$\pi_*\mathcal{O}_{T^*X}(U) \longrightarrow \mathrm{gr}D_X(U); \quad x_i \mapsto x_i, \quad dx_i \mapsto \overline{\partial_i},$$

where $\overline{\partial_i}$ is the image of ∂_i in the degree one part of $\mathrm{gr}D_X$. Therefore, globally, there is a description of the associated graded ring as the global sections of the pushforward of the cotangent bundle on X:

$$\mathrm{gr}D_X \cong \pi_*\mathcal{O}_{T^*X}.$$

Let M be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. Call M a **D-module** if, for each open set $U \subset X$, $M(U)$ is a finitely generated left $D_X(U)$-module. The category of D_X-modules that are quasi-coherent (or coherent) as $\mathcal{O}_X$-modules is denoted $\mathbf{qCoh}_{D_X}$ (or $\mathbf{coh}_{D_X}$).

The following lemma from [HTT08] gives a nice interpretation of D-modules when M is a locally free sheaf of $\mathcal{O}_X$-modules (which is the case for sheaves in $\mathbf{coh}_{D_X}$): they are integrable flat connections on the vector bundle corresponding to M.

Lemma 1.2.1. *Let M be a quasi-coherent $\mathcal{O}_X$-module. Extending the action of $\mathcal{O}_X$ to a D-module* *structure is equivalent to defining a $\mathbb{C}$-linear morphism of sheaves*

$$\nabla\colon \Theta_X \longrightarrow \mathcal{E}nd_{\mathbb{C}}(M)\,; \quad \theta \mapsto \nabla_\theta,$$

such that, for all $f \in \mathcal{O}_X$, $\theta, \theta' \in \Theta_X$ and $s \in M$,

(i) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$,

(ii) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$,

(iii) $\nabla_{[\theta,\theta']}(s) = [\nabla_\theta, \nabla_{\theta'}](s)$.

Proof. This is [HTT08, Lemma 1.2.1]. □

Example 1.2.1. *Let $X = \mathbb{A}^1$ so that $\Gamma(X, D_X) = \frac{\mathbb{C}\langle x,\partial\rangle}{[\partial,x]=1}$. Then $M = \mathcal{O}_X$ is a D_X-module; the sections of D_X are naturally differential operators which act on the sections of $\mathcal{O}_X$ by multiplication and differentiation.*

1.2.2 D-Affinity

One of the ways to characterise an affine space, X, is by saying that the global sections functor

$$\Gamma\colon \mathbf{qCoh}_X \longrightarrow {}_{\Gamma(X,\mathcal{O}_X)}\mathbf{Mod}$$

is exact (in which case it is an equivalence).

Definition 1.2.1. *A smooth variety X is **D-affine** if the functor*

$$\Gamma\colon \mathbf{qCoh}_{D_X} \longrightarrow {}_{\Gamma(X,D_X)}\mathbf{Mod}$$

is exact (in which case, by [HTT08, Proposition 1.4.4], it is an equivalence).

Affine spaces are D-affine; this allows us to think of D-modules on an affine space X as modules over $\Gamma(X, D_X)$. Therefore, by an abuse of notation, if X is affine then the symbol D_X will simultaneously stand for the coherent sheaf and the algebra of global sections.

Theorem 1.2.1. *(Beilinson–Bernstein [BB81]) Let G be a connected, simply connected, complex algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$. Let B be a Borel inside G and $X = G/B$ the flag variety. Then X is D-affine and there is a surjective map*

$$U(\mathfrak{g}) \longrightarrow \Gamma(X, D_X),$$

whose kernel is generated by the kernel of the central character, $\chi \in Z(\mathfrak{g})$, which kills the trivial module $\mathbb{C}$.

1.2.3 The Induced Action on Differential Operators

An action of an algebraic group, G, on a smooth complex variety, X, extends to an action on $\mathbb{C}[X]$ via

$$(g \cdot f)(x) := f(g^{-1} \cdot x)$$

for all $g \in G$, $f \in \mathcal{O}_X$ and $x \in X$. It also extends to an action on D_X via the formula

$$(g \cdot \theta)(f) := g \cdot (\theta(g^{-1} \cdot f))$$

for all $g \in G$, $f \in \mathcal{O}_X$ and $\theta \in D_X$.

1.2.4 The Fourier Transform and the Anti-Isomorphism $(-)^{\mathrm{op}}$

Let $V = \langle x_1, \ldots, x_n\rangle_{\mathbb{C}}$ be a vector space and choose a dual basis, $\{y_1, \ldots, y_n\}$ for V^*. For each $i = 1, \ldots, n$, let $\partial_i := \frac{\partial}{\partial x_i}$ and $\hat{\partial}_i := \frac{\partial}{\partial y_i}$. Then

$$D_V = \frac{\mathbb{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle}{\langle [\partial_i, x_j] = \delta_{ij},\ [\partial_i, \partial_j] = [x_i, x_j] = 0\rangle}$$

Define a ring isomomorphism, called the **Fourier transform**,

$$\mathcal{F}_{\mathsf{T}} \colon D_V \longrightarrow D_{V^*}; \quad x_i \mapsto \hat{\partial}_i, \quad \partial_i \mapsto -y_i.$$

Likewise, the mapping

$$(-)^{\mathrm{op}} \colon D_V \longrightarrow D_{V^*}; \quad x_i \mapsto \partial_i, \quad \partial_i \mapsto y_i,$$

extends to an anti-isomorphism of algebras and, after identifying $(V^*)^* \cong V$, $(-)^{\mathrm{op}} \circ (-)^{\mathrm{op}} = \mathrm{id}$.

1.2.5 Good Filtrations and Characteristic Varieties

Let M be a D-module. A **filtration** on M is an ascending sequence of quasi-coherent O_X-modules,

$$\cdots \subseteq F^i(M) \subseteq F^{i+1}(M) \subseteq \cdots$$

such that

(i) $F^i(M) = 0$ for $i \ll 0$,

(ii) $\bigcup_{i \in \mathbb{Z}} F^i(M) = M$ and

(iii) for any open set $U \subset X$, $F^i(D_X(U))F^j(M(U)) \subseteq F^{i+j}(M(U))$,

where D_X is filtered by the operator filtration. A filtration on M is called **good** if, for all open sets $U \subset X$, $\mathrm{gr}(M(U))$ is a finitely generated $\mathrm{gr}(D_X(U)) = \pi_* O_{T^*X}$-module.

Proposition 1.2.1. *Any coherent D-module admits a good filtration.*

Proof. This is [HTT08, Theorem 2.1.3]. □

Let M be a D-module with a good filtration. The **characteristic variety** of M is the subvariety of T^*X defined by

$$\mathrm{ch}(M) := \mathrm{Supp}(\mathrm{gr}(M)).$$

By [Ber, Proposition 8], this construction is independent of the choice of good filtration. When X is a vector space the characteristic variety is $V(\mathrm{Ann}_{\mathbb{C}[T^*X]}(\mathrm{gr}(M)))$.

1.2.6 Bernstein's Inequality and Holonomic D-Modules

Let $n = \dim X$. The dimension of the characteristic variety is bounded above by the dimension of the cotangent bundle it sits inside. A lower bound for the dimension is given by the Bernstein inequality [Ber, Theorem 10]

$$n \leq \dim \mathrm{ch}(M) \leq 2n.$$

A D-module, M, for which $\dim \mathrm{ch}(M) = n$ is called **holonomic**. If M is coherent as an O_X–module then $\mathrm{ch}(M) \subseteq X \subset T^*X$ so M is holonomic.

1.3 Rational Cherednik Algebras

The purpose of almost all the work in this thesis is, in one way or another, to better understand the module category of rational Cherednik algebras in certain cases. This section begins by defining them and then turns into an exposition of the strategies people have developed to understand them better. Rational Cheredik algebras are dependent on the data of a complex reflection group and a set of parameters.

1.3.1 Complex Reflection Groups

Definition 1.3.1. *Let W be a finite group acting on a finite-dimensional complex vector space $\mathfrak{h}$. A non-trivial element $s \in W$ is called a* ***complex reflection*** *if it fixes a hyperplane in $\mathfrak{h}$ pointwise. If W is generated by complex reflections then the pair $(W, \mathfrak{h})$ is called a* ***complex reflection group****.*

In 1954, Shephard and Todd classified all complex reflection groups, [ST54]. They showed that complex reflection groups fall into either an infinite family denoted $G(l, p, n)$, where l, p and n are certain integers, and a list of thirty-four exceptional cases. This thesis deals with complex reflection groups which belong to the infinite family. Let $l, n \in \mathbb{N}$ not both one, then the wreath product $\mu_l \wr \mathfrak{S}_n$ is denoted $G(l, 1, n)$; here, l is called the **degree** and n the **rank**. Within this collection lie the following families of reflection groups that will be of particular interest later.

- The symmetric group of degree n, $\mathfrak{S}_n = G(1, 1, n)$.
- The cyclic group of order l, $\mu_l = G(l, 1, 1)$.
- The Weyl group of type B, $B_i = G(2, 1, i)$.

Also of interest will be the dihedral groups of order $2d$, $I_2(d)$. In the classification these are the complex reflection groups, $G(d, d, 2)$ (which haven't been defined here for $d > 1$), and so are all rank two.

1.3.2 Rational Cherednik Algebras

Let $(W, \mathfrak{h})$ be a finite complex reflection group and define the following notation.

$$\mathcal{S} := \{\text{complex reflections in } W\}.$$

Given a hyperplane $H \subset \mathfrak{h}$, define

$$\begin{aligned} W_H &:= \text{ the pointwise stabiliser of } H \text{ in } W \\ v_H &:= \text{ a vector that spans a } W_H\text{-stable complement to } H \subset \mathfrak{h}, \\ \alpha_H &:= \text{ a linear functional in } \mathfrak{h}^* \text{ with kernel } H. \end{aligned}$$

For each reflection $s \in \mathcal{S}$, define,

$$\begin{aligned} H_s &:= \ker(1 - s) \subset \mathfrak{h}, & \mathcal{E} &:= \{H_s \subset \mathfrak{h} \,|\, s \in \mathcal{S}\}. \\ v_s &:= v_{H_s}, & \alpha_s &:= \alpha_{H_s} \end{aligned}$$

Definition 1.3.2. *The group W acts on $\mathcal{E}$ and on $\mathbb{C}W$ by conjugation. With respect to these actions, let $\gamma_{(-)} \colon \mathcal{E} \longrightarrow \mathbb{C}W; H \mapsto \gamma_H$ be a W-equivariant map such that, for each $H \in \mathcal{E}$, $\gamma_H \in \mathbb{C}W_H$ and the trace of γ_H, acting on the $\mathbb{C}W$-module $\mathbb{C}W_H$, is zero. The **Rational Cherednik algebra**, $H(W, \mathfrak{h}, \gamma)$, associated to the data $(W, \mathfrak{h}, \gamma)$ is defined to be the quotient of the smash product of the group, W, with the tensor algebra $T_{\mathbb{C}}(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations*

$$\begin{aligned} &[x, x'] = [y, y'] = 0 \\ &[y, x] = x(y) + \sum_{H \in \mathcal{E}} \frac{\alpha_H(y) x(v_H)}{\alpha_H(v_H)} \gamma_H \qquad \textit{for all } y, y' \in \mathfrak{h} \textit{ and } x, x' \in \mathfrak{h}^*. \end{aligned}$$

There are two ways to parametrise the map $\gamma_{(-)}$ which defines the Cherednik algebra. The first is by parameters indexed by conjugacy classes of reflections in $\mathcal{S}$, write these $\mathbf{c} := \{c_s \,|\, s \in \mathcal{S}\}$, so that $c_s = c_t$ whenever s and t are conjugate.

Convention. *Following [Chm06] (where she writes $\mathbf{k}(s)$ for c_s), given $\{c_s \in \mathbb{C} \,|\, s \in S\}$, such that $c_s = c_t$ whenever s and t are conjugate, define*

$$\gamma_H := -2 \sum_{s \in W_H \setminus \{1\}} c_s s.$$

A second way to parametrise $\gamma_{(-)}$ is by parameters, $\mathbf{k} := \{k_{H,i} \,|\, H \in \mathcal{E}, i = 0, \ldots, |W_H|\}$, such that $k_{H,0} = k_{H,|W_H|} = 0$ for all $H \in \mathcal{E}$. Then define

$$\gamma_H := \sum_{w \in W_H \setminus \{1\}} \left(\sum_{j=0}^{|W_H|-1} \det(w)^j (k_{H,j+1} - k_{H,j}) \right) w.$$

1.3.3 Parametrisations in the Literature for the Cherednik Algebra of $G(l, 1, n)$

The following notation is chosen to agree with [Val07, Definition 1.4.1]. Let $W = G(l, 1, n)$ and choose a basis $\mathfrak{h} = \langle y_1, \ldots, y_n \rangle_{\mathbb{C}}$ and a dual basis $\mathfrak{h}^* = \langle x_1, \ldots, x_n \rangle_{\mathbb{C}}$. There are two types of reflections. The first kind, called *cyclic*, are attached to μ_l. Let $\zeta = e^{\frac{2\pi\sqrt{-1}}{l}}$, a primitive l^{th} root of unity, $1 \leq i \neq j \leq n$ and $1 \leq t \leq l-1$, then the **cyclic reflection**, s_i^t, is defined by

$$s_i^t(y_i) = \zeta^t y_i \qquad\qquad s_i^t(y_j) = y_j.$$

The second kind, the **non-cyclic reflections**, are attached to the datum $\mathfrak{S}_n$. Let $0 \leq t \leq l-1$, $1 \leq i < j \leq n$ and $1 \leq k \leq n$ with $k \neq i, j$, then the reflection σ_{ij}^t is defined by

$$\sigma_{ij}^t(y_i) = \zeta^{-t} y_j \qquad\qquad \sigma_{ij}^t(y_j) = \zeta^t y_i \qquad\qquad \sigma_{ij}^t(y_k) = y_k.$$

For $j < i$, define

$$\sigma_{ij}^t := \sigma_{ji}^{-t}.$$

The reflection parameters

Now,

$$\mathcal{S} = \left\{ s_k^t, \sigma_{ij}^{t'} \,\middle|\, t = 1, \ldots, l-1,\ t' = 0, \ldots, l-1,\ 1 \leq i < j \leq n,\ k = 1, \ldots, n,\ k \neq i, j \right\},$$

but there are exactly l conjugacy classes of reflections corresponding to the parameters denoted as follows.

$$\begin{aligned} c_0 &:= c_{\sigma_{ij}^t} && \text{for all } i, j, t, \\ c_t &:= c_{s_k^t} && \text{for all } 1 \leq t \leq l-1 \text{ and all } k. \end{aligned}$$

The reflection parameters are $\mathbf{c} := (c_0, \ldots, c_{l-1})$. The corresponding Cheredik algebra is written $H_{\mathbf{c}}(W)$. When it is clear from the context which complex reflection group W is being used the Cherednik algebra will sometimes be written $H_{\mathbf{c}}$.

The hyperplane parameters

For each $i = 1, \ldots, n$ let H_i be the hyperplane corresponding to s_i^t for all t. Let H denote the hyperplane corresponding to any reflection σ_{ij}^t. Now define

$$\begin{aligned} k_{00} &:= k_{H,1} && \\ k_j &:= k_{H_i,j} && \text{for any } i \text{ and } 0 \leq j \leq l. \end{aligned}$$

It follows that $k_0 = k_l = 0$. The hyperplane parameters are $\mathbf{k} := (k_{00}, k_1, \ldots, k_{l-1})$, and the relationship between the two sets of parameters is as follows. For $1 \leq i \neq j \leq n$ and $1 \leq t \leq l-1$,

$$c_t := c_{s_i^t} = -\frac{1}{2} \sum_{p=0}^{l-1} (k_{p+1} - k_p) \zeta^{pt} \qquad\qquad c_0 := c_{\sigma_{ij}^t} = -k_{00}$$

$$k_{t+1} - k_t = -\frac{2}{l} \sum_{p=1}^{l-1} \zeta^{-pt} c_p$$

The commutation relations for the Cherednik algebra are now

$$[y_i, x_i] = 1 + \sum_{p=0}^{l-1}(k_{p+1} - k_p) \sum_{t=0}^{l-1} \zeta^{pt} s_i^t + k_{00} \sum_{j \neq i} \sum_{t=0}^{l-1} \sigma_{ij}^{(t)} \qquad (1 \leq i \leq n),$$

$$[y_i, x_j] = -k_{00} \sum_{t=0}^{l-1} \zeta^t \sigma_{ji}^t \qquad (1 \leq i \neq j \leq n).$$

Table A.1 in Appendix A gives a comparison of the differing conventions in the literature.

1.3.4 Properties of Rational Cherednik Algebras

There is a filtration, $\mathcal{F}$, of $H_{\mathbf{c}}(W)$, defined by putting the vectors of $\mathfrak{h}$ and $\mathfrak{h}^*$ in degree one and those of $\mathbb{C}W$ in degree zero. Note that this does not produce a grading on $H_{\mathbf{c}}(W)$ because the commutation relations aren't homogenous.

Proposition 1.3.1. *The following properties hold.*

(i) With respect to the filtration, $\mathcal{F}$,

$$\mathrm{gr}(H_{\mathbf{c}}(W)) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W.$$

(ii) Multiplication gives a vector space isomorphism

$$H_{\mathbf{c}}(W) \overset{\text{v.s.}}{\cong} \mathbb{C}[\mathfrak{h}] \otimes_k \mathbb{C}W \otimes_k \mathbb{C}[\mathfrak{h}^*].$$

(iii) The ring $H_{\mathbf{c}}(W)$ is noetherian and has finite global dimension.

Proof. The first two parts follow from the PBW Theorem, [EG02, Theorem 1.3]. The third is proved, for example, in [Bro03, Theorem 4.4]. □

Let $(W, \mathfrak{h})$ be a complex reflection group with dual bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ for $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. Define the **grading element** by

$$\mathsf{h} := -\sum_i (x_i y_i + y_i x_i).$$

Note that this is the negative of the grading element used in [GGOR03]. This is because the introduction of the negative sign makes the standard modules into highest—rather than lowest—weight modules. Now, for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$ and $w \in W$,

$$[\mathsf{h}, x] = -x \qquad [\mathsf{h}, y] = y \qquad [\mathsf{h}, w] = 0.$$

This produces an inner grading on $H_{\mathbf{c}}$—that is, $(H_{\mathbf{c}})_i = \{a \in H_{\mathbf{c}} \,|\, [\mathsf{h}, a] = ia\}$— with elements of $\mathfrak{h}^*$ in degree -1, $\mathbb{C}W$ in degree 0 and elements of $\mathfrak{h}$ in degree 1.

1.3.5 Relationship with Differential Operators

There is more to the PBW Theorem above. Each tensorand, $\mathbb{C}[\mathfrak{h}]$, $\mathbb{C}W$ and $\mathbb{C}[\mathfrak{h}^*]$ forms a subalgebra of $H_{\mathbf{c}}$. The rational Cherednik algebra can be described as the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$ generated by three collections of endomorphisms: The action by multiplication by $f \in \mathfrak{h}^* \subset \mathbb{C}[\mathfrak{h}]$ generates $\mathbb{C}[\mathfrak{h}]$; the action by W on $\mathfrak{h}$ (part of the data of a complex reflection group) extended to $\mathbb{C}[\mathfrak{h}]$ generates $\mathbb{C}W$; and, for each $y \in \mathfrak{h} \subset \mathbb{C}[\mathfrak{h}^*]$, the action by the **Dunkl operator**,

$$T_y := \partial_y + \sum_{H \in \mathcal{E}} \frac{\alpha_H(y)}{\alpha_H} \sum_{i=0}^{|W_H|-1} \sum_{w \in W_H} k_{H,j} \det{}^j(w) w \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}])$$

generates $\mathbb{C}[\mathfrak{h}^*]$. The Dunkl operators commute ([EG02, Proposition 4.5]) and the mapping $y \mapsto T_y$ extends to an injective algebra homomophism called the **Dunkl representation**,

$$\Theta\colon H_{\mathbf{c}} \longrightarrow D_{\mathfrak{h}^{\mathrm{reg}}} \rtimes W,$$

where $\mathfrak{h}^{\mathrm{reg}} := \mathfrak{h} \setminus \bigcup_{H \in \mathcal{E}} H$ and $D_{\mathfrak{h}^{\mathrm{reg}}}$ is the algebra of polynomial differential operators on $\mathfrak{h}^{\mathrm{reg}}$. When $\mathbf{c} = 0$ the terms of the Dunkl operators with negative powers of α_s disappear and the image of Θ is precisely the subalgebra $D_{\mathfrak{h}} \rtimes W$ and this algebra is a deformation of $\mathrm{gr}(H_{\mathbf{c}})$.

1.4 The Spherical Subalgebra

Let $e := \frac{1}{|W|}\sum_{w \in W} w$ be the trivial idempotent of W. The *spherical Cherednik algebra* is the subalgebra $U_{\mathbf{c}} := eH_{\mathbf{c}}e$ of $H_{\mathbf{c}}$. The spherical subalgebra $U_{\mathbf{c}}$ inherits the filtration from $H_{\mathbf{c}}$ and

$$\mathrm{gr}(U_{\mathbf{c}}) = e\mathrm{gr}(H_{\mathbf{c}})e = e\,(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \rtimes W)\,e \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W.$$

The ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ is a noetherian domain so $U_{\mathbf{c}}$ is a noetherian domain. $U_{\mathbf{c}}$ can be thought of as a deformation over U_0. The restriction of Θ to $U_{\mathbf{c}}$ gives an injective map

$$\Theta|_{U_{\mathbf{c}}}\colon U_{\mathbf{c}} \longrightarrow D^W_{\mathfrak{h}^{\mathrm{reg}}} \cong D_{\mathfrak{h}^{\mathrm{reg}}/W}.$$

Definition 1.4.1. *The parameter* $\mathbf{c}$ *is called* ***spherical*** *if* $H_{\mathbf{c}}eH_{\mathbf{c}} = H_{\mathbf{c}}$*—**aspherical** means not spherical.*

There are functors,

$$I_{\mathbf{c}}\colon {}_{H_{\mathbf{c}}}\mathbf{mod} \longrightarrow {}_{U_{\mathbf{c}}}\mathbf{mod}; \qquad M \mapsto eM$$

and

$$J_{\mathbf{c}}\colon {}_{U_{\mathbf{c}}}\mathbf{mod} \longrightarrow {}_{H_{\mathbf{c}}}\mathbf{mod}; \qquad M \mapsto H_{\mathbf{c}}e \otimes_{U_{\mathbf{c}}} M.$$

Theorem 1.4.1. *The following are equivalent.*

(i) The parameter, $\mathbf{c}$*, is spherical.*

(ii) The spherical subalgebra, $U_{\mathbf{c}}$*, has finite global dimension.*

(iii) The functors $I_{\mathbf{c}}$ *and* $J_{\mathbf{c}}$ *are mutally inverse equivalences.*

Proof. This is proved in [Eti12, Theorem 5.5]. □

For the complex reflection groups $G(l, 1, n)$, Dunkl and Griffeth give a complete characterisation of aspherical values of $\mathbf{c}$. (See Table A.1 for the conversion between notation.)

Theorem 1.4.2. *([DG10, Theorem 3.3]) Let* $W = G(l, 1, n)$ *so that* $\mathbf{c} = (c_0, \ldots, c_{l-1})$*. Let* ζ *be a primitive* l^{th} *root of unity. Determine numbers* $d_1, \ldots, d_{l-1}$ *from the equations,*

$$\sum_{j=0}^{l-1} (d_{j-1} - d_j)\zeta^{-jt} = -2lc_t,$$

for $t = 1, \ldots, l-1$*. Given a box,* b*, in a Young diagram, let* $\mathrm{ct}(b)$ *denote the content, see Section 3.1 for definitions of these. The set of aspherical values* $\mathbf{c}$ *for* $H_{\mathbf{c}}(W)$ *is the union of the hyperplanes* $c_0 = -k/m$ *for integers* $1 \leq k < m \leq n$ *and hyperplanes*

$$k = d_i - d_{i-k} + \mathrm{ct}(b)lc_0,$$

where b *is the lower right-hand corner box of a rectangular Young diagram with at most* n *boxes,* $0 \leq i \leq l-1$*,* $k \neq 0 \bmod l$ *and* $1 \leq k \leq i + (\mathrm{row}(b) - 1)l$*.*

1.5 Category $\mathcal{O}$ for Cherednik Algebras

As in the case of ${}_{U(\mathfrak{g})}\mathbf{Mod}$, for a semisimple Lie algebra $\mathfrak{g}$, the category ${}_{H_\mathbf{c}}\mathbf{Mod}$ is often too large for practical purposes. One solution is to impose finiteness conditions, which, in the case of $\mathfrak{g}$, give the BGG category $\mathcal{O}(\mathfrak{g})$ (see [Hum08] for more on this). A similar construction in the case of Cherednik algebra modules enjoys many of the same good properties.

Definition 1.5.1. *Let $\mathcal{O}_\mathbf{c}(W)$ denote the full subcategory of left $\mathbf{H}_\mathbf{c}(W)$-modules, M, such that the following two conditions hold.*

1. *M is finitely generated as a $\mathbb{C}[\mathfrak{h}]$-module.*
2. *M is locally nilpotent over $\mathbb{C}[\mathfrak{h}^*]$. That is, for any $m \in M$, $\mathbb{C}[\mathfrak{h}^*]m$ is a finite-dimensional subspace of M on which $\mathfrak{h}$ acts nilpotently.*

When the dependence on the underlying reflection group or parameters, $\gamma\colon \mathcal{E} \longrightarrow \mathbb{C}W$, is emphasised the category $\mathcal{O}$ will be written $\mathcal{O}(W, \mathfrak{h}, \gamma)$. Note that, by definition,

$$\mathrm{Hom}_{\mathcal{O}_\mathbf{c}}(M, N) = \mathrm{Hom}_{H_\mathbf{c}}(M, N),$$

for all $M, N \in \mathcal{O}_\mathbf{c}$.

Let $\rho\colon W \longrightarrow \mathrm{End}(V_\rho)$ be an irreducible complex representation of W. Let $\mathfrak{h}$ act as zero on V_ρ, so that V_ρ becomes a left $\mathbb{C}[\mathfrak{h}^*] \rtimes W$-module. The **standard** module attached to ρ is defined by

$$\Delta(\rho) := H_\mathbf{c}(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes \mathbb{C}W} V_\rho.$$

By the PBW Theorem, there is an isomorphism of left $\mathbb{C}[\mathfrak{h}] \rtimes W$-modules

$$\Delta(\rho) \xrightarrow{\simeq} \mathbb{C}[\mathfrak{h}] \otimes_\mathbb{C} V_\rho$$

where W acts diagonally on the tensor product.

Lemma 1.5.1. *For any non-zero $v \in V_\rho$, $1 \otimes v$ is an eigenvector and all such eigenvectors share a common eigenvalue, denoted c_ρ.*

Proof. Choose dual bases $\{x_i, y_i \mid i = 1, \dots, n\}$ for $\mathfrak{h}^*$ and $\mathfrak{h}$ as in the definition of the grading element. Then since the y_i's act as zero by definition,

$$\mathsf{h} \cdot 1 \otimes v = -\sum_{i=1}^{n} y_i x_i \otimes v = -\sum_{i=1}^{n} [y_i, x_i] \otimes v = -1 \otimes \sum_{i=1}^{n} [y_i, x_i] \cdot v.$$

Now, for each $t = 0, \dots, l-1$, the elements $\sum_{i=1} s_i^t$ and $\sum_{1\le i<j\le n} \sigma_{ij}^t + \sigma_{ji}^t$ commute with the complex reflections and these generate the group W. Thus, the element $\sum_{i=1}^n [y_i, x_i]$ belongs to the centre of the group algebra, $Z(\mathbb{C}W)$. Now, because ρ is an irreducible complex representation of W, Schur's Lemma implies that $\sum_{i=1}^n [y_i, x_i]$ acts as a scalar matrix on V_ρ. □

Now $\Delta(\rho)$ decomposes into h–weight spaces with highest weight c_ρ. Since the action of $\mathfrak{h}$ increases the degree, each of the standard modules are locally nilpotent and so $\Delta(\rho) \in \mathcal{O}_\mathbf{c}$. Let $\Lambda = \mathrm{Irr}(W)$ and define a partial ordering on Λ by

$$\rho < \lambda \Leftrightarrow c_\lambda - c_\rho \in \mathbb{Z}_{>0}.$$

1.5.1 The Naive Duality

Let $(-)^d\colon H(\mathfrak{h}, \gamma) \longrightarrow H(\mathfrak{h}^*, \gamma^d)^{\mathrm{op}}$ be the isomorphism defined by $y \mapsto -y$, $x \mapsto x$ and $w \mapsto w^{-1}$ for all $y \in \mathfrak{h}$, $x \in \mathfrak{h}^*$ and $w \in W$ (see [GGOR03, Section 4.2]). Given $M \in \mathcal{O}(\mathfrak{h}, \gamma)$, there is a right action of H on $\mathrm{Hom}_\mathbb{C}(M, \mathbb{C})$ by $(f \cdot h)(m) := f(hm)$ for $h \in H$, $f \in \mathrm{Hom}(M, \mathbb{C})$ and $m \in M$. Define $M^\vee$ to be the subset of elements in $\mathrm{Hom}_\mathbb{C}(M, \mathbb{C})$ on which $\mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently; it is a $\mathbb{C}$-vector space. Define a left action of $H(\mathfrak{h}^*, \gamma^d)$ on $M^\vee$ by

$$(x \cdot f)(m) := f(xm) \qquad (y \cdot f)(m) := -f(ym) \qquad (w \cdot f)(m) := f(w^{-1}m)$$

for all $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $w \in W$, $f \in M^\vee$ and $m \in M$ (see [GGOR03, Section 4.2]). This restricts to an equivalence

$$(-)^\vee : \mathcal{O}(\mathfrak{h}, \gamma) \longrightarrow \mathcal{O}(\mathfrak{h}^*, \gamma^d)^{\mathrm{op}}.$$

This is know as the **naive duality** for category $\mathcal{O}$.

Let ρ be an irreducible representation of W. Define the **costandard module**,

$$\nabla(\rho) := \{f \in \mathrm{Hom}_{\mathbb{C}[\mathfrak{h}] \rtimes W}(H, \rho) \,|\, \mathbb{C}[\mathfrak{h}^*] \text{ acts locally nilpotently}\}.$$

This is a cyclic $\mathbb{C}[\mathfrak{h}^*]$-module and so, by the PBW Theorem, is finitely generated as a left H-module. It follows that $\nabla(\rho)$ lies in $\mathcal{O}$.

Category $\mathcal{O}$ has many good properties as a result of the following theorem. Both the definition of a highest weight category with duality and the proof will be discussed in Section 1.6.

Theorem 1.5.1. *Let W be a complex reflection group with $\Lambda = \mathrm{Irr}(W)$ the poset with respect to the partial ordering $<$. Then $(\mathcal{O}_{\mathbf{c}}(W), \Lambda)$ is highest weight category with duality $(-)^\vee$ with the costandard modules defined above.*

This has the following consequences.

Corollary 1.5.1. *(i) For each irreducible representation $\lambda \in \Lambda$, $\Delta(\lambda)$ has a unique irreducible quotient $L(\lambda)$ and the set $\{L(\lambda) \,|\, \lambda \in \Lambda\}$ is a complete list of non–isomorphic simple modules in $\mathcal{O}_{\mathbf{c}}$.*

(ii) For each $\lambda \in \Lambda$ there is a indecomposable projective cover $P(\lambda)$ with a filtration by standard objects such that

$$(P(\lambda)\,;\Delta(\mu)) = [\Delta(\mu) : L(\lambda)].$$

(iii) For a generic choice of parameter, $\mathbf{c}$, the category $\mathcal{O}_{\mathbf{c}}$ is semisimple.

(iv) Every finite-dimensional left $H_{\mathbf{c}}$-module is in $\mathcal{O}_{\mathbf{c}}$.

Proof. The first two are direct consequences of Proposition 1.6.2, the third property follows from the second, given that, for a generic value of $\mathbf{c}$, $c_\lambda - c_\mu \in \mathbb{Z}$ if and only if $\lambda = \mu$ and the fourth property is true because, by decomposing the module into finitely many generalised weight spaces, one sees that the spectrum is finite and therefore bounded; so $\mathbb{C}[\mathfrak{h}^*]$ must act nilpotently. □

Say that a parameter $\mathbf{c}$ is **regular** if $\mathcal{O}_{\mathbf{c}}$ is semisimple.

Proposition 1.5.1. *([GGOR03, Lemma 2.9]) Let $\rho, \mu \in \mathrm{Irr}(W)$. If $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(\rho), \Delta(\mu)) \neq 0$ then $c_\mu - c_\rho$ is a positive integer.*

1.5.2 Hecke Algebras and the Knizhnik–Zamolodchikov Functor

Let $(W, \mathfrak{h})$ be a complex reflection group with a point $p \in \mathfrak{h}^{\mathrm{reg}} = \mathfrak{h} \setminus \bigcup_{H \in \mathcal{E}} H$. Following [GGOR03, Section 5.2.5], the **complex Hecke algebra** associated to W, denoted $\mathcal{H}_{\mathbf{c}}(W)$, is the quotient of the complex group algebra of the braid group, $B_W := \pi_1(\mathfrak{h}^{\mathrm{reg}}/W, p)$, by relations of the form

$$(T-1) \prod_{j=1}^{|W_H|-1} \left(T - \det(s)^{-j} e^{2\pi\sqrt{-1} k_{H,j}}\right) = 0,$$

for $H \in \mathcal{E}$, $s \in W$ the reflection around H with non-trivial eigenvalue, $e^{\frac{2\pi\sqrt{-1}}{|W_H|}}$, and T an s-generator of the monodromy around H. For almost[1] all complex reflection groups, $\dim \mathcal{H}_{\mathbf{c}} = |W|$. When the dependence on the underlying reflection group or parameters, $\gamma : \mathcal{E} \longrightarrow \mathbb{C}W$, is emphasised it will be written $\mathcal{H}(W, \gamma)$.

The KZ-functor turns modules in $\mathcal{O}_{\mathbf{c}}$ into modules over $\mathcal{H}_{\mathbf{c}}$. It is defined first on standard modules in $\mathcal{O}_{\mathbf{c}}$; by Corollary 1.5.1, for generic choices of $\mathbf{c}$ every module is a direct sum of these.

[1] it is still conjectural for some of the exceptional complex reflection groups [GGOR03].

[GGOR03, Theorem 5.13] then uses Tits Deformation Theorem to prove that it is defined for all objects for arbitrary $\mathbf{c}$. Let λ be an irreducible left $\mathbb{C}W$ module. The action of $H_{\mathbf{c}}$ on $\Delta(\lambda)$ extends to an action of $D_{\mathfrak{h}^{\text{reg}}} \rtimes W$ on the module

$$\Delta^{\text{reg}}(\lambda) := (D_{\mathfrak{h}^{\text{reg}}} \rtimes W) \otimes_{H_{\mathbf{c}}} \Delta(\lambda).$$

By the PBW Theorem, as a $\mathbb{C}[\mathfrak{h}^{\text{reg}}] \rtimes W$-module this is isomorphic to $\mathbb{C}[\mathfrak{h}^{\text{reg}}] \otimes \lambda$ and so is a trivial W-equivariant vector bundle on $\mathfrak{h}^{\text{reg}}$ of rank $\dim \lambda$. The action of the Dunkl operators gives a flat connection on $\Delta^{\text{reg}}(\lambda)$ (see [GGOR03, Proposition 5.7]). Let $\mathsf{KZ}(\Delta(\lambda))$ denote its horizontal sections. The braid group, B_W acts on the sections via monodromy and this action satisfies the complex Hecke algebra relations. This extends to a functor

$$\mathsf{KZ}\colon \mathcal{O}_{\mathbf{c}} \longrightarrow {}_{\mathcal{H}_{\mathbf{c}}}\mathbf{mod}.$$

Let $\mathcal{O}^{\text{tor}}$ denote the kernel of the localisation functor above.

Theorem 1.5.2. *The KZ functor has the following properties.*

(i) The category $\mathcal{O}^{\text{tor}}$ is the full subcategory of modules in $\mathcal{O}$ that are supported on the union of hyperplanes, $\mathfrak{h} \setminus \mathfrak{h}^{\text{reg}}$.

(ii) The KZ functor factors through an equivalence, $\mathcal{O}/\mathcal{O}^{\text{tor}} \xrightarrow{\simeq} {}_{\mathcal{H}_{\mathbf{c}}}\mathbf{mod}$, described above as considering the monodromy action of B_W; in this way, $\mathcal{O}^{\text{tor}} = \ker \mathsf{KZ}$.

*(iii) The KZ functor is **essentially surjective**: surjective on objects.*

(iv) The KZ functor is exact. Indeed, there is a projective module, denoted P_{KZ}, such that $\mathsf{KZ} = \mathrm{Hom}_{\mathcal{O}}(P_{\mathsf{KZ}}, -)$.

(v) (The Double Centraliser Property) Let P be a progenerator for $\mathcal{O}_{\mathbf{c}}$, then $\mathcal{O}_{\mathbf{c}} \cong \mathrm{End}_{\mathcal{O}}(P)^{\text{op}}$ and so $\mathrm{End}_{\mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}})}(\mathsf{KZ}(P))$ is Morita equivalent to $\mathcal{O}_{\mathbf{c}}$.

Proof. The first four properties follow from [GGOR03, Theorems 5.13–5.14]. The fifth is [GGOR03, Theorem 5.16]. □

Ginzburg, Guay, Opdam and Rouquier also give conditions for certain projective covers to appear as summands of P_{KZ}.

Theorem 1.5.3. *([GGOR03, Proposition 4.7 and Proposition 5.21]) Let $\rho \in \mathrm{Irr}(W)$. The following are equivalent*

(i) $P(\rho)$ is a summand of P_{KZ}

(ii) There exists some $\lambda \in \mathrm{Irr}(W)$ such that $L(\rho)$ is a submodule of $\Delta(\lambda)$.

(iii) $P(\rho) \cong I(\rho)$ the injective hull of $L(\rho)$.

The following two results will be useful later.

Proposition 1.5.2. *Let $\mathbf{c}\colon \mathcal{S} \longrightarrow \mathbb{C}$ be a class function and $\chi\colon W \longrightarrow \mathbb{C}^*$ be a character of W. Define the class function $\chi(\mathbf{c})\colon \mathcal{S} \longrightarrow \mathbb{C}$ by $\chi(\mathbf{c})(s) := \chi(s)(\mathbf{c}(s))$ for all $s \in \mathcal{S}$. There is an isomorphism of algebras, $H_{\mathbf{c}}(W) \cong H_{\chi(\mathbf{c})}(W)$, which is the identity on $\mathfrak{h}$ and $\mathfrak{h}^*$ and sends $w \in W$ to $\chi(w)w \in W$. This then gives an equivalence of categories*

$$\mathcal{O}_{\mathbf{c}} \simeq \mathcal{O}_{\chi(\mathbf{c})}.$$

Proof. Let $y \in \mathfrak{h}$ and $x \in \mathfrak{h}^*$. Then, under the mapping described above, $[y, x] \in H_{\mathbf{c}}(W)$ is sent to

$$x(y) + \sum_{s \in \mathcal{S}} \frac{\alpha_s(y)x(v_s)}{\alpha_s(v_s)} c_s \chi(s)s = x(y) + \sum_{s \in \mathcal{S}} \frac{\alpha_s(y)x(v_s)}{\alpha_s(v_s)} \chi(\mathbf{c})(s)s,$$

which is precisely the corresponding relation in $H_{\chi(\mathbf{c})}(W)$. □

Lemma 1.5.2. *Suppose a block of $\mathcal{O}(W)$ contains only one simple module. Then it is semisimple.*

Proof. Let L be the unique simple module in $\mathcal{O}(W)$. Since it is a highest weight category (see the next section), the simple modules do not have self-extensions. A projective cover of L must then be isomorphic to L. □

1.6 Highest Weight Categories

The BGG category $\mathcal{O}$ is a classical object in Lie theory and the inspiration for the Cherednik category $\mathcal{O}$. The essential structure these two categories share is that of a highest-weight category, introduced in [CPS88] by Cline, Parshall and Scott. There are other examples of highest-weight categories too: the module categories of Schur algebras, in particular.

Highest-weight categories have a theory of *standard modules* indexed by *weights* together with a partial ordering. This gives rise to a useful tool called *BGG reciprocity*. The definition is given for an abelian category; first though, three properties need to be defined.

Definition 1.6.1. *A partially ordered set, or **poset**, $(\Lambda, \leq)$, is said to be **locally finite** if for any $x, y \in \Lambda$, the set*

$$\{z \in \Lambda \mid x \leq z \text{ and } z \leq y\}$$

is finite.

Definition 1.6.2. *An abelian category $\mathbf{C}$ over $\mathbb{C}$, is **locally artinian** if it admits arbitrary direct limits of subobjects and every object is a union of its subobjects of finite length.*

Definition 1.6.3. *A locally artinian category $\mathbf{C}$ satisfies the **Grothendieck condition** if for all subobjects $\{A_\alpha\}$ and B of an object X,*

$$B \cap \left(\bigcup_\alpha A_\alpha \right) = \bigcup_\alpha (B \cap A_\alpha) .$$

Definition 1.6.4. *Let $\mathbf{C}$ be a locally artinian category with enough injectives that satisfies the Grothendieck condition. It is called a **highest weight category** if there is a locally finite poset, Λ, whose elements are called **weights**, subject to the following conditions.*

(HWC1) The weights index an exhaustive set of non-isomorphic simple objects in $\mathbf{C}$,

$$\{L(\lambda) \mid \lambda \in \Lambda\},$$

*(HWC2) The weights index a collection of objects, called **costandard objects**,*

$$\{\nabla(\lambda) \mid \lambda \in \Lambda\},$$

such that there exist embeddings $L(\lambda) \to \nabla(\lambda)$, where all the composition factors[2], $L(\mu)$, of $\nabla(\lambda)/L(\lambda)$ satisfy $\mu < \lambda$.

(HWC3) for all $\mu, \lambda \in \Lambda$, $\dim \mathrm{Hom}_A(\nabla(\lambda), \nabla(\mu))$ and the multiplicity with which $L(\mu)$ appears as a composition factor of $\nabla(\lambda)$ are finite.

*(HWC4) Each $L(\lambda)$ has an **injective hull**, $I(\lambda)$ in $\mathbf{C}$, with an increasing filtration of $I(\lambda)$ called a **costandard filtration**,*

$$0 = F_0(\lambda) \subseteq F_1(\lambda) \subseteq \cdots \subseteq F_r(\lambda) = I(\lambda),$$

such that

(i) $F_1(\lambda) \cong \nabla(\lambda)$,

(ii) for each $n > 1$, $F_n(\lambda)/F_{n-1}(\lambda) \cong \nabla(\mu_n)$ for some $\mu_n > \lambda$,

*If $\mathbf{C}$ is a subcategory of a module category then say that $\mathbf{C}$ is a **highest weight module category**.*

[2] Here, a composition factor of an object in $\mathbf{C}$ is defined to be a composition factor of one of its finite length subobjects.

The class of algebras whose module categories are highest weight with finitely many weights are known as **quasi-hereditary**. These algebras are defined independently of highest weight theory (see [CPS88, Section 3]); their definition is unimportant here. It suffices to remark that finite-dimensional hereditary algebras are quasi-hereditary (see [DR89, Theorem 1]).

Theorem 1.6.1. *([CPS88, Theorem 3.6]) Let A be a finite-dimensional $\mathbb{C}$-algebra. The following are equivalent.*

(i) A is quasi-hereditary.

(ii) A^{op} is quasi-hereditary.

(iii) ${}_A\mathbf{Mod}$ is highest weight.

(iv) ${}_A\mathbf{mod}$ is highest weight.

On the other hand, suppose C is a highest weight module category with finitely many simple modules. Then the full subcategory of finite length modules is equivalent to ${}_A\mathbf{mod}$ for some quasi-hereditary algebra A.

Corollary 1.6.1. *Suppose C is a highest weight module category with a finite poset of weights. Then C has enough projectives and every simple has a projective cover. Futhermore, every projective indecomposable is a projective cover.*

Proof. By the theorem, there is some finite-dimensional, quasi-hereditary algebra A such that the subcategory of finitely generated modules in C, C_f say, is equivalent to ${}_A\mathbf{mod}$. Such a module category has enough projectives: choosing generators for a module is equivalent to defining a surjective map from a free (and so projective) module. If C_f has enough projectives then so does C. Since A is finite-dimensional, ${}_A\mathbf{mod}$ is artinian so projective covers exist. Every projective indecomposable, P say, has a simple quotient, $L(\lambda)$ say. The surjections to $L(\lambda)$ lift respectively to maps in either direction between P and $P(\lambda)$. The defining property for projective covers forces these maps to be isomorphisms. □

This corollary allows one to talk about projective covers for highest weight module categories. Defining the dual notion to costandard modules relies on this. Let C be a highest weight module category with a poset of weights Λ. For each $\lambda \in \Lambda$, let $P(\lambda)$ be the projective cover of $L(\lambda)$. For each $\lambda \in \Lambda$, define the **standard module** $\Delta(\lambda)$ to be the largest quotient of $P(\lambda)$ such that every weight, μ, of every simple composition factor of $\Delta(\lambda)$ satisfies $\mu \leq \lambda$. By definition, $P(\lambda)$ is the projective cover of $\Delta(\lambda)$.

Definition 1.6.5. *Suppose $C \simeq {}_A\mathbf{mod}$ is a highest weight category for some quasi-hereditary algebra A with a finite poset of weights Λ. If there is a contravariant, additive functor δ from C to itself such that*

(i) δ^2 is equivalent to the identity functor,

(ii) δ induces a linear map on the vector spaces $Hom_C(M, N)$ for all $M, N \in C$

(iii) $\delta(\nabla(\lambda)) \cong \Delta(\lambda)$ for all $\lambda \in \Lambda$

then say that C is a ***highest weight category with duality****. A is also called a* ***BGG algebra****.*

Proposition 1.6.1. *Let (C, Λ) be a highest weight category with duality. Then each projective cover has a filtration by standard modules,*

$$P(\lambda) = F'_0(\lambda) \supseteq F'_1(\lambda) \supseteq \cdots \supseteq F'_r(\lambda) = 0,$$

such that

(i) $F_0(\lambda)/F_1(\lambda) \cong \Delta(\lambda)$,

(ii) for each $n > 0$, $F_n(\lambda)/F_{n+1}(\lambda) \cong \Delta(\mu_n)$ for some $\mu_n > \lambda$,

Furthermore, the multiplicities of standard composition factors in any such filtration are unique.

Proof. See [CPS89]. □

The multiplicity of $L(\lambda)$ as a composition factor of $M \in C$ is written $[M : L(\lambda)]$. When M has a filtration by standard modules, the multiplicity of $\Delta(\lambda)$ as a standard composition factor of M is written with a semicolon and parentheses, $(M; \Delta(\lambda))$. This is to avoid confusion in cases when $\Delta(\lambda)$ happens to be simple.

Here is a summary of the properties of highest weight categories that will be useful later.

Proposition 1.6.2. *([CPS88, Lemma 3.2, Lemma 3.5 and Lemma 3.8]) Let (C, Λ) be a highest weight category. Then*

(i) Each $\Delta(\lambda)$ has simple head $L(\lambda)$ which is also the socle of $\nabla(\lambda)$.

(ii) If either $\mathrm{Ext}^n_C(\nabla(\mu), \nabla(\lambda))$ or $\mathrm{Ext}^n_C(L(\mu), \nabla(\lambda))$ is non-zero for some $n > 0$ then $\mu > \lambda$. If $\mathrm{Ext}^1_C(L(\mu), L(\lambda)) \neq 0$ then either $\lambda > \mu$ or $\mu > \lambda$.

(iii) If M and N are finite length then $\mathrm{Ext}^n(M, N)$ is finite-dimensional for $n \leq 1$. If all $\nabla(\lambda)$ have finite length then $\mathrm{Ext}^n(M, N)$ is finite-dimensional for all n.

The following proposition is a consequence of the construction of projective covers; the hypothesis that C is a highest weight category can be weakened.

Proposition 1.6.3. *Let C be a highest weight category with a finite poset of weights Λ. Let $\lambda \in \Lambda$ and let M be a finite length module in C. Then*

$$\dim_k Hom_C(P(\lambda), M) = [M : L(\lambda)].$$

Proof. The module M has a quotient isomorphic to $L(\mu)$ with kernel K for some $\mu \in \Lambda$. This gives a short exact sequence,

$$0 \longrightarrow K \longrightarrow M \longrightarrow L(\mu) \longrightarrow 0.$$

Applying the exact functor, $\mathrm{Hom}_C(P(\lambda), -)$, and taking dimensions gives

$$\dim_k \mathrm{Hom}(P(\lambda), M) = \delta_{\lambda\mu} + \dim_k \mathrm{Hom}_C(P(\lambda), K).$$

Because M is finite length, K, has fewer simple composition factors and an induction gives the result. □

Theorem 1.6.2. *(BGG reciprocity, [CPS88, Theorem 3.11]) Let (C, Λ) be a highest weight category with duality. For any $\lambda, \mu \in \Lambda$,*

$$(P(\mu) ; \Delta(\lambda)) = [\Delta(\lambda) : L(\mu)].$$

Example 1.6.1. *[CPS88, Examples 3.3] show that the category of finitely generated left modules over the k-algebra of upper triangular matrices is a highest weight category as is the BGG category $O(\mathfrak{g})$ for a complex semisimple Lie algebra $\mathfrak{g}$.*

1.6.1 Proof of Theorem 1.5.1

This was proved by Ginzburg–Guay–Opdam–Rouquier in [GGOR03]. First, the poset of weights is finite (because W is a finite group) and so locally finite. Now, [GGOR03, Theorem 2.19] shows that $O(W)$ is a highest weight category and [GGOR03, Proposition 4.7] shows that the naive duality makes it a highest weight category with duality.

1.7 Quiver Representations

Let Q be a quiver with a finite vertex set I and a finite edge set E. Let $h, t \colon E \to I$ be the functions that associate the head and tail vertices to an edge respectively. The **path algebra**, $\mathbb{C}Q$, of Q is the $\mathbb{C}$-algebra defined as follows. Let $\mathbb{C}Q$ be the vector space generated by the set of all paths in Q. If two paths α and β in A have $h(\beta) = t(\alpha)$ then their product, written $\beta\alpha$, is defined to be their concatenation, otherwise $\beta\alpha$ is defined to be zero. This gives $\mathbb{C}Q$ the structure of an associative $\mathbb{C}$-algebra and coincides with the tensor algebra

$$A \cong T_{\mathbb{C}I}(\mathbb{C}E).$$

Let $\delta \in \mathbb{N}^{|I|}$ be an $|I|$-tuple, a so-called **dimension vector**. Then a **representation** of Q with dimension vector δ is a collection of matrices

$$\left\{M_\alpha\colon \mathbb{C}^{\delta_{t(\alpha)}} \longrightarrow \mathbb{C}^{\delta_{h(\alpha)}} \;\middle|\; \alpha \in E\right\}$$

Let $\{M_\alpha \,|\, \alpha \in E\}$ and $\{N_\alpha \,|\, \alpha \in E\}$ be two representations of Q with dimension vectors δ and δ' respectively. A collection of linear maps $\left\{f_i\colon \mathbb{C}^{\delta_i} \longrightarrow \mathbb{C}^{\delta'_i} \;\middle|\; i \in I\right\}$ is a **morphism of representations** of Q if, for all $\alpha\colon i \to j \in E$, $f_j \circ M_\alpha = N_\alpha \circ f_i$.

Together with these morphisms, the set of all representations of Q form the objects of a category, ${}_Q\mathbf{Rep}$. Given a representation, $\{M_\alpha\}$, of Q, let $V = \oplus_{i\in I}\mathbb{C}^{\delta_i}$. Each edge, $\alpha \in E$, defines an endomorphism of V by precomposing M_α with the projection from V and postcomposing with the inclusion into V,

$$V \longrightarrow \mathbb{C}^{\delta_{h(\alpha)}} \longrightarrow \mathbb{C}^{\delta_{t(\alpha)}} \longrightarrow V.$$

This extends to a left action of $\mathbb{C}Q$ on V.

In the other direction, given a left $\mathbb{C}Q$-module V, and an edge α, the action of α by left multiplication restricts to a linear map, M_α, between $\mathbb{C}^{\delta_{h(\alpha)}}$ and $\mathbb{C}^{\delta_{t(\alpha)}}$. These mutually inverse mappings are functorial and define an equivalence

$${}_Q\mathbf{Rep} \xrightarrow{\simeq} {}_{kQ}\mathbf{Mod}.$$

1.7.1 Calculations with Quivers

When Q has no oriented cycles $\mathbb{C}Q$ is finite-dimensional. Each simple is spanned by the trivial path at each vertex which produces a natural bijection between the vertices and irreducible representations. This produces a unique—up to isomorphism and reordering—decomposition of $\mathbb{C}Q$ into left projective indecomposables; the corresponding idempotents are right multiplication (precomposition) by the trivial paths at each vertex. Therefore, each left projective indecomposable has a vector space basis of all paths starting at a particular vertex; they are the left projective cover of the simple to which that vertex corresponds.

It is often more straightforward to do calculations with quiver representations rather than modules over the path algebra. In this spirit, it makes sense to ask *'when is an associative $\mathbb{C}$-algebra the path algebra of some quiver?'* This leads to the idea of an *Ext-quiver* which is a kind of inverse to the construction which takes a quiver and returns its path algebra.

Definition 1.7.1. *The Ext-quiver, Q^e, of a finite dimensional algebra A has vertices corresponding to isomorphism classes of simple left modules (or, equivalently, projective indecomposables). For each pair of vertices representing simples, S_i and S_j say, the number of arrows is equal to* $\dim \mathrm{Ext}^1_A(S_i, S_j)$.

Lemma 1.7.1. *Let P_i be the projective cover of S_i. Then*

$$\mathrm{Ext}^1_A(S_i, S_j) \cong \mathrm{Hom}_A(P_j, \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)).$$

Proof. The module $M := \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)$ is semisimple so can be decomposed into a direct sum $\oplus_k S_k^{m_k}$ for some non-negative numbers m_k. The sum of projective covers $\oplus_k P_k^{m_k}$ surjects onto this module by a map f and this map lifts over the surjection $\mathrm{rad}\colon \mathrm{rad}(P_i) \longrightarrow M$ to some ϕ such that $\mathrm{rad} \circ \phi = f$. Let

$$Q \xrightarrow{\alpha} \oplus_k P_k^{m_k} \xrightarrow{\beta} P_i \longrightarrow S_i \longrightarrow 0$$

be the end of a projective resolution of S_i. Then $\mathrm{Ext}^1(S_i, S_j)$ is the homology of the complex

$$\mathrm{Hom}(P_i, S_j) \xrightarrow{\beta^*} \mathrm{Hom}(\oplus P_k^{m_k}, S_j) \xrightarrow{\alpha^*} \mathrm{Hom}(Q, S_j).$$

But, $\alpha(Q) \subset \mathrm{rad}(\oplus_k P_k^{m_k})$ so if $g \in \mathrm{Hom}(\oplus P_k^{m_k}, S_j)$ then

$$\alpha^* g(Q) = g \circ \alpha(Q) \subset g(\mathrm{rad}(\oplus_k P_k^{m_k}) = 0,$$

and similarly, $\beta^* = 0$. Thus,

$$\begin{aligned}\mathrm{Ext}^1(S_i, S_j) &\cong \mathrm{Hom}(\oplus_k P_k^{m_k}, S_j)\\ &\cong \mathrm{Hom}(P_j, S_j)^{m_j}\\ &\cong \mathrm{Hom}(P_j, \oplus_k S_k^{m_k})\\ &\cong \mathrm{Hom}(P_j, \mathrm{rad}(P_i)/\mathrm{rad}^2(P_i)).\end{aligned}$$

□

Definition 1.7.2. *An algebra A is* ***hereditary*** *if one of the following equivalent conditions holds.*

(i) Every submodule of a projective module is projective

(ii) The global dimension of A is zero or one.

(iii) For all A-modules M and N, $\mathrm{Ext}^2_A(M, N) = 0$.

Example 1.7.1. *Path algebras of a finite quiver are hereditary. If Q is finite then the minimal projective resolution of any simple begins with its projective cover. Let this sequence be denoted*

$$\cdots \xrightarrow{f} P \longrightarrow S \longrightarrow 0\,.$$

Since there are no relations, each path of length two in $\mathbb{C}Q$ *factors uniquely into paths of length one. Thus the kernel of f is trivial so* $\mathrm{Ext}^2(-, S) = 0$.

Theorem 1.7.1. *(Gabriel) Let A be a finite-dimensional basic* $\mathbb{C}$*-algebra with Ext-quiver* Q^e*. Then there is a surjective map* $\pi\colon \mathbb{C}Q^e \longrightarrow A$ *whose kernel is contained in the ideal of all paths of length greater than two. If A is hereditary* π *is an isomorphism.*

Proof. See [Ben98, Theorem 4.1.7]. □

1.7.2 Quivers with Relations

For finite-dimensional algebras which aren't hereditary one needs to use *quivers with relations* to describe them up to Morita equivalence. A **quiver with relations** is a pair, (Q, R), where Q is a quiver and $R \subset \mathbb{C}Q$ is a subset. If (Q, R) is a quiver with relations then the **path algebra of** (Q, R) is the quotient of the path algebra of Q by the two sided ideal generated by R. Again, if $\mathbb{C}Q$ is finite-dimensional then it is perfect and $\mathbb{C}Q$ decomposes into projective indecomposables as before.

The following proposition allows one to immediately see the blocks of the path algebra of a quiver when it is finite-dimensional.

Proposition 1.7.1. *(Gabriel [Ben98, Proposition 4.1.7]) Let A be a finite-dimensional algebra over an algebraically closed field. Let* Q_A *be the Ext-quiver of A. Then the decomposition of A into blocks and the decomposition of Q into connected components both partition the set of isomorphism classes of irreducible modules in the same way.*

Corollary 1.7.1. *Let A be a quasi-hereditary* $\mathbb{C}$*-algebra, so that* ${}_A$**mod** *is a highest weight category with some set of weights* Λ*. Suppose that A is finite-dimensional and the category has a duality so that BGG reciprocity holds. Let* $L(\lambda)$ *and* $L(\mu)$ *be simple A-modules. Then* $L(\lambda)$ *and* $L(\mu)$ *belong to the same block if and only if there exists a sequence of weights* $\rho_1 = \lambda, \rho_2, \ldots, \rho_n = \mu \in \Lambda$ *such that, for each* $i = 1 \ldots n-1$*, either* $L(\rho_i)$ *appears as a composition factor of* $\Delta(\rho_{i+1})$ *or* $L(\rho_{i+1})$ *appears as a composition factor of* $\Delta(\rho_i)$*.*

Proof. One direction is quick to prove: Suppose $L(\mu)$ was a composition factor of $\Delta(\lambda)$. Because $\Delta(\lambda)$ is indecomposable, all its composition factors, (which include $L(\mu)$ and $L(\lambda)$) must belong to the same block. Now an induction gives the *only if* statement.

Conversely, suppose that $L(\mu)$ and $L(\lambda)$ belong to the same block. By the proposition, there is some path in the Ext-quiver that connects the two vertices. Assume this path has length one, so, without loss of generality, corresponds to a non-zero element of $\mathrm{Ext}^1(L(\lambda), L(\mu))$. By Lemma 1.7.1, this corresponds to a map

$$f\colon P(\mu) \longrightarrow \mathrm{rad}(P(\lambda))/\mathrm{rad}^2(P(\lambda));$$

that is, $L(\mu)$ appears as a composition factor of $P(\lambda)$ in the first radical layer. Let

$$p_\mu\colon \mathrm{rad}(P(\lambda))/\mathrm{rad}^2(P(\lambda)) \longrightarrow L(\mu)$$

be the projection and let

$$P(\lambda) = \begin{matrix} \Delta(\lambda) \\ \Delta(\lambda_1) \\ \vdots \\ \Delta(\lambda_n) \end{matrix}$$

be a decomposition of $P(\lambda)$ into standard composition factors. This corresponds to a filtration, $0 = F_{n+1} \subset F_n \subset \cdots \subset F_1 \subset P(\lambda)$, such that, for all $i = 1, \ldots, n$, $F_i/F_{i+1} \cong \Delta(\lambda_i)$. If $[\Delta(\lambda) : L(\mu)] \neq 0$ then the claim is proved, so assume that $[\Delta(\lambda) : L(\mu)] = 0$. This means that $[F_1 : L(\mu)] \neq 0$. Let i the largest index such that $[F_i : L(\mu)] \neq 0$; this implies $[F_{i+1} : L(\mu)] = 0$. Consider the composition,

$$F_i \longrightarrow \mathrm{rad}(P(\lambda)) \longrightarrow \mathrm{rad}(P(\lambda))/\mathrm{rad}^2(P(\lambda)),$$

of the inclusion with the quotient. By the maximality condition on i, composing this map with p_μ gives a non-zero map $F_i \longrightarrow L(\mu)$ with F_{i+1} in its kernel. Thus $\Delta(\lambda_i) = F_i/F_{i+1}$ surjects onto $L(\mu)$ and this implies that $\lambda_i = \mu$. Now $(P(\lambda); \Delta(\mu)) = [\Delta(\mu) : L(\lambda)] \neq 0$ as required. An induction gives the result when the path in the Ext-quiver has arbitrary length □

Using this corollary, one can see the block structure of the category just by looking at the simple composition factors of the standard modules. This will be useful in Section 2.2.

1.8 Koszul Algebras

For the whole section, assume that A is a non-negatively graded, Noetherian ring such that the subring, A_0, is a semisimple $\mathbb{C}$-algebra. Let A_+ be the **augmentation ideal** $\oplus_{i>0} A_i$. Then $A_0 \cong A/A_+$ is an A-A-bimodule. It is also a subring. Given graded modules M and N, let $\mathrm{Hom}^i(M, N)$ denote the set of all **degree** i **homomorphisms**; that is, homomorphisms f such that $f(M_j) \subset N_{i+j}$ for all j. Let $\underline{\mathrm{Hom}}(M, N)$ denote the vector space $\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i(M, N)$. If m is a homogenous element in a graded ring or module, let $|m|$ denote its degree. Let ${}_A\mathbf{gMod}$ (${}_A\mathbf{gmod}$) denote the category of (finitely generated) graded A-modules whose morphisms are degree zero module homomorphisms; that is, $\mathrm{gHom}(M, N) = \mathrm{Hom}^0(M, N)$. Given a graded module M and $i \in \mathbb{Z}$, let $M[i]$ denote the graded module with a **shift by** i, that is $M[i]_j := M_{j+i}$.

The algebras that will be considered in this thesis will often be path algebras of quivers with relations, in which case A_0 is a semisimple commutative algebra: the linear span of trivial paths with zero multiplication.

The following proposition shows that every morphism in ${}_A\mathbf{mod}$ is can be decomposed into homogenous components. Note that this is not true of ${}_A\mathbf{Mod}$.

Proposition 1.8.1. *Let* $M, N \in {}_A\mathbf{gmod}$. *Then*

$$\mathrm{Hom}(M, N) = \underline{\mathrm{Hom}}(M, N).$$

Proof. It suffices to prove the inclusion, $\subseteq$, in one direction. Let $e_1, \ldots, e_n$ be homogenous generators for M. Let $\phi \in \mathrm{Hom}(M, N)$ so that, for each $i = 1, \ldots, n$, $\phi(e_i) \in \oplus_{j=r_i}^{s_i} N_j$ for some r_i

and s_i. Thus,

$$\phi \in \bigoplus_{j=s}^{r} \mathrm{Hom}^j(M,N) \subseteq \underline{\mathrm{Hom}}(M,N),$$

where $r = \max_i\{|e_i| - r_i\}$ and $s = \min_i\{|e_i| - s_i\}$. □

The upshot of this proposition is that, each homogenous part of an arbitrary homomorphism of graded A-modules, $f : M \longrightarrow N$, can be thought of as morphism in ${}_A\mathbf{gmod}$ once the degrees of M and N have been fiddled with: any homogenous part of degree i is a graded homomorphism between $M[j]$ and $N[j+i]$ for all $j \in \mathbb{Z}$.

Definition 1.8.1. *A graded projective resolution of a graded module M,*

$$\cdots \xrightarrow{d_2} P^1 \xrightarrow{d_1} P^0 \xrightarrow{d_0} M \longrightarrow 0\ ,$$

is called ***linear*** *if each P^i is generated in degree i.*

For M finitely generated and graded, this is equivalent to the condition that entries of each matrix, d_i, lie exclusively in A_1.

1.8.1 Quadratic and Koszul Algebras

Suppose that A is a non-negatively graded algebra, with A_0 semisimple and commutative.

Definition 1.8.2. *The algebra A is said to be* ***quadratic*** *if it is generated over A_0 by A_1 with relations in degree two.*

In other words, A is quadratic if it can be written as a quotient of the tensor algebra, $T_{A_0}(A_1)$, by the ideal generated an A_0-bimodule $R \subset A_1 \otimes_{A_0} A_1$. Let $A_1^* = \mathrm{Hom}_{\mathbb{C}}(A_1, \mathbb{C})$, the vector space dual of A_1. Define a bimodule action of A_0 on A_1^* by the rule, if $f \in A_1^*$, $a \in A_0$ and $x \in A_1$ then $a \cdot f(x) := f(ax)$ and $f \cdot a(x) := f(xa)$. This makes sense because A_0 is commutative.

Definition 1.8.3. *Let A be a quadratic algebra. The* ***quadratic dual*** *of A is the algebra*

$$A^! := T_{A_0}(A_1^*)/R^{\perp},$$

where $R^{\perp}$ is the orthogonal complement of R in $A_1^* \otimes_{A_0} A_1^*$, that is,

$$R := \{f \in (A_1 \otimes_{A_0} A_1)^* \cong A_1^* \otimes_{A_0} A_1^* \mid f(R) = 0\}.$$

Definition 1.8.4. *The algebra A is* ***Koszul*** *if the subring A_0, as a left A-module, admits a linear projective resolution.*

Beilinson, Ginzburg and Soergel prove in [BGS96, Corollary 2.3.3] that Koszul algebras are always quadratic and that if there exists a grading on an algebra A which makes it Koszul then that grading comes from the expression of A as a quotient of a tensor algebra [BGS96, Corollary 2.5.2].

Let $A = T_{A_0}(A_1)/R$ be a quadratic algebra. For each $i \geq 2$, define the A_0-bimodules

$$K_i^i := \bigcap_{0 \leq \nu \leq i-2} A_1^{\otimes \nu} \otimes_{A_0} R \otimes_{A_0} A_1^{\otimes i-\nu-2} \subseteq A_1^{\otimes i},$$

where $A_1^{\otimes 0} := A_0$, and let $K_0^0 := A_0$ and $K_1^1 := A_1$. For each $i \geq 0$, define

$$K^i := A \otimes_{A_0} K_i^i$$

with a grading so that $\left(K^i\right)_i$ agrees with K_i^i. Define a graded differential $d_i \colon K^i \longrightarrow K^{i-1}$ to be the restriction of the map

$$A \otimes_{A_0} A_1^{\otimes i} \longrightarrow A \otimes_{A_0} A_1^{\otimes i-1}; \qquad a \otimes v_1 \otimes \cdots \otimes v_i \mapsto av_1 \otimes v_2 \otimes \cdots \otimes v_i.$$

This gives a complex of graded left A-modules,

$$\cdots \xrightarrow{d^2} K^1 \xrightarrow{d^1} K^0 \cong A \xrightarrow{(-)/A_+} A/A_+ \cong A_0 ,$$

called the **Koszul complex** of A.

1.8.2 The Koszul Dual

Let A be as above and define the non-negatively graded vector space

$$E(A) := \bigoplus_{i \geq 0} \mathrm{Ext}^i_A(A_0, A_0).$$

The cup product turns this into a graded ring. There is another grading on $E(A)$.

By Proposition 1.8.1, given $M, N \in {}_A\mathbf{gmod}$,

$$\mathrm{Hom}_A(M, N) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Hom}^j_A(M, N)$$

so that for each $i \geq 0$,

$$\mathrm{Ext}^i(M, N) = \bigoplus_{j \in \mathbb{Z}} \mathrm{Ext}^{i,j}_A(M, N),$$

where $\mathrm{Ext}^{i,j}_A(M, N)$ is the cohomology at the i^{th} position of the complex formed by applying the functor $\mathrm{Hom}^j_A(-, N)$ to a projective resolution of M. Let

$$E(A)_{ij} := \mathrm{Ext}^{i,j}_A(A_0, A_0)$$

The grading by $j \in \mathbb{Z}$ is called the **internal grading of** $E(A)$ and the grading by $i \geq 0$ is called the **external grading**. Both gradings are related by the following proposition.

Proposition 1.8.2. *Let A be a graded, noetherian algebra with A_0 a semisimple $\mathbb{C}$-algebra. Then $E(A)_{ij} = 0$ for $j < i$. The algebra A is Koszul if and only if $E(A)_{ij} = 0$ for $i < j$ so that*

$$E(A) = \bigoplus_{i \geq 0} E(A)_{ii}.$$

Proof. Choose a projective resolution $\{\phi_i : A^{b_i}[c_i] \longrightarrow A^{b_{i-1}}[c_{i-1}]\}$ of A_0. Because the entries of each ϕ_i are in A_+, the generator $1 \in A^{b_i}$ is in degree $-c_i \leq -i$ so $\mathrm{Hom}^j(A^{b_i}[c_i], A_0) = 0$ for all $j < i$. The Koszul condition now implies that $c_i = i$ for all i. □

Theorem 1.8.1. *Let A be quadratic. The following statements are equivalent.*

(i) A is Koszul.

(ii) The Koszul complex is a resolution of A_0.

(iii) $E(A)_{ij} = 0$ for all $i \neq j$.

(iv) $A^!$ is Koszul.

(v) A^{op} is Koszul.

(vi) $\left(A^!\right)^{\mathrm{op}} \cong E(A)$ as graded algebras.

Proof. See [BGS96, Lemma 2.1.2, Proposition 2.1.3, Proposition 2.2.1, Theorem 2.6.1, Proposition 2.9.1 and Theorem 2.10.1]. □

Lemma 1.8.1. *Let A be a finite-dimensional algebra over an algebraically closed field. Then A is Koszul if and only if each of its blocks are.*

Proof. Let $\mathrm{Irr}(A) = \{S_1, \dots, S_n\}$ be the set of all non-isomorphic simple A-modules and let $f_1, \dots, f_r$ be centrally primitive idempotents such that $f_1 + \cdots + f_r = 1$; by definition, $f_i f_j = \delta_{ij} f_i$. For each $i = 1 \dots r$, let $B_i := Af_i$ so that $A = B_1 \times \cdots \times B_n$ is a decomposition of A into blocks. Let L_i be the direct sum of all simples belonging to the block B_i. If each block is Koszul then there exists a linear resolution of each of the L_i. The direct sum of these resolutions gives a linear resolution of $S := \bigoplus_{i=1}^n S_i$ so that A is Koszul.

Conversely, suppose that $d^\bullet : P^\bullet \longrightarrow S$ is a linear projective resolution of S. Define $P^i_j := P^i f_j$. Suppose that there exists an $x \in P^i_j$ such that $d^i x \in P^{i-1}_k$ is non-zero. Then $d^i x = d^i(x f_j) = f_j(d^i x) = f_k f_j(d^i x) \neq 0$ so $j = k$. Therefore the differentials decompose into the sum of differentials between projectives in each block. Because, they have differentials with entries in A_1, each of these resolutions of L_i is linear, so each B_i is Koszul. □

Chapter 2

Koszul Duality for Rank Two

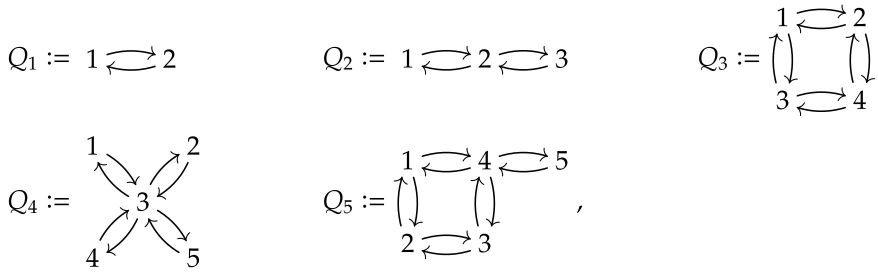
The main result of this chapter is that the category, $\mathcal{O}(W)$, associated to a dihedral group, W, is Koszul, in the sense that it is equivalent to a category of modules over a Koszul algebra. In some cases the algebra is Koszul self-dual. The work relies on the classification of standard modules of $\mathcal{O}(W)$ by Chmutova in [Chm06]. The idea is to find a progenerator, P, for the category and a quiver with relations, (Q, R), for which $A := \mathrm{End}_{\mathcal{O}}(P) \cong \mathbb{C}(Q, R)$, so that,

$$\mathcal{O}(W) \simeq {}_A\mathbf{mod} \simeq \mathbf{Rep}_{(Q,R)},$$

and then show that the category, $\mathbf{Rep}_Q$, is Koszul by explicitly calculating a linear resolution of each of the simple modules. As it turns out, as $\mathbf{c}$ runs over all possible parameters and W runs over all dihedral groups, there are only six different blocks up to equivalence. The chapter begins by defining some quivers with relations and choosing a basis for their corresponding path algebras, before a case by case examination of blocks of $\mathcal{O}_{\mathbf{c}}(W)$.

2.1 Classification of Blocks in the Category $\mathcal{O}$ of Dihedral Groups

Define the following quivers.



where the arrow from vertex i to vertex j is denoted a_{ji} and the trivial path at each vertex i is denoted e_i. For brevity, the following convention is used; if $a_{i_r i_{r-1}}, \dots, a_{i_2 i_1}$ are arrows then $a_{i_r \cdots i_1}$

denotes the path, $a_{i_r i_{r-1}} \circ \ldots \circ a_{i_2 i_1}$. Let $\lambda \in \mathbb{C}$ and $t := \lambda^2 - 1$ and define the following relations.

$$\begin{aligned}
R_1 &:= \{a_{121}\},\\
R_2 &:= \{a_{121}, a_{321}, a_{123}, a_{212} - a_{232}\},\\
R_3 &:= \begin{Bmatrix} a_{121}, & a_{342}, & a_{212} - a_{242}, & a_{313} - a_{343} \\ a_{131}, & a_{243}, & a_{421} - a_{431}, & a_{124} - a_{134} \end{Bmatrix},\\
R_4^\lambda &:= \begin{Bmatrix} a_{131}, & a_{231}, & a_{531}, & a_{234}, & \lambda^2 a_{313} + a_{323} - a_{343} \\ a_{132}, & a_{232}, & a_{432}, & a_{135}, & a_{313} + a_{323} - a_{353} \end{Bmatrix},\\
R_5^\lambda &:= \begin{Bmatrix} a_{141}, & a_{541}, & a_{341} - a_{321}, & a_{123} - a_{143}, & \lambda a_{412} - a_{432}, & t a_{414} - a_{434} + a_{454} \\ a_{121}, & a_{145}, & a_{212} - a_{232}, & \lambda a_{214} - a_{234}, & \lambda a_{323} - a_{343} & \end{Bmatrix}.
\end{aligned}$$

Next define the algebra B_i^λ (ignoring the superscript λ for $i \neq 4, 5$) to be the path algebra of Q_i with the relations R_i over the complex numbers. Let B_0 denote the algebra $\mathbb{C}$, thought of here as a simple block.

2.1.1 Bases for the Algebras B_i^λ

Proposition 2.1.1. *The following is a basis for* B_1*, written here as respective bases of the projective modules of paths starting from each vertex.*

$$\begin{aligned}
\mathsf{B}_1 e_1 &\overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{a_{21}\} \rangle_{\mathbb{C}}\\
\mathsf{B}_1 e_2 &\overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{a_{12}\} \cup \{a_{212}\} \rangle_{\mathbb{C}}
\end{aligned}$$

Proof. These are the only non–zero paths and they are linearly independent. □

Proposition 2.1.2. *The following is a basis for* B_2*, written here as respective bases of the projective modules of paths starting from each vertex.*

$$\begin{aligned}
\mathsf{B}_2 e_1 &\overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{a_{21}\} \rangle_{\mathbb{C}}\\
\mathsf{B}_2 e_2 &\overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{a_{12}, a_{32}\} \cup \{a_{212}\} \rangle_{\mathbb{C}}\\
\mathsf{B}_2 e_3 &\overset{\text{v.s.}}{\cong} \langle \{e_3\} \cup \{a_{23}\} \cup \{a_{323}\} \rangle_{\mathbb{C}}
\end{aligned}$$

Proof. The nine elements above are non–zero and linearly independent. There are exactly two paths of length 2 starting at each node for a total of six. Because the relations R_2 are linearly independent, $\dim(\mathsf{B}_2)_2 = 6 - 4 = 2$. Any continuation of these two paths is zero so $\dim(\mathsf{B}_2)_3 = 0$ and so $\dim \mathsf{B}_2 = 3 + 4 + 2 = 9$. □

Proposition 2.1.3. *The following is a basis for* B_3*, written here as respective bases of the projective modules of paths starting from each vertex.*

$$\begin{aligned}
\mathsf{B}_3 e_1 &\overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{a_{21}, a_{31}\} \cup \{a_{421}\} \rangle_{\mathbb{C}}\\
\mathsf{B}_3 e_2 &\overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{a_{12}, a_{42}\} \cup \{a_{212}, a_{312}\} \cup \{a_{4212}\} \rangle_{\mathbb{C}}\\
\mathsf{B}_3 e_3 &\overset{\text{v.s.}}{\cong} \langle \{e_3\} \cup \{a_{13}, a_{43}\} \cup \{a_{313}, a_{213}\} \cup \{a_{4213}\} \rangle_{\mathbb{C}}\\
\mathsf{B}_3 e_4 &\overset{\text{v.s.}}{\cong} \langle \{e_4\} \cup \{a_{24}, a_{34}\} \cup \{a_{124}, a_{424}, a_{434}\} \cup \{a_{2124}, a_{3124}\} \cup \{a_{42124}\} \rangle_{\mathbb{C}}
\end{aligned}$$

Proof. None of the twenty five elements of B_3 above is zero, as can be checked directly from the relations. If there is a subset whose paths are linearly dependent then they must have the same path length and start and end at the same vertex. So, to show that these are all linearly independent one needs only check that a_{424} and a_{434} are not multiples of one another, which is obvious from the relations. It suffices to check that the dimension of B_3 is twenty five. This is done by calculating the dimension of the subspaces $e_i\mathsf{B}_3e_j$ as $i, j = 1\dots4$.

Every non–zero path starting at 1 with length greater than one must begin $a_{421} = a_{431}$ and the only possible continuations of this are a_{2421} and a_{3431} which are both zero. Therefore, $\dim \mathsf{B}_3e_1 = 4$.

With the exception of e_2 and a_{42}, every path which starts at the vertex 2 can be rewritten so that it starts with a_{12} and precomposition by a_{12} does not act as zero on any of the paths in B_3e_1. Therefore,

$$\dim e_i\mathsf{B}_3e_2 = \begin{cases} 1 & \text{if } i = 1,3 \\ 2 & \text{if } i = 2,4. \end{cases}$$

Similarly, with the exception of e_3 and a_{43}, every path which starts at the vertex e_3 can be rewritten so that it starts with a_{13} and precomposition by a_{13} does not act as zero on any of the paths in $e_i\mathsf{B}_3e_1$. Therefore,

$$\dim e_i\mathsf{B}_3e_3 = \begin{cases} 1 & \text{if } i = 1,2 \\ 2 & \text{if } i = 3,4 \end{cases}$$

and so $\dim \mathsf{B}_3e_2 = \dim \mathsf{B}_3e_3 = 6$. With the exception of e_4, a_{24} and $a_{42}a_{24}$, every path which starts at the vertex e_4 can be rewritten so that it starts with a_{34} and precomposition by a_{34} does not act as zero on any of the paths in $e_i\mathsf{B}_3e_4$. Therefore,

$$\dim e_i\mathsf{B}_3e_4 = \begin{cases} \dim e_i\mathsf{B}_3e_3 & \text{if } i = 1,3 \\ \dim e_i\mathsf{B}_3e_3 + 1 & \text{if } i = 2 \\ \dim e_i\mathsf{B}_3e_3 + 2 & \text{if } i = 4 \end{cases}$$

and so $\dim \mathsf{B}_3e_4 = 9$. The arguments above mimic the fact that the projective covers inject into each other. The sum of these dimensions is twenty five so the twenty five elements above span B_3. □

Proposition 2.1.4. *The following is basis for* B_4*, written here as respective bases of the projective modules of paths starting from each vertex.*

$$\mathsf{B}_4e_1 \overset{\text{v.s.}}{\cong} \langle\{e_1\} \cup \{a_{31}\} \cup \{a_{431}\}\rangle_{\mathbb{C}}$$
$$\mathsf{B}_4e_2 \overset{\text{v.s.}}{\cong} \langle\{e_2\} \cup \{a_{32}\} \cup \{a_{532}\}\rangle_{\mathbb{C}}$$
$$\mathsf{B}_4e_3 \overset{\text{v.s.}}{\cong} \langle\{e_3\} \cup \{a_{13}, a_{23}, a_{43}, a_{53}\} \cup \{a_{313}, a_{323}\} \cup \{a_{4313}, a_{5323}\}\rangle_{\mathbb{C}}$$
$$\mathsf{B}_4e_4 \overset{\text{v.s.}}{\cong} \langle\{e_4\} \cup \{a_{34}\} \cup \{a_{134}, a_{434}, a_{534}\} \cup \{a_{3434}\} \cup \{a_{43434}\}\rangle_{\mathbb{C}}$$
$$\mathsf{B}_4e_5 \overset{\text{v.s.}}{\cong} \langle\{e_5\} \cup \{a_{35}\} \cup \{a_{235}, a_{435}, a_{535}\} \cup \{a_{3535}\} \cup \{a_{53535}\}\rangle_{\mathbb{C}}.$$

Proof. None of the twenty nine elements of B_4 above is zero, as can be checked directly from the maps. If there is a subset whose paths are linearly dependent then they must have the same path length and start and end at the same vertex. So, to show that these are all linearly independent one needs only check that a_{313} and a_{323} are not multiples of one another, which is obvious from the relations. It suffices to check that the dimension of B_4 is twenty nine. This is done by calculating the dimension of the subspaces B_4e_j for $j = 1\dots5$.

First, B_4e_1 and B_4e_2 are three-dimensional because a_{3431} and a_{3532} are both zero. Similarly, $\dim e_1\mathsf{B}_4 = \dim e_2\mathsf{B}_4 = 3$, so $\dim e_1\mathsf{B}_4e_3 = \dim e_2\mathsf{B}_4e_3 = 1$.

Suppose a path in $e_4\mathsf{B}_4e_3$ passes through vertex 1, then because $e_4\mathsf{B}_4e_1 = \{a_{431}\}$ and $e_1\mathsf{B}_4e_3 = \{a_{13}\}$ the only possibility is their concatenation, a_{4313}. I claim that the only other path is a_{43}. Suppose there is a third path, a say; it cannot pass through 1. Suppose a passes through 5. After the last time it leaves vertex 5 it must go straight to 3, otherwise it would eventually have to factor through the path $a_{432} = 0$. Therefore, if a passes through 5 it must factor through $a_{4353} = a_{4313}$; a contradiction. The only remaining possibilitiy is that it ends $a_{4343} = \lambda^2 a_{4313}$; a contradiction and the claim is proved. A similar argument shows that $e_5\mathsf{B}_4e_3$ is two-dimensional. Next, any path of length four in $e_3\mathsf{B}_4e_3$ must pass through one or two of the other vertices. The two relations with three terms allow any such path to be written in terms of paths which only pass through 1, 2 and 3. The relations immediately show that these are zero. This gives $\dim e_3 \mathsf{B}_4 e_3 = 1 + 2 = 3$ and so $\dim \mathsf{B}_4 e_3 = 1 + 1 + 3 + 2 + 2 = 9$.

Apart from e_4, every path starting at 4 must factor through a path in $\mathsf{B}_4 e_3$. Precomposition by a_{34} kills a_{23}, a_{323} and a_{5323} in $\mathsf{B}_4 e_3$ but no other paths, so $\dim \mathsf{B}_4 e_4 = 1 + 9 - 3 = 7$. A similar argument shows that $\dim \mathsf{B}_4 e_5 = 7$. The sum of these dimensions is twenty nine so the twenty nine elements above span B_4. □

Proposition 2.1.5. *The following is a basis for* B_5*, written here as respective bases of the projective modules of paths starting from each vertex.*

$$\mathsf{B}_5 e_1 \overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{a_{41}, a_{21}\} \cup \{a_{341}\} \rangle_{\mathbb{C}}$$
$$\mathsf{B}_5 e_2 \overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{a_{12}, a_{32}\} \cup \{a_{412}, a_{212}\} \cup \{a_{3412}\} \rangle_{\mathbb{C}}$$
$$\mathsf{B}_5 e_3 \overset{\text{v.s.}}{\cong} \langle \{e_3\} \cup \{a_{23}, a_{43}\} \cup \{a_{123}, a_{323}, a_{543}\} \cup \{a_{2123}, a_{4123}\} \cup \{a_{34123}\} \rangle_{\mathbb{C}}$$
$$\mathsf{B}_5 e_4 \overset{\text{v.s.}}{\cong} \langle \{e_4\} \cup \{a_{14}, a_{34}, a_{54}\} \cup \{a_{414}, a_{434}, a_{214}\} \cup \{a_{3414}, a_{5434}\} \rangle_{\mathbb{C}}$$
$$\mathsf{B}_5 e_5 \overset{\text{v.s.}}{\cong} \langle \{e_5\} \cup \{a_{45}\} \cup \{a_{345}, a_{545}\} \cup \{a_{4345}\} \cup \{a_{54345}\} \rangle_{\mathbb{C}}.$$

Proof. None of the thirty four elements of B_5 above is zero, as can be checked directly from the maps. If there is a subset whose paths are linearly dependent then they must have the same path length and start and end at the same vertex. So, to show that these are all linearly independent one needs only check that a_{414} and a_{434} are not multiples of one another, which is obvious from the relations. It suffices to check that the dimension of B_5 is thirty four. This is done by calculating the dimension of the subspaces $e_i \mathsf{B}_5 e_j$ as $i, j = 1 \dots 5$.

(i) The space $e_1 \mathsf{B}_5 e_1$ is one-dimensional. The two possible paths of length two, a_{141} and a_{121}, are both zero. Any longer non–zero path must begin $a_{341} = a_{321}$ and the only possible continuations of this are a_{4341} and a_{2341} which are both zero.

(ii) The space $e_2 \mathsf{B}_5 e_1$ is one-dimensional. Every path in $e_2 \mathsf{B}_5 e_1$ beginning with a_{41} either returns to 1 at some point (and so is zero), continues to 5 (and so is zero) or continues to 3. Now, the only possibility is paths that start with a_{4341} or a_{2341} both of which are zero. Therefore, if a path from 1 to 2 begins with a_{41} it is zero. Any path that begins a_{321} begins a_{341}. The only possible remaining non-zero path from 1 to 2 is a_{21}.

(iii) The space $e_3 \mathsf{B}_5 e_1$ is one-dimensional. By Remark (ii), a non-zero path from 1 to 3 can only pass through 2 once, in which case it begins $a_{321} = a_{341}$. If a path begins with a_{41} then its only non–zero continuation is a_{341}. Therefore, every path from 1 to 3 must begin a_{341} and cannot return to 1 or 2; the relation $a_{343} = \lambda a_{323}$ then implies it cannot return to 4.

(iv) The space $e_4 \mathsf{B}_5 e_1$ is one-dimensional. Any path from 1 to 4 that passes through 2 or 3 must begin a_{341} and then continue to $a_{4341} = a_{4121} = 0$. The only non-zero path that doesn't pass through 2 or 3 is a_{41} because a_{4141} and a_{4541} are zero.

(v) The space $e_5 \mathsf{B}_5 e_1$ is zero-dimensional. Any path must factor through a path in $e_4 \mathsf{B}_5 e_1$ and $a_{541} = 0$. Together, these give $\dim \mathsf{B}_5 e_1 = 1 + 1 + 1 + 1 + 0 = 4$.

(vi) Every path that starts at vertex 2, apart from e_2 and a_{32}, can be rewritten so that it starts with a_{12} and precomposition by a_{12} does not act as zero on any of the paths in $\mathsf{B}_5 e_1$. Therefore,

$$\dim e_i \mathsf{B}_5 e_2 = \begin{cases} \dim e_i \mathsf{B}_5 e_1 & \text{if } i = 1, 4, 5 \\ \dim e_i \mathsf{B}_5 e_1 + 1 & \text{if } i = 2, 3 \end{cases}$$

and $\dim \mathsf{B}_4 e_2 = 1 + 2 + 2 + 1 + 0 = 6$.

(vii) Every path which starts at vertex 3, apart from e_3, a_{43} and a_{543}, can be rewritten so that it starts with a_{23}. Indeed, $a_{143} = a_{123}$, $a_{343} = a_{323}$ and $a_{4543} = a_{4323} - t a_{4123}$. Precomposition by

a_{23} does not act as zero on any of the paths in B_5e_2. Therefore,

$$\dim e_i\mathsf{B}_5e_3 = \begin{cases} \dim e_i\mathsf{B}_5e_2 & \text{if } i = 1,2 \\ \dim e_i\mathsf{B}_5e_2 + 1 & \text{if } i = 3,4,5 \end{cases}$$

and, together with Remark (vi), this gives $\dim \mathsf{B}_4e_2 = 1 + 2 + 3 + 2 + 1 = 9$.

(viii) The space $e_1\mathsf{B}_5e_4$ is one-dimensional. By Remark (i), paths can only pass through 1 once, so a second non-zero path would have to pass through 3 or end $a_{145} = 0$. Remark (vii) shows that there is only one non-zero path from 3 to 1 so the path would have to end $a_{1234} = 0$.

(ix) The space $e_2\mathsf{B}_5e_4$ is one-dimensional. Any non-zero path from 4 to 2 must factor through a_{214} (since $a_{234} = \lambda a_{214}$) and so, because $\dim e_2\mathsf{B}_5e_1 = 1$, must finish a_{214}. Any path longer than a_{214} would have to end $a_{21414} = 0$, $a_{21434} = 0$ or $a_{21432} = 0$.

(x) The space $e_4\mathsf{B}_5e_4$ is three-dimensional. By Remarks (iv) and (viii), the only path from 4 to 4 that passes through 1 is a_{414}. The sixth and nineth remark show that the only path from 4 to 4 that passes through 2 must be $a_{43234} = 0$. It remains to consider paths which only pass through the vertices 3, 4 and 5, but, because of the the last relation, $a_{454} = a_{434} - ta_{414}$, one needs only consider paths that begin a_{434}. By the seventh remark, such a path would have to be a_{434} or $a_{41234} = 0$.

(xi) The space $e_3\mathsf{B}_5e_4$ is two-dimensional. The only path from 4 to 3 that passes through 1 is a_{3414} by Remarks (iii) and (viii). By Remark (vii), the only paths from 4 to 3 that begin a_{34} are a_{34} itself and $a_{3454} = a_{3414}$. The only other possibility is that a path begins with a_{454} in which case the last relation can be used to reduce to cases already considered.

(xii) The space $e_5\mathsf{B}_5e_4$ is two-dimensional. The Remarks (ii) and (vi) show that no path can pass through the vertices 1 or 2. Therefore, every path apart from a_{54} must end $a_{5454} = a_{5434}$. By Remarks (vii) and (xi), the only remaining possibilities are a_{5434} or zero. Together, these give $\dim \mathsf{B}_5e_4 = 1 + 1 + 3 + 2 + 2 = 9$.

(xiii) Precomposing the paths in B_5e_4 by a_{45} one sees that $\dim \mathsf{B}_5e_5 = 0 + 0 + 1 + 2 + 3 = 6$ and $\dim \mathsf{B}_5 = 4 + 6 + 9 + 9 + 6 = 34$; so the thirty four linearly independent elements above span B_5.

□

Proposition 2.1.6. *The algebras B_4^λ and B_4^μ are isomorphic if and only if $\lambda^2 = \mu^2$.*

Proof. First, note that any ring automorphism of a path algebra of a quiver must act as a permutation on paths of length zero. Indeed, in order that it be a ring homomorphism, the identity, which is the sum of all paths of length zero, must be invariant. Thus, each path of length zero must be mapped to a linear combination of paths of length zero; that is to say, the automorphism acts as a linear map on the vector space of paths of length zero. Consider the corresponding matrix. Each of the columns and rows must sum to one. Furthermore, since each path of length zero is an idempotent, the coefficients of this matrix can only be one or zero. The matrix must therefore be a permutation matrix.

Once the permutation of the length zero paths is chosen, σ say, each path of length one, a_{ij} say, must be sent to some scalar multiple of $a_{\sigma(i)\sigma(j)}$. For an isomorphism, such a scalar must be non-zero. Since the paths of length zero and one generate the algebra this determines the homomorphism.

In this particular example, I claim that σ could only be the identity permutation or (12)(45). Indeed, e_3 must remain fixed, otherwise the relation $a_{313} + a_{323} - a_{353}$ could not be sent to zero. Also, e_1 (or e_2) could not be sent to e_4 or e_5 or the relation a_{531} (or a_{432}) would not be killed. Likewise, e_4 or e_5 could not be sent to e_1 or e_2. When e_1 is mapped to e_2, the relation a_{135} is only killed when e_4 is sent to e_5. This proves the claim.

Suppose that an isomorphism, $f\colon \mathsf{B}_4^\lambda \longrightarrow \mathsf{B}_4^\mu$, fixes the vertices of the quiver. Let A_{ij} be the scalar by which f acts on a_{ij}. Then, because the relation $a_{313} + a_{323} - a_{353}$ must be mapped to a non-zero multiple of itself, one sees that $A_{31}A_{13} = A_{32}A_{23}$. Now, by examining the relation $\lambda^2 a_{313} + a_{323} - a_{343}$, one sees that $\lambda^2 A_{31}A_{13} = A_{32}A_{23}\mu^2$. Together these give the result.

When f acts as the permutation (12)(45) on the vertices, using the same argument as above one gets the conditions $A_{31}A_{13} = \lambda^{-2}A_{32}A_{23}$ and $A_{32}A_{23} = \mu^2 A_{13}A_{31}$. Again these imply the statement of the proposition. □

Proposition 2.1.7. *The algebras B_5^λ and B_5^μ are isomorphic only if $\lambda^2 = \mu^2$.*

Proof. Let $f\colon \mathsf{B}_5^\lambda \longrightarrow \mathsf{B}_5^\mu$ be an isomorphism of algebras. By the argument in the proof of Proposition 2.1.6, f must permute paths of length zero, thought of here as the vertices of the underlying quiver Q_5. I claim that f must fix each vertex. Indeed, f must send any vertex to another with the same number of arrows entering and leaving; this shows that vertices 4 and 5 must remain fixed. The relation a_{531} must be sent by f to a non-zero relation in R_5^μ, so f must also fix vertex 1. Finally, the relation a_{121} would not be killed by f if such a map exchanged vertices 2 and 3. This proves the claim.

Now, as in the proof of Proposition 2.1.6, let A_{ij} denote the scalar multiple by which f acts on a_{ij}. By examining the relations, one gets the following conditions.

$$\lambda A_{21}A_{14} = \mu A_{23}A_{34} \qquad \lambda A_{41}A_{12} = \mu A_{43}A_{32}$$
$$\lambda A_{32}A_{23} = \mu A_{34}A_{43} \qquad A_{21}A_{12} = \mu A_{23}A_{32}.$$

Together with the condition $(\lambda^2-1)A_{41}A_{14} = (\mu^2-1)A_{43}A_{34}$, these imply that $\frac{\lambda^2-1}{\lambda^4} = \frac{\mu^2-1}{\mu^4}$, which holds when $\lambda^2 = \mu^2$. □

2.1.2 Blocks of Category $\mathcal{O}$ for Dihedral Groups

The following theorem classifies the blocks of category $\mathcal{O}$ for dihedral groups up to Morita equivalence.

Theorem 2.1.1. *Let W be a dihedral group and B a block of $\mathcal{O}_{\mathbf{c}}(W)$ for some parameter $\mathbf{c}$. Then B is Morita equivalent to B_i^λ for some $i = 1\ldots5$ and $\lambda \in \mathbb{C}$.*

The next section is just a proof. It will examine different parameter values, case by case, repeatedly using the results of [Chm06].

2.2 Proof of Theorem 2.1.1

Proposition 2.2.1. *Let W be a complex reflection group and B a block in $\mathcal{O}_{\mathbf{c}}(W)$ with two irreducibles. B is Morita equivalent to B_1.*

Proof. Let L_1 and L_2 be the two irreducibles with standards Δ_1 and Δ_2 and projective covers P_1 and P_2 respectively. Without loss of generality, suppose that $1 < 2$ with respect to the highest weight structure; then Δ_1 is simple and Δ_2 is projective. By Theorem 1.5.3, this implies that P_1 is a summand of P_{KZ} and so must be injective and have a simple socle. Being projective, P_1 has a filtration by standard modules. Let M be the penultimate module in this filtration, so that P_1 is an extension of M by Δ_1. Since $(P_1; \Delta_1) = [\Delta_1 : L_1] = 1$, M has a standard filtration with only copies of the standard module Δ_2 appearing. Since Δ_2 is projective, this must be a direct sum; so the socle of P_1 is a direct sum of the respective socles of the copies of Δ_2 in M. But P_1 has a simple socle, so $(P_1; \Delta_2) = 1$ and Δ_2 must have a simple socle. This gives,

$$P_1 = \begin{matrix}\Delta_1\\ \Delta_2\end{matrix} = \begin{matrix}L_1\\ L_2\\ L_1\end{matrix} \qquad P_2 = \Delta_2 = \begin{matrix}L_2\\ L_1\end{matrix}.$$

From this description the ring structure of $B = \mathrm{End}_{\mathcal{O}}(P_1 \oplus P_2)$ is evident. Decompose $\mathrm{End}_{\mathcal{O}}(P_1 \oplus P_2) \overset{\text{v.s.}}{\cong} \mathrm{End}_{\mathcal{O}}(P_1) \oplus \mathrm{Hom}_{\mathcal{O}}(P_1, P_2) \oplus \mathrm{Hom}_{\mathcal{O}}(P_2, P_1) \oplus \mathrm{End}_{\mathcal{O}}(P_2) \overset{\text{v.s.}}{\cong} \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and let f and g be non-trivial homomorphisms from P_1 to P_2 and P_2 to P_1 respectively. Then $\mathrm{End}_{\mathcal{O}}(P_1)$ is spanned by the identity map id_1 and $g \circ f$ and $\mathrm{End}_{\mathcal{O}}(P_2)$ is spanned by the identity map id_2. Then, as a ring, $\mathrm{End}_{\mathcal{O}}(P_1 \oplus P_2)$ is generated by $\{\mathrm{id}_1, \mathrm{id}_2, f, g\}$ and mapping $\mathrm{id}_1 \mapsto e_2$, $\mathrm{id}_2 \mapsto e_1$, $f \mapsto a_{12}$ and $g \mapsto a_{21}$ gives an isomorphism of rings to B_1. □

2.2.1 The Complex Reflection Group $(I_2(d), \mathfrak{h})$

The number of conjugacy classes of reflections in a dihedral group depends on whether or not the order is twice an odd or even number, so each of these cases is examined individually. First, fix the following notation. Choose a presentation for the dihedral group of order $2d$,

$$I_2(d) = \langle s, t \,|\, s^2 = t^2 = (st)^d = 1 \rangle.$$

When d is even, s and t belong to different conjugacy classes. Let **triv** denote the trivial representation and **sgn** the determinant representation of W. When $d = 2m + 1$ for some positive integer m, all the reflections in $I_2(d)$ are conjugate, so these are the only two one-dimensional representations up to isomorphism. When $d = 2m$, let ϵ_1 be the representation defined by $\epsilon_1(s) := -1$ and $\epsilon_1(t) := 1$, and define $\epsilon_2 := \epsilon_1 \otimes \mathbf{sgn}$. Let ζ be a primitive m^{th} root of unity and let d be either even or odd. For $i = 1, \dots, m$, let τ_i be the two-dimensional representation defined by

$$\tau_i(s) := \begin{pmatrix} 0 & \zeta^i \\ \zeta^{-i} & 0 \end{pmatrix} \qquad \tau_i(t) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These are irreducible for $i = 1, \dots, m-1$. When $d = 2m + 1$, τ_m is also irreducible, but when $d = 2m$, $\tau_m = \epsilon_1 \oplus \epsilon_2$. Let $\mathfrak{h} \cong \tau_1$ as a $\mathbb{C}W$-module, so that $(I_2(d), \mathfrak{h})$ is a complex reflection group (in fact, a Coxeter group). When d is even the one-dimensional representations act on the parameters by

$$\epsilon_1((c_0, c_1)) = (c_0, -c_1) \qquad \epsilon_2((c_0, c_1)) = (-c_0, c_1).$$

For reference, the characters take the following values on the conjugacy classes of s and t. For

	s	t	st	$\cdots$	$(st)^{m-1}$
triv	1	1	1	$\cdots$	1
ϵ_1	-1	1	-1	$\cdots$	$(-1)^{m-1}$
ϵ_2	1	-1	-1	$\cdots$	$(-1)^{m-1}$
τ_i	0	0	$\zeta^i + \zeta^{-i}$	$\cdots$	$\zeta^{i(m-1)} + \zeta^{-i(m-1)}$
sgn	-1	-1	1	$\cdots$	1

each $i = 1, \dots, m$, $\tau_i \otimes \mathbf{sgn} = \tau_i$ and $\tau_i \otimes \epsilon_1 = \tau_i \otimes \epsilon_2 = \tau_{m-i}$.

2.2.2 Case A: Odd Index Dihedral Groups

Let $d = 2m + 1$. Then the reflections in W are all conjugate so the corresponding rational Cherednik algebra depends on a single parameter $c = c_0 \in \mathbb{C}$. There are three cases to consider.

Case A1: The parameter satisfies $c \in \mathbb{Z} + \frac{1}{2}$

Suppose that $c \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$. By [Chm06, Theorems 3.1.1 & 3.1.2], for $i = 1, \dots, m$, the standard modules have the following Jordan Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_i) = L(\tau_i)$$

By Corollary 1.7.1, $\mathcal{O}_{\mathbf{c}}$ has one non–simple block, B, corresponding to the subset $\{\mathbf{triv}, \mathbf{sgn}\} \subset \mathrm{Irr}(W)$. By Proposition 2.2.1, it is isomorphic to B_1. Applying Proposition 1.5.2 gives the same result when $c \in \mathbb{Z}_{<0} + \frac{1}{2}$.

Case A2: The parameter satisfies $dc \equiv \pm l$ (mod d)) for some $l = 1, \ldots, m$

Suppose that $dc \equiv \pm l$ (**mod** d)) for some $l = 1, \ldots, m$. Again, Proposition 1.5.2 allows one to assume that $c > 0$. By [Chm06, Theorem 3.1.1 & Theorem 3.1.2], the standard modules have the following Jordan Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\tau_l) \end{matrix} \qquad \Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_l) = \begin{matrix} L(\tau_l) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\tau_i) = L(\tau_i),$$

for $i \neq l$. Using BGG reciprocity, the projective indecomposable covers have the standard filtration,

$$P(\mathbf{triv}) = \Delta(\mathbf{triv}) \qquad P(\mathbf{sgn}) = \begin{matrix} \Delta(\mathbf{sgn}) \\ \Delta(\tau_l) \end{matrix} \qquad P(\tau_l) = \begin{matrix} \Delta(\tau_l) \\ \Delta(\mathbf{triv}) \end{matrix} \qquad P(\tau_i) = \Delta(\tau_i),$$

for $i \neq l$. There are $m-1$ simple blocks and one non-simple block, B, with three simples. Let P be the progenerator,

$$P := \bigoplus_{i=1}^{m} P(\tau_i) \oplus P(\mathbf{triv}) \oplus P(\mathbf{sgn}),$$

and let Q be the submodule that is a progenerator for B,

$$Q := P(\tau_l) \oplus P(\mathbf{triv}) \oplus P(\mathbf{sgn}).$$

The structure of $\mathrm{End}_B(Q)$ is difficult to calculate directly. Instead, the double centraliser property can be used as follows.

Let P_{KZ} denote the projective module which represents the KZ-functor. Using Theorem 1.5.3, it has the form

$$P_{\mathsf{KZ}} = \bigoplus_{i=1}^{m} P(\tau_i)^{n_i} \oplus P(\mathbf{sgn})^n$$

for some positive integers, $n, n_1, \ldots, n_m$. Consider the restriction, $\mathsf{KZ}|_B$, of the KZ-functor to the full subcategory of B-modules in $\mathcal{O}$. Now, $\mathsf{KZ}|_B$ is exact and so is represented by some projective module, B_{KZ}. Since P_{KZ} is a sum of projective indecomposables, B_{KZ} is a summand of P_{KZ} and the statements of Theorem 1.5.3 restricted to the module catgegory of B apply to B_{KZ}. Therefore,

$$B_{\mathsf{KZ}} = P(\mathbf{sgn})^a \oplus P(\tau_l)^b,$$

where $1 \leq a \leq n$ and $1 \leq b \leq n_l$.

Lemma 2.2.1. *With the notation above, $a = b = 1$.*

Proof. Because, $\mathcal{H}$, the corresponding Hecke algebra, is isomorphic to $\mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}})$, it has the same number of irreducible modules as there are non-isomorphic projective indecomposables in P_{KZ}: $m+1$. By [GP00, Theorem 8.3.1], there are at most two one-dimensional irreducible representations, the rest are two-dimensional. Now,

$$\begin{aligned} \dim \mathrm{Hom}_{\mathcal{O}}(P(\mathbf{sgn})^n, P(\mathbf{sgn})^n \oplus P(\tau_l)^{n_l}) &= n[P(\mathbf{sgn})^n : L(\mathbf{sgn})] + n[P(\tau_l)^{n_l} : L(\mathbf{sgn})] \\ &= 2n^2 + n_l n, \\ \dim \mathrm{Hom}_{\mathcal{O}}(P(\tau_l)^{n_l}, P(\mathbf{sgn})^n \oplus P(\tau_l)^{n_l}) &= n_l[P(\mathbf{sgn})^n : L(\tau_l)] + n_l[P(\tau_l)^{n_l} : L(\tau_l)] = n_l n + 2n_l^2. \end{aligned}$$

Therefore,

$$\begin{aligned} 4m + 2 = |W| = \dim \mathcal{H} = \dim \mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}}) && (2.1) \\ = \dim \mathrm{End}_{\mathcal{O}}(P(\mathbf{sgn})^n \oplus P(\tau_l)) + \sum_{i \neq l} \dim \mathrm{End}_{\mathcal{O}}(P(\tau_i)^{n_i}) && (2.2) \\ = 2(n^2 + n_l^2 + n_l n) + \sum_{i \neq l} n_i^2. && (2.3) \end{aligned}$$

Now, at most two of these integers must be one, the rest must be twos. This reduces to the following four cases.

(i) Two of the n_i for $i \neq l$ are ones.

(ii) One of the n_i for $i \neq l$ is a one and one of $\{n, n_l\}$ is a one.

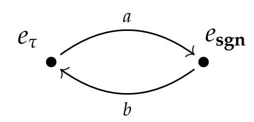
(iii) One of the n_i for $i \neq l$ is a one and both of $\{n, n_l\}$ are twos.

(iv) All the n_i for $i \neq l$ are twos.

The first three cases are incompatible with Equation 2.3. Case *(iv)* is only compatible if $n = n_l = 1$. Because B_{KZ} is a summand of P_{KZ}, this implies that $a = b = 1$. □

Let $\mathcal{H}'$ denote the image of $\mathsf{KZ}|_B$, $\mathrm{Hom}_{\mathcal{O}}(B_{\mathsf{KZ}}, P_{\mathsf{KZ}}) = \mathrm{End}_{\mathcal{O}}(B_{\mathsf{KZ}})$. Because KZ is surjective on objects, the fact that B is a block implies that $\mathcal{H}'$ is a block of $\mathcal{H}$.

Lemma 2.2.2. *The image of B under KZ, $\mathcal{H}'$, is isomorphic to the path algebra, C, of the following quiver with relations.*



$$bab = 0$$
$$aba = 0$$

Proof. Let a denote a non–zero homomorphism from $P(\tau_l)$ to $P(\mathbf{sgn})$ and b a non–zero homomorphism from $P(\mathbf{sgn})$ to $P(\tau_l)$. Both of these maps have images in the radicals of the codomains. There are two $L(\tau_l)$ composition factors in $P(\tau_l)$, one in the head and one in the socle. Let c be the endomorphism that sends the head to the socle. Similarly, let d be the endomorphism of $P(\mathbf{sgn})$ that sends the head to the socle. Now ba is an endomorphism of $P(\tau_l)$ whose image is in the radical and therefore is a multiple of c. Similarly ab must be a multiple of d.

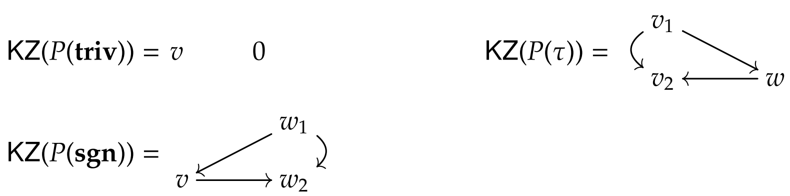
I claim that neither ba nor ab is zero. First, notice that since the socle is simple, every non-zero submodule of $P(\mathbf{sgn})$ must have socle $L(\mathbf{sgn})$. It follows that the image of a is isomorphic to $\Delta(\tau_l)$. Now, if ba were zero then $\Delta(\tau_l)$ would be in the kernel of b. This would imply that $P(\tau_l)$ had a submodule isomorphic to $P(\mathbf{sgn})/\Delta(\tau_l) \cong L(\mathbf{sgn})$. Since the socle of $P(\tau_l)$ is $L(\tau_l)$ this cannot be. Next, suppose $ab = 0$. Then, since b is non-zero, the head of $P(\mathbf{sgn})$, $L(\mathbf{sgn})$, must be a composition factor in the image of b and therefore the kernel of a. Now this accounts for the single $L(\mathbf{sgn})$ composition factor of $P(\tau_l)$ so the image of a cannot contain $L(\mathbf{sgn})$ as a submodule. This contradicts remark that every submodule of $P(\mathbf{sgn})$ contains the socle and so the claim is proved.

Rescale c and d so that $c = ba$ and $d = ab$. Now,

$$\dim \mathrm{End}_{\mathcal{O}}(B_{\mathsf{KZ}}) = [B_{\mathsf{KZ}} : L(\mathbf{sgn})] + [B_{\mathsf{KZ}} : L(\tau_l)] = 3 + 3 = 6,$$

so the maps corresponding to paths of length zero and one generate $\mathrm{End}_{\mathcal{O}}(B_{\mathsf{KZ}})$; since the quiver relations hold, $\mathrm{End}_{\mathcal{O}}(B_{\mathsf{KZ}})$ is isomorphic to a quotient of C. But $\dim C = 6$, so they are isomorphic. □

The following diagrams are the images of the projective indecomposables under the KZ-functor, written as quiver representations with respect to a suitable basis. The images of $P(\tau)$ and $P(\mathbf{sgn})$ are the projective indecomposables of $\mathcal{H}_{\mathbf{c}}$, which are identified with modules of all paths starting at each vertex. Here, an unlabled arrow means that the corresponding arrow on the Hecke algebra quiver acts by multiplication by one and no arrow means multiplication by zero.



Define homomorphisms of these quiver representations as follows.

$$\begin{aligned}
&f_{21}\colon \mathsf{KZ}(P(\mathbf{triv})) \longrightarrow \mathsf{KZ}(P(\tau)); && v \mapsto v_2 && \\
&f_{12}\colon \mathsf{KZ}(P(\tau)) \longrightarrow \mathsf{KZ}(P(\mathbf{triv})); && v_1 \mapsto v && \\
&f_{32}\colon \mathsf{KZ}(P(\tau)) \longrightarrow \mathsf{KZ}(P(\mathbf{sgn})); && v_1 \mapsto v && w \mapsto w_2 \\
&f_{23}\colon \mathsf{KZ}(P(\mathbf{sgn})) \longrightarrow \mathsf{KZ}(P(\tau)); && v \mapsto v_2 && w_1 \mapsto w
\end{aligned}$$

The four maps above are the only maps (up to scalar) between distinct pairs of the three quiver representations. The non–trivial endomorphisms of each indecomposable summand of P can be written as compositions of these four. Identifying the identity maps on $P(\mathbf{triv})$, $P(\tau)$ and $P(\mathbf{sgn})$ with the paths e_1, e_2 and e_3 respectively and f_{ij} with a_{ij} shows that $\mathrm{End}_{\mathcal{H}'}(\mathsf{KZ}(Q))$ is isomorphic to a quotient of B_2, because the relations, R_2, are satisfied. Proposition 2.1.2 and a dimension count shows that $\mathrm{End}_{\mathcal{H}'}(\mathsf{KZ}(Q)) \cong \mathsf{B}_2$. By the double centraliser property, B is Morita equivalent to B_2.

Case A3: Any other complex number

If $\mathbf{c}$ does not satify the conditions of Case 1 or Case 2 then, by [Chm06, Theorem 3.1.1 & Theorem 3.1.2], $\mathcal{O}_{\mathbf{c}}$ is semisimple.

2.2.3 Case B: Even Index Dihedral Groups

Let $d = 2m$ for $m > 1$. By [Chm06, Theorem 3.2.3 & Theorem 3.2.4][1], the structure of $\mathcal{O}_{\mathbf{c}}$ is determined by the position of $\mathbf{c}$ with respect to the following complex planes. Let r and i be integers with r not divisible by m.

$$E_r^+ := \{\mathbf{c} \in \mathbb{C}^2 \colon r = m(c_0 + c_1)\} \qquad E_r^- := \{\mathbf{c} \in \mathbb{C}^2 \colon r = m(c_0 - c_1)\}$$
$$L_i^1 := \left\{\mathbf{c} \in \mathbb{C}^2 \colon c_0 = i + \frac{1}{2}\right\} \qquad L_i^2 := \left\{\mathbf{c} \in \mathbb{C}^2 \colon c_1 = i + \frac{1}{2}\right\}$$

Even though they are really complex planes, they will be refered to as diagonal, vertical and horizontal lines respectively. See Figure 2.1 for an example.

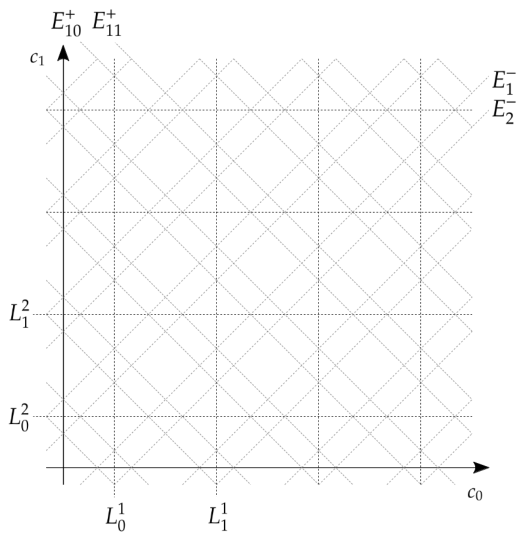


Figure 2.1: A projection of the exceptional lines of parameters from $\mathbb{C}^2$ to $\mathbb{R}^2$ for $m = 3$

[1]Note that [Chm06, Theorem 3.2.3] and its proof contain typos: Every occurence of $i > 0$ and $i' > 0$ should read $i \geq 0$ and $i' \geq 0$ respectively.

The non-trivial one-dimensional representations permute these lines as follows.

$$\epsilon_1: \begin{array}{l} E_r^+ \mapsto E_r^- \\ E_r^- \mapsto E_r^+ \\ L_i^1 \mapsto L_i^1 \\ L_i^2 \mapsto L_{-i}^2 \end{array} \qquad \epsilon_2: \begin{array}{l} E_r^+ \mapsto E_{-r}^- \\ E_r^- \mapsto E_{-r}^+ \\ L_i^1 \mapsto L_{-i}^1 \\ L_i^2 \mapsto L_i^2 \end{array} \qquad \mathbf{sgn}: \begin{array}{l} E_r^+ \mapsto E_{-r}^+ \\ E_r^- \mapsto E_{-r}^- \\ L_i^1 \mapsto L_{-i}^1 \\ L_i^2 \mapsto L_{-i}^2 \end{array}$$

Let $v_1 = e^{2\pi\sqrt{-1}c_0}$ and $v_2 = e^{2\pi\sqrt{-1}c_1}$. The complex Hecke algebra $\mathcal{H}_{\mathbf{c}}(W)$ is the quotient of the associative algebra generated by two variables with the following relations.

$$\mathcal{H}_{\mathbf{c}} := \cfrac{\mathbb{C}\langle T_s, T_t\rangle}{\left\langle \begin{array}{c} (T_s - v_2)(T_s+1) = 0 \\ (T_t - v_1)(T_t + 1) = 0 \\ \underbrace{T_sT_t\cdots T_sT_t}_{d \text{ factors}} = \underbrace{T_tT_s\cdots T_tT_s}_{d \text{ factors}} \end{array}\right\rangle}$$

Lemma 2.2.3. *Let* $\mathbf{c} = (c_0, c_1)$ *and* $\mathbf{c}' = (c_1, c_0)$. *Then* $\mathcal{H}_{\mathbf{c}} \cong \mathcal{H}_{\mathbf{c}'}$.

Proof. The map that exchanges the generators T_s and T_t gives an isomorphism: interchanging them creates a bijection between the relations on each side. □

Case B1.1: The parameter lies on two diagonals, but not a horizontal or vertical line

Suppose that $\mathbf{c}$ doesn't lie on a line L_i^1 or L_i^2 for any $i \in \mathbb{Z}$. If $\mathbf{c} \in E_r^+ \cap E_{r'}^-$ and l and l' are determined by $1 \le l, l' \le m-1$, $r \equiv \pm l$ (**mod** d)) and $r' \equiv m \pm l'$ (**mod** d)) then $l \neq l'$ (otherwise, $\mathbf{c}$ would lie on L_i^1 or L_i^2 for some i). Suppose that $r, r' > 0$, then the standard modules have the following Jordan Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\tau_l) \end{matrix} \qquad \Delta(\epsilon_1) = \begin{matrix} L(\epsilon_1) \\ L(\tau_{l'}) \end{matrix} \qquad \Delta(\epsilon_2) = L(\epsilon_2)$$

$$\Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_l) = \begin{matrix} L(\tau_l) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\tau_{l'}) = \begin{matrix} L(\tau_{l'}) \\ L(\epsilon_2) \end{matrix}$$

For i not equal to l or l', $\Delta(\tau_i) = L(\tau_i)$. By Corollary 1.7.1, there are two non-simple blocks corresponding to the subsets $\{\mathbf{triv}, \mathbf{sgn}, \tau_l\}$ and $\{\epsilon_1, \epsilon_2, \tau_{l'}\}$ of $\mathrm{Irr}(W)$. For each of these blocks, the argument that they are Morita equivalent to B_2 follows Case A2.

Remark 2.2.1. *When* $r > 0$ *and* $r' < 0$, *the argument is the same as B1.1 with the roles of* ϵ_1 *and* ϵ_2 *swapped. When* $r < 0$ *and* $r' > 0$, *the argument is the same as B1.1 with the roles of* **triv** *and* **sgn** *swapped. When* $r < 0$ *and* $r' < 0$ *the argument is the same as B1.1 with the roles of* ϵ_1 *and* ϵ_2 *swapped and the roles of* **triv** *and* **sgn** *swapped.*

Case B1.2.1: The parameter only lies on one diagonal: E_r^+

If $\mathbf{c} \in E_r^+$ for $r > 0$ and $\mathbf{c} \notin E_{r'}^-$ for any r', the standard modules have the following Jordan Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\tau_l) \end{matrix} \qquad \Delta(\epsilon_1) = L(\epsilon_1) \qquad \Delta(\epsilon_2) = L(\epsilon_2)$$

$$\Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_l) = \begin{matrix} L(\tau_l) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\tau_i) = L(\tau_i)$$

for all $i \neq l$. There is one non–simple block and the argument that it is Morita equivalent to B_2 follows Case A2. When $r < 0$, the argument is the same as B1.2.1 with the roles of **triv** and **sgn** swapped.

Case B1.2.2: The parameter lies on only one diagonal: $E^-_{r'}$

If $\mathbf{c} \in E^-_{r'}$ for $r' > 0$ and $\mathbf{c} \notin E^-_r$ for any r, then the standard modules have the following Jordan Hölder series.

$$\Delta(\mathbf{triv}) = L(\mathbf{triv}) \qquad \Delta(\epsilon_1) = \begin{matrix} L(\epsilon_1) \\ L(\tau_l) \end{matrix} \qquad \Delta(\epsilon_2) = L(\epsilon_2)$$

$$\Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_l) = \begin{matrix} L(\tau_l) \\ L(\epsilon_2) \end{matrix} \qquad \Delta(\tau_i) = L(\tau_i)$$

for all $i \neq l$. This is the same as B1.2.1 with the roles of the one-dimensional characters interchanged by tensoring with ϵ_1. When $r' < 0$, the argument is the same as B1.2.2 with the roles of ϵ_1 and ϵ_2 swapped.

Case B2.1: The parameter lies on a horizontal and vertical line

Suppose $\mathbf{c} \in L^1_i \cap L^2_{i'}$ for some $i, i' \in \mathbb{Z}$ and assume that $i, i' \geq 0$. The standard modules are as follows.

$$\Delta(\mathbf{triv}) = \begin{matrix} X \\ L(\mathbf{sgn})' \end{matrix} \qquad X := \begin{matrix} L(\mathbf{triv}) \\ L(\epsilon_1) \oplus L(\epsilon_2)' \end{matrix} \qquad \Delta(\epsilon_1) = \begin{matrix} L(\epsilon_1) \\ L(\mathbf{sgn})' \end{matrix}$$

$$\Delta(\epsilon_2) = \begin{matrix} L(\epsilon_2) \\ L(\mathbf{sgn})' \end{matrix} \qquad \Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_i) = L(\tau_i)$$

for all $i = 1, \ldots, m-1$. It follows that each $L(\tau_i)$ belongs to a simple block and the rest of the simples belong to a single block B. Let P be the progenerator

$$P = P(\mathbf{triv}) \oplus P(\epsilon_1) \oplus P(\epsilon_2) \oplus P(\mathbf{sgn})$$

for B. As in Case A2, let Q denote the summand of P_{KZ} that corresponds to the restriction of the KZ-functor to B. Notice that $L(\mathbf{sgn})$ is in the socle of every standard module in B, so if a particular projective cover is also an injective hull it must have a simple socle: $L(\mathbf{sgn})$. This means that Q is a direct sum of modules isomorphic to $P(\mathbf{sgn})$. It follows from Theorem 1.5.3 that neither $L(\epsilon_1)$ nor $L(\epsilon_2)$ could appear as submodules of $\Delta(\mathbf{triv})$; therefore the socle of $\Delta(\mathbf{triv})$ is $L(\mathbf{sgn})$.

Lemma 2.2.4. *The mulitiplicity of* $P(\mathbf{sgn})$ *in* Q *is one:* $Q \cong P(\mathbf{sgn})$.

Proof. There are m non-isomorphic projective modules in P_{KZ}, so there are m non-isomorphic irreducible $\mathcal{H}_{\mathbf{c}}$-modules. Because $v_1 = v_2 = -1$, by [GP00], the specialisation $\mathcal{H}_{\mathbf{c}}$ has only one one–dimensional representation and so $m-1$ two-dimensional representations.

$$P_{\mathsf{KZ}} = P(\mathbf{sgn})^n \oplus P(\tau_1)^{n_1} \oplus \cdots \oplus P(\tau_{m-1})^{n_{m-1}}$$

for $n, n_1, \ldots, n_{m-1}$ integers, one of which is one and the rest are two. Then, because of the block decomposition,

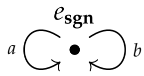
$$\mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}}) = \mathrm{End}_{\mathcal{O}}(P(\mathbf{sgn})^n) \oplus \mathrm{End}(P(\tau_i)^{n_i}) \oplus \cdots \oplus \mathrm{End}(P(\tau_{m-1})^{n_{m-1}})$$

and the number of $L(\mathbf{sgn})$ composition factors in $P(\mathbf{sgn})^n$ is $4n$. This gives

$$4m = |W| = \dim \mathcal{H}_{\mathbf{c}} = 4n^2 + \sum_{i=1}^{m-1} n_i^2.$$

The only solution to this is when $n = 1$ and $n_i = 2$ for all i. □

Lemma 2.2.5. *The algebra $\mathrm{End}_{\mathcal{O}}(Q)$ is isomorphic to the path algebra, C, of the quiver with relations*



$$ab = 0$$
$$ba = 0\,.$$
$$a^2 = b^2$$

Proof. Because $L(\mathbf{sgn})$ appears in exactly four of the standard modules and $P(\mathbf{sgn})$ is indecomposable, the algebra $\mathrm{End}_{\mathcal{O}}(P(\mathbf{sgn}))$ is four-dimensional and local. Because it is projective and appears as a subquotient, $\Delta(\mathbf{triv})$ must be a submodule of $P(\mathbf{sgn})$; together with the fact that $P(\mathbf{sgn})$ is injective, it follows that the socle of $P(\mathbf{sgn})$ coincides with the socle of $\Delta(\mathbf{triv})$. In other words, every submodule of $P(\mathbf{sgn})$ has socle $L(\mathbf{sgn})$.

I claim that the submodule generated by the subquotient $\Delta(\epsilon_1)$, M say, is isomorphic to $P(\epsilon_1)$. Indeed, let f be a homomorphism from $P(\epsilon_1)$ to $P(\mathbf{sgn})$ whose image is M. It follows that M must contain the composition factor $L(\mathbf{sgn})$, which is a submodule of the subquotient $\Delta(\epsilon_1)$. Also, as noted above, M must contain the submodule $L(\mathbf{sgn})$ of $\Delta(\mathbf{triv})$. This accounts for two $L(\mathbf{sgn})$ composition factors in M, but the domain, $P(\epsilon_1)$, only contains two such composition factors. Therefore, M contains no more than these two. Note, that every non-zero submodule of $P(\epsilon_1)$ must contain at least one of these $L(\mathbf{sgn})$ composition factors. Therefore, the kernel of f is zero and so $P(\epsilon_1)$ is isomorphic to M. A similar argument shows that $P(\epsilon_2)$ is isomorphic to a submodule of $P(\mathbf{sgn})$.

There is a map from $P(\mathbf{sgn})$ to $P(\epsilon_1)$ that corresponds to the $L(\mathbf{sgn})$ composition factor of $\Delta(\epsilon_1)$. In other words, the image of such a map in $P(\epsilon_1)$ is not contained in the submodule $\Delta(\mathbf{triv})$. Define an endomorphism, a, of $P(\mathbf{sgn})$ by the composition of this map from $P(\mathbf{sgn})$ to $P(\epsilon_1)$ followed by the injection of $P(\epsilon_1)$ into $P(\mathbf{sgn})$. Let $b\colon P(\mathbf{sgn}) \longrightarrow P(\epsilon_2) \longrightarrow P(\mathbf{sgn})$ be defined similarly for $P(\epsilon_2)$. Let c denote the endomorphism of $P(\mathbf{sgn})$ whose image is the simple socle.

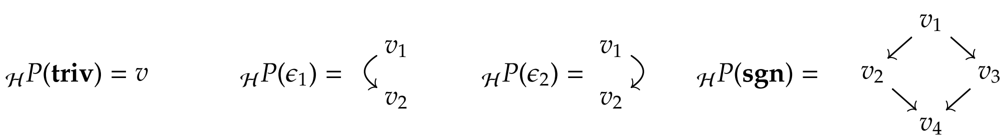
Now the image of a, M' say, is a submodule of M and it is not simple so it contains exactly two $L(\mathbf{sgn})$ composition factors: the head and the socle. Comparing composition factors in $P(\mathbf{sgn})$, the kernel of a cannot contain the whole of M' so a acts as a non–zero nilpotent endomorphism of M'. Therefore, a^2 is some non-zero multiple, λc, of c. A similar argument, with ϵ_1 replaced with ϵ_2 shows that b^2 is some non–zero multiple, μc, of c. Redefine a and b so that $\lambda = \mu = 1$.

Now, I claim that $ab = ba = 0$. Since the algebra $\mathrm{End}_{\mathcal{O}}(P(\mathbf{sgn}))$ is four-dimensional it suffices to show that $ab = ba = a^2$ cannot hold. Indeed, in that case, both a and b act as non-zero endomorphisms of $\mathsf{KZ}(P(\epsilon_1))$ so that there is only one homomorphism $\mathsf{KZ}(P(\mathbf{sgn})) \longrightarrow \mathsf{KZ}(P(\epsilon_1))$ (up to scalar) which sends the head of $\mathsf{KZ}(P(\mathbf{sgn}))$ to the socle of $\mathsf{KZ}(P(\epsilon_1))$. However, by the double centraliser property,

$$\begin{aligned}\dim \mathrm{Hom}_{\mathcal{H}}(\mathsf{KZ}(P(\mathbf{sgn})), \mathsf{KZ}(P(\epsilon_1))) &= \dim \mathrm{Hom}_{\mathcal{O}}(P(\mathbf{sgn}), P(\epsilon_1)) \\ &= [P(\epsilon_1) : L(\mathbf{sgn})] \\ &= 2;\end{aligned}$$

a contradiction. □

The images of projective indecomposables in B under the KZ-functor are calculated as representations of the quiver in Lemma 2.2.5 to which $\mathcal{H}_{\mathbf{c}}$ corresponds. The image of $P(\mathbf{triv})$ is the one-dimensional representation with the arrows all acting by multiplication by zero. From the discussion in the proof of Lemma 2.2.5, $P(\epsilon_1)$ and $P(\epsilon_2)$ are isomorphic to submodules of $P(\mathbf{sgn})$. Also, $P(\mathbf{sgn})$ is sent to the regular representation. Let $v_1, \ldots, v_4$ be the basis corresponding to the paths $e_{\mathbf{sgn}}, a, b, a^2$ respectively. In the diagrams below, an unmarked arrow means multiplication by one and no arrow means by the zero map.



Define endomophisms of $\mathsf{KZ}(P)$ as follows.

$$\begin{array}{lll}
f_{21}\colon \mathsf{KZ}(P(\mathbf{triv})) \longrightarrow \mathsf{KZ}(P(\epsilon_1)); & v \mapsto v_2 & \\
f_{12}\colon \mathsf{KZ}(P(\epsilon_1)) \longrightarrow \mathsf{KZ}(P(\mathbf{triv})); & v_1 \mapsto v & \\
f_{31}\colon \mathsf{KZ}(P(\mathbf{triv})) \longrightarrow \mathsf{KZ}(P(\epsilon_2)); & v \mapsto v_2 & \\
f_{13}\colon \mathsf{KZ}(P(\epsilon_2)) \longrightarrow \mathsf{KZ}(P(\mathbf{triv})); & v_1 \mapsto v & \\
f_{42}\colon \mathsf{KZ}(P(\epsilon_1)) \longrightarrow \mathsf{KZ}(P(\mathbf{sgn})); & v_1 \mapsto v_2 & v_2 \mapsto v_4 \\
f_{24}\colon \mathsf{KZ}(P(\mathbf{sgn})) \longrightarrow \mathsf{KZ}(P(\epsilon_1)); & v_1 \mapsto v_1 & v_2 \mapsto v_2 \\
f_{34}\colon \mathsf{KZ}(P(\mathbf{sgn})) \longrightarrow \mathsf{KZ}(P(\epsilon_2)); & v_1 \mapsto v_1 & v_3 \mapsto v_2 \\
f_{43}\colon \mathsf{KZ}(P(\epsilon_2)) \longrightarrow \mathsf{KZ}(P(\mathbf{sgn})); & v_1 \mapsto v_3 & v_2 \mapsto v_4.
\end{array}$$

To each identity map on $P(\mathbf{triv})$, $P(\epsilon_1)$, $P(\epsilon_2)$ and $P(\mathbf{sgn})$, associate trivial path at the vertex $1, 2, 3$ and 4 of Q_3 respectively. For each map f_{ji} associate the path a_{ji} in Q_3. The endomorphisms, $\{f_{ij}\}$, generate $\mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P))$ and satisfy relations corresponding to R_3; so $\mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P))$ is isomorphic to a quotient of B_3. Using the double centraliser property and BGG reciprocity to count the simple composition factors of the projective covers, $\dim \mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P)) = \dim \mathrm{End}_{\mathcal{O}}(P) = 25$; so this map is an isomorphism. It follows that B is Morita equivalent to B_3.

Case B2.2: The parameter lies on either a horizontal line or a vertical line, but not both

If $\mathbf{c} \in L^1_{i}$, and $i \geq 0$ then $\Delta(\tau_j) = L(\tau_j)$ for all j and the other standard modules have the following Jordan–Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\epsilon_2) \end{matrix} \qquad \Delta(\epsilon_2) = L(\epsilon_2) \qquad \Delta(\epsilon_1) = \begin{matrix} L(\epsilon_1) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\mathbf{sgn}) = L(\mathbf{sgn})$$

There are two non-simple blocks, each containing two simples. By Proposition 2.2.1, each of these is Morita equivalent to B_1.

If $\mathbf{c} \in L^2_i$ and $i \geq 0$ then $\Delta(\tau_j) = L(\tau_j)$ for all j and the other standard modules have the following Jordan–Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\epsilon_1) \end{matrix} \qquad \Delta(\epsilon_1) = L(\epsilon_1) \qquad \Delta(\epsilon_2) = \begin{matrix} L(\epsilon_2) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\mathbf{sgn}) = L(\mathbf{sgn})$$

This is the same as the case above with the roles of ϵ_1 and ϵ_2 interchanged.

In both cases if $i < 0$ then the same arguments apply with the roles of ϵ_1 and ϵ_2 swapped.

Case B3.1: The parameter lies on three lines: two diagonals and a horizontal

If three of the lines of parameters intersect then they must be E^+_r, $E^-_{r'}$ and either L^1_i or $L^2_{i'}$. This happens precisely when $r' \equiv m \pm r\ (\mathbf{mod}\ d))$. Suppose that $\mathbf{c} \in E^+_r \cap E^-_{r'} \cap L^2_i$ where $r \equiv l\ (\mathbf{mod}\ d))$, $r' \equiv m \pm l\ (\mathbf{mod}\ d))$, $i \in \mathbb{Z}$ and $1 \leq l \leq m-1$.

Remark 2.2.2. *As in Case B1.1, consider the four possible combinations of signs of r and r' seperately. In Case B3.1 (and later B3.2), the calculation is done using the assumption $r, r' > 0$. When $r > 0$ and $r' < 0$, the argument is the same with the roles of ϵ_1 and ϵ_2 swapped; when $r < 0$ and $r' > 0$, the argument is the same as with the roles of* **triv** *and* **sgn** *swapped; and when $r < 0$ and $r' < 0$ the argument is the same with the roles of ϵ_1 and ϵ_2 swapped and the roles of* **triv** *and* **sgn** *swapped.*

Assume $r, r' > 0$. By [Chm06], the standard modules have the following Jordan Hölder series.

$$\Delta(\epsilon_1) = \begin{matrix} L(\epsilon_1) \\ L(\tau_l) \end{matrix} \qquad \Delta(\epsilon_2) = \begin{matrix} L(\epsilon_2) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_l) = \begin{matrix} L(\tau_l) \\ L(\epsilon_2) \\ L(\mathbf{sgn}) \end{matrix}$$

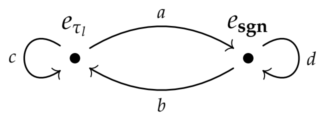
The standard module $\Delta(\mathbf{triv})$ contains a submodule which is a non–split extension of $\Delta(\epsilon_1)$ by $L(\epsilon_2)$ (that is, $\Delta(\epsilon_1)$ is a submodule). The quotient by this submodule is irreducible. Therefore,

$\Delta(\mathbf{triv})$ has head $L(\mathbf{triv})$ and has socle either $L(\tau_l)$ or $L(\tau_l) \oplus L(\epsilon_2)$. For all $j \neq l$ the standard module $\Delta(\tau_j) = L(\tau_j)$. Let B denote the non-simple block; it has the progenerator

$$P := P(\mathbf{triv}) \oplus P(\epsilon_1) \oplus P(\tau_l) \oplus P(\epsilon_2) \oplus P(\mathbf{sgn}).$$

Let P_{KZ} denote the projective module that represents the KZ-functor restricted to this block.

Lemma 2.2.6. *The block of the Hecke algebra $\mathcal{H}_{\mathbf{c}}$ corresponding to B is isomorphic to the path algebra of the following quiver with relations.*



$$\begin{array}{ll} da = ac = 0 & ab = d^2 \\ cb = bd = 0 & ba = c^2. \end{array}$$

Proof. By Theorem 1.5.3, the projective covers that appear as summands of P_{KZ} are $P(\mathbf{sgn})$ and $P(\tau_l)$. It follows that $P(\mathbf{sgn})$ and $P(\tau_l)$ must each have simple socles isomorphic to their heads. Let $P'_{\mathsf{KZ}} = P(\mathbf{sgn}) \oplus P(\tau_l)$ so that $\mathrm{End}_{\mathcal{O}}(P'_{\mathsf{KZ}})$ is Morita equivalent to $\mathrm{End}_{\mathcal{O}}(P_{\mathsf{KZ}})$.

Now, by BGG reciprocity, $P(\mathbf{sgn})$ has only one composition factor isomorphic to $L(\tau)$ and $P(\tau_l)$ has only one isomorphic to $L(\mathbf{sgn})$ so $\mathrm{Hom}_{\mathcal{O}}(P(\tau_l), P(\mathbf{sgn}))$ and $\mathrm{Hom}(P(\mathbf{sgn}), P(\tau_l))$ are both one–dimensional. By a similar argument, $\mathrm{End}_{\mathcal{O}}(P(\tau_l))$ is three–dimensional. Because $\Delta(\mathbf{triv})$ is projective it is a submodule (Lemma 1.1.2), so the standard filtration of $P(\tau_l)$ is

$$\begin{array}{c} \Delta(\tau_l) \\ \Delta(\epsilon_1) \\ \Delta(\mathbf{triv}) \end{array}$$

Because $P(\tau_l)$ has a simple socle $L(\tau_l)$, define y to be the (unique up to scalar) endomorphism of $P(\tau_l)$ that maps the head to the socle. Let x be the endomorphism of $P(\tau_l)$ whose image lies in the submodule $\begin{array}{c}\Delta(\epsilon_1)\\ \Delta(\mathbf{triv})\end{array}$ and is non–zero modulo $\Delta(\mathbf{triv})$. Each of the maps $\{\mathrm{id}, x, y\}$ maps the generator of $P(\tau_l)$ to a vector in a different radical layer so these form a basis for $P(\tau_l)$. Neither x nor y are isomorphisms so they must be nilpotent.

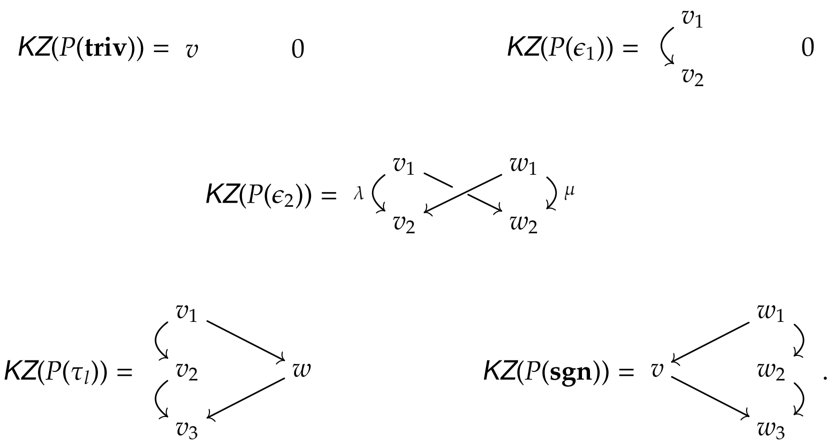
Now, the submodule, M, of $P(\tau_l)$, generated by $\mathrm{im}\, x$, contains the submodule $L(\tau_l)$ because $L(\tau_l)$ is the socle. Since x is linearly independent from y, M is not simple and so has two $L(\tau_l)$ composition factors: the head and the socle. If x were to act as zero on the generator for M then M (and therefore the socle $L(\tau_l)$) would be in the kernel of x, but no quotient of $P(\tau_l)$ with M in the kernel could contain two composition factors isomorphic to $L(\tau_l)$. Therefore x, being nilpotent, maps the generator of M to the socle. That is, x^2 is a multiple of y so that $\{\mathrm{id}, x, x^2\}$ is a basis for $\mathrm{End}_{\mathcal{O}}(P(\tau_l))$. An identical argument with τ_l replaced with $\mathbf{sgn}$ shows that $\mathrm{End}_{\mathcal{O}}(\mathbf{sgn})$ has a similar form. Let c and d be the endomorphisms of $P(\tau_l)$ and $P(\mathbf{sgn})$, respectively, that are represented by x in the above argument. We have $c^3 = d^3 = 0$. Let a be a non–zero homomorphism from $P(\tau_l)$ to $P(\mathbf{sgn})$ and b be a non–zero homomorphism between $P(\mathbf{sgn})$ and $P(\tau_l)$. Now, ac is also a map from $P(\tau_l)$ to $P(\mathbf{sgn})$ so ac must be a multiple of a. But, $ac^3 = 0$ so $ac = 0$. A similar argument shows that $da = cb = bd = 0$.

Suppose ba is zero. Then b would generate a one–dimensional submodule so the socle would contain the linearly independent elements c^2, b and d^2. The head is spanned by the two elements $\{\mathrm{id}_{P(\mathbf{sgn})}, \mathrm{id}_{P(\tau_l)}\}$. Hecke algebras are symmetric so, Lemma 1.1.3 gives a contradiction. Now, ba is a non–zero endomorphism of $P(\tau_l)$ so $ba = \alpha + \beta c + \gamma c^2$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. But then $0 = cba = \alpha c + \beta c^2$ so $\alpha = \beta = 0$. Redefine c so that $ba = c^2$. A similar argument shows that we can redefine d so that $d^2 = ab$.

Now $\mathcal{H}_{\mathbf{c}}$ is Morita equivalent to a quotient of the path algebra of the quiver, comparing dimensions with $\mathrm{End}_{\mathcal{O}}(P'_{\mathsf{KZ}})$ shows that they are isomorphic and $P'_{\mathsf{KZ}} = P_{\mathsf{KZ}}$. □

Now calculate the image of the projective modules under the KZ–functor as representations of this quiver.

Lemma 2.2.7. *The images of the projective covers in $\mathcal{O}$ under the KZ–functor have bases so that they can be represented by the following quiver representation diagrams. Here, an unlabelled arrow means* *that the corresponding arrow on the Hecke algebra quiver acts by multiplication by one and no arrow means multiplication by zero.*



Proof. First, note that, for any $\rho \in \mathrm{Irr}(W)$, because the KZ–functor is fully faithful on projectives, $\mathsf{KZ}(\mathrm{End}_{\mathcal{O}}(P(\rho)) \cong \mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P(\rho)))$ and, because functors preserve composition of maps, this is a ring isomorphism. In particular, localness is preserved so that $\mathsf{KZ}(P(\rho))$ must be indecomposable.

Now, $P(\mathbf{triv})$ has one $L(\tau_l)$ composition factor and no $L(\mathbf{sgn})$ so $\mathsf{KZ}(P(\mathbf{triv}))$ is the one - dimensional irreducible representation above. Next, $P(\epsilon_1)$ has two $L(\tau_l)$ composition factors and no $L(\mathbf{sgn})$. If c acts as zero on this representation then it is the direct sum of two copies of $\mathsf{KZ}(P(\mathbf{triv}))$, but $\mathsf{KZ}(P(\epsilon_1))$ must be indecomposable. Thus c cannot act as zero (or as an isomorphism of $\mathbb{C}^2$ because $c^3 = 0$), so it acts as a rank one matrix. There is a non–trivial endomorphism of $\mathsf{KZ}(P(\epsilon_1))$ that is not an isomorphism, denote this by f.

The images of $P(\tau_l)$ and $P(\mathbf{sgn})$ are the projective covers of the respective simple modules in $\mathcal{H}$. We identify these with left modules of all paths starting at each node.

Now, $P(\epsilon_2)$ has two of each type of composition factors so has dimension vector $(2, 2)$. First, if a were an isomorphism then, in order to satisfy $ac = da = 0$, both c and d would need to be zero. In that case, $ab = d^2$ would act as zero so b would need to act as zero and the representation would be reducible. A similar argument shows that b cannot be an isomorphism.

The spaces $\mathrm{Hom}(P(\mathbf{sgn}), P(\epsilon_2))$ and $\mathrm{Hom}(P(\tau_l), P(\epsilon_2))$ are two-dimensional. If a acts as zero on $\mathsf{KZ}(P(\epsilon_2))$ then $\mathrm{Hom}(\mathsf{KZ}(P(\mathbf{sgn})), \mathsf{KZ}(P(\epsilon_2)))$ is one-dimensional and if b acts as zero then $\mathrm{Hom}(\mathsf{KZ}(P(\tau_l)), \mathsf{KZ}(P(\epsilon_2)))$ is one-dimensional. Because the *kz*–functor is faithful, a and b cannot act as zero.

Therefore a and b have rank one. Let v_2 and w_2 be non–zero vectors in $\ker a$ and $\mathrm{im}\, a$ respectively. Since $c^3 = d^3 = 0$, c and d cannot act faithfully so their ranks must be less than two and $c^2 = ba$ and $d^2 = ab$ must act as zero. Therefore $\ker b$ is spanned by w_2 and $\mathrm{im}\, b$ is spanned by v_2. If c and d are both zero then the module would be reducible. Choose v_1 so that $a(v_1) = w_2$ and w_1 so that $b(w_1) = v_2$. Now there are complex numbers λ and μ (not both zero) such that c and d act on v_1 and w_1 by these constants respectively.

□

Lemma 2.2.8. *Let $M_\mu = \mathsf{KZ}(P(\epsilon_2))$, where $\mu \in \mathbb{C}$ is defined in the proof of the lemma above. Then $M_\mu \cong M_1$ as $\mathcal{H}$–modules.*

Proof. Define M to be the quotient of $P(\epsilon_2)$ by the submodule $\Delta(\mathbf{triv})$. Because it must have a simple head, M is a non–split extension of $\Delta(\tau_l)$ by $\Delta(\epsilon_2)$ and so corresponds to a non–zero element in $\mathrm{Ext}^1_{\mathcal{O}}(\Delta(\epsilon_2), \Delta(\tau_l))$ which we denote by x. Let i denote the inclusion of $L(\mathbf{sgn})$ into $\Delta(\epsilon_2)$ and π the quotient of M by $\Delta(\tau_l)$.

Let M' be the pullback of π along i, that is,

$$M' := \{(m, v) \in M \oplus L(\mathbf{sgn}) \mid \pi(m) = i(v)\}.$$

Let p_2 denote projection onto the second summand. The kernel of this map is the set of all

$m \in M$ such that $\pi(m) = i(0) = 0$ which is precisely the kernel, $\Delta(\tau_l)$, of π. We now have the commuting diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta(\tau_l) & \longrightarrow & M' & \xrightarrow{p_2} & L(\mathbf{sgn}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & \Delta(\tau_l) & \longrightarrow & M & \xrightarrow{\pi} & \Delta(\epsilon_2) & \longrightarrow & 0 \end{array}$$

and the right square is a pullback, so that the top short exact sequence is the image, i^*x, of the bottom short exact sequence x under the map

$$i^* \colon \mathrm{Ext}^1(\Delta(\epsilon_2), \Delta(\tau_l)) \longrightarrow \mathrm{Ext}^1(L(\mathbf{sgn}), \Delta(\tau_l))$$

induced by i.

Let N denote the quotient of $P(\mathbf{sgn})$ by the submodule $\Delta(\tau_l)$. Applying the functor $\mathrm{Hom}_{\mathcal{O}}(L(\epsilon_2), -)$ to this short exact sequence gives the exact sequence

$$\mathrm{Hom}(L(\epsilon_2), N) \to \mathrm{Ext}^1(L(\epsilon_2), \Delta(\tau_l)) \to \mathrm{Ext}^1(L(\epsilon_2), P(\mathbf{sgn})).$$

Now, N is an extension of $\Delta(\epsilon_2)$ by $\Delta(\mathbf{sgn})$ so $\mathrm{Hom}(L(\epsilon_2), N) = 0$ and $P(\mathbf{sgn})$ is injective so $\mathrm{Ext}^1(L(\epsilon_2), P(\mathbf{sgn})) = 0$ and so, by exactness, $\mathrm{Ext}^1(L(\epsilon_2), \Delta(\tau_l)) = 0$. Now apply the functor $\mathrm{Ext}^1(-, \Delta(\tau_l))$ to the short exact sequence $L(\mathbf{sgn}) \to \Delta(\epsilon_2) \to L(\epsilon_2)$. We have an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\Delta(\epsilon_2), \Delta(\tau_l)) \longrightarrow \mathrm{Ext}^1(L(\mathbf{sgn}), \Delta(\tau_l)).$$

The map on the right, i^*, is therefore injective so i^*x is non–zero and M' is a non–split extension of $\Delta(\tau_l)$ by $L(\mathbf{sgn})$. Now note that the kernel of p_1 is the set of all elements $(0, v)$ such that $i(v) = 0$ this is only satisfied by zero. Thus M' is isomorphic to a submodule of M and so there is only one copy of $L(\mathbf{sgn})$ in the socle of M.

The previous calculation shows that there is a map, which we denote ϕ, from $P(\mathbf{sgn})$ to M' whose image is not the socle of M'. The socle of M' is simple so must be contained in the image of any non–zero map into M'. Therefore the image of ϕ must contain two $L(\mathbf{sgn})$ composition factors and so the kernel of ϕ contains exactly one which must be in the socle of $P(\mathbf{sgn})$. It follows that the other two $L(\mathbf{sgn})$ composition factors of $P(\mathbf{sgn})$ cannot be in the kernel of ϕ so that the image, $d(P(\mathbf{sgn}))$, of d cannot be contained in $\ker \phi$. That is, $\phi \circ d \neq 0 \in \mathrm{Hom}_{\mathcal{O}}(P(\mathbf{sgn}), M) \subset \mathsf{KZ}(M)$

Applying the (exact) KZ–functor to the short exact sequence $0 \to \Delta(\mathbf{triv}) \to P(\epsilon_2) \to M \to 0$ gives us that $\mathsf{KZ}(M)$ is isomorphic to the quotient of $\mathsf{KZ}(P(\epsilon_2))$ by $\mathsf{KZ}(P(\mathbf{triv}))$. As a quiver representation, using the notation above, $\mathsf{KZ}(M)$ has the diagram

$$\mathsf{KZ}(P(\epsilon_2)) = \begin{array}{ccc} v_1 & & w_1 \\ & \searrow & \downarrow{\scriptstyle \mu} \\ & & w_2 \end{array}$$

Therefore, μ is the scalar by which precomposition by d acts on $\mathrm{Hom}_{\mathcal{O}}(P(\mathbf{sgn}), M)$. However, we have exhibited a non–zero element of this space, ϕ, which is not killed by d. Therefore μ cannot equal zero. We can now assume that $\mu = 1$ by multiplying w_2 by $\frac{1}{\mu}$, v_1 by μ and λ by $\frac{\lambda}{\mu}$. □

Extend the following maps to endomorphisms of $\mathsf{KZ}(P)$.

$f_{41}\colon \mathsf{KZ}(P(\mathbf{triv})) \longrightarrow \mathsf{KZ}(P(\epsilon_2))$	$v \mapsto v_2$		
$f_{14}\colon \mathsf{KZ}(P(\epsilon_2)) \longrightarrow \mathsf{KZ}(P(\mathbf{triv}))$	$v_1 \mapsto v$		
$f_{21}\colon \mathsf{KZ}(P(\mathbf{triv})) \longrightarrow \mathsf{KZ}(P(\epsilon_1))$	$v \mapsto v_2$		
$f_{12}\colon \mathsf{KZ}(P(\epsilon_1)) \longrightarrow \mathsf{KZ}(P(\mathbf{triv}))$	$v_1 \mapsto v$		
$f_{32}\colon \mathsf{KZ}(P(\epsilon_1)) \longrightarrow \mathsf{KZ}(P(\tau_l))$	$v_1 \mapsto v_2$	$v_2 \mapsto v_3$	
$f_{23}\colon \mathsf{KZ}(P(\tau_l)) \longrightarrow \mathsf{KZ}(P(\epsilon_1))$	$v_1 \mapsto v_1$	$v_2 \mapsto v_2$	
$f_{43}\colon \mathsf{KZ}(P(\tau_l)) \longrightarrow \mathsf{KZ}(P(\epsilon_2))$	$v_1 \mapsto v_1$	$v_2 \mapsto \lambda v_2$	$w \mapsto w_2$
$f_{34}\colon \mathsf{KZ}(P(\epsilon_2)) \longrightarrow \mathsf{KZ}(P(\tau_l))$	$v_1 \mapsto \lambda v_2$	$v_2 \mapsto v_3$	$w_1 \mapsto w$
$f_{54}\colon \mathsf{KZ}(P(\epsilon_2)) \longrightarrow \mathsf{KZ}(P(\mathbf{sgn}))$	$v_1 \mapsto v$	$w_1 \mapsto w_2$	$w_2 \mapsto w_3$
$f_{45}\colon \mathsf{KZ}(P(\mathbf{sgn})) \longrightarrow \mathsf{KZ}(P(\epsilon_2))$	$v \mapsto v_2$	$w_1 \mapsto w_1$	$w_2 \mapsto w_2$

To each identity map on $P(\mathbf{triv})$, $P(\epsilon_1)$, $P(\tau_l)$, $P(\epsilon_2)$ and $P(\mathbf{sgn})$, associate the vertex e_1, e_2, e_3, e_4 and e_5 of B_5 respectively. To each map f_{ji} associate the arrow a_{ji} of Q_5. Individually examining the vector spaces $\mathrm{Hom}_{\mathcal{H}}(\mathsf{KZ}(P(\lambda)), \mathsf{KZ}(P(\mu)))$ for all $\lambda, \mu \in \mathrm{Irr}(W)$, it is clear that the endomorphisms $\{f_{ij}\}$ generate $\mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P))$ and that they satisfy relations corresponding to R_5. Therefore, $\mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P))$ is isomorphic to a quotient of B_5. Using Theorem 1.5.2 and BGG reciprocity to count the simple composition factors of the projective covers, $\dim \mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P)) = \dim \mathrm{End}_{\mathcal{O}}(P) = 34$ so this map is an isomorphism. It follows that $\mathrm{End}_{\mathcal{O}}(P)$ is Morita equivalent to B_5^{λ} for some $\lambda \in \mathbb{C}$.

Case B3.2: The parameter lies on three lines: two diagonals and a vertical

Suppose that $\mathbf{c} \in E_r^+ \cap E_{r'}^- \cap L_i^1$ where $r \equiv l \,(\mathbf{mod}\ d))$, $r' \equiv m \pm l \,(\mathbf{mod}\ d))$, $i \in \mathbb{Z}$ and $1 \le l \le m-1$. With Remark 2.2.2 in mind, assume that $r, r' > 0$.

By [Chm06], the standard modules have the following Jordan Hölder series.

$$\Delta(\mathbf{triv}) = \begin{matrix} L(\mathbf{triv}) \\ L(\tau_l) \\ L(\epsilon_2) \end{matrix} \qquad \Delta(\epsilon_1) = \begin{matrix} L(\epsilon_1) \\ L(\tau_l) \\ L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\epsilon_2) = L(\epsilon_2)$$

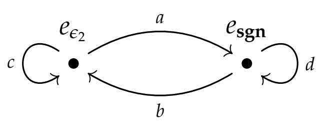
$$\Delta(\mathbf{sgn}) = L(\mathbf{sgn}) \qquad \Delta(\tau_l) = \begin{matrix} L(\tau_l) \\ L(\epsilon_2) \oplus L(\mathbf{sgn}) \end{matrix} \qquad \Delta(\tau_{\neq l}) = L(\tau_{\neq l}).$$

As in the previous case, there is one non-simple block B. Let P be the progenerator

$$P := P(\mathbf{triv}) \oplus P(\epsilon_1) \oplus P(\tau_l) \oplus P(\epsilon_2) \oplus P(\mathbf{sgn}).$$

Let P_{KZ} denote the projective module which represents the KZ-functor restricted to this block. Since the simple modules that appear as summands of standards are $L(\epsilon_2)$ and $L(\mathbf{sgn})$, the corresponding block of $\mathcal{H}$ is Morita equivalent to $\mathrm{End}_{\mathcal{O}}(P'_{\mathsf{KZ}})$ where $P'_{\mathsf{KZ}} := P(\epsilon_2) \oplus P(\mathbf{sgn})$.

Lemma 2.2.9. *The block of the Hecke algebra $\mathcal{H}_{\mathbf{c}}$ which is the image of $\mathsf{KZ}|_B$ is isomorphic to the path algebra of the following quiver with relations.*

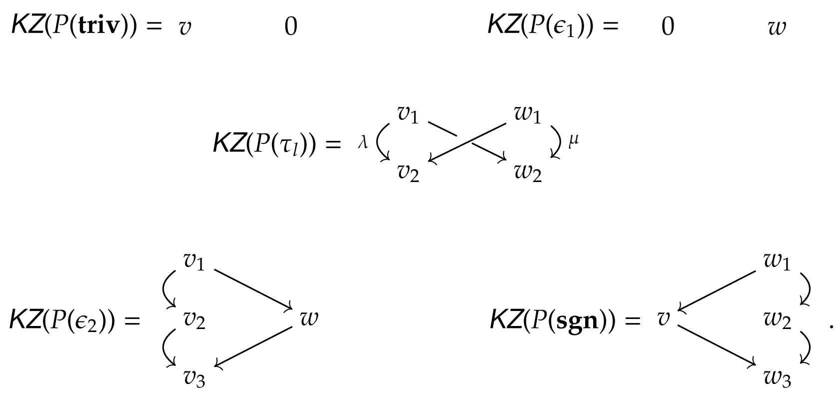


$$\begin{matrix} da = ac = 0 & ab = d^2 \\ cb = bd = 0 & ba = c^2 \end{matrix}.$$

Proof. Let $\mathbf{c}' = (c_1, c_0)$. By the Double Centraliser Theorem, the KZ-functor maps blocks of $\mathcal{O}_{\mathbf{c}}$ to blocks of $\mathcal{H}_{\mathbf{c}}$. Therefore, $\mathcal{H}_{\mathbf{c}}$ has no more than one non-simple block. Hence, the isomorphism of Lemma 2.2.3 that maps $\mathcal{H}_{\mathbf{c}}$ to $\mathcal{H}_{\mathbf{c}'}$ restricts to an isomorphism of the unique non-simple block in $\mathcal{H}_{\mathbf{c}}$ with the unique non-simple block in $\mathcal{H}_{\mathbf{c}'}$. This is the algebra of Lemma 2.2.6. □

Now calculate the images of the projective modules under the KZ–functor as representations of this quiver.

Lemma 2.2.10. *The images of the projective covers in $\mathcal{O}$ under the KZ–functor have bases so that they can be represented by the following quiver representation diagrams. Here, an unlabelled arrow means that the corresponding arrow on the Hecke algebra quiver acts by multiplication by one and no arrow means multiplication by zero.*



Proof. First, by BGG reciprocity, $P(\mathbf{triv})$ has one $L(\epsilon_2)$ composition factor and no $L(\mathbf{sgn})$ and $P(\epsilon_1)$ has one $L(\mathbf{sgn})$ composition factor and no $L(\epsilon_2)$ so $\mathsf{KZ}(P(\mathbf{triv}))$ and $\mathsf{KZ}(P(\epsilon_1))$ are the one-dimensional irreducible representations as above.

The images of $P(\epsilon_2)$ and $P(\mathbf{sgn})$ are the projective covers of the respective simple modules in $\mathcal{H}$. We identify these with left modules of all paths starting at each node.

Now, $P(\tau_l)$ has two of each type of composition factors so has dimension vector $(2, 2)$. First, if a were an isomorphism then, in order to satisfy $ac = da = 0$, both c and d would need to be zero. In that case, $ab = d^2$ would act as zero so b would need to act as zero and the representation would be reducible. A similar argument shows that b cannot be an isomorphism.

The spaces $\mathrm{Hom}(P(\mathbf{sgn}), P(\tau_l))$ and $\mathrm{Hom}(P(\epsilon_2), P(\tau_l))$ are two-dimensional. If a acts as zero on $\mathsf{KZ}(P(\tau_l))$ then $\mathrm{Hom}(\mathsf{KZ}(P(\mathbf{sgn})), \mathsf{KZ}(P(\tau_l)))$ is one-dimensional and if b acts as zero then $\mathrm{Hom}(\mathsf{KZ}(P(\epsilon_2)), \mathsf{KZ}(P(\tau_l)))$ is one -dimensional. Because the *kz*–functor is faithful, a and b cannot act as zero.

Therefore a and b have rank one. Let v_2 and w_2 be non–zero vectors in $\ker a$ and $\operatorname{im} a$ respectively. Since $c^3 = d^3 = 0$, c and d cannot act faithfully so their ranks must be less than two and $c^2 = ba$ and $d^2 = ab$ must act as zero. Therefore, $\ker b$ is spanned by w_2 and $\operatorname{im} b$ is spanned by v_2. If c and d are both zero then the module would be reducible. Choose v_1 so that $a(v_1) = w_2$ and w_1 so that $b(w_1) = v_2$. Now there are complex numbers λ and μ (not both zero) such that c and d act on v_1 and w_1 by these constants respectively. □

Lemma 2.2.11. *Let $M_\mu = \mathsf{KZ}(P(\epsilon_2))$, where $\mu \in \mathbb{C}$ is defined in the proof of the lemma above. Then $M_\mu \cong M_1$ as $\mathcal{H}$–modules.*

Proof. First note that, because they are projective, $\Delta(\mathbf{triv})$ and $\Delta(\epsilon_1)$ are both submodules of $P(\tau_l)$. Let M denote the quotient of $P(\tau_l)$ by $P(\mathbf{triv})$. Consider the map $f\colon P(\tau_l) \longrightarrow P(\mathbf{sgn})$ which sends a generator for $P(\tau_l)$ to the head of the submodule $\dfrac{\Delta(\tau_l)}{\Delta(\epsilon_1)} \subset P(\mathbf{sgn})$. Since the image contains no $L(\mathbf{triv})$ composition factors, $\Delta(\mathbf{triv})$ must be in the kernel of f, so f factors through M; indeed, it factors through a map g which sends a generator for the head of M to the head of the submodule of $P(\mathbf{sgn})$ generated by the standard subquotient $\Delta(\tau_l)$. Therefore, the kernel of g cannot contain the composition factors which belong to the standard subquotient $\Delta(\tau_l)$, so $\ker g \subset \Delta(\epsilon_1)$. If the kernel is non–zero it must contain the socle of $\Delta(\epsilon_1) \subset M$. However, the socle of $P(\mathbf{sgn})$ is simple and the image of g, being a submodule of $P(\mathbf{sgn})$ must then contain the socle $L(\mathbf{sgn}) \subset \Delta(\epsilon_1) \subset P(\mathbf{sgn})$. Therefore g is injective and so the socle of M is simple. It follows that the two $L(\mathbf{sgn})$ composition factors in M cannot both belong to the socle.

Define ϕ to be a map from $P(\mathbf{sgn})$ to M whose image is not in the socle. The socle of M is simple so must be contained in the image of any non–zero map into M. Therefore the image of ϕ must contain two $L(\mathbf{sgn})$ composition factors and so the kernel of ϕ contains exactly one which must be in the socle of $P(\mathbf{sgn})$. It follows that the other two $L(\mathbf{sgn})$ composition factors of $P(\mathbf{sgn})$ cannot be in the kernel of ϕ so that the image, $d(P(\mathbf{sgn}))$, of d cannot be contained in $\ker\phi$. That is, $\phi \circ d \neq 0 \in \mathrm{Hom}_{\mathcal{O}}(P(\mathbf{sgn}), M) \subset \mathsf{KZ}(M)$

Applying the (exact) KZ–functor to the short exact sequence $0 \to \Delta(\mathbf{triv}) \to P(\epsilon_2) \to M \to 0$ gives us that $\mathsf{KZ}(M)$ is isomorphic to the quotient of $\mathsf{KZ}(P(\tau_l))$ by $\mathsf{KZ}(P(\mathbf{triv}))$. As a quiver representation, using the notation above, $\mathsf{KZ}(M)$ has the diagram

$$\mathsf{KZ}(P(\epsilon_2)) = \begin{array}{ccc} v_1 & & w_1 \\ & \searrow & \downarrow \mu \\ & & w_2 \end{array}$$

Therefore, μ is the scalar by which precomposition by d acts on the basis of $\mathrm{Hom}_{\mathcal{O}}(P(\mathbf{sgn}), M)$. However, we have exhibited a non–zero element of this space, ϕ, which is not killed by d. Therefore μ cannot equal zero. We can now assume that $\mu = 1$ by multiplying w_2 by $\frac{1}{\mu}$, v_1 by μ and λ by $\frac{\lambda}{\mu}$. □

Extend the following maps to endomorphisms of $\mathsf{KZ}(P)$.

$$\begin{array}{llll}
f_{31}\colon \mathsf{KZ}(P(\mathbf{triv})) \longrightarrow \mathsf{KZ}(P(\tau_l)); & v \mapsto v_2 & & \\
f_{13}\colon \mathsf{KZ}(P(\tau_l)) \longrightarrow \mathsf{KZ}(P(\mathbf{triv})); & v_1 \mapsto v & & \\
f_{32}\colon \mathsf{KZ}(P(\epsilon_1)) \longrightarrow \mathsf{KZ}(P(\tau_l)); & w \mapsto w_2 & & \\
f_{23}\colon \mathsf{KZ}(P(\tau_l)) \longrightarrow \mathsf{KZ}(P(\epsilon_1)); & w_1 \mapsto w & & \\
f_{34}\colon \mathsf{KZ}(P(\epsilon_2)) \longrightarrow \mathsf{KZ}(P(\tau_l)); & v_1 \mapsto v_1 & v_2 \mapsto \lambda v_2 & w \mapsto w_2 \\
f_{43}\colon \mathsf{KZ}(P(\tau_l)) \longrightarrow \mathsf{KZ}(P(\epsilon_2)); & v_1 \mapsto \lambda v_2 & v_2 \mapsto v_3 & w_1 \mapsto w \\
f_{35}\colon \mathsf{KZ}(P(\mathbf{sgn})) \longrightarrow \mathsf{KZ}(P(\tau_l)); & v \mapsto v_2 & w_1 \mapsto w_1 & w_2 \mapsto w_2 \\
f_{53}\colon \mathsf{KZ}(P(\tau_l)) \longrightarrow \mathsf{KZ}(P(\mathbf{sgn})); & v_1 \mapsto v & w_1 \mapsto w_2 & w_2 \mapsto w_3
\end{array}$$

To each identity map on $P(\mathbf{triv})$, $P(\epsilon_1)$, $P(\tau_l)$, $P(\epsilon_2)$ and $P(\mathbf{sgn})$, associate the vertex e_1, e_2, e_3, e_4 and e_5 of B_5 respectively. To each map f_{ji} associate the arrow a_{ji} of Q_5. Individually examining the vector spaces $\mathrm{Hom}_{\mathcal{H}}(\mathsf{KZ}(P(\lambda)), \mathsf{KZ}(P(\mu)))$ for all $\lambda, \mu \in \mathrm{Irr}(W)$, it is clear that the endomorphisms $\{f_{ij}\}$ generate $\mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P))$ and that they satisfy relations corresponding to R_4. Therefore, $\mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P))$ is isomorphic to a quotient of B_4. Using Theorem 1.5.2 and BGG–reciprocity to count the simple composition factors of the projective covers, $\dim \mathrm{End}_{\mathcal{H}}(\mathsf{KZ}(P)) = \dim \mathrm{End}_{\mathcal{O}}(P) = 29$ so this map is an isomorphism. It follows that $\mathrm{End}_{\mathcal{O}}(P)$ is Morita equivalent to B_4. This concludes the proof of Theorem 2.1.1.

2.3 Koszul Properties

The algebras, B_i, being as they are path algebras, have a natural grading by path–length. They are also Koszul.

Theorem 2.3.1. *Each of the algebras* $\{\mathsf{B}_i^\lambda\}$ *is Koszul.*

2.3.1 The Block B_3 is Koszul

All the maps between these left modules are defined by precomposing paths by the stated path. One can show by dimension counts that the following homomorphisms of projective modules form an exact sequence.

Now define, for $j \geq 0$, $P^j := \oplus_i P_i^j$ and $d^j := \oplus_i d_i^j$. The long exact sequence

$$0 \longrightarrow P^4 \xrightarrow{d^4} P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} \mathsf{B}_3 \xrightarrow{/\mathsf{B}_{3+}} (\mathsf{B}_3)_0$$

is a projective resolution of $(\mathsf{B}_3)_0$. Each of its differentials are linear so B_3 is Koszul.

2.3.2 The Block B_4 is Koszul

All the maps between these left modules are defined by precomposing paths by the stated path. One can show by dimension counts that the following homomorphisms of projective modules form an exact sequence. Although B_4^λ is dependent on a parameter λ, the superscript will be dropped.

Now define, for $j \geq 0$, $P^j := \oplus_i P_i^j$ and $d^j := \oplus_i d_i^j$. The long exact sequence

$$0 \longrightarrow P^4 \xrightarrow{d^4} P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} \mathsf{B}_4 \xrightarrow{/\mathsf{B}_{4+}} (\mathsf{B}_4)_0$$

is a projective resolution of $(\mathsf{B}_4)_0$. Each of its differentials are linear so B_4 is Koszul.

2.3.3 The Block B_5 is Koszul

All the maps between these left modules are defined by precomposing paths by the stated path. One can show by dimension counts that the following homomorphisms of projective modules form an exact sequence. Although B_5^λ is dependent on a parameter λ, the superscript will be dropped.

Now define, for $j \geq 0$, $P^j := \oplus_i P_i^j$ and $d^j := \oplus_i d_i^j$. The long exact sequence

$$0 \longrightarrow P^4 \xrightarrow{d^4} P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} \mathsf{B}_5 \xrightarrow{/\mathsf{B}_{5+}} (\mathsf{B}_5)_0$$

is a projective resolution of $(\mathsf{B}_5)_0$. Each of its differentials are linear so B_5 is Koszul. This completes the proof of Theorem 2.3.1.

2.4 Koszul Duals

For each of the following algebras, B_i, Let $\{b_{ij}\}$ be a basis of $(\mathsf{B}_i)_1^*$ dual to $\{a_{ij}\}$, that is $b_{ij}(a_{kl}) = \delta_{ik}\delta_{jl}$.

2.4.1 The Algebra B_1 is Koszul Self-Dual

Proposition 2.4.1. *The algebras B_1 and $\mathsf{B}_1^!$ are isomorphic.*

Proof. The set of dual relations has the basis $R_1^\perp \overset{\text{v.s.}}{\cong} \langle\{b_{212}\}\rangle_{\mathbb{C}}$. It follows that

$$\mathsf{B}_1^! e_1 \overset{\text{v.s.}}{\cong} \langle\{e_1\} \cup \{b_{21}\} \cup \{b_{121}\}\rangle_{\mathbb{C}}$$
$$\mathsf{B}_1^! e_2 \overset{\text{v.s.}}{\cong} \langle\{e_2\} \cup \{b_{12}\}\rangle_{\mathbb{C}}$$

is a basis for $\mathsf{B}_1^!$. The $\mathbb{C}$–algebra homomorphism which exchanges e_1 and e_2 and maps a_{ij} to b_{ji} provides an isomorphism $\mathsf{B}_1^! \cong \mathsf{B}_1$. □

2.4.2 The Koszul Dual of B_2

The set of dual relations has the basis $R_2^\perp \overset{\text{v.s.}}{\cong} \{b_{323}, b_{212} + b_{232}\}$ since each of these elements kills all of R_2, they are all linearly independent, and there are

$$\dim(\mathsf{B}_2)_1 \otimes_{(\mathsf{B}_2)_0} (\mathsf{B}_2)_1 - \dim R_2 = 2$$

of them.

The Koszul dual, $\mathsf{B}_2^!$, has the following basis.

$$\begin{aligned}
\mathsf{B}_2^! e_1 &\overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{b_{21}\} \cup \{b_{321}, b_{121}\} \cup \{b_{2121}\} \cup \{b_{12121}\} \rangle_{\mathbb{C}} \\
\mathsf{B}_2^! e_2 &\overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{b_{12}, b_{32}\} \cup \{b_{212}\} \cup \{b_{1212}\} \rangle_{\mathbb{C}} \\
\mathsf{B}_2{}^! e_3 &\overset{\text{v.s.}}{\cong} \langle \{e_3\} \cup \{b_{23}\} \cup \{b_{123}\}. \rangle_{\mathbb{C}}
\end{aligned}$$

Each of the elements is linearly independent and non–zero, as can be checked from the relations. There are only $\dim(\mathsf{B}_2)_1 \otimes_{(\mathsf{B}_2)_0} (\mathsf{B}_2)_1 - \dim R_2^\perp = 6 - 2 = 4$ paths of length two in $B^!$. There are seven possible non–zero continuations of these four paths.

2.4.3 The Algebra B_3 is Koszul Self-Dual

The set of dual relations has the basis

$$R_3^\perp \overset{\text{v.s.}}{\cong} \left\langle \begin{Bmatrix} b_{434}, & b_{312}, & b_{212} + b_{242}, & b_{313} + b_{343} \\ b_{424}, & b_{213}, & b_{124} + b_{134}, & b_{421} + b_{431} \end{Bmatrix} \right\rangle_{\mathbb{C}},$$

since each of these elements kills all of R_3, they are all linearly independent, and there are $\dim(\mathsf{B}_3)_1 \otimes_{(\mathsf{B}_3)_0} (\mathsf{B}_3)_1 - \dim R_3 = 8$ of them.

Proposition 2.4.2. *The algebras $\mathsf{B}_3^!$ and B_3 are isomorphic.*

Proof. Let ϕ act as the permutation $\sigma = (14) \in \mathfrak{S}_4$ on the idempotents $\{e_1, e_2, e_3, e_4\}$ and define

$$\phi(a_{ij}) := \begin{cases} -b_{\sigma(i)\sigma(j)} & \text{if } (ij) = (31) \text{ or } (24) \\ b_{\sigma(i)\sigma(j)} & \text{otherwise} \end{cases}.$$

Extend ϕ to a ring homomorphism

$$\phi\colon T_{(\mathsf{B}_3)_0}((\mathsf{B}_3)_1) \longrightarrow \frac{T_{(\mathsf{B}_3)_0}((\mathsf{B}_3)_1^*)}{R^\perp}.$$

This is surjective because the generators for $\mathsf{B}_3{}^!$ lie in the image. It suffices to show that ϕ restricted to $R_3 \subset (\mathsf{B}_3)_2$ is a vector space isomorphism between R_3 and $R_3^\perp$, which follows because it is a bijection between the two bases given above. □

2.4.4 The Koszul Dual of B_4^λ

The set of dual relations has the basis

$$R_4^\perp \overset{\text{v.s.}}{\cong} \left\langle \begin{Bmatrix} b_{434}, & b_{534}, & b_{134} & b_{532}, & \lambda^2 b_{343} + b_{353} + b_{313} \\ b_{435}, & b_{535}, & b_{235}, & b_{431}, & b_{343} + b_{353} + b_{323} \end{Bmatrix} \right\rangle_{\mathbb{C}}$$

because each of these elements kills all of R_4, they are all linearly independent and there are $\dim(\mathsf{B}_4)_1 \otimes_{(\mathsf{B}_4)_0} (\mathsf{B}_4)_1 - \dim R = 10$ of them.

Proposition 2.4.3. *The algebra $\mathsf{B}_4^{\lambda\,!}$ has the following basis.*

$$\begin{aligned}
\mathsf{B}_4^! e_1 &\overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{b_{31}\} \cup \{b_{231}, b_{531}, b_{131}\} \cup \{b_{3131}\} \cup \{b_{13131}\} \rangle_{\mathbb{C}} \\
\mathsf{B}_4^! e_2 &\overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{b_{32}\} \cup \{b_{132}, b_{232}, b_{432}\} \cup \{b_{3232}\} \cup \{b_{23232}\} \rangle_{\mathbb{C}} \\
\mathsf{B}_4^! e_3 &\overset{\text{v.s.}}{\cong} \langle \{e_3\} \cup \{b_{13}, b_{23}, b_{43}, b_{53}\} \cup \{b_{343}, b_{353}\} \cup \{b_{1353}, b_{2343}\} \rangle_{\mathbb{C}} \\
\mathsf{B}_4^! e_4 &\overset{\text{v.s.}}{\cong} \langle \{e_4\} \cup \{b_{34}\} \cup \{b_{234}\} \rangle_{\mathbb{C}} \\
\mathsf{B}_4^! e_5 &\overset{\text{v.s.}}{\cong} \langle \{e_5\} \cup \{b_{35}\} \cup \{b_{135}\} \rangle_{\mathbb{C}}.
\end{aligned}$$

Proof. The proof follows the same argument for the proof of Proposition 2.1.4 with the names of the vertices permuted. □

Proposition 2.4.4. *The algebra B_4^λ is isomorphic to its Koszul dual if and only if $\lambda = 1$.*

Proof. A homomorphism of graded rings between $T_{(\mathsf{B}_4)_0}((\mathsf{B}_4)_1)$ and $\frac{T_{(\mathsf{B}_4)_0}((\mathsf{B}_4)_1^*)}{R_4^\perp}$ is determined by the image of $(\mathsf{B}_4)_0 \oplus (\mathsf{B}_4)_1$. An isomorphism $\phi\colon \mathsf{B}_4 \longrightarrow \mathsf{B}_4{}^!$ of rings preserves primitive idempotents so ϕ must act on $\{e_1, e_2, e_3, e_4, e_5\}$ by some permutation $\sigma \in \mathfrak{S}_5$. Then for each $a_{ij} \in (\mathsf{B}_4)_1$, $\phi(a_{ij}) = \lambda_{ij} b_{\sigma(i)\sigma(j)}$ for some collection of scalars λ_{ij}. These must be all non–zero because the relations in B_4 are in degree two.

Now, ϕ restricted to $R_5 \subset (\mathsf{B}_4)_2$ must be a vector space isomorphism between R_5 and $R_5^\perp$. The image of R_5 contains eight linearly independent elements of the form $b_{i\sigma(3)j}$ so ϕ must fix e_3. The image of the loops a_{131} and a_{232} must be in the span of b_{434} and b_{535} so $\sigma(1,2)$ is either $(4,5)$ or $(5,4)$. There are only two possibilities: If $\sigma(1) = 5$ then a_{531} must be mapped to $\lambda_{531} b_{235}$ so $\sigma = (1524)$. Otherwise, $\sigma(1) = 4$ and $a_{431} \mapsto \lambda_{431} b_{534}$ so $\sigma = (1452)$. In both of these two cases σ is a bijection between the monomial relations in R_5 and $R_5^\perp$.

Suppose that $\sigma = (1425)$. Consider the image of the two remaining relations in the basis for R_5, $v_1 := \lambda^2 a_{313} + a_{323} - a_{343}$ and $v_2 := a_{313} + a_{323} - a_{353}$. The image of these two vectors must span the same subspace as $w_1 := \lambda^2 b_{343} + b_{353} + b_{313}$ and $w_2 := b_{343} + b_{353} + b_{323}$ in $R_5^\perp$. That is, there must exist some invertible matrix $C \in \mathrm{GL}_2(\mathbb{C})$ such that $C\langle \phi(v_1), \phi(v_2)\rangle_{\mathbb{C}} = \langle w_1, w_2\rangle_{\mathbb{C}}$. Now $c_{11}\phi(v_1) + c_{12}\phi(v_2) = w_1$ implies that $c_{11} = 0$ and $c_{21}\phi(v_1) + c_{22}\phi(v_2) = w_2$ implies that $c_{22} = 0$. A solution exists if and only if $\lambda = 1$, in which case $c_{12} = c_{21} = \lambda_{313} = \lambda_{323} = -\lambda_{343} = -\lambda_{353}$. A similar argument shows that, when $\sigma = (1524)$, an isomorphism exists if and only if $\lambda = 1$. □

2.4.5 The Koszul Dual of B_5^λ

The set of dual relations has the basis

$$R_5^\perp \overset{\text{v.s.}}{\cong} \left\langle \begin{Bmatrix} b_{543}, & b_{214} + \lambda b_{234}, & b_{341} + b_{321}, & b_{212} + b_{232}, \\ b_{345}, & b_{123} + b_{143}, & b_{412} + \lambda b_{432}, & b_{323} + \lambda b_{343}, \\ b_{545}, & b_{414} + t b_{434}, & b_{434} + b_{454} & \end{Bmatrix} \right\rangle_{\mathbb{C}}$$

because these are linearly independent, non–zero, each kills all of R_5 and

$$\dim R_5^\perp = \dim(\mathsf{B}_5)_1 \otimes_{(\mathsf{B}_5)_0} (\mathsf{B}_5)_1 - \dim R_5 = 11.$$

Proposition 2.4.5. *The algebra $\mathsf{B}_5^{\lambda !}$ has the following basis.*

$$\begin{aligned}
\mathsf{B}_5^! e_1 &\overset{\text{v.s.}}{\cong} \langle \{e_1\} \cup \{b_{21}, b_{41}\} \cup \{b_{121}, b_{321}, b_{541}, b_{141}\} \cup \{b_{2121}, b_{4321}\} \cup \{b_{12121}\}\rangle_{\mathbb{C}} \\
\mathsf{B}_5^! e_2 &\overset{\text{v.s.}}{\cong} \langle \{e_2\} \cup \{b_{12}, b_{32}\} \cup \{b_{432}, b_{212}\} \cup \{b_{1212}, b_{3432}\} \cup \{b_{21212}\}\rangle_{\mathbb{C}} \\
\mathsf{B}_5^! e_3 &\overset{\text{v.s.}}{\cong} \langle \{e_3\} \cup \{b_{23}, b_{43}\} \cup \{b_{123}, b_{343}\} \cup \{b_{2343}\}\rangle_{\mathbb{C}} \\
\mathsf{B}_5^! e_4 &\overset{\text{v.s.}}{\cong} \langle \{e_4\} \cup \{b_{14}, b_{34}, b_{54}\} \cup \{b_{434}, b_{234}\} \cup \{b_{1234}\}\rangle_{\mathbb{C}} \\
\mathsf{B}_5^! e_5 &\overset{\text{v.s.}}{\cong} \langle \{e_5\} \cup \{b_{45}\} \cup \{b_{145}\}\rangle_{\mathbb{C}}.
\end{aligned}$$

Proof. The elements are linearly independent, and none is zero as can be checked directly from the relations. One can also check from the relations that

$$\left(\dim e_i B e_j\right)_{i,j} = \begin{pmatrix} 4 & 2 & 1 & 2 & 1 \\ F2 & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix},$$

so that $\dim B_5^! = 34$ as required. □

Part II

Combinatorics

Chapter 3

Combinatorial Prerequisites

3.1 Conventions and Terminology related to Multipartitions

3.1.1 Multipartitions and Young Diagrams

A **partition**, λ, is a non-increasing infinite sequence, $(\lambda_1, \lambda_2, \ldots)$, of non-negative integers, called **parts**, such that $\lambda_i = 0$ for all $i >> 1$. Two partitions are considered equal if they differ by a terminal string of zeroes and the number of non-zero parts is called the **length**. For a non-negative integer l, an l**-multipartition**, λ, is an l-tuple of partitions $\left(\lambda^{(1)}, \ldots, \lambda^{(l)}\right)$. The **degree** of an l-multipartition is the sum of all the parts of all the partitions in the l-tuple. If r is the degree of an l-multipartition, λ, then λ is said to be an l**-multipartition of** r.

Multipartitions are a natural generalisation of partitions. Just as partitions parametrise complex irreducible representations of the symmetric group, there is the following result of Schur.

Theorem 3.1.1. *(Schur) Let l and n be natural numbers, not both one. The irreducible characters of $G(l, 1, n)$ are in bijection with the l-multipartitions of n.*

Proof. See [Mac95, I. Appendix B]. □

A **Young diagram**[1] is a subset, $Y \subseteq \mathbb{N} \times \mathbb{N} \times [1, l]$ such that there exists an l-multipartition, λ, with

$$(i, j, k) \in Y \subset \mathbb{N} \times \mathbb{N} \times [1, l] \iff 1 \leq j \leq \lambda_i^{(k)}.$$

Elements of a Young diagram are called **nodes**. The **transpose** of a Young diagram is obtained by reflecting along the leading diagonal (from the top left corner). Given a partition λ, let λ^t denote its **transpose**; it is defined to be the partition corresponding to the transpose of the Young diagram of λ.

A **charged** l**-multipartion** is a pair, $|\lambda, \mathbf{s}\rangle$, where λ is an l-multipartition and $\mathbf{s} = (s_1, \ldots, s_l)$ is an l-tuple of integers. The **residue** of a node (i, j, k) in a Young diagram corresponding to a charged l-multipartition, $|\lambda, \mathbf{s}\rangle$, is

$$\mathrm{res}_{|\lambda,\mathbf{s}\rangle}(i, j, k) := j - i + s_k.$$

3.1.2 Addable and Removable Nodes and Hooks

A node, (i, j, k), in a Young diagram, Y, is said to be **removable** if $Y \setminus \{(i, j, k)\}$ is a Young diagram. A node, (i, j, k), in $\mathbb{N} \times \mathbb{N} \times [1, l] \setminus Y$ for some Young diagram, Y, is said to be **addable** if $Y \cup \{(i, j, k)\}$ is a Young diagram.

Let $e > 0$ be a positive integer. An e**-hook** is a subset of e nodes, $H \subseteq Y$, in a Young diagram, Y, that are arranged in the shape of a 'hook'. That is, there are natural numbers $j' \geq j$ and $i' \geq i$

[1] following the British convention

such that

$$H = \{(i, l, k) \in \mathbb{N} \times \mathbb{N} \times [1, l] \mid j \le l \le j'\} \cup \{(m, j, k) \in \mathbb{N} \times \mathbb{N} \times [1, l] \mid i \le m \le i'\}$$

and $j' - j + i' - i + 1 = e$. Given $e > 0$, a **removable e-hook** is an e-hook $H \subseteq Y$ such that $Y \setminus H$ is a Young diagram. Given a partition, λ, the **e-core** of λ is defined to be the partition corresponding to the Young diagram formed by removing all possible removable e-hooks. It is a well known result that the e-core is independent of the order in which one chooses to remove removable e-hooks (see, for example [Ols86, Lemma 3]).

Given a Young diagram $Y \subset \mathbb{N} \times \mathbb{N} \times [1, l]$ and a node, $(i, j, k) \in \mathbb{N} \times \mathbb{N} \times [1, l]$, the **hook length** of (i, j, k) is defined to be the integer

$$\mathbf{hl}(i, j, k) := \lambda_i^{(k)} - j + (\lambda^t)_j^{(k)} - i + 1.$$

Notice that this is defined for a node not lying inside the Young diagram corresponding to λ. In this case, the hook length would be negative. By interpreting the hook length as the (signed) number of nodes in a certain hook-shaped region, one can see that the function never takes the value zero.

Notation. *For a non-negative integer i, let τ_i be the 2-core with $\frac{i(i+1)}{2}$ boxes, that is,*

$\tau_0 = \varnothing,$ $\tau_1 = \square,$ $\tau_2 =$ ,

and so forth. When i is negative define $\tau_i = \varnothing$. Let σ_i denote the partition $(\tau_i, 1, 1)$.

3.2 Fock Space

Fock space was introduced in 1932 by V. Fock as an algebraic construction, describing quantum states of particles in a Hilbert space. It was later proved that they carry an action of the quantum affine Lie algebras defined in the following section. One can associate to a Fock space one of the categories $\mathcal{O}_{\mathbf{c}}(W)$ for some $\mathbf{c}$ and W. What is important from the point of view of this thesis is the definition of the weights of certain eigenvectors, which turn out to play a role in describing the effect of Koszul duality.

3.2.1 The Quantum Affine Lie Algebra

Let e and l be fixed natural numbers. For $s \in \mathbb{Z}$, let $\mathbb{Z}^l(s) := \{(s_1, \dots, s_l) \mid \sum_i s_i = s\}$. Let q be an indeterminate.

Definition 3.2.1. *Let $\widehat{\mathfrak{sl}}_n$ be the affine Lie algebra of the Lie group $\mathrm{SL}_n(\mathbb{C})$. Let $e_{i,j} \in \mathrm{Mat}_n(\mathbb{C})$ be the matrix with a 1 in position (i, j) and 0 elsewhere. For $i = 1, \dots, n-1$, define $h_i := e_{i,i} - e_{i+1,i+1}$ so that $\{h_1, \dots, h_{n-1}\}$ is a basis for the Cartan subalgebra of $\mathfrak{sl}_n$. Extend this to a basis $\{h_1, \dots, h_{n-1}, D\}$ of the Cartan subalgebra, $\mathfrak{h}$, of $\widehat{\mathfrak{sl}}_n$. Choose a dual basis of $\mathfrak{h}^*$, $\{\Lambda_1, \dots, \Lambda_{n-1}, \delta\}$ so that*

$$\langle \Lambda_i, h_j \rangle = \delta_{ij} \qquad \langle \Lambda_i, D \rangle = \langle \delta, h_i \rangle = 0 \qquad \langle \delta, d \rangle = 1.$$

The algebra $U_q(\widehat{\mathfrak{sl}}'_n)$ has a standard presentation with generators

$$\{e_i, f_i, t_i, t_i^{-1} \mid i = 0, \dots, n-1\}$$

*(see [Yvo05, §1.2.2.1] for the relations). The **q-deformed universal enveloping algebra of** $\widehat{\mathfrak{sl}}_n$, denoted $U_q(\widehat{\mathfrak{sl}}_n)$, is generated by the generators and relations of $U_q(\widehat{\mathfrak{sl}}'_n)$ together with the element ∂ and the relations*

$$[\partial, e_i] = \delta_{i,0} e_i \qquad [\partial, f_i] = -\delta_{i,0} f_i \qquad [\partial, t_i] = 0.$$

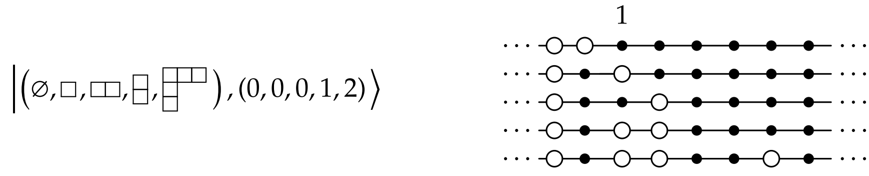


Figure 3.1: A charged 5-multipartition and its corresponding abacus.

3.2.2 Fock Space

Let $\mathrm{Mat}_{e,l}(\mathrm{Part})$ be the set of all $e \times l$ matrices whose coefficients are partitions (of any degree). Let s be an integer and $\mathrm{Mat}_{e,l}(s)$ denote the set of all $e \times l$ matrices whose coefficients are integers which sum to s.

Definition 3.2.2. *The **higher level Fock space** is the formal $\mathbb{C}(q)$-linear span of the following set of symbols.*

$$\Lambda^{s+\frac{\infty}{2}} \overset{\text{v.s.}}{\cong} \Big\langle \{|M_\lambda, M_{\mathbf{s}}\rangle \,\Big|\, M_\lambda \in \mathrm{Mat}_{e,l}(\mathrm{Part}),\ M_s \in \mathrm{Mat}_{e,l}(s)\} \Big\rangle_{\mathbb{C}(q)}.$$

This is an infinite dimensional vector space.

Let $p = q^{-1}$. The higher level Fock space has actions of $U_q(\hat{\mathfrak{sl}}_e)$ and $U_p(\hat{\mathfrak{sl}}_l)$ (to be defined in Section 3.4) such that the restricted actions of $U_q(\hat{\mathfrak{sl}}'_e)$ and $U_p(\hat{\mathfrak{sl}}'_l)$ commute. These actions are defined on two alternative indexings of the basis elements. The first is by charged l-multipartitions and the second by charged e-multipartitions. The dictionary between the indexing sets is provided by a combinatorial rule called *taking the e-quotient*, its inverse (proved in Lemma 3.3.1) is called *taking the l-quotient*; they are denoted $\overset{l}{\rightsquigarrow}$ and $\overset{e}{\rightsquigarrow}$ respectively. For example, if $|\lambda, \mathbf{s}\rangle$ is a charged l-multipartition whose e-quotient is $|\mu, \mathbf{t}\rangle$, a charged e-multipartition, then one writes

$$|\lambda, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu, \mathbf{t}\rangle,$$

and both $|\lambda, \mathbf{s}\rangle$ and $|\mu, \mathbf{t}\rangle$ correspond to the same basis element, $|M_\lambda, M_s\rangle$ say, of $\Lambda^{s+\frac{\infty}{2}}$ for $s = s_1 + \cdots + s_l = t_1 + \cdots + t_e$. The process of constructing quotients of charged multipartions is defined by encoding their combinatorial data in so-called 'abaci'.

3.3 Abaci and their Combinatorics

An **abacus with one runner** is a subset, $A \subset \mathbb{Z}$, such that there exists a $k \geq 0$ with $-i \in A$ and $i \notin A$ for all $i \geq k$. Elements of A are called **beads** and elements of $\mathbb{Z} \setminus A$ are called **notches**. An **abacus**, $A = (A_1, \ldots, \ldots, A_l) \subset \mathbb{Z} \times \{1, \ldots, l\}$, is a ordered collection of l of these abaci with one runner. An abacus can be represented diagramatically as a collection of horizontal integer number lines—each corresponding to a runner—with notches and beads marked at integral intervals. Each bead corresponds to an element of A and each notch marks an element of $\mathbb{Z} \setminus A$. Thus, each runner will eventually have an infinite uninterrupted string of beads to the left and an infinite uninterrupted string of notches to the right. See Figure 3.1 for an example.

To each charged l-multipartition, $|\lambda, \mathbf{s}\rangle$, associate the abacus with l-runners

$$A := \left\{ \left(\lambda_i^{(d)} + s_d + 1 - i, d \right) \,\middle|\, i \geq 1, 1 \leq d \leq l \right\}. \tag{3.1}$$

This correspondence is bijective. In the other direction, to each abacus with l runners, $A = (A_1, \ldots, A_l) \subset \mathbb{Z} \times \{1, \ldots, l\}$, there corresponds a charged l-multipartition, $|\lambda, \mathbf{s}\rangle$. The charge, s_j, associated to each runner, A_j, is defined as follows. Let k_j be the largest integer such that $i \in A_j$ for all $i \leq k_j$. Then there are a finite number of beads, p_j say, in A_j that are numbered strictly bigger than k_j. Define the **charge**, s_j, of A_j to be

$$s_j := k_j + p_j.$$

Now, each $\lambda_i^{(d)}$ is determined by Equation 3.1.

Let $A = (A_1, \ldots, A_l) \subset \mathbb{Z} \times \{1, \ldots, l\}$ be an abacus with l runners. Define functions ϕ_e and ψ_e as follows. Each integer a can be uniquely written as a sum of integers $a = \phi_e(a)e + \psi_e(a)$, where $1 \leq \psi_e(a) \leq e$. Notice that, because the convention that the charges of an l-multipartition run from $s_1, \ldots, s_l$, the functions ϕ_e and ψ_e aren't quite the same as those of the Euclidean algorithm. Define an abacus, A^e, with e runners by

$$A^e := \left\{ \left(\phi_e(a)l + b, \psi_e(a) \right) \middle| \, (a, b) \in A \right\}.$$

The charged multipartition corresponding to A^e is called the ***e*-quotient** of the charged multipartition corresponding to A. The relation, '**has the *e*-quotient**,' between charged multipartitions, will be denoted $\overset{e}{\rightsquigarrow}$. For an example of how to calculate quotients using diagrams see Figures 3.2 and 3.3 or [Yvo05, Section 1.1.5.2].

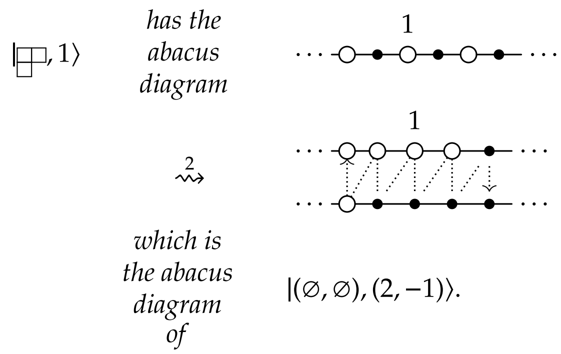


Figure 3.2: A calculation of the 2-quotient of $|\text{⊞}, 1\rangle$. Matching the beads at position 1, the remaining abacus is arranged as shown by the dotted line.

The next lemma shows that the operations $\overset{e}{\rightsquigarrow}$ and $\overset{l}{\rightsquigarrow}$ are mutually inverse.

Lemma 3.3.1. *Let $|\lambda, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu, \mathbf{t}\rangle$ be a charged l-multipartition and its e-quotient. Then $|\mu, \mathbf{t}\rangle \overset{l}{\rightsquigarrow} |\lambda, \mathbf{s}\rangle$.*

Proof. If A is the abacus corresponding to $|\lambda, \mathbf{s}\rangle$ then, using the two relations $\phi_e(a)l + b = \phi_l(\phi_e(a)l + b)l + \psi_l(\phi_e(a)l + b)$ and $\psi_l(\phi_e(a)l + b) = b$ for all $(a, b) \in A$,

$$\begin{aligned}
(A^e)^l &= \left\{ \left(\phi_e(a)l + b, \psi_e(a) \right) \middle| \, (a, b) \in A \right\}^l \\
&= \left\{ \left(\phi_l(\phi_e(a)l + b)e + \psi_e(a), \psi_l(\phi_e(a)l + b) \right) \middle| \, (a, b) \in A \right\} \\
&= \left\{ \left(\phi_e(a)e + \psi_e(a), b \right) \middle| \, (a, b) \in A \right\} \\
&= A.
\end{aligned}$$

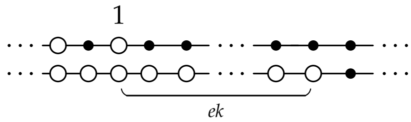
□

Shifting the charge, $\mathbf{s}$, of a charged multipartition by the l-tuple, $(1, \ldots, 1)$, affects the e-quotient in the following way.

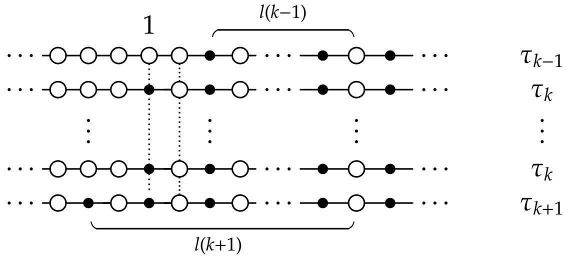
Lemma 3.3.2. *Let $|\lambda, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu, \mathbf{t}\rangle$ be charged multipartitions. Let permutations in $\mathfrak{S}_e$ act on e-multipartions by permuting partitions between entries, that is, if $\sigma \in \mathfrak{S}_e$ then $\sigma(\lambda) := \left(\lambda^{(\sigma^{-1}(1))}, \ldots, \lambda^{(\sigma^{-1}(e))} \right)$. Let $\sigma := (1 \cdots e) \in \mathfrak{S}_e$. Then if $\mathbf{s} + 1 := (s_1 + 1, \ldots, s_l + 1)$ then*

$$|\lambda, \mathbf{s} + 1\rangle \overset{e}{\rightsquigarrow} |\sigma(\mu), \sigma(\mathbf{t}) + (l, 0, \ldots, 0)\rangle.$$

$|(\square, \emptyset), (0, ke)\rangle$ *has the abacus diagram*



$\overset{e}{\rightsquigarrow}$



which is the abacus diagram of $|(\tau_{k-1}, \tau_k, \ldots, \tau_k, \tau_{k+1}), (k+1, k, \ldots, k, k-1)\rangle.$

Figure 3.3: A calculation of the e-quotient of $|(\square, \emptyset), (0, ke)\rangle$ for $k \geq 1$, where $l = 2$. Starting at position 1 and moving to the right, each column of the abacus is rotated 90° anticlockwise and stacked, one under the other, until an $e \times l$ 'bale' is formed after stacking column e as a row underneath $e - 1$. This first 'bale' is shown on the diagram by the dotted rectangle. Then, starting at column $e + 1$ a new $e \times l$ 'bale' is formed which is then concaternated to the right of the previous 'bale'. This process repeats itself until every bead to the right of position 1 in the input abacus has had its column rotated and stacked into a 'bale' of the output abacus. An infinite string of notches is then added to the right of each runner. Working to the left of position 1, start with the column 0 and perform the same procedure, rotating each column 90° anticlockwise, but this time stacking the resulting row on top of the previous one. The resulting $e \times l$ 'bale' formed from columns $1 - e$ to 0 is then concaterated to the left of the first 'bale'. Repeat this process until every notch to the left of position 1 has been dealt with and then add an infinite string of beads to the left of each runner.

Proof. First, note that for $a \in \mathbb{Z}$

$$(\phi_e(a+1), \psi_e(a+1)) = \begin{cases} (\phi_e(a) + 1, 1) & \text{if } a \equiv 0 \ (\mathbf{mod}\ e) \\ (\phi_e(a), \psi_e(a) + 1) & \text{otherwise.} \end{cases}$$

Let A denote the abacus corresponding to $|\lambda, \mathbf{s}\rangle$ and $A + 1$ the abacus associated to $|\lambda, \mathbf{s} + 1\rangle$.

$$A + 1 = \{(\lambda_i^{(d)} + s_d + 2 - i, d) \,|\, 1 \leq i, 1 \leq d \leq l\}.$$

Now,

$$\begin{aligned} (A+1)^e &= \{(\phi_e(a+1)l + b, \psi_e(a+1) \,|\, (a,b) \in A\} \\ &= \{(\phi_e(a)l + d, 1) + (l, 0) \,|\, (a,b) \in A \text{ and } a \equiv 0 \ (\mathbf{mod}\ l)\} \\ &\quad \cup \{(\phi_e(a)l + d, \psi_e(a) + 1) \,|\, (a,b) \in A \text{ and } a \not\equiv 0 \ (\mathbf{mod}\ l)\} \\ &= \{(x + l, \sigma(y)) \,|\, (x,y) \in A^e \text{ and } \sigma(y) = 1\} \\ &\quad \cup \{(x, \sigma(y)) \,|\, (x,y) \in A^e \text{ and } \sigma(y) \neq 1\}. \end{aligned}$$

Therefore, all the runners have been permuted by σ and the one that becomes shifted to the 1 position has had its charge shifted by l. □

And this shows that shifting the charge, $\mathbf{s}$, of a charged multipartition by the l-tuple, $(e, \ldots, e)$, affects the e-quotient as follows.

Corollary 3.3.1. *Let $|\lambda, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu, \mathbf{t}\rangle$ be a charged l-multipartition and its e-quotient. Then*

$$|\lambda, (s_1 + e, s_2 + e, \ldots, s_l + e)\rangle \overset{e}{\rightsquigarrow} |\mu, (t_1 + l, t_2 + l, \ldots, t_e + l)\rangle.$$

Corollary 3.3.2. *The degree of the multipartition in the quotient of a charged multipartition is invariant under shifting the charge by* $(1, \ldots, 1)$.

Given an l-multipartion $\lambda = \left(\lambda^{(1)}, \ldots, \lambda^{(l)}\right)$, define a new multipartition

$$\lambda' := \left(\lambda^{(l)^t}, \ldots, \lambda^{(1)^t}\right).$$

Given an l-tuple $\mathbf{s} \in \mathbb{Z}$, define a new l-tuple

$$\mathbf{s}' := (-s_l, \ldots, -s_1).$$

Lemma 3.3.3. *Let $|\lambda, s\rangle$ be a partition charged by $s \in \mathbb{Z}$. Let A be the corresponding abacus with one runner. Let A' be the abacus with one runner corresponding to $|\lambda^t, -s\rangle$. Then*

$$j + s + 1 \in A \iff -j - s \notin A'.$$

Proof. Suppose both $j + s + 1 \in A$ and $-j - s \in A'$. Then there exist $i_1, i_2 \in \mathbb{N}$ such that

$$\begin{aligned} 1 + j + s &= \lambda_{i_1} + 1 - i_1 + s \\ -j - s &= \lambda^t_{i_2} + 1 - i_2 - s. \end{aligned}$$

Adding these gives $(\lambda_{i_1} - i_1) + (\lambda^t_{i_2} - i_2) + 1 = 0$. The left-hand-side of this formula is the hook length formula for a node (i_1, i_2) on the Young diagram of λ and so cannot equal zero.

Define the subset of $\mathbb{Z}$,

$$I := (-\infty, -1] \cup \{1\} \subset \mathbb{Z}.$$

For any non-empty partition, given any $i \in I$, there exists a node on the Young diagram that has hook length i: if $i = 1$ then choose any removable node and if $i < 0$ choose $(\lambda^t_1 - i, 1)$. Suppose that j is such that $-j - s \notin A'$. Then for all $i_2 \geq 1$, $\lambda^t_{i_2} - i_2 \neq -j - 1$. If λ is empty this implies that $j \leq -1$ and so $\lambda_{-j} + j = j$ giving $1 + j + s \in A$.

Now suppose that λ is non-empty. Then, for all $(i_1, i_2) \in \mathbb{N} \times \mathbb{N}$, the hook length cannot be equal to $\lambda_{i_1} - i_1 - j$, so that $\lambda_{i_1} - i_1 - j \notin I$. But, for a large enough i_1, we have $\lambda_{i_1} - i_1 - j \leq 1$ so $\lambda_{i_1} - i_1 - j = 0$ which implies that $1 + j + s \in A$. □

Corollary 3.3.3. *Let $|\lambda, \mathbf{s}\rangle$ be a charged l-multipartition with abacus A. Let A' denote the abacus corresponding to $|\lambda', \mathbf{s}'\rangle$ then*

$$(j + s_d + 1, d) \in A \iff (-j - s_d, l + 1 - d) \notin A'.$$

The following proposition is the main result of the section.

Proposition 3.3.1. *Suppose that $|\lambda, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu, \mathbf{t}\rangle$. Then $|\lambda', \mathbf{s}'\rangle \overset{e}{\rightsquigarrow} |\mu', \mathbf{t}'\rangle$.*

Proof. First, I claim that for $a \in \mathbb{Z}$,

$$\phi_e(-a) = -1 - \phi_e(a + 1).$$

Note that

$$\psi_e(-a) = \begin{cases} e & \text{if } a \equiv 0 \ (\mathbf{mod}\ e) \\ e - \psi_e(a) & \text{otherwise.} \end{cases}$$

When $a \equiv 0$ ($\mathbf{mod}$ e), $\phi_e(a + 1) = \phi_e(a) + 1$ so that rearranging $\phi_e(a)e + \psi_e(a) = -\phi_e(-a)e - \psi_e(-a)$ gives the claim. When $a \not\equiv 0$ ($\mathbf{mod}$ e), $\phi_e(a + 1) = \phi_e(a)$ and the claim follows.

Next, I claim that, for all $j \in \mathbb{Z}$ and $1 \le r \le e$,

$$(j + t_r + 1, r) \in A^e \iff (-j - t_r, e + 1 - r) \notin A'^e.$$

Suppose that $(j + t_r + 1, r) \in A^e$, then there exists some $(a, d) \in A$ such that

$$j + t_r + 1 = \phi_e(a)l + d \qquad \text{and} \qquad r = \psi_e(a).$$

Writing $a = 1 + i + s_d$ allows one to use Corollary 3.3.3 to deduce that $(-i - s_d, l + 1 - d) \notin A'$. Therefore, $(\phi_e(-(i + s_d))l + l + 1 - d, \psi_e(-(i + s_d))) \notin A'^e$ and

$$\begin{aligned}\phi_e(-(i + s_d))l + l + 1 - d &= (-1 - \phi_e(i + s_d + 1))l + l + 1 - d\\ &= -\phi(a)l + 1 - d\\ &= -j - t_r.\end{aligned}$$

When $i + s_d \equiv 0$ (**mod** e), $\psi_e(a) = 1$ so $\psi_e(-(i + s_d)) = e = e + 1 - r$. When $i + s_d \not\equiv 0$ (**mod** e), $\phi_e(-(i+s_d)) = e-\phi_e(i+s_d) = e-\phi_e(a)+1 = e+1-r$. Each step of the argument holds in converse so the claim is proved.

Now, Corollary 3.3.3 together with the fact that the correspondence between abaci and charged multipartitions is a bijection give the following. If μ is the multipartiton corresponding to A^e then μ' is the multipartition corresponding to A'^e.

Let $n \in \mathbb{Z}$ be the largest integer such that $(n - i, d) \in A^e$ for all $i \ge 0$ and $1 \le d \le l$. Define n' similarly for A'^e. Let m be the smallest integer such that $(m + i, d) \notin A$ for all $i > 0$ and $1 \le d \le l$. Define m' similarly for A'^e. Corollary 3.3.3 and the claim above give $n = -m'$, $m = -n'$ and $m - n = m' - n'$. For each $1 \le i \le e$, define $I_i := \{(a, i) \in A^e \,|\, a > n(A)\}$ and I'_i similarly for A'^e. Then, by definition, $t_i = n + |I_i|$ and $t'_i = n' + |I'_i|$. Also, $|I'_i| = m - n - |I_{l+1-i}|$ so that

$$\begin{aligned}t'_i &= n' + |I'_i|\\ &= n' + m - n - |I_{l+1-i}|\\ &= m' - |I_{l+1-i}|\\ &= -n - |I_{l+1-i}|\\ &= -t_{l+1-i}.\end{aligned}$$

□

3.4 The $U_q(\widehat{\mathfrak{sl}}_e)$-Module Structure of Fock Space

Let $\mathbf{s} \in \mathbb{Z}^l$ and $\bar{s}_i := s_i$ (**mod** n) and define a function by the formula

$$\Delta(\mathbf{s}\,|\,n) := \frac{1}{2}\sum_{i=1}^{l}\left(\frac{s_i^2 - \bar{s}_i^2}{n} - \frac{s_i - \bar{s}_i}{n}\right).$$

The following result describes a module structure on $\Lambda^{s+\frac{\infty}{2}}$.

Proposition 3.4.1. *(Uglov) There is an action of $U_q(\widehat{\mathfrak{sl}}_e)$ on*

$$\Lambda^{s+\frac{\infty}{2}} \overset{\text{v.s.}}{\cong} \langle\{|\lambda, \mathbf{s}\rangle \,|\, \lambda \in \mathcal{P}^l, \mathbf{s} \in \mathbb{Z}^l(s)\}\rangle_{\mathbb{C}(q)},$$

such that each $|\lambda, \mathbf{s}\rangle$ is a weight vector and ∂ acts on $|\lambda, \mathbf{s}\rangle$ with the following weight.

$$\partial|\lambda, \mathbf{s}\rangle := -\Big(\Delta(\mathbf{s}\,|\,e) + N_0(|\lambda, \mathbf{s}\rangle, e)\Big)|\lambda, \mathbf{s}\rangle,$$

where $N_0(|\lambda, \mathbf{s}\rangle, e)$ denotes the number of e-residues of λ that are 0.

Proof. This is [Ugl00, Theorem 2.1]. □

Lemma 3.4.1. *(Yvonne) Let $|\lambda, \mathbf{s}\rangle$ be a charged l-multipartition with e-quotient $|\mu, \mathbf{t}\rangle$. Let ∂ denote the element of $U_q(\hat{\mathfrak{sl}}_e)$ as defined above and $\dot{\partial}$ the corresponding element of $U_p(\hat{\mathfrak{sl}}_l)$. Then the weight of ∂ acting on $|\lambda, \mathbf{s}\rangle$ is the same as the weight of $\dot{\partial}$ acting on $|\mu, \mathbf{t}\rangle$. That is,*

$$-\Big(\Delta(\mathbf{s}\,|\,e) + N_0(|\lambda, \mathbf{s}\rangle, e)\Big) = -\Big(\Delta(\mathbf{t}\,|\,l) + N_0(|\mu, \mathbf{t}\rangle, l)\Big)$$

and this number is denoted $\mathbf{wt}_e(|\lambda, \mathbf{s}\rangle) = \mathbf{wt}_l(|\mu, \mathbf{t}\rangle)$.

Proof. This is proved in [Yvo05, Proposition 3.24]. There, the author uses the letter n instead of e. □

Let $\mathcal{F}[\mathbf{s}]$ be the subspace of $\Lambda^{s+\frac{\infty}{2}}$ spanned by the elements $\left\{|\lambda, \mathbf{s}\rangle \;\middle|\; \lambda \in \mathcal{P}^l\right\}$. This is a weight space but not a submodule. Given $\mathbf{s} \in \mathbb{Z}^l(s)$ and $\mathbf{t} \in \mathbb{Z}^e(s)$ define the weight space

$$\mathcal{F}[\mathbf{s}, \mathbf{t}] := \left\langle \left\{ |\lambda, \mathbf{s}\rangle \in \mathcal{F}[\mathbf{s}] \;\middle|\; \text{there exists a } \mu \in \mathcal{P}^e \text{ such that } |\lambda, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu, \mathbf{t}\rangle \right\} \right\rangle_{\mathbb{C}(q)}.$$

Finally, let

$$\mathcal{F}[\mathbf{s}, \mathbf{t}]_w := \langle \{ |\lambda, \mathbf{s}\rangle \in \mathcal{F}[\mathbf{s}, \mathbf{t}] \mid \mathbf{wt}_e|\lambda, \mathbf{s}\rangle = w \} \rangle_{\mathbb{C}(q)}.$$

By Lemma 3.4.1, these are the weight spaces. It is these that are *categorified* by the blocks of category $\mathcal{O}$.

3.5 The Category $\mathcal{O}_{\mathbf{c}}(W)$ Corresponding to Fock Space

Let e and l be natural numbers, not both one. Let $\mathbf{s} \in \mathbb{Z}^l(s)$ and define $\mathcal{O}_m[\mathbf{s}]$ to be the category $\mathcal{O}_{\mathbf{c}}(W)$, where W is the complex reflection group $G(l, 1, m)$) acting on its natural representation $\mathfrak{h} \cong \mathbb{C}^m$. The parameters, $\mathbf{c}$, are dependent on $\mathbf{s}$ and e and are defined as follows.

Recall that $\mathcal{E}$ is the set of reflection hyperplanes of $(W, \mathfrak{h})$ and $\mathcal{S}$ is the set of conjugacy classes of non-trivial reflections. For $i = 0, \ldots, |W_H| - 1$, define $\epsilon_{H,i}$ to be the idempotent

$$\epsilon_{H,i} := \frac{1}{|W_H|} \sum_{w \in W_H} \det(w)^i w \in \mathbb{C}W_H.$$

Given $i \in \mathbb{Z}$, define $\bar{i} := i \;(\mathbf{mod}\; l)$. Then, for all $H \in \mathcal{E}$ and $i \in \mathbb{Z}$, define.

$$C_{H,i} := \begin{cases} k_i := \dfrac{s_{\bar{i}}}{e} - \dfrac{\bar{i}}{l} & \text{if } H \text{ is a cyclic hyperplane} \\ h_{i \;(\mathrm{mod}\; 2)} & \text{otherwise,} \end{cases}$$

where $s_0 := s_l$, $h_0 = 0$ and $h_1 = \dfrac{1}{e}$. The parameters $\mathbf{c} = (c_s)_{s \in \mathcal{S}}$ are then determined by the formula[2]

$$\gamma_H := |W_H| \sum_{i=0}^{|W_H|-1} (C_{H,i} - C_{H,i+1})\epsilon_{H,i} = -2 \sum_{s \in W_H \setminus \{1\}} c_s s.$$

Let $\mathcal{O}[\mathbf{s}]$ be the direct sum of all the modules categories $\mathcal{O}_m[\mathbf{s}]$ for all $m \geq 0$. Let $\mathbf{t} \in \mathbb{Z}^e(s)$ and $w \in \mathbb{Z}$. Given an l-multipartition of m, λ say, let $L(\lambda)$ denote the simple object in $\mathcal{O}(G(l, 1, m))$ corresponding to the character $\lambda \in \mathrm{Irr}(G(l, 1, m)$ (see Theorem 3.1.1). Let $P(\lambda)$ be the projective cover of $L(\lambda)$. Define an object $P(\mathbf{s}, \mathbf{t}, w)$ of $\mathcal{O}[\mathbf{s}]$ by

$$P(\mathbf{s}, \mathbf{t}, w) := \bigoplus_{|\lambda, \mathbf{s}\rangle \in \mathcal{F}[\mathbf{s}, \mathbf{t}]_w} P(\lambda)$$

[2] This differs by a sign from the definition given in [CM, Section 3.2]. Here, γ_H is chosen to agree with the end of [GGOR03, Remark 3.1], whereas in [CM, Section 3.2], γ_H is chosen to agree with [Ari08, Definition 4.5].

and then

$$\mathcal{O}[\mathbf{s},\mathbf{t}]_w := {}_{\mathrm{End}_{\mathcal{O}[\mathbf{s}]}(P(\mathbf{s},\mathbf{t},w))}\mathbf{mod}.$$

The following result provides the link between the representation theory of $\mathcal{O}[\mathbf{s},\mathbf{t}]_w$ and the combinatorics of $\mathcal{F}[\mathbf{s},\mathbf{t}]_w$.

Lemma 3.5.1. *Given $\mathbf{s} \in \mathbb{Z}^l(s)$, $\mathbf{t} \in \mathbb{Z}^e(s)$ and $w \in \mathbb{Z}$, there exists an $m \geq 0$ such that $\mathcal{O}[\mathbf{s},\mathbf{t}]_w$ is a single block of $\mathcal{O}_m[\mathbf{s}] = \mathcal{O}_{\mathbf{c}}(G(l,1,m))$ and the simple modules in $\mathcal{O}[\mathbf{s},\mathbf{t}]_w$ are parametrised by the l-multipartitions of m, λ_i, such that $\{|\lambda_i,\mathbf{s}\rangle\}$ is a basis of $\mathcal{F}[\mathbf{s},\mathbf{t}]_w$.*

Proof. See [CM, §3.4]. □

The Level-Rank Conjecture ([CM, Conjecture 6]) can now be stated.

Conjecture. *(Level-Rank Duality) Let $\mathbf{s} \in \mathbb{Z}^l$ and $\mathbf{t} \in \mathbb{Z}^e$, recall that $\mathbf{s}' = (-s_l,\dots,-s_1)$ (similarly for $\mathbf{t}'$). The category $\mathcal{O}[\mathbf{s},\mathbf{t}]_w$ is Koszul and its dual is equivalent to*

$$\mathcal{O}[\mathbf{s},\mathbf{t}]_w^! \simeq \mathcal{O}[\mathbf{t}',\mathbf{s}']_w.$$

Remark 3.5.1. *This has recently been proved by Rouquier–Shan–Varagnolo–Vasserot in [RSVV13] and Webster in [Web13].*

3.6 Ringel Duality

The transformation $\mathcal{O}[\mathbf{s},\mathbf{t}]_w \to \mathcal{O}[\mathbf{s}',\mathbf{t}']_w^!$ is a combination of both Koszul duality, $(-)^!$, and Ringel duality which is a derived equivalence. This point of this section is to show that Ringel duality, defined below, has the effect of transforming the charges $\mathbf{s} \mapsto \mathbf{s}'$.

Lemma 3.6.1. *Let $\mathbf{s} \in \mathbb{Z}^l$, $\mathbf{t} \in \mathbb{Z}^e$ and $w \in \mathbb{Z}$. Then $\mathcal{O}[\mathbf{s},\mathbf{t}]_w$ has Ringel dual $\mathcal{O}[\mathbf{s}',\mathbf{t}']_w$.*

Proof. From [GGOR03, Corollary 4.11], the Ringel duality functor R runs between

$$R\colon \mathcal{O}(V,\gamma) \longrightarrow \mathcal{O}(V^*,\gamma).$$

Recall the naïve duality which is an equivalence $(-)^\vee\colon \mathcal{O}(\mathfrak{h},\gamma) \simeq \mathcal{O}(\mathfrak{h}^*,\gamma^\dagger)^{\mathrm{op}}$ induced by the isomorphism of algebras defined by $(-)^\dagger\colon \mathbb{C}W \to \mathbb{C}W; w \mapsto w^{-1}$ and fixing $\mathfrak{h}$ and $\mathfrak{h}^*$. Composing, one gets the following functor

$$F := (-)^\vee \circ R\colon \mathcal{O}(\mathfrak{h},\gamma)\xrightarrow{R}\mathcal{O}(\mathfrak{h}^*,\gamma)\xrightarrow{(-)^\vee}\mathcal{O}(\mathfrak{h},\gamma^\dagger)^{\mathrm{op}}.$$

Note that

$$\begin{aligned}\epsilon_{H,j}^\dagger &= \frac{1}{|W_H|}\sum_{w\in W_H}\det(w)^j w^{-1}\\ &= \frac{1}{|W_H|}\sum_{w\in W_H}\det(w^{-1})^{-j} w^{-1}\\ &= \epsilon_{H,-j}\\ &= \epsilon_{H,|W_H|-j}.\end{aligned}$$

If H is a non-cyclic hyperplane then $|W_H| = 2$ and $\gamma_H = 2h_1(\epsilon_{H,0} - \epsilon_{H,1}) = \gamma_H^\dagger$, thus h_1 is invariant under F. If H is a cyclic hyperplane and $s' = (s'_1,\dots,s'_l)$ is the corresponding charge for $F\mathcal{O}(V,\gamma)$ then $|W_H| = l$ and

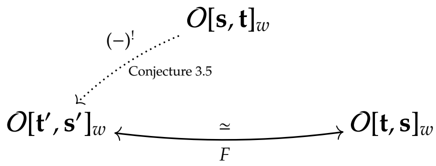
$$\begin{aligned}\tfrac{1}{l}\gamma_H^\dagger &= (k_l - k_1)\epsilon_{H,0} + (k_1 - k_2)\epsilon_{H,-1} + \cdots + (k_{l-1} - k_l)\epsilon_{H,1-l}\\ &= (k_l - k_1)\epsilon_{H,0} + (k_{l-1} - k_l)\epsilon_{H,1} + \cdots + (k_1 - k_2)\epsilon_{H,l-1}\end{aligned}$$

so that

$$\begin{aligned} s'_1 - s'_l &= s_1 - s_l \\ s'_2 - s'_1 &= s_l - s_{l-1} \\ &\vdots \\ s'_l - s'_{l-1} &= s_2 - s_1. \end{aligned}$$

Defining $\mathbf{s}' = (-s_l, \dots, -s_1)$ satisfies these equations. Proposition 3.3.1 shows that the e-quotient of the charge $\mathbf{s}'$ is $\mathbf{t}'$. $\square$

Here is a picture.



Chapter 4

The Level-Rank Duality Conjecture

The combinatorics developed in the previous chapter will now be used to prove some cases of the Level-Rank Duality Conjecture, Conjecture 3.5.

4.1 Nakayama Weights

Definition 4.1.1. *Let $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ be a multipartition of m charged by the l-tuple $\mathbf{s} = (s_1, \ldots, s_l)$. The **Nakayama e-weight** of $|\lambda, \mathbf{s}\rangle$ is the unordered set of l e-residues.*

Theorem 4.1.1. *(Nakayama Conjecture) The simple modules of $\mathcal{O}_m[\mathbf{s}]$ corresponding to two charged l-multipartions belong to the same block if and only if they have the same Nakayama e-weight.*

Proof. This was proved by Mathas in [Mat99, Corollary 5.38]. □

The following proposition provides a way to check Conjecture 3.5 in concrete cases.

Proposition 4.1.1. *Let $\mathcal{O}[\mathbf{s}, \mathbf{t}]_w$ be the block of $\mathcal{O}_m[\mathbf{s}]$ for some weight w with simple modules parametrised by the l-multipartitions $\{\lambda_1, \ldots, \lambda_n\}$. For each $i = 1, \ldots, n$, let $|\lambda_i, \mathbf{s}\rangle \overset{e}{\rightsquigarrow} |\mu_i, \mathbf{t}\rangle$ and let r be the degree of μ_1. Then $\{\mu'_1, \ldots, \mu'_n\}$ parametrises the simple modules of the block $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w \subseteq \mathcal{O}_r[\mathbf{t}']$.*

Proof. By Lemma 3.5.1, $\{|\lambda_i, \mathbf{s}\rangle \,|\, i = 1, \ldots, n\}$ forms a basis for $\mathcal{F}[\mathbf{s}, \mathbf{t}]_w$. Then, by Lemma 3.3.1, for each $i = 1, \ldots, n$, $|\mu_i, \mathbf{t}\rangle \overset{l}{\rightsquigarrow} |\lambda_i, \mathbf{s}\rangle$ so $|\mu_i, \mathbf{t}\rangle \in \mathcal{F}[\mathbf{t}, \mathbf{s}]$. By Lemma 3.4.1, $|\mu_i, \mathbf{t}\rangle \in \mathcal{F}[\mathbf{t}, \mathbf{s}]_w$. Because $\{|\lambda_1, \mathbf{s}\rangle, \ldots, |\lambda_n, \mathbf{s}\rangle\}$ is a basis for $\mathcal{F}[\mathbf{s}, \mathbf{t}]_w$ and $\overset{e}{\rightsquigarrow}$ and $\overset{l}{\rightsquigarrow}$ are bijections it follows that $\{|\mu_1, \mathbf{t}\rangle, \ldots, |\mu_n, \mathbf{t}\rangle\}$ is a basis for $\mathcal{F}[\mathbf{t}, \mathbf{s}]_w$. Applying Lemma 3.5.1 again implies that $\{\mu_i \,|\, i = 1, \ldots, n\}$ is a complete parametrising set for the simple modules in the block $\mathcal{O}[\mathbf{t}, \mathbf{s}]_w$. By Proposition 3.3.1, $|\mu'_i, \mathbf{t}'\rangle \in \mathcal{F}[\mathbf{t}', \mathbf{s}']_w$ and because $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ is the Ringel dual of $\mathcal{O}[\mathbf{t}, \mathbf{s}]_w$, the block $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ must contain the same number of simple modules, namely n. Therefore, the set $\{\mu'_i \,|\, i = 1, \ldots, n\}$ completely parametrises the simple modules in the block $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$. □

Corollary 4.1.1. *If $\mathcal{O}[\mathbf{s}, \mathbf{t}]_w$ is contains only one simple module then $\mathcal{O}[\mathbf{s}, \mathbf{t}]^!_w \simeq \mathcal{O}[\mathbf{t}', \mathbf{s}']_w$.*

Proof. The proposition implies that $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ has one simple module. By Lemma 1.5.2, both the blocks are simple and so Morita equivalent and Koszul self-dual. □

Proposition 4.1.2. *The Level-Rank Duality conjecture holds for $W = \mathfrak{S}_2$ and $W = \mu_2$ (it switches between these two cases) and in the following cases for $W = B_2$.*

(i) When $e = 2$ and $r = 2$.

(ii) When $e = 3$ and $r = 2$.

Proof. When W is a dihedral group of order two, $\mathcal{O}(W)$ is, by Proposition 2.2.1, either B_1 or semisimple. Since B_1 is Koszul self-dual, Theorem 4.1.1 immediately confirms the conjecture.

The proof for the cases of $W = B_2$ occupy the following section. □

	Values of $\mathbf{s}$	Values of $\mathbf{c}$		$\mathrm{End}_{\mathcal{O}}(P)$
$e=1$	$s_1-s_2\in\{0,1\}$	$c_0=-1$	$c_1\in\{-\frac{1}{2},\frac{1}{2}\}$	B_5
	otherwise	$c_0=-1$	$c_1\in\mathbb{Z}\setminus\{0,-1\}+\frac{1}{2}$	B_4
$e=2$	$s_1-s_2\in 2\mathbb{Z}$	$c_0=-\frac{1}{2}$	$c_1\in\mathbb{Z}+\frac{1}{2}$	$\mathsf{B}_3\oplus\mathsf{B}_0$
	$s_1-s_2=1$	$c_0=-\frac{1}{2}$	$c_1=0$	B_4
	otherwise	$c_0=-\frac{1}{2}$	$c_1\in\mathbb{Z}\setminus 0$	B_5
$e\geq 3$	$s_1-s_2\in e\mathbb{Z}\pm 1$	$c_0=-\frac{1}{e}$	$c_1\in\mathbb{Z}+\frac{1}{2}\pm\frac{1}{e}$	$\mathsf{B}_2\oplus\mathsf{B}_0^{\oplus 2}$
	$s_1-s_2\in e\mathbb{Z}$	$c_0=-\frac{1}{e}$	$c_1\in\mathbb{Z}+\frac{1}{2}$	$\mathsf{B}_1^{\oplus 2}\oplus\mathsf{B}_0$
	otherwise	$c_0=-\frac{1}{e}$	c_1 is generic	$\mathsf{B}_0^{\oplus 5}$

Table 4.1: The parameter values of $\mathcal{O}_{\mathbf{c}}(G(2,1,2))$ described by $\mathcal{O}_2[\mathbf{s}]$.

4.2 Level-Rank Duality for Low Rank Cases when $W = G(2,1,2)$

4.2.1 The category $\mathcal{O}(G(2,1,2))$

Fix $l=2$ and $m=2$. Then $W=G(2,1,2)$ and the parameters are a pair $\mathbf{c}=(c_0,c_1)\in\mathbb{Q}^2$ where c_0 corresponds to the non-cyclic hyperplanes and c_1 corresponds to the cyclic hyperplanes. Suppose H is a non-cyclic hyperplane corresponding to the reflection $t\in W$ say. Then

$$-2c_0t=\gamma_H=\frac{2}{e}(\epsilon_{H,0}-\epsilon_{H,1})=\frac{2}{e}t$$

so that $c_0=-\frac{1}{e}$. Let H be a cyclic hyperplane corresponding to the reflection $s\in W$. Then

$$-2c_1s=\gamma_H=2\left(\frac{s_2-s_1}{e}+\tfrac{1}{2}\right)s$$

so that $c_1=\frac{s_1-s_2}{e}-\frac{1}{2}$. The range of parameter values described by blocks of the form $\mathcal{O}[\mathbf{s},\mathbf{t}]_w$ is then

$$\left\{\left(-\frac{1}{e},\frac{s_1-s_2}{e}-\frac{1}{2}\right)\in\mathbb{Q}^2\,\middle|\,e\in\mathbb{N},\mathbf{s}\in\mathbb{Z}^2\right\}.$$

Recall the notation, B_i, from Section 2.1, to denote the blocks of $\mathcal{O}(W)$ for dihedral groups.

Proposition 4.2.1. *(Parameter Values for $W=G(2,1,2)$) The parameter values for $\mathcal{O}_{\mathbf{c}}(G(2,1,2))$ described by $\mathcal{O}_2[\mathbf{s}]$ are given in Table 4.1 along with the block decomposition.*

Proof. Recall the notation E_i^+, L_i^1 and L_i^2 defined in Section 2.2.3 and [Chm06, §3.2.1]. Consider the locus $C:=\bigcup_{i=1,2}E_i^+\cup L_i^1\cup L_i^2\subset\mathbb{C}^2$. The module $\Delta_{\mathbf{c}}(\mathbf{triv})$ is not semisimple if and only if $\mathbf{c}\in C$. Let

$$C':=\{\mathbf{c}\in C\,|\,\text{there exists an } i \text{ and } j \text{ such that } p\in E_i^+\cap L_j^1 \text{ or } p\in E_i^+\cap L_j^2\}$$

and

$$C'':=\{\mathbf{c}\in C\,|\,\text{there exists an } i \text{ and } j \text{ such that } p\in L_j^1\cap L_j^2\}.$$

Let V be the Klein 4-group acting on $\mathbb{C}^2$ by reflecting the coordinate axes. Count points in their V-orbit with multiplicity so that each orbit contains four points. This action is generated by the action of the one-dimensional characters so that considering the V-orbit of a point takes into account the behaviour of the modules $\Delta(\epsilon_1)$, $\Delta(\epsilon_2)$ and $\Delta(\mathbf{sgn})$.

The classification of parameters in [Chm06, Theorem 3.2.2] along with its analysis in Section 2.2.3 gives the following description of each block $\mathsf{B}_0,\ldots,\mathsf{B}_5$.

(i) If the V-orbit of a point is disjoint from C then the algebra is semisimple and so a direct sum of five copies of B_0.

(ii) If the V-orbit of a point is disjoint from E_i^+ for all $i \geq 0$ but intersects C at some L_i^1 or L_i^2 then the algebra has three simple projectives and one block, B_1, of order two.

(iii) If the V-orbit of a point is disjoint from L_i^1 and L_i^2 for all $i \geq 0$, but intersects C at some E_i^+ then the algebra has two simple projectives and the non-simple block is isomorphic to B_2.

(iv) The parameters for which the algebra is isomorphic to the block B_3 can be described as those points in $\mathbb{Q}^2$ whose V-orbit contains a point of C''.

(v) The parameters for which the algebra is isomorphic to the block B_4^λ are precisely those points whose V-orbit contains one point in P' and two points in $\mathbb{Q}^2 \setminus P$.

(vi) Parameters for the block B_5^λ are those points whose V-orbit contains exactly one point in P' and one point in $\mathbb{Q}^2 \setminus P$.

When $c_0 = -1$, a point has a V-orbit that intersects C if and only if $c_1 = k + \frac{1}{2}$ for some $k \in \mathbb{Z}$. Suppose $\mathbf{c}$ is one of these points with $c_1 > 0$. Then $(c_0, -c_1) \notin C$ and, also, $(-c_0, -c_1) \notin C$ if and only if $c_1 \neq \frac{1}{2}$. Therefore, $c_1 = \pm\frac{1}{2}$ are the only two points whose orbit contains only one point in $\mathbb{C}^2 \setminus C$.

When $c_0 = -\frac{1}{2}$, a point has a V-orbit that intersects C if and only if $c_1 = \frac{1}{2}k$ for some $k \in \mathbb{Z}$. The orbit of the point $\mathbf{c}$ can intersect C'' if and only if $c_1 = k + \frac{1}{2}$ for some $k \in Z$. Suppose $c_1 \neq 0$. Then $(-c_0, c_1)$ and $(-c_0, -c_1)$ both lie in $L_0^1 \subset C$ so the algebra is isomorphic to B_5. The point $(-1, 0) \in \mathbb{Q}^2 \setminus P$ is counted twice so the algebra is isomorphic to B_4.

When $e \geq 3$ and $c_1 = -\frac{1}{e}$ a point has a V-orbit that intersects C if and only if either $c_1 \in \mathbb{Z} + \frac{1}{2}$ or $c_1 \in \mathbb{Z} + \frac{1}{2} \pm \frac{1}{e}$. In the first case the orbit never contains a point in E_i^+ and in the second the orbit never contains a point in L_i^1 or L_i^2. □

4.2.2 Case 1: Low rank examples for $e = 2$

For $e = 2$ the block decompostion of $\mathcal{O}_2[\mathbf{s}]$ depends on the parity of $s_1 - s_2$.

Proposition 4.2.2. *Let $e = 2$ and $s_2 - s_1 = 2k$ for some $k \in \mathbb{Z}$. Then $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ is a block of $G(2, 1, k(k+1) + 2)$. When $k = 0, -1$ the Level-Rank Conjecture holds.*

Proof. The degree of the block $\mathcal{O}[\mathbf{t}', \mathbf{s}]_w$ is equal to the degree of the 2-quotient of $|(\square\square, \varnothing), (0, 2k)\rangle$. The quotient is

$$|(\square\square, \varnothing), (0, 2k)\rangle \overset{2}{\rightsquigarrow} \begin{cases} |(\tau_k, \sigma_k), (k, k)\rangle & \text{if } k \geq 0, \\ |(\tau_{-k-1}, \sigma^t_{-k-1}), (k, k)\rangle & \text{if } k \leq -1. \end{cases}$$

which has degree $r = k(k+1) + 2$. Using Corollary 3.3.2, for a general charge $\mathbf{s} = (s_1, s_1 + 2k)$, the charge of the 2-quotient is

$$\mathbf{t} = \begin{cases} (k + s_1, k + s_1) & \text{if } s_1 = 0 \ (\mathbf{mod}\ 2), \\ (k + s_1 + 1, k + s_1 - 1) & \text{otherwise.} \end{cases}$$

Therefore, $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ is a block of $\mathcal{O}_{\mathbf{c}'}(G(2, 1, r))$ where $\mathbf{c}' = (-\frac{1}{2}, \frac{1}{2})$ if s_1 is even or $\mathbf{c}' = (-\frac{1}{2}, -\frac{1}{2})$ if s_1 is odd.

When $k = 0, -1$ this gives $\mathcal{O}[\mathbf{s}, \mathbf{t}]_w \simeq \mathcal{O}[\mathbf{t}', \mathbf{s}']_w$. Because B_3 is Koszul self-dual this confirms the conjecture in these cases. □

4.2.3 Case 2: Low rank examples for $e = 3$

First, a calculation needs to be made for the Cherednik algebra corresponding to μ_3.

Cherednik Algebra for μ_3

Let $\mu_3 = \{1, s, s^2\}$, the cyclic group of order 3. Let $\omega = \exp(2\pi i/3)$, let ϵ be the irreducible representation which takes the value ω on s and let $\epsilon^2 := \epsilon \otimes \epsilon$ so that $\mathrm{Irr}(\mu_3) = \{\mathbf{triv}, \epsilon, \epsilon^2\}$. Let y and x be dual basis vectors for $\mathfrak{h}$ and $\mathfrak{h}^*$ respectively. Let $c_0 := c_s$ and $c_1 := c_{s^2}$. Then $H_{\mathbf{c}}$ has the relations

$$\begin{aligned}[s, x] &= \omega = [s, y]\\ [y, x] &= 1 - c_0 s - c_1 s^2.\end{aligned}$$

Let t, e_1 and e_2 be a generators for **triv**, ϵ and ϵ^2 respectively. The singular vectors in $\Delta(\mathbf{triv})$ are

$$\begin{cases} x^{3k+1} \otimes t & \text{if } c_0 + c_1 = -3k - 1 < 0 \\ x^{3k+2} \otimes t & \text{if } \omega^2 c_0 + \omega c_1 = 3k + 2 > 0, \end{cases}$$

for some $k \geq 0$, the singular vectors in $\Delta(\epsilon)$ are

$$\begin{cases} x^{3k+1} \otimes e_1 & \text{if } \omega c_0 + \omega^2 c_1 = -3k - 1 < 0 \\ x^{3k+2} \otimes e_1 & \text{if } c_0 + c_1 = 3k + 2 > 0 \end{cases}$$

for some $k \geq 0$ and the singular vectors in $\Delta(\epsilon^2)$ are

$$\begin{cases} x^{3k+1} \otimes e_2 & \text{if } \omega^2 c_0 + \omega c_1 = -3k - 1 < 0 \\ x^{3k+2} \otimes e_2 & \text{if } \omega c_0 + \omega^2 c_1 = 3k + 2 > 0. \end{cases}$$

for some $k \geq 0$.

Now $\mathcal{O}_1[(s_1, s_2, s_3)]$ with $e = 2$ has

$$\begin{aligned}\gamma_H = &\left(\left(\frac{s_1}{2} - \frac{1}{3}\right)(1 - \omega) + \left(\frac{s_2}{2} - \frac{2}{3}\right)(\omega - \omega^2) + \left(\frac{s_3}{2} - 1\right)(\omega^2 - 1)\right)s \\ &+ \left(\left(\frac{s_1}{2} - \frac{1}{3}\right)(1 - \omega^2) + \left(\frac{s_2}{2} - \frac{2}{3}\right)(\omega^2 - \omega) + \left(\frac{s_3}{2} - 1\right)(\omega - 1)\right)s^2.\end{aligned}$$

So

$$\begin{aligned} c_s + c_{s^2} &= 2 + \frac{3}{2}(s_1 - s_3) \equiv 2 \pmod 3 \Leftrightarrow s_1 - s_3 \equiv 0 \pmod 2 \\ \omega c_s + \omega^2 c_{s^2} &= -1 + \frac{3}{2}(s_3 - s_2) \equiv 2 \pmod 3 \Leftrightarrow s_2 - s_3 \equiv 0 \pmod 2 \\ \omega^2 c_s + \omega c_{s^2} &= -1 + \frac{3}{2}(s_2 - s_1) \equiv 2 \pmod 3 \Leftrightarrow s_1 - s_2 \equiv 0 \pmod 2 \end{aligned}$$

This agrees with the description of the blocks of $\mathcal{O}(\mu_3)$ given by Nakayama 2-weights:

Proposition 4.2.3. *Two irreducibles $L(\lambda_i)$ and $L(\lambda_j)$ are in the same block of $\mathcal{O}(\mu_3) = \mathcal{O}_1[\mathbf{s}]$ if and only if two of their corresponding charges s_i and s_j are equal modulo 2.*

The Level-Rank Conjecture

When $e \geq 3$, the block is semisimple when $s_1 - s_2$ is not congruent to -1, 0 or 1 modulo e. Following is an example from each of these cases for $e = 3$.

Proposition 4.2.4. *Let $s_1 - s_2 = 1 + ek$ for some $k \in \mathbb{Z}$. Then $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ is a block of $G(2, 1, \frac{e}{2}k^2 + \frac{e-2}{2}k + 1)$. When $k = 0$ and $e = 3$ the Level-Rank Conjecture holds.*

Proof. It suffices to show that the conjecture holds for the block isomorphic to B_2. By examining the Nakayama weights, the block of $\mathcal{O}_m[\mathbf{s}]$ isomorphic to B_2 is parametrised by the three two-partitions $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \varnothing)$, $(\square, \square)$ and $(\varnothing, \square\square)$. Therefore, the degree of the block $\mathcal{O}[\mathbf{t}', \mathbf{s}]_w$ is equal to the degree of the e-quotient of $|(\begin{smallmatrix}\square\\\square\end{smallmatrix},\varnothing),\mathbf{s}\rangle$. The quotient is

$$|(\begin{smallmatrix}\square\\\square\end{smallmatrix},\varnothing),(0,ek-1)\rangle \overset{e}{\rightsquigarrow} \begin{cases} |(\tau_{k-1},\tau_k,\ldots,\tau_k,\tau_{k+1},\tau_{k-1}), \\ \qquad (k+1,k,\ldots,k,k-1,k-1)\rangle & \text{if } k\geq 0, \\ |(\tau_{-k},\tau_{-k-1},\ldots,\tau_{-k-1},\tau_{-k-2},\tau_{-k}), \\ \qquad (k+1,k,\ldots,k,k-1,k-1)\rangle & \text{if } k\leq -1, \end{cases}$$

which has degree $r=\frac{e}{2}k^2+\frac{e-2}{2}k+1$.

When $k=0$, $\mathcal{O}[\mathbf{t}',\mathbf{s}']_w$ is a block of $\mathcal{O}_{\mathbf{c}}(\mu_e)$. For $e=3$ the corresponding parameters are $\mathbf{t}'=(1,1,-1)$. By Proposition 4.2.3, this implies that the three irreducibles of $\mathcal{O}_{\mathbf{c}}(\mu_3)$ belong to the same block. Because the block structure of $\mathcal{O}(\mu_3)$ depends on the residues of the charges modulo 2 the conjecture holds for all $\mathbf{s}=(j,j-1)$ for $j\in\mathbb{Z}$. □

Proposition 4.2.5. *Let $s_1-s_2=-1+ek$ for some $k\in\mathbb{Z}$. Then $\mathcal{O}[\mathbf{t}',\mathbf{s}']_w$ is a block of $G(2,1,\frac{e}{2}k^2+\frac{e-2}{2}k+1)$. When $k=-1$ and $e=3$ the Level-Rank Conjecture holds.*

Proof. It suffices to show that the conjecture holds for the block isomorphic to B_2. By examining the Nakayama weights, the block of $\mathcal{O}_m[\mathbf{s}]$ isomorphic to B_2 is parametrised by the three two-partitions $(\square\square,\varnothing)$, $(\square,\square)$ and $(\varnothing,\begin{smallmatrix}\square\\\square\end{smallmatrix})$. Therefore, the degree of the block $\mathcal{O}[\mathbf{t}',\mathbf{s}]_w$ is equal to the degree of the e-quotient of $|(\square\square,\varnothing),(0,ek+1)\rangle$. The quotient is

$$|(\square\square,\varnothing),(0,ek+1)\rangle \overset{e}{\rightsquigarrow} \begin{cases} |(\tau_{k+1},\tau_{k-1},\tau_k,\ldots,\tau_k,\tau_{k+1}), \\ \qquad (k+1,k+1,k,\ldots,k,k-1)\rangle & \text{if } k\geq 0, \\ |(\tau_{-k-1},\tau_{-k+1},\tau_{-k},\ldots,\tau_{-k},\tau_{-k-1}), \\ \qquad (k+1,k+1,k,\ldots,k,k-1)\rangle & \text{if } k\leq -1, \end{cases}$$

which has degree $r=\frac{e}{2}k^2+\frac{e+2}{2}k+2$.

When $k=-1$ so that $s_1-s_2=-1-e$, $\mathcal{O}[\mathbf{t}',\mathbf{s}']_w$ is a block of $\mathcal{O}_{\mathbf{c}}(\mu_e)$. For $e=3$ the corresponding parameters are $\mathbf{t}'=(2,0,0)$. By Proposition 4.2.3, this implies that the three irreducibles of $\mathcal{O}_{\mathbf{c}}(\mu_3)$ belong to the same block. Shifting the original charge, $(0,ek+1)$, by one has the effect of adding 2 to one of the coordinates of the charge $\mathbf{t}$. Because the block structure of $\mathcal{O}(\mu_3)$ depends on the residues of the charges modulo 2 the conjecture holds for all $\mathbf{s}=(j,j+1-e)$ for $j\in\mathbb{Z}$. □

Proposition 4.2.6. *Let $s_1-s_2=ek$ for some $k\in\mathbb{Z}$. Then $\mathcal{O}[\mathbf{t}',\mathbf{s}']_w$ is a block of $G(2,1,\frac{e}{2}(k^2+k)+1)$. When $k=-1$ and $e=3$ the Level-Rank Conjecture holds.*

Proof. The two blocks which are isomorphic to B_1 contain $(\square\square,\varnothing)$ and $(\begin{smallmatrix}\square\\\square\end{smallmatrix},\varnothing)$ respectively.

$$|(\square\square,\varnothing),(0,ek)\rangle \overset{e}{\rightsquigarrow} \begin{cases} |(\tau_k,\tau_{k-1},\tau_k,\ldots,\tau_k,\tau_{k+1}), \\ \qquad (k,k+1,k,\ldots,k,k-1)\rangle & \text{if } k\geq 0, \\ |(\tau_{-k-1},\tau_{-k},\tau_{-k-1},\ldots,\tau_{-k-1},\tau_{-k-2}), \\ \qquad (k,k+1,k,\ldots,k,k-1)\rangle & \text{if } k\leq -1, \end{cases}$$

$$|(\begin{smallmatrix}\square\\\square\end{smallmatrix},\varnothing),(0,ek)\rangle \overset{e}{\rightsquigarrow} \begin{cases} |(\tau_{k-1},\tau_k,\ldots,\tau_k,\tau_{k+1},\tau_k), \\ \qquad (k+1,k,\ldots,k,k-1,k)\rangle & \text{if } k\geq 0, \\ |(\tau_{-k},\tau_{-k-1},\ldots,\tau_{-k-1},\tau_{-k-2},\tau_{-k-1}), \\ \qquad (k+1,k,\ldots,k,k-1,k)\rangle & \text{if } k\leq -1, \end{cases}$$

The degree of the e-quotient is $r=\frac{e}{2}(k^2+k)+1$.

Thus $\mathcal{O}[\mathbf{t}',\mathbf{s}']_w$ is a block of $\mathcal{O}_{\mathbf{c}}(\mu_e)$. For $e=3$ the corresponding parameters are $\mathbf{t}'=(1,-1,0)$ for the block containing $(\square\square,\varnothing)$ and $\mathbf{t}'=(0,1,-1)$ for the block containing $(\begin{smallmatrix}\square\\\square\end{smallmatrix},\varnothing)$. By Proposition 4.2.3 these are both blocks with exactly two simples in $\mathcal{O}_{\mathbf{c}}(\mu_3)$. Since these blocks are unique up to Morita equivalence and Koszul self-dual the conjecture holds. Because the block structure of $\mathcal{O}(\mu_3)$ depends on the residues of the charges modulo 2 the conjecture holds for all $\mathbf{s}=(j,j)$ and $\mathbf{s}=(j,j-e)$ for $j\in\mathbb{Z}$.

□

4.2.4 Case 3: Low rank examples for $e = 1$

In order to study the case $e = 1$, algebras called q**-Schur algebras** will be used.

q-Schur Algebras

The definition of a q-Schur algebra is unimportant here; see [Mat99, Section 4.1]. It suffices to note that they are dependent on two parameters, $q \in \mathbb{C}$ and m a positive integer. They are written $S_m(q)$ and are closely related to Iwahori–Hecke algebras $\mathcal{H}_q(\mathfrak{S}_m)$; see Section 1.5.2. Also, only the case when $q = -1$ will be needed in this thesis. Their irreducible representations were classified by Dipper–James (see [Mat99, Theorem 4.15] for example): they are parametrised by partitions of m. By [Mat99, Theorem 5.37], the blocks of $S_m(-1)$ are parametrised by the 2-cores of the partitions corresponding to the simple modules they contain. Write the block corresponding to the 2-core τ_i, $D_{\tau_i,j}$, where j is the number of removable 2-hooks of a partition in the block.

The Koszul dual of $\mathcal{O}_2[\mathbf{s}] = \mathcal{O}_{\mathbf{c}}(W)$ is conjectured to be a block of $\mathcal{O}_{\mathbf{c}}(\mathfrak{S}_r)$ for $r = \frac{(s_2-s_1)(s_2-s_1+1)}{2} + 4$. Consider the cases $r = 5, 7, 10$.

For all $\mathbf{s}$, the corresponding algebra, $\mathrm{End}_{\mathcal{O}}(P)$, is either B_4 or B_5; indeed, all charged multipartions have the same Nakayama 1-weight so the five irreducibles belong in the same block regardless of their charge. Also, $\mathbf{t} = t_1 = s_1 + s_2$.

A computation shows that the 1-quotient of $|(\square\square, \varnothing), (0, s_2)\rangle$ is $|\mu_{s_2}, s_2\rangle$, where

$$\mu_i := \begin{cases} (\tau_{i-2}, 2, 2)^1 & \text{for } i \geq 2 \\ (2,2,1) & \text{for } i = 1 \\ (3,1) & \text{for } i = 0 \\ (4) & \text{for } i = -1 \\ (3-i, \tau_{-i-2}) & \text{for } i \leq -2. \end{cases}$$

Using Lemma 3.3.2, the 1-quotient of $|(\square\square, \varnothing), (s_1, s_2)\rangle$ is $|\mu_{s_2-s_1}, (s_1+s_2)\rangle$. Therefore, $\mathcal{O}[\mathbf{t}', \mathbf{s}']_w$ is a block of $\mathcal{O}_{\mathbf{c}'}(\mathfrak{S}_r)$, where r is the degree, $r := \frac{(s_2-s_1)(s_2-s_1+1)}{2} + 4$, and $\mathbf{c}' = -\frac{1}{2}$. By [CM, Remark 4], at level one, $\mathcal{O}_m[\mathbf{s}]$ is equivalent to the q-Schur algebra $S_r(-1)$. In [Mat99, Appendix B] the decomposition matrices for these algebras is given.

When s_2 and s_1 are chosen so that $s_2 - s_1$ is 1 or -2, $r = 5$ and $|(\square\square, \varnothing), \mathbf{s}\rangle$ has the 1-quotient with partition $(2^2, 1)$ or (5) respectively. When $s_2 - s_1$ is 2 or -3, $r = 7$ and $|(\square\square, \varnothing), \mathbf{s}\rangle$ has the 1-quotient with partition $(2^3, 1)$ or $(6, 1)$ respectively. When $s_2 - s_1$ is 3 or -4, $r = 10$ and $|(\square\square, \varnothing), \mathbf{s}\rangle$ has the 1-quotient with partition $(3, 2^3, 1)$ or $(7, 2, 1)$ respectively. Now, look at the blocks of $S_5(-1)$, $S_7(-1)$ and $S_{10}(-1)$ that contain simples corresponding to these partitions. They are parametrised as follows.

$$\begin{aligned} D_{\tau_1,2} &= \{(5), (3,2), (3,1^2), (2^2,1), (1^5)\} \subset S_5(-1) \\ D_{\tau_2,2} &= \{(6,1), (4,3), (4,1^3), (2^3,1), (2,1^5)\} \subset S_7(-1) \\ D_{\tau_3,2} &= \{(7,2,1), (5,4,1), (5,2,1^3), (3,2^3,1), (3,2,1^5)\} \subset S_{10}(-1) \end{aligned}$$

By [Mat99, Appendix B], each of these blocks have the same decomposition matrices as the block B_4 with respect to the basis $\{\mathbf{triv}, \epsilon_1, \tau, \mathbf{sgn}, \epsilon_2\}$, namely

$$\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ q & q & 1 & & \\ 0 & q^2 & q & 1 & \\ q^2 & 0 & q & 0 & 1 \end{pmatrix}.$$

Remark 4.2.1. *Unlike the blocks B_1, B_2 and B_3, the blocks B_4 and B_5 aren't determined up to Morita equivalence by their decomposition matrices. The problem is that they depend on an extra parameter λ; see Section 2.1. The proof that these blocks were Koszul ignored this by proving that all such blocks were*

[1] This is understood to mean the partition having reordered the parts so they are decreasing.

for all λ. This calculation gives supporting evidence—rather than proof—that the conjecture holds in this case.

Part III

Geometry

Chapter 5

Geometric Prerequisites

The purpose of this chapter is to introduce the necessary techniques from geometry to describe how the *localisation theorem* for rational Cherednik algebra is constructed by *deformation quantisation* and *quantum hamiltonian reduction*.

The Mittag–Leffler condition

Let $\mathcal{A} = \{A_i \,|\, i \in \mathbb{Z}\}$ be a set of abelian groups with homomorphisms $f_{ij} \colon A_j \longrightarrow A_i$ for all $i \leq j$ such that

(i) f_{ii} is the identity on A_i,

(ii) for all $i \leq j \leq k$, $f_{ik} = f_{ij} \circ f_{jk}$.

Then A is called an **inverse system of groups over** $\mathbb{Z}$ and the homomorphisms are called **transition morphisms**. The **inverse limit** of $\mathcal{A}$ is the subgroup of $\prod_{i\in\mathbb{Z}} A_i$ defined by

$$\varprojlim \mathcal{A} := \left\{ (a_i) \in \prod_{i\in\mathbb{Z}} A_i \,\middle|\, a_i = f_{ij}(a_j) \text{ for all } i \leq j \right\}.$$

Let $\mathbf{Ab}^{\mathbb{Z}}$ denote the category of inverse systems. Then $\lim_{\leftarrow}$ defines a functor $\mathbf{Ab}^{\mathbb{Z}} \longrightarrow \mathbf{Ab}$ (see [Wei94]).

Say that $\mathcal{A} \in \mathbf{Ab}^{\mathbb{Z}}$ satisfies the **Mittag–Leffler condition** if for all k there exists a $j \geq k$ such that, for all $i \geq j$, $f_{kj}(A_j) = f_{ki}(A_i)$.

Proposition 5.0.7. *If $\mathcal{A}$ satisfies the Mittag–Leffler condition then, given a short exact sequence in $\mathbf{Ab}^{\mathbb{Z}}$ of the form,*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow C \longrightarrow 0,$$

the quotient of $\overset{\lim}{\leftarrow} \mathcal{B}$ by $\overset{\lim}{\leftarrow} \mathcal{A}$ is isomorphic to $\overset{\lim}{\leftarrow} C$.

Proof. See [Wei94, Theorem 3.5.7]. □

5.1 Čech Cohomology

Let $\mathcal{U} := \{U_i \,|\, i \in I\}$ be an open cover of a topological space, X. For elements, $\alpha_0, \ldots, \alpha_p \in I$, write

$$U_{\alpha_0 \cdots \alpha_p} := \bigcap_{i=0}^{p} U_{\alpha_i}.$$

Let $\mathcal{F}$ be a sheaf of abelian groups on X. Define an abelian group,

$$\check{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{(\alpha_0,\ldots,\alpha_p)\in I^{p+1}} \mathcal{F}(U_{\alpha_0\cdots\alpha_p}).$$

Given $s \in \check{C}^p(\mathcal{U},\mathcal{F})$, let $s_{\alpha_0\cdots\alpha_p}$ be the value of s in $\mathcal{F}(U_{\alpha_0\cdots\alpha_p})$. Define the differential,

$$d\colon \check{C}^p(\mathcal{U},\mathcal{F}) \longrightarrow \check{C}^{p+1}(\mathcal{U},\mathcal{F}),$$

by the formula

$$d(s)_{\alpha_0\cdots\alpha_p} = \sum_{j=0}^{p+1}(-1)^j s_{\alpha_0\cdots\hat{\alpha}_j\cdots\alpha_{p+1}}|_{U_{\alpha_0\cdots\alpha_p}}.$$

Together, these form a complex, $\check{C}^\bullet(\mathcal{U},\mathcal{F})$, called the **Čech complex**. Its cohomology groups are called the **Čech cohomology groups**, written $\check{H}^i(\mathcal{U},\mathcal{F})$.

5.2 Local Cohomology

Let $S \subseteq X$ be a closed subset of a topological space and let $U := X\setminus S$ be its complement. Given a sheaf of abelian groups, $\mathcal{F}$, the restriction morphism of sheaves $\mathrm{Res}\colon \mathcal{F} \longrightarrow \mathcal{F}|_U$ has the kernel sheaf, $\Gamma_S(\mathcal{F}) = \{f \in \mathcal{F} \,|\, \mathrm{Supp}(f) \subseteq S\}$. The p^{th} right-derived functor of Γ_S gives cohomology groups denoted

$$H^p_S(X,\mathcal{F}),$$

the **local cohomology** of $\mathcal{F}$ along S.

5.2.1 Local and Čech Cohomology

Čech and local cohomology are related by the following lemma.

Lemma 5.2.1. *Let X be a complex variety, S a closed subset and let $U := X\setminus S$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then there is an open covering $\mathcal{U}$ of U such that, for all $p > 0$,*

$$\check{H}^p(\mathcal{U},\mathcal{F}) \cong H^{p+1}_S(X,\mathcal{F}).$$

Proof. This follows from [ILL+07, Theorem 12.41], where they define local cohomology from a purely algebraic point of view. The hypothesis that $\mathcal{F}$ is quasi-coherent allows the comparison. □

5.2.2 Local Cohomology and Completion

Let $S \subseteq X$ be a subvariety of a smooth projective variety X. Let $\hat{X}$ denote the completion of X along S. As topological spaces $\hat{X} \cong S$: it is the structure sheaf on $\hat{X}$ that differentiates it from S as a variety; see [Har77, Remark 9.3.1]. A consequence of this is the following lemma.

Lemma 5.2.2. *The local cohomology of X along S is isomorphic to the local cohomology of $\hat{X}$ along S; that is, for all integers $p \geq 0$,*

$$H^p_S(X,\mathcal{F}) \cong H^p_S(\hat{X},\hat{\mathcal{F}}).$$

Proof. This is proved in [Hun07, Proposition 2.14 (I)]. □

5.2.3 Computing Local Cohomology

Following [ILL+07], let R be a commutative noetherian domain and fix an $x \in R$. Let $R_x := R[x^{-1}]$, the localisation of R at the ideal generated by x. Let $K^\bullet(x;R) := 0 \to R \to R_x \to 0$ be the complex with R in degree zero and R_x in degree one and the inclusion map. This is called the **stable Koszul complex**. For $x_1,\ldots,x_n \in R$, let $K^\bullet(x_1,\ldots,x_n;R)$ denote the tensor product of complexes $K^\bullet(x_1;R)\otimes_R \cdots \otimes_R K^\bullet(x_n;R)$, where, if $(C^\bullet,d_C)$ and $(D^\bullet,d_D)$ are complexes, then their tensor product, $(C\otimes_R D,\Delta)$, is by definition the complex whose i^{th} graded piece is $\sum_{j+k=i} C_j\otimes D_k$ and whose differential Δ is determined by the map from $C_j\otimes D_k \longrightarrow (C_{j+1}\otimes D_k)\oplus(C_j\otimes D_{k+1})$

given by $\Delta(x \otimes y) := d_C(x) \otimes y + (-1)^j x \otimes d_D(y)$. Note that the differential corresponding to the tensor product of three complexes, $C^\bullet \otimes D^\bullet \otimes E^\bullet$, is

$$\Delta(x \otimes y \otimes z) = \delta_C x \otimes y \otimes z + (-1)^i x \otimes \delta_D y \otimes z + (-1)^{i+j} x \otimes y \otimes \delta_E z.$$

for $x \in C^i$, $y \in D^j$ and $z \in E^k$.

Let M be an R-module. Define $K^\bullet(x_1, \dots, x_n; M) := K^\bullet(x_1, \dots, x_n; R) \otimes_R M$ and let I be an ideal such that $\sqrt{I}$, the radical of I, is equal to the radical of the R-module generated by $x_1, \dots, x_n$. Then, by [ILL+07, Theorem 7.13], the local cohomology of $\mathrm{Spec}(R)$ with respect to the closed subvariety $V(I)$ is related by

$$H^i_{V(I)}(\mathrm{Spec}(R), M) \cong H^i(K^\bullet(x_1, \dots, x_n; M)),$$

for all $i \in \mathbb{Z}$.

Example 5.2.1. *Let $V, W \cong \mathbb{A}^n$ and let $X = V \times W$. Let $R = \mathbb{C}[X] = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ be the ring of regular functions on X, where the x_i's are coordinate functions on V and the y_i's are coordinate functions on W. Let I be the ideal of R generated by the x_i's: $I = \langle x_1, \dots, x_n \rangle_R$. Notice that $R_{x_1} \otimes_R R_{x_2} \cong R[x_1^{-1}, x_2^{-1}] = R[(x_1x_2)^{-1}] = R_{x_1x_2}$. Let $\hat{x}_i$ denote the monomial $x_1 \cdots x_{i-1}x_{i+1} \cdots x_n$. Then, for $M = \mathbb{C}[X]$ (thought of as an R-module), $K^\bullet(x_1, \dots, x_n; M) \cong K^\bullet(x_1, \dots, x_n; R)$, which is exact everywhere apart from degree n when the cohomology is the cokernel of the map,*

$$\bigoplus_{i=1}^n R_{\hat{x}_i} \longrightarrow R_{x_1, \dots, x_n},$$

that maps $(r_1, \dots, r_n) \mapsto r_1 - r_2 + \cdots + (-1)^{n-1} r_n$. This is the quotient of $R_{x_1 \cdots x_n} = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \otimes \mathbb{C}[W]$ by the R-submodule generated by the $\hat{x}_i$, which gives the space

$$H^n_{X^{\mathrm{us}}}(X, \mathcal{O}_X) = (x_1^{-1} \cdots x_n^{-1})\mathbb{C}[x_1^{-1}, \dots, x_n^{-1}] \otimes \mathbb{C}[W].$$

5.2.4 Decomposing Local Cohomology

Lemma 5.2.3. *Let X and Y be affine varieties and $Z \subseteq Y$ a closed subset. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on X and Y respectively. Then,*

$$H^p_{X \times Z}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes H^p_Z(Y, \mathcal{G}).$$

Proof. Because $Z \subseteq Y$,

$$(\mathcal{F} \boxtimes \mathcal{G})|_{X \times Y \setminus (X \times Z)} \cong \mathcal{F} \boxtimes (\ker(\mathcal{G} \longrightarrow \mathcal{G}|_{Y \setminus Z})).$$

Thus, the kernel sheaf,

$$\ker\left(\mathcal{F} \boxtimes \mathcal{G} \longrightarrow (\mathcal{F} \boxtimes \mathcal{G})|_{X \times Y \setminus (X \times Z)}\right) \cong \mathcal{F} \boxtimes (\ker(\mathcal{G} \longrightarrow \mathcal{G}|_{Y \setminus Z})),$$

This is a quasi-coherent sheaf on the affine space $X \times Y$, so taking global sections gives

$$H^0_{X \times Z}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \cong \Gamma\left(X \times Y, \ker\left(\mathcal{F} \boxtimes \mathcal{G} \longrightarrow (\mathcal{F} \boxtimes \mathcal{G})|_{X \times Y \setminus (X \times Z)}\right)\right) \cong \Gamma(X, \mathcal{F}) \otimes H^0_Z(Y, \mathcal{G}).$$

Now define an open cover of $X \times (Y \setminus Z)$ as follows. Choose generators, $z_1, \dots, z_m$, for the ideal of functions in $\mathbb{C}[Y]$ vanishing on Z. For each $i = 1, \dots, m$, let U_i be the open set in Y defined by $z_i \neq 0$ and set $\mathcal{U} = \{U_i \,|\, i = 1, \dots, m\}$ and $\mathcal{V} = \{X \times U_i \,|\, i = 1, \dots, m\}$. Then for $i_1, \dots, i_t \in \{1, \dots, m\}$,

$$\Gamma(X \times U_{i_1 \cdots i_t}, \mathcal{F} \boxtimes \mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes \Gamma(U_{i_1 \cdots i_t}, \mathcal{G});$$

therefore, for all $p \geq 0$,

$$\check{H}^p(\mathcal{V}, \mathcal{F} \boxtimes \mathcal{G}) \cong \Gamma(X, \mathcal{F}) \otimes \check{H}^p(\mathcal{U}, \mathcal{G}).$$

Applying Lemma 5.2.1 gives the result. □

5.3 Symplectic Geometry

A $\mathbb{C}$-algebra A is **Poisson** if it admits a $\mathbb{C}$-linear map,

$$\{-,-\}\colon A \wedge A \longrightarrow A,$$

called a **Poisson bracket**, such that it

1. is a derivation in each entry,
2. satisfies the Jacobi identity and so is a Lie bracket.

An affine variety, Spec(A), is **Poisson** if A is a Poisson algebra. If Spec(A) is Poisson, then there are canonical Poisson structures on localisations, the integral closure, completions and normalisations of A (see [Kal09]).

A smooth affine variety, X, with a closed, non-degenerate two-form ω_X is called a **symplectic variety**. Such a form induces an isomorphism, $\omega_X^*\colon TX \cong T^*X$, between the tangent and cotangent bundles. Affine symplectic varieties have a Poisson structure given by

$$\{f, g\} := \omega_X^{*\,-1}(df)(g),$$

for all $f, g \in \mathcal{O}_X$. Although, for a symplectic variety (X, ω_X), the Poisson structure, $\{-,-\}$, is determined and so introduces no extra information, it is often useful as a purely algebraic analogue of ω_X.

A smooth manifold, M, with a closed, non-degenerate two-form ω_M is called a **symplectic manifold**. An isomorphism of symplectic manifolds, (M, ω_M) and (N, ω_N), is called a **symplectomorphism**. It is a diffeomorphism $f\colon M \xrightarrow{\simeq} N$ such that $f^*\omega_N \simeq \omega_M$. A similar condition is required for a morphism between symplectic varieties, it is said that such a morphism **preserves the symplectic structure**.

5.4 Geometric Invariant Theory

Geometric Invariant Theory and Symplectic Reduction (also called Hamiltonian Reduction or Marsden-Weinsten reduction) are two ways of constructing quotients of spaces by group actions. They are defined for different classes of input data, but coincide in cases that will be considered later. This is the Kempf–Ness Theorem discussed in Section 5.5.2. First, one needs to understand the idea of quotients in a more general context.

The action of a group G on a set X determines (and is determined by) a group homomorphism from G to the group of all bijections from X to itself. More generally, if X belongs to some category C (for example, sets, vector spaces or topological spaces), an action of G on X is a group homomorphism from G to the group of automorphisms of X.

Let X be a topological space with a group action. The **topological quotient** is the set of orbits. A subset of orbits is open if the union of those orbits is open in X. Group actions on topological spaces often produce non-closed orbits and these are problematic if, for example, one wants the quotient to be separated (which varieties in this thesis are assumed to be).

The example, $\mathbb{C}^*$ acting on $T^*\mathbb{C}^2$ by $\lambda \cdot (x, y) = (\lambda x, \lambda y)$, produces orbits all of whose closures contain the closed orbit at the origin. Any function on the quotient that could distinguish between the orbit at the origin and any other would pullback to a function on $\mathbb{C}^2$ that would have to be discontinuous at the origin. One needs to discard or identify some of these badly-behaved orbits, so one must first develop criteria for the kind of quotient one wants. For example, if one starts with an affine or projective variety one might want that the quotient be affine or projective respectively. Or, one might want the notion of quotient to satisfy some property defined on the level of categories, so that comparisons can be made with other areas of mathematics. There are many excellent introductions to GIT; the exposition here follows [Dol03]. In that book, many of the results assume that G is a 'geometrically reductive' group. It is a theorem of Haboush that reductive groups are 'geometrically reductive', so the hypotheses here are at least as strong, see [Hab75].

Definition 5.4.1. *Let G be a group acting on an affine variety, X. An affine variety, Y, together with a morphism $q\colon X \to Y$ is a* ***good quotient*** *if the following hold.*

1. *q is surjective.*

2. *q is constant on orbits.*

3. *Given an open subset $U \subseteq Y$, the map $q^*\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(q^{-1}(U))$ induces an isomorphism,*

$$\mathcal{O}_Y(U) \cong \mathcal{O}_X\left(q^{-1}(U)\right)^G.$$

4. *q maps closed sets to closed sets.*

5. *q maps closed G-invariant disjoint sets to disjoint sets.*

If, instead, X and Y are projective varieties and $q\colon X \to Y$ is an affine morphism then the pair, Y and q, is a ***good quotient*** *if they also satisfy the conditions above.*

If, in addition, Y is an orbit space (each point in Y is the image of a unique orbit in X) then Y is said to be a ***geometric quotient****.*

Good quotients behave well with respect to a categorical notion of quotient; $q\colon X \to Y$ is said to be a **categorical quotient** if every morphism from X that is constant on orbits uniquely factors through q.

Proposition 5.4.1. *Let $q\colon X \to Y$ be a good quotient of varieties, X and Y. Then, if U is an open subset of Y with $V = q^{-1}(U)$,*

(i) $q|_V\colon V \to U$ is a categorical quotient,

(ii) if $x, y \in X$ then their orbits have disjoint closures if and only if $q(x) \neq q(y)$,

(iii) if all the orbits in X are closed then Y is a geometric quotient

To construct good quotients of reductive groups acting on projective varieties, one first reduces to the affine case. The projective variety is embedded into projective space by choosing an ample line bundle, the action of the group is then lifted to the line bundle; this is called *linearisation*. One then works locally, looking at invariant open subsets and glues them via the Proj functor to form a quotient.

Proposition 5.4.2. *([Dol03, Theorem 6.1]) Let G be a reductive algebraic group acting on an affine variety X. Then the map*

$$q\colon X \longrightarrow Y_0 := X/\!\!/_0 G := \operatorname{Spec}\left(\mathbb{C}[X]^G\right)$$

is a good quotient.

Definition 5.4.2. *Let X be a quasi-projective variety with a line bundle, $\pi\colon \mathcal{L} \longrightarrow X$. Let G be a reductive group acting on X via the map $\sigma\colon G \times X \longrightarrow X$. A* ***linearisation*** *of the action is a linear action of G on $\mathcal{L}$, defined by the map*

$$\overline{\sigma}\colon G \times \mathcal{L} \longrightarrow \mathcal{L},$$

such that

1. *the following diagram commutes*

$$\begin{array}{ccc} G \times \mathcal{L} & \xrightarrow{\overline{\sigma}} & \mathcal{L} \\ \Big\downarrow{\scriptstyle (\mathrm{id},\pi)} & & \Big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

2. *for all $g \in G$ and $x \in X$,*

$$\overline{\sigma}(g, x) = \sigma(g, x).$$

A linearisation determines an action of G on the sections of $\mathcal{L}$; given a section, $s \in \mathcal{L}$, a group element, $g \in G$, and a point, $x \in X$,

$$(g \cdot s)(x) := \overline{\sigma}\left(g, s(\sigma(g^{-1}, x))\right).$$

The second condition is equivalent to the zero section being invariant.

When $\mathcal{L}$ is a trivial line bundle, a choice of character, $\theta \in \mathbb{X}(G) := \mathrm{Hom}(G, \mathbb{C}^*)$, gives a linearisation defined by

$$\overline{\sigma}(g, (x, l)) = (\sigma(g, x), \theta(g)l) \in X \times \mathbb{A}^1.$$

The identity element, $\theta = 0 \in \mathbb{X}(G)$, gives the **trivial linearisation**, $\overline{\sigma}(g, (x, l)) = (\sigma(g, x), l)$, for all $l \in \mathcal{L}$ and $g \in G$. This linearisation acts trivially on sections:

$$(g \cdot s)(x) = \overline{\sigma}(g, s(\sigma(g^{-1}, x))) = s(x)$$

for all $x \in X$, $g \in G$ and $s \in \mathcal{L}$.

Lemma 5.4.1. *Given G-linearised line bundles $\mathcal{L}$ and $\mathcal{L}'$ on X there is a linearisation of the line bundle $\mathcal{L} \otimes \mathcal{L}'$*

Proof. For a proof see [Mai00, Section 1.12]. □

Let $\mathrm{Pic}^G(X)$ denote the set of line bundles $\mathcal{L}$ with some attached choice of linearisation. The previous lemma implies that the tensor product gives this set a group structure. Recall that, given a function $f \in \mathcal{O}_X$, the set X_f is defined to be the open subset of points at which f is non-zero.

Definition 5.4.3. *Let $\mathcal{L}$ be a linearised line bundle for a reductive group G acting on a quasi-projective variety X. A point $x \in X$ is said to be*

(i) ***semistable*** *if there exists an $n > 0$ and a G-invariant section $f \in \Gamma(X, \mathcal{L}^{\otimes n})^G$ such that $f(x) \neq 0$ and X_f is affine,*

(ii) ***stable*** *if it is semistable, the orbit, $G \cdot x$, is closed in the set of semistable points and the stabiliser, G_x, is finite,*

(iii) ***unstable*** *if it is not semistable,*

(iv) ***polystable*** *if, for some point $\tilde{x} \neq x$ in the fibre $\mathcal{L}_x^{\otimes -1}$, the orbit, $G \cdot \tilde{x}$, is closed in $\mathcal{L}^{\otimes -1}$.*

These loci of points are denoted $X^{\mathrm{ss}}(\mathcal{L})$, $X^{\mathrm{s}}(\mathcal{L})$, $X^{\mathrm{us}}(\mathcal{L})$ and $X^{\mathrm{ps}}(\mathcal{L})$ respectively. Polystable points will become important later in the discussion of the Kempf–Ness Theorem. The symbol, $\mathcal{L}$, will be dropped from notation when it is clear from the context which linearisation is being used.

See Section 5.10.1 for an example.

Definition 5.4.4. *Let G be a reductive algebraic group acting linearly on a quasi-projective variety X with a linearisation $\mathcal{L}$. The **GIT quotient** of X by G with respect to $\mathcal{L}$ is defined to be*

$$Y_{\mathcal{L}} := X /\!\!/_{\mathcal{L}} G := \mathrm{Proj}\left(\bigoplus_{s \geq 0} \Gamma(X, \mathcal{L}^{\otimes s})^G\right)$$

If the line bundle, $\mathcal{L}$, is trivial and corresponds, as in the case above, to a choice of character θ then the GIT quotient is denoted $/\!\!/_\theta$. Note that if $\mathcal{L}$ is trivially linearised then $Y_{\mathcal{L}} = \mathrm{Proj}\left(\Gamma(X, \mathcal{O}_X)^G\right) = \mathrm{Spec}\left(\mathbb{C}[X]^G\right) = Y_0$ so there is no ambiguity in the notation for the affine quotient $Y_0 = X /\!\!/_0 G$.

Proposition 5.4.3. *([Dol03, Propositon 8.1]) There is a map*

$$\pi \colon X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow Y_{\mathcal{L}}$$

which is a good quotient and its restriction to X^{s} is a geometric quotient.

The inclusion $\mathbb{C}[X]^G \subset \oplus_{s\geq 0}\Gamma(X, \mathcal{L}^{\otimes s})^G$ gives a projective morphism

$$p\colon Y_{\mathcal{L}} \longrightarrow Y_0.$$

5.4.1 GIT Walls and Chambers

Let $\mathcal{L}_1, \mathcal{L}_2 \in \mathrm{Pic}^G(X)$. If there is a connected variety S with a trivial G-action and a G-linearised line bundle $\mathcal{L} \in \mathrm{Pic}^G(S \times X)$ and points $s_1, s_2 \in S$ such that $\mathcal{L}|_{\{s_1\}\times X} \cong \mathcal{L}_1$ and $\mathcal{L}|_{\{s_2\}\times X} \cong \mathcal{L}_2$ then $\mathcal{L}_1$ is said to be **algebraically equivalent** to $\mathcal{L}_2$. This gives an equivalence relation on line bundles in $\mathrm{Pic}^G(X)$ which is compatible with the G-linearisation (see [Mai00, Definition 3.3]) and the quotient by this relation is called the **Néron–Severi group**, denoted $\mathrm{NS}^G(X)$.

In their paper [DH98], Dolgachev and Hu define the G-**ample cone**, $C^G(X)$, to be the convex cone in $\mathrm{NS}^G(X) \otimes \mathbb{R}$, spanned by ample line bundles $\mathcal{L}$ such that $X^{ss}(\mathcal{L}) \neq \emptyset$. They then define **GIT walls**, a collection of hyperplanes in $C^G(X)$ such that their union is the subset of $\mathcal{L}$ in $C^G(X)$ such that $X^{ss}(\mathcal{L}) \neq X^{s}(\mathcal{L})$. Each connected component of the complement of the walls in the G-ample cone is known as a **GIT chamber**.

Theorem 5.4.1. *([DH98, Theorem 0.2.3]) Suppose that there is some $\mathcal{L} \in C^G(X)$ such that $X^{ss}(\mathcal{L}) \neq X^{s}(\mathcal{L})$. Then*

(i) There are only finitely many chambers and walls,

(ii) The closure of a chamber is a rational polyhedral cone in the interior of $C^G(X)$.

(iii) Each GIT-equivalence class is either a chamber or a union of cells in the same wall.

Note that, when $X = \mathbb{A}^n$, the Picard group is trivial and the linearisation is determined by a character, so $C^G(X) \subset \mathbb{X}(G) \otimes_{\mathbb{Z}} \mathbb{R}$. The GIT quotients constructed later in this thesis will be of this form.

5.5 Moment Maps and Hamiltonian Reduction

Let M be a real, smooth, symplectic manifold with real dimension $2n$ with a closed, non-degenerate form $\omega \in \Omega^2(M)$. Let K be a real Lie group acting on M preserving ω such that the action map, $\sigma\colon K\times M \longrightarrow M$, is smooth. Let $\mathfrak{K}$ denote the corresponding real Lie algebra $\mathrm{Lie}(K)$. The *symplectic reduction* of M by K is a quotient with a symplectic structure. The differential of the action gives a homomorphism of Lie algebras

$$\sigma\colon \mathfrak{K} \longrightarrow \mathrm{Vect}(M).$$

Given a vector field $X \in \mathrm{Vect}(M)$, let ι_X denote the **contraction map**,

$$\iota_X\colon \Omega^p(M) \longrightarrow \Omega^{p-1}(M); \quad \nu \mapsto \Big((X_1, \ldots, X_{p-1}) \mapsto \nu(X, X_1, \ldots, X_{p-1})\Big).$$

For all $x \in \mathfrak{K}$, $\iota_{\sigma(x)}\omega$ is closed. These one forms are exact if, for each $x \in \mathfrak{K}$, there exists a smooth function $h_x \in \Omega^0(M)$ such that $dh_x = -\iota_{\sigma(x)}\omega$. If such functions, called **hamiltonians**, exist then one can also ask whether there exists a K-equivariant map,

$$\mu\colon M \longrightarrow \mathfrak{K}^*,$$

such that the smooth function, $\mu(-)(x) \in \Omega^0(M)$, is equal to h_x for all $x \in \mathfrak{K}$. If such a map exists it is called the **moment map** and the action of K on M is called a **hamiltonian action**.

Let $\alpha \in \mathfrak{K}^*$ be such that α is constant on the orbits of the adjoint action of K on $\mathfrak{K}$. Now, given any moment map μ, $\mu + \alpha$ is K-equivariant and $d(\mu + \alpha) = d\mu$ so $\mu + \alpha$ is also a moment map for the action of K. The different choices of moment maps are accounted for by the choice of fibre over which to take the quotient.

Theorem 5.5.1. *Let (M, ω) be a symplectic manifold with a hamiltonian action of a compact real Lie group, K. Assume that μ is a choice of moment map such that the action of K on $\mu^{-1}(0)$ is free and* *let $\iota\colon \mu^{-1}(0) \longrightarrow M$ denote the inclusion. Then, topological quotient, $\pi\colon \mu^{-1}(0) \longrightarrow \mu^{-1}(0)\big/K$, has the structure of a symplectic manifold with respect to a form, ω_Y, such that $\iota^*\omega = \pi^*\omega_Y$.*

The symplectic manifold, $\mu^{-1}(0)\big/K$, is called the **symplectic reduction** of M by K.

5.5.1 Twisted Moment Maps for Vector Spaces

For the purposes of this thesis it is enough to treat the case of actions on complex vector spaces. The exposition here closely follows [Hos12].

Let V be a complex vector space. Say that a group, G, acts **linearly** on V if it acts as a subgroup of $GL(V)$ and a group, K, acts **unitarily** on V if it acts as a subgroup of $U(V)$. Fix a hermitian inner product, $H\colon V \times V \longrightarrow \mathbb{C}$, on V and let K be a compact real Lie group acting unitarily on V. Identify $V \times V \cong TV$. Define a symplectic form on V,

$$\omega\colon V \times V \longrightarrow \mathbb{R},$$

by

$$(v, w) \longmapsto \frac{1}{2\pi i}(H(v, w) - H(w, v)) = \frac{1}{\pi}\mathrm{Im}(H(v, w)).$$

Let $\theta\colon K \longrightarrow U(1)$ be a character. By identifying $U(1) \cong S^1$ and $\mathrm{Lie}(S^1) \cong 2\pi i\mathbb{R}$, let $d\theta\colon \mathfrak{K} \longrightarrow \mathbb{R}$ denote the differential of θ. Using [Hos12, Lemma 3.2], the **θ-shifted moment map** for the action of K on V,

$$\mu_\theta\colon V \longrightarrow \mathfrak{K}^*,$$

is defined by,

$$\mu_\theta(v)(k) := \frac{1}{2\pi i}(H(kv, v) - d\theta(k)),$$

for $v \in V$ and $k \in \mathfrak{K}$.

5.5.2 The Kempf–Ness Theorem for Vector Spaces

The following version of the Kempf–Ness Theorem is presented by Hoskins in [Hos12, Theorem 4.2].

Theorem 5.5.2. *(The Kempf–Ness Theorem) Let G be the complexification of a compact real Lie group K and suppose that G acts linearly on V, whilst K acts unitarily on V. Let $\theta\colon G \longrightarrow \mathbb{G}_m$ be a character such that $\theta(K) \subset U(1)$, so that $\theta|_K$ is a compact character. Let $\mu := \mu_{\theta|_K}\colon V \longrightarrow \mathfrak{K}^*$ denote the moment map for this action, shifted by $\theta|_K$. Then*

(i) A G-orbit meets the preimage of 0 under μ if and only if it is θ-polystable; that is, $G \cdot \mu^{-1}(0) = V^{\mathrm{ps}}$.

(ii) A polystable G-orbit meets $\mu^{-1}(0)$ in a single K-orbit.

(iii) A point $v \in V$ is semistable if and only if its G-orbit closure meets $\mu^{-1}(0)$.

(iv) The inclusion $\mu^{-1}(0) \subseteq V^{\mathrm{ps}}$ induces a homeomorphism,

$$\mu^{-1}(0)\big/K \cong V/\!\!/_\theta G.$$

5.6 The Kirwan–Ness Stratification

In Section 5.9, sheaves are constructed on an affine space X with an action of a reductive group G; these also depend on some character $\chi \in (\mathfrak{g}^*)^G$. On the one hand, a functor called quantum hamiltonian reduction is applied, which produces, for each of these sheaves, a module for the spherical Cherednik algebra with a parameter dependent on χ. On the other, these sheaves can be restricted from X to the open subset X^{ss}_θ, for some GIT parameter θ. A second functor, which also produces spherical Cherednik algebra modules, is then applied to the result. In order to determine for which χ these two processes produce the same module, the unstable locus,

$X^{us}{}_\theta$, will be stratified using the *Kirwan–Ness stratification*. This allows one to study sheaves on each of the strata and, importantly, use the combinatorics associated to the stratification to determine the behaviour of the sheaves.

5.6.1 The Hilbert–Mumford Criterion

Let X be a quasi-projective smooth variety with the action of a reductive group G. Let $\mathcal{L}$ be the trivial line bundle on X with a choice of G-linearisation corresponding to a character $\theta\colon G \longrightarrow \mathbb{G}_m$. Let T be a maximal torus in G and $\Upsilon(T)_{\mathbb{R}}$ denote the tensor product $\Upsilon(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The Weyl group, $W = N_G(T)/T$, acts on $\Upsilon(T)$ and $\Upsilon(T)_{\mathbb{R}}$ by conjugation. Choose a W-invariant inner product $(-,-)$ on $\Upsilon(T)_{\mathbb{R}}$ and denote the associated norm by $\|\cdot\|_T$. Let $\Upsilon(G)$ denote the group of one-parameter subgroups of G,

$$\Upsilon(G) := \mathrm{Hom}(\mathbb{G}_m, G).$$

Suppose that the rank of G is n so that $T \cong (\mathbb{C}^*)^n$. When the image of $\lambda \in \Upsilon(G)$ lies inside T one can use the isomorphism

$$\Upsilon(T) \xrightarrow{\cong} \mathbb{Z}^n; \qquad \lambda \mapsto d\lambda(1)$$

to write λ as an n-tuple of integers. In order to simplify notation, the symbol λ will simultaneously stand for the map $\mathbb{G}_m \to G$ and the n-tuple of integers $d\lambda(1)$. With this in mind, one can talk about multiplying one-parameter subgroups by certain rational numbers: if a is some rational number whose denominator divides all of the entries of $d\lambda(1)$ then the symbol $a\lambda$ denotes the one-parameter subgroup corresponding to the n-tuple of integers $ad\lambda(1)$. Since the set of maximal tori forms a single conjugacy class, the image of a one-parameter subgroup comprises only semisimple elements, every λ is conjugate to some one-parameter subgroup of this form.

The stability of points of X can be detected by the restricted action of one-parameter subgroups of G. Let $\lambda\colon \mathbb{G}_m \longrightarrow G$ and $x \in X$. This gives a map,

$$\lambda_x\colon \mathbb{A} \setminus \{0\} \longrightarrow X; \qquad t \mapsto \lambda(t)\cdot x.$$

Being quasi-projective, X lies in some projective variety Y; so λ_x uniquely extends to a map, $\overline{\lambda}_x\colon \mathbb{P}^1 \longrightarrow Y$. Define,

$$\lim_{t\to 0} \lambda(t)\cdot x := \overline{\lambda}(0) \qquad\qquad \lim_{t\to\infty} \lambda(t)\cdot x := \overline{\lambda}(\infty),$$

and say that $\lim_{t\to 0}\lambda(t)\cdot x$ **exists** if it lies in X.

Let $x \in X$ and suppose there exists a $\lambda \in \Upsilon(G)$ such that $\lim_{t\to 0}\lambda(t)\cdot x = x_0 \in X$ exists. Then λ induces a $\mathbb{C}^*$-action on the fibre $\mathcal{L}_{x_0}$, given by $t\cdot(x_0, l) = (x_0, t^r l)$ for $t \in \mathbb{C}^*, l \in \mathcal{L}_{x_0}$ and some $r \in \mathbb{Z}$. Let $\mu(x,\lambda)$ be this integer r. If the limit does not exist then define $\mu(x,\lambda) = \infty$. To emphasise the dependence of μ on θ it will sometimes be written μ^θ. The following proposition is called the **Hilbert–Mumford criterion**.

Proposition 5.6.1. *([Hos12, Proposition 2.5]) A point, $x \in X$, is unstable if and only if there exists a one-parameter subgroup, λ, such that $\mu^\theta(x,\lambda) < 0$.*

See Section 5.10.3 for an example. Let

$$X^\theta_\lambda := \{x \in X \mid \mu^\theta(x,\lambda) < 0\}.$$

One can also use the Hilbert-Mumford criterion to measure the extent to which a point is unstable. If a is a natural number then $\mu(x, a\lambda) = a\mu(x,\lambda)$; so, in order to make the supremum meaningful, one needs a measure of the size of a one-parameter subgroup. Let $\lambda \in \Upsilon(G)$ and pick a g such that $\mathrm{Ad}(g)\cdot\lambda \in \Upsilon(T)$. Then define the norm of λ by

$$\|\lambda\| := \|\mathrm{Ad}(g)\cdot\lambda\|_T.$$

Define a function

$$M^{\theta}(x) := \sup_{\lambda \in \Upsilon(G)\setminus\{0\}} \left\{ \frac{-\mu(x,\lambda)}{\|\lambda\|} \right\},$$

this measures how unstable the point x is.

Multiplying λ by a natural number also preserves the ratio $\frac{\mu^{\theta}(x,\lambda)}{\|\lambda\|}$; so make the following definition.

Definition 5.6.1. *A one-parameter subgroup is called* ***primitive*** *(or* ***indivisible****) if it cannot be written as a positive multiple of another. A primitive one-parameter subgroup, λ, is called θ-**optimal** (or just **optimal**) for x if $M^{\theta}(x)$ realises its supremum, $\frac{-\mu^{\theta}(x,\lambda)}{\|\lambda\|}$, at λ. Finally, let λ and μ be one-parameter subgroups which are not a positive integer multiple of one another. Say that λ θ-**dominates** μ if $X^{\theta}_{\mu} \subseteq X^{\theta}_{\lambda}$ and $\frac{\mu^{\theta}(x,\mu)}{\|\mu\|} \geq \frac{\mu^{\theta}(x,\lambda)}{\|\lambda\|}$ for all $x \in X^{\theta}_{\mu}$.*

It follows that optimal subgroups are not dominated by any other. Let Γ_{θ} denote the set of all optimal one-parameter subgroups for the stability condition θ. The set Γ_{θ} is complicated: it varies even as one varies θ inside a single GIT chamber.

5.6.2 The Kirwan–Ness Stratification

The unstable locus, X^{us}, has the following **Kirwan–Ness stratification**. This was proved independently by Kirwan and Ness see [Kir84, Sections 12-13] and [Nes84] respectively.

(KN1) There is a decomposition into non-empty smooth locally closed strata

$$X^{\mathrm{us}} = \coprod_{d,\langle\lambda\rangle} S^{+}_{d,\langle\lambda\rangle},$$

where d is a positive real number and $\langle\lambda\rangle$ a conjugacy class of one-parameter subgroups of G.

(KN2) There is an enumeration of the one-parameter subgroups appearing in (KN1) by representatives $\lambda_1, \ldots, \lambda_p \in \Upsilon(T)$, such that $\overline{S}^{+}_{d,\langle\lambda_i\rangle} \cap S^{+}_{d',\langle\lambda_j\rangle} \neq \emptyset$ only if $i < j$ and $d < d'$.

(KN3) For $1 \leq i \leq p$, set

$$P(\lambda_i) := \left\{ g \in G \,\middle|\, \lim_{t\to 0} \lambda_i(t) g \lambda_i(t)^{-1} \text{ exists} \right\}.$$

There exists a smooth, locally-closed $P(\lambda_i)$-stable subvariety, S_{d,λ_i}, of X, such that the action map induces

$$G \overset{P(\lambda_i)}{\times} S_{d,\lambda_i} \cong S^{+}_{d,\langle\lambda_i\rangle}.$$

(KN4) For $1 \leq i \leq p$, set

$$Z_{d,\lambda_i} := \{ x \in S_{d,\lambda_i} \,|\, \lambda_i(t) \cdot x = x \text{ for all } t \in \mathbb{G}_m \},$$

and $Z_G(\lambda_i) := \left\{ g \in G \,\middle|\, \lambda_i(t) g \lambda_i(t)^{-1} = g \text{ for all } t \in \mathbb{G}_m \right\}$. Then the variety, Z_{d,λ_i}, is a $Z_G(\lambda_i)$-stable, smooth, locally closed, subvariety of X such that

$$S_{d,\lambda_i} := \left\{ x \in X \,\middle|\, \lim_{t\to 0} \lambda_i(t) \cdot x \in Z_{d,\lambda_i} \right\}.$$

(KN5) For $1 \leq i \leq p$, let

$$Z_{d,\lambda_i} := \coprod Z_{d,\lambda_i,j}$$

be the decomposition of Z_{d,λ_i} into connected components. For each j that appears, the morphism,

$$p_{d,\lambda_i,j} \colon S_{d,\lambda_i,j} \longrightarrow Z_{d,\lambda_i,j}; \quad x \mapsto \lim_{t\to 0} \lambda_i(t) \cdot x,$$

is a locally trivial fibration by affine spaces.

The strata can be described in terms of unstability by

$$S^+_{d,\langle\lambda\rangle} = \{x \in X \,\big|\, M(x) = d \text{ and there exists a } g \in G \text{ with } \mathrm{Ad}g \cdot \lambda \text{ optimal for } x\}$$

and

$$S_{d,\lambda} = \{x \in X \,\big|\, M(x) = d \text{ and } \lambda \text{ is optimal for } x\}.$$

The following lemmata will be useful later.

Lemma 5.6.1. *Let (X, ω_X) be a quasi-projective, smooth, symplectic variety with the action of a reductive group G. Then each space, $Z_{d,\lambda_i,j}$, in the Kirwan–Ness stratification of X^{us} is symplectic with a form given by the restriction of ω_X to $Z_{d,\lambda_i,j}$.*

Proof. Let $x \in Z_{d,\lambda_i,j}$. Since $Z_{d,\lambda_i,j}$ is connected, it is contained in a component of the smooth variety X^{λ_i}, the fixed points of the subgroup $\lambda_i(\mathbb{G}_m) \subseteq G$. Denote this component C. Since C is connected, the $\mathbb{C}^\times$-action on $\mathcal{L}|_C$ induced by λ_i acts by the same character on each fibre. Because they are unstable, $\mu(p, \lambda_i) < 0$ for all $p \in Z_{d,\lambda_i,j} \subseteq C$, so all the points $c \in C$ have the property that $\mu(c, \lambda_i) = d\|\lambda_i\| < 0$ and so are unstable.

By (KN1), one of the strata, S^+, parametrised by $d' \in \mathbb{R}$ and $\lambda_{i'} \in \Upsilon(T)$ say, must intersect C densely. Let $y \in S^+_{d',\langle\lambda_{i'}\rangle} \cap C$. There exists $\nu \in \Upsilon(G)$ conjugate to $\lambda_{i'}$ such that $\lim_{t\to 0} \nu(t) \cdot y$ exists. If $i \neq i'$ then

$$d' = \frac{\mu(y, \nu)}{\|\nu\|} < \frac{\mu(y, \lambda_i)}{\|\lambda_i\|} = d.$$

This contradicts (KN2). Thus $i = i'$ and $S^+_{d,\langle\lambda_i\rangle} \cap C$ is dense.

Take $y \in S^+_{d,\langle\lambda_i\rangle} \cap C$. Because $\mu(y, \lambda_i) = d\|\lambda_i\|$, $y \in S_{d,\lambda_i}$. Since it is fixed under the $\mathbb{C}^\times$-action induced by λ_i, $y \in Z_{d,\lambda_i}$. Thus $Z_{d,\lambda_i} \cap C$ is dense. Since the decomposition $Z_{d,\lambda_i} = \coprod Z_{d,\lambda_i,j}$ is disjoint into connected components, $Z_{d,\lambda_i,j}$ is dense in C.

It is therefore enough to prove that C is symplectic under the restriction of ω_X, or, indeed, that X^{λ_i} is. If $x \in X^{\lambda_i}$ then there is a decomposition

$$T_xX = \bigoplus_{j\in\mathbb{Z}} T_xX(j),$$

where $T_xX(j) := \left\{v \in T_xX \,\middle|\, \lambda_i(t) \cdot v = t^j v \text{ for all } t \in \mathbb{C}^*\right\}$. Since the action of T is hamiltonian it preserves the form, that is, for all $t \in T$,

$$\begin{aligned}\omega_X(T_xX(i), T_xX(j)) &= \omega_X(t \cdot T_xX(i), t \cdot T_xX(j)) \\ &= \omega_X(t^i T_xX(i), t^j T_xX(j)) \\ &= t^{i+j}\omega_X(T_xX(i), T_xX(j)).\end{aligned}$$

Therefore, unless $\omega_{X,x}(T_xX(i), T_xX(j)) = 0$, $i + j = 0$. It follows that the restriction of ω_X to $T_x(X^{\lambda_i}) = T_x(0)$ is non-degenerate. □

Lemma 5.6.2. *Let X and G be as above. The moment map, $\mu\colon X \longrightarrow \mathfrak{g}^*$, is constant on connected components of the set of fixed points.*

Proof. Let $\tau\colon \mathfrak{g} \longrightarrow \mathrm{Vect}(X)$ be the dual moment map; that is, for $X \in \mathfrak{g}$, $\tau(X)$ is the vector field on X such that, locally at a point $p \in X$, there is some function f, such that $\tau(X) = \{f, -\}$ and $\mu(p) = f(p)$. If p is a fixed point, then X acts as zero so $\tau(X) = \{f, -\}$ is zero at p. But, $\{f, -\} = 0$ implies that f is locally constant, so f must be constant along any path in the set of fixed points. □

Lemma 5.6.3. *Let X and G be as above. Suppose that there is an action of $F = \mathbb{G}_m$ on X that commutes with the action of G. Let $S = S_\lambda$ be one of the strata for some optimal $\lambda \in \Upsilon(G)$. Then S is F-stable.*

Proof. Let Z denote the set of λ-fixed points in X and Z_λ the set of λ-fixed points in S. Since the actions of G and F commute, F acts on Z; indeed, for $g \in \lambda(\mathbb{C})$, $a \in F$ and $z \in Z$, $g \cdot (a \cdot z) = a \cdot z$.

By the previous lemma, for $z \in Z$ and $a \in F$, $\mu(a \cdot z, \lambda) = \mu(z, \lambda)$; so if $z \in Z_\lambda$ then so is $a \cdot z$. Now given $a \in F$, $s \in S$,

$$\lim_{t \to 0} \lambda(t) \cdot (a \cdot s) = a \cdot \lim_{t \to 0} \lambda(t) \cdot s \in Z_\lambda,$$

which shows that $a \cdot s \in S$. □

Remark 5.6.1. *The index, d, for each Kirwan–Ness stratum is redundant in the rest of the thesis because, when X is affine, it is determined by λ and θ. That is,*

$$d = -\frac{(\theta, \lambda)}{\|\lambda\|}.$$

The subscript d will be dropped from the notation from now on.

5.7 Conical Symplectic Resolutions

The machinery that produces localisation by deform quantising sheaves on spaces requires a minimal amount of structure called a *conical symplectic resolution*.

Definition 5.7.1. *Let (Y, ω) be a complex, smooth, symplectic variety. It is a* ***conical symplectic resolution of degree*** *m if there is an action of $F = \mathbb{G}_m$ on Y such that*

(i) for all $t \in F$, $t^* \omega = t^m \omega$,

(ii) the induced action on $\mathbb{C}[Y]$ has only non-negative weights,

(iii) the invariants in $\mathbb{C}[Y]$ are only the constant functions; that is, $\mathbb{C}[Y]^F \cong \mathbb{C}$.

(iv) the induced map

$$p \colon Y \longrightarrow Y_0 := \operatorname{Spec}(\mathbb{C}[Y])$$

is a resolution of singularities; that is, Y is smooth and projective and p is an isomorphism over the smooth locus of Y_0.

It is called 'conical' because conditions (*ii*) and (*iii*) imply that the F-action on Y_0 contracts everything to a point, $\operatorname{Spec}(\mathbb{C}[Y]^F)$. The Nakajima quiver varieties that are about to be introduced are examples of conical symplectic resolutions.

5.8 Quiver Varieties

The aim of this section is to give a recipe for *Nakajima quiver varieties*, written Y_θ. These are examples of conical symplectic resolutions that will be used to produce a localisation theorem for Cherednik algebras. Fix $l, n \in \mathbb{N}$, not both one. The ingredients are

(i) a quiver with a dimension vector, based on l and n,

(ii) a GIT parameter, θ, for the resulting action of a group on its space of representations.

The exposition here follows [Gor08, Chapter 3].

5.8.1 The Affine Variety attached to a Quiver

Let Q be a quiver with a finite set of vertices I and a finite set of edges E. The set of all represenions of Q with a given dimension vector, ϵ, forms an affine space, $V = \operatorname{Rep}(Q, \epsilon)$, of dimension $\sum_{\alpha \in E} \epsilon_{t(\alpha)} \epsilon_{h(\alpha)}$. Two points of this space represent isomorphic left $\mathbb{C}Q$-modules if the corresponding matrices are simultaneously similar. The group

$$G := \prod_{i \in I} \mathrm{GL}_{\epsilon_i}(\mathbb{C})$$

acts on V as follows. Given $g \in G$ and $M_\alpha \in V$ define

$$g \cdot M_\alpha := g_{t(\alpha)} M_\alpha g_{h(\alpha)}^{-1}.$$

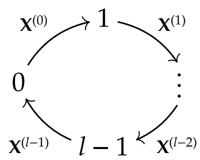
This corresponds to the action of changing the basis of the representations and so the quotient is a moduli space of representations of $\mathbb{C}Q$ with a constraint on the dimension. The fact that G is reductive allows us to take the quotient in the sense of GIT. Let $\overline{Q}$ denote the double of Q, constructed by adding, for each arrow in E, an arrow in the opposite direction called the **reverse**. Identify the map which kills the matrices corresponding to the reversed arrows of $\overline{Q}$ with the projection onto the zero section of the cotangent bundle,

$$\mathrm{Rep}(\overline{Q}, \epsilon) \cong T^*\mathrm{Rep}(Q, \epsilon).$$

5.8.2 Nakajima Quiver Varieties for $G(l, 1, 1)$

The case when $n = 1$ is special: for $G(l, 1, 1) = \mu_l$ there are two ways to construct Nakajima quiver varieties.

Let Q^l be the quiver with vertices $I = \{0, \ldots, l-1\}$ and, for each $i = 0, \ldots, l-1$ an arrow $\mathbf{X}^{(i)} : i \longrightarrow i + 1$ (**mod** l).

$Q^l :=$ 

Let $\epsilon_1 = (1, \ldots, 1)$, a dimension vector for Q^l. Consider $\overline{Q^l}$, the doubled quiver, with $\mathbf{Y}^{(i)}$ the reverse of $\mathbf{X}^{(i)}$. Let $X = T^*\mathrm{Rep}\left(Q^l, \epsilon_1\right) = \mathrm{Rep}\left(\overline{Q^l}, \epsilon_1\right)$. Typical elements of X look like

$$x = (X^{(0)}, \ldots, X^{(l-1)}, Y^{(0)}, \ldots, Y^{(l-1)}) \in X \overset{\text{v.s.}}{\cong} \mathbb{C}^{2l},$$

and typical elements of G look like

$$g = (g_0, \ldots, g_{l-1}) \in G = (\mathbb{C}^*)^l.$$

Now, the action of $g \in G$ on $x \in X$ is given by the formula

$$g \cdot x = \left(g_0^{-1} g_1 X^{(0)}, \ldots, g_{l-1}^{-1} g_0 X^{(l-1)}, g_0 g_1^{-1} Y^{(0)}, \ldots, g_{l-1} g_0^{-1} Y^{(l-1)}\right),$$

so that the subgroup $H := \{(t, \ldots, t) \in G \mid t \in \mathbb{C}^*\}$ acts trivially. Define

$$\hat{G} = G/H,$$

A typical element of $\mathbb{Y}(\hat{G})$ can be written

$$\lambda := (a_0, \ldots, a_{l-1}) \in \mathbb{Z}^l,$$

considered modulo adding $(1, \ldots, 1) \in \mathbb{Z}^l$. Use this relation to choose the unique member of each equivalence class with $a_0 = 0$. This identifies $\mathbb{Y}(\hat{G})$ with $\mathbb{Z}^{l-1}$. The characters of $\hat{G}$ form the group $\mathbb{X}(\hat{G}) \cong \mathrm{Hom}(\hat{G}, \mathbb{G}_{\mathbf{m}}) \cong \mathbb{Z}^{l-1}$ via the isomorphism that sends an $(l-1)$-tuple $(\theta_1, \ldots, \theta_{l-1}) \in \mathbb{Z}^{l-1}$ to the character $(g_0, \ldots, g_{l-1}) \mapsto g_0^{\theta_0} \cdots g_{l-1}^{\theta_{l-1}})$ where $\theta_0 := -\theta_1 - \cdots - \theta_{l-1}$.

Now, an arbitrary $\lambda = (a_1, \ldots, a_{l-1})$ acts on x with the weights

$$(a_1, a_2 - a_1, \ldots, a_{l-1} - a_{l-2}, -a_{l-1}, -a_1, a_1 - a_2, \ldots, a_{l-2} - a_{l-1}, a_{l-1})$$

Now X, being a cotangent bundle of an affine variety, is symplectic with respect to $\omega_X = \sum_{k=0}^{l-1} dX^{(k)} \wedge dY^{(k)}$. The tangent space at the identity of $\hat{G}$ is $\hat{\mathfrak{g}} := \mathrm{Lie}(\hat{G}) \cong \mathbb{C}^{l-1}$ which has an isomorphism to $\hat{\mathfrak{g}}^* \cong \mathbb{C}^{l-1}$ via the dot product. The differential of the $\hat{G}$ action gives a map

$\sigma\colon \hat{\mathfrak{g}} \longrightarrow \mathrm{Vect}(X)$ which is hamiltonian with respect to the moment map,

$$\mu\colon X \longrightarrow \hat{\mathfrak{g}}^* \cong \hat{\mathfrak{g}},$$

defined by,

$$\mu\left(X^{(0)},\dots,X^{(l-1)},Y^{(0)},\dots,Y^{(l-1)}\right)(t_1,\dots,t_{l-1}) = \sum_{i=1}^{l-1}(X^{(i)}Y^{(i)} - X^{(i-1)}Y^{(i-1)})t_i.$$

See [Gor08, Section 3.3]. The condition that a point of X be in $\mu^{-1}(0)$ is known as the **ADHM equation**.

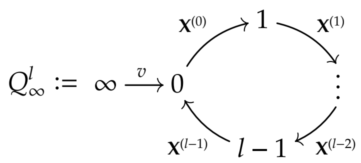
Define the **affine quiver variety** to be the GIT quotient $Y_0 = X/\!\!/_0\hat{G} = \mu^{-1}(0)/\!\!/_0\hat{G}$. Let $\theta \in \mathbb{X}(\hat{G})$. The **Nakajima quiver variety** is the GIT quotient

$$Y_\theta := \mu^{-1}(0)/\!\!/_\theta\hat{G}.$$

These are examples of **hypertoric varieties**: Nakajima quiver varieties such that the base change group, G, is commutative.

5.8.3 Nakajima Quiver Varieties for $G(l,1,n)$

Following Nakajima, when $n \geq 2$, one constructs a moduli space of representations of the preprojective algebra for the *framed* quiver. Let Q^l_∞ denote the quiver with vertex set $I \cup \{\infty\}$ and edge set $E \cup \{v\}$ where $v\colon \infty \to 0$.



This is called a **framing** of Q^l. Let $\epsilon_n = (n,\dots,n)$, a dimension vector for Q^l and let $\hat{\epsilon}_n = (1,n,\dots,n)$ be its extension to Q^l_∞.

Consider $\overline{Q^l_\infty}$, the doubled quiver, with $\mathbf{Y}^{(i)}$ the reverse of $\mathbf{X}^{(i)}$ and w the reverse of v. Let $X = T^*\mathrm{Rep}\left(Q^l_\infty,\hat{\epsilon}_n\right) = \mathrm{Rep}\left(\overline{Q^l_\infty},\hat{\epsilon}_n\right)$. In order to reduce confusion between the two indexing sets the following convention will be adopted. The indices corresponding to n will run from $1,\dots,n$ and the indices corresponding to l will run from $0,\dots,l-1$ and be in parentheses. Typical elements of X look like

$$x = (\mathbf{X}^{(0)},\dots,\mathbf{X}^{(l-1)},\mathbf{Y}^{(0)},\dots,\mathbf{Y}^{(l-1)};\mathbf{v},\mathbf{w}) \in X \overset{\text{v.s.}}{\cong} (\mathrm{M}_n(\mathbb{C})^l)^2 \times (\mathbb{C}^n)^2,$$

and typical elements of G look like

$$g = (g_{(0)},\dots,g_{(l-1)}) \in G = \mathrm{GL}_n(\mathbb{C})^l.$$

Let $T = T_0 \times \cdots \times T_{l-1} \overset{\text{v.s.}}{\cong} (\mathbb{C}^*)^n \times \cdots \times (\mathbb{C}^*)^n \cong (\mathbb{C}^*)^{nl}$ be the maximal torus of diagonal matrices inside G. A typical element of $\mathbb{Y}(T)$ can be written

$$\lambda := \left((a_1^{(0)},\dots,a_n^{(0)}),\dots,(a_1^{(l-1)},\dots,a_n^{(l-1)})\right) \in \mathbb{Z}^{nl}.$$

The characters of G form the group $\mathbb{X}(G) \cong \mathrm{Hom}(G,\mathbb{G}_\mathbf{m}) \cong \mathbb{Z}^l$ via the isomorphism that sends an l-tuple $(\theta^0,\dots,\theta^{l-1}) \in \mathbb{Z}^l$ to the character $(g_{(0)},\dots,g_{(l-1)}) \mapsto \det^{\theta^0}(g_{(0)})\cdots\det^{\theta^{l-1}}(g_{(l-1)})$.

Now, the action of $g \in G$ on $x \in X$ is given by the formula

$$g \cdot x = \Big(g_{(1)} \mathbf{X}^{(0)} {g_{(0)}}^{-1}, \ldots, g_{(0)} \mathbf{X}^{(l-1)} {g_{(l-1)}}^{-1}, \\ g_{(0)} \mathbf{Y}^{(0)} {g_{(1)}}^{-1}, \ldots, g_{(l-1)} \mathbf{Y}^{(l-1)} {g_{(0)}}^{-1}; \\ g_{(0)} \mathbf{v}, \mathbf{w} {g_{(0)}}^{-1} \Big), \quad (5.1)$$

so that λ acts on x with the weights

$$\Big(\big(a_i^{(1)} - a_j^{(0)} \big)_{i,j}, \ldots, \big(a_i^{(0)} - a_j^{(l-1)} \big)_{i,j}, \\ \big(a_i^{(0)} - a_j^{(1)} \big)_{i,j}, \ldots, \big(a_i^{(l-1)} - a_j^{(0)} \big)_{i,j}; \\ (a_1^{(0)}, \ldots, a_n^{(0)}), (-a_1^{(0)}, \ldots, -a_n^{(0)}) \Big). \quad (5.2)$$

Now X, being as it is a cotangent bundle of an affine variety, is symplectic with respect to $\omega_X = \sum_{i,j,k} d\mathbf{X}_{ij}^{(k)} \wedge d\mathbf{Y}_{ij}^{(k)} + \sum_i dv_i \wedge dw_i$. It has the structure of a conical symplectic resolution by setting the F-degree of each of the coordinate functions to be one. Then, for $t \in F$, $t^* \omega_X = t^2 \omega_X$ and the corresponding map, $X \longrightarrow X_0$, to its affinisation is the identity, so is trivially a resolution of singularities. The tangent space at the identity of G is $\mathfrak{g} \cong \mathrm{M}_n(\mathbb{C})^l$ which has an isomorphism to $\mathfrak{g}^*$ via the non-degenerate pairing

$$\mathrm{Tr}\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}; \qquad (X, Y) \mapsto \mathrm{Tr}(XY).$$

The differential of the G action gives a map $\sigma\colon \mathfrak{g} \longrightarrow \mathrm{Vect}(X)$ which is hamiltonian with respect to the moment map

$$\mu\colon X \longrightarrow \mathfrak{g} \cong \mathfrak{g}^*; \qquad (\mathbf{X}, \mathbf{Y}; v, w) \mapsto [\mathbf{X}, \mathbf{Y}] + \mathbf{v}\mathbf{w}.$$

See [Gor08, Section 3.3]. The condition that $(\mathbf{X}, \mathbf{Y}; v, w) \in \mu^{-1}(0)$ is known as the **ADHM equation**.

Define the **affine quiver variety** to be the GIT quotient $Y_0 = X /\!\!/_0 G = \mu^{-1}(0) /\!\!/_0 G$. Let $\theta \in \mathbb{X}(G)$. The **Nakajima quiver variety** is the GIT quotient

$$Y_\theta := \mu^{-1}(0) /\!\!/_\theta G$$

and the corresponding projective map,

$$p\colon Y_\theta \longrightarrow Y_0,$$

is a symplectic resolution of singularities when θ doesn't lie on a GIT wall (see [Gor08, Section 3.9]). The points of Y_θ parametrise equivalence classes of polystable quiver representations of $\overline{Q}_\infty^l$ that satisfy the ADHM equation. See Section 5.10.4 for an example. By [Nak99, Theorem 3.24], when $l = 1$ and $\theta \neq 0$, Y_θ is isomorphic to the Hilbert Scheme of n points on the plane $\mathbb{C}^2$.

The GIT walls in this case have been calculated by Gordon.

Proposition 5.8.1. *([Gor08, Lemma 4.3 and Remark 4.4]) The variety Y_θ is smooth if θ does not lie on one of the following hyperplanes, in which case θ belongs to a GIT chamber.*

$$\theta^i + \cdots + \theta^j - m(\theta^0 + \cdots + \theta^{l-1}) = 0 \qquad\qquad \theta^0 + \cdots + \theta^{l-1} = 0,$$

where $1 \leq i \leq j \leq l-1$ and $1 - n \leq m \leq n - 1$.

5.8.4 Nakajima quiver varieties as examples of conical symplectic resolutions

Let X, G and Y_θ be as in Section 5.8.3 above. In [Nak01, Section 2.7], Nakajima constructs an F-action on Y_θ that commutes with the action of G and makes Y_θ a conical symplectic resolution of degree two. Let F act on X so that $\mathbf{X}_{ij}^{(m)}, \mathbf{Y}_{ij}^{(m)}, v_i, w_i$ for all i, j, m, are eigenvectors with weight -1. Then F acts on the corresponding coordinate functions with weight 1, for all $t \in F$, $t^*\omega_X = t^2\omega_X$ and $X = X_0$: a trivial resolution of singularities. Thus X is also a conical symplectic resolution of degree two. The action of F preserves the zero fibre of the moment map since it is cut out by homogenous, degree two polynomials in the coordinate functions corresponding to the eigenvectors of F. For the same reason F acts on X^{ss}: it is defined by coordinate functions corresponding to eigenvectors. Now, the action of F commutes with the action of G, so there is an induced action of F on the quotient, $Y_\theta = (\mu^{-1}(0) \cap X^{\mathrm{ss}})/G$; this agrees with the action defined in [Nak01, Section 2.7].

The following results will be needed in Section 6.1. In particular, to apply Hartogs' Lemma to extend sections of sheaves over X^{ss} to X. First an innocuous-looking lemma.

Lemma 5.8.1. *Fix positive integers, $x_1 \leq \cdots \leq x_n \in \mathbb{N}$ and $y_1 \leq \cdots \leq y_n \in \mathbb{N}$, for some $n \geq 2$. The function*

$$f\colon \mathfrak{S}_n \longrightarrow \mathbb{N}; \quad \sigma \longmapsto \sum_{i=1}^{n} x_{\sigma(i)} y_i$$

is maximal at $\mathrm{id} \in \mathfrak{S}_n$.

Proof. If $n = 2$ and $x_1 \leq x_2$ then $0 \leq (x_2 - x_1)(y_2 - y_1)$ so $x_1y_1 + x_2y_2 \geq x_1y_2 + x_2y_1$. Now let $n \geq 2$. Suppose that $x_i \geq x_j$ for some $i < j$. Then, from the case for $n = 2$ deduce that $\sum_{k \neq i,j} x_k y_k + x_i y_i + x_j y_j \leq \sum_{k \neq i,j} x_k y_k + x_j y_i + x_i y_j$. Therefore, any sum in which the x_i's are arranged in non-ascending order can be improved by reordering. □

Proposition 5.8.2. *Let X and G be as above and θ a GIT parameter that is not on a GIT wall. Then the unstable points, X^{us}, have codimension at least two in X.*

Proof. Since there are only finitely many strata, there exists some optimal one-parameter subgroup, λ, such that $\dim S^+_{\langle\lambda\rangle} = \dim X^{\mathrm{us}}$. By (KN3) each strata can be decomposed as

$$S^+_{\langle\lambda\rangle} \cong G \overset{P(\lambda)}{\times} S_\lambda.$$

By counting dimensions,

$$\begin{aligned} \mathrm{codim}_X X^{\mathrm{us}} &= \dim X - \dim S^+_{\langle\lambda\rangle} \\ &= \dim X - \dim S_\lambda - \dim G + \dim P(\lambda). \end{aligned}$$

First, consider the case, $n = 1$. Here, $\dim G - \dim P(\lambda) = 0$, so the codimension is the number of negative weights of λ acting on T^V. This is the space of representations of the unframed quiver, $\mathrm{Rep}(\overline{Q}^l, \epsilon)$, with the action of $\hat{G} = (\mathbb{C}^*)^l/\mathbb{C}^*$. As an element of $\Upsilon(G)$, an optimal one-parameter subgroup is an l-tuple, $\lambda = (a_0, \ldots, a_{l-1})$. It must be non-zero in $\Upsilon(\hat{G})$, so there is some $i \neq j$ such that $a_i \neq a_{i-1}$ and $a_j \neq a_{j-1}$. This means that $X^{(i)}$ and $X^{(j)}$ are non-fixed eigenvectors in V, which implies that there are at least two negative weights of λ acting on T^*V, this implies that X^{ss} contains a subspace of dimension two.

Now let $n \geq 2$. Let $\mathbf{X}^\lambda$ denote the subspace of $\mathbf{X} = \mathbf{X}^{(0)} \oplus \cdots \oplus \mathbf{X}^{(l-1)}$ fixed by λ. Let G^λ denote the subspace of G fixed by λ (acting by conjugation). Note that

$$2\,(\dim G - \dim P(\lambda)) = ln^2 - \dim(G^\lambda).$$

I claim that $\dim(\mathbf{X}^\lambda) \leq \dim(G^\lambda)$.

Let $\lambda := (\lambda^{(0)}, \ldots, \lambda^{(l-1)})$, where, for each $k = 0, \ldots, l-1$, $\lambda^{(k)} := (a_1^{(k)}, \ldots, a_n^{(k)}) \in \mathbb{Z}^n$. By conjugating this by the action of the Weyl group, $\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n$, assume that the entries within each component are in ascending order; that is, for all $k = 0, \dots, l-1$ and $i = 1, \dots, n-1$, $a_i^{(k)} \leq a_{i+1}^{(k)}$. Rename these entries so that, for each $k = 0, \dots, l-1$, after removing duplicate entries, $\lambda^{(k)}$ would look like, $(b_1^{(k)}, \dots, b_{p_k}^{(k)})$ with $b_i^{(k)} < b_{i+1}^{(k)}$ for all $i = 1, \dots, p_k - 1$. Let $n_i^{(k)}$ be the number of times $b_i^{(k)}$ appears in $\lambda^{(k)}$. With this new notation each component of the subgroup looks like

$$\lambda^{(k)} = (\underbrace{b_1^{(k)}, \dots, b_1^{(k)}}_{n_1^{(k)}}, \underbrace{b_2^{(k)}, \dots, b_2^{(k)}}_{n_2^{(k)}}, \dots, \underbrace{b_{p_k}^{(k)}, \dots, b_{p_k}^{(k)}}_{n_{p_k}^{(k)}}).$$

Note that, for each $k = 0, \dots, l-1$, $n_1^{(k)} + \cdots + n_{p_k}^{(k)} = n$ partitions n. Now, for $k = 0, \dots, l-1$, the weights of $\lambda^{(k)}$ acting on $\mathrm{GL}_{n_1^{(k)} + \cdots + n_{p_k}^{(k)}}(\mathbb{C})$ are

$$\left(a_i^{(k)} - a_j^{(k)}\right)_{i,j},$$

and these are zero precisely on the square blocks, cut out by the $n_i^{(k)}$, that run down the diagonal. That is to say,

$$\dim(G^\lambda) = \sum_{i=1}^{p_k} \left(n_i^{(k)}\right)^2.$$

On the other hand, λ acts on $\mathbf{X}^{(k)}$ with weights

$$A := \left(a_i^{(k+1)} - a_j^{(k)}\right)_{i,j}.$$

Partition the rows of this matrix into $n = n_1^{(k+1)} + \cdots + n_{p_{k+1}}^{(k+1)}$ and the columns into $n = n_1^{(k)} + \cdots + n_{p_k}^{(k)}$. This divides the matrix of λ-weights into rectangular blocks inside each of which the weight is constant. There is more structure here though. Make a new $p_{k+1} \times p_k$ matrix, B, by treating each rectangular block of A as a single entry:

$$B := \left(b_i^{(k+1)} - b_j^{(k)}\right)_{i,j}.$$

Because $\lambda^{(k)}$ and $\lambda^{(k+1)}$ are assumed to be increasing, each column of B is strictly increasing as one moves down the column and each row is strictly decreasing as one moves left-to-right along the row. Clearly then, each row or column can contain at most one zero. For each (i, j) such that $B_{i,j} = 0$, A contains exactly $n_j^{(k)} n_i^{(k+1)}$ zeroes.

For each $k = 0, \dots, l-1$, put the positive integers, $\left\{n_1^{(k)}, \dots, n_{p_k}^{(k)}\right\}$, into increasing order and rename them $\{m_1^{(k)}, \dots, m_{p_k}^{(k)}\}$. Suppose, without loss of generality that $p_k \geq p_{k+1}$. Applying Lemma 5.8.1 to the two lists $\{m_1^{(k)}, \dots, m_{p_k}^{(k)}\}$ and $\{m_1^{(k+1)}, \dots, m_{p_{k+1}}^{(k+1)}\}$ shows that the number of zeroes of λ acting on $\mathbf{X}^{(k)}$ is bounded above by

$$\sum_{i=p_k - p_{k+1}+1}^{p_k} m_i^{(k)} m_{i-p_k+p_{k+1}}^{(k+1)}.$$

When $p_k \leq p_{k+1}$ the dummy variable runs from $p_{k+1} - p_k + 1$ to p_{k+1} and the subscript of $m^{(k)}$ and $m^{(k+1)}$ are adjusted appropriately. It follows that

$$\dim(\mathbf{X}^\lambda) \leq \sum_{k \,|\, p_k \geq p_{k+1}}^{l-1} \sum_{i=p_k-p_{k+1}+1}^{p_k} m_i^{(k)} m_{i-p_k+p_{k+1}}^{(k+1)} + \sum_{k \,|\, p_k < p_{k+1}}^{l-1} \sum_{i=p_{k+1}-p_k+1}^{p_{k+1}} m_{i-p_{k+1}+p_k}^{(k)} m_i^{(k+1)}.$$

But now note that

$$\{m_i^{(k)} \,|\, k = 0, \dots, l-1,\ i = 1, \dots, p_k\} = \{n_i^{(k)} \,|\, k = 0, \dots, l-1,\ i = 1, \dots, p_k\}.$$

Applying Lemma 5.8.1 again to two copies of this larger set gives

$$\sum_{k=0}^{l-1}\sum_i m_i^{(k)} m_i^{(k+1)} \le \sum_{k,i}\left(n_i^{(k)}\right)^2 = \dim(G^\lambda).$$

This proves the claim.

The corollary of this claim is the following. Let $\dim(\mathbf{X}\oplus\mathbf{Y})_-$ and $\dim(\mathbf{v}\oplus\mathbf{w})_-$ denote the number of negative weights of λ acting on $\mathbf{X}\oplus\mathbf{Y}$ and $\mathbf{v}\oplus\mathbf{w}$ respectively. Since the action of λ is hamiltonian, the weights of λ on $\mathbf{X}$ are the negatives of those on $\mathbf{Y}$. That means

$$\dim(\mathbf{X}\oplus\mathbf{Y})_- + \dim(\mathbf{X}^\lambda) = \dim\mathbf{X} = ln^2.$$

Putting everything together,

$$\begin{aligned}
\operatorname{codim}_X X^{\mathrm{us}} &= \dim X - \dim S^+_{\langle\lambda\rangle} \\
&= \dim X - \dim S_\lambda - \dim G + \dim P(\lambda) \\
&= \dim(\mathbf{X}\oplus\mathbf{Y})_- + \dim(\mathbf{v}\oplus\mathbf{w})_- - \dim G + \dim P(\lambda) \\
&= ln^2 - \dim(\mathbf{X}^\lambda) - \dim G + \dim P(\lambda) + \dim(\mathbf{v}\oplus\mathbf{w})_- \\
&\ge ln^2 - \dim(G^\lambda) - \dim G + \dim P(\lambda) + \dim(\mathbf{v}\oplus\mathbf{w})_- \\
&= \dim G - \dim P(\lambda) + \dim(\mathbf{v}\oplus\mathbf{w})_-.
\end{aligned}$$

This is a sum of two non-negative integers.

Now consider the worst case scenarios. If $\dim(\mathbf{v}\oplus\mathbf{w})_- = 1$ then $\lambda^{(0)} = (0,\dots,0,1)$, and the contribution from this component of λ gives $\dim G - \dim P(\lambda) \ge 1$. If $\dim(\mathbf{v}\oplus\mathbf{w})_- = 0$ then $\lambda^{(0)} = 0$. If $\dim G - \dim P(\lambda) = 0$ then $\operatorname{codim}_X X^{\mathrm{us}} = \dim X - \dim S_\lambda$ and the argument follows the case $n = 1$ above. Otherwise, suppose $\dim G - \dim P(\lambda) = 1$. There is only one possible form that λ can take now. First, λ must act trivially on all but one component of G, the k^{th} say. The block of the matrix of weights of $\lambda^{(k)}$ acting on $\mathrm{GL}_n(\mathbb{C})$ that contributes 1 to $\dim G - \dim P(\lambda)$ must be one-by-one, which implies, using the notation above that $n_1^{(k)} = n_2^{(k)} = 1$. Therefore, $n = 2$ and $a_1^{(k)} \ne a_2^{(k)}$, but for all $k' \ne k$, $a_1^{(k')} = a_2^{(k')}$. The weights of λ acting on $\mathbf{X}^{(k-1)} \times \mathbf{X}^{(k)}$ are

$$\begin{pmatrix} a_1^{(k)} - a_1^{(k-1)} & a_1^{(k)} - a_1^{(k-1)} \\ a_2^{(k)} - a_1^{(k-1)} & a_2^{(k)} - a_1^{(k-1)} \end{pmatrix}, \begin{pmatrix} a_1^{(k+1)} - a_1^{(k)} & a_1^{(k+1)} - a_2^{(k)} \\ a_1^{(k+1)} - a_1^{(k)} & a_1^{(k+1)} - a_2^{(k)} \end{pmatrix}.$$

At least four of these are non-zero, so there are at least four negative weights of λ acting on $\mathbf{X}^{(k-1)} \times \mathbf{X}^{(k)} \times \mathbf{Y}^{(k-1)} \times \mathbf{Y}^{(k)}$. Therefore, $\dim X - \dim S_\lambda - \dim G + \dim P(\lambda) \ge 4 - 1 = 3$. □

5.8.5 The King Criterion for the Stability of a Quiver Variety

Now that the language of quiver varieties has been introduced an alternative criterion for stability can be presented due to King. Let $\theta = (\theta^0,\dots,\theta^{l-1}) \in \mathbb{Z}^l$ be a GIT parameter. Extend this to a vector $\hat\theta = (\theta^\infty,\theta^0,\dots,\theta^{l-1})$ so that the dot product, $\hat\theta\cdot\hat\epsilon_n = 0$; that is, choose $\theta^\infty = -\sum_{i=0}^{l-1} n\theta^i$. Given a quiver representation V of $\overline{Q}_\infty^l$, let $\mathbf{dim}V$ be its dimension vector. Let $x \in X$. A proper subrepresentation, V, of x is said to **destabilise** x if $\hat\theta\cdot\mathbf{dim}V < 0$.

Theorem 5.8.1. *([Kin94, Proposition 3.1]) A point, $x \in X$, is semistable with respect to θ if and only if there does not exist a proper destabilising subrepresentation.*

Notice, that in the degenerate case $n = 1$, when the group G has been replaced with $\hat G$; it is automatic that $\theta\cdot\epsilon_1 = 0$.

5.9 W-Algebras

Let X be a conical symplectic resolution. This section introduces the machinery of W-algebras on X, an example of a more general process called *deformation quantisation*. Much of the exposition, which closely follows [BK12] and [KR08], is technical, so the section begins with the prototype for the more general theory: the case when X is the cotangent bundle of a complex vector space. The appropriate categories will then be cooked up once the following ingredients have been defined:

(i) a *W-algebra* with a G-action and an F-action (G is a complex Lie group and $F := \mathbb{G}_m$);

(ii) a *module* for the W-algebra which is

  (i) *good*,

  (ii) *quasi-G-equivariant*,

  (iii) *F-equivariant*;

(iii) a way to *twist* W-algebra modules by some character of $\mathfrak{g}$, $\chi \in (\mathfrak{g}^*)^G$.

5.9.1 A W-Algebra for the Cotangent Bundle of Affine Space

Let V be an n-dimensional complex vector space, $\mathbf{k} := \mathbb{C}((\hbar))$ the field of formal Laurent series in $\hbar$ and $\mathbf{k}(0)$ the subring $\mathbb{C}[\![\hbar]\!] \subset \mathbf{k}$ of formal power series. The following construction of $\mathcal{W}_{T^*V}(0)$ is a motivating example of a deformation quantisation of a complex symplectic manifold (in this case T^*V with its usual form ω_{T^*V}). Let $x_1, \ldots, x_n$ be a basis for V with a dual basis $y_1, \ldots, y_n$. By identifying T^*V with the vector space $V \times V^*$ one gets a basis for T^*V and $\omega_{T^*V} = \sum_i dx_i \wedge dy_i$. Let $\mathcal{O}^{\mathrm{hol}}_{T^*V}$ be the sheaf of holomorphic functions on T^*V. Holomorphic functions are locally defined by Taylor series, so, using the same notation for the coordinate functions on T^*V as the points, define a Poisson bracket on $\mathcal{O}^{\mathrm{hol}}_{T^*V}$ by extending the rule $\{x_i, x_j\} = \{y_i, y_j\} = 0$ and $\{y_i, x_j\} = \delta_{ij}$ for all $i, j = 1, \ldots, n$.

For each $m \in \mathbb{Z}$ let $\mathcal{W}_{T^*V}(m)$ be the following sheaf of formal power series.

$$\mathcal{W}_{T^*V}(m) := \left\langle \sum_{i \geq -m} \hbar^i a_i \,\middle|\, a_i \in \mathcal{O}^{\mathrm{hol}}_{T^*V} \right\rangle,$$

and define a sheaf of $\mathbf{k}$-algebras, $\mathcal{W}_{T^*V} := \bigcup_{m \in \mathbb{Z}} \mathcal{W}_{T^*V}(m)$; the ring multiplication is as follows. Let $a = \sum_{i \geq -m} \hbar^i a_i$ and $b = \sum_{i \geq -m'} \hbar^i b_i$ be arbitrary sections of $\mathcal{W}_{T^*V}(m)$ and $\mathcal{W}_{T^*V}(m')$ respectively. Given an n-tuple $\alpha \in \mathbb{Z}^n$, let $|\alpha| := \sum_i \alpha_i$, $\alpha! := \alpha_1! \cdots \alpha_n!$ and ∂_y^α be the differential operator $\frac{\partial^{|\alpha|}}{\partial^{\alpha_1} y_1 \cdots \partial^{\alpha_n} y_n}$. Now the following rule gives a deformed multiplication,

$$a * b := \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} \frac{\hbar^{|\alpha|}}{\alpha!} \partial_y^\alpha a \cdot \partial_x^\alpha b = \sum_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ i \geq -m \\ j \geq -m'}} \frac{\hbar^{|\alpha|+i+j}}{\alpha!} \partial_y^\alpha a_i \cdot \partial_x^\alpha b_j.$$

See [KR08, Section 2] for details. This example has several interesting properties.

First, there is a filtration of $\mathcal{W}_{T^*V}$ sheaves of $\mathbf{k}(0)$-algebras,

$$\cdots \subseteq \mathcal{W}_{T^*V}(i) \subseteq \mathcal{W}_{T^*V}(i+1) \subseteq \cdots.$$

The sheaf, $\mathcal{W}_{T^*V}(0)$, is called a *deformation quantisation* of T^*V. Define the **symbol map** to be the morphism of sheaves which takes the quotient

$$\sigma_m \colon \mathcal{W}_{T^*V}(m) \to \frac{\mathcal{W}_{T^*V}(m)}{\mathcal{W}_{T^*V}(m-1)} \cong h^{-m} \mathcal{O}^{\mathrm{hol}}_{T^*V}.$$

Second, there is an injective homomorphism of rings,

$$\iota \colon D_V \to \Gamma(T^*V, \mathcal{W}_{T^*V}),$$

defined by extending the map $x_i \mapsto x_i$ and $\partial_i \mapsto \hbar^{-1} y_i$.

Third, given sections $a = \sum_{i \geq 0} \hbar^i a_i, b = \sum_{j \geq 0} \hbar^j b_j \in \mathcal{W}_{T^*V}(0)$,

$$\begin{aligned}
\sigma_0(\hbar^{-1}[a,b]) &= \sigma_0 \left(\sum_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ i,j \geq 0}} \frac{\hbar^{|\alpha|+i+j-1}}{\alpha!} \left(\partial^\alpha_y a_i \cdot \partial^\alpha_x b_j - \partial^\alpha_y b_j \cdot \partial^\alpha_x a_i \right) \right) \\
&= \sum_{\substack{\alpha \in \mathbb{Z}^n_{\geq 0} \\ i,j \geq 0 \\ |\alpha|+i+j=1}} \frac{1}{\alpha!} \left(\partial^\alpha_y a_i \cdot \partial^\alpha_x b_j - \partial^\alpha_y b_j \cdot \partial^\alpha_x a_i \right) \\
&= \sum_{1 \leq k \leq n} \left(\frac{\partial}{\partial y_k} a_0 \cdot \frac{\partial}{\partial x_k} b_0 - \frac{\partial}{\partial y_k} b_0 \cdot \frac{\partial}{\partial x_k} a_0 \right) \\
&= \{a_0, b_0\} \\
&= \{\sigma_0(a), \sigma_0(b)\}.
\end{aligned}$$

5.9.2 W-Algebras in general

One now generalises by considering arbitrary complex symplectic manifolds which behave locally like the example above.

Definition 5.9.1. *Let X be a complex symplectic manifold with a symplectic form ω_X. A **W-algebra** is a sheaf of* **k***-algebras, $\mathcal{W}_X$, such that for any point $x \in X$ there exists an open neighbourhood U, a symplectic map $f \colon U \to T^*V$ and an isomorphism of sheaves of* **k***-algebras $g \colon \mathcal{W}_X|_U \to f^{-1}\mathcal{W}_{T^*V}$.*

The map g provides a subsheaf of $\mathbf{k}(0)$-algebras of $\mathcal{W}_X$, called a *deformation quantisation* of X, defined by $\mathcal{W}_X(0)(U) := g^{-1}\mathcal{W}_{T^*V}(0)(f(U))$. Guided by the example of $\mathcal{W}_{T^*V}$, define, for each $m \in \mathbb{Z}$, a sheaf, $\mathcal{W}_X(m)$, by $\mathcal{W}_X(m) := \hbar^{-m}\mathcal{W}_X(0)$.

5.9.3 Quantisations

Let X be a conical symplectic resolution with a Poisson bracket $\{-,-\}$. The following definition may be applied more generally to a Poisson variety over $\mathbb{C}$.

Definition 5.9.2. *A **quantisation** of X is a sheaf, $\mathcal{Q}$, of flat $\mathbf{k}(0)$-algebras on X, complete with respect to the $\hbar$-adic topology and with an isomorphism*

$$\frac{\mathcal{Q}}{\hbar\mathcal{Q}} \xrightarrow{\simeq} \mathcal{O}_X$$

such that $\sigma_0\left(\hbar^{-1}[f,g]\right) = \{\sigma_0(f), \sigma_0(g)\}$ for all $f, g \in \mathcal{Q}$.

Note that this last condition is a kind of compatibility condition between the quantisation and the symplectic form corresponding to the Poisson bracket $\{-,-\}$.

A W-algebra, $\mathcal{W}$, on X gives an example of a quantisation, $\mathcal{Q} = \mathcal{W}_X(0)$. Indeed,

$$\frac{\mathcal{W}_X(0)}{\hbar\mathcal{W}_X(0)} \cong \frac{\mathcal{W}_X(0)}{\mathcal{W}_X(-1)} \cong \mathcal{O}_X.$$

Given a quantisation, $\mathcal{Q}$, of X, set $\tilde{\mathcal{Q}} := \mathcal{Q}[\hbar^{-1}]$ and

$$\tilde{\mathcal{Q}}(m) := \hbar^{-m}\mathcal{Q},$$

a subsheaf of $\tilde{\mathcal{Q}}$. This gives a **lattice** of the quantisation

$$\cdots \subset \tilde{\mathcal{Q}}(m) \subset \tilde{\mathcal{Q}}(m+1) \subset \cdots$$

In the example above, $\tilde{\mathcal{Q}}_X(0)$ is by definition $\mathcal{W}_X$.

5.9.4 W-Algebras with G and F Actions

Let X be a conical symplectic resolution of degree $m \geq 1$ with a hamiltonian action of a complex Lie group G that commutes with the action of F. Let $\mathcal{W}_X$ be a W-algebra on X.

Definition 5.9.3. *A G-action on the W-algebra $\mathcal{W}_X$ is a collection of $\mathbf{k}$-algebra isomorphisms, $\rho_g \colon T_g^{-1}\mathcal{W}_X \xrightarrow{\simeq} \mathcal{W}_X$, parametrised by $g \in G$, such that,*

(i) for each section $a \in T_g^{-1}\mathcal{W}_X$, $\rho_g(a)$ depends holomorphically on g,

(ii) for all $g, h \in G$, $\rho_g \circ \rho_h = \rho_{gh}$.

Definition 5.9.4. *An F-**action of degree** m on $\mathcal{W}_X$ is an action in the sense of Definition 5.9.3 with the additional condition that, for all $t \in F$, $\rho_t(\hbar) = t^m\hbar$, where m is the degree of X as a conical symplectic resolution.*

Suppose that $\mathcal{W}_X$ has an F-action of degree m. Let $\mathbf{k}(\hbar^{1/m}) = \mathbb{C}((\hbar^{1/m}))$ be the field extension of $\mathbf{k}$ by $\hbar^{1/m}$.

$$\widetilde{\mathcal{W}_X} := \mathbf{k}(\hbar^{1/m}) \otimes_{\mathbf{k}} \mathcal{W}_X.$$

An F-action on $\mathcal{W}_X$ extends to an F-action on $\widetilde{\mathcal{W}_X}$ by setting $\rho_t(\hbar^{1/m}) := t\hbar^{1/m}$ for all $t \in F$.

Theorem 5.9.1. *(Losev) Let X be a conical symplectic resolution. Then the set of deformation quantisations with an F-action is in bijection with $H^2(X, \mathbb{C})$, the de Rham cohomology of X and the F-action on the deformation quantisation is unique.*

Proof. This is [Los12, Corollary 2.3.3]. There he calls quantisations with an F-action, graded quantisations. ◻

Let μ be a choice of moment map for the action of G on X. Given $A \in \mathfrak{g}$, let $\mu(A) := \mu(-)(A) \in \mathcal{O}_X^{\mathrm{hol}}$.

Definition 5.9.5. *A homomorphism of Lie algebras $\tau \colon \mathfrak{g} \longrightarrow \Gamma(X, \mathcal{W}_X(1))$ is a **quantised moment map** for the actions of F and G on $\mathcal{W}_X$ if the following conditions hold. For all $A \in \mathfrak{g}$, $a \in \mathcal{W}_X$, $g \in G$ and $t \in F$,*

(i) $[\tau(A), a] = \frac{\mathrm{d}}{\mathrm{dt}}\rho_{\exp(tA)}(a)|_{t=0}$,

(ii) $\sigma_0(\hbar\tau(A)) = \mu(A)$,

(iii) $\tau(\mathrm{Ad}(g)A) = \rho_g \circ \tau(A))$,

(iv) $t \cdot \tau(A) = \tau(A)$.

Lemma 5.9.1. *If τ is a quantised moment map for the actions of G and F then*

(i) the moment map, μ, is F-equivariant: $\mu(t \cdot x) = t^m\mu(x)$, for all $t \in F$ and $x \in X$,

(ii) the constant from Lemma 5.6.2 that μ takes on any component of the set of fixed points is zero

Proof. The first point follows from Properties (*ii*) and (*iv*) of quantised moment maps: If $\tau(A)$ is F-invariant for any $A \in \mathfrak{g}$, then $t \in F$ acts on $\hbar\tau(A)$ with weight m. For μ,

$$\mu(t \cdot x)(A) = \sigma_0(\hbar\tau(A))(t \cdot x) = \sigma_0(t \cdot (\hbar\tau(A)))(x) = t^m\sigma_0(\hbar\tau(A))(x) = t^m\mu(x)(A).$$

Now, because the actions of G and F commute, for any G-fixed point z and $t \in F$, $\mu(t \cdot z) = \mu(z')$ for some z' in the same connected component as z. Applying Lemma 5.6.2 gives $\mu(z) = \mu(z') = \mu(t \cdot z) = t^m\mu(z)$, which must be zero because $m \geq 1$. ◻

5.9.5 W-Modules, Goodness and Quasi-G-Equivariance

Next comes the construction of the categories of appropriate modules. Let X, G and $\mathcal{W}_X$ be as in the section above.

Definition 5.9.6. *Make the following definitions.*

1. *A sheaf, $\mathcal{M}$, is a $\mathcal{W}_X$-**module** if, for each open set $U \subseteq X$, $\mathcal{M}(U)$ is a left $\mathcal{W}_X(U)$-module.*
2. *A $\mathcal{W}_X$-module, $\mathcal{M}$, is **coherent** if, for each $x \in X$, $\mathcal{M}_x$ is a finitely generated $\mathcal{W}_{X,x}$-module.*
3. *A $\mathcal{W}(0)$-**lattice** of $\mathcal{M}$ is a $\mathcal{W}_X(0)$-submodule $\mathcal{N}$ of $\mathcal{M}$ such that*

$$\mathcal{W}_X \otimes_{\mathcal{W}_X(0)} \mathcal{N} \simeq \mathcal{M}.$$

4. *A $\mathcal{W}_X$-module, $\mathcal{M}$, is **good** if for every relatively compact open set, U, there exists a coherent $\mathcal{W}_X(0)|_U$-lattice for $\mathcal{M}|_U$.*
5. *Consider the category of $\mathcal{W}_X$-modules. Let $(\mathcal{W}_X)^{good}$ denote the full subcategory of good $\mathcal{W}_X$-modules.*

When $\mathcal{W}_X$ has an F-action, let $\left(\widetilde{\mathcal{W}}_X, F\right)^{\text{good}}$ denote the category of good F-equivariant $\widetilde{\mathcal{W}}_X$-modules.

Definition 5.9.7. *A $\mathcal{W}_X$-module, $\mathcal{M}$, is **quasi-G-equivariant** if there is a collection of* $\mathbf{k}$*-module isomorphisms, $\rho_g^{\mathcal{M}} \colon T_g^{-1}\mathcal{M} \xrightarrow{\simeq} \mathcal{M}$, for each of the elements $g \in G$ such that,*

(i) for each section $m \in T_g^{-1}\mathcal{M}$, the section $\rho_g^{\mathcal{M}}(m)$ depends holomorphically on g,

(ii) for all $g, h \in G$, $\rho_g \circ \rho_h = \rho_{gh}$,

(iii) for all $g \in G$, $a \in T_g^{-1}\mathcal{W}_X$ and $m \in T_g^{-1}\mathcal{M}$, $\rho_g(am) = \rho_g(a)\rho_g(m)$.

The category of good quasi-G-equivariant $\mathcal{W}_X$-modules is denoted $(\mathcal{W}_X, G)^{\text{good}}$ and the category of good quasi-G-equivariant $\widetilde{\mathcal{W}}_X$-modules is denoted $\left(\widetilde{\mathcal{W}}_X, G\right)^{\text{good}}$. Finally,

Definition 5.9.8. *Let*

$$(\mathcal{W}_X, F, G)^{good}$$

denote the category of good, quasi-G-equivariant, F-equivariant $\mathcal{W}_X$-modules such that, for all $g \in G$ and $t \in F$, the action of t and g on X commute, ρ_g and ρ_t commute, and τ is a quantised moment map for the action of F and G. Define $\left(\widetilde{\mathcal{W}}_X, F, G\right)^{good}$ similarly.

5.9.6 $\mathcal{W}$-Modules that are twisted by a Character

Now one needs to introduce $\mathcal{W}$-modules that are *twisted* by some character of $\mathfrak{g}$, the Lie algebra of G. Let X and $\mathcal{W}_X$ be as above, with τ a quantised moment map for the action of F and G. This definition follows [KR08, Section 2.4.2].

Definition 5.9.9. *Let $\mathcal{M}$ be a quasi-G-equivariant $\mathcal{W}_X$-module and fix $\chi \in (\mathfrak{g}^*)^G$. Then $\mathcal{M}$ is said to be χ-**twisted** if, for all $A \in \mathfrak{g}$ and $u \in \mathcal{M}$,*

$$A \cdot u = \tau(A)(u) - \chi(A)u,$$

where the action of A on u is the differential of the G-action: $\frac{d}{dt}\rho_{\exp(tA)}u|_{t=0}$. For each of the categories, C, of $\mathcal{W}_X$-modules above, let C_χ denote the full subcategory χ-twisted modules.

Note the sign of χ is different to [KR08].

Lemma 5.9.2. *If* $\mathcal{M} \in (\mathcal{W}_X, G)_\chi^{\text{good}}$ *then* $\text{supp}(\mathcal{M}) \subset \mu^{-1}(0)$.

Proof. The condition that the support of a sheaf belonging to some set can be tested locally. Since $\mathcal{M}$ is coherent there are local generators, $m_1, \dots, m_n$, such that $\mathcal{M} = \mathcal{W}m_1 + \dots + \mathcal{W}m_n$. Define $\mathcal{M}(0) := \mathcal{W}(0)m_1 + \dots + \mathcal{W}(0)m_n$ and for all $m \in \mathbb{Z}$, $\mathcal{M}(m) := \hbar^{-m}\mathcal{M}(0)$. Then $\mathcal{M}(0)$ is a $\mathcal{W}(0)$-module. Now, let V be the complex G-representation generated by $m_1, \dots, m_n$. Since $\mathcal{M}$ is quasi-G-equivariant it acts rationally so that V is finite-dimensional. Redefine $m_1, \dots, m_l$ to be a basis for V so that it is a G-stable generating set for $\mathcal{M}(0)$. Thus $\mathcal{M}(0)$ has an action of G which differentiates to an action of $\mathfrak{g}$. For each $A \in \mathfrak{g}$, let $\alpha(A)$ denote this endomorphism of $\mathcal{M}(0)$. Now $\mathcal{M}$ being χ-twisted for some $\chi \in (\mathfrak{g}^*)^G$ implies that the element $\alpha(A) - \tau(A) - \chi(A)$ acts as zero on $\mathcal{M}(0)$. Let $\mathcal{F} := \mathcal{M}(0)/\mathcal{M}(-1)$, a sheaf on X. Now, $\hbar\alpha$ and $\hbar\chi(A)$ act as zero on $\mathcal{F}$, so, by the second property of the quantised moment map τ, $\hbar(\alpha(A) - \tau(A) - \chi(A))$ acts as $-\hbar\tau(A)$ which acts as $-\mu(A)$ on $\mathcal{F}$. This implies that $\operatorname{supp}(\mathcal{F}) \subset \mu^{-1}(0)$. Let p be in the support of $\mathcal{M}$. Then it is in the support of $\mathcal{M}(0)$ so $\mathcal{M}(0)_p \neq 0$ and, by Nakayama's Lemma $\mathcal{M}(0)_p/\mathcal{M}(-1)_p \neq 0$. Since taking stalks is right exact, $\mathcal{M}(0)_p/\mathcal{M}(-1)_p = \mathcal{F}_p$, so p lies in the support of $\mathcal{F}_p$ and the lemma is proved. □

Following [KR08, Section 2.4.2], the embedding functor

$$I_\chi \colon (\mathcal{W}_X, G)_\chi \longrightarrow (\mathcal{W}_X, G)$$

has a left adjoint, $\Phi_\chi \colon (\mathcal{W}_X, G) \longrightarrow (\mathcal{W}_X, G)_\chi$, defined by

$$\Phi_\chi(\mathcal{M}) := \frac{\mathcal{M}}{\langle \{A \cdot (-) - \hat{\tau}(A) + \chi(A) \,|\, A \in \mathfrak{g}\} \rangle \mathcal{M}},$$

where the quotient is by the left submodule of $\mathcal{M}$ generated by the elements in brackets. Let

$$\mathcal{L}_\chi := \Phi_\chi(\mathcal{W}_X) \cong \frac{\mathcal{W}}{\mathcal{W}_X \langle \{\tau(A) - \chi(A) \,|\, A \in \mathfrak{g}\} \rangle},$$

so that $\mathcal{L}_\chi$ is a good quasi-G-equivariant $\mathcal{W}_X$-module with respect to the lattice

$$\mathcal{L}_\chi(0) := \frac{\mathcal{W}_X(0)}{\mathcal{W}_X(-1) \langle \{\tau(A) - \chi(A) \,|\, A \in \mathfrak{g}\} \rangle},$$

and, by Lemma 5.9.2, $\mathcal{L}_\chi$ is supported on the closed subset $\mu^{-1}(0) \subset X$.

5.9.7 Two Equivalent Constructions of W-Algebras on the GIT Quotient

Keep the same notation as above with the additional condition that G is reductive. Choose a character, θ, of G to get the open set X^{ss}. Restricting the sheaf, $\mathcal{W}_X$, from X to X^{ss} gives a W-algebra, $W_{X^{\mathrm{ss}}}$, on X^{ss}. Let $Y_\theta = X^{\mathrm{ss}} \cap \mu^{-1}(0)/G$ denote the quotient, which, by the Kempf–Ness Theorem (Theorem 5.5.2) is symplectic. Let $p \colon \mu^{-1}(0) \cap X^{\mathrm{ss}} \longrightarrow Y_\theta$ denote the quotient map. Define a sheaf of $\mathbf{k}$-algebras on Y_θ by,

$$\mathcal{W}_{Y_\theta} := \left(p_* \mathcal{E}nd_{\mathcal{W}_{X^{\mathrm{ss}}}}(\mathcal{L}_\chi)^G\right)^{\mathrm{op}}.$$

Note that this W-algebra is dependent on the choice of character χ via the definition of $\mathcal{L}_\chi$.

Proposition 5.9.1. *Suppose that θ does not lie on a GIT wall; so that G acts freely on X^{ss}. Then*

(i) The sheaf $\mathcal{W}_{Y_\theta}$ is a W-algebra on Y_θ; the corresponding quantisation is

$$\left(p_* \mathcal{E}nd_{\mathcal{W}_{X^{\mathrm{ss}}}(0)}(\mathcal{L}_\chi(0))^G\right)^{\mathrm{op}}.$$

(ii) There are quasi-inverse equivalences of categories

$$(\mathcal{W}_{Y_\theta})^{\mathsf{good}} \overset{\simeq}{\longleftrightarrow} (\mathcal{W}_{X^{ss}}, G)^{\mathsf{good}}_{X}$$
$$\mathcal{N} \longmapsto \mathcal{L}_X \otimes_{p^{-1}\mathcal{W}_{Y_\theta}} p^{-1}\mathcal{N}$$
$$\left(p_*\mathcal{H}om_{\mathcal{W}_{X^{ss}}}(\mathcal{L}_X, \mathcal{M})\right)^G \longleftarrow \mathcal{M}$$

(iii) If $\mathcal{W}_{X^{ss}}$ has an F-action then $\mathcal{W}_{Y_\theta}$ has an F-action and there is an equivalence of categories

$$\mathbb{E}\colon \left(\widetilde{\mathcal{W}}_{X^{ss}}, F, G\right)^{\mathsf{good}}_{X} \overset{\simeq}{\longrightarrow} \left(\widetilde{\mathcal{W}}_{Y_\theta}, F\right)^{\mathsf{good}}.$$

Proof. See [KR08, Proposition 2.8]. □

5.9.8 A Relationship between D-Modules and W-Algebra Modules

When M is smooth and $X = T^*M$, deformation quantisations can also be constructed via D-modules on M.

Theorem 5.9.2. *([KR08, Theorem 2.10]) Let M be a smooth variety, let $\pi\colon T^*M \longrightarrow M$ be the projection onto the zero section and $\mathcal{W}_X$ be a deformation quantisation of $X = T^*M$. Then there is a unique F-action on $\mathcal{W}_X$ such that there is an equivalence of categories*

$$(\widetilde{\mathcal{W}}_X, F)^{\mathsf{good}} \longrightarrow \mathbf{coh}_{D_M}; \quad \mathcal{F} \mapsto \mathcal{F}^F,$$

with quasi-inverse, $N \mapsto \widetilde{\mathcal{W}}_X \otimes_{\pi^{-1}D_M} \pi^{-1}N$.

5.9.9 Holomorphic versus algebraic functions

The definition of W-algebras uses the sheaf of holomorphic functions, $\mathcal{O}^{\mathsf{hol}}$. The analysis of W-algebras in the rest of the thesis will be on $T^*\mathbb{A}^n$, the cotangent bundle of some affine space with a hamiltonian action of some reductive group G and an F-action that gives it the structure of a conical symplectic resolution. The action of F induces a grading on the sections of $\mathcal{O}^{\mathsf{hol}}$ and each graded piece is finite-dimensional because of (*ii*) and (*iii*) in Definition 5.7.1. It follows that any holomorphic section that is homogenous with respect to F has a local taylor expansion that is polynomial. Therefore, the results in [KR08], in the context they will be used later, continue to hold if one replaces $\mathcal{O}^{\mathsf{hol}}$ with $\mathcal{O}$, the sheaf of algebraic functions.

5.10 Example

Let $V = \mathbb{C}^2$ with $G = \mathbb{C}^*$ acting on V with weights $(1,1)$ with respect to an eigenbasis x_1, x_2. Let y_1, y_2 be a dual eigenbasis for V^* so that G acts on $X = T^*V \cong V \times V^*$ with weights $(1,1,-1,-1)$. This action is hamiltonian with respect to the canonical symplectic form, $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Let F act with weights $(-1,-1,-1,-1)$ so that it commutes with the action of G, scales the form with weight one, lifts to an action with non-negative weights on the coordinate ring and the only F-invariants are constant functions. This gives X the structure of a conical symplectic resolution over itself with degree one.

5.10.1 Linearisation and GIT Walls

The polystable points of X are those points of X that don't lie on the coordinate axes and also the origin. Let $\mathcal{L}$ be the trivial line bundle on X. Now, identifying $\mathbb{X}(G)$ with $\mathbb{Z}$ gives a linearisation corresponding to $\theta \in \mathbb{Z}$, where $g \in G$ acts on $p = (p_1, p_2, p_3, p_4, l) \in X \times \mathcal{L}$ by $g \cdot p = (gp_1, gp_2, g^{-1}p_3, g^{-1}p_4, g^{\theta}l)$. Let $X_1, \ldots, Y_2, L$ be coordinate functions on $x_1, \ldots, y_2, l$. Then g acts on some monomial, $X_1^{a_1}X_2^{a_2}Y_1^{b_1}Y_2^{b_2}L^c$, with weight $b_1 + b_2 - a_1 - a_2 - \theta c$. Then sections in $\Gamma(X, \mathcal{L}^{\otimes n})^G$ are sums of monomials for which $b_1 + b_2 - a_1 - a_2 = nc\theta$.

If $\theta > 0$, then every monomial with $c > 0$ must be divisible by either Y_1 or Y_2. Thus the locus of unstable points is $X^{us} = \{Y_1 = Y_2 = 0\} = V \times \{0\}$. The GIT quotient is a smooth three-dimensional variety whose points parametrise the orbits in X which are disjoint from X^{us}. If $\theta < 0$, then every monomial with $c > 0$ must be divisible by X_1 or X_2 so $X^{us} = \{0\} \times V^*$ and the quotient is similar. If $\theta = 0$, a point x is unstable if $f(x) = 0$ for all

$$f \in \mathbb{C}[X]^G = \mathbb{C}[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2] = \frac{\mathbb{C}[A, B, C, D]}{\langle AD - BC\rangle},$$

so only the origin is unstable. The quotient is the singular affine variety $\mathrm{Spec}\left(\frac{\mathbb{C}[A,B,C,D]}{\langle AD-BC\rangle}\right)$.

Now, the G-ample cone is $\mathbb{Z}\otimes\mathbb{R} = \mathbb{R}$ and the set of semistable but not stable points is empty only when $\theta \neq 0$; so 0 is a wall and there are two chambers: $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$.

5.10.2 The Conical Symplectic Resolution of the Quotient

Let $\theta = 1$.

$$\mathfrak{X}_\theta = \mu^{-1}(0) \cap X_\theta^{ss} = \{(p_1, p_2, q_1, q_2) \in \mathbb{C}^4 \,|\, p_1q_1 = -p_2q_2,\ (q_1, q_2) \neq (0, 0)\}.$$

This naturally maps to $\mathbb{A}^2 \times \mathbb{P}^1$ by sending $x = (p_1, p_2, q_1, q_2) \mapsto (p_1, p_2), [q_1 : q_2]$. The orbits under the action of G map bijectively onto the points of $\mathbb{P}^1$ and the fibre over one of these points, $[q_1 : q_2]$ say, is

$$\{(p_1, p_2) \,|\, p_1q_1 = -p_2q_2\} \cong \mathbb{A}^1.$$

A similar argument shows that $Y_{\theta\neq 1} \cong T^*\mathbb{P}^1$. The singular affinisation is

$$Y_{\theta=0} = \mathrm{Spec}\left(\frac{\mathbb{C}[X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2]}{\langle X_1Y_1 + X_2Y_2\rangle}\right) \cong \mathrm{Spec}\left(\frac{\mathbb{C}[A, B, C]}{\langle A^2 - BC\rangle}\right),$$

where the isomorphism is given by $A \mapsto x_1y_1 - x_2y_2$, $B \mapsto -2x_1y_2$ and $C \mapsto 2x_2y_1$. See Figure 5.1 for a picture.

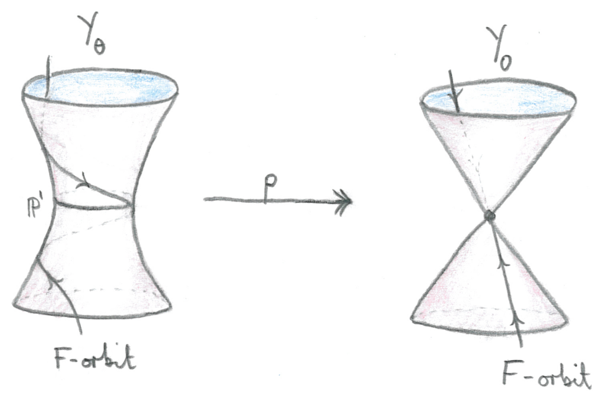


Figure 5.1: The cotangent bundle of $\mathbb{P}^1$ as a conical symplectic resolution of its affinisation. This resolution of singularities is the same as the projective morphism you get from considering Y_θ as a GIT quotient; see Section 5.4.

5.10.3 Hilbert–Mumford Criteria

The one-parameter subgroups of G form a group, isomorphic to $\mathbb{Z}$. Let $\lambda \in \mathbb{Z} \cong \Upsilon(G)$ and $(p_1, p_2, q_2, q_2, l) \in X \times \mathcal{L}$. Then $\lambda(t) \cdot (p_1, p_2, q_1, q_2, l) = (t^\lambda p_1, t^\lambda p_2, t^{-\lambda} q_1, t^{-\lambda} q_2, t^{\theta\lambda} l)$. Therefore, when $\theta > 0$, $\lambda > 0$ destabilises points for which $q_1 = q_2 = 0$ and when $\theta < 0$, any $\lambda < 0$ destabilises points for which $p_1 = p_2 = 0$. When $\theta = 0$ the function $\mu(-, \lambda) = 0$ so every point is semistable. This agrees with the calculation of unstable points above, but also shows that the Kirwan–Ness stratification is trivial: having only one stratum.

5.10.4 As a Quiver Variety

This example also fits into the framework of a quiver variety. Let

$$Q := \; 1 \underset{x_2}{\overset{x_1}{\rightrightarrows}} 2 \, ,$$

so that $\operatorname{Rep}(\overline{Q}, (1,1)) = T^*V$. Now $G' = \mathbb{C}^* \times \mathbb{C}^*$ with $\Upsilon(G') \cong \mathbb{Z}^2$ and

$$X' = \operatorname{Rep}(\overline{Q_\infty}, (1,1,1)) = \{p := ((p_1, p_2, q_1, q_2); (v, w))\}.$$

An element $g := (g_1, g_2)$ acts via

$$g \cdot p = (g_1^{-1} g_2 p_1, g_1^{-1} g_2 p_2, g_1 g_2^{-1} q_1, g_1 g_2^{-1} q_2; (g_1 v, g_1^{-1} w).$$

Then let $G = \mathbb{C}^*$ be identified by the image of the one-parameter subgroup, $(-1, 1)$, of G'. Here G acts with weights, $(1, 1, -1, -1)$, as before, on the components of X' to which X corresponds. If $\theta \in \mathbb{Z}$ is a character of G then it corresponds to the character $(-\theta, \theta)$ of G'. By the criteria of King, Theorem 5.8.1, a point p is unstable for $\theta > 0$ if there exists some destabilising subrepresentation with dimension vector $(0, 0, 1)$ which is only true if $q_1 = q_2 = 0$. Similarly, for $\theta < 0$ a point is unstable if it has a proper subrepresentation of type $(0, 1, 0)$, which is true if and only if $p_1 = p_2 = 0$. The ADHM equation holds if and only if either $v = 0$ or $w = 0$. Where $w = 0$, the action of G on $\mu^{-1}(0) \subset X'$ is the same as the action of G on $X \times \mathcal{L}$ and where $v = 0$, the action is the same as G on $X \times \mathcal{L}$ but with the opposite linearisation: $-\theta$.

5.10.5 Quantisation

Next, consider quantisation. Let F act on X with weights $(0, 0, 1, 1)$, so that $t \cdot \omega_X = t\omega_X$. Let $\chi \in \mathbb{X}(G) \cong \mathbb{C}$ be a character. Then the quantisations of X are isomorphic to $\mathcal{W}_{T^*V}(0)$ since $H^2(X, \mathbb{C}) = 0$. The sections of $\mathcal{W}_X(0)$ are generated by polynomials in $\hbar$ with holomorphic coefficients.

A quantised moment map is given by $\tau(1) = -x_1 y_1 \hbar^{-1} - x_2 y_2 \hbar^{-1}$, equivalently $\tau(1) = -x_1\partial_1 - x_2\partial_2$ as a vector field on V. Thus the quantisation is the non-commutative algebra

$$\frac{D_V}{D_V\langle \tau(1) - \chi(1)\rangle} = \frac{\langle x_1, x_2, \partial_1, \partial_2\rangle}{\langle x_1\partial_1 + x_2\partial_2 + \chi(1),\ [\partial_i, x_j] = \delta_{ij}\rangle}.$$

This has a rational G-action (G acts with weights $(1, 1, -1, -1)$ on $(x_1, x_2, \partial_1, \partial_2)$. Taking G invariants gives the algebra

$$\begin{aligned} {}''U_\chi'' &= \frac{\langle x_1\partial_1 - x_2\partial_2, x_1\partial_2, x_2\partial_1\rangle}{\langle x_1\partial_1 + x_2\partial_2 + \chi(1),\ [\partial_i, x_j] = \delta_{ij}\rangle} \\ &\cong \frac{\langle h, e, f\rangle}{[h, e] = 2e,\ [h, f] = -2f,\ [e, f] = h, h^2 - h - \chi^2 + 2\chi\rangle} \\ &= \frac{U(\mathfrak{sl}_2)}{\langle h(h-1) - \chi(\chi - 2)\rangle}, \end{aligned}$$

where the isomorphism is given by mapping $x_1\partial_1 - x_2\partial_2 \mapsto h$, $x_1\partial_2 \mapsto e$ and $x_2\partial_1 \mapsto f$.

Chapter 6

Localisation for Nakajima Quiver Varieties

Fix a parameter $\mathbf{c} = (c_0, \ldots, c_{l-1})$[1] for the Cherednik algebra corresponding to $W = G(l, 1, n)$. The goal of this chapter is to construct $U_{\mathbf{c}}(W)$-modules via sheaves on the space Y_θ defined in Section 5.8.3. Recall that θ is a character of $G = \mathrm{GL}_n(\mathbb{C})^{\times l}$ which acts on the space $X = T^*V$ where V is the vector space of representations of the quiver Q_∞^l. As in Section 5.8.3, the action of G on X is hamiltonian and commutes with an F action on X of degree two. Let $\mathcal{W}_X$ be the (by Theorem 5.9.1 unique) quantisation defined in Section 5.9.1.

By Theorem 5.9.2, there is a unique F action on $\mathcal{W}_X$ such that $\Gamma(X, \widetilde{\mathcal{W}_X})^F \cong D_V$. Each coordinate function corresponding to some arrow on the quiver has F-degree one, this implies that F acts with degree two on the symplectic form ω so that $\hbar$ has F-degree two. Recall the anti-isomorphism $(-)^{\mathrm{op}}$ of D_V. This produces an F-equivariant isomorphism,

$$\phi \colon \mathcal{W}_X \xrightarrow{\simeq} \mathcal{W}_X^{\mathrm{op}},$$

that acts on $\Gamma(X, \mathcal{W}_X(1))$ by exchanging $\mathbf{X}_{ij}^{(m)}$ and $\mathbf{Y}_{ij}^{(m)}$ for all i, j, m.

If $a \in \mathcal{W}_X$ and $g \in G$ then $g \cdot \phi(a) = \phi(g^{-1} \cdot a)$, so ϕ restricts to an isomorphism $\mathcal{W}_X^G \cong (\mathcal{W}_X^{\mathrm{op}})^G$. Fix some choice of quantised moment map, $\hat{\tau}$, and define the algebra

$$\begin{aligned} D^G_{\hat{\tau},\chi}(W) &:= \Gamma\left(\mathcal{W}_{X,\chi}\right)^{F,G} \\ &= \Gamma\left(\frac{\mathcal{W}_X}{\mathcal{W}_X\langle \hat{\tau}(A) - \chi(A) \,|\, A \in \mathfrak{g}\rangle}\right)^{F,G} \\ &\cong \left(\frac{D_V}{D_V\langle \hat{\tau}(A) - \chi(A) \,|\, A \in \mathfrak{g}\rangle}\right)^{G}. \end{aligned}$$

The deformation quantisations of X are of the form

$$\mathcal{W}^\beta := \left(\frac{\mathcal{W}_X}{\mathcal{W}_X\langle \hat{\tau} - \beta\rangle}\right)^G,$$

where $\beta \in (\mathfrak{g}^*)^G$ is some character. By Theorem 5.9.1, these are in bijection with $H^2(Y_\theta, \mathbb{C})$ via the period map. Under this bijection, if some quantisation, $\mathcal{W}$ say, is mapped to $\alpha \in H^2(Y_\theta, \mathbb{C})$ then $\mathcal{W}^{\mathrm{op}}$ is mapped to $-\alpha$. Let the cohomology class corresponding to $\mathcal{W}^\beta$ be denoted $\mathrm{Per}(\mathcal{W}^\beta)$.

Convention. *Adopt the convention that β is chosen so that $\phi(\hat{\tau} - \beta) = \hat{\tau} - \beta$ and call the corresponding quantised moment map*

$$\tau := \hat{\tau} - \beta.$$

[1]When $n = 1$, the parameter is of the form, $\mathbf{c} = (c_1, \ldots, c_{l-1})$.

This choice of β implies that $\phi(\mathcal{W}_X\tau) = \tau\mathcal{W}_X$, which in turn implies that

$$(\mathcal{W}^\beta)^{\mathrm{op}} = \phi\left(\left(\frac{\mathcal{W}_X}{\mathcal{W}_X\langle\tau\rangle}\right)^G\right) \cong \left(\frac{\mathcal{W}_X}{\mathcal{W}_X\langle\tau\rangle}\right)^G = \mathcal{W}^\beta.$$

Therefore, $\mathrm{Per}(\mathcal{W}^\beta) = -\mathrm{Per}(\mathcal{W}^\beta) = 0$.

Example 6.0.1. *Consider $G = \mathbb{C}^\times$ acting on an n-dimensional vector space, V, with weights $a_1, \dots, a_n$ on an eigenbasis $x_1, \dots, x_n$. Extend this to a hamiltonian action on $X = T^*V$ with $X_1, \dots, X_n, Y_1, \dots, Y_n$ coordinate functions as above. Then an arbitrary quantised moment map is of the form $\tau_a(1) = -\sum_i a_i X_i Y_i \hbar^{-1} + a\mathrm{tr}(1) = -\sum_i a_i X_i Y_i \hbar^{-1} + an$ for some constant $a \in \mathbb{C}$. Now, $-\frac{1}{2}\sum_i a_i(X_iY_i\hbar^{-1} + Y_iX_i\hbar^{-1})$ is invariant under ϕ, so set it to equal $\tau(1)$. Thus, $\tau = -\sum_i a_i X_i Y_i \hbar^{-1} - \frac{1}{2}\sum_i a_i = \tau_{-\frac{1}{2n}\sum_i a_i}$.*

It is now necessary to calculate the relationship between the character, χ, used to twist modules in $D^G_{\tau,\chi}(W)$ and the parameter $\mathbf{c}$ for the corresponding spherical Cherednik algebra $U_{\mathbf{c}}(W)$. First, a direct calculation for the case $n = 1$, $l = 2$.

Example 6.0.2. *Let $W = \mu_2$. Define the following G-invariants in D_V,*

$$A := X_0X_1 \qquad B := X_0\partial_0 \qquad C := X_1\partial_1 \qquad D := \partial_0\partial_1 \qquad E := v\partial_v.$$

Then $D^G_{\tau,\chi}$ is generated by A, B, C, D, E and the following relations hold.

$$B - C - E - \tfrac{1}{2} = \chi_0 \qquad\qquad C - B = \chi_1.$$

Let $Y := B + C$, and define

$$H := [A, D] = -Y - 1$$

Then $[H, A] = -2A$ and $[H, D] = 2D$. The final relation is given by

$$\begin{aligned} 4AD &= 4BC \\ &= 4B^2 + 4B\chi_1 \\ &= (Y - \chi_1)^2 + 2(Y - \chi_1)\chi_1 \\ &= Y^2 - \chi_1^2. \end{aligned}$$

Notice that since $E = -\chi_0 - \chi_1 - \frac{1}{2}$ (a scalar), it is redundant as a generator; in removing it from the list of generators, one can see that the algebra is independent of χ_0.

Next, conisder the corresponding spherical Cherednik algebra, $U_c(\mu_2)$. It is generated by the elements

$$E := \tfrac{1}{2}ex^2e \qquad\qquad F := \tfrac{1}{2}ey^2e$$

subject to the following relations. First, the commutation relation from the Cherednik algebra is $[y, x] = 1 - 2c_1 s$. Let

$$H' := [E, F] = -\tfrac{1}{2}e(xy + yx)e,$$

rearranging this gives $exye = -H' - \frac{1}{2}(1 - 2c_1)$ and taking the square gives

$$e(xy)^2e = H'^2 + (1 - 2c_1)H' + \tfrac{1}{4}(1 - 2c_1)^2.$$

On the other hand,

$$exyxye = ex^2y^2e + (1 + 2c_1)exye = 4EF + (1 + 2c_1)exye.$$

Together these give

$$\begin{aligned} 4EF &= H'^2 + (1 - 2c_1)H' + \tfrac{1}{4}(1 - 2c_1)^2 - (1 + 2c_1)exye \\ &= H'^2 + 2H' + \tfrac{3}{4} - c_1 - c_1^2. \end{aligned}$$

The isomorphism between $D^G_{\tau,\chi} \cong U_{c_1}(\mu_2)$ sends

$$A \mapsto E \qquad D \mapsto F \qquad H \mapsto H'.$$

This is a homomorphism provided $\chi_1 = c_1 + \frac{1}{2}$.

This generalises to a case of the following Theorem. For $i = 0, \dots, l-1$ let $I^{(i)} := (0, \dots, 0, I_n, 0, \dots, 0) \in \mathfrak{g}$. Define the characters of $\mathfrak{g}$,

$$\mathrm{tr}^{(i)}(A_0, \dots, A_{l-1}) := \mathrm{tr}(A_i)$$

for $(A_0, \dots, A_{l-1}) \in \mathfrak{g}$. In fact, the set, $\{\mathrm{tr}^{(0)}, \dots, \mathrm{tr}^{(l-1)}\}$, generates the characters $(\mathfrak{g}^*)^G$. Let

$$\{X^{(i)}_{s,t} \,|\, s,t = 1, \dots, n,\ i = 0, \dots, l-1\}$$

be the differential operators which multiply by coordinate functions on V and let $\partial^{(i)}_{s,t}$ be the corresponding partial derivatives.

Let ζ be a primitive l^{th} root of unity. Define a character, $\chi \in (\mathfrak{g}^*)^G$, by

$$\chi := \begin{cases} \left(c_0 + \frac{1}{2}\right)\mathrm{tr}^{(0)} & l = 1, \\ \frac{1}{l}\sum_{t=1}^{l-1}\left(1 - 2\sum_{k=1}^{l-1}\zeta^{kt}c_k\right)\mathrm{tr}^{(t)} & n = 1, \\ \left(c_0 + \frac{1}{2} + \frac{1}{l}\left(-1 + l - 2\sum_{i=1}^{l-1}c_i\right)\right)\mathrm{tr}^{(0)} + \frac{1}{l}\sum_{t=1}^{l-1}\left(-1 - 2\sum_{k=1}^{l-1}\zeta^{kt}c_k\right)\mathrm{tr}^{(t)} & l > 1,\ n > 1. \end{cases} \tag{6.1}$$

The following theorem is a combination of the results of [Gor06], [BK12] and [GGS09]. My contribution is a comparison of the differing conventions which is then used to derive a general formula.

Theorem 6.0.1. *(Gordon, Bellamy–Kuwabara, Ginzburg–Gordon–Stafford) The following algebras are isomorphic.*

$$D^G_{\tau,\chi}(W) \cong U_{\mathbf{c}}(W).$$

Proof. Suppose $l = 1$, so that $G = \mathrm{GL}_n(\mathbb{C})$ and $V = \mathfrak{g} \times U$, where $U = \mathbb{C}^n$. Let $x_1, \dots, x_n$ be coordinate functions on U and, for each $i = 1, \dots, n$, let ∂_i be the partial derivative with respect to x_i. In [GGS09], they define τ to be the quantised moment map corresponding to the action of $\mathfrak{g}$ on $\mathbb{C}[V]$ via derivations. Call the quantised moment map that they use $\hat{\tau}$. Let I_n denote the identity matrix in $\mathfrak{g}$. This acts on $\mathbb{C}[V]$ by the derivation $-\sum_i x_i\partial_i$; so $\hat{\tau}(I_n) = -\sum_i x_i\partial_i$.

For $g \in G$ and $(M; v) \in V = \mathrm{Mat}_n(\mathbb{C}) \times \mathbb{C}^n$, $g \cdot (M; v) = (gMg^{-1}; gv)$. Therefore, if g lies in the centre of G it acts trivially on $\mathrm{Mat}_n(\mathbb{C}) \times \{0\} \subset V$. It follows that the differential of the action of any scalar matrix acts by zero on $\mathrm{Mat}_n(\mathbb{C}) \times \{0\}$ and so the corresponding vector field given by τ along this subvariety must be zero. For this reason, Example 6.0.1 in the special case $a_1 = \cdots = a_n = 1$ must agree with $\tau(I_n)$. Thus

$$\tau(I_n) = \tau_{-\frac{1}{2n}\sum_i 1}(1) = -\sum_i x_i\partial_i - \frac{1}{2}\mathrm{tr}(I_n) = \hat{\tau}(I_n) - \frac{1}{2}\mathrm{tr}(I_n).$$

Because τ and $\hat{\tau} - \frac{1}{2}\mathrm{tr}$ agree at $I_n \in \mathfrak{g}$ and differ by a character they must be equal. Now [GGS09, Theorem 2.8] gives

$$D^G_{\tau,\chi} = \left(\frac{D_V}{D_V\langle \tau(A) - \chi(A) \,|\, A \in \mathfrak{g}\rangle}\right)^G = \left(\frac{D_V}{D_V\langle \hat{\tau}(A) - (\chi + \frac{1}{2})(A)\rangle}\right)^G \cong D^G_{\hat{\tau},\chi+\frac{1}{2}} \cong U_{\chi-\frac{1}{2}}.$$

Therefore, setting $\chi = c_0 + \frac{1}{2}$ gives the required result.

Suppose $n = 1$. In this case the first component of the character doesn't contribute to twisting $\mathcal{W}$-modules in the sense that if χ and χ' differ by $\mathrm{tr}^{(0)}$ then $D^G_{\tau,\chi}(W) = D^G_{\tau,\chi'}(W)$. Using the convention $\tau^{\mathrm{op}} = \tau$, the quantised moment map is

$$\begin{aligned}\tau(I^{(j)}) &= \tfrac{1}{2}(X^{(j)}\partial^{(j)} + \partial^{(j)}X^{(j)}) - \tfrac{1}{2}(X^{(j-1)}\partial^{(j-1)} + \partial^{(j-1)}X^{(j-1)}) - \tfrac{1}{2}(v\partial_v + \partial_v v)\delta_{j,0} \\ &= X^{(j)}\partial^{(j)} - X^{(j-1)}\partial^{(j-1)} - (v\partial_v + \tfrac{1}{2})\delta_{j,0},\end{aligned}$$

where $j = 0, \dots, l-1$. Let $\chi = \sum \chi_i \mathrm{tr}^{(i)}$, an arbitrary character. Then summing $\tau(I^{(j)}) - \chi_j$ over all j gives

$$v\partial_v + \sum_{i=0}^{l-1} \chi_i + \tfrac{1}{2} \in \langle \tau(A) - \chi(A) \,|\, A \in \mathfrak{g} \rangle.$$

For $i = 0, \dots, l-1$, let $B_i := X^{(i)}\partial^{(i)}$ and let $C := X^{(0)} \cdots X^{(l-1)}$ and $D := \partial^{(0)} \cdots \partial^{(l-1)}$. Then $B_0, \dots, B_{l-1}, C, D$ and $v\partial_v$ generate D_V^G and they satisfy the relation $B_0 \cdots B_{l-1} = CD$. Therefore, $B_0, \dots, B_{l-1}, C, D$ generate U_χ and the relations are

$$\begin{aligned} B_0 \cdots B_{l-1} &= CD, \\ B_1 - B_0 &= \chi_1, \\ &\vdots \\ B_{l-1} - B_{l-2} &= \chi_{l-1}. \end{aligned}$$

Now consider another algebra, A_χ, constructed in a different way. Define

$$V' := \mathrm{Rep}(Q^l, \delta),$$

the representations of the unframed quiver. The corresponding quantised moment map is

$$\tau'(I^{(j)}) = X^{(j)}\partial^{(j)} - X^{(j-1)}\partial^{(j-1)},$$

where $j = 0, \dots, l-1$. Note that this is $(-)^{\mathrm{op}}$-invariant. Let

$$A_\chi := \left(\frac{D_{V'}}{D_{V'}\langle \tau'(A) - \chi(A) \,|\, A \in \mathfrak{g} \rangle} \right)^G.$$

Now $B_0, \dots, B_{l-1}, C, D$ generate A_χ and the relations are

$$\begin{aligned} B_0 \cdots B_{l-1} &= CD, \\ B_0 - B_{l-1} &= \chi_0, \\ B_1 - B_0 &= \chi_1, \\ &\vdots \\ B_{l-1} - B_{l-2} &= \chi_{l-1}. \end{aligned}$$

Therefore, mapping χ_0 to $-\sum_{i=1}^{l-1} \chi_i$ gives an isomorphism $A_\chi \cong D^G_{\tau,\chi}$. Reducing to the algebra, A_χ, associated to the unframed quiver allows one to factor out the action of a copy of $\mathbb{C}^*$ in G.

Now consider the results of [BK12, Section 6.5]. They choose a basis $\langle u_1, \dots, u_l \rangle = V$ so that $(\lambda_1, \dots, \lambda_l) \in G$ acts on $(u_1, \dots, u_l)$ by

$$(\lambda_1\lambda_l^{-1}u_1, \dots, \lambda_l\lambda_{l-1}^{-1}u_l).$$

They also choose the basis $\langle v_1, \dots, v_l \rangle = \mathbb{X}(G)$, so that an arbitrary character is written $\phi = \sum_{i=1}^{l} \phi_i v_i$ for some $\phi_1, \dots, \phi_l$. They then factor out by the diagonal action of $\mathbb{C}^*$, let this group be denoted $\hat{G}$. The sublattice of characters such that $\sum_{i=1}^{l} \phi_i = 0$ gives a basis for $\mathbb{X}(\hat{G})$. They choose a new basis $w_i := v_i - v_{i+1}$ for $i = 1, \dots, l-1$ so that a general character is written $\phi = \sum_{i=1}^{l} \hat{\chi}_i w_i$ where $\hat{\chi}_i := \sum_{j=1}^{i} \phi_i$. The result is that the parameters $\hat{\chi}_1, \dots, \hat{\chi}_{l-1}$ are related to the hyperplane parameters by the formula

$$\hat{\chi}_i = h_i - h_0 + \frac{i-l}{l}$$

as $i = 1, \ldots, l-1$.

This is converted into the notation used in this thesis as follows. Identifying arbitrary elements $(t_0, \ldots, t_{l-1})$ and $(\lambda_1, \ldots, \lambda_l)$ by $t_i \leftrightarrow \lambda_{i+1}$ gives a G-equivariant map between the vector spaces denoted V by mapping $X_i \leftrightarrow u_{i+2}$, where the subscripts are taken modulo l. An arbitrary character $\chi = \sum_{i=0}^{l-1} \chi_i \mathrm{tr}^{(i)}$ is identified with ϕ by $\chi_i = \phi_{i+1}$ for $i = 0, \ldots, l-1$ and this gives

$$\chi_i = \hat{\chi}_{i+1} - \hat{\chi}_i$$

for $i = 0, \ldots, l-2$. Together this gives

$$\chi_i = h_{i+1} - h_i + \frac{1}{l} - \delta_{i,0}$$

for $i = 0, \ldots, l-2$, and under the isomorphism between A_χ and $D^G_{\tau,\chi}$ this gives

$$\chi_{l-1} = -\sum_{i=0}^{l-2} \chi_i = h_0 - h_{l-1} + \tfrac{1}{l}.$$

Converting the h's to k's by the formula

$$h_{i+1} - h_i = k_{1-i} - k_{-i},$$

for all $i = 0, \ldots, l-1$ gives

$$\chi_i = k_{1-i} - k_{-i} + \tfrac{1}{l}$$

for $i = 1, \ldots, l-1$.

Now, in order to convert the hyperplane paramaters to reflection parameters one uses the formula

$$k_{i+1} - k_i = \frac{-2}{l} \sum_{t=1}^{l-1} \zeta^{-it} c_t$$

for all $i = 1, \ldots, l-1$. The result is

$$\chi_i = \frac{1}{l}\left(1 - 2\sum_{t=1}^{l-1} \zeta^{it} c_t\right).$$

Now let $l > 1$ and $n > 1$, so that the results of [Gor06] apply. Let $\hat{\tau}$ be the quantised moment map chosen in that paper. As a consequence of having chosen $\hat{\tau}$ to agree with a paper of Oblomkov, in that paper it is defined differently: as the differential of the G-action on V—the negative of that in [GGS09] which is defined as the differential of the action of G on $\mathbb{C}[V]$. It follows that $-\tau$ and $\hat{\tau}$ differ by a character of $\mathfrak{g}$.

$$-\hat{\tau}(I^{(i)}) = \begin{cases} -\sum_{s,t} X^{(i-1)}_{s,t} \partial^{(i-1)}_{s,t} + \sum_{s,t} X^{(i)}_{s,t} \partial^{(i)}_{s,t} & \text{when } i \neq 0, \\ -\sum_{s,t} X^{(-1)}_{s,t} \partial^{(-1)}_{s,t} + \sum_{s,t} X^{(0)}_{s,t} \partial^{(0)}_{s,t} - \sum_j x_j \partial_j & \text{when } i = 0. \end{cases}$$

Comparing this with τ gives

$$\tau(I^{(i)}) = \begin{cases} -\hat{\tau}(I^{(i)}) & \text{when } i \neq 0, \\ -\hat{\tau}(I^{(0)}) - \frac{1}{2}\mathrm{tr}(I_n) & \text{when } i = 0; \end{cases}$$

so that $\tau = -\hat{\tau} - \frac{1}{2}\mathrm{tr}^{(0)}$.

Comparing $D^G_{\tau,\chi}$ with $D^G_{\hat{\tau},\chi_{c,k}}$ gives

$$\begin{aligned} D^G_{\tau,\chi} &\cong \left(\frac{D_V}{D_V\langle \tau(A) - \chi(A) \mid A \in \mathfrak{g}\rangle}\right)^G \\ &\cong \left(\frac{D_V}{D_V\langle -\hat{\tau}(A) - (\chi + \frac{1}{2}\mathrm{tr}^{(0)})(A) \mid A \in \mathfrak{g}\rangle}\right)^G \\ &\cong \left(\frac{D_V}{D_V\langle \hat{\tau}(A) - (-(\chi + \frac{1}{2}\mathrm{tr}^{(0)}))(A) \mid A \in \mathfrak{g}\rangle}\right)^G \\ &\cong D^G_{\hat{\tau},-(\chi+\frac{1}{2}\mathrm{tr}^{(0)})}. \end{aligned}$$

In [Gor06, Theorem 3.13], he proves that $D^C_{\hat{\tau},\chi_{k,c}} \cong U_{\mathbf{c}}$, where the character, $\chi_{k,c}$, is defined by

$$\chi_{k,c} := (k + C_0)\mathrm{tr}^{(0)} + \sum_{i=1}^{l-1} C_i \mathrm{tr}^{(i)},$$

and the C_i's and k are related to $\mathbf{c}$ using Table A:

$$k = -c_0 \qquad\qquad C_i = \frac{1}{l}\left(1 + 2\sum_{t=1}^{l-1} \zeta^{it} c_t\right) - \delta_{i,0}$$

for $i = 0, \dots, l-1$.

It follows that

$$\chi_0 = \tfrac{1}{2} - k - C_0 \qquad\qquad \chi_i = -C_i$$

for all $i = 1, \dots, l-1$. Together these give Equation 6.1. □

The parameter space for the spherical Cherednik algebra can now be thought of as $(\mathfrak{g}^*)^G$. Let U_χ denote the spherical rational Cherednik algebra $U_{\mathbf{c}}$ where χ and $\mathbf{c}$ are related as above.

6.1 Quantum Hamiltonian Reduction

Keep the same notation as above. That is, X is the affine plane of quiver representations for the quiver correponding to $W = G(l, 1, n)$ with its associated action of G.

The functor

$$\mathbb{H} := \Gamma(X, -)^{F,G} \colon \left(\widetilde{\mathcal{W}}_X, F, G\right)^{\mathrm{good}}_\chi \longrightarrow {}_{U_\chi}\mathbf{mod}$$

that takes F and G-invariant global sections is called **quantum hamiltonian reduction**. Because X is affine and F and G are reductive, $\mathbb{H}$ is exact.

Let $\mathfrak{X}_\theta = \mu^{-1}(0) \cap X^{\mathrm{ss}}_\theta$, the set of semistable points satisfying the ADHM equation. Let $\mathrm{Res}\colon \mathcal{W}_X \longrightarrow \mathcal{W}_{X^{\mathrm{ss}}_\theta}$ be the restriction of sheaves; it is exact because $X^{\mathrm{ss}}_\theta \subseteq X$ is open. It induces an exact functor

$$\mathrm{Res}\colon \left(\widetilde{\mathcal{W}}_X, F, G\right)^{\mathrm{good}}_\chi \longrightarrow \left(\widetilde{\mathcal{W}}_{X^{\mathrm{ss}}_\theta}, F, G\right)^{\mathrm{good}}_\chi.$$

Let $\pi\colon \mathfrak{X}_\theta \longrightarrow Y_\theta$ be the GIT quotient defined in Proposition 5.4.3. It is a resolution of singularities and because θ is not on a GIT wall, by definition, $X^{\mathrm{ss}} = X^{\mathrm{s}}$; therefore, the stabiliser of any point is trivial so G acts freely. Recall the definition of $\mathbb{E}$ from Proposition 5.9.1.

By Proposition 5.8.2, Hartogs' Extension Theorem, applied to an affine normal variety with a closed subvariety of codimension greater than two, gives

$$\Gamma(X, \widetilde{\mathcal{W}}_X)^{F,G} \cong \Gamma(X^{\mathrm{ss}}, \widetilde{\mathcal{W}}_{X^{\mathrm{ss}}})^{F,G}.$$

Because $\mathbb{E}$ is an equivalence, taking F-invariants of global sections gives a functor

$$\mathbb{T} := \Gamma(X, -)^F \colon \left(\widetilde{\mathcal{W}}_{Y_\theta}, F\right)^{\text{good}} \longrightarrow {}_{U_\chi}\mathbf{mod}.$$

Together, these functors fit into the following (not necessarily commutative) diagram.

$$\begin{array}{ccc}
(\widetilde{\mathcal{W}}_X, F, G)^{\text{good}}_\chi & & \\
\downarrow{\scriptstyle \text{Res}} & \searrow{\scriptstyle \mathbb{H}} & \\
(\widetilde{\mathcal{W}}_{X^{\text{ss}}_\theta}, F, G)^{\text{good}}_\chi & & \\
\downarrow{\scriptstyle \mathbb{E}} & & \\
(\widetilde{\mathcal{W}}_{Y_\theta}, F)^{\text{good}} & \xrightarrow{\ \mathbb{T}\ } & {}_{U_\chi}\mathbf{mod}
\end{array} \tag{6.2}$$

In [BPW12], when $\mathbb{T}$ induces a derived equivalence they say that **derived localisation holds for** χ. When $\mathbb{T}$ is an equivalence of abelian categories they say that **localisation holds for** χ.

Definition 6.1.1. *Say that a character, $\chi \in (\mathfrak{g}^*)^G$, is **bad for** $\theta \in \mathbb{X}(G)$ if* $\ker(\text{Res}) \not\subseteq \ker(\mathbb{H})$. *Say that χ is **bad** if it is bad for all θ and **good** if it is not bad; that is, there exists some θ such that* $\ker(\text{Res}) \subseteq \ker(\mathbb{H})$.

Recall that a parameter, $\mathbf{c}$, for the rational Cherednik algebra $H_{\mathbf{c}}(W)$ is called spherical if $H_{\mathbf{c}}(W)$ is Morita equivalent to $U_{\mathbf{c}}(W)$ and that this condition is equivalent to $U_{\mathbf{c}}(W)$ having finite global dimension; see Theorem 1.4.1. If a character χ is good for some θ and the corresponding parameter $\mathbf{c}$ is spherical then there are several nice consequences. The proof of the theorem below requires the following result of McGerty–Nevins.

Theorem 6.1.1. *(McGerty–Nevins) When U_χ has finite global dimension derived localisation holds for χ.*

Proof. This is [MN11, Corollary 7.5] and [MN11, Lemma 3.9], where the functor $\mathbb{T}$ is written f_*. □

Theorem 6.1.2. *Suppose that $\chi \in (\mathfrak{g}^*)^G$ is good for $\theta \in \mathbb{X}(G)$ and that the corresponding value of* $\mathbf{c}$ *given by Theorem 6.0.1 is spherical. Then*

(i) Diagram 6.2 commutes,

(ii) localisation holds for χ,

(iii) the kernel of $\mathbb{H}$ is precisely those sheaves supported on X^{us}, that is,

$$\ker \mathbb{H} = \ker \text{Res}.$$

Proof. First, if $\ker\text{Res} \subseteq \ker\mathbb{H}$, then $\mathbb{H} = \Gamma(X, -)^{F,G}$ factors through Res as

$$\mathbb{H} = \Gamma(X^{\text{ss}}, -)^{F,G} \circ \text{Res} = \mathbb{T} \circ \mathbb{E} \circ \text{Res}.$$

Secondly, I claim that if Diagram 6.2 commutes then the functor $\mathbb{T}$ is exact. Since $\mathbb{T}$ is the F-invariant global sections functor on the space Y_θ, it is automatically left exact; so it is sufficient to show that it is right exact.

The restriction functor, Res, has a left adjoint, $\text{Res}_!$, such that $\text{Res} \circ \text{Res}_! \cong \text{id}$. Indeed, define the **Kirwan functor**,

$$\kappa \colon \left(\widetilde{\mathcal{W}}_X, F, G\right)^{\text{good}} \longrightarrow \left(\widetilde{\mathcal{W}}_{Y_\theta}, F\right)^{\text{good}},$$

by

$$\kappa(\mathcal{N}) := p_* \mathcal{H}om\left(\mathcal{L}_\chi, \mathcal{N}|_{X^{\text{ss}}}\right),$$

where $\mathcal{L}_\chi$ is the twisted sheaf defined in Section 5.9.6. In [BPW12, Lemma 5.18], they show that it has a left adjoint $\kappa_!$ such that $\kappa \circ \kappa_! \cong \mathrm{id}$.

Recall that the forgetful functor $I_\chi \colon (\widetilde{\mathcal{W}}, F, G)^{\mathsf{good}}_\chi \longrightarrow (\widetilde{\mathcal{W}}, F, G)^{\mathsf{good}}$ also has a left adjoint: Φ_χ, defined in Section 5.9.6, and it satisfies $I_\chi \circ \Phi_\chi \cong \mathrm{id}$. Now define

$$\mathrm{Res}_! := \Phi_\chi \circ \kappa_! \circ \mathbb{E}.$$

Being the composition of two left adjoints and an equivalence, it is a left adjoint and, because Diagram 6.2 commutes,

$$\mathrm{Res} \circ \mathrm{Res}_! = \mathrm{Res} \circ \Phi_\chi \circ \kappa_! \circ \mathbb{E} \cong \mathbb{E}^{-1} \circ \kappa \circ \kappa_! \circ \mathbb{E} \cong \mathrm{id}.$$

Now, $\mathbb{T} \circ \mathbb{E}$ is right exact because

$$\mathbb{T} \circ \mathbb{E} \cong \mathbb{T} \circ \mathbb{E} \circ \mathrm{Res} \circ \mathrm{Res}_! = \mathbb{H} \circ \mathrm{Res}_!,$$

the composition of two right exact functors. It follows that $\mathbb{T} \circ \mathbb{E}$, and therefore $\mathbb{T}$, is right exact. This completes the proof of the claim.

The exactness of $\mathbb{T}$ now implies that $\mathbb{T}$ is an equivalence. Indeed, let $\mathbb{S} := \widetilde{\mathcal{W}}_{Y_\theta} \otimes_{U_\chi} -$, the left adjoint of $\mathbb{T} = \Gamma(Y_\theta, -)^F$. Being a left adjoint, $\mathbb{S}$ is right exact. Let $R\mathbb{T}$ denote the right derived functor of $\mathbb{T}$ and $L\mathbb{S}$ the left derived functor of $\mathbb{S}$. Because $\mathbb{T}$ is exact, as derived functors, $R\mathbb{T} = \mathbb{T}$. Let $\mathcal{M} \in (\widetilde{\mathcal{W}}_{Y_\theta}, F)^{\mathsf{good}}$. By Theorem 6.1.1, $R\mathbb{T}$ and $L\mathbb{S}$ are equivalences so

$$L\mathbb{S} \circ R\mathbb{T}\mathcal{M} = \widetilde{\mathcal{W}}_{Y_\theta} \overset{L}{\otimes}_{U_\chi} \Gamma(Y_\theta, \mathcal{M})^F \longrightarrow \mathcal{M}$$

is a quasi-isomorphism; therefore,

$$\mathbb{S} \circ \mathbb{T}\mathcal{M} = \widetilde{\mathcal{W}}_{Y_\theta} \otimes_{U_\chi} \Gamma(Y_\theta, \mathcal{M})^F \longrightarrow \mathcal{M}$$

is surjective.

Now, $\mathbb{T}\mathcal{M}$ is finitely generated, so choosing m generators gives a surjective map

$$U_\chi^{\oplus m} \longrightarrow \mathbb{T}\mathcal{M}.$$

Applying the right exact functor $\mathbb{S}$ gives a surjection which, when composed with the surjection above, gives a surjection

$$\mathbb{S}U_\chi^{\oplus m} \cong \widetilde{\mathcal{W}}_{Y_\theta}^{\oplus m} \longrightarrow \mathbb{S} \circ \mathbb{T}\mathcal{M} \longrightarrow \mathcal{M}.$$

This shows that any module is a quotient of some power of $\widetilde{\mathcal{W}}_{Y_\theta}$, in particular, it can be applied to the kernel to give an exact sequence,

$$\begin{array}{ccccc}
\widetilde{\mathcal{W}}_{Y_\theta}^{\oplus m'} & \longrightarrow & \widetilde{\mathcal{W}}_{Y_\theta}^{\oplus m} & \twoheadrightarrow & \mathcal{M} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S} \circ \mathbb{T}\widetilde{\mathcal{W}}_{Y_\theta}^{\oplus m'} & \longrightarrow & \mathbb{S} \circ \mathbb{T}\widetilde{\mathcal{W}}_{Y_\theta}^{\oplus m} & \twoheadrightarrow & \mathbb{S} \circ \mathbb{T}\mathcal{M},
\end{array}$$

where the vertical maps are from the natural transformation between $\mathbb{S} \circ \mathbb{T}$ and the identity functor. The first two of these are isomorphisms induced from

$$\widetilde{\mathcal{W}}_{Y_\theta} \otimes_{U_\chi} U_\chi \cong \widetilde{\mathcal{W}}_{Y_\theta},$$

which implies the third vertical map is an isomorphism. Now $\mathbb{S}$ and $\mathbb{T}$ are inverse equivalences, so localisation holds for χ.

Finally, since $\mathbb{T} \circ \mathbb{E} = \Gamma(X^{ss}, -)^{F,G}$ is now an equivalence it follows that

$$\ker \mathrm{Res} = \ker \mathbb{T} \circ \mathbb{E} \circ \mathrm{Res} = \ker \mathbb{H}.$$

□

In order to describe which $\chi \in (\mathfrak{g}^*)^G$ are good, the next section will be spent developing criteria for a parameter to be bad in various cases.

6.2 Case 1: A one-dimensional Torus acting on the Cotangent Bundle of a Vector Space

Let V be an n-dimensional complex vector space with a $G = \mathbb{C}_m$-action whose weights are $(a_1, \dots, a_n) \in \mathbb{Z}^n$ on an eigenbasis, $\{x_1, \dots, x_n\}$, of V. Let $X = T^*V \cong V \times V^*$ and let $\{y_1, \dots, y_n\}$ be a basis for V^* which is dual to $\{x_1, \dots, x_n\}$. Extend the action of G to X so that y_i has weight $-a_i$ for each i. This action is hamiltonian with respect to the symplectic form $\sum_{i=1}^n x_i \wedge y_i$. Let $F = \mathbb{C}_m$ act with weight -1 on V and V^* so that $\deg_F X_i = \deg_F Y_i = 1$ for all i.

Let $\mathcal{L}$ be the trivial line bundle on V with a linearisation corresponding to the character $\theta \in \mathbb{Z} \cong \mathbb{X}(G)$; that is, a weight one G-equivariant structure defined by $g \cdot ((v, w), l) := (g \cdot v, g \cdot w, g^\theta l)$ for $g \in G$. There is only one GIT wall—the point 0—in the space of parameters, $\mathbb{Q}$. Let $\theta \neq 0$.

The one-parameter subgroups are also in bijection with $\mathbb{Z}$ so that, for $\lambda \in \mathbb{Z} \cong \mathbb{Y}(G)$ and $t \in G$,

$$\lambda(t) \cdot (x_1, \dots, x_n, y_1, \dots, y_n) = \left(t^{\lambda a_1} x_1, \dots, t^{\lambda a_n} x_n, t^{-\lambda a_1} y_1, \dots, t^{-\lambda a_n} y_n\right).$$

Also, for all $p \in X$ such that $\mu^\theta(p, \lambda) \neq \infty$, $\mu^\theta(p, \lambda) = \lambda\theta$. Therefore, for point p to be unstable $\lim_{t\to 0} \lambda(t) \cdot p$ must exist, where λ has the same sign as $-\theta$. This gives

$$X^{\mathrm{us}} = \left\{ (x_1, \dots, x_n, y_1, \dots, y_n) \in X \;\middle|\; \begin{cases} x_i = 0 & \text{if } a_i\theta > 0 \\ y_i = 0 & \text{if } a_i\theta < 0 \end{cases} \right\}.$$

The stratification of X^{us} is trivial in either case, with one stratum only: $S_\lambda = X^{\mathrm{us}}$ corresponding to the one-parameter subgroup $\lambda = -\frac{\theta}{|\theta|}$.

Let $j : X^{\mathrm{ss}} \hookrightarrow X$ be the open embedding. The mapping defined by

$$\mu_X \colon X \cong V \times V^* \longrightarrow \mathfrak{g}^*; \quad (\mathbf{v}, \mathbf{w}) \mapsto -\sum_{i=1}^n a_i v_i w_i,$$

gives a moment map for the hamiltonian action of G.

Let $\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ be the coordinate functions on X. The mapping

$$\tau(1) = \frac{1}{2} \sum_{i=1}^n -a_i (X_i Y_i + Y_i X_i) \hbar^{-1},$$

gives a quantised moment map $\tau \colon \mathfrak{g} \to \widetilde{\mathcal{W}}_X(1)$. Notice that this particular choice of τ is invariant under the anti-isomorphism, $(-)^{\mathrm{op}}$ (defined in Section 1.2.4), and so satisfies the particular convention used in this thesis.

Given $\chi \in (\mathfrak{g}^*)^G$, define the $\widetilde{\mathcal{W}}_X$-modules

$$\widetilde{\mathcal{W}}_{X,\chi}[d] := \frac{\widetilde{\mathcal{W}}_X}{\widetilde{\mathcal{W}}_X \langle \tau(1) - \chi(1) - d \rangle},$$

and set $\widetilde{\mathcal{W}}_{X,\chi} := \widetilde{\mathcal{W}}_{X,\chi}[0]$.

Lemma 6.2.1. *For each $d \in \mathbb{Z}$, there are F and G actions on the module $\widetilde{\mathcal{W}}_{X,\chi}[d]$ so it lies in $\left(\widetilde{\mathcal{W}}_X, F, G\right)_\chi^{\mathrm{good}}$.*

Proof. Since the category of good $\widetilde{\mathcal{W}}_X$-modules is abelian, these are good modules. Define an action of G on $\widetilde{\mathcal{W}}_{X,\chi}[d]$ by letting $t \in G$ act on the coset $[1] := 1 + \widetilde{\mathcal{W}}_X \langle \tau(1) - \chi(1) - d \rangle$ by the rule

$t \cdot [1] := t^d[1]$. This makes sense because, by Property (iii) in Definition 5.9.5, τ is G-equivariant. For the differential of the action of G, [1] is an eigenvector with weight d so that, for $1 \in \mathfrak{g}$ and $u \in \widetilde{\mathcal{W}}_{X,\chi}[d]$, $\tau(1)u - \chi(1)u = du = 1 \cdot u$. This means that $\widetilde{\mathcal{W}}_{X,\chi}[d]$ is χ-twisted. Finally, the action of F on $\widetilde{\mathcal{W}}_{X,\chi}[d]$ is induced from the F action on $\widetilde{\mathcal{W}}_X$ with X_i and Y_i having degree 1 for all $i = 1, \ldots, n$ and $\hbar$ having degree 2. Note that $\deg_F(\tau(1)) = 0$ so that F acts trivially on $\tau(1) - \chi(1) - d$. □

Proposition 6.2.1. *Suppose that*

$$\chi(1) \notin \begin{cases} \left\{\sum_{i=1}^n |a_i|(n_i + \frac{1}{2}) \,\middle|\, n_i \geq 0\right\} & \text{if } \theta > 0 \\ \left\{-\sum_{i=1}^n |a_i|(n_i + \frac{1}{2}) \,\middle|\, n_i \geq 0\right\} & \text{if } \theta < 0. \end{cases}$$

Then, for all p, the natural homomorphism given by restriction

$$H^p(X, \widetilde{\mathcal{W}}_{X,\chi}[d])^{F,G} \longrightarrow H^p(X^{ss}, j^*\widetilde{\mathcal{W}}_{X,\chi}[d])^{F,G}$$

is an isomorphism.

When one of the weights is ± 1, this condition is equivalent to

$$\chi(1) \notin \begin{cases} \frac{1}{2}\sum_{i=1}^n |a_i| + \mathbb{Z}_{\geq 0} & \text{if } \theta > 0 \\ -\frac{1}{2}\sum_{i=1}^n |a_i| - \mathbb{Z}_{\geq 0} & \text{if } \theta < 0. \end{cases}$$

6.2.1 Proof of Proposition 6.2.1

By [Gro05, Exposé I, Corollaire 2.9], there is a (F, G)-equivariant long exact sequence

$$\cdots \longrightarrow H^i_{X^{us}}(X, \widetilde{\mathcal{W}}_{X,\chi}[d]) \longrightarrow H^i(X, \widetilde{\mathcal{W}}_{X,\chi}[d]) \longrightarrow H^i(X^{ss}, j^*\widetilde{\mathcal{W}}_{X,\chi}[d]) \longrightarrow \cdots;$$

so, in order to prove the proposition, it suffices to calculate when the F, G-equivariant local cohomology of $X^{us} \subset X$ with respect to $\widetilde{\mathcal{W}}_{X,\chi}[d]$ vanishes.

Assume that the weights $a_1, \ldots, a_n$ are strictly positive. Local and Čech cohomology can be used to calculate the groups in the proposition.

Local cohomology for $\widetilde{\mathcal{W}}_X$

Now, [KR08, Lemma 2.12] shows that if $\mathcal{M}$ is a coherent $\widetilde{\mathcal{W}}_X$-module and $\mathcal{M}(0)$ is a lattice for $\mathcal{M}$ then $H^p(X, \mathcal{M}(0)) = 0$ for all $p \geq 1$. Let $M := \mathcal{M}(0)$. Applying this to the long exact sequence in local cohomology gives an exact sequence

$$0 \longrightarrow H^0_{X^{us}}(X, M) \longrightarrow H^0(X, M) \to H^0(X^{ss}, j^*M) \longrightarrow H^1_{X^{us}}(X, M) \longrightarrow 0$$

and isomorphisms

$$H^p(X^{ss}, j^*M) \cong H^{p+1}_{X^{us}}(X, M),$$

for $p \geq 1$. Now there is a certain Čech complex that calculates $H^p(X^{ss}, j^*M)$ for any p. But this now depends on the sign of θ, so consider the two cases separately.

First, let $\theta > 0$ so that $X^{us} = \{0\} \times V^*$ and define the following open covering, $\mathcal{U}$, of the semistable locus. Let $U_i := T^*V \setminus \{x_i = 0\}$ for $i = 1, \ldots, n$, so that $U_{i_0,\ldots,i_s} := \cap_{j=0}^s U_{i_j}$ is affine for each $1 \leq i_0 < \cdots < i_s \leq n$ and $\cup_{i=1}^n U_i = X \setminus X^{us} = X^{ss}$. By [KR08, Lemma 2.12],

$$H^p(U_{i_0,\ldots,i_s}, j^*M) = 0$$

for any $p \geq 1$ and so by [Har77, Exercise III.4.11], the Čech cohomology with respect to the open cover, $\mathcal{U}$, of X^{ss} calculates $H^p(\cup U_i, j^*M) = H^p(X^{ss}, j^*M)$.

It follows that, for $p \geq 1$, $H^{p+1}_{X^{us}}(X, \widetilde{\mathcal{W}}_X(0))$ can be calculated by the Čech complex whose p^{th} term is

$$\prod_{1 \leq i_0 < \cdots < i_p \leq n} \widetilde{\mathcal{W}}_X(0)(U_{i_0,\ldots,i_p}).$$

By definition, $\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-1) \cong \hbar^{-m}\mathcal{O}_X$, a quasi-coherent $\mathcal{O}_X$-module. For $s \geq 0$, there is an exact sequence

$$0 \longrightarrow \hbar^{-m}\mathcal{O}_X \xrightarrow{\hbar^s} \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s-1) \longrightarrow \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s) \longrightarrow 0.$$

Example 5.2.1 shows that $H^p_{X^{us}}(X, \mathcal{O}_X) = 0$ for all $p \neq n$. Applying local cohomology induces exact sequences for all $p \neq n$,

$$H^p_{X^{us}}(\hbar^{-m}\mathcal{O}_X) \longrightarrow H^p_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s-1)\Big) \longrightarrow H^p_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\Big),$$

which implies that $H^p_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s-1)\Big)$ injects into $H^p_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\Big)$ for all $s \geq 0$. When $s = 0$ this implies both are zero, so by induction on s, $H^p_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s-1)\Big) = 0$ for all $s \geq 0$ when $p \neq n$.

When $p = n$ there is an exact sequence,

$$0 \longrightarrow H^n_{X^{us}}(\hbar^{-m}\mathcal{O}_X) \xrightarrow{\hbar^s} H^n_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s-1)\Big) \tag{6.3}$$

$$\xrightarrow{f_{s+1}} H^n_{X^{us}}\Big(\widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\Big) \longrightarrow 0.$$

It follows that

$$H^n_{X^{us}}\Big(X, \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\Big) \cong \hbar^{-m} H^n_{X^{us}}(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathbb{C}[\hbar]/\langle \hbar^s \rangle.$$

Example 5.2.1 then gives

$$H^n_{X^{us}}(X, \mathcal{O}_X) = (X_1^{-1} \cdots X_n^{-1})\mathbb{C}[X_1^{-1}, \ldots, X_n^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[V^*].$$

Using the Mittag–Leffler Condition

I claim that, for all $p \geq 1$,

$$H^p_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)\right) = \varprojlim_s H^p_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\right).$$

For each $s \in \mathbb{Z}$, let $\check{C}^\bullet_s$ be the Čech complex which calculates

$$\check{H}^p\left(\mathcal{U}, \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\right) \cong H^{p-1}_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\right).$$

Then, for all $r \geq 0$, the r^{th} terms of each complex in the sequence have the following inverse limit over $s \in \mathbb{Z}$.

$$\begin{aligned}
\varprojlim_s \check{C}^r_s &= \varprojlim_s \prod_{1 \leq i_0 < \cdots < i_r \leq n} \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)(U_{i_0,\ldots,i_r}) \\
&\cong \prod_{1 \leq i_0 < \cdots < i_r \leq n} \varprojlim_s \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)(U_{i_0,\ldots,i_r}) \\
&\cong \prod_{1 \leq i_0 < \cdots < i_r \leq n} \widetilde{\mathcal{W}}_X(m)(U_{i_0,\ldots,i_r}) \\
&\cong \check{C}^r\left(\mathcal{U}, \widetilde{\mathcal{W}}_X(m)\right);
\end{aligned}$$

these isomorphisms are natural so, as complexes, $\lim_{\overleftarrow{s}} \check{C}^\bullet_s = \check{C}^\bullet\left(\mathcal{U}, \widetilde{\mathcal{W}}_X(m)\right)$.

Now, applying the so-called 'Variant' immediately after the proof of [Wei94, Theorem 3.5.8], gives a surjective map,

$$H^p_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)\right) \longrightarrow \lim_{\overleftarrow{s}} H^p_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\right),$$

whose kernel is $\lim^1_{\leftarrow} H^{p-1}_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)/\widetilde{\mathcal{W}}_X(m-s)\right)$. This kernel is zero provided the sequence of chain complexes, $\check{C}^\bullet_s$, satisfies the Mittag–Leffler condition. This follows from the surjectivity of the maps $\cdot f_s$ in Equation 6.3 when $p = n$ and is trivially true when $p \neq n$. This proves the claim.

Now assume $p = 0$. Suppose f is an (F, G)-invariant global section of $\widetilde{\mathcal{W}}_X(m)$. Applying the symbol map, σ_m, gives a global section of $\mathcal{O}_X$ (multiplied by $\hbar^{-m}$). If $\hbar^m\sigma_m(f)$ is zero when restricted from X to X^{ss} then $\hbar^m\sigma_m(f)$ is zero everywhere on the dense subset $X^{ss} \subseteq X$. Therefore $\hbar^m\sigma_m(f) = 0$ which implies that $f \in \widetilde{\mathcal{W}}_X(m-1)$.

$$H^0_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)\right)^{F,G} = \Gamma\left(X, \ker\left(\widetilde{\mathcal{W}}_X(m) \longrightarrow \widetilde{\mathcal{W}}_{X^{ss}}(m)\right)\right)^{F,G} = 0.$$

Together with the claim, this implies that

$$H^p_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X(m)\right) = \begin{cases} 0 & \text{if } p \neq n, \\ \hbar^{-m}\left((X_1^{-1}\cdots X_n^{-1})\mathbb{C}[X_1^{-1}, \ldots, X_n^{-1}] \otimes \mathbb{C}[V^*]\right)[\![\hbar]\!] & \text{if } p = n. \end{cases}$$

Since $\widetilde{\mathcal{W}}_X = \bigcup_{m\geq 0} \widetilde{\mathcal{W}}_X(m)$, we have, for all p,

$$H^p_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X\right) = H^p_{X^{us}}\left(X, \cup_{m\geq 0}\widetilde{\mathcal{W}}_X(m)\right) \cong \lim_{\overrightarrow{m}} H^p_{X^{us}}\left(\widetilde{\mathcal{W}}_X(m)\right). \tag{6.4}$$

So (6.4) gives

$$H^p_{X^{us}}(X, \widetilde{\mathcal{W}}_X) = \begin{cases} 0 & \text{if } p \neq n, \\ \left((X_1^{-1}\cdots X_n^{-1})\mathbb{C}[X_1^{-1}, \ldots, X_n^{-1}] \otimes \mathbb{C}[V^*]\right)((\hbar)) & \text{if } p = n. \end{cases} \tag{6.5}$$

Moreover, thanks to the description via Čech cohomology, this is (F, G)-equivariant and right multiplication by any $f \in \widetilde{\mathcal{W}}_X$ induces right multiplication by f on $H^n_{X^{us}}\left(X, \widetilde{\mathcal{W}}_X\right)$.

Equivariant structure

Recall the action of G and of F on $\widetilde{\mathcal{W}}_X$, given by assigning the following bi-degrees to each variable.

$$\deg X_t = (-a_t, 1), \qquad \deg Y_t = (a_t, 1), \qquad \deg \hbar = (0, 2).$$

It follows that

$$\deg X_1^{i_1}\cdots X_n^{i_n} Y_1^{j_1}\cdots Y_n^{j_n}\hbar^k = \left(\sum_{t=1}^n a_t(j_t - i_t), \sum_{t=1}^n i_t + j_t + 2k\right).$$

In particular, (6.5) shows that all monomials in $H^n_{X^{us}}(X, \widetilde{\mathcal{W}}_X)$ have positive G-degree. Hence, for fixed integers m and m', the (F, G)-isotypic component, $H^n_{X^{us}}(X, \widetilde{\mathcal{W}}_X)^{m,m'}$ is a finite-dimensional subspace. Note that twisting by $d \in \Upsilon(G)$ shifts the first degree by d.

The cohomology groups, $H^n_{X^{us}}(X, \widetilde{\mathcal{W}}_{X,\chi}[d])^{F,G}$, are calculated using the long exact sequence of local cohomology induced by the short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{W}}_X[d] \xrightarrow{\cdot f} \widetilde{\mathcal{W}}_X[d] \longrightarrow \widetilde{\mathcal{W}}_{X,\chi}[d] \longrightarrow 0\,,$$

where $\cdot f$ denotes right multiplication by $f = \tau(1) - \chi(1) - d$. Since $\deg(\tau(1) - \chi(1)) = (0, 0)$, for any $m, m' \in \mathbb{Z}$, there is an exact sequence

$$0 \longrightarrow H^{n-1}_{X^{us}}(X, \widetilde{\mathcal{W}}_{X,\chi}[d])^{m,m'} \longrightarrow H^{n}_{X^{us}}(X, \widetilde{\mathcal{W}}_{X}[d])^{m,m'} \xrightarrow{\cdot f}$$

$$H^{n}_{X^{us}}(X, \widetilde{\mathcal{W}}_{X}[d])^{m,m'} \longrightarrow H^{n}_{X^{us}}(X, \widetilde{\mathcal{W}}_{X,\chi}[d])^{m,m'} \longrightarrow 0.$$

So $\cdot f$ is an isomorphism if and only if $H^{n-1}_{X^{us}}(\widetilde{\mathcal{W}}_{X,\chi}[d])^{m,m'}$ and $H^{n}_{X^{us}}(\widetilde{\mathcal{W}}_{X,\chi}[d])^{m,m'}$ are both zero. The other local cohomology groups, $H^{p}_{X^{us}}(\widetilde{\mathcal{W}}_{\chi}[d])$, are zero by Equation 6.5.

Suppose that $z := \sum c_{\mathbf{i},\mathbf{j},k} X_1^{i_1} \cdots X_n^{i_n} Y_1^{j_1} \cdots Y_n^{j_n} \hbar^k \in H^n_{X^{us}}(X, \widetilde{\mathcal{W}}_X)^{m,m'}$ belongs to the kernel of $\cdot f$. (Here the i_t's must be strictly negative and the j_t's must be non-negative.) In other words,

$$\begin{aligned}\left(\chi(1) + d + \frac{1}{2}\sum_i a_i\right) z &= \sum_{\mathbf{i},\mathbf{j},k}\sum_{t=1}^{n} c_{\mathbf{i},\mathbf{j},k} X_1^{i_1}\cdots X_n^{i_n} Y_1^{j_1}\cdots Y_n^{j_n}(-a_t X_t Y_t \hbar^{-1})\hbar^k \\ &= -\sum_{\mathbf{i},\mathbf{j},k}\sum_{t=1}^{n} a_t c_{\mathbf{i},\mathbf{j},k} X_1^{i_1}\cdots X_t^{i_t+1}\cdots X_n^{i_n} Y_1^{j_1}\cdots Y_t^{j_t+1}\cdots Y_n^{j_n}\hbar^{k-1} \\ &\quad -\sum_{i,j,k}\sum_{t=1}^{n} a_t j_t c_{\mathbf{i},\mathbf{j},k} X_1^{i_1}\cdots X_n^{i_n} Y_1^{j_1}\cdots Y_n^{j_n}\hbar^k.\end{aligned}$$

Comparing this term-by-term, for all $\mathbf{i}, \mathbf{j}, k$

$$c_{\mathbf{i},\mathbf{j},k}\left(\chi(1) + \sum_{t=1}^{n} a_t(j_t + \tfrac{1}{2}) + d\right) = \sum_{t=1}^{n} -a_t c_{\mathbf{i}-\epsilon_t,\mathbf{j}-\epsilon_t,k+1},$$

where, for $t = 1, \dots, n$, ϵ_t is the n-tuple with a 1 at place t and zeros everywhere else.

Let

$$S := \left\{(\mathbf{i}, \mathbf{j}, k) \,\middle|\, c_{\mathbf{i},\mathbf{j},k} \neq 0\right\}$$

and define a function on S that associates to any triple $(\mathbf{i}, \mathbf{j}, k)$ the sum $\sum_{l=1}^n j_l$. Since each j_l must be non-negative, the function has a global minimum at zero. Now it is possible to pick a minimal member of S with respect to this function. This is a triple $(\mathbf{i}', \mathbf{j}', k')$ such that $c_{\mathbf{i}',\mathbf{j}',k'} \neq 0$ and $c_{\mathbf{i}'-\epsilon_t,\mathbf{j}'-\epsilon_t,k'+1} = 0$ for all $t = 1, \dots, n$. Then, $c_{\mathbf{i}',\mathbf{j}',k'}\left(\chi(1) + \sum_{t=1}^n a_t(j'_t + \frac{1}{2}) + d\right) = 0$, so $\chi(1) + \sum_{t=1}^n a_t(j'_t + \frac{1}{2}) + d = 0$. Therefore, if, for all $j_t \geq 0$, $\chi(1) + \sum_{t=1}^n a_t(j_t + \frac{1}{2}) + d \neq 0$ then $\cdot f$ is injective.

For a monomial to belong to the space of G-invariants it must have G-degree zero; thus $\sum_{t=1}^n -a_t i_t + \sum_{t=1}^n a_t j_t + d = 0$ and the condition becomes $\chi(1) \neq -\sum_{t=1}^n a_t(i_t + \frac{1}{2})$ for all $i_t < 0$. This completes the proof when $\theta > 0$ and the weights are all positive.

When $\theta < 0$ the unstable locus is $V \times \{0\}$, so stratify the semistable locus by open sets, $U_i := T^*V \setminus \{Y_i = 0\}$ for $i = 1, \dots, n$, and set $U_{i_0,\dots,i_s} := \cap_{j=0}^s U_{i_j}$ for $1 \leq i_0 < \cdots < i_s \leq n$. The argument runs the same as before, except that now

$$H^p_{X^{us}}(\widetilde{\mathcal{W}}_X) = \begin{cases} 0 & p \neq n, \\ \left((Y_1^{-1}\cdots Y_n^{-1})\mathbb{C}[Y_1^{-1}, \dots, Y_n^{-1}] \otimes \mathbb{C}[V]\right)((\hbar)) & p = n. \end{cases} \tag{6.6}$$

Suppose that $z := \sum c_{\mathbf{i},\mathbf{j},k} Y_1^{i_1} \cdots Y_n^{i_n} X_1^{j_1} \cdots X_n^{j_n} \hbar^k \in H^{n-1}_{X^{us}}(X, \widetilde{\mathcal{W}}_X)^{m,m'}$ belongs to the kernel of $\cdot f$.

Again, the entries of **i** are negative while those of **j** are non-negative.

$$\left(\chi(1) + d + \frac{1}{2}\sum_i a_i\right) z = \sum_{\mathbf{i},\mathbf{j},k}\sum_{t=1}^n c_{\mathbf{i},\mathbf{j},k} Y_1^{i_1}\cdots Y_n^{i_n} X_1^{j_1}\cdots X_n^{j_n}(-a_t X_t Y_t h^{-1})h^k$$
$$= -\sum_{\mathbf{i},\mathbf{j},k}\sum_{t=1}^n a_t c_{\mathbf{i},\mathbf{j},k} Y_1^{i_1}\cdots Y_t^{i_t+1}\cdots Y_n^{i_n} X_1^{j_1}\cdots X_t^{i_t+1}\cdots X_n^{j_n} h^{k-1}$$
$$+ \sum\sum_{t=1}^n a_t(j_t+1) c_{\mathbf{i},\mathbf{j},k} Y_1^{i_1}\cdots Y_n^{i_n} X_1^{j_1}\cdots X_n^{j_n} h^k.$$

Comparing this term-by-term, for all $\mathbf{i}, \mathbf{j}, k$ we have

$$c_{\mathbf{i},\mathbf{j},k}\left(\chi(1) - \sum_{t=1}^n a_t(j_t + \tfrac{1}{2}) + d\right) = \sum_{t=1}^n -a_t c_{\mathbf{i}-\epsilon_t,\mathbf{j}-\epsilon_t,k+1}.$$

Therefore, if, for all $j_t < 0$, $\chi(1) - \sum_{t=1}^n a_t(j_t + \frac{1}{2}) + d \neq 0$ then $\cdot f$ is injective. Now z is G-invariant only if $\sum_{t=1}^n a_t i_t - \sum_{t=1}^n a_t j_t + d = 0$, so the condition becomes $\chi(1) \neq \sum_{t=1}^n a_t(i_t + \frac{1}{2})$ for all $i_t < 0$. This completes the proof when the weights are all positive.

Now suppose that $a_1, \ldots, a_n$ are arbitrary, non-zero integers. For $i = 1, \ldots, n$, define the vectors

$$s_i := \begin{cases} x_i & \text{if } a_i > 0 \\ y_i & \text{if } a_i < 0 \end{cases} \qquad t_i := \begin{cases} y_i & \text{if } a_i > 0 \\ -x_i & \text{if } a_i < 0 \end{cases},$$

and let $S_1, \ldots, S_n, T_1, \ldots, T_n$ be the corresponding coordinate functions. Let $\tilde{V} := \langle s_1, \ldots, s_n\rangle_{\mathbb{C}}$ so that $\tilde{V}^* = \langle t_1, \ldots, t_n\rangle_{\mathbb{C}}$ is a dual basis. These are related to X by a simple change of basis: $X = \tilde{V} \times \tilde{V}^*$. The unstable locus of T^*V is either $\{0\} \times \tilde{V}^*$ or $\tilde{V} \times \{0\}$ depending on whether θ is positive or negative respectively. Define, for each $i = 1, \ldots, n$, $b_i = |a_i|$. Then

$$\tau(1) = -\frac{1}{2}\sum_{i\,|\,a_i>0} a_i(X_iY_i + Y_iX_i)\hbar^{-1} - \frac{1}{2}\sum_{i\,|\,a_i<0} a_i(X_iY_i + Y_iX_i)\hbar^{-1},$$
$$= -\frac{1}{2}\sum_{i\,|\,a_i>0} a_i(S_iT_i + T_iS_i)\hbar^{-1} + \frac{1}{2}\sum_{i\,|\,a_i<0} a_i(S_iT_i + T_iS_i)\hbar^{-1},$$
$$= -\frac{1}{2}\sum_{i=1}^n b_i(S_iT_i + T_iS_i)\hbar^{-1}$$

This is precisely the case considered above, once the S_i's, T_i's and b_i's have been renamed.

It remains to consider case when some of the weights are zero. If all the weights are zero, the action is trivial, every point is unstable and so $\operatorname{codim}_X X^{\mathrm{us}} = 0$ which contradicts the assumption at the beginning of Section 6.1. So, suppose that the a_i are now arbitrary integers, with at least one zero and one non-zero. Let V_0 be the subspace of V spanned by fixed eigenvectors. Let V' be the complement of V_0 in V. By the argument above one can decompose T^*V' into a product $V_+ \times V_+^*$ so that V_+ is spanned by eigenvectors with a strictly positive weight. Thus $X = (T^*V_0) \times (T^*V_+)$ and

$$X^{\mathrm{ss}} = (T^*V_0) \times (T^*V_+)^{\mathrm{ss}}.$$

Rename the basis of V chosen at the beginning of the section so that $x_1, \ldots, x_c$ span V_0 and $x_{c+1}, \ldots, x_n$ span V_+. Rename the y_i's to be dual to the x_i's and rename $a_1, \ldots, a_n$ to be the associated eigenvalues of G acting on the x_i's. This gives $a_1 = \cdots = a_c = 0$ and $a_{c+1}, \ldots, a_n > 0$. Now, running through the argument of the previous section, stratifying the semistable locus by open sets $X^{ss} = U_{c+1} \cup \cdots \cup U_n$, gives

$$H^p_{X^{us}}(X, \widetilde{\mathcal{W}}_X) = \begin{cases} 0 & \text{if } p \neq n, \\ \left(\mathbb{C}[T^*V_0] \otimes (X_{c+1}^{-1} \cdots X_n^{-1})\mathbb{C}[X_{c+1}^{-1}, \ldots, X_n^{-1}] \otimes \mathbb{C}[V_+^*]\right)((\hbar)) & \text{if } p = n. \end{cases} \tag{6.7}$$

In the case for all the a_i's strictly positive, each G-isotypic component was finite-dimensional as a $\mathbb{C}((\hbar^{\frac{1}{m}}))$-vector space and, by restricting to an F-weight space, each (F, G)-isotypic component had to be finite-dimensional as a $\mathbb{C}$-vector space. Here, any polynomial in the coordinate functions on V_0 is G-invariant so the first claim is not even true; instead, I claim that each G-isotypic component is a finitely generated free module over $\widetilde{\mathcal{W}}_{T^*V_0}$. Indeed, the component with G-degree r is generated, as a $\mathbb{C}[T^*V_0]$-module (or, equivalently, an $\mathcal{O}_{T^*V_0}$-module), by elements of the form $1 \otimes X_{c+1}^{i_{c+1}} \cdots X_n^{i_n} Y_{c+1}^{j_{c+1}} \cdots Y_n^{j_n} \hbar^k$, where $\sum_{t=c+1}^n a_t(j_t - i_t) = r$, each component of $\mathbf{i}$ and $\mathbf{j}$ are negative and non-negative respectively and there is no restriction on k. Since $\widetilde{\mathcal{W}}_{T^*V_0} \cong \mathcal{O}_{T^*V_0}((\hbar^{\frac{1}{m}}))$ as $\mathcal{O}_{T^*V_0}$-modules, the claim follows. Let $\{b_1, \ldots, b_s\}$ be a basis for the finite-dimensional vector space $H^p_{X^{us}}(X, \widetilde{\mathcal{W}}_{T^*V_+})^{r,0}$, the isotypic component of G-degree r and F-degree zero. Then $b_1, \ldots, b_s$ are $\widetilde{\mathcal{W}}_{T^*V_0}$-generators of the G-isotypic component of $H^p_{X^{us}}(X, \widetilde{\mathcal{W}}_X)$ of degree r.

Next, notice that since G acts trivially on T^*V_0, $\tau(1) \in \widetilde{\mathcal{W}}_{T^*V_+}$ and so $f = \tau(1) - \chi(1) - d \in \widetilde{\mathcal{W}}_{T^*V_+}$. Since the sections of $\widetilde{\mathcal{W}}_{T^*V_+}$ commute with those of $\widetilde{\mathcal{W}}_{T^*V_0}$, $\cdot f$ is a morphism of $\widetilde{\mathcal{W}}_{T^*V_0}$-modules. Then $\cdot f$ is the morphism of $\widetilde{\mathcal{W}}_{T^*V_0}$-modules

$$\cdot f \colon \widetilde{\mathcal{W}}_{T^*V_0} \otimes_{\mathbb{C}} \langle b_1, \ldots, b_s \rangle_{\mathbb{C}} \xrightarrow{\cdot 1 \otimes f} \widetilde{\mathcal{W}}_{T^*V_0} \otimes_{\mathbb{C}} \langle b_1, \ldots, b_s \rangle_{\mathbb{C}},$$

and so is an isomorphism when the corresponding linear map

$$\cdot f \colon \langle b_1, \ldots, b_s \rangle_{\mathbb{C}} \xrightarrow{\cdot f} \langle b_1, \ldots, b_s \rangle_{\mathbb{C}}$$

is an isomorphism. That is, $\cdot f$ is an isomporphism when

$$\chi(1) \neq -\sum_{t=d+1}^{n} a_t(i_t + \tfrac{1}{2}) = -\sum_{t=1}^{n} a_t(i_t + \tfrac{1}{2}).$$

6.3 Case 2: A Torus Action on the Cotangent Bundle of a Vector Space

Let $G = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$, acting on a vector space V. Extend this to a hamiltonian action on $X = T^*V$. Let F act on X so that it commutes with the action of G and makes X into a conical symplectic resolution. Choosing a GIT parameter $\theta \in \mathbb{X}(G)$, gives a Kirwan–Ness stratification. Relabel the strata $S^+_{\langle \lambda_i \rangle} (= S_{\lambda_i})$ as S_k for $1 \leq k \leq q$ in such a way that $\overline{S}_{k'} \cap S_k \neq \emptyset$ only if $k \leq k'$. This gives a set,

$$\Gamma = \{\lambda_1, \ldots, \lambda_q\},$$

of optimal one-parameter subgroups. Each S_k can be described as the set of unstable points for the action of λ_k on T^*V that aren't unstable for the action of some $\lambda_{k'}$ with $k' < k$. Thus, each is a locally closed subset in X, but $\cup_{i=1}^k S_i$ is closed in X. For $0 \leq k \leq q$, let

$$X_k := X \setminus \bigcup_{i=1}^{k} S_i,$$

an open set in X. Let $j_k \colon X_k \longrightarrow X_{k-1}$ be the open embedding, so that $j_1 \circ \cdots \circ j_q = j \colon X^{ss} \to X$. For each $1 \leq k \leq q$, let Z_k be the locus of G-fixed points in S_k. Consider the normal bundle, $\mathcal{N}_{X_{k-1}|S_k}$, and pick a fixed point, $z \in Z_k$. Now λ_k acts on the fibre of this vector bundle over z

with some weights, $w^k_1, \dots, w^k_{m_k} \in \mathbb{Z}$. Let

$$K_k := \sum_{i=1}^{m_k} w^k_i$$

be the sum of these weights.

Choose a deformation quantisation with a G and F action, $\mathcal{W}_X$. Let τ be the quantisation of the moment map μ such that $\tau^{\mathbf{op}} = \tau$. Let $N \in \mathbb{X}(G)$ be a character of G, which will also denote the corresponding one-dimensional representation of G. Define

$$\widetilde{\mathcal{W}}_{X,\chi}(N) := \frac{\widetilde{\mathcal{W}}_X \otimes N}{\widetilde{\mathcal{W}}_X\langle \tau(A) \otimes 1 - \chi(A) \otimes 1 - 1 \otimes A \cdot 1 \,|\, A \in \mathfrak{g}\rangle},$$

with an $\mathfrak{g}$-action by, $A \cdot [1 \otimes n] := [1 \otimes A \cdot n]$, for $A \in \mathfrak{g}$, $f \in \widetilde{\mathcal{W}}_X$ and $n \in N$, where $[f \otimes n]$ denotes the coset containing $f \otimes n$. Note that when $l = 1$ and N is the character corresponding to $d \in \mathbb{Z} \cong \mathbb{X}(\mathbb{C}^*)$ then $\widetilde{\mathcal{W}}_{X,\chi}(N) = \widetilde{\mathcal{W}}_{X,\chi}[d]$.

Lemma 6.3.1. *Each $\widetilde{\mathcal{W}}_{X,\chi}(N)$ is projective and the collection of them, parametrised by $N \in \mathbb{X}(G)$, generate $(\widetilde{\mathcal{W}}_X, F, G)^{\mathsf{good}}_\chi$.*

Proof. By Theorem 5.9.2, it suffices to show that

$$D_{V,N,\chi} := \frac{D_V \otimes N}{D_V\langle \tau(A) \otimes 1 - \chi(A) \otimes 1 - 1 \otimes A \cdot (-) \,|\, A \in \mathfrak{g}\rangle}$$

is projective for the category, $(D_V, G)_\chi$, of coherent D_V-modules with a rational action of G such that $A \cdot (-)$ (the differentiated action) agrees with the action of $\tau(A) - \chi(A)$ for all $A \in \mathfrak{g}$. Given a module, $\mathcal{M}$, in this category, I claim that

$$\begin{aligned} \mathrm{Hom}_{(D_V,G)_\chi}(D_{V,N,\chi}, \mathcal{M}) &\cong \mathrm{Hom}_{(D_V,G)}(D_V \otimes N, \mathcal{M}) \\ &\cong \mathrm{Hom}_{D_V}(D_V \otimes N, \mathcal{M})^G \\ &\cong \mathrm{Hom}_{\mathbb{C}}(N, \mathcal{M})^G. \end{aligned}$$

The first isomorphism follows because, being an object of $(D_V, G)_\chi$, $\tau(A) \otimes 1 - \chi(A) \otimes 1 - 1 \otimes A \cdot (-)$ already acts by zero on the image of any homomorphism from $D_V \otimes N$. Since G is reductive and $\mathcal{M}$ and N are rational G-modules (so that $\mathrm{Hom}_{\mathbb{C}}(N, \mathcal{M})^G$ is finite-dimensional), $\mathrm{Hom}_{(D_V,G)_\chi}(D_{V,N,\chi}, -) \cong \mathrm{Hom}_{\mathbb{C}}(N, -)^G$ is an exact functor so the claim follows.

Because X is affine, choose a finite set of generators $m_1, \dots, m_s$ for $\mathcal{M}$ as in the proof of Lemma 5.9.2 so that the vector space, W, they span is a finite-dimensional G-module. Decomposing $W \cong N_1 \oplus \cdots \oplus N_t$ into irreducible G-representations then gives a surjection

$$\bigoplus_{i=1}^{t} \widetilde{\mathcal{W}}_{X,\chi}(N_i) \longrightarrow \mathcal{M}.$$

Therefore, $\{D_{V,N,\chi} \,|\, N \in \mathbb{X}(G)\}$ generate $(D_V, G)_\chi$. □

Hypothesis 6.3.1. *Suppose, for each $k = 1, \dots, q$, that the following hold.*

(i) S_k decomposes as a product $Z_k \times V_+$, where V_+ is the subspace of positive weights of λ_k acting on V.

(ii) Identify V_- with V_+^*, so that $V_+ \times V_- \cong T^*V_+$. Suppose that $Z_k \times T^*V_+$ is an open subset of T^*V.

Remark 6.3.1. *Hoskins, in [Hos12, Lemma 2.19], shows that (i) holds for $T^*\mathrm{Rep}(Q, \epsilon)$ where Q is a quiver and ϵ is a dimension vector. For the cases that will be considered later on, suppose V is the space of representations of a quiver with dimension vector $(1, \dots, 1)$ and $G = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$, acting by base change with the F-action defined in Section 5.8.3; that is, each of the coordinate functions has* *weight one. Fix a k and let V_+ and V_- be the subspaces of V with positive and negative λ_k-weights respectively. The weights of λ_k on S_k are non-negative so Z_k is the set of points in S_k cut out by the coordinate functions in V_+. Likewise, S_k is the set of points in X_{k-1} cut out by the coordinate functions in V_-. Since the functions that cut out these varieties are all eigenvectors for the action of F, F acts on the fibres V_+. Also, each of the fixed point sets, X^{λ_k}, for each k are closed in X. It follows that for all k, $\cup_{i=1}^{k} Z_i$ is closed in X. Now*

$$Z_k = X^{\lambda_k} \setminus \left(X^{\lambda_k} \cap \cup_{i=1}^{k-1} Z_i\right)$$

is an open set in X^{λ_k}, so $Z_k \times T^*V_+$ is open in $V^{\lambda_k} \times T^*V_+ = T^*V$. Note that in this case the $Z_k \times T^*V_+$ is the total space of the normal bundle, $\mathcal{N}_{X_{k-1}|S_k}$.

Theorem 6.3.1. *Suppose Hypothesis 6.3.1 holds. If*

$$\chi(d\lambda_i(1)) \notin \tfrac{1}{2}K_i + \sum_{i=1}^{q} w_i^k n_i$$

for some non-negative integers $n_1, \dots n_q$, then, for all p, the natural homomorphism given by restriction

$$H^p(X, \widetilde{\mathcal{W}}_{X,\chi}(N))^{F,G} \longrightarrow H^p(X^{ss}, j^*\widetilde{\mathcal{W}}_{X,\chi}(N))^{F,G}$$

is an isomorphism.

Remark 6.3.2. *Notice that the statement of the theorem implies Proposition 6.2.1. Indeed, in that case there is only one stratum. Suppose that all the weights of G acting on V are strictly positive. For $\theta > 0$ the optimal one-parameter subgroup λ has differential $-1 \in \mathbb{C}$. The normal bundle to $X^{us} = \{0\} \times V^*$ in X has fibre V over the fixed point $z = 0$ and on this fibre λ acts by the weights $-a_1, \dots, -a_n$. When $\theta < 0$, the optimal one-parameter subgroup has differential 1 and so acts on the fibre V^* of the normal bundle to $X^{us} = V \times \{0\}$ in X also with weights $-a_1, \dots, -a_n$. Thus, in either case, $K_1 = -\sum a_i$, and the formula above reduces to the one for Proposition 6.2.1.*

Remark 6.3.3. *The arguments in this section can be adapted to prove a more general statement for the action of a torus on an arbitrary symplectic variety X. In that case, an additional hypothesis is requied: one needs to assume that, for each $k = 1, \dots, q$, there is a vector space V on which λ_k acts such that the normal bundle decomposes as $Z_k \times T^*V_+$ where V_+ is the subspace of positive λ_k-weights. Also, this decomposition needs to be F-equivariant in the sense that F acts on each of the fibres. However, for the applications in this thesis it suffices to consider the more simplified case presented here.*

6.3.1 Proof of Theorem 6.3.1

For each $k = 1, \dots, q$, Lemma 5.6.3 shows that S_k is F-stable. By repeatedly restricting $\widetilde{\mathcal{W}}_X$ from X_{k-1} to X_k, the proof of the theorem can be reduced to the case of one restriction with the action of a single optimal one-parameter subgroup λ_k.

Fix $1 \le k \le q$. Define a module

$$\mathcal{M}_{X_{k-1}} := \frac{\widetilde{\mathcal{W}}_{X_{k-1}} \otimes N}{\widetilde{\mathcal{W}}_{X_{k-1}}\langle \tau(A) \otimes 1 - \chi(A) \otimes 1 - 1 \otimes A \cdot 1 \,|\, A \in \mathrm{Lie}(\lambda_k) \subset \mathfrak{g}\rangle}.$$

Restricting N to a character of λ_k, $d \in \mathbb{Z} \cong \mathbb{X}(\lambda_k)$ say, gives

$$\mathcal{M}_{X_{k-1}} \cong \widetilde{\mathcal{W}}_{X_{k-1}, \chi\circ\lambda_k}[d],$$

as quasi-λ_k-equivariant $\mathcal{W}_{X_{k-1}}$-modules.

I claim that, for all $p \ge 0$,

$$H^p_{X^{us}}(X, \mathcal{M}_{X_{k-1}})^{F,\lambda_k} = 0,$$

when χ satisfies the conditions of the theorem.

By Hypothesis 6.3.1, $Z_k \times T^*V_+$ is an open subset in $X = T^*V$. Since the action of λ_k is hamiltonian, decomposing $V = V_+ \times V_- \times V^{\lambda_k}$ shows the restriction of the symplectic form on

$X = T^*V = V \times V^*$ is the same as the form on $Z_k \times T^*V_+$. Therefore, restricting to an open subset containing S_k,

$$H^p_{S_k}(X_{k-1}, \mathcal{M}_{X_{k-1}}) \cong H^p_{S_k}\left(Z_k \times T^*V_+, \widetilde{\mathcal{W}}_{Z_k \times T^*V_+, \chi\circ\lambda_k}[d]\right).$$

Now, the action of λ_k on Z_k and $\hbar$ is trivial so one can now follow the argument at the end of Case 1, with G replaced with λ_k and T^*V_0 replaced with Z_k. The weights of the action of λ_k on V_+ are precisely the negatives of the weights of λ_k on a fibre of the normal bundle over a fixed point.

Generalising to the torus

For parameters satisfying the condition of the theorem, all the (F, λ_k)-equivariant local cohomology of the following module vanishes,

$$\mathcal{M} := \frac{\widetilde{\mathcal{W}}_{X_{k-1}} \otimes N}{\widetilde{\mathcal{W}}_{X_{k-1}}\langle \tau(d\lambda_k(1)) \otimes 1 - \chi(d\lambda_k(1)) \otimes 1 - 1 \otimes d\lambda_k(1) \cdot 1\rangle}.$$

This module is not only λ_k-equivariant: because G is commutative, it is G-equivariant. Therefore the (F, G)-equivariant cohomology vanishes. This implies the (F, G)-equivariant local cohomology of $\widetilde{\mathcal{W}}_{X,\chi}(N)$ vanishes as follows.

Choose a basis $\{v_1, \dots, v_r\}$ for $\mathfrak{g}$ whose first element is $d\lambda(1)$. Let

$$f_r := \tau(v_r) \otimes 1 - \chi(v_r) \otimes 1 - 1 \otimes v_r \cdot 1 \in \widetilde{\mathcal{W}}_X \otimes N,$$

and define, for each $i = 1, \dots, r$,

$$\widetilde{\mathcal{W}}^i = \frac{\widetilde{\mathcal{W}}_X \otimes N}{\widetilde{\mathcal{W}}_X\langle f_1, \dots, f_i\rangle}.$$

Now, by construction, $\widetilde{\mathcal{W}}^0 = \mathcal{M}$ and $\widetilde{\mathcal{W}}^r = \widetilde{\mathcal{W}}_{X,\chi}(N)$.

Note that, because $U(\mathfrak{g})$ is commutative, the f_i commute. So for all $1 \le i \le r$, consider the (F, G)-equivariant short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{W}}^{i-1} \xrightarrow{\times f_i} \widetilde{\mathcal{W}}^{i-1} \longrightarrow \widetilde{\mathcal{W}}^i \longrightarrow 0$$

The associated long exact sequence on (F, G)-equivariant local cohomology gives

$$H^p_{X^{\mathrm{us}}}(X, \widetilde{\mathcal{W}}^{i-1})^{F,G} = 0 \quad \Rightarrow \quad H^p_{X^{\mathrm{us}}}(X, \widetilde{\mathcal{W}}^i)^{F,G} = 0$$

for all p. An induction now shows that $H^p_{X^{\mathrm{us}}}(X, \widetilde{\mathcal{W}}^r)^{F,G} = 0$ when $H^p_{X^{\mathrm{us}}}(X, \widetilde{\mathcal{W}}^0)^{F,G} = 0$ for all p.

6.4 Case 3: The Nakajima Quiver Variety

Suppose now that X is the cotangent bundle of the affine space of quiver representations for the quiver Q^l_∞ as in Section 5.8.3. Give X the structure of a conical symplectic resolution as in Section 5.8.4. Choose a parameter $\theta \in \Upsilon(G)$ and let $\lambda_1, \dots, \lambda_q$ be the optimal one-parameter subgroups indexing the associated Kirwan–Ness strata. Now, for each $k = 1, \dots, q$, let $w^k_1, \dots, w^k_r$ be the weights of λ_k acting on a fibre of the normal bundle of $S^+_{\langle\lambda_k\rangle}$ sitting inside $X \setminus \bigcup_{i=1}^{k-1} S^+_{\langle\lambda_k\rangle}$ over a fixed point. Now McGerty–Nevins define

$$K_k := \sum_{i=1}^{r} w^k_i + \dim G - \dim P(\lambda_k).$$

Remark 6.4.1. *This formula reduces to the definition of K_k used in the cases when G is a torus, with or without the extra* $\dim G - \dim P(\lambda_k)$. *Originally, my definition of K_k didn't have this extra term and the calculations that followed produced peculiar formulae. Things look a lot nicer with this adjustment.*

In [MN13], which is not yet available, McGerty–Nevins prove the following theorem.

Theorem 6.4.1. *If, for all $k = 1, \dots, q$,*

$$\chi(d\lambda_k(1)) \notin \tfrac{1}{2}K_k + \sum_{i=1}^{q} w_i^k n_i$$

for some non-negative integers $n_1, \dots n_q$, then, χ is good for θ.

Chapter 7

The Kirwan–Ness Strata for $G(l, 1, n)$

7.1 The Kirwan–Ness Stratification of the Quiver Variety corresponding to $G(l, 1, n)$

The general definitions for optimal one-parameter subgroups and the Kirwan–Ness strata are now applied to the Nakajima quiver variety, Y_θ. The same notation is used. The space $X = T^*\mathrm{Rep}(Q_\infty^l, \hat{\epsilon})$ is an affine space so take $\mathcal{L}$ to be the trivial line bundle with the linearisation $g \cdot (x, l) := (g \cdot x, \theta(g)l)$ for $(x, l) \in \mathrm{Tot}(\mathcal{L})$ and $\theta \in \mathbb{X}(G)$. As before, write $\lambda = \left(a_k^{(m)}\right)_{k,m} \in \mathbb{Y}(T)$ and $\theta = (\theta^0, \dots, \theta^{l-1})$, so that, when $\lim_{t\to 0} \lambda(t) \cdot x$ exists,

$$\mu^\theta(x, \lambda) = \sum_{m,k} \theta^m a_k^{(m)}.$$

When it is understood from context which stability condition is being used, θ will be dropped from all the above notation.

Remark 7.1.1. *To what degree is X_λ^θ (the set of all points destabilised by λ) dependent on θ? Given a fixed λ,*

$$\begin{aligned} X_\lambda^\theta &= \{x \in X \mid \mu^\theta(x, \lambda) < 0\} \\ &= \begin{cases} \{x \in X \mid \lim_{t\to 0} \lambda(t) \cdot x \text{ exists}\} & \text{if } \sum_{i,k} \theta_i a_k^{(i)} < 0 \\ \varnothing & \text{otherwise.} \end{cases} \end{aligned}$$

So, given a fixed λ,

$$X_\lambda^\theta = X_\lambda^{\theta'}$$

for all θ, θ' such that X_λ^θ and $X_\lambda^{\theta'}$ are non-empty.

Remark 7.1.2. *In the hypertoric case, $n = 1$, so $\mu^\theta(x, \lambda)$ is the dot product of θ and λ, thought of as vectors in $\mathbb{Q}^l$. Let $\alpha(\lambda)$ be the angle between θ and λ that lies in the closed interval $[0, \pi]$. Then the function, M, has an alternative interpretation.*

$$M^\theta(x) = \|\theta\| \cos\left(\inf_{\lambda \in \mathbb{Y}(G)\setminus 0} \alpha(\lambda)\right) \leq \|\theta\|.$$

7.1.1 The General Case for $\theta = (-1, \dots, -1) \in \mathbb{Z}^l$

By Proposition 5.8.1, the GIT parameter $\theta = (-1, \dots, -1)$ never lies on a GIT wall. The aim of this subsection is to find the set, Γ_θ, of optimal one-parameter subgroups for this stability condition.

The adjoint action of $N(T)$ on $\mathbb{Y}(T)$ factors through the Weyl group $N(T)/T \cong (\mathfrak{S}_n)^l$. This permutes the entries of $\lambda = \left(a_k^{(m)}\right)_{k,m} \in \mathbb{Y}(T)$ so that, for an l-tuple of permutations, $(\sigma^{(m)}) \in (\mathfrak{S}_n)^l$,

$(\sigma^{(m)}) \cdot \lambda = \left(a^{(m)}_{\sigma^{(m)-1}(k)}\right)$.

First, there is a lemma that reduces the search for optimal one-parameter subgroups to include only those whose entries are either zero or one and not all zero. With this in mind, define

$$I := \left\{ \left(a_k^{(m)}\right)_{k,m} \,\middle|\, a_k^{(m)} \in \{0,1\} \right\} \setminus \{(0,\ldots,0)\}.$$

Lemma 7.1.1. *Let* $\lambda = \left(a_k^{(m)}\right)_{k,m} \in \Upsilon(T)$ *and* $\nu = \left(b_k^{(m)}\right)_{k,m}$ *where* $b_k^{(m)} := \begin{cases} 0 & \text{if } a_k^{(m)} \le 0 \\ 1 & \text{if } a_k^{(m)} > 0 \end{cases}$. *Then* ν *dominates* λ.

Proof. Suppose that there exists an $x \in X_\lambda \setminus X_\nu$. Then, either $\mu(x,\nu) = -\sum b_k^{(m)} \ge 0$, in which case $\mu(x,\lambda) = -\sum a_k^{(m)} \ge -\sum b_k^{(m)} = 0$, contradicting $x \in X_\lambda$, or $\lim_{t\to 0} \nu(t) \cdot x$ doesn't exist.

It follows that one of the coordinates of x has a negative weight with respect to ν. Suppose that it is $\mathbf{X}_{ij}^{(m)}$. Then $b_i^{(m+1)} < b_j^{(m)}$ so $b_i^{(m+1)} = 0$ and $b_j^{(m)} = 1$ which implies that $a_i^{(m+1)} \le 0 < a_j^{(m)}$ which contradicts x being unstable for λ. A similar argument shows that x cannot be unstable at a coordinate $\mathbf{Y}_{ij}^{(m)}$ for λ unless it is unstable for ν. Note that $\mathbf{v}_i$ cannot have a negative ν-weight. Suppose that $\mathbf{w}_i$ has a negative ν-weight. Then $b_i^{(0)} = 1$, so $a_i^{(0)} > 0$ and x is not unstable for λ. This contradiction implies $X_\lambda \subseteq X_\nu$.

It remains to prove that $\frac{\mu(x,\nu)}{\|\nu\|} \le \frac{\mu(x,\lambda)}{\|\lambda\|}$. This is done by two claims. First, increasing the negative entries of λ to zero decreases the value of this ratio. Second, once all the entries of λ are non-negative, changing those which are non-zero to one doesn't increase the value of the ratio.

Suppose $a_k^{(m)} < 0$ for some k, m. Let λ' be the one-parameter subgroup whose entries agree with λ everywhere except at the (k,m) position where it is zero; then $\|\lambda'\| < \|\lambda\|$. Now,

$$\frac{\mu(x,\lambda)}{\|\lambda\|} = \frac{\mu(x,\lambda')}{\|\lambda\|} - \frac{a_k^{(m)}}{\|\lambda\|} > \frac{\mu(x,\lambda')}{\|\lambda\|} > \frac{\mu(x,\lambda')}{\|\lambda'\|},$$

where the last inequality holds because $\mu(x,\lambda')$ is negative. By repeating this argument for each negative entry, the first claim is proved.

To prove the second claim, suppose that $\lambda = \left(a_k^{(m)}\right)_{k,m}$ is such that $\lim_{t\to 0} \lambda'(t) \cdot x$ exists and $a_k^{(m)} \ge 0$ for all k, m. Let N be the number of non-zero coordinates. Then, by the Cauchy-Schwartz inequality,

$$\left(\sum_{k,m} a_k^{(m)}.1\right)^2 \le \left(\sum_{k,m} a_k^{(m)2}\right)\left(\sum_{i=1}^{N} 1^2\right).$$

Since all the coordinates are non-negative, taking square roots gives

$$\sum_{k,m} a_k^{(m)} \le \sqrt{\sum_{k,m} a_k^{(m)2}}\,\sqrt{N};$$

so that

$$\frac{\mu(x,\lambda)}{\|\lambda\|} = \frac{-\sum a_k^{(m)}}{\sqrt{\sum a_k^{(m)2}}} \le -\sqrt{N} = \frac{\mu(x,\nu)}{\|\nu\|}.$$

□

Corollary 7.1.1. *If* $\lambda \in \Upsilon(T)$ *is optimal for* $x \in X_\lambda$ *then* $\lambda \in I$.

Proof. It suffices to check that the $b_k^{(m)}$ constructed in the lemma could not all be zero; if this were the case then $a_k^{(m)} \le 0$ for all k and m so $\mu(\lambda, x) \nless 0$: a contradiction. □

Next, the set of candidate optimal subgroups is reduced by removing those which are dominated by another. Define a relation, $\to$, on the set of pairs $\{1,\dots,n\}\times\mathbb{Z}_l$ by

$$(i,m)\to\begin{cases}(j,m-1) & \text{if and only if } \mathbf{X}_{ij}^{(m-1)}\neq 0\\ (j,m+1) & \text{if and only if } \mathbf{Y}_{ij}^{(m)}\neq 0\end{cases}.$$

Define a second relation, $\rightsquigarrow$, to be the transitive closure of $\to$; that is,

$$(i,m)\rightsquigarrow(j,m') \iff \begin{array}{c}\text{there exists a sequence of pairs,}\\ ((i_p,m_p)\,|\,p=1,\dots,r),\text{ such that}\\ (i,m)\to(i_0,m_0)\to\cdots\to(i_r,m_r)\to(j,m').\end{array}$$

Lemma 7.1.2. *Suppose that $x\in X_\lambda$ for some $\lambda=\left(a_k^{(m)}\right)_{k,m}$. If $(i,m)\rightsquigarrow(j,m')$ then $a_i^{(m)}\geq a_j^{(m')}$.*

Proof. Because $\rightsquigarrow$ is the transitive closure of $\to$ it suffices to consider the case when $(i,m)\to(j,m-1)$ or $(i,m)\to(j,m+1)$. The former case implies that $\mathbf{X}_{ij}^{(m-1)}\neq 0$ so that, in order for $\lim_{t\to 0}\lambda(t)\cdot x$ to exist, $a_i^{(m)}-a_j^{(m-1)}$ must be non-negative. The latter case implies that $\mathbf{Y}_{ij}^{(m)}\neq 0$ so that $a_i^{(m)}-a_j^{(m+1)}$ is non-negative. □

Let $x\in X^{\mathrm{us}}$ and let Λ_x be the set generated by $\{(i,0)\,|\,w_i\neq 0\}$ and the relation $\rightsquigarrow$. That is,

$$\Lambda_x := \{(i,0)\,|\,w_i\neq 0\}\cup\{(k,m)\in\{1,\dots,n\}\times\mathbb{Z}_l\,|\,(i,0)\rightsquigarrow(k,m)\text{ for some } i \text{ such that } w_i\neq 0\}.$$

Note that Λ_x is empty if and only if $\mathbf{w}=0$.

Lemma 7.1.3. *If $\lambda=\left(a_i^{(n)}\right)_{i,n}$ is optimal for x and $(k,m)\in\Lambda_x$ then $a_k^{(m)}=0$.*

Proof. Using Corollary 7.1.1 and Lemma 7.1.2, there is an i such that $0\leq a_k^{(m)}\leq a_i^{(0)}=0$. □

Next define a one-parameter-subgroup, λ_x, by

$$\lambda_x := \left(a_k^{(m)}\;\middle|\; a_k^{(m)}=\begin{cases}0 & \text{if } (k,m)\in\Lambda_x\\ 1 & \text{if } (k,m)\notin\Lambda_x\end{cases}\right).$$

Proposition 7.1.1. *If $x\in X^{\mathrm{us}}$ then $\mu(x,\lambda_x)<0$ and λ_x is optimal for x.*

Proof. Let $\left(a_k^{(m)}\right)_{k,m}$ denote the coordinates of λ_x as above. Suppose that $\mathbf{X}_{ij}^{(m)}\neq 0$. Then, whenever $(i,m+1)\in\Lambda_x$, $(j,m)\in\Lambda_x$; therefore,

$$\mathbf{wt}(\lambda_x(t)\cdot x)|_{\mathbf{X}_{ij}^{(m)}}=a_i^{(m+1)}-a_j^{(m)}\geq 0.$$

Suppose that $\mathbf{Y}_{ij}^{(m)}\neq 0$. Then, whenever $(i,m)\in\Lambda_x$, $(j,m+1)\in\Lambda_x$, so

$$\mathbf{wt}(\lambda_x(t)\cdot x)|_{\mathbf{Y}_{ij}^{(m)}}=a_i^{(m)}-a_j^{(m+1)}\geq 0.$$

For all i,

$$\mathbf{wt}(\lambda_x(t)\cdot x)|_{\mathbf{v}_i}=a_i^{(0)}\geq 0,$$

and if $w_i\neq 0$ then $(i,0)\in\Lambda_x$ so

$$\mathbf{wt}(\lambda_x(t)\cdot x)|_{\mathbf{w}_i}=a_i^{(0)}=0;$$

therefore, $\lim_{t\to 0}\lambda_x(t)\cdot x$ exists.

Suppose that $\mu(x,\lambda_x)=0$. Then $\Lambda_x=\{1,\dots,n\}\times\mathbb{Z}_l$. Since $x\in X^{\mathrm{us}}$, there exists some $\nu=\left(b_i^{(m)}\right)_{i,m}\in I$ such that $\mu(x,\nu)<0$. Now, for any (j,m) there exists an i such that $(i,0)\rightsquigarrow(j,m)$ and $w_i\neq 0$. By Lemma 7.1.2, $0=b_i^{(0)}\geq b_j^{(m)}$. This implies that $\nu=\lambda_x$ and so contradicts $\nu\in I$; therefore, $\mu(x,\lambda_x)<0$.

Suppose that $\nu = \left(b_k^{(m)}\right)_{k,m}$ is optimal for x. Then, by Corollary 7.1.1, $\nu \in I$ and by Lemma 7.1.3, $a_k^{(m)} = 0$ implies that $b_k^{(m)} = 0$. Let N_ν be the number of non-zero coordinates of ν and N_{λ_x} the number of non-zero coordinates of λ_x. Now, since ν and λ_x only have zeros and ones as entries,

$$\frac{\mu(x,\nu)}{||\nu||} = -\sqrt{N_\nu} \geq -\sqrt{N_{\lambda_x}} = \frac{\mu(x,\lambda_x)}{||\lambda_x||}$$

so $\lambda_x = \nu$. It follows that λ_x is optimal.

□

Given an unstable point, $x \in X$, one now has a recipe for producing optimal one-parameter-subgroups, λ_x; however, not all the $\lambda \in I$ appear in this way. A description of those that do is necessary to decribe the strata of the unstable locus.

Let $\lambda = \left(a_k^{(m)}\right)_{k,m} \in I$ and let

$$i(\lambda) := \begin{cases} \min\{m \in \mathbb{Z}_l \,|\, a_k^{(m)} = 1 \text{ for all } k = 1,\ldots,n\} & \text{if such a number exists} \\ -\infty & \text{otherwise,} \end{cases}$$

$$j(\lambda) := \begin{cases} \max\{m \in \mathbb{Z}_l \,|\, a_k^{(m)} = 1 \text{ for all } k = 1,\ldots,n\} & \text{if such a number exists} \\ \infty & \text{otherwise.} \end{cases}$$

The optimal one-parameter subgroups for $\theta = (1,\ldots,1)$ can now be described.

Theorem 7.1.1. *A one-parameter subgroup, $\lambda = \left(a_k^{(m)}\right)_{k,m} \in I$, is optimal if and only if it satisfies one of the following (mutually exclusive) conditions.*

(i) For all k and m, $a_k^{(m)} = 1$.

(ii) The number $i(\lambda)$ equals infinity.

(iii) Both $i(\lambda) > 0$ and $a_k^{(m)} = 1$ for all $i(\lambda) \leq m \leq j(\lambda)$ and all $k = 1,\ldots,n$.

Proof. Let $\lambda = \left(a_k^{(m)}\right)_{k,m}$ satisfy one of the three conditions above. A point $x \in X$ will be constructed so that $x \in X^{\text{us}}$ and $\lambda = \lambda_x$, the optimal one-parameter subgroup for x. Define x as follows.

$$\mathbf{X}_{ij}^{(m)} := \begin{cases} 0 & \text{if } a_j^{(m)} = 0 \text{ and } a_i^{(m+1)} = 1 \\ 1 & \text{otherwise,} \end{cases} \qquad \mathbf{Y}_{ij}^{(m)} := \begin{cases} 0 & \text{if } a_i^{(m)} = 1 \text{ and } a_j^{(m+1)} = 0 \\ 1 & \text{otherwise,} \end{cases}$$

$$v_i := 1 \qquad w_i := \begin{cases} 0 & \text{if } a_i^{(0)} = 1, \\ 1 & \text{if } a_i^{(0)} = 0. \end{cases}$$

Now, for all $t \in \mathbb{C}^*$, x has been defined so that $\lambda(t) \cdot x$ has non-negative weights and $\lambda \in I$ so $\mu(x,\lambda) < 0$. Therefore, x lies in the unstable locus.

First, I claim that whenever λ_x has a zero entry, the corresponding entry of λ must be zero. Second, I claim the converse: whenever an entry of λ is zero, so must the corresponding entry of λ_x.

Let $\left(b_k^{(m)}\right)_{k,m}$ denote the entries of λ_x and suppose that $b_k^{(m)} = 0$ for some k and m. Then $(k,m) \in \Lambda_x$ and so there exists an i such that $w_i \neq 0$ and a sequence

$$(i,0) \to (i_1,m_1) \to \ldots \to (i_p,m_p) \to (k,m).$$

This implies that $0 = a_i^{(0)}$ and, for each $r = 1,\ldots,p$, either $X_{i_r i_{r+1}}^{(m_r-1)} \neq 0$ or $Y_{i_r i_{r+1}}^{(m_r)} \neq 0$. Either way, $a_{i_r}^{m_r} \geq a_{i_{r+1}}^{m_{r+1}}$. It follows that

$$0 = a_i^{(0)} \geq a_{i_1}^{(m_1)} \geq \cdots \geq a_{i_p}^{(m_p)} \geq a_k^{(m)}.$$

Since $\lambda \in I$, $a_k^{(m)} = 0$ and this proves the first claim.

Next suppose that $a_k^{(m)} = 0$. Condition (i) doesn't hold now, so λ must satisfy one of the other two conditions. This means that $i(\lambda) \neq 0$ and so there must exist an i such that $a_i^{(0)} = 0$. Suppose that $i(\lambda)$ is infinite. Then for each $m' \in \mathbb{Z}^l$ there exists a $p_{m'}$ such that $a_{p_{m'}}^{(m')} = 0$ so that

$$(i, 0) \to (p_1, 1) \to \cdots \to (p_{m-1}, m-1) \to (k, m);$$

so $(i, 0) \rightsquigarrow (k, m)$ and hence $b_k^{(m)} = 0$.

Suppose, instead, that $i(\lambda) = r > 0$ and $j(\lambda) = s > 0$ are finite. Then Condition (iii) must hold, so every entry between $a_i^{(r)}$ and $a_j^{(s)}$ is 1 for all i and j; so either $m < r$ or $m > s$. As in the last case, for each $m' < r$ or $m' > s$, there exists a $p_{m'}$ such that $a_{p_{m'}}^{(m')} = 0$. If $m < r$ then, as before,

$$(i, 0) \to (p_1, 1) \to \cdots \to (p_{m-1}, m-1) \to (k, m),$$

whereas if $m > s$ then

$$(i, 0) \to (p_{l-1}, l-1) \to \cdots \to (p_{m+1}, m+1) \to (k, m).$$

In either case $(i, 0) \rightsquigarrow (k, m)$ so $b_k^{(m)} = 0$. This proves the second claim. Since λ was assumed to be in I, the entries of both λ and λ_x are one when they are non-zero. The two claims therefore show that $\lambda = \lambda_x$. This completes the proof in one direction.

It remains to show that an arbitrary optimal one-parameter subgroup in I must satisfy one of the three conditions. Prove this by contrapositive: Suppose that some $\lambda \in I$ does not satisfy any of the three conditions, then it suffices to show that $\lambda \neq \lambda_x$ for any $x \in X_\lambda$. There are two ways in which both Conditions (ii) and (iii) can fail to hold for λ. Treat these cases seperately.

First, suppose that $i(\lambda) = 0$, so that $a_i^{(0)} = 1$ for all $i = 1, \ldots, n$. Then any $x \in X_\lambda$ must have $\mathbf{w} = 0$, because the weights of λ on non-zero coordinates of $\mathbf{w}$ would be -1. This implies that $\Lambda_x = \emptyset$. However, if x is such that $\Lambda_x = \emptyset$ then $\lambda_x = (1, \ldots, 1)$ and, since λ is not allowed to satisfy Condition (i), $\lambda \neq \lambda_x$.

Second, if $i(\lambda) \neq 0$, then for Condition (iii) to fail there must exist an m such that $1 \leq i(\lambda) < m < j(\lambda) \leq l-1$ and an i such that $a_i^{(m)} = 0$. Let $x \in X_\lambda$ be an arbitrary point that is unstable for λ and define the entries $(b_i^{(n)}) := \lambda_x$. I claim that $\lambda \neq \lambda_x$. Suppose that $\lambda = \lambda_x$ so that $b_k^{(m)} = 0$. Then there exists some sequence

$$(i, 0) \to (i_1, m_1) \to \cdots \to (i_p, m_p) \to (k, m).$$

For each $r = 1, \ldots, p-1$, $m_{r+1} = m_r \pm 1$, so there must exist a k' such that $m_{k'} = r$ or $m_{k'} = s$. Assume that $m_{k'} = r$, then $(i, 0) \rightsquigarrow (i_{k'}, r)$, which by Lemma 7.1.2 implies that $0 = a_i^{(0)} \geq a_{i_{k'}}^{(r)} = 1$. This contradicts the assumption that $b_k^{(m)} = 0$, so the claim is proved. A similar argument provides a contradiction when there exists a k' such that $m_{k'} = s$. This completes the other direction of the proof.

□

7.1.2 The Case for other GIT Parameters

The following proposition shows that the optimal one-parameter subgroups for the stability condition $(1, \ldots, 1)$ are the same with all the entries multiplied by -1.

Proposition 7.1.2. *Let $\theta = (\theta_0, \ldots, \theta_{l-1})$ and λ a θ-optimal one-parameter subgroup. Then $-\lambda$ is $-\theta$-optimal.*

Proof. Suppose that λ is θ-optimal. Let $x \in X_\lambda^\theta$, a point that λ destabilises with respect to θ and define a point $\hat{x} = (\hat{\mathbf{X}}^{(m)}, \hat{\mathbf{Y}}^{(m)}; \hat{v}, \hat{w})$ by

$$\hat{\mathbf{X}}^{(m)}_{ij} := \begin{cases} 1 & \text{if } \mathbf{Y}^{(m)}_{ji} \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \hat{\mathbf{Y}}^{(m)}_{ij} := \begin{cases} 1 & \text{if } \mathbf{X}^{(m)}_{ji} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{v}_i := \begin{cases} 1 & \text{if } \mathbf{w}_i \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \hat{w}_i := \begin{cases} 1 & \text{if } \mathbf{v}_i \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now, by construction, the weights of any one-parameter subgroup, ρ say, acting on $\hat{x}$ are precisely the negatives of its weights on x so $\lim_{t\to 0} \rho(t) \cdot \hat{x} = \lim_{t\to 0} -\rho(t) \cdot x$, whenever it exists. This means that $\mu^{-\theta}(\hat{x}, \rho) = \mu^{\theta}(x, -\rho)$; in particular, $\mu^{-\theta}(x, -\lambda) = \mu^{\theta}(x, \lambda) < 0$. Hence $-\lambda$ destabilises $\hat{x}$, that is $\hat{x} \in X^{-\theta}_{-\lambda}$. Suppose that ν is $-\theta$-optimal for $\hat{x}$. Then, x is θ-unstable for $-\nu$ so

$$\frac{\mu^{-\theta}(\hat{x}, -\lambda)}{|| - \lambda||} = \frac{\mu^{\theta}(x, \lambda)}{||\lambda||} \leq \frac{\mu^{\theta}(x, -\nu)}{|| - \nu||} = \frac{\mu^{-\theta}(\hat{x}, \nu)}{||\nu||}$$

Since ν is assumed to be optimal, $\nu = -\lambda$ and the proposition is proved. □

Corollary 7.1.2. *The set of optimal one-parameter subgroups for $-\theta$ is in bijection with the set of those for θ. In fact,*

$$\Gamma_{-\theta} = \{-\lambda \mid \lambda \in \Gamma_{\theta}\}$$

Proof. Applying the proposition to Γ_{θ} and $\Gamma_{-\theta}$ gives inclusions of sets in both directions. □

Definition 7.1.1. *Say that a one-parameter subgroup is* ***essential*** *if it is optimal for either $\theta \pm (1, \ldots, 1)$.*

7.2 The Numbers, K_i, for $G(l, 1, n)$

Let $\lambda_{\mathbf{i}}$ be an optimal one-parameter subgroup for $\theta = 1$ so that its differential can be written

$$d\lambda_{\mathbf{i}}(1) := (0 \cdots 0 \underbrace{1 \cdots 1}_{i_0})(0 \cdots 0 \underbrace{1 \cdots 1}_{i_1}) \cdots (0 \cdots 0 \underbrace{1 \cdots 1}_{i_{l-1}})$$

for some $\mathbf{i} \in \mathbb{N}^l$, and define $K_{\mathbf{i}}$ to be the $\lambda_{\mathbf{i}}$-eigenvalue of acting on the determinant of the normal bundle of $S^+_{\langle\lambda_{\mathbf{i}}\rangle}$ in X_i. Let

$$I := \{\mathbf{i} \in \mathbb{N}^l \mid \lambda_{\mathbf{i}} \text{ is optimal for } \theta = 1\}.$$

Proposition 7.2.1. *The number, $K_{\mathbf{i}}$, is given by the formula*

$$K_{\mathbf{i}} := -i_0 + \sum_{t=0}^{l-1} (n - i_t)(2i_t - i_{t-1} - i_{t+1}),$$

where the subscripts of $\mathbf{i}$ are taken modulo l.

Proof. By (KN3), each stratum $S^+_{\langle\lambda_{\mathbf{i}}\rangle}$ can be decomposed as $G \overset{P(\lambda_{\mathbf{i}})}{\times} S_{\lambda_{\mathbf{i}}}$ which gives the decomposition of tangent spaces,

$$T_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}}|_{S_{\lambda_{\mathbf{i}}}} \cong \mathfrak{g}/\mathfrak{p}_{\lambda_{\mathbf{i}}} \times T_{S_{\lambda_{\mathbf{i}}}}.$$

The normal bundle fits into the short exact sequence

$$0 \longrightarrow T_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}} \longrightarrow T_{X|_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}}} \longrightarrow \mathcal{N}_{X|_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}}} \longrightarrow 0.$$

Let $z \in Z_{\mathbf{i}}$, a fixed point in $S_{\lambda_{\mathbf{i}}}$. Taking fibres of sheaves is exact, so taking the fibre at z gives a short exact sequence

$$0 \longrightarrow T_z S^+_{\langle\lambda_{\mathbf{i}}\rangle} \longrightarrow T_z X|_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}} \longrightarrow \mathcal{N}_z X|_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}} \longrightarrow 0.$$

The alternating sum of the dimension these vector spaces must be zero.

The eigenvalues of $\lambda_{\mathbf{i}}$ acting on the fibre of the normal bundle are all -1. So, by definition,

$$\begin{aligned} K_{\mathbf{i}} &= -\dim \mathcal{N}_z X|_{S^+_{\langle\lambda_{\mathbf{i}}\rangle}} + \dim G - \dim P(\lambda_{\mathbf{i}}) \\ &= \dim T_z S^+_{\langle\lambda_{\mathbf{i}}\rangle} - \dim X + \dim G - \dim P(\lambda_{\mathbf{i}}) \\ &= \dim S_{\lambda_{\mathbf{i}}} - \dim X + 2(\dim G - \dim P(\lambda_{\mathbf{i}})). \end{aligned}$$

Now $\dim X - \dim S_{\lambda_{\mathbf{i}}}$ is the number of negative weights of $\lambda_{\mathbf{i}}$ acting on X. For each $t = 0, \dots, l-1$, the number of negative weights of $\lambda_{\mathbf{i}}$ acting on $\mathbf{X}^{(t)}$ is $i_t(n - i_{t+1})$; on $\mathbf{Y}^{(t)}$ there are $i_{t+1}(n - i_t)$; on $\mathbf{v}$ there aren't any negative weights and on $\mathbf{w}$ there are i_0. Hence,

$$\dim X - \dim S_{\lambda_{\mathbf{i}}} = i_0 + \sum_{t=0}^{l-1} (n - i_t)(i_{t-1} + i_{t+1}).$$

Take an $n \times n$ matrix, for each $t = 0, \dots, l-1$, partition it into four blocks so it has size $(n - i_t) + i_t \times (n - i_t) + i_t$. Then, $\dim G - \dim P(\lambda_{\mathbf{i}})$ is the sum of the sizes of the upper-right-hand blocks as $i = 0, \dots, l-1$, for these are the positions in which entries of $P(\lambda_{\mathbf{i}})$ must be zero. Therefore, the formula for $\dim G - \dim P_{\lambda_{\mathbf{i}}}$ is

$$\dim G - \dim P_{\lambda_{\mathbf{i}}} = \sum_{t=0}^{l-1} (n - i_t) i_t.$$

The difference of these two gives the formula for $K_{\mathbf{i}}$. □

Now consider some concrete examples.

7.3 Localisation for $W = \mathfrak{S}_n$

Fix $l = 1$.

Theorem 7.3.1. *For $W = \mathfrak{S}_n$ there are no bad parameters.*

Proof. By Proposition 7.2.1, for $i = 1, \dots, n$, $K_i = -i$. For $\theta < 0$ $\{\lambda_i \,|\, i = 1, \dots, n\}$ gives a complete set of representatives for optimal one-parameter subgroups and for $\theta > 0$ the optimal one-parameter subgroups are $\{-\lambda_i\}$. For each stratum the weights on the fibres of the normal bundle are -1 so the formula for bad parameters reduces to

$$\chi \text{ is bad for } \theta \iff \chi(d\lambda_i(1)) \in \tfrac{1}{2} K_i + \mathbb{Z}_{\leq 0}.$$

For $\theta > 0$, $\chi(d\lambda_i(1)) = -i(c_0 + \frac{1}{2})$ so that c_0 is bad for $\theta > 0$ if

$$c_0 \in \frac{1}{i} \mathbb{Z}_{\geq 0}.$$

For $\theta < 0$, $\chi(d(-\lambda_i)(1)) = i(c_0 + \frac{1}{2})$ so that c_0 is bad if

$$c_0 \in -1 + \frac{1}{i} \mathbb{Z}_{\leq 0}.$$

Since these two sets are disjoint there are no bad parameters for this case. □

7.4 Localisation for $W = \mu_3$

Let $n = 1$. Define a one-parameter subgroup $\lambda_{i,j} := \lambda_{\mathbf{k}}$ where $k_t := \begin{cases} 0 & \text{if } 0 \le t < i \text{ or } j \le t < l \\ 1 & \text{if } i \le t < j. \end{cases}$.
For $\theta = (-1, \dots, -1)$, the set of one-parameter subgroups is $\{\lambda_{i,j} \,|\, 1 \le i < j \le l\} \cup \{\lambda_{0,l}\}$. Then

$$K_{i,j} = \begin{cases} -1 & \text{if } i = 0 \\ -2 & \text{if } i \neq 0 \end{cases}$$

Now, in terms of the hyperplane parameters $\{k_1, \dots, k_{l-1}\}$,

$$\chi(\lambda_{i,j}) = \sum_{t=i}^{j} \left(k_{1-t} - k{-}t + \frac{1}{l}\right) = \frac{j-i+1}{l} + k_{1-i} - k_{-j},$$

for $i > 0$, and

$$\chi(\lambda_{0,l-1}) = \frac{l-1}{l} - k_1.$$

Theorem 7.4.1. *For $W = \mu_3$ there are no bad parameters.*

The proof occupies the rest of this section. According to Theorem 6.0.1, given an optimal one parameter subgroup $\lambda = (a)(b)(c)$,

$$\chi(\lambda) = -ck_1 + (c-b)k_2 + \tfrac{1}{3}(b+c).$$

In order for a parameter to be bad for a particular θ, this number must belong to some set of positive real numbers. Divide $\mathbb{R}^2$ into the following subsets

$A := \{(k_1,k_2) \in \mathbb{R}^2 \,|\, k_2 \ge \frac{4}{3},\, k_2 - k_1 \ge \frac{2}{3},\, k_1 \le \frac{7}{6}\}$ $\quad B := \{(k_1,k_2) \in \mathbb{R}^2 \,|\, k_2 \le -\frac{2}{3},\, k_2 - k_1 \le -\frac{4}{3},\, k_1 \ge \frac{1}{6}\}$
$C := \{(k_1,k_2) \in \mathbb{R}^2 \,|\, k_2 - k_1 \ge \frac{2}{3},\, k_1 > \frac{7}{6}\}$ $\quad D := \{(k_1,k_2) \in \mathbb{R}^2 \,|\, k_2 - k_1 \le -\frac{4}{3},\, k_1 < \frac{1}{6}\}$.

Now, define the following subsets; each is an infinite union of two-dimensional real planes in $\mathbb{C}^2$.

$$B_1 := \left\{(k_1,k_2) \in \mathbb{C}^2 \,\middle|\, k_2 - k_1 \in -\tfrac{4}{3} + \mathbb{Z}_{\le 0}\right\} \qquad B_{-1} := \left\{(k_1,k_2) \in \mathbb{C}^2 \,\middle|\, k_2 - k_1 \in \tfrac{2}{3} + \mathbb{Z}_{\ge 0}\right\}$$
$$B_2 := \left\{(k_1,k_2) \in \mathbb{C}^2 \,\middle|\, k_2 \in \tfrac{4}{3} + \mathbb{Z}_{\ge 0}\right\} \qquad B_{-2} := \left\{(k_1,k_2) \in \mathbb{C}^2 \,\middle|\, k_2 \in \tfrac{-2}{3} - \mathbb{Z}_{\ge 0}\right\}$$
$$B_3 := \left\{(k_1,k_2) \in \mathbb{C}^2 \,\middle|\, k_1 \in \tfrac{7}{6} + \tfrac{1}{2}\mathbb{Z}_{\ge 0}\right\} \qquad B_{-3} := \left\{(k_1,k_2) \in \mathbb{C}^2 \,\middle|\, k_1 \in \tfrac{1}{6} - \tfrac{1}{2}\mathbb{Z}_{\ge 0}\right\}.$$

Remark 7.4.1. *These are the sets of bad parameters for the one parameter subgroups as follows.*

$$B_1 = B_{(0)(0)(1)} \qquad B_2 = B_{(0)(1)(0)} \qquad B_3 = B_{(0)(1)(1)} \cup B_{(1)(1)(1)}$$
$$B_{-1} = B_{(0)(0)(-1)} \qquad B_{-2} = B_{(0)(-1)(0)} \qquad B_{-3} = B_{(0)(-1)(-1)} \cup B_{(-1)(-1)(-1)}.$$

Lemma 7.4.1. *There are an inclusions of sets*

$$(B_1 \cup B_2 \cup B_3) \cap (B_{-1} \cup B_{-2} \cup B_{-3}) \subset A \cup B \cup C \cup D$$
$$(B_1 \cup B_{-2} \cup B_3) \cap (B_{-1} \cup B_2 \cup B_{-3}) \subset \mathbb{R}^2 \setminus (A \cup B).$$

Proof. First, note that the sets on the left hand side comprise only points in $\mathbb{R}^2 \subset \mathbb{C}^2$. Indeed, I claim that $B_i \cap B_j \in \mathbb{R}^2$ for any $i \neq j$. If $i = \pm 1$ then a point (k_1,k_2) must satisfy, $k_2 - k_1 \in \mathbb{R}$, if $i = \pm 2$ then $k_2 \in \mathbb{R}$ and if $i = \pm 3$ then $k_1 \in \mathbb{R}$. If $j = -i$ then the set is empty. Otherwise, for points in $B_i \cap B_j$, two out of three of these conditions must be met and so the claim follows.

Now it is a straight-forward check to see that,

$$B_1 \cap B_{-2},\ B_1 \cap B_{-3},\ B_2 \cap B_{-1},\ B_2 \cap B_{-3},\ B_3 \cap B_{-1},\ B_3 \cap B_{-2} \subset A \cup B \cup C \cup D,$$

and a similar check shows the second inclusion.

□

Corollary 7.4.1. *If a parameter* $(k_1, k_2) \in \mathbb{C}^2$ *is bad for every* θ *then it belongs to the set* $A \cup B \cup C \cup D \subset \mathbb{R}^2$.

Proof. I claim that if a point does not belong to this set then it is good for either $\theta = \pm(1,1,1)$. Indeed, each of the optimal one-parameter subgroups for these values of θ is accounted for in Remark 7.4.1 and so $(B_1 \cup B_2 \cup B_3) \cap (B_{-1} \cup B_{-2} \cup B_{-3})$ contains all points which are bad for $\theta = \pm(1,1,1)$. □

It remains to find, for each parameter $\mathbf{k} \in A \cup B \cup C \cup D$, some θ such that $\mathbf{k}$ is good for θ.

Proposition 7.4.1. *The optimal one parameter subgroups for* $(0,2,-1)$, $(0,-2,1)$, $(0,1,-2)$ *and* $(0,-1,2)$ *are as in Table 7.1.*

Proof. See Section C.1 in Appendix B. □

θ	$(-1,-1,-1)$	$(1,1,1)$	$(0,-2,1)$	$(0,2,-1)$	$(0,-1,2)$	$(0,1,-2)$
	(0)(0)(1)	(0)(0)(−1)	(0)(0)(−1)	(0)(0)(1)	(0)(0)(−1)	(0)(0)(1)
	(0)(1)(0)	(0)(−1)(0)	(0)(1)(0)	(0)(−1)(0)	(0)(1)(0)	(0)(−1)(0)
	(0)(1)(1)	(0)(−1)(−1)	(0)(1)(1)	(0)(−1)(−1)	(0)(−1)(−1)	(0)(1)(1)
	(1)(1)(1)	(−1)(−1)(−1)	(1)(1)(1)	(−1)(−1)(−1)	(−1)(−1)(−1)	(1)(1)(1)
			(0)(2)(−1)	(0)(−2)(1)	(0)(1)(−2)	(0)(−1)(2)
			(−1)(4)(1)	(1)(−4)(−1)	(1)(1)(−4)	(−1)(−1)(4)
			(1)(1)(−1)	(−1)(−1)(1)	(−1)(1)(−1)	(1)(−1)(1)

Table 7.1: A comparison of the optimal one-parameter subgroups for $W = \mu_3$ and various θ.

Lemma 7.4.2. *Parameters in* A *are good for* $\theta = (0,1,-2)$.

Proof. The space A is disjoint from $B_1 \cup B_{-2} \cup B_3$ so parameters in A are not bad for any of the essential one-parameter subgroups. It suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $-k_1 \geq -k_2 + \frac{2}{3}$, $k_2 \geq \frac{4}{3}$ and $-k_1 \geq -\frac{7}{6}$ for all $(k_1, k_2) \in A$.

$$\begin{aligned} \chi((0)(-1)(2)) &= -2k_1 + 3k_2 - \tfrac{1}{3} > 0, \\ \chi((-1)(-1)(4)) &= -4k_1 + 5k_2 + 1 > 0, \\ \chi((1)(-1)(1)) &= -k_1 + 2k_2 > 0. \end{aligned}$$

□

Lemma 7.4.3. *Parameters in* B *are good for* $\theta = (0,-1,2)$.

Proof. The space B is disjoint from $B_{-1} \cup B_2 \cup B_{-3}$ so parameters in B are not bad for any of the essential one-parameter subgroups. It suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $-k_2 \geq \frac{2}{3}$ and $k_1 \geq \frac{1}{6}$ for all $(k_1, k_2) \in B$.

$$\begin{aligned} \chi((0)(1)(-2)) &= 2k_1 - 3k_2 + \tfrac{1}{3} > 0, \\ \chi((1)(1)(-4)) &= 4k_1 - 5k_2 - 1 > 0, \\ \chi((-1)(1)(-1)) &= k_1 - 2k_2 > 0. \end{aligned}$$

□

Lemma 7.4.4. *Parameters in* C *are good for* $\theta = (0,2,-1)$.

Proof. The space C is disjoint from $B_1 \cup B_{-2} \cup B_{-3}$ so parameters in C are not bad for any of the essential one-parameter subgroups. It suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $k_2 \geq k_1 + \frac{2}{3}$ and $k_1 > \frac{7}{6}$ for all $(k_1, k_2) \in C$.

$$\begin{aligned}
\chi((0)(-2)(1)) &= -k_1 + 3k_2 - \tfrac{1}{3} \geq 2k_1 + 2 - \tfrac{1}{3} > 0,\\
\chi((1)(-4)(-1)) &= k_1 + 3k_2 - \tfrac{5}{3} \geq 4k_1 + 2 - \tfrac{5}{3} > 0,\\
\chi((-1)(-1)(1)) &= -k_1 + 2k_2 \geq k_1 + \tfrac{4}{3} > 0.
\end{aligned}$$

□

Lemma 7.4.5. *Parameters in D are good for $\theta = (0, -2, 1)$.*

Proof. The space A is disjoint from $B_{-1} \cup B_2 \cup B_3$ so parameters in D are not bad for any of the essential one-parameter subgroups. It suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $-k_2 \geq -k_1 + \frac{4}{3}$ and $-k_1 > -\frac{1}{6}$ for all $(k_1, k_2) \in D$.

$$\begin{aligned}
\chi((0)(2)(-1)) &= k_1 - 3k_2 + \tfrac{1}{3} \geq -2k_1 + 4 > 0,\\
\chi((-1)(4)(1)) &= -k_1 - 3k_2 + \tfrac{5}{3} \geq -4k_1 + 4 + \tfrac{5}{3} > 0,\\
\chi((1)(1)(-1)) &= k_1 - 2k_2 \geq -k_1 + \tfrac{8}{3} > 0.
\end{aligned}$$

□

This concludes the proof of Theorem 7.4.1.

7.5 Localisation for $W = G(2,1,2) = B_2$

This section is a proof of the following theorem.

Theorem 7.5.1. *There are no bad parameters for $G(2, 1, 2)$.*

Proposition 7.2.1 reduces to the following formula.

$$K_{i_0,i_1} := -i_0 + (n - i_0)(2i_0 - 2i_1) + (n - i_1)(2i_1 - 2i_0) = -i_0 - 2(i_1 - i_0)^2 < 0.$$

The GIT walls for $G(2, 1, 2)$ are cut out by the polynomials, $\{c_0, c_1, c_0 + c_1, c_0 + 2c_1\}$. Number the chambers $C_0^+, \ldots, C_3^+$ clockwise from the positive quadrant, with each C_i^- opposite C_i^+. That is,

$$\begin{aligned}
C_0^+ &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_0,\ 0 < \theta_1\right\} & C_0^- &:= \left\{\theta \in \mathbb{R}^2 \,|\, \theta_0 < 0,\ \theta_1 < 0\right\}\\
C_1^+ &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_0 + 2\theta_1,\ \theta_1 < 0\right\} & C_1^- &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_1,\ \theta_0 + 2\theta_1 < 0\right\}\\
C_2^+ &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_0 + \theta_1,\ \theta_0 + 2\theta_1 < 0\right\} & C_2^- &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_0 + 2\theta_1,\ \theta_0 + \theta_1 < 0\right\}\\
C_3^+ &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_0,\ \theta_0 + \theta_1 < 0\right\} & C_3^- &:= \left\{\theta \in \mathbb{R}^2 \,|\, 0 < \theta_0 + \theta_1,\ \theta_0 < 0\right\}.
\end{aligned}$$

Define the following subsets of $\mathbb{C}^2$.

$$\begin{aligned}
B_1 &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_1 \in -\tfrac{1}{2} + \tfrac{1}{2}\mathbb{Z}_{\leq 0}\right\} \setminus \{-1\} & B_{-1} &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_1 \in \tfrac{3}{2} + \tfrac{1}{2}\mathbb{Z}_{\geq 0}\right\} \setminus \{2\}\\
B_2 &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_0 \in -1 + \tfrac{1}{2}\mathbb{Z}_{\leq 0}\right\} & B_{-2} &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_0 \in \tfrac{1}{2}\mathbb{Z}_{\geq 0}\right\}\\
B_3 &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_0 - c_1 \in -\tfrac{5}{2} + \mathbb{Z}_{\leq 0}\right\} & B_{-3} &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_0 - c_1 \in \tfrac{1}{2} + \mathbb{Z}_{\geq 0}\right\}\\
B_4 &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_0 + c_1 \in -\tfrac{3}{2} + \mathbb{Z}_{\leq 0}\right\} & B_{-4} &:= \left\{\mathbf{c} \in \mathbb{C}^2 \,|\, c_0 + c_1 \in \tfrac{3}{2} + \mathbb{Z}_{\geq 0}\right\}
\end{aligned}$$

Given a one-parameter subgroup λ, let $B_{d\lambda(1)}$ be the set of parameters that are bad for λ.

Here,

$$\begin{aligned} B_1 &= B_{(00)(01)} \cup B_{(00)(11)}, & B_{-1} &= B_{(00)(0-1)} \cup B_{(00)(-1-1)}, \\ B_2 &= B_{(01)(01)} \cup B_{(11)(11)}, & B_{-2} &= B_{(0-1)(0-1)} \cup B_{(-1-1)(-1-1)}, \\ B_3 &= B_{(01)(00)}, & B_{-3} &= B_{(0-1)(00)}, \\ B_4 &= B_{(01)(11)}, & B_{-4} &= B_{(0-1)(-1-1)}, \end{aligned}$$

so that $B_1 \cup \cdots \cup B_4$ and $B_{-1} \cup \cdots \cup B_{-4}$ are the sets of bad points for the GIT parameters $(-1,-1)$ and $(1,1)$ respectively.

Define the following disjoint subsets of $\mathbb{R}^2$, thought of as sitting inside the copy of $\mathbb{C}^2$ above.

$$\begin{aligned} A &:= \left\{\mathbf{c} \in \mathbb{R}^2 \,\middle|\, c_0 > -1,\ c_1 - c_0 \geq \tfrac{5}{2}\right\} & B_+ &:= \left\{\mathbf{c} \in \mathbb{R}^2 \,\middle|\, c_0 \leq -1,\ c_0 + c_1 \geq \tfrac{1}{2}\right\} \\ B_- &:= \left\{\mathbf{c} \in \mathbb{R}^2 \,\middle|\, c_1 \geq \tfrac{3}{2},\ c_1 - c_0 > \tfrac{1}{2}\right\} & C &:= \left\{\mathbf{c} \in \mathbb{R}^2 \,\middle|\, c_0 < 0,\ c_0 - c_1 \geq \tfrac{1}{2}\right\} \\ D_+ &:= \left\{\mathbf{c} \in \mathbb{R}^2 \,\middle|\, c_1 \leq -\tfrac{1}{2},\ c_0 + c_1 \geq -\tfrac{1}{2}\right\} & D_- &:= \left\{\mathbf{c} \in \mathbb{R}^2 \,\middle|\, c_0 \geq 0,\ c_0 - c_1 > \tfrac{1}{2}\right\}. \end{aligned}$$

Let $B = B_+ \cup B_-$ and $D = D_+ \cup D_-$.

Lemma 7.5.1. *The set, $A \cup B \cup C \cup D$, gives a bound for the set of bad parameters when $W = G(2,1,2)$.*

Proof. If $\mathbf{c}$ is bad then it must be bad for $\theta = (1,1)$ and $\theta = (-1,-1)$. Therefore, it suffices to show that

$$(B_1 \cup B_2 \cup B_3 \cup B_4) \cap (B_{-1} \cup B_{-2} \cup B_{-3} \cup B_{-4}) \subseteq A \cup B \cup C \cup D.$$

Note that, for $i = 1, \ldots, 4$, $B_i \cap B_{-i} = \emptyset$. Also, if $i \neq j$ then $B_i \cap B_j \subset \mathbb{R}^2$. Then

$$\begin{aligned} &B_3 \cap B_{-1},\ B_3 \cap B_{-2},\ B_3 \cap B_{-4},\ B_2 \cap B_{-1},\ B_2 \cap B_{-4},\ B_4 \cap B_{-1} \subseteq A \cup B, \\ &B_1 \cap B_{-3},\ B_2 \cap B_{-3},\ B_4 \cap B_{-3},\ B_1 \cap B_{-2},\ B_1 \cap B_{-4},\ B_4 \cap B_{-2} \subseteq C \cup D. \end{aligned}$$

□

By the lemma above, it suffices to find, for each region, $A, \ldots, D_-$, some θ such that parameters in that region are good for that θ. Unfortunately, choosing $\theta = \pm(1,1)$ doesn't work for all the points in the regions above and the optimal one-parameter subgroups are way more complicated when θ belongs to different GIT chambers.

Proposition 7.5.1. *The optimal one-parameter subgroups for*

$$\theta = (1,-2), (-1,2), (3,-2), (-3,2), (3,-1), (-3,1)$$

are given in Table 7.5.

Proof. The optimals for $\theta = (1,-2), (3,-2), (-3,1)$ are calculated in Sections C.2, C.3 and C.4 respectively in Appendix C. Applying Corollary 7.1.2 gives the remaining optimals. □

By Theorem 6.0.1,

$$\chi((ab)(cd)) = (a+b)c_0 + (c+d-a-b)c_1 + (a+b-\tfrac{1}{2}(c+d)).$$

Lemma 7.5.2. *Parameters in A are good for $\theta = (1,-2)$.*

Proof. The GIT parameter θ lies in the GIT chamber C_3^+ for which $B_1 \cup B_2 \cup B_{-3} \cup B_4$ are the bad parameters for the essential one-parameter subgroups. But $(B_1 \cup B_2 \cup B_{-3} \cup B_4) \cap A = \emptyset$, so points in A are not bad for any of the essential one-parameter subgroups for C_3^+. It remains to check that points of A are not bad for any of the extra one-parameter subgroups. For this, it suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups

θ	$(-1,-1)$	$(1,1)$	$(-1,2)$	$(1,-2)$	$(-3,2)$	$(3,-2)$	$(-3,1)$	$(3,-1)$
	(00)(01)	(00)(-10)	(00)(-10)	(00)(01)	(00)(-10)	(00)(01)	(00)(-10)	(00)(01)
	(01)(00)	(-10)(00)	(01)(00)	(-10)(00)	(01)(00)	(-10)(00)	(01)(00)	(-10)(00)
	(01)(01)	(-10)(-10)	(-10)(-10)	(01)(01)	(01)(01)	(-10)(-10)	(01)(01)	(-10)(-10)
	(00)(11)	(00)(-1-1)	(00)(-1-1)	(00)(11)	(00)(-1-1)	(00)(11)	(00)(-1-1)	(00)(11)
	(01)(11)	(-10)(-1-1)	(-10)(-1-1)	(01)(11)	(-10)(-1-1)	(01)(11)	(01)(11)	(-10)(-1-1)
	(11)(11)	(-1-1)(-1-1)	(-1-1)(-1-1)	(11)(11)	(11)(11)	(-1-1)(-1-1)	(11)(11)	(-1-1)(-1-1)
			(11)(-2-2)	(-1-1)(22)	(33)(-2-2)	(-3-3)(22)	(33)(-1-1)	(-3-3)(11)
			(01)(-2-2)	(-10)(22)	(03)(-2-2)	(-30)(22)	(03)(-1-1)	(-30)(11)
			(-12)(-4-1)	(-21)(14)	(16)(-41)	(-6-1)(-14)	(03)(-10)	(-30)(01)
			(01)(-20)	(-10)(02)	(03)(-20)	(-30)(02)	(13)(-11)	(-3-1)(-11)
			(-10)(-4-1)	(01)(14)	(22)(-32)	(-2-2)(-23)	(19)(11)	(-9-1)(-1-1)
			(-11)(-1-1)	(-11)(11)	(-19)(-1-1)	(-91)(11)	(55)(-35)	(-5-5)(-53)
					(01)(-41)	(-10)(-14)	(01)(-11)	(-10)(-11)

Table 7.2: A comparison of the optimal one-parameter subgroups for $W = B_2$ and various θ.

λ. Note that $c_1 \geq c_0 + \frac{5}{2}$ and $c_0 > -1$ for all $\mathbf{c} \in A$.

$$\chi((-1-1)(22)) = -2c_0 + 6c_1 - 4 > 0, \qquad \chi((-21)(14)) = -c_0 + 6c_1 - \tfrac{7}{2} > 0,$$
$$\chi((01)(14)) = c_0 + 4c_1 - \tfrac{3}{2} > 0, \qquad \chi((-11)(11)) = 2c_1 - 1 > 0,$$
$$\chi((-10)(22)) = -c_0 + 3c_1 - 3 > 0, \qquad \chi((-10)(02)) = -c_0 + 3c_1 - 2 > 0.$$

□

Lemma 7.5.3. *Parameters in B_+ are good for $\theta = (3,-2)$.*

Proof. The GIT parameter θ lies in the GIT chamber C_2^+ for which $B_1 \cup B_{-2} \cup B_{-3} \cup B_4$ are the bad parameters for the essential one-parameter subgroups. But $(B_1 \cup B_{-2} \cup B_{-3} \cup B_4) \cap B_+ = \emptyset$, so points in B_+ are not bad for any of the essential one-parameter subgroups for C_2^+. It remains to check that points of B_+ are not bad for any of the extra one-parameter subgroups. For this, it suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $c_1 \geq -c_0 + \frac{1}{2}$ and $-c_0 \geq 1$ for all $\mathbf{c} \in B_+$.

$$\chi((-3-3)(22)) = -6c_0 + 10c_1 - 8 > 0, \qquad \chi((-30)(22)) = -3c_0 + 7c_1 - 5 > 0,$$
$$\chi((-6-1)(-14)) = -7c_0 + 10c_1 - \tfrac{17}{2} > 0, \qquad \chi((-30)(02)) = -3c_0 + 5c_1 - 4 > 0,$$
$$\chi((-2-2)(-23)) = -4c_0 + 5c_1 - \tfrac{9}{2} > 0, \qquad \chi((-91)(11)) = -8c_0 + 10c_1 - 9 > 0,$$
$$\chi((-10)(-14)) = -c_0 + 4c_1 - \tfrac{5}{2} > 0.$$

□

Lemma 7.5.4. *Parameters in B_- are good for $\theta = (3,-1)$.*

Proof. The GIT parameter θ lies in the GIT chamber C_1^+ for which $B_1 \cup B_{-2} \cup B_{-3} \cup B_{-4}$ are the bad parameters for the essential one-parameter subgroups. But $(B_1 \cup B_{-2} \cup B_{-3} \cup B_{-4}) \cap B_- = \emptyset$, so points in B_- are not bad for any of the essential one-parameter subgroups for C_1^+. It remains to check that points of B_- are not bad for any of the extra one-parameter subgroups. For this, it suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $-c_0 > c_1 - \frac{1}{2}$ and $c_1 \geq \frac{3}{2}$ for all $\mathbf{c} \in B_-$.

$$\chi((-3-3)(11)) = -6c_0 + 8c_1 - 7 > 0, \qquad \chi((-31)(-11)) = -2c_0 + 2c_1 - 2 > 0,$$
$$\chi((-5-5)(-53)) = -10c_0 + 8c_1 - 9 > 0, \qquad \chi((-30)(11)) = -3c_0 + 5c_1 - 4 > 0,$$
$$\chi((-30)(01)) = -3c_0 + 4c_1 - \tfrac{7}{2} > 0, \qquad \chi((-9-1)(-1-1)) = -10c_0 + 8c_1 - 9 > 0,$$
$$\chi((-10)(-11)) = -c_0 + c_1 - 1 > 0.$$

□

Lemma 7.5.5. *Parameters in C are good for $\theta = (-1, 2)$.*

Proof. The GIT parameter θ lies in the GIT chamber C_3^- for which $B_{-1} \cup B_{-2} \cup B_3 \cup B_{-4}$ are the bad parameters for the essential one-parameter subgroups. But $(B_{-1} \cup B_{-2} \cup B_3 \cup B_{-4}) \cap C = \emptyset$, so points in C are not bad for any of the essential one-parameter subgroups for C_3^-. It remains to check that points of C are not bad for any of the extra one-parameter subgroups. For this, it suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $-c_1 \geq -c_0 + \frac{1}{2}$, $-c_0 > 0$ and $-c_0 > 0$ for all $\mathbf{c} \in C$.

$$\chi((11)(-2-2)) = 2c_0 - 6c_1 + 4 > 0, \qquad \chi((-12)(-4-1)) = c_0 - 6c_1 + \tfrac{11}{2} > 0,$$
$$\chi((-10)(-4-1)) = -c_0 - 4c_1 + \tfrac{3}{2} > 0, \qquad \chi((-11)(-1-1)) = -2c_1 + 1 > 0,$$
$$\chi((01)(-2-2)) = c_0 - 5c_1 + 3 > 0, \qquad \chi((01)(-20)) = c_0 - 3c_1 + 2 > 0.$$

□

Lemma 7.5.6. *Parameters in D_+ are good for $\theta = (-3, 1)$.*

Proof. The GIT parameter θ lies in the GIT chamber C_1^- for which $B_{-1} \cup B_2 \cup B_3 \cup B_4$ are the bad parameters for the essential one-parameter subgroups. But $(B_{-1} \cup B_2 \cup B_3 \cup B_4) \cap D_+ = \emptyset$, so points in D_+ are not bad for any of the essential one-parameter subgroups for C_1^-. It remains to check that points of D_+ are not bad for any of the extra one-parameter subgroups. For this, it suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $c_0 \geq -c_1 - \frac{1}{2}$ and $-c_1 \geq \frac{1}{2}$ for all $\mathbf{c} \in D_+$.

$$\chi((33)(-1-1)) = 6c_0 - 8c_1 + 7 > 0, \qquad \chi((-13)(-11)) = 2c_0 - 2c_1 + 2 > 0,$$
$$\chi((55)(-35)) = 10c_0 - 8c_1 + 9 > 0, \qquad \chi((03)(-1-1)) = 3c_0 - 5c_1 + 4 > 0,$$
$$\chi((03)(-10)) = 3c_0 - 4c_1 + \tfrac{7}{2} > 0, \qquad \chi((19)(11)) = 10c_0 - 8c_1 + 9 > 0,$$
$$\chi((01)(-11)) = c_0 - c_1 + 1 > 0.$$

□

Lemma 7.5.7. *Parameters in D_- are good for $\theta = (-3, 2)$.*

Proof. The GIT parameter θ lies in the GIT chamber C_2^- for which $B_1 \cup B_{-2} \cup B_{-3} \cup B_4$ are the bad parameters for the essential one-parameter subgroups. But $(B_1 \cup B_{-2} \cup B_{-3} \cup B_4) \cap D_- = \emptyset$, so points in B_+ are not bad for any of the essential one-parameter subgroups for C_2^-. It remains to check that points of D_- are not bad for any of the extra one-parameter subgroups. For this, it suffices to show that $\chi(d\lambda(1)) > 0$ for all the non-essential optimal one-parameter subgroups λ. Note that $-c_1 > c_0 + \frac{1}{2}$ and $c_0 \geq 0$ for all $\mathbf{c} \in D_-$.

$$\chi((33)(-2-2)) = 6c_0 - 10c_1 + 8 > 0, \qquad \chi((03)(-2-2)) = 3c_0 - 7c_1 + 5 > 0,$$
$$\chi((16)(-41)) = 7c_0 - 10c_1 + \tfrac{17}{2} > 0, \qquad \chi((03)(-20)) = 3c_0 - 5c_1 + 4 > 0,$$
$$\chi((22)(-32)) = 4c_0 - 5c_1 + \tfrac{9}{2} > 0, \qquad \chi((-19)(-1-1)) = 8c_0 - 10c_1 + 9 > 0,$$
$$\chi((01)(-41)) = c_0 - 4c_1 + \tfrac{5}{2} > 0.$$

□

Appendices

Appendix A

Parameters for the Rational Cherednik Algebra in the Literature

	Parameter Space	Non-cyclic parameter	Cyclic parameters for $t = 1, \dots, l-1$	Parameter convention
Reflection	$\{c_0, \dots, c_{l-1}\}$	c_0	c_t	
Hyperplane	$\{k_{00}, k_0, \dots, k_{l-1}\}$	$k_{00} = -c_0$	$\sum_{j=0}^{l-1}(k_{j+1} - k_j)\zeta^{jt} = -2c_t$	$k_0 = 0$
[BK12, §6.1]	$\{h_0, \dots, h_{l-1}\}$		$\sum_{j=0}^{l-1}(h_{j+1} - h_j)\zeta^{-jt} = -2c_t$	
[Kuw08, §4.1]	$\{\kappa_0, \dots, \kappa_{l-1}\}$		$\sum_{j=0}^{l-1}(\kappa_{j+1} - \kappa_j)\zeta^{-jt} = -2c_t$	
[GGS09, §2.3]	$\{c\}$	$c = 2c_0$		
[Chm06, §2.1]	$\{\mathbf{k}(s) \mid s \in \mathcal{S}\}$	$\mathbf{k}(\sigma) = c_0$	$\mathbf{k}(s_i^t) = c_t$	
[GGOR03, §3.1]	$\left\{k_{H,0} \dots, k_{H,\lvert W_H\rvert} \middle\vert H \in \mathcal{E}\right\}$	$k_{H_\sigma,1} = -c_0$	$\sum_{j=0}^{l-1}(k_{H_s,j+1} - k_{H_s,j})\zeta^{jt} = -2c_t$	$k_{H,0} = k_{H,\lvert W_H\rvert} = 0$
[Gor06, §3.3]	$\{k, c_{\zeta^1}, \dots, c_{\zeta^{l-1}}\}$	$k = -c_0$	$c_{\zeta^t} = -2c_t$	
[Gor08, §2.6]	$\{h, H_0, \dots, H_{l-1}\}$	$h = -c_0$	$\sum_{j=0}^{l-1}\zeta^{-jt}H_j = -2c_t$	$H_0 + \cdots + H_{l-1} = 0$
[DG10, §1.4]	$\{c_0, d_0, \dots, d_{l-1}\}$	$c_0 = c_0$	$\sum_{j=0}^{l-1}(d_{j-1} - d_j)\zeta^{-jt} = -2lc_t$	$d_0 + \cdots + d_{l-1} = 0$
[Rou05, §5.1] & [GL11, §2.1.3]	$\left\{\mathbf{h}_{H,0} \dots, \mathbf{h}_{H,\lvert W_H\rvert} \middle\vert H \in \mathcal{E}\right\}$	$\mathbf{h}_{H_\sigma,1} = c_0$	$\sum_{j=0}^{l-1}(\mathbf{h}_j - \mathbf{h}_{j+1})\zeta^{-jt} = -2c_t$	

Table A.1: A comparison of parameter conventions in the literature for the rational Cherednik algebra associated to the complex reflection group $G(l, 1, n)$. The reflection parameters in this thesis are chosen to agree with [Chm06] and the hyperplane parameters with [Val07]. Here, σ is a non-cyclic simple reflection, s is a cyclic simple reflection and ζ is a primitive l^{th} root of unity. The indices for the cyclic parameters are always taken modulo l.

Appendix B

Koszul Resolutions

Each of the following figures give complexes of homomorphisms between the projective modules of paths starting at each vertex. Each map is defined to be precomposition by the stated path.

To define the resolution, for $j \geq 0$, let $P^j := \oplus_i P_i^j$ and $d^j := \oplus_i d_i^j$. For each $k = 1, \ldots, 5$, the long exact sequence

$$0 \longrightarrow P^4 \xrightarrow{d^4} P^3 \xrightarrow{d^3} P^2 \xrightarrow{d^2} P^1 \xrightarrow{d^1} \mathsf{B}_k \xrightarrow{/\mathsf{B}_{k+}} \!\!\!\!\rightarrow (\mathsf{B}_k)_0$$

is a projective resolution of $(\mathsf{B}_k)_0$.

Figure B.1: Koszul resolution for the block B_1

$$0 \longrightarrow \mathsf{B}_1 e_1 \xrightarrow{\begin{pmatrix} a_{12} \\ \cdot \end{pmatrix}} \begin{matrix} \mathsf{B}_1 e_2 \\ \oplus \\ \mathsf{B}_1 e_1 \end{matrix} \xrightarrow{\begin{pmatrix} a_{21} & \cdot \\ \cdot & a_{12} \end{pmatrix}} \begin{matrix} \mathsf{B}_1 e_1 \\ \oplus \\ \mathsf{B}_1 e_2 \end{matrix} \longrightarrow\!\!\!\!\rightarrow (\mathsf{B}_1)_0 \ .$$

Figure B.2: Koszul resolution for the block B_2

$$0 \longrightarrow P^4 := \mathsf{B}_2 e_1 \xrightarrow{d^4 := \begin{pmatrix} a_{12} \\ \cdot \end{pmatrix}} P^3 := \begin{matrix} \mathsf{B}_2 e_2 \\ \oplus \\ \mathsf{B}_2 e_1 \end{matrix} \xrightarrow{d^3 := \begin{pmatrix} a_{21} & \cdot \\ -a_{23} & \cdot \\ \cdot & a_{12} \\ \cdot & \cdot \end{pmatrix}} P^2 := \begin{matrix} \mathsf{B}_2 e_1 \\ \oplus \\ \mathsf{B}_2 e_3 \\ \oplus \\ \mathsf{B}_2 e_2 \\ \oplus \\ \mathsf{B}_2 e_1 \end{matrix}$$

$$\xrightarrow{d^2 := \begin{pmatrix} a_{12} & a_{32} & \cdot & \cdot \\ \cdot & \cdot & a_{21} & \cdot \\ \cdot & \cdot & -a_{23} & \cdot \\ \cdot & \cdot & \cdot & a_{12} \end{pmatrix}} P^1 := \begin{matrix} \mathsf{B}_2 e_2 \\ \oplus \\ \mathsf{B}_2 e_1 \\ \oplus \\ \mathsf{B}_2 e_3 \\ \oplus \\ \mathsf{B}_2 e_2 \end{matrix} \xrightarrow{d^1 := \begin{pmatrix} a_{21} & \cdot & \cdot & \cdot \\ \cdot & a_{12} & a_{32} & \cdot \\ \cdot & \cdot & \cdot & a_{23} \end{pmatrix}} \begin{matrix} \mathsf{B}_2 e_1 \\ \oplus \\ \mathsf{B}_2 e_2 \\ \oplus \\ \mathsf{B}_2 e_3 \end{matrix} \xrightarrow{/\mathsf{B}_{2+}} (\mathsf{B}_2)_0$$

Figure B.3: Koszul resolution for the block B_3

$$P_1^4 := \mathsf{B}_3e_1 \xrightarrow{d_1^4:=\begin{pmatrix} a_{12} \\ -a_{13} \end{pmatrix}} P_1^3 := \begin{matrix} \mathsf{B}_3e_2 \\ \oplus \\ \mathsf{B}_3e_3 \end{matrix} \xrightarrow{d_1^3:=\begin{pmatrix} a_{21} & a_{31} \\ -a_{24} & -a_{34} \\ -a_{21} & -a_{31} \end{pmatrix}} P_1^2 := \begin{matrix} \mathsf{B}_3e_1 \\ \oplus \\ \mathsf{B}_3e_4 \\ \oplus \\ \mathsf{B}_3e_1 \end{matrix} \xrightarrow{d_1^2:=\begin{pmatrix} a_{12} & a_{42} & \cdot \\ \cdot & -a_{43} & a_{13} \end{pmatrix}} P_1^1 := \begin{matrix} \mathsf{B}_3e_2 \\ \oplus \\ \mathsf{B}_3e_3 \end{matrix} \xrightarrow{d_1^1:=\begin{pmatrix} a_{21} & a_{31} \end{pmatrix}} \mathsf{B}_3e_1$$

$$P_2^3 := \mathsf{B}_3e_1 \xrightarrow{d_2^3:=\begin{pmatrix} a_{12} \\ -a_{13} \end{pmatrix}} P_2^2 := \begin{matrix} \mathsf{B}_3e_2 \\ \oplus \\ \mathsf{B}_3e_3 \end{matrix} \xrightarrow{d_2^2:=\begin{pmatrix} a_{21} & a_{31} \\ -a_{24} & -a_{34} \end{pmatrix}} P_2^1 := \begin{matrix} \mathsf{B}_3e_1 \\ \oplus \\ \mathsf{B}_3e_4 \end{matrix} \xrightarrow{d_2^1:=\begin{pmatrix} a_{12} & a_{42} \end{pmatrix}} \mathsf{B}_3e_2$$

$$P_3^3 := \mathsf{B}_3e_1 \xrightarrow{d_3^3:=\begin{pmatrix} a_{12} \\ -a_{13} \end{pmatrix}} P_3^2 := \begin{matrix} \mathsf{B}_3e_2 \\ \oplus \\ \mathsf{B}_3e_3 \end{matrix} \xrightarrow{d_3^2:=\begin{pmatrix} a_{21} & a_{31} \\ -a_{24} & -a_{34} \end{pmatrix}} P_3^1 := \begin{matrix} \mathsf{B}_3e_1 \\ \oplus \\ \mathsf{B}_3e_4 \end{matrix} \xrightarrow{d_3^1:=\begin{pmatrix} a_{13} & a_{43} \end{pmatrix}} \mathsf{B}_3e_3$$

$$P_4^2 := \mathsf{B}_3e_1 \xrightarrow{d_4^2:=\begin{pmatrix} a_{12} \\ -a_{13} \end{pmatrix}} P_4^1 := \begin{matrix} \mathsf{B}_3e_2 \\ \oplus \\ \mathsf{B}_3e_3 \end{matrix} \xrightarrow{d_4^1:=\begin{pmatrix} a_{24} & a_{34} \end{pmatrix}} \mathsf{B}_3e_4$$

Figure B.4: Koszul resolution for the block B_4

$$P_1^4 := \mathsf{B}_4 e_1 \xrightarrow{\begin{pmatrix} a_{13} \end{pmatrix}} P_1^3 := \mathsf{B}_4 e_3 \xrightarrow{\begin{pmatrix} a_{31} \\ a_{32} \\ -a_{35} \end{pmatrix}} P_1^2 := \begin{matrix} \mathsf{B}_4 e_1 \\ \oplus \\ \mathsf{B}_4 e_2 \\ \oplus \\ \mathsf{B}_4 e_5 \end{matrix} \xrightarrow{\begin{pmatrix} a_{13} & a_{23} & a_{53} \end{pmatrix}} P_1^1 := \mathsf{B}_4 e_3 \xrightarrow{\begin{pmatrix} a_{31} \end{pmatrix}} \mathsf{B}_4 e_1$$

$$P_2^4 := \mathsf{B}_4 e_2 \xrightarrow{\begin{pmatrix} a_{23} \end{pmatrix}} P_2^3 := \mathsf{B}_4 e_3 \xrightarrow{\begin{pmatrix} \lambda^2 a_{31} \\ a_{32} \\ -a_{34} \end{pmatrix}} P_2^2 := \begin{matrix} \mathsf{B}_4 e_1 \\ \oplus \\ \mathsf{B}_4 e_2 \\ \oplus \\ \mathsf{B}_4 e_4 \end{matrix} \xrightarrow{\begin{pmatrix} a_{13} & a_{23} & a_{43} \end{pmatrix}} P_2^1 := \mathsf{B}_4 e_3 \xrightarrow{\begin{pmatrix} a_{32} \end{pmatrix}} \mathsf{B}_4 e_2$$

$$P_3^3 := \begin{matrix} \mathsf{B}_4 e_2 \\ \oplus \\ \mathsf{B}_4 e_1 \end{matrix} \xrightarrow{\begin{pmatrix} a_{23} & \cdot \\ \cdot & a_{13} \end{pmatrix}} P_3^2 := \begin{matrix} \mathsf{B}_4 e_3 \\ \oplus \\ \mathsf{B}_4 e_3 \end{matrix} \xrightarrow{\begin{pmatrix} \lambda^2 a_{31} & a_{31} \\ a_{32} & a_{32} \\ -a_{34} & -a_{35} \end{pmatrix}} P_3^1 := \begin{matrix} \mathsf{B}_4 e_1 \\ \oplus \\ \mathsf{B}_4 e_2 \\ \oplus \\ \mathsf{B}_4 e_4 \\ \oplus \\ \mathsf{B}_4 e_5 \end{matrix} \xrightarrow{\begin{pmatrix} a_{13} & a_{23} & a_{43} & a_{53} \end{pmatrix}} \mathsf{B}_4 e_3$$

$$P_4^2 := \mathsf{B}_4 e_2 \xrightarrow{\begin{pmatrix} a_{23} \end{pmatrix}} P_4^1 := \mathsf{B}_4 e_3 \xrightarrow{\begin{pmatrix} a_{34} \end{pmatrix}} \mathsf{B}_4 e_4$$

$$P_5^2 := \mathsf{B}_4 e_1 \xrightarrow{\begin{pmatrix} a_{13} \end{pmatrix}} P_5^1 := \mathsf{B}_4 e_3 \xrightarrow{\begin{pmatrix} a_{35} \end{pmatrix}} \mathsf{B}_4 e_5$$

Figure B.5: Koszul resolution for the block B_5

$$\mathsf{B}_5e_1 \xrightarrow{d_1^4:=\begin{pmatrix} a_{14} \\ -a_{12} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_4 \\ \oplus \\ \mathsf{B}_5e_2 \end{matrix} \xrightarrow{d_1^3:=\begin{pmatrix} ta_{41} & \lambda a_{21} \\ -a_{43} & -a_{23} \\ a_{45} & \cdot \\ -\lambda a_{41} & -a_{21} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_1 \\ \oplus \\ \mathsf{B}_5e_3 \\ \oplus \\ \mathsf{B}_5e_5 \\ \oplus \\ \mathsf{B}_5e_1 \end{matrix} \xrightarrow{d_1^2:=\begin{pmatrix} a_{14} & a_{34} & a_{54} & \cdot \\ \cdot & -a_{32} & \cdot & a_{12} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_4 \\ \oplus \\ \mathsf{B}_5e_2 \end{matrix} \xrightarrow{d_1^1:=\begin{pmatrix} a_{41} & a_{21} \end{pmatrix}} \mathsf{B}_5e_1$$

$$\mathsf{B}_5e_2 \xrightarrow{d_2^4:=\begin{pmatrix} \lambda a_{21} \\ -a_{23} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_1 \\ \oplus \\ \mathsf{B}_5e_3 \end{matrix} \xrightarrow{d_2^3:=\begin{pmatrix} a_{12} & \lambda a_{32} \\ -a_{14} & -a_{34} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_2 \\ \oplus \\ \mathsf{B}_5e_4 \end{matrix} \xrightarrow{d_2^2:=\begin{pmatrix} a_{23} & a_{43} \\ -a_{21} & -\lambda a_{41} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_\tau \\ \oplus \\ \mathsf{B}_5e_1 \end{matrix} \xrightarrow{d_2^1:=\begin{pmatrix} a_{32} & a_{12} \end{pmatrix}} \mathsf{B}_5e_2$$

$$\mathsf{B}_5e_2 \xrightarrow{d_3^3:=\begin{pmatrix} \lambda a_{21} \\ -a_{23} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_1 \\ \oplus \\ \mathsf{B}_5e_3 \end{matrix} \xrightarrow{d_3^2:=\begin{pmatrix} a_{12} & \lambda a_{32} \\ -a_{14} & -a_{34} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_2 \\ \oplus \\ \mathsf{B}_5e_4 \end{matrix} \xrightarrow{d_3^1:=\begin{pmatrix} a_{23} & a_{43} \end{pmatrix}} \mathsf{B}_5e_3$$

$$\mathsf{B}_5e_1 \xrightarrow{d_4^3:=\begin{pmatrix} a_{14} \\ -a_{12} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_2 \\ \oplus \\ \mathsf{B}_5e_4 \end{matrix} \xrightarrow{d_4^2:=\begin{pmatrix} \lambda a_{21} & -ta_{41} \\ -a_{23} & a_{43} \\ \cdot & -a_{45} \end{pmatrix}} \begin{matrix} \mathsf{B}_5e_1 \\ \oplus \\ \mathsf{B}_5e_\tau \\ \oplus \\ \mathsf{B}_5e_5 \end{matrix} \xrightarrow{d_4^1:=\begin{pmatrix} a_{14} & a_{34} & a_{54} \end{pmatrix}} \mathsf{B}_5e_4$$

$$\mathsf{B}_5e_1 \xrightarrow{d_5^2:=\begin{pmatrix} a_{14} \end{pmatrix}} \mathsf{B}_5e_2 \xrightarrow{d_5^1:=\begin{pmatrix} a_{45} \end{pmatrix}} \mathsf{B}_5e_5$$

Appendix C

Proofs

C.1 Proof of Proposition 7.4.1

C.1.1 The Optimal Subgroups for $\theta = (0, -2, 1)$

Case	μ	Signs of weights of μ	Unstable points for μ	$m(\mu)^2$
1.1	$(0)(2)(-1)$	$((+)(-)(+), (-)(+)(-); (0)(0))$	$\{(*)(0)(*), (0)(*)(0); (*)(*)\}$	5
1.2	$(1)(1)(1)$	$((+)(-)(0), (-)(+)(0); (-)(+))$	$\{(*)(0)(*), (0)(*)(*); (0)(*)\}$	$4\frac{1}{2}$
1.3	$(-1)(4)(1)$	$((+)(-)(0), (-)(+)(0); (0)(0))$	$\{(*)(0)(*), (0)(*)(*); (*)(*)\}$	4
2	$(0)(1)(0)$	$((0)(-)(+), (0)(+)(-); (+)(-))$	$\{(*)(0)(*), (*)(*)(0); (*)(0)\}$	3
3	$(1)(1)(-1)$	$((0)(-)(+), (0)(+)(-); (0)(0))$	$\{(*)(0)(*), (*)(*)(0); (*)(*)\}$	1
4	$(0)(0)(-1)$	$((+)(0)(-), (-)(0)(+); (0)(0))$	$\{(*)(*)(0), (0)(*)(*); (*)(*)\}$	$\frac{1}{2}$
5	$(0)(1)(1)$	$((0)(0)(0), (0)(0)(0); (+)(-))$	$\{(*)(*)(*), (*)(*)(*); (*)(0)\}$	$\frac{1}{3}$

Table C.1: The optimal one-parameter subgroups for $W = \mu_3$ and $\theta = (0, -2, 1)$ along with the weights acting on an arbitrary point and the set of points which are unstable for that subgroup. The $*$ symbol represents any complex number.

First, a classification of the unstable points in X using the stability criteria of King, Theorem 5.8.1. With the notation defined in Section 5.8.5, $\theta \cdot \delta = -1$, so $\hat{\theta} = (1, 0, -2, 1)$. Then, in order for a proper subrepresentation to be destabilising, it must have one of the following dimension vectors.

$$(1,1,1,0) \qquad (1,0,1,0) \qquad (0,1,1,1) \qquad (0,1,1,0) \qquad (0,0,1,1) \qquad (0,0,1,0)$$

Let the phrase 'q **has type** (a, b, c, d)' be short-hand for 'q has a destablising subrepresentation of type (a, b, c, d)'.

Note the weights of $\lambda = (a)(b)(c)$ acting on the coordinates of a point $p \in X$.

$$\Big(\big(b-a \quad c-b \quad a-c\big), \big(a-b \quad b-c \quad c-a\big); \big(a\big), \big(-a\big)\Big).$$

Each of the cases below corresponds to some proposed optimal one-parameter subgroup μ. First, it is necessary to show that the cases below are exhaustive of unstable points.

Lemma C.1.1. *Let q be an unstable point. Then q satisfies one of the conditions in the cases that follow.*

Proof. Being of type $(0, 0, 1, 0)$ implies Case 1.1, 1.2 or 1.3. Being of type $(0, 1, 1, 0)$, $(1, 1, 1, 0)$, $(0, 0, 1, 1)$ and $(0, 1, 1, 1)$ is covered by the remaining cases. It now suffices to note that if a point is of type $(1, 0, 1, 0)$ then it must have type $(0, 0, 1, 0)$. □

Given an optimal one-parameter subgroup λ that destabilises a point p, define

$$m(\lambda) := \frac{-\mu(p,\lambda)}{\|\lambda\|}.$$

Then, λ is an optimal one-parameter subgroup for another point q if and only if λ destabilises q and $m(\lambda) = M(q)$.

Then, for each case the following argument will be used.

Lemma C.1.2. *Let q be a point in X^{us}, defined by its types. Let $\lambda = (a)(b)(c)$ be an optimal one-parameter subgroup for q. Then μ is optimal for q if the following hold.*

1. *One of the conditions on q creates a restriction that bounds the value of $m(\lambda)$ above.*
2. *The proposed subgroup μ achieves this bound.*
3. *The point q is destabilised by μ.*

Proof. The first two points show that $M(q) = m(\lambda) \leq m(\mu)$. If q is destablised by μ then $M(q) \geq m(\mu)$ so equality holds. □

Case 1.1: The point q has type $(0,0,1,0)$ and $(1,1,1,0)$

There are no restrictions imposed on the values a, b and c of λ. So,

$$\begin{aligned} 0 &\leq 5a^2 + (b+2c)^2 \\ &= 5(a^2+b^2+c^2) - (2b-c)^2, \\ &= 5\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

Taking square roots gives, $m(\lambda) \leq \sqrt{5} = m((0)(2)(-1))$. Being of type $(0,0,1,0)$ implies that $X^{(1)} = Y^{(0)} = 0$ and being of type $(1,1,1,0)$ implies that $Y^{(2)} = 0$. Thus $(0)(2)(-1)$ acts by non-zero weights on q and so is optimal for q.

Case 1.2: The point q has type $(0,0,1,0)$ and $(1,0,1,0)$ but not $(1,1,1,0)$

Being of type $(0,0,1,0)$ but not $(1,1,1,0)$ implies that $Y^{(2)} \neq 0$ so that $a \leq c$. Now,

$$\begin{aligned} 0 &\leq 2(2b-c)(c-a) + 5(c-a)^2 + (2a+b+2c)^2 \\ &= 9a^2 + b^2 + 7c^2 + 8bc \\ &= 9(a^2+b^2+c^2) - 2(2b-c))^2, \\ &= 9\|\lambda\|^2 - 2\mu(p,\lambda)^2, \end{aligned}$$

and $m(\lambda) \leq \sqrt{4\frac{1}{2}} = m((-1)(4)(-1))$. Being of type $(0,0,1,0)$ implies $X^{(1)} = Y^{(0)} = 0$ and being of type $(1,0,1,0)$ implies $v = 0$. Thus $(-1)(4)(-1)$ acts by non-zero weights on q and so is optimal for q.

Case 1.3: The point q has type $(0,0,1,0)$ but neither $(1,0,1,0)$ nor $(1,1,1,0)$

As in the previous case, $a \leq c$. But also not being of type $(1,0,1,0)$ implies that $v \neq 0$ so that $0 \leq a \leq c$. Now,

$$\begin{aligned} 0 &\leq 4a^2 + 5c^2 + 2c(2b-c) \\ &= 4a^2 + 3c^2 + 4bc \\ &= 4(a^2+b^2+c^2) - (2b-c)^2, \\ &= 4\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

and $m(\lambda) \leq \sqrt{4} = m((0)(1)(0))$. Being of type $(0,0,1,0)$ implies $X^{(1)} = Y^{(0)} = 0$ so $(0)(1)(0)$ acts by non-zero weights on q and so is optimal for q.

Case 2: The point q **has type** $(0,1,1,0)$ **but not type** $(0,0,1,0)$

These conditions together imply that $Y^{(0)} \neq 0$ so that $b \leq a$. Now,

$$\begin{aligned} 0 &\leq 6b(a-b) + 3(a-b)^2 + 2(b+c)^2 \\ &= 3a^2 - b^2 + 2c^2 + 4bc \\ &= 3(a^2+b^2+c^2) - (2b-c)^2 \\ &= 3\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

and $m(\lambda) \leq \sqrt{3} = m((1)(1)(-1))$. Being of type $(0,1,1,0)$ implies $X^{(1)} = Y^{(2)} = w = 0$. Thus $(1)(1)(-1)$ acts by non-zero weights on q and so is optimal for q.

Case 3: The point q **has type** $(1,1,1,0)$ **but not type** $(0,0,1,0)$ **or** $(0,1,1,0)$

Being of type $(1,1,1,0)$ but not of type $(0,1,1,0)$ implies that $w \neq 0$ and being of type $(1,1,1,0)$ but not of type $(0,1,1,0)$ implies that $Y^{(0)} \neq 0$. Together this gives $b \leq 0$. Now,

$$\begin{aligned} 0 &\leq a^2 + 5b^2 - 4b(2b-c) \\ &= a^2 - 3b^2 + 4bc \\ &= 3(a^2+b^2+c^2) - (2b-c)^2 \\ &= \|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

and $m(\lambda) \leq \sqrt{1} = m((0)(0)(-1))$. Being of type $(1,1,1,0)$ implies $X^{(1)} = Y^{(2)} = 0$. Thus $(0)(0)(-1)$ acts by non-zero weights on q and so is optimal for q.

Case 4: The point q **has type** $(0,0,1,1)$ **but not type** $(0,0,1,0)$**,** $(0,1,1,0)$ **or** $(1,1,1,0)$

Being of type $(0,0,1,1)$ but not of type $(0,0,1,0)$ implies $X^{(1)} \neq 0$ so $b \leq c$. If $b \geq 0$ then $\frac{2b-c}{b^2+c^2} \geq \frac{-2b-c}{b^2+c^2}$ so assume that $b \geq 0$. Then

$$\begin{aligned} 0 &\leq (c-b)(2b-c) + 5b(c-b) \\ &\leq (b^2+c^2) - 2(2b-c)^2 \\ &= (a^2+b^2+c^2) - 2(2b-c)^2 \\ &= \|\lambda\|^2 - 2\mu(p,\lambda)^2. \end{aligned}$$

and $m(\lambda) \leq \sqrt{\frac{1}{2}} = m((0)(1)(1))$. Being of type $(0,0,1,1)$ implies $X^{(2)} = Y^{(0)} = 0$. Thus $(0)(1)(1)$ acts by non-zero weights on q and so is optimal for q.

Case 5: The point q **only has type** $(0,1,1,1)$

I claim that $b \leq a, c$. Indeed, if $Y^{(0)} = 0$ then not being of type $(0,0,1,0)$ implies $X^{(1)} \neq 0$ and not being of type $(0,0,1,1)$ implies $X^{(2)} \neq 0$; these two imply the claim. Otherwise, $Y^{(0)} \neq 0$ and if $X^{(1)} \neq 0$ the claim follows. If $X^{(1)} = 0$ then not being of type $(1,1,1,0)$ implies that $Y^{(2)} \neq 0$ and the claim follows. Now,

$$\begin{aligned} 0 &\leq (a-b)^2 + 2(a-b)(2b-c) + 2(a-b)(c-b) + 8(2b-c)(c-b) + 6(c-b)^2 \\ &= (a^2+b^2+c^2) - 3(2b-c)^2 \\ &= \|\lambda\|^2 - 3\mu(p,\lambda)^2. \end{aligned}$$

and $m(\lambda) \leq \sqrt{\frac{1}{3}} = m((1)(1)(1))$. Being of type $(0,1,1,1)$ implies $w = 0$. Thus $(1)(1)(1)$ acts by non-zero weights on q and so is optimal for q.

C.1.2 The Optimal Subgroups for $\theta = (0,2,-1),(0,-1,2),(0,1,-2)$

The definitions and arguments used in the calculation of optimals for $\theta = (0,-2,1)$ are valid if one reformulates everything, exchanging vertices 1 and 2 on the quiver $Q_\infty^{3,1}$. Everywhere now $Y^{(2)}$, $Y^{(1)}$ and $Y^{(0)}$ play the roles of $X^{(0)}$, $X^{(1)}$ and $X^{(2)}$ respectively and in the definition of λ the roles of the coordinates b and c have swapped. In this way the optimal one-parameter subgroups for $(0,1,-2)$ can be calculated directly from those of $(0,-2,1)$ by swapping the second and third values. Applying Corollary 7.1.2, gives the result for $\theta = (0,2,-1)$ and $\theta = (0,-1,2)$.

C.2 The Optimal Subgroups for $\theta = (-1,2)$

Case	μ	Signs of weights of μ	Unstable points for μ	$m(\mu)^2$
1.1	$(11)(-2-2)$	$\left(\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right);(+\ +)(-\ -)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(\ast\ \ast)(0\ 0)\right\}$	10
1.2	$(01)(-2-2)$	$\left(\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right);(0\ +)(0\ -)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(\ast\ \ast)(\ast\ 0)\right\}$	9
1.3	$(00)(-1-1)$	$\left(\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right);(0\ 0)(0\ 0)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(\ast\ \ast)(\ast\ \ast)\right\}$	8
2.1	$(-12)(-4-1)$	$\left(\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right);(-\ +)(+\ -)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right);(0\ \ast)(\ast\ 0)\right\}$	$5\frac{1}{2}$
2.2	$(01)(-20)$	$\left(\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right);(0\ +)(0\ -)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right);(\ast\ \ast)(\ast\ 0)\right\}$	5
2.3	$(-10)(-4-1)$	$\left(\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right);(-\ 0)(+\ 0)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right);(0\ \ast)(\ast\ \ast)\right\}$	$4\frac{1}{2}$
2.4	$(00)(-10)$	$\left(\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right);(0\ 0)(0\ 0)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right);(\ast\ \ast)(\ast\ \ast)\right\}$	4
3.1	$(-11)(-1-1)$	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(-\ +)(+\ -)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(0\ \ast)(\ast\ 0)\right\}$	4
3.2	$(-10)(-1-1)$	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(-\ 0)(+\ 0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(0\ \ast)(\ast\ \ast)\right\}$	3
3.3	$(01)(00)$	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(0\ +)(0\ -)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(\ast\ \ast)(\ast\ 0)\right\}$	1
4	$(-1-1)(-1-1)$	$\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(-\ -)(+\ +)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\ 0)(\ast\ \ast)\right\}$	1
5	$(-10)(-10)$	$\left(\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right);(-\ 0)(+\ 0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right);(0\ \ast)(\ast\ \ast)\right\}$	$\frac{1}{2}$

Table C.2: The optimal one-parameter subgroups for $W = G(2,1,2)$ and $\theta = (-1,2)$ along with the weights acting on an arbitrary point and the set of points which are unstable for that subgroup. The $\ast$ symbol represents any complex number.

First, a classification of the unstable points in X using the stability criteria of King, Theorem 5.8.1. With the notation defined in Section 5.8.5, $\theta \cdot \delta = 2$, so $\hat{\theta} = (-2,-1,2)$. Then, in order for a proper subrepresentation to be destabilising, it must have one of the following dimension vectors.

$$(1,2,0) \qquad (0,2,0) \qquad (1,2,1) \qquad (1,1,0) \qquad (0,1,0) \qquad (1,0,0) \qquad (1,1,1)$$

Let the phrase 'q **has type** (a,b,c)' be short-hand for 'q has a destablising subrepresentation of type (a,b,c)'.

As in the last section, given an optimal one-parameter subgroup λ that destabilises a point p, define

$$m(\lambda) := \frac{-\mu(p,\lambda)}{\|\lambda\|}.$$

Let q be an unstable point for some one-parameter subgroup $\mu \in \mathbb{Y}(G)$. Because the image of μ in G lies inside a maximal torus, all of which are conjugate, there is some $g \in G = \mathrm{GL}_2 \times \mathrm{GL}_2$ that diagonalises $\mu(1)$. Define $\lambda := \mathrm{Ad}(g) \cdot \mu = (a_1 a_2)(b_1 b_2) \in \mathbb{Y}(T)$. Let $\mathbf{X}^{(0)}$, $\mathbf{Y}^{(1)}$, $\mathbf{v}$ and $\mathbf{w}$ be the coordinates of a point, $p = (\mathbf{X}^{(0)}, \mathbf{X}^{(1)}, \mathbf{Y}^{(0)}, \mathbf{Y}^{(1)}; \mathbf{v}, \mathbf{w})$, in the orbit of q which λ destabilises.

Then, if μ is optimal for q, $q \in S^+_{\langle\lambda\rangle}$. Let $V = V_\infty \oplus V_0 \oplus V_1$ be a decomposition of the quiver representation q at the nodes ∞, 0 and 1 repsectively and let $V_0 = \langle e_1, e_2\rangle$ and $V_1 = \langle e_3, e_4\rangle$. Choosing such a basis $\{e_1, e_2, e_3, e_4\}$ is equivalent to choosing a point r in the G-orbit of q. This notation will persist through the section: q will be some arbitrary point with some conditions, p will be a point in the same orbit that is destabilised by some $\lambda \in \mathbf{Y}(T)$ and r will be some point in the orbit of q that corresponds to choosing some basis $\{e_1, e_2, e_3, e_4\}$ of V as above.

Note the weights of $\lambda = (a_1a_2)(b_1b_2)$ acting on the coordinates of p.

$$\left(\left(b_i - a_j\right)_{ij}, \left(a_i - b_j\right)_{ij}, \left(a_i - b_j\right)_{ij}, \left(b_i - a_j\right)_{ij}; \left(a_i\right)_i, \left(-a_i\right)_i\right).$$

It follows that if either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ has a non-zero coordinate in the (i, j) position then $a_j \leq b_i$.

Each of the cases below corresponds to some proposed optimal one-parameter subgroup μ. First, it is necessary to show that the cases below are exhaustive of unstable points.

Lemma C.2.1. *Let q be an unstable point. Then q satisfies one of the conditions in the following cases.*

Proof. If q has type $(0, 2, 0)$ or $(1, 2, 0)$ then it is dealt with in Case 1 because being of type $(0, 2, 0)$ implies being of type $(1, 2, 0)$. Therefore, assume that q doesn't have type $(1, 2, 0)$. If q has type $(1, 2, 1)$ it is dealt with in Case 2 so assume that q doesn't have type $(1, 2, 1)$. If q has type $(1, 1, 0)$ or $(0, 1, 0)$ then it is treated in Case 3, so assume that q is not of type $(1, 1, 0)$ or $(0, 1, 0)$. This leaves q having type $(1, 0, 0)$ or $(1, 1, 1)$ which are treated in Cases 4 and 5 respectively. □

Then, for each case the following argument will be used.

Lemma C.2.2. *Let q be a point in X^{us}, defined by its types. Let p be a point in the orbit of q which is unstable for some $\lambda \in \Upsilon(G)$. Then μ is optimal for q if the following hold.*

1. *One of the conditions on q creates a restriction that bounds the value of $m(\lambda)$ above.*
2. *The proposed subgroup μ achieves this bound.*
3. *There is a point r in the orbit of q which is destabilised by μ.*

Proof. First note that the function M is invariant on G-orbits. The first two points show that $M(q) = M(p) = m(\lambda) \leq m(\mu)$. If r is destablised by μ then $M(r) \geq m(\mu)$ and so, because $M(r) = M(q) \leq m(\mu)$, $M(r) = m(\mu)$. It follows that μ is optimal for r. Now, since $S^+_{\langle\mu\rangle} = G \times_{P(\mu)} S_\mu$, it follows that $q \in S^+_{\langle\mu\rangle}$. □

Remark C.2.1. *The Weyl group, $\mathfrak{S}_2 \times \mathfrak{S}_2$, acts on subgroups in $\Upsilon(T)$, by permuting the coordinates, $(a_1a_2)(b_1b_2)$ say, in the obvious way. Therefore, for each case, depending on the multiplicity of the entries of λ, there will be up to four optimals. The corresponding action of the Weyl group on the point p has the effect of swapping the rows or columns (or both) of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ simultaneously.*

The condition that p is (or is not) of a certain type will put restrictions on the weights of any destabilising λ acting on p, for example, 'there exists an i and j such that $a_i \leq b_j$'. The function m is invariant with respect to the action of the Weyl group, that is to say if two subgroups are conjugate then m will take the same value on both. Therefore, it suffices to consider one case (maybe $i = 1$, $j = 2$ in the example) in order to show that m is bounded for all similar restrictions on λ.

Case 1.1: The point q has type $(1, 2, 0)$ and $\mathbf{w} = 0$

Being of type $(1, 2, 0)$ is equivalent to $\mathbf{X}^{(0)} = \mathbf{Y}^{(1)} = 0$ and

$$\begin{aligned}
0 &\leq (a_1 - a_2)^2 + (2a_1 + b_1)^2 + (2a_1 + b_2)^2 + (2a_2 + b_1)^2 + (2a_2 + b_2)^2 + 4(b_1 - b_2)^2 \\
&= \left(\sqrt{10}\sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2}\right)^2 - (a_1 + a_2 - 2(b_1 + b_2))^2, \\
&= 10\|\lambda\|^2 - \mu(p, \lambda)^2.
\end{aligned}$$

Because $-\mu(p, \lambda) = a_1 + a_2 - 2(b_1 + b_2) < 0$, taking square roots gives, $m(\lambda) \leq \sqrt{10} = m((11)(-2-2))$. Since $(11)(-2-2)$ acts by non-zero weights on q, it destabilises q and so $(11)(-2-2)$ optimal for q.

Case 1.2: The point q has type $(1,2,0)$ and exactly one of the entries of w is zero

If $w_1 \neq 0$ and $w_2 = 0$ then $a_1 \leq 0$, so

$$\begin{aligned} 0 &\leq 10a_1^2 - 2a_1(a_1 + a_2 - 2b_1 - 2b_2) + (2a_2 + b_1)^2 + (2a_2 + b_2)^2 + 4(b_1 - b_2)^2 \\ &= \left(3\sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2}\right)^2 - (a_1 + a_2 - 2(b_1 + b_2))^2, \\ &= 9\|\lambda\|^2 - \mu(p,\lambda)^2, \end{aligned}$$

and $m(\lambda) \leq \sqrt{9} = m((01)(-2-2))$. Since $(01)(-2-2)$ destabilises p, $(01)(-2-2)$ is optimal for q. A similar argument shows that $(10)(-2-2)$ is the optimal one-parameter subgroup when $w_1 = 0$ and $w_2 \neq 0$.

Case 1.3: The point q has type $(1,2,0)$ and neither entry of w is zero

If w_1 and w_2 are both non-zero then $a_1, a_2 \leq 0$ and so $b_1 + b_2 < 0$. Now,

$$\begin{aligned} 0 &\leq 9a_1^2 + 2a_1a_2 + 9a_2^2 - 2(a_1 + a_2)(a_1 + a_2 - 2b_1 - 2b_2) + 4(b_1 - b_2)^2, \\ &= \left(\sqrt{8}\sqrt{a_1^2 + a_2^2 + b_1^2 + b_2^2}\right)^2 - (a_1 + a_2 - 2(b_1 + b_2))^2, \\ &= 8\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

and so $m(\lambda) \leq \sqrt{8} = m((00)(-1-1))$. Since $(00)(-1-1)$ destablises p, it is optimal for q.

Case 2.1: The point q has type $(1,2,1)$, $(1,1,0)$ and $(0,1,0)$, but does not have type $(1,2,0)$

Either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ is non-zero, so one of them has at least one non-zero entry in the (j,i) position say. Therefore, $a_i \leq b_j$ for some $1 \leq i, j \leq 2$.

Assume that $i = j = 1$. Then $b_1 - a_1 \geq 0$ so

$$\begin{aligned} 0 &\leq 6(a_1 + a_2 - 2b_1 - 2b_2)(b_1 - a_1) + 10(b_1 - a_1)^2 + \tfrac{1}{3}(4a_2 - 2a_1 - 2b_1 + 3b_2)^2 + \tfrac{11}{3}(a_1 + b_1 + a_2)^2 \\ &= 9a_1^2 - 4a_1a_2 + 8a_1b_1 + 8a_1b_2 + 9a_2^2 + 8a_2b_1 + 8a_2b_2 + 3b_1^2 - 16b_1b_2 + 3b_2^2 \\ &= 11(a_1^2 + a_2^2 + b_1^2 + b_2^2) - 2(a_1 + a_2 - 2b_1 - 2b_2)^2 \\ &= 11\|\lambda\|^2 - 2\mu(p,\lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{\frac{11}{2}} = m((-12)(-4-1))$.

It remains to find a point r in the G-orbit of q which $(-12)(-4-1)$ destabilises. Specify r by choosing a basis for V. Because q is of type $(0,1,0)$, $\mathbf{X}^{(0)}$, $\mathbf{Y}^{(1)}$ and $\mathbf{w}$ have non-zero kernels containing a common vector, e_2 say. Because q has type $(1,2,1)$, their images must span a one-dimensional subspace, $\langle e_4\rangle_{\mathbb{C}}$ say. Extend e_2 to a basis, $\langle e_1, e_2\rangle_{\mathbb{C}}$, of V_0 and e_4 to a basis, $\langle e_3, e_4\rangle_{\mathbb{C}}$, of V_1. Then, for r, the coordinates $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are strictly lower triangular and $v_1 = w_2 = 0$. Using Table C.2, $(-12)(-4-1)$ acts on r with non-negative weights so $q \in S^+_{(-12)(-4-1)}$.

Case 2.2: The point q has type $(1,2,1)$ and $(0,1,0)$, but does not have type $(1,2,0)$ or $(1,1,0)$

The point has type $(1,1,0)$ if $\mathbf{X}^{(0)}\mathbf{v} = \mathbf{Y}^{(1)}\mathbf{v} = 0$. Therefore, there is some $1 \leq i, j \leq 2$ such that $X^{(0)}_{ji}v_i$ or $Y^{(1)}_{ji}v_i$ is not equal to zero; so $0 \leq a_i \leq b_j$.

Assume that $i = j = 1$. Then $b_1 - a_1 \geq 0$ and $a_1 \geq 0$ so

$$\begin{aligned} 0 &\leq 14a_1(b_1 - a_1) + 2a_1(a_1 + a_2 - 2b_1 - 2b_2) + 9(b_1 - a_1)^2 \\ &\quad + 4(b_1 - a_1)(a_1 + a_2 - 2b_1 - 2b_2) + 11a_1^2 + (2a_2 + b_2)^2 \\ &= 4a_1^2 - 2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 4a_2^2 + 4a_2b_1 + 4a_2b_2 + b_1^2 - 8b_1b_2 + 1b_2^2 \\ &= 5(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1 + a_2 - 2b_1 - 2b_2)^2 \\ &= 5\|\lambda\|^2 - \mu(p, \lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{5} = m((01)(-20))$. It remains to find a point r in the orbit of q which $(01)(-20)$ destabilises.

Let e_2 be a non-zero vector in $(0, 1, 0)$; it must be in the kernel of $\mathbf{w}$, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. Since q does not have type $(1, 2, 0)$ but does have type $(1, 2, 1)$, the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must span a one-dimensional subspace, $\langle e_4 \rangle_{\mathbb{C}}$ say. Then, after extending e_2 and e_4 to a basis, as coordinates of r, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are strictly lower triangular and $w_2 = 0$. Using Table C.2, $(01)(-20)$ acts on r with non-negative weights. Therefore, $q \in S^+_{(01)(-20)}$.

Case 2.3: The point q has type $(1, 2, 1)$ and $(1, 1, 0)$, but does not have type $(1, 2, 0)$ or $(0, 1, 0)$

Without loss of generality, assume that $\mathbf{X}^{(0)} \neq 0$. Now, because q has type $(1, 1, 0)$, the kernels of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must intersect non-trivially and $\ker \mathbf{X}^{(0)} \cap \ker \mathbf{Y}^{(1)} = \ker \mathbf{X}^{(0)}$. Note that the condition that q has type $(0, 1, 0)$ is now equivalent to $\ker \mathbf{X}^{(0)} \subseteq \ker \mathbf{w}$; so this must be false.

I claim that there exists some $1 \leq i, j, i' \leq 2$ with $i \neq i'$ such that $w_{i'} \neq 0 \neq X^{(0)}_{ji}$. Indeed, the claim is immediate if both entries of $\mathbf{w}$ were non-zero; otherwise, since $\mathbf{w}$ cannot be zero, it has a non-zero entry $w_{i'}$ say. Let $i \neq i'$ so $w_i = 0$. The claim could only fail to hold if column i of $\mathbf{X}^{(0)}$ were zero, but this would imply that $\ker \mathbf{w} = \ker \mathbf{X}^{(0)}$, a contradiction. The claim implies that $a_i \leq b_j$ and $a_{i'} \leq 0$.

Assume that $i = j = 1$. Then $b_1 - a_1 \geq 0$ and $a_2 \leq 0$ so

$$\begin{aligned} 0 &\leq 11a_2^2 - 6a_2(b_1 - a_1) - 4a_2(a_1 + a_2 - 2b_1 - 2b_2) + 9(b_1 - a_1)^2 \\ &\quad + 6(b_1 - a_1)(a_1 + a_2 - 2b_1 - 2b_2) + (2a_1 + 2b_1 - b_2)^2 \\ &= 4a_1^2 - 2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 4a_2^2 + 4a_2b_1 + 4a_2b_2 + b_1^2 - 8b_1b_2 + b_2^2 \\ &= 9(a_1^2 + a_2^2 + b_1^2 + b_2^2) - 2(a_1 + a_2 - 2b_1 - 2b_2)^2 \\ &= 9\|\lambda\|^2 - 2\mu(p, \lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{\frac{9}{2}} = m((-10)(-4-1))$.

Let e_2 be the non-zero vector in the kernel of $\mathbf{X}^{(0)}$ (and so in $\ker \mathbf{Y}^{(1)}$) and e_4 a non-zero vector in the image of both $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. As before, extend these to a basis of V so that the point r has $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ strictly lower triangular. Since the image of $\mathbf{v}$ is contained in $\langle e_2 \rangle_{\mathbb{C}}$, $v_1 = 0$. Using Table C.2, $(-10)(-4-1)$ acts on r with non-negative weights. Therefore, $q \in S^+_{(-10)(-4-1)}$.

Case 2.4: The point q has type $(1, 2, 1)$, but does not have type $(1, 2, 0)$, $(0, 1, 0)$ or $(1, 1, 0)$

Combinining the arguments of Case 2.2 and Case 2.3 gives $1 \leq i, j, j' \leq 2$ such that $j \neq j'$ and $a_{j'} \leq 0 \leq a_j \leq b_i$.

Assume that $i = j = 1$. Then $b_1 - a_1 \geq 0$ and $b_1 - a_2 \geq 0$ so

$$\begin{aligned} 0 &\leq 2(b_1 - a_1)(a_1 + a_2 - 2b_1 - 2b_2) + \tfrac{40}{11}(b_1 - a_2)^2 + 2(b_1 - a_2)(a_1 + a_2 - 2b_1 - 2b_2) \\ &\quad + \tfrac{4}{3}(a_1 + a_2 + b_1)^2 + \tfrac{11}{3}(\tfrac{10}{11}b_1 - a_1 + \tfrac{1}{11}a_2)^2 \\ &= 3a_1^2 - 2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 3a_2^2 + 4a_2b_1 + 4a_2b_2 - 8b_1b_2 \\ &= 4(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1 + a_2 - 2b_1 - 2b_2)^2 \\ &= 4\|\lambda\|^2 - \mu(p, \lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{4} = m((00)(-10))$.

Let e_4 be a non-zero vector in the image of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. Then r has $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ with zeros in the top row. Using Table C.2, $(00)(-10)$ acts on r with non-negative weights. Therefore $q \in S^+_{(00)(-10)}$.

Case 3.1: The point q has type $(1,1,0)$ and $(0,1,0)$, but does not have type $(1,2,1)$

Since q has type $(1,1,0)$, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must both have non-trivial kernels and so rank one or less. Since it does not have type $(1,2,1)$, the rank of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are both one and their images span a two-dimensional space. Also, this means that $\ker \mathbf{X}^{(0)} = \ker \mathbf{Y}^{(1)}$.

Lemma C.2.3. *If $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are rank one with trivially intersecting images and* $\ker \mathbf{X}^{(0)} = \ker \mathbf{Y}^{(1)}$ *then there is an i such that $a_i \leq b_1$ and $a_i \leq b_2$.*

Proof. If $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ had a column without zeros it is immediate. Otherwise, being rank one, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must both have a row of zeros and since their images are distinct, these rows cannot be at same place. Since their kernels are equal, they must have non-zero entries in the same column, i say. This proves the claim. □

Assume that $i = 1$. Then $b_1 - a_1 \geq 0$ and $b_2 - a_1 \geq 0$ so

$$\begin{aligned}
0 &\leq 2(b_1 - a_1)(a_1 + a_2 - 2b_1 - 2b_2) + 2(a_1 + a_2 - 2b_1 - 2b_2)(b_2 - a_1) + \tfrac{40}{11}(b_2 - a_1)^2 \\
&\quad + 12(\tfrac{1}{6}b_1 + \tfrac{1}{6}a_1 + \tfrac{1}{2}a_2 + \tfrac{1}{6}b_2)^2 + \tfrac{1}{33}(11b_1 - 10a_1 - b_2)^2 \\
&= 3a_1^2 - 2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 3a_2^2 + 4a_2b_1 + 4a_2b_2 - 8b_1b_2 \\
&= 4(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1 + a_2 - 2b_1 - 2b_2)^2 \\
&= 4\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

So $M(q) \leq \sqrt{4} = m((-11)(-1-1))$.

Let e_2 span the kernel of $\mathbf{X}^{(0)}$. Then $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have a column of zeros on the right and $v_1 = w_2 = 0$. Using Table C.2, $(-11)(-1-1)$ acts on r with non-negative weights. Therefore, $q \in S^+_{(-11)(-1-1)}$.

Case 3.2: The point q has type $(1,1,0)$, but does not have type $(1,2,1)$ or $(0,1,0)$

As in Case 3.1, because q is type $(1,1,0)$ but not $(1,2,1)$, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ satisfy the hypotheses of Lemma C.2.3. Not having type $(0,1,0)$ now implies that $\ker \mathbf{w}$ does not contain $\ker \mathbf{X}^{(0)}$; in particular, $\mathbf{w} \neq 0$. I claim that there is an i that satisfies the claim of Lemma C.2.3 and $w_{i'} \neq 0$ for $i' \neq i$. Indeed, if both $i = 1, 2$ satisfied the claim of the lemma then, choose i' so that $w_{i'} \neq 0$. Suppose, instead, that i satisfies the claim and i' doesn't. Then $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must have a zero at a common entry in column i'. Both would then have a column of zeros at position i' (they are rank one and have the same kernel). Now $w_{i'}$ being zero would contradict q not being type $(0,1,0)$.

Assume $i = 1$ and $j = 2$. Then $b_1 - a_1 \geq 0$, $b_2 - a_1 \geq 0$ and $a_2 \leq 0$.

$$\begin{aligned}
0 &\leq -2a_2(a_1 + a_2 - 2b_1 - 2b_2) - 2a_2(b_1 - a_1) - 2a_2(b_2 - a_1) + 2(a_1 + a_2 - 2b_1 - 2b_2)(b_1 - a_1) \\
&\quad + 2(a_1 + a_2 - 2b_1 - 2b_2)(b_2 - a_1) + 3(b_1 - a_1)^2 + 3(b_2 - a_1)^2 + 4a_2^2 \\
&= 2a_1^2 - 2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 2a_2^2 + 4a_2b_1 + 4a_2b_2 - b_1^2 - 8b_1b_2 - b_2^2 \\
&= 3(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1 + a_2 - 2b_1 - 2b_2)^2 \\
&= 3\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

So $M(q) \leq \sqrt{3} = m((-10)(-1-1))$.

Again, let e_2 span the kernel of $\mathbf{X}^{(0)}$ so that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have a column of zeros on the right and $v_1 = 0$. Using Table C.2, $(-10)(-1-1)$ acts on r with non-negative weights. Therefore, $q \in S^+_{(-10)(-1-1)}$.

Case 3.3: The point q has type $(0,1,0)$, but does not have type $(1,2,1)$ or $(1,1,0)$

Once again, having type $(0,1,0)$ but not type $(1,2,1)$ implies that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ satisfy the hypotheses of Lemma C.2.3. If q is not of type $(1,1,0)$ then the image of $\mathbf{v}$ cannot be contained in $\ker \mathbf{X}^{(0)} = \ker \mathbf{Y}^{(1)}$. I claim that i can be chosen to fulfil the claim of Lemma C.2.3 and $v_i \neq 0$.

If $\mathbf{X}^{(0)}$ has a column of zeros at position j and $i \neq j$ then i satisfies the claim; indeed, $v_i = 0$ implies that the image of $\mathbf{v}$ is contained in $\ker \mathbf{X}^{(0)}$. Otherwise, since $\mathbf{X}^{(0)}$ has rank one, it must have a row of non-zero entries. Since either $i = 1, 2$ fulfils the claim of Case 3.1 and $\mathbf{v} \neq 0$, choose i so that $v_i \neq 0$. This implies that $0 \leq a_i \leq b_1, b_2$.

Assume that $i = 1$ so that $b_1 - a_1 \geq 0$, $b_2 - a_1 \geq 0$ and $a_1 \geq 0$. Then,

$$
\begin{aligned}
0 &\leq 6a_1(a_1 + a_2 - 2b_1 - 2b_2) + 14a_1(b_1 - a_1) + 14a_1(b_2 - a_1) + 12a_1^2 + 4(a_1 + a_2 - 2b_1 - 2b_2)(b_1 - a_1) \\
&\quad + 5(b_1 - a_1)^2 + 8(b_2 - a_1)(b_1 - a_1) + 4(a_1 + a_2 - 2b_1 - 2b_2)(b_2 - a_1) + 5(b_2 - a_1)^2 \\
&= -2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 4a_2b_1 + 4a_2b_2 - 3b_1^2 - 8b_1b_2 - 3b_2^2 \\
&= (a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1 + a_2 - 2b_1 - 2b_2)^2 \\
&= \|\lambda\|^2 - \mu(p, \lambda)^2.
\end{aligned}
$$

So $M(q) \leq \sqrt{1} = m((01)(00))$.

Again, let e_2 span the kernel of $\mathbf{X}^{(0)}$ so that, as coordinates of r, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have a column of zeros on the right and $w_2 = 0$. Using Table C.2, $(01)(00)$ acts on r with non-negative weights. Therefore, $q \in S^+_{(01)(00)}$.

Case 4: The point q has type $(1,0,0)$ but does not have type $(1,2,1)$, $(0,1,0)$ or $(1,1,0)$

Being of type $(1,0,0)$ is equivalent to the condition $\mathbf{v} = 0$. In this case, not being of type $(1,1,0)$ is equivalent to the kernels of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ intersecting trivially and not being of type $(1,2,1)$ implies that their images span a two-dimensional space.

Lemma C.2.4. *Let $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ be rank one and have trivially intersecting images and kernels. Then there are $1 \leq i \neq i', j \neq j' \leq 2$ such that $a_i \leq b_j$ and $a_{i'} \leq b_{j'}$.*

Proof. Let $X_{j'i'}$ be a non-zero entry of $\mathbf{X}^{(0)}$ and define $i \neq i'$ and $j \neq j'$. Then $X_{ji'} = 0$ implies $X_{ji} = 0$ so, considering the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$, either $Y_{ji'}$ or Y_{ji} must be non-zero. If Y_{ji} is non-zero then the claim is proved. If $Y_{ji} = 0$ then $Y_{ji'} \neq 0$ and so, since $\mathbf{Y}^{(1)}$ has rank one, $Y_{j'i} = 0$. This determines the kernel of $\mathbf{Y}^{(1)}$, which implies that $X_{j'i} \neq 0$. Now $X_{j'i}$ and $Y_{ji'}$ are non-zero entries diagonally opposite as required. □

If $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ don't both have rank one then one of them has rank two. Either way, there is $1 \leq i \neq i' \leq 2$ such that $a_i \leq b_1$ and $a_{i'} \leq b_2$.

Assume that $i = 2$ and $j = 1$ so that $b_1 - a_2 \geq 0$ and $b_2 - a_1 \geq 0$.

$$
\begin{aligned}
0 &\leq 3(b_2 - a_1)(a_1 + a_2 - 2b_1 - 2b_2) + \tfrac{11}{4}(b_2 - a_1)^2 + \tfrac{9}{2}(b_2 - a_1)(b_1 - a_2) \\
&\quad + 3(a_1 + a_2 - 2b_1 - 2b_2)(b_1 - a_2) + \tfrac{11}{4}(b_1 - a_2)^2 + (\tfrac{1}{2}(b_2 + a_1 - a_2 - b_1))^2 \\
&= -2a_1a_2 + 4a_1b_1 + 4a_1b_2 + 4a_2b_1 + 4a_2b_2 - 3b_1^2 - 8b_1b_2 - 3b_2^2 \\
&= (a_1^2 + a_2^2 + b_1^2 + b_2^2) - (a_1 + a_2 - 2b_1 - 2b_2)^2 \\
&= \|\lambda\|^2 - \mu(p, \lambda)^2.
\end{aligned}
$$

So $M(q) \leq 1 = m((-1-1)(-1-1))$.

If $\mathbf{v} = 0$, $(-1-1)(-1-1)$ acts on q with non-negative weights. Therefore, $q \in S_{(-1-1)(-1-1)} = S^+_{(-1-1)(-1-1)}$.

Case 5: The point q does not have type $(1,2,1)$, $(1,1,0)$, $(0,1,0)$ or $(1,0,0)$

Since a point cannot be type $(0,2,0)$ or $(1,2,0)$ without being type $(1,2,1)$, this leaves $(1,1,1)$ as the only possible type. I claim that there are $1 \leq i \neq i', j \neq j' \leq 2$ such that $0 \leq a_i \leq b_j$ and $a_{i'} \leq b_{j'}$.

Suppose one of $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ has rank two. Since $\mathbf{v} \neq 0$, choosing i so that $v_i \neq 0$ gives the claim. If neither $\mathbf{X}^{(0)}$ nor $\mathbf{Y}^{(1)}$ has rank two then in order that $(1,2,1)$ not be a subrepresentation, both $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must have rank one and their images must be distinct. As in Case 4, because q is not of type $(1,1,0)$ their kernels must intersect trivially. Use Lemma C.2.4 to choose i and j. Since q has type $(1,1,1)$, the image of $\mathbf{v}$ must lie in either the kernel of $\mathbf{X}^{(0)}$ or the kernel of $\mathbf{Y}^{(1)}$ so the claim is proved.

Assume that $i = 2$ and $j = 1$ so that $b_1 - a_2 \geq 0$, $b_2 - a_1 \geq 0$ and $a_2 \geq 0$.

$$\begin{aligned}
0 &\leq 4a_2^2 + 6(a_1 + a_2 - 2b_1 - 2b_2)(b_2 - a_1) + 8(a_1 + a_2 - 2b_1 - 2b_2)(b_1 - a_2) + 4(a_1 + a_2 - 2b_1 - 2b_2)a_2 \\
&\quad + 5(b_2 - a_1)^2 + 12(b_2 - a_1)(b_1 - a_2) + 6(b_2 - a_1)a_2 + 9(b_1 - a_2)^2 + 10(b_1 - a_2)a_2 \\
&= -a_1^2 - 4a_1a_2 + 8a_1b_1 + 8a_1b_2 - a_2^2 + 8a_2b_1 + 8a_2b_2 - 7b_1^2 - 16b_1b_2 - 7b_2^2 \\
&= (a_1^2 + a_2^2 + b_1^2 + b_2^2) - 2(a_1 + a_2 - 2b_1 - 2b_2)^2 \\
&= \|\lambda\|^2 - 2\mu(p,\lambda)^2.
\end{aligned}$$

So $M(q) \leq \sqrt{\frac{1}{2}} = m((-10)(-10))$.

Let e_2 span the image of $\mathbf{v}$, so that for r, $v_1 = 0$. This is non-zero since q is not of type $(1,0,0)$. If $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ both kill e_2 then q would have type $(1,1,0)$. Let e_4 be the image of e_2. It follows that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are non-strictly lower triangular so $(-10)(-10)$ acts on r with non-negative weights. Therefore, $q \in S^+_{(-10)(-10)}$.

C.3 The Optimal Subgroups for $\theta = (-3, 2)$

Case	μ	Signs of weights of μ	Unstable points for μ	$m(\mu)^2$
1	(33)(−2−2)	$\left(\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right);(+\,+)(-\,-)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(\ast\,\ast)(0\,0)\right\}$	26
2.1	(03)(−2−2)	$\left(\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right);(0\,+)(0\,-)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(\ast\,\ast)(\ast\,0)\right\}$	17
3.1	(16)(−41)	$\left(\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right);(+\,+)(-\,-)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right);(\ast\,\ast)(0\,0)\right\}$	14
4.1	(03)(−20)	$\left(\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right);(0\,+)(0\,-)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right);(\ast\,\ast)(\ast\,0)\right\}$	13
3.2	(22)(−32)	$\left(\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right);(+\,+)(-\,-)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(0\,0)\right\}$	$9\frac{1}{3}$
4.2	(−19)(−1−1)	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(-\,+)(+\,-)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(0\,\ast)(\ast\,0)\right\}$	$9\frac{1}{3}$
4.3	(01)(00)	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(0\,+)(0\,-)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(\ast\,\ast)(\ast\,0)\right\}$	9
2.2	(00)(−1−1)	$\left(\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right);(0\,0)(0\,0)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(\ast\,\ast)(\ast\,\ast)\right\}$	8
5.1	(01)(−41)	$\left(\left(\begin{smallmatrix}-&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\+&0\end{smallmatrix}\right);(0\,+)(0\,-)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(\ast\,0)\right\}$	$4\frac{1}{2}$
5.2	(00)(−10)	$\left(\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right);(0\,0)(0\,0)\right)$	$\left\{\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(\ast\,\ast)\right\}$	4
6	(11)(11)	$\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(+\,+)(-\,-)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(0\,0)\right\}$	1
7	(01)(01)	$\left(\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right);(0\,+)(0\,-)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(\ast\,0)\right\}$	$\frac{1}{2}$
8	(−10)(−1−1)	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	$\frac{1}{3}$

Table C.3: The optimal one-parameter subgroups for $W = G(2,1,2)$ and $\theta = (-3,2)$ along with the weights acting on an arbitrary point and the set of points which are unstable for that subgroup. The $\ast$ symbol represents any complex number.

The proof is more-or-less the same as the proof in Section C.2. Recall, the notation and remarks. First, $\theta \cdot \delta = -2$, so $\hat{\theta} = (2, -3, 2)$. To be unstable, a point must have one of the following types

$$(1,2,0) \quad (0,2,0) \quad (1,2,1) \quad (1,1,0) \quad (0,1,0) \quad (0,2,1) \quad (0,2,2) \quad (0,1,1)$$

Lemma C.3.1. *Let q be an unstable point. Then q satisfies the conditions of one of the cases of the proof below.*

Proof. If q has type $(0,2,0)$ it belongs to Case 1, so assume that it doesn't. If q has type $(1,2,0)$ it belongs to Case 2.1 or 2.2 depending on $\mathbf{w}$, so assume that it doesn't have type $(1,2,0)$. If q has type $(0,2,1)$ then it either has type $(0,1,0)$ or not: these are Cases 3.1 and 3.2; assume that q doesn't have type $(0,2,1)$. If $(0,1,0)$ then if it has type $(1,2,1)$ then Case 4.1 applies, otherwise if it has type $(1,1,0)$ then it belongs to Case 4.2 and if it has neither type then it is covered by 4.3. Assume that q doesn't have type $(0,1,0)$. If q has type $(0,2,2)$ and $(1,2,1)$ then it belongs to either 5.1 or 5.2. If q has type $(0,2,2)$ but not type $(1,2,1)$ then Case 6 applies. Assume that q doesn't have type $(0,2,2)$. If q has type $(1,2,1)$ it belongs to either 5.1 or 5.2; assume it doesn't have type $(1,2,1)$. Now if q has type $(1,1,0)$ it belongs to Case 8 and if it doesn't it must have only have type $(0,1,1)$ and so belong to Case 7. □

Case 1: The point q has type $(0,2,0)$

Here,

$$\begin{aligned}0 &\le 9(a_1-a_2)^2+4(b_1-b_2)^2+(2a_1+3b_1)^2+(2a_1+3b_2)^2+(2a_2+3b_1)^2+(2a_2+3b_2)^2\\ &= 17a_1^2-18a_1a_2+12a_1b_1+12a_1b_2+17a_2^2+12a_2b_1+12a_2b_2+22b_1^2-8b_1b_2+22b_2^2\\ &= \left(\sqrt{26}\sqrt{a_1^2+a_2^2+b_1^2+b_2^2}\right)^2-(3a_1+3a_2-2(b_1+b_2))^2,\\ &= 26\|\lambda\|^2-\mu(p,\lambda)^2.\end{aligned}$$

Because $-\mu(p,\lambda)=3a_1+3a_2-2(b_1+b_2)<0$, taking square roots gives, $m(\lambda)\le\sqrt{26}=m((33)(-2-2))$. Having type $(0,2,0)$ means that $\mathbf{X}^{(0)}=\mathbf{Y}^{(1)}=\mathbf{w}=0$, so $(33)(-2-2)$ acts by non-zero weights on q and is optimal.

Case 2.1: The point q has type $(1,2,0)$, but not type $(0,2,0)$ and one coordinate of $\mathbf{w}$ is zero

Being of type $(1,2,0)$ implies that $\mathbf{X}^{(0)}=\mathbf{Y}^{(1)}=0$, so not being of type $(0,2,0)$ implies some coordinate of $\mathbf{w}$ is non-zero. So $w_i\neq 0$ and $w_{i'}=0$ for $i'\neq i$. Thus $a_i\le 0$.

Assume $a_1\le 0$,

$$\begin{aligned}0 &\le -6(3a_1+3a_2-2b_1-2b_2)a_1+26a_1^2+\tfrac{17}{2}(b_1-b_2)^2+\tfrac{1}{8}(8a_2+6b_1+6b_2)^2\\ &= 8a_1^2-18a_1a_2+12a_1b_1+12a_1b_2+8a_2^2+12a_2b_1+12a_2b_2+13b_1^2-8b_1b_2+13b_2^2\\ &= 17(a_1^2+a_2^2+b_1^2+b_2^2)-(3a_1+3a_2-2b_1-2b_2)^2\\ &= 17\|\lambda\|^2-\mu(p,\lambda)^2.\end{aligned}$$

It follows that $m(\lambda)\le\sqrt{17}=m((03)(-2-2))$. Since $(33)(-2-2)$ acts by non-zero weights on q, it is optimal.

Case 2.2: The point q has type $(1,2,0)$, but not type $(0,2,0)$ and neither coordinate of $\mathbf{w}$ is zero

The same argument as Case 2.1 applies, except that now $a_1,a_2\le 0$.

$$\begin{aligned}0 &\le \tfrac{266}{13}a_1^2+\tfrac{266}{13}a_2^2-10b_1^2+\tfrac{5}{13}(3a_1+3a_2-2b_1-2b_2)^2\\ &\quad-\tfrac{108}{13}(3a_1+3a_2-2b_1-2b_2)a_1-\tfrac{108}{13}(3a_1+3a_2-2b_1-2b_2)a_2+\tfrac{324}{13}a_1b+4(b_1-b_2)^2\\ &= -a_1^2-18a_1a_2+12a_1b_1+12a_1b_2-a_2^2+12a_2b_1+12a_2b_2+4b_1^2-8b_1b_2+4b_2^2\\ &= 8(a_1^2+a_2^2+b_1^2+b_2^2)-(3a_1+3a_2-2b_1-2b_2)^2\\ &= 8\|\lambda\|^2-\mu(p,\lambda)^2.\end{aligned}$$

It follows that $m(\lambda)\le\sqrt{8}=m((00)(-1-1))$. Since $(00)(-1-1)$ acts by non-zero weights on q, it is optimal.

Case 3.1: The point q has type $(0,2,1)$ and $(0,1,0)$, but not type $(1,2,0)$

Not being of type $(0,2,0)$ implies that some coordinate of $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ is non-zero so that $a_i \le b_j$ for some i, j.

Assume $a_1 \le b_1$.

$$\begin{aligned}
0 &\le \tfrac{364}{27}(b_1-a_1)^2 + \tfrac{140}{27}(3a_1+3a_2-2b_1-2b_2)(b_1-a_1) + \tfrac{1}{27}(3a_1+3a_2-2b_1-2b_2)^2 \\
&\quad + \tfrac{91}{2}(\tfrac{4}{13}a_2 + \tfrac{6}{13}b_2)^2 + \tfrac{378}{13}(\tfrac{13}{27}a_1 + \tfrac{13}{27}b_1 - \tfrac{1}{9}a_2 + \tfrac{2}{27}b_2)^2 \\
&= 5a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 + 5a_2^2 + 12a_2b_1 + 12a_2b_2 + 10b_1^2 - 8b_1b_2 + 10b_2^2 \\
&= 14(a_1^2+a_2^2+b_1^2+b_2^2) - (3a_1+3a_2-2b_1-2b_2)^2 \\
&= 14\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

It follows that $m(\lambda) \le \sqrt{14} = m((16)(-41))$.

Points of type $(0,2,1)$ also have type $(0,2,2)$ and $(1,2,1)$, this implies that $\mathbf{w} = 0$. Having type $(0,1,0)$ implies that $\ker \mathbf{X}^{(0)}$ and $\ker \mathbf{Y}^{(1)}$ contain a common element, e_2 say. Being of type $(0,2,1)$ implies that the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are contained in a one-dimensional space, $\langle e_4\rangle$ say. Extending e_2 and e_4 to a basis gives a point r whose $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ coordinates are strictly lower triangular. Now $(16)(-41)$ acts by non-zero weights on r, so $q \in S^+_{(16)(-41)}$.

Case 3.2: The point q has type $(0,2,1)$, but not type $(1,2,0)$ or $(0,1,0)$

First, $\mathbf{w} = 0$. So if the point is not type $(0,1,0)$ then the kernels of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must intersect trivially. Being of type $(0,2,1)$ also means that the images are contained in a one-dimensional space. Thus, the ranks of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must both be one.

Lemma C.3.2. *Suppose $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have rank one, trivially intersecting kernels and the same image. Then there exists an i such that $a_1, a_2 \le b_i$.*

Proof. If $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ had a row without zeros it is immediate. Otherwise, being rank one, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must both have a column of zeros and these columns cannot be at same place. Since their images coincide, they must have non-zero entries on the same row, i say, in different columns. This proves the claim. □

Assume that $a_1, a_2 \le b_1$.

$$\begin{aligned}
0 &\le \tfrac{728}{81}(b_1-a_2)^2 + \tfrac{10}{3}(b_1-a_1)(3a_1+3a_2-2b_1-2b_2) + \tfrac{10}{3}(b_1-a_2)(3a_1+3a_2-2b_1-2b_2) \\
&\quad + \tfrac{4}{3}(a_1+a_2+b_1+2b_2)^2 + (\tfrac{26}{9}b_1 - 3a_1 + \tfrac{1}{9}a_2)^2 \\
&= \tfrac{1}{3}a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 + \tfrac{1}{3}a_2^2 + 12a_2b_1 + 12a_2b_2 + \tfrac{16}{3}b_1^2 - 8b_1b_2 + \tfrac{16}{3}b_2^2 \\
&= \tfrac{28}{3}(a_1^2+a_2^2+b_1^2+b_2^2) - (3a_1+3a_2-2b_1-2b_2)^2 \\
&= \tfrac{28}{3}\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

It follows that $m(\lambda) \le \sqrt{9\tfrac{1}{3}} = m((22)(-32))$.

Let e_4 be a vector which spans the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. Then r has zeros in the top rows of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ and $\mathbf{w} = 0$, so $(22)(-32)$ acts on r with non-zero weights and so is optimal.

Case 4.1: The point q has type $(0,1,0)$ and $(1,2,1)$, but not type $(1,2,0)$ or $(0,2,1)$

I claim that there exists some i, j such that $a_i \le b_j$ and $a_i \le 0$. Since q is of type $(1,2,1)$, but not of type $(0,2,1)$, $w_i \ne 0$ for some i. Thus, the kernel of $\mathbf{w}$ must be $(0,1,0)$ and be contained in the kernel of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. However, at least one of $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ must be non-zero, otherwise q would be of type $(1,2,0)$. If both entries of $\mathbf{w}$ were non-zero then choosing i to be the column of any non-zero entry of $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ satisfies the claim. Otherwise, if the other entry of $\mathbf{w}$ zero then if one of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must have a non-zero entry in column i or their kernels would not coincide.

Assume that $a_1 \leq b_1, 0$.

$$\begin{aligned} 0 &\leq \tfrac{338}{17}a_1^2 + 4(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_1 - a_1) - 2(3a_1 + 3a_2 - 2b_1 - 2b_2)a \\ &\quad + (2a_2 + 3b_2)^2 + 17(b_1 - \tfrac{6}{17}a_1)^2 \\ &= 4a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 + 4a_2^2 + 12a_2b_1 + 12a_2b_2 + 9b_1^2 - 8b_1b_2 + 9b_2^2 \\ &= 13(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (3a_1 + 3a_2 - 2b_1 - 2b_2)^2 \\ &= 13\|\lambda\|^2 - \mu(p, \lambda)^2. \end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{13} = m((03)(-20))$.

Let e_4 be a vector which spans the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ and e_2 a non-zero vector in the kernel of $\mathbf{w}$. Then r has strictly lower-triangular $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$; and $w_2 = 0$ so $(03)(-20)$ acts on r with non-zero weights.

Case 4.2: The point q has type $(0,1,0)$ and $(1,1,0)$, but not type $(1,2,0)$, $(0,2,1)$ or $(1,2,1)$

Being type $(0,1,0)$ implies that both $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have non-zero kernels, so rank less than two. Not having type $(1,2,1)$ implies that the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must span a two-dimensional space which means they must both be rank one. Now it follows that $\ker \mathbf{X}^{(0)} = \ker \mathbf{Y}^{(1)}$, so applying Lemma C.2.3 gives an i such that $a_i \leq b_1, b_2$.

Assume that $a_1 \leq b_1, b_2$.

$$\begin{aligned} 0 &\leq \tfrac{10}{3}(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_1 - a_1) + \tfrac{10}{3}(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_2 - a_1) \\ &\quad + \tfrac{728}{81}(b_1 - a_1)^2 + \tfrac{1}{3}(3a_1 + a_2 + 3b_1 + 3b_2)^2 + 9(b_2 - \tfrac{26}{27}a_1 - \tfrac{1}{27}b_1)^2 \\ &= \tfrac{1}{3}a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 + \tfrac{1}{3}a_2^2 + 12a_2b_1 + 12a_2b_2 + \tfrac{16}{3}b_1^2 - 8b_1b_2 + \tfrac{16}{3}b_2^2 \\ &= \tfrac{28}{3}(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (3a_1 + 3a_2 - 2b_1 - 2b_2)^2 \\ &= \tfrac{28}{3}\|\lambda\|^2 - \mu(p, \lambda)^2. \end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{9\tfrac{1}{3}} = m((-19)(-1-1))$.

Let e_2 be a non-zero vector in the kernel of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. Then r has zeros in the right-hand column of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ and $w_2 = 0$. Having type $(1,1,0)$ implies that the image of $\mathbf{v}$ lies in the the kernel of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ so $v_1 = 0$ and $(-19)(-1-1)$ acts with non-negative weights on r.

Case 4.3: The point q has type $(0,1,0)$, but not type $(1,2,0)$, $(0,2,1)$, $(1,2,1)$ or $(1,1,0)$

I claim that there exists an i such that $0 \leq a_i \leq b_1, b_2$. Indeed, the argument in Case 4.2 shows that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ satisfy the hypothesis of Lemma C.2.3. Since q is not type $(1,1,0)$, the image of $\mathbf{v}$ cannot lie inside the kernel of $\mathbf{X}^{(0)}$, in particular $\mathbf{v} \neq 0$.

Suppose that $i = 1, 2$ both satisfy the claim of Lemma C.2.3. Then choosing i so that $v_i \neq 0$ does the trick. Otherwise, suppose that i satisfies the claim of the lemma and i' doesn't. It follows that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must have a common zero in column i', which implies that column i' has both entries zero. Thus $v_{i'} \neq 0$ and the claim is proved.

Assume that $0 \leq a_1 \leq b_1, b_2$.

$$\begin{aligned} 0 &\leq 28a_1^2 + 2(3a_1 + 3a_2 - 2b_1 - 2b_2)a_1 + 4(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_1 - a_1) \\ &\quad + 4(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_2 - a_1) + 22a(b_1 - a_1) + 22a(b_2 - a_1) \\ &\quad + 13(b_1 - a_1)^2 + 8(b_1 - a_1)(b_2 - a_1) + 13(b_2 - a_1)^2 \\ &= -18a_1a_2 + 12a_1b_1 + 12a_1b_2 + 12a_2b_1 + 12a_2b_2 + 5b_1^2 - 8b_1b_2 + 5b_2^2 \\ &= 9(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (3a_1 + 3a_2 - 2b_1 - 2b_2)^2 \\ &= 9\|\lambda\|^2 - \mu(p, \lambda)^2. \end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{9} = m((01)(00))$.

Choose e_2 so that $\langle e_2 \rangle = \ker \mathbf{X}^{(0)} = \ker \mathbf{Y}^{(1)} = \ker \mathbf{w}$. Then r has $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ with a column of zeros on the right and $w_2 = 0$, so (01)(00) acts on r with non-negative weights. Thus (01)(00) is optimal for r and so for q.

Case 5.1: The point q has type $(1,2,1)$ and $(0,1,1)$ but not type $(1,2,0)$, $(0,2,1)$ or $(0,1,0)$

I claim that there exists $1 \leq i, i', j \leq 2$ with $i \neq i'$ such that $a_{i'} \leq 0$ and $a_i \leq b_j$. As before, having type $(1,2,1)$ but not $(0,2,1)$ implies that $\mathbf{w} \neq 0$. Being of type $(0,1,1)$ but not of type $(0,1,0)$ implies that there is a vector in the kernel of $\mathbf{w}$ that is not in the kernel of either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$.

If neither entry of $\mathbf{w}$ is zero then choose i and j so that $X^{(0)}_{ji} \neq 0$ or $Y^{(1)}_{ji} \neq 0$. Otherwise, choose i so that $w_i = 0$, $w_{i'} \neq 0$ and it follows that there exists a j such that $X^{(0)}_{ji} \neq 0$ or $Y^{(1)}_{ji} \neq 0$.

Assume $a_1 \leq b_1$ and $a_2 \leq 0$.

$$\begin{aligned}
0 &\leq \tfrac{27}{2}a_2^2 + \tfrac{17}{2}(b_1 - a_1)^2 + 5(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_1 - a_1) \\
&\quad - 6(3a_1 + 3a_2 - 2b_1 - 2b_2)a_2 - 15(b_1 - a_1)a_2 + \tfrac{1}{2}(2a_1 + 2b_1 + b_2)^2 \\
&= -\tfrac{9}{2}a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 - \tfrac{9}{2}a_2^2 + 12a_2b_1 + 12a_2b_2 + \tfrac{1}{2}b_1^2 - 8b_1b_2 + \tfrac{1}{2}b_2^2 \\
&= \tfrac{9}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (3a_1 + 3a_2 - 2b_1 - 2b_2)^2 \\
&= \tfrac{9}{2}\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{4\tfrac{1}{2}} = m((01)(-41))$.

Let e_2 span the kernel of $\mathbf{w}$. Since q is of type $(1,2,1)$, choose e_4 to be a non-zero vector in the image of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. Then, as coordinates of r, $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have a row of zeros at the top and $w_2 = 0$. Also, since q is of type $(0,1,1)$ the image of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must lie inside the kernel of $\mathbf{w}$. Thus, as coordinates of r, $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must have zeros in the top-right-hand corner. Now (01)(−41) must act on r with non-negative weights so it is optimal for q.

Case 5.2: The point q has type $(1,2,1)$, but not type $(1,2,0)$, $(0,2,1)$, $(0,1,0)$ or $(0,1,1)$

First note that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ cannot both be zero or q would have type $(1,2,0)$. Being of type $(1,2,1)$ but not of type $(0,2,1)$ implies that $\mathbf{w} \neq 0$ and also that the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ lie inside some one-dimensional subspace. Let e_4 span this subspace; it is well-defined because if $\mathbf{X}^{(0)} = \mathbf{Y}^{(1)} = 0$ then q would have type $(1,2,0)$.

I claim that if $w_i = 0$ and $i \neq i'$ then there is some j' such that $0 \geq a_{i'} \geq b_{j'} \geq a_i$. If $w_i = 0$ then $w_{i'} \neq 0$. Now, either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ must have a non-zero entry in column i or q would have type $(0,1,0)$.

Not being of type $(0,1,1)$ implies that either $\mathbf{X}^{(1)}(e_4)$ or $\mathbf{Y}^{(0)}(e_4)$ doesn't belong to the kernel of $\mathbf{w}$. Suppose that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ both have a row of zeros in row j and $j' \neq j$, then $b_{j'} \geq a_i$ and either $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$ must have a non-zero entry in position (i', j'), so $a_{i'} \geq b_{j'}$. Otherwise, if either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ didn't have a row of zeros it would have four non-zero entries. Either $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$ must have a non-zero entry in row i'. Let j' be the column of this entry, then $a_{i'} \geq b_{j'}$ and the claim follows.

If the hypothesis of the claim ($w_i = 0$) is not satisfied then neither entry of $\mathbf{w}$ is zero and so $a_1, a_2 \leq 0$. Again $\mathbf{X}^{(0)}$ must have a non-zero entry, so this gives some i, j such that $a_1, a_2 \leq 0$ and $a_i \leq b_j$.

Assume $a_1, a_2 \leq 0$ and $a_2 \leq b_1$.

$$\begin{aligned}
0 &\leq 13a_1^2 + 5a_2^2 + 7(b_1 - a_2)^2 - 6(3a_1 + 3a_2 - 2b_1 - 2b_2)a_1 - 2(3a_1 + 3a_2 - 2b_1 - 2b_2)a_2 \\
&\quad + 4(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_1 - a_2) + 6a_1a_2 - 12a_1(b_1 - a_2) + (a_2 + b_1)^2 \\
&= -5a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 - 5a_2^2 + 12a_2b_1 + 12a_2b_2 - 8b_1b_2 \\
&= 4(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (3a_1 + 3a_2 - 2b_1 - 2b_2)^2 \\
&= 4\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{4} = m((00)(-10))$.

Extending e_4 to a basis gives an r for which $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have zeros in the top row. Thus $(00)(-10)$ acts on r with non-zero weights and so is optimal for q.

Case 6: The point q has type $(0,2,2)$ but not type $(1,2,0)$, $(0,2,1)$, $(0,1,0)$ or $(1,2,1)$

Not being type $(1,2,1)$ implies not being type $(1,2,0)$ or $(0,2,1)$: the images of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must span two dimensions. I claim that there exist $i \neq i'$ and $j \neq j'$ such that $a_i \leq b_j$ and $a_{i'} \leq a_{j'}$. Indeed, if either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ have rank two the claim is immediate. If not then, since $\mathbf{w} = 0$ (q has type $(0,2,2)$), not being of type $(0,1,0)$ implies that the kernels of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ intersect trivially and Lemma C.2.4 gives the claim.

Assume $a_1 \leq b_1$ and $a_2 \leq b_2$.

$$\begin{aligned}
0 &\leq 5(3a_1+3a_2-2b_1-2b_2)(b_1-a_1)+5(3a_1+3a_2-2b_1-2b_2)(b_2-a_2)\\
&\quad+\tfrac{27}{4}(b_1-a_1)^2+\tfrac{25}{2}(b_1-a_1)(b_2-a_2)+\tfrac{27}{4}(b_2-a_2)^2+(\tfrac{1}{2}a_1-\tfrac{1}{2}a_2+\tfrac{1}{2}b_1-\tfrac{1}{2}b_2)^2\\
&=-8a_1^2-18a_1a_2+12a_1b_1+12a_1b_2-8a_2^2+12a_2b_1+12a_2b_2-3b_1^2-8b_1b_2-3b_2^2\\
&=(a_1^2+a_2^2+b_1^2+b_2^2)-(3a_1+3a_2-2b_1-2b_2)^2\\
&=\|\lambda\|^2-\mu(p,\lambda)^2.
\end{aligned}$$

Hence $m(\lambda) \leq \sqrt{1} = m((11)(11))$.

Clearly, $(11)(11)$ destablilises any point such that $\mathbf{w} = 0$, which includes q, so it is optimal.

Case 7: The point q has type $(0,1,1)$ but not type $(1,2,0)$, $(0,2,1)$, $(0,1,0)$, $(1,2,1)$ or $(0,2,2)$

The claim of Case 6 holds. Indeed, one needs only check the case when $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are both rank one with non-trivially intersecting kernels because that was the only place the hypothesis involving type $(0,2,2)$ was used. In that case, notice that $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must act as zero on the representation $(0,1,1)$ because their images span a two-dimensional space. This implies that $\mathbf{w}$ must act as zero on the kernel of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. This, however, contradicts the assumption that q is not of type $(0,1,0)$. Therefore the claim follows. Because, $\mathbf{w} \neq 0$ there is an i such that $a_i \leq 0$.

Assume $a_1 \leq b_1, 0$ and $a_2 \leq b_2$.

$$\begin{aligned}
0 &\leq -a_1(3a_1+3a_2-2b_1-2b_2)+2(3a_1+3a_2-2b_1-2b_2)(b_1-a_1)\\
&\quad+\tfrac{5}{2}(3a_1+3a_2-2b_1-2b_2)(b_2-a_2)-\tfrac{3}{2}a_1(b_1-a_1)-\tfrac{5}{2}a_1(b_2-a_2)\\
&\quad+\tfrac{9}{4}(b_1-a_1)^2+5(b_1-a_1)(b_2-a_2)+\tfrac{13}{4}(b_2-a_2)^2+a_1^2+\tfrac{1}{4}(3a_1+3a_2-2b_1-2b_2)^2\\
&=-\tfrac{17}{2}a_1^2-18a_1a_2+12a_1b_1+12a_1b_2-\tfrac{17}{2}a_2^2+12a_2b_1+12a_2b_2-\tfrac{7}{2}b_1^2-8b_1b_2-\tfrac{7}{2}b_2^2\\
&=\tfrac{1}{2}(a_1^2+a_2^2+b_1^2+b_2^2)-(3a_1+3a_2-2b_1-2b_2)^2\\
&=\tfrac{1}{2}\|\lambda\|^2-\mu(p,\lambda)^2.
\end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{\frac{1}{2}} = m((01)(01))$.

Let e_2 be the kernel of $\mathbf{w}$ so that $w_2 = 0$ and $w_1 \neq 0$ as coordinates of r. Let e_4 be the vector belonging to the representation of type $(0,1,1)$. Then the representation is spanned by e_2 and e_4 so that $\mathbf{X}^{(0)}$, $\mathbf{X}^{(1)}$, $\mathbf{Y}^{(0)}$ and $\mathbf{Y}^{(1)}$ are lower triangular as coordinates of r.

Case 8: The point q has type $(1,1,0)$ but not $(1,2,0)$, $(0,2,1)$, $(0,1,0)$, $(0,2,2)$ and $(1,2,1)$

First, note that $\mathbf{w} \neq 0$ (not being type $(0,2,2)$), $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are not both zero (not being type $(1,2,0)$) and both $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have non-trivial kernels since they must act on the representation of type $(1,1,0)$ by zero. The intersection of the kernels of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ must be one-dimensional and intersect the kernel of $\mathbf{w}$ trivially since q is not type $(0,1,0)$. If either $\mathbf{X}^{(0)}$ or $\mathbf{Y}^{(1)}$ were zero then q would be of type $(1,2,1)$ so the rank of both must be one.

I claim that there is an $i \neq i'$ such that $a_i \leq b_1, b_2$ and $a_{i'} \leq 0$. The hypotheses of Lemma C.2.3 are satisfied. If $i = 1,2$ satisfies the claim of Lemma C.2.3 then choose i' so that $w_{i'} \neq 0$.

Otherwise, suppose that i satisfies the claim of Lemma C.2.3 and i' doesn't. Then $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have a common zero in column i' and since they are rank one and i satisfies the claim of Lemma C.2.3, column i' must have zeros in both entries for both matrices. Since the kernel of $\mathbf{w}$ cannot intersect the kernel of these matrices $w_i = 0$ so $w_{i'} \neq 0$ and the claim is proved.

Assume $a_1 \leq b_1, b_2$ and $a_2 \leq 0$.

$$\begin{aligned}
0 &\leq -6(3a_1 + 3a_2 - 2b_1 - 2b_2)a_2 + \tfrac{8}{3}(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_1 - a_1) \\
&\quad + \tfrac{8}{3}(3a_1 + 3a_2 - 2b_1 - 2b_2)(b_2 - a_1) - 8a_2(b_1 - a_1) - 8a_2(b_2 - a_1) \\
&\quad + 2(b_1 - a_1)^2 + \tfrac{10}{3}(b_1 - a_1)(b_2 - a_1) + 2(b_2 - a_1)^2 + \tfrac{28}{3}a_2^2 \\
&= -\tfrac{26}{3}a_1^2 - 18a_1a_2 + 12a_1b_1 + 12a_1b_2 - \tfrac{26}{3}a_2^2 + 12a_2b_1 + 12a_2b_2 - \tfrac{11}{3}b_1^2 - 8b_1b_2 - \tfrac{11}{3}b_2^2 \\
&= \tfrac{1}{3}(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (3a_1 + 3a_2 - 2b_1 - 2b_2)^2 \\
&= \tfrac{1}{3}\|\lambda\|^2 - \mu(p, \lambda)^2.
\end{aligned}$$

It follows that $m(\lambda) \leq \sqrt{\frac{1}{3}} = m((-10)(-1-1))$.

Let e_2 be a non-zero vector in the kernel of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$. Since the image of $\mathbf{v}$ lies in the kernel of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ this implies that $v_1 = 0$ for r and also $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have columns of zeros on the right. Hence $(-10)(-1-1)$ destablises r and so must be optimal for q.

C.4 The Optimal Subgroups for $\theta = (3, -1)$

Case	μ	Signs of weights of μ	Unstable points for μ	$m(\mu)^2$
1	$(-3-3)(11)$	$\left(\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right);(--)(++)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,0)(\ast\,\ast)\right\}$	20
2.1	$(-3-1)(-11)$	$\left(\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right);(--)(++)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right);(0\,0)(\ast\,\ast)\right\}$	12
2.2a	$(-30)(11)$	$\left(\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	11
2.4	$(-30)(01)$	$\left(\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&+\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	10
2.3	$(-9-1)(-1-1)$	$\left(\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right);(--)(++)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,0)(\ast\,\ast)\right\}$	$9\frac{1}{3}$
3	$(-5-5)(-53)$	$\left(\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right);(--)(++)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,0)(\ast\,\ast)\right\}$	$9\frac{1}{3}$
2.5	$(-10)(00)$	$\left(\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}+&0\\+&0\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	9
4	$(-1-1)(-1-1)$	$\left(\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right);(--)(++)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(0\,0)(\ast\,\ast)\right\}$	4
5.1	$(-10)(-11)$	$\left(\left(\begin{smallmatrix}0&-\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&+\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	3
5.2	$(-10)(-10)$	$\left(\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\+&0\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&\ast\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	2
2.2b	$(00)(11)$	$\left(\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}-&-\\-&-\end{smallmatrix}\right)\left(\begin{smallmatrix}+&+\\+&+\end{smallmatrix}\right);(0\,0)(0\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(\ast\,\ast)\right\}$	2
6	$(00)(01)$	$\left(\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right);(0\,0)(0\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right);(\ast\,\ast)(\ast\,\ast)\right\}$	1
7	$(-10)(-1-1)$	$\left(\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&0\\+&+\end{smallmatrix}\right)\left(\begin{smallmatrix}0&-\\0&-\end{smallmatrix}\right);(-\,0)(+\,0)\right)$	$\left\{\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&\ast\\\ast&\ast\end{smallmatrix}\right)\left(\begin{smallmatrix}\ast&0\\\ast&0\end{smallmatrix}\right);(0\,\ast)(\ast\,\ast)\right\}$	$\frac{1}{3}$

Table C.4: The optimal one-parameter subgroups for $W = G(2,1,2)$ and $\theta = (3,-1)$ along with the weights acting on an arbitrary point and the set of points which are unstable for that subgroup. The $\ast$ symbol represents any complex number.

Again the proof follows that of Section C.2. First, $\theta \cdot \delta = 4$, so $\hat{\theta} = (-4, 3, -1)$. To be unstable, a point must have one of the following types

$$(1,0,2) \quad (1,1,2) \quad (1,0,1) \quad (1,0,0) \quad (1,1,1) \quad (0,0,1) \quad (1,1,0) \quad (0,0,2)$$

Lemma C.4.1. *Let q be an unstable point. Then q satisfies the conditions of one of the cases of the proof below.*

Proof. If q has type $(1,0,2)$ is belongs to Case 1; assume it doesn't. Now if q has type $(1,1,2)$ it belongs to Case 2; indeed, if the point has type $(0,0,2)$, but not type $(1,0,2)$ then $\mathbf{v} \neq 0$, so Case

2.2 covers all q of type $(1,1,2)$ and $(0,0,2)$ but not $(1,0,2)$ or $(1,0,1)$. Assume that q doesn't have type $(1,1,2)$, this implies that q doesn't have type $(0,0,2)$. Now if q has type $(1,0,1)$ it belongs to Case 3; assume it doesn't. If q has type $(1,0,0)$ then it now belongs to Case 4; assume it doesn't. If q has type $(1,1,1)$ it belongs to Case 5; assume it doesn't. If q has type $(0,0,1)$ it belongs to Case 6; assume it doesn't. This implies that q has type $(1,1,0$ and so belongs to Case 7. □

Case 1: The point q has type $(1,0,2)$

Assume that $i = 2$ and $j = 1$ so that $b_1 - a_2 \geq 0$, $b_2 - a_1 \geq 0$ and $a_2 \geq 0$.

$$\begin{aligned}0 &\leq 9(a_1 - a_2)^2 + (a_1 + 3b_1)^2 + (a_1 + 3b_2)^2 + (a_2 + 3b_1)^2 + (a_2 + 3b_2)^2 + (b_1 - b_2)^2 \\ &= 20(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\ &= 20\|\lambda\|^2 - \mu(p, \lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{20} = m((-3-3)(11))$.

If q has type $(1,0,2)$ then $\mathbf{X}^{(1)} = \mathbf{Y}^{(1)} = \mathbf{v} = 0$ so $(-3-3)(11)$ acts on q with non-negative weights and is optimal.

Case 2.1: The point q has type $(1,1,2)$ and $(1,0,1)$ but not $(1,0,2)$

Being type $(1,0,1)$ but not $(1,0,2)$ implies that one of $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(1)}$ is non-zero which gives an i, j such that $a_i \geq b_j$.

Assume that $i = 1$ and $j = 1$ so that $a_1 - b_1 \geq 0$.

$$\begin{aligned}0 &\leq 10(a_1 - b_1)^2 + 4(a_1 - b_1)(-3a_1 - 3a_2 + b_1 + b_2) + \tfrac{6}{5}(a_2 + 3b_2)^2 + \tfrac{1}{5}(5b_1 + 5a_1 - 3a_2 + b_2)^2 \\ &= 3a_1^2 - 18a_1a_2 + 6b_1a_1 + 6a_1b_2 + 3a_2^2 + 6b_1a_2 + 6a_2b_2 + 11b_1^2 - 2b_1b_2 + 11b_2^2 \\ &= 12(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\ &= 12\|\lambda\|^2 - \mu(p, \lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{12} = m((-3-1)(-11))$.

Being type $(1,0,1)$ means that the kernels of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ contain a non-zero vector, e_4 say. It is unique up to scalar since they are not both zero. Also, their images must span a one-dimensional space, $\langle e_2 \rangle$ say, because q has type $(1,1,2)$. Extending this to a basis shows that $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ are strictly lower triangular as coordinates of r and since the image of $\mathbf{v}$ must coincide with the span of the images of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$, $v_1 = 0$ as a coordinate of r. Thus $(-3-1)(-11)$ acts on r with non-negative weights and so is optimal for q.

Case 2.2a: The point q has type $(1,1,2)$ and $(0,0,2)$ but not $(1,0,2)$ or $(1,0,1)$ and one of the entries of v is zero

Being of type $(0,0,2)$ implies that $\mathbf{X}^{(1)} = \mathbf{Y}^{(0)} = 0$, so that not being of type $(1,0,2)$ implies that $\mathbf{v} \neq 0$. Let $i \neq i'$ be such that $v_i \neq 0 = v_{i'}$. This gives $a_i \geq 0$.

Assume that $i = 1$ so that $a_1 \geq 0$.

$$\begin{aligned}0 &\leq 6a_1(-3a_1 - 3a_2 + b_1 + b_2) + 20a_1^2 + \tfrac{11}{2}(b_2 - b_1)^2 + \tfrac{1}{2}(2a_2 + 3b_1 + 3b_2)^2 \\ &= 2a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 + 2a_2^2 + 6a_2b_1 + 6a_2b_2 + 10b_1^2 - 2b_1b_2 + 10b_2^2 \\ &= 12(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\ &= 11\|\lambda\|^2 - \mu(p, \lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{11} = m((-30)(11))$.

Here $(-30)(11)$ acts by non-negative weights on q so it is optimal.

Case 2.2b: The point q has type $(1,1,2)$ and $(0,0,2)$ but not $(1,0,2)$ or $(1,0,1)$ and none of the entries of v is zero

Then $a_1, a_2 \geq 0$ so

$$\begin{aligned} 0 &\leq 18a_1a_2 + 6a_1(-3a_1 - 3a_2 + b_1 + b_2) + 6a_2(-3a_1 - 3a_2 + b_1 + b_2) + 11a_2^2 + 11a_1^2 + (b_1 - b_2)^2 \\ &= -7a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 - 7a_2^2 + 6a_2b_1 + 6a_2b_2 + b_1^2 - 2b_1b_2 + b_2^2 \\ &= 2(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\ &= 2\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{2} = m((00)(11))$.

As in Case 2.2a, $\mathbf{X}^{(1)} = \mathbf{Y}^{(0)} = 0$, so $(00)(11)$ acts by non-negative weights on q and is optimal.

Case 2.3: The point q has type $(1,1,2)$ and $(1,0,0)$ but not $(1,0,2)$, $(1,0,1)$ or $(0,0,2)$

Having type $(1,0,0)$ implies that $\mathbf{v} = 0$ and not having type $(0,0,2)$ implies that $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ are not both zero. Having type $(1,1,2)$ implies that their images span a one-dimensional space and not being type $(1,0,1)$ implies that their kernels intersect trivially. Thus $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ have rank one and Lemma C.3.2 gives an i such that $a_i \leq b_1, b_2$.

Assume $a_1 \leq b_1, b_2$ so that $a_1 - b_1, a_1 - b_2 \geq 0$.

$$\begin{aligned} 0 &\leq \tfrac{8}{3}(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_1) + \tfrac{8}{3}(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_2) \\ &\quad + \tfrac{70}{9}(a_1 - b_2)^2 + \tfrac{1}{3}(3a_1 - a_2 + 3b_1 + 3b_2)^2 + 8(\tfrac{5}{6}a_1 - b_1 + \tfrac{1}{6}b_2)^2 \\ &= \tfrac{1}{3}a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 + \tfrac{1}{3}a_2^2 + 6a_2b_1 + 6a_2b_2 + \tfrac{25}{3}b_1^2 - 2b_1b_2 + \tfrac{25}{3}b_2^2 \\ &= \tfrac{28}{3}(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\ &= \tfrac{28}{3}\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{9\tfrac{1}{3}} = m((-9-1)(-1-1))$.

Let e_2 span the image of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$. Then they have a row of zeros in the top row as coordinates of r and $\mathbf{v} = 0$. Thus $(-9-1)(-1-1)$ acts with non-negative weights on r and is optimal for q.

Case 2.4: The point q has type $(1,1,2)$ and $(0,0,1)$ but not $(1,0,2)$, $(1,0,1)$, $(0,0,2)$ or $(1,0,0)$

Not being of type $(1,0,0)$ or $(0,0,2)$ implies that $\mathbf{v} \neq 0$ and $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ are not both zero. Thus, for q to have type $(1,1,2)$, the image of $\mathbf{v}$ must coincide with the images of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$. I claim that there exists an i, j such that $a_i \geq b_j, 0$. Indeed, if both entries of $\mathbf{v}$ are non-zero then let (i, j) be the coordinates of a non-zero entry of $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$. Otherwise, choose i so that $v_i \neq 0$ and $v_i = 0$. Then $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must have a non-zero entry in row i, position (i, j) say. This satisfies the claim.

Assume $a_1 \geq b_1, 0$

$$\begin{aligned} 0 &\leq \tfrac{200}{11}a_1^2 + 4a_1(-3a_1 - 3a_2 + b_1 + b_2) + 2(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_1) \\ &\quad + (3b_2 + a_2)^2 + 11(\tfrac{3}{11}a_1 - b_1)^2 \\ &= a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 + a_2^2 + 6a_2b_1 + 6a_2b_2 + 9b_1^2 - 2b_1b_2 + 9b_2^2 \\ &= 10(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\ &= 10\|\lambda\|^2 - \mu(p,\lambda)^2. \end{aligned}$$

So $M(q) \leq \sqrt{10} = m((-30)(01))$.

Let e_2 span the image of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ and let e_4 span the representation of type $(0,0,1)$. Then $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ are strictly lower triangular and $v_1 = 0$ as coordinates of r. Thus $(-30)(01)$ acts with non-negative weights on r and so is optimal for q.

Case 2.5: The point q has type $(1,1,2)$ but not $(1,0,2)$, $(1,0,1)$, $(0,0,2)$, $(1,0,0)$ or $(0,0,1)$

Being type $(1,1,2)$ implies that the images of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(1)}$ span a one-dimensional space and so are rank at most one. If one of them was zero then the kernel of the other would give q type $(0,0,1)$ so they must both be rank one with trivially intersecting kernels; that is, the hypotheses of Lemma C.3.2 hold. Also not being $(1,0,0)$ implies that $\mathbf{v} \neq 0$.

I claim that there is an i such that $a_i \geq b_1, b_2, 0$. If $i = 1,2$ both satisfy the claim of the lemma then choose i so that $v_i \neq 0$. Otherwise, if i satisfies the lemma and i' doesn't then $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must have a common zero in row i'. It follows that they must both have zeros in both entries of row i' so that their image is row i. If v_i were zero this would contradict q having type $(1,1,2)$, since $\mathbf{v}$ and $\mathbf{X}^{(1)}$ are non-zero their images must coincide.

Assume $a_1 \geq b_1, b_2, 0$

$$\begin{aligned}
0 &\leq 2a_1(-3a_1 - 3a_2 + b_1 + b_2) + 2(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_1) \\
&\quad + \tfrac{15}{2}(a_1 - b_1)^2 + 2(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_2) \\
&\quad + 28(\tfrac{3}{7}a_1 + \tfrac{2}{7}b_1 + \tfrac{2}{7}b_2)^2 + \tfrac{54}{7}(\tfrac{5}{6}a_1 - b_2 + \tfrac{1}{6}b_1)^2 \\
&= -18a_1a_2 + 6a_1b_1 + 6a_1b_2 + 6a_2b_1 + 6a_2b_2 + 8b_1^2 - 2b_1b_2 + 8b_2^2 \\
&= 9(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\
&= 9\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

So $M(q) \leq \sqrt{9} = m((-10)(00))$.

Let e_2 span the image of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$. Then $v_1 = 0$ and $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must have zeros in the top row (as coordinates of r). Thus $(-10)(00)$ acts by non-zero weights on r and so is optimal for q.

Case 3: The point q has type $(1,0,1)$ and but not $(1,0,2)$, $(1,1,2)$ or $(0,0,2)$

Having type $(1,0,1)$ implies that $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ have non-trivially intersecting kernels. Not having type $(1,1,2)$ then implies that $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must both have rank one and trivially intersecting images. Now Lemma C.2.3 gives an i such that $a_1, a_2 \geq b_i$.

Assume $a_1 \geq b_1, b_2, 0$

$$\begin{aligned}
0 &\leq 20(a_1 - b_1)^2 + 20(a_2 - b_1)^2 + 8(a_1 - b_1)(-3a_1 - 3a_2 + b_1 + b_2) \\
&\quad + 8(a_2 - b_1)(-3a_1 - 3a_2 + b_1 + b_2) + (a_1 + a_2 + b_1 + 5b_2)^2 + (a_1 - a_2)^2 \\
&= \tfrac{1}{3}a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 + \tfrac{1}{3}a_2^2 + 6a_2b_1 + 6a_2b_2 + \tfrac{25}{3}b_1^2 - 2b_1b_2 + \tfrac{25}{3}b_2^2 \\
&= \tfrac{28}{3}(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\
&= \tfrac{28}{3}\|\lambda\|^2 - \mu(p,\lambda)^2.
\end{aligned}$$

So $M(q) \leq \sqrt{9\tfrac{1}{3}} = m((-5-5)(-53))$.

Let e_4 span the kernel of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ so that they have a column of zeros on the right as coordinates of r. Being of type $(1,0,1)$ implies that $\mathbf{v} = 0$ so $(-5-5)(-53)$ acts on r with non-negative weights and so is optimal for q.

Case 4: The point q has type $(1,0,0)$ and but not $(1,0,2)$, $(1,1,2)$, $(0,0,2)$ or $(1,0,1)$

Being of type $(1,0,0)$ and not of type $(1,0,1)$ implies that the kernels of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ intersect trivially and being of type $(1,1,2)$ implies that their images span two dimensions. I claim that there exists $i \neq i'$ and $j \neq j'$ such that $a_i \geq b_j$ and $a_{i'} \geq b_{j'}$. If either $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$ has rank two then the claim is immediate. Otherwise, they must both have rank one and Lemma C.2.4 applies to give the result.

Assume $a_1 \geq b_1$ and $a_2 \geq b_2$.

$$\begin{aligned}0 &\leq 4(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_1) + 4(-3a_1 - 3a_2 + b_1 + b_2)(a_2 - b_2)\\ &\quad + 6(a_1 - b_1)^2 + 8(a_1 - b_1)(a_2 - b_2) + 6(a_2 - b_2)^2 + (a_1 - a_2 + b_1 - b_2)^2\\ &= -5a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 - 5a_2^2 + 6a_2b_1 + 6a_2b_2 + 3b_1^2 - 2b_1b_2 + 3b_2^2\\ &= 4(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2\\ &= 4\|\lambda\|^2 - \mu(p,\lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{4} = m((-1-1)(-1-1))$.

Being of type $(1,0,0)$ means that $\mathbf{v} = 0$ so that $(-1-1)(-1-1)$ acts on q with non-negative weights and is optimal.

Case 5.1: The point q has type $(1,1,1)$ and $(0,0,1)$ but not $(1,0,2)$, $(1,1,2)$, $(0,0,2)$, $(1,0,1)$ or $(1,0,0)$

I claim there exists $i \neq i'$ and j such that $a_i \geq b_j$ and $a_{i'} \geq 0$. Indeed, not being of type $(1,0,0)$ implies that $\mathbf{v} \neq 0$. If both entries of $\mathbf{v}$ are non-zero then choose i, j so that (i, j) is a non-zero entry of $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$ (q has not got type $(0,0,2)$). Otherwise, let i be such that $v_i = 0$ and $v_{i'} \neq 0$. Being of type $(0,0,1)$ implies that $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ have rank at most one. The claim could only fail to hold if both $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ had zeros in row i. This would imply that the images of $\mathbf{X}^{(1)}$, $\mathbf{Y}^{(0)}$ and $\mathbf{v}$ coincided so contradicts q not being of type $(1,1,2)$. This proves the claim.

Assume $a_1 \geq b_1$ and $a_2 \geq 0$.

$$\begin{aligned}0 &\leq 12a_2^2 + 12a_2(a_1 - b_1) + 6a_2(-3a_1 - 3a_2 + b_1 + b_2) + \tfrac{11}{2}(a_1 - b_1)^2\\ &\quad + 4(a_1 - b_1)(-3a_1 - 3a_2 + b_1 + b_2) + 2(\tfrac{1}{2}a + \tfrac{1}{2}c + b_2)^2\\ &= -6a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 - 6a_2^2 + 6a_2b_1 + 6a_2b_2 + 2b_1^2 - 2b_1b_2 + 2b_2^2\\ &= 3(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2\\ &= 3\|\lambda\|^2 - \mu(p,\lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{3} = m((-10)(-11))$.

Let e_2 span the image of $\mathbf{v}$ and e_4 the kernel of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$. Then $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ have a column of zeros on the right and $v_1 = 0$. Notice that, because the images of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ must span two dimensions they must acts as zero on $(1,1,1)$. Hence $(1,1,1)$ contains both e_4 and e_2, the image of $\mathbf{v}$. It follows that $\mathbf{X}^{(0)}(e_2) = \mathbf{Y}^{(1)}(e_2) = e_4$ so $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ are (non-strictly) lower triangular. Now $(-10)(-11)$ acts on r with non-negative weights so it is optimal for q.

Case 5.2: The point q has type $(1,1,1)$ but not $(1,0,2)$, $(1,1,2)$, $(0,0,2)$, $(1,0,1)$, $(1,0,0)$ or $(0,0,1)$

First, not being type $(1,0,0)$ implies that $\mathbf{v} \neq 0$ and not being type $(0,0,1)$ implies that the kernels of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ intersect trivially. In fact, in one were zero then the other must have rank two. Or, both have rank one and trivially intersecting kernels and images.

I claim that there exists $i \neq i'$ and j such that $a_i \geq b_j$ and $a_{i'} \geq b_{j'}, 0$. Suppose both entries of $\mathbf{v}$ were non-zero. Then if one of $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$ had rank two the claim is immediate. Otherwise, they both have rank one and Lemma C.2.4 can be used to prove the claim.

Suppose instead that $v_i = 0$ and $v_{i'} \neq 0$. Then the claim could only fail if both $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ had zeros in row i. This would imply that they both had rank one and non-trivially intersecting images, a contradiction. This proves the claim.

Assume $a_1 \geq b_1$ and $a_2 \geq b_2, 0$.

$$\begin{aligned}0 &\leq 8a_2^2 + 4a_2(-3a_1 - 3a_2 + b_1 + b_2) + 8a_2(a_1 - b_1) + 4(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_1)\\ &\quad + 2(-3a_1 - 3a_2 + b_1 + b_2)(a_2 - b_2) + 5(a_1 - b_1)^2 + 4(a_1 - b_1)(a_2 - b_2) + 3(a_2 - b_2)^2\\ &= -7a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 - 7a_2^2 + 6a_2b_1 + 6a_2b_2 + b_1^2 - 2b_1b_2 + b_2^2\\ &= 2(a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2\\ &= 2\|\lambda\|^2 - \mu(p, \lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{2} = m((-10)(-10))$.

Let e_2 span the image of $\mathbf{v}$ and e_4 a non-zero vector in $(1, 1, 1)$ and V_1. Now $\mathbf{X}^{(0)}$, $\mathbf{X}^{(1)}$, $\mathbf{Y}^{(0)}$ and $\mathbf{Y}^{(1)}$ must be (non-strictly) lower triangular and $v_1 = 0$ as coordinates of r. Thus $(-10)(-10)$ acts on r with non-negative weights and so is optimal for q.

Case 6: The point q has type $(0, 0, 1)$ and but not $(1, 0, 2)$, $(1, 1, 2)$, $(0, 0, 2)$, $(1, 0, 1)$, $(1, 0, 0)$ or $(1, 1, 1)$

Having type $(0, 0, 1)$ implies the kernels of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ intersect non-trivially and not being type $(1, 1, 2)$ implies they are both rank one with trivially intersecting images. Not having type $(1, 0, 0)$ implies that $\mathbf{v} \neq 0$.

Suppose $v_i = 0$ so that $v_{i'} \neq 0$. I claim that there is some j such that $a_1, a_2 \geq b_j$ and $b_j \geq a_{i'} \geq 0$. Suppose this were not the case and choose j from Lemma C.2.3 applied to $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$. If both $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ had a zero in entry (j, i') then the composition $\mathbf{X}^{(0)}\mathbf{v}$ and $\mathbf{Y}^{(1)}\mathbf{v}$ would lie in the kernel of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ and have zeros in column j'. This would produce a subrepresentation of type $(1, 1, 1)$, a contradiction, and the claim is proved. Now $a_1, a_2 \geq b_{j'} \geq a_{i'} \geq 0$ so $b_{j'} = a_{j'}$.

Assume $a_1 \geq a_2 = b_1 \geq 0$.

$$\begin{aligned}0 &\leq 10a_1^2 + 6a_2^2 + 6(-3a_1 - 2a_2 + b_2)a_1 + 4(-3a_1 - 2a_2 + b_2)a_2 + 12a_1a_2\\ &= -8a_1^2 - 2a_2^2 - 12a_1a_2 + 6a_1b_2 + 4a_2b_2\\ &= (a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2\\ &= \|\lambda\|^2 - \mu(p, \lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{1} = m((00)(01))$.

Suppose that both entries of $\mathbf{v}$ were non-zero. Then $a_1, a_2 \geq 0$ and applying Lemma C.2.3 gives an i such that $a_1, a_2 \geq b_i$.

Assume $a_1, a_2 \geq b_1, 0$.

$$\begin{aligned}0 &\leq 6a_1^2 + 12a_1a_2 + 2a_1(a_1 - b_1) + 4a_1(-3a_1 - 3a_2 + b_1 + b_2) + 10a_2^2 + 6a_2(a_1 - b_1)\\ &\quad + 6a_2(-3a_1 - 3a_2 + b_1 + b_2) + 2(a_1 - b_1)(-3a_1 - 3a_2 + b_1 + b_2) + 2(a_1 - b_1)^2\\ &= -8a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 - 8a_2^2 + 6a_2b_1 + 6a_2b_2 - 2b_1b_2\\ &= (a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2\\ &= \|\lambda\|^2 - \mu(p, \lambda)^2.\end{aligned}$$

So $M(q) \leq \sqrt{1} = m((00)(01))$.

Let e_4 span the kernels of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$. Then $(00)(01)$ acts with non-negative weights on r so is optimal for q.

Case 7: The point q only has type $(1, 1, 0)$

Not being type $(1, 0, 0)$ implies that $\mathbf{v} \neq 0$, let e_2 be a non-zero vector in the image. Not being of type $(0, 0, 1)$ means that the kernels of $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ intersect trivially and not being type $(1, 1, 2)$ implies that their images span two dimensions. Being type $(1, 1, 0)$ means e_2 is in the kernels of $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$.

I claim that there is an i such that $a_i \geq b_1, b_2$ and $a_{i'} \geq 0$. Indeed, suppose both entries of $\mathbf{v}$

are non-zero and either $\mathbf{X}^{(1)}$ or $\mathbf{Y}^{(0)}$ has rank two. For the claim to fail in this case, $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ would need to have common zeros in both rows. If both entries of $\mathbf{v}$ are non-zero and $\mathbf{X}^{(1)}$ and $\mathbf{Y}^{(0)}$ both have rank one then apply Lemma C.3.2. Otherwise, $v_i = 0$ and $v_{i'} \neq 0$.

Assume $a_1 \geq b_1, b_2$ and $a_2 \geq 0$.

$$\begin{aligned}
0 &\leq 18(-3a_1 - 3a_2 + b_1 + b_2)a_2 + 8(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_1) \\
&\quad + 8(-3a_1 - 3a_2 + b_1 + b_2)(a_1 - b_2) + 24a_2(a_1 - b_1) + 24a_2(a_1 - b_2) \\
&\quad + 6(a_1 - b_1)^2 + 10(a_1 - b_1)(a_1 - b_2) + 6(a_1 - b_2)^2 + 28a_2^2 \\
&= -8a_1^2 - 18a_1a_2 + 6a_1b_1 + 6a_1b_2 - 8a_2^2 + 6a_2b_1 + 6a_2b_2 - 2b_1b_2 \\
&= (a_1^2 + a_2^2 + b_1^2 + b_2^2) - (-3a_1 - 3a_2 + b_1 + b_2)^2 \\
&= ||\lambda||^2 - 3\mu(p, \lambda)^2.
\end{aligned}$$

So $M(q) \leq \sqrt{\frac{1}{3}} = m((-10)(-1-1))$.

Completing e_2 to a basis shows that $v_1 = 0$ and $\mathbf{X}^{(0)}$ and $\mathbf{Y}^{(1)}$ have a column of zeros on the right as coordinates of r. Hence $(-10)(-1-1)$ acts on r with non-zero weights and so is optimal for r and so for q.

This completes the proof of Proposition 7.5.1.

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