CONTINUITY OF DERIVATIONS

AND

UNIFORM ALGEBRAS ON ODD SPHERES

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PREFACE

The material presented in this thesis is claimed as original with the exception of those sections where specific mention is made to the contrary.

ABSTRACT

The thesis is composed of two separate and distinct parts.

Part one is concerned with the problem of determining when certain linear mappings are necessarily continuous with particular attention being given to derivations.

Chapter 1 consists of a discussion of the separating space of a linear mapping. Chapter 2 contains a description of the Banach algebra $L^1[0,1]$ and some of its properties. In Chapter 3 we consider derivations on $L^1[0,1]$, proving in Theorem 3.1 that they are necessarily continuous. In Chapter 4 we extend this result to module derivations and in Theorem 4.2 we give sufficient conditions on a Banach algebra B such that every module derivation from B is continuous. When B is separable and commutative we can improve Theorem 4.2 and then it is easily seen that one of the sufficient conditions is best possible. In Chapter 5 we give sufficient conditions on a Banach algebra B such that certain higher derivations from any Banach algebra onto B are automatically continuous.

Part two is concerned with the recent result of D.E. Marshall and S-Y. A. Chang that every closed subalgebra of $L^\infty(T)$ (where T is the unit circle) containing $H^\infty(T)$ is a Douglas algebra. Using their techniques we give a proof of this result and discuss generalisations of these ideas and related concepts to higher dimensions.

Chapter 6 consists of a discussion of Douglas algebras, functions of vanishing mean oscillation (VMO), Carleson measures and other topics. In Chapter 7 we generalise the space of VMO and provide a characterisation of this new space in terms of Carleson measures. Using

these ideas we prove the Marshall-Chang theorem in Chapters 8 and 9. Chapter 10 discusses the subject of Douglas algebras in higher dimensions. Chapter 11 gives a description of a particular class of Hankel operators on $L^2(S)$ (where S is the unit sphere in $\underset{\sim}{C}^n$). In Chapter 12 we characterise the Toeplitz operators amongst operators on $H^2(S)$ in terms of an operator equation. In Chapters 10, 11 and 12 we pose several open questions.

CONTENTS

CHAPTER ONE

In this chapter we list some definitions and propositions which we shall need throughout the first part of this thesis.

Notation. X, Y, Z will denote (complex) Banach spaces, $\mathfrak{B}(X)$ will denote the algebra of bounded linear operators on X, and $\mathfrak{B}(X,Y)$ will denote the algebra of bounded linear operators from X to Y. For a set U in a Banach space the closure of U is given by $\bar{U}$. Throughout this thesis $\subset$ means strict inclusion.

Central in our approach to proving that certain linear mappings are continuous is the concept of the separating space which we now define.

Definition. Let S be a linear mapping from X into Y. The separating space, $\mathfrak{S}(S)$, of S is given by
$\mathfrak{S}(S) = \{y \in Y:$ there are $x_n \to 0$ in X with $Sx_n \to y$ in $Y\}$.

Some elementary properties of $\mathfrak{S}(S)$ are listed in the following lemma.

LEMMA 1.1 (a) $\mathfrak{S}(S)$ *is a closed linear subspace of* Y,

(b) S *is continuous if and only if* $\mathfrak{S}(S) = \{0\}$,

(c) *if* $U \in \mathfrak{B}(Y,Z)$ then $(U\mathfrak{S}(S))^- = \mathfrak{S}(US)$ *and there is a constant* M *(independent of* U *and* Z*) such that if* US *is continuous then* $\|US\| \leq M\|U\|$,

(d) *if* $T \in \mathfrak{B}(X)$, $R \in \mathfrak{B}(Y)$ *satisfy* $RS - ST \in \mathfrak{B}(X,Y)$

then $R\mathfrak{S}(S) \subseteq \mathfrak{S}(S)$ *and* $(R\mathfrak{S}(S))^- = \mathfrak{S}(ST)$.

Proof. (a), (b), (c) are well-known and proofs can be found in [28]. (b) is merely the closed graph theorem and it is this property of $\mathfrak{S}(S)$ that will provide us with a criteria for the continuity of S in particular situations. (c) is known when $RS - ST = 0$. We prove the case $RS - ST \in \mathfrak{B}(X,Y)$: let $y \in \mathfrak{S}(S)$ so that there are x_n in X, $x_n \to 0$ and $Sx_n \to y$. Then $Tx_n \to 0$ and $STx_n = (ST - RS)x_n + RSx_n \to Ry$ so that $Ry \in \mathfrak{S}(S)$. Hence $R\mathfrak{S}(S) \subseteq \mathfrak{S}(S)$. Also it is clear that $\mathfrak{S}(RS) = \mathfrak{S}(ST)$ and so (c) gives $(R\mathfrak{S}(S))^- = \mathfrak{S}(RS) = \mathfrak{S}(ST)$.

The next result and its following special case give stability lemmas for the separating space which yield the crucial property of the separating space which we shall appeal to in the proofs of our main results in part 1. The idea of the lemma is initially due to B.E. Johnson and A.M. Sinclair [16] and then A.M. Sinclair [29]. The form in which we state it is due to K. Laursen [19] and we give the proof for completeness.

LEMMA 1.2 *Let* S *be a linear mapping from* X *into* Y *and let* $\{T_n\}$ *be a sequence in* $\mathfrak{B}(X)$. *Then there exists an integer* N *such that* $\mathfrak{S}(ST_1\ldots T_n) = \mathfrak{S}(ST_1\ldots T_{n+1})$ *for* $n \geq N$.

Proof. Clearly $\mathfrak{S}(ST_1\ldots T_{n+1}) \subseteq \mathfrak{S}(ST_1\ldots T_n)$ for $n \geq 1$. If this inclusion is strict for infinitely many n, then by grouping the T_i's into finite products corresponding to the intervals of constancy of $\mathfrak{S}(ST_1\ldots T_n)$ we may assume that $\mathfrak{S}(ST_1\ldots T_{n+1}) \subset \mathfrak{S}(ST_1\ldots T_n)$ for all

$n \geq 1$. Let Q_n denote the natural quotient mapping from Y onto $Y / \mathscr{G}(ST_1 \ldots T_n)$ for each n. Then, by Lemma 1.1 (b), (c), $Q_n ST_1 \ldots T_n$ is continuous and $Q_n ST_1 \ldots T_{n-1}$ is discontinuous for all $n \geq 2$. Assuming, without loss of generality, that $\|T_n\| \leq 1$ for all n, we choose inductively a sequence $\{x_n\}$ from X so that

(i) $\|x_n\| \leq 2^{-n}$, and

(ii) $\|Q_n ST_1 \ldots T_{n-1} x_n\| \geq n + \|Q_n ST_1 \ldots T_n\| + \|Q_n S(\sum_{j=2}^{n-1} T_1 \ldots T_{j-1} x_j)\|$

for $n = 3, 4, \ldots$

Then let $z = \sum_{n=2}^{\infty} T_1 \ldots T_{n-1} x_n$ (the sum converges by (i)). For each positive integer n we have

$$\|Sz\| \geq \|Q_n Sz\| \quad \text{and}$$

$$\begin{aligned} \|Q_n ST_1 \ldots T_{n-1} x_n\| &= \|Q_n Sz - Q_n S(\sum_{j=2}^{n-1} T_1 \ldots T_{j-1} x_j) \\ &\quad - Q_n S(\sum_{j=n+1}^{\infty} T_1 \ldots T_{j-1} x_j)\| \\ &\leq \|Q_n Sz\| + \|Q_n S(\sum_{j=2}^{n-1} T_1 \ldots T_{j-1} x_j)\| \\ &\quad + \|Q_n ST_1 \ldots T_n\| \|x_{n+1} + \sum_{j=n+1}^{\infty} T_{n+1} \ldots T_j x_{j+1}\| \end{aligned}$$

and so

$$\begin{aligned} \|Sz\| &\geq \|Q_n ST_1 \ldots T_{n-1} x_n\| - \|Q_n S(\sum_{j=2}^{n-1} T_1 \ldots T_{j-1} x_j)\| \\ &\quad - \|Q_n ST_1 \ldots T_n\| \quad \text{(using (i))} \\ &\geq n \quad \text{by (ii)}. \end{aligned}$$

This contradiction proves the lemma.

We shall be interested in situations where operators T, R (in $\mathfrak{B}(X)$ and $\mathfrak{B}(Y)$, respectively) intertwine with S continuously, i.e. $ST - RS \in \mathfrak{B}(X,Y)$. In this situation Lemma 1.1 (d) enables us to put Lemma 1.2 in the form we shall need for our applications.

<u>LEMMA 1.3</u> *Let* $\{T_n\}$ *and* $\{R_n\}$ *be sequences in* $\mathfrak{B}(X)$ *and* $\mathfrak{B}(Y)$, *respectively. If* S *is a linear operator from* X *into* Y *such that* $R_n S - ST_n \in \mathfrak{B}(X,Y)$ *for all* n, *then there is an integer* N *such that* $(R_1 \ldots R_n \mathfrak{S}(S))^- = (R_1 \ldots R_N \mathfrak{S}(S))^-$ *for all* $n \geq N$.

Proof. By induction $R_1 \ldots R_n S - ST_1 \ldots T_n \in \mathfrak{B}(X,Y)$ for all n. Lemma 1.2 and Lemma 1.1 (d) then give the result.

<u>COROLLARY 1.4</u> *Let* X, Y, $\{T_n\}$, $\{R_n\}$, S, N *be as in Lemma 1.3. Let* $\{U_n\}$ *be a sequence in* $\mathfrak{B}(Y)$ *such that* $U_n R_1 \ldots R_n \mathfrak{S}(S) = \{0\}$ *for all* n. *Then* $U_n R_1 \ldots R_{n-1} \mathfrak{S}(S) = \{0\}$ *for all* $n > N$.

CHAPTER TWO

In this chapter we discuss the radical Banach algebra $L^1[0,1]$ and some of its properties. Throughout the next four chapters all ideals will be two-sided.

Definition. $L^1[0,1]$ is the Banach algebra of complex-valued functions which are (Lebesgue) integrable on the closed interval $[0,1]$ with pointwise addition and (convolution) multiplication given by

$$f * g(x) = \int_0^x f(x-t)g(t)\,dt \qquad (x \in [0,1])$$

and norm

$$\|f\| = \int_0^1 |f(t)|\,dt.$$

Remark. We take the usual liberty of referring to elements of $L^1[0,1]$ as functions whereas they are, in fact, equivalence classes of functions agreeing almost everywhere (a.e.) on $[0,1]$.

PROPOSITION 2.1 (1) $L^1[0,1]$ *is a radical Banach algebra which is singly-generated.*

(2) $L^1[0,1]$ *has a bounded approximate identity.*

Proof. (1) For $f \in L^1[0,1]$ let f^{*n} denote $f * f * \cdots * f$ (n times). Then (by induction) $1^{*n} = \frac{x^{n-1}}{(n-1)!}$, the norm of which is $\frac{1}{n!}$, where 1 denotes the function which takes the constant value 1 on $[0,1]$. Hence 1 generates the polynomials in x and so the continuous functions and so all of $L^1[0,1]$. We have $\|1^{*n}\|^{1/n} \to 0$ and so 1 is quasinilpotent. Since 1 generates $L^1[0,1]$ every element is thus quasinilpotent and so $L^1[0,1]$ is radical.

(2) Take a one-sided Dirac sequence e.g. $u_n = n\chi_{[0,1/n]}$.

<u>Definition</u>. Let V be the continuous operator on $L^1[0,1]$ given by (convolution) multiplication by 1, i.e.

$$Vf(x) = (1 * f)(x) = \int_0^x f(t)dt.$$

V is called the <u>Volterra integral operator</u>.

<u>Notation</u>. Let $\alpha, \beta \in [0,1]$. Then

$$M_\alpha = \{f \in L^1[0,1] : f \text{ vanishes a.e. on } [0,\alpha]\}.$$

f_β will denote the characteristic function of $[\beta,1]$ for each β in $[0,1]$.

<u>PROPOSITION 2.2</u> *The closed invariant subspaces of* V *are the subspaces* M_α *where* $0 \leq \alpha \leq 1$.

Proof. This proof is due to W.F. Donoghue, Jr. [12]. First, note that M_α is a closed invariant subspace of V for $0 \leq \alpha \leq 1$. The result is first established for $C[0,1]$, the space of continuous functions on $[0,1]$. It is clear that $C[0,1]$ is invariant under V. Let M be a non-trivial closed invariant subspace of V in $C[0,1]$. Let f be a non-zero element in M. Consider the sequence $f, Vf, V^2f, \ldots$. We choose a measure μ on $[0,1]$ orthogonal to every V^nf, i.e. $\int_0^1 V^nf(t)d\mu(t) = 0$ for all $n \geq 0$. A theorem, the most general version of which is due to J. Lions [20], asserts that for any two distributions on $\mathbb{R}^n$ with compact support, the convex hull of the support of the convolution is the vectorial sum of the convex hull of the supports of the factors. Thus if the convex hull of the support of f is (a,b) and the convex hull of the support of μ

is (c,d) it follows that the interval $(a-d,b-c)$ is the convex hull of the support of $f * \tilde{\mu}$ where $\tilde{\mu}$ is given by $\tilde{\mu}(t) = \mu(-t)$. But $f * \tilde{\mu}$ vanishes on the left half-axis. (We can assume that the functions and measures are defined throughout $\underset{\sim}{R}$ by defining them to be zero outside $[0,1]$). Therefore $d \leq a$ which implies that μ is orthogonal to M_a. The Hahn-Banach theorem and the Riesz representation theorem imply that the closed linear span of $\{V^n f\colon\ n \geq 0\}$ is M_a, unless $a = 0$, in which case the closed linear span will be the whole space if $f(0) \neq 0$. Thus any proper invariant subspace for V in $C[0,1]$ is a union of spaces of type M_s and is therefore a space of that type itself.

For the space $L^1[0,1]$ the same result follows from the observation that $VL^1[0,1] \subseteq C[0,1]$. For let M be a closed invariant subspace of V in $L^1[0,1]$ and let $f \in M$. If the smallest interval containing the support of Vf is $[a,b]$, then the sequence $\{V^n f\colon\ n \geq 1\}$ spans the subspace M_a of $C[0,1]$ as above and its closure in $L^1[0,1]$ is the corresponding M_a of that space. Evidently $f = 0$ a.e. on $[0,a]$ and so $\{V^n f\colon\ n \geq 0\}$ spans M_a in $L^1[0,1]$ and the result follows as before.

<u>Remark</u>. Initially J. Dixmier [11] found the invariant subspaces of V on real $L^1[0,1]$ by considering algebras generated by V and similar convolution operators. W.F. Donoghue, Jr. [12] and M.S. Brodski [7] independently discovered the invariant subspaces of V on complex $L^2[0,1]$. Donoghue's proof in fact works for $L^p[0,1]$ where $1 \leq p < \infty$.

<u>COROLLARY 2.3</u> *The closed ideals of* $L^1[0,1]$ *are the subspaces* M_α

where $0 \leq \alpha \leq 1$.

Proof. The corollary follows from Proposition 2.2 since $Vf = 1 * f$ and so the closed ideals of $L^1[0,1]$ are closed invariant subspaces of V.

PROPOSITION 2.4 *If* α *and* β *are positive and* $\alpha + \beta \leq 1$, *then* $(f_\beta * M_\alpha)^- = M_{\alpha+\beta}$.

Proof. $(f_\beta * M_\alpha)^-$ is a closed ideal of $L^1[0,1]$ and so by Corollary 2.3 we have $(f_\beta * M_\alpha)^- = M_\gamma$ for some $\gamma \in [0,1]$. We show that $\gamma = \alpha + \beta$. By the definition of convolution it is clear that $(f_\beta * M_\alpha)^- \subseteq M_{\alpha+\beta}$ and so $\gamma \geq \alpha + \beta$. If $\alpha + \beta = 1$, then $\gamma = 1 = \alpha + \beta$. So suppose $\alpha + \beta < 1$ and let $\varepsilon > 0$ be chosen such that $\alpha + \beta + \varepsilon \leq 1$. Consider $f_\beta * f_\alpha$:

$$(f_\beta * f_\alpha)(x) = \int_0^x f_\beta(x-t) f_\alpha(t)\,dt = \begin{cases} 0 & 0 \leq x \leq \alpha + \beta \\ \int_\alpha^{x-\beta} f_\beta(x-t)\,dt = \int_\alpha^{x-\beta} dt & \\ = x - (\alpha + \beta) & \alpha + \beta \leq x \leq 1 \end{cases}$$

From this it is clear that $f_\beta * f_\alpha \notin M_{\alpha+\beta+\varepsilon}$. But $f_\beta * f_\alpha \in M_\gamma$ and so $\gamma = \alpha + \beta$ (otherwise take $\varepsilon = \gamma - \alpha - \beta$).

It is clear that Corollary 2.3 shows that there do not exist any non-zero finite dimensional ideals in $L^1[0,1]$. In fact it is possible to prove this without appealing to the characterisation of the closed ideals.

PROPOSITION 2.5 (1) $L^1[0,1]$ *has no non-zero finite dimensional ideals.*

(2) *Let* I *be a non-zero ideal in* $L^1[0,1]$.
Then $f_\alpha I \subset I$ for any $\alpha \in (0,1]$, *and we can choose* α *so that* $f_\alpha I \neq \{0\}$.

Proof. (1) Let J be any non-zero ideal in $L^1[0,1]$ and choose $f \in J$ with $f \neq 0$. Then it is clear that if $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$, and neither $f_\alpha * f$ or $f_\beta * f$ is zero then $f_\alpha * f$ and $f_\beta * f$ are linearly independent and belong to J. Since for any non-zero f there is an infinite choice of distinct α's in $[0,1]$ with $f_\alpha * f \neq 0$ it follows that J is infinite dimensional.

(2) Let $\beta = \sup\{\gamma: \; f = 0$ a.e. on $[0,\gamma]$ for all $f \in I\}$. Then $\beta < 1$. For all $f \in I$, $f_\alpha * f = 0$ a.e. on $[0,\delta]$ where $\delta = \min(1,\alpha+\beta) > \beta$ if $\alpha > 0$. Hence $f_\alpha I \subset I$ by the definition of β, and by choosing α so that $\alpha + \beta < 1$ we have $f_\alpha I \neq \{0\}$.

CHAPTER THREE

In this chapter we prove that derivations on $L^1[0,1]$ are automatically continuous, and then show that the methods used in the proof can be extended to give other known results on the continuity of derivations.

Definition. Let B be an algebra. A derivation on B is a linear operator D on B satisfying $D(ab) = aD(b) + D(a)b$ for all a, b in B.

We note here that if B is a Banach algebra then D satisfies the hypothesis of Lemma 1.3 in the sense that D intertwines continuously with continuous operators on B. For if L_a denotes the operation of left multiplication by a on B and if we regard a as a fixed element of B then the definition of a derivation yields $DL_a - L_aD \in \mathfrak{B}(B)$ for any a in B. We also make the remark that when D is a derivation on B it is easy to see that $\mathfrak{S}(D)$ is a closed ideal in B.

In [17] B.E. Johnson and A.M. Sinclair proved that every derivation on a semi-simple Banach algebra is continuous. During a conference at the University of California, Los Angeles, in July, 1974 the related question of whether every derivation on the radical Banach algebra $L^1[0,1]$ is continuous was raised. Theorem 3.1 answers this question in the affirmative. First note that there do exist non-trivial derivations on $L^1[0,1]$, e.g. pointwise multiplication by the function h given by $h(x) = x$ is a continuous derivation on $L^1[0,1]$. For

$$h(f * g)(x) = x \int_0^x f(x - t)g(t) = \int_0^x (x - t + t)f(x - t)g(t)dt$$

$$= \int_0^x (x - t)f(x - t)g(t)dt + \int_0^x f(x - t)tg(t)dt$$

$$= (hf * g + f * hg)(x).$$

In fact H. Kamowitz and S. Scheinberg [18] have characterized the bounded derivations on $L^1[0,1]$ in terms of certain measures on $[0,1]$.

<u>THEOREM 3.1</u> *Let* D *be a derivation on* $L^1[0,1]$. *Then* D *is continuous.*

Proof. We consider $\mathfrak{S}(D)$ which is a closed ideal in $L^1[0,1]$. By Corollary 2.3 $\mathfrak{S}(D) = M_\alpha$ for some α with $0 \leq \alpha \leq 1$. To prove the continuity of D it suffices to show, by Lemma 1.1 (b), that $\alpha = 1$ which gives $\mathfrak{S}(D) = \{0\}$. We argue by contradiction. Suppose $\alpha < 1$. We choose a sequence $\{\beta_n\}$ of positive real numbers so that $\alpha + \beta_1 + \ldots + \beta_n < 1$ for all n. Then

$$(f_{\beta_1} * \ldots * f_{\beta_n} M_\alpha)^- = M_{\alpha+\beta_1+\ldots+\beta_n} \supset M_{\alpha+\beta_1+\ldots+\beta_{n+1}}$$
$$= (f_{\beta_1} * \ldots * f_{\beta_{n+1}} M_\alpha)^- \quad \text{for all } n$$

(by Proposition 2.4)

Lemma 1.3 gives us the required contradiction if we take $X = Y = L^1[0,1]$, $T_n = R_n$ = left multiplication by f_{β_n} and $S = D$.

<u>Remarks</u>. (1) The same result holds for $L^p[0,1]$, $1 < p < \infty$.

(2) The same method shows that any epimorphism from a Banach algebra A onto $L^1[0,1]$ is continuous since the separating space of an epimorphism is a closed ideal. The only modification required in the proof is that we choose $X = A$ and T_n = left multi-

plication by the preimage of f_{β_n} under the epimorphism in the application of Lemma 1.3.

(3) It is clear that the proof will show that any linear mapping S on $L^1[0,1]$ which intertwines with $L^1[0,1]$ continuously (i.e. $SL_f - L_fS \in \mathfrak{B}(L^1[0,1])$ for all f in $L^1[0,1]$) is continuous. However it is <u>not</u> enough to only assume that $SL_1 - L_1S = SV - VS \in \mathfrak{B}(L^1[0,1])$ even though 1 generates $L^1[0,1]$. Since (a) the spectrum of V is the single point 0, (b) V has no eigenvalues, and (c) V has a non-zero divisible subspace (a subspace Z of $L^1[0,1]$ is divisible for V if $(V - \mu I)Z = Z$ for all complex numbers μ) Theorem 4.1 of [28] shows that there exists a discontinuous linear operator S on $L^1[0,1]$ satisfying $SV = VS$.

<u>Examples</u>.

(1) We note here that it is possible to prove Theorem 3.1 without appealing to the characterization of the closed ideals of $L^1[0,1]$ by using Proposition 2.5. For Proposition 2.5 (1) shows that $\mathfrak{S}(D)$ must be infinite dimensional if it is non-zero and then we can construct an infinite descending chain of ideals contained in $\mathfrak{S}(D)$ as in the proof of Theorem 3.1 by using Proposition 2.5 (2). Lemma 1.3 again provides a contradiction which gives $\mathfrak{S}(D) = \{0\}$. This observation is useful when looking at the weighted convolution algebra $L^1_w[0,\infty)$ where it is not known, as far as we are aware, what all the closed ideals are like. $L^1_w[0,\infty)$ is the Banach algebra of complex-valued functions on the non-negative reals with the property that $\int_0^\infty |f(t)|w(t)dt$ exists where w is a continuous weight function mapping $\mathbb{R}^+ \to \mathbb{R}^+ \setminus \{0\}$ satisfying $w(s + t) \leq w(s)w(t)$. Addition is pointwise and multiplication is defined by convolution as before. The norm is

given by $\|f\| = \int_0^\infty |f(t)|w(t)dt$. If w is 'rapidly decreasing', e.g. if $w(x)^{1/x} \to 0$ as $x \to \infty$, then $L^1_w[0,\infty)$ is a radical Banach algebra and it is not hard to see that it has the same properties as $L^1[0,1]$ given in Proposition 2.5. Thus by the remarks above every derivation on $L^1_w[0,\infty)$ is continuous.

(2) $M[0,\infty)$ is the measure algebra of all complex-valued Borel measures on $[0,\infty)$ with convolution product. In [9] H.G. Diamond showed that derivations on $M[0,\infty)$ are continuous in the topology generated by the seminorms $\|\mu\|_x = |\mu|([0,x])$ for each x in $[0,\infty)$. We note here that this result (and the corresponding result for $M[0,1]$) follows from our methods since Lemma 1.3 can be extended to the case where X and Y are Frechet spaces and $M[0,\infty)$ has the properties of Proposition 2.5, i.e. it has no finite dimensional ideals and given a non-zero ideal I you can construct an infinite descending chain of ideals inside I where each ideal in the chain is obtained from the previous one by multiplication by a suitable element of $M[0,\infty)$.

(3) Let $\underset{\sim}{C}[[t]]$ denote the algebra of all formal power series over the complex field $\underset{\sim}{C}$ in a commutative indeterminate t with the weak topology determined by the projections $p_j: \sum \alpha_i t^i \to \alpha_j$. A subalgebra A of $\underset{\sim}{C}[[t]]$ is a Banach algebra of power series if it contains the polynomials and is a Banach algebra under a norm such that the inclusion map $A \subseteq \underset{\sim}{C}[[t]]$ is continuous. Let I be an ideal in A and let n be the smallest integer for which an element of the form $\sum_{j=n}^{\infty} a_j t^j$ (with $a_n \neq 0$) belongs to I. Then $tI = \{ta: a \in I\}$ is an ideal in A and $tI \subset I$ since no element of the form $\sum_{j=n}^{\infty} a_j t^j$ (with $a_n \neq 0$) belongs to tI. Also if $f = \sum_{j=n}^{\infty} a_j t^j$ $(a_n \neq 0)$ is in I then f and tf are linearly in-

dependent. So A has no finite dimensional ideals. Hence we have shown that A has equivalent properties to those described in Proposition 2.5 for $L^1[0,1]$ and so as before every derivation on A is continuous. This result was first proved using more technical methods by R.J. Loy [21].

(4) The final example is a radical Banach algebra which arises as a closed subalgebra of $\mathfrak{B}(H)$ where H is a Hilbert space. The example is due to G.R. Allan [1]. Let H be a separable Hilbert space and let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H. Let $T \in \mathfrak{B}(H)$ be a unilateral weighted shift operator given by $T(e_n) = \alpha_n e_{n+1}$ $(n = 1, 2, \ldots)$, where the weights $\{\alpha_n\}$ are elements of $\mathbb{C}$ such that $\alpha_n \to 0$. Now let B be the norm-closed subalgebra of $\mathfrak{B}(H)$ generated by T. Then B is a radical Banach algebra. Once again it is not hard to see that B has no finite dimensional ideals and given a non-zero ideal I in B there exists an operator S in B such that $\{0\} \subset SI \subset I$. Hence every derivation on B is continuous.

It is clear that in all the examples described the ideals have similar properties. We now give a theorem which appears in [15] and which provides sufficient conditions on the closed ideals of a Banach algebra B such that every derivation on B is continuous. The hypotheses of the theorem cover all the examples mentioned including $L^1[0,1]$.

<u>THEOREM 3.2</u> *Let* B *be a Banach algebra with the property that for each infinite dimensional closed ideal* J *in* B *there is a sequence* $b_1, b_2, \ldots$ *in* B *such that* $(b_1 \ldots b_n J)^- \supset (b_1 \ldots b_{n+1} J)^-$ *for all*

$n \in \mathbb{N}$. *If* B *contains no non-zero finite dimensional nilpotent ideal then every derivation on* B *is continuous.*

Proof. We give an outline of the proof for completeness. Let D be a derivation on B. It is clear from Lemma 1.3 as elsewhere in this chapter that the condition on the infinite dimensional closed ideals of B forces $\mathfrak{S}(D)$ to be finite dimensional. Thus $D|\mathfrak{S}(D)$ is continuous. If $y, z \in \mathfrak{S}(D)$ then there exist x_n in B, $x_n \to 0$ and $Dx_n \to y$. Then $x_n z \in \mathfrak{S}(D)$ and $x_n z \to 0$ which implies that $D(x_n z) \to 0$. Hence $yz = \lim_{n\to\infty} D(x_n)z = \lim_{n\to\infty} D(x_n z) - \lim_{n\to\infty} x_n D(z) = 0$ and so $\mathfrak{S}(D)$ is a nilpotent ideal. The hypothesis in the theorem gives $\mathfrak{S}(D) = \{0\}$ and so D is continuous.

Remarks. (1) For a commutative Banach algebra B the hypothesis in the theorem concerning infinite dimensional ideals may be replaced by the neater one that for each infinite dimensional closed ideal J in B there is an element b in B with $(bJ)^- \subset J$ and $(bJ)^-$ infinite dimensional.

(2) It can be shown [15] that semisimple Banach algebras satisfy the hypotheses of Theorem 3.2 and thus the theorem yields the result of Johnson and Sinclair mentioned at the beginning of the chapter.

(3) If B is a Banach algebra which satisfies the hypotheses of Theorem 3.2 then it can be shown [15] that an epimorphism of any Banach algebra onto B is necessarily continuous.

CHAPTER FOUR

Having proved that every derivation on a given Banach algebra B is continuous it is natural to ask whether every module derivation from B into a Banach-B-bimodule is continuous. (Of course this generalizes the case of derivations on B since the algebra B itself is a Banach-B-bimodule). For example after S. Sakai [27] had proved that every derivation on a C*-algebra is continuous, J.R. Ringrose [26] then generalized this by showing that every module derivation from a C*-algebra is continuous. In this chapter we extend Theorem 3.1 of Chapter 3 by proving that every module derivation from $L^1[0,1]$ is continuous. We obtain this result as a corollary of Theorem 4.8 which gives sufficient conditions on the closed ideals of a commutative separable Banach algebra B so that every module derivation from B is continuous. In Theorem 4.2 we obtain sufficient conditions on the closed ideals in the general case when B need not be commutative or separable.

<u>Definition</u>. Let B be a Banach algebra and M a Banach-B-bimodule. A linear map D: $B \to M$ is a <u>module derivation</u> from B if $D(ab) = a . D(b) + D(a) . b$ for all a, b in B (where . denotes the module operation on M).

We begin by discussing some ideals which are useful in the study of module derivations. Let B be a Banach algebra and M a Banach-B-bimodule, and let D: $B \to M$ be a module derivation. We define

$I_L = \{b \in B: \ b . \mathfrak{S}(D) = \{0\}\}$, $\quad I_R = \{b \in B: \ \mathfrak{S}(D) . b = \{0\}\}$.

We call I_L (and I_R) the <u>left</u> (and <u>right</u>) <u>continuity ideal</u> for D.

If B is commutative it is easily seen that $I_L = I_R$. In this case we will denote the ideal by I and refer to it as the continuity ideal for D.

LEMMA 4.1 *Let* D *be a module derivation from* B *to* M. *Then*

(1) I_L *and* I_R *are closed ideals of* B, *and*

(2) *if* I_L *has a bounded left (or right) approximate identity then* D *is continuous on* I_L.

Proof. (1) Let $a \in B$, $b \in I_L$. Then $ab \,.\, \mathfrak{S}(D) = \{0\}$ trivially. Also $a \,.\, \mathfrak{S}(D) \subseteq \mathfrak{S}(D)$ by Lemma 1.1 (d) and so $ba \,.\, \mathfrak{S}(D) = \{0\}$. Thus $ab \in I_L$ and $ba \in I_L$, i.e. I_L is an ideal. Similarly I_R is an ideal. It is clear that both I_L and I_R are closed.

(2) [3] Suppose I_L has a bounded left approximate identity and let $x_n \in I_L$ with $x_n \to 0$. By a well-known corollary to the Cohen factorization theorem [6] there exists a sequence $\{z_n\} \subseteq I_L$ and $y \in I_L$ such that $z_n \to 0$ and $x_n = yz_n$, $n \in \mathbb{N}$. By Lemma 1.1 (b) (c) the map $z \to y \,.\, D(z)$ is continuous since $y \in I_L$. Hence $D(x_n) = D(yz_n) = D(y) \,.\, z_n + y \,.\, D(z_n) \to 0$ as $n \to \infty$. Similarly D is continuous on I_L if I_L has a bounded right approximate identity.

THEOREM 4.2. *Let* B *be a Banach algebra which satisfies the following two conditions:*

(1) *if* K *is a closed ideal of infinite codimension in* B, *then there exist sequences* $\{b_n\}$, $\{c_n\}$ *in* B *satisfying* $c_n b_1 \ldots b_{n-1} \notin K$ *and* $c_n b_1 \ldots b_n \in K$ *for all* $n \geq 2$,

(2) *every closed ideal having finite codimension in* B *has a bounded left (or right) approximate identity.*

Then every module derivation from B *into a Banach-B-bimodule is continuous.*

Proof. Let M be a Banach-B-bimodule and let D be a module derivation from B to M and let I_L be the left continuity ideal for D. Suppose I_L is of infinite codimension in B. We obtain a contradiction using condition (1) by applying Corollary 1.4 with $X = B$, $Y = M$, $T_n x = b_n x$ for all x in B, $R_n y = b_n \cdot y$ and $U_n y = c_n \cdot y$ for all y in M. So I_L must have finite codimension in B, and so has a bounded left (or right) approximate identity by condition (2). Lemma 4.1 gives D continuous on I_L and so D is continuous on B.

Remark. We can replace condition (1) by the stronger one that every closed ideal K of infinite codimension in B has the property that given b in $B \setminus K$, there exists a, c in B such that $ab \notin K$, $bc \notin K$ but $abc \in K$. A simple inductive argument shows that this implies the condition in the theorem: we construct inductively two sequences $b_1, b_2, \ldots$ and $c_2, c_3, \ldots$ in B such that $b_1 \ldots b_n \notin K$, $c_n b_1 \ldots b_{n-1} \notin K$ and $c_n b_1 \ldots b_n \in K$ for all $n \geq 2$. To start the induction let b_1 be any element of $B \setminus K$, and then choose b_2, c_2 in B such that $b_1 b_2 \notin K$, $c_2 b_1 \notin K$ but $c_2 b_1 b_2 \in K$. Then, given $b_1, \ldots, b_r$, $c_2, \ldots, c_r$ satisfying the three conditions choose b_{r+1}, c_{r+1} in B such that $b_1 \ldots b_{r+1} \notin K$, $c_{r+1} b_1 \ldots b_r \notin K$ and $c_{r+1} b_1 \ldots b_{r+1} \in K$.

If B is commutative this condition is merely saying that for each b in $B \setminus K$, the annihilator of $b + K$ in the quotient algebra B / K is not prime.

In general C*-algebras do not satisfy this condition, e.g. take

B to be the Banach algebra of continuous functions on $[0,1] \cup \{2\}$ and let K be the zero ideal. However it is not hard to see that if B is a C*-algebra with the property that for every closed ideal K of infinite codimension in B, B/K has no non-trivial idempotents, then B satisfies this condition. A.M. Davie has also pointed out that for a Hilbert space H, $K = \mathcal{K}(H)$, the ideal of compact operators on H, does have this property in $\mathcal{B}(H)$.

We can show, however, that all C*-algebras satisfy the condition given in Theorem 4.2, thus obtaining Ringrose's result [26]. This is the result of the following corollary.

COROLLARY 4.3 *Every module derivation from a C*-algebra is continuous.*

Proof. Let A be a C*-algebra. Following the techniques used in Ringrose's proof [26] we show that A satisfies the two conditions of Theorem 4.2. Let K be a closed ideal of infinite codimension in A. Then the C*-algebra A / K contains an infinite dimensional closed commutative *-subalgebra B [25]. Since the carrier space X of B is infinite it follows from the isomorphism between B and $C_o(X)$ that there is a positive ~~operator~~ element T in B whose spectrum is infinite. Hence there exist non-negative continuous functions $b_1, b_2, \ldots, c_2, c_3, \ldots,$ defined on the positive real axis, such that

$$c_n b_1 \ldots b_{n-1}(T) \neq 0 \quad \text{and} \quad c_n b_1 \ldots b_n = 0 \quad \text{for all} \quad n \geq 2.$$

Let π denote the natural mapping from A onto A / K. Then there is a positive element S in A such that $\pi(S) = T$. If $P_j = b_j(S)$ $(j = 1, 2, \ldots)$, and $Q_j = c_j(S)$ $(j = 2, 3, \ldots)$, then $P_j, Q_j \in A$ and $\pi(Q_n P_1 \ldots P_{n-1}) = \pi(c_n(S))\pi(b_1(S)) \ldots \pi(b_{n-1}(S))$

$$= c_n(\pi(S))b_1(\pi(S))\dots b_{n-1}(\pi(S)) \neq 0 \quad (n \geq 2).$$

Thus $P_j, Q_j \in A$, $Q_n P_1 \dots P_{n-1} \notin K$, $Q_n P_1 \dots P_n \in K$ $(n \geq 2)$. So A satisfies condition (1). Now every closed ideal of a C*-algebra has a two-sided bounded approximate identity [10] and so A also satisfies condition (2). Theorem 4.2 then gives the result.

<u>COROLLARY 4.4</u> *Let* $L^1(G)$ *be the group algebra of a locally compact abelian group* G. *Then every module derivation from* $L^1(G)$ *is continuous.*

Proof. Again we show that $L^1(G)$ satisfies the two conditions of Theorem 4.2. First we note some well-known facts of harmonic analysis. $L^1(G)$ is a regular semi-simple commutative Banach algebra [14]. Let X denote the carrier space of $L^1(G)$. If F is a subset of X then define $\ker F = \{f \in L^1(G): \hat{f}(F) = \{0\}\}$, and $J(F) = \{f \in L^1(G): \hat{f}$ is zero in a neighbourhood of $F\}$. The hull of an ideal I in $L^1(G)$ is the set $\{\lambda \in X: \lambda(I) = \{0\}\}$. If an ideal I has hull F then the theory of regular semi-simple commutative Banach algebras implies that $J(F) \subseteq I$ [14]. Now let K be a closed ideal of finite codimension in $L^1(G)$ with hull F. We want to show that F is finite. So suppose F is infinite. By induction we choose two sequences $\{U_n\}$ and $\{V_n\}$ of open subsets in X such that $U_n \cap V_n = \emptyset$, $U_n \cap F \neq \emptyset$, and $U_n \subseteq V_j$ for $1 \leq j \leq n-1$. To ensure that the induction can proceed we also require that $V_1 \cap \dots \cap V_n$ contains infinitely many points of F for all n. Choose U_1, V_1 disjoint open sets so that $U_1 \cap F \neq \emptyset$ and V_1 contains an infinite number of points of F. Now suppose $U_1, \dots, U_n$ and $V_1, \dots, V_n$ have been chosen. We now choose disjoint open subsets W_{n+1} and V_{n+1} so that $W_{n+1} \cap (V_1 \cap \dots \cap V_n) \cap F \neq \emptyset$

and V_{n+1} contains an infinite number of points of $(V_1 \cap \ldots \cap V_n) \cap F$. Let $U_{n+1} = W_{n+1} \cap (V_1 \cap \ldots \cap V_n)$. This completes the inductive choice of $\{U_n\}$ and $\{V_n\}$. The regularity of $L^1(G)$ implies that there are $f_1, f_2, \ldots$ in $L^1(G)$ with $\hat{f}_j = 1$ at some point of F inside U_j and $\hat{f}_j$ zero outside U_j. Then, for each j, $f_j \notin K$ and the f_j's give rise to linearly independent elements in $L^1(G) / K$ which contradicts the fact that K has finite codimension in $L^1(G)$. Hence F is finite. $L^1(G)$ satisfies a strong Dytkin condition i.e. $\ker\{\lambda\}$ has a bounded approximate identity taken from $J(\{\lambda\})$ for each λ in X. An application of a result of M. Altman ([2]; see[6, p.58]) then shows that with F finite we can deduce that $\ker F$ has a bounded approximate identity from $J(F)$. Since $J(F) \subseteq K \subseteq \ker F$, K has a bounded approximate identity. Thus $L^1(G)$ satisfies condition (2).

Now suppose K is a closed ideal of infinite codimension in $L^1(G)$ with hull H. We show that H is infinite. For if H is finite then $\ker H$ has finite codimension in $L^1(G)$. Also, as remarked above, in this case $\ker H$ has a bounded approximate identity from $J(H)$ and so $J(H)^- = \ker H$. But $J(H) \subseteq K \subseteq \ker H$ and so $K = \ker H$ which has finite codimension. This contradiction shows that H must be infinite. As in the first part of the proof we choose two sequences $\{U_n\}$ and $\{V_n\}$ of open subsets in X such that $U_n \cap V_n = \emptyset$, $U_n \cap F \neq \emptyset$ and $U_n \subseteq V_j$ for $1 \leq j \leq n-1$. Again the regularity of $L^1(G)$ implies that for a sequence $\{\lambda_n\}$ with $\lambda_j \in U_j \cap F$ we have $b_1, b_2, \ldots, c_2, c_3, \ldots,$ in $L^1(G)$ with $\hat{b}_j(\lambda_k) = 1$ for $k > j$, $\hat{c}_j(\lambda_j) = 1$, $\hat{b}_j$ zero outside $\bigcup_{k>j} U_k$, and c_j zero outside U_j. These conditions and the semi-simplicity of $L^1(G)$ imply that $c_n b_1 \ldots b_{n-1} \notin K$ and $c_n b_1 \ldots b_n \in K$ for $n \geq 2$ so that $L^1(G)$ satisfies condition (1). An application of

Theorem 4.2 completes the proof.

Remark. The methods used in the proof of Corollary 4.4 in fact give the continuity of module derivations on any regular semi-simple commutative Banach algebra satisfying a strong Dytkin condition.

W.G. Bade and P.C. Curtis, Jr. [3] have also obtained sufficient conditions on the closed ideals of a Banach algebra B so that every module derivation from B is continuous. Their condition on the closed ideals of finite codimension is identical to condition (2) of Theorem 4.2. Their condition on the closed ideals of infinite codimension is as follows: if K is a closed ideal of infinite codimension in B, then there exists a sequence $\{x_n\}$ in B satisfying $x_n x_m = 0$ $(n \neq m)$ and $x_n^2 \notin K$ for all n. We remark here that the two theorems are in fact different and Theorem 4.2 appears to cover a wider class of algebras. Below we will show that $L^1[0,1]$ satisfies the conditions of Theorem 4.2 while it does not satisfy the conditions of Bade and Curtis. However we have not been able to find a Banach algebra which does the reverse, i.e. satisfy the conditions of Bade and Curtis while failing to satisfy those of Theorem 4.2, and we have tried to prove that the conditions of Theorem 4.2 follow from those of Bade and Curtis without success.

We now show that $L^1[0,1]$ satisfies the conditions of Theorem 4.2 (which implies that every module derivation from $L^1[0,1]$ is continuous - we obtain this result most easily as a corollary to Theorem 4.8 as will be shown). Let K be a closed ideal of infinite codimension in $L^1[0,1]$. Then $K = M(\alpha)$ by Corollary 2.3 where $\alpha > 0$. Let $g \in L^1[0,1]$ $g \notin K$. Let $p = \inf\{q: g \in M(q)\}$. Then $0 \leq p < \alpha$. We choose positive real numbers β, γ so that $p + \beta < \alpha$, $p + \gamma < \alpha$ but $\alpha < p + \beta + \gamma \leq 1$.

Then $f(\beta)g \notin M(\alpha)$, $f(\gamma)g \notin M(\alpha)$ but $f(\beta)gf(\gamma) \in M(\alpha)$ (see Proposition 2.4). The remark after Theorem 4.2 shows that condition (1) of Theorem 4.2 is satisfied. The only closed ideal of $L^1[0,1]$ having finite codimension in $L^1[0,1]$ is $L^1[0,1]$ itself which has a bounded approximate identity (Proposition 2.1 (2)) and so condition (2) of Theorem 4.2 is also satisfied.

However $L^1[0,1]$ does not satisfy the condition on closed ideals of infinite codimension given by Bade and Curtis and described above. For let $M(\alpha)$ be a closed ideal of $L^1[0,1]$ where $0 \leq \alpha < \frac{1}{2}$. Then $M(\alpha)$ is of infinite codimension. Suppose there exists a sequence $\{x_n\}$ in $L^1[0,1]$ with $x_n x_m = 0$ $(n \neq m)$ and $x_n^2 \notin M(\alpha)$ for all $n \geq 1$. Let $\beta_n = \inf\{\beta\colon\ x_n \in M(\beta)\}$ $(n \geq 1)$. It is clear that $0 < \beta_n \leq \alpha$ and $\beta_n + \beta_m \geq 1$ $(n \neq m)$. Let $\gamma = \liminf_{n\to\infty}\{\beta_n\}$. Then $\beta_j \geq 1 - \gamma$ for all $j \geq 1$ which shows that $\gamma \geq 1 - \gamma$, i.e. $\gamma \geq \frac{1}{2}$. But $0 < \beta_n \leq \alpha < \frac{1}{2} \Rightarrow \gamma < \frac{1}{2}$, which yields the required contradiction.

The next lemma, which is a consequence of Lemma 1.3, is due to W.G. Bade and P.C. Curtis, Jr. [4], and is closely related to Theorem 3.3 of [29].

<u>LEMMA 4.5</u> *Let* B *be a commutative Banach algebra with identity and let* M *be a Banach-B-bimodule. Let* $D\colon B \to M$ *be a discontinuous module derivation. Then there exists* x_0 *in* B *such that if* $D_0\colon B \to M$ *is given by* $D_0(b) = x_0 \cdot D(b)$ *for all* b *in* B *we have that* D_0 *is a discontinuous module derivation and* I_0, *the continuity ideal of* D_0, *is a closed prime ideal of* B.

Proof. Since B is commutative it is clear that D_0 is a module derivation and hence I_0 is a closed ideal of B by Lemma 4.1. We show that there exists x_0 in B such that $x_0 . \mathfrak{S}(D) \neq \{0\}$ and for every b in B either $bx_0 . \mathfrak{S}(D) = \{0\}$ or $\{bx_0 . \mathfrak{S}(D)\}^- = \{x_0 . \mathfrak{S}(D)\}^-$. This is sufficient to give us the required conclusion for then

$I_0 = \{b \in B: b . \mathfrak{S}(D_0) = \{0\}\} = \{b \in B: bx_0 . \mathfrak{S}(D) = \{0\}\}$ since $\mathfrak{S}(D_0) = \{x_0 . \mathfrak{S}(D)\}^-$ by Lemma 1.1 (c) and it is easy to see that I_0 will be prime. We now prove the existence of the element x_0 in B. Either there exists b_1 in B so that $\{0\} \neq \{b_1 . \mathfrak{S}(D)\}^- \subset \mathfrak{S}(D)$ or else for every b in B either $b . \mathfrak{S}(D) = \{0\}$ or $\{b . \mathfrak{S}(D)\}^- = \mathfrak{S}(D)$ in which case we can take x_0 be the identity of B (we assume that the module is unit-linked). If such an element b_1 exists then either there exists b_2 in B so that $\{0\} \neq \{b_2 b_1 . \mathfrak{S}(D)\}^- \subset \{b_1 . \mathfrak{S}(D)\}^-$ or for every b in B either $bb_1 . \mathfrak{S}(D) = \{0\}$ or $\{bb_1 . \mathfrak{S}(D)\}^- = \{b_1 . \mathfrak{S}(D)\}^-$ in which case we can take x_0 to be b_1. Lemma 1.3 tells us that this process must eventually stop; i.e. we shall have $b_1, \ldots, b_n$ such that $\{0\} \neq \{b_n b_{n-1} \ldots b_1 . \mathfrak{S}(D)\}^- \subset \{b_{n-1} \ldots b_1 \mathfrak{S}(D)\}^-$ and for every b in B either $bb_n \ldots b_1 . \mathfrak{S}(D) = \{0\}$ or $\{bb_n \ldots b_1 . \mathfrak{S}(D)\}^- = \{b_n \ldots b_1 . \mathfrak{S}(D)\}^-$. We can then take x_0 to be $b_n \ldots b_1$. Note that $\{0\} \neq \{b_n b_{n-1} \ldots b_1 . \mathfrak{S}(D)\}^-$ gives that D_0 is discontinuous by Lemma 1.1 (b).

<u>Remark</u>. We can assume that B does not have an identity by forming the algebra $B \oplus \mathbb{C}1$, extending D by $D(\lambda 1) = 0$, and allowing x_0 to be in $B \oplus \mathbb{C}1$. I_0 would then be a prime ideal in $B \oplus \mathbb{C}1$ with $I_0' = \{b \in B: (b,0) \in I_0\}$ a prime ideal in B.

Recently R.J. Loy [24] and J.R. Christensen [8] have exhibited some interesting consequences of the Borel graph theorem (see [31]). We will require some particular cases of their results which we now describe.

PROPOSITION 4.6 *Let* X_1, X_2, Y *be separable Banach spaces and let* $T: X_1 \times X_2 \to Y$ *be a continuous bilinear mapping. Suppose* Z *is a closed subspace of* Y *contained in the linear span of the range of* T. *Then there is a constant* K *and an integer* m *such that if* $z \in Z$ *there exist* $a_j \in X_1$, $b_j \in X_2$, $1 \leq j \leq m$, *satisfying*

(i) $z = \sum_{j=1}^{m} T(a_j, b_j)$,

(ii) $\sum_{j=1}^{m} \|a_j\| \|b_j\| \leq K\|z\|$.

Proof. See [24].

Notation. For a Banach algebra B, B^2 denotes the ideal spanned by two-fold products of elements of A.

PROPOSITION 4.7 *Let* B *be a separable Banach algebra such that* B^2 *is of finite codimension in* B. *Then* B^2 *is closed.*

Proof. See [24] or [8].

For commutative separable Banach algebras we can now prove the following theorem.

THEOREM 4.8 *Let* B *be a commutative separable Banach algebra such that* B^2 *is of finite codimension in* B *which satisfies the following two conditions:*

(1) *there are no closed prime ideals of infinite codimension,*

(2) *every maximal ideal* M *of* B *has* M^2 *of finite codimension in* B.

Then every module derivation from B *into a Banach-B-bimodule is continuous.*

Proof. Without loss of generality assume that B has an identity. Suppose that D is a discontinuous module derivation from B into some Banach-B-bimodule M. Let D_0, I_0 be as given in Lemma 4.5, so that D_0 is also discontinuous. I_0 is a closed prime ideal and so must be of finite codimension. But a prime ideal of non-zero finite codimension is maximal and so either $I_0 = B$ or I_0 is maximal and in both cases ${I_0}^2$ is of finite codimension in B. But then ${I_0}^2$ is closed by Proposition 4.7. We now obtain a contradiction by showing that D_0 is continuous on ${I_0}^2$. Let $f \in {I_0}^2$. We apply Proposition 4.6 with $X_1 = X_2 = Y = I_0$, $T(a,b) = ab$ for $a, b \in I_0$ and $Z = {I_0}^2$, to obtain $f = \sum_{j=1}^{m^2} g_j h_j$ where $\sum_{j=1}^{m} \|g_j\|\|h_j\| \leq K\|f\|$ for some constant K, and $g_j, h_j \in I_0$, $1 \leq j \leq m$. Then

$$\|D_0(f)\| = \|\sum_{j=1}^{m} D_0(g_j h_j)\| \leq \sum_{j=1}^{m} \|D(g_j) \cdot h_j + g_j \cdot D(h_j)\|$$

$$\leq \sum_{j=1}^{m} 2M\|g_j\|\|h_j\|$$

where M is a constant (by Lemma 1.1 (b) (c)),

and so $\|D_0(f)\| \leq 2M \sum_{j=1}^{m} \|g_j\|\|h_j\| \leq 2MK\|f\|$ which concludes the proof.

<u>Remarks</u>. (1) The condition that B^2 is of finite codimension in B is necessary since if B^2 is of infinite codimension in B we can construct a discontinuous module derivation from B. For let f be a discontinuous linear functional on B, chosen by Zorn's lemma, such that $f(B^2) = \{0\}$. Let M be any Banach-B-bimodule containing an element

$m \neq 0$ such that $B . m = m . B = \{0\}$. Define $D: B \to M$ by $D(b) = f(b)m$ for $b \in B$. Then D is a discontinuous module derivation for which $I = B$.

An example of such an algebra is the algebra of Hilbert-Schmidt operators on a Hilbert space.

(2) Given a particular module M we can weaken condition (1) slightly to "there are no closed prime ideals of infinite codimension in B which annihilate some non-trivial submodule of M".

We now show that condition (2) of Theorem 4.8 is best possible. Let B be a commutative Banach algebra. Suppose there exists a maximal ideal J of B such that J^2 is of infinite codimension in B. Then as in remark (1) of Theorem 4.8 we can construct a discontinuous module derivation from J to a Banach-J-bimodule. Of course this derivation can be raised to one mapping B to a Banach-B-bimodule.

Alternatively (see [28]) let $J = \ker\theta$ where θ is a character on B. Regard $\mathbb{C}$ as a Banach-B-bimodule by defining $b . \lambda = \lambda . b = \theta(b)\lambda$ for all b in B and λ in $\mathbb{C}$. Let f be a discontinuous linear functional on B, chosen by Zorn's lemma, such that $f(\mathbb{C}1 + J^2) = \{0\}$, where 1 is the identity of B (adjoined if necessary). From the decomposition

$$ab = (a - \theta(a)1)(b - \theta(b)1) + \theta(a)b + \theta(b)a - \theta(ab)1$$

we obtain

$$f(ab) = \theta(a)f(b) + \theta(b)f(a).$$

Hence f is a discontinuous module derivation from B into the Banach-B-bimodule $\mathbb{C}$.

Examples of Banach algebras B with this type of maximal ideal are $A \oplus \mathbb{C}1$ where A^2 is of infinite codimension in A such as

$C^n[0,1]$, the Banach algebra of all n times continuously differentiable complex-valued functions on $[0,1]$ with the norm

$$\|f\| = \max_{t\in[0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!} .$$

It is still open as to how near "best possible" condition (1) is. We pose the question: are there any commutative separable Banach algebras with closed prime ideals of infinite codimension on which all module derivations are continuous? Alternatively if we have a Banach algebra with a closed prime ideal of infinite codimension can we always construct a discontinuous module derivation? $A(D)$, the disc algebra of functions analytic on the open unit disc D in $\mathbb{C}$ and continuous on $\bar{D}$, is an example of a separable Banach algebra with a prime ideal of infinite codimension on which we can construct a discontinuous module derivation (see [28]).

The following corollary of Theorem 4.8 extends Theorem 3.1.

<u>COROLLARY 4.9</u> *Every module derivation from* $L^1[0,1]$ *is continuous.*

Proof. $L^1[0,1]$ is commutative and separable and has no closed prime ideals and no maximal ideals. Since $L^1[0,1]$ has a bounded approximate identity (Proposition 2.1 (2)) $L^1[0,1]^2 = L^1[0,1]$. Thus $L^1[0,1]$ satisfies the hypotheses of Theorem 4.8.

<u>Remark</u>. Bade and Curtis have proved the following result concerning singly-generated Banach algebras:

Let B be a singly-generated Banach algebra with generator z. Let M be a Banach-B-bimodule and let $\rho(z) \in \mathcal{B}(M)$ be the operator given by $\rho(z)(m) = z \cdot m$ for m in M. Then if (a) the spectrum of $\rho(z)$ is countable, (b) there are no non-zero $\rho(z)$-divisible subspaces and (c) $\rho(z)$ has no eigenvalues, we have that every module derivation from B into M is continuous.

The example $L^1[0,1]$ shows that condition (b) in this result is not necessary: for $L^1[0,1]$ is generated by 1 and, if we choose $M = L^1[0,1]$, $\rho(z)$ is the Volterra integral operator V (see Chapter 2). V has spectrum the single point 0 and has no eigenvalues. However although V has a non-zero divisible subspace (e.g. the set of $f \in L^1[0,1]$ such that f is infinitely differentiable, f has continuous derivatives and $f^{(n)}(0) = 0$, $n = 0, 1, 2, \ldots$) Corollary 4.9 (or Theorem 3.1) still shows that every derivation from $L^1[0,1]$ to $L^1[0,1]$ is continuous.

The methods of this chapter can be used to obtain some results on module homomorphisms. Recall that if B is a Banach algebra and M and N are Banach-B-bimodules then a linear mapping $\theta: M \to N$ is called a module homomorphism if $\theta(b \cdot x) = b \cdot \theta(x)$ and $\theta(x \cdot b) = \theta(x) \cdot b$ for all b in B and x in M. The continuity ideals for θ are defined as for module derivations e.g. $I_L(\theta) = \{b \in B: b \cdot \mathfrak{S}(\theta) = \{0\}\}$. Again it is clear that $I_L(\theta)$, $I_R(\theta)$ and $I(\theta)$ are all closed ideals. The theorem corresponding to Theorem 4.2 is as follows.

<u>THEOREM 4.10</u> *Let* B *be a Banach algebra which has the property that if* K *is a closed ideal of infinite codimension in* B *then there exist sequences* $\{b_n\},\{c_n\}$ *in* B *satisfying* $c_n b_1 \ldots b_{n-1} \notin K$ *and* $c_n b_1 \ldots b_n \in K$ *for all* $n \geq 2$. *Let* θ *be a module homomorphism between*

two Banach-B-bimodules. Then $I_L(\theta)$ *and* $I_R(\theta)$ *are of finite codimension in* B.

Proof. The proof is exactly analogous to the proof of Theorem 4.2.

Remarks. (1) Suppose $\theta: M \to N$ where M, N are B-bimodules. We can weaken the hypothesis that N be a Banach-B-bimodule by only demanding that N be a Banach space with continuous B-bimodule operations, i.e. for each b in B the operations $n \to b . n$ and $n \to n . b$ are continuous. This is important when considering algebra homomorphisms from B to other Banach algebras. In this situation $I_L(\theta)$ and $I_R(\theta)$ are no longer necessarily closed and the conclusion of the theorem is that $I_L(\theta)^-$ and $I_R(\theta)^-$ are of finite codimension in B. We prove this in a similar fashion to Theorem 4.2 obtaining a contradiction by using a slight adaption of Corollary 1.4. Essentially we require that, for $r_1, r_2, \ldots, u_2, u_3, \ldots$ in B,

$u_n r_1 \ldots r_n \in I_L(\theta)^-$ for $n \geq 2 \Rightarrow u_n r_1 \ldots r_{n-1} \in I_L(\theta)$ for $n > n_0$

where n_0 is some positive integer. It is easily seen that this follows from Lemma 1.3.

(2) If B has the property that every closed ideal of finite codimension has a bounded left approximate identity then it follows that θ is continuous on $I_L . M$ which is a closed submodule of M by the Banach moduleformofCohen's factorisation theorem ([14], Theorem 32.22 p. 268). For let $z \in I_L . M$; by Cohen's theorem we have $z = a . m$ where $a \in I_L$, $m \in M$ and $\|a\| \leq d$ where d is the bound of the approximate identity in I_L. Since θ is a module homomorphism $\theta(z) \in I_L . N$. Then there exists $b \in I_L$ such that $\|b . \theta(z) - \theta(z)\| \leq \|z\|$ where $\|b\| \leq d$, again by Cohen's theorem. Hence

$$\|\theta(z)\| \leq \|\theta(z) - b \,.\, \theta(z)\| + \|b \,.\, \theta(z)\|$$

$$\leq \|z\| + M\|b\|\|z\| \text{ by Lemma 1.1 (b), (c) since } b \in I_L$$

$$\leq (1 + Md)\|z\|$$

(3) From our earlier work we know that C*-algebras, $L^1[0,1]$, $L^1(G)$ and, in fact, any regular semi-simple commutative Banach algebra satisfying a strong Dytkin condition all satisfy the hypothesis of Theorem 4.10. So this theorem covers results for C*-algebras and regular semi-simple commutative Banach algebras obtained by A.M. Sinclair [30]. The result for $L^1[0,1]$ appears to be new.

As in Lemma 4.5 if B is a commutative Banach algebra with identity and M, N are Banach-B-bimodules with θ: $M \to N$ a discontinuous module homomorphism then there exists x_0 in B such that if θ_0: $M \to N$ is given by $\theta_0(m) = x_0 \,.\, \theta(m)$ for all m in M then θ_0 is a discontinuous module homomorphism and I_0, the continuity ideal for θ_0, is a closed prime ideal of B. If B has no closed prime ideals of infinite codimension this forces I_0 to be either all of B or maximal.

In the case where B is a separable Banach algebra, M and N are Banach-B-bimodules, and θ: $M \to N$ is a module homomorphism we can show that θ is continuous on the linear span of $I(\theta).M$ if this is a closed subspace of M. To do this we apply Proposition 4.6 with $X_1 = I(\theta)$, $X_2 = M$, Y = linear span of $I(\theta).M$, and $T(a,m) = a \,.\, m$ for $a \in I(\theta)$, $m \in M$. If z is in the linear span of $I(\theta).M$ this gives $z = \sum_{j=1}^{m} a_j \,.\, m_j$ where $\sum_{j=1}^{m} \|a_j\|\|m_j\| \leq K\|z\|$ for some constant K and $a_j \in I(\theta)$, $m_j \in M$, $1 \leq j \leq m$. Then

$$\|\theta(z)\| = \|\sum_{j=1}^{m} \theta(a_j \, . \, m_j)\| \leq \sum_{j=1}^{m} \|a_j \, . \, \theta(m_j)\|$$

$$\leq \sum_{j=1}^{m} M\|a_j\|\|m_j\|$$

where M is a constant (by Lemma 1.1 (b) (c)),

and so $\|\theta(z)\| \leq MK\|z\|$.

CHAPTER FIVE

In this chapter we employ the methods of previous chapters to obtain sufficient conditions on the closed ideals of a Banach algebra B so that certain higher derivations from any Banach algebra A onto B are necessarily continuous.

Definition. For m in $\mathbb{N}$, a higher derivation of rank m (respectively infinite rank) from an algebra A into an algebra B is a sequence $\{F_1, \ldots, F_m\}$' (resp. $\{F_1, F_2, \ldots\}$) of linear operators from A into B satisfying $F_n(ab) = \sum_{i=0}^{n} F_i(a)F_{n-i}(b)$ for each $n = 0, 1, \ldots, m$ (resp. $n = 0, 1, 2, \ldots$) and all a, b in A.

A higher derivation of rank m (resp. infinite rank) is said to be continuous if F_n is continuous on A for each $n = 0, 1, \ldots, m$ (resp. $n = 0, 1, 2, \ldots$). It is said to be onto if F_0 maps A onto B.

Another problem raised at the U.C.L.A. conference mentioned earlier was whether the result of B.E. Johnson and A.M. Sinclair [17] giving the automatic continuity of derivations on semi-simple Banach algebras could be extended to higher derivations. R.J. Loy pointed out subsequently that the result could be extended for higher derivations whose domain algebra is the same as the range algebra and where F_0 is the identity map. To do this he merely used results of N. Heerema [13] to express a higher derivation in terms of a derivation. We shall extend Loy's result

(1) by allowing the domain algebra to be any Banach algebra whatsoever,

(2) by allowing the range algebra to include a wider class than just semi-simple Banach algebras, and

(3) by weakening the condition that F_0 be the identity map.

<u>THEOREM 5.1</u> *Let* B *be a Banach algebra with the property that for each infinite dimensional closed ideal* J *in* B *there is a sequence* $\{b_n\}$ *in* B *such that* $(b_1 \ldots b_n J)^- \supset (b_1 \ldots b_{n+1} J)^-$ *for all positive integers* n. *Suppose also that* B *contains no non-zero finite dimensional nilpotent ideal. Let* $\{F_n\}$ *be a higher derivation of any rank from a Banach algebra* A *onto* B *such that* $\ker F_0 \subseteq \ker F_n$ *for all* n . *Then* $\{F_n\}$ *is continuous.*

Proof. We prove that F_n is continuous for all n by induction. From the definition of a higher derivation it is clear that F_0 is a homomorphism. Since F_0 is onto, $\mathfrak{S}(F_0)$ is a closed ideal in B. If $\mathfrak{S}(F_0)$ is infinite dimensional then there are $b_1, b_2, \ldots$ in B such that $(b_1 \ldots b_n \mathfrak{S}(F_0))^- \supset (b_1 \ldots b_{n+1} \mathfrak{S}(F_0))^-$ for all positive integers n. There are $a_1, a_2, \ldots$ in A such that $F_0(a_n) = b_n$ for all n. We obtain a contradiction by applying Lemma 1.3 with $X = A$, $Y = B$, $R_n b = b_n b$ for all b in B and $T_n a = a_n a$ for all a in A. Hence $\mathfrak{S}(F_0)$ is a closed finite dimensional ideal. We want to show that $\mathfrak{S}(F_0)$ is nilpotent and since $\mathfrak{S}(F_0)$ is finite dimensional it will be sufficient to show that $\mathfrak{S}(F_0)$ is contained in R, the radical of B. We could obtain this immediately from a corollary of B.E. Johnson's deep uniqueness of norm theorem (see [28, p. 40]) which states that a homomorphism from a Banach algebra onto a semi-simple Banach algebra is always continuous. However here we will argue in a more elementary fashion. The radical of an ideal is the intersection of the ideal and the radical of the algebra and so is an ideal in the algebra [6, p. 126]. Hence the radical of $\mathfrak{S}(F_0)$ is a finite dimensional nilpotent ideal in B, and so is zero by hypothesis. Then since $\mathfrak{S}(F_0)$ is a finite dimensional semi-simple algebra it has an identity e [6, p. 135]. Let Q be the natural map from B to B / R. QF_0 is a homomorphism from A onto B / R which is a semi-simple Banach algebra [6, p. 126]. Hence $\ker QF_0$ is closed [6, p. 131]. Define $\psi: A/(\ker QF_0) \to B / R$ by $\psi(a + \ker QF_0) = QF_0(a)$. Then ψ is an isomorphism of $A/(\ker QF_0)$ onto B / R. Also $Q\mathfrak{S}(F_0) \subseteq \mathfrak{S}(\psi)$. Now let $M = F_0^{-1}\{\mathfrak{S}(F_0)\}/(\ker QF_0)$. ψ maps M onto $\mathfrak{S}(F_0)/R$ which is finite dimensional and so M is a finite dimensional ideal in $A/(\ker QF_0)$. Now let $y \in \mathfrak{S}(F_0)$. There exist $x_n \in A/(\ker QF_0)$, $x_n \to 0$ with $\psi(x_n) \to Qy$ as $n \to \infty$. Also there exists $x \in M$ such that $\psi(x) = Qe$. So $\psi(xx_n) = \psi(x)\psi(x_n) \to ey + R = y + R$ in B / R as $n \to \infty$. But $xx_n \in M$, $xx_n \to 0$, and $\psi|M$ is continuous since M is finite dimensional and so $\psi(xx_n) \to 0$ in B / R. Hence $y \in R$. Thus we have shown that $\mathfrak{S}(F_0)$ is a finite dimensional ideal contained in the radical of B. It is thus nilpotent and hence is zero by hypothesis. Lemma 1.1 (b) then gives F_0 continuous. (An alternative way of showing that $\mathfrak{S}(F_0)$ is nilpotent is to appeal to a result of B. Barnes [5] which shows that each element of the separating space of a homomorphism has connected spectrum containing 0). Note that this proof of the continuity of F_0 justifies the remark made after the proof of Theorem 3.2.

We now assume that F_n is continuous for $0 \leq n \leq k-1$. We have $F_k(ab) = \sum_{i=0}^{k} F_i(a)F_{k-i}(b)$ for a, b in A. Hence $F_k(ab) - F_0(a)F_k(b) = \sum_{i=1}^{k} F_i(a)F_{k-i}(b)$. For a fixed a we then have $(F_kL(a) - L(F_0(a))F_k)(b) = C(b)$ where C is continuous by the inductive hypothesis and L(a) denotes the operation of left multiplication by a. (We use the same letter to denote this operation in A and B although, of course, they are different operators.) Now using the fact that F_0 is onto and the inductive hypothesis it is clear that $\mathfrak{S}(F_k)$ is a closed ideal in B. If $\mathfrak{S}(F_k)$ is infinite dimensional then, exactly as in the case of F_0, we obtain a contradiction by applying Lemma 1.3. Hence $\mathfrak{S}(F_k)$ is a closed finite dimensional ideal in B. We now show that $\mathfrak{S}(F_k) = \{0\}$ using a similar method to the one employed when dealing with F_0 although the situation is rather different since F_k is not necessarily a homomorphism. As argued in the case of F_0 the radical of $\mathfrak{S}(F_k)$ is zero and so $\mathfrak{S}(F_k)$ is a finite dimensional semi-simple algebra with identity f. Choose $h \in F_0^{-1}\{f\}$. $F_0(h^2 - h) = f^2 - f = 0$ and so $F_j(h) = F_j(h^2)$ $(j = 1, \ldots, k)$. This implies $F_j(h) = 0$ for $j = 1, \ldots, k$ since the identity of an ideal in an algebra is a central idempotent in this algebra.

A / ker F_0 is a Banach algebra and consider its subalgebra hA / ker F_0. Define F_0': hA / ker $F_0 \to$ fB by $F_0'(ha + \ker F_0) = fF_0(a)$. F_0' is one-one and onto fB which is finite dimensional and so hA / kerF_0 is finite dimensional. Define F_k': hA / ker $F_0 \to$ fB by $F_k'(ha + \ker F_0) = fF_k(a)$ which is well-defined since $\ker F_0 \subseteq \ker F_k$ and $F_j(h) = 0$ $(j = 1, \ldots, k)$. F_k' is continuous since hA / ker F_0 is finite dimensional. Now let $y \in \mathfrak{S}(F_k)$. There exist x_n in A, $x_n \to 0$ with $F_k(x_n) \to y$ as $n \to \infty$. $F_k'(hx_n + \ker F_0) = fF_k(x_n) \to fy = y$ as $n \to \infty$. But $F_k'(hx_n + \ker F_0) \to 0$. Hence $y = 0$ and so $\mathfrak{S}(F_k) = \{0\}$ which by Lemma 1.1 (b) gives F_k continuous and induction completes the proof.

<u>Remarks</u>. (1) The class of Banach algebras described by the hypotheses

in the theorem includes all the examples considered in Chapter 3 including $L^1[0,1]$ and semi-simple Banach algebras (see the remark after Theorem 3.2). For certain Banach algebras of power series the continuity of higher derivations under the restricted conditions of $A = B$, $F_0 =$ the identity map was first proved by R.J. Loy [22].

(2) The result for Banach algebras (such as $L^1[0,1]$, Banach algebras of power series and others described in Chapter 3) which satisfy the hypothesis on infinite dimensional closed ideals and for which there are no non-zero finite dimensional ideals can be proved without requiring the assumption on the kernels of the F_j's.

(3) Using the methods of [13] and [18] it is possible to classifyall the higher derivations acting on $L^1[0,1]$ where F_0 is the identity map.

(4) The methods of the proof also give the continuity of higher derivations on n indices of A into B (see [22]) under similar hypotheses to Theorem 5.1.

The following examples from Loy [23] show that the conditions on the F_j's are required.

<u>Examples</u>. We consider ℓ^2 with pointwise addition and product. Let θ be a discontinuous linear functional on ℓ^2 which vanishes on the dense subset $\ell^1 = (\ell^2)^2$.

(a) Take $A = B = \ell^2$. Then B is semi-simple and so satisfies the hypotheses of the theorem. Define F_0: $A \to B$ to be the unilateral shift $F_0(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ so that F is a one-one homomorphism of A into B. Given a positive integer n, define $F_i = 0$, $1 \leq i \leq n-1$ and $F_n(x) = (\theta(x), 0, 0, \ldots)$. Clearly

$\ker F_0 \subseteq \ker F_j$ for $1 \leq j \leq n$. Then $\{F_0, F_1, \ldots, F_n\}$ is a higher derivation of rank n of A into B and F_n is clearly discontinuous. In this example F_0 is not onto B.

(b) Take $A = \ell^2$ with identity e adjoined and $B = \underset{\sim}{C}$. It is trivial that B satisfies the hypotheses of the theorem. Let ϕ be a character on A with kernel ℓ^2 and extend θ to A by $\theta(e) = 0$ and linearity. Define $F_0 = \phi$ which is onto B. Then $F_0 = \phi$, $F_i = 0$, $1 \leq i \leq n-1$, $F_n = \theta$ is a higher derivation of rank n of A onto B with F_n discontinuous. Here $\ker F_0 \not\subseteq \ker F_n$.

Part Two

Uniform Algebras on Odd Spheres

CHAPTER SIX

In this chapter we shall introduce some basic definitions and concepts which we shall use throughout the second half of this thesis. We also list some well-known results which we shall need and give a brief introduction to the problem we shall be discussing in Chapters 7, 8, 9.

Notation. Let T denote the unit circle $\{z \in \mathbb{C}: |z| = 1\}$ and D the open unit disc $\{z \in \mathbb{C}: |z| < 1\}$. Lebesgue measure on T will usually be denoted by dt; for convenience, however, if E is a measurable subset of T, $|E|$ will also denote the Lebesgue measure of E. All functions discussed are complex-valued. C is the algebra of continuous functions on T and A is the algebra of continuous functions on $\bar{D}$ which are analytic on D. L^∞ will denote the Banach algebra of essentially bounded, Lebesgue measurable functions on T. The norm $\|f\|$ of a function f in L^∞ is the essential supremum of $|f|$ on T. The collection of boundary functions (via radial limits) of bounded analytic functions on D forms a closed subalgebra H^∞ of L^∞. L^p $(1 \leq p < \infty)$ denotes the Banach space of Lebesgue measurable functions f on T such that $\int_T |f|^p dt < \infty$. The maximal ideal space of any closed subalgebra B of L^∞ will be denoted by $\Phi(B)$.

For each f in L^1, $re^{i\theta} \in D$, let $f(re^{i\theta})$ denote the harmonic extension of f into D by means of its Poisson integral, i.e. $f(re^{i\theta}) = \frac{1}{2\pi} \int_T f(e^{it}) P(r, \theta - t)dt$ where P is the Poisson kernel given by

$$P(r,t) = \frac{1 - r^2}{1 - 2r \cos t + r^2} .$$

We shall often not distinguish between f in L^1 and its harmonic extension to D.

Definitions. A unimodular function is a function $f \in L^\infty$ for which $|f| = 1$ almost everywhere (a.e.) on T. An inner function is a unimodular function f in H^∞. A Blaschke product is an inner function of the form $B(z) = z^k \prod_{j=1}^{\infty} \frac{\overline{\lambda_j}}{|\lambda_j|} \frac{\lambda_j - z}{1 - \overline{\lambda}_j z}$ with k a non-negative integer, and $\{\lambda_j\}$ a sequence of non-zero complex numbers of modulus less than 1 such that $\sum_{j=1}^{\infty} (1 - |\lambda_j|) < \infty$; (this last condition insures the convergence of the infinite product).

A sequence $\{z_n\}$ in D is an interpolating sequence if for every bounded sequence $\{w_n\}$ in $\mathbb{C}$, there is an f in H^∞ such that $f(z_n) = w_n$ for all n. A Blaschke product whose zeros form an interpolating sequence is called an interpolating Blaschke product.

A useful property of interpolating Blaschke products is given by the following proposition. A proof can be found in K. Hoffman's book [44, p. 206].

PROPOSITION 6.1 *Let* B *be an interpolating Blaschke product with zero set* $\{z_n\}$. *Let* $\phi \in \Phi(H^\infty)$ and $\phi(B) = 0$ *then* ϕ *is in the closure of* $\{z_n\}$ *in* $\Phi(H^\infty)$.

We shall be interested in obtaining concise expressions for the relative size of a function . Thus for g defined on T and for each α consider the set where $|g|$ is greater than α, $\{x: |g(x)| > \alpha\}$. The function $\lambda(\alpha)$, defined to be the Lebesgue measure of this set,is called the distribution function of $|g|$. The decrease of $\lambda(\alpha)$ as α grows describes the relative size of the function - this is our main concern locally. Any quantity dealing solely with the size of g can be expressed in terms of the distribution function $\lambda(\alpha)$. For example, if $g \in L^p$, then $\int_T |g(e^{it})|^p dt = p \int_0^\infty \alpha^{p-1}\lambda(\alpha)d\alpha$.

We now introduce the Hardy-Littlewood maximal function . A description of this function and its properties can be found in E.M. Stein's excellent book [53] .

Definition. Let f be a function in L^1. We define

$$M(f)(e^{i\theta}) = \sup_{s>0} \frac{1}{2s} \int_{\theta-s}^{\theta+s} |f(e^{it})|dt.$$

$M(f)$ is the Hardy-Littlewood maximal function and a partial integration shows that there is an absolute constant A so that

$$f(re^{i\theta}) \leq AM(f)(e^{i\theta}) \qquad (re^{i\theta} \in D)$$

where we consider f as being defined on D by its harmonic extension.

The most useful theorem concerning the maximal function is the Hardy-Littlewood maximal theorem. The proof is not difficult but it involves a covering lemma of "Vitali-type". Readable accounts of the proof can be found in [53] or [33] .

THEOREM 6.2 *Let* f *be a given function defined on* T.

(1) *If* $f \in L^1$, *then for every* $\alpha > 0$

$$|\{e^{i\theta}: M(f)(e^{i\theta}) > \alpha\}| \leq \frac{B_0}{\alpha} \int_T |f(e^{it})|dt$$

where B_0 *is a constant.*

(2) *If* $f \in L^p$, $1 \leq p < \infty$, *then* $M(f) \in L^p$ *and* $\|M(f)\|_p \leq B_p \|f\|_p$ *where* B_p *depends only on* p.

If f is a function in L^1 and I is any subarc of T let $f_I = \frac{1}{|I|}\int_I f(t)dt$. For $0 < a \leq 2\pi$ we then define $S_a(f) = \sup_{|I|\leq a} \frac{1}{|I|}\int_I |f(t) - f_I|dt$, and we put

$$S_0(f) = \lim_{a\to 0} S_a(f), \qquad \|f\|_* = S_{2\pi}(f).$$

The function f is said to have bounded mean oscillation, or to belong to BMO, if $\|f\|_* < \infty$. The space BMO is a Banach space under the norm $\|\cdot\|_*$, provided that two functions differing by a constant are identified. A function f in BMO is said to have vanishing mean oscillation, or to belong to VMO, if $S_0(f) = 0$. It is clear from elementary considerations that VMO is a closed subspace of BMO. Intuitively a function is in VMO if its mean oscillation is locally small.

The concept of bounded mean oscillation was first introduced by F. John and L. Nirenberg [46] and vanishing mean oscillation was first described by D.E. Sarason in [52] where various characterizations of VMO are obtained. In John and Nirenberg's paper they prove various inequalities concerning functions in BMO one of which we now state as we shall require it later. Again the proof is not hard but it uses a rather technical and involved decomposition of integrable functions due to F. Riesz.

<u>LEMMA 6.3</u> *Suppose* f *is a function in BMO and* I *is a subarc of* T. *For each* $\alpha > 0$, *let* $\lambda(\alpha)$ *be the distribution function of* $|f - f_I|$. *Then there exist constants* c_1, c_2 *and* α_0 *(independent of* f*) such that*

$$\lambda(\alpha) \leq \frac{c_1}{\|f\|_*}\left(\int_I |f(t) - f_I|dt\right)e^{-c_2\alpha/\|f\|_*}$$

for all $\alpha \geq \|f\|_*\alpha_0$.

Definition. For any subarc I of T with centre e^{it} and measure $2\delta > 0$, let $R(I) = \{re^{i\theta} \in D: |\theta - t| \leq \delta,\ 1 - \delta \leq r < 1\}$. A finite positive measure μ on D is said to be a Carleson measure if there exists a constant c such that $\mu(R(I)) \leq c|I|$ for all subarcs I of T.

Any rectifiable curve $\Gamma \subseteq D$ induces a finite measure on D by defining the measure of any Borel set S to be the length of $\Gamma \cap S$. We say that Γ induces a Carleson measure if the induced measure is Carleson.

The following lemma is a version of Green's theorem which we shall use in the proof of Theorem 6.5 and in later chapters. In the form given it is due to D.E. Sarason.

LEMMA 6.4 *If* $f, g \in L^2$ *and* $f(0)g(0) = 0$, *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})g(e^{it})dt = \frac{1}{\pi}\iint_D \nabla f(re^{i\theta}) \cdot \nabla g(re^{i\theta}) r \log\frac{1}{r}\, dr d\theta$$

where $\nabla f(re^{i\theta}) = \left(\frac{\partial f}{\partial r}(re^{i\theta}), \frac{1}{r}\frac{\partial f}{\partial \theta}(re^{i\theta})\right) \in \mathbb{C}^2$.

Proof. Let $\sum_{-\infty}^{\infty} a_n e^{in\theta}$, $\sum_{-\infty}^{\infty} b_n e^{in\theta}$ be the Fourier series of f and g respectively. Then $f(re^{i\theta}) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$, $g(re^{i\theta}) = \sum_{-\infty}^{\infty} b_n r^{|n|} e^{in\theta}$ and by direct computation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})g(e^{it})dt = \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} a_n b_{-n} \qquad (a_0 b_0 = 0),$$

$$\nabla f(re^{i\theta}) = \left[\sum_{-\infty}^{\infty} a_n |n| r^{|n|-1} e^{in\theta},\ \sum_{-\infty}^{\infty} a_n r^{|n|-1} n i e^{in\theta}\right],$$

$$\nabla g(re^{i\theta}) = \left[\sum_{-\infty}^{\infty} b_n |n| r^{|n|-1} e^{in\theta},\ \sum_{-\infty}^{\infty} b_n r^{|n|-1} n i e^{in\theta}\right].$$

Again by direct computation, using the fact that $\int_0^1 r^n \log r\, dr = \frac{-1}{(n+1)^2}$

for $n \neq -1$, we obtain

$$\frac{1}{\pi}\iint_D \nabla f(re^{i\theta}) \cdot \nabla g(re^{i\theta}) r \log \frac{1}{r}\, dr d\theta = \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} a_n b_{-n}$$

and so the lemma is proved.

In their fundamental paper on BMO and H^p spaces of several variables C. Fefferman and E.M. Stein [41] proved the following theorem which exhibits the relationship between functions in BMO and Carleson measures. (Note that we have transferred their result from the real line to T).

THEOREM 6.5 *For a function* f *defined on* T *the following conditions are equivalent:*

(1) $f \in BMO$,

(2) $f \in L^1$ *and the measure* μ *on* D *defined by* $d\mu = (1 - r)|\nabla f(re^{i\theta})|^2 r dr d\theta$ *is a Carleson measure.*

Furthermore (if either condition holds), if $c = \sup_{|I| \leq 2\pi} \frac{1}{|I|} \mu(R(I))$, *then there exists a constant* A_1 *with* $c \leq A_1 \|f\|_*^2$.

The constant A_1 is independent of the function f.

Proof. We shall prove the equivalence only in the direction that we shall need later, i.e. (1) ⇒ (2). So suppose $f \in BMO$. We note first that a consequence of Lemma 6.3 is that

$$f \in BMO \Rightarrow \sup_{|I| \leq 2\pi} \frac{1}{|I|} \int_I |f(t) - f_I|^2 dt \leq c_3 \|f\|_*^2 \qquad \ldots(1)$$

where c_3 is a constant. This follows since $\int_T |f(t) - f_I|^p dt = p \int_0^\infty \alpha^{p-1} \lambda(\alpha) d\alpha$ where $\lambda(\alpha)$ is the distribution function of $|f - f_I|$.

Let I be any subarc of T with $|I| = 2\delta > 0$. We will assume without loss of generality that I has centre 1. Let $I_{4\delta} = \{e^{it} \in T: \ |t| \leq 4\delta\}$, and write χ for the characteristic function of $I_{4\delta}$, and $\tilde{\chi}$ for the characteristic function of the complement of $I_{4\delta}$ in T. We have

$$f = f_{I_{4\delta}} + (f - f_{I_{4\delta}})\chi + (f - f_{I_{4\delta}})\tilde{\chi} = f_1 + f_2 + f_3.$$

We also have $f(re^{i\theta}) = f_1(re^{i\theta}) + f_2(re^{i\theta}) + f_3(re^{i\theta})$ for the corresponding Poisson integrals where $re^{i\theta} \in D$. Now $\mu(R(I)) = \iint_{R(I)} (1 - r)|\nabla f(re^{i\theta})|^2 r dr d\theta$. In this integral f_1 contributes nothing since it is constant. Now

$$\begin{aligned}\iint_{R(I)} (1 - r)|\nabla f_2|^2 r dr d\theta &\le \iint_D (1 - r)|\nabla f_2|^2 r dr d\theta \\ &\le \iint_D |\nabla f_2|^2 r \log \frac{1}{r} \, dr d\theta \\ &\quad \text{since } 1 - r \le \log \frac{1}{r} \text{ for } 0 < r \le 1, \\ &= \tfrac{1}{2} \int_{-\pi}^{\pi} |f_2(e^{it})|^2 dt \quad \text{by Lemma 6.4} \\ &= \tfrac{1}{2} \int_{I_{4\delta}} |f - f_{I_{4\delta}}|^2 dt \\ &\le 4c_3\delta \|f\|_*^2 \quad \text{by (1)} \qquad \ldots(2)\end{aligned}$$

Also

$$\begin{aligned}|\nabla f_3(re^{i\theta})| &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |\nabla P(r,\theta-t)||f_3(e^{it})| dt \\ &= \frac{1}{\pi} \int_{T \setminus I_{4\delta}} \frac{|f(e^{it}) - f_{I_{4\delta}}|}{|e^{it} - re^{i\theta}|^2} dt \\ &\quad \text{since } |\nabla P(r,\theta-t)| = \frac{2}{|e^{it}-re^{i\theta}|^2}.\end{aligned}$$

Now if $\delta \ge \frac{\pi}{4}$ this integral is zero since $T \setminus I_{4\delta}$ is empty and if $\delta < \frac{\pi}{4}$ we have, for $e^{it} \in T \setminus I_{4\delta}$ and $re^{i\theta} \in R(I)$, that $|e^{it}-re^{i\theta}|^2 \ge k_1\delta^2 + k_2(\theta-t)^2$ where k_1 and k_2 are positive constants. Also it is clear (see [41, p.142]) that for $g \in BMO$ we have $\int_T \frac{|g(e^{it})-g_{I_{4\delta}}|}{k_1\delta^2+k_2(\theta-t)^2} dt \le \frac{C_4\|g\|_*}{\delta}$. Hence for any value of δ we have $|\nabla f_3 (re^{i\theta})| \le \frac{C_4}{\pi\delta} \|f\|_*$ which implies that

$$\iint_{R(I)} (1-r)|\nabla f_3|^2 r dr d\theta \le C_5\delta\|f\|_*^2 \text{ where } C_5 \text{ is a constant.} \qquad \ldots(3)$$

Since $|\nabla f|^2 \le 2(|\nabla f_2|^2+|\nabla f_3|^2)$ we deduce from (2) and (3) that

$\iint_{R(I)} (1 - r)|\nabla f|^2 r dr d\theta \leq c_6 \delta \|f\|_*^2$ where c_6 is a constant, i.e. $\mu(R(I)) \leq c|I|$ for some constant c and so μ is a Carleson measure.

The next theorem was a crucial part of L. Carleson's proof [32] of the corona theorem (i.e. the theorem which shows that D is dense in $\Phi(H^\infty)$). It is easily proved using the Hardy-Littlewood maximal theorem (as was shown in [53]).

<u>THEOREM 6.6</u> *Let* μ *be a Carleson measure on* D, *with* $\mu(R(I) \leq c|I|$ for all subarcs I of T. *Then for* $1 < p < \infty$,

$$\iint_D |f(z)|^p d\mu(z) \leq cA_p \|f\|_p^p, \quad \textit{for all } f \textit{ in } L^p,$$

where A_p *is a constant depending only on* p.

Proof. Let $\Psi(re^{i\theta})$ and $\psi(e^{i\theta})$ be non-negative functions on D and T respectively which are related by the non-tangential inequality $\sup_{|\theta-\phi|<1-r} \Psi(re^{i\phi}) \leq \psi(e^{i\theta})$. Then $\mu\{re^{i\theta}: \Psi > \alpha\} \leq c|\{e^{i\theta}: \psi > \alpha\}|$ for each α, and as a result $\iint_D \Psi^p d\mu \leq c \int_T \psi^p(e^{it})dt$. Once this is observed we need only take $\Psi(re^{i\theta}) = |f(re^{i\theta})|$, $\psi(e^{i\theta}) = AM(f)(e^{i\theta})$. The non-tangential inequality $\sup_{|\theta-\phi|<1-r} \Psi(re^{i\phi}) \leq \psi(e^{i\theta})$ is contained in the remark after the definition of the maximal function and the theorem then follows from Theorem 6.2 (2).

We now give an elementary measure theoretic lemma due to D.E. Sarason [52] which we shall require later.

<u>LEMMA 6.7</u> *Let* (X,ν) *be a probability measure space and* f *a function in* $L^\infty(\nu)$ *such that* $\|f\|_\infty \leq 1$ *and* $\int f d\nu = 1 - b^3$, *where* $0 < b < \frac{1}{2}$.

Let E *be the set of points in* X *where* $|1 - f| \geq b$. *Then* $\nu(E) \leq 2b$.

Proof. We have

$$1 - b^3 = \int_E \frac{f + \bar{f}}{2}\, d\nu + \int_{X \setminus E} \frac{f + \bar{f}}{2}\, d\nu \leq \int_E \frac{f + \bar{f}}{2}\, d\nu + \nu(X \setminus E).$$

By an elementary calculation, if $|\lambda| \leq 1$ and $|1 - \lambda| \geq b$ then $\frac{\lambda + \bar{\lambda}}{2} \leq 1 - \frac{b^2}{2}$. Hence $\int_E \frac{f + \bar{f}}{2}\, d\nu \leq (1 - \frac{b^2}{2})\nu(E)$ so that $1 - b^3 \leq (1 - \frac{b^2}{2})\nu(E) + \nu(X \setminus E) = 1 - \frac{b^2}{2}\nu(E)$. The desired inequality is now immediate.

<u>Notation</u>. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a collection of functions in L^∞. $[H^\infty, f_\lambda : \lambda \in \Lambda]$ will denote the (uniformly) closed subalgebra of L^∞ generated by H^∞ and the set $\{f_\lambda : \lambda \in \Lambda\}$.

We now turn to discuss the problem which is at the heart of the work in Chapters 7 to 9. We will be interested in the closed subalgebras of L^∞ which contain H^∞ properly. If A is such an algebra we let A_d denote the closed subalgebra of L^∞ generated by H^∞ and the complex conjugates of the inner functions that are invertible in A, i.e. $A_d = [H^\infty, \bar{b} : \bar{b} \in A$ and b is inner$]$. Clearly $A_d \subseteq A$; if $A_d = A$, A is called a <u>Douglas algebra</u>. R. Douglas [40] conjectured that equality is always the case for such A, i.e. $A_d = A$ for every closed subalgebra A containing H^∞. This conjecture has attracted much interest in the past few years and in particular it was soon shown that many natural examples of closed subalgebras of L^∞ containing H^∞ were Douglas algebras, e.g. L^∞ itself (see [51]). Recently the question has been answered in the affirmative, the proof being contained in papers by S-Y.A. Chang [34] and D.E. Marshall [47]. Chang proved that if A is a Douglas algebra and B is a closed subalgebra of L^∞ which contains H^∞

with $\Phi(B) = \Phi(A)$ then $B = A$, i.e. a Douglas algebra is uniquely determined amongst those closed subalgebras of L^∞ containing H^∞ properly by its maximal ideal space. Marshall proved that if A is a closed subalgebra of L^∞ containing H^∞ then $\Phi(A) = \Phi(A_d)$. It is clear that the two results together show that every closed subalgebra of L^∞ containing H^∞ is a Douglas algebra.

In Chapter 9 we shall give a direct proof of the Marshall-Chang theorem using the techniques of Chang and Marshall but avoiding almost entirely any reference to maximal ideal spaces. This shortens their proof a little and avoids using the corona theorem of Carleson (as Marshall does in his proof). We are grateful to A.M. Davie who suggested the possibility of tackling the proof in this way.

As Marshall pointed out his proof in fact yields the following stronger result which is the theorem we shall prove in Chapter 9.

<u>THEOREM 6.8</u> *Every closed subalgebra* A *of* L^∞ *containing* H^∞ *is given by* $A = [H^\infty, \bar{\underset{\sim}{B}} : \bar{\underset{\sim}{B}} \in A$ *and* $\underset{\sim}{B}$ *is an interpolating Blaschke product.*$]$

It is clear that this theorem shows that every closed subalgebra A of L^∞ containing H^∞ is a Douglas subalgebra.

At this point note that it is sufficient to prove Theorem 6.8 when $A = [H^\infty, u, \bar{u}]$ where u is a unimodular function in L^∞. For suppose A is a closed subalgebra of L^∞ containing H^∞. A is generated by its invertible elements; so suppose f is invertible in A and let $g = \exp[\log|f| + i(\log|f|)^\sim]$ where $(\log|f|)^\sim$ is the harmonic conjugate function of $\log|f|$. Then $|g| = |f|$ a.e. on T and g is invertible in H^∞. Therefore $u = fg^{-1}$ and $\bar{u} = f^{-1}g$ are unimodular functions in

L^∞ and this shows that A is generated by H^∞ and $\{u \in A: u$ is unimodular and $\bar{u} \in A\}$.

Marshall's construction of the relevant Blaschke products required for the proof of Theorem 6.8 is based on a construction due to L. Carleson which was used in his proof of the corona theorem [32]. In order to describe Marshall's construction (see Chapter 8) we now give some preliminary definitions.

Definition. The hyperbolic distance between two points in D is defined by $\rho(z,w) = \left|\frac{z - w}{1 - \bar{w}z}\right|$ $(z,w \in D)$. This defines a metric on D.

The relevance of the ρ-metric to our problem rests in the following characterisation of interpolation: a sequence $\{z_j\}$ in D is interpolating if and only if there is some $\gamma > 0$ for which $\rho(z_j,z_k) \geq \gamma$ for $j \neq k$ and the measure $\sum(1 - |z_j|)\delta_{z_j}$ is a Carleson measure (δ_{z_j} denotes the point mass at z_j) (for a proof of this fact see [42]).

Definition. Let V be a bounded domain bounded by a finite number of rectifiable Jordan curves Γ. Let $\Gamma = P \cup Q$, $\mathrm{Int}(P) \cap \mathrm{Int}(Q) = \phi$, where P and Q are finite sets of Jordan arcs. The function $\omega(z,P;V)$ which is harmonic in V and assumes the value 1 on P and the value 0 on Q is called the harmonic measure of P with respect to V, evaluated at the point z. (For the fact that the harmonic measure always exists in the above situation see [43]).

The following proposition is a form of the maximum modulus theorem. It is a special case of Theorem 1.6.3 of [48] and a proof of the result can be found there.

PROPOSITION 6.9 *Let* f *be a bounded analytic function on a domain* $V \subseteq \bar{D}$, *and let* X *be a subset of* ∂V *of harmonic measure zero. If* $\lim_{\substack{z\to\eta \\ \eta\in\partial V\setminus X}} |f(z)| \le K$, *then* $|f| \le K$ *in* V.

Finally we note two well-known theorems which we shall use in subsequent chapters.

THEOREM 6.10 *Every closed subalgebra of* L^∞ *which contains* H^∞ *properly also contains* C.

Proof. See [44].

THEOREM 6.11 *The quotient space* L^∞/H^∞ *is the dual of the space* H_0^1, *the space of functions in* L^1 *whose harmonic extension into* D *is analytic in* D *and has mean value* 0.

Proof. See [39].

CHAPTER SEVEN

In this chapter we extend the definition of VMO given in Chapter 6. We then characterise the generalised concept in terms of Carleson measures in a similar fashion to the way that BMO is characterised by Theorem 6.5. The techniques we use are extensions of those used by D.E. Sarason [52] and S-Y.A. Chang [34] to examine the particular case of VMO.

Suppose B is a closed subalgebra of L^∞ which contains H^∞. If B is generated by H^∞ and the complex conjugates of certain inner functions, then it is clear that $\Phi(B)$ consists precisely of the set of points in $\Phi(H^\infty)$ at which the Gelfand transforms of the inner functions involved all have unit modulus. Now let b be an inner function. Given $0 < \delta \leq 1$ we let $G_\delta = \{z \in D: \ |b(z)| \geq 1 - \delta\}$. We begin by looking at the Douglas algebra B generated by H^∞ and the complex conjugate of b, i.e. $[H^\infty, \bar{b}]$. Functions in $[H^\infty, \bar{b}]$ have the following asymptotic behaviour in the region G_δ:

$$\limsup_{\delta \to 0} \sup_{z \in G_\delta} |f(z)g(z) - (fg)(z)| = 0 \quad \text{for all } f, g \text{ in } [H^\infty, \bar{b}]. \qquad \ldots(1)$$

For if this does not hold for some functions f, g in $[H^\infty, \bar{b}]$, then there exists $\varepsilon > 0$ such that $|f(z_n)g(z_n) - (fg)(z_n)| \geq \varepsilon$ for some point $z_n \in G_{1/n}$, for $n = 2, 3, \ldots$. If we choose ϕ to be a limit point of $\{z_n\}$ in $\Phi(H^\infty)$, then $|\phi(b)| = 1$ and so $\phi \in \Phi([H^\infty, \bar{b}])$ by the comment above. But we have $|\phi(f)\phi(g) - \phi(fg)| \geq \varepsilon$, giving a contradiction.

This yields the following lemma.

LEMMA 7.1 *If* $f \in [H^\infty, \bar{b}]$ *is unimodular and invertible in* $[H^\infty, \bar{b}]$ *then for every* $\varepsilon > 0$, $\exists \delta > 0$ *such that* $|f(z)| \geq 1-\varepsilon^3$ *whenever* $z \in G_\delta$.

Proof. Take g to be the inverse of f in (1), i.e. $g = \bar{f}$.

In our definition of VMO(b) which we give below we will only be interested in a certain class, $\mathcal{P}$, of subarcs of T, which we now describe intuitively. We fix a $\delta > 0$ and choose any $z_0 = r_0 e^{i\theta_0} \in G_\delta$. Let I be a subarc of T, centred at $e^{i\theta_0}$. The "value" which determines whether $I \in \mathcal{P}$ or not is the proportion of the length of I to the distance of z_0 from the boundary of the unit circle. We now give a precise definition.

Definition. Let b be an inner function. If f is a function in BMO we say that f is in VMO(b) if for every $\varepsilon > 0$ and $\eta \geq 1$, there exists some $\delta > 0$ such that for $r_0 e^{i\theta_0} \in G_\delta$ and $1 \leq \psi \leq \eta$ we have

$$I = \{e^{it}: \frac{|\theta_0 - t|}{1-r_0} \leq \psi\} \Rightarrow \frac{1}{|I|} \int_I |f(e^{it}) - f_I|\, dt < \varepsilon.$$

Note that in the particular case when $b(z) = z$ $(z \in T)$ then VMO(b) is simply the space VMO defined in Chapter 6 since in this case $\mathcal{P}$ includes all subarcs of T.

We now adapt the methods of D.E. Sarason [52] and S-Y. Chang [34] to establish some of the properties of VMO(b).

THEOREM 7.2 *Let* f *be a unimodular function in* L^∞. *Then* $f \in$ VMO(b) $\Leftrightarrow \forall \varepsilon > 0$, $\exists \delta > 0$ *such that* $z \in G_\delta$ *implies* $|f(z)| \geq 1-\varepsilon^3$.

Proof. ($\Rightarrow$) Suppose $f \in VMO(b)$, and let $\varepsilon > 0$. Recall that for $z = re^{i\theta} \in D$ we have $f(re^{i\theta}) = \frac{1}{2\pi}\int_T P(r,\theta-t)f(e^{it})dt$ where P is the Poisson kernel. It is clear from the expression for the Poisson kernel that we can make $\int_{T\setminus F} P(r,\theta-t)dt$ as small as we like by choosing F to be a suitable arc centred at $e^{i\theta}$, i.e. there exists an $\eta \geq 1$ such that $\int_E P(r,\theta-t)dt < \frac{\varepsilon^3\pi}{4}$ where

$$E = \{e^{it}: \frac{|\theta-t|}{1-r} \geq \eta\} \qquad \ldots(2)$$

Note that η depends only on ε and not on r or θ. Now there exists $\delta > 0$ such that for $|b(r_0e^{i\theta_0})| \geq 1-\delta$ and $1 \leq \psi \leq \eta$ we have

$$I = \{e^{it}: \frac{|\theta_0-t|}{1-r_0} \leq \psi\} \Rightarrow \frac{1}{|I|}\int_I |f(e^{it}) - f_I|dt < \frac{\varepsilon^3}{4} \qquad \ldots(3)$$

Fix $z_0 = r_0e^{i\theta_0} \in G_\delta$. We have to prove that $|f(z_0)| \geq 1-\varepsilon^3$. Define a subarc J of T by $J = \{e^{it}: \frac{|\theta_0-t|}{1-r_0} \leq \eta\}$ so that

$$\frac{1}{|J|}\int_J |f(e^{it}) - f_J|dt < \frac{\varepsilon^3}{4} \quad \text{by (3)} \qquad \ldots(4)$$

We then have

$$1 = \frac{1}{|J|}\int_J |f(e^{it})|dt \leq \frac{1}{|J|}\int_J |f(e^{it}) - f_J|dt + \frac{1}{|J|}\int_J |f_J|dt$$

i.e.

$$1 \leq \frac{\varepsilon^3}{4} + |f_J| \Rightarrow |f_J| \geq 1 - \frac{\varepsilon^3}{4}. \qquad \ldots(5)$$

Also $\int_{T\setminus J} P(r_0,\theta_0-t)dt < \frac{\varepsilon^3\pi}{4}$ by (2) which implies that

$$1 - \frac{1}{2\pi}\int_J P(r_0,\theta_0-t)dt \leq \frac{\varepsilon^3}{8} \quad \text{since} \quad \frac{1}{2\pi}\int_T P(r_0,\theta_0-t)dt = 1 \qquad \ldots(6)$$

Collecting these inequalities together we have

$$\begin{aligned}
|f(z_0)-f_J| &= \left|\frac{1}{2\pi}\int_T [f(e^{it})-f_J]P(r_0,\theta_0-t)dt\right| \\
&\leq \frac{1}{2\pi}\int_{T\setminus J} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt \\
&\quad + \frac{1}{2\pi}\int_J |f(e^{it})-f_J|P(r_0,\theta_0-t)dt \\
&\leq \frac{\varepsilon^3}{4} + \frac{1}{2\pi}\int_J |f(e^{it})-f_J|P(r_0,\theta_0-t)dt \quad \text{by (2)} \\
&\leq \frac{\varepsilon^3}{4} + \left|\int_J \left[\frac{1}{|J|} - \frac{1}{2\pi}P(r_0,\theta_0-t)\right]|f(e^{it})-f_J|dt\right| \\
&\quad + \frac{1}{|J|}\int_J |f(e^{it})-f_J|dt
\end{aligned}$$

$$\leq \frac{\varepsilon^3}{4} + \frac{\varepsilon^3}{4} + 2[1 - \frac{1}{2\pi}\int_J P(r_0,\theta_0-t)dt] \quad \text{by (4)}$$
$$\leq \frac{\varepsilon^3}{2} + \frac{\varepsilon^3}{4} = \frac{3\varepsilon^3}{4} \quad \text{by (6)}.$$

So $1 - \frac{\varepsilon^3}{4} \leq |f_J| \leq |f(z_0)-f_J| + |f(z_0)| \leq \frac{3\varepsilon^3}{4} + |f(z_0)|$, by (5)
i.e. $|f(z_0)| \geq 1 - \varepsilon^3$.

($\Leftarrow$) Let $\varepsilon > 0$ and $\eta \geq 1$. There exists $\delta > 0$ such that $z \in G_\delta \Rightarrow |f(z)| \geq 1 - \left|\frac{\varepsilon}{2+8\pi(1+\eta^2)}\right|^3 = 1 - \alpha^3$, say. Without loss of generality we may suppose that $0 < \alpha < \frac{1}{2}$. Let $z_0 = r_0 e^{i\theta_0} \in G_\delta$ so that $|f(z_0)| \geq 1 - \alpha^3$. Suppose I is the subarc of T given by $I = \{e^{it}: \frac{|\theta_0 - t|}{1-r_0} \leq \psi\}$ where $1 \leq \psi \leq \eta$. We have to show that
$\frac{1}{|I|}\int_I |f(e^{it}) - f_I|dt < \varepsilon$.

By multiplying f by a constant of modulus one if necessary we may assume that $f(r_0e^{i\theta_0}) > 0$, say $f(r_0e^{i\theta_0}) = 1 - \beta^3$ where $\beta \leq \alpha$. Let E be the set of points on T where $|1 - f| \geq \alpha$. It follows from Lemma 6.7 that $\frac{1}{2\pi}\int_E P(r_0,\theta_0-t)dt \leq 2\alpha$. By a simple estimate based on the identity $P(r,t) = \frac{1-r^2}{(1-r)^2+4r\sin^2(\frac{t}{2})}$ it follows that $P(r_0,\theta_0-t) \geq \frac{1}{(1-r_0)(1+\psi^2)}$ for $e^{it} \in I$. Thus

$$\frac{1}{|I|}\int_{I\cap E} dt = \frac{1}{2\psi(1-r_0)}\int_{I\cap E} dt \leq \frac{1+\psi^2}{2\psi}\int_E P(r_0,\theta_0-t)dt$$
$$\leq \frac{(1+\eta^2)}{2} \cdot 4\pi\alpha = 2\pi\alpha(1+\eta^2)$$

We thus have

$$\frac{1}{|I|}\int_I |f(e^{it})-1|dt = \frac{1}{|I|}\int_{I\cap E}|f(e^{it})-1|dt + \frac{1}{|I|}\int_{I\setminus E}|f(e^{it})-1|dt$$
$$\leq \frac{2}{|I|}\int_{I\cap E} dt + \frac{\alpha}{|I|}\int_{I\setminus E} dt \leq \alpha[1 + 4\pi(1+\eta^2)],$$

and so

$$\frac{1}{|I|}\int_I |f(e^{it})-f_I|dt \leq \frac{1}{|I|}\int_I |f(e^{it})-1|dt + |1-f_I|$$
$$\leq 2\alpha[1 + 4\pi(1+\eta^2)] = \varepsilon.$$

LEMMA 7.3 $L^\infty \cap VMO(b)$ *is a C*-algebra. Also if* C_B *is the C*-algebra in* L^∞ *generated by the inner functions which are invertible in* $[H^\infty,\bar{b}]$ *then* $C_B \subseteq VMO(b)$.

Proof. It is clear that $L^\infty \cap VMO(b)$ is an algebra in L^∞ closed under uniform limits and complex conjugation and so is a C*-algebra. Suppose g is an inner function invertible in $[H^\infty,\bar{b}]$. Let $\varepsilon > 0$. g is unimodular and so, by Lemma 7.1, there exists $\delta > 0$ such that $|g(z)| \geq 1-\varepsilon^3$ whenever $|b(z)| \geq 1-\delta$. Hence Theorem 7.2 shows that $g \in VMO(b)$.

We are grateful to S-Y.A. Chang for allowing us to see a preprint [35] which has not yet appeared in publication. This enabled us to prove the following remarks which meant that we can replace an "ad hoc" argument in the proof of Lemma 7.4 (which is leading up to the proof of Theorem 7.7) by one which corresponds to the proof of Theorem 6.5 given by Fefferman and Stein.

Suppose $f \in VMO(b)$ and let $\varepsilon > 0$. Choose η_0 to be the smallest integer 2^N such that $2\sum_{n=N}^{\infty} \frac{1+2n}{2^n} \leq \varepsilon$. Now choose δ from the definition of $VMO(b)$ with $\eta = \eta_0$ so that if $z = re^{i\theta} \in G_\delta$ and $1 \leq \psi \leq \eta_0$ then $I = \{e^{it}: \frac{|\theta-t|}{1-r} \leq \psi\} \Rightarrow \frac{1}{|I|}\int_I |f(e^{it})-f_I|dt < \varepsilon$. Suppose $z_0 = r_0e^{i\theta_0} \in G_\delta$ and let $J = \{e^{it}: \frac{|\theta_0-t|}{1-r_0} \leq 1\}$. We want to show that

$$A(J) = \frac{1}{2\pi}\int_T |f(e^{it})-f_J|P(r_0,\theta_0-t)dt \leq C_7\varepsilon \qquad \ldots(7)$$

where C_7 is a constant depending only on $\|f\|_*$.

Let J_n be the arc with the same centre, $e^{i\theta_0}$, as J and with length $2^n|J|$. Suppose $\eta_0 = 2^N$. We will prove (7) for the case $2^N|J| < \pi$. (The same proof works in the contrary case with a slight change in the constant C_7.) We have

$$A(J) = \frac{1}{2\pi} \sum_{n=0}^{N} \int_{J_n \setminus J_{n-1}} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt$$
$$+ \frac{1}{2\pi} \int_{T \setminus J_N} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt,$$

where J_{-1} is taken to be the empty set.

The estimate

$$|f_{J_{n-1}} - f_{J_n}| \leq \frac{1}{|J_{n-1}|} \int_{J_{n-1}} |f - f_{J_n}|dt$$
$$\leq \frac{|J_n|}{|J_{n-1}|} \frac{1}{|J_n|} \int_{J_n} |f - f_{J_n}|dt \leq 2\varepsilon$$

valid for $n = 1, 2, \ldots, N$ gives

$$|f_J - f_{J_n}| \leq \sum_{k=1}^{n} |f_{J_{k-1}} - f_{J_k}| \leq 2n\varepsilon, \qquad n = 0,1,2,\ldots,N,$$

which together with an elementary estimate of $P(r_0,\theta_0-t)$ yields

$$\int_{J_n \setminus J_{n-1}} |f(e^{it}) - f_J|P(r_0\theta_0-t)dt$$
$$\leq \frac{\pi^2}{|J|} \frac{1}{2^{2n-1}} [\int_{J_n} (|f(e^{it})-f_{J_n}| + |f_{J_n}-f_J|)dt]$$
$$\leq \frac{\pi^2}{2^{n-1}} \frac{1}{|J_n|} \int_{J_n} (|f(e^{it})-f_{J_n}| + 2n\varepsilon)dt$$
$$\leq \pi^2 \frac{(1+2n)}{2^{n-1}} \varepsilon, \qquad n = 0,1,\ldots,N.$$

Hence $\sum_{n=0}^{N} \frac{1}{2\pi} \int_{J_n \setminus J_{n-1}} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt \leq (\sum_{n=0}^{N} \frac{1+2n}{2^n})\varepsilon\pi$. Using similar estimates we have, for N_1 the largest integer such that $2^{N_1}|J| < 2\pi$,

$$\frac{1}{2\pi} \int_{T \setminus J_N} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt$$
$$= \frac{1}{2\pi} \sum_{n=N}^{N_1} \int_{J_{n+1} \setminus J_n} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt$$
$$+ \frac{1}{2\pi} \int_{T \setminus J_{N_1}} |f(e^{it})-f_J|P(r_0,\theta_0-t)dt$$

$$\leq \pi \sum_{n=N}^{\infty} \frac{1+2n}{2^n} \|f\|_* + \pi\frac{(1+2N)}{2^N} \|f\|_*$$

$$\leq \varepsilon\pi\|f\|_* \text{ by the definition of } \eta_0.$$

Hence $A(J) \leq C_7\varepsilon$ where $C_7 = \pi(\sum_{n=0}^{\infty} \frac{1+2n}{2^n} + \|f\|_*)$.

We now prove three lemmas which will enable us to describe the bounded functions in VMO(b) in terms of Carleson measures.

LEMMA 7.4 *Suppose* $f \in \mathrm{VMO(b)}$ *and let* $0 < \varepsilon < 1$. *Then there exists* $\delta > 0$ *such that if* $z_0 = r_0e^{i\theta_0} \in G_\delta$ *then there exists a constant* C_8 *independent of* ε) *such that* $\iint_{S(\theta_0,r_0)} (1-r)|\nabla f|^2 r\,dr\,d\theta \leq c_8\varepsilon(1-r_0)$, *where* $S(\theta_0,r_0)$ *is the region* $\{re^{i\theta}: \frac{|\theta_0-\theta|}{1-r_0} \leq 4,\ r_0 \leq r < 1\}$.

Proof. Let $0 < \varepsilon < 1$, and, as before, let η_0 be the smallest integer 2^N such that $2\sum_{n=N}^{\infty} \frac{1+2n}{2^n} \leq \varepsilon$. From the definition of VMO(b) with $\eta = \max(5,\eta_0)$ choose δ so that if $z = re^{i\theta} \in G_\delta$ and $1 \leq \psi \leq \eta$ then $I = \{e^{it}: \frac{|\theta-t|}{1-r} \leq \psi\} \Rightarrow \frac{1}{|I|}\int_I |f(e^{it})-f_I|dt < \varepsilon$. Suppose $z_0 = r_0e^{i\theta_0} \in G_\delta$ and let J be the arc $\{e^{it}: \frac{|\theta_0-t|}{1-r_0} \leq 5\}$. Put $f_1 = \chi_J(f-f_J)$, $f_2 = \chi_{T\setminus J}(f-f_J)$ where χ_I denotes the characteristic function of the arc I.

We have $\frac{1}{|J|}\int_J |f(e^{it})-f_J|dt < \varepsilon$. Thus

$$\iint_{S(\theta_0,r_0)} (1-r)|\nabla f_1|^2 r\,dr\,d\theta \leq \iint_D (1-r)|\nabla f_1{}^2|r\,dr\,d\theta$$

$$\leq \iint_D |\nabla f_1|^2 r \log\frac{1}{r}\,dr\,d\theta$$

$$= \tfrac{1}{2}\int_T |f_1(e^{it})|^2 dt \quad \text{by Lemma 6.4}$$

$$= \tfrac{1}{2}\int_J |f-f_J|^2 dt \leq C_9\varepsilon|J|\,\|f\|_*$$

by Lemma 6.3

$$= C_{10}\varepsilon(1-r_0). \qquad \ldots(8)$$

(C_9, C_{10} are both constants).

Also $|\nabla f_2(re^{i\theta})| \le \frac{1}{2\pi} \int_T |\nabla P(r,\theta-t)||f_2(e^{it})|dt$

$$= \frac{1}{\pi} \int_{T\setminus J} \frac{|f(e^{it})-f_J|}{|e^{it}-re^{i\theta}|^2} dt$$

Hence if $re^{i\theta} \in S(\theta_0,r_0)$ we have, using an elementary estimate of $|e^{it}-re^{i\theta}|^2$,

$$|\nabla f_2(re^{i\theta})| \le \frac{1}{\pi} C_{11} \int_T \frac{|f(e^{it})-f_J|}{|e^{it}-z_0|^2} dt$$

$$\le \frac{C_{11}}{\pi} \left[\int_T \frac{|f(e^{it})-f_{J_0}|}{|e^{it}-z_0|^2} dt + 2\pi \frac{|f_{J_0}-f_J|}{1-r_0^2} \right]$$

where $J_0 = \{e^{it}: \frac{|\theta_0-t|}{1-r_0} \le 1\}$,

$\le C_{12} \frac{\varepsilon}{1-r_0}$ since $\int_T |f(e^{it})-f_{J_0}|P(r_0,\theta_0-t)dt \le 2\pi C_7\varepsilon$

by the remarks before the lemma.

(C_{12} is a constant).

Thus $\iint_{S(\theta_0,r_0)} (1-r)|\nabla f_2|^2 rdrd\theta \le \iint_{S(\theta_0,r_0)} \frac{C_{12}^2\varepsilon^2}{(1-r_0)^2}(1-r)rdrd\theta$

$$= 2C_{12}^2\varepsilon^2(1-r_0). \qquad \ldots(9)$$

Since $|\nabla f|^2 \le 2(|\nabla f_1|^2 + |\nabla f_2|^2)$ we obtain the desired conclusion from (8) and (9).

<u>LEMMA 7.5</u> *Suppose* $f \in VMO(b)$ *and let* $0 < \varepsilon < 1$. *Then there exists* $\delta > 0$ *such that the measure* μ_δ *on* D *defined by* $d\mu_\delta = \chi_{G_\delta}(1-r)|\nabla f|^2 rdrd\theta$ *is a Carleson measure with* $\mu_\delta(R(I)) \le C_8\varepsilon|I|$ *for all subarcs* I *of* T. *(*C_8 *is the same constant as in Lemma 7.4).*

Proof. (Chang [34]) By Lemma 7.4 we choose δ such that if $z_0 = r_0e^{i\theta_0} \in G_\delta$ then $\iint_{S(\theta_0,r_0)} (1-r)|\nabla f|^2 rdrd\theta \le C_8\varepsilon(1-r_0)$, where

$S(\theta_0,r_0)$ is the region $\{re^{i\theta}: \frac{|\theta_0-\theta|}{1-r_0} \leq 4, \ r_0 \leq r < 1\}$ and where C_8 is a constant. We assume without loss of generality that $I = \{e^{it}: -a \leq t \leq a\}$ for some $a \leq \pi$. To establish the result it suffices to find a collection F of regions of the form $S(\theta,r)$ with $re^{i\theta} \in G_\delta$, $\bigcup_{S(\theta,r)\in F} S(\theta,r) \supseteq R(I) \cap G_\delta$ and $\sum_{S(\theta,r)\in F} (1-r) \leq 2a$. We shall choose the collection F by the following inductive process. For each $n = 0,1,2,\ldots$ and $j = 1,2,3,\ldots,2^n$, let

$R_{n,j} = \{re^{i\theta}: \ \theta \in [-a+(j-1)a/2^{n-1}, -a+ja/2^{n-1}], \ 1-a/2^n \leq r < 1\}$,

with $R_{0,1} = R(I)$.

Let $r_0 = \inf\{r: \ re^{i\theta} \in R_{0,1} \cap G_\delta\}$ and choose θ_0 so that $r_0e^{i\theta_0} \in R_{0,1} \cap G_\delta$. Notice that if $1 - r_0 \geq a/2$, then $R(I) \cap G_\delta$ is contained in $S(\theta_0,r_0)$ by the definition of r_0 and so we can pick $S(\theta_0,r_0)$ in our collection F and stop the process. If $1 - r_0 < a/2$, let $r_{1,j} = \inf\{r: \ re^{i\theta} \in R_{1,j} \cap G_\delta\}$ for $j = 1, 2$. Choose $\theta_{1,j}$ so that $r_{1,j}e^{i\theta_{1,j}} \in R_{1,j} \cap G_\delta$. If $1-r_{1,j} \geq a/2^2$ then $R_{1,j} \cap G_\delta$ is contained in $S(\theta_{1,j},r_{1,j})$ and hence we can pick $S(\theta_{1,j},r_{1,j})$ in F and stop the procedure in the region $R_{1,j}$. If $1 - r_{1,j} < a/2^2$, then we continue the process in $R_{1,j}$ to the regions $R_{2,2j-1}$ and $R_{2,2j}$. It is clear that we can continue the above process inductively, and the collection F thus chosen satisfy our requirement.

Notation. Let X be a Banach space and suppose that E is a closed subspace of X. For $x \in X$ the distance of x to E, $d(x,E)$ is given by $d(x,E) = \inf\{\|x-y\|: \ y \in E\}$.

LEMMA 7.6 *Let* $f \in L^\infty$. *Suppose that for every* $\varepsilon > 0$ *there exists some* $\delta > 0$ *such that the measure* μ_δ *on* D *defined by*

$d\mu_\delta = \chi_{G_\delta}(1-r)|\nabla f|^2 r\,dr\,d\theta$ *is a Carleson measure with* $\mu_\delta(R(I)) \leq \varepsilon|I|$ *for all subarcs* I *of* T. *Then for every* $\varepsilon > 0$ *there is an absolute constant* C_{13} *such that* $d(fb^n, H^\infty) \leq C_{13}\varepsilon^{\frac{1}{2}}$ *for* n *sufficiently large.*

Proof. Let $\varepsilon > 0$ and choose δ so that μ_δ is a Carleson measure on D with $\mu_\delta(R(I)) \leq \varepsilon|I|$ for all subarcs I of T. First notethat without loss of generality we may assume that $G_\delta \subseteq \{z: \frac{1}{2} < |z| < 1\}$. For by Theorem 6.10 we deduce that $[H^\infty, \bar{b}] = [H^\infty, \overline{zb}]$ and so $d(f,[H^\infty,\bar{b}]) = d(f,[H^\infty,\overline{zb}])$. This implies that if $d(fz^n b^n, H^\infty) < k\varepsilon$ for some constant k and sufficiently large n then $d(fb^n, H^\infty) < k\varepsilon$ for sufficiently large n. Thus we could consider the inner function $zb(z)$ instead and clearly the region G_δ for the inner function $zb(z)$ satisfies our requirement so long as we choose $\delta < \frac{1}{2}$.

From this point on the proof follows Chang [34, Lemma 6].

Without loss of generality assume that $0 < \varepsilon < 1$. Since L^∞ / H^∞ is the dual of H_0^1 by Theorem 6.11 $d(fb^n, H^\infty)$ equals the norm of the functional that fb^n induces on H_0^1. It is therefore sufficient to show that, for all $g \in H^1$, $\left|\frac{1}{2\pi}\int_T f(e^{it})b^n(e^{it})g(e^{it})dt\right| \leq C_n\|g\|_1$ where C_n is a constant which is less than $C_{13}\varepsilon^{\frac{1}{2}}$ for some constant C_{13} as $n \to \infty$. Without loss of generality we can assume that g is in H^∞ since H^∞ is L^1-dense in H^1. We may also assume without loss of generality that $\|f\|_\infty \leq 1$ and $f(0) = 0$. By Lemma 6.4 we can write $\frac{1}{2\pi}\int_T f(e^{it})b^n(e^{it})g(e^{it})dt = \frac{1}{\pi}\iint_D \nabla f.\nabla(b^n g)r \log\frac{1}{r}\,dr\,d\theta$. Roughly speaking we shall estimate this integral by splitting it into two parts - first integrating over G_δ where we obtain our estimate by using the fact that $\chi_{G_\delta}(1-r)|\nabla f|^2 r\,dr\,d\theta$ is a Carleson measure, together with

Carleson's inequality (Theorem 6.6) - second integrating over $D \setminus G_\delta$ where we use the fact that $|b(z)| < 1-\delta$ to obtain our estimate.

Since b^n and g are both analytic functions we have $(b^n g)(z) = b^n(z)g(z)$ for all $z \in D$ and hence $\nabla(b^n g) = b^n \nabla g + g\nabla b^n$. We assume first that g is without zeros in D. Then there exists a function h, also analytic in D with $g = h^2$. We can then make an estimate:

$$\left|\frac{1}{\pi}\iint_D \nabla f.(b^n \nabla g) r \log\frac{1}{r}\,dr d\theta\right| \le \frac{1}{\pi}\iint_D |b^n||\nabla f||\nabla g| r \log\frac{1}{r}\,dr d\theta$$

$$= \frac{\sqrt{2}}{\pi}\iint_D |b^n||\nabla f||g'| r\log\frac{1}{r}\,dr d\theta,$$

$$\text{since } |\nabla g|^2 = 2|g'|^2$$

$$= \frac{\sqrt{2}}{\pi}\iint_D |b^n||\nabla f||g|^{\frac{1}{2}}|g|^{-\frac{1}{2}}|g'| r\log\frac{1}{r}\,dr d\theta$$

$$\le \sqrt{2}\left(\frac{1}{\pi}\iint_D |b^n|^2|\nabla f|^2|g| r\log\frac{1}{r}\,dr d\theta\right)^{\frac{1}{2}}$$

$$\times\left(\frac{1}{\pi}\iint_D |g|^{-1}|g'|^2 r\log\frac{1}{r}\,dr d\theta\right)^{\frac{1}{2}}.$$

For the second factor we have

$$\frac{1}{\pi}\iint_D |g|^{-1}|g'|^2 r\log\frac{1}{r}\,dr d\theta = \frac{4}{\pi}\iint_D |h'|^2 r\log\frac{1}{r}\,dr d\theta,$$

$$\text{since } |g'|^2 = 4|g||h'|^2$$

$$= \frac{2}{\pi}\iint_D |\nabla h|^2 r\log\frac{1}{r}\,dr d\theta$$

$$= \frac{4}{\pi}\int_T |h - h_T|^2 dt \quad \text{by Lemma 6.4}$$

$$= 8\|h-h_T\|_2^2 \le 8\|h\|_2^2 = 8\|g\|_1 \qquad \ldots(10)$$

To estimate the first factor we put

$$S_1 = \frac{1}{\pi}\iint_{G_\delta} |b^n|^2|g||\nabla f|^2 r\log\frac{1}{r}\,dr d\theta$$

$$S_2 = \frac{1}{\pi}\iint_{D\setminus G_\delta} |b^n|^2|g||\nabla f|^2 r\log\frac{1}{r}\,dr d\theta.$$

Now $\log\frac{1}{r} \le (2\log 2)(1-r)$ when $\frac{1}{2} \le r < 1$. So

$$S_1 \leq \frac{2\log 2}{\pi} \iint_{G_\delta} |b^n|^2 |h^2| |\nabla f|^2 r(1-r)\,dr\,d\theta$$

$$\leq \frac{2\log 2}{\pi} A_2 \|b^n\|_2{}^2 \|h\|_2{}^2 \varepsilon \quad \text{by Theorem 6.6}$$

$$= \frac{2\log 2}{\pi} A_2 \|h\|_2{}^2 \varepsilon \leq \frac{2A_2\varepsilon}{\pi} \|g\|_1 \qquad \ldots(11)$$

Also $S_2 \leq \frac{1}{\pi} \iint_{D\setminus G_\delta} (1-\delta)^{2n} |\nabla f|^2 |h|^2 r \log \frac{1}{r}\,dr\,d\theta.$

Since functions in L^∞ are clearly in BMO we have

$$S_2 \leq \frac{2\log 2}{\pi} (1-\delta)^{2n} \iint_D |\nabla f|^2 |h|^2 r(1-r)\,dr\,d\theta$$

$$\leq \frac{2\log 2}{\pi} (1-\delta)^{2n} A_1 \|f\|_*{}^2 A_2 \|h\|_2{}^2$$

by Theorems 6.5 and 6.6.

So

$$S_2 \leq \frac{2A_1A_2}{\pi} (1-\delta)^{2n} \|f\|_*{}^2 \|g\|_1 \qquad \ldots(12)$$

Combining (11) and (12) we have

$$\frac{1}{\pi} \iint_D |b^n|^2 |\nabla f|^2 |g| r \log \frac{1}{r}\,dr\,d\theta \leq \left(\frac{2A_2\varepsilon}{\pi} + \frac{2A_1A_2}{\pi} (1-\delta)^{2n} \|f\|_*{}^2 \right) \|g\|_1 \qquad \ldots(13)$$

Combining (10) and (13) we have

$$\left|\frac{1}{\pi} \iint_D \nabla f.(b^n \nabla g) r \log \frac{1}{r}\,dr\,d\theta\right| \leq \sqrt{2}\,(8\|g\|_1)^{\frac{1}{2}} \left(\frac{2A_2\varepsilon}{\pi} + \frac{2A_1A_2}{\pi}(1-\delta)^{2n} \|f\|_*{}^2 \right)^{\frac{1}{2}} \|g\|_1{}^{\frac{1}{2}}$$

$$= 4 \left(\frac{2A_2\varepsilon}{\pi} + \frac{2A_1A_2}{\pi}(1-\delta)^{2n} \|f\|_*{}^2 \right)^{\frac{1}{2}} \|g\|_1. \qquad \ldots(14)$$

To estimate $\frac{1}{\pi} \iint_D \nabla f.(g\nabla b^n) r \log \frac{1}{r}\,dr\,d\theta$ we set

$$S_3 = \frac{1}{\pi} \iint_{G_\delta} \nabla f.(g\nabla b^n) r \log \frac{1}{r}\,dr\,d\theta \text{ and}$$

$$S_4 = \frac{1}{\pi} \iint_{D\setminus G_\delta} \nabla f.(g\nabla b^n) r \log \frac{1}{r}\,dr\,d\theta.$$

Then $|S_3| \leq \left(\frac{1}{\pi} \iint_{G_\delta} |\nabla f|^2 |g| r \log \frac{1}{r}\,dr\,d\theta \right)^{\frac{1}{2}} \left(\frac{1}{\pi} \iint_{G_\delta} |\nabla b^n|^2 |g| r \log \frac{1}{r}\,dr\,d\theta \right)^{\frac{1}{2}}.$

By the same reasoning as we used in estimating S_1 and S_2 we obtain

$$\frac{1}{\pi}\iint_{G_\delta} |\nabla f|^2|g|\ r\log\frac{1}{r}\,drd\theta \le \frac{2A_2\varepsilon}{\pi}\|g\|_1, \qquad \text{and}$$

$$\frac{1}{\pi}\iint_{G_\delta} |\nabla b^n|^2|g| r\log\frac{1}{r}\,drd\theta \le \frac{8A_1A_2}{\pi}\|g\|_1$$

$$(\text{since } \|b^n\|_* \le 2\|b^n\|_\infty = 2).$$

Thus $|S_3| \le \left(\frac{2A_2\varepsilon}{\pi}\|g\|_1\right)^{\frac{1}{2}}\left(\frac{8A_1A_2}{\pi}\|g\|_1\right)^{\frac{1}{2}} = \frac{4A_1^{\frac{1}{2}}A_2}{\pi}\varepsilon^{\frac{1}{2}}\|g\|_1$ ...(15)

For S_4 we have

$$|S_4| \le \left(\frac{1}{\pi}\iint_{D\setminus G_\delta} |\nabla f|^2|g| r\log\frac{1}{r}\,drd\theta\right)^{\frac{1}{2}}\left(\frac{1}{\pi}\iint_{D\setminus G_\delta}|\nabla b^n|^2|g| r\log\frac{1}{r}\,drd\theta\right)^{\frac{1}{2}}$$

The same estimate as S_2 gives that

$$\frac{1}{\pi}\iint_{D\setminus G_\delta}|\nabla f|^2|g| r\log\frac{1}{r}\,drd\theta \le \frac{2A_1A_2}{\pi}\|f\|_*^2\|g\|_1 \qquad \text{and}$$

$$\frac{1}{\pi}\iint_{D\setminus G_\delta}|\nabla b^n|^2|g| r\log\frac{1}{r}\,drd\theta \le n^2(1-\delta)^{2n-2}\iint_D|\nabla b|^2|g| r\log\frac{1}{r}\,drd\theta$$

$$(\text{since } |\nabla b^n|^2 = n^2|b^{2(n-1)}||\nabla b|^2)$$

$$\le n^2(1-\delta)^{2(n-1)}\frac{8A_1A_2}{\pi}\|g\|_1.$$

Hence we have

$$|S_4| \le \left(\frac{2A_1A_2}{\pi}\|f\|_*^2\|g\|_1\right)^{\frac{1}{2}}\left(n^2(1-\delta)^{2(n-1)}\frac{8A_1A_2}{\pi}\|g\|_1\right)^{\frac{1}{2}}$$

$$= n(1-\delta)^{n-1}\frac{4A_1A_2}{\pi}\|f\|_*\|g\|_1. \qquad ...(16)$$

Combining (15) and (16) we obtain

$$\left|\frac{1}{\pi}\iint_D \nabla f.(g\nabla b^n) r\log\frac{1}{r}\,drd\theta\right| \le \frac{4A_1^{\frac{1}{2}}A_2}{\pi}\varepsilon^{\frac{1}{2}}\|g\|_1 + 4n(1-\delta)^{n-1}\frac{A_1A_2}{\pi}\|f\|_*\|g\|_1$$

$$= \frac{4A_1^{\frac{1}{2}}A_2}{\pi}\left[\varepsilon^{\frac{1}{2}} + n(1-\delta)^{n-1}A_1^{\frac{1}{2}}\|f\|_*\right]\|g\|_1.$$

...(17)

So from (14) and (17) we have

$$\left|\frac{1}{\pi}\iint_D \nabla f.\nabla(b^n g) r\log\frac{1}{r}\,drd\theta\right| \le 4\left[\frac{2A_2\varepsilon}{\pi} + \frac{2A_1A_2}{\pi}(1-\delta)^{2n}\|f\|_*^2\right]^{\frac{1}{2}}\|g\|_1$$

$$+ \frac{4A_1^{\frac{1}{2}}A_2}{\pi}\left[\varepsilon^{\frac{1}{2}} + n(1-\delta)^{n-1}A_1^{\frac{1}{2}}\|f\|_*\right]\|g\|_1.$$

Letting $n \to \infty$, we have proved, under the assumption that g has no zeros in D, that

$$\left|\frac{1}{\pi}\iint_D \nabla f.(\nabla b^n g)r \log\frac{1}{r}\,drd\theta\right| \le \frac{C_{13}}{3}\varepsilon^{\frac{1}{2}}\|g\|_1 \qquad \ldots(18)$$

when n is sufficiently large and C_{13} is a suitable constant.

For the general case, let v be the Blaschke factor of g and let $w = \frac{g}{v}$ so that $g = w + w(v-1)$. Since w and $w(v-1)$ are both functions in H^∞ without zeros in D we can apply (18) to the functions w and $w(v-1)$ to obtain $\left|\frac{1}{\pi}\iint_D \nabla f.(\nabla b^n w)r \log\frac{1}{r}\,drd\theta\right| \le \frac{C_{13}}{3}\varepsilon^{\frac{1}{2}}\|w\|_1 = \frac{C_{13}}{3}\varepsilon^{\frac{1}{2}}\|g\|_1$ and $\left|\frac{1}{\pi}\iint_D \nabla f.(\nabla b^n w(v-1))r \log\frac{1}{r}\,drd\theta\right| \le \frac{2C_{13}}{3}\varepsilon^{\frac{1}{2}}\|g\|_1$, when n is sufficiently large. Since $\nabla(b^n g) = \nabla(b^n w) + \nabla(b^n w(v-1))$ we obtain the desired inequality $\left|\frac{1}{\pi}\iint_D \nabla f.\nabla(b^n g)r \log\frac{1}{r}\,drd\theta\right| \le C_{13}\varepsilon^{\frac{1}{2}}\|g\|_1$ for n sufficiently large, and hence conclude the proof.

We can now give our characterisation of the bounded functions in VMO(b) in terms of Carleson measures.

<u>THEOREM 7.7</u> *Let* $f \in L^\infty$. *Then* $f \in \mathrm{VMO}(b)$ *if and only if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that the measure* μ_δ *defined by* $d\mu_\delta = \chi_{G_\delta}(1-r)|\nabla f|^2 r\,drd\theta$ *is a Carleson measure with* $\mu_\delta(R(I)) \le \varepsilon|I|$ *for all subarcs* I *of* T.

Proof. Denote the functions in L^∞ satisfying the property described by the second part of the statement of the theorem by $L(b)$. Suppose f is in VMO(b). Then by Lemma 7.5 it follows that there exists some $\delta > 0$ such that the measure μ_δ has the required properties and so $f \in L(b)$. On the other hand if f is in $L(b)$ then from Lemma 7.6 it follows that $f \in [H^\infty,\bar{b}]$. Similarly $\bar{f} \in [H^\infty,\bar{b}]$. Hence $L(b) \subseteq [H^\infty,\bar{b}] \cap \overline{[H^\infty,\bar{b}]}$. If f is unimodular in $[H^\infty,\bar{b}] \cap \overline{[H^\infty,\bar{b}]}$ then by Lemma 7.1 and Theorem 7.2 $f \in VMO(b)$. Since $[H^\infty,\bar{b}] \cap \overline{[H^\infty,\bar{b}]}$ is a C*-algebra spanned by its unimodular functions and $VMO(b) \cap L^\infty$ is a linear space it follows that $[H^\infty,\bar{b}] \cap \overline{[H^\infty,\bar{b}]} \subseteq VMO(b)$ and the result follows.

It is clear that we can generalise the concept of $VMO(b)$ and the results concerning this space where we are concerned with the single inner function b to a concept which involves an arbitrary collection of inner functions b_λ, for λ in some index set E.

Notation. For each finite subset F of the index set E, let b_F be the inner function $\prod_{\lambda \in F} b_\lambda$ and for $\delta > 0$ put
$G_\delta(F) = \{z \in D: \ |b_F(z)| \geq 1-\delta\}$.

Definition. Let $\{b_\lambda: \ \lambda \in E\}$ be a collection of inner functions indexed by the set E. If f is a function in BMO we say that $f \in VMO(b_\lambda: \ \lambda \in E)$ if for every $\varepsilon > 0$ and $\eta \geq 1$, there exists some $\delta > 0$ and some finite non-empty subset F of E such that for $r_0 e^{i\theta_0} \in G_\delta(F)$ and $1 \leq \psi \leq \eta$ we have

$$I = \{e^{it}: \ \frac{|\theta_0 - t|}{1-r_0} \leq \psi\} \Rightarrow \frac{1}{|I|} \int_I |f(e^{it}) - f_I|\,dt < \varepsilon.$$

We have the following results parallel to Theorem 7.2, Lemmas 7.3, 7.5, 7.6 and Theorem 7.7.

THEOREM 7.8 *Let* f *be a unimodular function in* L^∞. *Then*

$f \in VMO(b_\lambda: \lambda \in E) \Leftrightarrow \forall\varepsilon > 0, \exists\delta > 0$ *and some finite non-empty subset* F *of* E *such that* $z \in G_\delta(F) \Rightarrow |f(z)| \geq 1-\varepsilon^3$.

<u>LEMMA 7.9</u> $L^\infty \cap VMO(b_\lambda: \lambda \in E)$ *is a C*-algebra. Also let* C_B *be the C*-algebra in* L^∞ *generated by the inner functions which are invertible in* $[H^\infty, \bar{b}_\lambda: \lambda \in E]$. *Then* $C_B \subseteq VMO(b_\lambda: \lambda \in E)$.

<u>LEMMA 7.10</u> *Suppose* $f \in VMO(b_\lambda: \lambda \in E)$ *and let* $0 < \varepsilon < 1$. *Then there exists* $\delta > 0$ *and some finite non-empty subset* F *of* E *such that the measure* $\mu_\delta(F)$ *on* D *defined by* $d\mu_\delta(F) = \chi_{G_\delta(F)}(1-r)|\nabla f|^2 r\,dr\,d\theta$ *is a Carleson measure with* $\mu_\delta(F)(R(I)) \leq C_{14}\varepsilon|I|$ *for all subarcs* I *of* T. *(*C_{14} *is a constant independent of* ε*).*

<u>LEMMA 7.11</u> *Let* $f \in L^\infty$. *Suppose that for every* $\varepsilon > 0$ *there exists some* $\delta > 0$ *and some finite subset* F *of* E *such that the measure* $\mu_\delta(F)$ *on* D *defined by* $d\mu_\delta(F) = \chi_{G_\delta(F)}(1-r)|\nabla f|^2 r\,dr\,d\theta$ *is a Carleson measure with* $\mu_\delta(F)(R(I)) \leq \varepsilon|I|$ *for all subarcs* I *of* T. *Then for every* $\varepsilon > 0$, *there is an absolute constant* C_{15} *such that* $d(fb_F{}^n, H^\infty) \leq C_{15}\varepsilon^{\frac{1}{2}}$ *for* n *sufficiently large.*

<u>THEOREM 7.12</u> *Let* $f \in L^\infty$. *Then* $f \in VMO(b_\lambda: \lambda \in E)$ *if and only if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *and some non-empty finite subset* F *of* E *so that the measure* $\mu_\delta(F)$ *defined by* $d\mu_\delta(F) = \chi_{G_\delta(F)}(1-r)|\nabla f|^2 r\,dr\,d\theta$ *is a Carleson measure with* $\mu_\delta(F)(R(I)) \leq \varepsilon|I|$ *for all subarcs* I *of* T.

The proofs ofthese results are more or less identical with the proofs of the corresponding results given earlier with only minor alterations needed.

We conclude this chapter with a problem: in the definition of VMO(b) can we restrict attention to merely those arcs where $\psi = 1$? If not what sort of function provides a counter-example?

CHAPTER EIGHT

In this chapter we describe Marshall's construction of the interpolating Blaschke products which we shall need to prove Theorem 6.8.

There are two differences in our approach to the construction:

(a) we describe the construction on D rather than on the upper half-plane, and

(b) by using an argument due to A.M. Davie we avoid the use of harmonic measures in the construction.

Let u be a unimodular function in L^{∞} and let $A = [H^{\infty}, u, \bar{u}]$. By the remarks made in Chapter 6 it is sufficient to prove Theorem 6.8 when A is of this form and so we restrict our attention to this situation. For each α, $0 < \alpha < 1$, we wish to construct an interpolating Blaschke product $\underset{\sim}{B}_{\alpha}$ so that

(1) $\sup|u(z)| < 1$ where the supremum is taken over the zeros of $\underset{\sim}{B}_{\alpha}$;

(2) $|u(z)| < \alpha \Rightarrow |\underset{\sim}{B}_{\alpha}(z)| \leq \frac{1}{10}$.

The idea of the construction will be to surround the places where $|u| < \alpha$ by a contour Γ which is not 'too long'. That the contour is not 'too long' will mean that the arclength measure it induces is a Carleson measure. This construction is derived from the proof of the Corona theorem due to L. Carleson [32]. We then uniformly distribute, in the ρ-metric, a sequence $\{z_n\}$ on the contour, sufficiently separating the points of the sequence so that the remark on p.50 will tell us that the Blaschke product with $\{z_n\}$ as its zero set is interpolating. We will require for the proof of Theorem 6.8 that each $\underset{\sim}{B}_{\alpha}$ constructed is invertible in A. We obtain this as a consequence of Theorem 6.1 using (1).

First we need some technical lemmas prior to describing the construction. Throughout this chapter we take the liberty of using u to denote both the function in L^∞ and its harmonic extension to D.

LEMMA 8.1 *There exists an* $\alpha_1 < 1$ *such that if* $|u(a)| < \alpha$ *for* a *in some region of the form* $Q = \{re^{i\theta}: 1-2^{-n} \leq r < 1-2^{-n-2}, \theta_0 \leq \theta \leq \theta_1\}$ *then* $\sup_Q |u(z)| < \alpha_1$.

Proof. Let $\mathfrak{N}$ be the set of functions of modulus 1 a.e. on T. We shall think of functions in $\mathfrak{N}$ as extended harmonically to D. Suppose that for every $\alpha_1 < 1$, there exists a region Q of the type given in the statement of the lemma and there exists $f \in \mathfrak{N}$ such that $|f(a)| < \alpha$ for some $a \in Q$ but $\sup_Q |f(z)| \geq \alpha_1$. Then there exists a sequence $\{f_n\}_{n=1}^\infty$ of functions in $\mathfrak{N}$, a sequence of regions $\{Q_n\}$ of the given type and a set of points $\{a_n\}$ where $a_n \in Q_n$ $(n \geq 1)$ such that $|f_n(a_n)| < \alpha$ and $\sup_{Q_n} |f_n(z)| \geq 1 - \frac{1}{n}$. By a translation and dilation we may assume that each region is of type given in the statement of the lemma with $n = 0$; call this region Q. The sequence $\{f_n\}$ forms a normal family. Thus there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}$ and a function f, harmonic on D such that $f_{n_k} \to f$ uniformly on compact subsets of D and $a_{n_k} \to b$, say, where $b \in Q$. We then have $|f(b)| \leq \alpha < 1$ and $\sup_Q |f(z)| = 1$. But $\sup_{\bar{D}} |f(z)| = 1$ and so f is a harmonic function on D which attains its supremum inside D. Hence f is the constant function 1 which contradicts the fact that $|f(b)| < 1$. This contradiction proves the lemma.

Let $S(\theta_0, \delta) = \{re^{i\theta}: 1-\delta \leq r < 1, \theta_0 - \frac{\delta\pi}{2} \leq \theta \leq \theta_0 + \frac{\delta\pi}{2}\}$, where $\delta < 1$. Let $\nu = 1 + \frac{\alpha_1 + 5}{2\delta}$ where α_1 is obtained from Lemma 8.1, and define β by $1 - \beta = \left(\frac{1-\alpha_1}{1+\nu^2\pi^2}\right) \frac{1}{16\pi B_0}$ where B_0 is the constant appearing in Theorem 6.2 (1) which can be assumed greater than 1. Notice that $1 > \beta > \alpha_1$.

For a set $F \subseteq D$ let $F^* \subseteq T$ denote the projection of F onto T, i.e. $F^* = \{e^{it}: re^{it} \in F \text{ for some } r, 0 < r < 1\}$.

The proof of the following lemma is due to A.M. Davie.

<u>LEMMA 8.2</u> *Suppose* $|u(a)| > \beta$ *where* $a = pe^{iq}$ *is such that* $1-\delta \leq p \leq 1-\delta/2$ *and* $\theta_0 - \frac{\delta\pi}{2} \leq q \leq \theta_0 + \frac{\delta\pi}{2}$ *for some* δ, $0 < \delta < 1$ *and* θ_0, $0 \leq \theta_0 < 2\pi$. *Let* $E = \{re^{i\theta} \in S(\theta_0,\delta): |u(re^{i\theta})| < \alpha_1\}$. *Then* $|E^*| \leq \delta/2$.

Proof. Without loss of generality we may assume that $u(a)$ is real and positive. Let $g = 1 - \text{Re}u$ so that $g \geq 0$. Let $I = \{e^{it}: q-\nu\delta\pi \leq t \leq q+\nu\delta\pi\}$ and let $h = g\chi_I$ on T, (and extend into D using the Poisson integral). From the definition of ν,

$$|g(re^{i\theta}) - h(re^{i\theta})| \leq \frac{1-\alpha_1}{2} \quad \text{for} \quad re^{i\theta} \in S(\theta_0,\delta) \qquad \ldots(3)$$

Moreover $\frac{1}{2\pi}\int_I P_p(q-t)g(e^{it})dt \leq g(pe^{iq}) = 1-u(a) < 1-\beta$.

Now $p \leq 1-\delta/2$ and $P_p(q-t) = \dfrac{1-p^2}{(1-p)^2+4p\sin^2\frac{(q-t)}{2}} \geq \dfrac{1}{2\delta(1+\nu^2\pi^2)}$

for $e^{it} \in I$,

$$\text{and so} \quad \int_I g(e^{it})dt \leq 4\delta\pi(1+\nu^2\pi^2)(1-\beta) = \frac{(1-\alpha_1)}{4B_0}\delta \qquad \ldots(4)$$

Now, on E, $|u| < \alpha_1$ and so $g \geq 1-\alpha_1$ which implies $h \geq \frac{1-\alpha_1}{2}$ by (3). So, on E^*, $M(h) \geq 1 - \frac{\alpha_1}{2}$. Then, by the Hardy-Littlewood maximal theorem (Theorem 6.2),

$$|E^*| \leq |\{e^{it}: M(h)(e^{it}) \geq \frac{1-\alpha_1}{2}\}| \leq \frac{2B_0}{1-\alpha_1}\int_T h(e^{it})dt$$

$$= \frac{2B_0}{1-\alpha_1}\int_I g(e^{it})dt$$

$$\leq \frac{2B_0}{1-\alpha_1} \frac{(1-\alpha_1)}{4B_0}\delta = \delta/2 \text{ by } (4)$$

We now turn to the construction of the contour Γ. First we introduce some notation:

$$S_k^n = \{z = re^{i\theta}: \; 2^{-n-1} < 1-r \leq 2^{-n}, \; \frac{2k\pi}{2^{n+1}} \leq \theta < \frac{2(k+1)\pi}{2^{n+1}}\}$$

for $n = 0,1,\ldots;\; k = 0,1,\ldots,2^{n+1}-1$.

(see Figure 1)

For a region $S = \{re^{i\theta}: \; r_0 \leq r < r_1, \; \theta_0 \leq \theta < \theta_1\}$ let T_S be given by $T_S = \{re^{i\theta}: \; r_0 \leq r < r_1 - \frac{1}{2}(r_1 - r_0), \; \theta_0 \leq \theta < \theta_1\}$.

We describe two procedures which we apply to the regions of the form $S = S_k^n \cup \{z = re^{i\theta}: \; 0 < 1-r \leq 2^{-n-1}, \; \frac{2k\pi}{2^{n+1}} \leq \theta < \frac{2(k+1)\pi}{2^{n+1}}\}$. Denote this class of regions by $\mathcal{F}$.

<u>Case I</u>. If $\sup_{T_S}|u| > \beta$, shade the regions $\tilde{S} \in \mathcal{F}$ contained in S where $|u(z)| < \alpha$ for some z in $T_{\tilde{S}}$. Note that by Lemma 8.1, each $\tilde{S}$ will be in the 'radial quarter' of S 'nearest' T, and $\sup_{T_{\tilde{S}}}|u| < \alpha_1 < \beta$. By Lemma 8.2 $\sum_{\tilde{S} \text{ shaded}} |\tilde{S}^*| \leq \frac{1}{2}|S^*|$ ...(5)

<u>Case II</u>. If $\sup_{T_S}|u| \leq \beta$ shade the regions $\tilde{S} \in \mathcal{F}$ in S where $\sup_{T_{\tilde{S}}}|u| > \beta$, and let $R_S = S \setminus \bigcup_{\tilde{S} \text{ shaded}} \tilde{S}$. Note that

$|\partial R_S| \leq \frac{4+2\pi}{\pi}|S^*|$ ...(6)

We now proceed as follows: consider the two 'halves' of the disc, $\{z = re^{i\theta}: \; 0 \leq r < 1, \; 0 \leq \theta < \pi\}$ and $\{z = re^{i\theta}: \; 0 \leq r < 1, \; \pi \leq \theta < 2\pi\}$, separately. Apply the appropriate case to the top half of the disc first, obtaining shaded regions $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, \ldots$. On each $P_j^{(1)}$ apply the appropriate case obtaining doubly shaded regions $P_1^{(2)}, P_2^{(2)}, P_3^{(2)}, \ldots$. Repeat this process indefinitely. Observe that we alternate cases in passing from one shaded region to a shaded descendant. Carry out the same procedure for the bottom half of the disc and define Γ to be the union of all the boundaries of the R_S's obtained from applications of Case II in both halves of the disc; (see Figure 2). To see that Γ induces a Carleson measure, it suffices to check that $|\Gamma \cap S| \leq C|S^*|$ where C is some constant and S is a region in $\mathcal{F}$. By (5) and (6) we see that

$$|\Gamma \cap S| \leq \sum_{n=0}^{\infty} \frac{(4+2\pi)}{\pi} 2^{-n}|S^*| < 8|S^*|.$$

Note that any point in D for which $|u(z)| < \alpha$ will be in some R_S. Also $|\partial R_S \cap T| = 0$. This follows since u has unimodular radial limits a.e. and any point in $\partial R_S \cap T$ is a point where $\limsup_{r\to 1}|u(re^{i\theta})| \leq \beta < 1$.

We now consider the construction of the Blaschke product $\underset{\sim}{B}_\alpha$ whose zeros are located on $\Gamma \subseteq D$. Choose $\gamma < \frac{1}{10}$ and place points a_n $(n \geq 1)$ on Γ so that $\gamma \leq \rho(a_n, a_{n+1}) < 2\gamma$ where a_n and a_{n+1} are adjacent points on Γ and so that $\rho(a_n, a_m) \geq \gamma$ for $m \neq n$. The proof of the following lemma is due to S. Ziskind [55].

<u>LEMMA 8.3</u> $\{a_n\}$ *is an interpolating sequence in* D.

Proof. Since we have explicitly made $\rho(a_n, a_m) \geq \gamma$ for $m \neq n$, by the remark on p.50 we need only show that the measure $\mu = \sum(1-|z_j|)\delta_{z_j}$ is a Carleson measure. Since Γ is composed of various edges of the regions $S \in \mathcal{F}$ and since Γ induces a Carleson measure, we need only

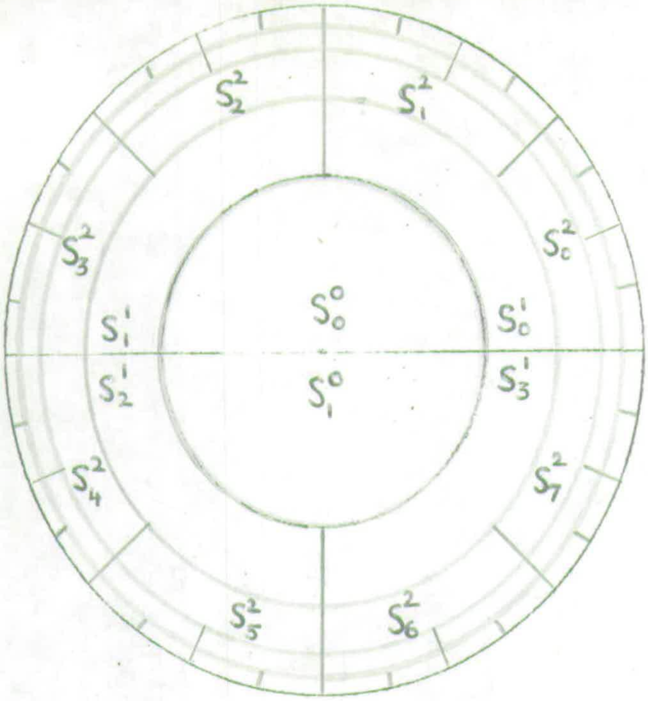
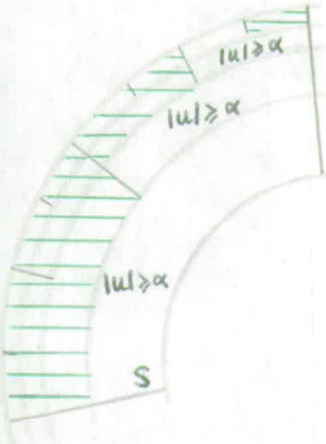
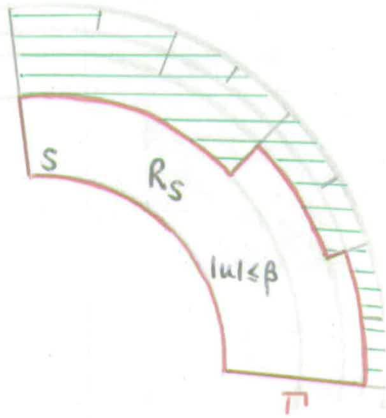


Figure 1



Case I



Case II

Figure 2

show that, whenever Λ is an edge of a region $S \in \mathcal{F}$ and $z_1, \ldots, z_k$ $(z_j = r_j e^{i\theta_j})$ are points on Λ for which the adjacent ρ-distance exceeds γ, then $\sum_{j=1}^{k} (1-|z_j|) \leq C|\Lambda|$ where the constant C may depend on γ. Consider first the case where Λ has fixed distance from the origin. Here $r_j = R$ is fixed $(1 \leq j \leq k)$ and $\theta_1 < \theta_2 < \ldots < \theta_k$, say. We then have

$$\gamma \leq \rho(z_j, z_{j+1}) = \left|\frac{z_{j+1}-z_j}{1-\bar{z}_{j+1}z_j}\right| \leq \frac{(\theta_{j+1}-\theta_j)}{1-R} .$$

Thus $\sum_{j=1}^{k} (1-r_j) \leq \sum_{j=1}^{k} \frac{1}{\gamma} (\theta_{j+1}-\theta_j) \leq \frac{1}{\gamma} |\Lambda|$.

In the case when Λ has fixed argument we have $\theta_j = \theta$ is fixed $(1 \leq j \leq k)$ and $r_1 < r_2 < \ldots < r_k$, say. Then

$$\gamma \leq \rho(z_j, z_{j+1}) = \frac{r_{j+1}-r_j}{1-r_{j+1}r_j} \leq \frac{r_{j+1}-r_j}{1-r_j}$$

so that $1-r_j \leq \frac{1}{\gamma} (r_{j+1}-r_j)$, giving $\sum_{j=1}^{k} (1-r_j) \leq \frac{1}{\gamma} |\Lambda|$.

We now wish to verify that (1) and (2) (see p.69) hold for the Blaschke product B_{α} whose zero sequence is $\{a_n\}$. By our construction (1) holds since $|u(z)| \leq \beta$ on Γ. If $z \in \bar{D}$ and $|u(z)| < \alpha$, then z is in some R_S. But $|B_{\alpha}| < \gamma$ on $\partial R_S \setminus T$ and $\partial R_S \cap T$ has harmonic measure zero as a subset of ∂R_S, since it has length zero. We conclude from Theorem 6.9 that $|B_{\alpha}| \leq \gamma < \frac{1}{10}$ on R_S and so (2) holds.

CHAPTER NINE

We are now in a position to give our proof of the Marshall-Chang theorem (Theorem 6.8) described in Chapter 6.

As noted in Chapter 6 it is sufficient to prove Theorem 6.8 for the case $A = [H^\infty, u, \bar{u}]$ where u is a unimodular function in L^∞. So suppose u is a unimodular function in L^∞. As described in Chapter 8 we construct Blaschke products $\underset{\sim}{B}_\alpha$ for each $\alpha \in (0,1)$ with the properties that for each α

(1) $\sup|u(z)| \leq \beta < 1$ where the supremum is taken over the zeros of $\underset{\sim}{B}_\alpha$, and

(2) $|u(z)| < \alpha \Rightarrow |\underset{\sim}{B}_\alpha(z)| \leq \frac{1}{10}$.

LEMMA 9.1 *For each* $\alpha \in (0,1)$, $\underset{\sim}{B}_\alpha$ *is invertible in* $A = [H^\infty, u, \bar{u}]$.

Proof. Suppose $\phi \in \Phi(A)$ and $\phi(\underset{\sim}{B}_\alpha) = 0$ for some α. By Theorem 6.1 ϕ is in the closure of the zeros $\{a_n\}$ of $\underset{\sim}{B}_\alpha$ in $\Phi(H^\infty)$. By (1) above $|u(a_n)| \leq \beta < 1$ for each $n \geq 1$ so that $|\phi(u)| \leq \beta < 1$. This contradicts the fact that $\phi \in \Phi(A)$. Thus each $\underset{\sim}{B}_\alpha$ is invertible in A.

Lemma 9.1 shows that $[H^\infty, \bar{\underset{\sim}{B}}_\alpha : 0 < \alpha < 1] \subseteq A$ since $\bar{\underset{\sim}{B}}_\alpha$ is the inverse of $\underset{\sim}{B}_\alpha$. To obtain the opposite inclusion and thus prove Theorem 6.8 (since from Chapter 8 each $\underset{\sim}{B}_\alpha$ is interpolating) we need only show that we can approximate u and $\bar{u}$ as close as we like in the uniform norm by functions from $[H^\infty, \bar{\underset{\sim}{B}}_\alpha : 0 < \alpha < 1]$. First note that with $0 < \varepsilon < 1$, and $1 > 1-\delta > \frac{1}{10}$ we have, by (2),

$$z \in G_\delta(\underset{\sim}{B}_{1-\varepsilon^3}) \Rightarrow |u(z)| \geq 1-\varepsilon^3.$$

So using Theorem 7.8 we deduce that $u \in VMO(\underset{\sim}{B}_\alpha: \ 0 < \alpha < 1)$. By Lemma 7.9 $\bar{u} \in VMO(\underset{\sim}{B}_\alpha: \ 0 < \alpha < 1)$ also. Then by combining Lemmas 7.10 and 7.11 we deduce that both u and $\bar{u}$ can be approximated as close as we like by functions from $[H^\infty, \bar{\underset{\sim}{B}}_\alpha: \ 0 < \alpha < 1]$. Hence u and $\bar{u}$ belong to $[H^\infty, \bar{\underset{\sim}{B}}_\alpha: \ 0 < \alpha < 1]$ and so $A \subseteq [H^\infty, \bar{\underset{\sim}{B}}_\alpha: \ 0 < \alpha < 1]$. So $A = [H^\infty, \bar{\underset{\sim}{B}}_\alpha: \ 0 < \alpha < 1]$ and Theorem 6.8 is proved.

We conclude this chapter by describing some recent results of S-Y.A. Chang concerning the structure of closed subalgebras of L^∞ containing H^∞. If A is a closed subalgebra of L^∞ containing H^∞ properly let C_A be the C*-algebra generated by inner functions invertible in A. Then in [35] Chang has shown that the linear space $H^\infty + C_A$ is a closed algebra which is equal to A. Thus she has shown that any closed subalgebra of L^∞ containing H^∞ properly is of the form H^∞ + some C*-algebra.

CHAPTER TEN

In the last four chapters we have been concerned with uniform algebras of functions on the unit sphere in $\mathbb{C}$ i.e. T. We now turn our attention to algebras of functions on the unit sphere in higher dimensions. In this chapter we consider the possibility of extending the idea of Douglas algebras into higher dimensions.

Notation. $\mathbb{C}^n$ denotes the n-dimensional complex Euclidean space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers z_i, with the inner product $\langle z,w \rangle = z_1\bar{w}_1 + \ldots + z_n\bar{w}_n$ and the corresponding norm $\|z\| = \langle z,z \rangle^{\frac{1}{2}}$. Let B denote the open unit ball $\{z \in \mathbb{C}^n: \|z\| < 1\}$ and S the unit sphere $\{z \in \mathbb{C}^n: \|z\| = 1\}$. From now on we will assume that $n > 1$ unless otherwise stated. σ denotes surface area measure on S. We write $L^\infty(S)$ for $L^\infty(\sigma)$ and $L^2(S)$ for $L^2(\sigma)$. $H^2(S)$ denotes the closure in $L^2(S)$ of the polynomials in the coordinate functions $z_1, \ldots, z_n$. $L^2(S)$ and $H^2(S)$ are Hilbert spaces and we also use angled brackets $\langle\ ,\ \rangle$ to denote the inner product in these spaces. We write C(S) for the algebra of all continuous functions on S.

The Poisson kernel is given by $P(u,z) = \dfrac{(1-|z|^2)^n}{|1-\langle z,u\rangle|^{2n}}$ $(z \in B,\ u \in S)$.

As in the case $n = 1$, if $f \in L^\infty(S)$ then the Poisson integral of f gives a bounded harmonic function F on B, and F has radial boundary limits equal to f a.e.. F is given by

$$F(z) = \frac{1}{2\pi^2}\int_S \frac{(1-|z|^2)^n}{|1-\langle z,u\rangle|^{2n}} f(u)\,d\sigma(u), \qquad (z \in B).$$

This correspondence gives an isometry between $L^\infty(S)$ and the space of bounded harmonic functions on B with the supremum norm. Under

this correspondence the algebra of bounded analytic functions on B corresponds to the closed subalgebra $H^{\infty}(S)$ of $L^{\infty}(S)$.
We denote by $H^{\infty}(S) + C(S)$ the set of all functions $f \in L^{\infty}(S)$ which can be expressed in the form $f = u + v$ where $u \in H^{\infty}(S)$ and $v \in C(S)$. W. Rudin [50] recently showed that $H^{\infty}(S) + C(S)$ is a closed subalgebra of $L^{\infty}(S)$.

Definition. A function $\phi \in L^{\infty}(S)$ is inner if $\phi \in H^{\infty}(S)$ and $|\phi| = 1$ a.e. on S.

It is not known whether any non-constant inner functions exist when $n > 1$. For a discussion of this and similar problems see L.A. Rubel and A.L. Shields [49]. In Chapter 6, for the case $n = 1$, we defined Douglas algebras in terms of inner functions. So in our attempt to extend this definition to algebras of functions on S we are immediately faced with this problem concerning the existence of inner functions. However one way of extending the definition is as follows.

Definition. Let A be a (uniformly) closed subalgebra of $L^{\infty}(S)$ which contains $H^{\infty}(S)$ properly. We say that A is a Douglas algebra if A is equal to the closed subalgebra of $L^{\infty}(S)$ generated by $H^{\infty}(S)$ and the inverses of those functions in H^{∞} which are invertible in A, (i.e. in our previous notation, if $A = [H^{\infty}, b^{-1} \in A: \ b \in H^{\infty}]$).

Because of the inner-outer factorization of functions in H^{∞} in the case $n = 1$ this definition applied to that case is equivalent to the definition of a Douglas algebra given in Chapter 6 (for with

$n = 1$, if a function in H^∞ is invertible in L^∞, then its outer factor is invertible in H^∞ and so the function itself is invertible in A if and only if its inner factor is).

However it is soon evident that we cannot hope to prove that every closed subalgebra of $L^\infty(S)$ containing $H^\infty(S)$ is a Douglas algebra in the sense of this definition - in fact, not even $H^\infty(S) + C(S)$ is a Douglas algebra in the sense given, as we now show.

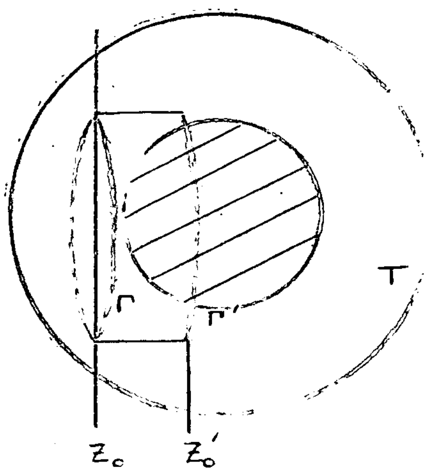
PROPOSITION 10.1 *Let* $h \in H^\infty(S)$ *be invertible in* $H^\infty(S) + C(S)$. *Then* h *is invertible in* $H^\infty(S)$.

Proof. By examining the form of the Poisson kernel it is clear that as $|z| \to 1$ the 'mass' of $P(u,z)$ inside a small neighbourhood V of w on S (where $z = rw$ for some r, $0 < r < 1$) tends to 1, i.e. $\frac{1}{2\pi^2} \int_V P(u,rw)d\sigma(u) \to 1$ as $r \to 1$.
(Compare the beginning of the proof of Theorem 7.2). From this it is clear that if $f \in C(S)$ and $g \in L^\infty(S)$ then $\|f_r g_r - (fg)_r\|_\infty \to 0$ as $r \to 1$ where $f_r(u) = f(ru)$, $g_r(u) = g(ru)$ for $0 < r \leq 1$ and $u \in S$ and f, g are considered as being extended to B via the Poisson integral. An immediate consequence of this is that if

$f, g \in H^\infty(S) + C(S)$ then $\|f_r g_r - (fg)_r\|_\infty \to 0$ as $r \to 1$. ...(1)

Now take $h \in H^\infty(S)$ with $h^{-1} \in H^\infty(S) + C(S)$. By putting $f = h$ and $g = h^{-1}$ in (1) it follows that $|h(z)| \geq \delta > 0$ for all z in a shell of the ball, T, near the sphere, i.e. $T = \{z: r_0 < |z| < 1\}$ for some $r_0 > 0$. Thus $\frac{1}{h}$ is analytic in T. A theorem of Hartogs (see Hormander [45]) tells us that given Ω, open in $\mathbb{C}^n$ where $n > 1$, and K, a compact subset of Ω such that $\Omega \setminus K$ is connected, then for every u analytic in $\Omega \setminus K$ we can find U analytic in Ω such that $u = U$ on $\Omega \setminus K$. We apply this result with $\Omega = B$, $K = B \setminus T$ and $u = \frac{1}{h}$ to obtain a bounded analytic function on B whose radial limits give a function which is the inverse of h, i.e. h is invertible in H^{∞}.

In fact in the case of a shell it is easy to see how to construct the analytic function U given by Hartogs' theorem. For if u is analytic in T then define U by $U(z_0,w) = \frac{1}{2\pi i}\int_{\Gamma} \frac{u(z_0,\eta)}{w-\eta}\,d\eta$ for $(z_0,w) \in B$ where Γ is described in the figure.



Proposition 10.1 implies that

$[H^{\infty}(S), b^{-1} \in H^{\infty}(S) + C(S) : b \in H^{\infty}(S)] = H^{\infty}(S)$. It also shows that a non-constant inner function (if one exists) cannot be invertible in $H^{\infty}(S) + C(S)$. This contrasts with the case when $n = 1$ where the function $f(z) = z$ is inner and $f^{-1} \in C$. This leads us to make two conjectures which we have been unable to prove:

(1) if $f \in H^{\infty}(S)$ and f is invertible in $L^{\infty}(S)$ then f is invertible in $H^{\infty}(S)$;

(2) if $f \in H^{\infty}(S) + C(S)$ and f is invertible in $L^{\infty}(S)$ then f is invertible in $H^{\infty}(S) + C(S)$.

Proposition 10.1 shows that (1) follows from (2). (1) implies that any inner function is constant. For a discussion of questions of this type see [49], [38].

There remains one further alternative method of defining a Douglas algebra in higher dimensions. Suppose we say that if A is a closed subalgebra of $L^\infty(S)$ containing $H^\infty(S)$ properly then A is a Douglas subalgebra if A is generated as a closed algebra by $H^\infty(S)$ and those complex conjugates of functions in $H^\infty(S)$ which are in A, i.e. if $A = [H^\infty(S), \bar{b} \in A: b \in H^\infty(S)]$. For the case $n = 1$ this is equivalent to our two previous definitions. For the case $n > 1$ $H^\infty(S) + C(S)$ is now a Douglas algebra in this sense. This is because $H^\infty(S) + C(S)$ is generated by $H^\infty(S)$ and the complex conjugates of the coordinate functions. However we conjecture that L^∞ is not a Douglas algebra in this sense, i.e. $L^\infty(S)$ is not generated as a C*-algebra by $H^\infty(S)$. Despite our intuitive feeling of the truth of this conjecture it may still be of interest to decide which subalgebras of $L^\infty(S)$ are Douglas algebras in this sense.

We conclude this chapter by pointing out that Hoffman and Singer's theorem (Theorem 6.10) is not true for $n > 1$. This theorem shows that when $n = 1$ every closed algebra which contains H^∞ properly also contains $H^\infty + C$, i.e. $H^\infty + C$ is the smallest closed subalgebra of L^∞ containing H^∞ properly and is minimal amongst such algebras. For $n > 1$ and $1 \leq i \leq n$ define C_i to be the following algebra of functions:

$C_i = \{f \in C(S)$: for each fixed value w_0 (with $|w_0| \leq 1$) of the coordinate function z_i we can extend f to an analytic function in the 'disc' $\{(z_1, \ldots, z_n): z_i = w_0, |z_1|^2+\ldots+|z_{i-1}|^2+|z_{i+1}|^2+\ldots+|z_n|^2 < 1-|w_0|^2\}\}$.

We now prove a well-known lemma which will enable us to show that $H^\infty(S) + C_i$ is closed.

LEMMA 10.2 *If* H *and* B *are closed subspaces of a Banach space* L *then the following assertions are equivalent:*

(1) *there is a constant* $a > 0$ *such that* $d(f, H \cap B) \leq a d(f,H)$ *for all* f *in* B*;*

(2) $H + B$ *is a closed subspace of* L.

Proof. The natural mappings $B \to L \to L/H$ induce a mapping $\alpha: B/H \cap B \to L/H$. By the open mapping theorem, there is a constant a such that $d(f, H \cap B) \leq a d(f,H)$ for all f in B if and only if the range B/H of α is closed. Since $H + B$ is the pre-image of B/H under the quotient map $L \to L/H$, the space B/H is closed if and only if $H + B$ is closed. This proves (1) and (2) are equivalent.

This lemma together with W. Rudin's result [50] that $H^\infty(S)+C(S)$ is a closed algebra allow us to prove that $H^\infty(S) + C_i$ is a closed algebra for $1 \leq i \leq n$.

PROPOSITION 10.3 *Let* $n > 1$. *For each* i, $1 \leq i \leq n$, $H^\infty(S) + C_i$ *is a closed subalgebra of* $L^\infty(S)$.

Proof. Let $1 \leq i \leq n$ and let $f \in C_i$. We have

$$\begin{aligned} d(f, H^\infty(S) \cap C_i) &= d(f, H^\infty(S) \cap C(S)) \\ &\leq a d(f, H^\infty(S)) \end{aligned}$$

for some constant $a > 0$, by Lemma 10.2 since $H^\infty(S) + C(S)$ is closed. Hence, by Lemma 10.2 again, $H^\infty(S) + C_i$ is a closed subspace of $L^\infty(S)$.

Now let $f \in H^\infty(S)$ and $g \in C_i$. Then, since $H^\infty(S) + C(S)$ is an algebra, $fg = h + k$ where $h \in H^\infty(S)$ and $k \in C(S)$. Fix $z_i = w_0$, where $|w_0| \leq 1$. Then $k = fg - h$ and $fg - h$ is analytic in the 'disc' $\{(z_1,\ldots,z_n): z_i = w_0, |z_1|^2+\ldots+|z_{i-1}|^2+|z_{i+1}|^2+\ldots+|z_n|^2<1-|w_0|^2\}$ since $f, h \in H^\infty(S)$ and $g \in C_i$. Hence $k \in C_i$ and so $fg \in H^\infty(S)+C_i$. Thus $H^\infty + C_i$ is a closed algebra.

Note that $\bigcap_{i=1}^{n} (H^\infty(S) + C_i) = H^\infty(S)$ and so, by symmetry, each $H^\infty(S) + C_i$ is properly contained in $H^\infty(S) + C(S)$. Thus $H^\infty(S)+C(S)$ is not the smallest closed subalgebra of $L^\infty(S)$ containing $H^\infty(S)$ properly. In fact there does not exist such a smallest closed algebra since if one existed it would be contained in $H^\infty(S) + C_i$ for each i, $1 \leq i \leq n$, and so would be contained in $H^\infty(S)$. We conjecture however, that, for each i, $1 \leq i \leq n$, $\bigcap_{j\neq i} (H^\infty(S) + C_j)$ is a minimal closed subalgebra of $L^\infty(S)$ containing $H^\infty(S)$ properly.

CHAPTER ELEVEN

The Toeplitz operators on the classical Hardy space H^2 on the unit circle have been the object of much study. They are operators of the form $T_\phi f = P(\phi f)$ where $\phi \in L^\infty$ and P denotes the projection of L^2 onto H^2. An account of this theory, which is concerned mainly with describing the spectra of these operators, and with operator algebras generated by them, can be found in Chapter 7 of R.G. Douglas' book [39]. Connected with the Toeplitz operators are the Hankel operators which are operators from H^2 to $L^2 \ominus H^2$ of the form $H_\phi f = (I-P)(\phi f)$, i.e. $H_\phi = M_\phi - T_\phi$ where M_ϕ denotes multiplication by ϕ on L^2. The object of the two remaining chapters of this thesis is to study some aspects of Toeplitz and Hankel operators on the unit sphere in $\underset{\sim}{C}^n$, in particular to determine how far the one-variable theory remains valid. In this context Toeplitz operators with continuous symbol have been studied by L.A. Coburn [36] and some related operators by R.R. Coifman, R. Rochberg and G. Weiss [37]. Some recent developments along these lines are described in [38].

Notation. For $\phi \in L^\infty(S)$ we denote by T_ϕ the operator on the Hilbert space $H^2(S)$ defined by $T_\phi f = P(\phi f)$ where P denotes the orthogonal projection of $L^2(S)$ onto $H^2(S)$. T_ϕ is called the Toeplitz operator with symbol ϕ. We denote by H_ϕ the operator from $H^2(S)$ to $L^2(S) \ominus H^2(S)$ defined by $H_\phi f = (I-P)(\phi f)$ where I is the identity operator on $L^2(S)$. H_ϕ is called the Hankel operator with symbol ϕ.

We will make use of many easily checked results such as thelinearity of the map $\phi \to T_\phi$ and the fact that $T_\phi^* = T_{\bar{\phi}}$. Moreover, if

$\psi \in H^{\infty}(S)$ we have $T_\phi T_\psi = T_{\phi\psi}$.

We will also use the natural orthonormal basis for $H^2(S)$ given by $e_k = \frac{1}{(2\pi^n)^{\frac{1}{2}}}\left[\frac{(n+|k|-1)!}{k!}\right]^{\frac{1}{2}} z^k$ where $k = (k_1, \ldots, k_n)$ is an n-tuple of non-negative integers and we take $|k| \equiv k_1 + \ldots + k_n$, $k! \equiv k_1!\, k_2! \ldots k_n!$, $z^k \equiv z_1^{k_1} z_2^{k_1} \ldots z_n^{k_n}$ where $z = (z_1, \ldots z_n)$.

For the sake of simplicity we assume for the moment that $n = 2$. We wish to make use of the following parametrisation of the unit sphere: $z = (z_1, z_2) = (\rho^{\frac{1}{2}}e^{i\theta}, (1-\rho)^{\frac{1}{2}}e^{i\psi})$ $\quad (0 \le \rho \le 1;\ 0 \le \theta,\ \psi < 2\pi)$. For $f \in C(S)$ define $\tilde{f}$ by $\tilde{f}(rz) = f(z)$ $\quad (r > 0,\ z \in S)$. Then $\int_S f d\sigma = \frac{d}{dr}\left(\int_{B_r} \tilde{f} d\mu_r\right)_{r=1}$ where B_r is the ball centred at the origin, of radius r, with volume measure $d\mu_r$. So

$$\int_S f d\sigma = \lim_{\delta\to 0} \frac{1}{\delta} \int_{1 \le r_1^2 + r_2^2 \le (1+\delta)^2} f\left(\frac{r_1}{(r_1^2+r_2^2)^{\frac{1}{2}}}, \theta_1, \theta_2\right) r_1 r_2\, dr_1\, dr_2\, d\theta_1\, d\theta_2$$

where $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$,

$$= \lim_{\delta\to 0} \frac{1}{4\delta} \int_{\substack{1 \le \rho_1 + \rho_2 \le (1+\delta)^2 \\ = 1+2\delta+\delta^2}} f\left(\frac{\rho_1^{\frac{1}{2}}}{(\rho_1+\rho_2)^{\frac{1}{2}}}, \theta_1, \theta_2\right) d\rho_1\, d\rho_2\, d\theta_1\, d\theta_2$$

putting $\rho_1 = r_1^2$, $\rho_2 = r_2^2$,

$$= \lim_{\delta\to 0} \frac{1}{4\delta} \int_0^{2\pi}\int_0^{2\pi} \int_{1 \le \rho_1+\rho_2 \le 1+2\delta+\delta^2} f\left(\frac{\rho_1^{\frac{1}{2}}}{(\rho_1+\rho_2)^{\frac{1}{2}}}, \theta_1, \theta_2\right) d\rho_1\, d\rho_2\, d\theta_1\, d\theta_2$$

$$= \lim_{\delta\to 0} \frac{1}{4\delta} \int_0^{2\pi}\int_0^{2\pi}\int_0^1 \left[\int_{1-\rho_1}^{1+2\delta+\delta^2-\rho_1} f\left(\frac{\rho_1^{\frac{1}{2}}}{(\rho_1+\rho_2)^{\frac{1}{2}}}, \theta_1, \theta_2\right) d\rho_2\right] d\rho_1\, d\theta_1\, d\theta_2$$

$$= \frac{1}{2} \int_0^{2\pi}\int_0^{2\pi}\int_0^1 f(\rho_1^{\frac{1}{2}}, \theta_1, \theta_2)\, d\rho_1\, d\theta_1\, d\theta_2.$$

Since the continuous functions are dense in $L^1(S)$ this shows that with respect to the (ρ, θ, ψ) set of coordinates the measure becomes: $d\sigma = \frac{1}{2} d\rho\, d\theta\, d\psi$.

The standard basis for $H^2(S)$ is now given by:

$$e_{k\ell} = \frac{1}{\pi 2^{\frac{1}{2}}} \left[\frac{(k+\ell+1)!}{k!\ell!} \right]^{\frac{1}{2}} \rho^{k/2} (1-\rho)^{\ell/2} e^{ik\theta} e^{i\ell\psi} \qquad (k,\ \ell \geq 0).$$

Note that this parametrisation extends to the case $n > 2$. In that situation we put

$$(z_1, z_2, \ldots, z_n) = (\rho_1^{\frac{1}{2}} e^{i\theta_1}, \rho_2^{\frac{1}{2}} e^{i\theta_2}, \ldots, \rho_{n-1}^{\frac{1}{2}} e^{i\theta_{n-1}}, (1-\rho_1-\ldots-\rho_{n-1})^{\frac{1}{2}} e^{i\theta_n}).$$

For the case $n = 1$, P. Hartman [57] proved that H_ϕ is compact if and only if $\phi \in H^\infty + C$. We now show that the corresponding theorem is not true for $n > 1$. We look at the case $n = 2$ and consider Toeplitz operators T_ϕ where the symbol ϕ depends on only the coordinate ρ, i.e.

$$\phi(z_1, z_2) = \phi(\rho^{\frac{1}{2}} e^{i\theta}, (1-\rho)^{\frac{1}{2}} e^{i\psi}) = g(\rho) \quad \text{where} \quad g \in L^\infty[0,1],$$

(where $L^\infty[0,1]$ denotes the space of complex-valued essentially bounded functions on $[0,1]$).

It is clear that this type of symbol cannot occur when $n = 1$ and it is among Toeplitz operators of this class that we discover some differences between properties in the cases $n = 1$ and $n > 1$. For example there exists symbols $\phi \in C(S)$ and $\psi \in L^\infty(S)$ of the above form for which the spectrum of T_ϕ is disconnected and the essential spectrum of T_ψ is disconnected [38]. This contrasts with the case $n = 1$ where the spectrum and essential spectrum of <u>any</u> Toeplitz operator is always connected (see [39]).

First we note when a symbol of the above form is in $H^\infty(S) + C(S)$.

<u>PROPOSITION 11.1</u> *Let* ϕ *be a symbol which depends only on* ρ. *Then* $\phi \in H^\infty(S) + C(S)$ *if and only if* g *is continuous on* $[0,1]$.

Proof. One implication is clear. Conversely if $\phi \in H^\infty(S) + C(S)$, write $\phi = u + v$ with $u \in H^\infty(S)$, $v \in C(S)$. Let

$p(z_1 z_2) = q(|z_1|^2)$ where $q \in L^\infty[0,1]$ and let q be orthogonal to $H^\infty(S)$; notice that this is equivalent to q annihilating the constant functions of ϕ, i.e. $\int_0^1 q(\rho)d\rho = 0$. Then $\int_S up d\sigma = 0$, i.e. $\int_S (\phi - v) q d\rho d\theta d\psi = 0$. Let $\tilde{v}(\rho) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} v(\rho,\theta,\psi)d\theta d\psi$, $0 \le \rho \le 1$. Then $\int_0^1 [g(\rho) - \tilde{v}(\rho)]q(\rho)d\rho = 0$. This is true for all such q. Hence $g - \tilde{v}$ is a constant. But $\tilde{v}$ is continuous and so g is continuous.

<u>PROPOSITION 11.2</u> *Let* ϕ *be a symbol which depends only on* ρ. *Then* T_ϕ *is a diagonal operator.*

Proof. We have

$$T_\phi e_{k\ell} = P\{\frac{1}{\pi 2^{\frac{1}{2}}}[\frac{(k+\ell+1)!}{k!\ell!}]^{\frac{1}{2}}\rho^{k/2}(1-\rho)^{\ell/2}e^{ik\theta}e^{i\ell\psi}g(\rho)\}$$

$$= [\frac{(k+\ell+1)!}{k!\ell!}\int_0^1 \rho^k(1-\rho)^\ell g(\rho)d\rho]\, e_{k\ell}$$

$$= \lambda_{k\ell}e_{k\ell} \quad \text{where} \quad \lambda_{k\ell} = \frac{(k+\ell+1)!}{k!\ell!}\int_0^1 \rho^k(1-\rho)^\ell g(\rho)d\rho.$$

We now consider the Hankel operator H_ϕ when ϕ is a symbol which depends only on ρ. Let $f \in H^2(S)$ have Fourier series $\sum_{k,\ell\ge 0} a_{k\ell} e_{k\ell}$. Then if $\phi(\rho,\theta,\psi) = g(\rho)$ we have

$$H_\phi f = (I-P)(M_\phi f) = \sum_{k,\ell\ge 0} a_{k\ell}[g(\rho)e_{k\ell} - \lambda_{k\ell}e_{k\ell}].$$

Thus H_ϕ is compact $\Leftrightarrow \|g(\rho)e_{k\ell} - \lambda_{k\ell}e_{k\ell}\|_2 \to 0$ as $k, \ell \to \infty$, i.e. if and only if $\frac{(k+\ell+1)!}{k!\ell!}\int_0^1 \rho^k(1-\rho)^\ell |g(\rho) - \lambda_{k\ell}|^2 d\rho \to 0$

$$\text{as } k, \ell \to \infty. \qquad \ldots(1)$$

Note that the $\lambda_{k\ell}$'s are 'weights' of g on $[0,1]$ against the functions $\frac{(k+\ell+1)!}{k!\ell!}\rho^k(1-\rho)^\ell$ and so the requirement (1) above for H_ϕ to be compact is that g satisfies a type of 'weighted VMO' condition on $[0,1]$. This observation leads to the following theorem.

THEOREM 11.3 *Let* $\phi(\rho,\theta,\psi) = g(\rho)$ *where* $g \in L^\infty[0,1]$. *Then* H_ϕ *is compact if and only if* $g \in VMO$ *on* $[0,1]$.

(Remark. By $g \in VMO$ on $[0,1]$ we mean that the function h defined be $h(e^{it}) = g(\frac{t}{2\pi})$ for $0 \le t < 2\pi$ belongs to VMO as defined in Chapter 6.)

Proof. (a) Let $g \in VMO$ and let $\varepsilon > 0$. Choose $a < \frac{1}{2}$ such that $S_a(g) < \varepsilon$ (where $S_a(g) = S_{2\pi a}(h)$ as defined in Chapter 6). Let $I_{k\ell}$ be the interval $[\frac{k}{k+\ell} - \frac{1}{2}a, \frac{k}{k+\ell} + \frac{1}{2}a] \cap [0,1]$. It is clear from the nature of the functions $\frac{(k+\ell+1)!}{k!\ell!}\rho^k(1-\rho)^\ell$ that

(a) $\sup\limits_{\rho\in[0,1]\setminus I_{k\ell}} \frac{(k+\ell+1)!}{k!\ell!}\rho^k(1-\rho)^\ell = \alpha_{k\ell} \to 0$ as $k, \ell \to \infty$, and

(b) $\frac{1}{|I_{k\ell}|}\left|\int_{I_{k\ell}} [\,|I_{k\ell}|\rho^k(1-\rho)^\ell \frac{(k+\ell+1)!}{k!\ell!} - 1]\,d\rho\right| = \beta_{k\ell} \to 0$ as $k, \ell \to \infty$.

Now choose k, ℓ large enough so that $\alpha_{k\ell} < \varepsilon$, $\beta_{k\ell} < \varepsilon$. Then

$$\begin{aligned}
|g_{I_{k\ell}} - \lambda_{k\ell}| &= \left|\frac{1}{|I_{k\ell}|}\int_{I_{k\ell}} g(\rho)\,d\rho - \frac{(k+\ell+1)!}{k!\ell!}\int_0^1 \rho^k(1-\rho)^\ell g(\rho)\,d\rho\right| \\
&\le \left|\frac{1}{|I_{k\ell}|}\int_{I_{k\ell}} [g(\rho) - \frac{(k+\ell+1)!}{k!\ell!}|I_{k\ell}|\rho^k(1-\rho)^\ell g(\rho)]\,d\rho\right| \\
&\quad + \frac{(k+\ell+1)!}{k!\ell!}\left|\int_{[0,1]\setminus I_{k\ell}} \rho^k(1-\rho)^\ell g(\rho)\,d\rho\right| \\
&\le \left|\frac{1}{|I_{k\ell}|}\int_{I_{k\ell}} g(\rho)[1 - \frac{(k+\ell+1)!}{k!\ell!}|I_{k\ell}|\rho^k(1-\rho)^\ell]\,d\rho\right| + \alpha_{k\ell}\|g\|_\infty \\
&\le \|g\|_\infty(\alpha_{k\ell} + \beta_{k\ell}) \le 2\varepsilon\|g\|_\infty \qquad \ldots(2)
\end{aligned}$$

Also
$$\begin{aligned}
\|g(\rho)e_{k\ell} - g_{I_{k\ell}}e_{k\ell}\|_2^2 &= \frac{(k+\ell+1)!}{k!\ell!}\int_0^1 \rho^k(1-\rho)^\ell|g(\rho)-g_{I_{k\ell}}|^2\,d\rho \\
&= \frac{(k+\ell+1)!}{k!\ell!}\int_{I_{k\ell}} \rho^k(1-\rho)^\ell|g(\rho)-g_{I_{k\ell}}|^2\,d\rho \\
&\quad + \frac{(k+\ell+1)!}{k!\ell!}\int_{[0,1]\setminus I_{k\ell}} \rho^k(1-\rho)^\ell|g(\rho)-g_{I_{k\ell}}|^2\,d\rho
\end{aligned}$$

$$\leq \frac{1}{|I_{k\ell}|} \int_{I_{k\ell}} |g(\rho) - g_{I_{k\ell}}|^2 d\rho$$

$$+ \frac{1}{|I_{k\ell}|} \int_{I_{k\ell}} [\frac{(k+\ell+1)!}{k!\ell!} |I_{k\ell}| \rho^k (1-\rho)^\ell - 1] \, |g(\rho) - g_{I_{k\ell}}|^2 \, d\rho$$

$$+ 4\alpha_{k\ell} \|g\|_\infty^2$$

$$\leq KS_a(g) + 4\beta_{k\ell} \|g\|_\infty^2 + 4\alpha_{k\ell} \|g\|_\infty^2$$

where K is a constant, using Lemma 6.3 (as in the proof of Theorem 6.5)

$$\leq K\varepsilon + 8\|g\|_\infty^2 \varepsilon \qquad \ldots(3)$$

Now $\|g_{I_{k\ell}} e_{k\ell} - \lambda_{k\ell} e_{k\ell}\|_2 = |g_{I_{k\ell}} - \lambda_{k\ell}| \leq 2\varepsilon\|g\|_\infty$ by (2). Hence

$$\|g(\rho)e_{k\ell} - \lambda_{k\ell} e_{k\ell}\|_2 \leq \|g(\rho)e_{k\ell} - g_{I_{k\ell}} e_{k\ell}\|_2 + \|g_{I_{k\ell}} e_{k\ell} - \lambda_{k\ell} e_{k\ell}\|_2$$

$$\leq (K + 8\|g\|_\infty^2)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} + 2\varepsilon\|g\|_\infty \quad \text{by (3)},$$

i.e. $\|g(\rho)e_{k\ell} - \lambda_{k\ell} e_{k\ell}\|_2 \to 0$ as $k, \ell \to \infty$. The remarks before the theorem show that this implies that H_ϕ is compact.

(b) Suppose H_ϕ is compact. Then, by (1),

$$\frac{(k+\ell+1)!}{k!\ell!} \int_0^1 \rho^k (1-\rho)^\ell |g(\rho) - \lambda_{k\ell}|^2 d\rho \to 0 \quad \text{as} \quad k, \ell \to \infty.$$

We wish to show that $g \in VMO$ on $[0,1]$.

Let $\varepsilon > 0$ and let $b \in [0,1]$ be rational. Let I be any interval contained in $[0,1]$ with centre b. Suppose $|I| = a$.

Amongst those k, ℓ that satisfy $\frac{k}{k+\ell} = b$ choose k, ℓ large enough so that $\alpha_{k\ell} < \varepsilon$ and $\beta_{k\ell} < \varepsilon$, and

$\frac{(k+\ell+1)!}{k!\ell!} \int_0^1 \rho^k (1-\rho)^\ell |g(\rho) - \lambda_{k\ell}|^2 d\rho < \varepsilon$. We then have

$$\frac{1}{|I|} \int_I |g(\rho) - \lambda_{k\ell}|^2 d\rho = \frac{1}{|I|} \int_I |g(\rho) - \lambda_{k\ell}|^2 [1 - |I| \rho^k (1-\rho)^\ell \frac{(k+\ell+1)!}{k!\ell!}] \, d\rho$$

$$+ \frac{(k+\ell+1)!}{k!\ell!} \int_I |g(\rho) - \lambda_{k\ell}|^2 \rho^k (1-\rho)^\ell d\rho$$

$$\leq 4\|g\|_\infty^2 \beta_{k\ell} + \varepsilon \leq \varepsilon(4\|g\|_\infty^2 + 1),$$

and $|g_I-\lambda_{k\ell}| \le \|g\|_\infty(\alpha_{k\ell}+\beta_{k\ell})$ as in (a)

$$\le 2\varepsilon\|g\|_\infty .$$

Therefore $\frac{1}{|I|}\int_I |g(\rho)-g_I|d\rho \le [\frac{1}{|I|}\int_I |g(\rho)-g_I|^2 d\rho]^{\frac{1}{2}}$

$$\le [\frac{1}{|I|}\int_I |g(\rho)-\lambda_{k\ell}|^2 d\rho]^{\frac{1}{2}}$$

$$+ [\frac{1}{|I|}\int_I |g_I-\lambda_{k\ell}|^2 d\rho]^{\frac{1}{2}}$$

$$\le \varepsilon^{\frac{1}{2}}(4\|g\|_\infty{}^2 + 1)^{\frac{1}{2}} + 2\varepsilon\|g\|_\infty .$$

Now since I was any interval contained in $[0,1]$ with rational centre and since the rationals are dense in $[0,1]$ we have shown that $f \in$ VMO on $[0,1]$.

Remarks. (1) Independent of the above work R.R. Coifman, R. Rochberg, and G. Weiss [37] have extended the definition of VMO from the circle to the sphere in $\mathbb{C}^n$ $(n > 1)$ and with this definition they essentially prove the result that, for $\phi \in L^\infty(S)$,

$$\phi \in \text{VMO} \Leftrightarrow PM_\phi - M_\phi P \text{ is compact on } L^2(S).$$

Now $H_\phi = M_\phi - PM_\phi : H^2(S) \to L^2(S)$ and so H_ϕ is essentially $M_\phi P - PM_\phi P$ acting on $L^2(S)$ which shows that

$$\phi \in \text{VMO} \Leftrightarrow H_\phi \text{ and } H_{\bar\phi} \text{ are both compact}$$

and so in particular their results contain Theorem 11.3. However their proofs are not easy and involve the study of commutators of singular integral operators.

(2) Theorem 11.3, together with Proposition 11.1, show that there exist symbols $\phi \in L^\infty(S)$ such that H_ϕ is compact but $\phi \notin H^\infty(S) + C(S)$.

The methods of L.A. Coburn [36] show that if $\phi \in C(S)$ then H_ϕ

is compact. One may ask: for what $\phi \in L^{\infty}(S)$ is H_{ϕ} compact? It is clear that the set A of such ϕ is a closed subalgebra of $L^{\infty}(S)$ containing $H^{\infty}(S)$. In the case $n = 1$, $A = H^{\infty} + C$. Remark (2) above shows that this is not true in general and remark (1) showsthat the largest C*-algebra contained in A is the algebra $L^{\infty}(S) \cap VMO = QC$. It is natural to ask whether $A = H^{\infty}(S) + QC$ - especially in the light of the results of Chang (mentioned at the end of Chapter 9) which show that, when $n = 1$, <u>any</u> closed subalgebra of L^{∞} containing H^{∞} is of the form $H^{\infty}+$ some C*-algebra. We can split this problem into three parts:

(a) Is $H^{\infty} + QC$ closed?

(b) Is $(H^{\infty} + QC)^{-}$ an algebra?

(c) Is A the closed algebra generated by $H^{\infty} + QC$?

We have been unable to answer any of these questions. This would seem to be a test case for extending Chang's work to the spheres in higher dimensions. A related problem is to describe the largest C*-algebra contained in $H^{\infty}(S) + C(S)$. When $n = 1$ it is QC (see [51]), but since $QC \not\subseteq H^{\infty}(S) + C(S)$ this is false for $n > 1$.

CHAPTER TWELVE

In the classical situation the Toeplitz operators are characterised among operators on H^2 by the operator equation $T_z{}^*TT_z = T$, where T_z is the Toeplitz operator with symbol ϕ where $\phi(z) = z$ on $\mathbb{T}$. It is well known that T_z acting on H^2 is the canonical model for the unilateral shift (of multiplicity one) acting on a separable Hilbert space. In this chapter we extend this result by characterising the Toeplitz operators on $H^2(S)$ by the operator equation $\sum_{s=1}^{n} T_{z_s}{}^*TT_{z_s} = T$, where T_{z_s} is the Toeplitz operator with symbol ϕ where $\phi(z_1,\ldots,z_n) = z_s$ on S, $(1 \leq s \leq n)$.

One part of the characterisation is easy: for if $T = T_\phi$ is a Toeplitz operator then $\sum_{s=1}^{n} T_{z_s}{}^*TT_{z_s} = \sum_{s=1}^{n} T_{\bar{z}_s\phi z_s} = T_{\sum_{s=1}^{n}\phi|z_s|^2} = T_\phi = T$.

We now want to show that if $\sum T_{z_s}{}^*TT_{z_s} = T$ then T is a Toeplitz operator.

First note that if ψ is a non-negative measurable function on $\mathbb{C}$, if $z \in S$ and if $F(\eta) = \psi(\langle z,\eta\rangle)$ $(\eta \in \mathbb{C}^n)$, then $\int_S F d\sigma(\eta)$ is independent of z. This follows since $d\sigma$ is a rotation-invariant measure. We want to use this fact in a particular situation, namely when $\psi(x) = |1+x|^{2m}$ $(x \in \mathbb{C},\ m \in \mathbb{N})$. This gives that $C_m = \int_S |1+\langle z,\eta\rangle|^{2m} d\sigma(\eta)$ is independent of z in S. So, since $|1 + \langle z,\eta\rangle|$ 'peaks' when $\eta = z$, for any neighbourhood U of z in S we have

$$C_m^{-1} \int_{S\setminus U} |1 + \langle z,\eta\rangle|^{2m} d\sigma(\eta) \to 0 \quad \text{as} \quad m \to \infty,$$

as the 'mass' of the integrand lies in U more and more as $m \to \infty$.

So, for any $g \in C(S)$, it follows that

$$C_m^{-1} \int_S g(\eta)|1 + \langle z,\eta\rangle|^{2m} d\sigma(\eta) \to g(z) \quad \text{as} \quad m \to \infty \quad (z \in S) \qquad \ldots(1)$$

Let $f_\eta^{(k)}(z) = C_k^{-\frac{1}{2}}(1 + \langle z,\eta\rangle)^k$. Then, for each $\eta \in S$, $k \in \underset{\sim}{N}$, $f_\eta^{(k)} \in H^2(S)$ and $\|f_\eta^{(k)}\|_2 = 1$.

Suppose $\sum_{s=1}^{n} T_{z_s}{}^*TT_{z_s} = T$ and put $\phi_k(\eta) = \langle Tf_\eta^{(k)}, f_\eta^{(k)}\rangle$.

Let ϕ be a weak*-limit point of ϕ_k in $L^\infty(S)$. Then, for any $g \in C(S)$,

$$\int_S g(\eta)\phi_{k_j}(\eta)d\sigma(\eta) \to \int_S g(\eta)\phi(\eta)d\sigma(\eta) \quad \text{as} \quad j \to \infty$$

for some subsequence ϕ_{k_j} of ϕ_k (since the weak*-topology on the closed unit ball of $L^\infty(S)$ is metrizable - see [56, p.426]) i.e. $\lim_{j\to\infty} \int_S g(\eta)\langle Tf_\eta^{(k_j)}, f_\eta^{(k_j)}\rangle d\sigma(\eta) = \int_S g(\eta)\phi(\eta)d\sigma(\eta)$. Now, by (1), the right hand side is given by

$$\int_S g(\eta)\phi(\eta)d\sigma(\eta) = \lim_{m\to\infty}\int_S\int_S C_m^{-1}\phi(\eta)g(z)|1+\langle\eta,z\rangle|^{2m}d\sigma(z)d\sigma(\eta)$$

$$= \lim_{m\to\infty}\int_S\int_S C_m^{-1}\phi(\eta)g(z)|1+\langle z,\eta\rangle|^{2m}d\sigma(z)d\sigma(\eta).$$

Therefore $\lim_{j\to\infty}\int_S g(\eta)\langle Tf_\eta^{(k_j)}, f_\eta^{(k_j)}\rangle d\sigma(\eta)$

$$= \lim_{m\to\infty}\int_S\int_S C_m^{-1}\phi(\eta)g(z)|1+\langle z,\eta\rangle|^{2m}d\sigma(z)d\sigma(\eta) \qquad \ldots(2)$$

For the sake of simplicity we will assume from now on that $n = 2$; the same results can be shown, in an identical fashion, for the general case $n > 1$.

By choosing g to be suitable continuous functions on S we will use (2) to evaluate $\langle Tz_1{}^p z_2{}^q, z_1{}^t z_2{}^u\rangle$ in terms of ϕ for all integers $p, q, t, u \geq 0$. However we have to be careful about the order in which we evaluate the inner products.

To start with choose $g(\eta) = \bar{\eta}_1{}^t\bar{\eta}_2{}^u$ with t, u non-negative integers (and from now on). Then $\int_S g(\eta)\langle Tf_\eta^{(k)}, f_\eta^{(k)}\rangle d\sigma(\eta)$

$$= C_k^{-1}\int_S \bar{\eta}_1{}^t\bar{\eta}_2{}^u \langle T(\sum_{i=0}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j} z_1{}^j\bar{\eta}_1{}^j z_2{}^{i-j}\bar{\eta}_2{}^{i-j}),$$

$$\sum_{p=0}^{k}\sum_{q=0}^{p}\binom{k}{p}\binom{p}{q} z_1{}^q\bar{\eta}_1{}^q z_2{}^{p-q}\bar{\eta}_2{}^{p-q}\rangle\, d\sigma(\eta)$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{p=0}^{k} \sum_{j=0}^{i} \sum_{q=0}^{p} \binom{k}{i}\binom{i}{j}\binom{k}{p}\binom{p}{q} \langle Tz_1^{j} z_2^{i-j}, z_1^{q} z_2^{p-q}\rangle \times$$
$$\times \int_S \bar{\eta}_1^{j+t} \bar{\eta}_2^{i-j+u} \eta_1^{q} \eta_2^{p-q} d\sigma(\eta)$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i}\binom{i}{j}\binom{k}{i+t+u}\binom{i+t+u}{j+t} \langle Tz_1^{j} z_2^{i-j}, z_1^{j+t} z_2^{i-j+u}\rangle \times$$
$$\times \int_0^1 \rho^{j+t}(1-\rho)^{i-j+u} d\rho$$

(where the bonimial coefficients here, and from now on, are taken to be zero if they have no meaning),

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i}\binom{i}{j}\binom{k}{i+t+u}\binom{i+t+u}{j+t} \frac{(j+t)!(i-j+u)!}{(i+t+u+1)!} \times$$
$$\times \langle Tz_1^{j} z_2^{i-j}, z_1^{j+t} z_2^{i-j+u}\rangle$$
$$= C_k^{-1} \sum_{i=0}^{k} \binom{k}{i}\binom{k}{i+t+u} \frac{1}{i+t+u+1} \sum_{j=0}^{i} \binom{i}{j} \langle Tz_1^{j} z_2^{i-j}, z_1^{j+t} z_2^{i-j+u}\rangle .$$

Now $T_{z_1}{}^*TT_{z_1} + T_{z_2}{}^*TT_{z_2} = T$, and by iterating this equation we obtain

$$\langle Tf,g\rangle = \sum_{\ell=0}^{m} \binom{m}{\ell} \langle Tz_1^{\ell} z_2^{m-\ell} f, z_1^{\ell} z_2^{m-\ell} g\rangle \quad \text{for all } f,g \text{ in } H^2(S)$$
$$\text{and any } m \text{ in } \tilde{N}. \qquad \ldots(3)$$

So $$\int_S \bar{\eta}_1^{t} \bar{\eta}_2^{u} \langle Tf_\eta^{(k)}, f_\eta^{(k)}\rangle \, d\sigma(\eta) = [C_k^{-1} \sum_{i=0}^{k} \binom{k}{i}\binom{i}{i+t+u} \frac{1}{i+t+u+1}] \times$$
$$\times \langle T1, z_1^{t} z_2^{u}\rangle . \qquad \ldots(4)$$

In an identical fashion the terms on the right hand side of (2) are given by

$$\int_S \int_S C_m^{-1} \phi(\eta) \bar{z}_1^{t} \bar{z}_2^{u} |1+\langle z,\eta\rangle|^{2m} d\sigma(z) d\sigma(\eta)$$
$$= [C_m^{-1} \sum_{i=0}^{m} \binom{m}{i}\binom{m}{i+t+u} \frac{1}{i+t+u+1}] \int_S \phi(\eta) \bar{\eta}_1^{t} \bar{\eta}_2^{u} d\sigma(\eta)$$
$$= [C_m^{-1} \sum_{i=0}^{m} \binom{m}{i}\binom{m}{i+t+u} \frac{1}{i+t+u+1}] \langle \phi, z_1^{t} z_2^{u}\rangle \qquad \ldots(5)$$

Now since $C_m^{-1} \sum_{i=0}^{m} \binom{m}{i}\binom{m}{i+t+u} \frac{1}{i+t+u+1} \not\to 0$ as $m \to \infty$ for any

$t, u \in \underset{\sim}{N}$, (2) together with (4) and (5) give

$$\langle T1, z_1^t z_2^u \rangle = \langle \phi, z_1^t z_2^u \rangle \qquad t, u \geq 0. \qquad \ldots(6)$$

We next evalaluate the inner products of T_{z_1} against the basis elements of $H^2(S)$. For this we choose $g(\eta) = \eta_1 \bar{\eta}_1^t \bar{\eta}_2^u$. Then, as before, $\int_S g(\eta) \langle Tf_\eta^{(k)}, f_\eta^{(k)} \rangle \, d\sigma(\eta)$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{p=0}^{k} \sum_{j=0}^{i} \sum_{q=0}^{p} \binom{k}{i} \binom{k}{p} \binom{i}{j} \binom{p}{q} \langle Tz_1^j z_2^{i-j}, z_1^q z_2^{p-q} \rangle \times$$
$$\times \int_S \eta_1 \bar{\eta}_1^t \bar{\eta}_2^u \bar{\eta}_1^j \bar{\eta}_2^{i-j} \eta_1^q \eta_2^{p-q} d\sigma(\eta)$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} \binom{k}{i+t+u-1} \binom{i+t+u-1}{t+j-1} \langle Tz_1^j z_2^{i-j}, z_1^{t+j-1} z_2^{i+u-j} \rangle \times$$
$$\times \int_0^1 \rho^{t+j} (1-\rho)^{i-j+u} d\rho$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} \binom{k}{i+t+u-1} \binom{i+t+u-1}{t+j-1} \frac{(t+j)!(i-j+u)!}{(i+t+u+1)!} \times$$
$$\langle Tz_1^j z_2^{i-j}, z_1^{t+j-1} z_2^{i+u-j} \rangle$$

$$= C_k^{-1} \sum_{i=0}^{k} \binom{k}{i} \binom{k}{i+t+u-1} \frac{(i+t+u-1)!}{(i+t+u+1)!} ([t \sum_{j=0}^{i} \binom{i}{j} + i \sum_{j=1}^{i} \binom{i-1}{j-1}] \times$$
$$\times \langle Tz_1^j z_2^{i-j}, z_1^{t+j-1} z_2^{i+u+j} \rangle)$$

Now (3) shows that

$$\langle Tz_1, z_1^t z_2^u \rangle = \sum_{\ell=0}^{m} \binom{m}{\ell} \langle Tz_1^{\ell+1} z_2^{m-\ell}, z_1^{\ell+t} z_2^{m+u-\ell} \rangle$$

for any m in $\underset{\sim}{N}$,

$$= \sum_{v=1}^{w} \binom{w-1}{v-1} \langle Tz_1^v z_2^{w-v}, z_1^{t+v-1} z_2^{w+u-v} \rangle$$

and so

$$\int_S \eta_1 \bar{\eta}_1^t \bar{\eta}_2^u \langle Tf_\eta^{(k)}, f_\eta^{(k)} \rangle \, d\sigma(\eta)$$

$$= [C_k^{-1} \sum_{i=0}^{k} \binom{k}{i}\binom{k}{i+t+u-1} \frac{(i+t+u-1)!}{(i+t+u+1)!}][t \langle T1, z_1^{t-1} z_2^{u}\rangle$$

$$+ i \langle Tz_1, z_1^{t} z_2^{u}\rangle] \quad \text{for any} \quad t,\ u \geq 0. \qquad \ldots(7)$$

Again, in an identical fashion we obtain

$$\int_S \int_S C_m^{-1} \phi(\eta) z_1 \bar{z}_1^{t} \bar{z}_2^{u} |1+\langle z,\eta\rangle|^{2m} d\sigma(z) d\sigma(\eta)$$

$$= [C_m^{-1} \sum_{i=0}^{m} \binom{m}{i}\binom{m}{i+t+u-1} \frac{(i+t+u-1)!}{(i+t+u+1)!}][t \langle \phi, z_1^{t-1} z_2^{u}\rangle$$

$$+ i \langle \phi z_1, z_1^{t} z_2^{u}\rangle] \quad \text{for any} \quad t,\ u \geq 0. \qquad \ldots(8)$$

So proceeding as before, by using (2) together with (7) and (8), <u>and</u> since we know the inner products of $T1$ from (6), we obtain

$$\langle Tz_1, z_1^{t} z_2^{u}\rangle = \langle \phi z_1, z_1^{t} z_2^{u}\rangle \qquad t,\ u \geq 0. \qquad \ldots(9)$$

By symmetry, with $g(\eta) = \eta_2 \bar{\eta}_1^{t} \bar{\eta}_2^{u}$ the same procedure gives

$$\langle Tz_2, z_1^{t} z_2^{u}\rangle = \langle \phi z_2, z_1^{t} z_2^{u}\rangle \qquad t,\ u \geq 0. \qquad \ldots(10)$$

Next we evaluate the inner products of Tz_1^2 by choosing $g(\eta) = \eta_1^2 \bar{\eta}_1^{t} \bar{\eta}_2^{u}$. Then, as before,

$$\int_S g(\eta) \langle Tf_\eta^{(k)}, f_\eta^{(k)}\rangle d\sigma(\eta)$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{p=0}^{k} \sum_{j=0}^{i} \sum_{q=0}^{p} \binom{k}{i}\binom{k}{p}\binom{i}{j}\binom{p}{q} \langle Tz_1^{j} z_2^{i-j}, z_1^{q} z_2^{p-q}\rangle \times$$

$$\times \int_S \eta_1^2 \bar{\eta}_1^{t} \bar{\eta}_2^{u} \bar{\eta}_1^{j} \bar{\eta}_2^{i-j} \eta_1^{q} \eta_2^{p-q} d\sigma(\eta)$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i}\binom{i}{j}\binom{k}{i+t+u-2}\binom{i+t+u-2}{t+j-2} \langle Tz_1^{j} z_2^{i-j}, z_1^{t+j-2} z_2^{i+u-j}\rangle \times$$

$$\times \int_0^1 \rho^{t+j}(1-\rho)^{i+u-j} d\rho$$

$$= C_k^{-1} \sum_{i=0}^{k} \sum_{j=0}^{i} \binom{k}{i}\binom{i}{j}\binom{k}{i+t+u-2}\binom{i+t+u-2}{t+j-2} \frac{(t+j)!(i+u-j)!}{(i+t+u+1)!} \times$$

$$\times \langle Tz_1^{j} z_2^{i-j}, z_1^{t+j-2} z_2^{i+u-j}\rangle$$

$$= C_k^{-1} \sum_{i=0}^{k} \binom{k}{i} \binom{k}{i+t+u-2} \frac{(i+t+u-2)!}{(i+t+u+1)!} \Big(\Big[(t^2-1) \sum_{j=0}^{i} \binom{i}{j} + (2t+1)i \sum_{j=1}^{i} \binom{i-1}{j-1}$$

$$+ i(i-1) \sum_{j=2}^{i} \binom{i-2}{j-2}\Big] \langle Tz_1^{\,j} z_2^{\,i-j}, z_1^{\,t+j-2} z_2^{\,i+u-j} \rangle \Big)$$

(3) shows that

$$\langle Tz_1^2, z_1^t z_2^u \rangle = \sum_{\ell=0}^{m} \binom{m}{\ell} \langle Tz_1^{\ell+2} z_2^{m-\ell}, z_1^{\ell+t} z_2^{m-\ell+u} \rangle$$

$$= \sum_{v=2}^{w} \binom{w-2}{v-2} \langle Tz_1^{v} z_2^{w-v}, z_1^{t+v-2} z_2^{u+w-v} \rangle$$

and so

$$\int_S \eta_1^2 \bar{\eta}_1^t \bar{\eta}_2^u \langle Tf_\eta^{(k)}, f_\eta^{(k)} \rangle \, d\sigma(\eta)$$

$$= [C_k^{-1} \sum_{i=0}^{k} \binom{k}{i} \binom{k}{i+t+u-2} \frac{(i+t+u-2)!}{(i+t+u+1)!}] [(t^2-1) \langle T1, z_1^{t-2} z_2^u \rangle$$

$$+ (2t+1)i \langle Tz_1, z_1^{t-1} z_2^u \rangle + i(i-1) \langle Tz_1^2, z_1^t z_2^u \rangle$$

As before, by comparing both sides of (2) and since we know the inner products of $T1$ and Tz_1 from (6) and (9) we obtain

$$\langle Tz_1^2, z_1^t z_2^u \rangle = \langle \phi z_1^2, z_1^t z_2^u \rangle \qquad t, u \geq 0 \qquad \ldots(11)$$

By symmetry, with $g(\eta) = \eta_2^2 \bar{\eta}_1^r \bar{\eta}_2^s$ the same procedure gives

$$\langle Tz_2^2, z_1^t z_2^u \rangle = \langle \phi z_2^2, z_1^t z_2^u \rangle \qquad t, u \geq 0 \qquad \ldots(12)$$

If $t + u \geq 1$ we can evaluate the inner products $\langle Tz_1 z_2, z_1^t z_2^u \rangle$ by using the operator equation and our previous results: for if, for example, $t \geq 1$ we have

$$\langle Tz_1 z_2, z_1^t z_2^u \rangle = \langle Tz_2, z_1^{t-1} z_2^u \rangle - \langle Tz_2^2, z_1^{t-1} z_2^{u+1} \rangle$$

$$= \langle \phi z_2, z_1^{t-1} z_2^u \rangle - \langle \phi z_2^2, z_1^{t-1} z_2^{u+1} \rangle \quad \text{by (10), (12),}$$

$$= \langle \phi z_1 z_2, z_1^t z_2^u \rangle .$$

If $t = u = 0$ we evaluate $\langle Tz_1 z_2, 1 \rangle$ as before by equating both sides of (2) with $g(\eta) = \eta_1 \eta_2$.

If we continue in this fashion, i.e. we next evaluate the inner products of Tz_1^3, then Tz_2^3, then $Tz_1^2 z_2$, then $Tz_1 z_2^2$, then Tz_1^4, etc., and collect all the identities such as (6), (9), (10), (11), (12), etc., we obtain

$$\langle Tz_1^{p}z_2^{q}, z_1^{r}z_2^{s}\rangle = \langle \phi z_1^{p}z_2^{q}, z_1^{r}z_2^{s}\rangle$$

for all non-negative integers p, q, r, s.

Since the polynomials in z_1, z_2 are dense in $H^2(S)$ this shows that $\langle Tf,g\rangle = \langle \phi f,g\rangle = \langle T_\phi f,g\rangle$ for all f, g in $H^2(S)$ and so $T = T_\phi$, i.e. T is the Toeplitz operator with symbol ϕ. We have thus proved the following theorem.

<u>THEOREM 12.1</u> *Let* $T \in \mathcal{B}(H^2)$. *Then* $T = T_\phi$ *for some* $\phi \in L^\infty(S)$ *if and only if* $\sum_{s=1}^{n} T_{z_s}{}^*TT_{z_s} = T$.

<u>Remarks</u>. (1) If T is a diagonal operator on $H^2(S)$ then T is the Toeplitz operator, T_ϕ, if and only if $\sum_{s=1}^{n} T_{z_s}{}^*TT_{z_s} = T$ and here the symbol ϕ is of the type described in Chapter 11, e.g. when $n = 2$, $\phi(z_1,z_2) = g(|z_1|^2) = g(\rho)$. This can be proved in an elementary fashion by appealing to the solution of a classical Hausdorff moment problem which gives necessary and sufficient conditions on a sequence $\{\mu_k\}$ such that $\mu_k = \int_0^1 \rho^k\phi(\rho)d\rho$ for some function $\phi \in L^\infty[0,1]$ (see [54, p.111]).

(2) An alternative approach to proving Theorem 12.1 is as follows: first prove that any T satisfying $\sum_{s=1}^{n} T_{z_s}{}^*TT_{z_s} = T$ can be 'lifted' to an operator S on $L^2(S)$ which satisfies $\sum_{s=1}^{n} M_{z_s}{}^*SM_{z_s} = S$ (where M_ϕ is multiplication by ϕ on $L^2(S)$) and such that T is the compression of S to $H^2(S)$. The proof of the theorem is completed by showing that this operator equation involving operators on $L^2(S)$ characterises the multiplication (by $L^\infty(S)$ functions) operators on $L^2(S)$. This can be achieved by elementary Hilbert space inner product calculations using the fact that an operator which commutes with M_ϕ for all ϕ in $L^2(S)$ is itself a multiplication operator.

For details of this argument see [38].

It is well-known in the classical case that 0 is the only compact Toeplitz operator on H^2. We extend this result to higher dimensions.

COROLLARY 12.2 0 *is the only compact Toeplitz operator on* $H^2(S)$.

Proof. Suppose T is a compact Toeplitz operator. Then by Theorem 12.1 $\sum_{s=1}^{n} T_{z_s}{}^{*}TT_{z_s} = T$ and iterating this equation we obtain

$$T = \sum_{s_1+\dots+s_n=m} \frac{m!}{s_1!\dots s_n!} T_{\bar{z}_1}^{s_1}\dots T_{\bar{z}_n}^{s_n} T T_{z_1}^{s_1}\dots T_{z_n}^{s_n} \qquad \dots(13)$$

for any $m \geq 1$.

Now for $1 \leq i \leq n$, $T_{z_i}^{k} \to 0$ weakly as $k \to \infty$ and so as $m \to \infty$ each of $T_{z_1}^{s_1}\dots T_{z_n}^{s_n} \to 0$ weakly where $s_1+\dots+s_n = m$. T is compact and so $TT_{z_1}^{s_1}\dots T_{z_n}^{s_n} \to 0$ strongly as $m \to \infty$ $(s_1+\dots+s_n = m)$. Hence the operator on the right hand side of (13) converges to zero strongly as $m \to \infty$. The identity (13) then gives $T = 0$.

There are many other interesting questions in the case $n > 1$ which arise by looking at the vast literature on Toeplitz operators in the case $n = 1$. We conclude this chapter by suggesting some further operator-theoretic generalizations to the case $n > 1$.

(1) For $1 \leq i \leq n$ it is not hard to see that T_{z_i} is the direct sum of a countable number of weighted shifts each of which is similar to the unilateral shift of multiplicity one. (T_{z_i} is not, however, similar to a countable direct sum of unilateral shifts of multiplicity one, i.e. a unilateral shift of countable multiplicity.) What are the invariant subspaces of such operators? Or, what is perhaps the proper

question to ask, what are the common invariant subspaces for T_{z_i}, $1 \leq i \leq n$? Is it possible to deduce anything concerning function theory on the sphere by examining such operators?

(2) $\{T_{z_i} : 1 \leq i \leq n\}$ is a set of n commuting contractions on a Hilbert space. Can we learn something by examining the dilation theory which exists for such sets of operators.

(3) In the abstract classical theory isometries play a crucial rôle, e.g. the Wold decomposition tells us that every isometry V has a unique reducing (closed) subspace M (i.e. invariant for V and V^*) such that $V|M$ is unitary and $V|M^{\perp}$ is unitarily equivalent to a unilateral shift operator. It seems plausible that a corresponding theory exists for commuting n-tuples of operators $\{T_1,\ldots T_n\}$ which satisfy $T_1^*T_1+\ldots+T_n^*T_n = I$, e.g. can we extend the Wold decomposition to the following result: if $\{T_1,\ldots T_n\}$ is a set of n commuting operators on a Hilbert space H with $\sum_{j=1}^{n} T_j^*T_j = I$, then there is a closed subspace M of H, reducing for each T_j, $1 \leq j \leq n$, such that $T_j|M$ is normal and $T_j|M^{\perp}$ is unitarily equivalent to a countable sum of weighted unilateral shift operators for each j, $1 \leq j \leq n$ (the weights of these operators being determined by the weights of the operator T_{z_j} on $H^2(S)$)?

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ABSTRACT OF THESIS

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Degree Doctor of Philosophy *Date* 8th May, 1976

Title of Thesis Continuity of Derivations and Uniform Algebras on Odd Spheres

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The thesis is composed of two separate and distinct parts.

Part one is concerned with the problem of determining when certain linear mappings are necessarily continuous with particular attention being given to derivations.

Chapter 1 consists of a discussion of the separating space of a linear mapping. Chapter 2 contains a description of the Banach algebra $L^1[0,1]$ and some of its properties. In Chapter 3 we consider derivations on $L^1[0,1]$, proving in Theorem 3.1 that they are necessarily continuous. In Chapter 4 we extend this result to module derivations and in Theorem 4.2 we give sufficient conditions on a Banach algebra B such that every module derivation from B is continuous. When B is separable and commutative we cam improve Theorem 4.2 and then it is easily seen that one of the sufficient conditions is best possible. In Chapter 5 we give sufficient conditions on a Banach algebra B such that certain higher derivations from any Banach algebra onto B are automaticall continuous.

Part two is concerned with the recent result of D.E. Marshall and S-Y.A. Chang that every closed subalgebra of $L^\infty(T)$ (where T is the unit circle) containing $H^\infty(T)$ is a Douglas algebra. Using their techniques we give a proof of this result and discuss generalisations of these ideas and related concepts to higher dimensions.

Chapter 6 consists of a discussion of Douglas algebras, functions of vanishing mean oscillation (VMO), Carleson measures and other topics. In Chapter 7 we generalise the space of VMO and provide a characterisation of the new space in terms of Carleson measures. Using these ideas we prove the Marshall-Chang theorem in Chapters 8 and 9. Chapter 10 discusses the subject of Douglas algebras in higher dimensions. Chapter 11

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gives a description of a particular class of Hankel operators on $L^2(S)$ (where S is the unit sphere in $\mathbb{C}^n$). In Chapter 12 we characterise the Toeplitz operators amongst operators on $H^2(S)$ in terms of an operator equation. In Chapters 10, 11 and 12 we pose several open questions.