PROVABLE AND UNPROVABLE CASES OF TRANSFINITE INDUCTION IN A THEORY OBTAINED BY ADDING TO HAω SO-CALLED "TERM-FORMS" OF THE KIND INTRODUCED BY M. YASUGI

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ABSTRACT: I begin by discussing several of the existing ways of proving the validity of transfinite induction up to ε_0 and argue that it is at least conceivable that there is room for a new proof that is more constructive than any of them. An attempt which I pay particular attention to is that made by Mariko Yasugi (1982). The centrepiece of her theory is the so-called "construction principle", a principle for defining computable functionals. I argue that, in principle, it ought to be possible to set up a theory whose terms denote or range over functionals of a sort constructed by a similar principle, in which the accessibility (a term to be defined below) of ε_0 is provable, yet which dispenses with quantifiers as well as with some strong axioms which she uses in order to achieve the same result. My theory, described in chapter 2, is called TF (for "term-forms"). In chapters 3, 4 and 5, a proof of the accessibility of ε_0 in TF is presented. This thesis ends (chapter 6) with a proof of the computability of the functionals that can be represented in TF.

DECLARATION: This thesis has been composed by me and is my work.

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Prefatory Remarks and Acknowledgements

By the "accessibility" of a totally ordered set let us understand the property that every strictly decreasing sequence of members of the set is finite. Then the main technical result of this thesis can be stated as follows: the accessibility of the standard representation of ε_0 by Cantor Normal Forms is provable in the theory TF which I set up in chapter 2. The program to which this thesis is intended as a first contribution therefore concerns provable and unprovable cases of accessibility and not, despite what was advertised in the title, transfinite induction. The problem is that there seems to be no word in universal use for the property which I call "accessibility", but from now on I shall use the word in the sense just defined. By the way, I do not see that the equivalence of a statement of the accessibility of an ordinal number and a statement of the validity of transfinite induction up to that number is provable in every theory; but it does not seem urgent to investigate this question here.

A great many interesting metamathematical theorems are proved by transfinite induction (or deduced from a statement of accessibility) alongside otherwise finitist methods (by which I understand, as is usual, methods that can be formalised in primitive recursive arithmetic). If there is any merit in constructivism at all, it therefore makes sense to ask what the best way is of proving the validity of initial cases of transfinite induction. I must immediately disclaim any pretence that I can argue in any very satisfactory way that my proof in TF is superior, from the point of view of constructiveness, to any of the existing proofs of the accessibility of ε_0, so I can only hope that I have indicated, in chapter 1 below especially, some reasons for thinking that some such view might be defensible. Primum vivere, deinde philosophari.

I am not now inclined to think that the use I have made of the theory TF is the best use that could possibly have been made of it. The theory contains terms which denote computable functionals of a slightly more versatile kind than the primitive recursive functionals -- as is conclusively shown by the fact that the accessibility of ε_0 can be proved within it. I can only hope that the reader will believe that this shows that the theory is an interesting one. But I have not had time, within this thesis, to explore other possible applications.

Semantic questions about TF have been entirely ignored. For philosophical purposes, I quite accept that I cannot do so for ever. But it would have been impossible to treat of every possible question that arises concerning the theory within this thesis.

Perhaps it is necessary to add a word or two about the style of my proofs. This is variable: some proofs are presented in an almost completely formalised style, while others are only sketches of proofs. While it is therefore possible, indeed likely, that the latter contain some mistakes, I do not see that, practically speaking, any other approach would have been preferable. I do not imagine that many readers would have had the stomach to read this thesis in its entirety if it contained twice as many fully formalised proofs as it does; indeed I shall be presently surprised if they have the stomach to read it even in its present state. I hope that the readers who do take the trouble to read the more formal sections will accept them as evidence that I can be formally correct when I want to be and, perhaps more important, I have very good intuitions about what can be proved in TF and what cannot be.

Expressions of indebtedness now follow, in approximate order of importance.

The spare time which made it possible to write this thesis is due to the exceptionally generous financial support that I have received -- from the Scottish Education Department, the Deutschen Akademischen Austauschdienst and the Cross Trust -- and I can only wish that the final result did more justice to the opportunities which I have enjoyed. I am also thankful to the people who have helped me obtain scholarships from these organisations:

Professor D.R.P. Wiggins, Mr. J. Broackes, Mr. S. Rasmussen and Dr. L. Briskman. Perhaps, at this point, I should single out Mr. Rasmussen for special thanks on account of an enormous amount of encouragement and emotional support as well as for numerous pints of ale which he has bought me..

Intellectually my main debt is to authors whom I only know through their writings. But as far as personal contact is concerned, I must thank Dr. Alan Smaill of the Department of Artificial Intelligence, who has generously spent many hours talking to me about logic and reading bits of this thesis and its ancestors. I have made many alterations in response to his criticisms and, since I am sure that the result has been in every case an improvement, it is possible that I should have made even more. Dr. P. Milne has undertaken what must have been for him the thankless task of reading a draft of three chapters and allowed me the satisfaction of talking to him about them. Both these men have made particular suggestions for improvements which I have adopted, but I think it is more important to stress that it is always helpful to be able to talk about a subject with someone other than oneself.

It was a great relief to be able to spend a year in a department which contains a very impressive concentration of logicians, namely the Institut fuer mathematische Logik und Grundlagenforschung in Muenster. Many people in that institute made some effort to make the time that I spent there pleasant. In particular, practically all members of the institute who have some interest in recursive functionals were good enough to attend a lecture which I gave there on 15 July 1992, and I would like to thank them heartily, especially Professor Diller and Professor Pohlers on account of the favourable remarks which they made afterwards. For hospitality I am indebted to Herrn E. Folkerts of the institute and Herrn H. Reiter, a citizen of Muenster. Dr. M. Rathjen has helpfully given me the benefit of his knowledge about the ordinal strength of the sub-system of second-order number theory discussed in the first chapter and, while I do not make use of that information in the present thesis, it will be indispensable to me in my further investigation of the properties of TF.

CHAPTER 1

SOME REFLECTIONS ON PROFESSOR YASUGI'S CONSTRUCTION PRINCIPLE

Logicians have often been concerned to find constructive proofs of the validity of initial cases of transfinite induction. The first important initial case is, of course, transfinite induction up to ε_0 and the first attempt of the kind I speak of was probably that of Gentzen, presented in the article in which he first proved the consistency of Peano number theory (1936, pp.554-6). This first attempt suffers from the disadvantage that the author makes no thorough attempt to analyze what axioms and rules of inference his proof is based on. The first satisfactory proof, so far as I know, was that of Bernays, presented in the last section of the book by Hilbert and Bernays. In fact, this proof was later turned into a more formal proof by Gentzen (1943, pp.146-151), who is therefore commonly described as the author of the proof. But this must be a mistake, due to ignoring the remarks at the beginning of the article (p.140), in which he says that he is merely presenting a formalised version of an existing proof. Actually Bernays presents two proofs, though they are related.

Probably most logicians regard this proof as the last word on the subject, as far as ε_0 is concerned, though of course there were infinitely many more extensive cases of transfinite induction which still had to be tackled. But a few have thought that the Bernays-Gentzen proof, which uses a weak sub-system of intuitionist second-order number theory, was not the most constructive proof possible. In this chapter I shall discuss in detail an article by Mariko Yasugi called "Construction Principle and Transfinite Induction up to ε_0", which begins with the following paragraph:

> It is well-known that the accessibility of the ordered structure which is a canonical representation of the ordinals below ε_0 (the first ε-number) cannot be proved in elementary number theory (see Gentzen (1943)), while it is provable if an analytic method is employed, namely it is provable in first-order arithmetic augmented by the Π^1_1-induction (see Gentzen (1943)). The full power of the Π^1_1-induction is not needed, however, and attempts have

been made to establish the accessibility along more concrete lines, for example in Gentzen (1936) and Takeuti (1975).

I would be interested to know what Professor Yasugi means by the "power" of the Π^1_1-induction rule. It is tempting to think that she thinks that the power of a theory -- in this case, second-order number theory with the Π^1_0 -comprehension and Π^1_1-induction rules -- is measured by the ordinal number of the theory, that is, the smallest standard well-ordering whose existence is not provable within the theory. But if this is what she means, then her article suffers from the disadvantage that she does not present any evidence that the same result is provable in a weaker theory. Indeed, in that article, she never properly formulates a theory, with a language, axioms and rules of inference, as far as I can tell. However it is quite possible that to improve on the Bernays-Gentzen proof, by setting up a less powerful theory (in the sense just explained) but in which the same result can be proved, was never her intention.

The paragraph which I quoted just above is followed by the following:

> In this article we are to propose a theory of "construction principle", a principle on the ground of which some functionals can be defined . . . The principle above is considered here as the basis of the functional interpretation of transfinite induction up to ε_0.

The phrase "construction principle" ought not to be too unfamiliar to the reader; it had been used by Goedel (1960, p.78) to denote the principle on the basis of which the functionals needed to interpret Heyting Arithmetic can be defined. Professor Yasugi's construction principle, presented in section 4 of her article (pp.15-19), yields a more inclusive class of functionals, however. By a "functional interpretation of transfinite induction up to ε_0", I think she means the production of the functional N (p.13), which has the following two properties: (1) it maps any strictly decreasing sequence of ordinal numbers smaller than ε_0 onto a strictly decreasing sequence of natural numbers; (2) it maps only finite sequences onto finite sequences. Thus the existence of N entails that ε_0 is accessible. Her article ends (pp.19-23) with a demonstration that N can be derived from the construction princple. I think, and will argue below, that there are other things one could understand by a "functional interpretation of transfinite induction up to ε_0"; but I am concerned at the moment with exegesis of the purpose of Professor Yasugi's article.

In the third paragraph she writes

> Section 2 consists in the interpretation of transfinite induction up to ε_0 in an arithmetic with infinite reasoning. Although the local technicalities used in this section are borrowed from section 2 of Gentzen (1943), our scheme is a "uniform version" of the provability demonstration, so to speak, thus reaching up to ε_0.

(I conjecture that by "scheme" she means "goal".) From this point onwards, I must admit that I find the purpose of her article seriously obscure. However the emphasis on uniformity is at least not surprising. Section 2 of Gentzen's article of 1943 (op.cit., pp.146-150) presents a method for constructing, within pure number theory, a proof of the validity of transfinite induction up to any number smaller than ε_0. He proved that, if the validity of transfinite induction up to ω_k is provable in this theory, the same goes for ω_{k+1}. However -- if we use $\mathbf{TI}(\alpha)$ as an abbreviation for the formula expressing validity of transfinite induction up to α -- the statement

$$\mathrm{TI}(\omega_k) \rightarrow \mathrm{TI}(\omega_{k+1}) \qquad\qquad (*)$$

where k is a variable, is not provable <u>within</u> pure number theory. Otherwise we could apply induction in order to get a proof of the validity of transfinite induction up to ε_0 within that theory.

If (*) is provable in some theory, then we can say that the proofs in that theory of the validity of transfinite induction up to each value of ω_k are uniform in k, in the sense that the proof for one value of k would be the same as the proof for another value, except that certain numerals occurring within the proof would be different. For there would be a proof, in which k occurs as a free variable, from which we can get a proof of the validity of transfinite induction up to any of these numbers by sutstituting some natural number for k throughout the proof.

I think that if Professor Yasugi's argument for the existence of N were to be semi-formalised, it would employ the ω-rule. She does not draw attention to this fact, although she is presumably aware of it, but a careful study of her proof has left me with no other impression. She defines a sequence of two-place predicates $G_0, G_1, G_2 \ldots$ (p.6f.) and she writes $G_i(\phi, x)$, ϕ here being a variable whose type depends upon i, as $G(i; \phi, x)$. In the course of proving proposition 3.2(2) in her article she wishes to prove the following sequent, in which i is a free variable, χ is a bound variable and $\tau_{(i,0)}$ a term-form already defined:

$$(4) \quad G(0; \chi, 0) \rightarrow G(0; \tau_{(i,0)}(\chi), \omega_{i+1})$$

To prove each instance of (4), she recommends that we start with the following sequent

$$(1) \quad G(i+1; \phi, 0) \rightarrow G(i+1; \tau_{(i,0)}(\phi), \omega_0)$$

and gradually derive the appropriate instance of (4) using the properties of τ. But this means that infinitely many instances of both (1) and (4) must be used in the proof in order to get (4) itself, as i may take infinitely many values. For we must recall that $G(i+1; \phi, 0)$ is not a formula of our theory but merely a schema from which formulae are derived by substituting particular numbers for i.

If I am not mistaken, the proofs of propositions 3.1(2), 3.1(3), 3.1(4) and 3.2(1) also cannot be formalised rather than merely semi-formalised. Taken by itself, this fact is not fatal to the argument as a whole. It could be that these results are stronger than any that need actually to be used in the remainder of the article. Because of this, it is not remarkable that the author, in her proof of proposition 3.2(1), seems to show some awareness of the fact that the proof of which she is giving an outline is not one that can be formalised. I suspect she manages to admit this with equanimity because, for the purpose of deriving each closed instance of (4) from the relevant closed instance of (1), not proposition 3.2(1) itself but only finitely many closed instances of it are required. Furthermore it may well be that, in order to prove each closed instance of 3.2(1), only finitely many closed instances of 3.1(2), 3.1(3) and 3.1(4) are required. Therefore it seems that Professor Yasugi makes only one essential use of the ω-rule. That use, however, is essential because undoubtedly (4) itself and not merely a finite number of closed instances of it are necessary for the argument as a whole.

It would be possible to interpret Professor Yasugi's proof as containing not infinitely many predicates G_i, but a single, inductively defined, three-place predicate G. However this is not only not her intention, as far as I can tell (see especially the remarks on the predicates A_i on p.4), but it is also well-known that you can get an extension of pure number theory, in which the accessibility of ε_0 is provable, by adding to it an inductive definition of such a predicate as an axiom. A proof of this kind will even be sketched in chapter 3 below. However it is equally well-known that you can get a theory with the same property by adding the ω-rule; indeed Professor Yasugi sketches such a proof herself in section 2 of her article. The possibility remains that she thinks that the proofs of the premisses of her application of the rule to prove sequent (4) are uniform in a way that the proofs of the premisses in the familiar demonstration are not. However, in view of the summary presented above of how (4) was to be derived, I see no justification for thinking this.

All in all, I therefore find it impossible to see what is gained by her talk about "uniformity" and, even more so, the idea, which the first paragraph of Professor Yasugi's article may have encouraged us to have, that she was going to present a proof of the accessibility of ε_0 which would use less "powerful" methods than the existing proofs seems to have been entirely left by the wayside. What I do think is of value in the article is the programme which the second paragraph (also quoted above) suggested: the construction principle itself and the idea of using the functionals defined by it in order to produce a functional interpretation of the statement that ε_0 is accessible.

By a functional interpretation, in some specified quantifier-free theory T_1, of some sequent in the language of some other theory T_2, I mean a proof in T_1 of a sequent from which the sequent, which we wanted to interpret, can be derived simply by applying the rules for the introduction of quantifiers in T_2. Obviously the formulae in any such sequent have to be in prenex form and it is usual to stipulate, as well, that the existential ones must precede the universal ones. I shall make this stipulation as well and I shall call sequents of the kind described "sequents in $\exists\forall$-form". Some theories have the convenient property that every sequent can be proved equivalent to a sequent in $\exists\forall$-form. When we are dealing with such a theory, therefore, it makes sense to talk about "the $\exists\forall$-form of a sequent" and, even when we are not dealing with a theory of this kind, it is often clear how one could extend the theory so as to make it of this kind, in which case it again makes sense to use that expression.

The statement that ε_0 is accessible can be formulated in the theory which I shall call HA -- by this I mean the theory HAω, which is described in the next chapter, but without quantifiers of higher types -- and it can be proved equivalent, within the theory which I call HAω^+, to a certain sequent in $\exists\forall$-form. I shall explain in chapter 3 below what the respective sequents look like. The important thing about the accessibility-statement, for our present purposes, is that it contains just one existential quantifier, which occurs within the scope of a universal quantifier and a free variable. On my understanding of the intuitionist tradition, a canonical proof of that statement will consist in the production of a functional which maps any values of the free variable and the universal variable onto an appropriate value for the existential variable together with a canonical proof that it does do so. Thus a canonical proof of the accessibility-statement will be almost the same thing as a functional interpretation of its $\exists\forall$-form. This statement will be supported by a slightly more detailed discussion in chapter 3 below, but the discussion there requires some technical apparatus which is not at present at our disposal.

It might be ojected that writers on functional interpretations, if they discuss the issue at all, generally go out of their way to deny that, in defining a translation of sequents into their $\exists\forall$-forms and then defining what would constitute a functional interpretation of these $\exists\forall$-forms, you are explaining what the original sequents "really mean", from an intuitionist point of view. In particular, Goedel, in the article where he first described the kind of functional interpretation which I am concerned with, wrote:

> Selbstverstaendlich wird nicht behauptet, dass die Definitionen 1-6 den Sinn der von Brouwer und Heyting eingefuehrten logischen Partikel wiedergeben. Wieweit sie diese ersetzen koennen, bedarf einer naeheren Untersuchung (1960, p.82).

but this does not contradict the claim I have made just above. To say that a canonical proof of the accessibility-statement is almost the same thing as a functional interpretation of its $\exists\forall$-form is not the same as to make this claim for all statements whatsoever, that occur within some branch of mathematics. The accessibility-statement (which, as I have already promised, will be discussed in detail in chapter 3) is an exceptionally simple statement: it contains only two quantifiers, and none of them occurs in the antecedent of a conditional.

The program which I intend to carry out for ε_0 resembles not so much Professor Yasugi's program in "Construction Principle and Transfinite Induction up to ε_0" as her program of proving the accessibility of any system of ordinal diagrams based on a pair of sets for which a constructive accessibility-proof is already given. This program is carried out in her (1985/6), in which she introduces a construction principle (the "hyper-principle") of a much stronger kind than that used to treat of ε_0. I shall quote some bits of the introductory section of that article to illustrate the similarity:

> Let $(\underline{C}, \prec)$ be a linearly ordered structure such that there is a method to determinewhether or not an object $\underline{x}$ belongs to $\underline{C}$ and let $\underline{c}$ be in $\underline{C}$. acc$(\underline{C}, \prec, \underline{c}, \underline{M})$ will express that $\underline{M}$ is a method such that, for every $\underline{f}$ a $\prec$-decreasing sequence from $\underline{C}$ led by $\underline{c}$, $\underline{M}(\underline{c}, \underline{f})$ gives a modulus of finiteness of $\underline{f}$, that is $\forall \underline{n} \leq \underline{M}(\underline{c}, \underline{f})(\underline{f}(\underline{n}) = \text{empty})$.
> . . . We are therefore naturally led to an intuitionistic system (ASOD) in the attempt to formalize the accessibility proof, and the nature of the accessibility can be embodied by the functional interpretation of the existential quantifiers. They occur in the form $\forall \underline{f} \exists \underline{n}\, \underline{P}(\underline{f}, \underline{n})$, where $\underline{P}(\underline{f}, \underline{n})$ is $\exists$-free. A functional $\underline{X}$ such that $\forall \underline{f}\, \underline{P}(\underline{f}, \underline{X}(\underline{f}))$ will represent the modulus of finiteness (of $\underline{f}$) (1985, p.227f.).

That is, she gives two proofs of the accessibility of systems of ordinal diagrams: the first is in the theory which she calls ASOD and uses quantifiers. Then she presents a method for functionally interpreting every statement of that theory which is in $\forall\exists$-form, and this yields a quantifier-free proof of a quantifier-free statement which closely resembles the original accessibility-statement. I think the remarks she makes towards the end of the introductory section (p.229) could be taken as meaning that she thinks the second proof will be more constructive than the first, though I am not sure about this. But in any case, for reasons explained above, I think it is plausible to maintain that the second proof will more closely resemble an intuitionist canonical proof.

There is another aspect of Professor Yasugi's thought which I have not yet done justice to. This is expressed in the following quotation:

> The essence of the diagrams can be characterized by its functional structure, that is, by determining the universe of functionals which produce "moduli of finiteness" for various decreasing sequences of the diagrams (ibid., p.227).

What she means by "modulus of finiteness" is explained in the last quotation. The quotation just cited seems to me to express the idea that, for each statement of accessibility, there is something to be gained by discovering some optimally weak (in ordinal-theoretic terms) system of functionals which can interpret that statement. What she does in that article for systems of ordinal diagrams of the kind described is almost the same as what I am going to do in this thesis for ε_0. It has to be admittted that neither she nor I presents any evidence that the system of functionals we use is optimally weak, but, on the other hand, there is at least some evidence that each system is weaker than the other systems that could be used for this purpose.

What precisely is the illumination to be gained from determining the "universe of functionals" associated with each accessibility-statement, I must admit I do not yet see. But I am prepared to take it on trust that Professor Yasugi knows what she is talking about.

To produce a functional interpretation of the $\exists\forall$-form of the statement that ε_0 is accessible is, by itself, not a novel achievment. By a functional interpretation of a theory, I mean an algorithmic method for transforming a proof of a statement of it in $\exists\forall$-form into a functional interpretation of its conclusion. There are many functional interpretations of theories in which that accessibility-statement can be proved and translated into $\exists\forall$-form as well. But they suffer from the disadvantage that the theories in question are all, so far as I know, very much stronger than ε_0 and that the functionals required in many cases do not admit of a constructive computability-proof. The proof of the computability of the functionals which I use in this thesis is presented in chapter 6 and it turns out to be remarkably simple.

A survey of the kinds of functional which can be used to interpret theories stronger than pure number theory, at least so far as this subject had been developed by 1973, is presented by Troelstra (1973, pp.81-4). A particularly interesting theory of functionals is Girard's system F (ibid., p.84; Girard 1989, pp.82-94). The theory was introduced for the purpose of interpreting second-order number theory and the types of Girard's terms are precisely the formulae of that theory. It would be interesting to know whether Girard's system can be divided into sub-systems, each of which can be used to interpret the corresponding sub-system of second-order number theory. But I do not know whether this has been done, or whether it would be as easy as it sounds. In any case, the comparative values of Girard's approach and the one adopted in this thesis is a matter I intend to discuss at a later date.

In this thesis I set up a theory of functionals which is much weaker than any of the theories just mentioned. Not only does it have the advantage of admitting a very simple computability-proof, I also guess that, since the theory was devised for the purpose of interpreting the statement that ε_0 is accessible, it will be better adapted to the study of the very weak sub-systems of second-order number theory than the strong theories I have just mentioned.

CHAPTER 2

THE THEORIES HAω, TF AND RELATED THEORIES

There is a cluster of theories in the logical literature which are called "HAω" or some similar name. In this chapter I shall set up a theory of this kind and prove that it at least contains one of the existing theories with that name. I am primarily interested in the quantifier-free part of the theory, qf.-HAω, from which my own theory TF, also without quantifiers, is obtained by generalising in a certain way.

There is a subject called "the philosophy of formal systems", whose name comes from H.B. Curry, though earlier logicians had also made contributions to it. When you read a logician's definition of a calculus, theory or formal system which he is setting up, it is necessary to know what his particular philosophy of formal systems is, in order to understand fully what he is doing. Many logicians do not say anything explicit on the subject. Fortunately, as a result of convention, there are not too many different ways of setting up a formal system in actual use, so it is generally possible to make sense of the procedures of logicians when they are doing just that. My own way of defining TF is not very different from the procedure of many other logicians. Someone who has no great interest in the philosophy of formal systems might therefore do better to skip the next few pages. But, especially as TF is a very unfamiliar theory, I believe I have some obligation to put my views on the subject on record.

My procedure is, on the whole, derived from Curry. I shall therefore quote part of his statement of how a definition of a particular formal system (its "primitive frame", as he calls it) proceeds, before adding a few remarks of my own.

He stipulates (1952, p.11) that a primitive frame shall consist of three parts. First rules are given for constructing terms of the system, then rules for constructing statements and, thirdly, rules for determining which statements are axioms and when a statement follows from certain other statements. The third part therefore obviously consists of two sub-parts. Concerning the first part of the primitive frame, Curry stipulates that it, in turn, shall consist of three sub-parts. That is, his list of the parts of which a primitive frame should consist begins as follows:

> I. TERMS
> A. Tokens, or primitive terms. This is simply a list, which may be infinite, of the terms of each kind. Nothing else is specified concerning them.
> B. Operations, i.e. modes of combination for forming new terms. There is a list of these with the number and kind of the arguments for each.
> C. Rules of formation, specifying how new terms are to be constructed. These will be of the form: If such and such an operation is applied to a sequence of terms of the proper number and kind, the result is a term of such and such a kind (1952, p.11).

In this chapter, I shall adopt the convention that the statements of a formal system are constructed by a procedure analogous to that by which the terms are constructed. By "objects of the formal system", or "formal objects" I mean all objects of the following four categories: terms, connectives, predicates and formulae. The formulae and statements (sequents) are constructed out of the terms, connectives and primitive predicates by operations similar to those used in the construction of the terms.

To return now to Curry's statement of how the terms of a formal system are constructed, I would like to comment in detail on each of his three clauses:

A. It is an important ingredient of Curry's philosophy (though here he only hints at it with the words 'Nothing else is specified concerning them') that it is quite unnecessary for a primitive frame to say what the primitive terms look like. Indeed, he is quite sympathetic to the idea that the terms should be regarded as abstract objects, which perhaps do not look like anything at all (ibid., p.29).

In this respect, Curry's attitude to a formal system contrasts with the more traditional attitude (exemplified by Kleene's Introduction to Metamathematics), according to which the terms of the formal system are symbols of an "object-theory" and actually appear on the page in the course of defining the formal system. However, if the system is defined in Curry's way, it is probably necessary that at least the names of the primitive terms appear on the page. It is probably also necessary -- Curry does not make this stipulation, but I wish to be understood as making it -- to stipulate that different names should, in general, be regarded as names of different terms. I shall give explicit rules for determining when two different names of terms should be regarded as names of the same term. Otherwise it should be assumed that different names denote different terms.

The list of primitive terms may, as Curry says, be infinite. If it is, we cannot give the names of all of them, but we can indicate a method for generating them: for example, we can say that $x_0, x_1, x_2, \ldots$ shall be one class of primitive terms.

Curry's use of the word "list" may require some comment. It is tempting to think that a list of primitive terms will be composed out of the terms which it lists. In fact, it will be composed of their names. This is in accordance with how we normally use the word "list": a shopping list, for example, does not consist of the items which one intends to buy, but of words.

This talk about "names" of terms impels me to draw another distinction. It is necessary to distinguish between names of terms, and other objects, of the formal system, on the one hand, and variables which range over such objects, on the other. Whereas one has to be careful to insure that all objects have names and that distinct objects have distinct names, one can be relatively ad hoc in the use of variables, indicating, on each occasion on which one introduces a variable, what sort of objects it is meant to range over.

B. Curry's attitude to operations illustrates another peculiarity of his approach. Traditionally, all objects (which were, traditionally, all symbols) are formed by writing down atomic objects in sequence: this way of forming expressions is necessitated by the two-dimensional nature of paper. But once we have abandoned the prejudice that the formal system has to consist of symbols, it makes sense to imagine that there is more than one way of combining objects to form new objects. This attitude ultimately makes it possible to make do with a smaller number of primitive objects than would otherwise be needed. That is, if M and N are terms of a formal system and we wish to imagine that the formal system contains a term which is formed by performing a certain operation on M and N, it is unnecessary to suppose that, in order to perform this operation, you need any other object besides M and N. The result of the operation might be denoted, in the language we use to talk about the formal system, by something like 'fMN', but it would not necessarily be correct to suppose that 'f' here is a name of anything in the formal system, for the reason just given. This is shown by the fact that the result of the operation could just as well be denoted by 'MN' or 'M^N' or even '${}^N M$', etc.

Because of this, I cannot agree with Curry's procedure when, in the chapter entitled 'Examples of Formal Systems' (1952, pp.17-27), he seems to treat expressions like 'f' in the above example as names of operations. Certainly it is useful to have names for the operations, but 'f' in the above example is not a name of anything: it looks like a name for something in the system, but the operations are not elements of the system but things we do to elements. So the sort of expression that it would be suitable to use for names of the operations would include verbal nouns. For example, if it is possible to perform an operation on a term that transforms it into another term called its "successor", it would be natural to call the operation "forming the successor".

There is admittedly a price to be paid for accepting Curry's innovations. Where the terms of the formal system are linguistic expressions of a familiar kind, such ideas as substitution, the length of a term and the number of occurrences of a sub-term within a term have an obvious meaning. But now these expressions have to be carefully defined, though how they are to be defined is still reasonably obvious. For definitions, I refer to Curry et. al. (1958, pp.44-59; 1972, pp.15-19).

For the sake of precision, it is necessary to stipulate that, if M and N are terms constructed by different processes, then M and N are different terms.

C. It is clear from Curry's examples that this part of the definition of the class of terms will contain stipulations like 'If $\mathfrak{A}$ and $\mathfrak{B}$ are terms, then $\neg\mathfrak{A}$ and $\mathfrak{A} \supset \mathfrak{B}$ are terms' (p.19). This means that, once the names of the primitive terms are given, we may substitute them for the German letters in that statement and get a true statement. In this particular formal system the primitive terms are called 'p_1', p_2', 'p_3', etc.

What sort of linguistic act is being performed by Curry's stipulation 'If $\mathfrak{A}$ and $\mathfrak{B}$ are terms, then $\neg\mathfrak{A}$ and $\mathfrak{A} \supset \mathfrak{B}$ are terms'? I think it is best to take it as an implicit definition of the symbols '$\neg$' and '$\supset$'. It implies, but does not say explicitly, that there is at least one way of getting a new term from any two terms and, for one of these ways, the name of the new term can be constructed by writing down the names of the old terms with '$\supset$' between them. I believe the definition could be given more explicitly as follows: "For any terms $\mathscr{u}$ and $\mathscr{b}$, the result of the one-place operation on $\mathscr{u}$ is denoted by '$\neg\mathscr{u}$' and the result of the two-place operation on $\mathscr{u}$ and $\mathscr{b}$ (in that order) is denoted by '$\mathscr{u} \supset \mathscr{b}$'". This, of course, involves quantifying into quotation marks, so an explanation of how that is to be interpreted will have to be added; though, in this particular case, I think I have already indicated how it is to be interpreted.

Partly because I reject Curry's idea that, in the example under consideration, '$\neg$' and '$\supset$' are names of operations, and partly to save space, I shall in practice run together parts B and C of the definition of the terms in a primitive frame. But, in describing each operation, I shall still have two distinct things to say about it. One is what sort of terms, and how many, the operation has to be performed on, and what sort of term results from the operation. Another is what the name of the term that results from the operation will look like, once the names of the terms upon which it is performed are given.

Now I have some more general remarks to make before I actually define the set of terms of HAω. By a 'definition' of the terms, I emphatically do not mean an explanation of what they mean -- that will come later -- but only a definition of the conditions that a system of objects will have to satisfy in order to be a plausible candidate for being the terms of HAω.

Every term of HAω has a type. I will define the types and the type-functors first, as if they were terms of a formal system themselves. They are not terms of HAω, but they could easily be terms of another formal system.

In reading a symbol for a type or object of HAω or TF, or ranging over such objects, it is necessary to know which sub-symbols of that symbol denote, or range over, the objects to which the last operation in the construction of the resulting object(s) was applied. That is, it is necessary to know which strings of symbols within the whole symbol actually denote, or range over, components of the objects described by the whole symbol. As is usual, I use brackets for this purpose: any expression which is enclosed within brackets does denote, or range over, a genuine component, or range of components. Other conventions will be described as I come to them.

A. The Formal System HAω.

There are many formulations in the literature of formal systems similar to this one, and some systems may have been formulated more than once. The formulations I have especially studied are those of Yasugi (1963), Troelstra (1973 chapter 1, especially p.46) and Schuette (1977 chapters 6 and 7, especially pp.149-151). Viewed in Curry's way, it would seem that Professor Yasugi's set of terms is built up from the primitive terms, which are 0 and variables, by means of four different operations, while Schuette's is built up from a more complicated set of primitive terms by means of just one two-place operation. My approach is closer to Yasugi's, but I use a no fewer than seven operations, of which the first four resemble her four and the last three correspond to the application of pairing and decoding operators to terms of suitable types (see Troelstra, p.47).

1.Types.

In the following, letters like 'ρ', 'σ', 'τ' range over types.

1.1. There is one primitive type, o (omikron).

1.2. There are two two-place operations on types: one operation on ρ and σ yields $\rho\sigma$, the other yields $\rho\times\sigma$. In reading a type-symbol, one should observe the following conventions: $\rho\sigma_1\sigma_2 \ldots \sigma_n$ should be thought of as formed from ρ and $\sigma_1\sigma_2 \ldots \sigma_n$ respectively while $\rho_1\rho_2 \ldots \rho_m\times\sigma_1\sigma_2 \ldots \sigma_n$ should be thought of as formed from $\rho_1\rho_2 \ldots \rho_m$ and $\sigma_1\sigma_2 \ldots \sigma_n$. The type oo is also called '1'. The <u>level</u> of a type σ, called $l(\sigma)$, is defined as follows: $l(o) = 0$; $l(\sigma\times\tau) = \max\{\sigma, \tau\}$; $l(\sigma\tau) = \max\{l(\sigma), l(\tau)+1\}$. The level of a term is the level of its type.

2.Terms.

Letters like 'M', 'N', 'P', with type-symbols in the superscript position where these are deemed necessary, range over terms of HAω of the indicated type, while the letters 'r', 's', 't' range over terms of type o.

2.1. There are two main kinds of primitive term, 0 and variables.

2.1.1. 0 is a term of type o.

2.1.2. For every type σ, there are denumerably many variables belonging to that type. They are called X_0^σ, X_1^σ, X_2^σ etc. If σ is o, the variables are also called $x_0, x_1, x_2, \ldots$ To save space, the variables X_i^σ, where $0 \leq i \leq 5$, are also called U^σ, V^σ, W^σ, X^σ, Y^σ and Z^σ. Likewise, x_i, for $0 \leq i \leq 5$, is also called u, v, w, x, y or z respectively. The numeral written in the subscript position of the name of a variable is said to denote the <u>shape</u> of the variable.

Underlined letters like '$\underline{X}^\sigma$', '$\underline{Y}^\sigma$' and '$\underline{Z}^\sigma$' shall be used as metamathematical variables ranging over variables of the indicated type.

2.2. The following are the operations that may be performed upon terms.

2.2.1. For every term of type o, there is an operation which is said to transform that term into its <u>successor</u>. The successor of s is s' The terms 0', 0'', 0''' etc are called <u>numerals</u> and are also denoted by '1', '2', '3' etc.

2.2.2. If M is a term of type τ and $\underline{X}$ a variable of type σ, there is an operation called 'λ-binding the variable $\underline{X}$ in M', which yields the result $\lambda\underline{X}.M$, a term of type $\sigma\tau$. This operation may only be applied if $\underline{X}$ is not already bound in M.

2.2.3. If M is of type $\sigma\tau$, for some σ and τ, and N is of type σ, there is an operation called 'application of M to N', which yields the result MN, of type τ. To economize on brackets, I stipulate that "$M_1M_2\ldots M_nN$" should be taken to denote the result of applying $M_1M_2\ldots M_n$ to N.

2.2.4. If M is of type $o\sigma\sigma$ and N is of type σ, $\rho[M, N, s]$ is of type σ. This operation is called 'primitive recursion'.

A term of HAω of level not greater than 1, in whose construction this operation is applied only to terms whose level is likewise not greater than 1 while the operations 2.2.5-7 are not used at all, is called a 'primitive recursive functor'. The letters 'f', 'g' etc. are used as metamathematical variables ranging over such functors while letters like 'k', 'm' and 'n' range over such as are of type o. The <u>order</u> of f, which we call O(f), is computed from its type, as follows: if the type is o, the order is 0 and if the type is $o\sigma$, for some σ, the order is greater by one than if the type were σ. (Semantically speaking, the order of a primitive recursive functor is the number of argument-places of the primitive recursive functions which it represents or ranges over.)

2.2.5. For any terms M and N, of types ρ and σ respectively, $\{M, N\}$ is a term of type $\rho\times\sigma$. This operation is called 'pairing' and the result of it is called the 'pair' of M and N.

2.2.6-7. If M is a term of type $\rho\times\sigma$, for some ρ and σ, there are two operations called 'decoding' which transform M into a term of type ρ and a term of type σ respectively. The results of the operations are M_0 and M_1 respectively.

<u>3.Formulae and Statements of HAω.</u>

I shall use letters like 'F', 'G' etc. as variables ranging over formulae. There is one predicate, =. Every atomic formula is formed from = and two terms of type o; the formula formed from s and t (in that order) is s=t. Molecular formulae are built up from atomic ones using the propositional connectives &, $\vee$, $\supset$ and $\sim$ and existential ($\exists$) and universal ($\forall$) quantifiers binding variables of every type. The formation-rules for formulae are the usual ones and all the connectives have their usual meaning. I write $F\equiv G$ as an abbreviation for $F\supset G.\&.G\supset F$.

In order to show the order in which the operations used in the construction of formulae are applied, I use brackets, as before, and dots. Brackets take precedence over dots, in the sense that any string of symbols enclosed within brackets denotes a sub-formula or ranges over some sub-formulae, regardless of how many dots occur at junctures within and without it. Each group of dots will be placed between a symbol denoting a connective and a symbol denoting, or ranging over, sub-formulae within the scope of that connective. The latter symbol will be bounded, on the one side, by the group of dots just mentioned, on the other by either a bracket or a larger group of dots or the end of the whole formula. For example

$$\overbrace{F \supset G}^{1} .\&. \overbrace{(\sim : \underbrace{\underbrace{F \supset G}_{4} .\&. \underbrace{G \supset F}_{5}}_{3})}^{2}$$

is made of the sub-formulae (1) and (2). (2), in turn, is composed out of (3), in the first instance, while (3) is composed out of (4) and (5).

The statements of my version of HAω are not formulae but sequents. Since it is meant to be an intuitionistically acceptable theory, I will impose the restriction that at most one formula containing quantifiers can occur in the succedent position of any sequent (this is a feature of Yasugi's formulation; see p.103). The sequent-connective is denoted by an arrow (→). As is usual, I use letters like Γ, Δ etc. as variables ranging over sequences of formulae.

4. Redexes and their Contraction.

In order to state the axioms of HAω, it is necessary to state which terms are redexes and how the contractum of a redex is determined. A term can only be a redex if the last operation in its construction is application, primitive recursion, or decoding.

4.1. A term of the shape MN is a redex only if M is formed by λ-binding some variable, say within some term, say Q. In that case, the contractum of MN is Q[X:=N], which notation means the result of substituting N for X within Q. The exact definition of substitution is a slightly complicated matter, and will be left until the end of this chapter.

4.2. ρ[M, N, s] is a redex if and only if s is 0 or a successor. If s is 0, it contracts to N. If s is a successor, say r', the term contracts to Mr(ρ[M, N, r]).

4.3. M_0 and M_1 are redexes if and only if M is a pair, say {N, Q}. If it is, then M_0 contracts to N and M_1 to Q.

5. α-Convertible Terms of HAω.

Let M and N be terms of the shape $\lambda X^{\ulcorner f \urcorner}_{g}$.P and $\lambda X^{\ulcorner f \urcorner}_{h}$.Q respectively. Suppose that, if we replace $X^{\ulcorner f \urcorner}_{g}$, wherever it occurs in P, and $X^{\ulcorner f \urcorner}_{h}$, wherever it occurs in Q, with some variable not occurring in either P or Q, we get the same term. Then M and N are said to be α-convertible to one another. The transitive closure of this relation is also called α-convertibility. Finally, any two terms which are formed from each other by replacing a sub-term of one with a term to which it is α-convertible are also α-convertible.

6. Mathematical Axioms of HAω.

All mathematical axioms are required to be free of logical symbols. They fall into four groups. The first two groups consist entirely of sequents with a single succedent formula.

3.1. Axioms of α-conversion. For any terms M and N which are α-convertible to one another, $\rightarrow M=N$ is an axiom of HAω.

3.2. Axioms of reduction. A term N is said to reduce to M in one step if M can be obtained from N by replacing some sub-term of N, which is a redex, with its contractum. The transitive closure of reduction in one step is called simply reduction. For any terms M and N, such that one reduces to the other, $\rightarrow M=N$ is an axiom of HAω.

3.3 Axioms of number. Under the interpretation of the terms of type o which follows from giving the symbol '0' and the general terms 'successor' and 'variable' their usual meanings, there are one or two truths about numbers which are not otherwise derivable in HAω, for example $t' = 0 \rightarrow$. Some people also take $s'=t' \rightarrow s=t$ as an axiom, but I guess it is probably derivable; see below. However, there is no particular reason to be parsimonious about which sequents one accepts as axioms of number, so I shall leave this set of axioms open-ended.

3.4. Axioms of equality.

$$\rightarrow t=t$$

$$s=t \quad \rightarrow t=s$$

$$r=s,\ s=t \rightarrow r=t$$

$$s=t \rightarrow r[\bar{q}:=s] = r[\bar{q}:=t]$$

In the fourth axiom-schema here, the symbol $\bar{q}$ denotes only a particular occurrence of a term q and $r[\bar{q}:=t]$ means that that occurrence of q is replaced with an occurrence of t. It is not necessary that every occurrence of q be replaced. If I wanted to indicate the result of replacing every occurrence, I would have written simply $r[q:=t]$.

7. Logical Axioms and Rules of Inference.

The logic is LK (introduced in Gentzen 1935, pp.191-3), subject to the restriction that only one succedent formula in any sequent may contain quantifiers. This logic is more liberal than LJ, but is a conservative extension of the theory with the logic LJ. This can be proved by proving

$$\rightarrow \sim\sim s = t :\supset: s = t$$

without any use of multiple succedents. A proof is given by Schuette (pp.139-141) for a corresponding statement in his negationless system of pure number theory. The proof can certainly be carried over to my version of HAω without multiple succedents. However, as it employs quantificational rules, it is a lot harder to adapt it to qf.-HAω. On the basis of a hasty inspection, I think it can be done, but the proof in qf.-HAω requires a number of technical tricks, notably theorem 4.6, which will not be introduced until later in this thesis. But informally speaking, the rule of double-negation elimination for equations is certainly justified, as all closed equations are decidable: see chapter 6.

Granted that the sequent written above is provable, the law of excluded middle and the law of double negation are presumably provable for all quantifier-free formulae in my language, by means of LJ. This has probably already been proved somewhere.

The induction-rule is formulated as follows

$$\frac{\Gamma \rightarrow \Delta, F(0) \qquad F(\underline{x}), \Theta \rightarrow \Lambda, F(\underline{x}')}{\Gamma, \Theta \rightarrow \Delta, \Lambda, F(t)}$$

subject to the condition that $\underline{x}$ should not occur in the main premiss except at the places indicated. t is any term of type o in the language.

Every instance of modus ponendo ponens

$$F, F \supset G, \Gamma \rightarrow \Delta, G$$

is provable. Sequents of this form will therefore be given axiomatic status and annotated with 'MPP'.

In the proofs which I shall present in chapters 4 and 5, the sequents shall be numbered and each will be followed by a brief indication of how it is derived. Mathematical Axioms and Logical Basic Sequents will be annotated with 'M' and 'L' respectively. Sequents derived by inference will be annotated with the numbers of the sequents from which they are derived and the names of the main inferences involved. Each introduction-rule is called by the name of the connective introduced, followed by 'E' or 'I', depending on whether the connective is introduced in the antecedent or succedent.

8. The Theory HAω$^+$

For certain purposes, it is desirable to consider the theory obtained by adding to HAω the following three rules of inference. The first rule is called the axiom of choice (AC)

$$\frac{\Gamma \longrightarrow \Delta, \forall \underline{X}^{\sigma} : \exists \underline{Y}^{\tau}. F(\underline{X}, \underline{Y})}{\Gamma \longrightarrow \Delta, \exists \underline{Y}^{\sigma\tau} : \forall \underline{X}^{\sigma}. F(\underline{X}, \underline{Y}\underline{X})}$$

the second is called 'the rule of the independence of premiss' (IP)(this rule is equivalent to Troelstra's IP_0'; see p.238)

$$\frac{\Gamma \longrightarrow \Delta, \quad H \supset \exists \underline{X}^{\sigma}. F(\underline{X})}{\Gamma \longrightarrow \Delta, \exists \underline{X}^{\sigma}(H \supset F(\underline{X}))}$$

where H must be in prenex form and contain no existential quantifiers, nor the variable $\underline{X}^{\sigma}$; and the third is called 'Markov's Principle' (MP), where H must be free of quantifiers:

$$\frac{\Gamma \longrightarrow \Delta, \sim \forall \underline{X}^{\sigma}. H(\underline{X})}{\Gamma \longrightarrow \Delta, \exists \underline{X}^{\sigma}. H(\underline{X})}$$

9. Some Special Combinators of HAω.

For every type σ, we can form a combinator J_σ, of type $o(\sigma\sigma)\sigma\sigma$, with the property that $J_\sigma 0 M^{\sigma\sigma} N^\sigma$ reduces to N^σ while $J_\sigma s' M^{\sigma\sigma} N^\sigma$ reduces to the same term as $M^{\sigma\sigma}(J_\sigma s M^{\sigma\sigma} N^\sigma)$. Namely, we define J_σ to be $\lambda y X^{\sigma\sigma} Y^\sigma . \rho[\lambda x. X^{\sigma\sigma}, Y^{\sigma\sigma}, y]$.

Using J_o, we may construct a term P, which represents the predecessor function and terms called $\lambda xy.x+y$ and $\lambda xy.x \dot{-} y$, which represent addition and subtraction terms called $\lambda xy.x+y$ and $\lambda xy.x \dot{-} y$, which represent addition and subtraction respectively. For proofs I refer to Schuette (pp.120-124).

I shall also use the (contextually) defined predicates $<$, $\leqslant$, $>$, and $\geqslant$. $s<t$, for example, may be regarded as a definitional abbreviation for $\sim t \dot{-} s = 0$.

B. The Formal System TF.

1. Type-functors.

A type-function is a primitive recursive function which takes as values types of terms of HAω. To make this conception precise, I must fix some way of encoding types by natural numbers. The code of σ is called $\#\sigma$; it is calculated as follows:

$$\#o = 1 \qquad \#\sigma\tau = 2^{\#\sigma}.3^{\#\tau} \qquad \#(\sigma\times\tau) = 3^{\#\sigma}.5^{\#\tau}$$

Given an n-place primitive recursive function which takes codes of types as values, the corresponding type-function is defined to be the function which maps n-tuples of numbers onto the types which are encoded by the values of the first function for the same n-tuples. Types count as 0-place type-functions.

If the terms of HAω are interpreted in the natural way, some of the primitive recursive functors I defined in section A can be interpreted as (if they are closed) representing, or otherwise ranging over, the numerical functions corresponding to type-functions. Hence it is possible also to think of them as representing, or ranging over, the type-functions themselves. However, if a primitive recursive functor is to be thought of in this way, I shall call it a type-functor and write its name within corners, so the type-functor corresponding to f is $\ulcorner f \urcorner$. The order of $\ulcorner f \urcorner$ is the same as the order of f.

I shall be especially interested in type-functors $\ulcorner h \urcorner$ which are derived from a term h of HAω so that either the following pair of equations, for all numerals $n_1, \ldots n_q$:

$$h0n_1 \ldots n_q = fn_1 \ldots n_m$$
$$hk'n_1 \ldots n_q = 2^{gkn_1 \ldots n_n} \cdot 3^{hkn_1 \ldots n_q}$$

or the following pair:

$$h0n_1 \ldots n_q = fn_1 \ldots n_m$$
$$hk'n_1 \ldots n_q = 3^{gkn_1 \ldots n_n} \cdot 5^{hkn_1 \ldots n_q}$$

for some g of order n+1 and f of order m, such that $\max\{m, n\} = q$, is provable in HAω. A term which satisfies one of the above pairs of equations, for some f and g, will be called $R^L fg$ or $R^C fg$ respectively.

In setting up the theory TF, when I wish to refer to a type, I shall allow myself to do so either by using a name for the type of the kind introduced in section A, or else by using a name for a numeral enclosed within corners. I shall call the type-functors $\ulcorner R^L g(\lambda \underline{x}.f) \urcorner 1$ and $\ulcorner R^C g(\lambda \underline{x}.f) \urcorner 1$ $\ulcorner f \urcorner \ulcorner g \urcorner$ and $\ulcorner f \urcorner \times \ulcorner g \urcorner$ respectively (here $\underline{x}$ must be a variable not occurring in f). If $\ulcorner f \urcorner$ and $\ulcorner g \urcorner$ are types, this idiom can easily be translated into that which I used to talk about types in section A. For if $\ulcorner f \urcorner$ is σ and $\ulcorner g \urcorner$ is τ, f will denote the number encoding σ while g denotes the number encoding τ. Therefore $R^L g(\lambda \underline{x}.f)1$, for example, will be the number $2^{\#\sigma} \cdot 3^{\#\tau}$. But this number is the code of the type στ, by our rules for coding. Therefore $\ulcorner f \urcorner \ulcorner g \urcorner$ is the same type as στ.

Furthermore, if f and g are both type-functors and f* and g* are corresponding primitive recursive functors, I shall write fg in place of $\ulcorner R^L g^*(\lambda \underline{x}.f^*) \urcorner 1$ and $\ulcorner f \times g \urcorner$ in place of $\ulcorner R^C g^*(\lambda \underline{x}.f^*) \urcorner 1$. This notation almost suggests itself, since it simply means that I combine names of type-functors in the same way as I formerly combined names of types. If the type-functors in question are types, then the notation means the same whichever way it is interpreted.

2. Shape-Functors.

The first difference between the terms of HAω and the terms of TF is that every variable (and consequently every term) of TF is required to have a type-functor whereas every variable of HAω is required to have a type. Every variable of TF also has a shape-functor, which is a primitive recursive functor of HAω of type o or 1. If a shape-functor is a numeral, then we may regard the variable in question as a variable of HAω and identify the shape-functor in question with what I previously called the 'shape' of the variable. Henceforth, therefore, if a shape-functor is a numeral, I shall also call it a 'shape'.

3. Computation of Type- and Shape-Functors.

At this point I would like to define what I mean by "equality". I shall use the symbol "$\bowtie$" in the following contexts. "$M \bowtie N$" shall mean, if M and N are of type o, M=N. If they have a common type-functor other than o, say $\ulcorner f \urcorner$, then it means $X^{\ulcorner f \urcorner} o M = X^{\ulcorner f \urcorner} o N$. In place of "$\rightarrow M \bowtie N$", I shall also say that M "is equal to" N. This should not be misunderstood as meaning that M is the same term as N, though it could reasonably be taken as meaning that M and N would have the same semantic value under a normal interpretation.

In the following, I shall in effect identify type- and shape-functors which are equal to each other. To be more exact, in stating the rules for the formation of term-forms and the determination of their type-functor, I shall give clauses along the lines of "If the type-functor of M is equal to $\ulcorner f \urcorner$ and if the type-functor of N is equal to $\ulcorner g \urcorner$, the type-functor of the result of such-and-such an operation on M and N is the result of such-and-such an operation on $\ulcorner f \urcorner$ and $\ulcorner g \urcorner$".

4. Terms of TF.

In this paragraph, we reach the actual terms of TF for the first time. The terms are 'term-forms', in the sense of Professor Yasugi, in that they are obtained by generalising the terms of HAω in a way similar to hers. I use letters like 'L', 'M', 'N' and 'P' as variables ranging over terms of TF in general and letters like 'r', 's' and 't' as variables ranging over terms whose type-functor is equal to o. Every term has a type-functor.

4.1. There are two kinds of primitive term, 0 and variables.

4.1.1. 0 is a term-form of type o.

4.1.2. For every type-functor $\ulcorner f \urcorner$ and shape-functor g there is a variable having that type-functor and that shape-functor. This variable is called '$X^{\ulcorner f \urcorner}_{g}$'. When $\ulcorner f \urcorner$ is o and g is a shape, the variables of type-functor $\ulcorner f \urcorner$ are also called 'x_0', 'x_1', 'x_2', etc. The same abbreviations apply as were introduced when we were dealing with HAω.

N.B. It is important to stipulate that a primitive recursive functor occurring within a type- or shape-functor also counts as occurring within any term which possesses that type- or shape-functor. If the primitive recursive functor in question is a variable, it may be λ-bound or used as the eigenvariable of an induction.

4.2. The following are the operations that may be performed on terms.

4.2.1. Successors can be formed just as in HAω and are denoted in the same way.

4.2.2. If M has type-functor $\ulcorner f \urcorner$ of order n and $X^{\ulcorner g \urcorner}_{h}$ is a variable which may or may not occur in M, then there is an operation called 'lambda-binding the variable $X^{\ulcorner g \urcorner}_{h}$' which may be applied to M to yield a new term of type-functor $\ulcorner g^{\urcorner} {}^{\ulcorner} f \urcorner$, provided that the variable $X^{\ulcorner g \urcorner}_{h}$ has not already been lambda-bound in the construction of M or of $\ulcorner f \urcorner$. The result of the operation is $\lambda X^{\ulcorner g \urcorner}_{h}.M$.

4.2.3. If M and N are terms and the type-functor of N is $\ulcorner f \urcorner$ and the type-functor of M is equal to $\ulcorner f \urcorner \ulcorner g \urcorner$ for some $\ulcorner g \urcorner$, then there is an operation called 'application of M to N' which yields a term of type-functor $\ulcorner g \urcorner$, unless M has the shape $\lambda x_i.Q$, for some Q and i, and x_i occurs in $\ulcorner g \urcorner$, in which case the result of the operation has the type-functor $\ulcorner g[x_i{:=}N] \urcorner$. The result of the operation is MN.

(In order that MN be well-formed we must ensure that $\lambda x_i.P$, where x_i occurs in the type-functor of P, only be applied to N when N is a possible value of k, m, n, . . .)

To economize on brackets, I stipulate that $M_1M_2...M_nN$ should be read in the same way as if it were a term of HAω.

4.2.4. If M and N are terms whose type-functors are equal to $o\ulcorner f\urcorner\ulcorner f\urcorner$ and $\ulcorner f\urcorner$ respectively, for some $\ulcorner f\urcorner$, then ρ[M, N, s] is the result of an operation on M, N and s having the type-functor $\ulcorner f\urcorner$.

4.2.5. For any term-forms M and N of type-functors f and g , the operation called 'pairing M and N' yields a new term-form called the 'pair of M and N', of type-functor $\ulcorner f\urcorner\times\ulcorner g\urcorner$. This is {M, N}.

4.2.6-7. If M is a term-form having type-functor $\ulcorner f\urcorner\times\ulcorner g\urcorner$, there are two operations, called 'decoding', applicable to M, which yield term-forms of type-functors $\ulcorner f\urcorner$ and $\ulcorner g\urcorner$ respectively. The results of the two operations are M_0 and M_1 respectively.

Remark: If, in clauses 4.1 and 4.2.1-7, all variables ranging over type-functors and shape-functors are restricted so as to range over types and shapes respectively, these clauses define the terms of HAω.

4.2.8. Let M be a term of type-functor $\ulcorner f\urcorner$ and let $\ulcorner g\urcorner$ be a type-functor and h a shape-functor so that $O(\ulcorner g\urcorner)\cdot O(h) = 1$. Then there is an operation which transforms M, $X^{\ulcorner g\urcorner}_h$ and any m into a term of type-functor $\ulcorner R^L fg\urcorner m$. The result of the operation is $([\lambda X^{\ulcorner g\urcorner}_h.M]; m)$. (N.B. this operation does not count as binding the variable $X^{\ulcorner g\urcorner}_h$ within M.)

4.2.9. Let M be a term of type-functor $\ulcorner f\urcorner$ and let L be a term of type-functor $\ulcorner g\urcorner$ of order 1. Then, for each such M and L, there is a term formed from them and any m having type-functor $\ulcorner R^C fg\urcorner m$. This term is ([L, M]; m).

4.2.10. Let M be a term whose type-functor is equal to $\ulcorner R^L fg\urcorner m+1$ and N one whose type-functor is equal to $\ulcorner R^C(g0)(\lambda x.gx')\urcorner m$; then ApMNm has type-functor $\ulcorner f\urcorner$.

4.2.11. Let M be a term whose type-functor is equal to $\ulcorner f\underline{v}\urcorner\ulcorner f\underline{v}'\urcorner$, for some f and $\underline{v}$, and N a term with a type-functor equal to $\ulcorner f0\urcorner$. Then $R_{\ulcorner f\underline{v}\urcorner}[\lambda\underline{v}.M, N, m]$ has the type-functor $\ulcorner fm\urcorner$.

5.Redexes and their Contraction.

In order to state the axioms of TF, it is necessary first to state which terms are redexes and how the contractum of each redex is determined.

Furthermore, it is necessary to introduce a new piece of notation. (M; m) shall denote the term determined by terms M and m in the following way (the following sub-paragraphs correspond to the sub-paragraphs of B.4 and the definition of (M; m) is by recursion on the complexity of M):-

1.1 (0; m) is identical to 0.

1.2 ($X^{\ulcorner f \urcorner}_{g}$; m) depends on the orders of f and g. If O(f)·O(g) > 0, then the term is identical to $X^{\ulcorner fm \urcorner}_{gm}$. If O(f) > 0 and O(g) = 0, it is $X^{\ulcorner fm \urcorner}_{g}$. If O(f) = 0 and O(g) = 1, it is $X^{\ulcorner f \urcorner}_{gm}$. If O(f) + O(g) = 0, it is $X^{\ulcorner f \urcorner}_{g}$.

2.1 (t'; m) is identical to (t; m)'.

2.2 ($\lambda X^{\ulcorner f \urcorner}_{g}$.N; m) is identical to λ(X; m).(N; m).

2.3 (MN; m) is identical to (M; m)(N; m).

2.4 - 2.7 are determined in the obvious way.

2.8 ((\[$\lambda X^{\ulcorner g \urcorner}_{h}$ .N]; m); n) is identical to (\[$\lambda X^{\ulcorner g \urcorner}_{h}$.(N; n)]; m).

2.9 ((\[L, M]; m); n) is identical to (\[L, (M; n)]; m).

2.10 (ApMNn; m) is identical to Ap(M; m)(N; m)n.

2.11 (R$^{\ulcorner}$f$\underline{v}$$^{\urcorner}$\[λ$\underline{v}$.M, N, m]; n) is identical to R$^{\ulcorner}$f$\underline{v}$$^{\urcorner}$\[(λ$\underline{v}$.M; n), (N; n), m].

Theorem 2.1: For any M and m, (M; m) is a well-formed term of TF and its type-functor is determined in the following way. If M has type-functor $\ulcorner f \urcorner$ and O(f) > 0, then (M; m) has the type-functor $\ulcorner fm \urcorner$. If M has type-functor $\ulcorner f \urcorner$ and O(f) = 0, then (M; m) has the type-functor $\ulcorner f \urcorner$.

Proof: By induction on the complexity of M. If M is 0 or a variable, the theorem is true by definition. For the induction-step, we must consider the various clauses in B.4.2. The following clauses are numbered correspondingly:-

2.2 We have stipulated that (λ$\underline{X}^{\ulcorner f \urcorner}$.N; m) is identical to λ($\underline{X}^{\ulcorner f \urcorner}$; m).(N; m). Let N have the type-functor $\ulcorner g \urcorner$; then λ$\underline{X}^{\ulcorner f \urcorner}$.N will have the type-functor $\ulcorner R^{L}g(\lambda \underline{x}.f)1 \urcorner$. The task is to deduce, given what the type-functors of ($\underline{X}^{\ulcorner f \urcorner}$; m) and (N; m) must be, that (λ$\underline{X}^{\ulcorner f \urcorner}$.N; m) will have the type-functor $\ulcorner R^L g(\lambda\underline{x}.f)1\urcorner$ or $\ulcorner R^L g(\lambda\underline{x}.f)1m\urcorner$ -- the former only if $O(R^L g(\lambda\underline{x}.f)1)$, which is $\max\{O(f), O(g)\}$, is 0. Now the basis of the induction tells us that $(\underline{X}^{\ulcorner f\urcorner}; m)$ has the type-functor $\ulcorner fm\urcorner$ or $\ulcorner f\urcorner$, depending on what O(f) is; and the induction-step tells us that (N; m) has the type-functor $\ulcorner gm\urcorner$ or $\ulcorner g\urcorner$ similarly. If $O(f) + O(g) = 0$, then $O(R^L g(\lambda\underline{x}.f)1)$ is also 0 and $\lambda(\underline{X}^{\ulcorner f\urcorner}; m).(N; m)$ has the type-functor $\ulcorner f\urcorner\ulcorner g\urcorner$, which is also, of course, the type-functor of $(\lambda\underline{X}^{\ulcorner f\urcorner}.N; m)$. If $\max\{O(f), O(g)\} > 0$, then $\lambda(\underline{X}^{\ulcorner f\urcorner}; m).(N; m)$ has the type-functor $\ulcorner fm\urcorner\ulcorner gm\urcorner$, $\ulcorner fm\urcorner\ulcorner g\urcorner$ or $\ulcorner f\urcorner\ulcorner gm\urcorner$, depending on the orders of f and g respectively. But these three terms all denote the same type-functor, viz., $\ulcorner R^L g(\lambda\underline{x}.f)1m\urcorner$.

2.3 Since MN is well-formed, M must have a type-functor of the shape $\ulcorner f\urcorner\ulcorner g\urcorner$, where $\ulcorner f\urcorner$ is the type-functor of N. If N is capable of being substituted into the type-functor of M, $\ulcorner f\urcorner$ must be a type, and otherwise the type-functor of MN is $\ulcorner g\urcorner$.

There are therefore four main cases to distinguish: (1) $O(f)\cdot O(g) > 0$; (2) $O(f) > 0$, $O(g) = 0$; (3) $O(f) = 0$, $O(g) > 0$; (4) $O(f) + O(g) = 0$; and the last two must again be divided into two sub-cases, depending on whether N does or does not require to be substituted in g. Now by the hypothesis of the induction, (M; m) will have the type-functor $\ulcorner fm\urcorner\ulcorner gm\urcorner$, $\ulcorner fm\urcorner\ulcorner g\urcorner$, $\ulcorner f\urcorner\ulcorner gm\urcorner$ or $\ulcorner f\urcorner\ulcorner g\urcorner$, depending on what the orders of f and g are; and (N; m) will have the type-functor $\ulcorner g\urcorner$ or $\ulcorner gm\urcorner$ similarly. If we work through all the cases, it turns out that (M; m)(N; m) has the type-functor $\ulcorner g\urcorner$ or $\ulcorner gm\urcorner$, modulo a possible substitution.

2.8 We have stipulated that $([\lambda\underline{X}^{\ulcorner f\urcorner}.N]; n)$ has the type-functor $\ulcorner R^L gfn\urcorner$, where $\ulcorner g\urcorner$ is the type-functor of N. The type-functor of (N; m) is $\ulcorner gm\urcorner$ or $\ulcorner g\urcorner$, depending on what the order of $\ulcorner g\urcorner$ is. But depending on exactly that, the type-functor of $([\lambda\underline{x}^{\ulcorner f\urcorner}.(N; m)]; n)$ will be either $\ulcorner R^L gfn\urcorner$ or $\ulcorner R^L gfnm\urcorner$, which is also the type-functor of $(([\lambda\underline{x}^{\ulcorner f\urcorner}.N]; n); m)$.

2.9 We have stipulated that ([L, M]; m) has type-functor $\ulcorner R^C fgm\urcorner$, where $\ulcorner f\urcorner$ and $\ulcorner g\urcorner$ (the last of order 1) are the type-functors of M and L respectively. The order of $\ulcorner RCfgm\urcorner$ is therefore the order of M. By the induction hypothesis, (M, n) has the type-functor $\ulcorner f\urcorner$ (if $O(f) = 0$)or $\ulcorner fn\urcorner$ (otherwise), so that ([L, (M; n)]; m) has the type-functor $\ulcorner R^C(fn)gm\urcorner$, which is equal, by paragraph B.1, to $\ulcorner R^C fgmn\urcorner$, if $O(f) > 0$; and $\ulcorner R^C fgm\urcorner$ otherwise. But this is by definition equal to the type-functor of (([L, M]; m); n).

2.10 By the conditions for ApMNm to be well-formed, together with the hypothesis of the induction we are making, (M; n) and (N; n) have the type-functors $\ulcorner R^{L}fg(m+1)n \urcorner$ and $\ulcorner R^{C}(g0)(\lambda \underline{x}.g\underline{x}')mn \urcorner$ respectively, provided that $O(f) \cdot O(g) > 1$. These are equal to $\ulcorner R^{L}(fn)(\lambda \underline{x}.g^{*}\underline{x})(m+1) \urcorner$ and $\ulcorner R^{C}(g^{*}0)(\lambda \underline{x}.g^{*}\underline{x}')m \urcorner$ (where g^{*} is defined to be $\lambda \underline{y}.g\underline{y}n$) respectively, so that Ap(M; n)(N; n)m has the type-functor $\ulcorner fn \urcorner$, where $O(f) > 0$, and $\ulcorner f \urcorner$ otherwise.

2.11 By the condition for $R_{\ulcorner f\underline{v} \urcorner}[\lambda \underline{v}.M, N, m]$ to be well-formed, together with the induction hypothesis, $(\lambda \underline{v}.M; n)$ and (N; n) have the type-fuctors $\ulcorner f\underline{v}n \urcorner \ulcorner f\underline{v}'n \urcorner$ and $\ulcorner f0n \urcorner$ respectively, so long as $O(f) > 1$ -- the case where $O(f) = 1$ is actually simpler. These type-functors are equal to $\ulcorner f^{*}\underline{v} \urcorner \ulcorner f^{*}\underline{v}' \urcorner$ and $f^{*}0$ respectively, where f^{*} is defined like g^{*} above. But that being so, $R_{\ulcorner f^{*}\underline{v} \urcorner}[(\lambda \underline{v}.M; n), (N; n), m]$ is well-formed and has the type-functor $\ulcorner f^{*}m \urcorner$, which is equal to $\ulcorner fmn \urcorner$, which is precisely the type-functor that we wanted $(R_{\ulcorner f\underline{v} \urcorner}[\lambda \underline{v}.M, N, m]; n)$ to have.

Now we can get around to stating which terms are redexes and how they are contracted. The following sub-paragraphs correspond, again, to the various sub-paragraphs of B.4.2.

2.3. MN is a redex if and only if M is a term formed by λ-abstraction, say $\lambda X^{\ulcorner f \urcorner}_{g}.P$. In that case, N will have type-functor $\ulcorner f \urcorner$. The contractum of MN is formed by taking P and substituting N for $X^{\ulcorner f \urcorner}_{g}$, wherever that variable occurs in P or in its type-functor.

2.4, 6, 7: Terms which have primitive recursion or decoding as the last stage in their construction are contracted under the same conditions, and in the same way, as similar terms of HAω. The only difference is that whereas the rules for the latter were stated using metamathematical variables ranging over types, variables ranging over type-functors must now be read in their place.

2.8. $([\lambda X^{\ulcorner f \urcorner}_{g}.M]; m)$ is only a redex if m is 0 or a successor, say n'. $([\lambda X^{\ulcorner f \urcorner}_{g}.M]; 0)$ contracts to M. $([\lambda X^{\ulcorner f \urcorner}_{g}.M]; n')$ contracts to $\lambda(X^{\ulcorner f \urcorner}_{g}; m)\underline{}([\lambda X^{\ulcorner f \urcorner}_{g}.M]; n)$.

2.9. ([L, M]; m) is only a redex if either m is 0 or m is a successor, say n'. In the first case it contracts to M, in the second to {(L; n), ([L, M]; n)}.

2.10. Ap(M)(N)m is a redex if and only if m is 0 or a successor. If 0, it contracts to MN.

If a successor, say n', it contracts to $Ap(MN_0)N_1n$.

2.11. If n is 0 then $R_{\ulcorner f\underline{v}\urcorner}[\lambda\underline{v}.M, N, n]$ contracts to N. If n is a successor, say p', it contracts to $(\lambda\underline{v}.M)p(R_{\ulcorner f\underline{v}\urcorner}[\lambda\underline{v}.M, N, p])$.

Theorem 2.2. Every contractum of a redex is itself a well-formed term and has a type-functor equal to that of the redex.

Proof: The theorem can only be proved by going through the rules for the formation of term-forms one-by-one and comparing them with the contraction-rules. I shall omit the treatment of some of the simpler cases.

2.3. We have stipulated that $(\lambda X^{\ulcorner f\urcorner}_{g}.M)N$ shall have the type-functor which is got from the type-functor of M by substituting N for $X^{\ulcorner f\urcorner}_{g}$, wherever $X^{\ulcorner f\urcorner}_{g}$ occurs in M. It is necessary to prove that $M[X^{\ulcorner f\urcorner}_{g}:=N]$ will have the same type-functor. I distinguish two cases, according as $X^{\ulcorner f\urcorner}_{g}$ does or does not occur within the type-functor of M. If it does not, the theorem obviously holds. If it does, we have to do an induction on the number of operations in the construction of M. The only non-vacuous basis case is where M is a variable. Obviously, M cannot itself be $X^{\ulcorner f\urcorner}_{g}$, so the only change that has to be made will be in the type-functor (and possibly the shape-functor) of M.

Remark: We see now why the rule for determining the type-functor of terms of the form LN in TF is so much more complicated than the corresponding rule for HAω. The problematic case is where L has the form $(\lambda\underline{x}.M)$ and $\underline{x}$ occurs in the type-functor of M. Then substituting N for $\underline{x}$ will change that type-functor, so we cannot simply stipulate that LN shall have the same type-functor as M, or this theorem would not hold.

2.8. $([\lambda X^{\ulcorner f\urcorner}_{g}.M]; 0)$ will have type-functor $\ulcorner f\urcorner$, where $\ulcorner f\urcorner$ is the type-functor of M. $([\lambda X^{\ulcorner f\urcorner}_{g}.M]; n')$ has a type-functor equal to $\ulcorner gn\urcorner\ulcorner(R^L fgn)\urcorner$, precisely the type-functor of $\lambda(X; n).([\lambda X^{\ulcorner f\urcorner}_{g}.M]; n)$.

It is clear that $([\lambda X^{\ulcorner f\urcorner}_{h}.M]; n'')$ reduces to $\lambda X^{\ulcorner fn'\urcorner}_{hn'}.\lambda X^{\ulcorner fn\urcorner}_{hn}.([\lambda X^{\ulcorner f\urcorner}_{h}.M]; n)$, to give an example. It is perfectly possible that the type-functors $\ulcorner f\urcorner n'$ and $\ulcorner f\urcorner n$ compute to the same type-functor, in which case, by our stipulation that a variable may only be lambda-bound once, we are in danger of transgressing the limits of well-formedness. However this can be avoided by ensuring that, when the two type-functors are equal, the shape-functors hn' and hn shall be nonequal.

2.9. ([M, N]; 0) has type-functor $\ulcorner R^{C}gf\urcorner 0$, where $\ulcorner f\urcorner$ and $\ulcorner g\urcorner$ are the type-functors of M and N respectively; but this computes to $\ulcorner g\urcorner$, while the term-form contracts to M.
([M, N]; n') has type-functor $\ulcorner R^{C}gf\urcorner n'$, which reduces to $\ulcorner f\urcorner n \times \ulcorner (R^{C}gf)n\urcorner$, but the latter is also the type-functor of {(M; n), ([M, N]; n)} (we use theorem 2.1 at this point).

2.10. By the condition for ApMN0 to be well-formed, M and N must have the type-functors $\ulcorner R^{L}fg1\urcorner$ and $\ulcorner R^{C}(g0)(\lambda x.gx')0\urcorner$ respectively. These reduce to $\ulcorner g0\urcorner\ulcorner f\urcorner$ and $\ulcorner g0\urcorner$ respectively, so that MN has the type-functor $\ulcorner f\urcorner$. But this is exactly what we stipulated ApMN0 would reduce to.

To treat terms of the shape of ApMNn', let n be formed by k applications of the successor operation to a term, say j, which is not a successor. We now do an induction on k. By our stipulations, M and N have type-functors equal to $\ulcorner gn\urcorner\ulcorner RLfgn\urcorner$ and $\ulcorner gn\urcorner \times \ulcorner R^{C}(g0)(\lambda x.gx')\urcorner n$. The contractum of ApMN(n') is the result of operation 4.2.10 on $M(N_0)$, N_1 and n. The first of these term-forms obviously has type-functor $\ulcorner R^{L}fg\urcorner n'$ while the second has $\ulcorner R^{C}(g0)(\lambda x.gx')\urcorner n$. But by definition of operation 4.2.10, the result of the operation on these two term-forms has the required type-functor.

6. Mathematical Axioms of TF.

These are generated by precisely the same schemata as the axioms of HAω, but they apply, of course, to a wider class of terms.

Now that the concept of reduction has been defined for terms of TF, it is time to prove the following important theorem.

Theorem 2.3 (Church-Rosser property): Let M be a term that reduces to both P and R.

Then there is another term which can be obtained from both P and R by reductions and α-conversions.

Proof:- We proceed by induction on the sum of the number of contractions used to get P from M and the number of contractions used to get R from M respectively, taking as the basis the first non-trivial case, namely, where this number is 2. The induction-step of the proof can be dealt with very easily, using standard methods (e.g. Curry and Feys 1958, pp. 110-115). The difficult part of the proof is the basis.

Let Q be the sub-term of M that is contracted to get P and let N be the sub-term of M that is contracted to get R. Since there may be more than one occurrence of either Q or N in M, we shall call the occurrences that are contracted $\bar{Q}$ and $\bar{N}$ respectively. Following Curry and Feys (ibid., pp.113-116), we define what it is for an (occurrence of a) subterm of P to be a <u>residual</u> of $\bar{N}$.

(1) If $\bar{N}$ is identical to $\bar{Q}$, there is no residual of $\bar{N}$ in P.

(2) If $\bar{N}$ is not a component of $\bar{Q}$ nor $\bar{Q}$ a component of $\bar{N}$, there will be an occurrence of a term-form identical to $\bar{N}$ in P, at a similar position to that of $\bar{N}$ in M, and it will be the residual of $\bar{N}$ in P.

(3) If $\bar{Q}$ is a component of $\bar{N}$, the residual of $\bar{N}$ in P is the component of P got from $\bar{N}$ by replacing $\bar{Q}$ within it with its contractum.

(4) We now treat the case where $\bar{N}$ is a component of $\bar{Q}$, dividing this case according to what was the last operation in the construction of Q.

4.2.3. Let Q be KL; then K was formed by lambda-binding some variable with respect to some term. Let the occurrences of K and L in $\bar{Q}$ be called $\bar{K}$ and $\bar{L}$ respectively. Then the contractum of Q will contain any number of occurrences of L; let these be called $\bar{L}_1$, $\bar{L}_2$, . . If $\bar{N}$ was in $\bar{L}$, there will be corresponding occurrences of N in each of $\bar{L}_1$, $\bar{L}_2$, . . . and all these occurrences will be residuals of $\bar{N}$ in P. If $\bar{N}$ was in $\bar{K}$, N will occur at a corresponding position in the contractum of Q and that occurrence of N will be the residual of $\bar{N}$.

4.2.4. If Q is ρ[K, L, s], then, if r is 0, $\bar{N}$ has a residual in the contractum of Q only if it occurs in L. If it does, the residual is the occurrence corresponding to $\bar{N}$. If Q contracts to K(s∸1)ρ[K, L, s∸1], then the residual of $\bar{N}$ is the occurrence within both occurrences of K resp. the occurrence of L that corresponds to $\bar{N}$ within K, resp. L, within Q.

These definitions ought to give the reader a general idea of what a residual is. The one further clause in the definition of "residual" to which I feel I had better draw attention is the one dealing with the case where Q is a redex formed by operation 2.9. In that case, if Q is ([L, M]; m') and N occurs within L, then the a residual of $\bar{N}$ in the contractum of Q occurs not only in L but also in (L; m).

To prove the basis of the theorem, it is necessary to show that it makes no difference whether you first contract $\bar{Q}$ and then the residuals of $\bar{N}$ in P, or whether you first contract $\bar{N}$ and then the residuals of $\bar{Q}$ in R. The cases where Q and N either coincide or are totally disjoint are easily dealt with; it is the other two cases which are difficult. They are, however, totally symmetrical, so it suffices to consider the case where $\bar{N}$ is a proper subterm of $\bar{Q}$.

We run through the various possibilities as to what was the final operation in the construction of Q. The following sub-paragraphs are numbered like the sub-paragraphs of section 4.

2.3. Q is $(\lambda \underline{X}.K)L$. This is the most difficult case. I distinguish two sub-cases: (a) N occurs within K; (b) N occurs within L. In treating both cases, I assume that α-conversions have been carried out to ensure that none of the bound variables in K concides with a free variable in L.

(a) Q contracts to $K[\underline{X}:=L]$ and the residual of N in this term is $N[\underline{X}:=L]$. On the other hand, if we contracted N first and if its contractum be called N*, the result of contracting the result of that reduction, namely $(\lambda \underline{X}.K[N:=N^*])L$ will be $K[N:=N^*][\underline{X}:=L]$. In the other case, the result of the second contraction is $K[\underline{X}:=L][N[\underline{X}:=L]:=(N[\underline{X}:=L])^*]$, where the asterisk after the name of a redex again shows that we are talking about the contractum of that redex. Since the respective terms that we are now considering differ, if at all, only in that in the one $N^*[\underline{X}:=L]$ occurs where, in the other, $(N[\underline{X}:=L])^*$ occurs, it suffices if we can show that these two symbols in fact denote the same term, modulo α-conversions.

It is now necessary to consider the form of N; either N is of the shape $(\lambda\underline{Y}.S)T$ or it is not. I again assume that none of the bound variables in S coincides with a free variable in T. If it is not, then it must be a redex formed by one of operations 2.4 - 2.11, and my treatment of the cases where Q is of one of these forms will show that $Q^*[\underline{X}:=L]$ is indeed identical to $(Q[\underline{X}:=L])^*$. For the moment, therefore, I shall only consider the other possibility. N^* is $S[\underline{Y}:=T]$ so that $N^*[\underline{X}:=L]$ is $S[\underline{Y}:=T][\underline{X}:=L]$while $(N[\underline{X}:=L])^*$ is $S[\underline{X}:=L][\underline{Y}:=T[\underline{X}:=L]]$.

The part of the proof currently in hand consists in showing that $S[\underline{Y}:=T][\underline{X}:=L]$ is identical to $S[\underline{X}:=L][\underline{Y}:=T[\underline{X}:=L]]$, at least modulo α-conversions. We do it by induction on the number of steps in the construction of S. If S is 0 or a variable other than $\underline{X}$ or $\underline{Y}$, the statement obviously holds. If S is $\underline{X}$ or $\underline{Y}$, the two expressions above either both denote L or both denote $T[\underline{X}:=L]$.

For the induction-step, let us suppose that S is formed by operation O from immediate sub-terms $S_1, \ldots . S_n$. Then, according to the obvious recursive definition of replacement, $S[\underline{Y}:=T][\underline{X}:=L]$ denotes $O(S_1[\underline{Y}:=T][\underline{X}:=L], \ldots S_n[\underline{Y}:=T][\underline{X}:=L])$ while $S[\underline{X}:=L][\underline{Y}:=T[\underline{X}:=L]]$ denotes $O(S_1[\underline{X}:=L][\underline{Y}:=T[\underline{X}:=L]], \ldots S_n[\underline{X}:=L][\underline{Y}:=T[\underline{X}:=L]])$. But the induction-hypothesis says that these two expressions denote exactly the same term.

(b) Q contracts to $K[\underline{X}:=L]$ and the further contraction of the residuals of N carries us to $K[:=L][S:=S^*]$, where "S" denotes all the residuals of N. Since these residuals all occur within L, it can be easily shown by induction on the construction of K that this is equivalent to $K[\underline{X}:=L[S:=S^*]]$.

2.4 - 2.11. When Q is of one of these forms, the theorm can be established much more simply, as the process of contracting Q can be described without reference to any such complicated operation as substitution.

For this stage of the proof it is essential to note that, with most of the reduction-rules, if

Q is formed by a sequence of operations O from sub-terms $Q_1, \ldots Q_n$, then the contractum of Q is formed by a different sequence of operations, say O_1, from at most the sub-terms $Q_1, \ldots Q_n$. The only exceptions to this generalisation are where Q is formed by operation 4.2.8 or 4.2.9, because in these cases its contractum will include a sub-term of the shape $(Q_i; n)$, which is not a sub-term of Q. I will leave these cases till last.

Let Q be formed by O from subterms $Q_1, \ldots Q_n$, so that we may describe it as $O(Q_1, \ldots Q_n)$. Let Q^- be the redex obtained from Q by replacing N therein with some variable not occurring in Q, say $\underline{Z}$ -- Q^- must be a redex because N cannot be 0 or a successor. Q^- may now be described as $O(Q_1^-, \ldots Q_n^-)$ (in fact, this is the obvious way of <u>defining</u> "the result of replacing N in Q with Z"). Since the relation "λuz.x is obtained from y by replacement of u by z" is transitive, the term obtained from Q by contracting N within it is identical to $Q^-[Z:=N^*]$. Similarly, Q^* is identical to $Q^{-*}[Z:=N]$. This is because the same rule holds for the contraction of $O(Q_1^-, \ldots Q_n^-)$ as for $O(Q_1, \ldots Q_n)$; that is, these two terms are composed out of their respective subterms by the same sequence of operations. Let us therefore call them $O_1(Q_1^-, \ldots Q_n^-)$ and $O_1(Q_1, \ldots Q_n)$ respectively. In virtue of the definition of replacement, $Q^{-*}[Z:=N]$ is identical to $O_1(Q_1^-[Z:=N], \ldots Q_n^-[Z:=N])$, is identical to $O_1(Q_1, \ldots Q_n)$, is identical to Q^*. But that was the claim just made.

Now $(Q[N:=N^*])^*$, that is, the result of contracting N within Q and then contracting the result, is identical to $(Q^-[Z:=N^*])^*$, is identical to $(O_1(Q_1^-[Z:=N^*], \ldots Q_n^-[Z:=N^*])$. But in view of the above, this is obviously equivalent to $O_1(Q_1, \ldots Q_n)[N:=N^*]$.

Let us finally consider the case where Q is formed by operation 4.2.8 or 4.2.9. We can ignore the former, however, because if it has the shape $([\lambda\underline{X}.M]; n')$ N necessarily occurs in M and not in $\underline{X}$. If Q has the shape $([L, M]; n')$ and N occurs in L, then the main thing required is to prove that $(L[N:=N^*]; n)$ is identical to $(L; n)[N':=N'^*]$, where N' is the residual of N in $(L; n)$.

<u>7. Logical Rules.</u>

The logic is the same as for HAω but without quantifiers. This means that the succedent position of a sequent may contain any number of formulae of any kind.

For heuristic purposes, it is sometimes useful to imagine a theory got by adding to HAω the the mathematical axioms of TF or, alternatively, adding to TF the quantificational rules of HAω as well (if we like) the rules peculiar to HAω$^+$. Such a theory might be called HAω+TF. But since I will not use it actually to prove anything, it seems unnecessary to formulate it exactly. For some purposes, I shall even be considering a theory which is like HAω+TF, but in which formulae may be infinitely long.

8. Induction.

The induction-rule, of course, is also a rule of TF, but I have decided to formulate it differently from the induction-rule of HAω. I am interested in the former theory on account of the initial cases of transfinite induction which may, and which may not, be proved valid in her. It is therefore necessary to suppose that she contain terms denoting transfinite ordinal numbers. It would be possible, but laborious, to select a subset of the closed terms already defined and state which numbers they are to stand for. It is more convenient simply to stipulate that new closed terms shall be added to the theory, to denote all transfinite numbers smaller than ε_0. These terms will be denoted by the usual symbols for transfinite ordinal numbers below ε_0, in Cantor Normal Form. Axioms of number governing these terms must also be added. Addition, multiplication and ordering among transfinite numbers shall be indicated by the same symbols as for natural numbers. For every term k, ω_k shall compute as follows: ω_o is equal to ω^o and ω_{n+1} is equal to ω^{ω_n}.

Like Gentzen (1943, p.142), I think this expansion of the set of terms is justified by the fact that not only the ordinal numbers smaller than ε_0 but also the functions of addition, multiplication and exponentiation upon them may be coded by natural numbers and (primitive recursive) functions of natural numbers. This expansion of the language I am using means that ordinary induction must now be viewed as an initial case of transfinite induction: that is, we add the formula $t < \omega$ to the antecedent of the conclusion of an induction.

Remark: This completes the definition of TF.

C. Some Important Properties of TF

1. Theorems on Equality.

Theorem 2.3. Every instance of the schema

$$s = t, F, \Gamma \rightarrow \Delta, F[s:=t]$$

is provable in TF.

Proof: by induction on the number of logical symbols in F. The basis is the case where F is an equation, say $q = r$. Then we prove $s = t, q = r \rightarrow q[s:=t] = r[s:=t]$ by the following derivation:

1/ $s = t \rightarrow q[s:=s] = q[s:=t]$ axiom on equality

2/ $s = t \rightarrow q[s:=t] = q$ 1, axiom on equality, cut

3/ $q[s:=t] = q, q = r \rightarrow q[s:=t] = r$ axiom on equality

4/ $s = t \rightarrow r = r[s:=t]$ axiom on equality

5/ $q[s:=t] = r, r = r[s:=t] \rightarrow q[s:=t] = r[s:=t]$ axiom on equality

6/ $q[s:=t] = q, q = r, r = r[s:=t] \rightarrow q[s:=t] = r[s:=t]$ 3, 5, cut

7/ $s = t, q = r, r = r[s:=t] \rightarrow q[s:=t] = r[s:=t]$ 2, 6, cut

8/ $r = r[s:=t], s = t, q = r \rightarrow q[s:=t] = r[s:=t]$ 7, interchanges

9/ $s = t, s = t, q = r \rightarrow q[s:=t] = r[s:=t]$ 4, 8, cut

As for the induction-step, let F_1, F_2 be the subformulae of F that are joined by the main connective: then, if the theorem holds for F_1 and F_2, the sequent we want can be derived by introduction-rules and structural inferences.

Remark: This theorem gives us in effect a new rule of inference. If some sequent of the shape $\Gamma \rightarrow \Delta$, s = t is provable, as is another of the shape $\Lambda \rightarrow \Theta$, F, then $\Gamma, \Lambda \rightarrow \Delta, \Theta$, F[s:=t] will also be. I shall therefore use the annotation 'theorem 2.3' whenever I have two sequents of the former shapes, to justify deriving one of the latter shape.

Theorem 2.3 does not enable us to replace a term occurring in F if that term has a type-functor other than o. But the following theorem does:

Theorem 2.4: Every sequent of the shape

$$M \bowtie N, F, \Gamma \rightarrow \Delta, F[M:=N]$$

is provable in TF.

Proof: Let M and N have the type-functor $\ulcorner f \urcorner$. Then "$M \bowtie N$" denotes the formula $X^{\ulcorner f \urcorner o}M = X^{\ulcorner f \urcorner o}N$. I shall treat first the case where F is s = t. If M does not occur in either s or t, the theorm is vacuously true. Suppose now that M occurs in s. Let $\underline{Y}$ be a variable having the same type-functor as M; then $(\lambda \underline{Y}.s[M:=Y)$ will have the type-functor $\ulcorner f \urcorner o$. Furthermore,

$$X^{\ulcorner f \urcorner o}M = X^{\ulcorner f \urcorner o}N \rightarrow (\lambda X^{\ulcorner f \urcorner o}.X^{\ulcorner f \urcorner o}M)(\lambda \underline{Y}.s[M:=\underline{Y}]) = (\lambda X^{\ulcorner f \urcorner o}.X^{\ulcorner f \urcorner o}N)(\lambda \underline{Y}.s[M:=\underline{Y}])$$

is an instance of the fourth axiom-schema on equality. Plainly
$(\lambda X^{\ulcorner f \urcorner o}.X^{\ulcorner f \urcorner o}M)(\lambda \underline{Y}.s[M:=\underline{Y}])$ reduces in two steps to s while
$(\lambda X^{\ulcorner f \urcorner o}.X^{\ulcorner f \urcorner o}N)(\lambda \underline{Y}.s[M:=\underline{Y}])$ reduces in the same number of steps to s[M:=N]. So from two axioms of reduction, together with the sequent written above, by a number of axioms on equality and structural inferences, we get the sequent

$$M \bowtie N, s = t, \Gamma \rightarrow \Delta, s[M:=N] = t.$$

By a repetition of the same argument we get

$$M \bowtie N, s = t, \Gamma \rightarrow \Delta, s[M:=N] = t[M:=N],$$

which is theorem 2.4 for the case where F is an equation. The induction-step proceeds as for theorem 2.3.

Remark: We now have another new rule of inference, of which the following is a special case: if $\Gamma \to \Delta$, F is provable and if M reduces to N, then $\Gamma \to \Delta$, F[M:=N] is provable.

I shall now give an example of a derivation in TF, which makes extensive use of theorem 2.4. For the definition of the type-functor $\ulcorner f \urcorner$, I must refer forward to the beginning of chapter 5. The sequent which I shall prove will be used extensively in chapter 5, in particular, as a premiss for applications of theorem 2.4. That sequent is

$$v<\omega \to X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(ev+1)\urcorner}\{X_{ev+1}^{\ulcorner f(v+1)ev+2\urcorner}, ([X_{\lambda x.dv+2+x}^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}, \{X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; dv)\}]; dv)\}$$
$$= X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(ev+1)\urcorner}([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; ev+1)$$

Proof: We derive the sequent written above by induction from two premisses, of which the basis is

$$\to X^{\ulcorner R^c(f19)(\lambda x.f1(9\dot{-}x'))4\urcorner}\{X_4^{\ulcorner f15\urcorner}, ([X_{\lambda x.3+x}^{\lambda x.\ulcorner f1(6\dot{-}x)\urcorner}, \{X_2^{\ulcorner f17\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f1(9\dot{-}x')\urcorner}, X_0^{\ulcorner f19\urcorner}]; 1)\}]; 1)\} = X^{\ulcorner R^c(f19)(\lambda x.f1(9\dot{-}x'))4\urcorner}([X_{\lambda x.x'}^{\lambda x.\ulcorner f1(9\dot{-}x')\urcorner}, X_0^{\ulcorner f19\urcorner}]; 4)$$

and the induction-step is derived as follows:

$$1/ \to X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+0)\urcorner}([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\dot{-}x\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}2\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; 0+1)$$
$$= X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+0)\urcorner}([X_{\lambda x.y+4+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}x\urcorner}, \{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)^{e(v+1)}\dot{-}y\dot{-}2}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0]; y+1)\}\}]; 0)$$

Both of the last two sequents can be proved by taking the terms on both sides of the equations and reducing them to normal form.

2/ $$X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+z)}\Big([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\dot{-}x\urcorner},$$

$$\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}z\urcorner},([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}],y+1)\}];z'\Big)$$

$$=X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+z)}\Big([X_{\lambda x.y+4+x},\{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\urcorner},$$

$$\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}z\urcorner},([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];y+1)\}\}];z\Big)$$

$$\rightarrow X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+z')}\{X_{y+4+z}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}z},$$

$$([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\dot{-}x\urcorner},\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}z\urcorner},([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];$$

$$y+1)\}];z')\}=X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+z')\urcorner}$$

$$\{X_{y+4+z}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}z\urcorner},([X_{\lambda x.y+4+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}x\urcorner},\{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\urcorner},\{X_{y+2}$$

$$([X_{\lambda x.x'},X_0];y+1)\}\}];z)\}$$

theorem 2.4

3/ $$\rightarrow X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+z')}\Big([X_{\lambda x.y+4+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}x\urcorner},$$

$$\{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\urcorner},\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}z\urcorner},([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];$$

$$y+1)\}\}];z+1\Big)=X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot{-}x')(y+3+z')\urcorner}$$

$$\{X_{y+4+z}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}z\urcorner},([X_{\lambda x.y+4+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}x\urcorner},\{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\urcorner},$$

$$\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; 2)\}$$

axiom of reduction

4/ $$\to X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1)\dot- x')(y+3+2')}([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- y\dot- 3\dot- x\urcorner},$$

$$\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; 2+2)$$

$$= X^{\ulcorner R^c(f(v+1)e(v+1))(\lambda x.\ulcorner f(v+1)e(v+1)\dot- x')(y+3+2')}\{X_{y+4+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2\dot- 2\urcorner},$$

$$([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- y\dot- 3\dot- x\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- x'\urcorner},$$

$$X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; 2+1)\}$$

axiom of reduction

5/ antecedent of 2 $\to ([X_{\lambda x.y+4+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- y\dot- 4\dot- x\urcorner}, \{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 3}$

$$\{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}\}]; 2+1) \bowtie$$

$$([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- y\dot- 3\dot- x\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2\urcorner}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- x'\urcorner},$$

$$X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; 2+2)$$

from 2, 3, 4 by two applications of theorem 2.4

6/ $$y < \omega \to ([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- y\dot- 3\dot- x\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot- y\dot- 2\urcorner}, ([$$

$$X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot- x'\urcorner}]; y+1)\}]; y+1) \bowtie$$

$$([X_{\lambda x.y+4+x}^{\ulcorner\lambda x.f(v+1)e(v+1)\dot{-}y\dot{-}4\dot{-}x\urcorner}, \{X_{y+3}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}2\urcorner},$$

$$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}\}]; y) \quad 1, 5, \text{ induction}$$

7/ $$\{X_{2(y+1)+2}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+1)\dot{-}2\urcorner}, ([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\dot{-}x\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}2\urcorner},$$

$$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; y+1)\} \bowtie$$

$$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; 2(y+1)+2) \rightarrow$$

$$\{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+2)\dot{-}2\urcorner}, \{X_{2(y+2)+1}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+2)\dot{-}1\urcorner},$$

$$\{X_{2(y+1)+2}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+1)\dot{-}2\urcorner}, ([X_{\lambda x.y+3+x}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}3\dot{-}x\urcorner}, \{X_{y+2}^{\ulcorner f(v+1)e(v+1)\dot{-}y\dot{-}2\urcorner},$$

$$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+1)\}]; y+1)\}\}\} \bowtie$$

$$\{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+2)\dot{-}2\urcorner}, \{X_{2(y+2)+1}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+2)\dot{-}1\urcorner}, ([$$

$$X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; 2(y+1)+2)\}\} \quad \text{theorem } 2.4$$

8/ $$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; 2(y+2)+2) \bowtie$$

$$\{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+2)\dot{-}2\urcorner}, \{X_{2(y+2)+1}^{\ulcorner f(v+1)e(v+1)\dot{-}2(y+2)\dot{-}1\urcorner}, ([$$

$$X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; 2(y+1)+2)\}\} \quad \text{axiom of reduction}$$

9/ antecedent of 7 $\to \{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 2^\urcorner}$, $\{X_{2(y+2)+1}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 1^\urcorner}$, $\{X_{2(y+1)+1}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+1) \dot- 2^\urcorner}$, $([X_{\lambda x. y+3+x}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- y \dot- 3 \dot- x^\urcorner}$, $\{X_{y+2}^{\ulcorner f(v+1)e(v+1) \dot- y \dot- 2^\urcorner}$, $([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- x'^\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)^\urcorner}]; y+1)\}]; y+1)\}\}$ $\bowtie$ $([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- x'^\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)^\urcorner}]; 2(y+2)+2)$

7, 8, theorem 2.4

10/ antecedent of 9 $\to \{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 2^\urcorner}$, $\{X_{2(y+2)+1}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 1^\urcorner}$, $\{X_{2(y+1)+2}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+1) \dot- 2^\urcorner}$, $([X_{\lambda x. y+4+x}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- y \dot- 4 \dot- x^\urcorner}$, $\{X_{y+3}^{\ulcorner f(v+1)e(v+1) \dot- y \dot- 3^\urcorner}$, $([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- x'^\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)^\urcorner}]; y+2)\}]; y)\}\}\}$ $\bowtie$ $([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- x'^\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)^\urcorner}]; 2(y+2)+2)$ by theorem 2.4 from 9 and another sequent which is got from 6 by an axiom of reduction and an application of theorem 2.4.

11/ $\to \{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 2^\urcorner}$, $([X_{\lambda x. y+4+x}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- y \dot- 4 \dot- x^\urcorner}$, $\{X_{y+3}^{\ulcorner f(v+1)e(v+1) \dot- y \dot- 3^\urcorner}$, $([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- x'^\urcorner}, X_0^{\ulcorner f(v+1)e(v+1) \urcorner}]; y+2)\}]; y+2)\}$ $\bowtie$ $\{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 2^\urcorner}$, $\{X_{2(y+2)+1}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+2) \dot- 1^\urcorner}$, $\{X_{2(y+1)+2}^{\ulcorner f(v+1)e(v+1) \dot- 2(y+1) \dot- 2^\urcorner}$, $([X_{\lambda x. y+4+x}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- y \dot- 4 \dot- x^\urcorner}$, $\{X_{y+3}^{\ulcorner f(v+1)e(v+1) \dot- y \dot- 3^\urcorner}$, $([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot- x'^\urcorner}$,

$$X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+2)\}]; y)\}\}$$ axiom of reduction

12/ antecedent of 10 $\rightarrow \{X_{2(y+2)+2}^{\ulcorner f(v+1)e(v+1) \dot{-} 2(y+2) \dot{-} 2\urcorner}, ([X_{\lambda x. y+4+x}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot{-} y \dot{-} 4 \dot{-} x\urcorner}, \{X_{y+3}^{\ulcorner f(v+1)e(v+1) \dot{-} y \dot{-} 3\urcorner}, ([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot{-} x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; y+2)\}]; y+2)\} \bowtie ([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot{-} x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; 2(y+2)+2)$

10, 11, theorem 2.4

This completes the derivation of the induction-step. Intuitively, we should expect the equation in question to be provable, because we should expect that, once any substitution of numerals for all the free variables in the type-functors has been made, both sides will reduce to the same normal form.

2. Theorems on Combinatory Completeness.

Traditionally (Curry and Feys 1958, pp.5 and 186f.; Barendregt 1981, p.30f.) a theory T is said to be "combinatorially complete" if it possesses the following property: for every term M of T and every sequence $N_1, \ldots N_m$ of terms, there is a term of T, say Q, containing none of $N_1, \ldots N_m$, so that $QN_1 \ldots N_m$ is equal within T to M.

It is obvious that TF is combinatorially complete in this sense. A more interesting question is whether we can find a generalisation of the property just defined so that TF, in contrast to HAω, will be combinatorially complete in the stronger sense as well. I have come up with the following: let N be made up of the components $N_1, \ldots N_k$, combined by means of operations 4.2.5 and 4.2.10. Let M be a term in which all occurrences of $N_1, \ldots N_k$ that occurred within N as the left component of a sub-term formed by operation 4.2.10 also occur in this context. Then there is a term of TF, which we might as well call Q, in which none of $N_1, \ldots N_k$ occur, so that QN is equal within TF to M.

Why do I call this property "combinatory completeness"? TF is meant to be a generalisation of qf.-HAω in the sense that properties which can be established in the latter theory for finitely long sequences of objects -- the objects denoted by its closed terms -- can now be shown to hold of some infinite sequences. Operation 4.2.10 is one of the operations by which we can construct terms that can be used to express properties of infintely long sequences. For ([M, N]; v) is a term which, whenever a numeral, say $0\underbrace{''\cdots'}_{k}$ is substituted for v, reduces to a sequence having k+1 components, combined by k applications of the pairing operation, 4.2.5. Thus it is reasonable to think that ([M, N]; v) is a term which can be used to express generalisations about all such sequences.

In the same way the term N, mentioned in the paragraph before the last, may "describe" (to use a suitably vague word) an infinitely long sequence. The terms $N_1, \ldots N_k$ may therefore be taken to describe either individual members or sub-sequences of that sequence.

To prove that TF is combinatorially complete in the sense defined would, I think, be quite a messy task. I shall therefore prove only a special case which will be important in chapter 5. Let the term M and the variables $\underline{X}_1, \ldots . \underline{X}_9$ be as described in definitions 5.3 in chapter 5. I would like to prove that, for each i so that $0 < i < 10$, there is a term

$\Lambda X_i^{(v)}$ M so that, for each i so that i is 2, 4, 5, 7, 9, $\Lambda X_i^{(v)}$ M is equal to M_i and, for each i so that i is 1, 3, 6 or 8 and for any K, ([$\Lambda X_i^{(v)}$ M, K]; dv) is equal to ([M_i, K]; dv).

I shall treat first the case where i is 2, 4, 5, 7 or 9. I shall begin by proving that $Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9]; dv)([M_8, M_9]; dv)dv$ is equal to M_9.

$$1/ \quad \rightarrow Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];0)([M_8,M_9];0)0 \bowtie M_9$$

axiom of reduction

$$2/ \quad \rightarrow Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];x')([M_8,M_9];x')x'$$
$$\bowtie Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];x)([M_8,M_9];x)x$$

This is an axiom of reduction: the reduction chain proceeds via $Ap((\lambda X_{x'}^{\ulcorner f(v+1)e(v+1)=x'\urcorner}.([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];x))(M_8;x))$ $([M_8,M_9];x)x$, to which the left-hand term of the equation reduces.

$$3/ \quad Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];x)([M_8,M_9];x)x \bowtie M_9 \rightarrow$$
$$Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];x')([M_8,M_9];x')x' \bowtie M_9$$

L, 2, theorem 2.4

$$4/ \quad dv < \omega \rightarrow Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_9];dv)([M_8,M_9];dv)dv \bowtie M_9$$

1, 3, induction

By gradually adding new components to the terms involved, I presume we can get such theorems as

$$\rightarrow Ap([\lambda \underline{X}_1.\lambda \underline{X}_2.([\lambda \underline{X}_3.\lambda \underline{X}_4.\lambda \underline{X}_5.([\lambda \underline{X}_6.\lambda \underline{X}_7.([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_i];dv)];dv)];dv)];dv)$$
$$([M_1,\{M_2,([M_3,\{M_4,\{M_5,([M_6,\{M_7,([M_8,M_9];dv)\}];dv)\}\}];dv)\}];dv)$$
$$(dv) \bowtie M_i \quad (i = 2,4,5,7,9)$$

Thus we see how to define $\Lambda X_i^{(v)}$ M for the case where i is 2, 4, 5, 7 or 9. To show how to define it for the other cases, I again begin with an over-simplified version of the theorem to be proved, viz., I shall prove that $([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.\underline{X}_8];dv)([M_8,M_9];dv)dv, K];dv)$ is equal to $([M_8,K];dv)$.

1/ $\rightarrow ([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.\underline{X}_8];0)([M_8,M_9];0)0, K];0) \bowtie ([M_8,K];0)$

This can be proved by reducing both sides of the equation to their normal forms.

2/ $\rightarrow ([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x', K];x')$

$\bowtie \{Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.(M_8;x)];x)([M_8,M_9];x)x,$

$([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x', K];x)\}$

axiom of reduction (the left-hand term reduces to the right-hand one).

3/ $\rightarrow Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.(M_8;x)];0)([M_8,M_9];0)0 \bowtie (M_8;x)$

axiom of reduction

4/ $\rightarrow Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.(M_8;x)];y')([M_8,M_9];y')y'$

$\bowtie Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.(M_8;x)];y)([M_8,M_9];y)y$ axiom of reduction, provided that $X_{y'}^{\ulcorner f(v+1)e(v+1)=y'\urcorner}$ does not occur in M_8 -- if it does, we need to choose another variable in place of y.

5/ $Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.(M_8;x)];y)([M_8,M_9];y)y \bowtie (M_8;x)$

$\rightarrow Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.(M_8;x)];y')([M_8,M_9];y')y' \bowtie (M_8;x)$

L, 4, theorem 2.4

6/ $x<\omega \rightarrow Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.(M_8;x)];x)([M_8,M_9];x)x \bowtie (M_8;x)$

3, 5, induction

7/ $\rightarrow([Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x', K];x')$

$\bowtie \{(M_8;x), ([Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x', K];x)\}$

2, 6, theorem 2.4

8/ $\rightarrow Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x'$

$\bowtie Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x)([M_8,M_9];x)x$ axiom of reduction

9/ $\rightarrow([Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x', K];x')$

$\bowtie \{(M_8,x), ([Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8],x)([M_8,M_9],x)x, K];x)\}$

7, 8, theorem 2.4

10/ $(([Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x)([M_8,M_9];x)x,,K];x) \bowtie ([M_8,K];x)$

$\rightarrow([Ap([\lambda \underline{X}_8.\lambda \underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x', K];x')$

$\bowtie \{(M_8;x),([M_8,K];x)\}$

theorem 2.4, interchange, cut with 9

11/ $\to ([M_8,K];x') \bowtie \{(M_8;x),([M_8,K];x)\}$

axiom of reduction

12/ $([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.X_8];x)([M_8,M_9];x)x,K];x)$
$\bowtie ([M_8,K];x) \to ([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.\underline{X}_8];x')([M_8,M_9];x')x',$
$K];x') \bowtie ([M_8,K];x')$ 10, 11, theorem 2.4

13/ $\mathrm{dv} < \omega \to ([Ap([\lambda\underline{X}_8.\lambda\underline{X}_9.\underline{X}_8];\mathrm{dv})([M_8,M_9];\mathrm{dv})\mathrm{dv},K];\mathrm{dv})$
$\bowtie ([M_8,K];\mathrm{dv})$ 1, 12, induction, Q.E.D.

By gradually adding new components to the terms involved, that is, $([\lambda\underline{X}_8.\underline{X}_9.\underline{X}_8];\mathrm{dv})$ and $([M_8,M_9];\mathrm{dv})$, we get such theorems as

$\to ([Ap([\lambda\underline{X}_6.\lambda\underline{X}_7.([\lambda\underline{X}_8.\lambda\underline{X}_9.X_i];\mathrm{dv})];\mathrm{dv}),K];\mathrm{dv})$
$\bowtie ([M_i,K];\mathrm{dv})$ $(i = 6, 8)$

so we see how to define $\Lambda X_i^{(v)} M$ for the case where i is 1, 3, 6 or 8.

I would now like to prove a theorem which is, approximately speaking, converse to the last one. Whereas the last theorem showed that it is possible to construct a term which, applied to a term formed from components $N_1, \ldots N_k$ by operations 4.2.5 and 4.2.10, splits that term up into its components, the theorem now in question shows that, when we have a term constructed in the way described, it is possible to form another term equal to it, but which is formed by means of just one application of operation 4.2.10. That is, I shall prove that, for every M, N and P of type-functors $\ulcorner f_1 \urcorner$, $\ulcorner f_2 \urcorner$, and $\ulcorner g \urcorner$, there is a term Q so that

$$\rightarrow ([M, (N, P]; u)]; v) \bowtie ([Q, P]; u+v)$$

is provable. Q will have a type-functor $\ulcorner f \urcorner$ where $x < u \rightarrow fx = f_2x$ and $u \leq x < v \rightarrow fx = f_1x$ are provable in HAω.

We define Q to be $Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}.X^{\ulcorner f \urcorner}_{\lambda x.x}]; u+v)([M, ([N, P]; u)]; v)(u+v)$ and construct the following derivation:

1/ $\rightarrow ([Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}.X^{\ulcorner f \urcorner}_{\lambda x.x}]; 0)([M, ([N, P]; 0)]; 0)(0), P]; 0) \bowtie P \bowtie ([N, P]; 0)$: this can be proved by reducing both sides to their normal forms.

2/ $x < u \rightarrow Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}.X^{\ulcorner fx \urcorner}_{x}]; x')([M, ([N, P]; x')]; 0)(x') \bowtie Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}.(N; x)]; x)([M, ([N, P]; x)]; 0)(x)$ ditto. The condition $x < u$ is necessary in order that $f \bowtie f_1$, i.e. that the terms of the equation be well-formed.

3/ $\rightarrow Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}.(N; x)]; x)([M, ([N, P]; x)]; 0)(x) \bowtie (N; x)$ This is proved in the course of the proof of the last theorem.

4/ $x < u \rightarrow ([Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}.X^{\ulcorner f \urcorner}_{\lambda x.x}]; x')([M, ([N, P]; x')]; 0)(x'),$

$P]; x') \bowtie \{ Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}. X^{\ulcorner fx \urcorner}_{x}]; x')([M, ([N, P]; x')]; 0)(x'),$
$([Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}. X^{\ulcorner f \urcorner}_{\lambda x.x}]; x')([M, ([N, P]; x')]; 0)(x'), P]; x) \}$

axiom of reduction

5/ $x < u \rightarrow$ l.h.s. of succedent of 4 $\bowtie \{ (N; x),$
$([Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}. X^{\ulcorner f \urcorner}_{\lambda x.x}]; x')([M, ([N, P]; x')]; 0)(x'), P]; x) \}$

2, 3, 4, theorem 2.4 (twice)

6/ $x' \leqslant u \rightarrow$ succedent of 5. By a cut from 5 and $x' \leqslant u \rightarrow x < u$, which is presumably provable.

7/ $x \leqslant u,\ x \leqslant u \supset ([Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}. X^{\ulcorner f \urcorner}_{\lambda x.x}]; x)([M, ([N, P]; x)]; 0), P]; x)$
$\bowtie \ , ([N, P]; x)$ $\rightarrow$ succedent sub-formula of the main antecedent formula MPP

8/ $x' \leqslant u, x \leqslant u$, main antecedent formula of 7 $\rightarrow$
$([Ap([\lambda X^{\ulcorner f \urcorner}_{\lambda x.x}. X^{\ulcorner f \urcorner}_{\lambda x.x}]; x')([M, ([N, P]; x')]; 0)(x'), P]; x') \bowtie$
$\{ (N; x), ([N, P]; x) \}$ 6, 7, theorem 2.4

9/ $x' \leqslant u \rightarrow x \leqslant u$ presumably provable

10/ $x' \leqslant u$, main antecedent formula of 8 $\rightarrow$ its succedent

8, 9, cut, contraction

11/ main antecedent formula of 10 $\rightarrow x' \leqslant u \supset$

(left-hand side of succedent of 8) $\bowtie \{(N;x), ([N,P];x)\}$

10, $\supset$I

12/ main antecedent formula of 11 $\bowtie x' \leq u \supset$ (left-hand side of succedent of 8) $\bowtie ([N,P];x')$

13/ $\rightarrow 0 \leq u \supset ([Ap([\lambda X^{\text{'}f\text{'}}_{\lambda x.x}.\, X^{\text{'}f\text{'}}_{\lambda x.x}];0)([M,([N,P];0)];0)(0), P];0)$

$\bowtie ([N,P];0)$ MPP, 1, $\supset$E, cut with $\rightarrow 0 \leq u$

14/ $\rightarrow u \leq u \supset ([Ap([\lambda X^{\text{'}f\text{'}}_{\lambda x.x}.\, X^{\text{'}f\text{'}}_{\lambda x.x}]; u)([M,([N,P];u)];0), P]; u)$

$\bowtie ([N,P];u)$ 12, 13, induction

15/ $\rightarrow ([Ap([\lambda X^{\text{'}f\text{'}}_{\lambda x.x}.\, X^{\text{'}f\text{'}}_{\lambda x.x}];u)([M,([N,P];u)];0), P];0)$

$\bowtie ([N,P];u)$ 14, MPP, cut, cut with $\rightarrow u \leq u$

16/ $\rightarrow$ left-hand side of succedent of 15

$\bowtie ([M,([N,P];u)];0)$ 15, axiom of reduction, theorem 2.4

Considering the definition of Q, the sequent just proved can also be written as

$$\rightarrow ([Q[v:=0], P]; u+0) \bowtie ([M,([N,P];u)];0)$$

which we can make into the basis of a new proof by induction, the conclusion of which is the theorem stated at the top of this section.

3. Remarks on Substitution.

I must begin by defining substitution. In the first place, it should be recalled that I identify variables whose type- and shape-functors are equal. Thus if $\rightarrow e = f$ and $\rightarrow g = h$ are provable in HAω, we may always write "$X^{\ulcorner e \urcorner}_{g}$" in place of "$X^{\ulcorner f \urcorner}_{h}$": this is merely replacing one name for a variable with another name for the same variable.

The obvious definition of substitution is as follows: the result of substituting N for X in M, which I shall denote by "$M[X^{\ulcorner f \urcorner}_{g}:=N]$", assuming that $X^{\ulcorner f \urcorner}_{g}$ is free in M and that it has the same type-functor as N, is N if M is $X^{\ulcorner f \urcorner}_{g}$; if M is 0 or a variable with a shape-functor which is not equal to g, it is M itself. These are the cases where M is a primitive term. If M is formed by an operation from components $P_1, \ldots P_k$, then $M[X^{\ulcorner f \urcorner}_{g}:=N]$ is the result of the same operation on $P_1[X^{\ulcorner f \urcorner}_{g}:=N], \ldots P_k[X^{\ulcorner f \urcorner}_{g}:=N]$.

In practice it is desirable to modify this definition somewhat. Following Curry and Feys (1958, p.94) I would like to define substitution so that no free variable within N becomes bound when the substitution is made. So if M is $\lambda X^{\ulcorner e \urcorner}_{h}.Q$, for some e, h and Q, then $M[X^{\ulcorner f \urcorner}_{g}:=N]$ is defined to be $\lambda X^{\ulcorner e \urcorner}_{d}.Q[X^{\ulcorner e \urcorner}_{h}:=X^{\ulcorner e \urcorner}_{d}][X^{\ulcorner f \urcorner}_{g}:=N]$, where d is the first shape-functor in some enumeration of the shape-functors so that $X^{\ulcorner e \urcorner}_{d}$ does not occur free in either Q or N. This enables us to state without qualification that, where $\rightarrow P = Q$ is an axiom of reduction in which Q is obtained from P or vice versa by one λβ-conversion, the result of substituting something for some variable occurring in P and Q is still an axiom.

I would now like to formulate a substitution-rule, that is, a statement of the conditions under which it is possible to substitute something for a variable occurring in a statement provable in TF and get another statement which is also provable in TF. First I need a definition of substitution within a formula, though this is easier than defining substitution within a term. If the formula is an equation, substitution of such-and-such a term (of the same type-functor) for such-and-such a variable is substitution of it for the variable in both sides of the equation. If the formula is complex, substitution is defined in the obvious recursive way.

It is now necessary to work through the axioms and rules of inference in order to see whether provability is always preserved under substitution. In fact it is not, unless we import some restrictions into our definition of substitution. Consider the following axiom of reduction, featuring a redex formed by operation 4.2.9:

$$\rightarrow([X_g^{\ulcorner f \urcorner}, X_e^{\ulcorner d \urcorner}]; m') \bowtie \{(X_g^{\ulcorner f \urcorner}; m), ([X_g^{\ulcorner f \urcorner}, X_e^{\ulcorner d \urcorner}]; m)\}$$

In this case it is obviously not possible to substitute an arbitrary term for the variable ($X_g^{\ulcorner f \urcorner}$; m) and another arbitrary term for the variable $X_g^{\ulcorner f \urcorner}$ and get a sequent which is still an axiom of TF -- indeed, I doubt if it is possible in this way to get a sequent which is provable in TF at all. The solution is to observie that the first variable is in a sense derived from the second and what we substitute for the first must be constrained by what we substitute for the second. I would therefore state the substitution-rule as follows: if $\underline{Z}$ has either a type- or a shape-functor whose order is greater than 0, and another variable, $\underline{Y}$, occurs in the statement in which the substitution is being made which is identical to ($\underline{Z}$; m) for some m, and N is the term which is being substituted for $\underline{Z}$, then (N; m) must be substituted for $\underline{Y}$ in the same statement. Similarly, if we want to make a substitution for $\underline{Y}$ and $\underline{Y}$ is identical to ($\underline{Z}$; m) for any m and for any $\underline{Z}$ occurring in the same statement, then we must make an appropriate substitution for $\underline{Z}$. It is clear that if substitution is defined in this way, axioms of reduction of the kind considered above are closed under it.

Axioms of reduction in which at least one of the redexes contracted is formed by operationi 4.2.9 are the only kind of axiom that consists in an equation in which a variable can occur on one side but not on the other. All the others are closed under substitution because the syntactic variables used in stating the contraction-rules range over arbitrary terms and terms which are identical continue to be so if we make a uniform substitution for variables occurring within them.

We must now consider rules of inference. The logical rules of inference preserve provability under substitution in the following sense: if we make a substitution in the conclusion of a rule, that conclusion continues to be provable so long as we make appropriate substitutions in the premisses. This follows from the way the logical rules (which in TF are all propositional) are stated, together with the definition of substitution in formulae.

In the case of the induction-rule, it is obviously correct to substitute anything for a variable in the conclusion that occupies the place of the eigenvariable in the premisses, just so long as the result is well-formed.

It is not clear to me whether it is effectively possible to decide whether any given proposed substitution is legitimate or not in TF. This does not matter for my purposes, though. It is essential only that, whenever I make a substitution in the proofs following, it is possible to check that there are no variables which I ought to have replaced and have not.

CHAPTER 3

PRELIMINARY SKETCH OF OUR PROOF OF THE ACCESSIBILITY OF ε_0

The argument of this thesis is complicated and some readers of the following chapters may feel, on account of the multitude and complexity of the trees, that they have well and truly lost sight of the wood. To minimise this feeling, I shall summarise here what I am going to do. Through chapters 4 and 5 I shall be working towards the conclusion that the accessibility of ε_0 can be proved in TF. Chapter 6 is devoted to proving a complementary result and does not require a knowledge of the intervening chapters. At this point, therefore, I shall merely give an outline of how our proof of the accessibility of ε_0 is going to proceed.

The proof bears some resemblance to the proof of Bernays and Gentzen discussed in chapter 1. That proof uses intuitionist second-order number theory with one application of the Π^1_0-comprehension rule and one application of the induction-rule to a Π^1_1-formula. I have already defined what I mean by a functional interpretation of a statement (in so far as it is in $\exists\forall$-form) and by a functional interpretation of a theory. It follows from these definitions that, if you have a proof of a sequent in $\exists\forall$-form in some theory, you can get a functional interpretation of that sequent by producing a functional interpretation of the entire theory. This is a task which I hope to accomplish one day but in the meantime I have concentrated only on producing a functional interpretation of a particular sequent, which will be exhibited shortly. Whether or not this is a simpler task than producing a functional interpretation of the whole theory I do not know. My original reason for concentrating on a single statement rather than an entire theory was that I conjectured that this task might require a weaker theory, in ordinal-theoretic terms, than any that could be used to interpret the whole of that sub-system of second-order number theory which has just been specified.

The fact that I do not know the answer to either of these questions makes it urgent to enquire whether that theory can be interpreted in TF or in a slight extension thereof. The task I have undertaken in this thesis should at least make good practice for an assault on those problems.

At this point I will sketch a proof of the accessibility of ε_0 in a theory which I call HA". I will begin by explaining how the theory HA" is derived from HA. Let us add to the objects of HA a new category of objects called <u>one-place complex predicates</u>. Each such object is the result of an operation upon a formula and a variable that occurs free in that formula. The result of the operation is denoted by sticking '$\lambda \underline{v}$', where $\underline{v}$ is the free variable in question, in front of the name of the formula. We now postulate that it is possible to apply a one-place predicate to a term of type o to get a formula; this is indicated just like application of terms to terms. Each such formula may be contracted, just like the result of applying a λ-term to a suitable argument, and the contractum is again a formula. Let X be a variable replaceable by one-place predicates and let us add X to the language of HA.

It is well-known that, in the theory so obtained, the validity of transfinite induction up to any number smaller than ε_0 is provable. In the proofs of this statement, the formula $\mathfrak{B}(X, z)$, defined as follows:

$$\mathfrak{B}(X, z) \equiv_{df.} \forall y (\forall x . x < y \supset X(x) :\supset: \forall x . x < y + 2^z \supset X(x))$$

may play an important role. The exact structure of the proofs suggest a method by which we may prove the validity of transfinite induction up to ε_0 itself, in an extension of HA. Let us introduce a new predicate, $\lambda v.\mathfrak{L}(v, z)$, in the context of the following pair of axioms:

$$\rightarrow \mathfrak{L}(0, z) \equiv \mathfrak{B}(X, z)$$

$$\rightarrow \mathfrak{L}(s', z) \equiv \mathfrak{B}(\lambda v.\mathfrak{L}(s, v), z)$$

In reading these axioms, we should take it that, for any one-place predicate F, $\mathfrak{L}(F, z)$ is the result of substituting F for X in $\mathfrak{L}(X, z)$.

I shall call the theory that is obtained by adding those axioms to HA HA'.

I take the validity of transfinite induction up to the ordinal number denoted by t to be expressed by the sequent

$$\forall y(\forall z . z < y \supset X(z) :\supset: X(y)) \rightarrow X(t)$$

of which I shall abbreviate the antecedent to 'Prog(X)'. In HA, one can prove

$$Prog(X) \rightarrow Prog(\lambda z.\mathfrak{L}(X, z)) \qquad (1)$$

the latter formula being the result of replacing X in Prog(X) with $\lambda z.\mathfrak{L}(X, z)$. Therefore one can prove

$$Prog(\lambda z.\mathcal{L}(u, z)) \rightarrow Prog(\lambda z.\mathfrak{L}(\lambda w.\mathcal{L}(u, v), z)) \quad (2)$$

in HA'. The succedent of this last sequent is equivalent to Prog($\lambda z.\mathcal{L}(u', z)$). Using (1) and (2) as the premisses of an induction, one gets

$$Prog(X) \rightarrow Prog(\lambda z.\mathcal{L}(u, z)) \qquad (3)$$

and, applying transfinite induction up to $\omega+1$ to the predicate $\lambda z.\mathcal{L}$ (u, z),

$$Prog(X) \rightarrow \mathcal{L}(u, \omega+1) \qquad (*)$$

Anyone familiar with the properties of the predicate $\lambda z.\mathfrak{L}(X, z)$ will know how to derive from this last sequent

$$Prog(X) \rightarrow \forall u . X(\omega_u)$$

in HA'.

Unlike HA', TF contains no predicate variables, but the sequent I described as asserting that transfinite induction up to ε_0 is valid contains the predicate-variable X. It is necessary, therefore, to consider which sequent of TF it is that we want to prove. I shall concentrate on proving that ε_0 is accessible, as this can be expressed in a theory which has function-variables of type 1, but no predicate-variables, by the formula

$$\forall v\{X'v > 0 \supset X'(v') < X'v . \& X'v = 0 \supset X'(v') = 0\} \supset:: X'0 < \varepsilon_0 .\supset. \exists y . X'y = 0$$

which says, translated into English:

X^1 is a function which enumerates a strictly decreasing sequence of numbers:⊃: the first term of the sequence enumerated by X^1 is smaller than ε_0 .⊃. at some point, X^1 has the value 0.

Let us abbreviate the formula to Acc(ε_0). Then, since Prog(λz.Acc(z)) is easily provable in HA, we may substitute the predicate λz.Acc(z) for the predicate-variable X and get a proof of the sequent

$$\rightarrow Acc(\varepsilon_0)$$

in a theory which is obtained from HA' by replacing the variable X, in the axioms peculiar to HA', with the predicate λz.Acc(z). I shall call the resulting theory HA" and the predicate that is introduced by the new pair of axioms $\lambda v.\mathfrak{L}^*(v, z)$.

It is well-known (Yasugi 1963, p.106) that every sequent of HAω is deductively equivalent, within HAω^+, to a sequent in $\exists\forall$-form. When I speak of the $\exists\forall$-form of a sequent in the language of HAω, I therefore mean the sequent got by the procedure she describes. The $\exists\forall$-form of $\rightarrow$Acc(ε_0) is

$$\rightarrow \exists Y^{1o} \exists X^{1o} \forall V^1 \{V(XV) > 0 \supset V((XV)') < V(XV) .\&. V(XV) = 0 \supset V((XV)') = 0 :\supset: (V0 < \varepsilon_0 \supset V(YV) = 0)\}$$

By a functional interpretation of this sequent in TF, we therefore mean a proof in TF of some sequent of the form

$$\rightarrow (V'(NV') > 0 \supset V'((NV')') < V'(NV') .\&. V'(NV') = 0 \supset V'((NV')') = 0) .\supset. (V'0 < \varepsilon_0 \supset V'(MV') = 0)$$

for some M and N which are terms of TF, both of type 1o, in which V^1 does not occur. The task of chapters 4 and 5 is to find terms M and N with the relevant properties and a proof in TF of some sequent of the form just presented. I shall call the last sequent $\rightarrow$Acc'(ε_0).

I shall now give an outline how the proof will go, emphasizing especially points of resemblance to the proofs in HA' and HA" that I have already sketched. In the proof in HA", the sequent which corresponds to the crucial sequent (*) will be

$$\rightarrow \mathcal{L}^*(u, \omega+1) \qquad\qquad (**)$$

so the main task that faces us at this point is to see what will correspond to (**) in a proof of $\rightarrow$Acc'(ε_0) in TF. There are no axioms in TF that resemble the axioms introducing $\lambda v.\mathcal{L}(v, z)$ in HA' or $\lambda v.\mathcal{L}^*(v, z)$ in HA". However it is clear that if, in (**), we replace u with a numeral, the succedent formula of the resulting sequent is equivalent, within HA", to a sequent of HA. For the numerals 1, 2, 3, . . . , I shall call the relevant formulae of HA $\mathcal{L}_1(\omega+1)$, $\mathcal{L}_2(\omega+1)$, $\mathcal{L}_3(\omega+1)$, ... Each of those formulae also has an $\exists\forall$ -form, in HAω. I shall call the formulae in the latter sequence $\mathcal{L}'_1(\omega+1)$, $\mathcal{L}'_2(\omega+1)$, $\mathcal{L}'_3(\omega+1)$, ... Now the sequents

$$\rightarrow \mathcal{L}'_1(\omega+1)$$
$$\rightarrow \mathcal{L}'_2(\omega+1)$$
$$\rightarrow \mathcal{L}'_3(\omega+1)$$
$$\vdots \qquad \vdots$$

can all be functionally interpreted in the quantifier-free part of HAω (hence in TF), by proofs ending with sequents of one succedent formula. Let the succedent formulae in question, corresponding to $\mathcal{L}'_1(\omega+1)$, $\mathcal{L}'_2(\omega+1)$, $\mathcal{L}'_3(\omega+1)$, . . ., be called $\mathcal{L}^*_1(\omega+1)$, $\mathcal{L}^*_2(\omega+1)$, $\mathcal{L}^*_3(\omega+1)$, ...

The situation now is as follows. Every one of the sequents

$$\rightarrow \mathcal{L}^*(1, \omega+1)$$

$$\rightarrow \mathcal{L}^*(2, \omega+1)$$

$$\rightarrow \mathcal{L}^*(3, \omega+1)$$

$$\vdots \qquad \vdots$$

can be derived, within HA", from $\rightarrow \mathcal{L}^*(u, \omega+1)$. Secondly, every one of the last series of sequents is, by itself, derivable in HA without any extra axioms. Thirdly, every one of the sequents

$$
\begin{aligned}
&\rightarrow \mathcal{L}_1^*(\omega+1)\\
&\rightarrow \mathcal{L}_2^*(\omega+1)\\
&\rightarrow \mathcal{L}_3^*(\omega+1)\\
&\quad\vdots \qquad\quad \vdots
\end{aligned}
$$

is provable in the quantifier-free part of HAω. What, however, we do not yet have is a theory which corresponds to the quantifier-free part of HAω (henceforth qf.-HAω) as HA" corresponds to HA. That is to say, we do not yet have a theory in which we can prove a sequent from which, just by replacing a variable with a numeral, we can get any one of the sequents in the sequence drawn above.

I maintain that TF is -- almost -- the theory we are looking for. Strictly speaking, though, the theory we are looking for does not actually exist, because each one of $\mathcal{L}_1^*(\omega+1)$, $\mathcal{L}_2^*(\omega+1)$, $\mathcal{L}_3^*(\omega+1)$, . . . , as well as containing different terms from all the others, also has a more complicated propositional structure than all its predecessors.

We can overcome this problem by considering, in place of each of the formulae $\mathcal{L}^*_1(\omega+1)$, $\mathcal{L}_2^*(\omega+1)$, $\mathcal{L}_3^*(\omega+1)$, . . . , an equation which is equivalent to it in qf.-HAω. The left-hand term in the equation will be a <u>characteristic term</u> of the corresponding formula, which contains, as subterms, all the terms which occur in the formula. The right-hand term will be 0. Let the characteristic terms be called $\chi_{\mathcal{L}1}(\omega+1)$, $\chi_{\mathcal{L}2}(\omega+1)$, $\chi_{\mathcal{L}3}(\omega+1)$, . . . For each of the formulae in question, it is provable in qf.-HAω that a term with the required properties exists. So now the task that faces us is to find an equation of the shape $\chi_{\mathcal{L}}(u, \omega+1) = 0$, provable in TF, with the property that, by substituting a numeral for u, we get something equivalent, within TF, to any one of the equations $\chi_{\mathcal{L}1}(\omega+1) = 0$, $\chi_{\mathcal{L}2}(\omega+1) = 0$, $\chi_{\mathcal{L}3}(\omega+1) = 0$, ...

In chapter 5 I shall present some discussion of the heuristic considerations that led to me picking on the term I did, but the main purpose of that chapter is to prove that the term I pick on has the required properties. The proof might be divided into four stages. First a matrix of terms $\chi_{\mathcal{L}}(\underline{Y})(u, z)$, where $\underline{Y}$ is a variable of suitable type-functor is defined. I immediately prove that, given some suitable substitution for $\underline{Y}$, the resulting term stands in the required relation to the characteristic terms $\chi_{\mathcal{L}1}(\omega+1)$, $\chi_{\mathcal{L}2}(\omega+1)$, $\chi_{\mathcal{L}3}(\omega+1)$, . . . The demonstration of this occupies lemmata 5.5 and 5.6 of chapter 5.

Secondly I find a term N of TF so that $\rightarrow\chi_{\mathcal{L}}(N)(v+2, 0) = 0$ is provable in TF. The proof works by induction on v. This demonstration occupies lemmata 5.7 - 5.9.

Thirdly I prove that there is a term Q so that $\rightarrow\chi_{\mathcal{L}}(Q)(v+1, \omega+1) = 0$ is a theorem of TF. This section of the proof uses the fact, which follows from lemma 5.6, that $\chi_{\mathcal{L}}(N)(v+2, 0) = 0$ implies that the predicate $\lambda u.(\chi_{\mathcal{L}}(P)(v+1, u) = 0)$ is progressive, for some P which depends on N. With this established, I now use something like transfinite induction up to $\omega+1$ as applied to that last predicate. In fact I use a special sort of induction-rule, which will be justified by theorem 4.5 in the next chapter.

In both of these last two sections of the proof, the terms N, P and Q are so constituted that every substitution of a numeral for v in $\chi_{\mathcal{L}}(Q)(v+1, \omega+1) = 0$ really does yield an equation equivalent within TF to one of $\mathcal{L}^*_1(\omega+1)$, $\mathcal{L}^*_2(\omega+1)$, $\mathcal{L}^*_3(\omega+1)$ . . .

It is this third section of the proof which gives us a sequent of TF which corresponds to the sequents (*) and (**) in HA' and HA" respectively. The fourth section of the proof, with which chapter 5 will end, consists in a derivation, from the conclusion of the third section, of the sequent $\rightarrow Acc'(\omega_v)$. This is a quantifier-free proof which imitates the corresponding section in the proof of $\rightarrow Acc(\omega_v)$ in HA".

I shall now add some remarks as to why it is reasonable to expect that a term with the properties of $\chi_{\mathcal{L}}(u, \omega+1)$ in TF can be constructed. $\chi_{\mathcal{L}1}(\omega+1)$, $\chi_{\mathcal{L}2}(\omega+1)$, $\chi_{\mathcal{L}3}(\omega+1)$, . . . are required to be characteristic terms of $\mathcal{L}^*_1(\omega+1)$, $\mathcal{L}^*_2(\omega+1)$, $\mathcal{L}^*_3(\omega+1)$, . . . , while the latter series of formulae are required to be the succedent formulae of the conclusions of functional interpretations of $\mathcal{L}'_1(\omega+1)$, $\mathcal{L}'_2(\omega+1)$, $\mathcal{L}'_3(\omega+1)$, ... Now every formula in the last sequence will contain more quantifiers and quantifiers of an (in general) higher type than all its predecessors. Consequently each formula in the series $\mathcal{L}^*_1(\omega+1)$, $\mathcal{L}^*_2(\omega+1)$, $\mathcal{L}^*_3(\omega+1)$, . . . , and hence each term in the series $\chi_{\mathcal{L}1}(\omega+1)$, $\chi_{\mathcal{L}2}(\omega+1)$, $\chi_{\mathcal{L}3}(\omega+1)$, . . . will contain more and more complex terms than its predecessors in the series. So can there really be a term with the properties we want $\chi_{c}(u, \omega+1)$ to have?

But precisely what distinguishes the terms of TF from the terms of HAω is that the former theory contains terms which, whenever a numeral is substituted for some variable occurring in them, reduce to some term of HAω whose number of components and their types depends on what numeral was substituted. In this connection, I would request the reader to reflect again on what happens when we substitue a numeral for v in a term of the shape ([M, N]; v).

CHAPTER 4

CHARACTERISTIC TERMS AND INDUCTION-RULES

In this chapter I shall establish two important properties of TF. First, every formula has a characteristic term (this notion will be defined below). Secondly, certain rules of inference, which are related to the induction-rule, are derivable. Connoisseurs of functional interpretations will know that, given functional interpretations of the premisses of a contraction or an induction inference in HAω, it is a by no means trivial task to derive from them a functional interpretation of the conclusion. Special theorems are required, which in effect constitute derived rules of qf.-HAω. In this chapter I find corresponding rules for TF, which is, after all, a generalisation of qf.-HAω.

Definition: For any formula F of TF, a characteristic term of F is a term χ_F such that all of the following four sequents are provable:

$$F \rightarrow \chi_F = 0 \qquad\qquad \chi_F = 0 \rightarrow F$$

$$\sim F \rightarrow \chi_F > 0 \qquad\qquad \chi_F > 0 \rightarrow \sim F$$

Theorem 4.1.: For every formula F of TF, let a corresponding term χ_F be defined as follows: $\chi_{s=t} = (s \dot{-} t)+(t \dot{-} s)$; $\chi_{\sim F} = \overline{sg}(\chi_F)$ (sg being a term for a function that maps 0 onto 0 and any nonzero number onto 1; and $\overline{sg}$ being a term for the opposite function); $\chi_{F_1 \& F_2} = \chi_{F_1} + \chi_{F_2}$; $\chi_{F_1 \vee F_2} = \chi_{F_1} \cdot \chi_{F_2}$; $\chi_{F_1 \supset F_2} = \overline{sg}(\chi_{F_1}) \cdot \chi_{F_2}$. Then χ_F is a characteristic term for F.

Proof: We prove the theorem by induction on the number of logical connectives in F. The case where F is an equation has been treated by Schuette (pp.129-131) for equations between terms of HAω, but in fact the proof works without further ado for terms of TF. As an example of how to prove the induction-step, I shall consider the case where F is

$F_1 \& F_2$. $F \rightarrow \chi_{F_1} = 0$ and $F \rightarrow \chi_{F_2} = 0$ are both obviously provable. $\rightarrow 0+0 = 0$ is a mathematical axiom. By two applications of theorem 2.3 to the last sequent, we get $\chi_{F_1} = 0, \chi_{F_2} = 0 \rightarrow \chi_{F_1} + \chi_{F_2} = 0$. By two cuts (with the first two sequents as the respective left-hand premiss) and a contraction, we get $F \rightarrow \chi_{F_1} + \chi_{F_2} = 0$.

We now have to prove the converse. $\chi_{F_1} = 0, \chi_{F_2} = 0 \rightarrow F$ follows from the induction hypothesis by an &-introduction. $\chi_{F_1} + \chi_{F_2} = 0 \rightarrow (\chi_{F_1} + \chi_{F_2}) \dot{-} \chi_{F_2} = 0 \dot{-} \chi_{F_2}$ follows from theorem 2.3. From this, using two applications of theorem 2.3 and two mathematical sequents, we get $\chi_{F_1} + \chi_{F_2} = 0 \rightarrow \chi_{F_1} = 0$. Similarly $\chi_{F_1} + \chi_{F_2} = 0 \rightarrow \chi_{F_2} = 0$. By two cuts, involving the last two sequents and the first one in this paragraph, and a contraction we get $\chi_{F_1} + \chi_{F_2} = 0 \rightarrow F$.

When we recall that $\chi_F > 0$ is defined to be equal to $\sim J_o 0 P \chi_F = 0$ and that $J_o 0 P \chi_F$ is equal to χ_F, it is clear that the sequents $\sim F \rightarrow \chi_F > 0$ and $\chi_F > 0 \rightarrow \sim F$ are derivable from $\chi_F = 0 \rightarrow F$ and $F \rightarrow \chi_F = 0$ respectively by a negation-introduction and elimination and an application of theorem 2.3.

Theorem 4.2.: If F, G, $\Gamma \rightarrow \Delta$ is provable, where G is obtainable from F by the replacement of certain terms in F -- let us call them $M_1, M_2, \ldots M_n$ -- by $N_1, N_2, \ldots N_n$; then there are terms $D(M_1, N_1), D(M_2, N_2), \ldots D(M_n, N_n)$ such that

$$F[M_i := D(M_i, N_i)], \Gamma \rightarrow \Delta \qquad (1 \leq i \leq n)$$

is provable.

Proof: $D(M_i, N_i)$, for each i, is defined to be $J(\overline{sg}\chi_F)(\lambda X.N_i)M_i$, where X has the same type-functor as M_i. Then, obviously, for each i

$$\rightarrow D(M_i, N_i) \bowtie M_i . \equiv . \sim F \qquad \rightarrow D(M_i, N_i) \bowtie N_i . \equiv F$$

are provable in TF. Therefore

$$(1) \rightarrow \sim F . \supset . F[M_i := D(M_i, N_i)] \equiv F$$

$$(2) \rightarrow F.\supset.\ F[M_i := D(M_i, N_i)] \equiv G$$

are provable, by n applications of theorem 2.4. Now from F, G, $\Gamma \rightarrow \Delta$ one gets $\Gamma \rightarrow \Delta, \sim F, \sim G$. But from sequent (2) above and $F \rightarrow F$, one can also prove $F, \sim G$, $F[M_i{:=}D(M_i, N_i)] \rightarrow$. By a cut with $\Gamma \rightarrow \Delta, \sim F, \sim G$, this yields Γ, F, $F[M_i{:=}D(M_i, N_i)] \rightarrow \Delta, \sim F$. By negation-introduction and a contraction, Γ, $F[M_i{:=}D(M_i, N_i)] \rightarrow \Delta, \sim F$. By an argument parallel to that by which we got $F, \sim G$, $F[M_i{:=}D(M_i, N_i)] \rightarrow$ we also get $\sim F$, $F[M_i{:=}D(M_i, N_i)] \rightarrow$. By a cut involving this and Γ, $F[M_i{:=}D(M_i, N_i)] \rightarrow \Delta, \sim F$ and another contraction we get the desired sequent.

Theorem 4.3: If $\Gamma \rightarrow \Delta$, F, G is provable where F and G are related as in the above theorem, then there are terms $D(M_i, N_i)$ so that $\Gamma \rightarrow \Delta, F[M_i{:=}D(M_i, N_i)]$ is also provable.

Proof: parallel to the proof for 4.2.

Remark: the theorems just proved are used in functionally interpreting instances of the contraction rules.

Lemma 4.4: Let two sequents of the forms

$$(1) \quad \rightarrow F(([X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m), 0)$$

and

$$(2)\ F(([Ap(Qz)([X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)m, Ap(Q_0z)([X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)m]; m), z) \rightarrow F(([X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m), z')$$

respectively be provable in TF, where $Ap(Q_0z)([X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)m$ has the same type-functor as $X^{\ulcorner f0\urcorner}_{g0}$ (i.e., $\ulcorner f\urcorner 0$) and $Ap(Qz)([X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)m$ has the same type-functor as $X^{\lambda x.\ulcorner fx\urcorner}_{\lambda x.gx'}$. These conditions are satisfied if and only if Q has type-functor

$o^{\ulcorner}RL(\lambda x.fx')(\lambda y.fy)m+1^{\urcorner}$ and Q_o has type-functor $o^{\ulcorner}RL(f0)(\lambda x.fx)m+1^{\urcorner}$. Thus 1 is the order of the type-functor of $Ap(Qz)([X^{\ulcorner\lambda x.fx'\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)m$, as is required by our definition of operation 4.2.10, in order that a term may be formed by it from those two components. Under the said circumstances, the sequent

$$\rightarrow F(([X^{\ulcorner\lambda x.fx'\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m), x)$$

is also provable in TF.

Proof: We define a term-form $T(\underline{v})$ of type-functor $\ulcorner RC(f0)(\lambda x.fx')m^{\urcorner}$, as follows:

$$\rho[\lambda z.\lambda \underline{X}.([Ap(Q(x \doteq z'))\underline{X}m, Ap(Q_o(x \doteq z'))\underline{X}m, ([X^{\ulcorner\lambda x.fx'\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m), \underline{v}]$$

Here $\underline{X}$ is of type-functor $\ulcorner RC(f0)(\lambda x.fx')m^{\urcorner}$ We can now establish that T(s) reduces as follows:

$$T(0) = ([X^{\ulcorner\lambda x.fx'\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)$$

$$T(s') = (\lambda z.\lambda \underline{X}.([ApQ(x \doteq z')\underline{X}m, ApQ_o(x \doteq z')\underline{X}m]; m))s(T(s))$$

$$= (\lambda \underline{X}.([ApQ(x \doteq s')\underline{X}m, ApQ_o(x \doteq s')\underline{X}m]; m))(T(s))$$

$$= ([ApQ(x \doteq s')T(s)m, ApQ_o(x \doteq s')T(s)m]; m)$$

By the first theorem on combinatory completeness, there are two terms which may be substituted for $X^{\ulcorner\lambda x.fx'\urcorner}_{\lambda x.gx'}$ and $X^{\ulcorner f0\urcorner}_{g0}$ respectively in $([X^{\ulcorner\lambda x.fx'\urcorner}_{\lambda x.gx'}, X^{\ulcorner f0\urcorner}_{g0}]; m)$ so that the result is equal to T(y). Upon making this substitution in (2) and applying theorem 2.4, we find that the following sequent will be derivable:

$$3/\quad F(([ApQzT(y)m, ApQ_oz(T(y))m]; m), z) \rightarrow F(T(y), z')$$

which entails, by theorem 2.3

4/ $x \dot{-} y' = z,\ F(([ApQ(x \dot{-} y')T(y)m,\ ApQ_0(x \dot{-} y')T(y)m]; m), z) \rightarrow F(T(y), x \dot{-} y)$

similarly by substituting in (1) we get

5/ $\rightarrow F(T(x), 0)$

We now make the following derivation:

6/ $\rightarrow T(y') \bowtie ([ApQ(x \dot{-} y')T(y)m,$
$ApQ_0(x \dot{-} y')T(y)m]; m)$ axiom of reduction

7/ $x \dot{-} y' = z,\ F(T(y'), x \dot{-} y') \rightarrow F(T(y), x \dot{-} y)$

4, 6, theorem 2.4

8/ $x \dot{-} y = z' \rightarrow x \dot{-} y' = z$ presumably provable

9/ $\rightarrow x \dot{-} y = z'. \supset. F(T(y'), x \dot{-} y') \supset F(T(y), x \dot{-} y)$

7, 8, cut, $\supset$I, $\supset$I.

10/ $x \dot{-} y = 0 \rightarrow F(T(y), x \dot{-} y)$

1, substitution of y for x, theorem 2.3

11/ $\rightarrow x \dot{-} y = x \dot{-} y. \supset. F(T(y'), x \dot{-} y') \supset F(T(y), x \dot{-} y)$

10, $\supset$I, 9, thinning, induction

12/ $F(T(y'), x \dot{-} y') \rightarrow F(T(y), x \dot{-} y)$

11, MPP, cut, cut with $\rightarrow x \dot{-} y = x \dot{-} y$, MPP, cut

13/ $F(T(y), x \dot{-} y),\ F(T(y), x \dot{-} y) \supset F(T(0), x)$

$\rightarrow F(T(0), x)$ MPP

14/ $F(T(y), x \dot{-} y) \supset F(T(0), x)$

$\rightarrow F(T(y'), x \dot{-} y') \supset F(T(0), x)$

12, 13, cut, $\supset I$

15/ $\rightarrow F(T(0), x \dot{-} 0) \supset F(T(0), x)$

theorem 2.3, cut with $\rightarrow x \dot{-} 0 = x$, $\supset I$

16/ $\rightarrow F(T(x), x \dot{-} x) \supset F(T(0), x)$

14, 15, induction

17/ $\rightarrow F(T(0), x)$

from 5 and 17 by various obvious inferences.
Q.E.D.

Theorem 4.5: Let two sequents of the forms

$$\to F(([N, ([X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}, X^{\ulcorner f0 \urcorner}_{g0}]; m)]; n), 0) \qquad (1)$$

and

$$F(([X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}, ([ApQ([X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}, ([X^{\ulcorner f \urcorner}_{g}, z]; m')]; n)(m+n+1),$$
$$ApQ_0([X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}, ([X^{\ulcorner f \urcorner}_{g}, z]; m')]; n)(m+n+1))]; m)]; n), z) \to$$
$$F(([ApP([X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}, z]; n)n, ([X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}, X^{\ulcorner f0 \urcorner}_{g0}]; m)]; n), z') \qquad (2)$$

respectively be provable in TF. Here ApP($[X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}$, z]; m)m has the type-functor $\ulcorner\lambda$x.f(m+x')$\urcorner$ as does N. This means that the type-functor of P must be determined as follows: let h be a primitive recursive functor so that $\to$ h0 = #o and $\to$ hx' = f(m+x') are provable in HAω. Then P has the type-functor $\ulcorner$RL(λx.f(m+x')h$\urcorner$n+1. ApQ$_0$($[X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}$, ($[X^{\ulcorner f \urcorner}_{g}$, z]; m+1)]; n)(m+n+1) must have type-functor $\ulcorner$f0$\urcorner$, so let d be be a functor so that $\to$ d0 = #o and $\to$ dx' = fx are provable, then Q$_0$ has the type-functor $\ulcorner$RL(f0)d$\urcorner$m+n+2, because ($[X^{\ulcorner \lambda x.f(m+x') \urcorner}_{\lambda x.g(m+x')}$, ($[X^{\ulcorner f \urcorner}_{g}$, z]; m+1)]; n) has the type-functor $\ulcorner$RC(d0)(λx.dx')$\urcorner$m+n+1, as the reader may check. Similarly Q must have the type-functor $\ulcorner$RL(λx.fx')d$\urcorner$m+n+2. When the conditions described here are fulfilled, the sequent

$$\to F(([S(z), ([X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}, X^{\ulcorner f0 \urcorner}_{g0}]; m)]; n), z)$$

is also provable, where S(x) is defined as follows:

$$S(x) =_{df.} \rho[\lambda z.\lambda \underline{X}(ApP([\underline{X}, z]; n)n, N, x]$$

Here $\underline{\text{X}}$ has the same type-functor as N and so does S(x).

Proof: Reflection on S will show that S(t) reduces as follows:

$$S(0) = N$$
$$S(s') = (\lambda z.\lambda \underline{X}.(ApP([\underline{X}, z]; n)n)s(S(s)))$$
$$= ApP([S(s), s]; n)n$$

The fact that S(0) reduces to N together with theorem 2.4 entails that (1) entails

$$\rightarrow F(([S(0), ([X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}, X^{\ulcorner f0 \urcorner}_{g0}]; m)]; n), 0) \quad (3)$$

Substituting S(z) for $X^{\ulcorner \lambda x.f(m+x')\urcorner}_{\lambda x.g(m+x')}$ in (2) we get

$$F(([S(z), ([ApQ([S(z), ([X^{\ulcorner f \urcorner}_{g}, z]; m')]; n)(m+n+1),$$

$$ApQ_0([S(z), ([X^{\ulcorner f \urcorner}_{g}, z]; m')]; n)(m+n+1)]; m)]; n), z)$$

$$\rightarrow F(([ApP([S(z), z]; n)n, ([X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}, X^{\ulcorner f0 \urcorner}_{g0}]; m)]; n), z') \quad (4)$$

Again using the fact that S(z') reduces to ApP([S(z), z]; n)n and theorem 2.4, we get

$$\text{antecedent of } 4 \rightarrow F(([S(z'), ([X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}, X^{\ulcorner f0 \urcorner}_{g0}]; m)]; n), z') (5)$$

Now the term-forms ApQ([S(z), ([$X^{\ulcorner f \urcorner}_{g}$, z]; m+1)]; n)(m+n+1) and ApQ$_0$([Sz, ([$X^{\ulcorner f \urcorner}_{g}$, z]; m+1)]; n)(m+n+1) may be rewritten as ApLz([$X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}$, $X^{\ulcorner f0 \urcorner}_{g0}$]; m)m and ApL$_0$z([$X^{\ulcorner \lambda x.fx' \urcorner}_{\lambda x.gx'}$, $X^{\ulcorner f0 \urcorner}_{g0}$]; m)m for some term-forms L and L$_0$. But once sequents (3) and (5) have been re-written according to this device, they turn out to have the same form as hypotheses (1) and (2) of the lemma, so we apply the lemma and get the desired conclusion.

Theorem 4.6: Let two sequents of the forms

$$\rightarrow F(N_1 0, N_2 0, \ldots N_m 0, \underline{Y}_1, \underline{Y}_2, \ldots \underline{Y}_n, 0)$$

and

$$F(\underline{X}_1, \underline{X}_2, \ldots \underline{X}_m, Q_1 \underline{X}_1 \ldots \underline{X}_m \underline{Y}_1 \ldots \underline{Y}_n z, \ldots Q_n \underline{X}_1 \ldots \underline{X}_m \underline{Y}_1 \ldots \underline{Y}_n z, z)$$

$$\rightarrow F(P\underline{X}_1 \ldots \underline{X}_m z, P_2 \underline{X}_1 \ldots \underline{X}_m z, \ldots P_m \underline{X}_1 \ldots \underline{X}_m z, \underline{Y}_1, \underline{Y}_2, \ldots \underline{Y}_n, z')$$

respectively be provable in TF. Then

$$\rightarrow F((Sz)_0, ((Sz)_1)_0, \ldots ((\ldots(Sz)\underbrace{_1 \ldots)_1}_{m-1})_1, \underline{Y}_1, \underline{Y}_2 \ldots \underline{Y}_n, z)$$

is also provable, where S is defined so that

$$S0 = \{N_1 0, \{N_2 0, \ldots \{N_{m-1} 0, N_m 0\} \ldots\}\}$$

$$Sz' = \{(ApP_1(S(z))m)z, \{(ApP_2(Sz)m)z, \ldots \{(ApP_{m-1}(Sz)m)z, (ApP_m(Sz)m)z\} \ldots\}\}$$

$= P_1(Sz)_0(Sz)_{10} \ldots (Sz)\ldots)_1$

that is, it is defined as

$$\lambda x.\rho[\lambda z.\lambda \underline{X}.\{(ApP_1\underline{X}(m \dot{-} 1))z, \{(ApP_2\underline{X}(m \dot{-} 1))z, \ldots \{(ApP_{m-1}\underline{X}(m \dot{-} 1))z,$$
$$(ApP_{m-1}\underline{X}(m \dot{-} 1))z\} \ldots\}\}, \{N_1 0, \{N_2 0, \ldots \{N_{m-1} 0, N_m 0\} \ldots\}\}, x]$$

where $\underline{X}$ has the same type-functor as Sz. The reader may check that, if $P_i(Sz)_0((Sz)_1)_0 \ldots ((\ldots (Sz)_1 \ldots)_1)_1$ is well-formed (which it must be), then so is $ApP_i(Sz)(m \dot{-} 1)$, which, in fact, reduces to it.

Remark: This theorem is familiar from the literature on HAω and is in fact the special case of which the preceding theorem is a generalisation. The proof uses the following lemma, of which lemma 4.4 is a generalisation.

Lemma 4.7: If two sequents with the shapes

$$\rightarrow F(\underline{Y}_1, \underline{Y}_2, \ldots \underline{Y}_n, 0)$$

and

$$F(Q_1\underline{Y}_1 \ldots \underline{Y}_n z, Q_2\underline{Y}_1 \ldots \underline{Y}_n z, \ldots Q_n\underline{Y}_1 \ldots \underline{Y}_n z, z) \rightarrow F(\underline{Y}_1, \ldots \underline{Y}_n, z')$$

are provable in TF, where $Q_i\underline{Y}_1\underline{Y}_2 \ldots \underline{Y}_n$ has the same type-functor as $\underline{Y}_i$, then so is

$$\rightarrow F(\underline{Y}_1, \ldots \underline{Y}_n, z)$$

Proof: We define a term T, containing, $\underline{Y}_1, \ldots \underline{Y}_n$, with the following properties

$$T0 = \{\underline{Y}_1, \{\underline{Y}_2, \ldots \{\underline{Y}_{n-1}, \underline{Y}_n\} \ldots\}\}$$

$$Tz' = \{Q_1(Tz)_0((Tz)_1)_0 \ldots ((\ldots(Tz)\underbrace{_1 \ldots)_1}_{n-1})_1, \{Q_2(Tz)_0((Tz)_1)_0 \ldots ((\ldots(Tz)\underbrace{_1 \ldots)_1}_{n-1})_1, \{, \ldots$$
$$\ldots, \{Q_{n-1}(Tz)_0((Tz)_1)_0 \ldots ((\ldots(Tz)_1 \ldots)_1)_1, Q_n(Tz)_0((Tz)_1)_0 \ldots ((\ldots(Tz)_1 \ldots)_1)_1\} \ldots\}\}$$

That is, we define T as

$$\lambda x.\rho[\lambda z.\lambda \underline{X}.\{Q_1(\underline{X})_0((\underline{X})_1)_0 \ldots ((\ldots(\underline{X}\underbrace{_1 \ldots)_1}_{n-1})_1, \{Q_2(\underline{X})_0((\underline{X})_1)_0 \ldots ((\ldots(\underline{X})_1 \ldots)_1)_1,$$
$$\{, \ldots, Q_n(\underline{X})_0((\underline{X})_1)_0 \ldots ((\ldots(\underline{X})_1 \ldots)_1)_1\} \ldots\}\}, \{\underline{Y}_1, \{, \ldots \underline{Y}_n\} \ldots\}\}, x]$$

The proof now proceeds rather like the proof of the preceding lemma. We prove

$$F((Tz')_0, ((Tz')_1)_0, \ldots ((\ldots(Tz')\underbrace{_1 \ldots)_1)_1}_{n-1}, x \dot{-} z')$$
$$\rightarrow F((Tz)_0, \ldots ((\ldots(Tz)_1 \ldots)_1)_1, x \dot{-} z)$$

which I shall abbreviate to $G(Tz', x \dot{-} z') \rightarrow G(Tz, x \dot{-} z)$. This sequent is derivable from (1) and (2) using properties of T and theorem 2.3 and it gives us

$$G(Tz, x \dot{-} z) \supset G(T0, x) \rightarrow G(Tz', x \dot{-} z') \supset G(T0, x)$$

which we can use as the right-hand premiss of an induction, with

$$\rightarrow G(T0, x) \supset G(T0, x)$$

as the left-hand premiss. The conclusion we derive is,

$$\rightarrow G(Tz, z \dot{-} z) \supset G(T0, z)$$

But the antecedent sub-formula can be got from (1) by substituting $(Tz)_0, ((Tz)_1)_0, \ldots$ $((\ldots(Tz)\underbrace{_1 \ldots)_1}_{n-1})_1$ for $\underline{Y}_1, \underline{Y}_2, \ldots \underline{Y}_n$ in (1). Thus $\rightarrow G(T0, z)$ is provable.

Proof of theorem 4.6: The first premiss can now be rewritten as

$$\rightarrow F((S0)_0, ((S0)_1)_0, \ldots ((\ldots(S0)\underbrace{_1\ldots)_1}_{m-1})_1, \underline{Y}_1, \ldots, \underline{Y}_n, 0)$$

and the second yields, after a substitution

$$F((Sz)_0, ((Sz)_1)_0, \ldots ((\ldots(Sz)_1\ldots)_1)_1, (ApQ_1(Sz)(m \dot{-} 1))\underline{Y}_1\ldots\underline{Y}_n z, \ldots$$
$$(ApQ_n(Sz)(m \dot{-} 1))\underline{Y}_1\ldots\underline{Y}_n z, z) \rightarrow$$
$$F((Sz')_0, ((Sz')_1)_0, \ldots, ((\ldots(Sz')_1\ldots)_1)_1, \underline{Y}_1, \ldots, \underline{Y}_n, z')$$

The terms $(ApQ_i(Sz)(m \dot{-} 1))\underline{Y}_i \ldots \underline{Y}_n z$ may now be re-written as $L_i\underline{Y}_i \ldots \underline{Y}_n z$ for some L_i. When this is done, we now have two sequents to which we can apply the lemma.

Remark (1): Grzegorczyk (1964, p.81) is sometimes credited with being the first logician to show that, in a formulation of HAω in which the operations of pairing and decoding (represented in our formulation by operations 5.2.5-7 upon terms) can be represented, so can all functionals of finite type defined by simultaneous primitive recursion. In constructing the terms S and T used in theorem 4.6 and lemma 4.7 respectively, I have followed his method (loc. cit.) pretty closely. For the actual proofs, on the other hand, I have followed Schuette (1977, pp.128f. and 163f.); Grzegorczyk does not supply a proof.

The reader who has understood how my operations 5.2.10 and 5.2.11 are mere generalizations of the operations of pairing and application will see easily enough that lemma 4.4 and theorem 4.5 are quite simple generalizations of the subsequent lemma and theorem. The term-forms T and S in that lemma and theorem respectively are constructed by a method exactly parallel to Grzegorzyk's, except that I have had more powerful resources to use (in fact I also use my operation 5.2.11 in constructing S in theorem 4.5, but this is just for greater conciseness; it would have been possible to do without it). The proofs themselves are practically identical.

Remark (2): Theorem 4.6 tells us, when we are given functional interpretations of the premisses of an induction inference in HAω, how to construct a functional interpretation of the conclusion and, in particular, how to construct the terms $(Sz)_o$, $((Sz)_1)_o$, . . . $((\ldots \underbrace{(Sz)_1 \ldots)_1)_1}_{m-1}$ which serve as witnesses for the existential quantifiers in the conclusion. Theorem 4.5 resembles an extension of this method to the case where the premisses and the conclusion to be interpreted contain infinitely many quantifiers, but subject to the constraint that all the terms which occupy the places of bound variables in the interpreting sequents can be derived by specifying the same term-form by successive numerals.

CHAPTER 5

PROOF OF THE ACCESSIBILITY OF ε_0

We start by defining some type-functions. Obviously there is a primitive recursive function, which I shall call ν, with the following properties: $\nu(0) = 3$; $\nu(1) = 9$; $\nu(x+2) = 2(\nu(x+1))+3$. There is another, which I shall call ϑ, with the following properties: $\vartheta(0) = 2$; $\vartheta(x+1) = \frac{1}{2}(\nu(x+1) \dot{-} 1)$.

Using notation similar to that employed in chapter 2, I shall stipulate that, where f and g are an m-place and an (m+1)-place function respectively:

$$R^L f g 0 n_1 \ldots n_m = f n_1 \ldots n_m$$

$$R^L f g k' n_1 \ldots n_m = 2^{g k n_1 \ldots n_m} \cdot 3^{R^L f g k n_1 \ldots n_m}$$

I shall now define three two-place primitive recursive functions, f, g_1 and g_2, simultaneously and by cases:

$$f(0,n) = \text{some arbitrary value}$$

$$\begin{aligned}
f(1,n) &= \#0(010)(010)010 && \text{if } 1 \leq n \leq 3 \\
&= \#0(010)(010)011 && \text{if } n = 4 \\
&= \#0 && \text{if } n = 5 \text{ or } 8 \\
&= \#010 && \text{if } n = 6 \text{ or } 7 \\
&= \#1 && \text{if } n = 9 \\
g_1(n,m) &= \#0 && \text{if } m = \vartheta(n) \\
&= 2^{\#0} \cdot 3^{f(n,\, \vartheta(n) \dot{-} m)} && \text{if } 0 \leq m < \vartheta(n)
\end{aligned}$$

$$\begin{aligned}
\mathcal{g}_2(n,m) &= \mathcal{f}(n, \nu(n) \dot{-} m) && \text{if } 0 \leq m \leq \vartheta(n) \\
&= \#0 && \text{if } m = \vartheta(n)+1 \text{ or } \nu(n)+1 \\
&= 2^{\#0} \cdot 3^{\mathcal{f}(n, \nu(n)+1 \dot{-} m)} && \text{if } \vartheta(n)+1 < m \leq \nu(n)
\end{aligned}$$

$$\begin{aligned}
\mathcal{f}(n+2, m) &= R^{L}\left(2^{\#0} \cdot 3^{\mathcal{f}(n+1,m)}\right)(\lambda x.\mathcal{g}_1(n+1,x))(\vartheta(n+1)+2) \\
&\qquad \text{if } 0 < m \leq \vartheta(n+1) \\
&= R^{L}(\#0)(\lambda x.\mathcal{g}_2(n+1,x))(\nu(n+1)+2) \quad \text{if } m = \vartheta(n+1)+1 \\
&= R^{L}(\mathcal{f}(n+1, m \dot{-} 1))(\lambda x.\mathcal{g}_2(n+1,m))(\nu(n+1)+2) \\
&\qquad \text{if } \vartheta(n+1)+1 < m \leq \nu(n+1)+1 \\
&= \#0 \quad \text{if } m = \nu(n+1)+2 \text{ or } \nu(n+1)+\vartheta(n+1)+3 \\
&= 2^{\#0} \cdot 3^{\mathcal{f}(n+1, m \dot{-} \nu(n+1) \dot{-} 2)} \quad \text{if } \nu(n+1)+2 < m \leq \nu(n+1) \\
&\qquad + \vartheta(n+1)+2 \\
&= \mathcal{f}(n+1, m \dot{-} \nu(n+1) \dot{-} 3) \quad \text{if } \nu(n+1)+\vartheta(n+1)+3 \\
&\qquad < m \leq \nu(n+2)
\end{aligned}$$

It can be checked that $\mathcal{f}$ is a primitive recursive function and, except where I have written 'some arbitrary value', takes only code-numbers of types as values. Let d, e, f, g_1 and g_2 be terms of HAω, of types 1, 1, o1, o1 and o1 respectively, which represent $\vartheta, \nu, \mathcal{f}, \mathcal{g}_1$ and $\mathcal{g}_2$ respectively. Then $\ulcorner f \urcorner$, $\ulcorner g_1 \urcorner$ and $\ulcorner g_2 \urcorner$ are type-functors of order 2.

On the basis of the way $\mathcal{f}$ was defined, it will be clear that the type-function represented by $\ulcorner f1 \urcorner$ enumerates the types o(o1o)(o1o)o1o @ 3, o(o1o)(o1o)o11, o, o1o @ 2, o and 1 respectively. $\ulcorner f(n+2) \urcorner$ enumerates types as follows:

$$\ulcorner f(n+2)1\urcorner = o(o\ulcorner f(n+1)1\urcorner)(o\ulcorner f(n+1)2\urcorner)\ldots(o\ulcorner f(n+1)d(n+1)\urcorner)o\ulcorner f(n+1)1\urcorner$$

$$\ulcorner f(n+2)2\urcorner = o(o\ulcorner f(n+1)1\urcorner)(o\ulcorner f(n+1)2\urcorner)\ldots(o\ulcorner f(n+1)d(n+1)\urcorner)o\ulcorner f(n+1)2\urcorner$$

$$\vdots$$

$$\ulcorner f(n+2)d(n+1)\urcorner = o(o\ulcorner f(n+1)1\urcorner)(o\ulcorner f(n+1)2\urcorner)\ldots(o\ulcorner f(n+1)d(n+1)\urcorner)$$
$$o\ulcorner f(n+1)d(n+1)\urcorner$$

$$\ulcorner f(n+2)d(n+1)+1\urcorner = o(o\ulcorner f(n+1)1\urcorner)(o\ulcorner f(n+1)2\urcorner)\ldots(o\ulcorner f(n+1)d(n+1)\urcorner)$$
$$o\ulcorner f(n+1)d(n+1)+1\urcorner\ulcorner f(n+1)d(n+1)+2\urcorner\ldots\ulcorner f(n+1)e(n+1)\urcorner o$$

$$\ulcorner f(n+2)d(n+1)+2\urcorner = o(o\ulcorner f(n+1)1\urcorner)(o\ulcorner f(n+1)2\urcorner)\ldots(o\ulcorner f(n+1)d(n+1)\urcorner)$$
$$o\ulcorner f(n+1)d(n+1)+1\urcorner\ulcorner f(n+1)d(n+1)+2\urcorner\ldots\ulcorner f(n+1)e(n+1)\urcorner\ulcorner f(n+1)d(n+1)+1\urcorner$$

$$\vdots$$

$$\ulcorner f(n+2)e(n+1)+1\urcorner = o(o\ulcorner f(n+1)1\urcorner)(o\ulcorner f(n+1)2\urcorner)\ldots(o\ulcorner f(n+1)d(n+1)\urcorner)$$
$$o\ulcorner f(n+1)d(n+1)+1\urcorner\ulcorner f(n+1)d(n+1)+2\urcorner\ldots\ulcorner f(n+1)e(n+1)\urcorner\ulcorner f(n+1)e(n+1)\urcorner$$

$$\ulcorner f(n+2)e(n+1)+2\urcorner = o$$

$$\ulcorner f(n+2)e(n+1)+3\urcorner = o\ulcorner f(n+1)1\urcorner$$

$$\ulcorner f(n+2)e(n+1)+4\urcorner = o\ulcorner f(n+1)2\urcorner$$

$$\vdots$$

$$\ulcorner f(n+2)e(n+1)+d(n+1)+2\urcorner = o\ulcorner f(n+1)d(n+1)\urcorner$$

$$\ulcorner f(n+2)e(n+1)+d(n+1)+3\urcorner = o$$

$$\ulcorner f(n+2)e(n+1)+d(n+1)+4\urcorner = \ulcorner f(n+1)d(n+1)+1\urcorner$$

$$\ulcorner f(n+2)e(n+1)+d(n+1)+5\urcorner = \ulcorner f(n+1)d(n+1)+2\urcorner$$

$$\vdots$$

$$\ulcorner f(n+2)e(n+2)\urcorner = \ulcorner f(n+1)e(n+1)\urcorner$$

Now I shall investigate some of the properties of the term-forms constructed by rules 4.2.8-12 from chapter 2B. Let us suppose that $M_1, \ldots M_9$ are a nonuple of term-forms having the following type-functors:

$$\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner$$

$$\ulcorner f(v+1)dv+1\urcorner$$

$$\lambda x.\ulcorner f(v+1)ev \dot{-} x\urcorner$$

$$\ulcorner f(v+1)ev+1\urcorner$$

$$\ulcorner f(v+1)ev+2\urcorner$$

$$\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x\urcorner$$

$$\ulcorner f(v+1)ev+dv+3\urcorner$$

$$\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x'\urcorner$$

$$\ulcorner f(v+1)e(v+1)\urcorner$$

Lemma 5.1: For $2 \leq i \leq 4$, the terms

$$\mathrm{Ap}\, M_i\, \{M_5, ([M_6, \{M_7, ([M_8, M_9]; dv)\}]; dv)\}(ev+1)$$

are well-formed and have the type-functors o, $\lambda x.\ulcorner fv(ev \dot{-} x')\urcorner$ and $\ulcorner fvev\urcorner$ respectively.

Proof: Referring to the definition of operation 4.2.11 in chapter 2B, we see that it suffices if, for some g of order 1, (1) M_i has a type-functor equal to $\ulcorner R^L\#o\, g\urcorner(ev+2)$, $\ulcorner R^L(\lambda x.fv(ev \dot{-} x'))g\urcorner(ev+2)$ or $\ulcorner R^L(fvev)g\urcorner(ev+2)$ respectively; (2) the type-functor of

$$\{M_5, ([M_6, \{M_7, ([M_8, M_9]; dv)\}]; dv)\} \qquad (1)$$

is equal to $\ulcorner R^C(g0)(\lambda x.gx')\urcorner(ev+1)$, for the aforesaid g.

If we consult the definition of $\mathfrak{f}$, we see that, since f represents $\mathfrak{f}$,

$$dv+1 < x.\&.x \leq ev+1 \rightarrow f(v+1)ev \dot{-} x \;=\; R^{L}(fv(ev \dot{-} x')) \, g_2 v \, (ev+2)$$

ought to be provable in TF. On the other hand, the type-functor of (1) can be shown to be equal to

$$\ulcorner 3^{f(v+1)ev+2} \cdot 5^{R^{C}\left(3^{f(v+1)ev+dv+3} \cdot 5^{R^{C}(f(v+1)e(v)+1)(\lambda x.f(v+1)e(v+1) \dot{-} x')}\right)(\lambda x.f(v+1)ev+dv+2 \dot{-} x)dv} \urcorner$$

by applying the definition of operation 4.2.11 so as first to determine the type-functor of ([M_8, M_9]; dv) and then working leftwards. By induction on dv, it can be proved that that type-functor is equal to $\ulcorner R^C \, h0 \, (\lambda xhx')\urcorner(ev+1)$, where h is a type-functor of order 2 so defined that the following sequents are provable:

$$\begin{aligned}
&\rightarrow h(v+1)ev+1 = f(v+1)ev+2 \\
dv+1 < x.\&.x \leq ev \; &\rightarrow h(v+1)x = f(v+1)ev+2+x \\
&\rightarrow h(v+1)dv+1 = f(v+1)ev+dv+3 \\
0 < x.\&.x \leq dv \; &\rightarrow h(v+1)x = f(v+1)ev+2+x \\
&\rightarrow h(v+1)0 = f(v+1)e(v+1)
\end{aligned}$$

Consulting the definition of g_2, we now see that

$$0 \leq x.\&.x \leq ev+1 \rightarrow h(v+1)x = g_2vx$$

is provable. Consequently the type-functor of (1) is equal to $\ulcorner R^{C}(g_2v0)(\lambda x.g_2vx')(ev+1)\urcorner$. So the two conditions we set out to satisfy are satisfied, taking g_2v as g.

Remark: I shall henceforth, relative to any given nonuple of the kind described, abbreviate

$$\mathrm{Ap}\ M_i\ \{M_5, ([M_6, \{M_7, ([M_8, M_9]; dv)\}]; dv)\}(ev+1)$$

to $\tilde{M}_2$, $\tilde{M}_3$ or $\tilde{M}_4$ accordingly.

Lemma 5.2: The term-form

$$\mathrm{Ap}\ M_1\ \{M_4\ ([M_6, M_7]; dv)\}\ dv+1$$

is well-formed and has type-functor $\lambda x.o\ \ulcorner fv(dv \dot{-} x)\urcorner$.

Proof: As for 5.1. I shall henceforth abbreviate the term-form in question to $\underset{\sim}{M}_1$.

Definitions 5.3: for every nonuple of the kind described

$([M_1, \{M_2, ([M_3, \{M_4, \{M_5, ([M_6, \{M_7, ([M_8, M_9]; dv)\}]; dv)\}\}]; dv\}]; dv)$,
which has type-functor $\ulcorner R^C(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1) \dot{-} x')\urcorner(e(v+1) \dot{-} 1)$, shall be called simply M and its type-functor shall be abbreviated to $\ulcorner h(v+1)\urcorner$, of order 0.

Remark: We should recall the first theorem on combinatory completeness (from chapter 2, section C2) and the terms $\Lambda X_i^{(v)}M$, for i between 1 and 9, defined there.

We can now begin to construct the term $\lambda vz.\chi_{\mathcal{L}}(v+1, z)$, discussed in chapter 3. As was already announced there, I shall approach the task in stages, first defining a matrix of terms called $\lambda vz.\chi_{\mathcal{L}}(Y)(v+1, z)$, where Y is of the same type-functor as M and where all occurrences of it are, as usual, fully indicated in this notation. Then I shall find an appropriate M to be substituted for Y, to yield the term we are looking for.

Definition 5.4: The term $\lambda Y^{1o}X^{1o}V^{1}z.\chi_{Acc}(Y^{1o}, X^{1o}, V^{1})(z)$ of type (1o)(1o)1oo (from which, when suitable terms of types 1o and a variable of type 1 not occurring in them are plugged in, we get the characteristic term of the accessibility-predicate discussed in chapter 3) is defined to be:

$$\lambda Y^{1o} X^{1o} V^{1} z.\{\overline{sg}(V(XV)) + [V((XV)') \dot{-} V(XV)]\}\cdot$$
$$\{[V(XV) + \overline{sg}(V((XV)'))]\cdot[(\overline{sg}(V0 \dot{-} z))\cdot(V(YV) \dot{-} 0)]\}$$

Lemma 5.5: There is a term of TF with the properties I require of $\chi_{\mathcal{L}}(Y)(v+1, z)$; namely that $\chi_{\mathcal{L}}(M[v:=0])(1, z)$ reduce to

$$\{(\tilde{M}_2 \dot{-} M_5) + \overline{sg}\,\chi_{Acc}((M_6\tilde{M}_2;1),(M_6\tilde{M}_2;0),\tilde{M}_4)(\tilde{M}_2)\}\cdot$$
$$\{(M_7 \dot{-} M_5 + 2^{z})\cdot\chi_{Acc}((\underset{\sim}{M}_1;1),(\underset{\sim}{M}_1;0), M_9)(M_7)\}$$

and that $\chi_{\mathcal{L}}(M[v: = n+1])(n+2, z)$ reduce to

$$\{(\tilde{M}_2 \dot{-} M_5) + \overline{sg}\,\chi_{\mathcal{L}}([M_6\tilde{M}_2, ([\tilde{M}_3, \tilde{M}_4]; d(n+1))]; d(n+1))(n+1, \tilde{M}_2)\}\cdot$$

$$\{(M_7 \dot{-} M_5 + 2^{z})\cdot\chi_{\mathcal{L}}([\underset{\sim}{M}_1, ([M_8, M_9]; d(n+1))]; d(n+1))(n+1, M_7)\}$$

Proof: To get a term that solves these equations, we use operation .2.12. First we define a term-form N, of type-functor $o(\ulcorner h(v+1)\urcorner oo)\ulcorner h(v+2)\urcorner oo$ as follows:

$$\lambda v\, Z^{\ulcorner h(v+1)\urcorner oo}\, y^{\ulcorner h(v+2)\urcorner}\, y.\{(\Lambda X_2^{\widetilde{(v+1)}} y \dot{-} \Lambda X_5^{(v+1)} y) + \overline{sg}\, Z$$
$$([(\Lambda X_6^{(v+1)} y)\Lambda X_2^{\widetilde{(v+1)}} y, ([\Lambda X_3^{\widetilde{(v+1)}} y, \Lambda X_4^{\widetilde{(v+1)}} y]; d(v+1))]; d(v+1))$$
$$(\Lambda X_2^{\widetilde{(v+1)}} y)\}\cdot\{(\Lambda X_7^{(v+1)} y \dot{-} \Lambda X_5^{(v+1)} y + 2^{y})\cdot Z([\Lambda \underset{\sim}{X}_1^{(v+1)} y,$$
$$([\Lambda X_8^{(v+1)} y, \Lambda X_9^{(v+1)} y]; d(v+1))]; d(v+1))(\Lambda X_7^{\widetilde{(v+1)}} y)\}$$

and another, Q, of type-functor $\ulcorner h1 \urcorner oo$, as follows:

$$\lambda y^{\ulcorner h1 \urcorner} y. \{(\widetilde{\Lambda X_2^{(0)} y} \dot{-} \Lambda X_5^{(0)} y) + \overline{sg}\, \chi_{Acc}((((\Lambda X_6^{(0)} y) \widetilde{\Lambda X_2^{(0)} y}; 1),$$
$$((\Lambda X_6^{(0)} y) \widetilde{\Lambda X_2^{(0)} y}; 0), \widetilde{\Lambda X_4^{(0)} y})(\widetilde{\Lambda X_2^{(0)} y})\} \cdot \{(\Lambda X_7^{(0)} y \dot{-} \Lambda X_5^{(0)} y$$
$$+ 2^y) \cdot \chi_{Acc}((\Lambda X_{\underset{\sim}{1}}^{(0)} y; 1), (\Lambda X_{\underset{\sim}{1}}^{(0)} y; 0), \Lambda X_9^{(0)} y)(\Lambda X_7^{(0)} y)\}$$

$\lambda Y^{\ulcorner h(u+1) \urcorner} y.\chi_{\mathcal{L}}(Y^{\ulcorner h(u+1) \urcorner})(u+1, y)$ is now defined to be $R_{\ulcorner h(v+1) \urcorner oo}$ [N, Q, u]. By the definition of operation 4.2.11, this term has the type-functor $\ulcorner h(u+1) \urcorner oo$. We prove by induction on u that $(\lambda Y^{\ulcorner h(u+1) \urcorner} y.\chi_{\mathcal{L}}(Y^{\ulcorner h(u+1) \urcorner})(u+1, y))M[v:=u]z$ has the properties we require of it. When u is 0 it reduces to Q M[v:=0] z, which, considering the definition of Q, reduces by two $\lambda\beta$-conversions to

$$\{(\widetilde{\Lambda X_2^{(0)} M} \dot{-} \Lambda X_5^{(0)} M) + \overline{sg}\, \chi_{Acc}((((\Lambda X_6^{(0)} M) \widetilde{\Lambda X_2^{(0)} M}; 1),$$
$$((\Lambda X_6^{(0)} M) \widetilde{\Lambda X_2^{(0)} M}; 0), \widetilde{\Lambda X_4^{(0)} M})(\widetilde{\Lambda X_2^{(0)} M})\} \cdot \{(\Lambda X_7^{(0)} M \dot{-} \Lambda X_5^{(0)} M$$
$$+ 2^z) \cdot \chi_{Acc}((\Lambda X_{\underset{\sim}{1}}^{(0)} M; 1), (\Lambda X_{\underset{\sim}{1}}^{(0)} M; 0), \Lambda X_9^{(0)} M)(\Lambda X_7^{(0)} M)\}$$

which further reduces, considering that we may replace $\Lambda X_i^{(0)}$ M[v:=0] etc. with M_i etc.,to

$$\{(\tilde{M}_2 \dot{-} M_5) + \overline{sg}\, \chi_{Acc}((M_6 \tilde{M}_2; 1), (M_6 \tilde{M}_2; 0), \tilde{M}_4)(\tilde{M}_2))\} \cdot$$

$$\{(M_7 \dot{-} M_5 + 2^z) \cdot \chi_{Acc}((\underset{\sim}{M}_1; 1), (\underset{\sim}{M}_1; 0), M_9)(M_7)\}$$

which is exactly what we wanted. When u is n+1 for some n, it reduces to $Nn(R_{\ulcorner h(v+1) \urcorner oo}$ [N, Q, n]) M[v:=n+1] z which, applying the induction hypothesis, is equal to Nn $(\lambda Y^{\ulcorner h(n+1) \urcorner} y.\chi_{\mathcal{L}}(Y^{\ulcorner h(n+1) \urcorner})(n+1, y))$ M[v:=n+1] z. If we replace the variables $Y^{\ulcorner h(n+1) \urcorner}$ and y by the rule of α-conversion and then do a couple of $\lambda\beta$-conversions, we get

$$(\lambda y^{\ulcorner h(n+2)\urcorner} y.\{(\Lambda X_2^{\widetilde{(n+1)}} y \dot{-} \Lambda X_5^{(n+1)} y) + \bar{sg}(\lambda Z^{\ulcorner h(n+1)\urcorner} x.\ \chi_{\mathcal{L}}(Z^{\ulcorner h(n+1)\urcorner})$$

$$(n+1, x))([[(\Lambda X_6^{(n+1)} y)\Lambda X_2^{\widetilde{(n+1)}} y, ([\Lambda X_3^{\widetilde{(n+1)}} y, \Lambda X_4^{\widetilde{(n+1)}} y]; d(n+1))];$$

$$d(n+1))(\Lambda X_2^{\widetilde{(n+1)}} y)\}\cdot\{(\Lambda X_7^{(n+1)} y \dot{-} \Lambda X_5^{(n+1)} y + 2^y).$$

$$\lambda Z^{\ulcorner h(n+1)\urcorner} x.\chi_{\mathcal{L}}(Z^{\ulcorner h(n+1)\urcorner})(n+1, x))([\Lambda X_{\underset{\sim}{1}}^{(n+1)} y, ([\Lambda X_8^{(n+1)} y,$$

$$\Lambda X_9^{(n+1)} y]; d(n+1))]; d(n+1))(\Lambda X_7^{(n+1)} y)\})\, M\ z$$

and then, by five more $\lambda\beta$-conversions

$$\{(\Lambda X_2^{\widetilde{(n+1)}} M \dot{-} \Lambda X_5^{(n+1)} M) + \bar{sg}\chi_{\mathcal{L}}([(\Lambda X_6^{(n+1)} M)\Lambda X_2^{\widetilde{(n+1)}} M,$$

$$([\Lambda X_3^{\widetilde{(n+1)}} M, \Lambda X_4^{\widetilde{(n+1)}} M]; d(n+1))]; d(n+1))(n+1, \Lambda X_2^{\widetilde{(n+1)}} M)\}.$$

$$\{(\Lambda X_7^{(n+1)} M \dot{-} \Lambda X_5^{(n+1)} M + 2^2)\cdot \chi_{\mathcal{L}}([\Lambda X_{\underset{\sim}{1}}^{(n+1)} M,$$

$$([\Lambda X_8^{(n+1)} M, \Lambda X_9^{(n+1)} M]; d(n+1))]; d(n+1))(\Lambda X_7^{(n+1)} M)\}$$

which is equivalent, when we simplify the terms $\Lambda X_i^{(n+1)} M[v:=n+1]$ etc., to

$$\{(\tilde{M}_2 \dot{-} M_5) + \bar{sg}\chi_{\mathcal{L}}([M_6\tilde{M}_2, ([\tilde{M}_3, \tilde{M}_4]; d(n+1))]; d(n+1))$$

$$(\tilde{M}_2)\}\cdot\{(M_7 \dot{-} M_5 + 2^2)\cdot\chi_{\mathcal{L}}([\underset{\sim}{M}_1, ([M_8, M_9]; d(n+1))];$$

$$d(n+1))(M_7)\}$$

Lemma 5.6: For any M, the equivalences

$$\rightarrow \chi_{\mathcal{L}}(M[v:=0])(1, z) = 0 :\equiv: \tilde{M}_2 < M_5 \supset Acc((M_6\tilde{M}_2;1), (M_6\tilde{M}_2;0), \tilde{M}_4)(\tilde{M}_2) .\supset. M_7 < M_5 + 2^2 \supset Acc((\underset{\sim}{M}_1;1), (\underset{\sim}{M}_1;0), M_9)(M_7)$$

and

$$\rightarrow \chi_{\mathcal{L}}(M[v:=n+1])(n+2, z) = 0 :\equiv: \tilde{M}_2 < M_5 \supset \chi_{\mathcal{L}}([M_6\tilde{M}_2, ([\tilde{M}_3, \tilde{M}_4];d(n+1))];d(n+1))(\tilde{M}_2) = 0 .\supset. M_7 < M_5 + 2^2 \supset \chi_{\mathcal{L}}([\underset{\sim}{M}_1, ([M_8, M_9];d(n+1))];d(n+1))(M_7) = 0$$

are provable in TF.

Proof: These statements hold in virtue of the way the matrix of terms $\chi_{\mathcal{L}}(Y^{\ulcorner h(v+1)\urcorner})(v+1, z)$ was defined. To verify the first statement, one should substitute 0 for v and then M[v:=0] for $Y^{\ulcorner h1 \urcorner}$. The resulting term will then reduce to a characteristic term of the formula on the right-hand side, considering that $\chi_{Acc}(M^{1o}, N^{1o}, Q^1)(z) = 0$ is also equivalent to $Acc(M^{1o}, N^{1o}, Q^1)(z)$. Similar remarks apply to the second statement. By theorem 4.1 a term related to a formula in the relevant way really will be a characteristic term, that is, the two sequents written above must be provable.

Heuristic Remarks on Lemma 5.6: Where $M_5, \ldots M_9$ are variables, the right-hand sub-formula of the biconditional in the first sequent entails $\mathcal{L}_1'(z)$ in HAω (see chapter 3), so a proof of the sequent consisting of that formula may be taken as a functional interpretation of $\rightarrow \mathcal{L}_1'(z)$ and its conclusion may be identified with $\rightarrow \mathcal{L}_1^*(z)$. In virtue of the equivalence just proved, we also know that a proof of $\rightarrow \chi_{\mathcal{L}}(M[v:=0])(1, z) = 0$ can be transformed into a proof of $\rightarrow \mathcal{L}^{+}_{1}(z)$ in qf.-HAω and hence into a proof of $\rightarrow \mathcal{L}'_{1}(z)$ and therefore of $\rightarrow \mathcal{L}_{1}(z)$ in the full HAω.

The important point about the formula $\chi_{\mathcal{L}}(M[v:=n])(n+1, z) = 0$ is that, where n is a numeral, it entails $\mathcal{L}'_{n+1}(z)$, within HAω, and therefore also $\mathcal{L}_{n+1}(z)$. I shall now indicate briefly how this could be proved. First it should be noted that, where n is a numeral, term-forms of the shape of ([N, M]; dn) reduce to sequences of $\mathcal{C}(n)+1$ terms combined by means of the pairing operation. Therefore a term of the shape of $\chi_{\mathcal{L}}([N_1, ([N_2, N_3]; dn)]; dn)(n, z)$ reduces to a term containing $\nu(n)$ sub-terms obtained by specifying the terms N_1, N_2, N_3.

My definitions of the function ν and the type-function $\mathfrak{f}$ have been motivated by the properties of $\mathcal{L}'_{n}(z)$. It can be calculated (see chapter 3) that it will begin with $\mathcal{C}(n)$ existential followed by $\mathcal{C}(n)+1$ universal quantifiers and that the quantifiers, all together, will have the types $\mathfrak{f}(n, 1)$, $\mathfrak{f}(n, 2)$, $\mathfrak{f}(n, 3)$, ... $\mathcal{L}'_{n+2}(z)$ will be obtained by taking the formula

$$\forall y\{\forall x. x<y \supset \mathcal{L}'_{n+1}(x) :\supset: \forall x. x<y+2^2 \supset \mathcal{L}'_{n+1}(x)\} \qquad (\mathcal{L})$$

and translating it into ∃∀-form. Let us suppose, as an induction hypothesis, that, if n is a numeral,

$$\exists X^{\ulcorner\mathfrak{f}(n+1)1\urcorner}_{e(n+1)\dot{-}1}\ \exists X^{\ulcorner\mathfrak{f}(n+1)2\urcorner}_{e(n+1)\dot{-}2} \ldots \exists X^{\ulcorner\mathfrak{f}(n+1)en+1\urcorner}_{en+2}\ \forall X^{\ulcorner\mathfrak{f}(n+1)en+2\urcorner}_{en+1}\ \forall X^{\ulcorner\mathfrak{f}(n+1)en+3\urcorner}_{en} \ldots, \forall X^{\ulcorner\mathfrak{f}(n+1)e(n+1)\urcorner}_{0} :$$

$$\chi_{\mathcal{L}}(\{X^{\ulcorner\mathfrak{f}(n+1)1\urcorner}_{e(n+1)\dot{-}1}, \{X^{\ulcorner\mathfrak{f}(n+1)2\urcorner}_{e(n+1)\dot{-}2}, \ldots X^{\ulcorner\mathfrak{f}(n+1)e(n+1)\urcorner}_{0}\}\ldots\}\})(n+1, z) = 0$$

is equivalent within HAω to $\mathcal{L}'_{n+1}(z)$. Here $\{X^{\ulcorner\mathfrak{f}(n+1)1\urcorner}_{e(n+1)\dot{-}1}, X^{\ulcorner\mathfrak{f}(n+1)2\urcorner}_{e(n+1)\dot{-}2}, \ldots X^{\ulcorner\mathfrak{f}(n')e(n')\urcorner}_{0}\}$ is the sequence of variables got by replacing $M_1[v:=n], \ldots M_9[v:=n]$ within $M[v:=n]$ with variables of the same type-functors and then reducing the result to its normal form. By this induction hypothesis, $(\mathcal{L})$ is equivalent to

$$\forall y\{\forall x: x<y \supset \exists X^{\ulcorner\mathfrak{f}(n+1)1\urcorner}_{e(n+1)\dot{-}1} \ldots \forall X^{\ulcorner\mathfrak{f}(n+1)e(n+1)\urcorner}_{0} . \chi_{\mathcal{L}}(\{X^{\ulcorner\mathfrak{f}(n+1)1\urcorner}_{e(n+1)\dot{-}1}, \ldots X^{\ulcorner\mathfrak{f}(n+1)e(n+1)\urcorner}_{0}\}\ldots\})$$

$$(n+1, x) = 0 :: \supset :: \forall x : x < y + 2^2 \supset \exists X^{\ulcorner \xi(n+1)1 \urcorner} \ldots \forall X_0^{\ulcorner \xi(n+1)e(n+1) \urcorner} . \chi_{\mathcal{L}}$$

$$(\{X \quad , \ldots . X_0^{\ulcorner \xi(n+1)e(n+1) \urcorner}\})(n+1, x) = 0\}$$

so that the ∃∀-form of this last formula will be equivalent to $\mathcal{L}'_{n+2}(z)$. We now have to consider in detail what the ∃∀-form of this last formula will look like, and what the conclusion of a functional interpretation of it will look like. In fact the latter will be of the shape

$$\tilde{M}_2 < M_5 \supset \chi_{\mathcal{L}}([M_6 \tilde{M}_2, ([\tilde{M}_3, \tilde{M}_4]; d(n+1))]; d(n+1))(n+1, \tilde{M}_2)$$

$$= 0 . \supset . M_7 < M_5 + 2^2 \supset \chi_{\mathcal{L}}([\underset{\sim}{M}_1, ([M_8, M_9]; d(n+1))]; d(n+1))(n+1, M_7) = 0$$

subject to the requirement that $M_5, \ldots M_9$ be variables. This is in virtue of what $([M_6\tilde{M}_2, ([\tilde{M}_3, \tilde{M}_4]; en+1)]; en+1)$ and $([\underset{\sim}{M}_1, ([M_8, M_9]; en+1)]; en+1)$ reduce to, when n is a numeral.

Therefore, if we succed in proving a sequent of the shape of

$$\rightarrow \chi_{\mathcal{L}}(M)(v+1, z) = 0 \qquad (***)$$

where $M_5, \ldots M_9$ are variables, it will be possible to derive from that sequent any one of $\rightarrow \mathcal{L}'_1(z)$, $\rightarrow \mathcal{L}'_2(z)$, $\rightarrow \mathcal{L}'_3(z)$, ... within HAω once the appropriate numeral has been substituted for v. Thus the program set out in chapter 3 shall have been fulfilled. If we achieve this goal, it will suggest that, since $\rightarrow Acc(\omega_2)$, $\rightarrow Acc(\omega_3)$, $\rightarrow Acc(\omega_4)$, ... may be derived from

$\rightarrow \mathcal{L}'_2(0)$, $\rightarrow \mathcal{L}'_3(0)$, $\rightarrow \mathcal{L}'_4(0)$, ... respectively, (***) will entail $\rightarrow Acc(\omega_v)$. To prove (***) is therefore the main task of this chapter and, once it is proved, $\rightarrow Acc(\omega_v)$ follows reasonably easily.

Lemma 5.7: (This yields the basis for an induction inference that has (***) as conclusion) there are terms, which we shall call $M_1, \ldots M_4$, of TF of type-functors ⌜f11⌝, . . . ⌜f14⌝ so that

$$\rightarrow \chi_\alpha(\{M_1, \{M_2, \{M_3, \{M_4, \{X_4^{\ulcorner f15 \urcorner}, \{X_3^{\ulcorner f16 \urcorner}, \ldots X_0^{\ulcorner f19 \urcorner}\}\ldots\}\}\}\}\}\})(1,0) = 0$$

is provable in TF.

Proof: We can take the following four terms for $M_1, \ldots M_4$:

$$M_1 =_{df.} \lambda y X^{olo} Y^{olo}{}_2 X'.D(0, (X^{olo}(X'0)(\lambda v.X'v'))')$$

$$M_2 =_{df.} \lambda y X^{olo} Y^{olo}{}_2 X'.D(0, Y^{olo}(X'0)(\lambda v.X'v'))$$

$$M_3 =_{df.} \lambda y X^{olo} Y^{olo}{}_2 X'.X'0$$

$$M_4 =_{df.} \lambda y X^{olo} Y^{olo}{}_2 X'v.X'v'$$

Here $D(0, (X^{olo}(X^1 0)(\lambda v.X^1 v'))')$ and $D(0, Y^{olo}(X^1 0)(\lambda v.X^1 v'))$, both terms of type o, are of the kind which, in virtue of theorems 4.2-3, we may use to interpret instances of the contraction rules in HAω, the former enabling us to contract the formulae $X^1 0 = 0$ and $X^1(X^{olo}(X^1 0)(\lambda v.X^1 v'))' = 0$ in the succedent position of a sequent, the latter enabling us to contract Decr.$(\lambda X^1.0, X_0)$ and Decr.$(\lambda X^1.X_2 (X^1 0)(\lambda v.X^1 v'), X_0)$ in the antecedent.

I have introduced "Decr.(N^{1o}, Q^1)" (meaning "N functionally interprets the statement that Q of type 1 enumerates a strictly decreasing sequence") as an abbreviation for

$$Q(NQ) > 0 \supset Q((NQ)') < Q(NQ) . \& . Q(NQ) = 0 \supset Q((NQ)') = 0$$

The derivation now goes as follows:

$$1/\ X_0'0 > 0,\ X_0'0 > 0 \supset X_0'1 < X_0 0 . \& . X_0 0 = 0 \supset X_0 1 = 0$$

$$\rightarrow X_0 1 < X_0 0 \qquad \text{MPP, thinning, \&E}$$

2/ $X_0 0 > 0$, $\mathrm{Decr.}\left(\left(\lambda y X^{olo} y^{olo}{}_2 X'.0\right) X_4 \ldots X_1, X_0\right) \rightarrow$

$X_0 1 < X_0 0$ 1, definition of Decr., theorem 2.4

3/ antecedent of 2 $\rightarrow M_4 X_4 \ldots X_0 0 < X_0 0$

2, definition of M_4, theorem 2.4

4/ $X_0 0 < X_4 \rightarrow M_3 X_4 \ldots X_0 0 < X_4$

L, theorem 2.3

5/ $M_3 X_4 \ldots X_0 0 < X_4 :\supset: \mathrm{Decr.}\left(X_2 \left(M_3 X_4 \ldots X_0 0\right), M_4 X_4 \ldots X_0\right)$

$.\supset. M_4 X_4 \ldots X_0 0 < M_3 X_4 \ldots X_0 0 \supset$

$M_4 X_4 \ldots X_0 \left(X_3 \left(M_3 X_4 \ldots X_0 0\right) M_4 X_4 \ldots X_0\right) = 0$,

antecedent of 2, $X_0 0 < 4$, $\mathrm{Decr.}\left(X_2 \left(M_3 X_4 \ldots X_0 0\right), M_4 X_4 \ldots X_0\right)$

$\rightarrow M_4 X_4 \ldots X_0 \left(X_3 \left(M_3 X_4 \ldots X_0 0\right) M_4 X_4 \ldots X_0\right) = 0$

MPP, 4, cut, MPP, cut, 2, theorem 2.4, cut

6/ $M_4 X_4 \ldots X_0 \left(X_2 \left(M_3 X_4 \ldots X_0 0\right) M_4 X_4 \ldots X_0\right) = 0 \rightarrow$

$X_0 \left(\left(\lambda y X^{olo} y^{olo}{}_2 X'. \left(X_2^{olo}\left(X' 0\right)\left(\lambda v. X' v'\right)\right)' X_4 \ldots X_0\right) = 0$

L, theorem 2.4

7/ antecedent of 5 $\rightarrow$ succedent of 6   5, 6, cut

8/ $\rightarrow \left(\left(\lambda y X^{olo} y^{olo}{}_2 X'. Y^{olo}\left(X' 0\right)\left(\lambda v. X' v'\right)\right) X_4 \ldots X_1\right) X_0$

$= \left(X_2 \left(M_3 X_4 \ldots X_0 0\right)\right) M_4 X_4 \ldots X_0$

by reductions and rules on equality

9/ $\mathrm{Decr.}((\lambda y X^{olo} y^{olo}{}_2 X'. y^{olo}(X'0)(\lambda v. X'v'))X_4 \ldots X_1, X_0)$

$\rightarrow \mathrm{Decr.}(X_2(M_3 X_4 \ldots X_0 0), M_4 X_4 \ldots X_0)$

L, 8, theorem 2.3

10/ $\mathrm{Decr.}((\lambda y X^{olo} y^{olo}{}_2 X'. y^{olo}(X'0)(\lambda v. X'v'))X_4 \ldots X_1, X_0)$,
$\mathrm{Decr.}((\lambda y X^{olo} y^{olo}{}_2 X'. 0)X_4 \ldots X_1, X_0)$, $X_0 0 > 0$, $X_0 0 < X_4$,
$M_3 X_4 \ldots X_0 0 < X_4 :\supset: \mathrm{Decr.}(X_2(M_3 X_4 \ldots X_0 0), M_4 X_4 \ldots X_0)$
$.\supset. M_4 X_4 \ldots X_0 0 < M_3 X_4 \ldots X_0 0 \supset$
$M_4 X_4 \ldots X_0 (X_3(M_3 X_4 \ldots X_0 0) M_4 X_4 \ldots X_0) = 0 \rightarrow$
succedent of 7   9, 7 (with interchanges), cut

11/ $\mathrm{Decr.}(M_2 X_4 \ldots X_1, X_0)$, $X_0 0 > 0$, $X_0 0 < X_4$, main antecedent formula of 10 $\rightarrow$ succedent of 10

10, theorem 4.2

12/ $\rightarrow X_0 0 > 0 \vee X_0 0 = 0$ proved in chapter 4

13/ $X_0 0 = 0 \rightarrow X_0((\lambda y X^{olo} y^{olo}{}_2 X'. 0)X_4 \ldots X_0) = 0$

L, theorem 2.3

14/ $X_0 0 > 0 \vee X_0 0 = 0$, $\mathrm{Decr.}(M_2 X_4 \ldots X_1, X_0)$, $X_0 0 < X_4$, main antecedent formula of 11 $\rightarrow$ succedent of 11, succedent of 13   11,13, VE

15/ antecedent of 14 $\to M_1 X_4 \ldots X_1 = 0$

14, theorem 2.4, theorem 4.3

16/ $\mathrm{Decr.}(M_2 X_4 \ldots X_1, X_0)$, main antecedent formula of 15, $X_0 0 < X_4 \to M_1 X_4 \ldots X_1 = 0$

12, 15, cut

17/ $M_3 X_4 \ldots X_0 0 < X_4 \supset \mathrm{Acc}\big(X_3(M_3 X_4 \ldots X_0 0), X_2(M_3 X_4 \ldots X_0 0), M_4 X_4 \ldots X_0\big)(M_3 X_4 \ldots X_0 0) \to \mathrm{Decr.}(M_2 X_4 \ldots X_1, X_0) . \supset . X_0 0 < X_4 \supset M_1 X_4 \ldots X_1 = 0$ 16, definition of Acc, $\supset$I, $\supset$I

From now on let us abbreviate $M_1 X_4 \ldots X_1$ to $\underset{\sim}{M}_1$, $M_2 X_4 \ldots X_1$ to $\underset{\sim}{M}_2$, $M_3 X_4 \ldots X_0$ to $\tilde{M}_3$ and $M_4 X_4 \ldots X_0$ to $\tilde{M}_4$. Then 17 can be rewritten as

$$\tilde{M}_3 0 < X_4 \supset \mathrm{Acc}\big(X_3(\tilde{M}_3 0), X_2(\tilde{M}_3 0), \tilde{M}_4\big)(\tilde{M}_3 0) \to \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_4)$$

18/ $X_1 < X_4 + 2^\circ \rightarrow X_1 < X_4 \vee X_1 = X_4$ property of $<$

19/ $X_0 0 < X_1, X_1 < X_4 \rightarrow X_0 0 < X_4$ similar

20/ $X_1 < X_4, X_0 0 < X_4 \supset X_0(\underset{\sim}{M}_1 X_0) = 0 \rightarrow$
$X_0 0 < X_1 \supset X_0(\underset{\sim}{M}_1 X_0) = 0$ MPP, cut with 19, $\supset$I

21/ $\mathrm{Decr.}(\underset{\sim}{M}_2, X_0), \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_4)$
$X_0 0 < X_4 \supset X_0(\underset{\sim}{M}_1 X_0) = 0$ MPP

22/ antecedent of 21, $X_1 < X_4 \rightarrow X_0 0 < X_1 \supset X_0(\underset{\sim}{M}_1 X_0) = 0$

20, 21, cut

23/ $X_1 < X_4, \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)X_4 \rightarrow \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_1)$

22, $\supset$I

24/ $X_1 = X_4, \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_4) \rightarrow \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_1)$

theorem 2.3

25/ $X_1 < X_4 + 2^\circ, \mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_4) \rightarrow$
$\mathrm{Acc}(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_1)$

23, 24, $\vee$E, cut with 18

26/ $Acc(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_4) \rightarrow$

$X_1 < X_4 + 2^{\circ} \supset Acc(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_1)$

25, interchange, $\supset I$

27/ $\tilde{M}_3 0 < X_4 \supset Acc(X_3(\tilde{M}_3 0), X_2(\tilde{M}_3 0), \tilde{M}_4)$

$(\tilde{M}_3 0) \rightarrow X_1 < X_4 + 2^{\circ} \supset Acc(\underset{\sim}{M}_1, \underset{\sim}{M}_2, X_0)(X_1)$

17, 26, cut

By the second theorem on combinatory completeness, there are terms $\bar{M}$ and $\bar{X}$ so that $(\bar{M};0)$ is equal to M_2, $(\bar{M};1)$ is equal to M_1, $(\bar{X};0)$ is equal to X_2 and $(\bar{X};1)$ to X_3. Then, from 27 by an $\supset$-introduction and several applications of theorem 2.4 involving these equalities, we get

$$\rightarrow \tilde{M}_3 0 < X_4 \supset Acc((\bar{X}(\tilde{M}_3 0);1), (\bar{X}(\tilde{M}_3 0), 0), \tilde{M}_4)$$
$$(X_4) . \supset . X_1 < X_4 + 2^{\circ} \supset Acc((\underset{\sim}{\bar{M}};1), (\underset{\sim}{\bar{M}};0), X_0)(X_1)$$

If we translate this into an equation using lemma 5.6 and then eliminate all occurrences of operation 4.2.11, using theorem 2.4, in favour of 4.2.5, we get the sequent which was to be proved.

Lemma 5.8: There is a term N(v+2), of type-functor $\lambda x.\ulcorner f(v+2)e(v+1)+1 \dot{-} x\urcorner$, in which none of the variables $X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'}$, $X^{\ulcorner f(v+1)e(v+1)\urcorner}_{o}$, $X^{\lambda x.\ulcorner f(v+2)e(v+2)\dot{-}x'\urcorner}_{\lambda x.x'}$ or $X^{\ulcorner f(v+2)e(v+2)\urcorner}_{o}$ occurs, so that the following sequent is provable in TF:

$$\chi_\alpha([X^{\lambda x.\ulcorner f(v+1)d(v+1)\dot{-}x\urcorner}_{\lambda x.e(v+2)+x}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{o}], dv')]; dv')(v+1, 0) = 0$$
$$\rightarrow \chi_\alpha([N(v+2), ([X^{\lambda x.\ulcorner f(v+2)e(v+2)\dot{-}x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+2)e(v+2)\urcorner}_{o}]; d(v+2)]; d(v+2))(v+2, 0) = 0$$

Proof: We start by defining some term-forms. ϕ is short for $\phi(X^{o}_{dv+1}, X^{o}_{ev+1}, z)$, that is, a term of type o containing the three variables indicated, and with the properties that the sequents $\rightarrow 2^{z} < 2^{\phi+1}$ and $z > 0 \rightarrow \phi < z$ are provable in qf.-HAω (that such a term exists was proved by Schuette; see his (1950)). We define $\underset{\cdot}{U}$ of type-functor $\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x\urcorner$ to be

$$Ap X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.ev+dv+4+x}\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x}, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}]; dv)\}(dv+1)$$

and $\underset{\cdot}{X}$ of type-functor o to be

$$Ap X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+3}\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} + 2^{\phi}, ([\underset{\cdot}{U}, \{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1},$$
$$([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{o}]; dv)\}]; dv)\}(ev+1)$$

$\underset{\cdot}{V}_1$ of type-functor $\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x\urcorner$ and $\underset{\cdot}{V}_2$ of type-functor $\ulcorner f(v+1)e(v+1)\urcorner$ are defined like $\underset{\cdot}{X}$, but with $X^{\lambda x.\ulcorner f(v')ev\dot{-}x\urcorner}_{\lambda x.ev+3+x}$ and $X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}$ respectively in place of $X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+3}$. We now construct the following derivation:

$$1/\ \underset{\cdot}{X} < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} + 2^{\phi} \supset \chi_\alpha([\underset{\cdot}{U}\underset{\cdot}{X}, ([\underset{\cdot}{V}_1, \underset{\cdot}{V}_2]; dv)]; dv)(v, \underset{\cdot}{X}) = 0$$
$$.\supset.\ X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1} < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} + 2^{\phi+1} \supset \chi_\alpha([Ap X^{\lambda x.\ulcorner f(v+1)dv\dot{-}x\urcorner}_{\lambda x.ev+dv+4+x}$$
$$\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} + 2^{\phi}, ([\underset{\cdot}{U}, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}]; dv)\}(dv+1), ([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},$$
$$X^{\ulcorner f(v+1)e(v+1)\urcorner}_{o}]; dv)]; dv)(v, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}) = 0$$

, antecedent sub-formula of this formula $\rightarrow$ its succedent Mpp

2/ $\rightarrow 2^2 < 2^{\phi+1}$ property of ϕ

3/ $X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner} < 2^2 \rightarrow X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner} < 2^{\phi+1}$

2, properties of $<$

4/ $X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner} < 2^{\phi+1}, X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner} < 2^{\phi+1} \supset$
$\chi_{\mathcal{L}}(([Ap\, X_{\lambda x.ev+dv+4+x}^{\lambda x.\ulcorner f(v+1)dv \dot{=} x\urcorner}\{X_{ev+1}^{\ulcorner f(v+1)ev+2\urcorner}, ([U, X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner}]; dv)\}dv+1,$
$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x'\urcorner}, X_0]; dv)]; dv)(v, X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner}) = 0$

$\rightarrow \chi_{\mathcal{L}}(([Ap\, X_{\lambda x.ev+dv+4+x}^{\lambda x.\ulcorner f(v+1)dv \dot{=} x\urcorner}\{X_{ev+1}^{\ulcorner f(v+1)ev+2\urcorner}, ([U, X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner}]; dv)\}dv+1,$
$([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x'\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; dv)]; dv)(v, X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner}) = 0$

MPP

5/ $X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner} < 2^2$, main antecedent formula of 4
$\rightarrow$ succedent formula of 4 3, 4, cut

6/ main antecedent formula of 4, $X_{dv+1}^{\ulcorner f(v+1)ev+dv+3\urcorner} < 2^2$,
$\rightarrow$ succedent formula of 4 5, interchange

7/ antecedent of 1 $\rightarrow X_{dv+1} < 2^2 \supset \chi_{\mathcal{L}}(([$ etc.

1, 6, cut, $\supset$I

8/ as 7, but with $2^\phi + 2^\phi$ in place of $2^{\phi+1}$ in the antecedent -- 7, theorem 2.4

9/
$$Ap\, X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4} \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X,$$
$$([V_1, V_2]; dv)\}]; dv)\}ev+1 < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} \supset \chi_\alpha$$
$$([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x}_{\lambda x.dv+2+x} (Ap\, X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4} \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x},$$
$$\{X, ([V_1, V_2]; dv)\}]; dv)\}ev+1\}, ([Ap\, X^{\lambda x.\ulcorner f(v+1)ev \dot- x\urcorner}_{\lambda x.ev+3+x} \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1},$$
$$([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2]; dv)\}]; dv)\}ev+1,$$
$$Ap\, X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2} \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2];$$
$$dv)\}]; dv)\}ev+1]; dv)]; dv)(v, Ap\, X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4} \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1},$$
$$([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2]; dv)\}]; dv)\}ev+1) = 0$$
$$. \supset . X < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} + 2^\phi \supset \chi_\alpha([U X, ([V_1, V_2]; dv)]; dv)$$
$$(v, X) = 0,$$

antecedent sub-formula of the last formula

$\rightarrow$ its succedent sub-formula MPP

10/ antecedent of 9, main antecedent formula of 8

$\rightarrow$ succedent formula of 8 -- 9, 8, cut

We must now inspect the two main antecedent formulae of 10, with a view to contracting them into one by an application of theorem 4.2. Reading the two formulae, and referring to the definitions of $\underset{\cdot}{U}$, $\underset{\cdot}{X}$, $\underset{\cdot}{V}_1$, and $\underset{\cdot}{V}_2$, we see that the two formulae are identical, apart from the fact that there are some five terms occurring in either, each of which differs from the term occurring at the same position in the other formula. The following table shows the five pairs of terms which are responsible for the differences between the two formulae. The left-hand column contains the relevant terms which occur in the main antecedent formula of 8, the right-hand column shows the differing terms occurring at corresponding positions in the main antecedent formula of 9:

$$
\begin{array}{ll}
X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner} + 2^{\phi} & X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner} \\
\underset{\cdot}{U} & X_{\lambda x.dv+2+x}^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner} \\
X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} & \underset{\cdot}{X} \\
X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x' \urcorner} & \underset{\cdot}{V}_1 \\
X_{0}^{\ulcorner f(v+1)e(v+1) \urcorner} & \underset{\cdot}{V}_2
\end{array}
$$

By theorem 4.2, we can construct five new terms, of the same type-functors as the terms in either of these columns, which can be used to replace the terms in both columns. Let these new terms be called $D_1, \ldots D_5$. Then the following sequent will be provable, by an application of theorem 4.2 and a $\supset$-introduction:

$$
\begin{aligned}
11/\ & Ap\, X_{ev+dv+3}^{\ulcorner f(v+1)dv+1 \urcorner}\{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}(ev+1) < D_1 \supset \\
& \chi_\alpha([D_2 (Ap\, X_{ev+dv+3}^{\ulcorner f(v+1)dv+1 \urcorner}\{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}(ev+1)), \\
& ([Ap\, X_{\lambda x.ev+3+x}^{\lambda x.\ulcorner f(v+1)ev \dot{-} x \urcorner}\{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}(ev+1), \\
& Ap\, X_{ev+2}^{\ulcorner f(v+1)ev+1 \urcorner}\{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}(ev+1)]; dv)]; dv) \\
& (v, Ap\, X_{ev+dv+3}^{\ulcorner f(v+1)dv+1 \urcorner}\{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}(ev+1)) = 0 \ :\supset: \\
& D_3 < D_1 + 2^{\phi} \supset \chi_\alpha([Ap\, X_{\lambda x.ev+dv+4+x}^{\lambda x.\ulcorner f(v+1)dv \dot{-} x \urcorner}\{D_1, ([D_2, D_3]; dv)\}(dv+1), \\
& ([D_4, D_5]; dv)]; dv)(v, D_3) = 0 \quad \rightarrow
\end{aligned}
$$

$$Ap\, X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+3}\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2]; dv)\}];$$

$$dv)\}(ev+1) < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} \supset \chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x}_{\lambda x.dv+2+x}(Ap\, X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4}$$

$$\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2]; dv)\}]; dv)\}(ev+1)),$$

$$([Ap\, X^{\lambda x.\ulcorner f(v+1)ev\dot{-}x\urcorner}_{\lambda x.ev+3+x}\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x}, \{X,$$

$$([V_1, V_2]; dv)\}]; dv)\}(ev+1), Ap\, X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([$$

$$X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2]; dv)\}]; dv)\}(ev+1)]; dv)]; dv)$$

$$(v, Ap\, X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+3}\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x}, \{X, ([V_1, V_2];$$

$$dv)\}]; dv)\}(ev+1)) = 0 \quad . \supset .$$

$$X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1} < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} + 2^2 \supset \chi_{\mathcal{L}}([Ap\, X^{\lambda x.\ulcorner f(v+1)dv\dot{-}x\urcorner}_{\lambda x.ev+dv+4+x}$$

$$\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1} +2^{\phi}, ([U, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}]; dv)\}dv+1, ([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},$$

$$X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}]; dv)]; dv)(v, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}) = 0$$

12/ $\chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)dv\dot{-}x\urcorner}_{\lambda x.ev+dv+4+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+3}, ([X^{\lambda x.\ulcorner f(v+1)ev\dot{-}x\urcorner}_{\lambda x.ev+3+x}\{X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}, \{D_1,$

$([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}\}]; dv)\}]; dv)(v+1, \phi) = 0$

$\rightarrow$ succedent of 11 $\qquad$ 11, lemma 5.6, propositional rules

13/ $\phi < z,\ \phi < z \supset \chi_\alpha([X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.ev+dv+5+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4}, ([X^{\lambda x.\ulcorner f(v+1)ev \dot{-} x\urcorner}_{\lambda x.ev+2+x}, \{X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}, \{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}\}]; dv)\}]; dv)$

$(v+1, \phi) = 0 \rightarrow \chi_\alpha([X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.ev+dv+5+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4}$, etc. MPP

14/ $z > 0 \rightarrow \phi < z$

property of ϕ

15/ $z > 0$, main antecedent formula of 13 $\rightarrow$ succedent of 13

13, 14, cut

16/ antecedent of 15 $\rightarrow$ succedent of 12

12, 15, cut

By the first theorem on combinatory completeness, we can construct terms $L_1, \ldots L_4$ so that the sequents ending with the following four equations (respectively) are provable in TF:

$$Ap L_1 \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x\urcorner}_{\lambda x.dv+2+x}, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}]; dv)\}(dv+1)$$

$$\bowtie Ap X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.ev+dv+5+x} \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([\underset{\cdot}{Y}, X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}]; dv)\}(dv+1)$$

$$Ap L_2 \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x\urcorner}_{\lambda x.dv+2+x}, \{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}]; dv)\}]; dv)\}(ev+1) \quad \bowtie$$

$$Ap\, X^{\ulcorner f(v+1)dv+1 \urcorner}_{ev+dv+4} \{ X^{\ulcorner f(v+1)ev+2 \urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner}_{\lambda x.dv+2+x}, \{ \underset{\cdot}{X}, ([\underset{\cdot}{V}_1, \underset{\cdot}{V}_2]; dv)\}]; dv)\}(ev+1)$$

$$Ap\, L_3 \{ X^{\ulcorner f(v+1)ev+2 \urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner}_{\lambda x.dv+2+x}, \{ X^{\ulcorner f(v+1)ev+dv+3 \urcorner}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x^{-}}_{\lambda x.x^{-}}, X_0 \quad]; dv)\}]; dv)\}(ev+1)$$

$$\bowtie\ Ap\, X^{\lambda x.\ulcorner f(v+1)ev \dot{-} x \urcorner}_{\lambda x.ev+3+x} \{ X^{\ulcorner f(v+1)ev+2 \urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner}_{\lambda x.dv+2+x}, \{ \underset{\cdot}{X}, ([\underset{\cdot}{V}_1, \underset{\cdot}{V}_2]; dv)\}]; dv)\}(ev+1)$$

$$Ap\, L_4 \{ X^{\ulcorner f(v+1)ev+2 \urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner}_{\lambda x.dv+2+x}, \{ X^{\ulcorner f(v+1)ev+dv+3 \urcorner}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x^{-} \urcorner}_{\lambda x.x^{-}}, X_0^{\ulcorner f(v+1)e(v+1) \urcorner}]; dv)\}]; dv)\}(ev+1)$$

$$\bowtie\ Ap\, X^{\ulcorner f(v+1)ev+1 \urcorner}_{ev+2} \{ X^{\ulcorner f(v+1)ev+2 \urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv \dot{-} x \urcorner}_{\lambda x.dv+2+x}, \{ \underset{\cdot}{X}, ([\underset{\cdot}{V}_1, \underset{\cdot}{V}_2]; dv)\}]; dv)\}(ev+1)$$

Things can also be arranged so that $L_1 \ldots L_4$ contain none of the variables $X^{\ulcorner f(v+1)ev+2 \urcorner}_{ev+1}$, $X^{\lambda x.\ulcorner f(v+1)ev+2 \dot{-} x}_{\lambda x.dv+2+x}$, $X^{\ulcorner f(v+1)ev+dv+3 \urcorner}_{dv+1}$, $X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x^{-} \urcorner}_{\lambda x.x^{-}}$ or $X_0^{\ulcorner f(v+1)e(v+1) \urcorner}$.

On the basis of these equalities, by multiple applications of theorem 2.4 to sequent 16, we get

$$17/ z>0,\ \phi<z \supset \chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.ev+dv+5+x},\ \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4},\ ([X^{\lambda x.\ulcorner f(v+1)ev \dot{-} x\urcorner}_{\lambda x.ev+3+x},\ \{X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2},\ \{D_1,\ ([D_2,\{D_3,([D_4,D_5];dv)\}];dv)\}\}];dv)\}];dv)(v+1,\phi)=0 \rightarrow$$

$$ApL_2\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1},([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x}_{\lambda x.dv+2+x},\{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1},([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];dv)\}];dv)\}(ev+1) < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}$$

$$\supset \chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x}_{\lambda x.dv+2+x}(ApL_2\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1},([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x},\{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1},([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];dv)\}];dv)\}(ev+1)),$$

$$([ApL_3\{X^{\ulcorner f(v+1)e(v)+2\urcorner}_{ev+1},([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x},\{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1},([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];dv)\}];dv)\}(ev+1),$$

$$ApL_4\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1},([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x},\{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1},([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},X_0^{\ulcorner f(v+1)e(v+1)\urcorner}];dv)\}];dv)\}(ev+1)];dv)];dv)$$

$$(v,ApL_2\{X^0_{ev+1},([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x\urcorner}_{\lambda x.dv+2+x},\{X^0_{dv+1},([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},X_0^1];dv)\}];dv)\})=0 \quad .\supset.$$

$$X^{\ulcorner f(v+1)e(v)+dv+3\urcorner}_{dv+1} < X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}+2^2 \supset \chi_{\mathcal{L}}([ApL_1\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1},([X^{\lambda x.\ulcorner f(v+1)ev+dv+2\dot{-}x}_{\lambda x.dv+2+x},X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}];dv)\}(dv+1),([X^{\lambda x.\ulcorner f(v+1)e(v+1)\dot{-}x'\urcorner}_{\lambda x.x'},X_0];dv)];dv)(v,X^0_{dv+1})$$

$$=0$$

18/ $y < z+1,\ X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} < X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner} + 2^y \rightarrow$

$X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} < X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner} + 2^z$ properties of $<$ and $+$

19/ $X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} < X_{ev+1}^{\ulcorner f(v+1)ev+2} + 2^z,$

$X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} < X_{ev+1}^{\ulcorner f(v+1)ev+2} + 2^z \supset \chi_\alpha([ApL_1\{X_{ev+1}^{0}, ([X_{\lambda x. dv+2+x}^{\lambda x. \ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner},$

$X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner}]; dv)\}dv+1, ([X_{\lambda x. x'}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot{-} x'}, X_0^{\ulcorner f(v+1)e(v+1) \urcorner}]; dv)]; dv)$

$(v, X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner}) = 0 \quad \rightarrow \chi_\alpha([ApL_1$ etc. MPP

20/ $y < z+1,\ X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} < X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner} + 2^y,$

main antecedent formula of 19 $\rightarrow$ succedent of 19

18, 19, cut

21/ $y < z+1,\ X_{dv+1}^{\ulcorner f(v+1)ev+dv+3 \urcorner} < X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner} + 2^z \supset \chi_\alpha([$

$ApL_1\{X_{ev+1}^{\ulcorner f(v+1)ev+2}$ etc. $\rightarrow X_{dv+1}^{\ulcorner f(v+1)ev+dv+3} < X_{ev+1}^{\ulcorner f(v+1)ev+2} + 2^y \supset$

$\chi_\alpha([ApL_1\{X_{ev+1}^{\ulcorner f(v+1)ev+2}$ etc. 20, $\supset I$

22/ antecedent of 17, $y < z+1 \rightarrow$ succedent of 17, but with y in place of z --

by propositional inferences from 17 and 21

Applying the equivalence described in lemma 5.6 to the complicated succedent formula of 22, we get, by propositional inferences from that equivalence and from 22:

23/ $z > 0,\ \phi < z \supset \chi_\delta([X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x}_{\lambda x.ev+dv+5+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4}, ([X^{\lambda x.\ulcorner f(v+1)ev \dot{-} x\urcorner}_{\lambda x.ev+3+x}, \{X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}, \{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)\}\}]; dv)\}]; dv)(v+1, \phi) = 0,\ y < z+1 \rightarrow$
$\chi_\delta([L_1, \{L_2, ([L_3, \{L_4, \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x\urcorner}_{\lambda x.dv+2+x}, \{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_0]; dv)\}]; dv)\}\}]; dv)\}]; dv)(v+1, y) = 0$

24/ $z = 0,\ y < z+1 \rightarrow y = 0$ properties of $<$ and $+$

25/ $y = 0,\ \chi_\delta([X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.e(v+2)+dv+2+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{e(v+2)+dv+1}, ([X^{\lambda x.\ulcorner f(v+1)ev \dot{-} x\urcorner}_{\lambda x.e(v+2)+x'}, \{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot{-} x\urcorner}_{\lambda x.dv+2+x}, \{X^{\ulcorner f(v+1)ev+dv+3}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_0]; dv)\}]; dv)\}\}]; dv)\}]; dv)\}]; dv)(v+1, 0) = 0 \rightarrow \chi_\delta([X^{\lambda x.\ulcorner f(v+1)dv \dot{-} x\urcorner}_{\lambda x.e(v+2)+dv+2+x}, \ldots\ldots dv)(v+1, y) = 0$

theorem 2.3

26/ $z = 0,\ y < z+1$, main antecedent formula of 25 $\rightarrow$ succedent of 25 24, 25, cut

27/ $\rightarrow z = 0 \vee z > 0$ proved in chapter 4

28/ $z = 0 \vee z > 0$, $y < z+1$, main antecedent formula of 26, main antecedent formula of 23, $y < z+1$
$\rightarrow$ succedent of 26, succedent of 23

23, 26, VE

29/ main antecedent formula of 26, $y < z+1$, main antecedent formula of 23 $\rightarrow$ succedent of 26, succedent of 23

27, 28, structural inferences

In virtue of the resemblance between the two succedent formulae of 29, we may apply theorem 4.3 to contract them into one formula. That is, there are four term-forms, say $C_1, \ldots C_4$, of the same type-functors as $L_1, \ldots L_4$, so that the following sequent is provable in TF:

30/ antecedent of 29 $\rightarrow X_\alpha([C_1, \{C_2, ([C_3, \{C_4, \{X_{ev+1}^{\ulcorner f(v+1)ev+2 \urcorner}, ([X_{\lambda x.dv+2+x}^{\lambda x. \ulcorner f(v+1)ev+dv+2 \dot{-} x \urcorner}, \{X_{dv+2}^{\ulcorner f(v+1)ev+dv+3 \urcorner}, ([X_{\lambda x.x}^{\lambda x. \ulcorner f(v+1)e(v+1) \dot{-} x \urcorner}, X_0];$
$dv)\}];dv)\}\}];dv)\}];dv)(v+1, y)$

31/ $\chi_\alpha([X^{\lambda x.\ulcorner f(v+1)dv \dot- x\urcorner}_{\lambda x.e(v+2)+dv+2+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{e(v+2)+dv+1}, ([X^{\lambda x.\ulcorner f(v+1)ev \dot- x\urcorner}_{\lambda x.e(v+2)+x'}, \{X^{\ulcorner f(v+1)ev+1\urcorner}_{e(v+2)},$
$\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot- x'\urcorner}_{\lambda x.x'},$
$X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}];dv)\}];dv)\}\}];dv)\}];dv)(v+1,0)=0$

$\rightarrow \phi < z \supset \chi_\alpha([X^{\lambda x.\ulcorner f(v+1)dv \dot- x\urcorner}_{\lambda x.ev+dv+5+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4}, ([X^{\lambda x.\ulcorner f(v+1)ev \dot- x\urcorner}_{\lambda x.ev+3+x},$
$\{X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}, \{D_1, ([D_2, \{D_3, ([D_4, D_5];dv)\}];dv)\}\}];dv)\}];$
$dv)(v+1,\phi)=0 :\supset: y < z+1 \supset \chi_\alpha([C_1, \{C_2, ([C_3,$
$\{C_4, X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1},$
$([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot- x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}];dv)\}];dv)\}\}];dv)\}];dv)$
$(v+1,y)=0$ 30, interchange, $\supset$I, $\supset$I

I would now like to make some replacements, justified by theorem 2.4, in the above sequent. We saw in section C1 of chapter 2 that the sub-term $\{X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, ([X^{\lambda x.\ulcorner f(v+1)ev+dv+2 \dot- x\urcorner}_{\lambda x.dv+2+x}, \{X^{\ulcorner f(v+1)ev+dv+3\urcorner}_{dv+1}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot- x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}];dv)\}];dv)\}$ may be replaced with $([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot- x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+2)e(v+2)\urcorner}_{0}];d(v+1))$. Similarly the left-hand term of the equation which is the antecedent formula may be replaced with

$$\chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{=} x\urcorner}_{\lambda x.e(v+2)+x}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}]; d(v+1))];$$
$$d(v+1))(v+1, 0)$$

It is now time to simplify the terms occurring in 31 by using the second theorem on combinatory completeness. We can construct two terms, D_3 and D_4, of type-functors $\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x\urcorner$ and $\ulcorner f(v+1)e(v+1)\urcorner$ respectively, so that $([X^{\lambda x.\ulcorner f(v+1)dv \dot{=} x\urcorner}_{\lambda x.ev+dv+5+x}, \{X^{\ulcorner f(v+1)dv+1\urcorner}_{ev+dv+4}, ([X^{\lambda x.\ulcorner f(v+1)ev \dot{=} x\urcorner}_{\lambda x.ev+3+x},$ $\{X^{\ulcorner f(v+1)ev+1\urcorner}_{ev+2}, \{D_1, ([D_2, \{D_3, ([D_4, D_5]; dv)\}]; dv)]\}\}]; dv)\}];$ $dv)$ may be replaced with $([X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{=} x\urcorner}_{\lambda x.ev+2+x}, ([D_3,$ $D_4]; d(v+1))]; d(v+1))$; and another, D_1, of type-functor $\lambda x.\ulcorner f(v+1)ev+1 \dot{=} x\urcorner$, so that $([C_1, \{C_2,$ $([C_3, \{C_4, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}]; d(v+1))\}]; dv)\}]; dv)$ may be replaced with $([D_1, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}];$ $d(v+1))]; d(v+1))$.

As a result of these replacements, we get from 31

$$32/\ \chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{=} x\urcorner}_{\lambda x.e(v+2)+x}, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{=} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}];$$
$$d(v+1))]; d(v+1))(v+1, 0) = 0 \rightarrow$$

$$\phi < z \supset \chi_\alpha([X^{\lambda x.\ulcorner f(v+1)e(v)+1 \dot{-} x\urcorner}_{\lambda x.ev+2+x}, ([\underset{\cdot}{D}_3, \underset{\cdot}{D}_4]; d(v+1))]; d(v+1))$$

$$(v+1, \phi) = 0 :\supset: y < z+1 \supset \chi_\alpha([\underset{\cdot}{D}_1, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x\urcorner}_{\lambda x.x},$$

$$X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; d(v+1))]; d(v+1))(v+1, y) = 0$$

We now make three substitutions (though in two cases they could be regarded just as re-namings of some variable). z is replaced with $X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1}$ and y with $X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}$. Under these substitutions ϕ becomes $\phi(X^{\ulcorner f(v+1)ev+dv+3}_{dv+1}, X^{\ulcorner f(v+1)ev+2\urcorner}_{ev+1}, X^{\ulcorner f(v+1)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1})$, which term shall henceforth be denoted simply by "ϕ". The third substitution is of $X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2 \dot{-} x\urcorner}_{\lambda x.d(v+1)+2+x}$ ϕ for $X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{-} x}_{\lambda x.ev+2+x}$.

I shall henceforth use the names "$\underset{\cdot}{D}_1$", "$\underset{\cdot}{D}_3$" and "$\underset{\cdot}{D}_4$" to denote the terms obtained from $\underset{\cdot}{D}_1$, $\underset{\cdot}{D}_3$ and $\underset{\cdot}{D}_4$ by these substitutions — this usage will lead to no ambiguities, as I have no use for the "$\underset{\cdot}{D}_1$", "$\underset{\cdot}{D}_3$", "$\underset{\cdot}{D}_4$" in their old sense any more.

By the first theorem on combinatory completeness there are terms, which shall be called $\tilde{D}_1$, $\tilde{\phi}$, $\tilde{D}_3$ and $\tilde{D}_4$, having type-functors $\lambda x.\ulcorner f(v+2)d(v+1) \dot{-} x\urcorner$, $\ulcorner f(v+2)d(v+1)+1\urcorner$, $\lambda x.\ulcorner f(v+2)e(v+1) \dot{-} x\urcorner$ and $\ulcorner f(v+2)e(v+1)+1\urcorner$

respectively, so that the four sequents ending in the following equations are provable in TF:

$$Ap\,\tilde{D}_1\{X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1},([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x},X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1}];$$
$$d(v+1))\}d(v+1)+1 \quad \bowtie \underset{\cdot}{D}_1$$

$$Ap\,\tilde{\phi}\{X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1},([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x},\{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1},$$
$$([X^{\lambda x.\ulcorner f(v+2)e(v+2)\dot{-}x\urcorner}_{\lambda x.x},X^{\ulcorner f(v+2)e(v+2)\urcorner}_{0}];d(v+1))\}];d(v+1))\}e(v+1)+1$$
$$\bowtie \phi$$

$$Ap\,\tilde{D}_3\{X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1},([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x},\{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1},$$
$$([X^{\lambda x.\ulcorner f(v+2)e(v+2)\dot{-}x\urcorner},X^{\ulcorner f(v+2)e(v+2)\urcorner}_{0}];d(v+1))\}];d(v+1))\}e(v+1)+1$$
$$\bowtie \underset{\cdot}{D}_3$$

$$Ap\,\tilde{D}_4 \text{ etc.} \quad \bowtie \underset{\cdot}{D}_4$$

Furthermore $\tilde{D}_1$ need contain none of the variables $X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}$, $X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x}$ and $X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1}$ while $\tilde{\phi}$, $\tilde{D}_3$ and $\tilde{D}_4$ contain neither these nor $X^{\lambda x.\ulcorner f(v+2)e(v+2)\dot{-}x\urcorner}_{\lambda x.x}$ nor $X^{\ulcorner f(v+2)e(v+2)\urcorner}_{0}$.

Let the term $([\tilde{D}_1,\{\tilde{\phi},([\tilde{D}_3,\{\tilde{D}_4,\{X^{\ulcorner f(v+1)e(v+1)+2\urcorner}_{e(v+1)+1},$ $([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x},\{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3\urcorner}_{d(v+1)+1},([X^{\lambda x.\ulcorner f(v+2)e(v+2)\dot{-}x\urcorner}_{\lambda x.x},$

$X_0^{\ulcorner f(v+2)e(v+2)\urcorner}];d(v+1))\}];d(v+1))\}\}];d(v+1))\}];d(v+1))$,

which obviously has the type-functor $\ulcorner h(v+2)\urcorner$, be abbreviated to N, the nine components shown being called $N_1, \dots N_9$.

Then from sequent 32, by the substitutions just described and from the equalities just described using theorem 2.4, we get

33/ antecedent of 32 $\rightarrow \tilde{N}_2 < N_5 \supset$

$\chi_\alpha([N_6\tilde{N}_2,([\tilde{N}_3,\tilde{N}_4];d(v+1))];d(v+1))(v+1,\tilde{N}_2)=0$

$.\supset.\ N_7 < N_5+2^0 \supset \chi_\alpha([\underset{\sim}{N}_1,([N_8,N_9];d(v+1))];d(v+1))$

$(v+1,N_7)=0$

By lemma 5.6:

34/ antecedent of 33 $\rightarrow \chi_\alpha(N)(v+2,0)=0$

However, considering the constitution of N, by theorem 2.4 and the second theorem on combinatory completeness, 34 may be replaced by a sequent of precisely the shape that we are looking for.

Heuristic Discussion of Lemmata 5.7 and 5.8: The reader will have noticed that, while the deductions required to prove these two lemmata are quite short and the propositional complexity of the formulae involved is not great, some of the terms occurring in these formulae are very complicated. My choice of terms arises from the need to satisfy the conditions for the applications of theorems 4.2 and 4.3. For example, if we have two formulae, F and G, occurring in the antecedent position of some sequent and we wish to replace them with a single formula, this is only possible if the terms occurring in F stand in a certain relation to the terms occurring in G. In order to construct F and G in such a way that this condition is satisfied, it is sometimes necessary to use quite complicated terms.

I shall now try to summarise the considerations that I have used in constructing these deductions.

It has long since been known that, given a proof of a sequent in HA, there is an algorithmic method for finding a functional interpretation within qf.-HAω of the ∃∀-form of that sequent. I have used these established methods to find the terms $M_1, \ldots M_4$ in the proof of lemma 5.7. Since we know in advance that such terms as $M_1, \ldots M_4$ must exist, I am not now inclined to think that it was really necessary actually to exhibit them. But having discovered them, I thought I might as well include in this thesis the calculations by which I did so, in case they subsequently turn out to be of interest.

Lemma 5.8 is much more remarkable. I have found out the deduction contained in its proof by imagining that I am doing a functional interpretation of a proof in a certain theory of a highly fictitious kind. Let us recall the theory HA" defined in chapter 3. Then the sequent

$$\mathcal{L}^*(n, 0) \to \mathcal{L}^*(n+1, 0)$$

is equivalent to a sequent of HA when (and only when) n is a numeral. The ∃∀-form of the relevant sequent of HA will have the shape

$$\exists \underline{X}_1 \ldots \exists \underline{X}_{\omega(n)} \forall \underline{Y}_1 \ldots \forall \underline{Y}_{\omega(n)+1} . F(\underline{X}_1, \ldots \underline{Y}_{\omega(n)+1}, 0) \to$$
$$\exists \underline{V}_1 \ldots \exists \underline{V}_{\omega(n+1)} \forall \underline{Z}_1 \ldots \forall \underline{Z}_{\omega(n+1)+1} . G(\underline{V}_1, \ldots \underline{Z}_{\omega(n+1)+1}, 0)$$

Let us now imagine that, in place of $\vartheta(n)$ and $\vartheta(n+1)$, I had written names of variables of HA. The resulting expression might be incoherent nonsense -- there may be something illogical about the idea of a sequent containing a variable number of quantifiers -- but it is heuristically valuable. In the first place, the accessibility of ε_0 will be provable in the theory so created. Furthermore, what I have imagined myself to be doing, in proving lemma 5.8, is functionally interpreting a sequent of the kind just described. The functional interpretation would use imaginary terms of the shape of

$$\{M_1, \{M_2, \ldots \{M_{v \dot{-} 1}, M_v\} \ldots \}\}$$

where v is a variable. And then I replace imaginary terms of HAω with genuine terms of TF; for example the above-named imaginary term is replaced with

$$([M, N]; v \dot{-} 1)$$

In summary, by using firstly the standard method for interpreting the ∃∀-forms of sequents provable in HA in qf.-HAω, and secondly the method just described for replacing imaginary terms of qf.-HAω with genuine terms of TF, we can replace an imaginary formula containing a variable number of quantifiers with a genuine formula containing no quantifiers at all. We can do something similar to the propositional connectives, as I have already explained in chapter 3. Using characteristic terms, we can likewise replace an imaginary formula containing a variable number of propositional connectives with a genuine formula containing no propositional connectives.

Lemma 5.9: There is a term $\hat{M}(v+1)$ of type-functor $\lambda x.\ulcorner f(v+1)(ev+1 \dot{-} x)\urcorner$ so that

$$\rightarrow \chi_{\alpha}\left([\hat{M}(v+2), ([X_{\lambda x.x}^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x\urcorner}, X_0^{\ulcorner f(v+2)e(v+2)\urcorner}]; d(v+2))]; d(v+2)\right)(v+2, 0) = 0$$

is provable in TF.

Proof: It suffices if we can define $\widehat{M}(v+1)$ so that it reduces as follows: $\widehat{M}(1)$ reduces to $\bar{M}$, where $\bar{M}$ is any term so defined that $(\bar{M}; i)$ for $1 \leq i \leq 4$ is equal to M_i as used in the proof of lemma 5.7 (by the second theorem on combinatory completeness, there must be such a term). Secondly $\widehat{M}(v+2)$ reduces to

$$N(v+2)[X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{-} x\urcorner}_{\lambda x.e(v+2)+x} := \hat{M}(v+1)]$$

where N(v+2) is the term which appeared in the statement of lemma 5.8. For the sequent we want to derive can be derived by induction in TF from two premisses. The first

$$\rightarrow \chi_{\mathcal{L}}([\hat{M}(1), ([X^{\lambda x.\ulcorner f1(8 \dot{-} x)\urcorner}_{\lambda x.x'}, X^{\ulcorner f1 9\urcorner}_{0}]; d1)]; d1)(1, 0) = 0$$

was proved in lemma 5.7. Let us now take the basic sequent

$$\chi_{\mathcal{L}}([\hat{M}(v+1), ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x'\urcorner}_{\lambda x.x'}, X^{\ulcorner f(v+1)e(v+1)\urcorner}_{0}]; d(v+1))]; d(v+1))$$

$$(v+1, 0) = 0 \quad \rightarrow \quad id.$$

as the left-hand premiss of a cut, of which the right-hand premiss is the sequent got by substituting $\widehat{M}(v+1)$ for $X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{-} x\urcorner}_{\lambda x.e(v+2)+x}$ in the sequent proved in lemma 5.8. We now have the two premisses required for an induction.

We define $\widehat{M}(u+1)$ to be

$$R_{\ulcorner \lambda x.\ulcorner f(v+1)ev+1 \dot{-} x\urcorner}[\lambda v X^{\lambda x.\ulcorner f(v+1)ev+1 \dot{-} x\urcorner}_{\lambda x.e(v+2)+x}.N(v+2), \bar{M}, u]$$

It can now be proved by a metamathematical induction on the values of u that $\hat{M}(u+1)$ has the properties we require of it. $\hat{M}(1)$ reduces to $\overline{M}$; $\hat{M}(n+2)$ reduces to $N(n+2)[X_{\lambda x.e(v+2)+x}^{\lambda x.\ulcorner f(v+1)ev+1 \dot{-} x\urcorner} := \hat{M}(n+1)]$, by the induction hypothesis.

Lemma 5.10: There is a term S(v+1), of type-functor $\lambda x.\ulcorner f(v+1)ev+1 \dot{-} x\urcorner$, so that

$$\rightarrow \chi_{\mathfrak{L}}([S(v+1), ([X_{\lambda x.x}^{\lambda x.\ulcorner f(v+1)e(v+1) \dot{-} x\urcorner}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; d(v+1))]; d(v+1))(v+1, \omega+1) = 0$$

is provable in TF.

Remark: I conjecture that the reader of this thesis, like its author, is becoming tired of reading proofs that are almost fully formalised. From now on I shall merely sketch how, given lemma 5.9, one can produce a functional interpretation in TF of the $\exists\forall$-form of the statement that ε_0 is accessible.

Proof of lemma 5.10: By the first theorem on combinatory completeness together with the equation proved in chapter 2C1, there is a term

$$([N_1, \{N_2, ([N_3, \{N_4, \{N_5, ([N_6, \{N_7, ([N_8, N_9]; d(v+1))\}]; d(v+1))\}\}]; d(v+1))\}]; d(v+1))$$

where N5, . . . N9 are variables (they will in fact be $X_{e(v+1)+1}^{\ulcorner f(v+2)e(v+1)+2}$, $X_{\lambda x.d(v+1)+2+x}^{\lambda x.\ulcorner f(v+2)\urcorner e(v+1)+d(v+1)+2 \dot{-} x\urcorner}$ etc.), which term we shall abbreviate to N, so that the sequent whose provability was established by lemma 5.9 is equivalent to $\rightarrow \chi_{\mathfrak{L}}(N)(v+2, 0) = 0$

By lemma 5.6 this last sequent is equivalent to

$$\tilde{N}_2 < X_{e(v+1)+1}^{\ulcorner f(v+2)e(v+1)+2 \urcorner} \supset \chi_\alpha\left(\left[X_{\lambda x.d(v+1)+2+x}^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2 \dot{-} x \urcorner} \tilde{N}_2, \left([\tilde{N}_3,\right.\right.$$

$$\left.\left.\tilde{N}_4]; d(v+1)\right)\right]; d(v+1)\Big)(v+1, \tilde{N}_2) = 0 \rightarrow X_{d(v+1)+1}^{\ulcorner f(v+2)e(v+1)+d(v+1)+3 \urcorner} <$$

$$X_{e(v+1)+1}^{\ulcorner f(v+2)e(v+1)+2 \urcorner} + 1 \supset \chi_\alpha\left(\left[\underset{\sim}{N}_1, \left([N_8, N_9]; d(v+1)\right)\right]; d(v+1)\right)\left(v+1, X_{d(v+1)+1}^{c}\right) = 0 \quad (1)$$

If we use, in particular, the operation of generalised λ-abstraction (5.2.9), it is clear that we may construct a matrix of formulae F(Y, z), where Y has the type-functor $\ulcorner R^C(f(v+1)e(v+1))(\lambda x.f(v+1)e(v+1) \dot{-} x')(ev)\urcorner$ and all occurrences of Y and z are fully indicated, so that the above sequent is equivalent to

$$F\left(\left([N_6, \{\tilde{N}_2, ([\tilde{N}_3, \tilde{N}_4]; d(v+1))\}]; d(v+1)\right), X_{e(v+1)+1}^{\ulcorner f(v+2)e(v+1)+2 \urcorner}\right)$$

$$\rightarrow F\left(\left([ApN_1([X_{\lambda x.d(v+1)+3+x}^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1 \dot{-} x \urcorner}, X_{d(v+1)+2}^{\ulcorner f(v+2)e(v+1)+d(v+1)+2 \urcorner}];\right.\right.$$

$$d(v+1) \dot{-} 1)(d(v+1) \dot{-} 1), \{X_{d(v+1)+1}^{\ulcorner f(v+2)e(v+1)+d(v+1)+3}, ([X_{\lambda x.x'}^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x'},$$

$$\left.\left.X_0'];d(v+1))\}];d(v+1)\right), X_{e(v+1)+1}^{\ulcorner f(v+2)e(v+1)+2 \urcorner} + 1\right) \quad (2)$$

For example, if we abbreviate $([X_{\lambda x.e(v+1)+1+x}^{\lambda x.\ulcorner f(v+2)\,etc.}, Y]; d(v+1)+1)$ to $\tilde{Y}$, an acceptable definition of F(Y, z) would be

$$\Lambda X_7^{(v)} \tilde{Y} < z \supset \chi_\alpha\left(\left[(\Lambda X_6^{(v)} \tilde{Y})\Lambda X_7^{(v)} \tilde{Y}, \left([\Lambda X_8^{(v)} \tilde{Y},\right.\right.\right.$$

$$\left.\left.\Lambda X_9^{(v)} \tilde{Y}]; d(v+1)\right)\right]; d(v+1)\Big)\left(v+1, \Lambda X_7^{(v)} \tilde{Y}\right) = 0$$

I shall now show that, using the same operation, we may construct a term-form P, of type-functor $\lambda x.\ulcorner f(v+2)d(v+1) \dot{-} x\urcorner$, in which $X^{\lambda x.\ulcorner f(v+2) etc.\urcorner}_{\lambda x.d(v+1)+2+x}$ and $X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}$ do not occur, so that

$$\to Ap\ P([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x}, X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}]; d(v+1))d(v+1)$$

$$\bowtie Ap N_1([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}x\urcorner}_{\lambda x.d(v+1)+3+x}, X^{\ulcorner f(v+2)e(v+1)+d(v+1)+2\urcorner}_{d(v+1)+2}]; d(v+1))d(v+1)$$

is provable in TF. The proof proceeds by our finding a proof of

$$\to Ap([\lambda X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}x\urcorner}_{\lambda x.e(v+2)+d(v+1)+3+x}.\lambda X^{\ulcorner f(v+2)e(v+1)+d(v+1)+2\urcorner}_{e(v+2)+d(v+1)+2}.\lambda X^{\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}z\urcorner}_{e(v+2)+d(v+1)+3+z}.(Ap N_1$$
$$\{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}z\urcorner}_{e(v+2)+d(v+1)+3+z}, ([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}x\urcorner}_{\lambda x.e(v+2)+d(v+1)+3+x}, X^{\ulcorner f(v+2) etc.\urcorner}_{e(v+2)+d(v+1)+2}]; z)\}z+1)]; z)$$
$$([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x}, X^{\ulcorner f(v+2) etc.\urcorner}_{d(v+1)+3+z}]; z+1)(z+1) \quad \bowtie$$

$$Ap N_1([X^{\lambda x.\ulcorner f(v+2)+e(v+1) etc.\urcorner}_{\lambda x.d(v+1)+3+x}, X^{\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}z\urcorner}_{d(v+1)+3+z}]; z+1)(z+1)$$

in which we then substitute $d(v+1) \dot{-} 1$ for z. I get this, in turn, by proving

$$\to Ap([\lambda X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}x\urcorner}_{\lambda x.e(v+2)+d(v+1)+3+x}.\lambda X^{\ulcorner f(v+2)e(v+1)+d(v+1)+2\urcorner}_{e(v+2)+d(v+1)+2}.\lambda X^{\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}z\urcorner}_{e(v+2)+d(v+1)+3+z}.$$
$$(Ap N_1\{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}z\urcorner}_{e(v+2)+d(v+1)+3+z}, ([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}(z\dot{-}y)\dot{-}x}_{\lambda x.d(v+1)+3+(z\dot{-}y)+x}, ([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+1\dot{-}x\urcorner}_{\lambda x.e(v+2)+d(v+1)+3+(z\dot{-}y)+x},$$
$$X^{\ulcorner f(v+2)e(v+1)+d(v+1)+2\urcorner}_{e(v+2)+d(v+1)+2}]; (z\dot{-}y))]; z\dot{-}(z\dot{-}y))\}(z+1))]; z\dot{-}y)$$
$$([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2\dot{-}x\urcorner}_{\lambda x.d(v+1)+2+x}, X^{\ulcorner f(v+2) etc.\urcorner}_{d(v+1)+3+z}]; (z\dot{-}y)+1)(z\dot{-}y+1)$$

$$\bowtie Ap N_1 \text{ etc.}$$

for, if this is proved for the case where y = 0, the equation becomes equivalent to the theorem to be proved, using an axiom of reduction and theorem 2.4.

We get the last-written sequent by induction on $z \dot{-} y$. Finding a derivation of the two inductive premisses requires little more, so far as I can see, than a knowledge of the reduction-rules of TF.

I take it that by a similar argument one can show that there are terms Q_1, Q_2 and Q_3 so that the following three sequents are provable:

$$\rightarrow Ap\,Q_1([X^{\ulcorner\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}]; e(v+1)+1)(e(v+1)+1)$$

$$\bowtie\ \tilde{N}_2$$

$$\rightarrow Ap\,Q_2([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}]; e(v+1)+1)e(v+1)+1$$

$$\bowtie\ \tilde{N}_3$$

$$\rightarrow Ap\,Q_3 \quad \text{etc.}$$

$$\bowtie\ \tilde{N}_4$$

Then sequent (2) is equivalent to

$$F(([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1) \dot{-} x\urcorner}_{\lambda x.d(v+1)+2+x}, \{Ap\,Q_1([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}];$$
$$e(v+1)+1)(e(v+1)+1), ([Ap\,Q_2([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}];$$
$$e(v+1)+1)(e(v+1)+1), Ap\,Q_3([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x\urcorner}_{\lambda x.x}, X^{\ulcorner f(v+2)e(v+1)+2\urcorner}_{e(v+1)+1}];$$

$$e(v+1)+1)(e(v+1)+1)]; d(v+1))\}]; d(v+1)), X^{\ulcorner f(v+2)e(v+1)+2 \urcorner}_{e(v+1)+1}) \rightarrow$$

$$F(([App([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2 \dot{-} x \urcorner}_{\lambda x.d(v+1)+2+x}, X^{\ulcorner f(v+2)e(v+1)+2 \urcorner}_{e(v+1)+1}]; d(v+1))$$

$$(d(v+1)), \{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3 \urcorner}_{d(v+1)+1}, ([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x \urcorner}_{\lambda x.x},$$

$$X^{\ulcorner f(v+2)e(v+2) \urcorner}_{0}]; d(v+1))\}]; d(v+1)), X^{\ulcorner f(v+2)e(v+1)+2 \urcorner}_{e(v+1)+1} + 1)$$

At the same time, it is clear that

$$\rightarrow F(([X^{\lambda x.\ulcorner f(v+2)e(v+1)+d(v+1)+2 \dot{-} x \urcorner}_{\lambda x.d(v+1)+2+x}, \{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3 \urcorner}_{d(v+1)+1},$$

$$([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x \urcorner}_{\lambda x.x}, X^{\ulcorner f(v+2)e(v+2) \urcorner}_{0}]; d(v+1))\}]; d(v+1)), 0)$$

is provable in TF (because $X^{\ulcorner f(v+2) etc. \urcorner}_{d(v+1)+1} < 0 \rightarrow$ is provable). Thus we have the two premisses required for an application of theorem 4.5, of which the result is

$$z < \omega \rightarrow F(([S(z), \{X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3 \urcorner}_{d(v+1)+1}, ([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x \urcorner}_{\lambda x.x},$$

$$X^{\ulcorner f(v+2)e(v+2) \urcorner}_{0}]; d(v+1))\}]; d(v+1)), z)$$

where S is constructed according to the prescription given in the proof of theorem 4.5. The above sequent is equivalent to

$$\rightarrow z < \omega \supset \chi_{\alpha}([S(z) X^{\ulcorner f(v+2)e(v+1)+d(v+1)+3 \urcorner}_{d(v+1)+1}, ([X^{\lambda x.\ulcorner f(v+2)e(v+2) \dot{-} x \urcorner}_{\lambda x.x},$$

$$X^{\ulcorner f(v+2)e(v+2) \urcorner}_{0}]; d(v+1))]; d(v+1))(v+1, z) = 0$$

in virtue of the way F was defined. We now want four term-forms, s, C, D and D_0, the last three having the type-functors $\lambda x.\ulcorner f(v+1)e(v)+1 \dot- x\urcorner$, $\lambda x.\ulcorner f(v+1)e(v+1) \dot- x\urcorner$ and $\ulcorner f(v+1)e(v+1)\urcorner$, so that

$$s < y \supset \chi_{\mathcal{L}}([X^{\lambda x.\ulcorner f(v+1)ev+1 \dot- x\urcorner}_{\lambda x.ev+2+x}, ([D, D_0]; d(v+1))]; d(v+1))(v+1, s) = 0$$

$$\to \chi_{\mathcal{L}}([C, ([X^{\lambda x.\ulcorner f(v+1)e(v+1) \dot- x\urcorner}_{\lambda x.x}, X_0^{\ulcorner f(v+1)e(v+1)\urcorner}]; d(v+1))]; d(v+1))(v+1, y) = 0$$

is provable. In fact, a sequent having nearly this shape (sequent 17) appeared in the proof of lemma 5.8, so I do not think it is necessary to repeat the proof. We make some substitutions in the last two sequents and then do a cut. We then take sequent (1) above, make some substitutions and do another cut, thus getting the sequent which was to be proved.

Remark: this lemma concludes what I described in chapter 3 as the "third section" of the proof of the accessibility of ε_0. That is, I have now found the sequent of TF which corresponds to (*) of HA' and (**) of HA".

Lemma 5.11: There is a term Q(u) having type-functor $\lambda x.\ulcorner f(v+1 \dot- u)e(v \dot- u)+1 \dot- x\urcorner$ and in which all occurrences of u are indicated, so that

$$\chi_{\mathcal{L}}([Q(u), ([O^{\lambda x.\ulcorner f(v+1 \dot- u)e(v+1 \dot- u) \dot- x\urcorner}, X_0^{\ulcorner f(v+1 \dot- u)e(v+1 \dot- u)\urcorner}]; d(v+1 \dot- u))];$$
$$d(v+1 \dot- u))(v+1 \dot- u, 2_u(\omega+1)) = 0 \to$$
$$\chi_{\mathcal{L}}([Q(u+1), ([O^{\lambda x.\ulcorner f(v \dot- u)e(v \dot- u) \dot- x\urcorner}, X_0^{\ulcorner f(v \dot- u)e(v \dot- u)\urcorner}]; d(v \dot- u))];$$
$$d(v \dot- u))(v \dot- u, 2_{u+1}(\omega+1)) = 0$$

is provable in TF. Here $0\lambda x.\ulcorner f(v+1\dot{-}u)e(v+1\dot{-}u)\dot{-}x'\urcorner$ and $0\lambda x.\ulcorner f(v\dot{-}u)e(v\dot{-}u)\dot{-}x'\urcorner$ are new terms, closed apart from the variables indicated, whose properties will be described below.

Proof: In virtue of lemma 5.6, the formula

$$\chi_\alpha([N_1, \{N_2, ([N_3, \{N_4, \{N_5, ([N_6, \{N_7, ([N_8, N_9]; d(v\dot{-}u))\}]; d(v\dot{-}u))\}\}]; d(v\dot{-}u))\}]; d(v\dot{-}u))(v+1\dot{-}u, 2_u(\omega+1)) = 0$$

is equivalent to

$$\tilde{N}_2 < N_5 \supset \chi_\alpha([N_6\tilde{N}_2, ([\tilde{N}_3, \tilde{N}_4]; d(v\dot{-}u))]; d(v\dot{-}u))(v\dot{-}u, \tilde{N}_2) = 0 \rightarrow N_7 < N_5 + 2_u(\omega+1) \supset \chi_\alpha([\underset{\sim}{N}_1, ([N_8, N_9]; d(v\dot{-}u))]; d(v\dot{-}u))(v\dot{-}u, N_7)$$

So, substituting 0 for N_5 we get

$$\chi_\alpha([N_1, \{N_2, ([N_3, \{N_4, \{0, ([N_6, \{N_7, ([N_8, N_9]; d(v\dot{-}u))\}]; d(v\dot{-}u))\}\}]; d(v\dot{-}u))\}]; d(v\dot{-}u))(v+1\dot{-}u, 2_u(\omega+1)) = 0 \rightarrow \chi_\alpha([ApN_1\{0, ([N_6, N_7]; d(v\dot{-}u))\}(d(v\dot{-}u)+1), ([N_8, N_9]; d(v\dot{-}u))]; d(v\dot{-}u))(v\dot{-}u, 2_{u+1}(\omega+1)) = 0$$

It is at this point that I make use of the terms $0\lambda x.\ulcorner f(v+1\dot{-}u)e(v+1\dot{-}u)\dot{-}x'\urcorner$ and $0\lambda x.\ulcorner f(v\dot{-}u)e(v\dot{-}u)\dot{-}x'\urcorner$ etc. All that is important about the way they are defined is that, when a substitution for the variables v or u and a plugging in of some argument into the type-functor makes the type-functor equal to o, the term should be equal to 0. Then, if we substitute terms of this kind for some of the variables, the result of the substitution is

$$\chi_{\alpha}([N_1, \{N_2, ([N_3, \{N_4, ([O^{\lambda x.\ulcorner f(v+1\dot- u)e(v+1\dot- u)\dot- x\urcorner}, X_0^{\ulcorner f(v+1\dot- u)e(v+1\dot- u)\urcorner}];$$
$$d(v+1\dot- u)\}]; d(v\dot- u))\}]; d(v\dot- u))(v+1\dot- u, 2_u(\omega+1)) = 0$$
$$\rightarrow \chi_{\alpha}([Ap N_1 ([O^{\lambda x.\ulcorner f(v+1\dot- u)e(v\dot- u)+d(v\dot- u)+2\dot- x\urcorner}, 0]; d(v\dot- u)+1)$$
$$(d(v\dot- u)+1), ([O^{\lambda x.\ulcorner f(v\dot- u)e(v\dot- u)\dot- x\urcorner}, X_0^{\ulcorner f(v\dot- u)e(v\dot- u)\urcorner}]; d(v\dot- u))];$$
$$d(v\dot- u))(v\dot- u, 2_{u+1}(\omega+1)) = 0$$

The lemma is therefore proved if we can define Q(u) so that Q(0) is equal to S(v+1) (from the previous lemma) and, if

$$([Q(u), ([O^{\lambda x.\ulcorner f(v+1\dot- u)e(v+1\dot- u)\dot- x\urcorner}, X^{\ulcorner f(v+1\dot- u)e(v+1\dot- u)\urcorner}]; d(v+1\dot- u))], d(v+1\dot- u))$$

is equal to

$$([Q_1(u), \{Q_2(u), ([Q_3(u), \{Q_4(u), ([O^{\lambda x.\ulcorner f(v+1\dot- u)e(v+1\dot- u)\dot- x\urcorner},$$
$$X_0^{\ulcorner f(v+1\dot- u)e(v+1\dot- u)\urcorner}]; d(v+1\dot- u))\}]; d(v\dot- u))\}]; d(v\dot- u))$$

then Q(u+1) is equal to

$$Ap\, Q_1(u)([O^{\lambda x.\ulcorner f(v+1\dot- u)e(v\dot- u)+d(v\dot- u)+2\dot- x\urcorner}, 0]; d(v\dot- u)+1) d(v\dot- u)+1$$

These provisions can be realised by defining Q(u) to be

$$R_{\ulcorner \lambda x.f(v+1\dot- z)d(v+1\dot- z)\dot- x\urcorner}[\lambda z.\lambda X^{\lambda x.\ulcorner f(v+1\dot- z)d(v+1\dot- z)\dot- x\urcorner}.(Ap(\Lambda X_1^{(v\dot- z)}$$

$$([X^{\lambda x.\ulcorner f(v+1\dot{-}z)d(v+1\dot{-}z)\dot{-}x\urcorner}, ([X^{\lambda x.\ulcorner f(v+1\dot{-}z)e(v+1\dot{-}z)\dot{-}x\urcorner}, X^{\ulcorner f(v+1\dot{-}z)e(v+1\dot{-}z)\urcorner}]; d(v+1\dot{-}z))];$$

$$d(v+1)\dot{-}z))([0^{\lambda x.\ulcorner f(v+1\dot{-}z)e(v\dot{-}z)+d(v\dot{-}z)+2\dot{-}x\urcorner} 0]; d(v\dot{-}z)+1)(d(v\dot{-}z)+1), S(v+1), u]$$

Here the two main terms to which the operation is applied are of the following type-functors respectively: (1) $o^{\ulcorner}\lambda x.f(v+1\dot{-}z)d(v+1\dot{-}z)\dot{-}x^{\urcorner\ulcorner}\lambda x.f(v\dot{-}z)d(v\dot{-}z)\dot{-}x^{\urcorner}$, which is equal to $o((\lambda zx.^{\ulcorner}f(v+1\dot{-}z)d(v+1\dot{-}z)\dot{-}x)z)(((\lambda zx.\ f(v+1\dot{-}z)d(v+1\dot{-}z)\dot{-}x^{\urcorner})z')$; (2)$\lambda x.^{\ulcorner}f(v+1)d(v+1)\dot{-}x^{\urcorner}$, which is equal to $(\lambda zx.^{\ulcorner}f(v+1\dot{-}z)d(v+1\dot{-}z)\dot{-}x^{\urcorner})0$. The term just defined is therefore well-formed and has the type-functor $\lambda x.^{\ulcorner}f(v+1\dot{-}u)d(v+1\dot{-}u)\dot{-}x^{\urcorner}$.

Conclusion to chapter 5: We can now take the two sequents whose provability was established in lemmata 5.10 and 5.11 respectively and use them as premisses of an induction-inference. Introducing v in place of u in the conclusion, we get

$$\rightarrow \chi_\alpha([Q(v+1), \{0, \{0^{\ulcorner f16\urcorner}, \{0^{\ulcorner f17\urcorner}, \{0^{\ulcorner f18\urcorner}, X_0^{\ulcorner f19\urcorner}\}\}\}\}]; d(1))$$
$$(1, 2_v(\omega+1)) = 0$$

If we expand the last formula here in accordance with lemma 5.6, it is obvious that we get a formula from which a functional interpretation of the $\exists\forall$-form of the accessibility-statement discussed in chapter 3 can be derived.

CHAPTER 6

THE COMPUTABILITY OF THE TERM-FORMS

That every term of TF can be reduced to a normal form is established by a very simple generalisation of methods standardly used to prove the same result for HAω. For an example of a proof of the latter kind, I refer to Troelstra (1973, p.103f.).

I shall define a concept of computability for terms of TF. First, though, it is necessary to explain what I mean by the <u>applicative complexity</u> of a term. If the type-functor of the term is not of the form $\ulcorner \ulcorner f \urcorner \ulcorner g \urcorner \urcorner$ i.e., not equal to $2^{\ulcorner f \urcorner} \cdot 3^{\ulcorner g \urcorner}$ for some f and g, its applicative complexity is 0. If the type-functor is equal to $\ulcorner \ulcorner f \urcorner \ulcorner g \urcorner \urcorner$, for some f and g, then its applicative complexity is 1 greater than the sum of that of any term with the type-functor $\ulcorner f \urcorner$ and any term with the type-functor $\ulcorner g \urcorner$.

I do not know whether the applicative complexity of a term can be effectively established. I do not think that this matters, though. Unless I know that the applicative complexity of M is greater than 0, I will not imagine that there is such a term as MN, for any N. So the following computability-proof at least establishes the existence of a normal form for every term that anyone will ever actually form. Computability is defined as follows: if the applicative complexity of a term is 0, it is computable if and only if it has a normal form. If the term has a type-functor of the shape $\ulcorner \ulcorner f \urcorner \ulcorner g \urcorner \urcorner$, then the term is computable if and only if it has a normal form and, when applied to a computable term of type-functor $\ulcorner f \urcorner$, the result is a computable term. This definition is well-founded because the computability of a term of applicative complexity n is defined in terms of the computability of terms having a lower applicative complexity.

To show that every term has a normal form it obviously suffices to show that every term is computable. Predictably, the proof proceeds by showing first that the primitive terms are computable and then that the property of computability is preserved under each of the operations by which molecular terms are formed from others.

0 is obviously computable. Every variable is already in normal form. Let $\underline{X}$ be a variable and let $M_1, \ldots M_k$ be a sequence of computable terms so that $\underline{X}M_1 \ldots M_k$ has

the applicative complexity 0. Let the normal forms of $M_1, \ldots M_k$ be $M_1', \ldots M_k'$; then examination of the forms of redexes will show that $\underline{X}M_1' \ldots M_k'$ neither is nor contains a redex. Therefore $\underline{X}$ is computable.

I shall now run through the operations by which terms are formed from other terms. Obviously the successor and application operations transform computable terms into computable terms. $\lambda\underline{X}.M$ will be computable if and only if every term obtained from M by substituting a computable term for all occurrences of $\underline{X}$ is computable, so the problem reduces to showing that computability is invariant under the other operations.

That $\rho[M, N, t]$ is computable if its immediate components are can be shown by familiar methods. Computability is also obviously invariant under operations 4.2.5-7.

Let M and m be computable; then m has a normal form, say $n\underbrace{''\cdots'}_{k}$, where n is not a successor. Then $([\lambda\underline{X}.M]; m)$ reduces to $\lambda\underline{Y}_1 \ldots \lambda\underline{Y}_k.([\lambda\underline{X}.M]; n)$. Here $([\lambda\underline{X}.M]; n)$ has the applicative complexity 0. The task therefore again reduces to showing that that term remains computable whenever computable terms are substituted for the variables $\underline{Y}_1, \ldots \lambda\underline{Y}_k$ wherever they occur in M.

$([M, N]; n)$ is equal to N if n is equal to 0; otherwise it has the applicative complexity 0, so it is computable if M, N and n are.

Let the normal form of m be $n\underbrace{''\cdots'}_{k}$ again. We prove by induction on k that ApMNm is computable if M and N are. If k is 0, ApMNm is either equal to MN or else has the applicative complexity 0. If k is equal to l+1, N is equal to a pair of computable terms and ApMNm reduces to $Ap(MN_0)N_1(n''\cdots')$. But from the assumptions that M and N are computable, it follows that MN_0 and N_1 are.

$R^{\ulcorner}{}_{f\underline{v}}{}^{\urcorner}[\lambda\underline{v}.M, N, m]$ is treated pretty similarly to $\rho[M, N, t]$.

The existence of normal forms for all terms of type o entails that TF is consistent, by a familiar argument (Schuette 1977, p.116f.).

It is difficult to extract any precise additional information, of the sort that can be expressed by numbers, from the above computability-proof. However I think it shows that TF is as acceptable a theory, to a constructivist, as HAω. The computability-predicate used in the proof of the computability of the terms of HAω that I alluded to earlier is Π^1_1, in terms of the recursion-theoretic hierarchies (for a proof of this, see Troelstra (1973, p.119f.) and Hinman (1978, p.82)). The computability-predicate I have defined in this chapter, although rather different, does not appear to be any more complicated.

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