ORTHOGONAL POLYNOMIALS

expressed as

BORDERED DETERMINANTS and CONTINUANTS

by

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INTRODUCTION

The purpose of this Thesis is to treat the Theory of Orthogonal Polynomials by an approach based on Matrices and Determinants.

In Part I, general Orthogonal Polynomials are considered for any positive weight function $w(x)$ over any range (a,b), with the restriction that in each polynomial the coefficient of the highest degree term is unity. Of great importance to the theory are the positive definite persymmetric matrix of moments, M_n , and the cofactors N_{nr} of the elements of the last column of $|M_n|$.

The orthogonal polynomials, $P_n(x)$, are expressed in several determinantal forms - the Bordered Moment Determinant (Theorem 5), The Secular Polynomial (Theorem 20), The Characteristic Determinant of a Matrix in Rational Canonical Form (Theorem 21), and three Continuant Determinants (Theorems 14, 15, 16), one being normalised to a Characteristic Polynomial of a Symmetric Continuant (Theorem 16). A summary of these six different forms is on pp. 15 & 16, and the Matrix Transformations connecting them are given in Theorems 22 - 26.

In Theorems 17 - 19, well known results about the zeros of Orthogonal Polynomials are given - in some cases standard proofs are quoted, in others the theorems are proved by matrix methods. The recurrence relation for three consecutive polynomials is established in Theorem 12.

In Part III, seven special cases are considered - Uncentred Legendre Polynomials, Legendre Polynomials, two types of Uncentred Jacobi Polynomials, Laguerre Polynomials, polynomials allied to the Laguerre Polynomials and Hermite Polynomials.

In each case, the following are calculated - m_n, N_{nr}, $|M_n|$, the coefficient of x^r in $P_n(x)$, the recurrence relation, and P_0, P_1, P_2, P_3, P_4. In addition, the matrices associated with three of the determinantal forms for $P_4(x)$ (viz. the Bordered Determinant, the Secular Polynomial, and the Characteristic Polynomial of a Symmetric Continuant) and also the Matrix of Cofactors K_4 are displayed.

The calculation of N_{nr} for the Special Cases proves rather lengthy, and these calculations are therefore separated from the Special Cases, and kept together in Part II.

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Part I - The General Theory.

Part II - Calculation of N_{nr} for Special Cases.

Part III - Special Cases.

PART I

THE GENERAL THEORY

DEFINITIONS

Let $w(x)$ be a function of x, positive throughout (a,b).

Let its moments be

$$m_r \equiv \int_a^b x^r w(x)\,dx \qquad (r=0,1,2,\ldots)$$

Let M_n be the $n+1 \times n+1$ matrix of moments,

$$M_n \equiv \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n} \end{bmatrix} \qquad (n=0,1,2,\ldots)$$

$$\text{Let } L_n \equiv \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ m_2 & m_3 & m_4 & \cdots & m_{n+2} \\ m_3 & m_4 & m_5 & \cdots & m_{n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n+1} & m_{n+2} & m_{n+3} & \cdots & m_{2n+1} \end{bmatrix} \qquad (n=0,1,2,\ldots)$$

Let $|M_{-1}| \equiv 1$

Let N_{nr} be the $n \times n$ cofactor of the m_{n+r} which occurs in the last column of M_n,

$$N_{nr} \equiv (-)^{n+r} \begin{Vmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_1 & m_2 & m_3 & \cdots & m_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{r-1} & m_r & m_{r+1} & \cdots & m_{n+r-2} \\ m_{r+1} & m_{r+2} & m_{r+3} & \cdots & m_{n+r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-1} \end{Vmatrix} \qquad \begin{matrix}(n=1,2,3,\ldots \\ r=0,1,2,\ldots n)\end{matrix}$$

Let $p_n(x)$ be orthogonal polynomials of degree n in x , such that
$$\int_a^b p_i(x)\, p_j(x)\, w(x)\, dx \begin{cases} = 0 & \text{for } i \neq j \\ \neq 0 & \text{for } i = j \end{cases}$$

PROPERTIES OF M_n

Theorem 1 (i) $N_{nn} = |M_{n-1}|$ (ii) $N_{no} = (-)^n\,|L_{n-1}|$

These results are clear from the definitions.

Theorem 2 M_n is positive definite

The quadratic form $y'M_ny$ (where $y = \{y_0\ y_1\ y_2 \cdots y_n\}$)

$$= \sum_{i=o}^{n} \sum_{j=o}^{n} m_{i+j}\, y_i\, y_j$$

$$= \sum_{i=o}^{n} \sum_{j=o}^{n} \int_a^b x^{i+j}\, w(x)\, dx\ \ y_i\, y_j$$

$$= \int_a^b \left[\sum_{i=o}^{n} \sum_{j=o}^{n} x^i\, y_i\, x^j\, y_j \right] w(x)\, dx$$

$$= \int_a^b \left[\sum_{i=o}^{n} x^i\, y_i \right]^2 w(x)\, dx \ > \ 0$$

since the integrand is positive throughout (a,b) .

ORTHOGONAL POLYNOMIALS AS BORDERED DETERMINANTS

Definition Let $Q_n(x) \equiv \begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \end{vmatrix}$

$$= (-)^n \sum_{r=0}^{n} N_{nr}\, x^r$$

Theorem 3
$$\int_a^b Q_n(x)\, x^r\, w(x)\, dx = \begin{cases} 0 & \text{if } r < n \\ (-)^n |M_n| & \text{if } r = n \\ (-)^{n+1} N_{n+1,n} & \text{if } r = n+1 \end{cases}$$

For $n = 0, 1, 2, \ldots$.

$$\text{LHS} = \begin{vmatrix} m_r & m_{r+1} & m_{r+2} & \cdots & m_{r+n} \\ m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \end{vmatrix}$$

If $r < n$ two rows are identical and determinant vanishes.

If $r = n$ moving top row to bottom, det $= (-)^n |M_n|$.

If $r = n+1$ moving top row to bottom, det $= (-)^{n+1} N_{n+1,n}$.

Theorem 4 $\int_a^b Q_n(x)\, Q_m(x)\, w(x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ |M_n||M_{n-1}| & \text{if } m = n \end{cases}$

for $m, n = 0, 1, 2, \ldots$

(i) $n > 0$: Either $m=n$ or we may assume without loss of generality that $m < n$

$$\text{LHS} = \int_a^b Q_n(x)\, (-)^m \left[\sum_{r=0}^{m} N_{mr}\, x^r\right] dx$$

$$= (-)^m \sum_{r=0}^{m} \left[N_{mr} \int_a^b Q_n(x)\, w(x)\, x^r\, dx \right]$$

$$= \begin{cases} 0 & \text{if } m < n \quad \text{by Thm 3} \\ (-)^n N_{nn} (-)^n |M_n| & \text{if } m=n \quad \text{by Thm 3} \end{cases}$$

$$= \begin{cases} 0 & \text{if } m < n \\ |M_{n-1}|\,|M_n| & \text{if } m=n \text{ by Thm 1.} \end{cases}$$

(ii) $m=n=0$

$$\text{LHS} = \int_a^b Q_o{}^2(x)\, w(x)\, dx = \int_a^b w(x)\, dx = m_o = |M_o|\,|M_{-1}|$$

since $|M_{-1}|$ was defined as 1 .

Theorem 5 The orthogonal polynomials having coefficient of x^n unity, are

$$P_n(x) \equiv (-)^n |M_{n-1}|^{-1} Q_n(x)$$

$$= (-)^n |M_{n-1}|^{-1} \begin{vmatrix} 1 & x & x^2 & \ldots & x^n \\ m_o & m_1 & m_2 & \ldots & m_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m_{n-1} & m_n & m_{n+1} & \ldots & m_{2n-1} \end{vmatrix}$$

By Thm 4 the $P_n(x)$ satisfy the orthogonal conditions and the coefft of x^n is 1 by Thm 1.

SOME PROPERTIES OF THE ORTHOGONAL POLYNOMIALS

Theorem 6 $\int_a^b P_n(x)\, x^r\, w(x)\, dx = \begin{cases} 0 & \text{if } r < n \\ |M_n|\,|M_{n-1}|^{-1} & \text{if } r = n \\ -|M_{n-1}|^{-1} N_{n+1,n} & \text{if } r = n+1 \end{cases}$

for $n = 0,1,2,\ldots$

These results are obtained by direct substitution of $P_n(x) \equiv (-)^n |M_{n-1}|^{-1} Q_n(x)$ in Thm 3

Theorem 7 $\int_a^b P_n^2(x)\, w(x)\, dx = |M_{n-1}|^{-1}|M_n| \qquad (n = 0, 1, 2\ldots)$

By substitution in Thm 4.

Theorem 8 $\int_a^b P_n(x)\, P_m(x)\, w(x)\, dx = 0$ for $m \neq n$

$(m,n = 0,1,2,\ldots)$

By substitution in Thm 4.

Theorem 9 $\int_a^b P_n^2(x)\, w(x)\, x\, dx = |M_{n-1}|^{-2}\left[N_{n,n-1}|M_n| - N_{n+1,n}|M_{n-1}|\right]$

$(n = 1,2,3,\ldots)$

Writing $x P_n(x)$ as a polynomial in x,

$$\text{LHS} = |M_{n-1}|^{-1}\int_a^b P_n(x) \sum_{r=0}^{n} N_{nr}\, x^{r+1}\, w(x)\, dx$$

$$= |M_{n-1}|^{-1}\int_a^b P_n(x)\left[N_{n,n-1}\, x^n + N_{nn}\, x^{n+1}\right] w(x)\, dx$$

$$= |M_{n-1}|^{-1}\left[N_{n,n-1}|M_n|\,|M_{n-1}|^{-1} - N_{nn}\, N_{n+1,n}|M_{n-1}|^{-1}\right]$$

by Thm 6

$$= |M_{n-1}|^{-2}\left[N_{n,n-1}|M_n| - N_{n+1,n}|M_{n-1}|\right]$$ by Thm 1 .

<u>Theorem 10</u> $\int_a^b P_n(x)\, P_{n-1}(x)\, w(x)\, x\, dx = |M_{n-1}|^{-1}|M_n|$

$(n = 1, 2, 3, \ldots)$

Writing $x\, P_{n-1}(x)$ as a polynomial in x,

$$\text{LHS} = |M_{n-2}|^{-1}\int_a^b P_n(x)\left[\sum_{r=0}^{n-1} N_{n-1,r}\, x^{r+1}\right] w(x)\, dx$$

$$= |M_{n-2}|^{-1}\int_a^b P_n(x)\, N_{n-1,n-1}\, x^n\, w(x)\, dx$$

$$= |M_{n-2}|^{-1}\, N_{n-1,n-1}\, |M_n|\, |M_{n-1}|^{-1} \qquad \text{by Thm 6}$$

$$= |M_n|\, |M_{n-1}|^{-1} \qquad \text{by Thm 1.}$$

<u>Theorem 11</u> $\int_a^b P_n(x)\, P_m(x)\, w(x)\, x\, dx = 0$, for $m < n-1$

$(m = 0, 1, 2, \ldots,\ n = 2, 3, 4, \ldots)$

The highest power of x in $x\, P_m(x)$ is $m+1 < n$ and so, by Thm 6, expression is zero.

RECURRENCE RELATION FOR ORTHOGONAL POLYNOMIALS

Theorem 12

$$-P_{n+1}(x) + \left[x + \frac{N_{n+1,n}}{|M_n|} - \frac{N_{n,n-1}}{|M_{n-1}|}\right]P_n(x) - \frac{|M_n|\,|M_{n-2}|}{|M_{n-1}|^2} P_{n-1}(x) = 0$$

$(n = 1,2,3,\ldots)$

$P_{n+1}(x) - x\,P_n(x)$ is of degree $\leqslant n$, and, by Thm 6,

$$\int_a^b \left[P_{n+1}(x) - x\,P_n(x)\right] x^r\, w(x)\,dx = 0 \quad \text{for } r=0,1,2,\ldots,\overline{n-2}.$$

Hence $P_{n+1}(x) - x\,P_n(x)$ can be expressed in the form

$u_n\,P_n(x) - v_n\,P_{n-1}(x)$ where u_n and v_n are constants.

i.e. $-P_{n+1}(x) + (x+u_n)\,P_n(x) - v_n\,P_{n-1}(x) = 0$ $(n=1,2,\ldots)$

Multiplying by $w(x)P_{n-1}(x)$, integrating, and using Thm 8,

$$\int_a^b P_n(x)P_{n-1}(x)w(x)x\,dx - v_n\int_a^b P_{n-1}^2(x)w(x)\,dx = 0$$

i.e. $v_n = |M_{n-1}|^{-2}\,|M_n|\,|M_{n-2}|$ for $n = 1,2,3\ldots$

by Thms 7 & 10.

Multiplying by $w(x)P_n(x)$, integrating, and using Thm 8,

$$\int_a^b P_n^2(x)w(x)x\,dx + u_n\int_a^b P_n^2(x)w(x)\,dx = 0$$

i.e. $u_n = \dfrac{N_{n+1,n}}{|M_n|} - \dfrac{N_{n,n-1}}{|M_{n-1}|}$ for $n = 1,2,3,\ldots$

by Thms 7 and 9

Since M_n is positive definite $v_n > 0$

Theorem 13 $\underline{-P_1(x) + \left[x + \frac{N_{10}}{|M_o|}\right] P_o(x) = 0}$

$$P_o(x) = 1$$

$$P_1(x) = -\frac{1}{m_o}\begin{vmatrix} 1 & x \\ m_o & m_1 \end{vmatrix} = x - \frac{m_1}{m_o} = x + \frac{N_{10}}{|M_o|}$$

Hence $\underline{-P_1(x) + (x+u_o)P_o(x) = 0}$ where $\underline{u_o = \frac{N_{10}}{|M_o|}}$

ORTHOGONAL POLYNOMIALS AS CONTINUANTS

Theorem 14 $P_n(x) =$

$$\begin{vmatrix} x + \frac{N_{10}}{|M_0|} & -1 & \cdot & \cdots & \cdot \\ -\frac{|M_1||M_{-1}|}{|M_0|^2} & x + \frac{N_{21}}{|M_1|} - \frac{N_{10}}{|M_0|} & -1 & \cdots & \cdot \\ \cdot & -\frac{|M_2||M_0|}{|M_1|^2} & x + \frac{N_{32}}{|M_2|} - \frac{N_{21}}{|M_1|} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdots & -\frac{|M_{n-1}||M_{n-3}|}{|M_{n-2}|^2} & & x + \frac{N_{n,n-1}}{|M_{n-1}|} - \frac{N_{n-1,n-2}}{|M_{n-2}|} \end{vmatrix}$$

From Thms 12 & 13 :

$$\begin{aligned} (x+u_o)P_o(x) - P_1(x) &= 0 \\ -v_1P_o(x) + (x+u_1)P_1(x) - P_2(x) &= 0 \\ -v_2P_1(x) + (x+u_2)P_2(x) - P_3(x) &= 0 \\ \cdots\cdots\cdots\cdots & \\ -v_{n-1}P_{n-2}(x) + (x+u_{n-1})P_{n-1}(x) - P_n(x) &= 0 \end{aligned}$$

so

$$\frac{(-)^n P_n(x)}{\begin{vmatrix} x+u_0 & -1 & \cdot & \cdots & \cdot \\ -v_1 & x+u_1 & -1 & \cdots & \cdot \\ \cdot & -v_2 & x+u_2 & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & & -v_{n-1} & x+u_{n-1} \end{vmatrix}} = \frac{P_0(x)}{\begin{vmatrix} -1 & \cdot & \cdot & \cdots & \cdot \\ x+u_1 & -1 & \cdot & \cdots & \cdot \\ -v_2 & x+u_2 & -1 & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & \cdots & -1 \end{vmatrix}}$$

i.e. $$P_n(x) = \begin{vmatrix} x+u_0 & -1 & \cdot & \cdots & \cdot \\ -v_1 & x+u_1 & -1 & \cdots & \cdot \\ \cdot & -v_2 & x+u_2 & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & x+u_{n-1} \end{vmatrix}$$ which is result required.

<u>Theorem 15</u>

$$P_n(x) = \frac{\left\{|M_0||M_1||M_2|\cdots|M_{n-2}|\right\}^{-2} \; |M_{n-1}|^{-1} \quad \times}{\begin{vmatrix} |M_{-1}||M_0|x + N_{10}|M_{-1}| & -|M_{-1}||M_1| & \cdot & \cdots & \cdot \\ -|M_{-1}||M_1| & |M_0||M_1|x + N_{21}|M_0| - N_{10}|M_1| & -|M_0||M_2| & \cdots & \cdot \\ \cdot & -|M_0||M_2| & \cdots & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & \cdots & -|M_{n-3}||M_{n-1}| & \begin{matrix}|M_{n-2}||M_{n-1}|x \\ +N_{n,n-1}|M_{n-2}| \\ -N_{n-1,n-2}|M_{n-1}|\end{matrix} \end{vmatrix}}$$

This form of $P_n(x)$ is obtained from the form of Thm 14 by multiplying the i^{th} row by $|M_{i-2}|$

the i^{th} col by $|M_{i-1}|$ $\quad (i = 1,2,3,..n)$

THE ZEROS OF $P_n(x)$

Theorem 16 $P_n(x) = \left| xI - D_n \right|$ where $D_n \equiv$

$$
\begin{bmatrix}
-\frac{N_{10}}{|M_0|} & \frac{\sqrt{|M_{-1}||M_1|}}{|M_0|} & \cdot & \cdots & \cdot \\
\frac{\sqrt{|M_{-1}||M_1|}}{|M_0|} & -\frac{N_{21}}{|M_1|} + \frac{N_{10}}{|M_0|} & \frac{\sqrt{|M_0||M_2|}}{|M_1|} & \cdots & \cdot \\
\cdot & \frac{\sqrt{|M_0||M_2|}}{|M_1|} & -\frac{N_{32}}{|M_2|} + \frac{N_{21}}{|M_1|} & \cdots & \cdot \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdot & \cdot & \cdots & & -\frac{N_{n,n-1}}{|M_{n-1}|} + \frac{N_{n-1,n-2}}{|M_{n-2}|}
\end{bmatrix}
$$

This form of $P_n(x)$ is obtained from the form of Thm 15 by dividing the i^{th} row and column by $\sqrt{(|M_{i-2}||M_{i-1}|)}$ $(i = 1,2,3,\ldots n)$

These divisors are real since $|M_n|$ is positive by Thm 2.

Theorem 17 The zeros of $P_n(x)$ are real and distinct and lie in the interval (a,b).

In Thm 16, $P_n(x)$ is exhibited as the characteristic polynomial of a real symmetric continuant.
Hence the zeros of $P_n(x)$ are real.

If the zeros are not distinct, there is a repeated root $x = k$ (say)

Then $P_n(x) \,/\, (x-k)^2$ is a polynomial of degree $n-2$

and so $\int_a^b P_n(x) \left[P_n(x) \; / \; (x-k)^2 \right] w(x) \, dx = 0$ by Thm 6

but $\int_a^b \left[P_n(x) \; / \; (x-k) \right]^2 w(x) \, dx > 0$ since the integrand $\geqslant 0$ in (a,b) and is not identically 0.

So we have a contradiction, and therefore the zeros of $P_n(x)$ are distinct.

To show that the zeros lie in (a,b) let k be any zero. Then $P_n(x) \; / \; x-k$ is a polynomial of degree $n-1$.

Hence $\int_a^b P_n(x) \left[P_n(x) \; / \; x-k \right] w(x) \, dx = 0$ by Thm 6

but $P_n^2(x) \; w(x) \geqslant 0$ in (a,b) and is not identically 0.

therefore $x-k$ must change sign in (a,b)

i.e. all the zeros of $P_n(x)$ lie in the interval.

<u>Theorem 18</u> <u>$P_n(x)$ and $P_{n-1}(x)$ cannot have a common zero.</u>

By Thm 12 :

$$- P_n(x) + (x+u_{n-1})P_{n-1}(x) - v_{n-1}P_{n-2}(x) = 0 \qquad (n = 2,3,\ldots)$$

where v_{n-1} is not 0.

Therefore, if $P_n(x)$ and $P_{n-1}(x)$ have a common zero k (say), k is also a zero of $P_{n-2}(x)$, and, step by step, it is a zero of $P_{n-3}(x), \ldots, P_2(x), P_1(x)$, and $P_0(x)$.

But $P_0(x)$ is a constant 1, therefore hypothesis is wrong, i.e. $P_n(x)$ and $P_{n-1}(x)$ do not have a common zero.

Theorem 19 The zeros of $P_{n-1}(x)$ interspace those of $P_n(x)$

In Thm 16 we exhibited $P_n(x)$ in the form $|xI - D_n|$

The leading principal minor of $|xI - D_n|$ is $P_{n-1}(x)$

This minor may be reduced orthogonally to diagonal canonical form thus :

$$\begin{bmatrix} H' & . \\ . & 1 \end{bmatrix}\begin{bmatrix} xI - D_n \end{bmatrix}\begin{bmatrix} H & . \\ . & 1 \end{bmatrix} = \begin{bmatrix} x-x_1 & . & . & \cdots & . & d_1 \\ . & x-x_2 & . & \cdots & . & d_2 \\ . & . & x-x_3 & \cdots & . & d_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & \cdots & x-x_{n-1} & d_{n-1} \\ d_1 & d_2 & d_3 & \cdots & d_{n-1} & x+d_n \end{bmatrix}$$

where $x_1, x_2, \ldots x_{n-1}$ are the zeros of $P_{n-1}(x)$ which we have already proved to be real and distinct, and we may assume that $x_1 < x_2 < x_3 \cdots < x_{n-1}$.

Now $d_i \neq 0$, $i = 1,2,3,\ldots n-1$, since if $d_i = 0$, $x = x_i$ would make $P_n(x)$ vanish and we have proved in Thm 18 that $P_n(x)$ and $P_{n-1}(x)$ do not have a common zero.

When $x=\infty$, $|xI-D_n| = \infty^n > 0$

" x_{n-1} " $-d_{n-1}{}^2(x_{n-1}-x_1)(\ldots(x_{n-1}-x_{n-2}) < 0$

" x_{n-2} " $-d_{n-2}{}^2(x_{n-2}-x_1)(\ldots(x_{n-2}-x_{n-1}) > 0$

. . . .

" x_1 " $-d_1{}^2(x_1 - x_2)(\ldots(x_1-x_{n-1})$ whichhas the sign of $(-)^{n-1}$

" $-\infty$ " $(-\infty)^n$ which has the sign of $(-)^n$

i.e. $|xI - D_n|$ has signs $+,-,+,-,\ldots$ at $x=\infty, x_{n-1}, \ldots, x_1, -\infty$ and therefore vanishes at n places intermediately.

Hence the zeros of $P_{n-1}(x)$ interspace those of $P_n(x)$

THE SECULAR POLYNOMIAL

(A SYMMETRICAL FORM DERIVED FROM THE BORDERED DETERMINANT)

Theorem 20 $P_n(x) = |M_{n-1}|^{-1}|M_{n-1}x - L_{n-1}|$

Applying to $Q_n(x)$ the transformations $col_i - col_{i-1}$ $(i = 2,3,4,\dots,n+1)$

we get

$$Q_n(x) = \begin{vmatrix} 1 & \cdot & \cdot & \cdots & \vdots \\ m_0 & m_1-m_0x & m_2-m_1x & \cdots & m_n-m_{n-1}x \\ m_1 & m_2-m_1x & m_3-m_2x & \cdots & m_{n+1}-m_nx \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n-m_{n-1}x & \cdot & \cdots & m_{2n-1}-m_{2n-2}x \end{vmatrix}$$

$$= (-)^n|M_{n-1}x - L_{n-1}|$$

Hence $P_n(x) = |M_{n-1}|^{-1}|M_{n-1}x - L_{n-1}|$

THE RATIONAL CANONICAL FORM

Theorem 21 $P_n(x) = \begin{vmatrix} x & \cdot & \cdot & \cdots & N_{n0}/|M_{n-1}| \\ -1 & x & \cdot & \cdots & N_{n1}/|M_{n-1}| \\ \cdot & -1 & x & \cdots & N_{n2}/|M_{n-1}| \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots\ x & N_{n,n-2}/|M_{n-1}| \\ \cdot & \cdot & \cdot & \cdots -1 & x + N_{n,n-1}/|M_{n-1}| \end{vmatrix}$

Applying the transformation $row_1 + x\,row_2 + x^2 row_3 + \dots + x^{n-1} row_n$ to RHS :

$$\begin{vmatrix} 0 & \cdot & \cdot & \cdots & \sum_{i=0}^{n-1}\left\{\frac{N_{ni}}{|M_{n-1}|}x^i\right\} + x^n \\ -1 & x & \cdot & \cdots & N_{n1}/|M_{n-1}| \\ \cdot & -1 & x & \cdots & N_{n2}/|M_{n-1}| \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & x + N_{n,n-1}/|M_{n-1}| \end{vmatrix}$$

$$= \sum_{i=0}^{n-1}\left\{\frac{N_{ni}}{|M_{n-1}|}x^i\right\} + x^n = P_n(x)$$

SUMMARY OF THE DIFFERENT FORMS FOR $P_n(x)$

(Thm 5) $\underline{P_n(x) = (-)^n |M_{n-1}|^{-1} |A_n|}$

where
$$A_n \equiv \begin{bmatrix} 1 & x & x^2 & \cdots & x^n \\ m_0 & m_1 & m_2 & \cdots & m_n \\ m_1 & m_2 & m_3 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-1} \end{bmatrix}$$

(Thm 14) $\underline{P_n(x) = |B_n|}$ where $B_n \equiv$

$$\begin{bmatrix} x + \frac{N_{10}}{|M_0|} & -1 & \cdot & \cdots & \cdot \\ -\frac{|M_1||M_{-1}|}{|M_0|^2} & x + \frac{N_{21}}{|M_1|} - \frac{N_{10}}{|M_0|} & -1 & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & -\frac{|M_{n-1}||M_{n-3}|}{|M_{n-2}|^2} & & x + \frac{N_{n,n-1}}{|M_{n-1}|} - \frac{N_{n-1,n-2}}{|M_{n-2}|} \end{bmatrix}$$

(Thm 15) $\underline{P_n(x) = \left[|M_0||M_1||M_2|\cdots|M_{n-2}|\right]^{-2} |M_{n-1}|^{-1} |C_n|}$ where $C_n \equiv$

$$\begin{bmatrix} |M_{-1}||M_0| x + N_{10}|M_{-1}| & -|M_{-1}||M_1| & \cdot & \cdots & \cdot \\ -|M_{-1}||M_1| & |M_0||M_1| x + N_{21}|M_0| - N_{10}|M_1| & -|M_0||M_2| & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdots & -|M_{n-3}||M_{n-1}| & & \begin{matrix} |M_{n-2}||M_{n-1}| x \\ + N_{n,n-1}|M_{n-2}| \\ - N_{n-1,n-2}|M_{n-1}| \end{matrix} \end{bmatrix}$$

(Thm 16) $\underline{P_n(x) = |xI - D_n|}$ where $D_n \equiv$

$$
\begin{bmatrix}
-\frac{N_{10}}{|M_0|} & \frac{\sqrt{|M_{-1}||M_1|}}{|M_0|} & \cdot & \cdots & \cdot \\
\frac{\sqrt{|M_{-1}||M_1|}}{|M_0|} & -\frac{N_{21}}{|M_1|}+\frac{N_{10}}{|M_0|} & \frac{\sqrt{|M_0||M_2|}}{|M_1|} & \cdots & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdots & & -\frac{N_{n,n-1}}{|M_{n-1}|}+\frac{N_{n-1,n-2}}{|M_{n-2}|}
\end{bmatrix}
$$

(Thm 20) $\underline{P_n(x) = |M_{n-1}|^{-1} |E_n|}$ where $E_n \equiv M_{n-1}x - L_{n-1} =$

$$
\begin{bmatrix}
m_0x - m_1 & m_1x - m_2 & m_2x - m_3 & \cdots & m_{n-1}x - m_n \\
m_1x - m_2 & m_2x - m_3 & m_3x - m_4 & \cdots & m_nx - m_{n+1} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
m_{n-1}x - m_n & m_nx - m_{n+1} & \cdots & \cdots & m_{2n-2}x - m_{2n-1}
\end{bmatrix}
$$

(Thm 21) $\underline{P_n(x) = |F_n|}$

where $F_n \equiv$

$$
\begin{bmatrix}
x & \cdot & \cdot & \cdots & N_{n0}/|M_{n-1}| \\
-1 & x & \cdot & \cdots & N_{n1}/|M_{n-1}| \\
\cdot & -1 & x & \cdots & N_{n2}/|M_{n-1}| \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdots & x & N_{n,n-2}/|M_{n-1}| \\
\cdot & \cdot & \cdots & -1 & x + N_{n,n-1}/|M_{n-1}|
\end{bmatrix}
$$

TRANSFORMATIONS CONNECTING DIFFERENT FORMS OF $P_n(x)$

Theorem 22 $\begin{bmatrix} 1 & 0 \\ m & -E_n \end{bmatrix} = A_n G_n$ where $m \equiv \{m_0, m_1, m_2, \dots m_{n-1}\}$

$0 \equiv \{0, 0, 0, \dots 0\}$

and $G_n \equiv \begin{bmatrix} 1 & -x & \cdot & \cdots & \cdot \\ \cdot & 1 & -x & \cdots & \cdot \\ \cdot & \cdot & 1 & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & 1 \end{bmatrix}$

This result is clear from the proof of Thm 20, since the transformations $col_i - x\, col_{i-1}$ $(i = 2, 3, 4, \dots n+1)$ are equivalent to post-multiplication by G_n.

Theorem 23 $M_{n-1} F_n = E_n$

$$M_{n-1}F_n = M_{n-1}x + M_{n-1}\begin{bmatrix} \cdot & \cdot & \cdot & \cdots & N_{no}/|M_{n-1}| \\ -1 & \cdot & \cdot & \cdots & N_{n1}/|M_{n-1}| \\ \cdot & -1 & \cdot & \cdots & N_{n2}/|M_{n-1}| \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & N_{n,n-1}/|M_{n-1}| \end{bmatrix}$$

$$= M_{n-1}x - \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_{n-1} & \sum_0^{n-1} m_r N_{nr}/|M_{n-1}| \\ m_2 & m_3 & m_4 & \cdots & m_n & \sum_0^{n-1} m_{r+1} N_{nr}/|M_{n-1}| \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-2} & \sum_0^{n-1} m_{r+n-1} N_{nr}/|M_{n-1}| \end{bmatrix}$$

$$= M_{n-1}x - L_{n-1}$$

/since

since $\sum_{r=0}^{n-1} m_{r+i} N_{nr} / |M_{n-1}|$

$$= |M_{n-1}|^{-1} \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} & m_i \\ m_1 & m_2 & m_3 & \cdots & m_n & m_{i+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-2} & m_{i+n-1} \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-1} & 0 \end{vmatrix}$$

$$= |M_{n-1}|^{-1} \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} & 0 \\ m_1 & m_2 & m_3 & \cdots & m_n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-2} & 0 \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-1} & -m_{i+n} \end{vmatrix}$$

(applying $col_{n+1} - col_{i+1}$) for $i = 0,1,2,\ldots,n-1$.

$$= \quad - m_{i+n} \ .$$

<u>Theorem 24</u> $C_n = \operatorname{diag}(|M_{-1}|, |M_0|, |M_1|, \ldots |M_{n-2}|)\ B_n\ \operatorname{diag}(|M_0|, |M_1|, |M_2|, \ldots |M_{n-1}|)$

This result is clear from Thm 15 since the transformations $|M_{i-2}|\, row_i$ and $|M_{i-1}|\, col_i$ are equivalent to pre- and post-multiplication by the matrices indicated.

<u>Theorem 25</u> $[xI - D_n] = H_n C_n H_n$

where $H_n = \operatorname{diag}\left[\sqrt{|M_{-1}||M_0|}, \sqrt{|M_0||M_1|}, \ldots \sqrt{|M_{n-2}||M_{n-1}|}\right]^{-1}$

This result is clear from Thm 16 since dividing the i^{th} row and column by $\sqrt{|M_{i-2}||M_{i-1}|}$ is equivalent to pre- and post-multiplication by H_n .

Theorem 26 $C_n = K_n' E_n K_n$

where $K_n = \begin{bmatrix} N_{00} & N_{10} & N_{20} & N_{30} & \cdots & N_{n-1,0} \\ \cdot & N_{11} & N_{21} & N_{31} & \cdots & N_{n-1,1} \\ \cdot & \cdot & N_{22} & N_{32} & \cdots & N_{n-1,2} \\ \cdots & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdots & N_{n-1,n-1} \end{bmatrix}$

$$K_n' E_n K_n = K_n' M_{n-1} K_n x - K_n' L_{n-1} K_n$$

$$K_n' M_{n-1} = \begin{bmatrix} N_{00} & \cdot & \cdot & \cdots & \cdot \\ N_{10} & N_{11} & \cdot & \cdots & \cdot \\ N_{20} & N_{21} & N_{22} & \cdots & \cdot \\ \cdots & & & & \\ N_{n-1,0} & N_{n-1,1} & & \cdots & N_{n-1,n-1} \end{bmatrix} \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ m_1 & m_2 & m_3 & \cdots & m_n \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} \\ \cdots & & & & \\ m_{n-1} & m_n & m_{n+1} & \cdots & m_{2n-2} \end{bmatrix}$$

The $(i,j)^{th}$ element in this product $= \sum_{r=0}^{i-1} m_{r+j-1} N_{i-1,r}$

$$= \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_{i-2} & m_{j-1} \\ m_1 & m_2 & m_3 & \cdots & m_{i-1} & m_j \\ m_2 & m_3 & m_4 & \cdots & m_i & m_{j+1} \\ \cdots & & & & & \\ m_{i-1} & m_i & m_{i+1} & \cdots & m_{2i-3} & m_{j+i-2} \end{vmatrix} = \begin{cases} 0 & \text{if } j \leqslant i-1 \\ m_{0246\ldots(2i-4)(j+i-2)} & \text{if } j \geqslant i \end{cases}$$

where $m_{abcd\ldots k}$ is defined as the determinant whose diagonal is $m_a\ m_b\ m_c \ldots m_k$ and whose columns have suffixes which are consecutive from row to row.

$$\therefore K_n' M_{n-1} K_n = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{n-1} \\ \cdot & m_{02} & m_{03} & \cdots & m_{0n} \\ \cdot & \cdot & m_{024} & \cdots & m_{02\overline{n+1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & m_{02..\overline{2n-2}} \end{bmatrix} \begin{bmatrix} N_{00} & N_{10} & N_{20} & \cdots & N_{n-1,0} \\ \cdot & N_{11} & N_{21} & \cdots & N_{n-1,1} \\ \cdot & \cdot & N_{22} & \cdots & N_{n-1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & N_{n-1,n-1} \end{bmatrix}$$

$$= \begin{bmatrix} m_0 N_{00} & ? & ? & \cdots & ? \\ \cdot & m_{02} N_{11} & ? & \cdots & ? \\ \cdot & \cdot & m_{024} N_{22} & \cdots & ? \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & m_{024..2n-2} N_{n-1,n-1} \end{bmatrix}$$

Since M_{n-1} is symmetrical, so also is $K_n' M_{n-1} K_n$, and using the results $m_{024..2n} = |M_n|$ and $N_{nn} = |M_{n-1}|$,

$$K_n' M_{n-1} K_n = \begin{bmatrix} |M_{-1}||M_0| & \cdot & \cdot & \cdots & \cdot \\ \cdot & |M_0||M_1| & \cdot & \cdots & \cdot \\ \cdot & \cdot & |M_1||M_2| & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & |M_{n-2}||M_{n-1}| \end{bmatrix}$$

$$K_n' L_{n-1} = \begin{bmatrix} N_{00} & \cdot & \cdot & \cdots & \cdot \\ N_{10} & N_{11} & \cdot & \cdots & \cdot \\ N_{20} & N_{21} & N_{22} & \cdots & \cdot \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ N_{n-1,0} & N_{n-1,1} & \cdots & & N_{n-1,n-1} \end{bmatrix} \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_n \\ m_2 & m_3 & m_4 & \cdots & m_{n+1} \\ m_3 & m_4 & m_5 & \cdots & m_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_n & m_{n+1} & m_{n+2} & \cdots & m_{2n-1} \end{bmatrix}$$

The $(i,j)^{th}$ element in this product $= \sum_{r=0}^{i-1} m_{r+j} N_{i-1,r}$

$$= \begin{vmatrix} m_0 & m_1 & m_2 & \cdots & m_{i-2} & m_j \\ m_1 & m_2 & m_3 & \cdots & m_{i-1} & m_{j+1} \\ m_2 & m_3 & m_4 & \cdots & m_i & m_{j+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{i-1} & m_i & m_{i+1} & \cdots & m_{2i-3} & m_{j+i-1} \end{vmatrix} = \begin{cases} 0 & \text{if } j \leqslant i-2 \\ m_{0246\ldots(2i-4)(j+i-1)} & \text{if } j \geqslant i-1 \end{cases}$$

$\therefore K_n' L_{n-1} K_n =$

$$\begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_{n-1} & m_n \\ m_{02} & m_{03} & m_{04} & \cdots & m_{0n} & m_{0(n+1)} \\ \cdot & m_{024} & m_{025} & \cdots & m_{02\overline{n+1}} & m_{02(n+2)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & m_{024.\overline{2n-2}} & m_{024..\overline{2n-1}} \end{bmatrix} \begin{bmatrix} N_{00} & N_{10} & N_{20} & \cdots & N_{n-1,0} \\ \cdot & N_{11} & N_{21} & \cdots & N_{n-1,1} \\ \cdot & \cdot & N_{22} & \cdots & N_{n-1,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & N_{n-1,n-1} \end{bmatrix}$$

$$= \begin{bmatrix} m_1N_{00} & ? & ? & \cdots & ? & ? \\ m_{02}N_{00} & m_{02}N_{10}+m_{03}N_{11} & ? & \cdots & ? & ? \\ \cdot & m_{024}N_{11} & m_{024}N_{21}+m_{025}N_{22} & \cdots & ? & ? \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdots & m_{024.2n-2}N_{n-2,n-2} & * \end{bmatrix}.$$

where $* = m_{024..2n-2}N_{n-1,n-2}+m_{024..2n-1}N_{n-1,n-1}$

Since L_{n-1} is symmetrical, so also is $K_n' L_{n-1} K_n$, and using the results $m_{024\ldots 2n} = |M_n|$, $N_{nn} = |M_{n-1}|$, and

$m_{024\ldots(2n-2)(2n+1)} = -N_{n+1,n}$,

$$K_n' L_{n-1} K_n = \begin{bmatrix} -N_{10}|M_{-1}| & |M_{-1}||M_1| & \cdot & \cdots & \cdot \\ |M_{-1}||M_1| & N_{10}|M_1| - N_{21}|M_0| & |M_0||M_2| & \cdots & \cdot \\ \cdot & |M_0||M_2| & N_{21}|M_2| - N_{32}|M_1| & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdots & N_{n-1,n-2}|M_{n-1}| - N_{n,n-1}|M_{n-2}| \end{bmatrix}$$

Hence $\quad K_n' E_n K_n = K_n' M_{n-1} K_n x - K_n' L_{n-1} K_n$

$$= C_n$$

PART II

CALCULATION OF N_{nr} FOR THE SPECIAL CASES.

CALCULATION OF CERTAIN DETERMINANTS

Theorem 27

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a+1} & \frac{1}{a+2} & \cdots & \frac{1}{a+n-1} \\ \frac{1}{a+1} & \frac{1}{a+2} & \frac{1}{a+3} & \cdots & \frac{1}{a+n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a+r-1} & & \cdots & & \frac{1}{a+n+r-2} \\ \frac{1}{a+r+1} & & \cdots & & \frac{1}{a+n+r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a+n} & & \cdots & & \frac{1}{a+2n-1} \end{vmatrix} =$$

$$\frac{(1!2!\ldots(n-1)!)^2\, n!\, ((a-1)!a!\ldots(a+n-2)!)\, (a+n+r-1)!}{((a+n)!(a+n+1)!\ldots(a+2n-1)!)\, (n-r)!\, (a+r-1)!\, r!}$$

By Cauchy's Double Alternant ,

$$\begin{vmatrix} \frac{1}{x_1-y_1} & \frac{1}{x_1-y_2} & \cdots & \frac{1}{x_1-y_k} \\ \frac{1}{x_2-y_1} & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{x_k-y_1} & \frac{1}{x_k-y_2} & \cdots & \frac{1}{x_k-y_k} \end{vmatrix} = (-)^{\frac{1}{2}k(k-1)} \times \frac{(x_k-x_{k-1})\ldots(x_2-x_1)(y_k-y_{k-1})\ldots(y_2-y_1)}{(x_k-y_k)\ldots\ldots(x_1-y_1)}$$

/writing

Writing

$$x_1, x_2, \ldots x_k = a+n, a+n+1, \ldots a+n+r-1, a+n+r+1, \ldots a+2n$$

$$y_1, y_2, \ldots y_k = n,\ n-1,\ n-2,\ \ldots\ 3,\ 2,\ 1 \quad ,$$

LHS of Theorem

$$= \frac{\left[(-)^{\frac{1}{2}n(n-1)}\right]^2 \dfrac{n!(n-1)!\ldots(r+1)!(r-1)!\ldots 2!1!}{(n-r)!}(n-1)!(n-2)!\ldots 2!1!}{\dfrac{(a+2n-1)!}{(a+n-1)!}\dfrac{(a+2n-2)!}{(a+n-2)!}\ldots\dfrac{(a+n+r)!}{(a+r)!}\dfrac{(a+n+r-2)!}{(a+r-2)!}\ldots\dfrac{(a+n-1)!}{(a-1)!}}$$

$$= \frac{(1!2!3!\ldots(n-1)!)^2\ n!\ \left((a-1)!a!\ldots(a+n-2)!\right)\ (a+n+r-1)!}{\left((a+n)!(a+n+1)!\ldots(a+2n-1)!\right)\ (n-r)!\ (a+r-1)!\ r!}$$

Theorem 28

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{r} & \frac{1}{r+1} & & \cdots & \frac{1}{n+r-1} \\ \frac{1}{r+2} & \frac{1}{r+3} & & \cdots & \frac{1}{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+1} & \frac{1}{n+2} & & \cdots & \frac{1}{2n} \end{vmatrix} = \frac{(1!2!3!\ldots(n-1)!)^3\ n!\ (n+r)!}{\left((n+1)!(n+2)!\ldots(2n)!\right)\ (n-r)!\ (r!)^2}$$

This result is obtained by writing $a = 1$ in Thm 27.

Theorem 29

$$\begin{vmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{r+1} & \frac{1}{r+2} & & \cdots & \frac{1}{n+r} \\ \frac{1}{r+3} & \frac{1}{r+4} & & \cdots & \frac{1}{n+r+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+2} & \frac{1}{n+3} & & \cdots & \frac{1}{2n+1} \end{vmatrix} = \frac{(1!2!3!\ldots(n-1)!)^3\ (n!)^2\ (n+r+1)!}{((n+2)!(n+3)!\ldots(2n+1)!)(n-r)!r!(r+1)!}$$

This result is obtained by putting $a = 2$ in Thm 27.

PROPERTIES OF n?

Definition

$$\text{Let} \quad n? \equiv \begin{cases} n(n-2)(n-4)\ldots4.2 & \text{if } n \text{ is even} \\ n(n-2)(n-4)\ldots3.1 & \text{if } n \text{ is odd .} \end{cases}$$

The choice of the symbol ? is based on the following idea:

factorial ! could be a corruption of $\underset{\bullet}{1}$ indicating that the factors decrease by 1 , and similarly ? would be a corruption of $\underset{\bullet}{2}$ indicating that the factors decrease by 2.

Theorem 30

(i) $n?\,(n-1)? = n!$

(ii) If n is even, $n? = 2^{\frac{1}{2}n}\,(\tfrac{1}{2}n)!$

(iii) If n is odd, $n? = \dfrac{n!}{2^{\frac{1}{2}(n-1)}\,(\tfrac{1}{2}n-\tfrac{1}{2})!}$

These results are evident from the definition of n? .

CALCULATION OF CERTAIN DETERMINANTS (continued)

Theorem 31

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a+2} & \frac{1}{a+4} & \cdots & \frac{1}{a+2n-2} \\ \frac{1}{a+2} & \frac{1}{a+4} & \frac{1}{a+6} & \cdots & \frac{1}{a+2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a+2r-2} & \frac{1}{a+2r} & & \cdots\cdots & \frac{1}{a+2n+2r-4} \\ \frac{1}{a+2r+2} & \frac{1}{a+2r+4} & & \cdots\cdots & \frac{1}{a+2n+2r} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a+2n} & \frac{1}{a+2n+2} & & \cdots\cdots & \frac{1}{a+4n-2} \end{vmatrix} =$$

$$\frac{(2?4?\ldots(2n-2)?)^2\ (2n)?\ ((a-2)?a?\ldots(a+2n-4)?)\ (a+2n+2r-2)?}{((a+2n)?(a+2n+2)?\ldots(a+4n-2)?)\ (2n-2r)?\ (a+2r-2)?\ (2r)?}$$

In Cauchy's Double Alternant, writing

$x_1, x_2, \ldots x_k = a+2n, a+2n+2, \ldots a+2n+2r-2, a+2n+2r+2, \ldots a+4n$

$y_1, y_2, \ldots y_k = 2n,\ 2n-2,\ 2n-4,\ \ldots\ 6,\ 4,\ 2,$

and working in a manner similar to the proof of Thm 27, we get the RHS required - which is the same as the RHS of Thm 27 with ! replaced by ? , and everything except a doubled.

Theorem 32

$$\frac{\begin{vmatrix} 1 & \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2n-1} \\ \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2r-1} & \frac{1}{2r+1} & & \cdots & \frac{1}{2n+2r-3} \\ \frac{1}{2r+3} & \frac{1}{2r+5} & & \cdots & \frac{1}{2n+2r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & & \cdots & \frac{1}{4n-1} \end{vmatrix}}{} =$$

$$\frac{2^{2n^2-2n+1}\,(1!2!\ldots(2n-1)!)(2!4!\ldots(2n-2)!)\;n!\;(2n+2r-1)!}{((2n+1)!(2n+3)!\ldots(4n-1)!)\;(n-r)!\;(2r)!\;(n+r-1)!}$$

Putting $a = 1$ in Thm 31 ,

$$\text{LHS} = \frac{(2?4?\ldots(2n-2)?)^2\;(2n)?\;(1?3?\ldots(2n-3)?)\;(2n+2r-1)?}{((2n+1)?(2n+3)?\ldots(4n-1)?)\;(2n-2r)?\;(2r-1)?\;(2r)?}$$

$$= \frac{\left(2^{\frac{1}{2}n(n-1)}1!2!\ldots(n-1)!\right)\;2^n n!\;(2!4!\ldots(2n-2)!)\dfrac{(2n+2r-1)!}{2^{n+r-1}(n+r-1)!}}{\dfrac{(2n+1)!}{n!}\;\dfrac{(2n+3)!}{(n+1)!}\;\cdots\;\dfrac{(4n-1)!}{(2n-1)!}\quad 2^{-\frac{1}{2}n(3n-1)}\,2^{n-r}(n-r)!\,(2r)!}$$

by Thm 30

$= \text{RHS}$

<u>Theorem 33</u>

$$\begin{vmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2n+1} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2r+1} & \frac{1}{2r+3} & \cdots 1 \cdots & \cdots & \frac{1}{2n+2r-1} \\ \frac{1}{2r+5} & \frac{1}{2r+7} & \cdots & \cdots & \frac{1}{2n+2r+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \cdots & \frac{1}{4n+1} \end{vmatrix} =$$

$$\frac{2^{2n^2-n}\,(1!2!3!\ldots(2n)!)(1!3!5!\ldots(2n-1)!)\,(2n+2r+1)!}{((2n+3)!(2n+5)!\ldots(4n+1)!)\,(n-r)!\,(2r+1)!\,(n+r)!}$$

Putting $a = 3$ in Thm 31 ,

$$\text{LHS} = \frac{(2?4?\ldots(2n-2)?)^2\,(2n)?\,(1?3?5?\ldots(2n-1)?)\,(2n+2r+1)?}{((2n+3)?(2n+5)?\ldots(4n+1)?)\,(2n-2r)?\,(2r+1)?\,(2r)?}$$

$$= \frac{(2^{\frac{1}{2}n(n-1)}1!2!\ldots(n-1)!)\,2^n n!\,(1!3!5!\ldots(2n-1)!)\,\frac{(2n+2r+1)!}{(n+r)!\,2^{n+r}}}{\frac{(2n+3)!}{(n+1)!}\,\frac{(2n+5)!}{(n+2)!}\ldots\frac{(4n+1)!}{(2n)!}\,2^{-\frac{1}{2}n(3n+1)}\,2^{n-r}\,(n-r)!\,(2r+1)!}$$

by Thm 30

$= \text{RHS}$

Theorem 34

$$\begin{vmatrix} 1 & x_1 & x_1(x_1+d) & \dots & x_1(x_1+d)(x_1+2d)\dots(x_1+\overline{k-2}d) \\ 1 & x_2 & x_2(x_2+d) & \dots & x_2(x_2+d)(x_2+2d)\dots(x_2+\overline{k-2}d) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_k & x_k(x_k+d) & \dots & x_k(x_k+d)(x_k+2d)\dots(x_k+\overline{k-2}d) \end{vmatrix}$$

$$= (x_k-x_{k-1})(x_k-x_{k-2})\dots(x_k-x_1)((x_{k-1}-x_{k-2})\dots\dots(x_2-x_1)$$

The LHS can be reduded by linear transformations on the columns to

$$\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & \dots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \dots & x_2^{k-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_k & x_k^2 & x_k^3 & \dots & x_k^{k-1} \end{vmatrix} = \text{RHS}$$

Theorem 35

$$\begin{vmatrix} \frac{1}{2.3} & \frac{1}{3.4} & \frac{1}{4.5} & \cdots & \frac{1}{(n+1)(n+2)} \\ \frac{1}{3.4} & \frac{1}{4.5} & \frac{1}{5.6} & \cdots & \frac{1}{(n+2)(n+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(r+1)(r+2)} & & \cdots\cdots & & \frac{1}{(n+r)(n+r+1)} \\ \frac{1}{(r+3)(r+4)} & & \cdots\cdots & & \frac{1}{(n+r+2)(n+r+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(n+2)(n+3)} & & \cdots\cdots & & \frac{1}{(2n+1)(2n+2)} \end{vmatrix} =$$

$$\frac{(1!2!3!\ldots n!)^3\ (n+1)!\ (n+r+2)!}{((n+2)!(n+3)!\ldots(2n+2)!)\ (r+1)!\ r!\ (n-r)!}$$

$$\text{LHS} = \begin{vmatrix} \frac{2!}{2.3.4} & \frac{2!}{3.4.5} & \cdots & \frac{2!}{n(n+1)(n+2)} & \frac{1}{(n+1)(n+2)} \\ & & \text{etc.} & & \end{vmatrix}$$

(applying $col_i - col_{i+1}$ $\quad i = 1,2,\ldots(n-1)$)

$$= \begin{vmatrix} \frac{3!}{2.3.4.5} & \frac{3!}{3.4.5.6} & \cdots & \frac{3!}{(n-1)n(n+1)(n+2)} & \frac{2!}{n(n+1)(n+2)} & \frac{1}{(n+1)(n+2)} \\ & & \text{etc.} & & & \end{vmatrix}$$

(applying $col_i - col_{i+1}$ $\quad i = 1,2,\ldots(n-2)$)

$$= \begin{vmatrix} \frac{n!}{2.3.4\ldots(n+2)} & \frac{(n-1)!}{3.4.5\ldots(n+2)} & \cdots & \frac{1}{(n+1)(n+2)} \\ \frac{n!}{3.4.5\ldots(n+3)} & \cdots\cdots & & \\ \cdots\cdots\cdots\cdots & & & \\ \frac{n!}{(r+1)\ldots(n+r+1)} & \cdots & \text{etc.} & \\ \frac{n!}{(r+3)\ldots(n+r+3)} & \cdots & & \\ \cdots\cdots\cdots\cdots & & & \\ \frac{n!}{(n+2)\ldots(2n+2)} & \cdots\cdots\cdots\cdots & & \frac{1}{(2n+1)(2n+2)} \end{vmatrix}$$

$$= \frac{n!\ (n-1)!\ (n-2)!\ \ldots\ 2!\ 1!}{\frac{(n+2)!}{1!}\ \frac{(n+3)!}{2!}\ \cdots\ \frac{(n+r+1)!}{r!}\ \frac{(n+r+3)!}{(r+2)!}\ \cdots\ \frac{(2n+2)!}{(n+1)!}} \times$$

$$\begin{vmatrix} 1 & 2 & 2.3 & \cdots & 2.3.4.\ldots.n \\ 1 & 3 & 3.4 & \cdots & 3.4.5\ldots(n+1) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & r+1 & (r+1)(r+2) & \cdots\cdots & \\ 1 & r+3 & (r+3)(r+4) & \cdots\cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & n+2 & (n+2)(n+3) & \cdots & (n+2)(n+3)\ldots 2n \end{vmatrix}$$

$$= \frac{(1!2!\ldots n!)^2\ (n+1)!\ (n+r+2)!}{((n+2)!(n+3)!\ldots(2n+2)!)\ (r+1)!} \times \frac{n!(n-1)!\ldots 2!1!}{(n-r)!\ r!}$$

by Thm 34

$= \quad$ RHS

Theorem 36

$$\frac{\begin{vmatrix} a! & (a+1)! & \cdots & (a+n-1)! \\ (a+1)! & (a+2)! & \cdots & (a+n)! \\ \cdots & \cdots & \cdots & \cdots \\ (a+r-1)! & & \cdots\cdots & (a+n+r-2)! \\ (a+r+1)! & & \cdots\cdots & (a+n+r)! \\ \cdots & \cdots & \cdots & \cdots \\ (a+n)! & & \cdots\cdots & (a+2n-1)! \end{vmatrix}}{} = \frac{(1!2!3!\ldots n!)(a!(a+1)!\ldots(a+n)!)}{(n-r)!\ \ r!\ \ (a+r)!}$$

$$\text{LHS} = a!(a+1)!\ldots(a+r-1)!(a+r+1)!\ldots(a+n)! \times \begin{vmatrix} 1 & (a+1) & (a+1)(a+2) & \cdots\cdots \\ 1 & (a+2) & (a+2)(a+3) & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (a+r) & \cdots\cdots & \cdots\cdots \\ 1 & (a+r+1) & \cdots\cdots & \cdots\cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & (a+n+1) & \cdots\cdots & \cdots\cdots \end{vmatrix}$$

$$= \frac{a!(a+1)!\ldots(a+n)!}{(a+r)!} \times \frac{n!(n-1)!\ldots 2!1!}{(n-r)!\ \ r!} \qquad \text{by Thm 34}$$

$$= \text{RHS}$$

Theorem 37

$$\begin{vmatrix} 0! & 1! & 2! & \dots & (n-1)! \\ 1! & 2! & 3! & \dots & n! \\ \dots & \dots & \dots & \dots & \dots \\ (r-1)! & \dots & \dots & \dots & \dots \\ (r+1)! & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ n! & \dots & \dots & \dots & (2n-1)! \end{vmatrix} = \frac{(1!2!3!\dots n!)^2}{(n-r)!\ (r!)^2}$$

This result is obtained by putting $a = 0$ in Thm 36

Theorem 38

$$\begin{vmatrix} 1! & 2! & 3! & \dots & n! \\ 2! & 3! & 4! & \dots & (n+1)! \\ \dots & \dots & \dots & \dots & \dots \\ r! & \dots & \dots & \dots & \dots \\ (r+2)! & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (n+1)! & \dots & \dots & \dots & \dots \end{vmatrix} = \frac{(1!2!3!\dots n!)^2\ (n+1)!}{(n-r)!\ (r+1)!\ r!}$$

This result is obtained by putting $a = 1$ in Thm 36

Theorem 39

$$
\frac{\begin{vmatrix}
a? & (a+2)? & (a+4)? & \cdots & (a+2n-2)? \\
(a+2)? & (a+4)? & (a+6)? & \cdots & (a+2n)? \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
(a+2r-2)? & \cdots & \cdots & \cdots & \cdots \\
(a+2r+2)? & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
(a+2n)? & \cdots & \cdots & \cdots & (a+4n-2)?
\end{vmatrix}}{} = \frac{2^{\frac{1}{2}n(n-1)}\,(1!2!3!\ldots n!)(a?(a+2)?\ldots(a+2n)?)}{(n-r)!\;\; r!\;\; (a+2r)?}
$$

$$
\text{LHS} = a?(a+2)?\ldots(a+2r-2)?(a+2r+2)?\ldots(a+2n)? \times
\begin{vmatrix}
1 & a+2 & (a+2)(a+4) & \cdots \\
1 & a+4 & (a+4)(a+6) & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
1 & a+2r & \cdots & \cdots \\
1 & a+2r+4 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
1 & a+2n+2 & \cdots & \cdots
\end{vmatrix}
$$

$$
= \frac{a?(a+2)?\ldots(a+2n)?}{(a+2r)?} \times \frac{(2n)?(2n-2)?\ldots4?2?}{(2n-2r)?\;(2r)?} \quad \text{by Thm34}
$$

$$
= \frac{a?(a+2)?\ldots(a+2n)?}{(a+2r)?} \times \frac{n!(n-1)!\ldots2!1!}{(n-r)!\;r!} \times 2^{\frac{1}{2}n(n-1)} \quad \text{by Thm 30}
$$

$$
= \text{RHS}
$$

Theorem 40

$$\begin{vmatrix} 1 & 1? & 3? & \dots & (2n-3)? \\ 1? & 3? & 5? & \dots & (2n-1)? \\ \dots & \dots & \dots & \dots & \dots \\ (2r-3)? & \dots & \dots & \dots & \dots \\ (2r+1)? & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (2n-1)? & ,,,,, & \dots & \dots & (4n-3)? \end{vmatrix} = \frac{2^{r-1}\,(1!3!\dots(2n-1)!)\;n!\;(r-1)!}{(n-r)!\;\;r!\;\;(2r-1)!}$$

Putting $a = -1$ in Thm 39,

$$\text{LHS} = \frac{2^{\frac{1}{2}n(n-1)}\;(1!2!\dots n!)\;(1?3?5?\dots(2n-1)?)}{(n-r)!\;\;r!\;\;(2r-1)?}$$

$$= \frac{2^{\frac{1}{2}n(n-1)}\;(1!2!\dots n!)\;(1!3!5!\dots(2n-1)!)\;2^{r-1}\;(r-1)!}{(n-r)!\;r!\qquad 2^{\frac{1}{2}n(n-1)}\;(1!2!\dots(n-1)!)\;\;(2r-1)!}$$

$= \text{RHS}$ by Thm 30

Theorem 41

$$\begin{vmatrix} 1? & 3? & 5? & \dots(2n-1)? \\ 3? & 5? & 7? & \dots(2n+1)? \\ \dots & \dots & \dots & \dots \\ (2r-1)? & \dots & \dots & \dots \\ (2r+3)? & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ (2n+1)? & \dots & \dots & (4n-1)? \end{vmatrix} = \frac{2^{-n+r}\;(1!3!5!\dots(2n+1)!)}{(n-r)!\;\;(2r+1)!}$$

Putting $a = 1$ in Thm 39,

$$\text{LHS} = \frac{2^{\frac{1}{2}n(n-1)}\;(1!2!\dots n!)(1!3!5!\dots(2n+1)!)\;2^{r}\;r!}{(n-r)!\quad r!\quad 2^{\frac{1}{2}n(n+1)}\;(1!2!\dots n!)(2r+1)!} \quad \text{by Thm 30}$$

$= \text{RHS}$

EVALUATION OF N_{nr} WHEN $m_{2k+1} = 0$

Theorem 42

When $m_{2k+1} = 0$, $N_{nr} =$

(i) 0 when n is even, r is odd.

(ii) 0 when n is odd, r is even.

(iii) $$(-)^{\frac{1}{2}(n-r)} \begin{vmatrix} m_0 & \cdots & m_{n-2} \\ m_2 & \cdots & m_n \\ \cdots & \cdots & \cdots \\ m_{r-2} & \cdots & m_{n+r-4} \\ m_{r+2} & \cdots & m_{n+r} \\ \cdots & \cdots & \cdots \\ m_n & \cdots & m_{2n-2} \end{vmatrix} \begin{vmatrix} m_2 & \cdots & m_n \\ m_4 & \cdots & m_{n+2} \\ \cdots & \cdots & \cdots \\ m_n & \cdots & m_{2n-2} \end{vmatrix}$$

when n,r are even.

(iv) $$(-)^{\frac{1}{2}(n-r)} \begin{vmatrix} m_0 & \cdots & m_{n-1} \\ m_2 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots \\ m_{n-1} & \cdots & m_{2n-2} \end{vmatrix} \begin{vmatrix} m_2 & \cdots & m_{n-1} \\ m_4 & \cdots & m_{n+1} \\ \cdots & \cdots & \cdots \\ m_{r-1} & \cdots & m_{n+r-4} \\ m_{r+3} & \cdots & m_{n+r} \\ \cdots & \cdots & \cdots \\ m_{n+1} & \cdots & m_{2n-2} \end{vmatrix}$$

when n,r are odd.

/(i)

(i) $$N_{nr} = - \begin{vmatrix} m_o & 0 & \cdots & m_{n-2} & 0 \\ 0 & m_2 & \cdots & 0 & m_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{r-1} & 0 & \cdots & m_{n+r-3} & 0 \\ m_{r+1} & 0 & \cdots & m_{n+r-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & m_n & \cdots & 0 & m_{2n-2} \\ m_n & 0 & \cdots & m_{2n-2} & 0 \end{vmatrix} = \pm \begin{vmatrix} \begin{matrix} m_o & \cdots & m_{n-2} \\ \cdots & \cdots & \cdots \\ m_n & \cdots & m_{2n-2} \end{matrix} & 0 \\ 0 & \begin{matrix} m_2 & \cdots & m_n \\ \cdots & \cdots & \cdots \\ m_{r-1} & \cdots & \cdots \\ m_{r+3} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ m_n & \cdots & m_{2n-2} \end{matrix} \end{vmatrix}$$

$= 0$ since diagonal blocks are not square.

(ii) $$N_{nr} = - \begin{vmatrix} m_o & 0 & \cdots & 0 & m_{n-1} \\ 0 & m_2 & \cdots & m_{n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & m_{r+1} & \cdots & m_{n+r-3} & 0 \\ 0 & m_{r+3} & \cdots & m_{n+r-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n-1} & 0 & \cdots & 0 & m_{2n-2} \\ 0 & m_{n+1} & \cdots & m_{2n-2} & 0 \end{vmatrix} = \pm \begin{vmatrix} \begin{matrix} m_o & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots \\ m_{r-1} & \cdots & \cdots \\ m_{r+3} & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ m_{n-1} & \cdot & m_{2n-2} \end{matrix} & 0 \\ 0 & \begin{matrix} m_2 & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots \\ m_{n+1} & \cdot\cdot & m_{2n-2} \end{matrix} \end{vmatrix}$$

$= 0$ since diagonal blocks are not square.

(iii) N_{nr} =

$$
\begin{vmatrix}
m_0 & 0 & \cdots & m_{n-2} & 0 \\
0 & m_2 & \cdots & 0 & m_n \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & m_r & \cdots & 0 & m_{n+r-2} \\
0 & m_{r+2} & \cdots & 0 & m_{n+r} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & m_n & \cdots & 0 & m_{2n-2} \\
m_n & 0 & \cdots & m_{2n-2} & 0
\end{vmatrix}
= (-)^{\frac{1}{4}n^2-\frac{1}{2}r}
\begin{vmatrix}
\begin{matrix} m_0 & \cdots & m_{n-2} \\ \cdots & \cdots & \cdots \\ m_{r-2} & \cdots & \\ m_{r+2} & \cdots & \\ \cdots & \cdots & \cdots \\ m_n & \cdots & m_{2n-2} \end{matrix} & 0 \\
0 & \begin{matrix} m_2 & \cdots & m_n \\ \cdots & \cdots & \cdots \\ m_n & \cdots & m_{2n-2} \end{matrix}
\end{vmatrix}
$$

= result given, since $(-)^{\frac{1}{4}n(n-2)} = +1$ when n is even.

(iv) N_{nr} =

$$
\begin{vmatrix}
m_0 & 0 & \cdots & 0 & m_{n-1} \\
0 & m_2 & \cdots & m_{n-1} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
m_{r-1} & 0 & \cdots & 0 & m_{n+r-2} \\
m_{r+1} & 0 & \cdots & 0 & m_{n+r} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
m_{n-1} & 0 & \cdots & 0 & m_{2n-2} \\
0 & m_{n+1} & \cdots & m_{2n-2} & 0
\end{vmatrix}
= (-)^{\frac{1}{4}n^2-\frac{1}{2}n+\frac{1}{2}r-\frac{1}{4}}
\begin{vmatrix}
\begin{matrix} m_0 & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots \\ m_{n-1} & \cdots & m_{2n-2} \end{matrix} & 0 \\
0 & \begin{matrix} m_2 & \cdots & m_{n-1} \\ \cdots & \cdots & \cdots \\ m_{r-1} & \cdots & \\ m_{r+3} & \cdots & \\ \cdots & \cdots & \cdots \\ m_{n+1} & \cdots & m_{2n-2} \end{matrix}
\end{vmatrix}
$$

= result given, since $(-)^{\frac{1}{4}(n-1)(n+1)} = +1$ when n is odd

PART III

SPECIAL CASES

UNCENTRED LEGENDRE POLYNOMIALS

Let $w(x) = 1, \quad (a,b) = (0,1)$

Then $m_n = \frac{1}{n+1}$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{r} & & \cdots & & \frac{1}{n+r-1} \\ \frac{1}{r+2} & & \cdots & & \frac{1}{n+r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+1} & & \cdots & & \frac{1}{2n} \end{vmatrix} = \frac{(-)^{n+r}\,(1!2!\ldots(n-1)!)^3\; n!\;(n+r)!}{(n-r)!\,((n+1)!(n+2)!\ldots(2n)!)\;(r!)^2}$$

by Thm 28

$$|M_n| = N_{n+1,n+1} = \frac{(1!2!3!\ldots n!)^3}{(n+1)!(n+2)!\ldots(2n+1)!}$$

$$N_{n,n-1} = -\frac{(1!2!3!\ldots(n-2)!)^3(n-1)!\;n!}{((n+1)!(n+2)!\ldots(2n-2)!)\;(2n)!}$$

$$P_n(x) = \sum_0^n N_{nr}|M_{n-1}|^{-1}x^r = \sum_0^n (-)^{n+r}\frac{(n+r)!\;(n!)^2}{(n-r)!(2n)!(r!)^2}\;x^r$$

$P_0(x) = 1$

$P_1(x) = x - \frac{1}{2}$

$P_2(x) = x^2 - x + \frac{1}{6}$

$P_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$

$P_4(x) = x^4 - 2x^3 + \frac{9}{7}x^2 - \frac{2}{7}x + \frac{1}{70}$

$$u_n = \frac{N_{n+1,n}}{|M_n|} - \frac{N_{n,n-1}}{|M_{n-1}|} = -\tfrac{1}{2}$$

$$v_n = \frac{|M_{n-2}|\,|M_n|}{|M_{n-1}|^2} = \frac{n^2}{4(4n^2-1)}$$

$$-P_{n+1}(x) + (x-\tfrac{1}{2})P_n(x) - \frac{n^2}{4(4n^2-1)}P_{n-1}(x) = 0$$

$$A_4 = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}$$

$$E_4 = \begin{bmatrix} x-\frac{1}{2} & \frac{1}{2}x-\frac{1}{3} & \frac{1}{3}x-\frac{1}{4} & \frac{1}{4}x-\frac{1}{5} \\ \frac{1}{2}x-\frac{1}{3} & \frac{1}{3}x-\frac{1}{4} & \frac{1}{4}x-\frac{1}{5} & \frac{1}{5}x-\frac{1}{6} \\ \frac{1}{3}x-\frac{1}{4} & \frac{1}{4}x-\frac{1}{5} & \frac{1}{5}x-\frac{1}{6} & \frac{1}{6}x-\frac{1}{7} \\ \frac{1}{4}x-\frac{1}{5} & \frac{1}{5}x-\frac{1}{6} & \frac{1}{6}x-\frac{1}{7} & \frac{1}{7}x-\frac{1}{8} \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x - \frac{1}{2} & -\frac{1}{2\sqrt{3}} & 0 & 0 \\ -\frac{1}{2\sqrt{3}} & x - \frac{1}{2} & -\frac{2}{2\sqrt{15}} & 0 \\ 0 & -\frac{2}{2\sqrt{15}} & x - \frac{1}{2} & -\frac{3}{2\sqrt{35}} \\ 0 & 0 & -\frac{3}{2\sqrt{35}} & x - \frac{1}{2} \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{72} & -\frac{1}{43200} \\ 0 & 1 & -\frac{1}{12} & \frac{1}{3600} \\ 0 & 0 & \frac{1}{12} & -\frac{1}{1440} \\ 0 & 0 & 0 & \frac{1}{2160} \end{bmatrix}$$

LEGENDRE POLYNOMIALS

Let $w(x) = \frac{1}{2}$, $(a,b) = (-1,1)$

Then
$$m_n = \begin{cases} \dfrac{1}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \cdots \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 & \cdots \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} & \cdots \\ & & \text{etc.} & & & \end{vmatrix}$$

When $n+r$ is odd $N_{nr} = 0$ by Thm 42

When $n+r$ is even
$$N_{nr} = (-)^{\frac{1}{2}(n-r)}\, 2^{n^2-n+1} \times \frac{(1!2!\ldots(n-1)!)^3\; n!\; (n+r-1)!}{((n+1)!\ldots(2n)!)\; r!\; (\frac{1}{2}n-\frac{1}{2}r)!(\frac{1}{2}n+\frac{1}{2}r-1)!}$$

by Thms 42, 32, 33, & 30.

$$|M_n| = \frac{2^{n^2+n+1}\,(1!2!\ldots(n-1)!)^3\,(n!)^2}{((n+2)!(n+3)!\ldots(2n)!)\;(2n+2)!}$$

$$N_{n,n-1} = 0$$

$$P_n(x) = \sum_{0}^{n} (-)^{\frac{1}{2}n-\frac{1}{2}r} \frac{n!\,(n-1)!\,(n+r-1)!}{(2n-1)!\,r!\,(\frac{1}{2}n-\frac{1}{2}r)!\,(\frac{1}{2}n+\frac{1}{2}r-1)!} x^r$$

for even values of n+r only.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = x^2 - \frac{1}{3}$$

$$P_3(x) = x^3 - \frac{3}{5}x$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$$

$$u_n = 0 \qquad\qquad v_n = \frac{n^2}{4n^2-1}$$

$$-P_{n+1}(x) + x\,P_n(x) - \frac{n^2}{4n^2-1} P_{n-1}(x) = 0$$

$$A_4 = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 0 & \frac{1}{3} & 0 & \frac{1}{5} \\ 0 & \frac{1}{3} & 0 & \frac{1}{5} & 0 \\ \frac{1}{3} & 0 & \frac{1}{5} & 0 & \frac{1}{7} \\ 0 & \frac{1}{5} & 0 & \frac{1}{7} & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} x & -\frac{1}{3} & \frac{1}{3}x & -\frac{1}{5} \\ -\frac{1}{3} & \frac{1}{3}x & -\frac{1}{5} & \frac{1}{5}x \\ \frac{1}{3}x & -\frac{1}{5} & \frac{1}{5}x & -\frac{1}{7} \\ -\frac{1}{5} & \frac{1}{5}x & -\frac{1}{7} & \frac{1}{7}x \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x & -\frac{1}{\sqrt{3}} & 0 & 0 \\ -\frac{1}{\sqrt{3}} & x & -\frac{2}{\sqrt{15}} & 0 \\ 0 & -\frac{2}{\sqrt{15}} & x & -\frac{3}{\sqrt{35}} \\ 0 & 0 & -\frac{3}{\sqrt{35}} & x \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 1 & 0 & -\frac{4}{225} \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{4}{135} \end{bmatrix}$$

UNCENTRED JACOBI POLYNOMIALS - TYPE I

Let $w(x) = x, \qquad (a.b) = (0,1)$

Then $\qquad m_n = \dfrac{1}{n+2}$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{r+1} & & \cdots & & \frac{1}{n+r} \\ \frac{1}{r+3} & & \cdots & & \frac{1}{n+r+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+2} & & \cdots & & \frac{1}{2n+1} \end{vmatrix}$$

$$= (-)^{n+r} \frac{(1!2!\ldots(n-1)!)^3 \; (n!)^2 \; (n+r+1)!}{((n+2)!(n+3)!\ldots(2n+1)!) \; (n-r)! \; r! \; (r+1)!}$$

by Thm 29

$$|M_n| = \frac{(1!2!\ldots n!)^3 \; (n+1)!}{(n+2)!(n+3)!\ldots(2n+2)!}$$

$$N_{n,n-1} = - \frac{(1!2!\ldots(n-2)!)^3 \; ((n-1)!)^2 \; n!}{((n+2)!(n+3)!\ldots(2n-1)!) \; (2n+1)!}$$

$$P_n(x) = \sum_{0}^{n} (-)^{n+r} \frac{n!\ (n+1)!\ (n+r+1)!}{(2n+1)!\ (n-r)!\ r!\ (r+1)!} x^r$$

$$P_o(x) = 1$$

$$P_1(x) = x - \frac{2}{3}$$

$$P_2(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$$

$$P_3(x) = x^3 - \frac{12}{7}x^2 + \frac{6}{7}x - \frac{4}{35}$$

$$P_4(x) = x^4 - \frac{20}{9}x^3 + \frac{5}{3}x^2 - \frac{10}{21}x + \frac{5}{126}$$

$$u_n = -\frac{2(n+1)^2}{(2n+1)(2n+3)} \qquad v_n = \frac{n(n+1)}{4(2n+1)^2}$$

$$-P_{n+1}(x) + \left(x - \frac{2(n+1)^2}{(2n+1)(2n+3)}\right)P_n(x) - \frac{n(n+1)}{4(2n+1)^2} P_{n-1}(x) = 0$$

$$A_4 = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}$$

$$E_4 = \begin{bmatrix} \frac{1}{2}x-\frac{1}{3} & \frac{1}{3}x-\frac{1}{4} & \frac{1}{4}x-\frac{1}{5} & \frac{1}{5}x-\frac{1}{6} \\ \frac{1}{3}x-\frac{1}{4} & \frac{1}{4}x-\frac{1}{5} & \frac{1}{5}x-\frac{1}{6} & \frac{1}{6}x-\frac{1}{7} \\ \frac{1}{4}x-\frac{1}{5} & \frac{1}{5}x-\frac{1}{6} & \frac{1}{6}x-\frac{1}{7} & \frac{1}{7}x-\frac{1}{8} \\ \frac{1}{5}x-\frac{1}{6} & \frac{1}{6}x-\frac{1}{7} & \frac{1}{7}x-\frac{1}{8} & \frac{1}{8}x-\frac{1}{9} \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x-\frac{2}{3} & -\frac{\sqrt{2}}{6} & 0 & 0 \\ -\frac{\sqrt{2}}{6} & x-\frac{8}{15} & -\frac{\sqrt{6}}{10} & 0 \\ 0 & -\frac{\sqrt{6}}{10} & x-\frac{18}{35} & -\frac{\sqrt{12}}{14} \\ 0 & 0 & -\frac{\sqrt{12}}{14} & x-\frac{32}{63} \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & -\frac{1}{3} & \frac{1}{240} & -\frac{1}{37800} \\ 0 & \frac{1}{2} & -\frac{1}{60} & \frac{1}{50400} \\ 0 & 0 & \frac{1}{72} & -\frac{1}{25200} \\ 0 & 0 & 0 & \frac{1}{43200} \end{bmatrix}$$

UNCENTRED JACOBI POLYNOMIALS - TYPE II

Let $w(x) = x(1-x)$, $\quad (a,b) = (0,1)$

Then $$m_n = \frac{1}{(n+2)(n+3)}$$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} \frac{1}{2.3} & \frac{1}{3.4} & \frac{1}{4.5} & \cdots & \frac{1}{(n+1)(n+2)} \\ \frac{1}{3.4} & \frac{1}{4.5} & \frac{1}{5.6} & \cdots & \frac{1}{(n+2)(n+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(r+1)(r+2)} & & \cdots\cdots & & \frac{1}{(n+r)(n+r+1)} \\ \frac{1}{(r+3)(r+4)} & & \cdots\cdots & & \frac{1}{(n+r+2)(n+r+3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{(n+2)(n+3)} & & \cdots\cdots & & \frac{1}{(2n+1)(2n+2)} \end{vmatrix}$$

$$= \frac{(-)^{n+r} \; (1!2!\ldots n!)^3 \; (n+1)! \; (n+r+2)!}{((n+2)!(n+3)!\ldots(2n+2)!) \; (r+1)! \; r! \; (n-r)!} \quad \text{by Thm 35}$$

$$|M_n| = \frac{(1!2!\ldots n!)^3 \; ((n+1)!)^2}{(n+3)!(n+4)!\ldots(2n+3)!}$$

$$N_{n,n-1} = -\frac{(1!2!\ldots(n-2)!)^3((n-1)!n!)^2(n+1)!}{((n+2)!(n+3)!\ldots(2n)!) \quad (2n+2)!}$$

$$P_n(x) = \sum_{o}^{n} (-)^{n+r} \frac{n!\ (n+1)!\ (n+r+2)!}{(2n+2)!\ r!\ (r+1)!\ (n-r)!} x^r$$

$$P_o(x) = 1$$

$$P_1(x) = x - \tfrac{1}{2}$$

$$P_2(x) = x^2 - x + \frac{1}{5}$$

$$P_3(x) = x^3 - \frac{3}{2}x^2 + \frac{9}{14}x - \frac{1}{14}$$

$$P_4(x) = x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{42}$$

$$u_n = \tfrac{1}{2} \qquad\qquad v_n = \frac{n(n+2)}{4(2n+1)(2n+3)}$$

$$- P_{n+1}(x) + (x - \tfrac{1}{2})\, P_n(x) - \frac{n(n+2)}{4(2n+1)(2n+3)} P_{n-1}(x) = 0$$

$$A_4 = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ \frac{1}{2.3} & \frac{1}{3.4} & \frac{1}{4.5} & \frac{1}{5.6} & \frac{1}{6.7} \\ \frac{1}{3.4} & \frac{1}{4.5} & \frac{1}{5.6} & \frac{1}{6.7} & \frac{1}{7.8} \\ \frac{1}{4.5} & \frac{1}{5.6} & \frac{1}{6.7} & \frac{1}{7.8} & \frac{1}{8.9} \\ \frac{1}{5.6} & \frac{1}{6.7} & \frac{1}{7.8} & \frac{1}{8.9} & \frac{1}{9.10} \end{bmatrix}$$

$$E_4 = \begin{bmatrix} \frac{1}{6}x - \frac{1}{12} & \frac{1}{12}x - \frac{1}{20} & \frac{1}{20}x - \frac{1}{30} & \frac{1}{30}x - \frac{1}{42} \\ \frac{1}{12}x - \frac{1}{20} & \frac{1}{20}x - \frac{1}{30} & \frac{1}{30}x - \frac{1}{42} & \frac{1}{42}x - \frac{1}{56} \\ \frac{1}{20}x - \frac{1}{30} & \frac{1}{30}x - \frac{1}{42} & \frac{1}{42}x - \frac{1}{56} & \frac{1}{56}x - \frac{1}{72} \\ \frac{1}{30}x - \frac{1}{42} & \frac{1}{42}x - \frac{1}{56} & \frac{1}{56}x - \frac{1}{72} & \frac{1}{72}x - \frac{1}{90} \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x-\frac{1}{2} & -\sqrt{\frac{1}{20}} & 0 & 0 \\ -\sqrt{\frac{1}{20}} & x-\frac{1}{2} & -\sqrt{\frac{2}{35}} & 0 \\ 0 & -\sqrt{\frac{2}{35}} & x-\frac{1}{2} & -\sqrt{\frac{5}{84}} \\ 0 & 0 & -\sqrt{\frac{5}{84}} & x-\frac{1}{2} \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & -\frac{1}{12} & \frac{1}{3600} & -\frac{1}{21168000} \\ 0 & \frac{1}{6} & -\frac{1}{720} & \frac{1}{2352000} \\ 0 & 0 & \frac{1}{720} & -\frac{1}{1008000} \\ 0 & 0 & 0 & \frac{1}{1512000} \end{bmatrix}$$

LAGUERRE POLYNOMIALS

Let $w(x) = e^{-x}$, $(a,b) = (0,\infty)$

Then $m_n = n!$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} 0! & 1! & 2! & \dots & (n-1)! \\ 1! & 2! & 3! & \dots & n! \\ \dots & \dots & \dots & \dots & \dots \\ (r-1)! & & \dots & & (n+r-2)! \\ (r+1)! & & \dots & & (n+r)! \\ \dots & \dots & \dots & \dots & \dots \\ n! & & \dots & & (2n-1)! \end{vmatrix}$$

$$= (-)^{n+r} \frac{(1!2!\dots n!)^2}{(n-r)! \ (r!)^2} \qquad \text{by Thm 37.}$$

$$|M_n| = (1!2!\dots n!)^2$$

$$N_{n,n-1} = -(1!2!\dots(n-2)!)^2 \ (n!)^2$$

$$P_n(x) = \sum_{0}^{n} (-)^{n+r} \frac{(n!)^2}{(n-r)!\ (r!)^2} x^r$$

$$P_o(x) = 1$$

$$P_1(x) = x - 1$$

$$P_2(x) = x^2 - 4x + 2$$

$$P_3(x) = x^3 - 9x^2 + 18x - 6$$

$$P_4(x) = x^4 - 16x^3 + 72x^2 - 96x + 24$$

$$u_n = 2n+1 \qquad v_n = n^2$$

$$- P_{n+1}(x) + (x - 2n - 1) P_n(x) - n^2 P_{n-1}(x) = 0$$

$$A_4 = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 0! & 1! & 2! & 3! & 4! \\ 1! & 2! & 3! & 4! & 5! \\ 2! & 3! & 4! & 5! & 6! \\ 3! & 4! & 5! & 6! & 7! \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1-x & 1-2x & 2-6x & 6-24x \\ 2-2x & 2-6x & 6-24x & 24-120x \\ 2-6x & 6-24x & 24-120x & 120-720x \\ 6-24x & 24-120x & 120-720x & 720-5040x \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x-1 & -1 & 0 & 0 \\ -1 & x-3 & -2 & 0 \\ 0 & -2 & x-5 & -3 \\ 0 & 0 & -3 & x-7 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & -1 & 2 & -24 \\ 0 & 1 & -4 & 72 \\ 0 & 0 & 1 & -36 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

POLYNOMIALS ASSOCIATED WITH THE LAGUERRE POLYNOMIALS

Let $w(x) = x e^{-x}$, $(a,b) = (0,\infty)$

Then $m_n = (n+1)!$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} 1! & 2! & 3! & \cdots & n! \\ 2! & 3! & 4! & \cdots & (n+1)! \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ r! & \cdots & \cdots & \cdots & (n+r-1)! \\ (r+2)! & \cdots & \cdots & \cdots & (n+r+1)! \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n+1)! & \cdots & \cdots & \cdots & (2n)! \end{vmatrix}$$

$$= (-)^{n+r} \frac{(1!2!\ldots n!)^2 \, (n+1)!}{(n-r)! \;\; r! \;\; (r+1)!} \qquad \text{by Thm 38}$$

$$|M_n| = (1!2!\ldots n!)^2 \, (n+1)!$$

$$N_{n,n-1} = - (1!2!\ldots(n-2)!)^2 \, (n-1)! \; n! \; (n+1)!$$

$$P_n(x) = \sum_{0}^{n} (-)^{n+r} \frac{n!\ (n+1)!}{(n-r)!\ r!\ (r+1)!} x^r$$

$$P_o(x) = 1$$
$$P_1(x) = x - 2$$
$$P_2(x) = x^2 - 6x + 6$$
$$P_3(x) = x^3 - 12x^2 + 36x - 24$$
$$P_4(x) = x^4 - 20x^3 + 120x^2 - 240x + 120$$

$$u_n = 2n+2 \qquad\qquad v_n = n(n+1)$$

$$- P_{n+1}(x) + (x - 2n - 2) P_n(x) - n(n+1) P_{n-1}(x) = 0$$

$$A_4 = \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1! & 2! & 3! & 4! & 5! \\ 2! & 3! & 4! & 5! & 6! \\ 3! & 4! & 5! & 6! & 7! \\ 4! & 5! & 6! & 7! & 8! \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1-2x & 2-6x & 6-24x & 24-120x \\ 2-6x & 6-24x & 24-120x & 120-720x \\ 6-24x & 24-120x & 120-720x & 720-5040x \\ 24-120x & 120-720x & 720-5040x & 5040-40320x \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x-2 & -\sqrt{2} & 0 & 0 \\ -\sqrt{2} & x-4 & -\sqrt{6} & 0 \\ 0 & -\sqrt{6} & x-6 & -\sqrt{12} \\ 0 & 0 & -\sqrt{12} & x-8 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & -2 & 12 & -576 \\ 0 & 1 & -12 & 864 \\ 0 & 0 & 2 & -288 \\ 0 & 0 & 0 & 24 \end{bmatrix}$$

HERMITE POLYNOMIALS

Let $w(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $(a,b) = (-\infty, \infty)$

Then

$$m_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (n-1)? & \text{if } n \text{ is even} \end{cases}$$

$$N_{nr} = (-)^{n+r} \begin{vmatrix} 1! & 0 & 1? & 0 & 3? & \cdots \\ 0 & 1? & 0 & 3? & 0 & \cdots \\ 1? & 0 & 3? & 0 & 5? & \cdots \\ & & & \text{etc.} & & \end{vmatrix}$$

When $n+r$ is odd $N_{nr} = 0$ by Thm 42.

When $n+r$ is even

$$N_{nr} = \frac{(-)^{\frac{1}{2}(n-r)} \; 2^{-\frac{1}{2}(n-r)} \; (1!2!\ldots n!)}{r! \; (\frac{1}{2}(n-r))!}$$ by Thms 42, 40, 41, & 30.

$$|M_n| = 1!2!3!\ldots n!$$

$$N_{n,n-1} = 0$$

$$P_n(x) \quad = \quad \sum_{o}^{n} \quad (-)^{\frac{1}{2}(n-r)} \quad \frac{2^{-\frac{1}{2}(n-r)} \quad n!}{r! \quad (\frac{1}{2}(n-r)!)} \quad x^r$$

for even values of [redacted] n+r only.

$$P_o(x) \quad = \quad 1$$
$$P_1(x) \quad = \quad x$$
$$P_2(x) \quad = \quad x^2 - 1$$
$$P_3(x) \quad = \quad x^3 - 3x$$
$$P_4(x) \quad = \quad x^4 - 6x^2 + 3$$

$$u_n \quad = \quad 0 \qquad\qquad v_n \quad = \quad n$$

$$- \quad P_{n+1}(x) \quad + \quad x\,P_n(x) \quad - \quad n\,P_{n-1}(x) \quad = \quad 0$$

$$A_4 \quad = \quad \begin{bmatrix} 1 & x & x^2 & x^3 & x^4 \\ 1 & 0 & 1? & 0 & 3? \\ 0 & 1? & 0 & 3? & 0 \\ 1? & 0 & 3? & 0 & 5? \\ 0 & 3? & 0 & 5? & 0 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -x & 1 & -x \\ -x & 1 & -x & 3 \\ 1 & -x & 3 & -x \\ -x & 3 & -x & 15 \end{bmatrix}$$

$$\left[xI - D_4\right] = \begin{bmatrix} x & -1 & 0 & 0 \\ -1 & x & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & x & -\sqrt{3} \\ 0 & 0 & -\sqrt{3} & x \end{bmatrix}$$

$$K_4 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

LITERATURE OF ORTHOGONAL POLYNOMIALS

The standard work on Orthogonal Polynomials is

G. Szegö : Orthogonal Polynomials , American Mathematical Society Colloquium Publications, Vol. XXIII, New York, 1939. This contains a comprehensive list of references, pp.378-393, to which we refer in general. The very great majority of the papers cited treat orthogonal polynomials from the analytical standpoint, and only refer incidentally to determinants

Muir, History of Determinants , Vol. II (period 1860-1880), pp. 354-356, describes the paper of Rouché below.

Rouché, E. Sur les fonctions X_n de Legendre, Comptes Rendus, Paris, Vol. XLVII (1858), pp. 917-921, gives the Legendre polynomials in bordered determinant form, which he transforms to secular polynomial form.

Turnbull & Aitken, Canonical Matrices, London and Glasgow, 1932, p. 102, Examples 4 and 5, give the Legendre and Hermite polynomials in continuant form, and indicate the transformation to the characteristic polynomial of a symmetric continuant for these types.

The work of Tchebichef and of Stieltjes on continued fractions in relation to orthogonal polynomials naturally implies the continuant form for the latter.

The specific problem of inter-relating by matrix theory the various determinantal forms, the essential subject of the present thesis, would appear not to have been investigated before.