AXIALLY SYMMETRIC GRAVITATIONAL FIELDS.

by

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A thesis

presented in accordance with

the requirements for the degree of

Doctor of Philosophy

at the

University of Edinburgh.

May, 1937.

PART I.

INTRODUCTION.

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1. Arrangement of the thesis.

The present thesis consists of three parts. In the first part we give a brief historical outline of the most important results that have been obtained in the theory of axially symmetric gravitational fields. Parts II and III are devoted to two contributions by the writer to this branch of the general theory of relativity. A bibliography of papers dealing with the subject is given at the end. This bibliography has been arranged in chronological order, and references to it are made by quoting the authors name followed by the year of publication.

2. Statement of the problem.

The general theory of relativity is a more comprehensive theory than ~~tk~~ Newton's theory of gravitation. Nevertheless in so far as it replaces newtonian theory, it is legitimate to concentrate attention on the narrower aspect of the general theory of relativity as a theory of gravitation. Comparing from that point of view the general theory of relativity with newtonian theory, we notice that the main trend of researches carried out in the two theories is very different. While in the general theory of relativity the interest is mainly concentrated on cosmological solutions, in newtonian theory the object of research is the gravitational field of a finite distribution of matter, leading to a boundary condition problem. The reason is not far to seek, for one of the defects of newtonian theory is the absence of cosmological solutions other than the trivial solution of an empty universe , namely three dimensinal space and universal time. The interest of the new theory is ofcourse greatest in those fields uncovered by the older newtonian theory.

. In the present thesis however we are concerned with that aspect of the general theory of relativity which is parallel to the classical newtonian theory of gravitation. We are not concerned with the homogeneous cosmological solutions but with non-homogeneous solutions involving boundary conditions.

Corresponding to any newtonian gravitational field there exists an einsteinian gravitational field. The correspondence is not unique, for the newtonian statement of the problem is not sufficiently explicit. In the general theory of relativity it is not enough to define the shape and density of the matter producing the gravitational field, we must also state the nature of the internal stresses and strains in the matter, before the problem of determining the field becomes definite. If we suppose this is done for any newtonian gravitational field, then we are presented with the problem of determining the exact solution of Einstein's gravitational equations corresponding to this field. The boundary conditions that we lay down in this problem are that the fundamental form must nowhere be singular and that the coefficients of the fundamental form and their first derivatives must be continuous throughout the field. We do not lay down the condition that the field must tend to be galilean at infinity corresponding to the newtonian boundary condition of the vanishing of the potential at infinity, for in part II an example of a field will be given in which the above boundary conditions determine the coefficients uniquely, but which is not galilean at infinity.

This problem has been solved exactly in only a very small number of cases. There is first of all the Schwarzschild solution, corresponding to the newtonian ~~th~~ theory of an attracting particle, the problem having been made definite by supposing the attracting mass to be a spherically symmetric homogeneous incompressible liquid. In part II another solution of this type is given, corresponding to the newtonian ~~solution~~ theory of the attraction of an infinite cylinder. Here the problem has been made definite by supposing the cylinder to be a liquid, rotating with just that angular velocity which reduces the pressure everywhere to zero. The present writer is not aware of any other non-homogeneous solution of Einstein's gravitational equations.

In the case of the Schwarzschild solution the gravitational equations were found to be soluble owing to the great simplicity introduced by the assumption of spherical symmetry. In looking therefore for more general solutions it is obviously indicated to attempt to introduce a measure of symmetry less than that of complete spherical symmetry. One is thus led to the study of axially symmetric solutions of the gravitational equations. The definition of axial symmetry is not as simple as that of spherical symmetry. Though there has been no disagreement among different investigators about the canonicalisation of the fundamental form appropriate to the case of axial symmetry, no very satisfactory definition of the conception has been given. A geometrical definition has been attempted by the writer in the second paragraph of part II.

In the case of spherical symmetry it is a consequence of the gravitational equations that space-time is of the normal statical type. By a static solution we mean one in which the line-element can be put in the form

(2.1) $$ds^2 = d\sigma^2 - g_{44}\,dt^2,$$

where

$$d\sigma^2 = g_{\alpha\beta}\,dx^\alpha\,dx^\beta \qquad (\alpha, \beta = 1, 2, 3)$$

and where all the $g_{\alpha\beta}$, g_{44} are independent of t . In the case of axial symmetry this is not so, and consequently we have statical and stationary axially symmetric fields. In the stationary case the coefficients are also independent of t , but the line-element cannot be reduced to the form (2.1).

3. Statical solutions.

Weyl and independently Levi-Civita (1919), obtained the general solution for a static axially symmetric gravitational field in a region where the energy tensor vanishes. The solution may be expressed as follows. The line-element is given by

$$(3.1)\quad ds^2 = e^{2\lambda - 2\nu}(dr^2 + dz^2) + r^2 e^{-2\nu} d\phi^2 - e^{2\nu} dt^2,$$

where

$$(3.2)\quad \frac{\partial^2 \nu}{\partial r^2} + \frac{\partial^2 \nu}{\partial z^2} + \frac{1}{r}\frac{\partial \nu}{\partial r} = 0,$$

$$(3.3)\quad \frac{\partial \lambda}{\partial r} = r\left\{\left(\frac{\partial \nu}{\partial r}\right)^2 - \left(\frac{\partial \nu}{\partial z}\right)^2\right\}$$

$$(3.4)\quad \frac{\partial \lambda}{\partial z} = 2r\frac{\partial \nu}{\partial r}\frac{\partial \nu}{\partial z}$$

The equation (3.2) is seen to be the condition of integrability of (3.3) and (3.4), and hence we have field satisfying the gravitational equations corresponding to any solution of (3.2). The equation (3.2) is seen to be Laplace's equation in cylindrical coordinates, with the term $\frac{\partial^2 \nu}{\partial \phi^2}$ missing.

The axis of symmetry is given by $r = 0$, and if this axis is not to be a singular line in the geometry of space-time, we must have $\lambda = 0$ on the axis. This is easily seen by considering a small circle round the axis in a z-plane, and determining whether the ratio of its circumference to its radius approaches 2π as r approaches zero.

Levi-Civita (1919) applied the theory to obtain the field of an infinite cylinder. Assuming ν to be a function of r only, he found

$$(3.6) \quad ds^2 = \left(\frac{r}{r_0}\right)^{2(k^2-k)} (dr^2 + dz^2) + r^2 \left(\frac{r}{r_0}\right)^{-2k} d\phi^2 - \left(\frac{r}{r_0}\right)^{2k} dt^2$$

He also showed that in this field the gravitational force varies proportionally to the power α of the geodesic distance from the axis, where

$$(3.7) \quad \alpha = -1 + \frac{2k}{k^2 - k + 1}$$

To obtain further cases of his general solution, Weyl introduced an auxiliary space by interpreting r, z, ϕ

as cylindrical coordinates in a euclidean space. Given a particular distribution of matter in the auxiliary space, we find the ordinary newtonian potential and this gives an exact solution when substituted in (3.1).

Weyl's solution is the most general one for this type of field, and hence must contain the Schwarzschild solution as a particular case. Weyl showed that the Schwarzschild solution is obtained by considering the potential of a uniform bar along the z-axis in the auxiliary space. Analytically this is expressed as follows. If we transform (3.1) with the transformation

$$(3.7) \qquad r + iz = a \sinh(x + iy),$$

then the function ν giving rise to the external Schwarzschild solution is

$$(3.8) \qquad \nu = \log \tanh \frac{x}{2}.$$

Several investigators have applied themselves to the problem of determining the field due to two spheres. It is obvious that a statical solution of this type does not exist, and the problem therefore is of rather academic interest, but we can give it a certain measure of physical interest by supposing the spheres to be kept at rest by a rigid structure of negligible mass.

The basis of these attempts has been the generalisation of the preceding result. The field of one mass centre is given by the potential of a uniform bar in the auxiliary space. This suggests finding the line-element arising from two uniform bars in the ~~canonical space~~ auxiliary space. Three arbitrary constants will appear m_1, m_2, and d, which are to be interpreted as the masses and the distance between them. Solutions have been given by Bach (1922), Palatini (1923), and Chazy (1924), ~~of~~ of which Bach's solution appears to be the best.

Bach obtains the form (3.1) where

$$(3.9)\quad e^{2\lambda} = \frac{(r_1^2+r_2^2)-4m_1^2}{4r_1r_2}\cdot\frac{(r_3+r_4)-4m_2^2}{4r_3r_4}\left\{\frac{d(m_1+d)r_1+d(m_1+m_2+d)r_2-m_1dr_4}{d(m_1+d)r_1+(m_1+d)(m_2+d)r_2-m_1(m_2+d)r_4}\right\}$$

and

$$(3.10)\quad e^{2\nu} = \frac{r_1+r_2-2m_1}{r_1+r_2+2m_1}\cdot\frac{r_3+r_4-2m_2}{r_3+r_4+2m_2}.$$

and where r_1, r_2, r_3, r_4, are the distances of the point r, z, ϕ, to the extremities of the two bars in the canonical space. The two bars are situated on the z-axis the first bar between z_1 and z_2, the second bar between z_3 and z_4. We see that on the axis $\lambda = 0$ for $z > z_1$ or $z < z_4$ but on the stretch $z_2 > z > z_3$, λ has a constant value

$$\lambda_0 = \log \frac{d(m_1+m_2+d)}{(m_1+d)(m_2+d)} \tag{3.11}$$

and hence the line-element is singular at all points on the axis between the mass centres.

Weyl (1922), in an addition to Bach's paper, supposed that the singularity was removed by filling the region in the neighbourhood of the portion of the axis between the masses with a medium capable of retaining them in equilibrium with suitable stresses. Calculating the flux of force across a surface normal to the axis between the masses, he found for the force of attraction K betweem them the formula

$$K = \frac{2\pi}{K} \log \frac{(m_1+d)(m_2+d)}{d(m_1+m_2+d)}. \tag{3.12}$$

If 2d is much greater than m_1 or m_2, this givesa approximately

$$K = \frac{G m_1 m_2}{(2d)^2}, \tag{3.13}$$

the correct newtonian expression, if 2d is the distance between the masses.

Other papers dealing with Weyl's theory are not of very much interest. Chou (1931) has attempted to apply Weyl's theory to obtain the field of a spheroidal homeoid. Chou himself however remarks that the line-element which he obtains can be transformed into the Schwarzschild form, but making the observation that "the interpretation of the coordinates is not the same". Further we may mention three papers by Straneo (1924), devoted to a discussion of certain difficulties which appear to have occurred to the writer in connection with Weyl's theory.

An excellent review of the development of the subject up to 1927 will be found in Darmois' article in the Memorial des Sciences Mathematiques (1927).

4. Stationary solutions.

The stationary case differs from the static case in the presence of an additional coefficient in the canonical form of the line-element. A general exact solution of the gravitational equations for the stationary case was obtained by Lewis. This solution contains Weyl's solution as a particular case, but unlike Weyl's solution it is not the most general solution. A simple derivation of Lewis' solution is given in paragraph 9 of part II. Lewis used his solution to obtain the external field of a rotating cylinder. The gravitational equations in the interior of rotatating non-interacting matter, are considered by the present writer in part II. A general exact solution is found, and the field in the interior of an infinite rotating cylinder are deduced as a particular case. Associating the internal solution thus obtained with Lewis' external solution, the problem of the field of a rotating cylinder can be completely solved, the boundary conditions determining uniquely the arbitrary constants occurring in the solution in terms of the physical dimensions of the system.

Exact solutions in the case of a rotating liquid have not so far been obtained. In this case we must content ourselves with approximate solutions.

The external field of a rotating sphere was first investigated by Lense and Thirring (1918). They used their result to ~~obtain~~ determine the effect of the rotation of a central body on the advance of the perihelion of ~~x~~ a planetary body. They found a slight increase in the advance of the perihelion but below the limit of observation.

Second approximations to the field of a rotating sphere were obtained by Bach (1922) and by Lewis (1932). A second approximation to the more general case of the field of a rotating MacLaurin ellipsoid was obtained by Akeley (1931).

In part III of this thesis the ~~general~~ problem of the gravitational field of a general rotating liquid is discussed, and the general solution given correct to the second order.

The theory is applied to determine the field of a slowly rotating sphere. Comparison of the results here obtained with those of Bach and Lewis is not easy, as these writers were not able to express the arbitrary constants occurring in their solutions in terms of the physical dimensions of the system. The result we obtain is however checked by comparison with the Schwarzschild solution, to which it must ofcourse tend when the angular velocity tends to zero.

PART II.

THE GRAVITATIONAL FIELD OF A DISTRIBUTION OF PARTICLES ROTATING ABOUT AN AXIS OF SYMMETRY.

IX.—The Gravitational Field of a Distribution of Particles Rotating about an Axis of Symmetry. By W. J. van Stockum, Mathematical Institute, University of Edinburgh. *Communicated by* Professor E. T. WHITTAKER, F.R.S.

(MS. received July 2, 1936. Revised MS. received December 9, 1936. Read November 2, 1936.)

§ 1. INTRODUCTION.

THE generalisation to the stationary case of the solution given by Weyl (1918) of Einstein's gravitational equations in a statical axially symmetric universe, has been attempted by various writers. It has not, however, so far been possible to reduce the equations to linear form and thus to find the most general solution. A set of special solutions has been obtained by Lewis (1932), valid in a region free of matter, which contain Weyl's solution as a particular case. They depend upon an arbitrary solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{\partial^2 V}{\partial z^2}+\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

In the first part of the present paper the gravitational equations are considered in the interior of an axially symmetric distribution of particles rotating with constant angular velocity about its axis of symmetry. Solutions of the equations are found depending upon an arbitrary solution of the equation

$$\frac{\partial^2 V}{\partial r^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{r}\frac{\partial V}{\partial r}=0.$$

In the second part the field of an infinite rotating cylinder is considered, and the internal solution obtained by the method of the present paper is associated with the external solution given by Lewis. The boundary conditions, namely the continuity of the coefficients of the fundamental form and their first derivatives across the surface of the cylinder, determine uniquely the constants occurring in Lewis's solution in terms of the physical dimensions of the system. It appears that there are two essentially different types of external field of a rotating cylinder, according as the radius of the cylinder is less or greater than a certain critical value. In the first case, the geodesic planes normal to the axis of symmetry are infinite and tend to euclidean planes at infinity; in the second case, these planes are finite and closed. The external field of a rotating cylinder given by Lewis corresponds to the case where the radius is less than the critical value.

§ 2. Definition of Axial Symmetry.

It is customary to define stationary axially symmetric space-time to be such that by a suitable choice of co-ordinates the fundamental form assumes a certain simple expression. As however, in what follows, we shall allow ourselves to be guided by geometrical intuition in defining the energy tensor of a rotating system of particles, it may be more consistent to give a geometrical definition of axial symmetry, and to deduce the particularisation of the fundamental form from our definition.

Stationary axially symmetric space-time we define as follows. The universe contains a privileged observer O, whose world-line is a time-like geodesic g. The observer O separates space-time into space and time by referring to the 3-spaces S formed by all geodesics at O normal to g as space and to some suitably chosen parameter t defining his position on g as time. The universe is said to be stationary if with the passing of time the observer O detects no change in the intrinsic geometry of the space S. If the parameter t is used as one of the co-ordinates, it follows that the coefficients of the fundamental form must be independent of t. We now say that, in addition to being stationary, the universe is axially symmetric if at any instant there exists in S a privileged geodesic a, passing through O, which is such that at any point of it all directions in S normal to a are intrinsically indistinguishable. We have then at every point of g a privileged geodesic a normal to it. We shall now show that it follows from the definition of axial symmetry that the unit tangent vectors at g to the geodesics a are parallel in Levi-Civita's sense. For suppose this is not the case; then selecting the S, passing through the point O on g, we obtain a cone of directions at O by propagating the unit tangent vectors to the geodesics a parallelly along g to O. Considering the geodesics in S defined by this cone of directions, we obtain a surface in S, containing the privileged geodesic a of S. But this surface would define, at all points of a, privileged directions in S normal to a, namely those tangent to the surface, which contradicts the assumption of axial symmetry.

We may now choose a system of co-ordinates as follows. At an arbitrary point of g we select two unit vectors in S which, with the unit tangent vector to a, form an orthogonal triad. We can use the triad to set up in this S a system of geodesic polar co-ordinates, r being the length of the geodesic joining an arbitrary point to O, θ the angle between the tangent to this geodesic and the tangent to a at O, and ϕ the azimuthal angle. Propagating the triad parallelly along g, we have defined a triad in every S and r, θ, ϕ, t, can now be used as co-ordinates of space-time. From the definition of axial symmetry it follows at once that the coefficients of the fundamental form must be independent of ϕ. We can show that in the present system of co-ordinates the t-lines will be normal to the r-lines. This is seen by noting that the t-line of a point, the r, θ, ϕ co-ordinates of which are kept fixed, is described by propagating the geodesic radius vector, which is normal to g, parallelly along it, and it is well known that its extremity then describes a curve which is at all points normal to the geodesic radius vector. It can further be shown to follow from the definition of axial symmetry that the t-lines are normal to the θ-lines. Consider the surface Σ in S formed by all the points geodesically equidistant from O, and let this surface intersect a in a point P. The directions of the t-lines at points of S can be projected on to S. Suppose this is done at all points of S lying on Σ. Then since the t-lines are normal to the r-lines the projections will lie in Σ, forming a congruence on Σ. Now, since all directions at P in Σ must be equivalent, this congruence must lie symmetrically about P, and hence must cut the θ-lines orthogonally. It follows that the t-lines are normal to the θ-lines. From these last two results it follows that the product terms $drdt$ and $d\theta dt$ are absent in the fundamental form, which will then be of the form

$$ds^2 = dr^2 + Ad\theta^2 + Bd\phi^2 + Cd\phi dt + Ddt^2$$

where the coefficients are functions or r and θ only.

§ 3. Calculation of Ricci Tensor.

For analytical purposes the co-ordinate system established in the preceding section is not the most convenient. We therefore apply the transformation

$$(3.1) \quad . \quad . \quad x^1 = x^1(r, \theta), \qquad x^2 = x^2(r, \theta), \qquad x^3 = \phi, \qquad x^4 = t,$$

thus obtaining the slightly more general form

$$(3.2) \quad . \quad . \quad ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{mn}dx^m dx^n, \qquad \begin{pmatrix} \alpha, \beta = 1, 2 \\ m, n = 3, 4 \end{pmatrix}$$

where the $g_{\alpha\beta}$ and the g_{mn} are functions of the co-ordinates x^1 and x^2 only. In what follows, unless the contrary be explicitly stated, it will be understood that Greek indices are to have the range 1, 2 and Roman indices the range 3, 4. The summation convention is adhered to, with the understanding that repeated Greek and Roman indices are to be summed through the ranges 1, 2 and 3, 4 respectively.

We note that the transformation (3.1) can always be chosen so that $g_{11} = g_{22}$ and $g_{12} = 0$. If we suppose this to have been done, we may write (3.2)

$$(3.3) \quad ds^2 = e^{2\psi}(dx^1dx^1 + dx^2dx^2) + l dx^3dx^3 + 2m dx^3dx^4 - f dx^4dx^4.$$

It will be convenient first of all to obtain the expressions for the components of the Ricci tensor of the form (3.2), in tensor form with regard to transformations of the type (3.1). We notice that with respect to this group of transformations the functions g_{mn}, g^{mn}, D, where D is defined by

$$D^2 = -|g_{mn}|,$$

transform like invariants, and that their partial derivatives transform like vectors in the (x^1, x^2)-surfaces. We denote partial differentiation by small Greek suffixes. These suffixes can then be raised and lowered in the usual way with respect to the form

$$(3.4) \quad d\sigma^2 = g_{\alpha\beta} dx^\alpha dx^\beta.$$

Covariant derivation with respect to (3.4) we denote by a comma preceding the suffix, or alternatively by the symbol $\frac{\delta}{\delta x^\alpha}$. For the Christoffel symbols of the form (3.2), containing the indices 3 or 4, we have the formulæ

$$(3.5) \quad \begin{cases} \begin{Bmatrix} l \\ mn \end{Bmatrix} = 0, & \begin{Bmatrix} m \\ \alpha\beta \end{Bmatrix} = 0, \quad \begin{Bmatrix} \alpha \\ \beta m \end{Bmatrix} = 0, \\ \begin{Bmatrix} m \\ \alpha n \end{Bmatrix} = \tfrac{1}{2} g^{ms} g_{ns\alpha}, & \begin{Bmatrix} \alpha \\ mn \end{Bmatrix} = -\tfrac{1}{2} g^{\alpha}_{mn}. \end{cases}$$

Substituting these values in the usual expressions for the components of the Ricci tensor in terms of the Christoffel symbols, we obtain, after some simplification, the following formulæ for the non-vanishing components

$$(3.6) \quad \begin{cases} R^\alpha_\beta = K^\alpha_\beta + D^{-1} D^\alpha_{,\beta} - D^{-1} D^\alpha D_\beta - \tfrac{1}{4} g^\alpha_{mn} g^{mn}_\beta, \\ R^m_n = \tfrac{1}{2} D^{-1} \dfrac{\delta}{\delta x^\alpha}(D g^{ms} g^\alpha_{ns}), \end{cases}$$

where K^α_β is the Ricci tensor of (3.4). Writing these formulæ explicitly for the form (3.3), we obtain

$$(3.7) \quad \begin{cases} \sqrt{(-g)} R^\alpha_\beta = D \Delta\psi \delta^\alpha_\beta + D_{\alpha,\beta} - \tfrac{1}{4} D^{-1}(l_\alpha f_\beta + l_\beta f_\alpha + 2 m_\alpha m_\beta), \\ \sqrt{(-g)} R^3_3 = \dfrac{1}{2} \dfrac{\partial}{\partial x^\alpha} \left[\dfrac{f l_\alpha + m m_\alpha}{D} \right], \\ \sqrt{(-g)} R^3_4 = \dfrac{1}{2} \dfrac{\partial}{\partial x^\alpha} \left[\dfrac{f m_\alpha - m f_\alpha}{D} \right], \\ \sqrt{(-g)} R^4_3 = \dfrac{1}{2} \dfrac{\partial}{\partial x^\alpha} \left[\dfrac{m l_\alpha - l m_\alpha}{D} \right], \\ \sqrt{(-g)} R^4_4 = \dfrac{1}{2} \dfrac{\partial}{\partial x^\alpha} \left[\dfrac{l f_\alpha + m m_\alpha}{D} \right], \\ \sqrt{(-g)}(R^3_3 + R^4_4) = \Delta D, \end{cases}$$

where Δ is the ordinary Laplacian operator in the variables x^1 and x^2.

§ 4. The Energy Tensor.

We consider now the energy tensor of a distribution of particles rotating round the axis of symmetry. We suppose that from the point of view of the observer O of § 2, the particles are describing the ϕ-lines in the space S. It follows then that the first two components of the unit tangent vector to their world-lines vanish. We furthermore suppose that the particles are describing their paths without mutual interaction, the world-line of each particle being a geodesic in space-time. The energy tensor of such a system of particles is of the form

$$T^r_s = \mu\lambda^r\lambda_s, \qquad (r, s = 1, 2, 3, 4)$$

where μ is the density of the particles and λ^r is the unit tangent vector to their world-lines. In order to satisfy the condition of axial symmetry, it is of course necessary that the density μ be a function of x^1 and x^2 only. In the present case we have $\lambda^1 = \lambda^2 = 0$. If we write $\Omega = \lambda^3/\lambda^4$, we find for the components of the unit tangent vector

$$\lambda^3 = \frac{\Omega}{\sqrt{(f - 2\Omega m - \Omega^2 l)}},$$
$$\lambda^4 = \frac{1}{\sqrt{(f - 2\Omega m - \Omega^2 l)}}.$$

We obtain for the non-vanishing components of the energy tensor the expressions

$$(4.1) \quad \begin{cases} T^3_3 = \mu\dfrac{\Omega^2 l + \Omega m}{f - 2\Omega m - \Omega^2 l}, & T^3_4 = \mu\dfrac{\Omega^2 m - \Omega f}{f - 2\Omega m - \Omega^2 l}, \\ T^4_3 = \mu\dfrac{\Omega l + m}{f - 2\Omega m - \Omega^2 l}, & T^4_4 = \mu\dfrac{\Omega m - f}{f - 2\Omega m - \Omega^2 l}. \end{cases}$$

The vanishing of the divergence of the energy tensor gives the equations

$$(4.2) \quad T^{rs}_{\cdot\cdot, s} = \frac{\partial\mu}{\partial x^s}\lambda^r\lambda^s + \mu\lambda^r_{\cdot, s}\lambda^s + \mu\lambda^r\lambda^s_{\cdot, s} = 0. \quad (r, s = 1, 2, 3, 4).$$

Since $\lambda^a = 0$, and $\dfrac{\partial\mu}{\partial x^m} = 0$, the first term in (4.2) disappears. Using the fact that $\lambda^a = \lambda_a = 0$ and that λ^m is a function of x^1 and x^2 only, we find that

$$(4.3) \quad \begin{cases} \lambda^m{}_{,n} = \lambda^a{}_{,\beta} = 0, \\ \lambda^m{}_{,a} = \dfrac{\partial\lambda^m}{\partial x^a} + \begin{Bmatrix} m \\ an \end{Bmatrix}\lambda^n, \\ \lambda^a{}_{,m} = \begin{Bmatrix} a \\ mn \end{Bmatrix}\lambda^n. \end{cases}$$

Substituting from (4.3) in (4.2), the latter reduce to the two equations

$$\mu\begin{Bmatrix} a \\ mn \end{Bmatrix}\lambda^m\lambda^n = 0 \tag{4.4}$$

which, using (3.5), may be written

$$\mu(f_a - 2\Omega m_a - \Omega^2 l_a) = 0. \tag{4.5}$$

We now suppose that Ω is a constant, and that $\mu \neq 0$, then the equations (4.5) can be integrated at once, giving

$$f - 2\Omega m - \Omega^2 l = \text{constant}. \tag{4.6}$$

It is important to notice that the equation (4.6) holds only when $\mu \neq 0$. We now adopt a new system of co-ordinates with the transformation

$$\bar{x}^1 = x^1, \qquad \bar{x}^2 = x^2, \qquad \bar{x}^3 = x^3 - \Omega x^4, \qquad \bar{x}^4 = x^4. \tag{4.7}$$

This transformation constitutes a change to a rotating system of reference, relative to which the matter is at rest. If the bars be omitted, which may be done without danger of ambiguity, the fundamental form becomes

$$ds^2 = e^{2\psi}(dx^1dx^1 + dx^2dx^2) + Ldx^3dx^3 + 2Mdx^3dx^4 - Fdx^4dx^4, \tag{4.8}$$

where

$$\begin{cases} L = l, \\ M = m + \Omega l, \\ F = f - 2\Omega m - \Omega^2 l. \end{cases} \tag{4.9}$$

If we denote the components of the energy tensor in the new system of co-ordinates by accented letters, we find

$$\begin{cases} 'T^3_3 = T^3_3 - \Omega T^4_3, \\ 'T^3_4 = \Omega T^3_3 + T^3_4 - \Omega^2 T^4_3 - \Omega T^4_4, \\ 'T^4_3 = T^4_3, \\ 'T^4_4 = \Omega T^4_3 + T^4_4, \end{cases} \tag{4.10}$$

the remaining components being zero as before. If we now substitute from (4.1), using (4.9), we find that the quantity Ω disappears from the expressions, and we obtain, finally,

$$\begin{cases} 'T^3_3 = 0, & 'T^3_4 = 0, \\ 'T^4_3 = \mu\dfrac{M}{F}, & 'T^4_4 = -\mu. \end{cases} \tag{4.11}$$

The equation (4.6), yielded by the vanishing of the divergence of the energy tensor, becomes in the present system of co-ordinates

$$F = \text{constant}. \tag{4.12}$$

§ 5. The Gravitational Equations.

Since the fundamental form (4.8) is of exactly the same type as (3.3), we obtain the expressions for the components of the Ricci tensor in the system of co-ordinates of § 4, by writing F, L, M, for f, l, m, in (3.7). We write the gravitational equations in the form

$$(5.1) \quad . \quad . \quad . \quad R^i_j = -\kappa(T^i_j - \tfrac{1}{2}T\delta^i_j), \qquad (i, j = 1, 2, 3, 4)$$

where we may, without ambiguity, omit the accents relating to the present system of co-ordinates. We have, from (4.11),

$$T = T^4_4 = -\mu,$$

and hence we deduce from (5.1),

$$R^3_3 + R^4_4 = 0,$$

and this gives, by (3.7),

$$(5.2) \quad . \quad . \quad . \quad . \quad . \quad \Delta D = 0.$$

On the basis of this equation we may proceed with the introduction of Weyl's canonical co-ordinates, the application of which to the stationary case was first noted by Lewis (*loc. cit.*). We define a transformation

$$(5.3) \quad . \quad . \quad . \quad . \quad r = D, \qquad z = D',$$

where D' is a function of x^1 and x^2 such that

$$D + iD' = f(x^1 + ix^2),$$

which, if (5.2) is satisfied, is always possible. This transformation leaves the fundamental form of the (x^1, x^2)-surfaces in the isothermal form, and hence occasions no change in the expressions for the Ricci tensor. We suppose the transformation to the co-ordinates r, z to have been effected, but we will retain the indicial notation whenever convenient, suffixes 1 and 2 referring to differentiation with respect to r and z respectively. We may then put $D = r$ in all formulæ thus far calculated. For $D_{\alpha,\beta}$ we have

$$(5.4) \quad . \quad . \quad D_{1,1} = -\psi_1, \qquad D_{1,2} = -\psi_2, \qquad D_{2,2} = \psi_1.$$

Considering now the remaining gravitational equations, if we substitute from (3.7), (4.11), and (5.4) in (5.1), and write for convenience $\rho = \kappa\mu\sqrt{(-g)}$, we obtain after some simplification:

$$(5.5) \quad . \quad . \quad . \quad \Delta\psi = \frac{1}{4r^2}(L_1F_1 + L_2F_2 + M_1^2 + M_2^2) - \frac{\rho}{2r},$$

$$(5.6) \qquad \psi_1 = -\frac{1}{4r}(L_1F_1 + M_1^2 - L_2F_2 - M_2^2),$$

$$(5.7) \qquad \psi_2 = -\frac{1}{4r}(L_1F_2 + L_2F_1 + 2M_1M_2),$$

$$(5.8) \qquad \frac{\partial}{\partial x^a}\left[\frac{FM_a - MF_a}{r}\right] = 0,$$

$$(5.9) \qquad \frac{\partial}{\partial x^a}\left[\frac{FL_a - LF_a}{r}\right] = -2\rho,$$

$$(5.10) \qquad \frac{\partial}{\partial x^a}\left[\frac{ML_a - LM_a}{r}\right] = -2\rho\frac{M}{F}.$$

§ 6. Solution of the Gravitational Equations.

We first of all remark that the equations (5.8) to (5.10) are not independent; (5.10) may be deduced from (5.8) and (5.9). To obtain the general solution of the equations we make use of the relation (4.12). Putting

$$(6.1) \qquad F = 1,$$

the equations (5.5) to (5.9) become

$$(6.2) \qquad \Delta\psi = \frac{1}{4r^2}(M_1^2 + M_2^2) - \frac{\rho}{2r},$$

$$(6.3) \qquad \psi_1 = -\frac{1}{4r}(M_1^2 - M_2^2),$$

$$(6.4) \qquad \psi_2 = -\frac{1}{2r}M_1M_2,$$

$$(6.5) \qquad M_{11} + M_{22} - \frac{1}{r}M_1 = 0,$$

$$(6.6) \qquad L_{11} + L_{22} - \frac{1}{r}L_1 = -2r\rho.$$

We notice that the equation (6.5) expresses the condition of integrability of the equations (6.3) and (6.4). We may therefore attempt to obtain solutions of the equations by choosing a function M satisfying (6.5). The equations (6.3) and (6.4) being integrable then determine ψ. We are then left with the equations (6.2) and (6.6) to determine L and ρ.

The functions L and M, however, are not independent. Owing to the particular choice of the co-ordinate system we have from (5.3)

$$r^2 = FL + M^2, \tag{6.7}$$

and hence by (6.1)

$$L = r^2 - M^2. \tag{6.8}$$

Substituting from (6.8) in (6.6), and using the fact that M satisfies (6.5), the equation (6.6) becomes

$$M_1^2 + M_2^2 = r\rho. \tag{6.9}$$

We may consider this equation to define the density distribution, and then the equations (6.3) to (6.6) are all satisfied. If we now calculate $\Delta\psi$ from (6.3) and (6.4), and substitute the result in (6.2), this equation becomes identical with (6.9), so that it also is satisfied. We have therefore shown that the general solution of the equations depends upon an arbitrary solution of the equation (6.5).

If we write $\phi = x^3$, $ct = x^4$, we have for the fundamental form

$$ds^2 = e^{2\psi}(dr^2 + dz^2) + (r^2 - M^2)d\phi^2 + 2Mcd\phi dt - c^2dt^2, \tag{6.10}$$

where M is any solution of (6.5), ψ is determined from (6.3) and (6.4), and where the density μ is given by the equation

$$\kappa\mu = \frac{1}{r^2}e^{-2\psi}(M_1^2 + M_2^2). \tag{6.11}$$

We cannot deduce solutions for the external field from the present solution by putting $\mu = 0$, for we then see from (6.11) that this implies $M =$ constant, and the resulting solution is trivial, space-time being then galilean. The reason is that the equation (6.1) was deduced from the vanishing of the divergence of the energy tensor on the supposition $\mu \neq 0$, and hence does not necessarily hold when the energy tensor vanishes.

§ 7 The Field of an Infinite Rotating Cylinder.

We now consider the particular solution of the equations which is obtained by supposing M to be a function of r only. The equation (6.5) then reads

$$M_{11} - \frac{1}{r}M_1 = 0,$$

and this yields on integration

$$M = ar^2, \tag{7.1}$$

where a is a constant of integration. Determining the function ψ from (6.3) and (6.4), we find

$$e^{2\psi}=e^{-a^2r^2}.$$

The equation (6.11) for the density now reads

$$\kappa\mu=4a^2e^{-a^2r^2}. \tag{7.2}$$

We obtain therefore the fundamental form

$$ds^2=H(dr^2+dz^2)+Ld\phi^2+2Md\phi dt-Fdt^2, \tag{7.3}$$

where

$$\begin{cases} H=e^{-a^2r^2}, & L=r^2(1-a^2r^2), \\ M=acr^2, & F=c^2. \end{cases} \tag{7.4}$$

The present system of co-ordinates constitutes, as we have seen, a system of reference relative to which the matter composing the cylinder is at rest. We define the angular velocity of the cylinder to be the angular velocity relative to a non-rotating system of reference associated with an observer on the axis of symmetry, using Walker's definition of non-rotating (1935). Walker defines a non-rotating system of reference for an observer to be such that in it the acceleration of a free isolated particle in the neighbourhood of the observer is independent of its velocity. This clearly corresponds to what is meant by a non-rotating system of reference in Newtonian dynamics. We may call such a system a dynamical rest frame for the observer. Walker has shown that for such a system the unit tangent vectors in the direction of the space axes must be defined by Fermi-transport along the world-line of the observer, which, since the world-line in the present instance is a geodesic, reduces to ordinary parallel transport.

Let us denote the unit tangent vector to the r-lines by ξ^i $(i=1, 2, 3, 4)$. Then if ξ^i is transported parallelly along the world-line of the observer the equations

$$\frac{\partial\xi^i}{\partial x^a}\frac{dx^a}{ds}+\left\{ {i \atop ab} \right\}\xi^a\frac{dx^b}{ds}=0 \quad (i, a, b=1, 2, 3, 4) \tag{7.5}$$

must be satisfied at the origin. We have

$$\begin{cases} \xi^i=(H^{-\frac{1}{2}}, 0, 0, 0), \\ \dfrac{dx^a}{ds}=(0, 0, 0, F^{-\frac{1}{2}}). \end{cases} \tag{7.6}$$

Substituting from (7.6) in (7.5), and using (3.5), we obtain the two equations

$$D^{-2}(FM_1-MF_1)H^{-\frac{1}{2}}F^{-\frac{1}{2}}=0, \tag{7.7}$$

$$D^{-2}(MM_1+LF_1)H^{-\frac{1}{2}}F^{-\frac{1}{2}}=0. \tag{7.8}$$

We now apply the transformation (4.7), where we write Ω' instead of Ω, to the form (7.3), and calculate the expressions on the left-hand side of (7.7) and (7.8) for this new form, obtaining the equations

$$r(c - a\Omega' r^2) = 0, \tag{7.9}$$

$$\frac{\Omega' + ac}{r} - 2a^2 r\Omega' + a^3 r^3 c^{-1}\Omega'^2 = 0. \tag{7.10}$$

These equations are satisfied at the origin only if

$$\Omega' = -ac. \tag{7.11}$$

It may be shown similarly that the unit tangent vector to the z-lines is propagated parallelly along the world-line of the observer, independent of the value of Ω'. We therefore obtain the expression for the fundamental form in a non-rotating system of co-ordinates, by applying the transformation (4.7), where we substitute for Ω the value given by (7.11). When this is done we obtain the form (7.3), where

$$\begin{cases} H = e^{-a^2 r^2}, & L = r^2(1 - a^2 r^2), \\ M = a^3 c r^4, & F = c^2(1 + a^2 r^2 + a^4 r^4). \end{cases} \tag{7.12}$$

§ 8. Interpretation of the Solution.

We see from (7.11) that the angular velocity ω of the cylinder is given by

$$\omega = ac. \tag{8.1}$$

If we denote the density of the cylinder on the axis of symmetry by μ_0, we find from (7.2), if we substitute for κ its value in terms of Newton's gravitational constant γ, that

$$a^2 c^2 = 2\pi\gamma\mu_0, \tag{8.2}$$

and hence we have for the angular velocity

$$\omega = \sqrt{2\pi\gamma\mu_0}. \tag{8.3}$$

If we suppose the cylinder to be of the density of water on the axis of symmetry, so that $\mu_0 = 1$, the period of rotation of the cylinder is approximately 2 hours 42 minutes. If we denote by R the value of r on the boundary of the cylinder, then we see from (7.12) that we must have

$$aR < 1,$$

otherwise the coefficient of $d\phi^2$ in the fundamental form is negative in the interior of the cylinder. For a cylinder of given density therefore there is an upper limit for the radius. Since $\omega = ac$, the inequality may also be written

$$\omega R < c,$$

so that the upper limit of R is the same as the upper limit of the radius of a rotating cylinder in the special theory of relativity. The quantity R, however, is not the radius of the cylinder but connected with it by the equation

$$R' = \int_0^R e^{-\frac{1}{2}a^2r^2} dr,$$

where R' denotes the radius. If $\mu_0 = 1$ we find that the maximum radius is approximately $3 \cdot 5 \times 10^8$ K.M.

We now consider with what angular velocity a particle is to describe a ϕ-line if its world-line is to be a geodesic in space-time. We write the equations of the geodesics in the Lagrangian form

$$\frac{d}{ds}\frac{\partial T}{\partial x'^i} - \frac{\partial T}{\partial x^i} = 0, \qquad (i = 1, 2, 3, 4)$$

where

$$T = H(r'^2 + z'^2) + L\phi'^2 + 2M\phi' t' - Ft'^2,$$

dashes denoting differentiation with respect to the arc s. We attempt to satisfy these equations by putting r = constant, z = constant. The only pertinent equation is then seen to be

$$\frac{d}{ds}\frac{\partial T}{\partial r'} - \frac{\partial T}{\partial r} = 0,$$

which gives a quadratic for $d\phi/dt$, namely

$$L_1 d\phi^2 + 2M_1 d\phi dt - F_1 dt^2 = 0. \tag{8.4}$$

Substituting from (7.12) in (8.4), and solving the quadratic, we obtain the roots

$$\omega_1 = ac, \tag{8.5}$$

$$\omega_2 = -\frac{1 + 2a^2r^2}{1 - 2a^2r^2} ac. \tag{8.6}$$

The first root gives the angular velocity of the cylinder, verifying that the world-lines of the particles composing the cylinder are geodesics. If we suppose a thin tube hollowed out in the cylinder along a ϕ-line, the second root gives the angular velocity with which a particle must be endowed in order to traverse the tube in a sense contrary to the sense of rotation of the cylinder.

The unit tangent vectors to the world-lines of the particles composing the cylinder must always be time-like. To investigate whether this is the case, we consider the null-directions in the (ϕ, t)-surfaces. These are given by

$$L d\phi^2 + 2M d\phi dt - F dt^2 = 0. \tag{8.7}$$

Solving this quadratic we obtain the values of $d\phi/dt$ corresponding to the null-directions. They are found to be

$$\Omega_1 = \frac{1 - a^3r^3}{1 - a^2r^2}\frac{ac}{ar}, \tag{8.8}$$

$$\Omega_2 = -\frac{1 + a^3r^3}{1 - a^2r^2}\frac{ac}{ar}. \tag{8.9}$$

We see from (8.8) that, for all values of r, we have $\Omega_1 > ac$, and hence the tangent vectors to the world-lines of the particles are always time-like. Comparing (8.6) and (8.9), we see that when $ar = \frac{1}{2}$, we have $\Omega_2 = \omega_2$. Hence when $ar \longrightarrow \frac{1}{2}$, the velocity with which a particle must be projected in order to describe a ϕ-line, in a sense opposite to the sense of rotation of the cylinder, tends to the velocity of light. At points where r exceeds this value it will be impossible for a particle to describe a ϕ-line in this sense. If we consider the case of a cylinder whose density and radius are such that $aR = \frac{1}{2}$, then on the boundary a light signal sent out in the direction of a ϕ-line, and in a sense opposite to the sense of rotation of the cylinder, will travel along the ϕ-line. An observer therefore on the surface of the cylinder will be able to look right round the cylinder. Assuming $\mu_0 = 1$, we find that if the observer is at rest on the surface of the cylinder, a ray of light sent out by him returns in approximately 40 minutes.

Returning to equation (8.6), we see that when $ar = \frac{1}{\sqrt{2}}$, the angular velocity ω_2 becomes infinite. The ϕ-line is then a space-like geodesic. The length l of a ϕ-line is given by

$$l = \int_0^{2\pi} r(1 - a^2r^2)^{\frac{1}{2}} d\phi = 2\pi r(1 - a^2r^2)^{\frac{1}{2}},$$

and we see that l is a maximum when $ar = \frac{1}{\sqrt{2}}$. As r increases beyond this value the length of successive ϕ-lines diminishes. When $ar \longrightarrow 1$ the length of the corresponding ϕ-line$\longrightarrow 0$. If the cylinder is such that $aR = 1$, then all geodesics issuing from the origin in the planes $z = \text{constant}$, meet again on the boundary, which then reduces to a line, the antipodal line of the axis of symmetry. We will return to this point when we consider the external solution.

Before proceeding to the question of the external field and boundary conditions, we may consider the Newtonian analogue of the present solution. If in Newtonian potential theory an infinite liquid cylinder of uniform density is endowed with a constant angular velocity Ω, there exists a definite value for Ω, which will reduce the pressure everywhere to zero. If μ is the density of the cylinder and γ the gravitational constant, we find for Ω the value

$$\Omega = \sqrt{2\pi\mu\gamma}.$$

The present paper is concerned with the gravitational field of such a cylinder according to the general theory of relativity. We see that we obtain the same value for the angular velocity, but in the present case the density is not constant, it decreases with increasing distance from the axis. If the radius of the cylinder is small, however, the density does not vary very much from its value on the axis of symmetry.

§ 9. Lewis's Solution for the External Field.

In order to complete the solution we must now consider the external field and the boundary conditions. We make use of Lewis's solution for the external field of a rotating cylinder (Lewis, 1932). The form in which Lewis gives this solution is not a convenient one from the point of view of determining the constants occurring in it by means of the boundary conditions. We will therefore here obtain this solution by a different method, and in a form more convenient for our purposes.

We return to the equations (5.5) to (5.10), where we put $\rho = 0$. With Lewis we remark that the conditions of integrability of (5.6) and (5.7), and the condition of compatibility of these equations with (5.5), are contained in the system (5.8) to (5.10). These last three equations are not independent, any one being a consequence of the remaining two. We consider (5.9) and (5.10), and make the substitution

$$u = \frac{F}{L}, \qquad v = \frac{M}{L}. \tag{9.1}$$

Using the relation (6.7), the equations (5.9) and (5.10) become

$$\frac{\partial}{\partial x^a}\left[\frac{r u_a}{u + v^2}\right] = 0, \qquad \frac{\partial}{\partial x^a}\left[\frac{r v_a}{u + v^2}\right] = 0. \tag{9.2}$$

We attempt to obtain solutions of these equations by putting

$$u_a = \Theta_a(u + v^2), \qquad v_a = \Phi_a(u + v^2). \tag{9.3}$$

Substituting from (9.3) in (9.2), we see that the functions Θ and Φ must satisfy the equation

$$\Theta_{11} + \Theta_{22} + \frac{1}{r}\Theta_1 = 0. \tag{9.4}$$

Choosing two arbitrary solutions of this equation, we now attempt to obtain u and v from the system of equations (9.3). This system of first order partial differential equations is complete, that is to say, the conditions of integrability are identically satisfied if

$$\Theta_1\Phi_2 - \Theta_2\Phi_1 = 0,$$

as may easily be verified. This relation involves

$$\Phi = f(\Theta),$$

and since Θ and Φ are both solutions of (9.4), it is easily seen that we have

$$\Phi = A\Theta + B,$$

where A and B are constants. It is then evident from (9.3) that this implies

$$v = Au + B. \tag{9.5}$$

The two sets of equations in (9.3) are then equivalent, and they reduce to an ordinary differential equation which may be written

$$\frac{du}{A^2u^2 + (2AB + 1)u + B^2} = d\Theta. \tag{9.6}$$

In the integration of (9.6) three cases arise, according as $4AB + 1$ is $>$, $=$ or < 0. In the case $4AB + 1 < 0$, we obtain, introducing different constants,

$$-u = \alpha^2 + \beta^2 + 2\alpha\beta \coth 2\Theta,$$
$$-v = \alpha \coth 2\Theta + \beta,$$

and these give

$$\begin{cases} r^{-1}L = \dfrac{1}{\alpha} \sinh 2\Theta, \\ r^{-1}M = -\cosh 2\Theta - \dfrac{\beta}{\alpha} \sinh 2\Theta, \\ r^{-1}F = -2\beta \cosh 2\Theta - \dfrac{\alpha^2 + \beta^2}{\alpha} \sinh 2\Theta. \end{cases} \tag{9.7}$$

Substituting from (9.7) in (5.6) and (5.7) we obtain for ψ the equations

$$\begin{cases} \psi_1 = -\dfrac{1}{4r} + r(\Theta_1^2 - \Theta_2^2), \\ \psi_2 = 2r\Theta_1\Theta_2. \end{cases} \tag{9.8}$$

In the case $4AB + 1 < 0$, we have similarly

$$\begin{cases} r^{-1}L = \dfrac{1}{\alpha} \cos 2\Theta, \\ r^{-1}M = \sin 2\Theta - \dfrac{\beta}{\alpha} \cos 2\Theta, \\ r^{-1}F = 2\beta \sin 2\Theta + \dfrac{\alpha^2 - \beta^2}{\alpha} \cos 2\Theta. \end{cases} \tag{9.9}$$

The equations for ψ are now

$$(9.10) \qquad \begin{cases} \psi_1 = -\dfrac{1}{4r} - r(\Theta_1^2 - \Theta_2^2), \\ \psi_2 = -2r\Theta_1\Theta_2. \end{cases}$$

The case $4AB + 1 = 0$ does not, in the present instance, require separate treatment.

§ 10. Boundary Conditions. The Case $aR < \frac{1}{2}$.

The external field of the cylinder is now obtained by choosing a solution of (9.4) which is a function of r alone. We find

$$(10.1) \qquad \Theta = n \log\left(\frac{r}{r_0}\right),$$

We consider first of all the solution given by (9.7). Obtaining ψ from (9.8), we find for H

$$(10.2) \qquad H = k\left(\frac{r}{R}\right)^{2n^2 - \frac{1}{2}}.$$

We suppose that on the boundary of the cylinder we have $r = R$. We write $2\Theta = \theta_0 + \theta$, where

$$(10.3) \qquad \begin{cases} \theta_0 = 2n \log\left(\dfrac{R}{r_0}\right), \\ \theta = 2n \log\left(\dfrac{r}{R}\right), \end{cases}$$

so that $\theta = 0$ on the boundary. Substituting in (9.7) we equate the values of the coefficients on the boundary with the internal boundary values given by (7.4). If we write for convenience $ct = x^4$, we obtain the equations

$$(10.4) \qquad \begin{cases} \dfrac{R}{\alpha} \sinh \theta_0 = R^2(1 - a^2R^2), \\ -\cosh \theta_0 - \dfrac{\alpha}{\beta} \sinh \theta_0 = aR, \\ -2\beta \cosh \theta_0 - \dfrac{\alpha^2 + \beta^2}{\alpha} \sinh \theta_0 = R^{-1}. \end{cases}$$

We now calculate the derivatives of the coefficients of the form and equate their values on the boundary, obtaining the equations

$$(10.5) \qquad \begin{cases} \dfrac{1}{\alpha}(\sinh \theta_0 + 2n \cosh \theta_0) = 2R(1 - 2a^2R^2), \\ -(\cosh \theta_0 + 2n \sinh \theta_0) - \dfrac{\alpha}{\beta}(\sinh \theta_0 + 2n \cosh \theta_0) = 2aR, \\ -2\beta(\cosh \theta_0 + 2n \sinh \theta_0) - \dfrac{\alpha^2 + \beta^2}{\alpha}(\sinh \theta_0 + 2n \cosh \theta_0) = 0. \end{cases}$$

$$F = \frac{rc^2}{R} \frac{\sinh(\epsilon - \theta)}{\sinh \epsilon}, \tag{10.14}$$

and where θ is defined by

$$\theta = \sqrt{(1 - 4a^2R^2)} \log \frac{r}{R}. \tag{10.15}$$

The present system of co-ordinates is, as we have seen, to be interpreted as a rotating system. In order to obtain the field in a system of co-ordinates which is a dynamical rest frame for the observer on the axis of symmetry, we must apply the transformation (4.7) where $\Omega = -ac$. When this is done it will be found that for sufficiently great values of r, the coefficient of dt^2 changes its sign. This does not mean that the fundamental form changes its signature, for by (6.7) the determinant of the quadratic (8.7) is positive for all values of r, and hence, as long as L is positive a transformation of the type (4.7) can always be found which will transform the quadratic into the difference of two squares. We see, however, that the unit tangent vector to the world-line of a particle with fixed space co-ordinates in Walker's dynamical rest frame is space-like at sufficiently great distances from the axis of symmetry. The separation of space-time into space and time by means of Walker's rest frame therefore holds only in the neighbourhood of the observer. We can, however, find a system of co-ordinates in which the coefficient F remains positive throughout space. We apply the transformation (4.7), where we put

$$\Omega = -\frac{c}{R} \frac{\sinh \epsilon + \cosh \epsilon}{\sinh 3\epsilon + \cosh 3\epsilon} 2 \cosh \epsilon. \tag{10.16}$$

We then obtain the form (7.3), where H is given by (10.11) and

$$\left\{\begin{aligned} L &= R^2\left[\frac{\sinh 3\epsilon + \cosh 3\epsilon}{4 \sinh 2\epsilon \cosh \epsilon}\left(\frac{r}{R}\right)^{1+2n} + \frac{\sinh 3\epsilon - \cosh 3\epsilon}{4 \sinh 2\epsilon \cosh \epsilon}\left(\frac{r}{R}\right)^{1-2n}\right], \\ M &= -\frac{Rc}{\sinh 3\epsilon + \cosh 3\epsilon}\left(\frac{r}{R}\right)^{1-2n}, \\ F &= c^2 \frac{4 \sinh 2\epsilon}{\sinh 3\epsilon + \cosh 3\epsilon}\left(\frac{r}{R}\right)^{1-2n}, \end{aligned}\right. \tag{10.17}$$

and where n is defined by (10.7). We easily verify that in the present system of co-ordinates the cosine of the angle between the ϕ-lines and the t-lines tends to zero as r tends to infinity. Furthermore the angular velocities with which a particle must describe a ϕ-line in order that its world-line may be a geodesic in space-time, tend to become equal and opposite and both tend to zero as r tends to infinity. Hence the present system of co-ordinates may be described as one which is not rotating with respect to the fixed stars. We call the present system of co-ordinates an astronomical rest frame for the observer on the axis of symmetry.

The angular velocity ω' with which the cylinder is rotating in the present system of co-ordinates is obtained from (10.16). Expressing ω' in terms of R by using (10.10) we obtain

$$\begin{aligned} \omega' &= \frac{ac}{2a^4R^4}[1 - 2a^2R^2 - \sqrt{(1-4a^2R^2)}], \\ &= ac(1 + 2a^2R^2 + 10a^4R^4 + \ldots). \end{aligned} \tag{10.18}$$

Comparing (10.18) and (8.1), we see that Walker's dynamical rest frame rotates with an angular velocity ω'' relative to the astronomical rest frame, where ω'' is given by

$$\omega'' = 2a^2R^2(1 + 5a^2R^2 + \ldots)ac. \tag{10.19}$$

If aR is small compared with unity, so that the radius of the cylinder is small compared with its maximum permissible value, then ω'' will be very small compared with ω'. If we suppose the cylinder to be of the density of water on the axis of symmetry, and its radius to be that of the earth, then Walker's system of reference will complete a revolution relative to the fixed stars in $7{\cdot}7 \times 10^5$ years.

§ 11. Boundary Conditions. The Case $aR \geqslant \frac{1}{2}$.

The equations (10.8) and (10.9) show that the solution of the preceding section for the external field is valid only if $aR < \frac{1}{2}$, otherwise the constants occurring in the solution are imaginary. The external field in the case $aR = \frac{1}{2}$ is simply obtained from the equations (10.11) to (10.14) by a limiting process. We find

$$\left\{\begin{aligned} H &= e^{-\frac{1}{4}}\left(\frac{r}{R}\right)^{-\frac{1}{2}}, \\ L &= \tfrac{1}{4}Rr\left(3 + \log\frac{r}{R}\right), \\ M &= \tfrac{1}{2}rc\left(1 + \log\frac{r}{R}\right), \\ F &= \frac{rc^2}{R}\left(1 - \log\frac{r}{R}\right). \end{aligned}\right. \tag{11.1}$$

To obtain the external field in the case $aR > \frac{1}{2}$, we return to the solution (9.9). Proceeding exactly as before, we find

$$\left\{\begin{aligned} H &= e^{-a^2R^2}\left(\frac{r}{R}\right)^{-2a^2R^2}, \\ L &= Rr\frac{\sin(3\epsilon + \theta)}{2\sin 2\epsilon\cos\epsilon}, \\ M &= rc\frac{\sin(\epsilon + \theta)}{\sin 2\epsilon}, \\ F &= \frac{rc^2}{R}\frac{\sin(\epsilon - \theta)}{\sin\epsilon}, \end{aligned}\right. \tag{11.2}$$

where

$$\theta = \sqrt{(4a^2R^2 - 1)} \log \frac{r}{R}, \tag{11.3}$$

$$\tan \epsilon = \sqrt{(4a^2R^2 - 1)}. \tag{11.4}$$

The present solution only holds when $aR > \frac{1}{2}$.

We see from (11.2) that L, the coefficient of $d\phi^2$ in the fundamental form, is zero when

$$3\epsilon + \theta = \pi. \tag{11.5}$$

Denoting the value of r at this point by r', we find

$$r' = R \exp \left[\frac{\pi - 3 \tan^{-1}\sqrt{(4a^2R^2 - 1)}}{\sqrt{(4a^2R^2 - 1)}} \right]. \tag{11.6}$$

When $r > r'$ the coefficient of $d\phi^2$ becomes negative and the fundamental form changes its signature. It is easy to show, however, that all geodesics in the surfaces z = constant, issuing from the origin, meet again at the point $r = r'$. It follows that these surfaces are closed, the point $r = r'$ being the antipodal point of the origin. We see from (11.6) that when $aR \longrightarrow \frac{1}{2}$, then $r' \longrightarrow \infty$, and hence the external region becomes infinite as the radius of the cylinder approaches its critical value. When $aR \longrightarrow 1$ it is seen that $r' \longrightarrow R$. Hence as the radius of the cylinder increases, the external region diminishes and finally vanishes when the radius reaches its maximum value. The cylinder then fills space completely. We see from (8.5) and (8.8) that both ω_1 and Ω_1 remain finite when $aR \longrightarrow 1$. It follows that the world-lines of the particles on the antipodal line are null geodesics. Hence as $aR \longrightarrow 1$, the velocity of the particles on the boundary tends to the velocity of light. The cylinder can therefore never fill space completely, there must always remain a small filament of empty space surrounding the cylinder.

REFERENCES TO LITERATURE.

LEWIS, T., 1932. "Some Special Solutions of the Equations of Axially Symmetric Fields," *Proc. Roy. Soc.*, A, vol. cxxxvi, pp. 176–192.

WALKER, A. G., 1935. "Note on Relativistic Mechanics," *Proc. Edin. Math. Soc.*, vol. iv, pp. 170–174.

WEYL, H., 1918. "Zur Gravitationstheorie," *Ann. Phys. Lpz.*, pp. 117–145.

(*Issued separately May* 5, 1937.)

PART III.

THE GRAVITATIONAL FIELD OF A ROTATING LIQUID.

THE GRAVITATIONAL FIELD OF A ROTATING LIQUID.

1. Introduction.

The gravitational field of a rotating liquid was first investigated by Lense and Thirring (1918). These writers obtained a first approximation to the external field of a slowly rotating sphere by using Einstein's method of the retarded potential for the approximate solution of the gravitational equations. Bach (1922) and Lewis (1932) have attempted to obtain closer approximations to the external field by direct analysis of the equations To a first approximation the fields of Bach and Lewis agree and are identical with that obtained by Lense and Thirring. The second approximations do not however agree. Lewis accounts for the discrepancy by observing that owing to a different choice of coordinate system, Bach's definition of a sphere does not correspond exactly with his own. Doubt has been thrown on the validity of Bach's second approximation by Akely (1931), who calculated the internal field corresponding to Bach's external solution and found that the boundary conditions could not be satisfied. The second approximation obtained by Lewis does not appear to be free from criticism either, like Bach's method it suffers from the disadvantage of concentrating attention exclusively on the external field. In these methods a point is ultimately reached where a solution of a partial differential equation must be chosen, and in the case of general relativity no less than in newtonian potential theory, the gravitational equationsd do not determine the solution uniquely without reference to the boundary conditions.

The more general problem of the gravitational field of a rotating liquid in the internal and external case has been treated by Akely (1931). The method adopted by Akely consists in assuming expansions of the type

$$\Psi = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \Psi_{\alpha,2\beta}\, \rho^{\alpha}\, \omega^{2\beta}$$

where Ψ is one of the coefficients of the fundamental form, ρ is the density and ω the angular velocity. These series are then substituted in the gravitational equations for the internal case, and equations for the different coefficients $\Psi_{\alpha,2\beta}$ are obtained. Solutions of these equations are then chosen which satisfy the necessary boundary conditions.

While no criticism is made of the validity of Akely's process, it appears to the writer to be unnecessarily laborious and not in a form which admits a simple statement of the results of the analysis. Each particular figure of equilibrium requires independent investigation and the resulting formulae are extremely complicated.

From the point of view of the present writer however there is another objection to the methods so far discussed. It appeared in part II of this thesis that in the case of a rotating cylinder there were two essentially different non-rotating systems of reference, which we refferred to as the dynamical and the astronomical rest frame, and we obtained the relative angular velocity of these two frames. We are therefore naturally led to ask the question whether this phenomena is associated particularly with the case of an infinite cylinder or whether it is of more universal interest. The present writer therefore proposed to investigate whether the two frames were distinct in the case of a rotating sphere and, if so, to find the formula for the relative angular velocity of the two frames.

Now none of the preceding methods is adapted to the solution of this problem, for the relation between the two frames is obtained through the boundary conditions, and consequently it is necessary to have a method by means of which the boundary conditions can be easily dealt with.

The method adopted by the present writer consists in attempting to reduce the gravitational equations to linear equations. When this has been achieved the question of satisfying the boundary conditions reduces to a well-known type of problem, which can, in certain cases at any rate, be solved.

It is well-known that to a first approximation the gravitational equations reduce to Poisson's equation. It is here shown that to a second approximation the gravitational equations reduce to Poisson's equation and three further linear partial differential equations. The explicit expression of the coefficients of the fundamental form in term of these four potentials is given. We thus obtain a general result applicable to all figures of equilibrium. The question of non-rotating frames of reference is then discussed and it is shown that the dynamical and the astronomical rest frames are distinct and the general formula for the relative angular velocity of the two is given. We next proceed to the case of a slowly rotating sphere. There is a point of difference between the results here obtained and those of Bach and Lewis, for though it appears that the eccentricity may be neglected in calculating the second order terms, regard must be had to the terms arising out of the eccentricity in the first order terms. Both Bach and Lewiss neglect the eccentricity altogether.

The angular velocity of the dynamical rest frame relative to the fixed stars is calculated for the case of the earth. It is found to be very small, amounting to about 1min. 5 sec. of arc per century. One of the physical interpretations of this result is that the invariable plane of a Foucauld pendulum suspended at the North Pole precesses with that angular velocity relative to the fixed stars. Theoretically this furnishes another test of the general theory of relativity.

It is quite clear that to a first approximation the results here established agree with newtonian analysis, it is desirable however to test the accuracy of the second order terms. To this end the value we obtain for the pressure in the interior of a rotating sphere is compared with the assoc pres- sure in the interior of the Schwarzschild solution and it is shown that when the angular velocity tends to zero the two formulae agree to the required degree of approximation.

2. Energy tensor of a rotating liquid.

As in part II we assume for the fundamental form the two alternative expressions

$$(2.1)\quad ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta + g_{mn}\, dx^m dx^n,$$

or

$$(2.2)\quad ds^2 = e^{2H}(dr^2 + dz^2) + l\, d\phi^2 + 2m\, d\phi\, dt - f\, dt^2,$$

where we adopt the same convention about Greek and Roman indices.

The energy tensor of a perfect fluid in the general theory of relativity is of the form

$$T^r_s = \left(\mu + \frac{p}{c^2}\right)\lambda^r \lambda_s + \frac{p}{c^2}\delta^r_s, \quad (r, s = 1,2,3,4)$$

where μ is the density, p the pressure and λ^r the unit tangent vector to the world -lines of mean motion of the particles composing the liquid. We suppose that μ is a constant, so that the liquid is homogeneous and incompressible. As in (II,3) the particles composing the liquid are describing the ϕ-lines of the space associated with an observer on the axis of symmetry. We have then $\lambda^1 = \lambda^2 = 0$, and writing $\Omega = \lambda^3/\lambda^4$ we find

$$\lambda^3 = \frac{\Omega}{\sqrt{(f - 2\Omega m - \Omega^2 \ell)}}$$

(2.3)

$$\lambda^4 = \frac{1}{\sqrt{(f - 2\Omega m - \Omega^2 \ell)}}$$

We obtain then for the non-vanishing components of the energy tensor the expressions

$$T^{\alpha}_{\beta} = \frac{p}{c^2}\delta^{\alpha}_{\beta},$$

$$T^3_3 = \left(\mu + \frac{p}{c^2}\right)\frac{\Omega^2 \ell + \Omega m}{f - 2\Omega m - \Omega^2 \ell} + \frac{p}{c^2},$$

(2.4)

$$T^3_4 = \left(\mu + \frac{p}{c^2}\right)\frac{\Omega^2 m - \Omega f}{f - 2\Omega m - \Omega^2 \ell},$$

$$T^4_3 = \left(\mu + \frac{p}{c^2}\right)\frac{\Omega \ell + m}{f - 2\Omega m - \Omega^2 \ell},$$

$$T^4_4 = \left(\mu + \frac{p}{c^2}\right)\frac{\Omega m - f}{f - 2\Omega m - \Omega^2 \ell} + \frac{p}{c^2}.$$

The vanishing of the divergence of the energy tensor gives the equations

(2.5)

$$T^{rs}_{\cdot\cdot,s} = \frac{\partial}{\partial x^s}\left(\mu + \frac{p}{c^2}\right)\lambda^r \lambda^s + \left(\mu + \frac{p}{c^2}\right)\lambda^r_{\cdot,s}\lambda^s + \left(\mu + \frac{p}{c^2}\right)\lambda^r \lambda^s_{\cdot,s} + g^{rs}\frac{\partial}{\partial x^s}\frac{p}{c^2} = 0 \qquad (r,s = 1,2,3,4)$$

Now we have

$$(2.6)\qquad \begin{aligned} \lambda^{m}_{\cdot,n} &= \lambda^{\alpha}_{\cdot\beta} = 0 , \\ \lambda^{\alpha}_{\cdot,m} &= -\tfrac{1}{2}\, g^{\alpha}_{mn}\,\lambda^{n} . \end{aligned}$$

Substituting from (2.4) in (2.3) and using the fact that μ is a constant and p a function of x^1 and x^2 only, we obtain

$$(2.7)\qquad -\tfrac{1}{2}\left(\mu + \frac{p}{c^2}\right) g^{\alpha}_{mn}\,\lambda^{m}\lambda^{n} + \frac{p^{\alpha}}{c^2} = 0 .$$

Which becomes in the notation of (2.2)

$$(2.8)\qquad \frac{f_{\alpha} - 2\Omega m_{\alpha} - \Omega^2 l_{\alpha}}{f - 2\Omega m - \Omega^2 l} = -\,\frac{2\,\frac{p_{\alpha}}{c^2}}{\mu + \frac{p}{c^2}}$$

Integrating this equation we have

$$(2.9)\qquad f - 2\Omega m - \Omega^2 l = k\left(\mu + \frac{p}{c^2}\right)^{-2},$$

where k is a constant of integration. We now transform to a rotating system of coordinates with the transformation

$$(2.10)\qquad \begin{aligned} \bar{x}^1 &= x^1 , & \bar{x}^2 &= x^2 , \\ \bar{x}^3 &= x^3 - \Omega x^4 , & \bar{x}^4 &= x^4 . \end{aligned}$$

The fundamental form then becomes, ommitting the bars which may be done without danger of ambiguity,

$$(2.11)\qquad ds^2 = e^{2H}\,(dr^2 + dz^2) + L\,d\phi^2 + 2M\,d\phi\,dt - F\,dt^2 ,$$

where

$$L = l,$$

(2.12)
$$M = m + \Omega l.$$

$$F = f - 2\Omega m - \Omega^2 l.$$

Denoting the components of the energy tensor in the new system of coordinates by accented letters, we find

$$'T^\alpha_\beta = \frac{p}{c^2}\delta^\alpha_\beta$$

(2.13)
$$'T^3_3 = \frac{p}{c^2}, \quad 'T^3_4 = 0,$$

$$'T^4_3 = \left(\mu + \frac{p}{c^2}\right)\frac{M}{F}, \quad 'T^4_4 = -\mu.$$

The equation (2.9), derived from the vanishing of the divergence of the energy tensor now reads

(2.14)
$$F = k\left(\mu + \frac{p}{c^2}\right)^{-2}$$

If the system of coordinates is such that $F = c^2$ on the bounding surface of the liquid where $p = 0$, then

(2.15)
$$F = \frac{\mu^2 c^2}{\left(\mu + \frac{p}{c^2}\right)^2} = c^2\left(1 + \frac{p}{\mu c^2}\right)^{-2}.$$

The equation (2.15) holds in a system of coordinates relative to which the liquid is at rest. As a particular case it must therefore hold for the Schwarzschild internal solution, and it may easily be verified to be true for that case.

3. The gravitational equations.

As in part II we find for the components of the Ricci tensor of the form (2.11) the expressions

$$\sqrt{(-g)}\, R^{\alpha}_{\ \beta} = D \Delta H \delta^{\alpha}_{\ \beta} + D_{\alpha,\beta} - \frac{1}{4D}\left(L_{\alpha}F_{\beta} + L_{\beta}F_{\alpha} + 2 M_{\alpha} M_{\beta}\right),$$

$$\sqrt{(-g)}\; R^{3}_{\ 3} = \frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\left(\frac{F L_{\alpha} + M M_{\alpha}}{D}\right),$$

$$(3.1) \qquad \sqrt{(-g)}\; R^{3}_{\ 4} = \frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\left(\frac{F M_{\alpha} - M F_{\alpha}}{D}\right),$$

$$\sqrt{(-g)}\; R^{4}_{\ 3} = \frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\left(\frac{M L_{\alpha} - L M_{\alpha}}{D}\right),$$

$$\sqrt{(-g)}\; R^{4}_{\ 4} = \frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\left(\frac{L F_{\alpha} + M M_{\alpha}}{D}\right).$$

We write the gravitational equations in the form

$$(3.2) \qquad R^{i}_{\ j} = -\kappa\left(T^{i}_{\ j} - \tfrac{1}{2} T \delta^{i}_{\ j}\right), \quad (i, j = 1, 2, 3, 4)$$

Substituting in (3.2) from (2.13) and (3.1) and taking simple linear combinations of the equations we obtain

$$(3.3) \quad 4 D \Delta H + \Delta D - \frac{1}{D}\left(L_1 F_1 + M_1^2 + L_2 F_2 + M_2^2\right) = -2\kappa\mu D e^{2H}$$

(3.4) $$D_{11} - D_{22} - 2H_1 = \frac{1}{2D}\left(L_1 F_1 + M_1^2 - L_2 F_2 - M_2^2\right),$$

(3.5) $$D_{12} - H_2 = \frac{1}{4D}\left(L_1 F_2 + L_2 F_1 + 2 M_1 M_2\right),$$

(3.6) $$\Delta D = 2\kappa \frac{p}{c^2} D e^{2H},$$

(3.7) $$\frac{\partial}{\partial x^\alpha}\left(\frac{FM_\alpha - MF_\alpha}{D}\right) = 0,$$

(3.8) $$\frac{\partial}{\partial x^\alpha}\left(\frac{FL_\alpha - LF_\alpha}{D}\right) = -2\kappa\left(\mu + \frac{p}{c^2}\right) D e^{2H},$$

(3.9) $$\frac{\partial}{\partial x^\alpha}\left(\frac{ML_\alpha - LM_\alpha}{D}\right) = -2\kappa\left(\mu + \frac{p}{c^2}\right) D e^{2H} \frac{M}{F}.$$

The equations (3.7) to (3.9) are not independent, (3.9) can be deduced from (3.7) and (3.8). The system therefore reduces to the six equations (3.3) to (3.8). We see that we cannot here proceed with the introduction of Weyl's canonical coordinates, for their admissibility depends on the equation $\Delta D = 0$, which, as (3.6) shows, is not satisfied in the present case. Exact solutions of the above system of equations being very hard to obtain we now proceed to the discussion of approximate solutions.

4. Approximate solutions.

The case we investigate is that of a finite quantity of incompressible liquid of uniform density, rotating with constant angular velocity about an axis of symmetry. If d be the greatest linear dimension and μ the density of the liquid, we shall attempt to find approximate solutions of the gravitational equations, on the assumption that the dimensionless quantity

$$(4.1) \qquad \varepsilon = \kappa \mu d^2 ,$$

is small compared with unity. In the case of the sun we have $\varepsilon = 2.03 \times 10^{-4}$ so that the theory will apply to rotating masses of liquid comparable in size and density with the sun. We call a quantity of the order of magnitude of ε a quantity of the first order, one of the order of magnitude ε^2 of the second order, and so on.

If we assume for the fundamental form of such a field the expression

$$ds^2 = g_{ij}\, dx^i\, dx^j , \qquad (i, j = 1, 2, 3, 4)$$

where

$$g_{ij} = \delta^i_j + \gamma_{ij}$$

then, as is well-known, the quantities γ_{ij} are of the first or higher orders of magnitude. We will suppose that the γ_{ij} are expressible as series, each temm in the series being of a higher order by one than the preceding term. We then attempt to determine the first two terms in each of these series. The method applies ofcourse equally well to a system of coordinates which instead of being approximately cartesian, is approximately cylindrical. We therefore assume the field to be given by the fundamental form (2.11), where we put

$$H = h + \psi, \tag{4.2}$$

$$L = r^2(1 + 2\ell + 2\lambda), \tag{4.3}$$

$$F = c^2(1 - 2f - 2\eta), \tag{4.4}$$

small Roman letters on the right hand sides of (4.2) to (4.4) denoting quantities of the first order and small Greek letters denoting quantities of the second order. The question of the proper assumption to be made in the case of M requires some discussion. The present system of coordinates is one relative to which the liquid is at rest. To obtain a non-rotating system of coordinates we apply the transformation (2.10), where we change the sign of Ω , the transformation here proceeding in the revese direction. The new system of coordinates is then non-rotating.

rotating (in Walkers sense), if the unit tangent vectors to the coordinates lines at the origin are paralelly propagated along the world-line of the origin. If now by the same method that was used in (II,7), we investigate when this is the case, it will be found that the transformed coefficient of $d\phi\, dt$ must not contain powers of r less than r^3. But after the transformation we find for the the coefficient of $d\phi\, dt$

$$\bar{M} = M - \Omega L = M - \Omega r^2(1 + 2\ell + 2\lambda)$$

It is clear therefore that M must contain the term Ωr^2, otherwise the transformed coefficient will contain a power of r less than r^3. We therefore assume for M the expression

$$M = \omega r^2 (1 + 2m + 2\nu), \tag{4.5}$$

where m is a quantity of the first order and ν a quantity of the sedond order.

We consider now the order of magnitude of the angular velocity of the liquid. There is ofcourse a relation between the angular velocity and the density of a rotating liquid. The solution of part II may clearly be regarded as the limiting case of a rotating liquid, moee rapid rotation being impossible without the pressure becoming negative in the interior of the liquid, and we there obtained the formula

$$\omega^2 = \kappa \mu c^2,$$

which may be written

$$\frac{\omega d}{c} = \varepsilon^{1/2}. \tag{4.6}$$

In the present case therefore the quantity $\frac{\omega d}{c}$ is of the order $\frac{1}{2}$ or higher.

The order of magnitude of the pressure may be simply obtained by comparing (2.15) and (4.4). We obtain from (2.15)

$$F = c^2\left(1 - 2\frac{p}{\mu c^2}\right) \tag{4.7}$$

approximately, and hence we see

$$\frac{p}{\mu c^2} = O(\varepsilon).$$

We deduce

$$\frac{\kappa p}{c^2} d^2 = \kappa \mu d^2 \cdot \frac{p}{\mu c^2} = O(\varepsilon^2). \tag{4.8}$$

In what follows it will not always be convenient to write the equations in such a way that only dimensionless quantities occur, and hence we may occasionally loosely refer to the equations (4.1), (4.6) and (4.8) by saying that

ω is of the order $\frac{1}{2}$, $\kappa\mu$ of the order 1, and $\kappa\rho$ of the order 2. It will always however be easy to justify the argument by a rearrangement of the equation.

5. First approximation.

We now proceed to the appraximation by substituting from (4.2) to (4.5) in the gravitational equations. It will be convenient for the purposes of calculation to choose a time unit so that $c = 1$. We begin by neglecting terms of the second order and higher, but retaining terms of the order $\frac{3}{2}$. We have then

$$H = h,$$

(5.1)
$$L = r^2(1+2l),$$
$$M = \Omega r^2(1+2m),$$
$$F = 1-2f,$$

where

(5.2) $$\Omega = \frac{\omega}{c}.$$

Then we have

(5.3) $$D = r\left(1+l-f+\tfrac{1}{2}\Omega^2 r^2\right),$$

and

(5.4)
$$F_1 L_1 + M_1^2 = -4rf_1 + 4\Omega^2 r^2,$$
$$F_2 L_2 + M_2^2 = 0,$$
$$F_1 L_2 + F_2 L_1 + 2M_1 M_2 = -4rf_2.$$

We write for convenience

$$g = l - f. \tag{5.5}$$

Now substituting from (5.1) to (5.4) in the equations (3.3) to (3.8), remembering that $\kappa\rho$ is of the second order, these become respectively

$$4\Delta h + g_{11} + g_{22} + \frac{2}{r}g_1 + \frac{4}{r}f_1 = -2\kappa\mu + \Omega^2, \tag{5.6}$$

$$2l_1 - 2h_1 = -r(g_{11} - g_{22}) - \Omega^2 r, \tag{5.7}$$

$$2l_2 - 2h_2 = -2r g_{12}, \tag{5.8}$$

$$g_{11} + g_{22} + \frac{2}{r}g_1 = -3\Omega^2, \tag{5.9}$$

$$m_{11} + m_{22} + \frac{3}{r}m_1 + \left(l_{11} + l_{22} - \frac{1}{r}l_1\right) = 0, \tag{5.10}$$

$$l_{11} + l_{22} + \frac{2}{r}l_1 + f_{11} + f_{22} = \Omega^2 - \kappa\mu. \tag{5.11}$$

The condition of integrability of the equations (5.7) and (5.8) is found to be

$$g_{112} + g_{222} + \frac{2}{r}g_{12} = 0$$

and is therefore satisfied by (5.9). Calculating Δh from (5.7) and (5.8) we find

$$\Delta h = l_{11} + l_{22} + \frac{1}{r}g_1 - \frac{1}{2}g_{11} - \frac{1}{2}g_{22} + \frac{1}{2}\Omega^2.$$

Substituting this value of Δh in (5.6) we obtain

(5.12) $$2\left(\ell_{11}+\ell_{22}+\frac{1}{r}\ell_1\right) = g_{11}+g_{22}+\Omega^2-\kappa\mu,$$

and using (5.5) this equation is easily seen to be equivalent (5.11). We now choose a particular solution of (5.9) namely

(5.13) $$g = -\frac{1}{2}\Omega^2 r^2$$

The equations (5.7) and (5.8) then give

(5.14) $$h = \ell$$

while (5.12) becomes

(5.15) $$\ell_{11}+\ell_{22}+\frac{1}{r}\ell_1 = -\frac{1}{2}\kappa\mu .$$

If we write $m = \ell + n$ and substitute in (5.10) and use (5.15), we obtain for n the equation

(5.16) $$n_{11}+n_{22}+\frac{3}{r}n = \kappa\mu .$$

For the first approximation we obtain therefore

(5.17)
$$\begin{aligned} H &= \ell, \\ L &= r^2(1+2\ell), \\ M &= \Omega r^2(1+2\ell+2n), \\ F &= 1-2\ell-\Omega^2 r^2 \end{aligned}$$

where ℓ and n satisfye the equations (5.15) and

(5.16). Although the terms involving the pressure have been omitted in the gravitational equations, we can use the formula (4.7) to obtain an approximation to the pressure. Comparing (4.7) with the expression for F in (5.17) we see that

$$\frac{p}{\mu} = \ell + \tfrac{1}{2}\Omega^2 r^2. \tag{5.18}$$

The first approximation gives therefore the newtonian expression for the pressure.

6. Second approximation.

To obtain the second approximation we now put

$$\begin{aligned} H &= \ell + \psi, \\ L &= r^2(1 + 2\ell + 2\lambda), \\ M &= \Omega r^2(1 + 2\ell + 2n + 2\nu), \\ F &= 1 - 2\ell - \Omega^2 r^2 - 2\eta. \end{aligned} \tag{6.1}$$

When these quantities are substituted in the gravitational equations, it will be found that ν is a quantity of the second order. The resulting term in the expression for M is therefore of the order $\frac{5}{2}$ and since in the present investigation we neglect quantities of higher order than the second, we may neglect ν .

Considering now the right hand sides of the equations

(3.3) to (3.8), in calculating these correct to the second order, we use the equation (5.18) giving a first approximation for the quantity $\frac{p}{\mu}$. This approximation is clearly sufficiently good to obtain these expressions correct to the second order, the quantity $n\frac{p}{\mu}$ always occurring multiplied by a quantity of the first order.

We find from (6.1)

$$(6.2)\qquad D = r(1+\lambda-\eta-2l^2-\Omega^2r^2l+2\Omega^2r^2m),$$

the first order terms in the expression for D cancelling. It will be convenient to introduce three new quantities all of the second order, defined by

$$(6.3)\qquad \begin{aligned} \alpha &= \lambda-\eta-2l^2-\Omega^2r^2l+2\Omega^2r^2m, \\ \beta &= \lambda-l^2, \\ \gamma &= \eta+l^2+\Omega^2r^2l-2\Omega^2r^2m. \end{aligned}$$

We have then

$$(6.4)\qquad \begin{aligned} D &= r+r\alpha, \\ \alpha &= \beta-\gamma. \end{aligned}$$

We may note the following formulae

$$(6.5)\qquad \begin{aligned} \frac{L_1F_1+M_1^2}{4D} &= -l_1-rl_1^2-\beta_1, \\ \frac{L_2F_2+M_2^2}{4D} &= -rl_2^2, \\ \frac{L_1F_2+L_2F_1+2M_1M_2}{4D} &= -l_2-2rl_1l_2-\beta_2. \end{aligned}$$

The equation (3.3) then becomes

$$4r\Delta\ell + 4r\Delta\psi + r(\alpha_{11}+\alpha_{22}+\tfrac{2}{r}\alpha_1) + 4\ell_1 + 4r(\ell_1^2+\ell_2^2) + 4\beta_1 = -2\kappa\mu r(1+2\ell)$$

or using (5.15)

$$(6.6)\qquad \Delta\psi + \tfrac{1}{4}(\alpha_{11}+\alpha_{22}+\tfrac{2}{r}\alpha_1) + (\ell_1^2+\ell_2^2) + \tfrac{1}{r}\beta_1 = -\kappa\mu\ell .$$

The equations (3.4) and (3.5) become after some rearrangement

$$(6.7)\qquad \psi_1 = \beta_1 + \tfrac{r}{2}(\alpha_{11}-\alpha_{22}) + r(\ell_1^2-\ell_2^2),$$

$$(6.8)\qquad \psi_2 = \beta_2 + r\alpha_{12} + 2r\ell_1\ell_2 .$$

From (3.6) we obtain

$$(6.9)\qquad \alpha_{11} + \alpha_{22} + \tfrac{2}{r}\alpha_1 = 2\kappa\mu(\ell + \tfrac{1}{2}\Omega^2 r^2).$$

The equation (3.7) is satisfied if we neglect ν. The reduction of equation (3.8) is a little laborious. We first of all note the formulae

$$\frac{Fl_1 - LF_1}{D} = 2 + 4r\ell_1 + 2\beta - 2\gamma + 2r\beta_1 + 2r\gamma_1 + 4\Omega^2 r^3 m_1 - 4\Omega^2 r^3 \ell_1,$$

$$(6.10)\qquad \frac{Fl_2 - Lf_2}{D} = 4r\ell_2 + 2r\beta_2 + 2r\gamma_2 + 4\Omega^2 r^3 m_2 - 4\Omega^2 r^3 \ell_2 .$$

Substituting from (6.10) in (3.8) and using (5.10) and

(5.15) the equation becomes

$$(6.11)\qquad \beta_{11}+\beta_{22}+\frac{2}{r}\beta_1+\gamma_{11}+\gamma_{22}=-3\kappa\mu\ell-\frac{5}{2}\kappa\mu\Omega^2r^2.$$

The condition of integrability of the equations (6.7) and (6.8) is found to be

$$(6.12)\qquad \alpha_{112}+\alpha_{222}+\frac{2}{r}\alpha_{12}+4\ell_2\left(\ell_{11}+\ell_{22}+\frac{1}{r}\ell_1\right)=0,$$

and is easily seen to be satisfied by using (5.15) and (6.9). Calculating $\Delta\Psi$ from (6.7) and (6.8) we obtain

$$(6.13)\qquad \Delta\Psi=\beta_{11}+\beta_{22}-\frac{1}{2}\left(\alpha_{11}+\alpha_{22}-\frac{2}{r}\alpha_1\right)+\kappa\mu\Omega^2r^2-\left(\ell_1^2+\ell_2^2\right),$$

Substituting this expression for $\Delta\Psi$ in (6.6) it is found that (6.6) becomes identical with (6.11). The system of equations therefore reduces to the two equations (6.9) and (6.11) to determine α,β,γ , only two of these three quantities being independent by (6.4). Having determined functions α and β satisfying these equations, then the equations (6.7) and (6.8) are integrable and determine Ψ The quantities λ and η are then obtained in terms of α,β,γ from (6.3).

If in (6.9) we write $\alpha=\beta-\gamma$ and eliminate the quantity β from the resulting equation and (6.11) we obtain

$$(6.14)\qquad \gamma_{11}+\gamma_{22}+\frac{1}{r}\gamma_1=-\frac{5}{2}\kappa\mu\ell-\frac{7}{4}\kappa\mu\Omega^2r^2.$$

(5.15) the equation becomes

$$(6.11)\qquad \beta_{11}+\beta_{22}+\frac{2}{r}\beta_1+\gamma_{11}+\gamma_{22}=-3\kappa\mu\ell-\frac{5}{2}\kappa\mu\Omega^2r^2.$$

The condition of integrability of the equations (6.7) and (6.8) is found to be

$$(6.12)\qquad \alpha_{112}+\alpha_{222}+\frac{2}{r}\alpha_{12}+4\ell_2\left(\ell_{11}+\ell_{22}+\frac{1}{r}\ell_1\right)=0,$$

and is easily seen to be satisfied by using (5.15) and (6.9). Calculating $\Delta\psi$ from (6.7) and (6.8) we obtain

$$(6.13)\qquad \Delta\psi=\beta_{11}+\beta_{22}-\frac{1}{2}\left(\alpha_{11}+\alpha_{22}-\frac{2}{r}\alpha_1\right)+\kappa\mu\Omega^2r^2-(\ell_1^2+\ell_2^2),$$

Substituting this expression for $\Delta\psi$ in (6.6) it is found that (6.6) becomes identical with (6.11). The system of equations therefore reduces to the two equations (6.9) and (6.11) to determine α, β, γ , only two of these three quantities being independent by (6.4). Having determined functions α and β satisfying these equations, then the equations (6.7) and (6.8) are integrable and determine ψ The quantities λ and η are then obtained in terms of α, β, γ from (6.3).

If in (6.9) we write $\alpha=\beta-\gamma$ and eliminate the quantity β from the resulting equation and (6.11) we obtain

$$(6.14)\qquad \gamma_{11}+\gamma_{22}+\frac{1}{r}\gamma_1=-\frac{5}{2}\kappa\mu\ell-\frac{7}{4}\kappa\mu\Omega^2r^2.$$

The solution for the second approximation depends therefore on the two linear partial differential equations (6.9) and (6.14), together with the system (6.7) and (6.8) to determine ψ .

The coefficients of the fundamental form may now be expressed in terms of l, n, α, γ. We find

$$\text{(6.15)} \quad \begin{aligned} H &= 2l + 2\psi, \\ L &= r^2(1 + 2l + 2l^2 + 2\alpha + 2\gamma), \\ M &= \Omega r^2(1 + 2l + 2n), \\ F &= 1 - 2l - \Omega^2 r^2 + 2l^2 - 2\Omega^2 r^2 l - 4\Omega^2 r^2 n - 2\gamma \end{aligned}$$

We now re-introduce the ordinary time unit, and substitute for κ its value in terms of Newton's gravitational constant, namely

$$\text{(6.16)} \qquad \kappa = \frac{8\pi G}{c^2} .$$

If we make the substitution

$$\text{(6.17)} \qquad \begin{aligned} l &= \frac{U}{c^2}, \quad n = \frac{V}{c^2}, \quad \psi = \frac{W}{c^4}, \\ \alpha &= \frac{P}{c^4}, \quad \gamma = \frac{Q}{c^4}, \end{aligned}$$

and write

$$\text{(6.18)} \qquad \sigma = \mu G$$

then, remembering (5.2), the equations (5.15), (5.16)

(6.9), (6.14) and (6.7), (6.8) become respectively

(6.19) $$U_{11} + U_{22} + \frac{1}{r} U_1 = -4\pi\sigma,$$

(6.20) $$V_{11} + V_{22} + \frac{3}{r} V_1 = 8\pi\sigma,$$

(6.21) $$P_{11} + P_{22} + \frac{2}{r} P_1 = 16\pi\sigma U + 8\pi\sigma\omega^2 r^2,$$

(6.22) $$Q_{11} + Q_{22} + \frac{1}{r} Q_1 = -20\pi\sigma U - 14\pi\sigma\omega^2 r^2,$$

(6.23) $$W_1 = P_1 + Q_1 + \frac{r}{2}(P_{11} - P_{22}) + r(U_1^2 - U_2^2),$$

(6.24) $$W_2 = P_2 + Q_2 + r P_{12} + 2r U_1 U_2 .$$

Substituting from (6.17) and (6.15) in (2.11), we obtain the fundamental form

(6.25) $$ds^2 = K(dr^2 + dz^2) + L\,d\phi^2 + 2M\,d\phi\,dt - F\,dt^2,$$

where

(6.26)
$$K = 1 + \frac{2U}{c^2} + \frac{2U^2 + 2W}{c^4},$$
$$L = r^2\left(1 + \frac{2U}{c^2} + \frac{2U^2 + 2P + 2Q}{c^4}\right),$$
$$M = \omega r^2\left(1 + \frac{2U + 2V}{c^2}\right),$$
$$F = c^2\left(1 - \frac{2U + \omega^2 r^2}{c^2} - \frac{2Q - 2U^2 + 2\omega^2 r^2 U + 4\omega^2 r^2 V}{c^4}\right).$$

We may now obtain a second approximation to the pressure in the interior of the liquid by using equation (2.15). We find

$$F = c^2\left(1 - 2\frac{p}{\mu c^2} + 3\frac{p^3}{\mu^2 c^4}\right) \tag{6.27}$$

Comparing (6.26) and (6.27), we find

$$\frac{p}{\mu} = U + \tfrac{1}{2}\omega^2 r^2 + \frac{\tfrac{1}{2}U^2 + Q + \tfrac{5}{2}\omega^2 r^2 U + 2\omega^2 r^2 V + \tfrac{3}{8}\omega^4 r^4}{c^2} \tag{6.28}$$

The actual determination of the field in any particular case now proceeds as follows. The rotating liquid divides the r,z plane into two regions, which we call the internal and the external regions ~~respectively~~. If we suppose the bounding curve of the liquid as given, then the equation (6.19) determines uniquely, apart from an additive constant, two functions U_i, and U_e, which together with their first derivatives are continuous across the bounding surface, and such that U_i, satisfies (6.19) in the internal region and U_e satisfies the same equation, where we put $\sigma = 0$, in the external region. The function U having thus been determined, we obtain solutions V_i, V_e, P_i, P_e ; Q_i, Q_e of the equations (6.20) to (6.22) respectively inthe same way and satisfying the same conditions. Substituting the values of the functions thus obtained in (6.29), we obtain the coefficients of the fundamental form K_i, K_e, ; L_i, L_e, etc. From the conditions which we have imposed on the solutions U, V, P, Q, it is clear that the ~~fundamenta~~ internal values of the coefficients of the fundamental form together with their first derivatives will be continuous across the surface of the liquid with their external values.

In order that the solution thus obtained may correspond with an equilibrium configuration it is ofcourse necessary that when the solutions that have been obtained for U, V, Q are substituted in (6.28), it is found that the pressure vanishes on the bounding surface of the liquid. There is nothing to guide us in our original choice of a bounding surface of the liquid, only by trial can we determine whether the chosen configuration is actually an equilibrium configuration. The same difficulty exists in the ordinary newtonian analysis of the rotating liquid. The results of newtonian analysis can clearly be used in the present investigation to ensure at least first order vanishing of the pressure on the bounding surface.

7. Non-Rotating frames of reference.

The formulae (6.26) give the field in a system of coordinates relative to which the liquid is at rest. In order to obtain the field in a non-rotating system of coordinatesw we apply the transformation (2.10), where we change the sign of Ω , as the transformation is here proceeding in the reverse direction, so that we write

$$(7.1)\qquad \bar{x}^1 = x^1, \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3 + \Omega x^4, \quad \bar{x}^4 = x^4.$$

After this transformation we have that the new coefficient of $d\phi\, dt$ is given by

$$(7.2)\qquad \bar{M} = M - \Omega L$$

In order that the frame of reference may be a dynamical rest frame, it is necessary, as we have already noted, that M does not contain any powers of r less than the third.

The functions ~~XXXX~~ U,V,P,Q are determined uniquely by the conditions of the problem apart from an arbitrary additive constant. We may choose if we like solutions which are zero at infinity, but this is not always the most convenient choice. For the present however we suppose that the solution of V which has been chosen isis such that $V = 0$ at infinity. In this case the function V will not in general be zero at the origin, it will contain a constant term which we denote by V_0. Substituting from (6.26) in (7.2) we have

$$(7.3)\qquad \bar{M} = \omega r^2\left(1 + \frac{2U}{c^2} + \frac{2V}{c^2}\right) - \Omega r^2\left(1 + \frac{2U}{c^2}\right).$$

If we now determine Ω so that at the origin the terms involving r^2 disappear we find

$$(7.4) \qquad \Omega = \omega\left(1 + 2\frac{V_0}{c^2}\right).$$

The formula (7.4) now gives the angular velocity of the liquid relative to the dynamical rest frame. We must now find the interpretation of the quantity ω. Let us apply the transformation (7.1) where we put $\Omega = \omega$. In that case the transformed coefficient M is given by

$$(7.5) \qquad \bar{M} = 2\frac{\omega r^2}{c^2}V$$

and as we have chosen V so as to be zero at infinity we see that in the present system $\bar{M} = 0$ at infinity. This is clearly a frame of reference relative to which the fixed stars are at rest and is the frame which we called the astronomical rest frame in part II. It is seen therefore that in order to be able to interpret the constant ω as the angular velocity relative to the astronomical rest frame it is necessary to obtain a solution V which which vanishes at infinity.

We see now that in the case of a general rotating liquid we have the same situation as in the case of a rotating cylinder, there are two essetially different non-rotating frames of reference, the dynamical and the astronomical rest frame. The relative angular velocity ~~of~~ ω' of the two frames is given by

$$(7.6) \qquad \omega - \Omega = \omega' = -2\frac{\omega}{c^2}V_0.$$

We obtain the actual expressions of the coefficients of the fundamental form in the astronomical rest frame by carrying out the transformation (7.1) where we write $\Omega = \omega$. We find

$$K = 1 + \frac{2U}{c^2} + \frac{2U^2 + 2W}{c^4},$$

$$(7.7) \qquad L = r^2\left(1 + \frac{2U}{c^2} + \frac{2U^2 + 2P + 2Q}{c^4}\right),$$

$$M = \frac{2\omega r^2 V}{c^2},$$

$$F = c^2\left(1 - \frac{2U}{c^2} - \frac{2Q - 2U^2}{c^4}\right).$$

The formula (6.28) for the pressure is ofcourse unaltered by the transformation.

The formulae (7.7) will give the expression for the coefficients of the fundamental form in the dynamical rest frame if we choose a solution V which is zero at the origin. We see therefore that the interpretation of the constant ω depends on the particular solution chosen for V. The relative angular velocity of the two frames is however quite independent of the arbitrary additive constant in V. For an arbitrary choice of this constant the equation (7.6) must be written, as is easily seen,

$$(7.8) \qquad \omega' = \frac{2\omega}{c^2}\left(V_0 - V_\infty\right),$$

and we see that ω' is independent of this constant.

7. The field of a rotating sphere.

We now apply the results obtained to determine correct to the second order the field of a slowly rotating sphere. We suppose that the bounding surface of the liquid is given by

(8.1) $$\frac{r^2}{a^2} + \frac{z^2}{b^2} = 1 .$$

We suppose further that the shape of the liquid is so nearly spherical that we may neglect higher powers than the second of the eccentricity e of the conic (8.1). The equation of the bounding surface may then be written correct to this order

(8.2) $$R^2 - a^2 = e^2(r^2 - a^2)$$

where we write

(8.3) $$R^2 = r^2 + z^2 .$$

We determine first of all the solution of (6.19) appropriate to the present assumptions. This is given by the ordinary classical potential theory. We have

(8.4) $$U = \pi a^2 b \sigma \int_u^\infty \left(1 - \frac{r^2}{a^2+\lambda} - \frac{z^2}{b^2+\lambda}\right)\frac{d\lambda}{(a^2+\lambda)(b^2+\lambda)^{1/2}}$$

where $u = 0$ for U_i, and u is the positive root of

$$(8.5) \qquad \frac{r^2}{a^2+u} + \frac{z^2}{b^2+u} - 1 = 0$$

int the case of U_e. Writing

$$W = a^2 + \lambda$$

and neglecting higher powers of e than the second, the equation (8.4) becomes

$$(8.6) \quad U = \pi a^2 b \sigma \int_u^\infty \left(1 - \frac{R^2}{W}\right) \frac{dW}{W^{3/2}} + \frac{1}{2} e^2 \pi \sigma a^4 b \int_u^\infty \left(1 - \frac{R^2}{W}\right) \frac{dW}{W^{5/2}}$$

where $u = a^2$ int the case of U_i, and

$$(8.7) \qquad u = R^2 + e^2 \frac{a^2 z^2}{R^2}$$

in the case of U_e. Calculating the definite integrals in (8.6) and adding a constant so as to make the term in the expression for U which is independent of e^2 vanish on the boundary, we obtain

$$(8.8) \quad U_i = \frac{2}{3}\pi\sigma(a^2 - R^2) - \frac{2}{3}\pi\sigma e^2 a^2 + \frac{2}{15}\pi\sigma e^2 (r^2 - 2z^2),$$

and

$$(8.9) \quad U_e = \frac{4}{3}\pi\sigma a^2\left(\frac{a}{R} - 1\right) - \frac{2}{3}\pi\sigma e^2 \frac{a^3}{R} + \frac{2}{15}\pi\sigma e^2 \left(\frac{a}{R}\right)^5 (r^2 - 2z^2).$$

The condition that

$$U_i + \tfrac{1}{2}\omega^2 r^2 = 0$$

on the boundary, gives a relation between the angular velocity and the eccentricity. This is easily seen to be

(8.10) $$\omega^2 = \frac{8}{15}\pi\sigma e^2 .$$

Hence we may write (8.8) and (8.9)

(8.11) $$U_i = \frac{2}{3}\pi\sigma(a^2 - R^2) - \frac{5}{4}\omega^2 a^2 + \frac{1}{4}\omega^2(r^2 - 2z^2),$$

(8.12) $$U_e = \frac{4}{3}\pi\sigma a^2\left(\frac{a}{R} - 1\right) - \frac{5}{4}\omega^2 a^2\left(\frac{a}{R}\right) + \frac{1}{4}\omega^2\left(\frac{a}{R}\right)^5(r^2 - 2z^2).$$

Proceeding now to the solution of the equations (6.20) to (6.22), we remark first of all that in calculating the second order terms we may neglect the angular velocity. The formulae were calculated on the assumption that the angular velocity is of the order one half. Here we suppose that the angular velocity is small compared with its maximum permissible value for a rotating liquid. We therefore assume that ω is of the order one or higher and it is easily seen that the terms involving ω in the second order terms are then really of the third order at least, and may therefore be neglected.

We transform the equations (6.20) to (6.22) into polar coordinates, with the transformation

(8.13) $$R \sin \theta = r, \quad R \cos \theta = z.$$

The solutions of these equations which we require will be functions of R only, hence we may omit the terms arising from differentiation with respect to θ in the transformed equations, and we obtain

(8.14) $$\frac{\partial^2 V}{\partial R^2} + \frac{4}{R}\frac{\partial V}{\partial R} = 8\pi\sigma,$$

(8.15) $$\frac{\partial^2 P}{\partial R^2} + \frac{3}{R}\frac{\partial P}{\partial R} = 16\pi\sigma U,$$

(8.16) $$\frac{\partial^2 Q}{\partial R^2} + \frac{2}{R}\frac{\partial Q}{\partial R} = -20\pi\sigma U.$$

We choose a solution of (8.14) which tends to zero as R tends to infinity and has no singularities and which together with its first derivative is continuous across the surface $R = a$. The solution is then unique and is easily found to be given by

(8.17) $$V_i = \frac{4}{5}\pi\sigma R^2 - \frac{4}{3}\pi\sigma a^2,$$

(8.18) $$V_e = -\frac{8}{15}\pi\sigma a^2 \left(\frac{a}{R}\right)^3.$$

We obtain solutions P and Q of the equations (8.15) and (8.16) satisfying the same conditions, but we determine the additive arbitrary constant so that $P = 0$ and

$Q = 0$ on the boundary. We find

(8.19) $$P_i = -\frac{4}{9}\pi^2\sigma^2\left(R^4 - 3a^2R^2 + 2a^4\right),$$

(8.20) $$P_e = \frac{8}{27}\pi^2\sigma^2 a^4\left(1 - \frac{a^3}{R^3}\right),$$

(8.21) $$Q_i = \frac{2}{9}\pi^2\sigma^2\left(3R^4 - 10a^2R^2 + 7a^4\right),$$

(8.22) $$Q_e = \frac{16}{9}\pi^2\sigma^2 a^4\left(1 - \frac{a}{R}\right).$$

To obtain the function W we transform the equations (6.23) and (6.24) to polar coordinates, obtaining

(8.23) $$\frac{\partial W}{\partial R} = \frac{\partial P}{\partial R} + \frac{\partial Q}{\partial R} + \frac{1}{2}R\sin^2\theta\left\{\frac{\partial^2 P}{\partial R^2} - \frac{1}{R}\frac{\partial P}{\partial R} + 2\left(\frac{\partial U}{\partial R}\right)^2\right\},$$

(8.24) $$\frac{\partial W}{\partial \theta} = -\frac{1}{2}R\sin\theta\cos\theta\left\{\frac{\partial^2 P}{\partial R^2} - \frac{1}{R}\frac{\partial P}{\partial R} + 2\left(\frac{\partial U}{\partial R}\right)^2\right\}.$$

Solving these we find

(8.25) $$W_i = P_i + Q_i - \frac{1}{4}R\sin^2\theta\left\{\frac{\partial^2 P_i}{\partial R^2} - \frac{1}{R}\frac{\partial P_i}{\partial R} + 2\left(\frac{\partial U_i}{\partial R}\right)^2\right\},$$

(8.26) $$W_e = P_e + Q_e - \frac{1}{4}R\sin^2\theta\left\{\frac{\partial^2 P_e}{\partial R^2} - \frac{1}{R}\frac{\partial P_e}{\partial R} + 2\left(\frac{\partial U_e}{\partial R}\right)^2\right\}.$$

Substituting the values we have found for U,V,W,P,Q, in (7.5), we have the expression for the internal and external field of a slowly rotating sphere, correct to the second order.

We may now find the relation between the astronomical and the dynamical rest frame in the case of a rotating sphere. Substituting from (8.17) in (7.4), we find

(8.27) $$\omega' = \omega - \Omega = \frac{8\pi\sigma a^2 \omega}{3c^2}$$

Hence we see that the dynamical rest frame is rotating slowly relative to the fixed stars, in the same direction as the rotating sphere.

The formula (8.27) furnishes theoretically another test of the truth of the general theory of relativity. Consider a Foucault pendulum suspended at one of the poles of the earth. Such a pendulum, being a small dynamical system in the neighbourhood of the observer at the origin is exactly the kind of system which the observer uses to specify the dynamical rest frame. We now see that this plane is rotating relatively to the fixed stars. As the pendulum is not at the centre of the earth the angular velocity is not given by the formula (8.27), but by the formula

(8.28) $$\omega' = \frac{2\omega}{c^2}\left(V_\infty - V_r\right)$$

where V_r is the value of V at the point where the the pendulum is suspended. Hence we find in the case of a pendulum suspended at the surface of the earth

$$(8.29) \qquad \omega' = \frac{16\pi\sigma a^2 \omega}{15 c^2}$$

This clearly is a result which, theoretically at any rate is capable of verification. Actually the angular velocity ω' is far too small to be observed, but the formula (8.29) nevertheless appears to the writer to be of some interest in giving a better insight into the question of rotation in general relativity.
Calculating the value of ω' for the case of the earth, we find that the plane of a Foucault pendulum rotates through an angle of about 26 sec. of arc per century. The effect is ofcourse very much larger in the case of the sun. Assuming the sun to rotate throughout with its equatorial angular velocity, we find that the plane of the pendulum in the case of the sun will rotate through about 52 min. per century.

Another physical consequence of the theory is ~~that~~ derived from a consideration of planetary motion round a rotating sphere. Consider the orbital plane of the planet. From the definition of the dynamical rest frame, it is evident that this plane is fixed relative to the dynamical rest frame. It follows therefore that the orbital plane will ratate relative to the fixed stars.

If the plane is inclined to the axis of rotation this rotation will be observable. The angular velocity will be given by the formula (8.28). Substituting from (8.18) we find

$$\omega' = \frac{16\pi\sigma a^2\omega}{15c^2}\left(\frac{a}{R}\right)^3 \tag{8.30}$$

In the case of the system formed by the moon and the earth we find on calculation that ω' is about 1.2×10^{-4} secs. of arc per century. This is far below the limit of observation. In the case of the planet mercury and the sun ω' is about 5.3×10^{-3} secs. of arc per century. This again is far too small to be observable.

Somewhat better values are found in the case of some of the satellites. We see from (8.30) that the effect varies as the inverse cube of the distance of the planetary body from the rotating sphere. The greatest effect will therefore be found in cases where the planetary body is near the rotating sphere. As another example we take the satellite Mimas of Saturn. The orbital plane of this satellite is inclined at an angle of 1 deg. 36 min. to the equator of Saturn, The distance of Mimas ~~xx~~ from Saturn is only about three times the radius of Saturn. On calculation we find that the orbital plane of Mimas rotates through about 19 secs. of arc per century.

A still greater effect is obtained in the case of Jupiter V, but it would be more difficult of observation as the angle which the plane of the orbit makes with the equatorial plane of Jupiter is only 27 min. On calculation we find that the orbital plane rotates through an angle of 59 secs. per century.

9. Comparison with the Schwarzschild solùtion.

In order to check the accuracy of the second order terms calculated by the present method, we will now compare the formulae which we have obtained for the pressure with that given by the Schwarzschild solution, and show that as the angular velocity tends to zero our formula agrees to the required degree of accuracy with the Schwarzschild formula.

Substituting from (8.11) and (8.20) in (6.28), we obtain for the pressure in the interior of a slowly rotating sphere, the formula

$$(9.1) \quad \frac{p}{\mu} = \frac{2}{3}\pi\sigma(a^2-R^2) + \frac{16\pi^2\sigma^2}{9c^2}(a^2-R^2)(a^2-\tfrac{1}{2}R^2) + \tfrac{1}{4}\omega^2(3r^2-2z^2-5a^2).$$

Now in the Schwarzschild solution we have the fundamental form

$$(9.2)\quad ds^2 = \frac{dr^2}{1-\frac{1}{3}\kappa\mu r^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{c^2}{4}\left[3\sqrt{1-\tfrac{1}{3}\kappa\mu r_0^2} - \sqrt{1-\tfrac{1}{3}\kappa\mu r^2}\right]^2 dt^2$$

and the pressure is given by

$$(9.3)\quad \frac{p}{\cancel{\mu}} = \frac{1}{3}\mu c^2 \,\frac{\sqrt{1-\frac{1}{3}\kappa\mu r^2} - \sqrt{1-\frac{1}{3}\kappa\mu r_0^2}}{\sqrt{1-\frac{1}{3}\kappa\mu r_0^2} - \frac{1}{3}\sqrt{1-\frac{1}{3}\kappa\mu r^2}}$$

Calculating this formula correct to the first order, and expressing the result in terms of Newton's gravitational constant, we obtain

$$(9.4)\quad \frac{p}{\cancel{\mu}} = \frac{2}{3}\pi\sigma(r_0^2 - r^2) + \frac{16\pi^2\sigma^2}{9c^2}(r_0^2 - r^2)\, r_0^2 .$$

Comparing (9.4) and (9.1), we see that for $\omega = 0$, the formulae are not quite identical. The reason is that the quantities r and R accurring in the formulae are not the same to a first approximation. If we transform the form (9.2) with the transformation

$$(9.5)\qquad r = \rho\left[1 + \frac{2\pi\sigma}{3c^2}(\rho_0^2 - \rho^2)\right]$$

then the form (9.2) becomes to a first approximation

$$(9.6)\quad ds^2 = \left[1 + \frac{2\pi\sigma}{3c^2}(\rho_0^2 - \rho^2)\right](d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta\, d\phi^2) - c^2\left[1 - \frac{2\pi\sigma}{3c^2}(\rho_0^2 - \rho^2)\right] dt^2 .$$

and hence comparing with (7.5), we see that now , to a first approximation

$$\rho = R. \tag{9.7}$$

Substituting from (9.5) in (9.4), we obtain , noting that $r_0 = \rho_0$,

$$\frac{p}{\mu} = \tfrac{2}{3}\pi\sigma(\rho_0^2-\rho^2) + \frac{16\pi^2\sigma^2}{9c^2}(\rho_0^2-\rho^2)(\rho_0^2-\tfrac{1}{2}\rho^2). \tag{9.8}$$

Now comparing (9.8) and (9.1), we see that the formula we have obtained for the pressure in the interior of a rotating sphere agrees with the Schwarzschild expression to the required degree of accuracy, in the case of zero angular velocity.

BIBLIOGRAPHY.

BIBLIOGRAPHY.

1918

Lense and Thirring: "Uber die wirkung rotierender ferner Massen in der Einsteinschen Gravitationstheorie". (Physikalischer Zeitschrift, vol. 19, pp 33-39).

Lense and Thirring: "Uber den Einfluss der Eigenrotation der Centralkörper auf die bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie". (Physikalischer Zeitschrift vol. 19 pp. 156)

H. Weyl: "Zur Gravitationstheorie". (Annalen der Physik pp 117-145)

1919

T. Levi-Civita: "Soluzione binarie di Weyl". (Atti Reale Academia dei Lincei 1919, pp3-13)

T. Levi-Civita: L'analoga del potenziale logarithmico". (ibid. pp. 101-109)

H. Weyl: "Bemerkung uber die axialsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen". (Annalen der Physik 1919, pp 185-188).

1922

R.Bach: "Explicite Aufstellung statischer axialsymmetrischer Felder, mit einem Zusatz uberb das statische zwei-korper Problem, von H. Weyl. "
(Mathematische Zeitshrift 1922 pp. 134-145)

1923

A. Palatini: "Sopra i potenziale simetrici che conducono alle soluzione longitudinali delli equazioni gravitationale di Einstein."
(Atti Reale ACCademia dei Lincei 1923 pp. 263-267)

1924

J. Chazy:"Sur le champ de gravitation de deux masses fixes dans la theorie de la relativité".
(Bulletin de la Societe Mathematique de France 1924, pp. 17-38)
~~404-410~~

P. Straneo: "Intorno alla teoria dei Campi einsteiniana a simmetria assiale".
(Atti Reale Accademia dei Lincei, 1924, pp. 404-410)

P. Straneo: "Deduzione e interpretazione di qualche ds einsteiniano simmetrico intorno ad un asse".
(ibid. pp. 468-474).

P.Straneo: "Considerazioni generali sui campi einsteiniani a simmetria assiale".
(ibid. pp. 547-552)

1927

G. Darmois.: "Les equations de la Gravitation Einsteinienne". (Memorial des Sciences Mathematiques. Fasc. XXV.)

1931

E.S. Akeley: "The axially symmetric stationary gravitational field".
(Philosophical Magazine, VII. 11, 1931 pp. 322-330)

E.S. Akeley: "The rotating fluid in Relativity theory".
(ibid. pp. 331-344)

P.Y. Chou: "The gravitational field of a body with rotational symmetry in Einstein's theory of gravitation".
(American Journal of Mathematics, 53, 1931, pp. 289-308)

1932

T. Lewis: "Some special solutions of the equations of axially symmetric gravitational fields".
(Proceedings Royal Society, London, A 136. 1932, pp176-192)

1935

A.G. Walker: "Note on Relativistic Mechanics".
(Proc. Edinburgh Mathematical Society, vol iv, pp170-174)