

THE APPLICATION OF UNITARITY AND DISPERSION RELATIONS TO
DELBRÜCK SCATTERING: AND A NEW APPROACH TO PION-NUCLEON
SCATTERING.

A Thesis submitted in partial fulfilment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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PREFACE

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All the material presented in this thesis is original except in those places where explicit references are made to published papers.

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THE APPLICATION OF UNITARITY AND DISPERSION RELATIONS TO
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A NEW APPROACH TO PION-NUCLEON SCATTERING USING TWO TIME

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THE APPLICATION OF UNITARITY AND DISPERSION RELATIONS TO

..
DELBURCK SCATTERING

1. General Interaction of Gamma-rays with Atoms.

Gamma-ray Attenuation with the various atomic constituents may be classified in the following manner.

Kinds of Interaction

- I Basic interaction with atomic electrons: the electric field of the gamma-rays exerts an oscillating electric force on the charge of the atomic electrons and a smaller magnetic torque on their spin.
- II Basic interaction with nuclear particles: similar to I.
- III Interaction with the electric field surrounding the charged nuclei and electrons.
- IV Interaction with the meson field surrounding the nucleons.

Each of the above sections may be further divided according to whether absorption or scattering takes place.

A. Absorption

B. Elastic (coherent) scattering

If the atomic system responds as a whole to the photon impact, its internal energy is not increased and the scattering is elastic. The effects of the elastic gamma ray interaction with different parts of the system combine "coherently" (i.e. by addition of the amplitudes).

C. Inelastic (incoherent) scattering

If the scattering of a photon causes an atomic particle to recoil with respect to the others, the internal energy of the

atomic system is increased, and the photon energy correspondingly decreased. The effects of inelastic gamma ray interactions with different parts of the system combine "incoherently", (i.e. by addition of the intensities of the effects).

As the optical theorem and unitarity relate these processes it is perhaps useful to describe them briefly.

IA. Photoelectric effect

This effect is most important at low energies, and for high Z materials. The cross-section decreases roughly as ω^{-3} for $\omega < 0.5$ Mev and like ω^{-1} for $\omega > 0.5$ Mev where ω is the energy of the incoming gamma ray. The ejected electron emerges approximately sideways to the photon beam for $\omega \ll 0.5$ Mev, but almost in the forward direction for high energies.

IIIA. Pair Production

Pair production predominates for high photon energies, especially for high Z. The threshold is approximately 1 Mev and the cross-section rises monotonically above this energy until it levels off near 50 Mev, depending on the atomic number of the scatterer. An important fact is that the electron and positron are projected predominantly in the direction of the incident photon, especially when the photon energy, and hence its momentum, is very large. Most of the electrons and positrons are confined to directions within $0.5 \frac{\text{Mev}}{\omega}$ radians from the direction of the incoming photon.

IB Rayleigh Scattering

Gamma ray scattering at small angles imparts only a small recoil to the scatterer. The recoil is then often absorbed by a whole atom or molecule, so that the scattering cross-sections of different electrons combine coherently. This effect is obviously greatest at low energies. Even for photon energies of above 0.1 Mev Rayleigh scattering is only less probable than Compton scattering by one or two orders of magnitude. This scattering increases with Z.

IIIB Delbrück Scattering

This effect is small in the region 0.5 to 3 Mev where most experiments have been carried out. A description of experimental and theoretical results will be given later.

IC Compton Scattering

A gamma ray is scattered inelastically and an atomic electron recoils out of an atom as though it had been initially free. (This scattering is elastic if one considers the electron as an isolated particle, and inelastic if the electron is considered as a part of the whole material). Compton scattering is most important for gamma rays of 1-5 Mev in high Z-materials. The cross-section decreases to less than 0.1 barn at 10 Mev. The recoil electron flies off nearly in the direction of the incident photon.

All the other effects such as the nuclear photoelectric effect (IIA) and nuclear scattering (IIB) have very small cross-sections. Mesonic effects (IV) only become appreciable around 150 Mev and then have cross sections of the order of a few millibarns.

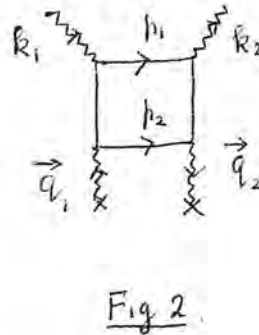
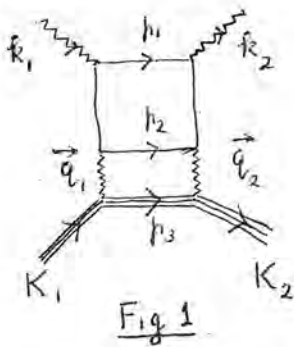
During experiments performed to measure the Delbrück effect one thus sees that it occurs coherently with Rayleigh scattering and scattering from the atomic nucleus. (We will throughout neglect any mesonic effects). For not too high energies the scattering from the nucleus can be considered as ordinary Thomson classical scattering and this amplitude is real and known. The amplitudes for Delbrück and Rayleigh scattering are complex and so the experimentally measurable cross section is given by:

$$\frac{d\sigma}{d\Omega} = \left| a^{(T)} + \text{Re } a^{(D)} + \text{Re } a^{(R)} + i \text{Im } a^{(D)} + i \text{Im } a^{(R)} \right|^2$$

The lack of knowledge of accurate values of these amplitudes makes it very difficult to analyse experimental data. Rayleigh scattering is more important than Delbrück scattering at the low energies where most experiments have been carried out. The fact that monochromatic gamma ray sources are not readily available at higher energies is unfortunate, as experiments done in the region of 5-10 Mev would have a better chance of confirming the existence of Delbrück scattering.

2. Review of Experimental and Theoretical Results

So far we have only discussed general effects. Let us now be more specific and attend to some points in detail. From the field theoretical point of view Delbruck scattering is the elastic scattering of γ -rays via the formation, and subsequent annihilation, of an intermediate electron-positron state. Thus Delbrück scattering may be considered as the first order radiative correction to Compton scattering on a nucleus when the nucleus is held fixed and has infinite mass. The Feynman diagrams for the process are,



where k_1 and k_2 are the four momenta of the incoming and outgoing photons ($k_1^2 = k_2^2 = 0$) and \vec{q}_1 and \vec{q}_2 are the three momenta transfer to the Coulomb field of the nucleus. They have no fourth component as the scattering is elastic.

A line representing the incoming, virtual, and outgoing nucleus of momenta K_1 , p_3 and K_2 respectively has been added to Figure 1. When the nucleus has infinite mass and is considered as fixed the first diagram transforms into the second where the crosses denote the action of the external field.

This scattering, according to Figure 2, is a particular case of the scattering of light by light with two lines not on the mass shell, and so is part of the fourth order tensor $T_{\mu\nu\alpha\beta}(k_1, k_2; q_1, q_2)$. However, the general light-light scattering tensor has never been

computed.

Interest in this process has been shown for many years. Delbrück originally proposed this type of scattering in 1935, but the computation of the cross-section was so complicated, using old fashioned perturbation theory, that only rough estimates of the forward scattering cross-section for high energies and energies comparable to 0.5 Mev could be found. (See the work of Kemmer, 1937 and Akhiezer and Pomeranchuk 1938). The effect is of fundamental importance as it is a direct proof of the polarization of the vacuum surrounding a nuclear charge. As such it is a purely quantum mechanical effect, and can only be predicted from classical theory if Maxwell's equations are altered and made non-linear. That vacuum polarization exists is shown very well by Lamb shift experiments, and Delbrück scattering would be another good check on the present day theory of quantum electrodynamics.

Anyone interested in the early experiments and theoretical work should read the thesis of Toll, 1952, who first used dispersion relations in connection with the optical theorem to find the forward scattering amplitude. Estimates were also made by him of non-forward scattering, using the Weizsäcker-Williams method, but these are rather rough and it is difficult to see exactly how large the possible errors may be.

An exact calculation of the forward scattering amplitude for the graph in Fig. II was also made by Rohrlich and Gluckstern in 1952, by writing down the Feynman integral and evaluating it directly. They also used the optical theorem together with dispersion relations and found the same answer.

The result can be seen in any text book on quantum electrodynamics and is reproduced here for completeness.

$$\frac{\mathcal{D}}{\text{Im}} \alpha(\omega) = \alpha^2 Z^2 r_0 \left\{ \frac{\eta}{\pi} \left[2C_1(\eta) - D_1(\eta) \right] + \frac{1}{27\pi\eta} \left[(104 + 64\eta^2) E_1(\eta) - (67 + 6\eta^2)(1 - \eta^2) F_1(\eta) \right] - \frac{\eta^2}{9} - \frac{9}{4} \right\}.$$

$$\frac{\mathcal{D}}{\text{Re}} \alpha(\omega) = \alpha^2 Z^2 r_0 \left\{ \frac{\eta}{\pi} \left[2C_2(\eta) - D_2(\eta) \right] - \frac{1}{27\pi\eta} \left[(104 + 64\eta^2) E_2(\eta) - (67 + 6\eta^2)(1 - \eta^2) F_2(\eta) \right] \right\}.$$

where

$$C_1(\eta) = \text{Re} \int_0^{1/\eta} \frac{\sin^{-1} x}{x} \frac{\cosh^{-1} \frac{1}{\eta x}}{\eta x} dx \quad (\eta > 0).$$

$$D_1(\eta) = \text{Re} \int_0^{1/\eta} \frac{\cosh^{-1} \left(\frac{1}{\eta x} \right)}{\sqrt{1-x^2}} dx. \quad (\eta > 0).$$

$$C_2(\eta) = \int_i^{1/\eta} \frac{\cosh^{-1} x}{x} \cdot \frac{\cosh^{-1} \frac{1}{\eta x}}{\eta x} dx \quad (\eta \leq 1).$$

$$D_2(\eta) = \int_i^{1/\eta} \frac{\cosh^{-1} \left(\frac{1}{\eta x} \right)}{\sqrt{x^2 - 1}} dx \quad (\eta \leq 1).$$

E_1, E_2, F_1, F_2 are related to the complete elliptic functions of the first and second kind and $\eta = \frac{2m}{\omega}$.

This shows that even in the forward direction, where considerable simplifications occur, the integrals cannot be carried out completely. One therefore expects that several integrals cannot be performed analytically in the non forward scattering case.

Any corrections due to sixth-order graphs are expected to be large on the basis that $(Z\alpha)^2$ is around one third, for high Z . However, Rohrlich, in 1957, showed that such large corrections only occur in the imaginary part of the amplitude where it is small compared to the real part. The overall correction due to sixth-order graphs is thus less than ten per cent. Note that the theorem due to Furry excludes all graphs with an odd number of corners.

A calculation was made by Bethe and Rohrlich in 1952 to find the cross-section for high energies and small angles. They found a diffraction peak type of behaviour for the differential scattering cross-section, which has since been checked experimentally by Moffat and Stringfellow in 1960. The latter experiment is the only exception to the general rule, all others being carried out for gamma ray energies of only a few Mev. Moffat and Stringfellow used 87 Mev gamma rays and found reasonable agreement with theory.

However, such calculations are not of great importance when comparison with experimental results in the low energy region have to be made, because one needs to know the differential scattering cross-section for intermediate values of the energy and for finite scattering angles, (say 1-5 Mev and $0-30^\circ$). All references to such experiments, and quite a number have been made in the last ten years, may be found in the paper of Standing and Jovanovitch 1962. A complete list of theoretical papers on the subject is given in Appendix II.

The extension of calculations from forward to finite angle scattering is possible, using unitarity. However, when one realises that the unitarity condition involves three particle intermediate

states (i.e., three three-dimensional integrals over \vec{p}_1, \vec{p}_2 and \vec{p}_3 with one four dimensional delta function, in all a five dimensional integral), and the particles involved have spins, one sees that any exact calculation would be extremely complicated and sooner or later involve non-integrable functions. The fact that one is working in a fixed coordinate system with $\vec{K}_1 = \vec{K}_2 = 0$ and $M = \infty$ leaves no freedom, and the three dimensional integral over \vec{p}_3 is merely replaced by one over \vec{q}_1 .

Claesson in 1957 first wrote down the unitarity condition but left it in rather an impractical form. Kessler completed this work in 1958 by evaluating the various traces of the matrices, thereby giving a five dimensional representation for the amplitude. He did not carry out any of the five-dimensional integrations analytically but was content to integrate his result numerically for gamma ray energies of 2.62 Mev. The results were disappointing and showed that the real part of the scattering amplitude must be larger than the imaginary part to explain the experimental curves. This fact can be predicted from the exact forward scattering values, as the imaginary part has a threshold at 1 Mev and only becomes comparable in magnitude to the real part at roughly 10 Mev. This general trend may be expected to hold for finite angles also.

Zernik in 1960 evaluated numerically the imaginary part at 2.52 and 6.14 Mev. His results are more accurate than those of Kessler. Further numerical evaluation using his programme for higher energies seemed impossible due to the difficulty of using five dimensional grids of smaller and smaller dimension. With increase of energy, the intermediate electron positron pair become peaked into

narrower angles entailing the use of a smaller grid in the computer programme. Zernik also stated that "the analytical evaluation of Kessler's integral does not appear to be feasible." Nevertheless some further integrations must be made if one is to find any more information from the unitarity plus dispersion relations approach.

The calculations given here show that it is possible to reduce Kessler's five dimensional integral to a three dimensional one. This result was found in 1962 before the publication of a paper by Sannikov in March 1963. His work goes even further than the author's and is discussed in Appendix I. A three-dimensional notation is used here and the work simplified by using a model which gives the essential mathematical complications without bringing in spin. The introduction of spin could be taken into account and the whole of the imaginary part found as a three dimensional integral, but this would mean a fantastically long and complicated calculation.

The approach used in most text-books of writing down the Feynman integral would be even more difficult. The extra separation of the real and the imaginary part is automatically given by unitarity and dispersion relations.

3. The Imaginary Part of the Delbrück Scattering Amplitude

a) The Unitarity Condition

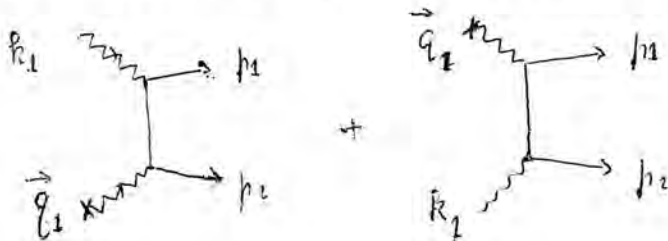
The elastic unitarity condition relates the imaginary part of the Delbrück scattering amplitude to the product of the amplitudes for pair-production and pair-annihilation. Let us use the same notation as Kessler, except for the replacement of his undashed and dashed vectors by vectors with suffixes one and two respectively. Our metric is given by $p^2 = E^2 - \vec{p}^2 = m^2$ and we shall use as Dirac equation $(\gamma_\mu \vec{p}^\mu + m) u = 0$. All three dimensional vectors will be written with an arrow above them. If the T matrix is connected to the S matrix by

$$S = 1 + iT$$

then unitarity, i.e. $SS^\dagger = 1$, becomes, for the T matrix,

$$\text{Im} T_{12} = \frac{1}{2} \sum_n T_{1n} T_{n2}^\dagger \quad (1)$$

T_{1n} is given, in second order perturbation theory, by the pair production amplitude described by following graphs,



which is

$$T_{1n} = \delta(\omega_1 - \omega_n) \frac{iZ\alpha^{3/2}}{2\pi^2} \frac{m}{\sqrt{\omega_1 E_1 E_2}} \frac{1}{|\vec{q}_1|^2} \bar{u} (Q_1^a + Q_1^b) v \quad (2)$$

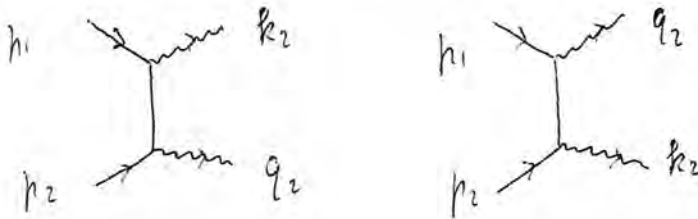
where

$$Q_1^a = \gamma_0 \frac{(\gamma \cdot k_1)(\gamma \cdot \epsilon_1) - 2 \vec{p}_1 \cdot \vec{\epsilon}_1}{2(k_1 \cdot p_1)}$$

and

$$Q_1^b = \gamma_0 \frac{(\gamma \cdot k_1)(\gamma \cdot \epsilon_1) + 2 \vec{p}_1 \cdot \vec{\epsilon}_1}{2(k_1 \cdot p_1)}$$

The pair annihilation cross-section can be found from this by substitution, or written down from the following graphs.



Thus

$$T_{n_2}^+ = \delta(\omega_m - \omega_2) \frac{-iZ\alpha^{3/2}}{2\pi^2} \frac{m}{\sqrt{\omega_2 \epsilon_1 \epsilon_2}} \frac{1}{|\vec{q}_2|^2} \bar{v} (\bar{Q}_2^a + \bar{Q}_2^b) u, \quad (3)$$

where

$$\bar{v} = v^\dagger \gamma_0, \quad \bar{Q}_2^a = \gamma_0 Q_2^a \gamma_0,$$

and

$$Q_2^a = \gamma_0 \frac{(\gamma \cdot k_2)(\gamma \cdot \epsilon_2) - 2 \vec{p}_1 \cdot \vec{\epsilon}_2}{2(k_2 \cdot p_1)}$$

$$Q_2^b = \gamma_0 \frac{(\gamma \cdot k_2)(\gamma \cdot \epsilon_2) + 2 \vec{p}_1 \cdot \vec{\epsilon}_2}{2(k_2 \cdot p_1)}$$

The substitution of (2) and (3) into (1) and evaluation of the traces is rather long and tedious. One finally arrives at the result of Kessler, which is split up into two parts, depending on whether the polarisation vectors of the incident and scattered photons are

parallel or perpendicular to the plane of scattering. The three dimensional coordinate systems used are shown below.

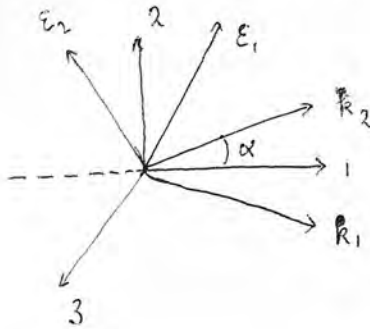


Fig. (a)

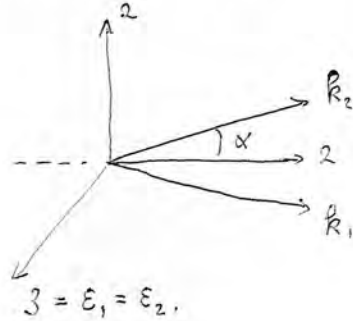


Fig. (b)

If we use the notation that p_{1j} means the j -th component of the three dimensional vector p_1 , then,

$$a_{Im}^{(D)}(\vec{k}_1, \vec{k}_2) = \frac{Z^2 \alpha^2 r_0}{16\pi^3} \int \frac{m p_1 p_2 dE_1 d\Omega_1 d\Omega_2}{|\vec{q}_1|^2 |\vec{q}_2|^2} \bar{X}(k_1, k_2, p_1, p_2, q_1, q_2, E_1, E_2)$$

where

$$\begin{aligned} \bar{X} = & (|\vec{q}_1|^2 + |\vec{q}_2|^2) \left(\frac{\vec{h}_1 \cdot \vec{E}_1}{h_1 \cdot k_1} - \frac{\vec{h}_2 \cdot \vec{E}_1}{h_2 \cdot k_1} \right) \left(\frac{\vec{h}_1 \cdot \vec{E}_2}{h_1 \cdot k_2} - \frac{\vec{h}_2 \cdot \vec{E}_2}{h_2 \cdot k_2} \right) - \\ & - 8 \left(E_2 \frac{\vec{h}_1 \cdot \vec{E}_1}{h_1 \cdot k_1} + E_1 \frac{\vec{h}_2 \cdot \vec{E}_1}{h_2 \cdot k_2} \right) \left(E_2 \frac{\vec{h}_1 \cdot \vec{E}_2}{h_1 \cdot k_2} + E_1 \frac{\vec{h}_2 \cdot \vec{E}_1}{h_2 \cdot k_1} \right) + \\ & + \omega^2 \left(\frac{1}{h_1 \cdot k_1} + \frac{1}{h_2 \cdot k_1} \right) \left((h_{12} + h_{22})^2 + (h_{13} + h_{23})^2 - (E_1 - E_2)^2 \sin^2 \alpha \right) + \\ & + 2\omega^2 \sin^2 \alpha \left(\frac{1}{h_1 \cdot k_1} - \frac{1}{h_2 \cdot k_2} \right) \left(\frac{1}{h_1 \cdot k_2} - \frac{1}{h_2 \cdot k_1} \right) (h_{13} h_{23} - m^2) + \end{aligned}$$

$$\begin{aligned}
 & + 2\omega \sin \alpha \left\{ \frac{\omega \mu_{11} \vec{h}_2 \cdot \vec{E}_2 + \mu_{12} \mu_2 \cdot k_2 - \omega \mu_{11} \vec{h}_1 \cdot \vec{E}_1 - \mu_{12} \mu_1 \cdot k_1}{(\mu_1 \cdot k_1)(\mu_2 \cdot k_2)} + \right. \\
 & \quad \left. + \frac{\omega \mu_{11} \vec{h}_1 \cdot \vec{E}_2 + \mu_{22} \mu_1 \cdot k_2 - \mu_{11} \omega \vec{h}_2 \cdot \vec{E}_1 - \mu_{12} \mu_2 \cdot k_1}{(\mu_1 \cdot k_2)(\mu_2 \cdot k_1)} \right\} + \\
 & + \omega^2 \sin \alpha \left[\left(\frac{\vec{h}_1 \cdot \vec{E}_1}{\mu_1 \cdot k_1} - \frac{\vec{h}_2 \cdot \vec{E}_1}{\mu_2 \cdot k_1} \right) \left(\frac{E_1 \cos \alpha + \mu_{11}}{\mu_2 \cdot k_2} - \frac{E_2 \cos \alpha + \mu_{21}}{\mu_1 \cdot k_2} \right) - \right. \\
 & \quad \left. - \left(\frac{\vec{h}_1 \cdot \vec{E}_2}{\mu_1 \cdot k_2} - \frac{\vec{h}_2 \cdot \vec{E}_2}{\mu_2 \cdot k_2} \right) \left(\frac{E_1 \cos \alpha + \mu_{11}}{\mu_2 \cdot k_1} - \frac{E_2 \cos \alpha + \mu_{21}}{\mu_1 \cdot k_1} \right) \right].
 \end{aligned}$$

(4)

for the figure (a).

The expression for the amplitude when the polarization vectors are perpendicular to the plane is just as long and complicated and, for that reason, it is not reproduced here.

Conservation of energy and momentum is expressed by

$$\vec{k}_1 + \vec{q}_1 = \mu_1 + \mu_2 = \vec{k}_2 + \vec{q}_2$$

Now rewrite equation (4) as

$$A_{Im}^{(D)}(\vec{k}_1, \vec{k}_2) = \frac{Z^2 \omega^2 \gamma_0 m}{16\pi^3} \int \frac{d^3 p_1 d^3 p_2 d^3 q_1 d^3 q_2}{E_1 E_2 |\vec{q}_1|^2 |\vec{q}_2|^2} \delta^{(w)}(\vec{k}_1 + \vec{q}_1 - \vec{k}_2 - \vec{q}_2) \delta^{(w)}(\vec{k}_1 + \vec{q}_1 - \mu_1 - \mu_2) X$$

and split off the integrations over the momenta transfer variables i.e.

$$a_{Im}^{(D)}(\vec{k}_1, \vec{k}_2) = \frac{Z^2 \alpha^2 \tau_0 m}{16 \pi^3} \int \frac{d^3 q_1}{|\vec{q}_1|^2} \cdot \frac{d^3 q_2}{|\vec{q}_2|^2} \delta^{(4)}(\vec{k}_1 + \vec{q}_1 - \vec{k}_2 - \vec{q}_2) A(k_1, k_2, q_1, q_2). \quad (5)$$

where

$$A(k_1, k_2, q_1, q_2) = \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(4)}(\vec{k}_1 + \vec{q}_1 - \vec{p}_1 - \vec{p}_2) X(k_1, k_2, p_1, p_2, q_1, q_2), \quad (6)$$

which is the unitarity condition for photon photon scattering with two particles off the mass shell, $q_1^2 \neq q_2^2 \neq 0$. So long as we fix \vec{q}_1 and \vec{q}_2 , we can then carry out the integrations over \vec{p}_1 and \vec{p}_2 .

X is a very complicated expression but is made up of the contribution from all channels. The terms can be split off according to the products of p_1, p_2 with k_1, k_2 in the denominator. Let us define channel I by

$$A_1 : \quad \vec{k}_1 + \vec{q}_1 \rightarrow \vec{k}_2 + \vec{q}_2$$

so that the denominator is the product $(p_1 \cdot k_1)(p_1 \cdot k_2)$. The other terms then arise from the channel II

$$A_2 : \quad -\vec{k}_2 + \vec{q}_1 \rightarrow -\vec{k}_1 + \vec{q}_2$$

and channel III

$$A_3 : \quad -\vec{q}_2 + \vec{q}_1 \rightarrow \vec{k}_2 - \vec{k}_1$$

If we define s, t and u variables for these three channels (see later) then we can limit our calculations to channel I and find analogous results for channels II and III from crossing

symmetry. The system of integrations becomes schematically

$$\alpha_{\text{Im}}^{(D)}(\omega, t) = \int \frac{d^3 q_1}{|\vec{q}_1|^2} \frac{d^3 q_2}{|\vec{q}_2|^2} \delta^{(4)}(k_1 + \vec{q}_1 - k_2 - \vec{q}_2) A(s, t, q_1, q_2, \omega). \quad (7)$$

with

$$A(s, t, q_1, q_2, \omega) = \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta^{(4)}(k_1 + \vec{q}_1 - p_1 - p_2) X(s, t, q_1, q_2, \omega, p_1, p_2). \quad (8)$$

As a ready expression is available for X in three dimensional space, we shall carry out integrations in three dimensions.

Let

$$\vec{p}_1 = \frac{\vec{u} + \vec{v}}{2}, \quad E_1 = \frac{u_0 + v_0}{2}$$

$$\vec{p}_2 = \frac{\vec{u} - \vec{v}}{2}, \quad E_2 = \frac{u_0 - v_0}{2}$$

and use

$$\frac{1}{E_1 E_2} = 4 \int dE_1 dE_2 \delta(\vec{p}_1^2 + m^2 - E_1^2) \delta(\vec{p}_2^2 + m^2 - E_2^2) \theta(E_1) \theta(E_2).$$

these new delta-functions become, as functions of u and v,

$$\delta(\vec{p}_1^2 + m^2 - E_1^2) = \delta(\vec{u}^2 + \vec{v}^2 + 2\vec{u} \cdot \vec{v} + 4m^2 - u_0^2 - v_0^2 - 2u_0 v_0)$$

$$\delta(\vec{p}_2^2 + m^2 - E_2^2) = \delta(\vec{u}^2 + \vec{v}^2 - 2\vec{u} \cdot \vec{v} + 4m^2 - u_0^2 - v_0^2 + 2u_0 v_0)$$

$$\therefore \delta(\vec{p}_1^2 + m^2 - E_1^2) \delta(\vec{p}_2^2 + m^2 - E_2^2) = \delta(\vec{u} \cdot \vec{v} - u_0 v_0) \delta(u_0^2 + v_0^2 - \vec{u}^2 - \vec{v}^2 - 4m^2)$$

$$\therefore A(k_1, k_2, q_1, q_2) = \int d^3u d^3v du_0 dv_0 \delta(\vec{u} \cdot \vec{v} - u_0 v_0) \delta(u_0^2 + v_0^2 - \vec{u}^2 - \vec{v}^2 - 4m^2) \\ \delta(\vec{u} - \vec{k}_1 + \vec{q}_1) \delta(u_0 - \omega) X(k_1, k_2, \frac{u+v}{2}, \frac{u-v}{2}, q_1, q_2).$$

Introduce the following variables

$$\left. \begin{aligned} \vec{K} &= \vec{k}_1 + \vec{q}_1 = \vec{k}_2 + \vec{q}_2 \\ \vec{L} &= \vec{k}_1 - \vec{q}_1 \\ \vec{M} &= \vec{k}_2 - \vec{q}_2 \end{aligned} \right\} \therefore \begin{aligned} \vec{k}_1 &= \frac{\vec{K} + \vec{L}}{2} \\ \vec{k}_2 &= \frac{\vec{K} + \vec{M}}{2} \end{aligned}$$

and complete the integrations over d^3u and du_0 .

$$\therefore A(k_1, k_2, q_1, q_2) = \int d^3v dv_0 \delta(\omega v_0 - \vec{K} \cdot \vec{v}) \delta(v_0^2 - \vec{K}^2 - \vec{v}^2 + \omega^2 - 4m^2).$$

$$X\left(\frac{\vec{K} + \vec{L}}{2}, \frac{\vec{K} + \vec{M}}{2}, \omega, \frac{\vec{K} + \vec{v}}{2}, \frac{\vec{K} - \vec{v}}{2}, \frac{\omega + v_0}{2}, \frac{\omega - v_0}{2}, \vec{\epsilon}_1, \vec{\epsilon}_2\right)$$

The important terms are now the denominators. Collect together all the terms with the factor $(p_1 \cdot k_1)(p_1 \cdot k_2)$ in the denominator and define

$$X_I = \frac{f_I}{p_1 \cdot k_1 p_1 \cdot k_2}$$

This means that we are only considering channel I.

Then

$$A_I = \int \frac{d^3v dv_0 \delta(\omega v_0 - \vec{K} \cdot \vec{v}) \delta(v_0^2 + \omega^2 - 4m^2 - \vec{K}^2 - \vec{v}^2)}{\left[\left(\frac{\omega + v_0}{2}\right)\omega - \left(\frac{\vec{K} + \vec{v}}{2}\right)\left(\frac{\vec{K} + \vec{L}}{2}\right)\right] \left[\left(\frac{\omega + v_0}{2}\right)\omega - \left(\frac{\vec{K} + \vec{v}}{2}\right)\left(\frac{\vec{K} + \vec{M}}{2}\right)\right]} \\ \times f_I\left(\frac{\vec{K} + \vec{L}}{2}, \frac{\vec{K} + \vec{M}}{2}, \frac{\vec{K} + \vec{v}}{2}, \frac{\vec{K} - \vec{v}}{2}, \vec{\epsilon}_1, \vec{\epsilon}_2, \omega, \frac{\omega + v_0}{2}, \frac{\omega - v_0}{2}\right)$$

One delta function fixes the value of ψ_0 , which may be integrated over immediately. If we next introduce polar and azimuthal angles θ and ϕ with respect to the direction \vec{K} , the value of $\cos \theta$ is fixed by the second delta function. The final two integrals are over the modulus of ψ and ϕ .

$$A_I = \frac{\int v^2 dv d(\omega\theta) d\phi \delta(\cos\theta - \frac{\omega}{|K||v|} \sqrt{v^2 + K^2 + 4m^2 - \omega^2})}{2|K||v| \sqrt{v^2 + |K|^2 + 4m^2 - \omega^2} \left\{ \omega^2 - \frac{K^2}{2} - \frac{\vec{K} \cdot \vec{L}}{2} + \frac{(\vec{K} - \vec{L}) \cdot \vec{v}}{2} \right\} \left\{ \omega^2 - \frac{K^2}{2} - \frac{\vec{K} \cdot \vec{M}}{2} + \frac{\vec{v} \cdot (\vec{K} - \vec{M})}{2} \right\}}$$

$$\star \int_I \left(\frac{\vec{K} + \vec{L}}{2}, \frac{\vec{K} + \vec{M}}{2}, \frac{\vec{K} + \vec{v}}{2}, \frac{\vec{K} - \vec{v}}{2}, \epsilon_1, \epsilon_2, \omega, \frac{\omega^2 + \vec{K} \cdot \vec{v}}{2\omega}, \frac{\omega^2 - \vec{K} \cdot \vec{v}}{2\omega} \right)$$

(9)

Obviously the basic integrals we need to know are those over the denominators. Any terms in the numerator are reducible using partial fractions, so we put $f_I = 1$ for the time being. The terms in the numerator only appear when spin is involved, so if we neglect spin, we are justified in replacing f_I by unity. The dimension of $|v|$ which comes into the numerator in an exact calculation is at most four, so, for a complete knowledge of this integral we need also other integrals involving the components of \vec{v} along the vectors \vec{K} , \vec{L} and \vec{M} . These other integrals can sometimes be reduced by partial fractions but can all be calculated exactly if enough time and patience is available.

B) Kinematics

Define the invariants as follows, (metric $p^2 = E^2 - p^2 = m^2$).

$$\begin{aligned}
 s &= (\vec{k}_1 + \vec{q}_1)^2 = -2 \vec{k}_1 \cdot \vec{q}_1 - \vec{q}_1^2 = -2 \vec{k}_2 \cdot \vec{q}_2 - \vec{q}_2^2. \\
 t &= +(\vec{k}_1 - \vec{k}_2)^2 = 2\omega^2(1 - \cos\phi) \\
 u &= (\vec{k}_1 - \vec{q}_2)^2 = 2 \vec{k}_1 \cdot \vec{q}_2 - \vec{q}_2^2 = 2 \vec{k}_2 \cdot \vec{q}_1 - \vec{q}_1^2.
 \end{aligned} \tag{10}$$

$$\therefore s - t + u = -\vec{q}_1^2 - \vec{q}_2^2.$$

We now express the products of K, L, M in terms of s, t, q_1^2, q_2^2 and ω .

$$\vec{K}^2 = \omega^2 - s$$

$$\vec{L}^2 = \omega^2 + 2\vec{q}_1^2 + s.$$

$$\vec{M}^2 = \omega^2 + 2\vec{q}_2^2 + s.$$

$$\vec{K} \cdot \vec{L} = \omega^2 - \vec{q}_1^2.$$

$$\vec{K} \cdot \vec{M} = \omega^2 - \vec{q}_2^2.$$

$$\vec{L} \cdot \vec{M} = s - 2t + \vec{q}_1^2 + \vec{q}_2^2 + \omega^2.$$

The final result for the Delbrück scattering amplitude is obviously only a function of ω and the scattering angle ϕ or, alternatively, of ω and t . However, before integration over d^3q_1 the amplitude A_I is a function of s, t, q_1^2 and q_2^2 . Hence the integral can be replaced by three integrals over $d(q_1^2), d(q_2^2)$ and ds .

Let us fix the value of $\vec{q}_2 = \vec{k}_1 + \vec{q}_1 - \vec{k}_2$. This leaves \vec{q}_1 as the momentum transfer variable. It is important to note that $|\vec{q}_1|$ can never vanish, or scattering does not occur. The largest and smallest values of q_1 are given by the maximum and minimum of the relation,

$$\vec{q}_1 = \vec{p}_1 + \vec{p}_2 - \vec{k}_1$$

which depend on the vectors being antiparallel and parallel respectively.

$$\begin{aligned} |\vec{q}_1| &= |p_1 + p_2 \pm k_1| \\ &= \sqrt{E_1^2 - m^2} + \sqrt{E_2^2 - m^2} \pm \omega \quad (\omega = E_1 + E_2) \end{aligned}$$

$$|\vec{q}_1|^2 = \pm 2\omega q_1 - 2m^2 + 2\sqrt{E_1^2 - m^2}\sqrt{E_2^2 - m^2} - 2E_1E_2$$

Now assume that $E_1 \approx E_2$ and both are larger than m . We expand the square roots and solve the resulting quadratic for $|\vec{q}_1|$.

$$|\vec{q}_1| = \pm \omega \pm \sqrt{\omega^2 - 4m^2}$$

Hence as the modulus of $|\vec{q}_1|$ must be positive

$$\omega - \sqrt{\omega^2 - 4m^2} \leq |\vec{q}_1| \leq \omega + \sqrt{\omega^2 - 4m^2} \quad (11)$$

These upper and lower bounds on $|\vec{q}_1|$, combined with the fact that the energy ω must be greater than $2m$, give restrictions on the values of S . Let ρ be the angle between the vectors \vec{q}_1 and \vec{k}_1 . Then

$$S = -2\omega|\vec{q}_1| \cos \delta - |\vec{q}_1|^2 \quad \text{and} \quad S \geq 4m^2.$$

and so

$$\left. \begin{aligned} S_{\max} &= -|\vec{q}_1|^2 + 2\omega|\vec{q}_1| \\ S_{\min} &= -|\vec{q}_1|^2 - 2\omega|\vec{q}_1| \geq 4m^2 \end{aligned} \right\} \quad (12)$$

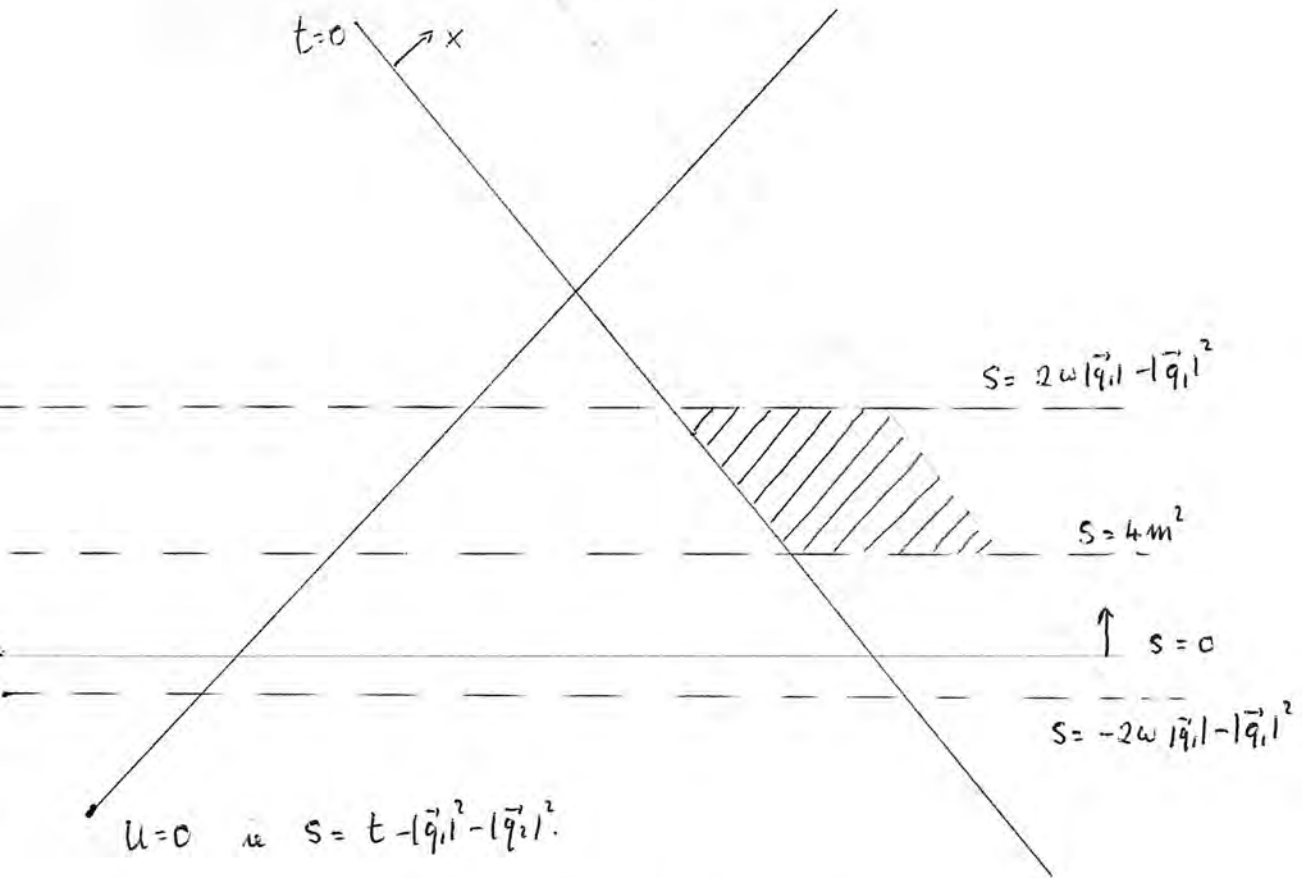
Alternatively if we integrate over the angle ρ instead of S , the limits of integration are

$$-1 \leq \cos \delta \leq \frac{-(4m^2 + |\vec{q}_1|^2)}{2\omega|\vec{q}_1|}. \quad (13)$$

For forward scattering $t = 0$, and for backward scattering $t = +4\omega^2$. The transition from channel I to channel II does not alter the t variable but changes s into u . Unfortunately, the amplitude in the third channel is not simply related to the amplitudes in channels one or two and will have to be calculated separately. If we use homogeneous coordinates, the range of s and t for fixed ω and q_1^2 for channel I is given by the following diagram.

$$t = 4\omega^2$$

$$t=0 \rightarrow x$$



C) Integration of the Unitarity Condition

The problem in hand is to carry out the integration of equation (1). Choose the following three dimensional coordinate system

$$\vec{r} = |\vec{r}| (\cos \theta, \sin \theta \cos \varphi, \sin \theta \sin \varphi).$$

$$\vec{K} = |\vec{K}| (1, 0, 0).$$

$$\vec{L} = |\vec{L}| (\cos \alpha, \sin \alpha, 0).$$

$$\vec{M} = |\vec{M}| (\cos \beta, \sin \beta \cos \delta, \sin \beta \sin \delta).$$

where the values of α , β and γ are easily found to be,

$$\cos \alpha = \frac{\vec{K} \cdot \vec{L}}{|\vec{K}| |\vec{L}|} = \frac{\omega^2 - q_1^2}{\sqrt{\omega^2 - s} \sqrt{\omega^2 + 2q_1^2 + s}}$$

$$\cos \beta = \frac{\vec{K} \cdot \vec{M}}{|\vec{K}| |\vec{M}|} = \frac{\omega^2 - q_2^2}{\sqrt{\omega^2 - s} \sqrt{\omega^2 + 2q_2^2 + s}}$$

$$\cos \gamma = \frac{2\omega^2(q_1^2 + q_2^2 - t) + 2st - (s + q_1^2)(s + q_2^2)}{\sqrt{4q_1^2\omega^2 - (s + q_1^2)^2} \sqrt{4q_2^2\omega^2 - (s + q_2^2)^2}}$$

Let

$$a_1^2 = 4q_1^2\omega^2 - (s + q_1^2)^2$$

$$a_2^2 = 4q_2^2\omega^2 - (s + q_2^2)^2$$

and

$$\cos \gamma = \frac{A}{a_1 a_2}$$

The projections of \vec{v} along the directions of \vec{K} , \vec{L} and \vec{M} are

$$\vec{v} \cdot \vec{K} = \omega \sqrt{v^2 + 4m^2 - s}$$

$$\vec{v} \cdot \vec{L} = \frac{\omega(\omega^2 - q_1^2)}{\omega^2 - s} \sqrt{v^2 + 4m^2 - s} + \frac{a_1 \sqrt{\omega^2 s - v^2 s - 4m^2 \omega^2}}{\omega^2 - s} \cos \varphi \quad (14)$$

$$\begin{aligned} \vec{v} \cdot \vec{M} = & \frac{\omega(\omega^2 - q_2^2)}{\omega^2 - s} \sqrt{v^2 + 4m^2 - s} + \frac{A}{a_1} \frac{\sqrt{\omega^2 s - v^2 s - 4m^2 \omega^2}}{\omega^2 - s} \cos \varphi + \\ & + \frac{\sqrt{a_1^2 a_2^2 - A^2}}{(\omega^2 - s) a_1} \sqrt{\omega^2 s - v^2 s - 4m^2 v^2} \sin \varphi. \end{aligned}$$

The introduction of a new variable

$$X^2 = v^2 + 4m^2 - S$$

gives

$$A_I = \int \frac{2 dx d\varphi}{\sqrt{\omega^2 - S}} \cdot \frac{1}{(S + q_1^2 + \vec{v} \cdot (\vec{K} - \vec{L}))} \cdot \frac{1}{(S + q_2^2 + \vec{v} \cdot (\vec{K} - \vec{M}))}$$

with

$$\vec{v} \cdot \vec{K} = \omega x.$$

$$\vec{v} \cdot \vec{L} = \frac{\omega(\omega^2 - q_1^2)}{\omega^2 - S} x + \frac{a_1 W \cos \varphi}{\omega^2 - S}$$

$$\begin{aligned} \vec{v} \cdot \vec{M} = \frac{\omega(\omega^2 - q_2^2)}{\omega^2 - S} x + \frac{A W \cos \varphi}{(\omega^2 - S)} + & (15) \\ & + \frac{\sqrt{a_1^2 a_2^2 - A^2}}{a_1 (\omega^2 - S)} W \sin \varphi. \end{aligned}$$

where

$$W = + \sqrt{(\omega^2 - S)(S - 4m^2) - x^2 S}.$$

Now define

$$J_1 = \int \frac{2 dx d\varphi}{\sqrt{\omega^2 - S} [S + q_1^2 + \vec{v} \cdot (\vec{K} - \vec{L})]} \quad (16)$$

$$J_2 = \int \frac{2 dx d\varphi}{\sqrt{\omega^2 - S} [S + q_2^2 + \vec{v} \cdot (\vec{K} - \vec{M})]} \quad (17)$$

$$J_3 = \int \frac{2 dx d\varphi}{\sqrt{\omega^2 - s} \{s + q_1^2 + \vec{e} \cdot (\vec{K} - \vec{L})\} \{s + q_2^2 + \vec{e} \cdot (\vec{K} - \vec{M})\}} \quad (18)$$

Consider first J_1 . The integration over $d\varphi$ is a standard one.

$$\int_0^{2\pi} \frac{d\varphi}{A + B \cos \varphi} = \frac{2\pi}{\sqrt{A^2 - B^2}} \quad (A^2 > B^2)$$

$$\therefore J_1 = 2\pi \int_{-\sqrt{e}}^{+\sqrt{e}} \frac{dx (\omega^2 - s)^{3/2}}{C^2(x)}$$

where $e = (\omega^2 - s) \left(1 - \frac{4m^2}{s}\right)$

and $C^2(x)$ is a quadratic function in x .

$$C^2(x) = (\omega^2 - s)(s + q_1^2)^2 x^2 + 2(\omega^2 - s)(s + q_1^2)(q_1^2 - s)\omega x + (\omega^2 - s) \left[(s + q_1^2)^2 (\omega^2 - 4m^2) - 4q_1^2 \omega^2 (s - 4m^2) \right] \quad (19)$$

Now use

$$\int_{-\sqrt{e}}^{+\sqrt{e}} \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \ln \left(2\sqrt{a} \sqrt{ax^2 + bx + c} + 2ax + b \right) \Big|_{-\sqrt{e}}^{+\sqrt{e}}$$

where a , b , and c are the coefficients of x in equation (19).

Substituting the values of a , b , c and e leads to a long complicated expression. The following values are given for reference.

$$2a\sqrt{e+b} + 2\sqrt{a}\sqrt{ae+b\sqrt{e}+c} = 2(\omega^2-s)(s+q_1^2) \left[(s+q_1^2)\sqrt{\omega^2-s} + \omega(q_1^2-s) \right] \left[1 + \sqrt{1 - \frac{4m^2}{s}} \right],$$

$$-2a\sqrt{e+b} + 2\sqrt{a}\sqrt{ae-b\sqrt{e}+c} = 2(\omega^2-s)(s+q_1^2) \left[(s+q_1^2)\sqrt{\omega^2-s} + \omega(q_1^2-s) \right] \left[1 - \sqrt{1 - \frac{4m^2}{s}} \right].$$

The amount of cancellation in this result is surprising. Finally, we find

$$\overline{J}_1 = \frac{4\pi}{s+q_1^2} \ln \left| \frac{1 + \sqrt{1 - \frac{4m^2}{s}}}{1 - \sqrt{1 - \frac{4m^2}{s}}} \right| \quad (20)$$

and, similarly,

$$\overline{J}_2 = \frac{4\pi}{s+q_2^2} \ln \left| \frac{1 + \sqrt{1 - \frac{4m^2}{s}}}{1 - \sqrt{1 - \frac{4m^2}{s}}} \right| \quad (21)$$

which are independent of ω .

The evaluation of J_3 is considerably more complicated. Use first

$$s+q_1^2 + \vec{v} \cdot (\vec{K} - \vec{L}) = \frac{(\omega^2-s)(s+q_1^2) + \omega x(q_1^2-s) + a_1 \sqrt{(s-4m^2)(\omega^2-s) - x^2 s} \cos\varphi}{(\omega^2-s)}$$

$$s+q_2^2 + \vec{v} \cdot (\vec{K} - \vec{M}) = \frac{(\omega^2-s)(s+q_2^2) a_1 + \omega x(q_2^2-s) a_1 - A \sqrt{(s-4m^2)(\omega^2-s) - x^2 s} \cos\varphi}{(\omega^2-s) a_1}$$

$$- \frac{\sqrt{a_1^2 a_2^2 - A^2} \sqrt{(s-4m^2)(\omega^2-s) - x^2 s} \sin\varphi}{(\omega^2-s) a_1}.$$

so that the ϕ integration is of the standard type

$$\int_0^{\pi} \frac{d\phi}{(A_1 + B_1 \cos \phi)(A_2 + B_2 \cos \phi + C_2 \sin \phi)} =$$

$$= \frac{B_1 (B_1 A_2 - A_1 B_2)}{(A_1 B_2 - A_2 B_1)^2 - (B_1 C_2)^2 + (A_1 C_2)^2} \cdot \frac{\pi}{\sqrt{A_1^2 - B_1^2}} +$$

$$+ \frac{C_2 A_1 C_2 + B_2 (A_1 B_2 - B_1 A_2)}{(A_1 B_2 - A_2 B_1)^2 - (B_1 C_2)^2 + (A_1 C_2)^2} \cdot \frac{\pi}{\sqrt{A_2^2 - B_2^2 - C_2^2}} \quad (22)$$

for $A_1^2 > B_1^2$; $A_2^2 > B_2^2 + C_2^2$.

The expression for $A_1^2 - B_1^2$ has already been given in equation (9). We need now that

$$A_2^2 - B_2^2 - C_2^2 = a_1^2 \left\{ x^2 (\omega^2 (q_1^2 - s) + a_2^2 s) + 2\omega x (q_1^2 - s) (\omega^2 - s) (s + q_2^2) + (\omega^2 - s) [(\omega^2 - s)(s + q_2^2) - (s - 4m^2) a_2^2] \right\} \quad (23)$$

After substitution of all the values of A_1 , B_1 , A_2 , B_2 and C_2 in equation (10) the integrals over x belong to standard types involving the square root of a quadratic in x in the denominator. One needs the following results.

$$\int \frac{(A + Bx) dx}{(g^2 + hx + i) \sqrt{ax^2 + sx + d}} = \frac{A - \beta B}{g - \beta} \frac{1}{2 \sqrt{(d - \beta)^2 + \alpha \beta^2}}$$

$$\left[\ln \left\{ \frac{2(d - \beta)^2 + \alpha \beta^2 + (x + \beta)(s - 2\alpha \beta) - 2 \sqrt{(ax^2 + sx + d)(d - \beta)^2 + \alpha \beta^2}}{2(d - \beta)^2 + \alpha \beta^2 + (x + \beta)(s - 2\alpha \beta) + 2 \sqrt{(ax^2 + sx + d)(d - \beta)^2 + \alpha \beta^2}} \right\} \right] +$$

$$\rightarrow \frac{A - qB}{h - q} \cdot \frac{1}{2} \frac{1}{\sqrt{\gamma - \beta q + \alpha q^2}}$$

$$\cdot \ln \left| \frac{2(\gamma - \beta q + \alpha q^2) + (x+q)(\beta - 2\alpha q) - 2\sqrt{(\alpha x^2 + \beta x + \delta)(\gamma - \beta q + \alpha q^2)}}{2(\gamma - \beta q + \alpha q^2) + (x+q)(\beta - 2\alpha q) + 2\sqrt{(\alpha x^2 + \beta x + \delta)(\gamma - \beta q + \alpha q^2)}} \right|$$

with
$$h = \frac{-h + \sqrt{h^2 - 4ig}}{2g}, \quad q = \frac{-h - \sqrt{h^2 - 4ig}}{2g}, \quad (24)$$

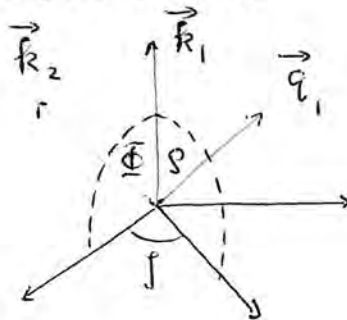
The amount of algebra involved in the substitution of the various values of A, B, g, h, i, α , β and γ into (11) is rather large. However, many cancellations occur, and the final answer reduces to

$$J_3 = \frac{4\pi}{2\sqrt{t}\sqrt{\xi}} \ln \left| \frac{\sqrt{st(s-4m^2)} + 2m(s+q_1^2) + \sqrt{\xi} \cdot \sqrt{st(s-4m^2) - 2m(s+q_1^2)} + \sqrt{\xi}}{\sqrt{st(s-4m^2)} + 2m(s+q_1^2) - \sqrt{\xi} \cdot \sqrt{st(s-4m^2) - 2m(s+q_1^2)} - \sqrt{\xi}} \right| \quad (25)$$

with
$$\xi = (s-4m^2)st + (s+q_1^2)(s+q_2^2)$$

a result which also does not depend explicitly on ω .

Suppose for a trial we substitute (23) for A into (5). The integration over d^3q_2 is immediate, and we are then left with three integrations over d^3q_1 . Take a three dimensional vector system with \vec{k}_1 as pole vector.



Denote the angles of \vec{q}_1 with respect to \vec{k}_1 and the \vec{k}_1, \vec{k}_2 plane by ϑ and \int respectively. Then equation (5) becomes

$$A_{Im}^D(\vec{k}_1, \vec{k}_2) = \frac{Z^2 \alpha^2 r_0 m}{16 \pi^3} \int \frac{dq_1 q_1^2 d(\cos \rho) d\int}{|\vec{q}_1|^2 |\vec{k}_1 + \vec{q}_1 - \vec{k}_2|^2} J_3(\vec{k}_1, \vec{k}_2, \vec{q}_1, \vec{q}_2). \quad (26)$$

where we have neglected the spins. The angle ρ is limited by (13) i.e.

$$-1 \leq \cos \rho \leq -\frac{(4m^2 + q_1^2)}{2\omega |\vec{q}_1|} \quad (13)$$

and the modulus of q_1 is restricted to lie between

$$\omega - \sqrt{\omega^2 - 4m^2} \leq |q_1| \leq \omega + \sqrt{\omega^2 - 4m^2}. \quad (11)$$

Define new variables, to remove the mass of the electron from our expressions:

$$Q = \frac{q}{2m}, \quad \gamma = \frac{\omega}{2m}, \quad S = \frac{s}{4m^2}, \quad T = \frac{t}{4m^2}. \quad (27)$$

Hence (11) and (13) become respectively

$$\gamma - \sqrt{\gamma^2 - 1} \leq Q \leq \gamma + \sqrt{\gamma^2 - 1}. \quad (28)$$

$$-1 \leq \cos \rho \leq -\frac{1+Q^2}{2\gamma Q} \quad (29)$$

$$\text{corresponding to } 1 \leq S \leq 2\gamma\delta - Q^2. \quad (30)$$

If Φ is the scattering angle

$$\cos \bar{\Phi} = 1 - \frac{T}{2\delta^2} \quad , \quad \sin^2 \bar{\Phi} = \frac{T}{\delta^2} \left(1 - \frac{T}{4\delta^2} \right)$$

and from

$$|\vec{q}_2|^2 = |\vec{k}_1 + \vec{q}_1 - \vec{k}_2|^2 \quad , \quad \text{we find}$$

$$Q_2^2 = T - S + (S + Q_1^2) \left(1 - \frac{T}{2\delta^2} \right) + \sqrt{\left(4\delta^2 Q_1^2 - (S + Q_1^2)^2 \right) \left(\frac{T}{\delta^2} \left[1 - \frac{T}{4\delta^2} \right] \right)} \cos \int$$

$$\begin{aligned} \therefore d\int &= \frac{-d(Q_2^2)}{\sqrt{\frac{T}{\delta^2} \left(1 - \frac{T}{4\delta^2} \right) \left(4\delta^2 Q_1^2 - (S + Q_1^2)^2 \right) - \left[(S + Q_2^2) - T + (S + Q_1^2) \left(1 - \frac{T}{2\delta^2} \right) \right]^2}} \\ &= \frac{-d(Q_2^2)}{\sqrt{B(S, T, Q_1^2, Q_2^2)}} \end{aligned}$$

and

$$d(\cos \int) = \frac{-dS'}{2Q_1 \delta}$$

Hence in (26) we replace the $d(\cos \int)$ and $d\int$ integrations by integrations over dS' and $d(Q_2^2)$ respectively

$$a_{Im}^{(D)}(\gamma, T) = \frac{Z^2 \alpha^2 \gamma_0 m^2}{16\pi^3} \frac{1}{\delta} \frac{4\pi}{2\sqrt{T}} \int \frac{d(Q_1^2)}{Q_1} \frac{d(Q_2^2)}{\sqrt{B(S, T, Q_1^2, Q_2^2)}} \frac{dS'}{Q_1 \delta} \frac{J_3(S, Q_1^2, Q_2^2, T)}{Q_2^2} \frac{2\sqrt{T}}{4\pi}$$

Written out in full,

$$a_{Im}^{(D)}(\gamma, T) = \frac{Z^2 \alpha^2 \gamma_0 m^2}{16\pi^3} \frac{1}{\delta} \frac{4\pi}{2\sqrt{T}} \int \frac{d(Q_1^2)}{Q_1^2} \frac{d(Q_2^2)}{Q_2^2} \frac{dS'}{\dots}$$

$$e_{Im} \left\{ \frac{ST(S-1) + (S+Q_1^2) + \sqrt{ST(S-1) + (S+Q_1^2)(S+Q_2^2)}}{ST(S-1) + (S+Q_1^2) - \sqrt{ST(S-1) + (S+Q_1^2)(S+Q_2^2)}} \cdot \frac{ST(S-1) - (S+Q_1^2) + \sqrt{ST(S-1) + (S+Q_1^2)(S+Q_2^2)}}{ST(S-1) - (S+Q_1^2) - \sqrt{ST(S-1) + (S+Q_1^2)(S+Q_2^2)}} \right\} \\ \sqrt{ST(S-1) + (S+Q_1^2)(S+Q_2^2)} \sqrt{\frac{T}{\delta^2} \left(1 - \frac{T}{4\delta^2} \right) \left(4\delta^2 Q_1^2 - (S+Q_1^2)^2 \right) - \left[(S+Q_2^2) - T - (S+Q_1^2) \left(1 - \frac{T}{2\delta^2} \right) \right]^2}$$

where

$$T - S + (S + \varphi_1^2) \left(1 - \frac{T}{2\gamma^2}\right) - \sqrt{4\gamma^2 \varphi_1^2 - (S + \varphi_1^2)^2} \sqrt{\frac{T}{\delta^2} \left(1 - \frac{T}{4\delta^2}\right)} \leq \varphi_2^2 \leq T - S + (S + \varphi_1^2) \left(1 - \frac{T}{2\delta^2}\right) + \sqrt{4\gamma^2 \varphi_1^2 - (S + \varphi_1^2)^2} \sqrt{\frac{T}{\delta^2} \left(1 - \frac{T}{4\delta^2}\right)}.$$

$$1 \leq S \leq 2\varphi_1\gamma - \varphi_1^2.$$

$$\text{and } \gamma - \sqrt{\gamma^2 - 1} \leq \varphi_1^2 \leq \gamma + \sqrt{\gamma^2 - 1}.$$

Any further integrations are impossible as there are just too many square roots in (31). There are now no essential difficulties involved in finding an exact expression for the imaginary part from equation (9). One needs to express the vectors shown in the polarization diagram in terms of the vectors \vec{K} , \vec{L} and \vec{M} , which is elementary

$$\vec{\varepsilon}_1 = (\varepsilon_1, \varphi_1) \frac{S + \varphi_1^2 - 2\omega^2}{4\varphi_1^2\omega^2 - (S + \varphi_1^2)^2} \vec{L} + \frac{S + \varphi_1^2 + 2\omega^2}{4\varphi_1^2\omega^2 - (S + \varphi_1^2)^2} (\varepsilon_1, \varphi_1) \vec{K}$$

$$\vec{\varepsilon}_2 = (\varepsilon_2, \varphi_2) \frac{S + \varphi_2^2 + 2\omega^2}{4\varphi_2^2\omega^2 - (S + \varphi_2^2)^2} \vec{K} + \frac{S + \varphi_2^2 - 2\omega^2}{4\varphi_2^2\omega^2 - (S + \varphi_2^2)^2} (\varepsilon_2, \varphi_2) \vec{M}.$$

so in principle we know all the scalar products of $\vec{k}_1, \vec{k}_2, \vec{p}_1, \vec{p}_2, \vec{\varepsilon}_1$ and $\vec{\varepsilon}_2$ in terms of the other vectors $\vec{v}, \vec{K}, \vec{L}, \vec{M}$.

However, it is obvious that the substitution of these scalar products into (9) and evaluation of the integrals involved would be fantastically long so no further work was done in this direction.

The Real Part of the Delbrück Scattering Amplitude.

The total amplitude for Delbrück scattering is a function of k_1 and k_2 only. It must be made up of the following invariants

$$T_{\mu\nu}(k_1, k_2) = A \delta_{\mu\nu} + B k_{1\mu} k_{2\nu} + C k_{1\nu} k_{2\mu}.$$

However gauge invariance imposes the following restrictions;

$$k_{1\mu} T_{\mu\nu}(k_1, k_2) = k_{2\nu} T_{\mu\nu}(k_1, k_2) = 0.$$

Thus we find

$$e_{1\mu} e_{2\nu} T_{\mu\nu}(k_1, k_2) = g(\omega, \cos \Phi) \left\{ (e_1 \cdot e_2) - \frac{(k_1 \cdot e_2)(k_2 \cdot e_1)}{(k_1 \cdot k_2)} \right\}.$$

where Φ is the scattering angle and e_1, e_2 are the polarization vectors of the incoming and outgoing photons respectively.

A knowledge of the imaginary part of $g(\omega, \cos \Phi)$ would be sufficient to yield the real part through one dispersion relation at fixed momentum transfer. If one wanted to find the real part of the virtual photon photon scattering amplitude, one integration would not be sufficient. The amplitude is then a function of several variables

$$T_{\mu\nu\alpha\beta}(k_1, k_2, q_1, q_2) = A \delta_{\mu\nu} + B k_{1\mu} k_{2\nu} + C k_{2\mu} k_{1\nu} + D q_{1\mu} k_{1\nu} + \dots$$

and gauge invariance only reduces this number slightly. A separate dispersion relation would then be required for each independent coefficient. Unfortunately, this approach cannot be carried out

analytically as the dispersion integrals would involve functions like J_3 integrated over S with an extra $S' - S$ in the denominator. Such integrals are also not expressible in terms of analytic functions.

Conclusion

The conclusion of this work is really very small. It is that the five dimensional integral for the imaginary part of the Delbrück scattering amplitude allows two further integrations to be carried out analytically, before it is absolutely necessary to resort to machine computation. Whether it would be quicker to carry out the work fully, i.e. obtain a three dimensional integral representation for the imaginary part, and then integrate numerically, or to find another computing programme, which allows one to integrate the five dimensional integral at higher energies is another question.

APPENDIX I

A paper appeared by Sannikov in March 1963 in which he also used the unitarity plus dispersion relations approach. He evaluated the integrals given previously, using only four dimensional vectors and has given an expression for the high energy behaviour of the imaginary part of the amplitude. This was done by taking only terms in the numerator which do not involve the mass. The dispersion relation in g can be done analytically under the assumption that $\Phi \gg \frac{m}{\omega}$, (large angle scattering). He finally arrived at expressions for the real and imaginary part of the amplitude for large angle scattering and found that the real part of the amplitude is greater than the imaginary part in this region. This result is interesting but not particularly useful as, at high energies, it is the forward diffraction peak which can be measured experimentally. The backward scattering is much smaller and no experiments have ever been performed to measure it.

The work of Sannikov goes further than that of the author as he has evaluated more terms in the unitarity condition. However, they will all have to be calculated if information is to be found on the behaviour of the amplitude for intermediate energies and angles.

APPENDIX II

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A NEW APPROACH TO PION-NUCLEON SCATTERING USING TWO TIME

GREEN'S FUNCTIONS

INTRODUCTION

The description of bound states in a relativistic field theory involves considerable complications not met in the non-relativistic limit. Bethe and Salpeter derived an equation which has since been investigated by several authors. Their equation, although being completely relativistic, suffers from a serious defect, because the dependence of the amplitude on the relative time coordinate of the two incoming particles is not understood. Hence the usual procedure is to work in the instantaneous interaction approximation where this time difference is put equal to zero. Even in this approximation the Bethe Salpeter equation has only been solved for certain special cases, and no complete solution is known.

The calculations reported here are based on the idea of reducing the four dimensional Bethe Salpeter equation to a simpler equation which is then applied to pion nucleon scattering. Using the orthogonality relation for two time Green's functions, which automatically exclude the relative time coordinate, a three dimensional equation may be derived.

Two time Green's functions have been used in many body theory and were first applied to problems in quantum field theory by A.A. Logunov and A.N. Tavkhelidze (1963). When the author was on leave of absence from the University of Edinburgh at the Joint Institute for Nuclear Research in the Soviet Union, the application of two time Green's function methods to pion nucleon scattering was suggested to him as an interesting problem by A.A. Logunov and he then carried out the work reported here.

Chapter 1 contains the definition of two time Green's functions and the reduction of the Bethe Salpeter equation to a new three dimensional equation called the "generalized" Lippmann-Schwinger equation. Edwards and Matthews applied the usual Lippmann-Schwinger equation to pion nucleon scattering in 1957, taking for their potential the Born term scattering amplitude from Chew-Low theory. In this fixed source approximation, the integral equation becomes a system of algebraic equations for the partial wave amplitudes, and exhibits a resonance in the scattering state where the isotopic spin and the angular momentum have values of three halves. However, even after one subtraction has been performed, the integrals involved require cut-offs. The resonance predicted here is, of course, the famous three-three resonance of pion nucleon scattering and occurs at an energy of approximately 190 Mev in the laboratory system of coordinates.

A drawback against using the usual Lippmann-Schwinger equation is that it does not satisfy the restriction of unitarity, which is now recognized to play an important role in strong interaction physics. That our generalized Lippmann-Schwinger equation does satisfy the unitarity condition is shown in the second chapter. Unitarity is also not satisfied by the equation of Tamm and Dancoff, so we expect our equation to yield better values of the phase shifts than those derived by the above mentioned equation.

What connection is there between our approach and that of applying dispersion relations? When combined with unitarity, dispersion relations provide a system of non-linear coupled integral equations for the partial wave amplitudes of scattering

processes. Leaving aside the question of uniqueness of their solution, non-linear equations are technically more difficult to solve than linear equations. Our new equation, being linear, splits up into independent linear equations for each of the partial wave amplitudes and so does not seem to be connected at all with the dispersion theoretic approach. However, upon solving this system of equations by the usual determinantal method of Fredholm, one essentially arrives at a dispersion relation for the denominator function. This dispersion relation is different to the dispersion relation for the denominator function, which follows from the determinantal method of Baker, (applied to pion nucleon scattering by Bali, Garibotti, Giambiagi and Pignotti in 1961) and the "bootstrap" philosophy of Chew and Frautschi. Chapter 3 contains a more detailed discussion of the partial wave decomposition, while a comparison of the various dispersion relations is left to the final chapter, Chapter 5.

We must also consider the possible choices of our "potential" to be used in solving the generalized Lippmann-Schwinger equation. This potential is identified with certain graphs whose field theoretical partial wave amplitudes are calculated in Chapter 4. A general fact known to theoreticians working with pion nucleon scattering is that the crossed nucleon pole is responsible for the major part of the three-three resonance and it is suspected that the finer features follow from higher order graphs. We may simulate the effect of these higher order graphs by adding the contribution of a spin three halves intermediate particle and, rather than replacing the cuts in the complex plane of the total energy in the

centre of mass system by a set of suitably chosen poles, prefer to solve the equation completely. Unfortunately, at the time of writing this thesis, the results of the numerical computations are not available. However, as the author is not involved in the programming, it is a pity to hold up the submission of this thesis, when his particular part of the work is completed. A detailed paper will be published in due course, together with P.S. Isaev, who kindly checked through these calculations, and undertook the supervision of the computing work when it was time for the author to return to the United Kingdom.

If the final results for the position and width of the three-three resonance are satisfactory it is an easy matter to find the actual phase shifts for pion nucleon scattering. A slight change of the isotopic spin indices then allows one to discuss pion, hyperon, or any meson baryon scattering. Nevertheless, if the results are encouraging, the author feels that this approach has inherent difficulties. The presence of a square root seriously complicates the analytic properties of the scattering amplitude and he suspects that the scattering amplitude is just not a simple enough function to satisfy a linear integral equation. In other words, the approach used here is equivalent to saying that, in certain regions of the complex energy plane, the difference between linear and non-linear effects is small, so that the replacement of a non-linear equation by a linear one is permissible. In general this is not true.

1. DERIVATION OF A GENERALIZED LIPPMANN-SCHWINGER EQUATION FROM THE BETHE-SALPETER EQUATION

We will take as our starting point the Bethe Salpeter equation in integral form, which is a relativistic equation suitable for dealing with bound state or scattering problems. The Bethe Salpeter equation for the propagation kernel of a two particle system

$$G(x_3, x_4; x_1, x_2) = K^{(2)}(x_3, x_4; x_1, x_2) + \frac{(\Phi_0, T(S \psi(x_3) \psi(x_4) \bar{\psi}(x_1) \bar{\psi}(x_2)) \Phi_0)}{(\Phi_0, S \Phi_0)}$$

is

$$G(x_3, x_4; x_1, x_2) = G_0(x_3, x_4; x_1, x_2) + i \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 G_0(x_3, x_4; x_5, x_6) K(x_5, x_6; x_7, x_8) G(x_7, x_8; x_1, x_2) \quad (1)$$

where $K(x_5, x_6; x_7, x_8)$ is the sum of all the irreducible graphs in the particular scattering process, G_0 is the free particle Green's function, T is the time ordering symbol and S is the asymptotic value of the transition matrix, i.e. $S = U(+\infty, -\infty)$. Bethe and Salpeter also define a wave function $\chi(x_1, x_2)$ by the expansion of the two particle Green's function over intermediate states of total energy momentum four vector p , and quantum number α .

$$G(x_3, x_4; x_1, x_2) = \sum_{p, \alpha} \chi_{p\alpha}(x_3, x_4) \chi_{p\alpha}^+(x_1, x_2).$$

The main difficulty in dealing with such an equation is that the "relative time coordinate" $x_{20} - x_{10}$ is ambiguous. By defining two time Green's functions,

$$\begin{aligned} \tilde{G}(\vec{x}_3, \vec{x}_4, x_{30}; \vec{x}_1, \vec{x}_2, x_{10}) &= \int d^4x_{40} d^4x_{20} \varepsilon(x_{40} - x_{30}) \delta(x_{20} - x_{10}), \\ G(\vec{x}_3, x_{30}, \vec{x}_4, x_{40}; \vec{x}_1, x_{10}, \vec{x}_2, x_{20}) &. \end{aligned} \quad (2)$$

we eliminate the relative time coordinate.

The calculations which follow are easier to carry out in momentum space, so let us define the Fourier transfer of a function F by

$$F(x) = \frac{1}{(2\pi)^4} \int F(p) e^{-i p \cdot x} d^4p$$

In momentum space the Fourier transform of equation (1) is

$$G(p_3, p_4; p_1, p_2) = G_0(p_3, p_4; p_1, p_2) + \frac{i}{(2\pi)^6} \int d^4p_5 d^4p_6 d^4p_7 d^4p_8.$$

$$G_c(p_3, p_4; p_5, p_6) K(p_5, p_6; p_7, p_8) G(p_7, p_8; p_1, p_2) \quad (3)$$

Employing the integral representation of the delta function,

$$\delta(x_{20} - x_{10}) = \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} e^{-i\varepsilon(x_{20} - x_{10})} d\varepsilon$$

together with the Fourier transform, we replace equation (2) by

$$\tilde{G}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_1, \vec{p}_2, p_{10}) = \frac{1}{(2\pi)^7} \int d\varepsilon d\varepsilon' G(\vec{p}_3, p_{30} - \varepsilon, \vec{p}_4, \varepsilon; \vec{p}_1, p_{10} - \varepsilon', \vec{p}_2, \varepsilon') \quad (4).$$

In future work we need an explicit form for the two time free particle Green's function. This can be found as follows. The free particle Green's function

$$\begin{aligned} G_0(\vec{p}_3, p_{30}-\epsilon, \vec{p}_4, \epsilon; \vec{p}_1, p_{10}-\epsilon', \vec{p}_2, \epsilon') &= \\ &= \delta(\vec{p}_3-\vec{p}_1) \delta(\vec{p}_2-\vec{p}_4) \delta(p_{30}-p_{10}) \delta(\epsilon-\epsilon') G_0(\vec{p}_1, p_{10}-\epsilon; \vec{p}_2, \epsilon) \end{aligned}$$

when substituted into equation (4), gives

$$\tilde{G}(\vec{p}_1, p_{10}, \vec{p}_2) = \frac{1}{(2\pi)^2} \int d\epsilon \int d\epsilon' \delta(\epsilon-\epsilon') G_0(\vec{p}_1, p_{10}-\epsilon; \vec{p}_2, \epsilon)$$

However, $G(\vec{p}_1, p_{10}-\epsilon; \vec{p}_2, \epsilon)$ is known to be the product of the propagation functions for the pion and the nucleon respectively, i.e.

$$\begin{aligned} \tilde{G}(\vec{p}_1, p_{10}; \vec{p}_2) &= \frac{1}{(2\pi)^2} \int d\epsilon \frac{-1}{(2\pi)^8} \frac{\delta_0(p_{10}-\epsilon) - \vec{\delta} \cdot \vec{p}_1 + M}{(p_{10}-\epsilon)^2 - (\vec{p}_1^2 + M^2)} \frac{1}{\epsilon^2 - (\vec{p}_2^2 + M^2)} \\ &= \frac{-1}{(2\pi)^8} \int d\epsilon \frac{\delta_0(p_{10}-\epsilon) - \vec{\delta} \cdot \vec{p}_1 + M}{(p_{10}-\epsilon + \sqrt{\vec{p}_1^2 + M^2})(p_{10}-\epsilon - \sqrt{\vec{p}_1^2 + M^2})(\epsilon + \sqrt{\vec{p}_2^2 + M^2})(\epsilon - \sqrt{\vec{p}_2^2 + M^2})} \end{aligned}$$

According to the Feynman prescription for handling the poles, a small imaginary part must be added in the denominators. This displaces two of the poles into the lower half plane and two into the upper half plane. A contour integral, taken over a large semi-circle in the upper half plane of the ϵ plane, gives

contributions from only two of the poles so we find that

$$\begin{aligned} \tilde{G}(\vec{p}_1, p_{10}; \vec{p}_2) &= \frac{-\pi i}{2(2\pi)^{10}} \frac{1}{\sqrt{\vec{p}_2^2 + \mu^2}} \frac{\delta_0(p_{10} - \sqrt{\vec{p}_2^2 + \mu^2}) - \vec{x} \cdot \vec{p}_1 + M}{(p_{10} - \sqrt{\vec{p}_2^2 + \mu^2})^2 - (\vec{p}_1^2 + M^2)} + \\ &+ \frac{-\pi i}{2(2\pi)^{10}} \frac{1}{\sqrt{\vec{p}_1^2 + M^2}} \frac{\delta_0(\sqrt{\vec{p}_1^2 + M^2} - \vec{x} \cdot \vec{p}_1 + M)}{(p_{10} - \sqrt{\vec{p}_1^2 + M^2})^2 - (\vec{p}_2^2 + \mu^2)}. \end{aligned} \quad (5)$$

Let us now solve the Bethe Salpeter equation by iteration. We replace G on the right hand side of equation (3) by

$$G_0 + G_0 K G_0 + \dots$$

Now transform the result into the two time formalism by using the appropriate transform in momentum space, equation (4), yielding

$$\begin{aligned} \tilde{G}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_1, \vec{p}_2, p_{10}) &= \delta(\vec{p}_3 - \vec{p}_1) \delta(\vec{p}_4 - \vec{p}_2) \delta(p_{30} - p_{10}) \tilde{G}_0(\vec{p}_1, p_{10}; \vec{p}_2) + \\ &+ \frac{i}{(2\pi)^{16}} \int d\varepsilon d\varepsilon' d\sigma d\sigma' d^4p_5 d^4p_6 d^4p_7 d^4p_8 G_0(\vec{p}_3, p_{30} - \varepsilon, \vec{p}_4, \varepsilon; \vec{p}_5, p_{50} - \sigma, \vec{p}_6, \sigma) \\ &\cdot K(\vec{p}_5, p_{50} - \varepsilon, \vec{p}_6, \sigma; \vec{p}_7, p_{70} - \sigma', \vec{p}_8, \sigma') G_0(\vec{p}_7, p_{70} - \sigma', \vec{p}_8, \sigma'; \vec{p}_1, p_{10} - \varepsilon', \vec{p}_2, \varepsilon') \\ &+ \dots \end{aligned}$$

which, using the δ -functions in the free time Green's functions becomes,

$$\begin{aligned} \tilde{G}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_1, \vec{p}_2, p_{10}) &= \delta(\vec{p}_3 - \vec{p}_1) \delta(\vec{p}_2 - \vec{p}_4) \delta(p_{30} - p_{10}) \tilde{G}_0(\vec{p}_1, p_{10}; \vec{p}_2) + \\ &+ \frac{i}{(2\pi)^{16}} \int d\varepsilon d\varepsilon' G_0(\vec{p}_3, p_{30} - \varepsilon; \vec{p}_4, \varepsilon) K(\vec{p}_3, p_{30} - \varepsilon, \vec{p}_4, \varepsilon; \vec{p}_1, p_{10} - \varepsilon', \vec{p}_2, \varepsilon') \\ &\cdot G_0(\vec{p}_1, p_{10} - \varepsilon'; \vec{p}_2, \varepsilon') + \dots \end{aligned} \quad (6)$$

The definition of the inverse of the two time Green's function is

$$\int d\vec{r}_{50} d\vec{r}_5 d\vec{r}_6 \tilde{G}^{-1}(\vec{r}_3, \vec{r}_4, t_{30}; \vec{r}_5, \vec{r}_6, t_{50}) \tilde{G}(\vec{r}_5, \vec{r}_6, t_{50}; \vec{r}_1, \vec{r}_2, t_{10}) = \delta(t_{10} - t_{30}) \delta(\vec{r}_1 - \vec{r}_3) \delta(\vec{r}_2 - \vec{r}_4). \quad (7)$$

But we can find G^{-1} from (6).

Symbolically, write

$$\tilde{G} = \check{G}_0 + \widetilde{G_0 K G_0} + \dots$$

and $\tilde{G}^{-1} \tilde{G} = 1.$

Suppose there exists an expansion in powers of the coupling constant for G^{-1} , i.e.

$$\tilde{G}^{-1} = A + B + C + \dots$$

Then

$$(A + B + C + \dots)(\check{G}_0 + \widetilde{G_0 K G_0} + \dots) = 1.$$

and equating terms in the same power of the coupling constant

$$A = \check{G}_0^{-1}$$

$$B = -\check{G}_0^{-1} \widetilde{G_0 K G_0} \check{G}_0^{-1}$$

Hence,

$$\check{G}^{-1} = \check{G}_0^{-1} - \check{G}_0^{-1} \widetilde{G_0 K G_0} \check{G}_0^{-1} + \dots$$

which, when written out fully, takes the following form.

$$\begin{aligned} \tilde{G}^{-1}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_1, \vec{p}_2, p_{10}) &= \delta(\vec{p}_3 - \vec{p}_1) \delta(\vec{p}_4 - \vec{p}_2) \delta(p_{30} - p_{10}) \tilde{G}_0^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) - \\ &- \tilde{G}_0^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) \left[\frac{i}{(2\pi)^4} \int d\varepsilon d\varepsilon' G_\circ K G_\circ \right] \tilde{G}_0^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) - \\ &= \delta(\vec{p}_3 - \vec{p}_1) \delta(\vec{p}_4 - \vec{p}_2) \delta(p_{30} - p_{10}) \tilde{G}_0^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) - V(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_1, \vec{p}_2, p_{10}), \end{aligned}$$

where the function V takes into account all higher terms in the expansion.

Substitute the above expression into (7), giving

$$\begin{aligned} \int d p_{50} d \vec{p}_5 d \vec{p}_6 \left\{ \delta(\vec{p}_3 - \vec{p}_5) \delta(\vec{p}_4 - \vec{p}_6) \delta(p_{30} - p_{50}) \tilde{G}_0^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) - \right. \\ \left. - V(p_{30}, \vec{p}_3, \vec{p}_4; \vec{p}_1, \vec{p}_2, p_{10}) \right\} \tilde{G}(\vec{p}_5, \vec{p}_6, p_{50}; \vec{p}_1, \vec{p}_2, p_{10}) = \\ = \delta(p_{30} - p_{10}) \delta(\vec{p}_3 - \vec{p}_1) \delta(\vec{p}_4 - \vec{p}_2), \end{aligned}$$

so that

$$\begin{aligned} \tilde{G}_0^{-1}(p_{30}, p_3, p_4) \tilde{G}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_1, \vec{p}_2, p_{10}) &= \delta(p_{30} - p_{10}) \delta(\vec{p}_3 - \vec{p}_1) \delta(\vec{p}_4 - \vec{p}_2) + \\ &+ \int d p_{50} d \vec{p}_5 d \vec{p}_6 V(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_5, \vec{p}_6, p_{50}) \tilde{G}(\vec{p}_5, \vec{p}_6, p_{50}; \vec{p}_1, \vec{p}_2, p_{10}). \end{aligned}$$

A similar, but homogeneous equation, is then satisfied by the Bethe Salpeter wave function $\chi(\vec{p}_3, \vec{p}_4, p_{30})$

$$\begin{aligned} \tilde{G}^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) \chi(\vec{p}_3, \vec{p}_4, p_{30}) &= \int d p_{50} d \vec{p}_5 d \vec{p}_6 V(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_5, \vec{p}_6, p_{50}) \cdot \\ &\chi(\vec{p}_5, \vec{p}_6, p_{50}). \end{aligned} \tag{8}$$

Now transform to the centre of mass system of coordinates where

$$\chi(\vec{p}_3, \vec{p}_4, p_{30}) = \delta(\vec{p}_3 + \vec{p}_4) \delta(p_{30} - W) \varphi(\vec{p}_3).$$

W being the total energy, and

$$\tilde{V}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_5, \vec{p}_6, p_{50}) = \delta(p_{30} - p_{50}) \delta(\vec{p}_3 + \vec{p}_4 - \vec{p}_5 - \vec{p}_6) \tilde{V}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_5, \vec{p}_6, p_{50}).$$

so that equation (8) becomes

$$\tilde{G}_0^{-1}(\vec{p}_3, \vec{p}_4, p_{30}) \delta(\vec{p}_3 + \vec{p}_4) \delta(p_{30} - W) \varphi(\vec{p}_3) = \int d\vec{p}_5 d\vec{p}_6 \delta(p_{30} - p_{50})$$

$$\delta(\vec{p}_3 + \vec{p}_4 - \vec{p}_5 - \vec{p}_6) \tilde{V}(\vec{p}_3, \vec{p}_4, p_{30}; \vec{p}_5, \vec{p}_6, p_{50}) \delta(p_{50} - W) \delta(\vec{p}_5 + \vec{p}_6) \varphi(\vec{p}_5)$$

$$= \int d\vec{p}_5 \delta(p_{30} - W) \delta(\vec{p}_3 + \vec{p}_4) \tilde{V}(p_{30}, \vec{p}_3, \vec{p}_4; p_5, -p_5, W) \varphi(\vec{p}_5)$$

A final integration of both sides over $d\vec{p}_4$ and $d p_{30}$, gives us

$$\tilde{G}_0^{-1}(W, \vec{p}_3) \varphi(\vec{p}_3) = \int d\vec{p}_5 \tilde{V}(W, \vec{p}_3, \vec{p}_5) \varphi(\vec{p}_5) \quad (9a)$$

or changing the notation slightly

$$\varphi(\vec{p}) = \tilde{G}_0(W, \vec{p}) \int d\vec{p}' \tilde{V}(W, \vec{p}, \vec{p}') \varphi(\vec{p}') \quad (9b)$$

The integration variable obviously cannot be on the energy shell so if

$$E = \sqrt{\vec{p}^2 + M^2}, \quad \omega = \sqrt{\vec{p}^2 + \mu^2}, \quad E' = \sqrt{\vec{p}'^2 + M^2}, \quad \omega' = \sqrt{\vec{p}'^2 + \mu^2}, \quad \text{then}$$

$$W = E + \omega = E' + \omega' .$$

Equation (9a) is a generalization of the usual Schrödinger equation, and has an energy dependent potential.

This three dimensional equation for the wave function ψ may be transformed in the usual way to an equation for the Lippmann Schwinger amplitude T. Define

$$\psi(\vec{p}) = \delta(\vec{p} - \vec{p}'') + \tilde{G}_0(\vec{p}, W) T(\vec{p}, \vec{p}'')$$

then for $\vec{p} \neq \vec{p}'$ equation (9) gives

$$T(\vec{p}, \vec{p}'') = V(\vec{p}, \vec{p}'') + \int d\vec{p}' V(W, \vec{p}, \vec{p}') \tilde{G}_0(W, \vec{p}') T(\vec{p}', \vec{p}'') \quad (10)$$

where all the vectors are three dimensional. It is now permissible to place the vectors \vec{p} and \vec{p}'' on the energy shell ^{so} that this equation is being used to define one particular extrapolation off the energy shell. Nevertheless, this is the best approach if we use a potential chosen from field theory. Suppose we return to the expression for G_0 , equation (5), and replace this by

$$\tilde{G}_0 = \frac{\pi i}{2} \frac{-1}{(2\tau)^{10}} \left\{ \frac{1}{\sqrt{\vec{p}^2 + \mu^2}} \frac{\gamma_0(W - \omega) - \vec{\gamma} \cdot \vec{p}' + M}{(W - \omega')^2 - \vec{p}'^2 - M^2} + \frac{1}{\sqrt{\vec{p}'^2 + M^2}} \frac{\gamma_0 E' - \vec{\gamma} \cdot \vec{p}' + M}{(W - E')^2 - \vec{p}'^2 - \mu^2} \right\}$$

assuming that $|\vec{p}_1|^2 = |\vec{p}_2|^2$.

Define spinors normalized to unity by

$$u(\vec{p}) = \sqrt{\frac{E+M}{2M}} \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+M} \end{pmatrix}$$

so that

$$\begin{aligned}
 u(\vec{p}) \bar{u}(\vec{p}) &= \frac{1}{2M} \begin{pmatrix} E+M & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E+M \end{pmatrix} \\
 &= \frac{1}{2M} (\gamma_0 E - \vec{\gamma} \cdot \vec{p} + M)
 \end{aligned}$$

where

$$\gamma_0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\gamma} = \beta \alpha = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$

are the usual Dirac matrices. If we now turn a blind eye to the fact that \vec{p}' in equation (10) is not actually on the energy shell, we may replace \tilde{G}_0 by

$$\tilde{G}_0 = c \frac{2M}{\sqrt{\vec{p}'^2 + M^2}} \frac{u(\vec{p}') \bar{u}(\vec{p}')}{\omega^2 - \vec{p}'^2 - \mu^2} + \frac{c}{\sqrt{\vec{p}'^2 + \mu^2}} \frac{2M}{E^2 - \vec{p}'^2 - M^2} u(\vec{p}') \bar{u}(\vec{p}')$$

with c some constant. Equation (10) now becomes

$$\begin{aligned}
 T(\vec{p}, \vec{p}'') &= V(\vec{p}, \vec{p}'') + c \int \frac{V(\vec{p}, \vec{p}') 2M u \bar{u} T(\vec{p}', \vec{p}'') d\vec{p}'}{\sqrt{\vec{p}'^2 + M^2} (\omega^2 - \vec{p}'^2 - \mu^2)} + \\
 &+ c \int \frac{V(\vec{p}, \vec{p}') 2M u \bar{u} T(\vec{p}', \vec{p}'') d\vec{p}'}{\sqrt{\vec{p}'^2 + \mu^2} (E^2 - \vec{p}'^2 - M^2)} \quad (11)
 \end{aligned}$$

which is similar to the non-relativistic Lippmann-Schwinger equation. The presence of the additional square root means that the volume element is a relativistic invariant, so we shall refer to equation (11) as the "generalized" Lippmann Schwinger equation. Obviously,

if one takes the limit of $M \rightarrow \infty$, i.e., fixed source approximation, one term is much smaller than the other so our equation reduces to the usual Lippmann-Schwinger equation for a pion moving in a certain nuclear potential.

2. THE UNITARITY CONDITION FOR PION-NUCLEON SCATTERING

One important restriction on equation (11) is that it must satisfy the unitarity condition. The connection between the T matrix and the S matrix is usually taken to be

$$S_{fi} = 1 - i(2\pi)^4 \delta^{(4)}(p_1 + q_1 - p_2 - q_2) \frac{M}{\sqrt{2\omega_1 2\omega_2 E_1 E_2}} T_{fi} \quad (12)$$

Unitarity gives $SS^\dagger = 1$, or

$$\begin{aligned} i(2\pi)^4 \sum_n \frac{\delta^{(4)}(p_1 + q_1 - p_n - q_n)}{2\omega_n E_n} M T_{fn} T_{ni}^\dagger &= T_{fi}^\dagger - T_{fi} \\ &= -2i \operatorname{Im} T_{fi} \end{aligned}$$

$$\therefore \operatorname{Im} T_{fi} = -\frac{1}{2} (2\pi)^4 \sum_n \frac{\delta^{(4)}(p_1 + q_1 - p_n - q_n)}{2\omega_n E_n} M T_{fn} T_{ni}^\dagger$$

For two particle intermediate states, we split this sum up artificially, into two parts

$$\begin{aligned} \operatorname{Im} T_{fi} &= -\frac{1}{2} \frac{(2\pi)^4}{2} \int \frac{d^3 p_3}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{\delta^{(4)}(p_1 + q_1 - p_3 - q_3)}{2\omega_3 E_3} M T_{fn} T_{ni}^\dagger \\ &\quad - \frac{1}{2} \frac{(2\pi)^4}{2} \int \frac{d^3 p_3}{(2\pi)^3} \frac{d^3 q_3}{(2\pi)^3} \frac{\delta^{(4)}(p_1 + q_1 - p_3 - q_3)}{2\omega_3 E_n} M T_{fn} T_{ni}^\dagger \end{aligned}$$

The first term can be transformed into a seven dimensional integral using

$$(2E_3)^{-1} = \int d^4 p_3 \delta(E_3^2 - \vec{p}_3^2 - M^2) \theta(E_3)$$

and then the four dimensional integral over $d^4 p_3$ may be carried out

using the four-dimensional delta-function. A similar trick is easily applied to the $2\omega_3$ appearing in the denominator of the second term giving

$$\begin{aligned} \text{Im } T_{fi} &= -\frac{1}{2} \frac{1}{(2\pi)^2} \frac{1}{2} \int d^3 p_3 \frac{2M \delta(E_3^2 - \vec{p}_3^2 - M^2)}{2\sqrt{\vec{p}_3^2 + M^2}} T_{fn} T_{ni}^+ \\ &\quad - \frac{1}{2} \frac{1}{(2\pi)^2} \frac{1}{2} \int d^3 q_3 \frac{2M \delta(\omega_3^2 - \vec{q}_3^2 - M^2)}{2\sqrt{\vec{q}_3^2 + M^2}} T_{fn} T_{ni}^+ \end{aligned}$$

and of course $\vec{q}_3^2 = \vec{p}_3^2$ in the centre of mass system. If we insert the intermediate state spinors and use dashed variables we find,

$$\begin{aligned} \text{Im } T_{fi} &= -\frac{1}{4} \frac{1}{(2\pi)^2} \int d^4 p' \frac{2M \delta(E'^2 - \vec{p}'^2 - M^2)}{2\sqrt{\vec{p}'^2 + M^2}} T_{fn} u \bar{u} T_{ni}^+ \\ &\quad - \frac{1}{4} \frac{1}{(2\pi)^2} \int d^4 p' \frac{2M \delta(\omega'^2 - p'^2 - M^2)}{2\sqrt{\vec{p}'^2 + M^2}} T_{fn} u \bar{u} T_{ni}^+ \end{aligned} \quad (13)$$

Now the imaginary part of the amplitude T , which satisfies equation (10) i.e.

$$T(\vec{p}, \vec{p}') = V(\vec{p}, \vec{p}') + c \int V(\vec{p}, \vec{p}') \tilde{G}_0(\vec{p}') T(\vec{p}', \vec{p}') d^3 p'$$

is found in the following fashion. Assume that $V(\vec{p}, \vec{p}')$ and c are real, which is true for elastic scattering, then



$$\begin{aligned}
 \text{Im } T(p, p'') &= c \int V(p, p') \left[\text{Re } \tilde{G}_0(p') \text{Im } T(p', p'') + \text{Im } \tilde{G}_0(p') \text{Re } T(p', p'') \right] d^3 p' \\
 &= c \int V(p, p') \left[\text{Re } \tilde{G}_0(p') \text{Im } T(p', p'') + \text{Im } \tilde{G}_0(p') \left\{ \text{Re } T(p', p'') + \right. \right. \\
 &\quad \left. \left. + i \text{Im } T(p', p'') - i \text{Im } T(p', p'') \right\} \right] d^3 p' \\
 &= c \int V(p, p') \left[\tilde{G}_0(p') \text{Im } T(p', p'') + \text{Im } \tilde{G}_0(p') T^\dagger(p', p'') \right] d^3 p'
 \end{aligned}$$

and therefore,

$$\begin{aligned}
 \int \left\{ \delta(p-p') - c V(p, p') \tilde{G}_0(p') \right\} \text{Im } T(p', p'') d^3 p' &= \\
 = c \int V(p, p') \text{Im } \tilde{G}_0(p') T^\dagger(p', p'') d^3 p'.
 \end{aligned}$$

However, from equation (10),

$$\int \left\{ \delta(p-q) - c V(p, q) \tilde{G}_0(q) \right\} T(q, p') d^3 q = V(p, p')$$

$$\begin{aligned}
 \therefore \int \left\{ \delta(p-p') - c V(p, p') \tilde{G}_0(p') \right\} \text{Im } T(p', p'') d^3 p' &= \\
 = c \iint \left[\delta(p-q) - c V(p, q) \tilde{G}_0(q) \right] T(q, p') d^3 q \text{Im } \tilde{G}_0(p') T^\dagger(p', p'') d^3 p'
 \end{aligned}$$

from which it follows that

$$\text{Im } T(p, p'') = c \int T(p, p') \text{Im } \tilde{G}_0(p') T^\dagger(p', p'') d^3 p'$$

The imaginary part of the operator $\tilde{G}_0(p')$ is known from equation (11).

Hence

$$\begin{aligned} \text{Im } T(p, p'') &= 2M\pi c \int \frac{d^3 p' \delta(E'^2 - p'^2 - M^2)}{\sqrt{p'^2 + \mu^2}} T(p, p') u(p') \bar{u}(p') T^\dagger(p', p'') + \\ &+ 2M\pi c \int \frac{d^3 p' \delta(\omega'^2 - p'^2 - \mu^2)}{\sqrt{p'^2 + M^2}} T(p, p') u(p') \bar{u}(p') T^\dagger(p', p'') \end{aligned}$$

which, upon comparison with equation (13), gives

$$c = -\frac{1}{8\pi} \cdot \frac{1}{(2\pi)^2}$$

Hence we have a final equation for the scattering amplitude which satisfies unitarity

$$\begin{aligned} T(p, p'') &= V(p, p'') + \frac{1}{8\pi(2\pi)^2} \int \frac{V(p, p') 2M u(p') \bar{u}(p') T(p', p'') d^3 p'}{\sqrt{p'^2 + M^2} (p'^2 + \mu^2 - \omega^2)} + \\ &+ \frac{1}{8\pi(2\pi)^2} \int \frac{V(p, p') 2M u(p') \bar{u}(p') T(p', p'') d^3 p'}{\sqrt{p'^2 + \mu^2} (p'^2 + M^2 - E^2)} \end{aligned}$$

Let us now introduce new spinors $2M u = \omega$ multiply the equation by $\bar{\omega}$ from the left and ω from the right, and define

$$\bar{\omega}(p'') T(p, p'') \omega(p) = t(p, p'') ; \quad \bar{\omega}(p'') V(p, p'') \omega(p) = v(p, p'')$$

then

$$\begin{aligned} t(p, p'') &= v(p, p'') + \frac{1}{8\pi(2\pi)^2} \int \frac{v(p, p') t(p', p'') d^3 p'}{\sqrt{p'^2 + M^2} (p'^2 + \mu^2 - \omega^2)} + \\ &+ \frac{1}{8\pi(2\pi)^2} \int \frac{v(p, p') t(p', p'') d^3 p'}{\sqrt{p'^2 + \mu^2} (p'^2 + M^2 - E^2)} \end{aligned}$$

3. REDUCTION OF THE GENERALIZED LIPPMANN-SCHWINGER EQUATION TO EQUATIONS FOR THE PARTIAL WAVE AMPLITUDES

In the centre of mass system of coordinates, the spin decomposition of the amplitude $t(p', p'')$ depends only on the Pauli spin vector and the three dimensional vectors p' and p'' . From invariance arguments

$$t(p', p'') = F_N + i \sigma \cdot \hat{p}' \wedge \hat{p}'' F_S \quad (15)$$

where the carat denotes unit vectors, and F_S and F_N are the spin flip and non-spin flip amplitudes respectively. In order to find the decomposition of F_N and F_S into the partial wave amplitudes

$$f_{J, \ell} = \frac{1}{|\vec{p}'|} e^{i\delta_{J, \ell}} \sin \delta_{J, \ell} \quad (16)$$

we must change from the momentum representation of the amplitude to a representation based on the quantum numbers $\vec{J} = \vec{\ell} + \frac{1}{2} \vec{\sigma}$, M , the magnetic quantum number, and ℓ . This is done conveniently in two steps using $|\ell, m, \sigma_3\rangle$ representation as intermediary. Hence

$$\langle \vec{p}', \sigma_3' | t | \vec{p}'', \sigma_3'' \rangle = \sum_{J M \ell} \langle p' \sigma_3' | \vec{p}' | J M \ell \rangle t_{J \ell} \langle \vec{p}'' | J M \ell | \vec{p}'' \sigma_3'' \rangle$$

where, after some algebra, the Clebsch Gordan coefficients are found to be

$$\begin{aligned}
\langle \vec{j}, \sigma_3 | J, M, \ell \rangle &= \sqrt{\frac{J+M+1}{2(J+1)}} \delta_{\ell, J+\frac{1}{2}} \delta_{\sigma_3, -1} Y_{J+\frac{1}{2}}^{M+\frac{1}{2}}(\vec{j}) \\
&- \sqrt{\frac{J-M+1}{2(J+1)}} \delta_{\ell, J+\frac{1}{2}} \delta_{\sigma_3, +1} Y_{J+\frac{1}{2}}^{M-\frac{1}{2}}(\vec{j}) \\
&+ \sqrt{\frac{J-M}{2J}} \delta_{\ell, J-\frac{1}{2}} \delta_{\sigma_3, -1} Y_{J-\frac{1}{2}}^{M-\frac{1}{2}}(\vec{j}) \\
&+ \sqrt{\frac{J+M}{2J}} \delta_{\ell, J-\frac{1}{2}} \delta_{\sigma_3, +1} Y_{J-\frac{1}{2}}^{M+\frac{1}{2}}(\vec{j}).
\end{aligned}$$

The addition theorem permits one to sum over M . (The matrix elements cannot depend on M as there is no privileged direction in space).

$$\sum_{m=-\ell}^{m=+\ell} Y_{\ell}^m(\vec{j}') Y_{\ell}^{*m}(\vec{j}'') = \frac{2\ell+1}{4\pi} P_{\ell}(\cos \theta), \quad \cos \theta = \hat{j}' \cdot \hat{j}''.$$

We then find that

$$\begin{aligned}
t_J &= \frac{2J+1}{8\pi} \left[T_{J, J+\frac{1}{2}} P_{J+\frac{1}{2}} + T_{J, J-\frac{1}{2}} P_{J-\frac{1}{2}} \right] \\
&+ \frac{2i}{8\pi} \left[T_{J, J+\frac{1}{2}} P'_{J+\frac{1}{2}} - T_{J, J-\frac{1}{2}} P'_{J-\frac{1}{2}} \right] \frac{\sigma \cdot \hat{j}'' \hat{j}'}{|\hat{j}'|^2}.
\end{aligned}$$

Properties of the Legendre polynomials allow one to alter the last expression and obtain an expansion of F_N and F_S in terms of the partial wave amplitudes. (Details of such a decomposition

are given in text-books so they are not reproduced here). We finally arrive at the following formulae.

$$F_N = \sum_e \left((l+1) f_{l+\frac{1}{2},e} + e f_{l-\frac{1}{2},e} \right) P_e(\cos \theta).$$

$$F_S = \sum_e \left(f_{l-\frac{1}{2},e} - f_{l+\frac{1}{2},e} \right) P_e'(\cos \theta).$$
(17)

$P_e'(\cos \theta)$ is the derivative of $P_e(\cos \theta)$ with respect to $\cos \theta$.

Thus, substituting into equation (15),

$$\begin{aligned} \ddagger &= \sum_e \left[(l+1) f_{l+\frac{1}{2},e} + e f_{l-\frac{1}{2},e} \right] P_e(\cos \theta) \\ &\quad + \sum_e \left[f_{l-\frac{1}{2},e} - f_{l+\frac{1}{2},e} \right] P_e'(\cos \theta) i\sigma \hat{h}'' \hat{h}' \\ &= \sum_e f_{l+\frac{1}{2},e} \left\{ (l+1) P_e(\cos \theta) - i\sigma \hat{h}'' \hat{h}' P_e'(\cos \theta) \right\} \\ &\quad + \sum_e f_{l-\frac{1}{2},e} \left\{ e P_e(\cos \theta) + i\sigma \hat{h}'' \hat{h}' P_e'(\cos \theta) \right\} \\ &= \sum_e f_{l+\frac{1}{2},e} L_e^+ + \sum_e f_{l-\frac{1}{2},e} L_e^- \end{aligned}$$
(18)

where

$$\begin{cases} L_e^+ = (l+1) P_e(\cos \theta) - i\sigma \hat{h}'' \hat{h}' P_e'(\cos \theta). \\ L_e^- = e P_e(\cos \theta) + i\sigma \hat{h}'' \hat{h}' P_e'(\cos \theta). \end{cases}$$

The operators L_e^\pm satisfy the following properties

$$\int d\Omega_{p'} L_i^\pm(h, h') L_j^\pm(h', h'') = 4\pi \delta_{ij} L_i^\pm(h, h'').$$

(19)

$$\int d\Omega_{p'} L_i^\pm(h, h') L_j^\mp(h', h'') = 0.$$

and are thus the required projection operators to find individual equations for the partial wave amplitudes. Expand t and v in terms of these operators and substitute into (14).

$$\begin{aligned} \sum_e \left(f_{e+\frac{1}{2}, e} L_e^+ + f_{e-\frac{1}{2}, e} L_e^- \right) &= \sum_e \left(v_{e+\frac{1}{2}} L_e^+ + v_{e-\frac{1}{2}} L_e^- \right) + \\ &+ \frac{1}{8\pi} \frac{1}{(2\pi)^2} \int \frac{\sum_e \left(v_{e+\frac{1}{2}, e} L_e^+ + v_{e-\frac{1}{2}, e} L_e^- \right) \sum_{k'} \left(f_{k+\frac{1}{2}, k} L_k^+ - f_{k-\frac{1}{2}, k} L_k^- \right) h'^2 dh' d\Omega_{p'}}{\sqrt{h'^2 + M^2} (h'^2 + \mu^2 - \omega^2)} \\ &+ \frac{1}{8\pi} \frac{1}{(2\pi)^2} \int \frac{\sum_e \left(v_{e+\frac{1}{2}, e} L_e^+ + v_{e-\frac{1}{2}, e} L_e^- \right) \sum_{k'} \left(f_{k+\frac{1}{2}, k} L_k^+ + f_{k-\frac{1}{2}, k} L_k^- \right) h'^2 dh' d\Omega_{p'}}{\sqrt{h'^2 + \mu^2} (h'^2 + M^2 - E^2)} \end{aligned}$$

The orthogonality of L^+ and L^- is now exploited giving

$$\begin{aligned} \sum_e \left(f_{e+\frac{1}{2}, e} L_e^+ + f_{e-\frac{1}{2}, e} L_e^- \right) &= \sum_e \left(v_{e+\frac{1}{2}, e} L_e^+ + v_{e-\frac{1}{2}, e} L_e^- \right) + \\ &+ \frac{1}{8\pi} \frac{4\pi}{(2\pi)^2} \int \frac{\sum_e \left(v_{e+\frac{1}{2}, e} f_{e+\frac{1}{2}, e} L_e^+ + v_{e-\frac{1}{2}, e} f_{e-\frac{1}{2}, e} L_e^- \right) h'^2 dh'}{\sqrt{h'^2 + M^2} (h'^2 + \mu^2 - \omega^2)} \\ &+ \frac{1}{8\pi} \frac{4\pi}{(2\pi)^2} \int \frac{\sum_e \left(v_{e+\frac{1}{2}, e} f_{e+\frac{1}{2}, e} L_e^+ + v_{e-\frac{1}{2}, e} f_{e-\frac{1}{2}, e} L_e^- \right) h'^2 dh'}{\sqrt{h'^2 + \mu^2} (h'^2 + M^2 - E^2)} \end{aligned}$$

which upon multiplication by either L^\pm and another angular integration yields,

$$f_{e\pm}(l|p) = \mathcal{V}_{e\pm}(l|p) + \frac{1}{2} \frac{1}{(2\pi)^2} \int \frac{\mathcal{V}_{e\pm}(l|p, p') f_{e\pm}(l|p', p) d^3 p' p'^2}{\sqrt{p'^2 + \mu^2} (p'^2 + m^2 - E^2)} +$$

$$+ \frac{1}{2} \frac{1}{(2\pi)^2} \int \frac{\mathcal{V}_{e\pm}(l|p, p') f_{e\pm}(l|p', p) p'^2 d^3 p'}{\sqrt{p'^2 + m^2} (p'^2 + \mu^2 - \omega^2)} \quad (20)$$

The factor $(2\pi)^2$ is irrelevant and could be removed by altering the definition of the T matrix

Now

$$t = \sum (f_{e+\frac{1}{2}, e} L_e^+ + f_{e-\frac{1}{2}, e} L_e^-)$$

so

$$\int t L_e^+ d\Omega = \int f_{e+\frac{1}{2}, e} L_e^+ L_e^+ d\Omega.$$

or

$$f_{e+\frac{1}{2}} = \frac{1}{4\pi L_e^+(p, p)} \int t(p, p') L_e^+(p', p) d\Omega_{p'}.$$

Therefore

$$\left. \begin{aligned} f_{e+} = f_{e+\frac{1}{2}, e} &= \frac{1}{4\pi(e+1)} \int t(p, p') L_e^+(p', p) d\Omega_{p'} \\ f_{e-} = f_{e-\frac{1}{2}, e} &= \frac{1}{4\pi e} \int t(p, p') L_e^-(p', p) d\Omega_{p'} \end{aligned} \right\} \quad (21)$$

Simpler expressions for $f_{e\pm}$ are found by replacing t and L_e^\pm in (21) as follows.

$$\begin{aligned}
 f_{\ell+} &= \frac{1}{4\pi(\ell+1)} \int (f_1 + f_2 \hat{h} \cdot \hat{h}' + i f_2 \sigma \cdot \hat{h} \wedge \hat{h}') ((\ell+1) P_\ell + i \sigma \cdot \hat{h} \wedge \hat{h}' P_\ell') d\Omega_{h'} \\
 &= \frac{1}{2(\ell+1)} \int (f_1 P_\ell + (\ell+1)z f_2 P_\ell - (1-z^2) f_2 P_\ell') dz \\
 &= \frac{1}{2} \int \left[f_1 P_\ell + \frac{(\ell+1)z P_\ell - (1-z^2) P_\ell'}{(\ell+1)} f_2 \right] dz \quad z = \cos \theta \\
 &= \frac{1}{2} \int (f_1 P_\ell + f_2 P_{\ell+1}) dz
 \end{aligned}$$

A similar relation holds for $f_{\ell-}$ so that

$$f_{\ell \pm \frac{1}{2}, z}(|h|) = \frac{1}{2} \int \left\{ f_1(h, h') P_\ell(z) + f_2(h, h') P_{\ell \pm 1}(z) \right\} dz \quad (22)$$

which is the usual relation quoted in papers on pion nucleon scattering.

Equation (14) is a system of uncoupled linear integral equations for the partial wave amplitudes. Dispersion relations lead to a system of coupled non-linear equations for the partial wave amplitudes, and, in this respect our equation is easier. It is very similar to the Lippmann-Schwinger equation as it involves an energy dependent potential which must be taken from perturbation theory. Before discussing the choice of the potential let us assume that it is well behaved and so allows us to solve equation (20) by the determinantal method of Fredholm. Take, for example the equation

$$f(p, p'') = v(p, p'') + \int K(p, p', p'') f(p', p'') dp'$$

with K a given kernel and p'' some parameter. For fixed p'' the Fredholm solution of the above equation is

$$f(p, p'') = v(p, p'') + \frac{1}{D(p'')} \int \Delta(p, p', p'') v(p', p'') dp'$$

where the denominator function is

$$D(p'') = 1 - \int K(p', p', p'') dp' + \iint \begin{vmatrix} K & K \\ K & K \end{vmatrix} \dots$$

and the new kernel is

$$\Delta(p, p', p'') = K(p, p', p'') - \iint \begin{vmatrix} K & K \\ K & K \end{vmatrix} \dots$$

Thus, to first order, the solution of the above equation is

$$f(p, p'') = v(p, p'') + \frac{\int K(p, p', p'') v(p', p'') dp'}{1 - \int K(p', p', p'') dp'}$$

If a resonance appears in a particular scattering state then the denominator function must have a zero. Before actually solving our equation for the phase shifts it is enough to test our equation by finding the position of the first resonance and comparing it with the experimental position of the three-three resonance.

The expression for this determinant is

$$\begin{aligned}
 D_{e\pm}(p'') &= 1 - \frac{1}{2} \frac{1}{(2\pi)^2} \int \frac{p'^2 v_{e\pm}(p', p') dp'}{\sqrt{p'^2 + M^2} (p'^2 + \mu^2 - \omega^2(p''))} \\
 &\quad - \frac{1}{2} \frac{1}{(2\pi)^2} \int \frac{p'^2 v_{e\pm}(p', p') dp'}{\sqrt{p'^2 + \mu^2} (p'^2 + M^2 - E^2(p''))}
 \end{aligned} \tag{23}$$

Let us now transform this equation to be a function of W only using the following relations

$$4W^2 p^2 = W^4 - 2(M^2 + \mu^2)W^2 + (M^2 - \mu^2)^2$$

$$\therefore dp = \frac{W^4 - (M^2 - \mu^2)^2}{2W^2} \frac{dW}{\sqrt{W^4 - 2(M^2 + \mu^2)W^2 + (M^2 - \mu^2)^2}}$$

$$2W(E \pm M) = (W \pm M)^2 - \mu^2$$

$$2WE = W^2 + M^2 - \mu^2$$

$$2W\omega = W^2 - M^2 + \mu^2$$

To save writing dashes everywhere, let W denote the energy of the intermediate state, i.e. the integration variable, and x be the energy of the incoming particle.

$$\therefore D_{e\pm}(x) = 1 - \frac{x^2}{(2\pi)^2} \int_{M+\mu}^{\infty} \frac{W \sqrt{W^4 - 2(M^2 + \mu^2)W^2 + (M^2 - \mu^2)^2} v_{e\pm}(W) dW}{[W^2 - x^2][W^2 x^2 - (M^2 - \mu^2)^2]}$$

In general this integral does not converge so, we must carry out a subtraction and we do so at the point $X = M$. The solution is then normalized by defining $D_{e\pm}(M) = 1$. After some algebra

we find

$$\begin{aligned}
 D_{e\pm}(w) &= 1 - \frac{(x^2 - M^2)(x^2 M^2 - (M^2 - \mu^2)^2)}{(2\pi)^2} \\
 &\quad \cdot \int \frac{w^3 \sqrt{w^4 - 2(M^2 + \mu^2)w^2 + (M^2 - \mu^2)^2} \mathcal{V}_{e\pm}(w) dw}{(w^2 - x^2)(w^2 - M^2)(w^2 x^2 - (M^2 - \mu^2)^2)(w^2 M^2 - (M^2 - \mu^2)^2)} \\
 &= (x - w_R) + i \frac{\Gamma}{2} \quad (24)
 \end{aligned}$$

Thus the real part of the determinant function has a zero at resonance while the imaginary part, at this particular energy value, gives us its width. If $\mathcal{V}_{e\pm}(w)$ is negative in the particular scattering state then the whole expression is positive, so no resonance occurs. However, if the potential is positive for a particular scattering state there will be a zero of and hence a resonance in the partial wave amplitude.

5. THE CHOICE OF THE POTENTIAL

(a) The Nucleon Pole

There are actually two possibilities for the choice of the potential. We may use the exact expression for V , i.e.

$$V = \tilde{G}_0^{-1} \widetilde{G_0 K G_0} \tilde{G}_0^{-1} + \dots \quad (25)$$

which is a rather complicated function of the spin variables, or we may solve equation (20) by iteration. The second choice is simpler and tells us that the Born approximation for the potential is just the partial wave amplitude for a particular graph. To lowest order in the pion nucleon coupling constant, the potential is thus given by the nucleon pole graphs

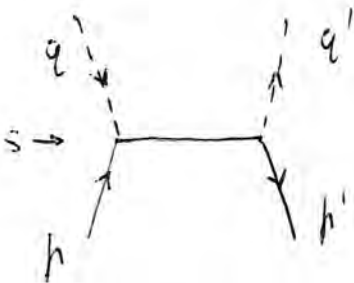


Fig. 1.



Fig. 2.

It is convenient at this point to give definitions of the Dirac equation, metric and scalar variables to be used. These are,

Dirac equation : $\hat{h} u(\vec{h}) = \gamma_\mu h^\mu u(\vec{h}) = M u(\vec{h})$

metric : $h^2 = M^2 = E^2 - \vec{h}^2$

and $s = (h+q)^2, \quad t = (h-h')^2, \quad u = (h-q')^2$

so that $S + t + u = 2M^2 + 2p^2$

In the centre of mass system, the invariants become

$$S = W^2 = (E + \omega)^2, \quad t = -2p^2(1 - \cos\theta)$$

$$u = -2p^2(1 + \cos\theta) + (E - \omega)^2$$

$$\text{where } \cos\theta = 1 - \frac{S + u - (2M^2 + 2p^2)}{2p^2}$$

We shall use a pseudoscalar interaction between the pions and the nucleons,

$$ig \bar{\Psi} \gamma_5 \tau_i \Psi \varphi^i$$

where the value of $g^2/4\pi$ is equal to about fourteen. A pseudovector interaction may be used in principle, but the extra differentiation increases the power of the terms in the numerator and makes certain integrals require cut-offs, which are free of divergences for the corresponding power of the coupling constant in pseudoscalar theory.

The Feynman rules for Figures 1 and 2 yield,

$$T = \frac{g^2}{(2\pi)^2} \bar{u}(p') \left[T_\beta T_\alpha \frac{\hat{q} + \hat{q}'}{s - M^2} - T_\alpha T_\beta \frac{\hat{q} + \hat{q}'}{u - M^2} \right] u(p) \quad (26)$$

$$\text{so if } T = \bar{u}(p') \left[A + B \frac{\hat{q} + \hat{q}'}{2} \right] u(p)$$

only B is finite.

The isotopic spin decomposition follows the standard procedure

$$\begin{aligned}
 B_{\beta\alpha} &= \delta_{\beta\alpha} B^{(+)} + \frac{1}{2} [\tau_{\beta}, \tau_{\alpha}] B^{(-)} \\
 &= \tau_{\beta} \tau_{\alpha} \left(\frac{B^{(+)} + B^{(-)}}{2} \right) + \tau_{\alpha} \tau_{\beta} \left(\frac{B^{(+)} - B^{(-)}}{2} \right)
 \end{aligned}$$

and using

$$B^{(+)} = \frac{1}{3} \left(B^{(\frac{1}{2})} + 2 B^{(3/2)} \right)$$

$$B^{(-)} = \frac{1}{3} \left(B^{(\frac{1}{2})} - B^{(3/2)} \right)$$

$$B_{\beta\alpha} = \tau_{\beta} \tau_{\alpha} \left(\frac{2 B^{(\frac{1}{2})} + B^{(3/2)}}{6} \right) + \tau_{\alpha} \tau_{\beta} \frac{B^{(3/2)}}{2}$$

$$\therefore B^{(\frac{1}{2})} = \frac{g^2}{(2\pi)^2} \left(\frac{-3}{s-M^2} - \frac{1}{u-M^2} \right)$$

$$\therefore B^{(3/2)} = \frac{g^2}{(2\pi)^2} \cdot \frac{2}{u-M^2}$$

The connection between f_1 f_2 and A and B can be found by multiplying out the spinors, so that, using two component spinors normalized to unity

$$T = \chi^* \left\{ \frac{E+M}{2W} \left(\frac{A+(W-M)B}{4\pi} \right) + \frac{E-M}{2W} \left(\frac{-A+(W+M)B}{4\pi} \right) \frac{\sigma \cdot q' \sigma \cdot q}{|q|^2} \right\} \chi$$

and, when $A = 0$

$$f_1 = \frac{E+M}{2W} \cdot \frac{W-M}{4\pi} B(s_i, t)$$

$$f_2 = \frac{E-M}{2W} \cdot \frac{W+M}{4\pi} \cdot B(s_i, t)$$

$$f_1^{3/2} = \frac{g^2}{(2\pi)^2} \frac{(E+M)(W-M)}{8\pi W} \frac{2}{u-M^2}$$

$$f_2^{3/2} = \frac{g^2}{(2\pi)^2} \frac{(E-M)(W+M)}{8\pi W} \frac{2}{u-M^2}$$

Hence

$$f_{\pm}^{3/2} = \frac{g^2}{(2\pi)^2} \frac{(E+M)(W-M)}{8\pi W} \int \frac{2}{u-M^2} P_e(z) dz$$

$$+ \frac{g^2}{(2\pi)^2} \frac{(E-M)(W+M)}{8\pi W} \int \frac{2}{u-M^2} P_{\pm 1}(z) dz.$$

Now use

$$u - M^2 = 2p^2 \left(\frac{-W^2 + 2j^2 + 2p^2 + M^2}{2p^2} - \cos\theta \right)$$

and

$$\int_{-1}^{+1} \frac{P_e(z) dz}{x-z} = 2 G_e(x).$$

giving

$$f_{\pm}^{3/2}(w) = v_{\pm}^{3/2}(w) = \frac{g^2}{(2\pi)^2} \left\{ \frac{(E+M)(W-M)}{8\pi W j^2} G_e(j) + \frac{(E-M)(W+M)}{8\pi W j^2} G_{\pm 1}(j) \right\} \quad (27)$$

$$j = 1 - \frac{w^2 - 2p^2 - M^2}{2p^2} = \frac{j^2 - 2EW}{2j^2}$$

The expression for $f_{\pm}^{1/2}(w)$ is found in a similar fashion and gives

$$\begin{aligned}
f_{e\pm}^{\frac{1}{2}}(W) &= \mathcal{V}_{e\pm}^{\frac{1}{2}}(W) = -\frac{3}{2} \frac{g^2}{(2\pi)^2} \frac{(E+M)(W-M)}{8\pi W} \frac{1}{S-M^2} \delta(S_{\frac{1}{2}}) \\
&\quad - \frac{3}{2} \frac{g^2}{(2\pi)^2} \frac{(E-M)(W+M)}{8\pi W} \frac{1}{S-M^2} \delta(P_{\frac{1}{2}}) \\
&\quad - \frac{f_{e\pm}^{3/2}}{2}.
\end{aligned}
\tag{28}$$

These expressions are the first order approximations for our potentials $\mathcal{V}_{e\pm}^{\frac{1}{2}}(W)$. We must now check whether these potentials can give rise to a resonance in the three-three scattering state. Take the limit of the partial wave amplitudes for small p^2 .

$$Q_e\left(\frac{k^2 - 2Ew}{2p^2}\right) \approx Q_e\left(\frac{-2Ew}{2p^2}\right)$$

But for $f \gg 1$

$$Q_0(f) = \frac{1}{2} \ln \left| \frac{1+f}{1-f} \right| \approx \frac{1}{f}.$$

$$Q_1(f) = \frac{1}{2} f \ln \left| \frac{1+f}{1-f} \right| \approx 0.$$

$$Q_2(f) = \frac{1}{4} (3f^2 - 1) \ln \left| \frac{1+f}{1-f} \right| - \frac{3}{2} f \approx -\frac{1}{2f}.$$

$$\therefore Q_0\left(\frac{k^2 - 2Ew}{2p^2}\right) \approx \frac{-k^2}{2Ew} ; \quad Q_1\left(\frac{k^2 - 2Ew}{2p^2}\right) \approx 0$$

$$\therefore Q_2\left(\frac{k^2 - 2Ew}{2p^2}\right) \approx \frac{k^2}{2Ew}.$$

Substituting these values into the potentials (27) and (28) we find

$$V_{\rho^{\frac{3}{2}}}^{\bar{T}=\frac{3}{2}} \approx \frac{g^2}{(2\pi)^2} \frac{1}{8\pi M} \frac{\phi^2}{2M\omega}, \quad V_{\rho^{\frac{3}{2}}}^{\bar{T}=\frac{1}{2}} \approx -\frac{g^2}{(2\pi)^2} \frac{1}{8\pi M} \frac{\phi^2}{4M\omega}$$

$$V_{\rho^{\frac{1}{2}}}^{\bar{T}=\frac{3}{2}} \approx -\frac{g^2}{(2\pi)^2} \frac{1}{8\pi M} \frac{\phi^2}{M\omega}, \quad V_{\rho^{\frac{1}{2}}}^{\bar{T}=\frac{1}{2}} \approx -\frac{3}{2} \frac{g^2}{(2\pi)^2} \frac{\phi^2}{8\pi M} \frac{1}{M\omega}$$

$$V_{\rho^{\frac{1}{2}}}^{\bar{T}=\frac{3}{2}} \approx -\frac{g^2}{(2\pi)^2} \frac{1}{8\pi M} \frac{\phi^2}{M\omega}, \quad V_{\rho^{\frac{1}{2}}}^{\bar{T}=\frac{1}{2}} \approx -\frac{3}{2} \frac{g^2}{(2\pi)^2} \frac{1}{8\pi M}$$

Thus only the state with $J = \frac{3}{2}$ and $T = \frac{3}{2}$ is positive, allowing the possibility of a resonance.

Rewrite this potential as a function of W and the masses M and μ and substitute it in the dispersion relation for D . It is obviously sufficient to do this for the $J = \frac{3}{2}$, $T = \frac{3}{2}$ scattering state, where, if we use the abbreviations,

$$\alpha = W^4 - 2(M^2 + \mu^2)W^2 + (M^2 - \mu^2)^2$$

$$\beta = (M^2 - \mu^2)^2 + 2W^2 - W^4$$

$$V_{\rho^{\frac{3}{2}}}^{\bar{T}=\frac{3}{2}} = \frac{g^2}{(2\pi)^2} \frac{1}{4\pi} \cdot \frac{[(W+M)^2 - \mu^2][W-M]}{\alpha} \left[\frac{1}{2} \frac{\beta}{\alpha} \ln \left| \frac{W^4 - 2W^2 - W^2 M^2}{(M^2 - \mu^2)^2 - W^2 M^2} \right| - 1 \right] +$$

$$+ \frac{g^2}{(2\pi)^2} \frac{1}{4\pi} \cdot \frac{[(W-M)^2 - \mu^2][W+M]}{\alpha} \left\{ \frac{3}{4} \left[\left(\frac{\beta}{\alpha} \right)^2 - \frac{1}{3} \right] \ln \left| \frac{W^4 - 2W^2 - W^2 M^2}{(M^2 - \mu^2)^2 - W^2 M^2} \right| - \frac{2}{3} \frac{\beta}{\alpha} \right\} \quad (29)$$

The resulting formula for $D(x)$ has a pole at the point $x = W$, and the real part of $D(x)$ is the principal value of the integral at this pole. It is convenient to use the pion mass as the unit of

energy, when $M = 6.7$. Our problem is now to find the value of x for which

$$\operatorname{Re} D(x) = 1 - \frac{g^2(x^2 - M^2)(x^2 M^2 - (M^2 - 1)^2)}{4(2\pi)^5} \text{P.V.} \int_{M+1}^{\infty} \frac{Y(W) dW}{(W^2 - x^2)(W^2 x^2 - (M^2 - 1)^2)} \quad (30)$$

with

$$Y(W) = \left\{ \frac{W^3 [(W+M)^2 - 1] [(W-M)]}{(W^2 - M^2)(W^2 M^2 - (M^2 - 1)^2) \sqrt{(W^2 - (M+1)^2)(W^2 - (M-1)^2)}} \right\}_x$$

$$\times \left\{ \frac{(M^2 - 1)^2 + 2W^2 - W^4}{(W^2 - (M+1)^2)(W^2 - (M-1)^2)} \ln \left| \frac{W^4 - 2W^2 - W^2 M^2}{M^4 + 1 - 2M^2 - W^2 M^2} \right| - 2 \right\} +$$

$$+ \left\{ \frac{3}{4} \frac{W^3 [(W+M)^2 - 1] [(W+M)]}{(W^2 - M^2)(W^2 M^2 - (M^2 - 1)^2) \sqrt{(W^2 - (M+1)^2)(W^2 - (M-1)^2)}} \right\}_x$$

$$\times \left\{ \left[\left(\frac{(M^2 - 1)^2 + 2W^2 - W^4}{(W^2 - (M+1)^2)(W^2 - (M-1)^2)} \right)^2 - \frac{1}{3} \right] \ln \left| \frac{W^4 - 2W^2 - W^2 M^2}{M^4 + 1 - 2M^2 - W^2 M^2} \right| - 2 \frac{(M^2 - 1)^2 + 2W^2 - W^4}{(W^2 - (M+1)^2)(W^2 - (M-1)^2)} \right\}$$

(30a)

has a zero. The known relationship between the total energy in the centre of mass system and the energy of the incident pion in the laboratory system

$$W^2 = (M + \mu)^2 + 2 E_{\text{lab}} M$$

predicts a value of $W \approx 8.8 = M + 2.1$ for the energy of the three three resonance. The zero of equation (30) is at present

being found on an electronic computer at the Joint Institute for Nuclear Research at Dubna. However, at the time of writing this thesis the results are not known.

Fortunately the power of W involved in the integrand is

$$\int_{-\infty}^{\infty} \frac{dW \ln(W^2)}{W^4} \quad (31)$$

which converges without the use of a cut-off. Let the position of the zero be at $x = W_R$, say. The imaginary part of the integral arises from $(W - W_R)$ so that the width of the resonance is also known as a function of W_R .

$$\Gamma(W_R) = \frac{\pi g^2 (W_R^2 - M^2)(W_R^2 M^2 - (M^2 - 1)^2)}{4(2\pi)^5} \frac{\gamma(W_R)}{2W_R [W_R^4 - (M^2 - 1)^2]} \quad (32)$$

and may be expressed in Mev or pion mass units once W_R is known.

(b) Isobar Pole: (corrections to the first order contribution).

The nucleon pole in the crossed channel is responsible for the rough features of the three-three resonance. To improve upon this it is obviously necessary to take into account higher order contributions to the potential, which are compatible with elastic unitarity. One method would be to calculate the potential due to the exchange of a pion and a nucleon in the scattering channel. A knowledge of such contributions would require long calculations

and involve renormalization effects, so it is simpler to consider the pion and nucleon as a spin three halves isobar of some fixed mass. The coupling constant of such a "particle" with the nucleon and the pion is not known but can be roughly estimated from unitarity.

The spin three halves "particle", which is usually denoted by N^* , may be described by a spin-vector field ψ_μ . Properties of such objects are to be found in the book by Umezawa on Quantum Field Theory, and are briefly sketched out here. First we need the field equation, which is satisfied by ψ_μ , and we derive this from the generalized Dirac equation for spinors of rank three,

$$\partial_{is} \varphi_{\dot{i}}^{st} = im \chi_{r\dot{i}}^t$$

$$\partial_{\dot{i}}^{sr} \varphi_{r\dot{i}}^t = im \varphi_{\dot{i}}^{st}$$

where β and χ are symmetric spinors in all their indices.

The definitions of ∂_{is} and $\partial^{\dot{i}sr}$ are

$$\partial_{is} = \sigma_{\mu rs} \partial_\mu, \quad \partial^{\dot{i}sr} = \sigma_\mu^{sr} \partial^\mu$$

where

$$\sigma_{krs} = -\sigma_k^{rs} = [\sigma_k]_{rs}$$

$$\sigma_{4rs} = \sigma_4^{rs} = -i[\mathbb{I}]_{rs}$$

and the σ_k are the Pauli spin matrices with components $[\sigma_k]_{rs}$. The r and s labels run over one and two. Now, if we define the spin vector wave function ψ_μ by

$$\Psi_\mu = \begin{pmatrix} \psi_r^s \\ \psi_{\mu r} \end{pmatrix}$$

where $\psi_r^s = \frac{1}{2} \sigma_{\mu t}^i \varphi_{\dot{u}}^{st}$, $\psi_{\mu r} = \frac{1}{2} \sigma_{\mu t}^i \chi_{r\dot{u}}^t$,

the field equations for ψ and χ become

$$(i\gamma_\mu \partial_\mu + m)\psi_\nu = 0, \quad \partial_\nu \psi_\nu = 0, \quad \gamma_\nu \psi_\nu = 0,$$

which, in momentum space, are equivalent to

$$\gamma_\mu p_\mu \psi_\nu = m \psi_\nu$$

$$p_\nu \psi_\nu = 0$$

$$\gamma_\nu \psi_\nu = 0$$

These restrictions on ψ_ν reduce the number of independent components from sixteen to four, so, using the relation that $2s + 1$ is equal to the number of linearly independent field components, it follows that the spin of the particle described by ψ_μ is three halves.

The mass of the isobar is the same as the mass value of the resonance which will be denoted by W_R . However, such a particle is unstable, so its mass must be given a small imaginary part, which is equivalent to assuming a Breit Wigner resonance form for the appropriate partial wave.

$$f_\ell = \frac{R_\ell}{(W - W_R) + i\frac{\Gamma}{2}}. \quad (33)$$

The partial wave amplitudes are related to phase shifts by

$$f_e = \frac{e^{i\delta_e} \sin \bar{\delta}_e}{p}$$

which gives

$$\text{Im } f_e = p f_e f_e^* \tag{34}$$

as the unitarity condition. If we substitute equation (33) into equation (34), at the resonance value of p , we find the following connection between the full width at half maximum Γ and the residue of the pole R .

$$\Gamma_e = 2p_R R_e \tag{35}$$

Let us now find the partial wave amplitudes for the following two graphs.

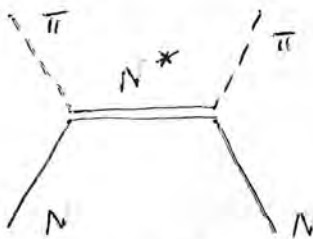


Fig. 3.

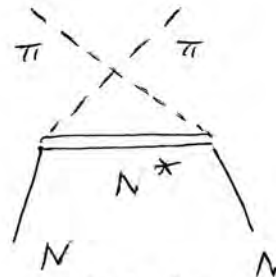


Fig. 4.

The propagator for the N^* is not unique as the Green's function definition of $P_\mu(k)$ is ambiguous for $k^2 \neq W_R^2$. Only the mass shell value has physical meaning and is connected with the residue of the pole. The form of the propagator usually employed in dispersion theoretic calculations is

$$P_{\mu\nu}(k) = \left(3g_{\mu\nu} - \gamma_\mu \gamma_\nu + \frac{4k_\mu k_\nu}{W_R^2} + \frac{\gamma_\mu \hat{k} k_\nu + k_\mu \hat{k} \gamma_\nu}{W_R^2} \right) \frac{\hat{k} + W_R}{k^2 - W_R^2 + i\epsilon} \quad (36)$$

Next we need to know the form of the coupling between the pion, nucleon and spin vector fields, which must be of odd parity. Let us choose

$$\left(\frac{\chi(W_R)}{\pi N N^*} \right) (q_\pi + p_N)^\mu \bar{\psi}_N \psi_\mu \quad (37)$$

This form of the coupling constants, which is in general a function of W_R , turns out to be convenient. The actual value of $\chi/\pi N N^*(W_R)$ is found from the residue of the partial wave amplitude, which, through equation (35), is connected to the width of the resonance. Isotopic spin factors have not been included in equation (37) and are inserted into the equations below. Now we follow essentially the steps of the preceding paragraph. The amplitude for Figure 3 is

$$\bar{u} \left[A^{(3/2)} + \frac{\hat{q}_1 + \hat{q}_2}{2} B^{(3/2)} \right] u = \bar{u} \frac{3\chi(W_R)}{4p_R^2(E_R+M)} (p_2 + q_2)^\mu \hat{P}_{\mu\nu}(k) (p_1 + q_1)^\nu u$$

from which, after some algebra, $A^{(3/2)}$ and $B^{(3/2)}$ can be found. The amplitudes for $A^{(1/2)}$ and $B^{(1/2)}$ are zero. Actually the complete expressions for $A^{(3/2)}$ and $B^{(3/2)}$ need not be given as we only require the value of the three partial wave amplitude at $W = W_R$, to find the coupling constant.

This gives

$$f_{33} = \frac{3 \gamma(W_R)}{\pi N N^*} \frac{1}{2 W_R (W - W_R)}$$

so by comparison with equation (33), we see that

$$\frac{\gamma(W_R)}{\pi N N^*} = \frac{W_R \Gamma(W_R)}{3 p_R} \quad (38)$$

As usual, the crossed pole contribution is responsible for a resonance in the three three scattering state. The amplitude for this wave is

$$f_{3/2}^{3/2} = \frac{\gamma(W_R)}{\pi N N^*} \frac{1}{8 \pi^2 W p^2} \left\{ (E+M) Q_1(\xi) \left(\frac{3\gamma(W-2M-W_R)}{E_R+M} + \frac{2M-W_R-W}{E_R-M} \right) \right. \\ \left. + (E-M) Q_2(\xi) \left(\frac{3\gamma(W+2M+W_R)}{E_R+M} + \frac{W_R-2M-W}{E_R-M} \right) \right\} \quad (39)$$

where

$$\left. \begin{aligned} \xi &= 1 - \frac{W^2 + W_R^2 - 2(M^2 + \mu^2)}{2p^2} \\ \eta &= 1 - \frac{W^2 + W_R^2 - 2(M^2 + \mu^2)}{2p_R^2} \end{aligned} \right\} \quad (40)$$

Hence we find a rather complicated function for the potential as a function of W , W_R and $\Gamma(W_R)$.

Let

$$\alpha = W^4 - 2(M^2 + \mu^2)W^2 + (M^2 - \mu^2)^2 \\ \alpha_R = W_R^4 - 2(M^2 + \mu^2)W_R^2 + (M^2 - \mu^2)^2$$

$$\rho = (M^2 - \mu^2)^2 + 2W^2 - W^4 + 2(M^2 - W_R^2)W^2,$$

$$\sigma = (M^2 - \mu^2)^2 + 2(M^2 + \mu^2)W_R^2 - W_R^4 - 2W^2W_R^2,$$

$$Z = \frac{W^4 - 2M^2W^2 - 2W^2\mu^2 + W_R^2W^2}{(M^2 - \mu^2)^2 - W^2W_R^2}$$

then

$$V^{3/2}(W, W_R, \Gamma(W_R)) = \frac{\Gamma(W_R) W^2 W_R^3}{6(2\pi)^2} \frac{1}{\alpha} \frac{1}{\sqrt{4R}}$$

$$\left\{ \left[\frac{(W+M)^2 - \mu^2}{\alpha_R} \right] \left(\frac{3\sigma(W-2M-W_R)}{\alpha_R((W_R+M)^2 - \mu^2)} - \frac{(W+W_R-2M)}{((W_R-M)^2 - \mu^2)} \right) \left(\frac{\rho}{\alpha} \ln|Z| - 2 \right) + \right. \\ \left. + \frac{3}{2} \left[\frac{(W-M)^2 - \mu^2}{\alpha_R} \right] \left(\frac{3\sigma(W+2M+W_R)}{\alpha_R((W_R+M)^2 - \mu^2)} - \frac{(W-W_R+2M)}{((W_R-M)^2 - \mu^2)} \right) \left(\left[\left(\frac{\rho}{\alpha} \right)^2 - \frac{1}{3} \right] \ln|Z| - 2 \frac{\rho}{\alpha} \right) \right\} \quad (41)$$

This potential is an additive small correction to equation (29).

The substitution of this equation into the dispersion integral for D leads to an even more complicated expression for its real part.

We must now find the zero of the following equation

$$\text{Re } D(y) = 1 - \frac{(y^2 - M^2)(y^2 M^2 - (M^2 - 1)^2)}{4(2\pi)^4} \text{P.V.} \int_{M+\mu}^{\infty} \frac{\left\{ \frac{y^2}{2\pi} Y(W) + \frac{2}{3} \Gamma(W_R) W_R^3 Y'(W) \right\} dW}{(W^2 - y^2)(W^2 y^2 - (M^2 - 1)^2)} \quad (42)$$

where $Y(W)$ is given by equation (30a), $\Gamma(W_R)$ by equation (32) and

$$Y'(W) = \frac{W^3[(W+M)^2-1] \left\{ \frac{3[(M^2-1)^2+2(M^2+1)W_R^2-W_R^4-2W_R^2W^2][W+2M-W_R]}{[W_R^2-(M+1)^2][W_R^2-(M-1)^2][(W_R+M)^2-1]} + \frac{2M-W-W_R}{(W_R-M)^2-1} \right\}}{(W^2-M^2)(W^2M^2-(M^2-1)^2) \sqrt{(W^2-(M+1)^2)(W^2-(M-1)^2)}}$$

$$\left\{ \frac{(M^2-1)^2+2W^2-W^4+2(W^2-W_R^2)W^2}{(W^2-(M+1)^2)(W^2-(M-1)^2)} \ln \left| \frac{W^4-2M^2W^2-2W^2+W_R^2W^2}{(M^2-1)^2-W_R^2W^2} \right| - 2 \right\} +$$

$$+ \frac{3}{4} \frac{W^3[(W-M)^2-1] \left\{ \frac{3[(M^2-1)^2+2(M^2+1)W_R^2-W_R^4-2W_R^2W^2][W+2M+W_R]}{[W_R^2-(M+1)^2][W_R^2-(M-1)^2][(W_R+M)^2-1]} + \frac{W_R-2M-W}{(W_R-M)^2-1} \right\}}{(W^2-M^2)(W^2M^2-(M^2-1)^2) \sqrt{(W^2-(M+1)^2)(W^2-(M-1)^2)}}$$

$$\left\{ \left[\frac{(M^2-1)^2+2W^2-W^4+2(M^2-W_R^2)W^2}{(W^2-(M+1)^2)(W^2-(M-1)^2)} \right]^2 - \frac{1}{3} \right\} \ln \left| \frac{W^4-2M^2W^2-2W^2+W_R^2W^2}{(M^2-1)^2-W^2W_R^2} \right| - 2 \frac{(M^2-1)^2+2W^2-W^4+2(M^2-W_R^2)W^2}{(W^2-(M+1)^2)(W^2-(M-1)^2)} \quad (43)$$

Let the position of the new resonance be given by $y = W_R'$, then the new width of the resonance is given by

$$\Gamma(W_R') = \frac{\pi(W_R'^2-M^2)(W_R'^2M^2-(M^2-1)^2)}{4(2\pi)^4} \left\{ \frac{g^2}{2\pi} Y(W_R') + \frac{2\Gamma(W_R)}{3} W_R^3 Y'(W_R') \right\} \frac{1}{2W_R' [W_R'^4 - (M^2-1)^2]}$$

The evaluation of the new position of the resonance and its width is also being carried out at Dubna. Due to the vector coupling between the spin three halves particle and the pion and nucleon, an extra power of W^2 is added to the numerator so the integral now diverges as

$$\int \frac{dW \ln(W^2)}{W^2}$$

at the upper limit of integration. This fact that the integral converges is important and represents an advantage of this potential approach over the other methods. Ordinarily, the use of dispersion relations, the determinantal approach, or Chew-Low theory, would give divergent integrals which then require an arbitrary cut-off, i.e. introduce an additional undetermined parameter. This theory does not require such a cut off.

The result of this calculation could, of course, be modified by taking into account the effect of a pion pion interaction. Graphs for such an interaction would be



Fig. 5.

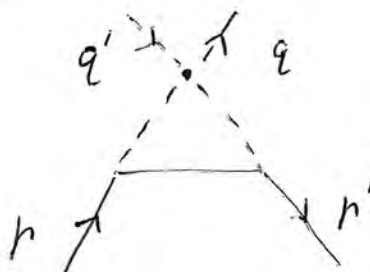


Fig. 6.

in the case of a four point interaction and degenerate into

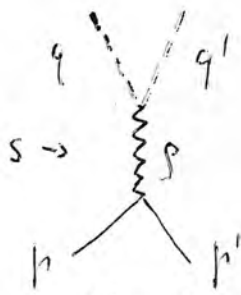


Fig. 7.

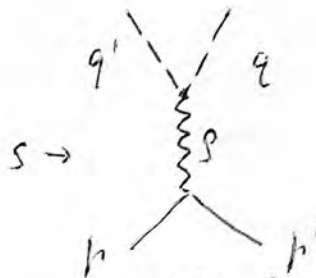


Fig. 8.

if an intermediate vector boson, the ρ meson, is postulated. Such diagrams, which are known to have an important effect on the S-wave scattering amplitudes, only slightly influence the p-wave amplitudes, so they were not taken into account. Our theory cannot predict the mass and width of the ρ meson from Figures 5 and 6, as the $T = 1, J = 1$ scattering amplitude in the t channel is zero. One would have to consider higher order graphs.

5. COMPARISON OF RESULTS WITH THOSE PREDICTED BY THE
"DETERMINANTAL" METHOD AND "BOOTSTRAPS"

As mentioned in the introduction, the determinantal method has been applied to pion nucleon scattering by Bali et al. (1961). The three resonance predicted by them arises from finding the first zero of the function

$$\operatorname{Re} D(x) = 1 - (x-M) \text{P.V.} \int_{M+\mu}^{\infty} \frac{\nu(w) dw}{(w-M)(w-x)} \quad (45)$$

where $\nu(w)$ is given by equation (29). Effectively, they derive a double dispersion relation, without cut-offs, for D . Equation (45) is obviously simpler than (24). Bali et al. predict a resonance at about 500 Mev in the laboratory system of the incident pion, which is too high. No one has solved the problem of finding the zero of equation (45) taking into account both the nucleon and isobar pole so it is not known how much the isobar pole may decrease the figure of 500 Mev.

The "bootstrap" philosophy current in present day S-matrix theory was originally proposed by Chew and Frautschi. Forces arising from the exchange of a particle give rise to a resonance at approximately the mass value of this particle and so it has, in a crude way, produced itself. In a recent paper, Abers and Zemach (1963) applied such an idea to pion nucleon scattering. In more detail, the partial wave dispersion relations have right

hand cuts from unitarity and left hand cuts from exchange diagrams. The left hand cuts are called the force cuts and may be associated with our potential. These force cuts are then supposed to give rise to resonances in the direct channel. Suppose we solve the partial wave dispersion relations by the N/D method, with N the numerator function containing the left hand cuts and D the denominator function containing the right hand cuts. Then we find a system of coupled integral equations for D and N.

$$D(x) = 1 - \frac{x-M}{\pi} \int \frac{\rho N(w) \rho(w) dw}{(w-M)(w-x)} \quad (46)$$

$$N(w) = B(w) + \frac{i}{\pi} \int \frac{\rho N(w') \rho(w')}{w'-w} \left(B(w') - \frac{w-M}{w'-M} B(w) \right) dw' \quad (47)$$

where $B(w)$ is our potential and $\rho(w)$ is the kinematical factor in the unitarity condition. One then solves equations (46) and (47) using for the potential the sum of the contributions from the exchange of a nucleon, a spin three-halves isobar and a ρ meson. For variable values of the masses and coupling constants a self-consistent solution arises at a resonance value of $W_R \approx 1160 \text{ Mev} \approx 100 \text{ Mev}$ in the laboratory system of the incident pion. This value is now too low but more accurate than that of Bali et al.

It is difficult to see how the extra factors in equation (24) will influence our results as compared to those of Bali and Abers, but we may tentatively state the following. The value of the resonance predicted by equation (24), with potentials (29) and (41) should be nearer to 190 Mev than that predicted by Bali, as we have taken into account higher order potentials. We are solving equation (24) for fixed values of the masses and coupling constants so probably our result will not be as good as that of Abers and Zemach, who use several variable parameters and pick out one self-consistent value. The actual results will be published in due course.

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