A THEORY OF FRACTIONAL INTEGRATION FOR GENERALISED FUNCTIONS WITH APPLICATIONS

by

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Summary

A theory of fractional integration is developed for certain spaces $F_{p,\mu}$ of testing-functions and the corresponding generalised functions $F'_{p,\mu}$. Some properties of the spaces are first developed and some elementary mappings discussed . There is a close examination of the operators $I^{\eta,\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ defined by

$$I^{\eta,\alpha}_{x^m} \phi(x) = \frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m-u^m)^{\alpha-1} u^{m\eta+m-1} \phi(u)\, du$$

$$K^{\eta,\alpha}_{x^m} \phi(x) = \frac{mx^{m\eta}}{\Gamma(\alpha)} \int_x^\infty (u^m-x^m)^{\alpha-1} u^{-m\eta-m\alpha+m-1} \phi(u)\, du$$

and their mapping properties relative to $F_{p,\mu}$ are derived . The operators are extended to $F'_{p,\mu}$ using adjoints and corresponding results obtained .

Three applications are given . The operator

$$L_\nu \equiv \frac{d^2}{dx^2} + \frac{2\nu + 1}{x} \frac{d}{dx}$$

is discussed and certain connections with fractional integration established . A generalised Hankel transform is developed on $F'_{p,\mu}$ and similar connections with fractional integration obtained . Finally, certain integral operators involving hypergeometric functions are studied , a typical operator being

$$(H_1(a,b,c)\, \phi)(x) = \int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} F(a,b,c,1-\frac{x}{t})\, \phi(t)\, dt$$

Existence and uniqueness theorems are established for various integral equations in $F'_{p,\mu}$.

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I also wish to express my gratitude to the Carnegie Trust for the Universities of Scotland for the award of a Scholarship for my period of study .

Contents

ERRATA

Page 11, Line 14. For ' γ_0 ' read ' γ_0^p ' .

Page 13, Line 9, should begin ' By completeness of L^p, ... '

Page 14, Line 12. For ' $\delta^k \phi_n(x_1)$ ' read ' $\delta^k \phi_n(x)$ '

do. Lines 25 and 26, should end

$$' = | \delta^k \phi_n - \chi_k |_p = | \delta^k \phi_n - \psi_k |_p \to 0, \ .... '$$

Page 16, Line 6. Read ' ..., $x\{\phi(x)\}^p$ remains bounded as $x \to 0+$ or $x \to \infty$. '

Page 17 Last line. For ' L_q ' read ' L^q ' .

Page 23 2nd and 3rd last lines . Read

$$' \leq \gamma_{k+1}^p(\psi) + (k+|\mu|)\gamma_k^p(\psi) \leq \gamma_{k+1}^{p,\mu}(\phi) + (k+|\mu|)\,{}_k^{p,\mu}(\phi) '.$$

Page 25, Line 15. Lower limit in the second integral is 0 .

Page 27, Last line . Read

$$I_{x^m}^{\eta,\alpha} \phi(x) = \frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^x \ldots\ldots$$

Page 35, line 2 .'Since multiplication .... ' sentence should read ' Since $\frac{m}{\Gamma(\alpha)}$ is an entire function of α , we need only prove that T_α is analytic with respect to α on $F_{p,\mu}$ for Re $\alpha > 0$ ' .

Page 35, Line 16. For μ, η , α read Re μ ,Re η , Re α .

Page 35 last two lines and Page 36 first three should read

$$' \gamma_k^{p,\mu} \left(\frac{1}{h} [T_{\alpha+h}\phi - T_\alpha \phi] - \frac{\partial T}{\partial \alpha}{}_\alpha \phi \right)$$

$$\leq \sup_{0 \leq t \leq 1} |f_h(1-t^m)| \ \gamma_0^p \left(I_{x^m}^{\mathrm{Re}(\eta+\frac{\mu}{m}),\ \mathrm{Re}\,\alpha-\epsilon} \left| x^k \frac{d^k \psi}{dx^k} \right| \right) \to 0$$

as $h \to 0$ where $\phi(x) = x^\mu \psi(x)$

Page 44, Line 17. For 'Re $\mu < \frac{1}{p}$', read 'Re $(\mu+m\alpha) < \frac{1}{p}$ ' .

Page 55, Line 6. For ' $F_{p,\mu+1}$ ' read ' $F_{p,\mu-1}$ '

Page 67 , Lines 1,4,5. For ' $(\delta+1)\ \psi(\frac{u}{x})$, $(\delta+1)\ \psi(A)$ 'read ' $[(\delta+1)\psi](\frac{u}{x})$ $[(\delta+1)\ \psi](A)$ '

Page 68, Line 11. For '(6)' read '(5)' .

ERRATA (continued)

Page 64, Line 15 should read ' $\leq \sum M_i \binom{k}{i} \left| x^{k-1} \frac{d^{k-1}\phi}{dx^{k-1}} \right|$ ' .

Page 70, fifth from bottom.

For '$(M-1)!$' read '$M!$'

Page 73, Line 6. For 'F_p' read 'F_2' .

Page 75, Line 13. Read ' $-1 < \operatorname{Re} \nu_0 \leq -\frac{1}{2}$.

Line 19. For ' $| \psi_{2,\nu}(x) |_2$ ' read

$$' \int_0^1 \left| \sqrt{y} \frac{\partial^2 J_\nu}{\partial \nu^2} (y) \right| \left| \phi \left(\frac{y}{x} \right) \right| \frac{dy}{x} '$$

Line 21 For '$\frac{1}{2} + \operatorname{Re} \nu_0 - 2\epsilon > \frac{1}{2}$' read '$\frac{1}{2} + \operatorname{Re} \nu_0 - 2\epsilon > -\frac{1}{2}$'

Page 76, Line 13 reads ' $| \psi_{1,\nu} |_2 \leq C_1 | \phi |_2$ '

Page 79 Last line . Read ' analytic functions of α '

Page 84. Title of Chapter :

' Hypergeometric Integral Equations '

Page 87, Line 8. For ' entire ' read ' analytic ' .

CHAPTER 1

INTRODUCTION

§1.1 Background and Summary

Let $\phi(x)$ be a complex-valued function defined for $0 < x < \infty$. Under fairly mild restrictions on ϕ, an n^{th} order indefinite integral of ϕ is given by

$$I^n\phi(x) = \frac{1}{\Gamma(n)} \int_0^x (x-t)^{n-1} \phi(t)\, dt \qquad (1)$$

($n = 1,2,3...$) This formula is sometimes ascribed to Cauchy. We can use (1) to motivate the definition of a fractional integral of ϕ ; namely, for any complex number α, with Re $\alpha > 0$ (to ensure convergence), we define $I^\alpha\phi$ by

$$I^\alpha\phi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi(t)\, dt \qquad (2)$$

$I^\alpha\phi$ is often called the Riemann-Liouville integral of order α of the function ϕ. Similarly, we are led to consider the operator K^α defined for Re $\alpha > 0$ by

$$K^\alpha\phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} \phi(t)\, dt \qquad (3)$$

$K^\alpha\phi$ is often called the Weyl integral of order α of ϕ (with origin ∞).

It is well known that fractional integration is an important tool in many areas of analysis, for example in connection with ordinary and partial differential equations. Likewise the theory of generalised functions or distributions plays an important role in analysis, a notable instance again being the theory of differential equations. It would therefore seem worthwhile to attempt to connect these two concepts by developing a theory of fractional integration for generalised functions. In this thesis, we develop such a theory and indicate some applications of it. Our theory is more general than that in [8] and distinct from that in [7].

In attempting to define a fractional integral of a generalised function, two approaches suggest themselves. The first is based on the theory of convolution of distributions as described in [11]. Let $f(x)$ be a locally integrable function on $0 \leq x < \infty$ extended to the real line by setting $f(x) = 0 \quad (x < 0)$. Then writing

$$p_\alpha(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & (x > 0) \\ 0 & (x \leq 0) \end{cases}$$

(2) becomes

$$I^\alpha f(x) = \int_{-\infty}^{\infty} p_\alpha(x-t)\, f(t)\, dt$$

or

$$I^\alpha f = p_\alpha * f \tag{4}$$

where * denotes convolution. When Re $\alpha > 0$, p_α is locally integrable and hence p_α and f generate regular distributions with supports in the half-open interval $[0,\infty)$. Then interpreting * as distributional convolution, (4) defines $I^\alpha f$ as a distribution. With this motivation we could use (4) to define $I^\alpha f$ for <u>any</u> distribution f with support in $[0,\infty)$.

However, it is also necessary to consider certain extensions and modifications of I^α and K^α. For example, we shall be concerned with operators, first studied extensively by Kober [13], of the form

$$I_x^{\eta,\alpha} f = x^{-\eta-\alpha} I^{\alpha} x^{\eta} f(x) \tag{5}$$

$$K_x^{\eta,\alpha} f = x^{\eta} K^{\alpha} x^{-\eta-\alpha} f(x) \tag{6}$$

x^{η} and $x^{-\eta-\alpha}$ are not smooth (infinitely differentiable) functions so that (5) and (6) are meaningless for distributions. Further complications arise when we integrate with respect to x^n rather than x obtaining

$$I_{x^n}^{\eta,\alpha} f = \frac{n}{\Gamma(\alpha)} x^{-n\eta-n\alpha} \int_0^x (x^n-u^n)^{\alpha-1} u^{n\eta+n-1} f(u)\, du \tag{7}$$

and analogous operators $K_{x^n}^{\eta,\alpha}$. These turn out to have important applications in connection with singular differential operators, integral equations and integral transforms. (Some references are given in [8]). In order to define these operators for generalised functions we will pursue a second approach based on adjoint operators.

Under certain restrictions on f, ϕ,

$$\int_0^{\infty} I^{\alpha} f(x)\, \phi(x)\, dx = \int_0^{\infty} f(x)\, K^{\alpha}\phi(x)\, dx \tag{8}$$

or

$$(I^{\alpha} f, \phi) = (f , K^{\alpha}\phi) \tag{9}$$

(8) was proved by Love and Young in their paper on fractional integration by parts [20]. If S ,T are spaces of functions such that K^{α} is a continuous linear mapping of S into T , we can use (9) to define I^{α} as a continuous linear mapping of T' into S' where S',T' are the spaces of generalised functions corresponding to S , T . (9) shows that I^{α} and K^{α} are adjoints and similarly $I_{x^n}^{\eta,\alpha}$ and $K_{x^n}^{\eta',\alpha}$ are adjoints , where $\eta' = \eta + 1 - \frac{1}{n}$ Thus the first task is to devise suitable classes of testing funct-

ions on which the operators can meaningfully be defined.

In Chapter 2 we introduce such classes denoted by $F_{p,\mu}$. The elements of $F_{p,\mu}$ are smooth functions defined for $0 < x < \infty$, which satisfy certain integrability conditions involving the L^p norms. The motivation for the choice of testing functions is provided by Kober's work in [13] on $I_x^{\eta,\alpha}$ and $K_x^{\eta,\alpha}$. The corresponding spaces $F'_{p,\mu}$ of generalised functions are then introduced. The remainder of the chapter is devoted to the proof of some elementary facts about $F_{p,\mu}$ and $F'_{p,\mu}$ together with a discussion of some elementary mappings defined on them. In particular the spaces are so constructed that the operation of multiplication by x^λ for any complex λ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$, thus obviating one of the major snags of the first approach. Throughout this chapter we follow closely the treatment in Chapter 1 of Zemanian [25].

In Chapter 3 we introduce the operators $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$ The case $m = 1$ has been thoroughly investigated by Kober in [13] By a simple change of variable we can easily obtain the corresponding mapping properties of the operators relative to the spaces $F_{p,\mu}$ Further, an argument involving analytic continuation enables us to remove the restriction $\operatorname{Re} \alpha > 0$, although for $\operatorname{Re} \alpha < 0$, we will have operators of fractional differentiation rather than integration. We then define the operators on $F'_{p,\mu}$ using adjoints. Although the spaces $F'_{p,\mu}$ are primarily geared to the ' homogeneous ' operators $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$, results concerning $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$ are obtained incidentally. A similar programme was carried through in [8] but the spaces under consideration here are much more general.

Although the spaces $F'_{p,\mu}$ were designed with fractional integration in mind, many other operators can meaningfully be defined on them. In the remaining chapters we consider some of these operators with special reference to their connection with fractional integration.

In Chapter 4 we consider the singular differential operator L_ν defined by

$$L_\nu\phi(x) = \frac{d^2\phi}{dx^2} + \frac{2\nu+1}{x}\frac{d\phi}{dx} \qquad (10)$$

If we replace ν by $\frac{1}{2}n - 1$ and x by $r = \sqrt{x_1^2 + \ldots + x_n^2}$ we obtain the Laplacian of a spherically symmetric function ϕ of n space variables $x_1, \ldots, x_n$. L_ν is also connected with axially symmetric potentials and differential equations such as the Euler - Poisson - Darboux equation [6]. There are various connections between L_ν and fractional integration with respect to x^2, which have been studied by, for example, Erdélyi [5] and Lions [15] and [16]. We establish two of these relations, both for $F_{p,\mu}$ and $F'_{p,\mu}$. Again our results are more general than those in [8].

In Chapter 5 we develop a generalised Hankel transform. A Hankel transform for generalised functions has been developed in [25] by Zemanian. There he introduces certain spaces of generalised functions which are tailor-made for the Hankel transform and he is able to develop quite an extensive theory culminating in an operational calculus for a class of differential equations. Our generalised functions are, not surprisingly, less amenable to the Hankel transform. Nevertheless, they can be used to bring out the connection between Hankel transforms and fractional integration which have been studied, for ordinary functions, by Kober and Erdélyi in [13] and [14].

We begin by developing the Hankel transform on $F_{p,0}$ but soon specialise to $F_{2,0}$ where much more can be proved. Kober [13] established connections between $I_x^{\eta,\alpha}$ and $K_x^{\eta,\alpha}$ and the Hankel transform in Tricomi's form. By a simple change of variable, we transcribe these to produce relations between $I_{x^2}^{\eta,\alpha}$ and $K_{x^2}^{\eta,\alpha}$ and the Hankel transform in its usual form for L_2. These we then establish on $F_{2,0}$, and finally, by taking adjoints, we obtain the corresponding theory on $F'_{2,0}$ Analytic continuation is again involved and this entails a fair amount of analysis involving asymptotic expansions.

It should be mentioned at this stage that a similar theory can be developed for the K transform on $F'_{p,0}$ using, for example, results of Okikiolu [22]. Zemanian [25] has also developed a generalised K transform by considering specially-devised testing-function spaces. However, using $F'_{p,0}$ it is possible to establish again the connections between K transforms and fractional integration such as have been examined, for ordinary functions, by Erdélyi [4] and others. Nevertheless, because of the similarity with the Hankel transform we shall not discuss the K transform theory here.

Instead in Chapter 6, we consider a completely different application of our theory, namely to some hypergeometric integral equations. Connections between the hypergeometric function ${}_2F_1(a,b,c,z)$ and fractional integration are legion; see [2], [3]. It is not surprising therefore that operators involving ${}_2F_1(a,b,c,z)$ are closely linked with fractional integration also. Such operators have been studied by e.g. Higgins [12] and notably by Love [17] and [18]. Many other authors have studied particular cases of these operators. A list of references for these can be found in [17] or [18].

Typical of Love's operators is

$$(H_1(\ a,b,c\ )\phi)(x) \ = \int_0^x \frac{(\ x-t\ )^{c-1}}{\Gamma(c\ )} F(\ a,b,c,1-\tfrac{x}{t})\phi(t)dt \quad (11)$$

where we write $F(\ a,b,c,z\ ) \ = \ {}_2F_1(\ a,b,c,z\ )$. We obtain an expression for $H_1(\ a,b,c\ )$ in terms of operators of the form $I_x^{\eta,\alpha}$ and hence derive the mapping properties of $H_1(\ a,b,c\ )$ relative to the spaces $F_{p,\mu}$. We show in particular that under restrictions on the parameters, $H_1(\ a,b,c\ )$ is invertible and

$$[\ H_1(\ a,b,c\ )\ ]^{-1} \ = \ x^{-a}\,H_1(\ -a,b-c,-c\ )\ x^{a} \qquad (12)$$

We next consider $H_2(\ a,b,c\ )$ which is given by (11) except that $F(\ a,b,c,1-\frac{x}{t}\ )$ is replaced by $F(\ a,b,c,1-\frac{t}{x}\ )$. We proceed as for $H_1(\ a,b,c\ )$ and note that

$$H_2(\ a,b,c\ ) \ = \ x^{a}\,H_1(\ a,c-b,c\ )\ x^{-a} \qquad (13)$$

Finally, we discuss the operators $H_3(\ a,b,c\ )$ and $H_4(\ a,b,c\ )$ which are, in fact, the adjoints of $H_2(\ a,b,c\ )$ and $H_1(\ a,b,c\ )$ respectively, and can therefore be expressed in terms of the operators $K_x^{\eta,\alpha}$.

Having obtained the properties of the four operators on $F_{p,\mu}$, we take adjoints to prove the corresponding results for $F'_{p,\mu}$. In particular, we obtain existence and uniqueness theorems for the integral equations

$$H_i(\ a,b,c,)\ f \ = \ g \qquad (14)$$

where f and g are generalised functions and $i = 1,2,3,4$.

§1.2 Conventions and Notation

At this stage we make certain conventions which will be adhered to throughout the thesis. Generalised functions will be denoted by letters such as f, g etc. while testing-functions will be denoted by Greek letters such as ϕ, ψ etc. The value assigned to a testing-function ϕ by a functional f will be denoted by (f , ϕ).

We shall be concerned with complex-valued functions ϕ of a positive real variable x. $(0 , \infty)$ will denote the open interval $\{ x : 0 < x < \infty \}$. ϕ is called smooth if it is infinitely differentiable at each point $x \in (0, \infty)$. The set of all smooth functions on $(0, \infty)$ will be denoted by C^{∞}. For each p, $1 \leq p \leq \infty$, L_p denotes the set of (measurable) functions ϕ on $(0, \infty)$ for which

$$| \phi |_p = \left(\int_0^{\infty} |\phi(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

L^p will denote the corresponding space of equivalence classes of such functions which differ on a set of measure zero. Similarly, L_∞ denotes the set of (measurable) functions ϕ for which $|\phi|_\infty$, the essential supremum of ϕ over $(0, \infty)$ is finite. L^∞ denotes the corresponding space of equivalence classes. The numbers p and q will always be connected by the relation

$$\frac{1}{p} + \frac{1}{q} = 1$$

and unless otherwise stated, $1 \leq p \leq \infty$.

Some remarks are in order concerning the numbering system adopted. Lemmas, theorems, corollaries etc. in any one chapter are numbered in a continuous sequence. A statement in a chapter about Theorem 2 refers to Theorem 2 of that chapter while Theorem 2 of Chapter 2 will be referred to in chapter 3 as Theorem 2.2 . Similar remarks apply to numbered formulae.

§1.3 Standard results

Finally in this introductory chapter we quote some standard theorems which will be used frequently in the following chapters. These results fall into two groups. The first contains results concerning dual spaces and adjoint operators. The terminology is that of Zemanian [25] Chapter 1 where the proofs may also be found on the pages indicated.

Theorem 1

If V is a complete countably multinormed space, then its dual V' is also complete.

Proof on pp. 21-3 of [25].

Theorem 2

If U and V are countably multinormed spaces and T is a continuous linear mapping of U into V, then the adjoint operator T^* is a continuous linear mapping of V' into U'. If T is an isomorphism of U onto V, then T^* is an isomorphism of V' onto U' and

$$(T^*)^{-1} = (T^{-1})^*$$

Proof on pp. 28-9.

The other results concern the legality of differentiating under the integral sign. Here we refer to Luxemburg [21].

Theorem 3

Let $f(x,t)$ be defined for all t in an interval I

and $A \leq x \leq B$, and let $f(x,t)$ be integrable over I for all $A \leq x \leq B$. If $f(x,t)$ is a differentiable function of x for all $A \leq x \leq B$, (one-sided at the end-points) almost everywhere in I, then

$$F(x) = \int_I f(x,t)\,dt$$

is differentiable for all $A \leq x \leq B$ (one-sided at the end-points) and

$$F'(x) = \int_I \frac{\partial f}{\partial x}(x,t)\,dt$$

provided there exists an integrable function $g(t)$ such that

$$\left| \frac{\partial f}{\partial x}(x,t) \right| \leq g(t)$$

almost everywhere on I and for $A \leq x \leq B$.

This is Corollary 19.1, p. 174 of [21]. On examining the proof of the latter, we see that a similar theorem will hold when x is replaced by a complex variable z and the inequalities involving x are replaced by corresponding inequalities involving $\mathrm{Re}\, z$. This generalisation will also be required.

CHAPTER 2

The Spaces $F_{p,\mu}$ and $F'_{p,\mu}$

§2.1 The testing-function spaces F_p

For each p, $1 \leq p \leq \infty$, we define F_p to be the set of those smooth functions ϕ such that, for each non- negative integer k, $x^k \frac{d^k\phi}{dx^k} \in L_p$, i.e.

$$F_p = \{ \phi : \phi \in C^\infty \text{ and } x^k \frac{d^k\phi}{dx^k} \in L_p \ (k = 0,1,2,\dots)\} \quad (1)$$

With the usual pointwise operations of addition of functions and multiplication of a function by a complex number, F_p becomes a complex linear space.

For $\phi \in F_p$ and $k = 0,1,2,\dots$, define γ_k^p by

$$\gamma_k^p(\phi) = \left| x^k \frac{d^k\phi}{dx^k} \right|_p \quad (2)$$

Using the properties of $|\ |_p$, we see that, for each k, γ_k^p is a seminorm on F_p, while γ_0 is a norm. Hence the collection

$$M_p = \{ \gamma_k^p : k = 0,1,2,\dots \} \quad (3)$$

is a countable multinorm [25] and with the topology generated by M_p, F_p becomes a countably multinormed space.

We require a notion of sequential convergence as follows. A sequence $\{ \phi_n \}$ converges to ϕ in F_p (or in the topology of F_p) if and only if

(i) $\phi_n \in F_p$ ($n = 1,2,\dots$)

(ii) $\phi \in F_p$

and (iii) for each $\gamma_k^p \in M_p$, $\gamma_k^p(\phi_n-\phi) \to 0$ as $n \to \infty$.

Clearly, convergence in F_p implies convergence in L_p. Similarly,

$\{ \phi_n \}$ is called a fundamental sequence (or a Cauchy sequence) in F_p if and only if

(i) $\phi_n \in F_p$ ($n = 1,2,\ldots..$)

(ii) for each $\gamma_k^p \in M_p$, given $\epsilon > 0$, $\exists\, N_k$ such that

$$\gamma_k^p (\phi_m - \phi_n) < \epsilon \qquad (m,n > N_k)$$

As usual, it is easy to prove that if $\{ \phi_n \}$ converges to ϕ in F_p, then $\{ \phi_n \}$ is a fundamental sequence in F_p. It is not immediately obvious whether the converse is true i.e. whether F_p is complete. However, we will now prove, with the help of an easy lemma, that F_p is complete.

Define an operator δ on F_p by

$$(\delta\phi)\ (x) = x \frac{d\phi}{dx} = x\, \phi'(x)$$

or

$$\delta = x \frac{d}{dx} \tag{4}$$

<u>Lemma 1</u>

$$\phi \in F_p \text{ if and only if } \phi \in C^\infty \text{ and } \delta^k\phi \in L_p \ (k = 0,1,..)$$

Proof : We have only to show that

$$\delta^k\phi \in L_p \ (k=0,1,2,..) \iff x^k \frac{d^k\phi}{dx^k} \in L_p \ (k=0,1,2...)$$

$\Rightarrow$ When $k = 0$, $x^k \dfrac{d^k\phi}{dx^k} = \phi = \delta^0\phi \in L_p$

For $k = 1,2,\ldots$ we can prove by induction that

$$x^k \frac{d^k\phi}{dx^k} = \delta\,(\delta - 1)\ \ldots..\ (\delta - k + 1)\ \phi\ (x)$$

so that $x^k \dfrac{d^k\phi}{dx^k}$ is a linear combination of $\delta\phi$, $\delta^2\phi$, $\delta^3\phi$ ..... $\delta^k\phi$.

and thus $\in L_p$.

$\Leftarrow$ We can prove by induction that $\delta^k\phi$ is a linear combination of ϕ, $x\dfrac{d\phi}{dx}$, ...$x^k \dfrac{d^k\phi}{dx^k}$ and thus belongs to L_p.

Thus we may write

$$F_p = \{ \phi : \phi \in C^\infty \text{ and } \delta^k\phi \in L_p \ (k = 0,1,2....) \} \tag{5}$$

Theorem 2

F_p is complete, $1 \leq p \leq \infty$.

Proof: The proof is closely related to that given in Zemanian [25] pp. 253-4.

Let $\{ \phi_n \}$ be a fundamental sequence in F_p. Then, by definition of the seminorms γ_k^p, $x^k \dfrac{d^k \phi_n}{dx^k}$ is a fundamental sequence in L_p, for each $k = 0,1,2,\ldots$.

$\Rightarrow$ $\delta^k \phi_n$ is a fundamental sequence in L_p for $k = 0,1,2\ldots$.

By completeness of L_p $\exists \psi_k$: $\delta^k \phi_n \to \psi_k$ in L_p as $n \to \infty$. We show that $\exists\, \chi_0 \in F_p$: $\psi_k = \delta^k \chi_0$ a.e. on $(0,\infty)$ $(k = 0,1,2..)$

Let x_1 be a fixed point in $(0,\infty)$, x a variable point in $(0,\infty)$. Write $D \equiv \dfrac{d}{dx}$ and let D^{-1} denote the integration operator

$$D^{-1} = \int_{x_1}^{x} \ldots\ldots dt$$

For any smooth function $\zeta(x)$ on $(0,\infty)$,

$$D^{-1} D\zeta(x) = \zeta(x) - \zeta(x_1) \tag{6}$$

Now recall that $\delta \equiv xD$. Let us write $\delta^{-1} = D^{-1}x^{-1}$. Also, let $\dfrac{1}{p} + \dfrac{1}{q} = 1$, where we assume first that $1 < p < \infty$.

$$| \delta^{-1} \delta^{k+1} (\phi_m - \phi_n) | = | D^{-1} x^{-1} \delta^{k+1} (\phi_m - \phi_n) |$$

$$= \left| \int_{x_1}^{x} \frac{1}{t} \, \delta^{k+1} (\phi_m - \phi_n)(t) \, dt \right|$$

$$\leq \left| \int_{x_1}^{x} \left| \frac{1}{t} \right|^q dt \right|^{\frac{1}{q}} \left| \int_{x_1}^{x} | \delta^{k+1} (\phi_m - \phi_n)(t) |^p \, dt \right|^{\frac{1}{p}}$$

(by Hölder's inequality applied to the interval with end-points x_1 and x)

$$\leq \left| \int_{x_1}^{x} \left| \frac{1}{t} \right|^q dt \right|^{\frac{1}{q}} \left(\int_{0}^{\infty} \left| \delta^{k+1} (\phi_m - \phi_n)(t) \right|^p dt \right)^{\frac{1}{p}}$$

Let Ω denote an arbitrary open interval whose closure is compact in $(0,\infty)$. Since $\frac{1}{t} \neq 0$ on $(0,\infty)$, $\int_{x_1}^{x} |\frac{1}{t}|^q \, dt$ is a bounded smooth function on Ω. Hence

$$| \delta^{-1}\delta^{k+1}(\phi_m - \phi_n) | \leq M\gamma_{k+1}^{p}(\phi_m - \phi_n)$$

on Ω for some constant M. Since $\{\phi_n\}$ is a fundamental sequence in F_p, $\{\delta^{-1}\delta^{k+1}\phi_n\}$ is a fundamental sequence in the sup norm and hence $\delta^{-1}\delta^{k+1}\phi_n$ is uniformly convergent on Ω as $n \to \infty$.

Now, from (6),

$$\delta^{-1}\delta^{k+1}\phi_n(x) = D^{-1}x^{-1}xD\,\delta^k\phi_n(x) = D^{-1}D\,\delta^k\phi_n(x)$$

$$= \delta^k\phi_n(x) - \delta^k\phi_n(x_1)$$

or $$\delta^k\phi_n(x) = \delta^{-1}\delta^{k+1}\phi_n(x) + \delta^k\phi_n(x_1) \tag{7}$$

As $n \to \infty$, $\delta^k\phi_n(x)$ converges in $L_p \Rightarrow \delta^k\phi_n(x_1)$ converges in $L_p(\Omega)$. Hence $\delta^k\phi_n(x_1)$ tends to a limit as $n \to \infty$. Since $\delta^{-1}\delta^{k+1}\phi_n$ is uniformly convergent on Ω, we can now conclude from (7) that $\delta^k\phi_n$ converges uniformly on Ω as $n \to \infty$. The uniform limit, χ_n say, of $\delta^k\phi_n$ is a continuous function on $(0,\infty)$. From (7),

$$\chi_n(x) = \delta^{-1}\chi_{n+1}(x) + \chi_n(x_1) \tag{8}$$

Using (8), we conclude that χ_n is a smooth function and that $\chi_n = \delta^n\chi_0$. Now since ψ_k is the L_p limit of $\delta^k\phi_n$ and χ_k is the uniform limit of $\delta^k\phi_n$ on every Ω as $n \to \infty$, $\chi_k(x) = \psi_k(x)$ a.e. on $(0,\infty)$ Hence, for $k = 0,1,2\dots$,

$$| \delta^k\chi_0 |_p = | \chi_k |_p = | \psi_k |_p < \infty$$

Thus χ_0 is smooth, $\delta^k\chi_0 \in L_p$ for each $k = 0,1,2\dots$; so by Lemma 1 , $\chi_0 \in F_p$. Further for each k,

$$| \delta^k(\phi_n - \chi_0) |_p = | \delta^k\phi_n - \delta^k\chi_0 |_p = | \delta^k\phi_n - \chi_n |_p$$

$$= | \delta^k\phi_n - \psi_n |_p \to 0 \text{ as } n \to \infty,$$

from which it easily follows that $\{\phi_n\}$ converges to χ_0 in F_p as $n \to \infty$. This completes the proof for the case $1 < p < \infty$.

The cases $p = 1$ and $p = \infty$ are similar except that in the application of Hölder's Inequality, one of the integrals is replaced by a supremum over the interval with end-points x and x_1.

Summarising, we now have

Theorem 3

For $1 \leq p \leq \infty$, F_p is a complete countably multinormed space (and hence a Fréchet space) .

By an argument similar to that used in Theorem 2, we can show that if $\{ \phi_n \}$ converges to zero in F_p as $n \to \infty$, then, for each non-negative integer k, $\{ D^k \phi_n \}$ converges to zero uniformly on every compact subset of $(0,\infty)$ as $n \to \infty$. ($D \equiv \frac{d}{dx}$) It follows that, for each p, $1 \leq p \leq \infty$, F_p is a testing-function space in the sense of Zemanian [25] p. 39 and we will call the elements of F_p testing-functions.

In §2.3 we shall compare the spaces F_p with other important testing- function spaces. For the moment, we conclude this section by proving a lemma which will be used frequently in the sequel.

Lemma 4

$$\phi \in F_p \Rightarrow x^{\frac{1}{p}} \phi(x) \text{ is bounded on } (0,\infty) \quad (1 \leq p \leq \infty)$$

Proof : Suppose first that $1 \leq p < \infty$.

Choose a,b with $0 < a < b < \infty$. Integrating by parts, we have

$$\int_a^b x\phi'(x)\{\phi(x)\}^{p-1}dx = \frac{1}{p}[x\{\phi(x)\}^p]_a^b - \frac{1}{p}\int_a^b \{\phi(x)\}^p\,dx \qquad (9)$$

$\phi \in F_p \Rightarrow x\phi'(x) \in L_p$. Also $\{ \phi(x) \}^{p-1} \in L_q$, since

$$\int_0^\infty |\phi(x)|^{(p-1)q}\,dx = \int_0^\infty |\phi(x)|^p\,dx < \infty$$

Hence, by Hölder's Inequality $\int_0^\infty x\phi'(x)\{\phi(x)\}^{p-1}dx$ is absolutely convergent so that the left-hand side of (9) is bounded as $a \to 0+$ or $b \to \infty$. Similarly, since $\phi \in L_p$, the integral on the right-hand side of (9) remains bounded as $a \to 0+$ or $b \to \infty$. It follows that $x\{\phi(x)\}^p$ remains bounded as $a \to 0+$ or $b \to \infty$. The result follows in this case.

The result is trivial in the case $p = \infty$, since then $x^{\frac{1}{p}}\phi(x) = \phi(x)$ is essentially bounded and hence bounded on $(0,\infty)$.

This completes the proof.

It follows, in particular, that if $\phi \in F_p$, $1 \leq p < \infty$, then $\phi(x) \to 0$ as $x \to \infty$.

§2.2 The generalised- function spaces F_p'

In this section we consider functionals on F_p, i.e. mappings from F_p into the complex numbers.

A functional on F_p is linear if

$$(f, \alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1(f, \phi_1) + \alpha_2(f, \phi_2)$$

for all complex numbers α_1, α_2 and ϕ_1, ϕ_2 in F_p. f is (sequentially) continuous if whenever $\phi_n \to \phi$ in F_p, $(f,\phi_n) \to (f,\phi)$ as $n \to \infty$. We note that a linear functional is continuous if and only if $(f,\phi_n) \to 0$ as $n \to \infty$ whenever $\{\phi_n\}$ converges to zero in F_p. The set of all continuous linear functionals on F_p is called the dual of F_p and will be denoted by F_p'. The elements of F_p' are generalised functions (in the sense of Zemanian [25]) .

Given $f, g \in F_p'$ we define a functional $f + g$ on F_p by

$$(f + g, \phi) = (f, \phi) + (g, \phi) \qquad (\phi \in F_p)$$

It is easy to see that, in fact, $f + g \in F'_p$. Similarly, given a complex number α and $f \in F'_p$, we can define an element αf of F'_p by

$$(\alpha f, \phi) = \alpha(f, \phi) \qquad (\phi \in F_p)$$

With these definitions of addition and scalar multiplication, F'_p becomes a (complex) linear space.

We assign to F'_p the topology of weak convergence. A sequence $\{ f_n \}$ converges to f in F'_p if and only if

(i) $f_n \in F'_p$ $\qquad (n = 1,2,3\ldots\ldots)$

(ii) $f \in F'_p$

and (iii) for each $\phi \in F_p$, $(f_n, \phi) \to (f, \phi)$ as $n \to \infty$, in the sense of complex numbers. Similarly, $\{ f_n \}$ is a fundamental sequence in F'_p if and only if

(i) $f_n \in F'_p$ $\qquad (n = 1,2,3\ldots\ldots)$

(ii) for each $\phi \in F_p$, $\{ (f_n, \phi) \}$ is a Cauchy sequence of complex numbers. Theorem 1.1 immediately gives

Theorem 5

F'_p is complete for $1 \leq p \leq \infty$.

Certain elements of F'_p can be identified with conventional functions; in particular, let $f \in L_q$. We can define a functional $\tilde{f}$ by

$$(\tilde{f}, \phi) = \int_0^\infty f(x)\, \phi(x)\, dx \qquad (\phi \in F_p) \quad (1.0)$$

The integral exists by Hölder's Inequality. $\tilde{f}$ is clearly linear. Further since convergence in $F_p \Rightarrow$ convergence in L_p it follows easily that $\tilde{f}$ is continuous; i.e. $\tilde{f} \in F'_p$ Identifying functions which differ on a set of measure zero, we can therefore imbed L_q

in F'_p by means of (10).

Generalised functions with an integral representation of the form (10) are called regular; those which have no such representation are called singular. An example of a singular element of F'_p ($1 \leq p \leq \infty$) is provided by δ_a ($a > 0$) a translated delta-function, defined by

$$(\delta_a, \phi) = \phi(a) \qquad (\phi \in F_p)$$

We shall use regular functionals to motivate the definition of various operators on F'_p in the sequel.

§2.3 Relationship of F'_p to $\mathcal{D}'$ and $\mathcal{E}'$

It is interesting to compare the spaces F'_p with other spaces of generalised functions on $(0,\infty)$, in particular with distributions and distributions with compact support.

Let $\mathcal{D}$ be the linear space of all complex-valued smooth functions ϕ defined on $(0,\infty)$ whose support is a compact subset of $(0,\infty)$. A sequence $\{\phi_n\}$ converges in $\mathcal{D}$ to ϕ if and only if

(i) $\phi_n \in \mathcal{D}$ ($n = 1,2,3......$)

(ii) $\phi \in \mathcal{D}$

(iii) all the ϕ_n and ϕ have their supports inside a fixed compact subset of $(0,\infty)$ (the subset being independent of n)

and (iv) for $k = 0,1,2....$, $D^k\phi_n \to D^k\phi$ uniformly on $(0,\infty)$

The space of continuous linear functionals on $\mathcal{D}$ is denoted by $\mathcal{D}'$ and the elements of $\mathcal{D}'$ are called distributions on $(0,\infty)$ or simply distributions.

Let $\mathcal{E}$ be the linear space of all smooth complex-valued functions ϕ defined on $(0,\infty)$. A sequence $\{\phi_n\}$ converges in $\mathcal{E}$ to ϕ if and only if

(i) $\phi_n \in \mathcal{E}$ $\quad (n = 1,2,3,\dots)$

(ii) $\phi \in \mathcal{E}$

and (iii) for $k = 0,1,2,\dots$, $D^k\phi_n \to D^k\phi$ uniformly on each compact subset of $(0,\infty)$.

The space of continuous linear functionals on $\mathcal{E}$ is denoted by $\mathcal{E}'$.

It is clear that $\mathcal{D} \subset \mathcal{E}$. Furthermore, if $\{\phi_n\}$ converges to ϕ in $\mathcal{D}$, then $\{\phi_n\}$ converges to ϕ in $\mathcal{E}$. It follows that $\mathcal{E}' \subset \mathcal{D}'$. It can be proved that $\mathcal{E}'$ consists of those elements of $\mathcal{D}'$ which have compact support (in the sense of distributions) and hence the elements of $\mathcal{E}'$ are called distributions with compact support.

From the definition of the spaces F_p, it is clear that

$$\mathcal{D} \subset F_p \subset \mathcal{E}$$

for each p, $1 \leq p \leq \infty$. Further, both inclusions are strict; for the first, we note that the function ϕ given by

$$\phi(x) = e^{-x} \qquad (0 < x < \infty)$$

belongs to F_p for each p, but not to $\mathcal{D}$, while for the second, we note that the function ψ given by

$$\psi(x) = x \qquad (0 < x < \infty)$$

belongs to $\mathcal{E}$ but not to any of the F_p spaces. However, since $\mathcal{D}$ is dense in $\mathcal{E}$, [25] p. 37, it follows immediately that F_p is dense in $\mathcal{E}$.

It is easy to prove that if $\{\phi_n\}$ converges to ϕ in $\mathcal{D}$, then $\{\phi_n\}$ converges to ϕ in F_p. For, the supports of ϕ_n, ϕ are all contained in the closed interval $[a,b]$ for some $0 < a < b < \infty$, so that

$$\gamma_k^p(\phi_n - \phi) = \left(\int_0^\infty \left| x^k \frac{d^k}{dx^k}(\phi_n - \phi) \right|^p dx \right)^{\frac{1}{p}}$$

$$= \left(\int_a^b \left| x^k \frac{d^k}{dx^k}(\phi_n - \phi) \right|^p dx \right)^{\frac{1}{p}}$$

$$\leq b^k \sup_{a \leq x \leq b} \left| \frac{d^k}{dx^k}(\phi_n - \phi) \right| (b-a)^{\frac{1}{p}}$$

$$\to 0 \text{ as } n \to \infty.$$

It follows that any element of F'_p, when restricted to $\mathfrak{D}$, is a member of $\mathfrak{D}'$ so that $F'_p \subset \mathfrak{D}'$

On the other hand, suppose $\{\phi_n\}$ converges to ϕ in F_p. We proved in Lemma 2 that $\delta^k \phi_n$ converges uniformly on any compact subset Ω of $(0, \infty)$ and hence $D^k \phi_n$ converges uniformly on such an Ω. Hence $\{\phi_n\}$ converges to a limit in $\mathcal{E}$ and clearly this limit is ϕ. Thus

Convergence in F_p $\Rightarrow$ Convergence in $\mathcal{E}$

and hence $\mathcal{E}' \subset F'_p$. We therefore have

<u>Theorem 6</u>

For $1 \leq p \leq \infty$, $\mathcal{E}' \subset F'_p \subset \mathfrak{D}'$

<u>§2.4 The spaces $F_{p,\mu}$ and $F'_{p,\mu}$</u>

In order to obtain meaningful definitions of various operations such as multiplication by powers of x and differentiation, we must introduce a generalisation of the F_p and F'_p spaces.

For any complex number μ and $1 \leq p \leq \infty$, we define $F_{p,\mu}$ by

$$F_{p,\mu} = \{ \phi : \phi(x) = x^{\mu}\psi(x) \text{ where } \psi \in F_p \}$$

With the usual pointwise operations of addition and scalar multiplication, $F_{p,\mu}$ becomes a (complex) linear space.

For $k = 0,1,2....$, define $\gamma_k^{p,\mu}$ on $F_{p,\mu}$ by

$$\gamma_k^{p,\mu}(\phi) = \gamma_k^{p} (\psi) \tag{11}$$

where $\phi(x) = x^{\mu} \psi(x) \in F_{p,\mu}$ and γ_k^p is given by (2). Each $\gamma_k^{p,\mu}$ is a seminorm on $F_{p,\mu}$ and $\gamma_0^{p,\mu}$ is a norm, so that the collection

$$M_{p,\mu} = \{ \gamma_k^{p,\mu} : k = 0,1,2... \} \tag{12}$$

is a countable multinorm and with the topology generated by $M_{p,\mu}$, $F_{p,\mu}$ becomes a countably multinormed space. Convergent and fundamental sequences are defined analogously to those for F_p.

From the definition of the seminorms, it is clear that multiplication by x^{μ} is an isomorphism of F_p onto $F_{p,\mu}$. The following result is then immediate.

Theorem 7

For each μ and $1 \leq p \leq \infty$, $F_{p,\mu}$ is a complete countably multinormed space and hence a Fréchet space.

Of course $F_{p,0}$ is simply our original F_p and we shall continue to write

$$F_{p,0} \equiv F_p.$$

Suppose $\{ \phi_n \}$ converges to zero in $F_{p,\mu}$ with $\phi_n(x) = x^{\mu}\psi_n(x)$. Then $\{\psi_n \}$ converges to zero in F_p as $n \to \infty$. Hence by the remarks following Theorem 3, for $k = 0,1,2....$,

$\{ D^k \psi_n \}$ converges to zero uniformly on each compact subset of $(0,\infty)$ as $n \to \infty$. It follows that $\{ D^k \phi_n \}$ converges uniformly to zero on each compact subset of $(0,\infty)$ as $n \to \infty$. Hence for each p and μ, $F_{p,\mu}$ is a testing-function space in the sense of Zemanian [25] p. 39.

Analogously to the F'_p spaces, we can construct $F'_{p,\mu}$ the space of continuous linear functionals on $F_{p,\mu}$. The elements of $F'_{p,\mu}$ will also be called generalised functions. With the topology of weak convergence (pointwise convergence) we have by Theorem 1.1,

Theorem 8

For each complex μ and $1 \leq p \leq \infty$, $F'_{p,\mu}$ is complete .

§2.5 Operators on $F_{p,\mu}$

We now consider some operators on $F_{p,\mu}$. The terminology used will be that of [25], Chapter 1.

For any complex number λ, we define the operator x^λ on $F_{p,\mu}$ by

$$(x^\lambda \phi) (x) = x^\lambda \phi(x) \qquad (0 < x < \infty)$$

No confusion will arise from using the same symbol for the function x^λ and the operation of multiplying by this function. We have already remarked that, for any μ, x^μ is an isomorphism of F_p onto $F_{p,\mu}$; and the inverse operator is $x^{-\mu}$. It now follows at once that, for any λ, μ x^λ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\lambda+\mu}$ with inverse $x^{-\lambda}$.

Next, we consider again the operator δ defined by (4), i.e. $(\delta\phi) (x) = x \frac{d\phi}{dx}$.

Let $\phi \in F_p$ .We have

$$x^k \frac{d^k}{dx^k}(\delta\phi) = x^k \frac{d^k}{dx^k}\left(x\frac{d\phi}{dx}\right)$$

$$= x^k \left(x\frac{d^{k+1}\phi}{dx^{k+1}} + k\frac{d^k\phi}{dx^k} \right)$$

$$= x^{k+1}\frac{d^{k+1}\phi}{dx^{k+1}} + kx^k\frac{d^k\phi}{dx^k} \qquad (13)$$

Hence $x^k \frac{d^k}{dx^k}(\delta\phi) \in L_p$ ($k = 0,1,2\dots$) i.e. $\delta\phi \in F_p$ so that δ maps F_p into F_p. δ is linear. Also from (13),

$$\gamma^p_k(\delta\phi) \leq \gamma^p_{k+1}(\phi) + k\,\gamma^p_k(\phi) \qquad (14)$$

so that δ is continuous at 0. Hence δ is a continuous linear mapping of F_p into F_p. If $1 \leq p < \infty$, δ is one-to-one, since

$$x\frac{d\phi}{dx} = 0 \Rightarrow \frac{d\phi}{dx} = 0 \Rightarrow \phi = c, \text{ a constant on } (0,\infty).$$

But $\phi \in L_p \Rightarrow c = 0$. If $p = \infty$, δ is not one-to-one, since all constant functions are mapped to zero. It can also be proved, e.g. using the theory of fractional integration developed in Chapter 3 , that δ is onto if $1 \leq p < \infty$, but not if $p=\infty$.

Now suppose $\phi \in F_{p,\mu}$, $\phi(x) = x^\mu\,\psi(x)$, $\psi \in F_p$.

$$\delta\phi(x) = x\frac{d}{dx}\left(x^\mu\,\psi(x)\right) = x^\mu\left(\mu\psi + x\frac{d\psi}{dx}\right)$$

Now by the above, $\psi \in F_p \Rightarrow x\frac{d\psi}{dx} \in F_p \Rightarrow \delta\phi \in F_{p,\mu}$, so δ maps $F_{p,\mu}$ into $F_{p,\mu}$. Linearity is again clear. For continuity, we have

$$\gamma^{p,\mu}_k(\delta\phi) = \gamma^p_k\left(\mu\psi + x\frac{d\psi}{dx}\right) = \gamma^p_k(\mu\psi + \delta\psi)$$

$$\leq \gamma^{p+1}_k(\psi) + (k + |\mu|)\,\gamma^p_k(\psi) \qquad \text{by (14)}$$

$$= \gamma^{p+1,\mu}_k(\phi) + (k + |\mu|)\,\gamma^{p,\mu}_k(\phi)$$

Hence δ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$

for all μ and $1 \leq p \leq \infty$. As before, we can obtain other results using the theory of Chapter 3 (See § 3.6) .

We define the operator δ' by

$$\delta'\phi(x) = \frac{d}{dx}(x\phi(x)) \tag{15}$$

Since $\frac{d}{dx}(x\phi(x)) = x\frac{d\phi}{dx} + \phi(x)$, we may write

$$\delta' = \delta + I$$

where I is the identity operator. It follows that δ' is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$.

Finally, we have the differentiation operator D defined by $D\phi(x) = \frac{d\phi}{dx}$. Since

$$D\phi(x) = x^{-1}\delta\phi(x)$$

it follows from the above that D is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-1}$.

For reference purposes, we gather together the results of this section in the following theorem.

<u>Theorem 9</u>

Let μ be any complex number, and let $1 \leq p \leq \infty$,

(i) x^{λ} is an isomorphism of $F_{p,\mu}$ onto $F_{p,\lambda+\mu}$ with inverse $x^{-\lambda}$

(ii) δ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$ (automorphism of F_p if $1 \leq p < \infty$)

(iii) δ' is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$

(iv) D is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-1}$

Other operators on $F_{p,\mu}$ will be dealt with as they arise .

§2.6 Operators on $F'_{p,\mu}$

In this section, we define operators on $F'_{p,\mu}$ corresponding to those of the previous section. The definitions are motivated by consideration of regular functionals.

For regular functionals, proceeding formally,

$$(x^{\lambda}f, \phi) = \int_0^\infty x^{\lambda}f(x)\ \phi(x)\ dx = \int_0^\infty f(x)\ x^{\lambda}\phi(x)\ dx$$

$$\text{or} \quad (x^{\lambda}f, \phi) = (f, x^{\lambda}\phi) \tag{16}$$

The right-hand side is meaningful if $f \in F'_{p,\mu}$ and $\phi \in F_{p,\mu-\lambda}$. We use (16) to define $x^{\lambda}f$ for any λ and $f \in F'_{p,\mu}$ and denote the mapping so obtained by x^{λ}. In fact, $x^{\lambda}f$ is a continuous linear functional on $F_{p,\mu-\lambda}$ and by Theorems 9 and 1.2, x^{λ} is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$ with inverse $x^{-\lambda}$

Let $\phi \in F_{p,\mu}$, $f \in F'_{p,\mu}$ be regular with compact support. Formally, we have

$$\int_0^\infty \delta f(x)\ \phi(x)\ dx = \int x\ f'(x)\ \phi(x)\ dx$$

$$= \Big[x\ f(x)\phi(x) \Big]_0^\infty - \int_0^\infty f(x)\ \frac{d}{dx}\ (x\ \phi(x))\ dx$$

The integrated terms vanish to give

$$(\delta f, \phi) = (f, -\delta'\phi) \tag{17}$$

We use (17) to define δ on $F'_{p,\mu}$ as the adjoint of $-\delta'$. As before, δ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu}$ by Theorems 9 and 1.2 .

Similarly, we define δ' on $F'_{p,\mu}$ by

$$(\delta' f, \phi) = (f, -\delta\phi) \tag{18}$$

where $f \in F'_{p,\mu}$ and $\phi \in F_{p,\mu}$. δ' is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu}$.

Finally, we define the differentiation operator D on $F'_{p,\mu}$ by

$$(Df, \phi) = (f, -D\phi) \qquad (19)$$

where $f \in F'_{p,\mu}$ and $\phi \in F_{p,\mu+1}$. D is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+1}$.

We therefore have the following theorem.

Theorem 10

Let λ, μ be complex numbers, $1 \leq p \leq \infty$.

(i) x^{λ} is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$ with inverse $x^{-\lambda}$

(ii) δ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu}$

(iii) δ' is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu}$

(iv) D is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+1}$

It should always be clear from the context whether the operators x^{λ}, δ, δ', D are being applied to testing-functions or generalised functions.

CHAPTER 3

Fractional Integration in $F_{p,\mu}$ and $F'_{p,\mu}$

§3.1 Introduction.

For $\operatorname{Re}\alpha > 0$, and a suitable function ϕ, we define $I^{\alpha}\phi$, a fractional integral of order α of ϕ (sometimes called the Riemann- Liouville integral of order α) by

$$I^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}\phi(u)\,du \qquad (1)$$

It is possible to modify the operator I^{α} in two stages. Firstly, we may integrate with respect to x^m ($m > 0$) rather than x by means of the operator $I^{\alpha}_{x^m}$ defined by

$$I^{\alpha}_{x^m}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_0^x (x^m-u^m)^{\alpha-1}u^{m-1}\phi(u)\,du \qquad (2)$$

so that I^{α}_{x} is just I^{α} again. On the other hand, there are ' homogeneous ' operators $I^{\eta,\alpha}$ introduced and discussed by Kober and Erdélyi in [15] and [14] defined by

$$\begin{aligned} I^{\eta,\alpha}\phi(x) &= x^{-\eta-\alpha} I^{\alpha} x^{\eta}\phi(x) \\ &= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}u^{\eta}\phi(u)\,du \qquad (3) \end{aligned}$$

where η is any complex number. Finally, combining these two steps, we obtain the operator $I^{\eta,\alpha}_{x^m}$ defined by

$$\begin{aligned} I^{\eta,\alpha}_{x^m}\phi(x) &= x^{-m\eta-m\alpha} I^{\alpha}_{x^m} x^{m\eta}\phi(x) \\ &= \frac{m\,x^{-m\eta-m\alpha}}{\Gamma(\alpha)}\int_0^x (x^m-u^m)^{\alpha-1}u^{m\eta+m-1}\phi(u)\,du \qquad (4) \end{aligned}$$

In this chapter, we develop the theory of $I^{\eta,\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ (defined below) for our spaces $F_{p,\mu}$ and $F'_{p,\mu}$. We will, of course, obtain incidentally properties of the ' inhomogeneous ' operators $I^{\alpha}_{x^m}$ and $K^{\alpha}_{x^m}$.

We begin by generalising a theorem of Kober concerning fractional integrals of functions in L_p.

§3.2 Action of $I^{\eta,\alpha}_{x^m}$ on L_p.

Kober [13] has proved the following theorem.

Theorem 1.

Let $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $I^{\eta,\alpha}$ (as defined by (3)) is a continuous linear mapping of L_p into L_p provided $\operatorname{Re} \eta > -\frac{1}{q}$.

Using this, we can prove the following extension.

Theorem 2.

If $m \operatorname{Re} \eta + m > \frac{1}{p}$, $I^{\eta,\alpha}_{x^m}$ is a continuous linear mapping of L_p into L_p ($1 \leq p \leq \infty$).

Proof: Suppose first that $1 \leq p < \infty$. Using (3) and (4), we can show that

$$x^{-\left(\frac{m-1}{mp}\right)} \left(I^{\eta,\alpha}_{x^m} \phi \right) \left(x^{\frac{1}{m}} \right) = I^{\eta',\alpha}_{x} \psi(x) \tag{5}$$

where

$$\eta' = \eta + \frac{m-1}{mp} \quad \text{and} \quad \psi(x) = x^{-\left(\frac{m-1}{mp}\right)} \phi\left(x^{\frac{1}{m}}\right) \quad \ldots$$

Now by a change of variable we can easily show that

$$| \psi |_p = m^{\frac{1}{p}} | \phi |_p \tag{6}$$

Hence $\psi \in L_p$. Similarly we have

$$\left| x^{-\left(\frac{m-1}{mp}\right)} I_{x^m}^{\eta,\alpha} \phi(x^{\frac{1}{m}}) \right|_p = m^{\frac{1}{p}} \left| I_{x^m}^{\eta,\alpha} \phi \right|_p \qquad (7)$$

From (5) and (7), we now have

$$m^{\frac{1}{p}} \left| I_{x^m}^{\eta,\alpha} \phi \right|_p = \left| I_x^{\eta',\alpha} \psi \right|_p \qquad (8)$$

Now if $m \operatorname{Re} \eta + m > \frac{1}{p}$, $\operatorname{Re} \eta > \frac{1}{mp} - 1 \Rightarrow \operatorname{Re} \eta' > -\frac{1}{q}$.
Hence by Theorem 1, $\exists$ K, independent of $\psi \in L_p$, such that

$$\left| I_x^{\eta',\alpha} \psi \right|_p \leqslant K \left| \psi \right|_p \qquad (9)$$

Combining (6), (8) and (9) gives

$$\left| I_{x^m}^{\eta,\alpha} \phi \right|_p \leqslant K \left| \phi \right|_p$$

where K is independent of $\phi \in L_p$. The result follows.

It remains to consider the case $p = \infty$. Proceeding as before, we obtain

$$(I_{x^m}^{\eta,\alpha} \phi)(x^{\frac{1}{m}}) = I_x^{\eta,\alpha} \psi (x) \qquad (10)$$

where $\psi(x) = \phi (x^{\frac{1}{m}})$. Clearly $\left| \psi \right|_\infty = \left| \phi \right|_\infty$, and also

$$\left| (I_{x^m}^{\eta,\alpha} \phi)(x^{\frac{1}{m}}) \right|_\infty = \left| (I_{x^m}^{\eta,\alpha} \phi)(x) \right|_\infty$$

Since we are now assuming that $\operatorname{Re} \eta > -1$, we can now complete the proof using Theorem 1 as above.

§3.3 Action of $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$ on F_p and $F_{p,\mu}$ -

Since F_p is a subspace of L_p, we know from Theorem 2 that if $m \operatorname{Re} \eta + m > \frac{1}{p}$, $I_{x^m}^{\eta,\alpha}$ maps F_p into L_p. We will now show that, in fact, under the same restriction on η, $I_{x^m}^{\eta,\alpha}$ maps F_p into F_p. This involves justifying differentiation under the integral sign for which we use Theorem 1.3.

We recall the definition of the operator δ given by equation (2.4) and the fact that δ is a continuous linear mapping of F_p into F_p. (Theorem 2.9 (ii)).

Theorem 3.

Let $\phi \in F_p$, $1 \leq p \leq \infty$, $m \operatorname{Re} \eta + m > \frac{1}{p}$, $\operatorname{Re} \alpha > 0$.

Then

(i) $I_{x^m}^{\eta,\alpha} \phi \in C^{\infty}$

(ii) For each $k = 0,1,2....$,

$$\delta^k I_{x^m}^{\eta,\alpha}\phi (x) = I_{x^m}^{\eta,\alpha} \delta^k\phi (x)$$

(iii) For each $k = 0,1,2....$,

$$x^k \frac{d^k}{dx^k} (I_{x^m}^{\eta,\alpha}\phi(x)) = I_{x^m}^{\eta,\alpha} (x^k \frac{d^k\phi}{dx^k})$$

(iv) $I_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of F_p into F_p.

Proof:

(i) We have

$$I_{x^m}^{\eta,\alpha} \phi (x) = \frac{m}{\Gamma(\alpha)} x^{-m\eta-m\alpha} \int_0^x (x^m-u^m)^{\alpha-1} u^{m\eta+m-1} \phi(u)\, du$$

$$= \frac{m}{\Gamma(\alpha)} \int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+m-1} \phi(xt)\, dt .$$

We apply Theorem 1.3 with $I = (0,1)$, and with $0 < A \leq x \leq B < \infty$ ($A < B$). Also we take

$$f (x,t) = \frac{m}{\Gamma(\alpha)} (1-t^m)^{\alpha-1} t^{m\eta+m-1} \phi(xt) .$$

Now by Lemma 2.4 ,

for some constant C_1, $|\phi(xt)| \leq C_1 (xt)^{-\frac{1}{p}}$ for x and t in the above intervals. Hence,

$$| f(x,t) | \leq C_2 (1-t^m)^{\alpha-1} t^{m\eta+m-1-\frac{1}{p}}$$

where

$$C_2 = \frac{m}{|\Gamma(\alpha)|} A^{-\frac{1}{p}} C_1$$

$$\Rightarrow \int_0^1 f(x,t) < \infty, \text{ since } \mathrm{Re}\alpha > 0 \text{ and } m\mathrm{Re}\eta + m > \frac{1}{p}.$$

Now $f_x(x,t) = \frac{m}{\Gamma(\alpha)} (1-t^m)^{\alpha-1} t^{m\eta+m-1} x^{-1}(xt)\phi'(xt)$ and since $\delta\phi \in F_p$, we have as before that

$$| f_x(x,t) | \leq C_3 (1-t^m)^{\alpha-1} t^{m\eta+m-1-\frac{1}{p}}$$

for some constant C_3. The right-hand side is integrable over (0,1). Hence by Theorem 1.3, $I_{x^m}^{\eta,\alpha}\phi$ is differentiable and

$$\frac{d}{dx}\left(I_{x^m}^{\eta,\alpha}\phi(x) \right) = x^{-1} I_{x^m}^{\eta,\alpha}\delta\phi(x) \tag{11}$$

Since δ maps F_p into F_p, we can proceed from (11) to show that $I_{x^m}^{\eta,\alpha}\phi$ is infinitely differentiable.

(ii) From (11) we have

$$\delta I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,\alpha}\delta\phi$$

Again, since $\delta\phi \in F_p$, we may use induction to prove the result.

(iii) We have that for $k = 0,1,2,\dots$,

$$x^k \frac{d^k}{dx^k} = \delta(\delta-1)\dots\dots(\delta-k+1)$$

The right-hand side is a polynomial in δ, say $P(\delta)$. Clearly, from (ii),

$$P(\delta) I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,\alpha} P(\delta)\phi$$

so that the result follows.

(iv) We have shown in (i) that $I_{x^m}^{\eta,\alpha}\phi$ is smooth. Also, by (iii)

$$x^k \frac{d^k}{dx^k}\left(I_{x^m}^{\eta,\alpha}\phi\right) = I_{x^m}^{\eta,\alpha}\left(x^k \frac{d^k\phi}{dx^k}\right)$$

$$\Rightarrow \gamma_k^p\left(I_{x^m}^{\eta,\alpha}\phi\right) = \left| I_{x^m}^{\eta,\alpha}\left(x^k \frac{d^k\phi}{dx^k}\right)\right|_p$$

By Theorem 2, there is a constant K_k independent of ϕ, such that

$$\left| I_{x^m}^{\eta,\alpha}\left(x^k \frac{d^k\phi}{dx^k}\right)\right|_p \leq K_k \left| x^k \frac{d^k\phi}{dx^k}\right|_p$$

Hence, we have for $k = 0,1,2,\ldots$,

$$\gamma_k^p\left(I_{x^m}^{\eta,\alpha}\phi\right) \leq K_k\, \gamma_k^p(\phi) \tag{12}$$

(12) shows that $I_{x^m}^{\eta,\alpha}$ is a continuous mapping of F_p into F_p. As linearity is obvious, the proof is complete.

We next extend Theorem 3 to the spaces $F_{p,\mu}$.

<u>Theorem 4</u>

Let $1 \leq p \leq \infty$, Re $\alpha > 0$. Then $I_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$ provided that

$$\text{Re}\,(m\eta + \mu) + m > \frac{1}{p}$$

<u>Proof</u> : Let $\phi \in F_{p,\mu}$, $\phi(x) = x^\mu \psi(x)$, with $\psi \in F_p$.

$$\begin{aligned} I_{x^m}^{\eta,\alpha}\phi(x) &= I_{x^m}^{\eta,\alpha}\left(x^\mu\psi(x)\right) \\ &= \frac{m}{\Gamma(\alpha)}\int_0^1 (1-t^m)^{\alpha-1}(xt)^\mu\, \psi(xt)\, t^{m\eta+m-1}\, dt \\ &= x^\mu \frac{m}{\Gamma(\alpha)}\int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+\mu+m-1}\, \psi(xt)\, dt \end{aligned}$$

$$\Rightarrow I_{x^m}^{\eta,\alpha}\phi(x) = x^\mu\, I_{x^m}^{\eta+\frac{\mu}{m},\,\alpha}\,\psi(x) \tag{13}$$

Now $\psi \in F_p$ and by hypothesis,

$$m \operatorname{Re}\left(\eta + \frac{\mu}{m}\right) + m = \operatorname{Re}(m\eta + \mu) + m > \frac{1}{p}$$

Hence, by Theorem 3, $I_{x^m}^{\eta+\frac{\mu}{m},\alpha} \psi \in F_p$. Further we can write (13) in the form

$$I_{x^m}^{\eta,\alpha} \phi(x) = x^{\mu} I_{x^m}^{\eta+\frac{\mu}{m},\alpha} x^{-\mu}\phi(x)$$

Using Theorem 3 again and also Theorem 2.9 (i), we see that $I_{x^m}^{\eta,\alpha}$, being the composition of three continuous linear mappings , is itself a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$.

Theorem 5

Let $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, $\operatorname{Re}(m\eta + \mu) + m > \frac{1}{p}$, $\phi \in F_{p,\mu}$. Then

$$I_{x^m}^{\eta+\alpha,\beta} I_{x^m}^{\eta,\alpha} \phi(x) = I_{x^m}^{\eta,\alpha+\beta} \phi(x)$$

Proof : Note first that both sides belong to $F_{p,\mu}$ by Theorem 4.

$$I_{x^m}^{\eta+\alpha,\beta} I_{x^m}^{\eta,\alpha} \phi(x) \qquad \ldots\ldots (14)$$

$$= \frac{m}{\Gamma(\beta)} x^{-m\eta-m\alpha-m\beta}\int_0^x (x^m-t^m)^{\beta-1} t^{m-1} dt \, \frac{m}{\Gamma(\alpha)}\int_0^t (t^m-u^m)^{\alpha-1} u^{m\eta+m-1}\phi(u)\, du$$

We wish to interchange the order of integration in (14). To this end, we note that also

$$I_{x^m}^{\eta+\alpha,\beta} I_{x^m}^{\eta,\alpha} \phi(x)$$

$$= \frac{m}{\Gamma(\alpha)}\frac{m}{\Gamma(\beta)}\int_0^1\int_0^1 (1-t^m)^{\beta-1} t^{m\eta+m\alpha+m-1} (1-u^m)^{\alpha-1} u^{m\eta+m-1} \phi(xtu)\, dt\, du$$

The double integral converges absolutely since $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, $\operatorname{Re}(m\eta + \mu) + m > \frac{1}{p}$ and $|\phi(y)| \leq M y^{\operatorname{Re}\mu - \frac{1}{p}} (0 < y < \infty)$ for some constant M by Lemma 2.4 .Thus, by Fubini's Theorem, we may interchange the order of integration in (14) to get

$$\frac{m x^{-m\eta-m\alpha-m\beta}}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^x u^{m\eta+m-1}\,\phi(u)\,du \int_u^x (x^m-t^m)^{\beta-1}(t^m-u^m)^{\alpha-1} m t^{m-1}\,dt$$

On putting

$$z = \frac{t^m - u^m}{x^m - u^m}$$

the t-integral becomes

$$(x^m-u^m)^{\alpha+\beta-1}\int_0^1 (1-z)^{\beta-1} z^{\alpha-1}\,dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,(x^m-u^m)^{\alpha+\beta-1}$$

Finally,

$$I_{x^m}^{\eta+\alpha,\beta}\, I_{x^m}^{\eta,\alpha}\,\phi(x) = \frac{m x^{-m\eta-m\alpha-m\beta}}{\Gamma(\alpha+\beta)} \int_0^x (x^m-u^m)^{\alpha+\beta-1} u^{m\eta+m-1}\,\phi(u)\,du$$

$$= I_{x^m}^{\eta,\,\alpha+\beta}\,\phi(x) \qquad \text{as required}$$

All results so far concerning $I_{x^m}^{\eta,\alpha}$ have been proved under the hypothesis $\mathrm{Re}\ \alpha > 0$. We now show that this restriction can be removed by the process of analytic continuation.

Definition For each α in some domain D of the complex plane let T_α be an operator on $F_{p,\mu}$. We shall say that T_α is analytic with respect to α in D if there exists an operator $\frac{\partial T}{\partial\alpha}\alpha$ on $F_{p,\mu}$ such that for each fixed ϕ in $F_{p,\mu}$, $0 < x < \infty$,

$$\frac{1}{h}\,[\,T_{\alpha+h}\,\phi(x) - T_\alpha\,\phi(x)\,] - \frac{\partial T}{\partial\alpha}\alpha\,\phi(x) \to 0$$

in the topology of $F_{p,\mu}$ as $h \to 0$ in any manner (h being complex)

Theorem 6

On $F_{p,\mu}$, $I_{x^m}^{\eta,\alpha}$ is analytic with respect to α for $\mathrm{Re}\ \alpha > 0$, provided that $\mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$.

Proof: We fix η with $\mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$ and fix $\phi \in F_{p,\mu}$. We have $I_{x^m}^{\eta,\alpha}\phi = \frac{m}{\Gamma(\alpha)}\,T_\alpha\phi$ where

$$(T_\alpha \phi)(x) = \int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+m-1} \phi(xt)\, dt \qquad (15)$$

Since multiplication by $\frac{m}{\Gamma(\alpha)}$ is easily seen to be an analytic operator on $F_{p,\mu}$ in the sense of the above definition, and **since** the composition of two analytic operators is analytic, we need only prove that T_α is analytic with respect to α on $F_{p,\mu}$ for $\operatorname{Re}\alpha > 0$. Differentiating (15) formally with respect to α gives

$$\left(\frac{\partial T_\alpha}{\partial \alpha} \phi\right)(x) = \int_0^1 (1-t^m)^{\alpha-1} \log (1-t^m)\, t^{m\eta+m-1} \phi(xt)\, dt$$

With this expression for $\frac{\partial T_\alpha}{\partial \alpha}$,

$$\frac{1}{h} [T_{\alpha+h}\phi (x) - T_\alpha \phi(x)] - \frac{\partial T_\alpha}{\partial \alpha} \phi (x)$$

$$= \int_0^1 f_h (1-t^m) (1-t^m)^{\alpha-\epsilon -1} t^{m\eta+m-1} \phi(xt)\, dt \qquad (16)$$

where $f_h(x) = \{ \frac{x^h - 1}{h} - \log x \} x^\epsilon$ and $0 < \epsilon < \operatorname{Re} \alpha$

We **show** below that as $h \to 0$, in any manner,

$$\sup_{0\leqslant t\leqslant 1} | f_h(1-t^m) | = \sup_{0\leqslant x\leqslant 1} |f_h(x)| \to 0 .$$

It then follows from (16) that

$$\gamma_0^{p,\mu} [\frac{1}{h} (T_{\alpha+h}\phi - T_\alpha\phi) - \frac{\partial T_\alpha}{\partial \alpha} \phi]$$

$$\leqslant \sup_{0\leqslant t\leqslant 1} | f_h(1-t^m) | \, \gamma_0^{p,\mu} (I_{x^m}^{\eta,\alpha-\epsilon} |\phi|) \to 0$$

/Re η /Re α?

as $h \to 0$ in any manner. Proceeding as in Theorem 3, we may differentiate under the integral sign in (16) to deduce that for each $k = 1,2,3\ldots$,

$$x^k \frac{d^k}{dx^k} [\frac{1}{h} (T_{\alpha+h}\phi - T_\alpha\phi) - \frac{\partial T_\alpha}{\partial \alpha} \phi]$$

$$= \int_0^1 f_h (1-t^m) (1-t^m)^{\alpha-\epsilon-1} t^{m\eta+m-1} (xt)^\mu \chi_k(xt)\, dt$$

where $x^{\mu}\chi_k(x) = x^k \dfrac{d^k\phi}{dx^k}$ so that $\chi_k \in F_p$. Hence,

$$\gamma_k^{p,\mu}\left[\frac{1}{h}\left(T_{\alpha+h}\phi - T_{\alpha}\phi\right) - \frac{\partial T}{\partial\alpha}_{\alpha}\phi\right]$$

$$\leqslant \sup_{0\leqslant t\leqslant 1} \left| f_h(1-t^m)\right| \times \left| I_{x^m}^{\eta+\frac{\mu}{m},\alpha-\epsilon}\chi_k\right|_p \to 0 \text{ as } h\to 0.$$

The result would then follow.

Thus it only remains to prove that $\sup_{0\leqslant x\leqslant 1}|f_h(x)| \to 0$ as $h \to 0$ in any manner.

$$|f_h(x)| = \left|\frac{x^h-1}{h} - \log x\right| x^{\epsilon}$$

Now $\left|\dfrac{x^h-1}{h} - \log x\right| = \left|\dfrac{1}{h}\{\exp(h\log x) - 1 - h\log x\}\right|$

$$= \left|\frac{1}{h}\sum_{n=2}^{\infty}\frac{(h\log x)^n}{n!}\right| \leqslant \frac{1}{|h|}\sum_{n=2}^{\infty}\frac{|h|^n\,|\log x|^n}{n!}$$

$$= \frac{1}{|h|}\sum_{n=2}^{\infty}\frac{(-|h|\log x)^n}{n!} \qquad (\text{ since } \log x < 0)$$

$$= -\left\{\frac{x^{-|h|}-1}{-|h|} - \log x\right\}, \text{ reversing a previous step}$$

$$= \frac{x^{-|h|}-1}{|h|} + \log x \qquad = x^{-\epsilon}g_h(x) \text{ say.}$$

We have that $|f_h(x)| \leqslant |g_h(x)|$. We prove that $\sup_{0\leqslant x\leqslant 1}|g_h(x)| \to 0$ as $h \to 0$ in any manner. Since g_h is a real function involving only $|h|$, we may use calculus to locate turning values. Suppose as we may that $0 < |h| < \epsilon$. Then $g_h(x) \to 0$ as $x \to 0+$ and $g_h(x) \to 0$ as $x \to 1-$.

$$g_h'(x) = \epsilon x^{\epsilon-1}\left\{\frac{x^{-|h|}-1}{|h|} + \log x\right\} + \left\{-x^{-|h|-1} + \frac{1}{x}\right\}x^{\epsilon}$$

$$= \left(\frac{\epsilon}{|h|} - 1\right)\left(x^{-|h|} - 1\right)x^{\epsilon-1} + \epsilon x^{\epsilon-1}\log x$$

In $0 < x < 1$, $x^{\epsilon-1} \neq 0 \Rightarrow g_h'(x) = 0$ when

$$\log x = -\frac{\epsilon-|h|}{|h|\,\epsilon}\left\{x^{-|h|} - 1\right\}$$

Then $\log x^{-|h|} = \frac{\epsilon - |h|}{\epsilon} \{ x^{-|h|} - 1 \}$

or $\log y_h = \frac{\epsilon - |h|}{\epsilon} \{ y_h - 1 \}$

From the convexity of $\log y$ and its derivatives at $y = 1$, it is easily seen that $y_h \to 1$ as $h \to 0$. Hence

$$\sup_{0 \leq x \leq 1} |g_h(x)| = \sup_{0 \leq x \leq 1} \left| \frac{x^{-|h|} - 1}{|h|} + \log x \right| x^{\epsilon}$$

$$\leq 1 \left| \frac{y_h - 1}{|h|} - \frac{\epsilon - |h|}{|h| \epsilon} (y_h - 1) \right|$$

$$= (y_h - 1) \left(\frac{1}{|h|} - \frac{\epsilon - |h|}{|h| \epsilon} \right) = \frac{1}{\epsilon} (y_h - 1) \to 0 \text{ as } h \to 0.$$

This finally completes the proof of Theorem 6.

We also note in passing that a similar argument proves the following result.

Theorem 7

On $F_{p,\mu}$, and with fixed α, $\mathrm{Re}\alpha > 0$, $I_{x^m}^{\eta,\alpha}$ is analytic with respect to η provided $\mathrm{Re} (m\eta + \mu) + m > \frac{1}{p}$.

We shall mainly be concerned with analytic continuation of $I_{x^m}^{\eta,\alpha}$ with respect to α for which we use the following lemma.

Lemma 8

Let $\mathrm{Re}\ \alpha > 0$, $\phi \in F_{p,\mu}$, $\mathrm{Re} (m\eta + \mu) + m > \frac{1}{p}$. Then

$$\delta I_{x^m}^{\eta,\alpha+1} \phi = I_{x^m}^{\eta,\alpha+1} \delta\phi = m I_{x^m}^{\eta,\alpha} \phi - (m\eta + m\alpha + m) I_{x^m}^{\eta,\alpha+1} \phi \tag{17}$$

Proof : That $\delta I_{x^m}^{\eta,\alpha+1}\phi = I_{x^m}^{\eta,\alpha+1}\delta\phi$ follows from Theorem 3 (ii).

Now, we have $\frac{d}{dx} I_{x^m}^{\eta,\alpha+1}\phi(x)$

$$= \frac{d}{dx}\left\{ \frac{m}{\Gamma(\alpha+1)} x^{-m\eta-m\alpha-m} \int_0^x (x^m-t^m)^{\alpha} t^{m\eta+m-1} \phi(t)\, dt \right\}$$

$$= -(m\eta+m\alpha+m)\, x^{-m\eta-m\alpha-m-1}\, I + x^{-m\eta-m\alpha-m} \frac{d}{dx} I \quad , \text{ where}$$

$$I(x) = \frac{m}{\Gamma(\alpha+1)} \int_0^x (x^m-t^m)^{\alpha}\, t^{m\eta+m-1}\, \phi(t)\, dt$$

Using Theorem 1.3 , we can differentiate under the integral sign to obtain

$$\frac{d}{dx} I(x) = \frac{m}{\Gamma(\alpha)} m x^{m-1} \int_0^x (x^m-t^m)^{\alpha-1}\, t^{m\eta+m-1}\, \phi(t)\, dt$$

$$\Rightarrow \quad x \frac{d}{dx} (I_{x^m}^{\eta,\alpha+1} \phi(x))$$

$$= -(m\eta+m\alpha+m)\, x^{-m\eta-m\alpha-m} \frac{m}{\Gamma(\alpha+1)} \int_0^x (x^m-t^m)^{\alpha}\, t^{m\eta+m-1}\, \phi(t)\, dt$$

$$+ m\, x^{-m\eta-m\alpha} \frac{m}{\Gamma(\alpha)} \int_0^x (x^m - t^m)^{\alpha-1}\, t^{m\eta+m-1}\, \phi(t)\, dt$$

$$\Rightarrow \quad \delta I_{x^m}^{\eta,\alpha+1}\phi = -(m\eta+m\alpha+m)\, I_{x^m}^{\eta,\alpha+1}\phi + m\, I_{x^m}^{\eta,\alpha}\phi$$

as required.

We can arrange the formula of Lemma 8 to give

$$m\, I_{x^m}^{\eta,\alpha} \phi(x) = (m\eta+m\alpha+m)\, I_{x^m}^{\eta,\alpha+1} \phi(x) + I_{x^m}^{\eta,\alpha+1} \delta\phi(x) \qquad (18)$$

For fixed x and η with $\mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$, the right-hand side of (18) is, by Theorem 6, an analytic function of α on $F_{p,\mu}$ provided $\mathrm{Re}\,\alpha > -1$. We can therefore use (18) to extend the definition of $I_{x^m}^{\eta,\alpha}$, in the first instance to $-1 < \mathrm{Re}\,\alpha \leq 0$, and hence, step by step , to the whole α -plane. The extended operator on $F_{p,\mu}$ is an entire function of α .

By sufficiently many applications of (18) together with the result of Theorem 4, we have

Theorem 9

Let $1 \leq p \leq \infty$. For any complex number α and $\mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$, $I_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$.

We shall shortly prove much more about the mapping properties of $I_{x^m}^{\eta,\alpha}$ under these conditions.

Putting $\alpha = 0$ in (18) we have for $\mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$

$$m\, I_{x^m}^{\eta,0}\, \phi(x) = (m\eta+m)\, I_{x^m}^{\eta,1}\, \phi(x) + I_{x^m}^{\eta,1}\, \delta\phi(x)$$

$$\Rightarrow\; m\, I_{x^m}^{\eta,0}\, \phi(x) = m x^{-m\eta-m}\int_0^x u^{m\eta+m-1}[(m\eta+m)\,\phi(u) + u\,\frac{d\phi}{du}]\, du$$

$$= m x^{-m\eta-m}\int_0^x \frac{d}{du}\,(\phi(u)\, u^{m\eta+m})\, du$$

$$= m x^{-m\eta-m}\,[\,\phi(u)\, u^{m\eta+m}\,]_0^x$$

$$= m x^{-m\eta-m}\,\phi(x)\, x^{m\eta+m} \quad ,\ \text{since } \mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$$

$$= m\,\phi(x)$$

$$\Rightarrow\; I_{x^m}^{\eta,0}\,\phi(x) = \phi(x) \qquad (19)$$

so that $I_{x^m}^{\eta,0}$ is the identity operator on $F_{p,\mu}$ in this case.

Next, we can use analytic continuation with respect to α and β to remove the restrictions $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\beta > 0$ in Theorem 5. However, we must insert the extra condition

$$\mathrm{Re}\,(m\eta + m\alpha + \mu) + m > \frac{1}{p}$$

which was redundant before. We therefore have

Theorem 10

Let $\mathrm{Re}\,(m\eta+\mu) + m > \frac{1}{p}$, $\mathrm{Re}\,(m\eta+m\alpha+\mu) + m > \frac{1}{p}$ $\phi \in F_{p,\mu}$. Then

$$I_{x^m}^{\eta+\alpha,\beta}\, I_{x^m}^{\eta,\alpha}\, \phi(x) = I_{x^m}^{\eta,\alpha+\beta}\, \phi(x) \qquad (20)$$

This immediately leads to the following very important result.

Corollary 11

Let $\mathrm{Re}\,(m\eta+\mu)+m>\frac{1}{p}$, $\mathrm{Re}\,(m\eta+m\alpha+\mu)+m>\frac{1}{p}$. Then $I^{\eta,\alpha}_{x^m}$ is an automorphism of $F_{p,\mu}$ and

$$\left(I^{\eta,\alpha}_{x^m}\right)^{-1} = I^{\eta+\alpha,-\alpha}_{x^m} \tag{21}$$

Proof : We know, from Theorem 9, that under the given conditions $I^{\eta,\alpha}_{x^m}$ and $I^{\eta+\alpha,-\alpha}_{x^m}$ are continuous linear mappings of $F_{p,\mu}$ into itself. Let $\phi \in F_{p,\mu}$. Taking $\beta = -\alpha$ in (20) gives

$$I^{\eta+\alpha,-\alpha}_{x^m} I^{\eta,\alpha}_{x^m}\phi = I^{\eta,0}_{x^m}\phi = \phi \quad \text{by (19)}$$

Replacing η by $\eta+\alpha$, α by $-\alpha$ and β by α in (20) gives

$$I^{\eta,\alpha}_{x^m} I^{\eta+\alpha,-\alpha}_{x^m}\phi = I^{\eta+\alpha,0}_{x^m}\phi = \phi$$

The result follows at once.

We derive next some results, similar to that of Lemma 8, which will be used in the sequel. We assume $\phi \in F_{p,\mu}$ and that $\mathrm{Re}\,(m\eta+\mu)+m>\frac{1}{p}$.

$$\delta I^{\eta,\alpha+1}_{x^m}\phi = I^{\eta,\alpha+1}_{x^m}\delta\phi$$

$$= \frac{m}{\Gamma(\alpha+1)} x^{-m\eta-m\alpha-m}\int_0^x (x^m-u^m)^{\alpha} u^{m\eta+m-1} u\phi'(u)\,du$$

$$= \frac{m}{\Gamma(\alpha+1)} x^{-m\eta-m\alpha-m}\left\{ \left[(x^m-u^m)^{\alpha} u^{m\eta+m}\phi(u)\right]_0^x - \int_0^x \phi(u)\frac{d}{du}\left[(x^m-u^m)^{\alpha}u^{m\eta+m}\right]du \right\}$$

$$= m\,\frac{m}{\Gamma(\alpha)} x^{-m\eta-m\alpha-m}\int_0^x (x^m-u^m)^{\alpha-1} u^{m\eta+m}\phi(u)\,u^{m-1}du$$

$$- (m\eta+m) \frac{m}{\Gamma(\alpha+1)} x^{-m\eta-m\alpha-m} \int_0^x (x^m-u^m)^\alpha u^{m\eta+m-1} \phi(u) du$$

$$\Rightarrow I_{x^m}^{\eta,\alpha+1}\delta\phi = \delta I_{x^m}^{\eta,\alpha+1}\phi = mI_{x^m}^{\eta+1,\alpha} \phi - (m\eta+m) I_{x^m}^{\eta,\alpha+1} \phi \quad (22)$$

With δ' defined as before by

$$\delta' \phi(x) = \frac{d}{dx} (x\phi) = \delta\phi (x) + \phi (x)$$

(17) and (22) immediately give

$$\delta' I_{x^m}^{\eta,\alpha+1}\phi = I_{x^m}^{\eta,\alpha+1}\delta'\phi = m I_{x^m}^{\eta,\alpha}\phi - (m\eta+m\alpha+m-1) I_{x^m}^{\eta,\alpha+1}\phi \quad (23)$$

$$I_{x^m}^{\eta,\alpha+1} \delta'\phi = \delta' I_{x^m}^{\eta,\alpha+1}\phi = m I_{x^m}^{\eta+1,\alpha}\phi - (m\eta+m-1) I_{x^m}^{\eta,\alpha+1}\phi \quad (24)$$

We conclude this section by stating the mapping properties of the ' inhomogeneous ' operators $I_{x^m}^{\alpha}$ as given by (2). From (4) we see that for $\operatorname{Re} \alpha > 0$,

$$I_{x^m}^{\alpha} \phi(x) = x^{m\alpha}x^{-m\alpha} I_{x^m}^{\alpha} \phi(x) = x^{m\alpha} I_{x^m}^{0,\alpha}\phi(x) \quad (25)$$

It follows from Theorem 2.9 (i) and Theorem 4 that $I_{x^m}^{\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$ provided $\operatorname{Re} \mu + m > \frac{1}{p}$. The right-hand side of (25) is, by Theorem 6, an analytic function of α for fixed x and ϕ. We can use (25) to define $I_{x^m}^{\alpha}$ for $\operatorname{Re} \alpha \leq 0$ on $F_{p,\mu}$ provided only $\operatorname{Re}\mu + m > \frac{1}{p}$. Using the properties of $I_{x^m}^{0,\alpha}$, we can deduce the following theorem.

<u>Theorem 12</u>

Let $\operatorname{Re} \mu + m > \frac{1}{p}$. Then $I_{x^m}^{\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$, $I_{x^m}^{0}$ is the identity operator on $F_{p,\mu}$. If, further, $\operatorname{Re} (m + m\alpha + \mu) > \frac{1}{p}$,

$I^{\alpha}_{x^m}$ is an isomorphism with inverse

$$(I^{\alpha}_{x^m})^{-1} = I^{-\alpha}_{x^m} \qquad (26)$$

If $\frac{1}{p} - m - \text{Re}\,\mu < \min(0, m\,\text{Re}\alpha, m\,\text{Re}\beta)$ $\phi \in F_{p,\mu}$

$$I^{\alpha}_{x^m} I^{\beta}_{x^m} \phi = I^{\alpha+\beta}_{x^m}\phi = I^{\beta}_{x^m} I^{\alpha}_{x^m} \phi \qquad (27)$$

(27) is sometimes called the first index law for the operators $I^{\alpha}_{x^m}$. The second index law for I^{α}_{x} will be discussed in Chapter 6 where it arises in connection with hypergeometric integral operators.

§3.4 Action of $K^{\eta,\alpha}_{x^m}$ on F_p and $F_{p,\mu}$

For Re $\alpha > 0$, and a suitable function ϕ, we define $K^{\alpha}\phi$, sometimes called the Weyl integral of order α of ϕ by

$$K^{\alpha}\phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (u-x)^{\alpha-1} \phi(u)\, du \qquad (28)$$

As in the case of I^{α}, we have the operators $K^{\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ defined by

$$K^{\alpha}_{x^m} \phi(x) = \frac{m}{\Gamma(\alpha)} \int_x^{\infty} (u^m-x^m)^{\alpha-1} u^{m-1} \phi(u)du \qquad (29)$$

$$\begin{aligned} K^{\eta,\alpha}_{x^m} \phi(x) &= x^{m\eta} K^{\alpha}_{x^m} x^{-m\eta-m\alpha} \phi(x) \\ &= \frac{mx^{m\eta}}{\Gamma(\alpha)} \int_x^{\infty} (u^m-x^m)^{\alpha-1} u^{-m\eta-m\alpha+m-1} \phi(u)du \\ &= \frac{m}{\Gamma(\alpha)} \int_1^{\infty} (t^m-1)^{\alpha-1} t^{-m\eta-m\alpha+m-1} \phi(xt)dt \end{aligned} \qquad (30)$$

As is to be expected, the development of the theory of $K^{\eta,\alpha}_{x^m}$ on $F_{p,\mu}$ is similar to that of $I^{\eta,\alpha}_{x^m}$. We shall state the results without giving the full details of the proofs.

Using Theorem 1.3 and results of Kober [13] we can prove

Theorem 13

Let $1 \leq p \leq \infty$. $K^{\eta,\alpha}_{x^m}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided $\operatorname{Re}\alpha > 0$, $\operatorname{Re}(m\eta-\mu) > -\frac{1}{p}$.

Proof : Kober proved that $K^{\eta,\alpha}_{x}$ maps L_p into itself if $\operatorname{Re}\eta > -\frac{1}{p}$ and $\operatorname{Re}\alpha > 0$. By a change of variable, we can show that $K^{\eta,\alpha}_{x^m}$ maps F_p into L_p if $\operatorname{Re}\alpha > 0$, $\operatorname{Re} m\eta > -\frac{1}{p}$. Theorem 1.3 completes the proof for $\mu = 0$. A little manipulation then proves the general case.

As in the case of $I^{\eta,\alpha}_{x^m}$, we can remove the restriction $\operatorname{Re}\alpha > 0$ by means of analytic continuation. Let η be fixed with $\operatorname{Re}(m\eta-\mu) > -\frac{1}{p}$. Then, on $F_{p,\mu}$ $K^{\eta,\alpha}_{x^m}\phi(x)$ is, for each fixed x and ϕ, an analytic function of α. (We can actually prove a stronger result involving convergence in the topology of $F_{p,\mu}$ as in Theorem 6.) Under the hypotheses of Theorem 13, integration by parts gives

$$K^{\eta,\alpha+1}_{x^m}\delta\phi = \delta K^{\eta,\alpha+1}_{x^m}\phi = (m\eta+m\alpha)K^{\eta,\alpha+1}_{x^m}\phi - m K^{\eta,\alpha}_{x^m}\phi \qquad (31)$$

which is an analogue of (17). Rearranging gives

$$m K^{\eta,\alpha}_{x^m}\phi(x) = (m\eta+m\alpha)K^{\eta,\alpha+1}_{x^m}\phi(x) - K^{\eta,\alpha+1}_{x^m}\delta\phi(x) \qquad (32)$$

For fixed x and ϕ, we can use (32) to extend $K^{\eta,\alpha}_{x^m}\phi(x)$ to an entire function of α. We can then drop the restriction $\operatorname{Re}\alpha > 0$. in Theorem 13.

Putting $\alpha = 0$ in (32) shows that

$$K^{\eta,0}_{x^m}\phi = \phi \qquad (33)$$

for $\phi \in F_{p,\mu}$ provided $\mathrm{Re}\,(m\eta-\mu) > -\frac{1}{p}$. If, in addition, $\mathrm{Re}\,(m\eta+m\alpha-\mu) > -\frac{1}{p}$, we can prove by interchanging the order of integration that

$$K_{x^m}^{\eta,\alpha}\; K_{x^m}^{\eta+\alpha,\beta}\,\phi = K_{x^m}^{\eta,\alpha+\beta}\,\phi \tag{34}$$

This leads to the following theorem.

Theorem 14

Let $1 \le p \le \infty$. $K_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided $\mathrm{Re}\,(m\eta-\mu) > -\frac{1}{p}$. If, in addition, $\mathrm{Re}\,(m\eta+m\alpha-\mu) > -\frac{1}{p}$, $K_{x^m}^{\eta,\alpha}$ is an automorphism of $F_{p,\mu}$ with inverse $K_{x^m}^{\eta+\alpha,-\alpha}$.

If $\phi \in F_{p,\mu}$ and $\mathrm{Re}\,(m\eta-\mu) > -\frac{1}{p}$, the following results analogous to (22), (23), (24) hold.

$$\delta\, K_{x^m}^{\eta,\alpha+1}\,\phi = K_{x^m}^{\eta,\alpha+1}\,\delta\phi = m\eta\, K_{x^m}^{\eta,\alpha+1}\phi - m\, K_{x^m}^{\eta+1,\alpha}\,\phi \tag{35}$$

$$\delta' K_{x^m}^{\eta,\alpha+1}\,\phi \equiv K_{x^m}^{\eta,\alpha+1}\,\delta'\phi = (m\eta+1)\, K_{x^m}^{\eta,\alpha+1}\phi - mK_{x^m}^{\eta+1,\alpha}\,\phi \tag{36}$$

$$K_{x^m}^{\eta,\alpha+1}\,\delta'\phi = \delta' K_{x^m}^{\eta,\alpha+1}\,\phi = (m\eta+m\alpha+1)\, K_{x^m}^{\eta,\alpha+1}\phi - mK_{x^m}^{\eta,\alpha}\,\phi \tag{37}$$

Finally, we mention some properties of $K_{x^m}^{\alpha}$ on $F_{p,\mu}$. From (30), for any η, $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\mu < \frac{1}{p}$ and $\phi \in F_{p,\mu}$,

$$K_{x^m}^{\alpha}\,\phi(x) = K_{x^m}^{\alpha}\,x^{-m\alpha}x^{m\alpha}\,\phi(x) = K_{x^m}^{0,\alpha}\,x^{m\alpha}\,\phi(x) \tag{38}$$

For each fixed x and ϕ, the right-hand side of (38) is an analytic function of α and we may use (38) to extend the definition of $K_{x^m}^{\alpha}$ to $\mathrm{Re}\,\alpha \le 0$.

Proceeding as for $I_{x^m}^{\alpha}$, we can prove

Theorem 15

If $\mathrm{Re}\,(\mu + m\alpha) < \frac{1}{p}$, $K^{\alpha}_{x^m}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$ .If also $\mathrm{Re}\,\mu < \frac{1}{p}$, $K^{\alpha}_{x^m}$ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+m\alpha}$ and

$$(K^{\alpha}_{x^m})^{-1} = K^{-\alpha}_{x^m}$$

If $\frac{1}{p} - \mathrm{Re}\,\mu > \max\,(m\,\mathrm{Re}\,\alpha\,,\, m\,\mathrm{Re}\,\beta\,,\, m\,\mathrm{Re}\,(\alpha+\beta)\,)$, $\phi \in F_{p,\mu}$

$$K^{\alpha}_{x^m} K^{\beta}_{x^m} \phi = K^{\alpha+\beta}_{x^m} \phi = K^{\beta}_{x^m} K^{\alpha}_{x^m} \phi \qquad (39)$$

(39) is the first index law for the operators $K^{\alpha}_{x^m}$. The second index law will be discussed in Chapter 6 .

§3.5 The action of $I^{\eta,\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ on $F'_{p,\mu}$ -

We are now ready to develop the theory of fractional integration on the spaces $F'_{p,\mu}$ of generalised functions . The definitions of $I^{\eta,\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ are motivated by considering regular functionals .

Let $\phi \in F_{p,\mu}$ and let $f \in F'_{p,\mu}$ be a regular functional .Proceeding formally , with $\mathrm{Re}\,\alpha > 0$, we have

$$(I^{\eta,\alpha}_{x^m} f\,,\,\phi) = \int_0^{\infty} I^{\eta,\alpha}_{x^m} f\,(x)\,\phi\,(x)\,dx$$

$$= \int_0^{\infty} \phi(x)\,dx\ \frac{m}{\Gamma(\alpha)}\,x^{-m\eta-m\alpha} \int_0^{x} (x^m-u^m)^{\alpha-1}\,u^{m\eta+m-1}\,f(u)\,du$$

$$= \int_0^{\infty} f(u)\,du\ \frac{m}{\Gamma(\alpha)}\,u^{m\eta+m-1} \int_u^{\infty} (x^m-u^m)^{\alpha-1}\,x^{-m\eta-m\alpha}\,\phi(x)\,dx$$

$$= (f\,,\,K^{\eta',\alpha}_{x^m} \phi)$$

where $\eta' = \eta + 1 - \frac{1}{m}$. With this motivation , we define $I_{x^m}^{\eta,\alpha}$ on $F'_{p,\mu}$ for <u>any</u> α by

$$(I_{x^m}^{\eta,\alpha} f, \phi) = (f , K_{x^m}^{\eta+1-\frac{1}{m}, \alpha} \phi) \qquad (40)$$

where $\phi \in F_{p,\mu}$ and f is any member of $F'_{p,\mu}$. By Theorem 14 , $K_{x^m}^{\eta+1-\frac{1}{m}, \alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided that $\operatorname{Re} \{ m (\eta+1-\frac{1}{m}) - \mu \} > -\frac{1}{p}$, that is , provided $\operatorname{Re} (m\eta-\mu) + m > \frac{1}{q}$, where as usual $\frac{1}{p} + \frac{1}{q} = 1$. Hence , by Theorem 1.2 , we find that $I_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself under the same condition .

Using (40) and (34) , we can immediately deduce the following theorem analogous to Theorem 10 .

<u>Theorem 16</u>

Let $\operatorname{Re} (m\eta-\mu) + m > \frac{1}{q}$, $\operatorname{Re} (m\eta+m\alpha-\mu) + m > \frac{1}{q}$ $f \in F'_{p,\mu}$. Then

$$I_{x^m}^{\eta+\alpha,\beta} I_{x^m}^{\eta,\alpha} f = I_{x^m}^{\eta,\alpha+\beta} f \qquad (41)$$

Analogous to (19), we have, for $f \in F'_{p,\mu}$, and $\operatorname{Re} (m\eta-\mu) + m > \frac{1}{q}$, that

$$I_{x^m}^{\eta,0} f = f \qquad (42)$$

This follows by replacing η by $\eta + 1 - \frac{1}{m}$ in (33) and taking adjoints . From Theorem 16 or from the general theory of adjoints it also follows that , if $\operatorname{Re} (m\eta-\mu) + m > \frac{1}{q}$, $\operatorname{Re} (m\eta+m\alpha-\mu) + m > \frac{1}{q}$, $I_{x^m}^{\eta,\alpha}$ is an automorphism of $F'_{p,\mu}$ with inverse $I_{x^m}^{\eta+\alpha,-\alpha}$. We have therefore proved the following theorem .

Theorem 17

Let $1 \leq p \leq \infty$. $I^{\eta,\alpha}_{x^m}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu}$ provided $\text{Re}\,(m\eta-\mu) + m > \frac{1}{q}$. If , in addition , $\text{Re}\,(m\eta+m\alpha-\mu) + m > \frac{1}{q}$, $I^{\eta,\alpha}_{x^m}$ is an automorphism of $F'_{p,\mu}$ with inverse $I^{\eta+\alpha,-\alpha}_{x^m}$.

Comparing Theorem 17 with Theorem 9 and Corollary 11 , we see that the restrictions on η and α are obtained by replacing μ by $-\mu$ and p by q . This is to be expected from consideration of Hölder's Inequality. If $\phi \in F_{p,\mu}$,

$$\int_0^\infty f(x)\ \phi(x)\,dx$$

will converge if $f(x) = x^{-\mu} g(x)$ with $g \in L_q$; in particular, if $f \in F_{q,-\mu}$ which is imbedded in $F'_{p,\mu}$.

We note in passing

Theorem 18

For $\text{Re}\,(m\eta-\mu) + m > \frac{1}{q}$, $(I^{\eta,\alpha}_{x^m} f , \phi)$ is an analytic function of α for each fixed $f \in F'_{p,\mu}$ and $\phi \in F_{p,\mu}$

This follows easily from the remark made after Theorem 13 that, for fixed η satisfying the hypotheses of the theorem, $K^{\eta,\alpha}_{x^m}\phi$ is, for fixed ϕ, an analytic function of α and $\frac{\partial}{\partial\alpha} K^{\eta,\alpha}_{x^m}\phi$ exists as a limit in the topology of $F_{p,\mu}$. We can prove similarly that, for fixed α, $(I^{\eta,\alpha}_{x^m} f , \phi)$ is an analytic function of η in the half-plane $\text{Re}\,\eta > \frac{1}{m}(\text{Re}\,\mu + \frac{1}{q} - m)$.

We recall from Chapter 2 that the adjoint of δ' is $-\delta$. Replacing η by $\eta + 1 - \frac{1}{m}$ in (31), (35), (36) and (37) and taking adjoints, we obtain the following results analogous to (17), (22), (23) and (24), valid for $f \in F'_{p,\mu}$ with $\mathrm{Re}\,(m\eta-\mu) + m > \frac{1}{q}$.

$$\delta\, I_{x^m}^{\eta,\alpha+1} f = I_{x^m}^{\eta,\alpha+1}\, \delta f = m\, I_{x^m}^{\eta,\alpha} f - (m\eta+m\alpha+m)\, I_{x^m}^{\eta,\alpha+1} f \tag{43}$$

$$I_{x^m}^{\eta,\alpha+1}\, \delta f = \delta\, I_{x^m}^{\eta,\alpha+1} f = m\, I_{x^m}^{\eta+1,\alpha} f - (m\eta+m)\, I_{x^m}^{\eta,\alpha+1} f \tag{44}$$

$$\delta'\, I_{x^m}^{\eta,\alpha+1} f = I_{x^m}^{\eta,\alpha+1}\, \delta' f = m\, I_{x^m}^{\eta,\alpha} f - (m\eta+m\alpha+m-1)\, I_{x^m}^{\eta,\alpha+1} f \tag{45}$$

$$I_{x^m}^{\eta,\alpha+1}\, \delta' f = \delta'\, I_{x^m}^{\eta,\alpha+1} f = m\, I_{x^m}^{\eta+1,\alpha} f - (m\eta+m-1)\, I_{x^m}^{\eta,\alpha+1} f \tag{46}$$

Let $\phi \in F_{p,\mu}$, $f \in F'_{p,\mu}$. By consideration of regular functionals, we are led to define $K_{x^m}^{\eta,\alpha}$ on $F'_{p,\mu}$ by

$$(\, K_{x^m}^{\eta,\alpha} f\, ,\, \phi\,) = (\, f\, ,\, I_{x^m}^{\eta-1+\frac{1}{m},\alpha}\, \phi\,) \tag{47}$$

We know from Theorem 4 that the operator $I_{x^m}^{\eta-1+\frac{1}{m},\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided that $\mathrm{Re}\,\{\, m\,(\,\eta-1+\frac{1}{m}\,) + \mu\,\} + m > \frac{1}{p}$, i.e. if $\mathrm{Re}\,(\, m\eta+\mu\,) > -\frac{1}{q}$. Hence, under this condition, $K_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself . Proceeding similarly, we obtain

<u>Theorem 19</u>

For $1 \le p \le \infty$, $K_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself provided $\mathrm{Re}\,(\, m\eta+\mu\,) > -\frac{1}{q}$. If, in addition, $\mathrm{Re}\,(\, m\eta+m\alpha+\mu\,) > -\frac{1}{q}$, $K_{x^m}^{\eta,\alpha}$ is an automorphism of $F'_{p,\mu}$ and its inverse is $K_{x^m}^{\eta+\alpha,-\alpha}$. In this case also , for $f \in F'_{p,\mu}$,

$$K_{x^m}^{\eta,\alpha}\, K_{x^m}^{\eta+\alpha,\beta} f = K_{x^m}^{\eta,\alpha+\beta} f \tag{48}$$

Again the restrictions on the parameters are obtained from those in Theorem 14 by replacing μ by $-\mu$ and p by q.

Replacing η by $\eta - 1 + \frac{1}{m}$ in (17), (22), (23) and (24) gives the following results analogous to (31), (35), (36) and (37) valid for $f \in F'_{p,\mu}$ with $\operatorname{Re}(m\eta+\mu) > -\frac{1}{q}$

$$K_{x^m}^{\eta,\alpha+1}\delta f = \delta K_{x^m}^{\eta,\alpha+1} f = (m\eta+m\alpha)\, K_{x^m}^{\eta,\alpha+1} f - m\, K_{x^m}^{\eta,\alpha} f \qquad (49)$$

$$\delta\, K_{x^m}^{\eta,\alpha+1} f = K_{x^m}^{\eta,\alpha+1}\delta f = m\eta\, K_{x^m}^{\eta,\alpha+1} f - m\, K_{x^m}^{\eta+1,\alpha} f \qquad (50)$$

$$\delta' K_{x^m}^{\eta,\alpha+1} f = K_{x^m}^{\eta,\alpha+1}\delta' f = (m\eta+1)\, K_{x^m}^{\eta,\alpha+1} f - m\, K_{x^m}^{\eta+1,\alpha} f \qquad (51)$$

$$K_{x^m}^{\eta,\alpha+1}\delta' f = \delta' K_{x^m}^{\eta,\alpha+1} f = (m\eta+m\alpha+1)\, K_{x^m}^{\eta,\alpha+1} f - m\, K_{x^m}^{\eta,\alpha} f \qquad (52)$$

We conclude this section with a brief discussion of the operators $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$ on $F'_{p,\mu}$. Proceeding formally we have, for a regular functional f,

$$(I_{x^m}^{\alpha} f, \phi) = (x^{m\alpha} I_{x^m}^{0,\alpha} f, \phi) \qquad \text{by (25)}$$

$$= (I_{x^m}^{0,\alpha} f, x^{m\alpha}\phi) \qquad \text{by (2.16)}$$

$$= (f, K_{x^m}^{1-\frac{1}{m},\alpha}\, x^{m\alpha}\phi) \qquad \text{by (40)}$$

Hence, we define $I_{x^m}^{\alpha}$ for any complex number α on $F'_{p,\mu}$ by

$$(I_{x^m}^{\alpha} f, \phi) = (f, K_{x^m}^{1-\frac{1}{m},\alpha}\, x^{m\alpha}\phi) \qquad (53)$$

$$= (f, x^{m-1} K_{x^m}^{\alpha}\, x^{-m+1}\phi)$$

the definition being meaningful if $\phi \in F_{p,\mu-m\alpha}$. Using the theory of $K^{\eta,\alpha}_{x^m}$ together with Theorem 1.2 , we can easily prove

Theorem 20

$I^{\alpha}_{x^m}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$ provided that $\mathrm{Re}\,(m-\mu) > \frac{1}{q}$. If, in addition , $\mathrm{Re}\,(m+m\alpha-\mu) > \frac{1}{q}$, $I^{\alpha}_{x^m}$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ with inverse $I^{-\alpha}_{x^m}$. If $\frac{1}{q} - m + \mathrm{Re}\,\mu < \min\,(0, m\,\mathrm{Re}\alpha , m\,\mathrm{Re}\beta)$,

$$I^{\alpha}_{x^m} I^{\beta}_{x^m} f = I^{\alpha+\beta}_{x^m} f = I^{\beta}_{x^m} I^{\alpha}_{x^m} f \qquad (54)$$

for $f \in F'_{p,\mu}$.

Similarly we are led to define $K^{\alpha}_{x^m}$ on $F'_{p,\mu}$ by

$$(K^{\alpha}_{x^m} f , \phi) = (f , x^{m\alpha} I^{-1+\frac{1}{m},\alpha}_{x^m} \phi) \qquad (55)$$

$$= (f , x^{m-1} I^{\alpha}_{x^m} x^{-m+1} \phi)$$

where $f \in F'_{p,\mu}$ and $\phi \in F_{p,\mu-m\alpha}$. We then have the following theorem .

Theorem 21

$K^{\alpha}_{x^m}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$ provided that $\mathrm{Re}\,(m\alpha-\mu) < \frac{1}{q}$. If, in addition, $\mathrm{Re}(-\mu) < \frac{1}{q}$, $K^{\alpha}_{x^m}$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and ...

$$(K^{\alpha}_{x^m})^{-1} = K^{-\alpha}_{x^m}$$

If $\frac{1}{q} + \mathrm{Re}\,\mu > \max\,(m\,\mathrm{Re}\,\alpha , m\,\mathrm{Re}\,\beta , m\,\mathrm{Re}\,(\alpha+\beta))$, $f \in F'_{p,\mu}$

$$K^{\alpha}_{x^m} K^{\beta}_{x^m} f = K^{\alpha+\beta}_{x^m} f = K^{\beta}_{x^m} K^{\alpha}_{x^m} f \qquad (56)$$

Although we have developed the theory of $I^{\alpha}_{x^m}$ and $K^{\alpha}_{x^m}$ on $F'_{p,\mu}$, we shall mainly be concerned in the sequel with the homogeneous operators $I^{\eta,\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ to which the spaces are best suited.

§3.6 Further properties of δ and δ' on $F_{p,\mu}$ and $F'_{p,\mu}$

The theory developed in the previous sections can be used to obtain further information about δ and δ' which were first discussed in Chapter 2.

Let $\phi \in F_{p,\mu}$ with $\text{Re}\,\mu > \frac{1}{p}$. We may put $m = 1$, $\eta = -1$ and $\alpha = 0$ in (17) to obtain

$$\delta\, I_x^{-1,1}\phi = I_x^{-1,1}\delta\phi = I_x^{-1,0}\,\phi = \phi \qquad \text{by (19)}$$

Thus, if $\text{Re}\,\mu > \frac{1}{p}$, δ is an automorphism of $F_{p,\mu}$ and

$$\delta^{-1} = I_x^{-1,1} \tag{57}$$

This in turn implies that on $F_{p,\mu}$ with $\text{Re}\,\mu > \frac{1}{p}$,

$$\delta = I_x^{0,-1} \tag{58}$$

We could also have obtained these results using (22).

Suppose on the other hand that $\phi \in F_{p,\mu}$ with $\text{Re}\,\mu < \frac{1}{p}$. We now put $m = 1$, $\eta = \alpha = 0$ in (31) to get

$$K_x^{0,1}\,\delta\phi = \delta\, K_x^{0,1}\,\phi = -\,K_x^{0,0}\,\phi = -\,\phi \qquad \text{by (33)}$$

So, if $\text{Re}\,\mu < \frac{1}{p}$, δ is an automorphism of $F_{p,\mu}$ and

$$\delta^{-1} = -\,K_x^{0,1} \tag{59}$$

or

$$\delta = -\,K_x^{-1,1} \tag{60}$$

As regards the limiting case $\text{Re}\,\mu = \frac{1}{p}$, we have already seen in §2.5 that with $\mu = 0$, $p = \infty$, δ is not invertible on $F_{p,\mu}$.

A similar procedure can be carried out for δ' using (23) or (24) when $\text{Re}\ \mu > \frac{1}{p}$ and (36) or (37) when $\text{Re}\ \mu < \frac{1}{p}$. We have the following theorem.

Theorem 22

(i) Let $\text{Re}\ \mu > \frac{1}{p}$. Then δ and δ' are automorphisms of $F_{p,\mu}$ and

$$\delta^{-1} = I_x^{-1,1} \quad ; \quad (\delta')^{-1} = I_x^{0,1}$$

(ii) Let $\text{Re}\ \mu < \frac{1}{p}$. Then δ and δ' are automorphisms of $F_{p,\mu}$ and

$$\delta^{-1} = -K_x^{0,1} \quad ; \quad (\delta')^{-1} = -K_x^{-1,1}$$

Taking adjoints, or using (43)-(46) and (49)-(52), we obtain the corresponding results for $F'_{p,\mu}$.

Theorem 23

(i) Let $\text{Re}\ \mu < -\frac{1}{q}$. Then δ and δ' are automorphisms of $F'_{p,\mu}$ and

$$\delta^{-1} = I_x^{-1,1} \quad ; \quad (\delta')^{-1} = I_x^{0,1}$$

(ii) Let $\text{Re}\ \mu > -\frac{1}{q}$. Then δ and δ' are automorphisms of $F'_{p,\mu}$ and

$$\delta^{-1} = -K_x^{0,1} \quad ; \quad (\delta')^{-1} = -K_x^{-1,1}$$

Let $\phi \in F_{p,\mu}$. We can use (58) to prove by induction that, (for $n = 0,1,2\ldots$), provided $\text{Re}\ \mu - n > -\frac{1}{q}$,

$$I_x^{-n}\phi(x) = \frac{d^n\phi}{dx^n} \qquad (61)$$

Similarly, for $\phi \in F_{p,\mu}$ with $\text{Re}\ \mu < \frac{1}{p}$ and $n = 0,1,2\ldots$,

$$K_x^{-n}\phi(x) = (-1)^n \frac{d^n\phi}{dx^n} \qquad (62)$$

as might be expected.

There are similar results for $F'_{p,\mu}$

CHAPTER 4

Fractional Integration and Singular Differential Operators

§4.1 The singular differential operator L_ν

We consider the operator L_ν defined for suitable functions ϕ and any complex number ν by

$$L_\nu \phi(x) = \frac{d^2\phi}{dx^2} + \frac{2\nu+1}{x}\frac{d\phi}{dx} \qquad (1)$$

Such an operator arises naturally in many situations. For example, if $\nu = \frac{1}{2}n - 1$ and we replace x by r, L_ν becomes the Laplacian for spherically symmetric functions on R^n. Other references are given in the introduction.

It has long been known that there is a close connection between the operator L_ν and operators of fractional integration, particularly those of the form $I^{\eta,\alpha}_{x^2}$ and $K^{\eta,\alpha}_{x^2}$. The connection was explored in [8] for a certain space of testing-functions and extended to the corresponding space of generalised functions. In this chapter, we establish similar results for the spaces $F_{p,\mu}$ and $F'_{p,\mu}$.

We recall from Chapter 2 that the operator

$$D \equiv \frac{d}{dx}$$

is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-1}$ for every complex number μ, and $1 \leq p \leq \infty$. Also $\frac{1}{x}$ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$. We therefore have

Theorem 1

Let $1 \leq p \leq \infty$. For each complex μ and ν,

L_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-2}$.

It is clear that, for each fixed $\phi \in F_{p,\mu}$ and $0 < x < \infty$, $L_\nu \phi(x)$ is an entire function of ν with derivative

$$\frac{\partial}{\partial \nu} L_\nu \phi(x) = \frac{2}{x} \frac{d\phi}{dx}$$

In fact, since for any complex h,

$$\frac{1}{h} [L_{\nu+h}\phi - L_\nu \phi] - \frac{2}{x} \frac{d\phi}{dx}$$

vanishes identically for $0 < x < \infty$, we can immediately deduce

<u>Theorem 2</u>

Let $\phi \in F_{p,\mu}$. For each fixed x, $L_\nu \phi(x)$ is an entire function of ν and furthermore, the derivative $\frac{\partial}{\partial \nu} L_\nu \phi$ exists as a limit in the topology of $F_{p,\mu-2}$

Further mapping properties of L_ν are derived below. It is easy to prove that, for any suitable function ϕ,

$$x^2 L_\nu \phi(x) = \delta^2 \phi + 2\nu \delta\phi \qquad (2)$$

$$x L_\nu x \phi(x) = \delta'^2 \phi + 2\nu \delta' \phi \qquad (3)$$

where δ, δ' are defined as in Chapter 2.

We next define L_ν on $F'_{p,\mu}$. Let $\phi \in F_{p,\mu+2}$ and let f be a twice-differentiable function such that f and $L_\nu f$ generate regular functionals. Proceeding formally,

$$\begin{aligned}
(L_\nu f , \phi) &= (x^{-2} (\delta^2 f + 2\nu \delta f) , \phi) && \text{by (2)} \\
&= ((\delta^2 + 2\nu\delta) f , x^{-2}\phi) && \text{by (2.16)} \\
&= (f , (\delta'^2 - 2\nu\delta') x^{-2}\phi) && \text{by (2.17)} \\
&= (f , x L_{-\nu} x^{-1} \phi) && \text{by (3)}
\end{aligned}$$

$\phi \in F_{p,\mu+2} \Rightarrow x L_{-\nu} x^{-1} \phi \in F_{p,\mu}$ so that the right-hand side is meaningful if $f \in F'_{p,\mu}$. Thus, we define L_ν on $F'_{p,\mu}$ by

$$(L_\nu f , \phi) = (f , x L_{-\nu} x^{-1} \phi) \tag{4}$$

where $\phi \in F_{p,\mu+2}$. Since $x L_{-\nu} x^{-1}$ is a continuous linear mapping of $F_{p,\mu+2}$ into $F_{p,\mu}$, we have by Theorem 1.2 that L_ν is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+2}$. Now, by Theorem 2, $\frac{\partial}{\partial\nu} L_{-\nu} x^{-1} \phi$ exists as a limit in the topology of $F_{p,\mu+1}$. Hence, we have that $\frac{\partial}{\partial\nu} x L_{-\nu} x^{-1} \phi$ exists as a limit in the topology of $F_{p,\mu}$. We have therefore proved

Theorem 3

Let $1 \leq p \leq \infty$. For each complex μ and ν, L_ν is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+2}$. Further, for each fixed $\phi \in F_{p,\mu+2}$, $(L_\nu f , \phi)$ is an entire function of ν.

§4.2 Connections between L_ν and fractional integration

In this section, we establish some relations on $F_{p,\mu}$ involving L_ν and the homogeneous operators of fractional integration. We then obtain the corresponding results for $F'_{p,\mu}$.

Theorem 4

(i) Let $\phi \in F_{p,\mu}$, Re $(2\nu+\mu) > \frac{1}{p}$, $1 \leq p \leq \infty$. Then

$$I^{\nu,\alpha}_{x^2} L_\nu \phi = L_{\nu+\alpha} I^{\nu,\alpha}_{x^2} \phi \tag{5}$$

(ii) Let $\phi \in F_{p,\mu}$, Re $(2\nu-\mu) > -\frac{1}{p}$, $1 \leq p \leq \infty$. Then

$$L_{-\nu} K^{\nu,\alpha}_{x^2} \phi = K^{\nu,\alpha}_{x^2} L_{-\nu-\alpha} \phi \tag{6}$$

Proof : (i) Under the given restrictions on the parameters, both sides of (5) belong to $F_{p,\mu-2}$. Using (2) we have

$$I_{x^2}^{\nu,\alpha} \ L_\nu \phi = I_{x^2}^{\nu,\alpha} x^{-2} (\delta^2 + 2\nu \delta) \phi$$

$$= x^{-2} I_{x^2}^{\nu-1,\alpha} (\delta + 2\nu) \delta\phi$$

where we use the definition of $I_{x^2}^{\nu,\alpha}$ for Re $\alpha > 0$ and analytic continuation for Re $\alpha \leqslant 0$. Then, using (3.22),

$$I_{x^2}^{\nu,\alpha} \ L_\nu \phi$$

$$= x^{-2} [2 I_{x^2}^{\nu,\alpha-1} \delta\phi - (2\nu-2+2) I_{x^2}^{\nu-1,\alpha}\delta\phi + 2\nu I_{x^2}^{\nu-1,\alpha} \delta\phi]$$

$$= 2 x^{-2} I_{x^2}^{\nu,\alpha-1} \delta\phi$$

On the other hand, using (2) and (3.17)

$$L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} \phi = x^{-2} (\delta+2\nu+2\alpha)\delta I_{x^2}^{\nu,\alpha} \phi$$

$$= x^{-2} (\delta+2\nu+2\alpha) I_{x^2}^{\nu,\alpha} \delta\phi$$

$$= x^{-2} [2 I_{x^2}^{\nu,\alpha-1} \delta\phi - (2\nu+2\alpha-2+2) I_{x^2}^{\nu,\alpha}\delta\phi + (2\nu+2\alpha) I_{x^2}^{\nu,\alpha}\delta\phi]$$

$$= 2 x^{-2} I_{x^2}^{\nu,\alpha-1} \delta\phi$$

The result follows.

(ii) The proof of this part is similar and uses (3.31) and (3.35) The details are omitted.

Equations (5) and (6) give perhaps the neatest relations between the differential operators L_ν and fractional integration on $F_{p,\mu}$. We now prove the corresponding results on $F'_{p,\mu}$.

Theorem 5

Let $f \in F'_{p,\mu}$, $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

(i) If Re $(2\nu-\mu) > \frac{1}{q}$,

$$I^{\nu,\alpha}_{x^2} L_{\nu} f = L_{\nu+\alpha} I^{\nu,\alpha}_{x^2} f \tag{7}$$

(ii) If Re $(2\nu +\mu) > -\frac{1}{q}$,

$$L_{-\nu} K^{\nu,\alpha}_{x^2} f = K^{\nu,\alpha}_{x^2} L_{-\nu-\alpha} f \tag{8}$$

Proof : By Theorem 3 and Theorem 3.17, both sides of (7) belong to $F'_{p,\mu+2}$. Let $\phi \in F_{p,\mu+2}$.

$$(I^{\nu,\alpha}_{x^2} L_{\nu} f, \phi) = (L_{\nu} f , K^{\nu+\frac{1}{2},\alpha}_{x^2} \phi) \qquad \text{by } (3.40)$$

$$= (f , x L_{-\nu} x^{-1} K^{\nu+\frac{1}{2},\alpha}_{x^2} \phi) \qquad \text{by } (4)$$

$$= (f , x L_{-\nu} K^{\nu,\alpha}_{x^2} x^{-1} \phi)$$

using the definition of $K^{\nu,\alpha}_{x^2}$ for Re $\alpha > 0$, and analytic continuation otherwise . On the other hand ,

$$(L_{\nu+\alpha} I^{\nu,\alpha}_{x^2} f , \phi) = (f , K^{\nu+\frac{1}{2},\alpha}_{x^2} x L_{-\nu-\alpha} x^{-1} \phi)$$

$$= (f , x K^{\nu,\alpha}_{x^2} L_{-\nu-\alpha} x^{-1} \phi)$$

Thus, we have only to show that

$$x L_{-\nu} K^{\nu,\alpha}_{x^2} x^{-1} \phi = x K^{\nu,\alpha}_{x^2} L_{-\nu-\alpha} x^{-1} \phi$$

or

$$L_{-\nu} K^{\nu,\alpha}_{x^2} x^{-1} \phi = K^{\nu,\alpha}_{x^2} L_{-\nu-\alpha} x^{-1} \phi$$

But, since $x^{-1}\phi \in F_{p,\mu+1}$, the result follows from Theorem 4 (ii) with μ replaced by $\mu+1$ and ϕ by $x^{-1}\phi$.

The proof of (8) is similar. Once again, we see that the restrictions on the parameters in Theorem 5 are obtained from those in Theorem 4 by replacing μ by $-\mu$ and p by q.

§4.3 Further properties of L_ν

We recall from (3.57) that if $\mathrm{Re}\,\mu > \frac{1}{p}$, δ is an automorphism of $F_{p,\mu}$ and

$$\delta^{-1} = I_x^{-1,1} \qquad (9)$$

Now, from (2), we have

$$L_\nu = x^{-2}(\delta^2 + 2\nu\delta)$$

and in particular $L_0 = x^{-2}\delta^2$. It follows that, if $\mathrm{Re}\,\mu > \frac{1}{p}$, L_0 is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu-2}$ and

$$L_0^{-1} = I_x^{-1,1} I_x^{-1,1} x^2 \qquad (10)$$

If also $\mathrm{Re}(2\nu+\mu) > \frac{1}{p}$, we may put $\alpha = -\nu$ in (5) to get

$$I_{x^2}^{\nu,-\nu} L_\nu \phi = L_0 I_{x^2}^{\nu,-\nu} \phi$$

for $\phi \in F_{p,\mu}$. Since we are also assuming $\mathrm{Re}\,\mu > \frac{1}{p}$, we may, by Corollary 3.11, apply $(I_{x^2}^{\nu,-\nu})^{-1}$ to both sides to obtain

$$L_\nu \phi = I_{x^2}^{0,\nu} L_0 I_{x^2}^{\nu,-\nu} \phi$$

It follows that in this case L_ν is invertible on $F_{p,\mu}$ and

$$\begin{aligned} L_\nu^{-1}\psi &= I_{x^2}^{0,\nu} L_0^{-1} I_{x^2}^{\nu,-\nu} \psi \\ &= I_{x^2}^{0,\nu} I_x^{-1,1} I_x^{-1,1} x^2 I_{x^2}^{\nu,-\nu} \psi \\ &= x^2 I_{x^2}^{1,\nu} I_x^{1,1} I_x^{1,1} I_{x^2}^{\nu,-\nu} \psi \end{aligned} \qquad (11)$$

for $\psi \in F_{p,\mu-2}$.

If, on the other hand, Re $\mu < \frac{1}{p}$, we have from (3.59) that δ is again an automorphism of $F_{p,\mu}$ and

$$\delta^{-1} = - K_x^{0,1} \tag{12}$$

so again L_0 is invertible on $F_{p,\mu}$ and

$$L_0^{-1} = K_x^{0,1} K_x^{0,1} x^2$$

Putting $\nu = 0$ and replacing α by $-\alpha$ in (6), we obtain for $\phi \in F_{p,\mu}$

$$L_0 K_{x^2}^{0,-\alpha} \phi = K_{x^2}^{0,-\alpha} L_\alpha \phi$$

Restoring ν in place of α,

$$L_0 K_{x^2}^{0,-\nu} \phi = K_{x^2}^{0,-\nu} L_\nu \phi$$

If also, Re $(2\nu+\mu) < \frac{1}{p}$, we may, by Theorem 3.14, apply $(K_{x^2}^{0,-\nu})^{-1}$ to both sides to obtain

$$L_\nu \phi = K_{x^2}^{-\nu,\nu} L_0 K_{x^2}^{0,-\nu} \phi$$

Again, L_ν is invertible and for $\psi \in F_{p,\mu-2}$

$$L_\nu^{-1} \psi = K_{x^2}^{-\nu,\nu} K_x^{0,1} K_x^{0,1} K_{x^2}^{1,-\nu} x^2 \psi \tag{13}$$

We have therefore proved

Theorem 6

(i) If min (Re μ , Re ($2\nu+\mu$)) $> \frac{1}{p}$, L_ν is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu-2}$ and for $\psi \in F_{p,\mu-2}$, the equation

$$L_\nu \phi = \psi$$

has a unique solution ϕ in $F_{p,\mu}$ given by (11)

(ii) If max (Re μ , Re ($2\nu+\mu$)) $< \frac{1}{p}$, L_ν is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu-2}$ and for $\psi \in F_{p,\mu-2}$, the equation

$$L_\nu \phi = \psi$$

has a unique solution ϕ in $F_{p,\mu}$ given by (13) .

Similarly, taking adjoints , or proceeding via (4), (7) and (8), we can obtain the corresponding results for $F'_{p,\mu}$.

Theorem 7

(i) If $\min(-\mathrm{Re}\,\mu, \mathrm{Re}(2\nu-\mu)) > \frac{1}{q}$, L_ν is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu+2}$ and for $g \in F'_{p,\mu+2}$, the equation

$$L_\nu f = g$$

has a unique solution f in $F'_{p,\mu}$ given by

$$f = x^2 \, I^{1,\nu}_{x^2} \, I^{1,1}_{x} \, I^{1,1}_{x} \, I^{\nu,-\nu}_{x^2} \, g$$

(ii) If $\max(-\mathrm{Re}\,\mu, \mathrm{Re}(2\nu-\mu)) < \frac{1}{q}$, L_ν is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu+2}$ and for $g \in F'_{p,\mu+2}$, the equation

$$L_\nu f = g$$

has a unique solution f in $F'_{p,\mu}$ given by

$$f = K^{-\nu,\nu}_{x^2} \, K^{0,1}_{x} \, K^{0,1}_{x} \, K^{1,-\nu}_{x^2} \, x^2 \, g$$

CHAPTER 5

Fractional Integration and the Hankel Transform

§5.1 Introduction

In this chapter, we consider the connection between the homogeneous operators of fractional integration and the Hankel transform. In its usual form for L_p (as opposed to Tricomi's form which we shall consider later) the Hankel transform of order ν is defined on L_p by

$$(H_\nu \phi)(x) = \underset{n \to \infty}{\mathrm{l.i.m}\,(q)} \int_0^n \sqrt{xt}\, J_\nu(xt)\, \phi(t)\, dt \qquad (1)$$

for any complex number ν . Here, l.i.m (q) denotes the limit in the L_q norm , $\frac{1}{p} + \frac{1}{q} = 1$ as usual, and J_ν is the Bessel function of the first kind and order ν . We require the following result.

Theorem 1

Let $1 < p \leq 2$, $\mathrm{Re}\,\nu > -\frac{1}{2} - \frac{1}{q}$, $\phi \in L_p$. Then $H_\nu\phi$ exists almost everywhere on $(0,\infty)$ and H_ν is a continuous linear mapping of L_p into L_q.

Since $F_p \subset L_p$, it follows that H_ν maps F_p into L_q , under the hypotheses of Theorem 1. We will show that, in fact, H_ν is a continuous linear mapping of F_p into F_q.

If the function $\phi(x)$ vanishes for x sufficiently large, then

$$(H_\nu\phi)(x) = \int_0^\infty \sqrt{xt}\, J_\nu(xt)\, \phi(t)\, dt \qquad (2)$$

the integral actually being over a finite interval. (2) is easier to handle than (1) from the point of view of differentiation. Thus, to show that, for any $\phi \in F_p$, $H_\nu\phi$ is smooth, we do not use (1). Instead, we first approximate to ϕ by a sequence $\{\phi_n\}$ in F_p, each $\phi_n(x)$ vanishing for x sufficiently large so that we may use (2); we then use Theorem 1.3 to differentiate under the integral sign and finally use the continuity of H_ν on L_p. The details follow in Section 2.

There appears to be no easy way to deal with H_ν on the spaces $F_{p,\mu}$ with $\mu \neq 0$. Okikiolu [22] has proved some results for operators of the form

$$(\mathcal{J}_\nu\phi)(x) = \underset{n\to\infty}{\text{l.i.m}}\,(q) \int_0^n (xt)^{\frac{1}{2}-\nu} J_{\nu-\frac{1}{2}}(xt)\,\phi(t)\,dt$$

and we could use these to obtain some results for elements $\phi(x)$ in $F_{p,\mu}$ which vanish for x sufficiently large. But the limit in mean prevents us dealing with general elements of $F_{p,\mu}$ and we shall not pursue this. Thus, all our results in this chapter will be for the spaces F_p.

§5.2 The Hankel Transform on F_p

As indicated above, we begin by approximating to an arbitrary function $\phi \in F_p$. We assume throughout this section that $1 < p \leq 2$.

Lemma 2

For any $\phi \in F_p$, there exists a sequence $\{\phi_n\}$ of elements of F_p, each vanishing for x sufficiently large, such that ϕ_n converges to ϕ in the topology of F_p.

Proof : Let λ_1 be an arbitrary smooth function such that

$$\lambda_1(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & x \geq 2 \end{cases}$$

and for each positive integer n define λ_n by

$$\lambda_n(x) = \lambda_1\left(\frac{x}{n}\right)$$

Clearly λ_n is smooth for each n and

$$\lambda_n(x) = \begin{cases} 1 & 0 < x < n \\ 0 & x \geq 2n \end{cases}$$

Given $\phi \in F_p$, define ϕ_n by

$$\phi_n(x) = \lambda_n(x) \ \phi(x)$$

Clearly $\phi_n(x) = 0$ for $x \geq 2n$. Also, since $\lambda_n^{(k)}$,the k^{th} derivative of λ_n, is bounded on $(0,\infty)$ for each $k = 0,1,2\ldots$, it follows easily that for each k

$$x^k \frac{d^k}{dx^k} \phi_n(x) \in L_p$$

so that $\phi_n \in F_p$. We show that ϕ_n converges to ϕ in F_p as $n \to \infty$.

We must show that for each $k = 0,1,2\ldots$, $\gamma_k^p (\phi-\phi_n)$ converges to zero as $n \to \infty$, with γ_k^p given by (2.2).

$$\{ \gamma_k^p(\phi-\phi_n) \}^p = \int_0^\infty \left| x^k \frac{d^k}{dx^k} (\phi-\phi_n) \right|^p dx$$

$$= \int_n^{2n} \left| x^k \frac{d^k}{dx^k} (\phi-\phi_n) \right|^p dx + \int_{2n}^\infty \left| x^k \frac{d^k\phi}{dx^k} \right|^p dx$$

$\phi \in F_p \Rightarrow x^k \frac{d^k\phi}{dx^k} \in L_p$ so that the second integral on the right tends to zero as $n \to \infty$. We now consider the first integral.

$$x^k \frac{d^k}{dx^k} (\phi-\phi_n) = x^k \frac{d^k}{dx^k} \{ (1-\lambda_n(x)) \phi(x) \}$$

$$= \sum_{l=0}^{k} x^l \frac{d^l}{dx^l} (1 - \lambda_n(x)) x^{k-l} \frac{d^{k-l}\phi}{dx^{k-l}} \binom{k}{l}$$

$$= x^k \frac{d^k\phi}{dx^k} (1 - \lambda_n(x)) + \sum_{l=1}^{k} x^l \frac{d^l}{dx^l} (1-\lambda_n(x)) x^{k-l} \frac{d^{k-l}\phi}{dx^{k-l}} \binom{k}{l} \quad (3)$$

We are concerned with the value of the right-hand side for $n \leq x \leq 2n$.

$$|1-\lambda_n(x)| \leq 1 + | \lambda_n(x)| = 1 + | \lambda_1(\frac{x}{n}) |$$

$$\Rightarrow \sup_{n\leq x\leq 2n} |1-\lambda_n(x)| \leq 1 + \sup_{n\leq x\leq 2n} |\lambda_1(\frac{x}{n})| = 1 + \sup_{1\leq t\leq 2} |\lambda_1(t)|$$

$$= M_0 \quad \text{say}$$

where M_0 is a constant , independent of n . Also, for $l \geq 1$,

$$x^l \frac{d^l}{dx^l} (1-\lambda_n(x)) = - x^l \frac{d^l}{dx^l} \lambda_n(x) = - x^l \frac{d^l}{dx^l} \lambda_1(\frac{x}{n})$$

$$= - x^l \frac{1}{n^l} \lambda_1^{(l)} (\frac{x}{n})$$

$$\Rightarrow \sup_{n\leq x\leq 2n} | x^l \frac{d^l}{dx^l} (1 - \lambda_n(x)) | = \sup_{n\leq x\leq 2n} |(\frac{x}{n})^l \lambda_1^{(l)} (\frac{x}{n})|$$

$$= \sup_{1\leq t\leq 2} | t^l \lambda_1^{(l)}(t) | = M_l \quad \text{say}$$

where again M_l is a constant, independent of n . It follows from (3) that

$$| x^k \frac{d^k}{dx^k} (\phi-\phi_n) | \leq \sum_{l=0}^{k} M_l | x^{k-l} \frac{d^{k-l}\phi}{dx^{k-l}} |$$

Now the right-hand side belongs to L_p so that

$$\int_n^{2n} | x^k \frac{d^k}{dx^k} (\phi-\phi_n) |^p \, dx \to 0 \qquad \text{as } n \to \infty$$

Hence $\gamma_k^p (\phi-\phi_n) \to 0$ as $n \to \infty$ for each $k = 0,1,2.....$, i.e. ϕ_n converges to ϕ in F_p as required

We use our sequence $\{\phi_n\}$ again in the next lemma.

Lemma 3

If $\phi \in F_p$, ϕ_n is constructed as in Lemma 2, and $\operatorname{Re} \nu > -\frac{1}{2} - \frac{1}{q}$, then

$$\underset{n \to \infty}{\text{l.i.m}}\,(q) \int_0^n \sqrt{xt}\, J_\nu(xt)\, \phi(t)\, dt = \underset{n \to \infty}{\text{l.i.m}}\,(q) \int_0^\infty \sqrt{xt}\, J_\nu(xt) \phi_n(t)\, dt$$

Proof : Note first that the integral on the right side is over a finite interval, namely $(0, 2n)$, for each fixed n. Write

$$\psi_n(x) = \int_0^n \sqrt{xt}\; J_\nu(xt)\; \phi(t)\; dt$$

$$\chi_n(x) = \int_0^{2n} \sqrt{xt}\; J_\nu(xt) \phi_n(t)\; dt$$

We must show that

$$\underset{n \to \infty}{\text{l.i.m}}\,(q)\; \psi_n = \underset{n \to \infty}{\text{l.i.m}}\,(q)\; \chi_n$$

$$\psi_n - \chi_n = \int_n^{2n} \sqrt{xt}\; J_\nu(xt) \phi_n(t)\; dt$$

$$= \underset{N \to \infty}{\text{l.i.m}}\,(q) \int_0^N \sqrt{xt}\; J_\nu(xt) \omega_n(t)\; dt$$

$$= H_\nu(\omega_n)$$

where $\omega_n(t) = \phi_n(t)$ $(n \leq t \leq 2n)$ and $= 0$ otherwise.

Hence by Theorem 1,

$$|\psi_n - \chi_n|_q \leqslant K_1 \left[\int_n^{2n} |\phi_n(t)|^p\, dt \right]^{\frac{1}{p}} \tag{4}$$

where K_1 is a constant, independent of n and ϕ. Also, proceeding as in the proof of Lemma 2, we have

$$|\phi_n(t)| \leqslant |\phi(t)| \sup_{1 \leq t \leq 2} |\lambda_1(t)| = M\, |\phi(t)|$$

where M is independent of n. Hence, from (4) with $K_2 = K_1 M$, we have

$$| \psi_n - \chi_n |_q \leq K_2 \left[\int_n^{2n} | \phi(t) |^p \, dt \right]^{\frac{1}{p}} \to 0 \text{ as } n \to \infty .$$

since $\phi \in L_p$. This completes the proof .

Lemma 3 tells us that

$$H_\nu \phi = \underset{n \to \infty}{\text{l.i.m (q)}} \; H_\nu \phi_n \qquad (5)$$

So far we only know that for each n, $H_\nu \phi_n \in L_q$ by Theorem 1. That each $H_\nu \phi_n$ actually belongs to F_q will be a consequence of the next lemma.

Lemma 4

Let $\psi \in F_p$ and suppose $\psi(x) = 0$ for x sufficiently large. Also, let $\operatorname{Re} \nu > -\frac{1}{2} - \frac{1}{q}$ .Then $H_\nu \psi \in F_q$.

Proof : Let $\psi(x) = 0$ for $x \geq A$. We are able to use the form (2) for $H_\nu \psi$ so that

$$(H_\nu \psi) (x) = \int_0^A \sqrt{xt} \, J_\nu(xt) \, \psi(t) \, dt$$

$$= \int_0^{Ax} \sqrt{u} \, J_\nu(u) \, \psi\left(\frac{u}{x} \right) \frac{du}{x}$$

For fixed $x > 0$, choose B : $0 < B < Ax$. Let

$$I_1(x) = \int_0^B \sqrt{u} \, J_\nu(u) \, \psi\left(\frac{u}{x} \right) \frac{du}{x}$$

$$I_2(x) = \int_B^{Ax} \sqrt{u} \, J_\nu(u) \, \psi\left(\frac{u}{x} \right) \frac{du}{x}$$

By Lemma 2.4, for some constant M,

$$\left| \sqrt{u} \, J_\nu(u) \quad \psi\left(\frac{u}{x} \right) \frac{1}{x} \right| \leq M \, u^{\frac{1}{2} + \operatorname{Re}\nu - \frac{1}{p}} \, x^{-1 + \frac{1}{p}}$$

Since $\frac{1}{2} + \operatorname{Re}\nu - \frac{1}{p} > -1$ by hypothesis, we can use Theorem 1.3 to deduce that I_1 is differentiable and

$$\delta\, I_1(x) = -\int_0^B \sqrt{u}\, J_\nu(u)\,(\delta+1)\,\psi\left(\frac{u}{x}\right)\frac{du}{x}$$

((δ+1)ψ)(u/x)?

As regards I_2, the integrand is continuous and hence using a standard theorem (e.g. Widder [24] p.353) I_2 is differentiable and since $(\delta+1)\,\psi(A) = 0$, we obtain

$$\delta\, I_2(x) = -\int_B^{Ax} \sqrt{u}\, J_\nu(u)\,(\delta+1)\,\psi\left(\frac{u}{x}\right)\frac{du}{x}$$

same

Consequently, $H_\nu\psi$ is differentiable and

$$\delta H_\nu\psi = -H_\nu(\delta+1)\psi$$

Since $\psi \in F_p \Rightarrow (\delta+1)\psi \in F_p$ by Theorem 2.9, we may proceed by induction to prove that $H_\nu\psi$ is smooth and

$$\delta^k H_\nu\psi = (-1)^k H_\nu(\delta+1)^k\psi \quad \in L_q \tag{6}$$

by Theorem 1. Since $x^k \frac{d^k}{dx^k} H_\nu\psi$ is a linear combination of $H_\nu\psi$, $\delta H_\nu\psi$, ..., $\delta^k H_\nu\psi$, it follows that $H_\nu\psi \in F_q$ as required.

To reach our goal using (5), we need one more lemma.

<u>Lemma 5</u>

With ϕ, ϕ_n defined as in Lemma 2, the function $\underset{n\to\infty}{\text{l.i.m}}\,(q)\ H_\nu\phi_n$ belongs to F_q provided $\mathrm{Re}\,\nu > -\frac{1}{2} - \frac{1}{q}$.

<u>Proof :</u> Let us write $\psi_n = H_\nu\phi_n$. Then, by Theorem 1,

$$|\psi_n - \psi_m|_q \leq K_0\,|\phi_n - \phi_m|_p$$

for some constant K_0 independent of m, n and ϕ. Similarly, using (6), for $k = 1,2,3,\ldots$, $\exists$ constants K_k such that

$$|\delta^k(\psi_n - \psi_m)|_q \leq K_k\,|(\delta+1)^k(\phi_n - \phi_m)|_p$$

It now follows that

$$| \, x^k \frac{d^k}{dx^k} (\psi_n - \psi_m) \, |_q \leq \sum_{l=0}^{k} C_l \, | \, x^l \frac{d^l}{dx^l} (\phi_n - \phi_m) \, |_p$$

for some constants C_l ; i.e.

$$\gamma_k^q (\psi_n - \psi_m) \leq \sum_{l=0}^{k} C_l \, \gamma_l^p (\phi_n - \phi_m) \qquad (7)$$

Now since $\{ \phi_n \}$ converges to ϕ in F_p by Lemma 2 , $\{ \phi_n \}$ is a fundamental sequence in F_p . (7) now implies that $\{ \psi_n \}$ is a fundamental sequence in F_q . By completeness, $\exists \, \psi \in F_q$ such that ψ_n converges to ψ in F_q . But, since convergence in F_q implies convergence in L_q, it follows that

$$\psi = \underset{n \to \infty}{\text{l.i.m}} (q) \, \psi_n = \underset{n \to \infty}{\text{l.i.m}} (q) \, H_\nu \phi_n \in F_q$$

The lemma is proved .

From (6) we now have

Corollary 6

$$\phi \in F_p \Rightarrow H_\nu \phi \in F_q \quad \text{provided} \quad \text{Re}\,\nu > -\frac{1}{2} - \frac{1}{q} .$$

That H_ν is a linear mapping is obvious. We now prove

Lemma 7

If $\text{Re}\,\nu > -\frac{1}{2} - \frac{1}{q}$, H_ν is a continuous mapping of F_p into F_q .

Proof : We have with the previous notation

$$\delta^k H_\nu \phi_n = (-1)^k H_\nu \, (\delta + 1)^k \phi_n$$

Passing to the limit and using the fact that δ is continuous on F_p , F_q and H_ν is continuous on L_p,

$$\delta^k H_\nu \phi = (-1)^k H_\nu \, (\delta + 1)^k \phi$$

$$\Rightarrow \quad x^k \frac{d^k}{dx^k} H_\nu \phi = \sum_{l=0}^{k} C_l H_\nu \left(x^l \frac{d^l \phi}{dx^l} \right)$$

for some constants C_l . Hence, by Theorem 1, for some constants D_l ,

$$\gamma_k^q (H_\nu \phi) \leq \sum_{l=0}^{k} D_l \gamma_l^p (\phi)$$

The result follows .

We summarise our results in a theorem .

Theorem 8

If $1 < p \leq 2$, $\operatorname{Re} \nu > -\frac{1}{2} - \frac{1}{q}$, H_ν is a continuous linear mapping of F_p into F_q .

Although we have established Theorem 8 for $1 < p \leq 2$, we shall , in fact , be primarily concerned with the case $p = 2$. For $1 < p < 2$, a characterisation of the range of H_ν in L_q is not known so that the question of inversion cannot be dealt with. However, when $p = 2$, much more is known . Indeed, if $\operatorname{Re} \nu > -1$, H_ν maps L_2 into L_2 and is both one-to-one and onto with inverse $H_\nu^{-1} = H_\nu$. Combining this with Theorem 8 , we immediately obtain

Theorem 9

If $\operatorname{Re} \nu > -1$, H_ν is an automorphism of F_2 and

$$H_\nu^{-1} = H_\nu$$

§5.3 Asymptotic expansion of $\frac{\partial^2 J_\nu}{\partial \nu^2}$

In Section 4 we will consider further the action of H_ν on F_2. However, in order to perform an analytic continuation, we must derive asymptotic expansions for the derivatives of the Bessel function J_ν with respect to ν. This section is devoted to these derivations.

We shall follow closely the methods in [10], Chapter 7. That is to say, we first obtain asymptotic expansions for derivatives with respect to ν of K_ν, the modified Bessel function of the third kind and order ν, and then proceed via the Hankel functions.

From [10], p.23, we have that, if $\operatorname{Re}\nu > -\frac{1}{2}$,

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left[\sum_{m=0}^{M-1} \frac{\Gamma(\nu+\frac{1}{2}+m)}{m!\,\Gamma(\nu+\frac{1}{2}-m)} (2x)^{-m} + R_M\right] \quad (8)$$

where

$$R_M(x) = \frac{(2x)^{-M}}{(M-1)!\,\Gamma(\nu+\frac{1}{2}-M)} \int_0^\infty e^{-t} t^{\nu-\frac{1}{2}+M} dt \int_0^1 (1-u)^{M-1} \left(1+\frac{ut}{2x}\right)^{\nu-\frac{1}{2}-M} du \quad (9)$$

Let ν_0 be fixed, $\operatorname{Re}\nu_0 > -\frac{1}{2}$ and let $0 < \epsilon < \operatorname{Re}\nu_0+\frac{1}{2}$. We can choose M such that $\operatorname{Re}\nu - \frac{1}{2} - M < 0$ whenever $|\nu-\nu_0| \leqslant \epsilon$. Then

$$\left|\left(1+\frac{ut}{2x}\right)^{\nu-\frac{1}{2}-M}\right| = \left(1+\frac{ut}{2x}\right)^{\operatorname{Re}\nu-\frac{1}{2}-M} \leqslant 1$$

$$\Rightarrow \quad |R_M(x)| \leqslant \frac{(2x)^{-M}}{(M-1)!\,|\Gamma(\nu+\frac{1}{2}-M)|} \int_0^\infty e^{-t} t^{\operatorname{Re}\nu-\frac{1}{2}+M} dt \int_0^1 (1-u)^{M-1}\, du$$

$$= \frac{(2x)^{-M}}{(M-1)!\,|\Gamma(\nu+\frac{1}{2}-M)|}\, \Gamma(\operatorname{Re}\nu+\tfrac{1}{2}+M)$$

$$\leqslant C(M,\epsilon)\, x^{-M}$$

where $$C(M,\epsilon) = 2^{-M} \sup_{|\nu-\nu_0|\leqslant\epsilon} \left|\frac{\Gamma(\operatorname{Re}\nu+\frac{1}{2}+M)}{M!\,\Gamma(\operatorname{Re}\nu+\frac{1}{2}-M)}\right|$$

It now follows easily from (8) that for any $M \geqslant 1$, $\exists\, C(M,\epsilon)$ independent of x such that for $|\nu-\nu_0| \leqslant \epsilon$, $x \geqslant 1$,

$$| R_M(x) | \leqslant C(M, \epsilon)\ x^{-M}$$

By differentiating (9) and proceeding as above, we can deduce similarly that, for each $k = 0,1,2....$, $\exists$ a constant $C(M,\epsilon,k)$ such that

$$\left| \frac{\partial^k R_M}{\partial \nu^k} (x) \right| \leqslant C(M,\epsilon,k)\ x^{-M}$$

where again $C(M,\epsilon,k)$ is independent of ν, x for $| \nu-\nu_0 | \leqslant \epsilon$ and $x \geqslant 1$. Using Hankel's symbol

$$(\nu, m) = \frac{2^{-2m}}{m!} \{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\dots(4\nu^2 - (2m-1)^2)\}$$

$$= \frac{\Gamma(\nu + \frac{1}{2} + m)}{m!\ \Gamma(\nu + \frac{1}{2} - m)}$$

we obtain

$$\frac{\partial^k K_\nu}{\partial \nu^k} (x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} e^{-x} \left[\sum_{m=0}^{M-1} \frac{\partial^k}{\partial \nu^k} (\nu, m)(2x)^{-m} + R_{M,k} \right] \quad (10)$$

where $| R_{M,k}(x) | \leqslant C(M,\epsilon,k)\ x^{-M}$ (11)

uniformly for $| \nu-\nu_0 | \leqslant \epsilon$ and $x \geqslant 1$.

Suppose now that $\mathrm{Re}\ \nu_0 < \frac{1}{2}$. Then $\nu_0 = -\mu_0$ with $\mathrm{Re}\ \mu_0 > -\frac{1}{2}$. Also, $K_\nu(x) = K_{-\nu}(x)$ so that

$$\frac{\partial^k}{\partial \nu^k} K_\nu(x) = (-1)^k \frac{\partial^k}{\partial \mu^k} K_\mu(x) \qquad (\mu = -\nu)$$

By applying the previous case with ν, ν_0 replaced by μ, μ_0 we can deduce that (10) and (11) hold for <u>any</u> ν_0 and sufficiently small ϵ.

For our purposes, it will be sufficient to take $M = 1$ and we shall write R for remainder terms such that , for ϵ sufficiently small

$$| R | \leqslant C(\epsilon,k)\ x^{-1} \quad \text{for } | \nu-\nu_0 | \leqslant \epsilon \text{ and } x \geqslant 1.$$

We now proceed via Hankel functions. The Hankel function $H_\nu^{(1)}$

of the first kind and order ν satisfies

$$H_\nu^{(1)}(x) = -\frac{2i}{\pi} e^{-\frac{1}{2}i\nu\pi} K_\nu(-ix) \qquad (12)$$

$$\Rightarrow \quad \frac{\partial^k}{\partial\nu^k} H_\nu^{(1)}(x) = -\frac{2i}{\pi} e^{-\frac{1}{2}i\nu\pi} \sum_{l=0}^{k} \binom{k}{l} (-\tfrac{1}{2}i\pi)^{k-l} \frac{\partial^l K_\nu}{\partial\nu^l}(-ix)$$

$$= -\frac{2i}{\pi} e^{-\frac{1}{2}i\nu\pi} \left(\frac{\pi}{-2ix}\right)^{\frac{1}{2}} e^{ix} \sum_{l=0}^{k} \binom{k}{l} (-\tfrac{1}{2}i\pi)^{k-l} \left[\frac{\partial^l}{\partial\nu^l}(\nu,0) + R \right]$$

Since $(\nu,0) = 1$, we have

$$\frac{\partial^k}{\partial\nu^k} H_\nu^{(1)}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\frac{1}{2}\nu\pi-\frac{\pi}{4})} \left[(-\tfrac{1}{2}i\pi)^k + R \right]$$

In particular, for $k = 2$, we have

$$\frac{\partial^2}{\partial\nu^2} H_\nu^{(1)}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{i(x-\frac{1}{2}\nu\pi-\frac{\pi}{4})} \left[-\frac{\pi^2}{4} + R \right] \qquad (13)$$

The second Hankel function $H_\nu^{(2)}(x)$ of order ν satisfies

$$H_\nu^{(2)}(x) = \frac{2i}{\pi} e^{\frac{1}{2}i\nu\pi} K_\nu(ix) \qquad (14)$$

Hence replacing i by $-i$ throughout we obtain

$$\frac{\partial^2}{\partial\nu^2} H_\nu^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i(x-\frac{1}{2}\nu\pi-\frac{\pi}{4})} \left[-\frac{\pi^2}{4} + R \right] \qquad (15)$$

Finally, since $J_\nu(x) = \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x) \right]$, adding (13) and (15) gives

$$\frac{\partial^2}{\partial\nu^2} J_\nu(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\left(x-\tfrac{1}{2}\nu\pi-\frac{\pi}{4}\right) \left[-\frac{\pi^2}{4} + R \right]$$

$$\Rightarrow \quad \sqrt{x}\, \frac{\partial^2}{\partial\nu^2} J_\nu(x) = -\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \cos\left(x-\tfrac{1}{2}\nu\pi-\frac{\pi}{4}\right) + R$$

We have thus proved

<u>Lemma 10</u>

For any ν_0 and $\epsilon > 0$ sufficiently small,

$$\sqrt{x}\, \frac{\partial^2}{\partial\nu^2} J_\nu(x) = -\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \cos\left(x-\tfrac{1}{2}\nu\pi-\frac{\pi}{4}\right) + R_\nu$$

where $|R_\nu(x)| \leq Cx^{-1}$, uniformly in $|\nu-\nu_0| \leq \epsilon$, $x \geq 1$.

Here C depends only on ϵ . It follows that for any compact subset K of the complex plane, $\exists$ C (depending on K) such that

$$| R_{\nu} (x) | \leq C x^{-1}$$

uniformly for $\nu \in K$ and $x \geq 1$

§5.4 Analyticity of the Hankel Transform on F_p .

We are now ready to discuss the analyticity of H_ν on F_2 . Throughout this section, ' l.i.m ' will denote the limit in the L_2 norm. By definition,

$$H_\nu \phi (x) = \underset{n \to \infty}{\text{l.i.m}} \int_0^n \sqrt{xt}\, J_\nu (xt)\, \phi (t)\, dt$$

Assuming x and ϕ fixed, we may regard $H_\nu \phi(x)$ as a function of ν . Recalling that J_ν is an entire function of ν , we may differentiate formally to obtain

$$\frac{\partial}{\partial \nu} H_\nu \phi (x) = \underset{n \to \infty}{\text{l.i.m}} \int_0^n \sqrt{xt}\, \frac{\partial J_\nu}{\partial \nu} (xt)\, \phi (t)\, dt$$

We will show that, for fixed ϕ , $\frac{\partial}{\partial \nu} H_\nu \phi$ exists as a limit in the topology of F_2 whenever $\text{Re}\,\nu > -1$.

We first consider the operator T_ν defined by

$$T_\nu \phi(x) = \underset{n \to \infty}{\text{l.i.m}} \int_0^n \sqrt{xt}\, \frac{\partial^2 J_\nu}{\partial \nu^2} (xt)\, \phi(t)\, dt \qquad (16)$$

so that $T_\nu \phi(x)$ is obtained by differentiating $H_\nu \phi(x)$ formally twice with respect to ν .

Lemma 11

Let ν_0 be fixed with $\text{Re}\, \nu_0 > -1$. For ϵ sufficiently small, there exists a constant K , independent of ν such that

$$| T_\nu \phi |_2 \leq K | \phi |_2 \qquad (\phi \in F_2)$$

whenever $|\nu-\nu_0| \leq \epsilon$

<u>Proof :</u> We follow closely the method of Bochner [1] pp.227-8.

As in Lemma 10, we may write

$$\sqrt{y} \ \frac{\partial^2 J}{\partial \nu^2}\nu (y) = -\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \cos\left(y-\tfrac{1}{2}\nu\pi - \frac{\pi}{4}\right) + R_\nu(y) \qquad (17)$$

We also write

$$\psi_{1,\nu}(x) = -\underset{n\to\infty}{\text{l.i.m}} \left(\frac{\pi}{2}\right)^{\frac{3}{2}} \int_0^n \cos(xt-\tfrac{1}{2}\nu\pi-\tfrac{\pi}{4})\ \phi(t)\ dt$$

$$\psi_{2,\nu}(x) = \int_0^{\frac{1}{x}} R_\nu(xt)\phi(t)\ dt$$

$$\psi_{3,\nu}(x) = \underset{n\to\infty}{\text{l.i.m}} \int_{\frac{1}{x}}^n R_\nu(xt)\ \phi(t)\ dt$$

Let us consider first $\psi_{2,\nu}(x)$. We have

$$|\psi_{2,\nu}(x)| \leq \int_0^{\frac{1}{x}} |R_\nu(xt)|\ |\phi(t)|\ dt$$

$$= \int_0^1 |R_\nu(y)|\ |\phi(\tfrac{y}{x})|\ \frac{dy}{x}$$

If Re $\nu_0 > -\frac{1}{2}$, we choose ϵ : Re $\nu_0 - 2\epsilon > -\frac{1}{2}$. Now, there exists a constant M_1 such that

$$\left| \sqrt{y} \ \frac{\partial^2 J}{\partial \nu^2}\nu (y) \right| \leq M_1\, y^{\frac{1}{2} + \mathrm{Re}\ \nu - \epsilon} \qquad (18)$$

uniformly for $0 < y \leq 1$ and $|\nu-\nu_0| \leq \epsilon$. But if $|\nu-\nu_0| \leq \epsilon$ Re $(\frac{1}{2}+\nu-\epsilon) \geq$ Re $(\frac{1}{2}+\nu_0-2\epsilon) > 0$ so that

$$\left| \sqrt{y} \ \frac{\partial^2 J}{\partial \nu^2}\nu (y) \right| \leq M_1$$

uniformly for $0 < y \leq 1$ and $|\nu-\nu_0| \leq \epsilon$. Further, since

$\cos (y - \frac{1}{2}\nu\pi - \frac{\pi}{4})$ is uniformly bounded on this set , $\exists\ M_2$ such that

$$| R_\nu (y)| \leqslant M_2$$

uniformly for $0 < y \leq 1$ and $| \nu-\nu_0 | \leqslant \epsilon$. Hence, in this case ,

$$| \psi_{2,\nu}(x) | \leqslant \frac{M}{x}2 \int_0^1 | \phi (\frac{y}{x})| \, dy = \frac{M}{x}2 (I_x^{0,1}|\phi|)(\frac{1}{x})$$

Now $\phi(x) \in F_2 \Rightarrow (I_x^{0,1}|\phi|)(x) \in F_2$ by Theorem 3.3 .

$$\Rightarrow | \psi_{2,\nu} (x)|_2 \leqslant M_2 | \frac{1}{x} (I_x^{0,1} |\phi|)(\frac{1}{x})|_2$$

$$= M_2 | (I_x^{0,1} |\phi|)(x) |_2$$

$$\leqslant K_1 | \phi |_2$$

by Theorem 3.3 , where K_1 is independent of ϕ and of ν in $| \nu-\nu_0 | \leqslant \epsilon$.

Suppose , on the other hand , that $-1 < \text{Re } \nu_0 < -\frac{1}{2}$. We now choose ϵ : $-1 < \text{Re } \nu_0 - 2\epsilon$. As in (18) , $\exists\ M_3$ such that

$$| \sqrt{y} \frac{\partial^2 J}{\partial \nu^2}\nu (y) | \leqslant M_3 y^{\frac{1}{2} + \text{Re } \nu - \epsilon}$$

uniformly for $0 < y \leq 1$ and $| \nu-\nu_0 | \leqslant \epsilon$. Thus , if $| \nu-\nu_0 | \leqslant \epsilon$

$$| \sqrt{y} \frac{\partial^2 J}{\partial \nu^2}\nu (y) | \leqslant M_3 y^{\frac{1}{2} + \text{Re } \nu_0 - 2\epsilon}$$

$$\Rightarrow | \psi_{2,\nu} (x) |_2 \leqslant M_3 \int_0^1 y^{\frac{1}{2}+ \text{Re } \nu_0-2\epsilon} |\phi (\frac{y}{x}) | \frac{dy}{x}$$

$$= M_3 \frac{1}{x} (I_x^{\frac{1}{2}+ \text{Re } \nu_0-2\epsilon ,1}|\phi|)(\frac{1}{x})$$

Since $\frac{1}{2} + \text{Re } \nu_0 - 2\epsilon > \frac{1}{2}$, we can use Theorem 3.3 as before to deduce that $\exists\ K_2$ such that $-\frac{1}{2}$?

$$| \psi_{2,\nu} |_2 \leqslant K_2 | \phi |_2$$

where K_2 is independent of ϕ and of ν in $| \nu-\nu_0 | \leqslant \epsilon$.

So in either case , there is a constant C_2 , independent of ϕ in F_2 , such that

$$| \ \psi_{2,\nu} \ |_2 \leq C_2 \ | \phi |_2 \tag{19}$$

Re $\nu_0 = -\frac{1}{2}$?

uniformly in $| \ \nu-\nu_0 \ | \leq \epsilon$, for ϵ small enough .

Suppose ϵ fixed in accordance with the above. We now consider $\psi_{1,\nu}$. Since

$$\cos \left(xt-\tfrac{1}{2}\nu\pi-\frac{\pi}{4} \right) = \cos xt \cos \left(\tfrac{1}{2}\nu\pi + \frac{\pi}{4}\right) + \sin xt \sin \left(\tfrac{1}{2}\nu\pi + \frac{\pi}{4}\right),$$

$$\psi_{1,\nu}(x) = - \left(\frac{\pi}{2} \right)^{\frac{3}{2}} \cos \left(\tfrac{1}{2}\nu\pi + \frac{\pi}{4}\right) \underset{n \to \infty}{\text{l.i.m}} \int_0^n \cos xt \ \phi(t) \ dt$$

$$- \left(\frac{\pi}{2} \right)^{\frac{3}{2}} \sin \left(\tfrac{1}{2}\nu\pi + \frac{\pi}{4}\right) \underset{n \to \infty}{\text{l.i.m}} \int_0^n \sin xt \ \phi(t) \ dt$$

Now $\cos \left(\tfrac{1}{2}\nu\pi + \frac{\pi}{4}\right)$ and $\sin \left(\tfrac{1}{2}\nu\pi + \frac{\pi}{4}\right)$ are bounded on $| \ \nu-\nu_0 \ | \leq \epsilon$. Hence Theorem 1 applied to the Fourier sine and cosine transforms shows that $\exists \ C_1$, independent of ϕ in F_2 such that

$$| \ \psi_{1,\nu} \ |_2 \leq \quad | \phi |_2 \tag{20}$$

uniformly in $| \ \nu-\nu_0 \ | \leq \epsilon$.

Finally , for $\psi_{3,\nu}$, we note that by Lemma 10 $\exists \ M_4$ such that

$$| \ R_\nu (y) | \leq \frac{M_4}{y}$$

uniformly in $| \ \nu-\nu_0 \ | \leq \epsilon$. $\psi_{3,\nu}(x)$ exists as an improper integral and

$$| \ \psi_{3,\nu}(x) \ | \leq \int_{\frac{1}{x}}^{\infty} | \ R_\nu (xt) | \ | \phi (t) | \ dt$$

$$= \int_1^{\infty} | \ R_\nu (y) | \ | \ \phi \left(\frac{y}{x}\right) | \ \frac{dy}{x}$$

$$\leq \frac{M_4}{x} \int_1^{\infty} \frac{1}{y} \ | \ \phi \left(\frac{y}{x}\right) | \ dy$$

$$= \frac{M_4}{x} \left(K_x^{0,1} \ |\phi| \right) \left(\frac{1}{x} \right)$$

We may now apply Theorem 3.13 to deduce that

$$| \psi_{3,\nu} |_2 = M_4 \, | K_x^{0,1} |\phi| \, |_2 \leq C_3 | \phi |_2 \tag{21}$$

where C_3 is independent of ϕ and of ν in $| \nu-\nu_0 | \leq \epsilon$.

Finally, since

$$T_\nu \phi (x) = \psi_{1,\nu}(x) + \psi_{2,\nu}(x) + \psi_{3,\nu}(x)$$

the result follows from (19), (20) and (21).

We now prove

Theorem 12

Let $\operatorname{Re} \nu > -1$, $\phi \in F_2$. Then

$$\frac{H_{\nu+h} \phi - H_\nu \phi}{h} - \frac{\partial}{\partial \nu} H_\nu \phi$$

converges to zero in the topology of F_2 as $h \to 0$ in any manner. In particular, for each fixed x, $H_\nu \phi(x)$ is an analytic function of ν.

Proof : $\frac{1}{h} [H_{\nu+h} \phi - H_\nu \phi] - \frac{\partial}{\partial \nu} F_\nu \phi$ $\frac{\partial}{\partial \nu} H_\nu$?

$$= \underset{n \to \infty}{\mathrm{l.i.m}} \int_0^n \sqrt{xt} \, \{ \frac{1}{h} [J_{\nu+h}(xt) - J_\nu(xt)] - \frac{\partial}{\partial \nu} J_\nu(xt) \} \, \phi(t) \, dt$$

For $|h|$ sufficiently small, we may apply a local mean value theorem for complex variables [23] to deduce that

$$\frac{1}{h} [H_{\nu+h} \phi - H_\nu \phi] - \frac{\partial}{\partial \nu} H_\nu \phi = h \, T_{\nu'} \phi$$

where $\nu' = \nu + \theta h$, $|\theta| < 1$, and T_ν is given by (16). The theorem will be proved if we show that, for $k = 0,1,2....$,

$$\gamma_k^2 (h \, T_{\nu'} \phi) \to 0 \quad \text{as} \quad h \to 0 \text{ in any manner .}$$

But

$$\gamma_k^2 (h \, T_{\nu'} \phi) = |h| \, \gamma_k^2 (T_{\nu'} \phi) = |h| \, | x^k \frac{d^k}{dx^k} T_{\nu'} \phi |_2$$

By arguments analogous to those for H_ν , we can justify differentiating $T_{\nu'}\phi$ under the integral sign with respect to x to deduce that , for some constants c_0, c_1 , ... , c_k ,

$$x^k \frac{d^k}{dx^k} (T_{\nu'}\phi) = T_{\nu'} (\sum_{l=0}^{k} c_l \delta^l \phi)$$

Then applying Lemma 11 , we obtain

$$| x^k \frac{d^k}{dx^k} (T_{\nu'}\phi) |_2 \leq \sum_{l=0}^{k} d_l \gamma_l^2(\phi)$$

where the constants d_l are independent of ν in $| \nu-\nu' | \leq \epsilon$, for ϵ sufficiently small. Hence, if $|h|$ is sufficiently small,

$$\gamma_k^2 (h T_{\nu'} \phi) \leq |h| \sum_{l=0}^{k} d_l \gamma_l^2(\phi) \to 0 \text{ as } h \to 0$$

and the result follows .

We shall require Theorem 12 in the next section .

§5.5 Connections between H_ν and fractional integration in F_2

In [13], Kober established connections between the Hankel transform in Tricomi's form and the operators $I_x^{\eta,\alpha}$ and $K_x^{\eta,\alpha}$. In this section we translate these into corresponding results for H_ν and the operators $I_{x^2}^{\eta,\alpha}$ and $K_{x^2}^{\eta,\alpha}$ applied to functions in L_2 .

Let $\phi \in L_2$ and Re $\nu > -1$. $\mathfrak{h}_\nu$, the Hankel transform of order ν in Tricomi's form is defined by

$$\mathfrak{h}_\nu \phi (x) = \underset{n \to \infty}{\text{l.i.m}} \int_0^n J_\nu (2\sqrt{xt}) \phi(t) \, dt \qquad (22)$$

By standard results $\mathfrak{h}_\nu$ is a continuous linear mapping of L_2 into L_2 when Re $\nu > -1$. Kober proved the following theorem .

Theorem 13

Let $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > -1$, $\phi \in L_2$. Then

(i) $$I_x^{\frac{1}{2}\nu,\alpha} \mathfrak{H}_\nu \phi = \mathfrak{H}_{\nu+2\alpha} I_x^{\frac{1}{2}\nu,\alpha} \phi$$

(ii) $$\mathfrak{H}_\nu K_x^{\frac{1}{2}\nu,\alpha} \phi = K_x^{\frac{1}{2}\nu,\alpha} \mathfrak{H}_{\nu+2\alpha} \phi$$

We deduce

Theorem 14

Let $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > -1$, $\phi \in L_2$. Then

(i) $$I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha} H_\nu \phi = H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha} \phi \qquad (23)$$

(ii) $$H_\nu K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4},\alpha} \phi = K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4},\alpha} H_{\nu+2\alpha} \phi \qquad (24)$$

Proof : (i) Let $\psi \in L_2$. Then $\sqrt{t}\,\psi(\frac{1}{2}t^2) \in L_2$ also and by a change of variable it is easy to show that

$$\left(I_x^{\frac{1}{2}\nu,\alpha} \mathfrak{H}_\nu \psi\right)(\tfrac{1}{2}x^2) = x^{-\frac{1}{2}} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha} H_\nu\left(\sqrt{x}\,\psi(\tfrac{1}{2}x^2)\right)$$

$$\left(\mathfrak{H}_{\nu+2\alpha} I_x^{\frac{1}{2}\nu,\alpha} \psi\right)(\tfrac{1}{2}x^2) = x^{-\frac{1}{2}} H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha}\left(\sqrt{x}\,\psi(\tfrac{1}{2}x^2)\right)$$

Hence, by Theorem 13(i) ,

$$I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha} H_\nu\left(\sqrt{x}\,\psi(\tfrac{1}{2}x^2)\right) = H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha}\left(\sqrt{x}\,\psi(\tfrac{1}{2}x^2)\right) \qquad (25)$$

Now, given $\phi \in L_2$, let $\psi(x) = (2x)^{-\frac{1}{4}} \phi(\sqrt{2x})$. Then $\psi \in L_2$ and $\sqrt{x}\,\psi(\frac{1}{2}x^2) = \phi(x)$; so, by (25) ,

$$I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha} H_\nu \phi(x) = H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4},\alpha} \phi(x)$$

as required . (24) follows similarly from Theorem 13(ii)

Theorem 14 will hold in particular when $\phi \in F_2$. However, for fixed ϕ and x , provided $\operatorname{Re}\nu > -1$, and also $\operatorname{Re}(\nu+2\alpha) > -1$, both sides of (23) are analytic functions of ν

by Theorem 12 and results in Chapter 3 . Similarly for (24). Hence we may remove the restriction Re $\alpha > 0$ and substitute Re $(\nu+2\alpha) > -1$ to obtain the following theorem.

<u>Theorem 15</u>

Let $\phi \in F_2$, Re $\nu > -1$, Re $(\nu+2\alpha) > -1$. Then

(i) $$I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, \alpha} H_\nu \phi = H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, \alpha} \phi$$

(ii) $$H_\nu K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, \alpha} \phi = K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, \alpha} H_{\nu+2\alpha} \phi$$

We can use these results to express H_ν in terms of the Fourier sine and cosine transforms $H_{\frac{1}{2}}$ and $H_{-\frac{1}{2}}$. Suppose that the hypotheses of Theorem 15 are satisfied . We may then invert $I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, \alpha}$ by Corollary 3.11 to get

$$H_\nu \phi = I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4} + \alpha, -\alpha} H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, \alpha} \phi$$

If $\nu+2\alpha = \frac{1}{2}$ and Re $\nu > -1$,

$$H_\nu \phi = I_{x^2}^{0, \frac{1}{2}\nu - \frac{1}{4}} H_{\frac{1}{2}} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, -(\frac{1}{2}\nu - \frac{1}{4})} \phi \quad (26)$$

On the other hand , from (ii) with ν replaced by $\nu-2\alpha$,

$$H_\nu \phi = K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, -\alpha} H_{\nu-2\alpha} K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4} - \alpha, \alpha} \phi$$

if Re $\nu > -1$, Re $(\nu-2\alpha) > -1$. In particular, if $\nu-2\alpha = -\frac{1}{2}$, Re $\nu > -1$,

$$H_\nu \phi = K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, -(\frac{1}{2}\nu + \frac{1}{4})} H_{-\frac{1}{2}} K_{x^2}^{0, \frac{1}{2}\nu + \frac{1}{4}} \phi \quad (27)$$

Thus, knowledge of the Fourier sine and cosine transforms and the operators $I_{x^2}^{\eta, \alpha}$ and $K_{x^2}^{\eta, \alpha}$ is sufficient to study H_ν on F_2 when Re $\nu > -1$.

§5.6 The Hankel transform on F_2'

We come now to the definition of H_ν on F_2'. As usual, the definition is motivated from consideration of regular functionals. Let $f \in L_2$ and $\phi \in F_2$ vanish for sufficiently large values of the argument so that $H_\nu f$ and $H_\nu \phi$ are given by integrals. Proceeding formally, we have

$$(H_\nu f , \phi) = \int_0^\infty \left(\int_0^\infty \sqrt{xt}\, J_\nu(xt)\, f(t)\, dt \right) \phi(x)\, dx$$

$$= \int_0^\infty f(t)\, dt \int_0^\infty \sqrt{tx}\, J_\nu(tx)\, \phi(x)\, dx$$

$$= (f , H_\nu \phi)$$

Hence, for arbitrary $f \in F_2'$, we define $H_\nu f$ by

$$(H_\nu f , \phi) = (f , H_\nu \phi) \qquad (28)$$

for $\phi \in F_2$. By Theorems 9 and 12 and Theorem 1.2 we obtain immediately

Theorem 16

For $\operatorname{Re} \nu > -1$, H_ν is an automorphism of F_2' and

$$H_\nu^{-1} = H_\nu$$

Further, H_ν is analytic on F_2' in the sense that, for fixed $f \in F_2'$ and $\phi \in F_2$, $(H_\nu f , \phi)$ is an analytic function of ν

The connection with fractional integration is exhibited by

Theorem 17

Let $f \in F_2'$, $\operatorname{Re} \nu > -1$, $\operatorname{Re} (\nu+2\alpha) > -1$. Then

$$\text{(i)} \quad I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, \alpha} H_\nu f = H_{\nu+2\alpha} I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, \alpha} f \qquad (29)$$

(ii) $$H_\nu \, K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, \alpha} f = K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, \alpha} \, H_{\nu+2\alpha} f \qquad (30)$$

The theorem is immediate on taking adjoints in Theorem 15 .

We can use (29) and (30) to prove the following results , analogous to (26) and (27), valid for $f \in F_2'$, and $\mathrm{Re}\, \nu > -1$.

$$H_\nu f = I_{x^2}^{0, \frac{1}{2}\nu - \frac{1}{4}} \, H_{\frac{1}{2}} \, I_{x^2}^{\frac{1}{2}\nu - \frac{1}{4}, -(\frac{1}{2}\nu - \frac{1}{4})} f \qquad (31)$$

$$H_\nu f = K_{x^2}^{\frac{1}{2}\nu + \frac{1}{4}, -(\frac{1}{2}\nu + \frac{1}{4})} \, H_{-\frac{1}{2}} \, K_{x^2}^{0, \frac{1}{2}\nu + \frac{1}{4}} f \qquad (32)$$

§5.7 Comparison of the spaces F_p' and $\mathcal{H}_\mu'$

To end this chapter, we compare our spaces F_p' with the spaces $\mathcal{H}_\mu'$ on which Zemanian develops his generalised Hankel transform in [25]

For any complex number μ , $\mathcal{H}_\mu$ is the space of all smooth functions ϕ on $(0, \infty)$ such that, for each pair of non-negative integers m and k,

$$\gamma_{m,k}^{\mu}(\phi) = \sup_{0<x<\infty} | x^m (x^{-1}D)^k x^{-\mu-\frac{1}{2}} \phi(x) | < \infty$$

($D \equiv \frac{d}{dx}$) . With the topology generated by the semi-norms $\gamma_{m,k}^{\mu}$ $\mathcal{H}_\mu$ is a complete countably multinormed space .

Proceeding as in the proof of Lemma 5.2-1 , on p. 130 of [25] , we can show that, if $1 \leq p \leq \infty$, $\mathrm{Re}\, \mu > -\frac{1}{2} - \frac{1}{p}$,

$$\mathcal{H}_\mu \subset F_p \; .$$

The inclusion is strict as is seen by considering $\gamma_{0,1}^{\mu}(e^{-x})$, which is infinite for every μ .

It is not hard to show that the identity mapping is continuous from $\mathcal{H}_{\mu}$ to F_p . Thus, if $\phi_n \to 0$ in $\mathcal{H}_{\mu}$, as $n \to \infty$, $\phi_n \to 0$ in F_p also as $n \to \infty$. It follows at once that

$$F_p' \subset \mathcal{H}_{\mu}'$$

for $\mathrm{Re}\ \mu > -\frac{1}{2} - \frac{1}{p}$. We therefore have a smaller class of generalised functions than Zemanian . This is hardly surprising since the spaces $\mathcal{H}_{\mu}'$ were constructed with the Hankel transform specifically in mind , whereas the spaces F_p' were constructed for an investigation of fractional integration . Nor are our spaces as flexible as $\mathcal{H}_{\mu}'$. Nevertheless they do serve to bring out the connection between H_{ν} and fractional integration .

CHAPTER 6

§6.1 Introduction

In this chapter, we return to the spaces $F_{p,\mu}$ to discuss four hypergeometric integral operators studied in [17] and [18] by Love , and we extend the operators to the generalised function spaces $F'_{p,\mu}$.

For any complex numbers a, b, c with $c \neq 0,-1,-2, \ldots$, and for $|z| < 1$, Gauss's hypergeometric function F(a,b,c,z) is defined by

$$F(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{1}$$

where $(a)_0 = 1, \quad (a)_n = a(a+1)\ldots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ for $n \geq 1$. Similarly for $(b)_n$ and $(c)_n$. For fixed a, b, c ($c \neq 0, -1, -2, \ldots$), the power series in (1) converges absolutely for $|z| < 1$. For brevity, we shall write

$$F^*(a,b,c,z) = \frac{1}{\Gamma(c)} F(a,b,c,z) \tag{2}$$

For $|z| < 1$, we have, in the first instance for $c \neq 0,-1,-2,\ldots$,

$$F^*(a,b,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\Gamma(c+n)} \frac{z^n}{n!} \tag{3}$$

However, since the reciprocal of the gamma function is entire, the right-hand side is meaningful for any c . Indeed, (3) shows that, for fixed z , $|z| < 1$, F*(a,b,c,z) is an entire function of a, b and c .

We can extend F*(a,b,c,z) to the half-plane $\operatorname{Re} z < \frac{1}{2}$, by means of Kummer's formula ,[9],

$$F^*(a,b,c,z) = (1-z)^{-a} F^*\left(a,c-b,c,\frac{z}{z-1}\right) \tag{4}$$

where we use the principal branch of $(1-z)^{-a}$. For each fixed z with $\operatorname{Re} z < \frac{1}{2}$, the extended function $F^*(a,b,c,z)$ is an entire function of a, b, c. We note also that

$$F^*(a,b,c,z) = F^*(b,a,c,z) \tag{5}$$

The first integral operator which we consider is $H_1(a,b,c)$ defined for any complex numbers a, b and c with $\operatorname{Re} c > 0$ and for suitable functions ϕ by

$$(H_1(a,b,c)\,\phi)(x) = \int_0^x (x-t)^{c-1} F^*(a,b,c,1-\frac{x}{t})\phi(t)\,dt \tag{6}$$

$$= x^c \int_0^1 (1-v)^{c-1} F^*(a,b,c,\ 1-\frac{1}{v})\ \phi(xv)\,dv \tag{7}$$

In order to discuss the mapping properties of $H_1(a,b,c)$ on $F_{p,\mu}$, we first discuss the behaviour of $F^*(a,b,c,\ 1-\frac{1}{v})$ as $v \to 0+$.

Provided $b-a$ is not an integer, we have, by [9], p. 109,

$$F(a,b,c,\ 1-\frac{1}{v})$$

$$= v^a \frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)} F(a,c-b,a-b+1,v) + v^b \frac{\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)} F(c-a,b,b-a+1,v)$$

$$= \frac{\pi}{\sin(b-a)\pi} \left\{ \frac{v^a F(a,c-b,a-b+1,v)}{\Gamma(c-a)\Gamma(b)\Gamma(a-b+1)} - \frac{v^b F(c-a,b,b-a+1,v)}{\Gamma(c-b)\Gamma(a)\Gamma(b-a+1)} \right\} \tag{8}$$

so that, in this case, as $v \to 0+$,

$$F^*(a,b,c,\ 1-\frac{1}{v}) = 0\left(v^{\min(\operatorname{Re} a,\ \operatorname{Re} b)}\right) \tag{9}$$

However, if $b-a$ is an integer, we must use a limiting argument in (8). Let $b = a + h$. Then the right-hand side of (8) becomes

$$\frac{\pi v^a}{\sin h\pi} \left\{ \frac{F(a,c-a-h,1-h,v)}{\Gamma(c-a)\Gamma(a+h)\Gamma(1-h)} - \frac{v^h F(c-a,a+h,1+h,v)}{\Gamma(c-a-h)\Gamma(a)\Gamma(1+h)} \right\} \tag{10}$$

For fixed a, c, v, the expression $\{\ \}$ is an analytic function of h. By expanding in powers of h and letting $h \to 0$, we can prove that, as $v \to 0+$,

$$F^*(a,a,c,1-\frac{1}{v}) = O(v^{\text{Re } a}\log v) \tag{11}$$

Suppose now that $b = a+m$, where m is a positive integer. Putting $b = a+m+h$ in the right-hand side of (8) gives

$$\frac{(-1)^m \pi v^a}{\sin h\pi}\left\{\frac{F(a,c-a-m-h,1-m-h,v)}{\Gamma(c-a)\Gamma(a+m+h)\Gamma(1-m-h)} - \frac{v^{m+h}F(c-a,a+m+h,m+h+1,v)}{\Gamma(c-a-m-h)\Gamma(a)\Gamma(m+h+1)}\right\}$$

$$= \frac{(-1)^m \pi v^a}{\sin h\pi}\left\{\sum_{n=0}^{m-1}\frac{(a)_n(c-a-m-h)_n}{\Gamma(1-m-h+n)\Gamma(c-a)\Gamma(a+m+h)}\frac{v^n}{n!}\right.$$

$$+\sum_{n=0}^{\infty}\frac{(a)_{n+m}(c-a-m-h)_{n+m}}{\Gamma(c-a)\Gamma(a+m+h)\Gamma(1-h+n)}\frac{v^{n+m}}{(n+m)!}$$

$$\left.-\sum_{n=0}^{\infty}\frac{(c-a)_n(a+m+h)_n}{\Gamma(c-a-m-h)\Gamma(a)\Gamma(m+h+1+n)}\frac{v^{n+m+h}}{n!}\right\}$$

Letting $h \to 0$, the terms in the finite sum all vanish, while those in the two infinite sums cancel in pairs and, proceeding as before, we deduce (11) again in this case. A similar argument holds if m is negative. Hence, for any a,b,c and $\delta > 0$, we have that, as $v \to 0+$,

$$F^*(a,b,c,1-\frac{1}{v}) = O(v^{\min(\text{Re } a, \text{ Re } b)-\delta}) \tag{12}$$

By studying the partial derivatives of $F^*(a,b,c\ 1-\frac{1}{v})$ with respect to a,b and c, we can prove similarly

<u>Lemma 1</u>

Let a, b, c be any complex numbers and let $\delta > 0$. Then there exists a constant M such that, for $0 < v < 1$,

$$|F^*(a,b,c,1-\frac{1}{v})| \leq M v^{\min(\text{Re } a, \text{ Re } b)-\delta}$$

$$|\frac{\partial}{\partial a}F^*(a,b,c,1-\frac{1}{v})| \leq M v^{\min(\text{Re } a, \text{ Re } b)-\delta}$$

$$|\frac{\partial}{\partial b}F^*(a,b,c,1-\frac{1}{v})| \leq M v^{\min(\text{Re } a, \text{ Re } b)-\delta}$$

$$|\frac{\partial}{\partial c}F^*(a,b,c,1-\frac{1}{v})| \leq M v^{\min(\text{Re } a, \text{ Re } b)-\delta}$$

§6.2 The action of $H_1(a,b,c)$ on $F_{p,\mu}$

We are now ready to discuss the action of $H_1(a,b,c)$ as defined by (6) or (7) on functions of $F_{p,\mu}$.

Theorem 2

Let a,b be any complex numbers, Re c > 0 and $\phi \in F_{p,\mu}$ with $-\operatorname{Re}\mu - \frac{1}{q} < \min(\operatorname{Re} a, \operatorname{Re} b)$. Then

(i) $H_1(a,b,c)\phi$ exists and is continuous on $(0,\infty)$

(ii) For each fixed x, $0 < x < \infty$, $(H_1(a,b,c)\phi)(x)$ is an entire function of a and b and an analytic function of c for Re c > 0.

Proof : (i) We have, from (7),

$$(H_1(a,b,c)\phi)(x) = x^c \int_0^1 (1-v)^{c-1} F^*(a,b,c,1-\tfrac{1}{v})\phi(xv)\,dv \quad (13)$$

By Lemma 1 and Lemma 2.4, for $0 < v < 1$, $\delta > 0$, $\exists$ M, dependent on δ, such that

$$|(1-v)^{c-1} F^*(a,b,c,1-\tfrac{1}{v})\phi(xv)| \leq M(1-v)^{\operatorname{Re} c - 1}(xv)^{\operatorname{Re}\mu - \frac{1}{p}} \times v^{\min(\operatorname{Re} a, \operatorname{Re} b) - \delta}$$

Under the given restrictions, the right-hand side is an integrable function of v over (0,1) for δ sufficiently small. Hence, the integral on the right of (13) converges uniformly on compact subsets of $(0,\infty)$. Hence, $H_1(a,b,c)\phi$ is continuous.

(ii) To prove analyticity with respect to a,b,c, it is merely necessary to justify differentiation under the integral sign. For example

$$\frac{\partial}{\partial c}\{(1-v)^{c-1} F^*(a,b,c,1-\tfrac{1}{v})\phi(xv)\} = (1-v)^{c-1}\frac{\partial}{\partial c}F^*(a,b,c,1-\tfrac{1}{v})\phi(xv)$$
$$+ (1-v)^{c-1}\log(1-v)\, F^*(a,b,c,1-\tfrac{1}{v})\,\phi(xv)$$

The hypotheses of the complex form of Theorem 1.3 can be satisfied using Lemma 1 and the result follows. Similarly for a,b.

We shall, in fact, be able to prove much more about $H_1(a,b,c)\phi$ shortly. To this end, we next establish a connection between the operator $H_1(a,b,c)$ and fractional integration.

Lemma 3

Let $\phi \in F_{p,\mu}$, $-\operatorname{Re}\mu - \frac{1}{q} < \min(\operatorname{Re}\xi, \operatorname{Re}\eta)$

$\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta \geq 0$. Then

$$I_x^{\eta,\alpha} I_x^{\xi,\beta} \phi = x^{-\eta-\alpha} H_1(\xi+\beta-\eta, \beta, \alpha+\beta)\, x^{\eta-\beta}\phi \qquad (14)$$

Proof : $I_x^{\eta,\alpha} I_x^{\xi,\beta} \phi(x)$

$$= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-u)^{\alpha-1} u^{\eta}\, du\, \frac{u^{-\xi-\beta}}{\Gamma(\beta)} \int_0^u (u-t)^{\beta-1} t^{\xi} \phi(t)\, dt$$

We may justify inverting the order of integration by means of Fubini's Theorem so that

$$I_x^{\eta,\alpha} I_x^{\xi,\beta} \phi(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x t^{\xi}\phi(t)\, dt \int_t^x (x-u)^{\alpha-1}(u-t)^{\beta-1} u^{\eta-\xi-\beta} du$$

On putting $w = \frac{u-t}{x-t}$, the inner integral becomes

$$(x-t)^{\alpha+\beta-1}\, t^{\eta-\xi-\beta} \int_0^1 (1-w)^{\alpha-1} w^{\beta-1} \left[1-w\left(1-\frac{x}{t}\right)\right]^{\eta-\xi-\beta} dw$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-t)^{\alpha+\beta-1}\, t^{\eta-\xi-\beta}\, F\left(\xi+\beta-\eta, \beta, \alpha+\beta, 1-\frac{x}{t}\right)$$

using Formula (10), p.59 of [9]. Thus, finally

$$I_x^{\eta,\alpha} I_x^{\xi,\beta} \phi(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha+\beta)} \cdot \int_0^x (x-t)^{\alpha+\beta-1} F\left(\xi+\beta-\eta, \beta, \alpha+\beta, 1-\frac{x}{t}\right) t^{\eta-\beta}\phi(t)\, dt$$

$$= x^{-\eta-\alpha} H_1(\xi+\beta-\eta, \beta, \alpha+\beta)\, x^{\eta-\beta}\phi(x)$$

as required.

It now follows that for $\phi \in F_{p,\mu}$,

$$H_1(\xi+\beta-\eta, \beta, \alpha+\beta)\phi = x^{\eta+\alpha} I_x^{\eta,\alpha} I_x^{\xi,\beta} x^{\beta-\eta}\phi(x)$$

provided $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $-\operatorname{Re}(\mu+\beta-\eta) - \frac{1}{q} < \min(\operatorname{Re}\xi, \operatorname{Re}\eta)$ and hence we obtain

Corollary 4

Let $\phi \in F_{p,\mu}$, $-\operatorname{Re}\mu - \frac{1}{q} < \min(\operatorname{Re} a, \operatorname{Re} b)$, $\operatorname{Re} c > \operatorname{Re} b > 0$. Then for any η,

$$(H_1(a,b,c)\phi)(x) = x^{\eta+c-b} I_x^{\eta,c-b} I_x^{\eta+a-b,b} x^{b-\eta} \phi(x) \tag{15}$$

The fact that the right side is independent of η is seen by writing it in the form

$$x^{\eta+c-b} x^{-\eta-c+b} I_x^{c-b} x^{\eta} x^{-\eta-a} I_x^{b} x^{\eta+a-b} x^{b-\eta} \phi(x)$$

$$= I_x^{c-b} x^{-a} I_x^{b} x^{a} \phi(x) \tag{16}$$

Using the theory of Chapter 3, we can continue the right side of (15) analytically with respect to b to remove the restriction $\operatorname{Re} c > \operatorname{Re} b > 0$. We then use (15) to define $H_1(a,b,c)$ on $F_{p,\mu}$ provided only that $-\operatorname{Re}\mu - \frac{1}{q} < \min(\operatorname{Re} a, \operatorname{Re} b)$. By Theorem 2(ii) and uniqueness of analytic continuation, the new definition coincides with the old when $\operatorname{Re} c > 0$.

We now use results in Chapter 3 to derive more properties of $H_1(a,b,c)$ on $F_{p,\mu}$. From (15) we see at once that if $-\operatorname{Re}\mu - \frac{1}{q} < \min(\operatorname{Re} a, \operatorname{Re} b)$, $H_1(a,b,c)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+c}$. If, in addition, we have $-\operatorname{Re}\mu - \frac{1}{q} < \min(\operatorname{Re} c, \operatorname{Re}(a+b))$, $H_1(a,b,c)$ is invertible and, for any η and $\psi \in F_{p,\mu+c}$,

$$([H_1(a,b,c)]^{-1}\psi)(x) = x^{\eta-b} I_x^{\eta+a,-b} I_x^{\eta+c-b,b-c} x^{-\eta-c+b}\psi(x) \tag{17}$$

$$= x^{\eta-b-a} I_x^{\eta,-b} I_x^{\eta+c-b-a,b-c} x^{-\eta-c+a+b} \psi(x)$$

$$= x^{-a} H_1(-a, b-c, -c)\, x^{a} \psi(x) \tag{18}$$

from (15)

We gather our results together in the form of a theorem.

Theorem 5

H_1 (a,b,c) is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+c}$ provided $- \text{Re}\,\mu - \frac{1}{q} <$ min(Re a, Re b). If in addition $- \text{Re}\,\mu - \frac{1}{q} <$ min (Re c, Re (a+b)) , H_1 (a,b,c) is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+c}$ with inverse given by (17). Further, for $\psi \in F_{p,\mu+c}$, (18) holds .

An interesting point emerges at this stage. From (5) it follows that for any a, b, c and $- \text{Re}\,\mu - \frac{1}{q} <$ min(Re a,Re b)

$$(H_1(a,b,c)\phi)(x) = (H_1(b,a,c)\phi)(x)$$

for all $\phi \in F_{p,\mu}$. Then from (16) we have for $\phi \in F_{p,\mu}$,

$$I_x^{c-b} \; x^{-a} \; I_x^{b} \; x^{a} \; \phi(x) = I_x^{c-a} \; x^{-b} \; I_x^{a} \; x^{b} \; \phi(x)$$

the restriction Re c > Re b > 0 being removed by analytic continuation . Thus, if $- \text{Re}\,\mu - \frac{1}{q} <$ min (0, Re (a-b)), $\phi \in F_{p,\mu}$

$$I_x^{c-b} \; x^{-a} \; I_x^{b} \; x^{a-b} \; \phi(x) = I_x^{c-a} \; x^{-b} \; I_x^{a} \; \phi(x)$$

Taking c = b , we can use (3.25) and the fact that $- \text{Re}\,\mu - \frac{1}{q} < 0$ to deduce that

$$x^{-a} \; I_x^{b} \; x^{a-b} \; \phi(x) = I_x^{b-a} \; x^{-b} \; I_x^{a} \; \phi(x)$$

Finally, writing $\alpha = -a$, $\beta = b$ and $\gamma = a-b$ we obtain

Theorem 6

Let $\phi \in F_{p,\mu}$, $- \text{Re}\,\mu - \frac{1}{q} <$ min (0, Re γ) , $\alpha + \beta + \gamma = 0$. Then

$$x^{\alpha} \; I_x^{\beta} \; x^{\gamma} \; \phi(x) = I_x^{-\gamma} \; x^{-\beta} \; I_x^{-\alpha} \; \phi(x) \qquad (19)$$

(19) is a form of the second index law for fractional integrals which has been discussed by Erdélyi [7] and Love [19] .

(The first index law for I_x^{α} is (3.27) with m = 1)

§6.3 Other hypergeometric integral operators on $F_{p,\mu}$

We now consider three more hypergeometric integral operators closely related to $H_1 (a,b,c)$

For any complex numbers a, b and Re c > 0 and for suitable functions ϕ , we define $H_2 (a,b,c) \phi$ by

$$(H_2 (a,b,c)\phi)(x) = \int_0^x (x-t)^{c-1} F^*(a,b,c,1 - \frac{t}{x})\phi(t)dt \quad (20)$$

Proceeding as in [17], p.195, we deduce that

$$H_1(a,c-b,c) x^{-a} \phi(x) = x^{-a} H_2 (a,b,c) \phi(x)$$

$$\Rightarrow \quad H_2 (a,b,c) \phi (x) = x^a H_1 (a,c-b,c) x^{-a} \phi(x) \quad (21)$$

Since the right-hand side is meaningful for Re c $\leq$ 0 , we can use (21) to extend the definition of $H_2 (a,b,c)$. Let $\phi \in F_{p,\mu}$. Then, provided $- \mathrm{Re}\,\mu - \frac{1}{q} < \min (0, \mathrm{Re} (c-b-a))$, we have that $(H_2 (a,b,c)\phi)(x)$ is, for each fixed x and ϕ, an analytic function of a, b, c , and maps $F_{p,\mu}$ into $F_{p,\mu+c}$. We see from Theorem 5 that under the same conditions

$$([H_1(-a, -b, -c)]^{-1}\phi)(x) = x^a H_1(a,c-b,c) x^{-a} \phi(x) \quad (22)$$

for $\phi \in F_{p,\mu}$ so that in this case

$$H_2 (a,b,c) = [H_1 (-a, -b, -c)]^{-1} \quad \text{on } F_{p,\mu}$$

We can express $H_2 (a,b,c)$ in terms of fractional integrals by means of Theorem 5 . We gather the results together in

Theorem 7

Let $- \mathrm{Re}\,\mu - \frac{1}{q} < \min (0, \mathrm{Re} (c-b-a))$. Then $H_2 (a,b,c)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+c}$ and, for any η ,

$$(H_2(a,b,c) \phi)(x) = x^{\eta+a+b} I_x^{\eta,b} I_x^{\eta+a+b-c,c-b} x^{c-b-a-\eta} \phi(x) \quad (23)$$

If, in addition, $- \mathrm{Re}\,\mu - \frac{1}{q} < \min (\mathrm{Re} (c-a), \mathrm{Re} (c-b))$,

$H_2(a,b,c)$ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+c}$, and for $\psi \in F_{p,\mu+c}$,

$$(H_2(a,b,c)^{-1}\psi)(x) = x^{\eta+a+b-c} I_x^{\eta+a,b-c} I_x^{\eta+b,-b} x^{-\eta-a-b}\psi(x) \quad (24)$$

In this case, the integral equation

$$H_2(a,b,c)\phi = \psi$$

has, for each $\psi \in F_{p,\mu+c}$, a unique solution $\phi \in F_{p,\mu}$ given by (24). Also, on $F_{p,\mu+c}$

$$[H_2(a,b,c)]^{-1} = x^a H_2(-a,b-c,-c)\, x^{-a}$$

$$[H_2(a,b,c)]^{-1} = H_1(-a,-b,-c)$$

and on $F_{p,\mu}$ $\quad H_2(a,b,c) = [H_1(-a,-b,-c)]^{-1}$

Our other two operators are the adjoints of $H_1(a,b,c)$ and $H_2(a,b,c)$ and are discussed by Love in [18]. For complex numbers a,b,c with $\operatorname{Re} c > 0$, we define $H_3(a,b,c)$ and $H_4(a,b,c)$ by

$$(H_3(a,b,c)\phi)(x) = \int_x^\infty (t-x)^{c-1} F^*(a,b,c, 1-\frac{x}{t})\phi(t)\,dt \quad (25)$$

$$(H_4(a,b,c)\phi)(x) = \int_x^\infty (t-x)^{c-1} F^*(a,b,c, 1-\frac{t}{x})\phi(t)\,dt \quad (26)$$

We can treat $H_4(a,b,c)$ in a similar fashion to $H_1(a,b,c)$. Let $\phi \in F_{p,\mu}$, $\operatorname{Re} \mu - \frac{1}{p} < \min(\operatorname{Re}\xi, \operatorname{Re}\eta)$ and $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$. Then by interchanging the order of integration we can prove that

$$K_x^{\xi,\beta} K_x^{\eta,\alpha}\phi = x^{\eta-\beta} H_4(\xi+\beta-\eta, \beta, \alpha+\beta)\, x^{-\eta-\alpha}\phi(x) \quad (27)$$

from which we deduce that if $\operatorname{Re}\mu - \frac{1}{p} < \min(\operatorname{Re}(a-c), \operatorname{Re}(b-c))$, $\operatorname{Re} c > \operatorname{Re} b > 0$,

$$(H_4(a,b,c)\phi)(x) = x^{b-\eta} K_x^{\eta+a-b,b} K_x^{\eta,c-b} x^{\eta+c-b}\phi(x) \quad (28)$$

Now, under the given restrictions, for fixed x and ϕ, we can show that $(H_4(a,b,c)\phi)(x)$ is an analytic function of a, b, c. Further the right-hand side of (28) can be continued analytically using the theory of Chapter 3. We can therefore extend the definition of $H_4(a,b,c)$ using (28), the new and old definitions coinciding, by analytic continuation, when $\operatorname{Re} c > 0$. $H_4(a,b,c)$ when thus extended is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+c}$ provided only $\operatorname{Re}\mu - \frac{1}{p} < \min(\operatorname{Re}(a-c), \operatorname{Re}(b-c))$. Proceeding as before we can derive the following results.

Theorem 8

Let $\operatorname{Re}\mu - \frac{1}{p} < \min(\operatorname{Re}(a-c), \operatorname{Re}(b-c))$.

Then $H_4(a,b,c)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+c}$, and for $\phi \in F_{p,\mu}$, $H_4(a,b,c)\phi$ is given by (28). If, in addition, $\operatorname{Re}\mu - \frac{1}{p} < \min(0, \operatorname{Re}(a+b-c))$, $H_4(a,b,c)$ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+c}$ and for $\psi \in F_{p,\mu+c}$,

$$[(H_4(a,b,c))^{-1}\psi](x) = x^{-\eta-c+b} K_x^{\eta+c-b,b-c} K_x^{\eta+a,-b} x^{\eta-b} \psi(x) \quad (29)$$

$$\text{or} \quad [(H_4(a,b,c))^{-1}\psi](x) = x^{a} H_4(-a,b-c,-c)\, x^{-a} \psi(x) \quad (30)$$

In this case, the integral equation

$$H_4(a,b,c)\phi = \psi \qquad (\psi \in F_{p,\mu+c})$$

has a unique solution $\phi \in F_{p,\mu}$ given by (29).

From (28), we deduce that

$$(H_4(a,b,c)\phi)(x) = x^{a} K_x^{b}\, x^{-a}\, K_x^{c-b} \phi(x) \quad (31)$$

Since, using (5) again, $H_4(a,b,c)\phi = H_4(b,a,c)\phi$, we deduce that if $\operatorname{Re}\mu - \frac{1}{p} < \min(\operatorname{Re}(a-c), \operatorname{Re}(b-c))$ and $\phi \in F_{p,\mu}$,

$$x^{a} K_x^{b} x^{-a} K_x^{c-b} \phi(x) = x^{b} K_x^{a} x^{-b} K_x^{c-a} \phi(x)$$

$$\Rightarrow x^{a-b} K_x^{b} x^{-a} K_x^{c-b} \phi(x) = K_x^{a} x^{-b} K_x^{c-a} \phi(x)$$

Taking $c = b$ and using the theory of K_x^{α} developed in Chapter 3, we deduce that if $\mathrm{Re}\ \mu - \frac{1}{p} < \min(0, \mathrm{Re}(a-b))$,

$$x^{a-b} K_x^{b} x^{-a} \phi(x) = K_x^{a} x^{-b} K_x^{b-a} \phi(x)$$

Writing $\alpha = -a$, $\beta = b$ and $\gamma = a-b$, we obtain

Theorem 9

Let $\phi \in F_{p,\mu}$, $\mathrm{Re}\ \mu - \frac{1}{p} < \min(0, \mathrm{Re}\ \gamma)$, $\alpha + \beta + \gamma = 0$. Then

$$x^{\gamma} K_x^{\beta} x^{\alpha} \phi(x) = K_x^{-\alpha} x^{-\beta} K_x^{-\gamma} \phi(x) \qquad (32)$$

which is a form of the second index law for the operators K_x^{α}.

(The first index law is (3.39) .

Finally, we consider $H_3(a,b,c)$. For $\mathrm{Re}\ c > 0$, we may use (25) and (26) and proceed as in [18], pp. 1073-4, to prove that for suitable functions ϕ,

$$(H_3(a,b,c)\phi)(x) = x^{-a} H_4(a, c-b, c)\, x^{a} \phi(x) \qquad (33)$$

We may use (33) to extend the definition of $H_4(a,b,c)$ to $\mathrm{Re}\ c \leq 0$. If $\phi \in F_{p,\mu}$ and $\mathrm{Re}\ \mu - \frac{1}{p} < \min(-\mathrm{Re}\ c, -\mathrm{Re}(a+b))$, $(H_3(a,b,c)\phi)(x)$ is, for fixed x and ϕ, an analytic function of a, b and c. From (33) and Theorem 8, we deduce

Theorem 10

Let $\mathrm{Re}\ \mu - \frac{1}{p} < \min(-\mathrm{Re}\ c, -\mathrm{Re}(a+b))$. Then $H_3(a,b,c)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+c}$ and, for $\phi \in F_{p,\mu}$ and any η,

$$(H_3(a,b,c)\phi)(x) = x^{c-a-b-\eta} K_x^{\eta+a+b-c,\,c-b} K_x^{\eta,b} x^{\eta+a+b} \phi(x) \qquad (34)$$

If, in addition, $\text{Re}\,\mu - \frac{1}{p} < \min(-\text{Re}\,a, -\text{Re}\,b)$, $H_3(a,b,c)$ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+c}$ and, for $\psi \in F_{p,\mu+c}$,

$$[(H_3(a,b,c))^{-1}\psi](x) = x^{-\eta-a-b} K_x^{\eta+b,-b} K_x^{\eta+a,b-c} x^{\eta+a+b-c}\psi(x) \quad (35)$$

$$= x^{-a} H_3(-a,b-c,-c)\, x^{a}\,\psi(x)$$

$$= (H_4(-a,-b,-c)\psi)(x)$$

In this case, the integral equation

$$H_3(a,b,c)\phi = \psi$$

has, for each $\psi \in F_{p,\mu+c}$, a unique solution $\phi \in F_{p,\mu}$ given by (35) .

§6.4 Hypergeometric Integral Operators on $F'_{p,\mu}$

We are now ready to discuss the operators $H_i(a,b,c)$ $(i = 1,2,3,4)$ on $F'_{p,\mu}$.

We begin again with $H_1(a,b,c)$. Assuming that f and $H_1(a,b,c)f$ generate regular functionals, we have, proceeding formally,

$$(H_1(a,b,c)f, \phi) = \int_0^\infty \phi(x)dx \int_0^x (x-t)^{c-1}F^*(a,b,c,1-\tfrac{x}{t})\, f(t)dt$$

$$= \int_0^\infty f(t)dt \int_t^\infty (x-t)^{c-1}F^*(a,b,c,1-\tfrac{x}{t})\, \phi(x)dx$$

$$\text{or} \quad (H_1(a,b,c)f, \phi) = (f, H_4(a,b,c)\phi) \quad (36)$$

By Theorem 8, the right-hand side is meaningful if $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu-c}$ and $\text{Re}\,\mu - \frac{1}{p} < \min(\text{Re}\,a, \text{Re}\,b)$. In this case, we use (36) to define $H_1(a,b,c)f$ for $f \in F'_{p,\mu}$. By Theorem 1.2, $H_1(a,b,c)f \in F'_{p,\mu-c}$ and taking adjoints in Theorem 8 we obtain

Theorem 11

Let $\text{Re}\,\mu - \frac{1}{p} < \min(\text{Re}\,a, \text{Re}\,b)$. Then $H_1(a,b,c)$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-c}$ and, for $f \in F'_{p,\mu}$

$$H_1(a,b,c)f = x^{\eta+c-b} I_x^{\eta,c-b} I_x^{\eta+a-b,b} x^{b-\eta} f \qquad (37)$$

for any complex η. If, further, $\text{Re}\,\mu - \frac{1}{p} < \min(\text{Re}\,c, \text{Re}(a+b))$, $H_1(a,b,c)$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-c}$ and, for $g \in F'_{p,\mu-c}$,

$$[H_1(a,b,c)]^{-1} g = x^{\eta-b} I_x^{\eta+a,-b} I_x^{\eta+c-b,b-c} x^{b-c-\eta} g \qquad (38)$$

$$\text{or} \quad [H_1(a,b,c)]^{-1} g = x^{-a} H_1(-a,b-c,-c) x^{a} g \qquad (39)$$

In this case, the integral equation

$$H_1(a,b,c)\, f = g$$

has, for each $g \in F'_{p,\mu-c}$, a unique solution $f \in F'_{p,\mu}$ given by (38).

From consideration of regular functionals, we are led to the following definitions of $H_2(a,b,c)$, $H_3(a,b,c)$ and $H_4(a,b,c)$ for $f \in F'_{p,\mu}$ and $\phi \in F_{p,\mu-c}$.

$$(H_2(a,b,c)f, \phi) = (f, H_3(a,b,c)\phi) \qquad (40)$$

$$(H_3(a,b,c)f, \phi) = (f, H_2(a,b,c)\phi) \qquad (41)$$

$$(H_4(a,b,c)f, \phi) = (f, H_1(a,b,c)\phi) \qquad (42)$$

By taking adjoints in Theorems 10, 7 and 5 we obtain the following results.

Theorem 12

Let $\text{Re}\,\mu - \frac{1}{p} < \min(0, \text{Re}\,(c-a-b))$. Then

$H_2(a,b,c)$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-c}$ and, for $f \in F'_{p,\mu}$,

$$H_2(a,b,c)\, f = x^{\eta+a+b}\, I_x^{\eta,b}\, I_x^{\eta+a+b-c,c-b}\, x^{c-a-b-\eta}\, f \qquad (43)$$

for any complex η. If, further, $\operatorname{Re}\, \mu - \frac{1}{p} < \min(\operatorname{Re}(c-a), \operatorname{Re}(c-b))$; $H_2(a,b,c)$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-c}$ and, for $g \in F'_{p,\mu-c}$,

$$[H_2(a,b,c)]^{-1} g = x^{a+b-c+\eta}\, I_x^{\eta+a,b-c}\, I_x^{\eta+b,-b}\, x^{-\eta-a-b}\, g \qquad (44)$$

Also, $[H_2(a,b,c)]^{-1} g = x^a H_2(-a,b-c,-c)\, x^{-a} g$

$$= H_1(-a,-b,-c)\, g$$

In this case, the integral equation

$$H_2(a,b,c)\, f = g$$

has, for each $g \in F'_{p,\mu-c}$, a unique solution $f \in F'_{p,\mu}$ given by (44).

<u>Theorem 13</u>

Let $-\operatorname{Re}\, \mu - \frac{1}{q} < \min(-\operatorname{Re}\, c, -\operatorname{Re}(a+b))$. Then $H_3(a,b,c)$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-c}$ and for $f \in F'_{p,\mu}$, and any complex η,

$$H_3(a,b,c)\, f = x^{c-a-b-\eta}\, K_x^{\eta+a+b-c,c-b}\, K_x^{\eta,b}\, x^{\eta+a+b}\, f \qquad (45)$$

If, further, $-\operatorname{Re}\, \mu - \frac{1}{q} < \min(-\operatorname{Re}\, a, -\operatorname{Re}\, b)$, $H_3(a,b,c)$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-c}$ and for $g \in F'_{p,\mu-c}$,

$$[H_3(a,b,c)]^{-1} g = x^{-\eta-a-b}\, K_x^{\eta+b,-b}\, K_x^{\eta+a,b-c}\, x^{\eta+a+b-c}\, g \qquad (46)$$

$$= x^{-a}\, H_3(-a,b-c,-c)\, x^a\, g$$

In this case, the integral equation

$$H_3(a,b,c) f = g \qquad (g \in F'_{p,\mu-c})$$

has a unique solution $f \in F'_{p,\mu}$ given by (46).

Theorem 14

$H_4(a,b,c)$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-c}$ provided $-\text{Re}\ \mu - \frac{1}{q} < \min(\text{Re}\ (a-c)\ ,\ \text{Re}\ (b-c)\ )$, and for $f \in F'_{p,\mu}$, and any complex η,

$$H_4(a,b,c)f = x^{b-\eta}\, K_x^{\eta+a-b,b}\, K_x^{\eta,c-b}\, x^{\eta+c-b}\, f \tag{47}$$

If also $-\text{Re}\ \mu - \frac{1}{q} < \min(\ 0,\ \text{Re}\ (a+b-c)\ )$, $H_4(a,b,c)$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-c}$ and for $g \in F'_{p,\mu-c}$,

$$[\ H_4(a,b,c)\ ]^{-1} g = x^{-\eta-c+b}\, K_x^{\eta+c-b,b-c}\, K_x^{\eta+a,-b}\, x^{\eta-b}\, g \tag{48}$$

$$= x^{a}\, H_4(\ -a,b-c,-c\ )\, x^{-a}\, g = H_3(\ -a,-b,-c\ )\, g$$

In this case, the integral equation

$$H_4(a,b,c)f = g \qquad (\ g \in F'_{p,\mu-c}\ )$$

has a unique solution $f \in F'_{p,\mu}$ given by (48) .

Lastly, we have the second index laws for I_x^{α} , K_x^{α} on $F'_{p,\mu}$ obtained by taking adjoints in Theorems 6 and 9 .

Theorem 15

(i) Let $f \in F'_{p,\mu}$, $\text{Re}\ \mu - \frac{1}{p} < \min(\ 0,\ \text{Re}\ \gamma\ )$, $\alpha + \beta + \gamma = 0$. Then

$$x^{\alpha}\, I_x^{\beta}\, x^{\gamma}\, f = I_x^{-\gamma}\, x^{-\beta}\, I_x^{-\alpha}\, f \tag{49}$$

(ii) Let $f \in F'_{p,\mu}$, $-\text{Re}\ \mu - \frac{1}{q} < \min(0,\ \text{Re}\ \gamma\ )$, $\alpha + \beta + \gamma = 0$. Then

$$x^{\gamma}\, K_x^{\beta}\, x^{\alpha}\, f = K_x^{-\alpha}\, x^{-\beta}\, K_x^{-\gamma}\, f \tag{50}$$

Once again, we note that the restrictions on the parameters in Theorems 11-15 are obtained from those in the corresponding results for $F_{p,\mu}$ by interchanging μ and $-\mu$, p and q .

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