# Integration on Surreal Numbers

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# Abstract

The thesis concerns the (class) structure No of Conway's surreal numbers. The main concern is the behaviour on No of some of the classical functions of real analysis, and a definition of integral for such functions.

In the main texts on No, most definitions and proofs are done by transfinite recursion and induction on the complexity of elements. In the thesis I consider a general scheme of definition for functions on No, generalising those for sum, product and exponential. If a function has such a definition, and can live in a Hardy field, and satisfies some auxiliary technical conditions, one can obtain in No a substantial analogue of real analysis for that function. One example is the sign-change property, and this (applied to polynomials) gives an alternative treatment of the basic fact that No is real closed. I discuss the analogue for the exponential.

Using these ideas one can define a generalisation of Riemann integration (the indefinite integral falling under the recursion scheme). The new integral is linear, monotone, and satisfies integration by parts.

For some classical functions (e.g. polynomials) the integral yields the traditional formulae of analysis. There are, however, anomalies for the exponential function. But one can show that the logarithm, defined as the inverse of the exponential, is the integral of 1/x as usual.

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# Contents

# Introduction

Surreal numbers were discovered by J. Conway and described in the $0^{\text{th}}$ part of his book [6]. He showed that from a remarkably simple set of rules is possible to extract a rich algebraic structure, the class **No** of surreal number. It is, among other things, an elementary extension of $\mathbb{R}_{an}$, the structure given by the reals with $+, \cdot, <$ and restricted analytic functions. Later, H. Gonshor (see [10]) and M. Kruskal added a full exponential function, making **No** an elementary extension of $\mathbb{R}_{an}(\exp)$. Moreover, **No** contains, in a natural way, all ordinal numbers; therefore, it is possible to give meaning to expressions such as $\sqrt{\omega - 1}$, and much more.

Central to both [6] and [10] are the notion of complexity of a surreal number, and the idea of defining functions on **No** by transfinite recursion over the complexity of the argument; the value of such a function $f$ at a point $x$ is determined by the value of $f$ at simpler points. Strictly correlated is the notion of uniformity of such a definition.

One of the aims of this thesis is to give a precise meaning to both ideas of recursively definable function on **No** and of uniformity. I will show that, together with some "finiteness" condition, they have some striking consequences.

I have sometimes the notion of complexity of a function: loosely speaking, if $f$ is recursive over $\mathfrak{A}$ then $f$ in general is more complicated than the functions in $\mathfrak{A}$; I can then use induction on the complexity of a function to give some important definitions and to prove some basic theorems.

My second objective is to define the Riemann integral of a recursive function $f$, as another function

$$\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t.$$

Suppose that we have a family of function $\mathfrak{A}$ containing $+$, and a function $f$ over the reals, and let $\mathfrak{B}$ be the family of the primitives of $\mathfrak{A}$. Assume that we succeeded in extending every function in $\mathfrak{A}$ and $\mathfrak{B}$ and $f$ to all of **No**. Call $\mathscr{L}$ the first order language given by $(<, \mathfrak{A}, \mathfrak{B}, f)$. Suppose the following:

1. $(\mathbb{R}, \mathscr{L})$ is an elementary substructure of $(\mathbf{No}, \mathscr{L})$;

2. $f$ is recursive over $\mathfrak{A}$.

My aim is to find a function $\mathcal{F}$ which extends to all **No** the primitive on $\mathbb{R}$ of $f$, such that:

1. $\mathcal{F}$ is recursive over $\mathfrak{A} \cup \mathfrak{B}$;

2. $(\mathbb{R}, \mathscr{L}, \mathcal{F})$ is an elementary substructure of $(\mathbf{No}, \mathscr{L}, \mathcal{F})$.

At every point $a \in \mathbf{No}$, $\mathcal{F}(a)$ will have to satisfy a type $T_a(x)$ in the language $\mathscr{L}$, over the set

$$S := \{a\} \cup \{c, \mathcal{F}(c) : c \text{ is simpler than } a\}.$$

For instance, we know that if $f(x) > 0$ for every $x$ and $c < a$, then $\mathcal{F}(c) < \mathcal{F}(a)$. If the class $I_a$ of surreal numbers satisfying $T_a(x)$ is convex and nonempty, the natural choice is to define $\mathcal{F}(a)$ as the simplest element in $I_a$. I will give a possible choice of formulae for $T_a$, all of the form $x > d$ or $x < d$ for some $d \in \mathbf{No}$ definable over $S$.

S. Norton did also give a definition of integral for function on **No**, which was later improved by Kruskal. Their definition is similar to the present one, but I am not acquainted enough with their work to give a full account of it.

In the first chapter I will recall some elementary basic definitions and theorems on surreal numbers, and give the formal definition of recursive functions, which will be used throughout all of the thesis. I will assume that the reader is familiar with the basic definitions and properties of **No**, as described in [6], and that he feels comfortable with "one line proofs" employed in it.

In the second I will define the integration, and prove some of its properties. I will find it necessary to assume some further properties about $f$ and $\mathfrak{A}$ in order to be able to define a meaningful notion of integral. One of the assumptions I could make is that the functions in $\mathfrak{A}$ form a Hardy field over **No**, or even an o-minimal structure; nevertheless, I will try to avoid using such a strong hypothesis.

In the third chapter I will prove the integration by part formula. While for content it is a direct continuation of the second, the complexity of the necessary algebraic manipulations distinguishes it. The result will not be used later in the thesis; even the integration of polynomials, which could be obtained directly from it, will instead be proved in a different way.

The fourth chapter will deal with definition and integration of polynomials and restricted analytic functions. I will show that the integral is exactly what we expect. Moreover, I will give a proof of the real closure of **No** different from the one in [6], and show how it can be generalised to recursive functions.

Finally, the last chapter will deal with logarithm and exponential function. In their discussion I will assume a certain acquaintance on the part of the reader with

Gonshor's treatment, even if I will try to quote all the necessary definitions and the theorems. I will show that, for $x > 0$,

$$\log x = \int_1^x \frac{1}{t}\,\mathrm{d}t.$$

On the other hand, the integral of $\exp x$ is not what we would expect.

# Chapter 1

# Definitions and basic properties

In this chapter I will recall the basic properties of the class **No**. I will also introduce the basic notion of a function $f : \mathbf{No} \to \mathbf{No}$ defined recursively over a family of functions $\mathfrak{A}$. Moreover, I will give some general notions on ordered sets, which will be useful in later chapters.

## 1.1 Basic definitions

Let **No** be Conway's field of surreal numbers (see [6] and [10]).

**No** can be identified with the class of all possible maps with domain an ordinal and codomain the set $\{+,-\}$. The identification of a surreal number $x$ with a function is called the *sign sequence* of $x$, and the domain of this function is $\ell(x)$, the *length* of $x$.

On this class we put a linear order in the following way:

**Definition 1.1.** **On** is the class of ordinal numbers. Let $x,y \in \mathbf{No}$. Suppose that $x \neq y$, and let $\gamma \in \mathbf{On}$ be the smallest ordinal such that $x(\gamma) \neq y(\gamma)$. Then, $x < y$ iff

- $x(\gamma) = -$ and $y(\gamma) = +$ or is undefined, or
- $x(\gamma)$ is undefined and $y(\gamma) = +$.

Equivalently, **No** is ordered lexicographically, with $- <$ undefined $< +$.

For $x,y \in \mathbf{No}$, I will write $x \preceq y$ ($x$ is *simpler* than $y$) if the sign sequence of $x$ is the restriction of the sign sequence of $y$ to some initial segment of $\ell(y)$. The relation $\preceq$ is a well-founded partial order on **No**. I say that $x$ is an *ancestor* of $y$, in symbols $x \prec y$, iff $x \preceq y$ and $x \neq y$.

Given two surreal number $x,y$, their *common length* is either $\ell(x)$ if $x = y$ or the smallest ordinal $\alpha$ such that $x(\alpha) \neq y(\alpha)$. The restriction of $x$ (or equivalently of $y$) to the common length of $x,y$ is the *common ancestor* of $x,y$.

A subclass $S \subseteq \mathbf{No}$ is *convex* iff

$$\forall x,y \in S\, \forall z \in \mathbf{No}\ (x \leq z \leq y) \rightarrow z \in S.$$

The following are the fundamental properties connecting $<$ with $\preceq$.

**Lemma 1.2.** *Let S be a non-empty convex subclass of* **No**. *Then, there exists a unique $s \in S$ which is a minimum for $\preceq$ in S (the* simplest element *of S).*

*Proof.* The relation $\preceq$ is well founded. Suppose for contradiction that there are two elements $s \neq s'$ which are minimal for $\preceq$ in $S$. Without loss of generality, $s \leq s'$; let $\alpha$ be their common length. If $s(\alpha)$ was undefined then $s \prec s'$, contradicting the minimality of $s'$. Since $s'(\alpha)$ is defined too, we have $s(\alpha) = -$ and $s'(\alpha) = +$. Let $c$ be the restriction of $s$ to $\alpha$. Then, $s < c < s'$. Therefore, by convexity, $c \in S$, contradicting the minimality of $s$ (and of $s'$). □

*Remark* 1.3. Let $a \in \mathbf{No}$. Then $\{x \in \mathbf{No} : a \preceq x\}$ is a convex subclass of **No**.

An open (closed) *interval* is a subclass of **No** of the form $(a,b)$ (of the form $[a,b]$) for some $a < b \in \mathbf{No}$. The common ancestor of $x,y$ is also the simplest element in the interval $[x,y]$.

The *concatenation* of two surreal numbers $x,y$ is the surreal number $x:y$, given by the sign sequence of $x$ followed by the sign sequence of $y$. Every ordinal number $\alpha$ can be identified canonically with the surreal number given by a sign sequence of only pluses of length $\alpha$. In particular, 0 is the simplest element of **No**.

The opposite of a surreal number $x$ is $-x$; it has the same length as $x$, and its sign sequence is obtained exchanging all pluses in the sign sequence of $x$ with minuses and all minuses with pluses.

Let $L,R$ be subsets of **No**. Then $L < R$ means $\forall x \in L, \forall y \in R\ x < y$. Moreover, $(L \mid R)$ is the cut

$$(L \mid R) := \{x \in \mathbf{No} : L < x < R\}.$$

**Theorem 1.** *$(L \mid R)$ is non-empty if $L < R$.*

*Proof.* (See [10], Theorem 2.1 for a different proof).

Suppose, for contradiction, that $(L \mid R)$ is empty. I will construct a sequence of surreal numbers $(x_\alpha)_{\alpha \in \mathbf{On}}$ such that $\forall \alpha < \beta \in \mathbf{On}\ x_\alpha \prec x_\beta$. Moreover, I will construct two sequences $L_\alpha$ and $R_\alpha$ of subsets of **No** such that $\forall \beta < \alpha \in \mathbf{On}$

$$L_\beta \supseteq L_\alpha \text{ and } R_\beta \supseteq R_\alpha$$

and $\forall \alpha \in \mathbf{No}$

$$(L_\alpha \mid R_\alpha) = \emptyset \ \text{ and } \forall y \in L_\alpha \cup R_\alpha\ x_\alpha \preceq y$$

$(L \mid R)$ is empty. Therefore, for every $x \in \mathbf{No}$ either

$$\exists y \in L \text{ such that } y \geq x \text{ or}$$
$$\exists y \in R \text{ such that } y \leq x.$$

In the first case, I say that $x$ is excluded by $L$, in the second by $R$.

Let $x_0 = 0$, $L_0 = L$, $R_0 = R$. Suppose that we have already defined $x_\gamma, L_\gamma, R_\gamma$ for every $\gamma < \alpha$.

There are two possibilities: $\alpha$ is a successor ordinal, or a limit one.

• If $\alpha = \beta + 1$ and $x_\beta \notin (L \mid R)$ then $x_\beta$ is excluded either by $L$, or by $R$.

In the first case, let $\delta$ be the smallest ordinal such that $(x_\beta : \delta)$ (the concatenation of $x_\beta$ and $\delta$) is excluded by $R$ (I will prove later that it exists). Let

$$x_\alpha = x_\beta : \delta$$
$$R_\alpha = \left\{ y \in R_\beta : x_\alpha \preceq y \right\}$$
$$L_\alpha = \left\{ y \in L_\beta : x_\alpha \preceq y \right\}.$$

In the second, let let $\delta$ be the smallest ordinal such that $x_\beta : (-\delta)$ is excluded by $L$, and let

$$x_\alpha = x_\beta : (-\delta),$$
$$L_\alpha = \left\{ y \in L_\beta : x_\alpha \preceq y \right\},$$
$$R_\alpha = \left\{ y \in R_\beta : x_\alpha \preceq y \right\}.$$

I remind that in the first case I must also prove that $\exists \lambda \in \mathbf{On}$ such that $(x_\alpha : -\lambda)$ is excluded by $L$ (and similarly for the second case). Suppose not; then $(x_\alpha : -\lambda)$ is excluded by $R$, i.e.

$$\forall \lambda \in \mathbf{On}\ \exists t_\lambda \in R\ (x_\alpha : -\lambda) \geq t_\lambda.$$

But $R$ is a *set*, therefore we can find $t \in R$ such that

$$\forall \lambda \in \mathbf{On}\ (x_\alpha : -\lambda) > t.$$

Let $c$ be the common ancestor of $x_\alpha, t$. Then $t \leq c \leq x_\alpha$. Moreover, $x_\beta < c < x_\alpha$ and $x_\beta \prec x_\alpha$. Therefore, $x_\beta \prec c \prec x_\alpha$, so $c = x_\beta : \delta'$ for some $\delta' < \delta \in \mathbf{On}$. This contradicts the definition of $\delta$ as the smallest ordinal such that $x_\beta : \delta$ is excluded by $R$. The second case is similar.

• If $\alpha$ is a limit ordinal, let

$$L_\alpha = \bigcap_{\beta < \alpha} L_\beta$$
$$R_\alpha = \bigcap_{\beta < \alpha} R_\beta$$
$$x_\alpha = \bigcup_{\beta < \alpha} x_\beta.$$

Again, I must prove that if $x_\alpha$ is excluded by $R$ there exists $\lambda \in \mathbf{On}$ such that $x_\alpha : -\lambda$ is excluded by $L$. Suppose not; then

$$\exists t \in R \, \forall \lambda \in \mathbf{On} \, x_\alpha : -\lambda > t.$$

Let $c$ be the common ancestor of $t$ and $x_\alpha$. Then $c \prec x_\alpha$ and $x_\alpha = \bigcup_{b<\alpha} x_\beta$, therefore $c \preceq x_\beta$ for some $\beta < \alpha$.

If $x_\beta < c$, then $x_\alpha < c$, a contradiction. If $x_\beta > c$, then $\forall z \succeq x_\beta \; z > c$, while we can find $x_{\beta+1} < c$. If $x_\beta = c$ then $x_\alpha = x_\beta : 1 : z$ for some $z \in \mathbf{No}$; in particular, $x_{\beta+1} = x_\beta : \delta$ for some $\delta \in \mathbf{On}$. But $x_\beta$ is excluded by $R$, therefore $x_{\beta+1} < x_\beta$, a contradiction.

Finally, by hypothesis, $L \cup R$ is a set, while $\{x_\alpha : \alpha \in \mathbf{On}\}$ is a proper class, which is impossible. □

If $L$ or $R$ are proper classes, the construction of $(x_\alpha)_{\alpha \in \mathbf{On}}$ in the previous proof may not terminate, or for some $\beta \in \mathbf{On}$ I may not be able to find $\delta \in \mathbf{On}$ such that $(x_\beta : \pm\delta)$ is excluded by $R$ (or by $L$). In either case, I construct the sign sequence of the cut $(L \mid R)$. In the first case it is given by $x = \bigcup_{\alpha \in \mathbf{On}} x_\alpha$, in the second by $x_\beta$ followed by infinitely many (i.e. a proper class of) pluses (or minuses).

An example of the second case is $L = \mathbb{N}$ and $R = \{x \in \mathbf{On} : x > \mathbb{N}\}$. The sign sequence of $\langle L \mid R \rangle$ is given by $\omega$ pluses followed by infinitely many minuses.

The *simplest element* in $(L \mid R)$ is written $\langle L \mid R \rangle$.

**Lemma 1.4.** *Every $x \in \mathbf{No}$ can be written in a canonical way as $\langle L \mid R \rangle$, choosing*

$$\begin{aligned} L &= \{y \in \mathbf{No} : y < x \,\&\, y \prec x\} \\ R &= \{y \in \mathbf{No} : y > x \,\&\, y \prec x\} \end{aligned} \tag{1.1}$$

*Proof.* I have to verify that $x$ is the simplest element in the cut $(L \mid R)$. First, by definition of $L, R$, $x$ is in this cut. Let $c$ be the simplest surreal number in it. Suppose, for contradiction, that $c \neq x$, for instance that $c < x$. Therefore, $c \prec x$, so $c \in L$ and $c \notin (L \mid R)$. □

If $x = \langle L \mid R \rangle$, I will say that an element of $L$ (of $R$) is a left (right) *option* of $x$. The options of the canonical representation of $x$ are called *canonical options*.

I will write $x = \langle x^L \mid x^R \rangle_{\substack{x^L \in L \\ x^R \in R}}$, or simply $x = \langle x^L \mid x^R \rangle$ instead of $x = \langle L \mid R \rangle$.

*Remark* 1.5. Let $x, y \in \mathbf{No}$. Let $x = \langle x^L \mid x^R \rangle$, $y = \langle y^L \mid y^R \rangle$ be some representation of $x, y$. Then,

- $x < y$ iff $\exists y^L$ such that $x \leq y^L$ or $\exists x^R$ such that $x^R \leq y$
- $x \leq y$ iff $\forall y^R \forall x^L \; x^L < y$ and $x < y^R$.

*Remark* 1.6. Let $x, y \in \mathbf{No}$. The following are equivalent:

- $x \preceq y$.
- There exists $x = \langle x^L \mid x^R \rangle$, a representation of $x$ such that

$$\forall x^L \forall x^R \, x^L < y < x^R.$$

- If $x = \langle x^L \mid x^R \rangle$ is the canonical representation of $x$,

$$\forall x^L \forall x^R \, x^L < y < x^R.$$

The last definitions and theorems of this section are taken from [10].

**Definition 1.7.** Let $L, R, L', R'$ be subsets of **No**, with $L < R$ and $L' < R'$. Then $(L, R)$ is *cofinal* in $(L', R')$ iff

$$(\forall r' \in R' \, \exists r \in R \, r \leq r') \;\; \& \;\; (\forall l' \in L' \, \exists l \in L \, l \geq l').$$

*Example* 1.8. Let $L, L'$ be sets of ordinal numbers. $(L, \emptyset)$ cofinal in $(L', \emptyset)$ is equivalent to $L$ cofinal in $L'$ as set of ordinals.

**Theorem 2 (Cofinality theorem).** *Suppose that $x = \langle L \mid R \rangle$, $L' < x < R'$ and $(L', R')$ is cofinal in $(L, R)$. Then, $\langle L' \mid R' \rangle = x$.*

**Theorem 3 (Cofinality theorem b).** *Let $(L, R)$ and $(L', R')$ be mutually cofinal in each other. Then, $\langle L \mid R \rangle = \langle L' \mid R' \rangle$.*

**Theorem 4 (Inverse cofinality theorem).** *Let $x \in$ **No**, let $\langle L \mid R \rangle$ be the canonical representation of $x$, let $\langle L' \mid R' \rangle$ be another representation. Then, $(L', R')$ is cofinal in $(L, R)$.*

## 1.2 Further structure on No

In this section, every algebraic structure (Group, Field, etc.), unless otherwise specified, may have a proper class as domain.

**No** is endowed with further algebraic structure, via definition schemata, which I will introduce with an example about the definition of sum of two surreal numbers.

First, let $x = \langle x^L \mid x^R \rangle$ and $y = \langle y^L \mid y^R \rangle$ be the canonical representations of $x, y$. Suppose that I have already defined $x + y^{\mathbf{o}}$ and $x^{\mathbf{o}} + y$ for every $x^{\mathbf{o}}, y^{\mathbf{o}}$ canonical options of $x, y$ respectively. Then, define

$$x + y = \langle x + y^L, x^L + y \mid x + y^R, x^R + y \rangle. \tag{1.2}$$

There is something to prove, namely that in the above definition every left option is less than every right option (this, and much else, is proved in [6]).

The definition is recursive. The recursion is done on $x$ and $y$ (with the well-founded partial order $\preceq$). Let $x$ minimal such that there exists $z$ such that $x+z$ is undefined. Let $y$ be a minimal such $z$. Therefore, $x+y^{\mathbf{o}}$ and $x^{\mathbf{o}}+y$ are defined for every option $x^{\mathbf{o}}$ and $y^{\mathbf{o}}$, so all the options in (1.2) are defined and, by the already mentioned lemma, every left option is less than every right one, so $x+y$ is defined.

But something more is true: suppose that $x = \langle x^L \mid x^R \rangle$ and $y = \langle y^L \mid y^R \rangle$ are any representations (not necessarily canonical) of $x,y$. The options in (1.2) are defined for these representations too. Not only it is true that every left option is less than every right option, but the number $\langle x+y^L, x^L+y \mid x+y^R, x^R+y \rangle$ is still $x+y$. In this case I say that the definition of $x+y$ is uniform.

$(\mathbf{No},+,<)$ is an ordered Abelian group, the neutral element is 0, the simplest element of $\mathbf{No}$, and

$$-x = \langle -x^R \mid -x^L \rangle.$$

On $\mathbf{No}$ there is also a multiplication, recursively defined as

$$xy = \langle x^Ly+xy^L-x^Ly^L,\ x^Ry+xy^R-x^Ry^R \mid x^Ly+xy^R-x^Ly^R,\ x^Ry+xy^L-x^Ry^L \rangle.$$

Again, every left option is less than every right option, and the value of $xy$ is independent from the choices of the representations of $x$ and $y$.

With these definitions of order, sum and multiplication, $\mathbf{No}$ is a real closed field.

There is a canonical embedding from the class $\mathbf{On}$ of ordinals into $\mathbf{No}$: an ordinal $\alpha$ goes into the constant function with domain $\alpha$ and value $+$. For $x,y \in \mathbf{On}$, $x+y$ and $xy$ are ordinal numbers too. The sum and product of two ordinals as surreal numbers are not the usual sum and product on ordinals (which are not commutative), but the *natural* addition and multiplication(1). A proof can be found in [10], chapter 4D.

### 1.2.1 Valuation and power series in No

**Definition 1.9.** A *valued field* is a triple $(\mathbb{F}, G, v)$, with $\mathbb{F}$ is a field, $G$ a linearly ordered Abelian group (written additively), and $v$ a map

$$v : \mathbb{F} \to G \cup \{\infty\}$$

such that

1. $\forall x \in \mathbb{F}\ v(x) = \infty$ iff $x = 0$

(1)The natural addition and multiplication of two ordinals $\alpha, \beta$ are defined in terms of their Cantor normal forms.

2. $v$ is surjective

3. $v(x+y) \leq \max\{v(x), v(y)\}$

4. $v(xy) = v(x) + v(y)$.

The map $v$ is called the *valuation*, $G$ the *value group*. The convention is that $\forall g \in G\, g + \infty = \infty$ and $\infty < g$.

Usually, a valuation satisfying the previous definition is called non-Archimedean, but I will consider only such valuations.

In literature, the order of $G$ is often reversed, i.e. instead of 3 they often write

$$v(x+y) \geq \min\{v(x), v(y)\},$$

but this would result in the anti-intuitive fact that infinitesimal elements have "large" value. Moreover, working on **No** is easier with this convention.

**Definition 1.10.** Given a valued field $(\mathbb{F}, G, v)$,

$$\mathcal{O} := \{x \in \mathbb{F} : v(x) \leq 0\}$$

is a local subring of $\mathbb{F}$, the *valuation ring*. Its only maximal ideal is

$$\mathfrak{M} := \{x \in \mathbb{F} : v(x) < 0\}.$$

The quotient $\mathcal{O}/\mathfrak{M}$ is the *residue field*. An element of $\mathcal{O}$ is called *bounded* or *finite*, an element of $\mathfrak{M}$ *infinitesimal*.

Two valuations $(\mathbb{F}, G, v)$ and $(\mathbb{F}, G', v')$ on the same field $\mathbb{F}$ are *equivalent* iff there is an isomorphism of ordered groups $\phi : G \to G'$ such that $\forall x \in \mathbb{K}^*\ v'(\phi(x)) = v(x)$.

Suppose that on the field $\mathbb{F}$ there is also an order $<$ such that $(\mathbb{F}, <)$ is an ordered field. The valuation $<$ is *compatible* with the order iff $\mathcal{O}$ is a convex subclass of $\mathbb{K}$.

In the following, I will mostly consider fields of characteristic 0.

**Definition 1.11.** Let $\mathbb{F}$ be a linearly ordered group, $x, y \in \mathbb{F}$.

- $x \gg y$ iff for every natural number $n$, $|x| \geq n|y|$
- $x \sim y$ iff neither $x \ll y$ nor $y \ll x$
- $x \simeq y$ iff $x = y$ or $x \gg x - y$ (or equivalently $y \gg x - y$).

$x \sim y$ is equivalent to $\exists n \in \mathbb{N}\ \frac{|x|}{n} \leq y \leq n|x|$.

On an ordered field $\mathbb{F}$ we can define the *natural valuation*. The value group is the quotient $\mathbb{K}^*/_\sim$ and the valuation the quotient map. The natural valuation is compatible with the order. An ordered field is said to be *Archimedean* iff the natural valuation is trivial (i.e. its value group is $\{0\}$) iff (in a unique way) it is an ordered subfield of $\mathbb{R}$.

**No** is an ordered field and therefore it contains $\mathbb{Q}$. If $(L,R)$ is a Dedekind cut of rational numbers, it determines an unique element $\langle L \mid R\rangle \in \mathbf{No}$. The resulting set (an element for every Dedekind cut of $\mathbb{Q}$) is a subfield of **No** that can be identified canonically with $\mathbb{R}$.

**Definition 1.12.** There is a valuation $v : \mathbf{No}^* \longrightarrow \mathbf{No}$, which, for $x > 0$, is defined as

$$v(x) = \langle \{ v(x^L) : 0 < x^L \ll x \} \mid \{ v(x^R) : x^R \gg x \,\&\, x^R > 0 \} \rangle. \tag{1.3}$$

The map $\omega : \mathbf{No} \longrightarrow \mathbf{No}^{>0}$ is specified in the following way

$$\omega^x := \langle \{0\} \cup \left\{ q\omega^{x^L} : 0 < q \in \mathbb{Q} \right\} \mid \left\{ q\omega^{x^R} : 0 < q \in \mathbb{Q} \right\} \rangle.$$

The map $\omega$ is well defined, and $x < y$ iff $\omega^x \ll \omega^y$. Moreover, $\forall x,y \in \mathbf{No}$ $\omega^{x+y} = \omega^x\omega^y$. For the proof, see [10], chapter 5B.

**Lemma 1.13.** *The definition of $v$ is sound, independent of the representation of $x$, and $v$ is a valuation equivalent to the natural one. Moreover,*

$$\forall x \in \mathbf{No}\ v(\omega(x)) = x.$$

*Proof.* By induction: let $x,y$ be positive surreal numbers, and suppose that the result is true for every $x^{\mathbf{o}}, y^{\mathbf{o}}$ which are canonical options of $x,y$.

Since $x$ and $y$ are positive, they do not have negative options in their canonical representations.

If $x^L \ll x \ll x^R$ then by inductive hypothesis $v(x^L) < v(x^R)$, so the definition of $v(x)$ is sound.

*Claim* 1. Suppose that $x \ll y$. Then, $v(x) < v(y)$.

If $x \prec y$ then $v(x) < v(y)$ by definition of $v(y)$. Similarly for $y \prec x$. Otherwise, let $z$ be the common ancestor of $x,y$. Then, $x < z < y$ and $x \ll y$, therefore $x \ll z$ or $z \ll y$.

If $x \ll z$, by inductive hypothesis $v(x) < v(z)$, and similarly $v(z) \leq v(y)$, therefore $v(x) < v(y)$. Similarly for $z \ll y$.

*Claim* 2. Suppose that $v(x) < v(y)$. Then, $x \ll y$.

Either $v(x) \leq v(y)^L$ or $v(x)^R \leq v(y)$ for some $v(y)^L$ or $v(x)^R$ options of $v(y), v(x)$ in the definition 1.12. Suppose that the first case happens. Then, $v(x) \leq v(y^L)$ for some canonical options $y^L$ such that $y^L \ll y$. By inductive hypothesis, either $y^L \ll y$ and $x \sim y^L$, or $x \ll y^L$. Therefore, $x \ll y$. Similarly for the second case.

Now I will prove that the definition of $v(x)$ is uniform. Let $x = \langle t^L \mid t^R \rangle$ be another representation of $x > 0$, and let

$$w = \langle \{ v(t^L) : 0 < t^L \ll x \} \mid \{ v(t^R) : t^R \gg x \,\&\, t^R > 0 \} \rangle.$$

First, $v(t^L) < v(x) < v(t^R)$ for every $t^L, t^R$ mentioned in the definition of $w$, therefore $w \preceq v(x)$.

On the other hand, by the inverse cofinality theorem, for every $x^L$ there exists $t^L$ such that $x^L \leq t^L < x$. Therefore, $0 < x^L \ll x$ implies that $x^L \ll t^L$ or $x^L \sim t^L$, implying $v(x^L) \leq v(t^L)$. Similarly, for every $x^R$ there exists $t^R$ such that $v(x^R) \geq v(t^R)$. So, $v(x) \preceq w$.

The fact that $v(x+y) \leq \max\{ v(x), v(y) \}$ is a direct consequence of what I have already proved.

*Claim* 3. If $x > 0$, $v(\omega^x) = x$.

By induction on $x$. First, $v(\omega(x)) < x^R$. In fact, by induction, $x^R = v(\omega^{x^R}) = v(q\omega^{x^R})$ for every $q > 0 \in \mathbb{Q}$, and the latter is a right option of $v(\omega^x)$. Similarly, $v(\omega(x)) > x^L$.

Second, $x < v(\omega^x)^R$. In fact, the right option is equal to $v((\omega^x)^R) = v(q\omega^{x^R}) = v(\omega^{x^R}) = x^R > x$. Similarly for left options.

It remains to show that $v(xy) = v(x) + v(y)$ for $x, y > 0$. But this is a consequence of the fact that $\omega^{x+y} = \omega^x \omega^y$ □

**Definition 1.14.** Given a field $\mathbb{K}$ and a linearly ordered additive group $G$, $\mathbb{K}((G))$ is the field of formal power series with coefficient in $\mathbb{K}$ and exponents in $G$. Its elements are given as formal sums

$$x = \sum_{g \in G} r_g t^g,$$

where $r_g$ are elements of $\mathbb{K}$ and such that the *support* of $x$

$$\text{supp}(x) = \{ g \in G : r_g \neq 0 \}$$

is an anti-well-ordered *set*. In the previous definition, $G$ might be a proper class.

The sum of two elements in $\mathbb{K}((G))$ is defined "point-wise":

$$(\sum_g r_g t^g) + (\sum_g s_g t^g) = \sum_g (r_g + s_g) t^g.$$

Their product is the Cauchy product:

$$(\sum_{h\in G} r_h t^h)(\sum_{k\in G} s_k t^k) = \sum_{g\in G}(\sum_{h+k=g} r_h s_k) t^g.$$

By Neumann's lemma (see for instance [1], 7.20), the previous definition is sound: given $x, y \in \mathbb{K}((G))$, the support of $x+y$ is anti-well-founded. Moreover, for every $g \in G$ there are at most finitely many $h \in \mathrm{supp}(x)$ and $k \in \mathrm{supp}(y)$ such that $h+k=g$, and the support of the resulting $xy$ is well founded.

With these operations, $\mathbb{K}((G))$ is not only a ring, but a field.

Moreover, if $\mathbb{K}$ is an algebraically closed field and $G$ is a divisible group, $\mathbb{K}((G))$ is an algebraically closed field.

Let $x \neq 0 \in \mathbb{K}((G))$. If $x = \sum_{g\in G} r_g t^g$ then the *leading coefficient* of $x$ is $r_h$, where $h$ is the maximum of the support of $x$

If $\mathbb{K}$ is an ordered field, $\mathbb{K}((G))$ is an ordered field too: $x > 0$ iff the leading coefficient of $x$ is greater than 0.

If $\mathbb{K}$ is a real closed field and $G$ is divisible, then $\mathbb{K}((G))$ is real closed too.

On $\mathbb{K}((G))$ there is a canonical valuation (the *Hahn valuation*)

$$v : \mathbb{K}((G))^* \to G,$$

with $v(x)$ the maximum of the support of $x$. The value group of $v$ is all of $G$, and the residue field is $\mathbb{K}$. If $\mathbb{K}$ is an Archimedean ordered field, $v$ coincides with the natural valuation.

**Definition 1.15.** Let $(\mathbb{F}, v, G)$ be a valued field. A *sequence* (of elements in $\mathbb{F}$) is a function from some limit ordinal into $\mathbb{F}$. A sequence $(x_\alpha)_{\alpha<\lambda}$ is *pseudo-convergent* (or *pseudo-Cauchy*) iff

$$\forall \alpha < \beta < \gamma < \lambda, v(x_\alpha - x_\beta) > v(x_\beta - x_\gamma).$$

$x \in \mathbb{F}$ is a *pseudo-limit* of the pseudo-convergent sequence $(x_\alpha)_{\alpha<\lambda}$ iff

$$\forall \alpha < \lambda \;\; v(x_\alpha - x) = v(x_\alpha - x_{\alpha+1}).$$

$\mathbb{F}$ is *pseudo-complete* iff every pseudo-convergent sequence has a pseudo-limit.

An *extension* of $\mathbb{F}$ is given by a valued field $(\mathbb{F}', G', v')$ and pair of maps $\iota : \mathbb{F} \to \mathbb{F}'$ and $\phi : G \to G'$ such that $\iota$ is a field embedding, $\phi$ is an embedding of ordered groups, and the diagram commutes: $\forall x \in \mathbb{F}\; v'(\iota(x)) = \phi(v(x))$.

Such an extension is *immediate* iff $\phi$ and the induced map between the residue fields are isomorphisms.

$\mathbb{K}((G))$ with the canonical valuation is pseudo-complete. If $\mathbb{F}$ is an ordered valued field, and the valuation on $\mathbb{F}$ is compatible with the order, then the class of pseudo-limits of a given sequence $(x_\alpha)_{\alpha\in\lambda}$ is a convex subclass of $\mathbb{F}$.

If $\mathbb{F}$ is a set, then $\mathbb{F}$ is pseudo-complete iff it is maximal[(2)] (i.e. it does not admit non-trivial immediate extensions), iff it is isomorphic to $\mathbb{K}((G))$[(3)], with $G$ the value group and $\mathbb{K}$ the residue field of $\mathbb{F}$. See [12] for a proof of this fact.

In **No**, by theorem 1, every pseudo-convergent sequence $(x_\alpha)_{\alpha<\lambda}$ has a pseudo-limit, and the simplest pseudo-limit of the sequence is *the* pseudo-limit of $(x_\alpha)_{\alpha<\lambda}$.

Let $\lambda$ be an ordinal, $(r_\alpha)_{\alpha<\lambda}$ be a sequence non zero real numbers, and $(a_\alpha)_{\alpha<\lambda}$ be a strictly decreasing sequence of surreal numbers. The formal expression

$$\sum_{\alpha<\lambda} r_\alpha \omega^{a_\alpha}$$

determines the unique surreal number $x = \sum_{\alpha<\lambda} r_\alpha \omega^{a_\alpha}$. It is defined by induction on $\lambda$:

- If $\lambda = \gamma + 1$, then

$$x = \Big(\sum_{\alpha<\gamma} r_\alpha \omega^{a_\alpha}\Big) + r_\gamma \omega^{a_\gamma}.$$

- If $\lambda$ is a limit ordinal, then $x$ is the pseudo-limit of the pseudo-convergent sequence

$$\Big(\sum_{\alpha<\gamma} r_\alpha \omega^{a_\alpha}\Big)_{\gamma<\alpha}.$$

Conversely, every non-zero surreal number $x$ is represented by a unique sum

$$x = \sum_{\alpha<\lambda} r_\alpha \omega^{a_\alpha},$$

called the *normal form* of $x$.

*Proof.* The uniqueness is obvious: if $x = \sum_{a\in\mathbf{No}} r_a \omega^a$ and $y = \sum_{a\in\mathbf{No}} s_a \omega^a$ are two distinct formal sums, and $c$ is the largest surreal number $b$ such that $r_b \neq s_b$, then $v(x-y) = c$.

Given $x \neq 0$, I will find its normal form. Let $a_0 := v(x)$ and $r_0$ be the unique real number such that $(x - r_0\omega^{a_0}) \ll x$. Given an ordinal $\lambda$ and sequences $(a_\alpha)_{\alpha<\lambda}$ and $(r_\alpha)_{\alpha<\lambda}$, let

$$x_\lambda = \sum_{\alpha<\lambda} r_\alpha \omega^{a_\alpha}.$$

(2) This is no longer true for classes, unless we change the definition of pseudo-complete. For instance, **No** is pseudo-complete, but not maximal.

(3) This is true only if the characteristic of the residue field is 0. Otherwise, additional hypothesis are required.

If $x_\lambda = x$, we proved the conclusion. Otherwise, define $a_\lambda := v(x - x_\lambda)$ and $r_\lambda$ the unique real such that $x - (x_\lambda + r_\lambda \omega^{a_\lambda}) \ll x$. The sequence $(x_\lambda)_{\lambda \in \mathbf{On}}$ is defined for every ordinal number $\lambda$, and the $x_\lambda$ are are all distinct (because $\forall \lambda \in \mathbf{On}\ r_\lambda \neq 0$). But for every limit ordinal $\lambda$, $x$ is a pseudo-limit of $(x_\alpha)_{\alpha<\lambda}$, therefore $x_\lambda \preceq x$, and this is impossible. □

Therefore, **No** can be identified in a canonical way with $\mathbb{R}((\mathbf{No}))$. Since **No** is a field, $\mathbf{No} \cong \mathbb{R}((\mathbf{No}))$ is a real closed field. This is essentially Conway's proof of the fact that **No** is real closed, starting from the knowledge that **No** is an ordered field. I will give later a different proof of this fact.

Given a field $\mathbb{K}$ and an ordered group $G$, every power series

$$f \in \mathbb{K}[[x_1, \dots, x_n]]$$

defines a function $f : \mathfrak{M}^n \to \mathbb{K}((G))$ by formal substitution, which, by Neumann's lemma, is well defined.

Let $\mathbb{K}$ be an ordered field containing $\mathbb{R}$, and $f$ be a real analytic function converging in a neighbourhood of $[-1,1]^n$. The Taylor expansion of $f$ determines a power series $f_p$ with real coefficients around every $p \in [-1,1]^n \subset \mathbb{R}^n$, which induces a function $f : [-1,1]^n \to \mathbb{K}((G))$

$$f(p+\varepsilon) = f(p) + f_p(\varepsilon), p \in \mathbb{R}^n, \varepsilon \in \mathfrak{M}^n$$

called a *restricted analytic function* ($[-1,1]$ is the interval in $\mathbb{K}((G))$). The first order language given by the ordered field language $(0,1,+,\cdot,<)$ plus a function symbol for every real analytic function $f$ converging in a neighbourhood of $[-1,1]^n$ is called $\mathscr{L}_{an}$. If $\mathbb{K}$ is real closed and $G$ is divisible, the resulting $\mathscr{L}_{an}$ structure on $\mathbb{K}((G))$ is elementary equivalent to $\mathbb{R}$ (see [19]). In particular, **No** is elementary equivalent to $\mathbb{R}$ in $\mathscr{L}_{an}$.

Gonshor defined a total exponential function $\exp : \mathbf{No} \to \mathbf{No}^{>0}$ (see § 1.5 for more details). With this definition, **No** is an elementary extension of $\mathbb{R}$ in the language $\mathscr{L}_{an}(\exp)$.

## 1.3 Recursive definitions

The sum of two element $x, y \in \mathbf{No}$ is defined as:

$$x + y = \langle x + y^L, x^L + y \mid x + y^R, x^R + y \rangle$$

where $x^L, x^R$ are generic left and right options of $x$, and similarly for $y$. The definition of $x+y$ is recursive. We must know the value of $x' + y$ and $x + y'$ for any elements

$x' \prec x$ and $y' \prec y$ in order to compute $x+y$. In the following, I will try to give a precise meaning to the notion of *recursive definition* of a function on **No**, which is general enough to encompass the definition of functions such as $x+y, xy, 1/x, \exp x$, but at the same time allows me to prove some nontrivial results.

I will write $x^{\mathbf{o}}$ for a generic (left or right) option of $x$. Given two sets of functions $\{f^L\}_{f^L \in A}$ and $\{f^R\}_{f^R \in B}$ in the variables $X, Y_1, Y_2, Z_1, Z_2$, I will write

$$f = \langle f^L \mid f^R \rangle_{\substack{f^L \in A \\ f^R \in B}}$$

or simply $f = \langle f^L \mid f^R \rangle$ if for any $x \in \mathbf{No}$, written in the canonical form $\langle x^L \mid x^R \rangle$, a generic left option of $f(x)$ is

$$f^L(x, x^L, x^R, f(x^L), f(x^R))$$

and similarly for a right option. This means that $\forall x \in \mathbf{No}$

$$f(x) = \langle f^L(x, x^L, x^R, f(x^L), f(x^R) \mid f^R(x, x^L, x^R, f(x^L), f(x^R)) \rangle_{\substack{x^L \prec x \;\&\; x^L < x \\ x^R \prec x \;\&\; x^R > x \\ f^L \in A \\ f^R \in B}}$$

For a shorthand, I will also write

$$f^L(x, x^{\mathbf{o}}, f(x^{\mathbf{o}})) \text{ or } f^{\mathbf{o}}(x, x^{\mathbf{o}}, f(x^{\mathbf{o}})) \text{ or even } f^{\mathbf{o}}(x, x^L, x^R)$$

and say that $f$ is defined recursively (or simply recursive) over the family of functions $A \cup B$. Of course, for $f(x)$ to be defined, it is necessary that

$$f^L(x, x^{\mathbf{o}}, f(x^{\mathbf{o}})) < f^R(x, x^{\mathbf{o}}, f(x^{\mathbf{o}}))$$

for any $f^L, f^R$ and for any $x^L, x^R$ canonical options of $x$.

$f^L$ and $f^R$ are options of $f$.

If $f : \mathbf{No}^n \longrightarrow \mathbf{No}$, it is still possible to say what it means to be defined recursively. Write each coordinate of $\vec{x} \in \mathbf{No}^n$ as $x_i = \langle x_i^L \mid x_i^R \rangle$. I will say that

$$\vec{x}^{\mathbf{o}} = (y_1, \dots, y_n)$$

is a canonical option of $\vec{x}$ (and similarly for right options) if for each $i = 1, \dots, n$ $y_i \preceq x_i$ and, for at least one $j \leq n$, we have $y_j \prec x_j$. Alternatively, I order $\mathbf{No}^n$ with the well-founded and set-like partial order bnd induced by $\preceq$ (see definition 1.56), and $\vec{x}^{\mathbf{o}}$ is an element which precedes $\vec{x}$ in this order.

Then

$$f(\vec{X}) = \langle f^L \mid f^R \rangle$$

if for each $\vec{x} \in \mathbf{No}^n$ written in the canonical form, a generic left option of $f(\vec{x})$ is

$$f^L(\vec{x},\vec{x}^{\mathbf{o}},f(\vec{x}^{\mathbf{o}}))$$

where $f^L$ is a function of as many variables as needed.

For instance, if $f(x_1,x_2) = x_1x_2$, among the left options of $f(\vec{x})$ there is

$$f(x_1,x_2^L)+f(x_1^L,x_2)-f(x_1^L,x_2^L).$$

Note that $f(\vec{x}^{\mathbf{o}})$ can appear many times as an argument of $f^{\mathbf{o}}$; once for each possible combination

$$x_i \lesseqgtr x_i^{\mathbf{o}}, i=1,\ldots,n.$$

If $f = \langle f^L \mid f^R \rangle$, the value of $f(x)$ depends in general on the form I choose to write $x$ (that is the reason why I had to specify that I use the canonical form of $x$).

**Definition 1.16.** I say that the recursive definition of $f$ is *uniform* if $f(x)$ does not depend on the form I choose for $x$, for any $x \in \mathbf{No}$. This means that:

- For every $f^L$ and $f^R$ options of $f$, for every $x,x',x'' \in \mathbf{No}$ such that $x' < x < x''$

$$f^L(x,x',x'',f(x'),f(x'')) < f^R(x,x',x'',f(x'),f(x''))$$

- $\forall x \in \mathbf{No}$ and for any representation $x = \langle L \mid R \rangle$,

$$f(x) = \langle f^L(x,x^L,x^R,f(x^L),f(x^R) \mid f^R(x,x^L,x^R,f(x^L),f(x^R)) \rangle_{\substack{x^L \in L \\ x^R \in R}}.$$

In the following I will mostly consider uniform definitions.

## 1.4 Canonical form on intervals

Suppose that $L$ and $R$ are two subclasses of $\mathbf{No}$, with $L < R$. If $L,R$ are both proper sets (and not classes), I say that $(L \mid R)$ is a *set-bounded convex subclass* of $\mathbf{No}$.

**Lemma 1.17.** *Let $S = (A \mid B)$ be a set-bounded convex subclass of* $\mathbf{No}$. *Then, as an ordered tree, $S$ is isomorphic to* $\mathbf{No}$ *in a unique way. Moreover, this isomorphism $s : \mathbf{No} \to S$ is recursively definable, and uniformly so.*

*Proof.* I will define the isomorphism $s : \mathbf{No} \to S$ by induction. Uniqueness will follow from the definition itself. Obviously, $s(0)$ is the simplest element of $S$. As $A$ and $B$ are both proper sets, $(A \mid s(0))$ and $(s(0) \mid B)$ are non empty, so there is a simplest element in each of them: $s(-1)$ and $s(1)$.

In general, if $x = \langle x^L \mid x^R \rangle_{x^L \in L, x^R \in R}$ is the canonical form of $x$, and I have already defined $s(x^\circ)$ for every option of $x$, then $s(x)$ is the simplest element in $S$ satisfying

$$s(x^L) < s(x) < s(x^R)$$

for every option of $x$. There are three cases:

1. $R$ is empty: $s(x) = \langle s(x^L) \mid B \rangle$
2. $L$ is empty: $s(x) = \langle A \mid s(x^R) \rangle$
3. $L, R$ are both non empty: $s(x) = \langle s(x^L) \mid s(x^R) \rangle$.

The general formula for $s$ is $s(x) = \langle A, s(x^L) \mid B, s(x^R) \rangle$. It remains to show that $s$ is uniformly defined, i.e. that if $x = \langle y^L \mid y^R \rangle$ is any representation of $x$, then

$$s(x) = \langle A, s(y^L) \mid B, s(y^R) \rangle.$$

By cofinality, for every $x^L$ canonical left option for $x$ there exists $y^L$ such that $x^L \leq y^L < x$. Then $s(x^L) \leq s(y^L) < s(x)$ (and similarly for right options), so $s(x) = \langle A, s(y^L) \mid B, s(y^R) \rangle$. □

*Examples* 1.18.
- $(0,1)$ is isomorphic to **No**: $s(0) = 0$, $s(1) = 1/2$, $s(2) = 3/4$, $s(\omega) = 1 - 1/\omega, \ldots$
- Let $\eta$ be the cut between infinitesimal and positive finite numbers. Then $(-\eta, \eta)$ is isomorphic to **No**.
- Let $a$ be the cut between finite and infinite positive numbers. Then $(-\infty, a)$ is not isomorphic to **No**, because $s(\omega)$ is not defined.

So, given $S = (A \mid B)$ a set-bounded convex subclass of **No**, to every $x \in S$ I can associate through $s$ a canonical form: if $s^{-1}(x) = y$ and $y = \langle y^L \mid y^R \rangle$ is the canonical form of $y$, then $x = \langle A, s(y^L) \mid s(y^R), B \rangle$ is the canonical form of $x$ with respect to $S$, and $s(y^\circ)$ are the canonical options of $x$ w.r.t. $S$.

Until now, I have supposed that all the functions are total. How do the definitions change for functions defined only on a subset of **No**? For instance, power series are defined for infinitesimal elements. If $S = (L \mid R)$ is a set-bounded convex subclass of **No**, and $x \in S$, I use the canonical form of $x$ with respect to $S$ to give a meaning to $f(x)$ for a recursively defined function $f = \langle f^L \mid f^R \rangle$ with domain $S$, i.e. by writing $f = \langle f^L \mid f^R \rangle$ I mean that each $f^\circ(x, y_1, y_2, z_1, z_2)$ is defined (at least) on the set

$$\{x, y_1, y_2, z_1, z_2 : x, y_1, y_2 \in S, y_1 < x < y_2, z_1, z_2 \in \mathbf{No}\}$$

and for each $x \in S$

$$f(x) = \langle f^L(x, x^L, x^R, f(x^L), f(x^R)) \mid f^R(x, x^L, x^R, f(x^L), f(x^R)) \rangle$$

where $x^L, x^R$ are canonical options of $x$ w.r.t. $S$.

*Example* 1.19.

$$\left(\frac{1}{x}\right)^{\circ} = \frac{1 - \left(1 - \frac{x}{x^L}\right)^{\alpha} \left(1 - \frac{x}{x^R}\right)^{\beta}}{x}$$

where the denominator simplifies formally with the numerator, is defined on the intervals $(0, +\infty)$ and $(-\infty, 0)$.

## 1.5 Exponential function

On **No** it is possible to define a total *exponential* function exp : $\mathbf{No} \to \mathbf{No}^{>0}$ (the domain of exponential induced by the analytic structure is only $\mathcal{O}$). For proofs of the theorems stated in this section and other properties of the exponential function, see [10].

Given $n \in \mathbb{N}$, let $[z]_n$ be the $n$-truncation of the Taylor expansion of $\exp z$ at 0:

$$[z]_n := \sum_{i=0}^{n} \frac{z^i}{i!}.$$

The recursive definition of $\exp x$ is the following:

$$\exp x = \left\langle 0,\ (\exp x^L)[x - x^L]_n,\ (\exp x^R)[x - x^R]_{2n+1} \;\middle|\; \frac{\exp x^R}{[x^R - x]_n},\ \frac{\exp x^L}{[x^L - x]_{2n+1}} \right\rangle,$$

where if $z < 0$ then $[z]_n$ must be positive.

The definition is uniform, and the resulting structure satisfies the following axioms:

- exp is surjective.
- $\forall x, y \in \mathbf{No}$ $\exp(x + y) = \exp x \exp y$.
- For every $|x| \leq 1$ $\exp x$ coincides with the restricted analytic function $e(x)$.
- $\exp(x) > x^n$ for all positive infinite $x$.
- $\forall x > y$ $\exp x > \exp y$.

The previous axioms imply that **No** is an elementary extension of $\mathbb{R}$ in the language $\mathscr{L}_{an}(\exp)$ (see [19]). Note that $\omega^x$ is *not* $\omega$ raised to the power $x$.

Gonshor gave other properties of the function exp. Let $x > 0 \in \mathbf{No}$. Define

$$g(x) := \langle\, v(x),\ g(x^L) \mid g(x^R) \,\rangle,$$

where $x^L$ varies among the *positive* left options of $x$. The definition of $g$ is also uniform; moreover, $g$ is a monotone increasing bijection from $\mathbf{No}^{>0}$ onto $\mathbf{No}$.

**Theorem 5.** *Let* $z = \sum_{i<\alpha} r_i \omega^{a_i}$ *be the normal form of* $z \in \mathbf{No}$. *If* $a_i > 0$ *for every* $i < \alpha$ *and* $z > 0$, *then*

$$\exp z = \omega^y \text{ where } y := \sum_{i<\alpha} r_i \omega^{g(a_i)}.$$

*Proof.* It is Theorem 10.13 of [10]. □

## 1.6 Useful formulae

The following formulae are especially useful in dealing with surreal numbers. Proofs can be found in [6] or [10].

$$x+y = \langle x^L+y, x+y^L \mid x^R+y, x+y^R \rangle \tag{1.4}$$

$$\begin{aligned} xy = \langle & x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid \\ & x^R y + x y^L - x^R y^L, x^L y + x y^R - x^L y^R \rangle \end{aligned} \tag{1.5}$$

$$(x^n)^{\mathbf{o}} = x^n - (x-x^L)^i (x-x^R)^j \tag{1.6}$$

where $0 \le i \le n \in \mathbb{N}$, $i+j=n$ and $(x^n)^{\mathbf{o}}$ is a left option if $j$ is even, a right option if it is odd.

$$\omega^x = \langle 0, r\omega^{x^L} \mid r\omega^{x^R} \rangle_{r>0\in\mathbb{R}} \tag{1.7}$$

$$v(x) = \langle \{ v x^R : 0 < x^L \ll x \} \mid \{ v(x^R) : x^R \gg x \ \&\ x^R > 0 \} \rangle \tag{1.8}$$

$$\left(\frac{1}{x}\right)^{\mathbf{o}} = \frac{1 - \left(1 - \frac{x}{x^L}\right)^i \left(1 - \frac{x}{x^R}\right)^j}{x} \tag{1.9}$$

$$\left(\sum_{i<\alpha} r_i \omega^{a_i}\right)^{\mathbf{o}} = \sum_{i\le\beta} r_i \omega^{a_i} \pm \varepsilon \omega^{a_\beta} \tag{1.10}$$

where $\alpha \in \mathbf{On}$ is an ordinal number, $(a_i)_{i<\alpha}$ is a strictly decreasing sequence of surreal numbers, $r_i$ are real numbers, $\beta$ varies among the ordinal numbers strictly less than $\alpha$ and $\varepsilon$ is any positive real.

$$\exp x = \left\langle 0, \exp(x^L)[x-x^L]_n, \exp(x^R)[x-x^R]_{2n+1} \;\middle|\; \frac{\exp(x^R)}{[x^R-x]_n}, \frac{\exp(x^L)}{[x^L-x]_{2n+1}} \right\rangle, \tag{1.11}$$

see §1.5. All the previous recursive definitions are uniform. The next one (the concatenation of $x$ and $y$) is uniform in $y$, but *not* in $x$;

$$x:y = \langle x^L, x:y^L \mid x^R, x:y^R \rangle \tag{1.12}$$

## 1.7 O-minimality

An ordered field is a commutative field $A$ with a linear order $<$ such that

$$\left.\begin{aligned} x \leq y \rightarrow x+z \leq y+z \\ (z > 0 \,\&\, x \leq y) \rightarrow xz \leq yz \end{aligned}\right\} \forall x,y,z \in A.$$

An *interval* in $A$ is a subset of $A$ of the form $(a,b)$ or $[a,b]$ or $(a,b]$ or $[a,b)$, where $a < b \in A \cup \{\pm\infty\}$.

Let $\mathscr{L}$ be a first order language expanding the language $\mathscr{L}' := (0,1,+,\cdot,<)$ of ordered rings. Suppose that $A$ is an $\mathscr{L}$-structure, such that the restriction of $A$ to $\mathscr{L}'$ is an ordered field.

$A$ is *o-minimal* iff every subset of $A$ definable in $\mathscr{L}$ (with parameters) is a finite union of intervals and of points[(4)].

For a good account on the subject, see [17]. Here, I will only recall some of the properties of an o-minimal structure, which will be useful later.

Of course, the restriction of an o-minimal structure to some smaller language is also o-minimal. If $A$ is elementary equivalent to $B$, then $A$ is o-minimal iff $B$ is. Hence, I can talk of o-minimal (first order) theories.

Every o-minimal theory admits definable Skolem functions. This means that if $C \subseteq A^{n+m}$ is a subset definable (with parameters $a_1,\dots,a_n$), and $D$ is the projection of $C$ over $A^n$, then I can find a function $\phi : D \rightarrow A^m$ definable (with the same parameters) such that $\forall x \in D$, $(x,\phi(x)) \in C$. For every $x \in D$, the function $\phi$ "chooses" an element in the fibre over $x$.

Every complete o-minimal theory T has a prime model, i.e. $P \models$ T such that for every $A \models$ T, $P$ is an elementary substructure of $A$. In particular, if $A \models T$ and I take a subset $S \subseteq A$, I can talk of the model of T generated by $S$.

If $a \in A$ and $C \preceq A$, the type of $a$ over $C$ is determined completely by the cut of $a$ over $C$, i.e. by formulae of the kind $x < c$, $x = c$ and $x > c$ as $c$ varies in $C$. The model generated by $C \cup \{a\}$ is also determined (up to isomorphisms fixing $C$) by this cut.

An o-minimal structure $(A,\mathscr{L})$ is $\kappa$-saturated for some cardinal $\kappa$ iff its restriction $(A,<)$ is $\kappa$-saturated.

[(4)] O-minimality has been defined also for structures which are simply expansions of linear orders, or of linearly ordered groups.

An ordered field is o-minimal iff it is real closed. The theory of real closed fields is complete, and its prime model is given by the field of real algebraic numbers.

As an example of what I said before, suppose that $A$ is a real closed field, and $S \subseteq A$ is a subfield. Then, the model generated by $S$ is $\overline{S}$, the real closure of $S$. The type of $a \in A$ over $S$ is determined by the cut of $a$ over $\overline{S}$, and if $a, a'$ are in the same cut, then $\overline{S(a)}$ is isomorphic to $\overline{S(a')}$, with an unique isomorphism fixing $S$ (and hence $\overline{S}$) and sending $a$ to $a'$.

In the following discussion, the main example of o-minimal theory is $\mathrm{T}_{an}(\exp)$, the theory of $\mathbb{R}$ in the language $\mathscr{L}_{an}(\exp)$, the language of ordered rings plus restricted analytic functions and the total exponentiation.

## 1.8 Other properties

All lemmas in this section are quite elementary. The most interesting result is lemma 1.41, stating some conditions under which the Cauchy completion of a substructure of **No** is again a substructure of **No**.

Given $\alpha \in \mathbf{No}$, let

$$\mathbf{No}(\alpha) := \{x \in \mathbf{No} : \ell(x) < \alpha\} \tag{1.13}$$

$$\mathbf{B}(\alpha) := \{x \in \mathbf{No} : \ell(x) = \alpha\}. \tag{1.14}$$

Given $S \subset \mathbf{No}$, I define

$$\ell(S) := \min\{\alpha \in \mathbf{On} : \alpha > \ell(x) \, \forall x \in S\}.$$

Equivalently, $\ell(S) = \min\{\alpha \in \mathbf{On} : S \subseteq \mathbf{No}(\alpha)\}$. Note that $\ell(\{x\}) = \ell(x) + 1$, and if $x = \langle L \mid R \rangle$, then

$$\ell(x) \leq \ell(L \cup R),$$

with equality holding if $\langle L \mid R \rangle$ is the canonical representation of $x$.

I recall that, given $x, y \in \mathbf{No}$, the concatenation of $x, y$, in symbols $x : y$, is defined as the surreal number obtained juxtaposing $x$ with $y$, considered as functions from some ordinal into $\{+, -\}$. Then, $\ell(x : y) = \ell(x) \oplus \ell(y)$ (ordinal sum). The operation $:$ is associative, 0 is its neutral element, and if $\alpha, \beta \in \mathbf{On}$, then $\alpha : \beta = \alpha \oplus \beta$. If $\alpha \in \mathbf{No}$ is infinite positive, $1 : (-\alpha)$ is an infinitesimal positive surreal number. A surreal number is positive infinite iff it is of the form $\omega : y$ for some $y \in \mathbf{No}$.

*Remark* 1.20. If $x, y \in \mathbf{No}$, $x = \langle x^L \mid x^R \rangle$ is the canonical representation of $x$ and $y = \langle y^L \mid y^R \rangle$ is *any* representation of $y$, then

$$x : y = \langle x^L, x : y^L \mid x^R, x : y^R \rangle.$$

Note that the previous recursive definition is *not* uniform.

**Definition 1.21.** Let $S \subseteq \mathbf{No}$ be a subclass of $\mathbf{No}$. I say that $S$ is initial in $\mathbf{No}$ iff

$$\forall x \in S\, \forall y \in \mathbf{No}\; y \preceq x \rightarrow y \in S.$$

*Example* 1.22. $\mathbf{No}(\alpha)$ is an initial subset of $\mathbf{No}$ for every ordinal $\alpha$.

**Lemma 1.23.** *Given a family of functions (which is a proper set)*

$$\mathfrak{A} = \left\{ f_i \colon \mathbf{No}^{n_i} \rightarrow \mathbf{No} \right\}$$

*and a set $S \subset \mathbf{No}$, there exists a set $\mathbb{K}$ such that $S \subset \mathbb{K} \subset \mathbf{No}$ and $f(\mathbb{K}^n) \subseteq \mathbb{K}$ for every $f \in \mathfrak{A}$. Moreover, I can choose $\mathbb{K} = \mathbf{No}(\alpha)$ for some cardinal number $\alpha$.*

I call such $\mathbb{K}$ a fixed set for $\mathfrak{A}$.

*Proof.* For any ordinal number $\alpha$, let

$$g(\alpha) := \ell\big(\bigcup_{f \in \mathfrak{A}} f(\mathbf{No}(\alpha))\big)$$

i.e.

$$g(\alpha) := \min\left\{ \beta \in \mathbf{No} : \beta > \ell(f(x))\ \forall f \in \mathfrak{A}, \forall x \in \mathbf{No} \text{ s.t. } \ell(x) < \alpha \right\}.$$

$g\colon \mathbf{On} \rightarrow \mathbf{On}$ is a continuous increasing function, therefore it has an arbitrarily large fixed point $\alpha$, which can be taken a cardinal number. $\mathbf{No}(\alpha)$ satisfies the conclusion. □

**Lemma 1.24 (Löwenheim-Skolem).** *Given $\mathfrak{A}$ and $S$ as before, there exists $\mathbb{K}$ which is a fixed set for $\mathfrak{A}$, and such that $(\mathbb{K},\mathfrak{A}) \prec (\mathbf{No},\mathfrak{A})$. I can suppose that $\mathbb{K} = \mathbf{No}(\alpha)$ for some cardinal number $\alpha$, or that $|\mathbb{K}| \leq \aleph_0 + |\mathfrak{A}| + |S|$.*

*Proof.* The condition on the cardinality of $\mathbb{K}$ is the classical Löwenheim-Skolem theorem. Otherwise, apply lemma 1.23 to the skolemization of the structure $(\mathbf{No},\mathfrak{A})$. □

### 1.8.1 Cauchy sequences

**Definition 1.25.** Let $\mathbb{K} \subset \mathbf{No}$ be a divisible subgroup of $\mathbf{No}$. I say that $x \in \mathbf{No}^n$ is $\mathbb{K}$-infinitesimal iff $|x| < \varepsilon$ for every $\varepsilon > 0 \in \mathbb{K}$ (with an analogous definition for $\mathbb{K}$-bounded).

*Example* 1.26. Let $f: \mathbf{No}^n \to \mathbf{No}$ be a continuous function and

$$(\mathbb{K}, f, +, <) \prec (\mathbf{No}, f, +, <).$$

Then $\forall x \in \mathbb{K}^n, y \in \mathbf{No}^n$ if $x - y$ is $\mathbb{K}$-infinitesimal, then $f(x) - f(y)$ is $\mathbb{K}$-infinitesimal.

In the following lemma, for $A \subseteq \mathbf{No}$ to be $\mathbb{K}$-dense in $B \subseteq \mathbf{No}$ means

$$\forall x \in B \, \forall \varepsilon > 0 \in \mathbb{K} \, \exists y \in A \, |x - y| < \varepsilon$$

**Lemma 1.27.** *Let* $\mathbb{K} \subset \mathbf{No}$ *be a subgroup of* $\mathbf{No}$ *and a proper set. If* $\mathbb{K}$ *is* $\mathbb{K}$*-dense in the interval* $(0,1)$ *then* $\mathbb{K} \subseteq \mathbb{R}$.

*Proof.* Suppose that $\mathbb{K}$ is $\mathbb{K}$-dense in $(0,1)$ and that it contains an infinitesimal element $\varepsilon > 0$. Let $\varepsilon \sim \omega^{-c}$ for some $c > 0 \in \mathbf{No}$. Consider the sequence

$$x_\alpha := \omega^{-\frac{c}{\alpha}}$$

where $\alpha$ runs through all non zero ordinal numbers. For each $\alpha$ there is a $z_\alpha \in \mathbb{K}$ such that $|z_\alpha - x_\alpha| < \varepsilon$. But then the $z_\alpha$ are all distinct, so $\mathbb{K}$ cannot be a proper set. □

*Question* 1.28. Under which conditions I can talk of "$\mathbb{K}$-standard part" of a $\mathbb{K}$-bounded element? I.e. when can I say that for every $y \in \mathbf{No}$ which is $\mathbb{K}$-bounded there exists $x \in \mathbb{K}$ such that $x - y$ is $\mathbb{K}$-infinitesimal?

*Answer* 1.28. If and only if $\mathbb{K} = \mathbb{R}$. In fact the existence of a standard part implies that $\mathbb{K}$ is dense in $(0,1)$.

*Question* 1.29. Given $\mathbb{K} \subset \mathbf{No}$ and $x \in \mathbf{No}$ that can be approximated by $\mathbb{K}$, i.e.

$$\forall \varepsilon > 0 \in \mathbb{K} \, \exists y \in \mathbb{K} \, |x - y| < \varepsilon,$$

when does exists a standard part of $x$? To be more precise, for which $\mathbb{K}$ every element of $\mathbf{No}$ which can be approximated by $\mathbb{K}$ has a standard part?

**Definition 1.30 (Cauchy).** Let $\mathbb{K}$ be a linearly ordered Abelian group, $\alpha$ be an ordinal. A sequence $(x_i)_{i<\alpha}$ of elements of $\mathbb{K}$ is a *Cauchy sequence* iff

$$\forall \varepsilon > 0 \in \mathbb{K} \, \exists n \in \alpha \, \forall i, j > n \, |x_i - x_j| < \varepsilon.$$

$\mathbb{K}$ is *Cauchy complete* iff every Cauchy sequence has a limit (in the order topology).

Cauchy sequences and completeness can be defined in a wider context than ordered group.

*Question* 1.31. For which ordinal $\alpha$ is $\mathbf{No}(\alpha)$ Cauchy complete?

If $\mathbb{K}$ is a Cauchy complete field and $G$ is an ordered group, then $\mathbb{K}((\Gamma))$ is Cauchy complete.

**Lemma 1.32.** *If $\alpha$ is an epsilon number, then $\mathbf{No}(\alpha)$ is not complete, nor pseudo-complete.*

*Proof.* If $x \in \mathbf{No}(\alpha)$, then $\omega^x \in \mathbf{No}(\alpha)$ too, because $\alpha$ is an epsilon number. Consider the surreal number

$$y := \sum_{i<\alpha} \omega^{-i}.$$

Every partial sum

$$y_\beta := \sum_{i<\beta} \omega^{-i}$$

is in $\mathbf{No}(\alpha)$ for every $\beta < \alpha$ (see [18]). Moreover, $(y_\beta)_{\beta<\alpha}$ is a Cauchy and pseudo-Cauchy sequence, but $\ell(y) = \alpha$, so it has no limit nor pseudo-limit in $\mathbf{No}(\alpha)$. □

**Definition 1.33.** Let $\mathbb{K} \subset \mathbf{No}$ be a divisible subgroup of $\mathbf{No}$ and $\alpha \in \mathbf{On}$. Let $(x_i)_{i<\alpha}$ be a sequence in $\mathbb{K}$ and $x \in \mathbf{No}$. Then, $(x_i)_{i<\alpha}$ has $\mathbb{K}$-limit $x$ (or simply limit if $\mathbb{K}$ is clear from the context), $x_i \to x$, means that

$$\forall \varepsilon > 0 \in \mathbb{K}\ \exists n < \alpha\ \forall i > n\ |x_i - x| < \varepsilon.$$

The limit is not unique. The class of possible limits of $x_i$ is a convex subclass of $\mathbf{No}$, and I call the simplest element in such class the simplest limit (if it exists).

The Cauchy completion of $\mathbb{K}$ is $\overline{\mathbb{K}}$, the set of simplest limits of all sequences in $\mathbb{K}$.

Note that if $x \in \mathbb{K}$, $x_i \to x$ is the usual notion of convergence in the order topology of $\mathbb{K}$.

**Lemma 1.34.** *Let $\mathbb{K} \subset \mathbf{No}$ be a divisible subgroup of $\mathbf{No}$. Then $\mathbb{K}$ is Cauchy complete iff every element of $\mathbf{No}$ which can be approximated by $\mathbb{K}$ has a $\mathbb{K}$-standard part.*

*Proof.* If $x \in \mathbf{No}$ can be approximated by $\mathbb{K}$,

$$\forall \varepsilon > 0 \in \mathbb{K}\ \exists x_\varepsilon \in \mathbb{K}\ |x_\varepsilon - x| < \varepsilon$$

Consider the sequence $x_\varepsilon$. It is Cauchy, therefore it has a limit $y$ in $\mathbb{K}$, which is the standard part of $x$.

For the converse, consider any Cauchy sequence $(x_i)_{i\in\alpha}$ in $\mathbb{K}$. Then

$$\forall \varepsilon > 0 \in \mathbb{K}\ \exists n(\varepsilon) < \alpha\ \forall i,j \geq n\ |x_i - x_j| < \varepsilon.$$

Without loss of generality, $n(t)$ is monotone decreasing. Take

$$x = \langle x_{n(\varepsilon)} - \varepsilon \mid x_{n(\varepsilon)} + \varepsilon \rangle_{\varepsilon>0\in\mathbb{K}}.$$

$x$ exists. In fact, if $0 < a < b \in \mathbb{K}$, then $n(a) \geq n(b)$. Therefore, $|x_{n(a)} - x_{n(b)}| < b$, so $x_{n(b)} - b < x_{n(a)} < x_{n(b)} + b$. This implies that

$$x_{n(a)} - a < x_{n(b)} + b \text{ and } x_{n(b)-b} < x_{n(a)} + a.$$

Moreover, $x$ is obviously a limit of $x_i$. Therefore, $x$ has a standard part $y \in \mathbb{K}$, which is a limit of $x_i$. □

*Remark* 1.35. The Cauchy completion of $\mathbb{K}$ is Cauchy complete.

*Proof.* Usual diagonalisation proof. Let $(x_i)_{i<\alpha} \in \overline{\mathbb{K}}$, $x \in \mathbf{No}$ such that $x_i \to x$. Let $\varepsilon > 0 \in \overline{\mathbb{K}}$. Let $n(\varepsilon) < \alpha$ such that $\forall i > n \; |x_i - x| < \varepsilon$.

$\varepsilon$ is not $\mathbb{K}$-infinitesimal, therefore we can suppose that $\varepsilon \in \mathbb{K}$.

For every $x_i$ there is a sequence $(x_{i,j})_{j<\beta_i} \in \mathbb{K}$ with limit $x_i$. Let $m(i,\varepsilon)$ be such that $\forall j > m \; |x_i - x_{i,j}| < \varepsilon$. Then

$$z_\varepsilon := x_{n(\varepsilon),m(n(\varepsilon),\varepsilon)}$$

has limit $x$ (I use $\mathbb{K}$ as index set instead of an ordinal, but it does not make any difference, as long as $\mathbb{K}$ is a set). □

**Lemma 1.36.** *Let $\mathbb{K} \subset \mathbf{No}$ be an initial divisible subgroup of $\mathbf{No}$, $x \in \overline{\mathbb{K}}$.*

1. *Let $x = \langle x^L \mid x^R \rangle$ in some representation of $x$. If $x_i$ is the sequence of left options of $x$ (ordered from the farthest to the nearest), and $x \notin \mathbb{K}$ then $x_i \to x$.*

2. *Conversely, if $x_i \to x$ then $\forall x^{\mathbf{o}}$ options of $x$ $\exists i < \alpha$ arbitrary large such that*

$$|x^{\mathbf{o}} - x| \geq |x_i - x|.$$

3. *There exists $x_i \to x$ such that $\forall i \; x_i < x$. More strongly, there exists $x_i \to x$ that is cofinal in the canonical representation of $x$, i.e. $\forall x^L, x^R$ canonical options of $x$ there exist $i, j$ such that $x^L \leq x_i < x < x_j \leq x^R$.*

*Proof.*

1. Suppose not. Then there exists $\varepsilon > 0 \in \mathbb{K}$ such that $\forall i \; x - x_i > 2\varepsilon$. Let $z \in \mathbb{K}$ s.t. $|z - x| < \varepsilon$. Then $x^L < z - \varepsilon < x < x^R$ for all options $x^L, x^R$, therefore $x \preceq z - \varepsilon$. But $z - \varepsilon \in \mathbb{K}$, and $\mathbb{K}$ is initial in $\mathbf{No}$: contradiction.

2. Suppose that there exists $x^{\mathbf{o}}$ a canonical option of $x$ such that $\forall i \; |x^{\mathbf{o}} - x| < |x_i - x|$. But then $x^{\mathbf{o}}$ is a limit of $x_i$, which is absurd.

3. If $x \notin \mathbb{K}$, use 1. Otherwise, $(x - \varepsilon)_{\varepsilon > 0 \in \mathbb{K}} \to x$.

□

**Definition 1.37.** Let $\mathbb{K} \subset \mathbf{No}$ be a subgroup of $\mathbf{No}$, $f\colon \mathbf{No}^n \to \mathbf{No}$ such that $f(\mathbb{K}^n) \subseteq \mathbb{K}$. I say that $f$ preserves $\mathbb{K}$-limits iff for every sequence $(x_i)_{i<\alpha} \in \mathbb{K}^n$ converging to $x \in \overline{\mathbb{K}}^n$ we have $f(x_i) \to f(x)$.

*Remark* 1.38. Suppose that $f\colon \mathbf{No}^n \to \mathbf{No}$ is uniformly continuous on bounded intervals, that $\mathbb{K} \subset \mathbf{No}$ is a subgroup of $\mathbf{No}$ closed under $f$, and that $\mathbb{K} \preceq \mathbf{No}$ in the language $(<,+,f)$. Then $f$ preserves $\mathbb{K}$-limits.

*Question* 1.39. Does $f$ point-wise continuous suffice?

**Lemma 1.40.** *If* $\mathbb{K} \subset \mathbf{No}$ *is an initial subfield of* $\mathbf{No}$*, then* $\overline{\mathbb{K}}$ *is a subfield of* $\mathbf{No}$.

*Proof.* Let $x,y \in \overline{\mathbb{K}}$, let $x_i \to x$, $y_i \to y$, $x_i,y_i \in \mathbb{K}$. By lemma 1.36, I can suppose that $x_i$ and $y_i$ are cofinal in the canonical representation of $x$ and $y$.

If $z$ is the simplest limit of $x_i + y_i$, I must prove that $x+y=z$.

$x_i + y_i \to x+y$, therefore $z \preceq x+y$ by definition of $z$. It is also easy to see that $x_i + y_i$ is cofinal in the canonical form of $x+y$, therefore $x+y \preceq z$.

Proceed similarly for the product, using $xy - (x-x_i)(y-y_i)$ instead of $x_i + y_i$.

For the inverse $1/x$, note that either $x$ is not $\mathbb{K}$-infinitesimal, and in that case proceed as for product, or $x=0$, and there is no need to invert it. □

In general, we have the following:

**Lemma 1.41.** *Let* $f\colon \mathbf{No} \to \mathbf{No}$ *be recursively definable over a family of functions* $\mathfrak{A}$. *Let* $\mathbb{K} \subset \mathbf{No}$ *be an initial divisible subgroup of* $\mathbf{No}$*, closed under* $f$. *Let* $\overline{\mathbb{K}}$ *be the Cauchy completion of* $\mathbb{K}$. *Suppose that* $\overline{\mathbb{K}}$ *is closed under* $\mathfrak{A}$.

*Suppose moreover that* $\forall t \in \overline{\mathbb{K}}, \forall \varepsilon > 0 \in \mathbb{K}$ *there exist* $f^L \in \mathfrak{A}$ *a left option of* $f$ *and* $a,b \in \mathbb{K}$ *such that* $a<t<b$ *and* $\forall t', t''$ *such that* $a \le t' < t < t'' \le b$ $|f^L(t,t',t'') - f(t)| < \varepsilon$*, and the same for right options.*

*Suppose moreover that* $f$ *and every* $g \in \mathfrak{A}$ *preserve* $\mathbb{K}$*-limits. Then* $\overline{\mathbb{K}}$ *is closed under* $f$.

*The same result holds if* $f$ *is a function of many variables whose definition is uniform.*

*Proof.* Let $x \in \overline{\mathbb{K}} \setminus \mathbb{K}$. By lemma 1.36, if $x_i$ are the canonical options of $x$, $x_i \to x$. Moreover $f = \langle f^L \mid f^R \rangle$, so

$$f(x)^{\mathbf{o}} = f^{\mathbf{o}}(x, x_i, x_j, f(x_i), f(x_j)) = z_k,$$

$k$ in some ordinal $\gamma$ (order the $z_k$ from the farthest to the nearest to $z$). Therefore, $z_k \in \overline{\mathbb{K}}$ (because $\overline{\mathbb{K}}$ is closed under $\mathfrak{A}$) and $f(x)$ is a limit of $z_k$ (because of hypothesis of the lemma) and the simplest such.

In many variables I need uniformity, because if $\vec{x} = (x^{(1)},\dots,x^{(n)}) \in \overline{\mathbb{K}}^n \setminus \mathbb{K}^n$, I cannot suppose that all coordinates $x^{(i)} \notin \mathbb{K}$, but only some of them. Therefore, I cannot say that $\vec{x}$ is the $\mathbb{K}$-limit of its canonical options. □

*Remark* 1.42. For a generic ordered field $\mathbb{K}$ (which is a set), the Cauchy completion $\mathbb{K} \subseteq \overline{\mathbb{K}}$ can also be defined by the following universal property:

1. $\mathbb{K}$ is dense in $\overline{\mathbb{K}}$.
2. If $\mathbb{K}$ is dense in the field $\mathbb{F}$, then $\mathbb{F}$ is isomorphic to a subfield of $\overline{\mathbb{K}}$ by a unique isomorphism that is the identity on $\mathbb{K}$.

Therefore, $\overline{\mathbb{K}}$ is unique up to isomorphisms.

**Definition 1.43.** The cofinality of a linearly ordered set $\mathbb{K}$ is $cf(\mathbb{K})$, the smallest ordinal $\alpha$ such that there exists a sequence $(x_i)_{i<\alpha}$ with domain $\alpha$ cofinal in $\mathbb{K}$, i.e.

$$\forall y \in \mathbb{K}\ \exists i < \alpha\ x_i > y.$$

If $\mathbb{K}$ is an ordered field, then $cf(\mathbb{K})$ is always a non-zero limit ordinal. If moreover $\mathbb{K}$ is dense in the field $\mathbb{F}$, then $\mathbb{K}$ and $\mathbb{F}$ have the same cofinality.

**Lemma 1.44.** *The set $\overline{\mathbb{K}}$ can be defined as a quotient of the sets of all Cauchy sequences in $\mathbb{K}$ with domain $\alpha := cf(\mathbb{K})$. The equivalence relation is given by*

$$(x_i)_{i<\alpha} \sim (y_i)_{i<\alpha} \text{ iff } \forall \varepsilon > 0 \in \mathbb{K}\ \exists n < \alpha\ \forall i > n\ |x_i - y_i| < \varepsilon.$$

*The sum and product are given point-wise. The order is given by*

$$(x_i)_{i<\alpha} < (y_i)_{i<\alpha} \text{ iff } (x_i)_{i<\alpha} \nsim (y_i)_{i<\alpha}\ \&\ \exists n < \alpha\ \forall i > n\ x_i < y_i.$$

*The inclusion $\mathbb{K} \subseteq \overline{\mathbb{K}}$ is given by the diagonal map. Moreover, $\overline{\mathbb{K}}$ is Cauchy complete, and if $\mathbb{F}$ is another Cauchy complete ordered field $\mathbb{F}$ containing $\mathbb{K}$ and such that $\mathbb{K}$ is dense in $\mathbb{F}$, then there exists a unique isomorphism between $\mathbb{F}$ and $\overline{\mathbb{K}}$ that is the identity on $\mathbb{K}$.*

*Sketch of proof.* The proof follows the usual one for $\mathbb{K} = \mathbb{Q}$.

Suppose that $\mathbb{K}$ is dense in $\mathbb{F}$. Any map from $\mathbb{F}$ to $\overline{\mathbb{K}}$ assigning to an element $x \in \mathbb{F}$ a Cauchy sequence in $\mathbb{K}$ converging to $x$ induces an isomorphism between $\mathbb{F}$ and a subfield of $\overline{\mathbb{K}}$. □

D. Scott in [16] gives (at least) two other constructions of $\overline{\mathbb{K}}$. In particular, he writes:

> *The element of $\overline{\mathbb{K}}$ are in one-to-one correspondence with the [Dedekind] cuts in $\mathbb{K}$ that are* never *invariant under a nonzero translation by an element of $\mathbb{K}$.*

For a fuller account on the subject, see for instance [9].

### 1.8.2 Properties of $\mathbf{No}(\alpha)$

*Remark* 1.45. Let $a < b \in \mathbf{No}$, $c = \langle a \mid b \rangle$. Then either $c \prec a$ and $c \prec b$, or $a \prec b$ or $b \prec a$.

If $a \prec b$ or $c \prec b$, then $b:(-1) \in (a,b)$. If $b \prec a$ or $c \prec a$, then $a:1 \in (a,b)$.

**Lemma 1.46.** *Let $\alpha \in \mathbf{On}$. Then, $\mathbf{No}(\alpha)$ is discrete iff $\alpha$ is finite. Moreover, $\alpha$ is a limit ordinal iff $\mathbf{No}(\alpha)$ is densely ordered iff*

$$\forall a < b \in \mathbf{No}(\alpha)\ \langle a \mid b \rangle \in \mathbf{No}(\alpha).$$

*Proof.* Density: The second $\Leftrightarrow$ is easy. I will prove the first $\Leftrightarrow$.

$\Leftarrow$ Suppose that $\alpha = \beta + 1$. Obvious if $\alpha$ is finite. If $\alpha$ is infinite, let

$$b = 1:(-\beta).$$

$\ell(b) = 1 \oplus \beta = \beta < \alpha$, where $\oplus$ is the ordinal sum. $0, b \in \mathbf{No}(\alpha)$, but $c = \langle 0 \mid b \rangle \notin \mathbf{No}(\alpha)$.

$\Rightarrow$ Let $a < b \in \mathbf{No}(\alpha)$, $c = \langle a \mid b \rangle$. Then either $c \prec a$, so $c \in \mathbf{No}(\alpha)$, or $a \prec b$, so $(b:-1) \in (a,b)$, and $\ell(b:-1) = \ell(b) \oplus 1 < \alpha$, or similarly if $b \prec a$.

Discreteness:

$\Leftarrow$ Obvious.

$\Rightarrow$ Suppose that $\alpha$ infinite. If $\alpha$ is a limit ordinal, then $\mathbf{No}(\alpha)$ is dense, so it cannot be discrete. Otherwise, $\alpha = \lambda + n$, $\lambda \in \mathbf{On}$ is an infinite limit ordinal, $0 < n \in \mathbb{N}$. Let $b = 1:(-\lambda):(n-1)$. Then, $\ell(b) = \lambda + (n-1) < \alpha$. Suppose that $c$ is the successor of $b$ in the discrete order of $\mathbf{No}(\alpha)$. Then,

$$\forall \gamma < \lambda + (n-1) \in \mathbf{On}, b < c < 1:(-\gamma).$$

Therefore, $b \prec c$, so $\ell(c) > \ell(b)$, and $\ell(c) \geq \alpha$, a contradiction. □

**Lemma 1.47.** *Let $\alpha$ be an ordinal number. $\mathbf{B}(\alpha)$ is discrete iff $\alpha$ is finite.*

*Proof.* $\Leftarrow$ Obvious.

$\Rightarrow$ If $\alpha$ is a limit infinite ordinal, then $b = 1.(-\alpha)$ has no successor. Otherwise, $\alpha = \lambda + n$, with $\lambda \in \mathbf{On}$ an infinite limit, $n \in \mathbb{N}$ and use example 1.55 in the next section. □

**Lemma 1.48.** $\mathbf{No}(\lambda)$ *is a subfield of* $\mathbf{No}$ *iff $\lambda$ is an epsilon number. Moreover, in this case* $\mathbf{No}(\lambda)$ *is also an elementary substructure of* $\mathbf{No}$ *in the language $\mathcal{L}_{an}(\exp)$ and for every $a \in \mathbf{No}(\alpha)$, $\omega^a \in \mathbf{No}(\alpha)$.*

*Proof.* See [18]. □

**Lemma 1.49.** *Let $\alpha, \beta$ be two infinite cardinal numbers. Then, $(\mathbf{No}(\alpha), <)$ is $\beta$-saturated iff $\beta \le cf(\alpha)$, where $cf(\alpha)$ is the cofinality of $\alpha$.*

*Proof.* Let $\mathbb{K} = (\mathbf{No}(\alpha), <)$.

$\Leftarrow$ Let $S \subset \mathbb{K}$ be as set of cardinality less than $cf(\alpha)$, let $T = T_S(x)$ be a 1-type over it. The important fact is that $\ell(S) < \alpha$. I want to prove that $T$ has a realisation in $\mathbb{K}$. Suppose that no $s \in S$ realises $T$ (otherwise the conclusion follows). By lemma 1.46, $\mathbb{K}$ is densely ordered without endpoints, so its theory has elimination of quantifiers. Therefore, without loss of generality every formula in $T$ is of the type either $(x > s)$ or $(x < s)$ for some $s \in S$. Let

$$L := \{s \in S : (x > s) \in T\}$$
$$R := \{s \in S : (x < s) \in T\}.$$

It is obvious that $L < R$, and that $a = \langle L \mid R \rangle$ realises $T$. Moreover, $\ell(L \cup R) \le \ell(S) < \alpha \Rightarrow \ell(a) \le \ell(S) \Rightarrow a \in \mathbb{K}$.

$\Rightarrow$ Suppose for contradiction that $\mathbb{K}$ is $\beta$-saturated and that $cf(\alpha) < \beta$. In that case, for some $\gamma < \beta$ there exists a $\gamma$-sequence of ordinal numbers $\lambda_i$ which is cofinal in $\alpha$. But then, by saturation, there exists $c \in \mathbb{K}$ that is greater than each of the $\lambda_i$; therefore $\alpha$, the simplest such $c$, is in $\mathbb{K}$, which is impossible. □

**Corollary 1.50.** *Let $\alpha$ be an uncountable cardinal. Let $\mathbb{K}$ be an o-minimal expansion of $(\mathbf{No}(\alpha), <)$. If $2^\beta = \beta^+ = \alpha$, or if $\alpha$ is inaccessible, then $\mathbb{K}$ is saturated.*

*Moreover, **No** itself is $\alpha$-saturated for every cardinal $\alpha$.*

**Lemma 1.51 (Compactness).** *Let $\mathfrak{A}$ be a set of subintervals of **No**, where every interval has endpoints in $\mathbf{No} \cup \{\pm\infty\}$. If $\mathfrak{A}$ is a covering of **No**, then it has a finite sub-covering.*

Note that $\mathfrak{A}$ must be a proper set. The lemma can be generalised to other $|\mathfrak{A}|^+$-saturated ordered groups $\mathbb{K}$.

*Proof.* Suppose not. Then, the type in $x$ (in the language $(+, <)$) given by formulae

$$x \notin A_1 \cup \cdots \cup A_n,$$

for $n \in \mathbb{N}$ and $A_i \in \mathfrak{A}$, $i = 1, \dots n$, would be consistent, and by saturation there would be $x \in \mathbf{No}$ satisfying it, contradicting the fact that $\mathfrak{A}$ is a covering of **No**. □

## 1.9 Orderings

This section deals with some basic properties of partial orders and of ordered trees. It ends with a characterisation of **No** (remark 1.73).

### 1.9.1 Set theoretic background

The discussion in this thesis takes place in the axiomatic system NBG of von Neumann, Bernays and Gödel, with global choice. Its main features is that it treats classes as legitimate objects; sets are classes which are member of some other class. The main distinguishing axioms are the following.

**Axiom of predicative comprehension for classes.** For any condition(5) $\phi(x)$ that contains only quantifiers over sets (and not classes), there exists a class $A$ which consists exactly of those sets $x$ which satisfy $\phi(x)$.

**Axiom of global choice.** There exists a function(6) $F$ whose domain contains all non-void sets, and such that for every non-void set $x$, $F(x) \in x$.

See [8] for more details.

I will also talk of a *collection* of classes as an abbreviation device for a condition $\phi(x)$ without quantifiers over classes. Similarly, given two collections $C$ and $D$ (determined by a condition $\phi_C(x)$ and $\phi_D(x)$), a function from $C$ into $D$ is given by a formula $\psi(x,y)$ (with only $x,y$ as free variables), such that

$$\forall x \in C\, \exists! y \in D\ \psi(x,y).$$

A relation over a collection $C$ is again given by a formula $\phi(x,y)$ (without quantifiers over classes). For instance, I will define $\mathbf{No}^{\mathscr{D}}$, the Dedekind completion of **No**: it is a collection and not a class.

Given a well founded partial and set-like order $(A,\leq)$ (see 1.9.2 for the definition), it is possible to give definitions by transfinite recursion on $A$, without needing to go outside NBG. Moreover, it is possible to prove formulae without bounded class variables by induction on $A$.

### 1.9.2 Partial Orders

**Definition 1.52.** A quasi-ordered class (or *quasi-order* for short) is a pair $(A,\leq)$, where $A$ is a set or a class, and $\leq$ is a binary relation on $A$ satisfying the following axioms:

(5) A condition is a formula $\phi(x)$ with only $x$ as free variable.

(6) A function is a class whose elements are ordered pairs and satisfying the usual properties.

**Transitivity** $\forall x,y,z \in A\ x \le y$ and $y \le z$ imply $x \le z$.

**Reflexivity** $\forall a \in A\ a \le a$.

$a \sim b$ means $a \le b$ and $b \le a$. $\sim$ is an equivalence relation on $A$. If $(A, \le)$ satisfies also

**Antisymmetry** $\forall a, b \in A\ a \sim b$ iff $a = b$,

the order is called a *partial order*.

If neither $a \le b$ nor $b \le a$ then $a$ and $b$ are *incomparable*, in symbols: $a \parallel b$. By $a < b$ I mean $a \le b$ and $b \not\le a$.

A quasi-order is *total* iff no two elements are incomparable, i.e. for every $a, b \in A$ either $a \le b$ or $b \le a$. A *linear* order is a total partial order.

A *chain* is a subclass of $A$ which is linearly ordered by $\le$.

A quasi-order is *well-founded* iff there is no infinite sequence $(x_i)_{i \in \mathbb{N}}$ in it such that $x_{i+1} < x_i$.

Given a quasi-order $(A, \le)$, $\le$ induces a partial order on the quotient $A/_\sim$, the *canonical quotient* of $(A, \le)$.

*In the following, all orders will be partial orders, unless explicitly stated otherwise.*

A quasi-ordered class $A$ is *set-like* iff it is well-founded and for every $a \in A$ the class of predecessors of $a$

$$\mathscr{P}(a) := \{x \in A : x < a\}$$

is a proper set.

Given a well-founded partial order $(A, <)$ and $a \in A$, the *length* of $a$ is inductively defined as

$$\ell(a) = \min\{\, \alpha \in \mathbf{On} : \alpha > \ell(x)\ \forall x < a\}$$

(or $+\infty$ if the minimum does not exist). If $A$ is set-like, then $\ell(a)$ is defined $\forall a \in A$.

If $S \subset A$, let

$$\ell(S) := \min\{\, \alpha \in \mathbf{On} : \alpha > \ell(x)\ \forall x \in S\},$$

or $+\infty$ if the minimum does not exist. If $a \in A$, let $\mathscr{P}(a) := \{x \in A : x < a\}$. Then, $\ell(a) = \ell(\mathscr{P}(a))$.

*Remark* 1.53. Let $\alpha, \beta \in \mathbf{On}$. The natural sum $\alpha + \beta$ is the smallest ordinal strictly greater than $\alpha + \beta'$ and $\alpha' + \beta$ for every $\alpha' < \alpha$ and $\beta' < \beta$[7].

[7]The natural sum of two ordinals can be defined either via their Cantor normal form, or using this remark as a definition.

In particular, the natural sum of two ordinals coincides with their sum as surreal numbers.

Given two partial orders $(A, <_A)$ and $(B, <_B)$, how can I induce an order on the product $A \times B$?

**Definition 1.54** (lex)**.** The *lexicographic product* (or *ordinal product*) of $A, B$ is the partial order$(A \times B, \underset{\text{lex}}{\leq})$ defined by

$$(a,b) \underset{\text{lex}}{\leq} (a',b') \text{ iff } a < a' \vee (a = a' \ \& \ b \leq b').$$

If $A$ and $B$ are both linear, $C := (A \times B, \underset{\text{lex}}{\leq})$ is linear too. If $A$ and $B$ are both well-founded, $C$ is well founded too. But $(\mathbf{On} \times \mathbf{On}, \text{lex})$ is not set-like, even if $\mathbf{On}$ is. If $A$ and $B$ are both well founded, and $a \in A, b \in B$, then

$$\ell(a,b) = \big(\ell(a) \otimes \ell(B)\big) \oplus \ell(b).$$

where $\oplus$ and $\otimes$ are the ordinal sum and product.

*Example* 1.55. For every $\alpha, \beta \in \mathbf{On}$

$$(\mathbf{B}(\alpha) \times \mathbf{B}(\beta), \text{lex}) \simeq \mathbf{B}(\alpha \oplus \beta) \tag{1.15}$$

with the isomorphism given by concatenation.

**Definition 1.56** (bnd)**.** The *cardinal product*[8] $(A \times B, \underset{\text{bnd}}{\leq})$ is the partial order

$$(a,b) \underset{\text{bnd}}{\leq} (a',b') \text{ iff } a \leq a' \ \& \ b \leq b'.$$

The cardinal product can be easily generalised to the product of more than two factors.

If $A$ and $B$ are both well-founded, $C$ is well founded too. If $A$ and $B$ are both set-like, so is $C$. But $C$ is almost never linear, even if $A, B$ are. For $A$ and $B$ well-founded, $\ell(a,b) = \ell(a) + \ell(b)$.

*Proof.* Induction on $(a,b)$. By definition of length and inductive hypothesis,

$$\begin{aligned} \ell(a) + \ell(b) &= \min\{\gamma : \gamma > \ell(a',b') \ \forall a' \leq a, b' \leq b, (a',b') \neq (a,b)\} \\ &= \min\{\gamma : \gamma > \ell(a') + \ell(b') \ \forall a' \leq a, b' \leq b, (a',b') \neq (a,b)\}, \end{aligned}$$

and the conclusion follows from remark 1.53. □

[8] It coincides with the product in the category of partial orders.

**Definition 1.57** (sym). $(A \times A, \underset{\text{sym}}{\leq})$ is the *symmetric product*:

$$(a,b) \underset{\text{sym}}{\leq} (c,d) \text{ iff } (a,b) \underset{\text{bnd}}{\leq} (c,d) \vee (a,b) \underset{\text{bnd}}{\leq} (d,c).$$

This ordering is only a quasi-ordering: $(a,b) \sim (b,a)$. $C$ is almost never total. If $A$ is well founded, so is $C$; if $A$ is set-like, so is $C$, and again $\ell(a,b) = \ell(a) + \ell(b)$.

*Proof.* Induction on $(a,b)$. By definition and inductive hypothesis,

$$\ell(a,b) = \min\left\{ \gamma : \gamma > \ell(c) + \ell(d)\ \forall (c,d) \text{ s.t. } (c,d) \underset{\text{bnd}}{<} (a,b) \vee (c,d) \underset{\text{bnd}}{<} (b,a) \right\},$$

and the conclusion follows from the commutativity of $+$. □

**Definition 1.58** (bsym). If $A$ is a linear order, then bsym and $\text{bsym}_2$ are the *bounded symmetric* orderings on $A \times A$:

$$(a,b) \underset{\text{bsym}}{\leq} (c,d) \text{ iff } \max(a,b) < \max(c,d) \vee$$
$$\vee \left(\max(a,b) = \max(c,d)\ \&\ \min(a,b) \leq \min(c,d)\right)$$
$$(a,b) \underset{\text{bsym}_2}{\leq} (c,d) \text{ iff } \min(a,b) < \min(c,d) \vee$$
$$\vee \left(\min(a,b) = \min(c,d)\ \&\ \max(a,b) \leq \max(c,d)\right)$$

In the general case, when $A$ is not necessarily linear, bsym is defined as:

$$(a,b) \underset{\text{bsym}}{\leq} (c,d) \text{ iff } (a,b) \underset{\text{sym}}{\leq} (c,d) \vee (a < c\ \&\ b < c) \vee (a < d\ \&\ b < d).$$

In general, neither bsym nor $\text{bsym}_2$ are partial orders, only quasi-orders. Both are well-founded (total) if $A$ is. bsym is set-like, but $\text{bsym}_2$ is not. The formula for $\ell(a,b)$ is quite complicated in both cases. (It would be interesting to define $\text{bsym}_2$ for any partial order $A$).

*Example* 1.59. $\ell(\mathbb{N} \times \mathbb{N}, \text{bsym}) = \omega$:

$$(0,0) < (1,0) < (1,1) < (2,0) < (2,1) < (2,2) <$$
$$< (3,0) < (3,1) < (3,2) < (3,3) < \cdots$$

$\ell(\mathbb{N} \times \mathbb{N}, \text{bsym}_2) = \omega \times \omega$:

$$(0,0) < (0,1) < (0,2) < \cdots < (1,1) < (1,2) < (1,3) < \cdots < (2,2) < \cdots$$

A well founded order $\leq$ on $A \times B$ gives the means of doing induction on pairs $(a,b)$. The greater is $\ell(a,b)$, the more powerful is the induction (i.e. the stronger is the inductive hypothesis). On the other hand, the smaller is $\ell(a,b)$, the more efficient is a recursive definition of a function $f$ on $A \times B$ (i.e. I need to know $f(a',b')$ for less values before being able to compute $f(a,b)$). In the case of **No**, if $<$ is not set-like, there is a danger that $f$ is not defined for some input: this is the reason why I had to use the cardinal product bnd instead of the lexicographic one lex in the definition of a function of many variables.

The bounded symmetric order bsym is quite important: when I do induction on pairs on functions $f, g : \mathbf{No} \to \mathbf{No}$, I often use bsym. For induction on pairs of elements of **No** I use the cardinal product bnd (what Gonshor calls induction on the natural sum of $a, b$).

**Lemma 1.60.** *Let $(A, \leq)$ be a quasi-ordered set, $\Gamma$ a group of automorphisms of $(A, \leq)$. Suppose that*

$$\forall x \in A\ \forall \gamma \in \Gamma\ \left(\gamma x \sim x \vee \gamma x \parallel x\right).$$

*Introduce on $A$ the relation $R$ given by $xRy$ iff $x \sim \gamma y$ for some $\gamma \in \Gamma$. Then, $R$ is an equivalence relation. Let $B := A/_\Gamma$ be the quotient of $A$ under $R$. Then $\leq$ induces a partial order on $B$, which is well-founded if $A$ is.*

*Proof.* The definition of $\leq$ on $B$ is $\bar{a} \leq \bar{b}$ iff $a \leq \gamma b$ for some $\gamma \in \Gamma$.

*Equivalence relation*: Reflexivity is obvious. For transitivity, let $\bar{a}R\bar{b}$ and $\bar{b}R\bar{c}$, i.e. $a \sim \gamma b$, $b \sim \lambda c$ for some $\gamma, \lambda \in \Gamma$. Then, $a \sim \gamma\lambda c$.

*Good definition of* $\leq$: If $\bar{a} = \bar{b}$ and $\bar{a} \leq \bar{c}$, I have to prove that $\bar{b} \leq \bar{c}$. The hypothesis means that $a \sim \gamma b$ and $a \leq \lambda c$ for some $\gamma, \lambda \in \Gamma$. This implies that $b \leq \gamma^{-1}\lambda c$.

*Reflexivity*: Obvious.

*Antisymmetry*: If $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{a}$, then $a \leq \gamma b$, $b \leq \lambda a$. Therefore, $a \leq \gamma b \leq \gamma\lambda a$. So, by hypothesis on $\Gamma$, $a \sim \gamma\lambda a$, therefore $a \sim \gamma b$, i.e. $\bar{a} = \bar{b}$.

*Transitivity*: $\bar{a} \leq \bar{b} \leq \bar{c}$. Then $a \leq \gamma b$, $b \leq \lambda c$. Therefore, $a \leq \gamma\lambda c$.

*Foundation*: Suppose that $\forall i \in \mathbb{N}\ \bar{a}_{i+1} < \bar{a}_i$. Then, $a_i > \gamma_i a_{i+1}$ for some $\gamma_i \in \Gamma$. Therefore,

$$a_0 > \gamma_0 a_1 > \gamma_0\gamma_1 a_2 > \gamma_0\gamma_1\gamma_2 a_3 \cdots$$

and $A$ is not well-founded. □

*Example* 1.61. Let $A = (B \times B, \underset{\text{bnd}}{\leq})$ and let $\Gamma$ be the group generated by the swapping of coordinates. The induced ordering on the quotient is (the canonical quotient of) sym.

**Definition 1.62.** Let $(A_i, \leq)_{i \in S}$ be a family of partially ordered sets. Its direct product is the set $\prod_{i \in S} A_i$, given by the direct product of the $A_i$, endowed with the order

$$(x_i)_{i \in S} \leq (y_i)_{i \in S} \Leftrightarrow \forall i \in S\, x_i \leq y_i.$$

Suppose that every $A_i$ has a minimum 0. The *support* of $x := (x_i)_{i \in S} \in \prod_{i \in S} A_i$ is the set

$$\text{supp}(x) := \{ i \in S : x_i \neq 0 \}.$$

If $(S, <)$ is a linearly ordered set, the *lexicographic product* $\Gamma_{i \in S} A_i$ is the set of all elements of $\prod_{i \in S} A_i$ with well-ordered support, with the partial order defined by

$$x := (x_i)_{i \in S} < (y_i)_{i \in S} \Leftrightarrow x_{i_0} < y_{i_0},$$

where $i_0$ is the smallest $i \in S$ such that $x_i \neq y_i$. If all factors are the same $A$, I call the lexicographic power of $A$ (over the base $S$) the corresponding lexicographic product.

If $S$ is anti well-ordered and each $A_i$ is well-founded, then $\Gamma_{i \in S} A_i$ is well-founded too. If all factors are linearly ordered, the lexicographic product is also linear.

*Example* 1.63. The order on $\mathbf{No}[x]$ introduced in the proof of lemma 4.2 is the lexicographic power of **No** over the base $\mathbb{N}$ with reversed order.

**Definition 1.64.** Let $(A, <)$ be a quasi-ordered set. Let $A^{<\omega}$ be the class of all $n$-tuples of elements $A$ as $n \in \mathbb{N}$, with the quasi-order defined by $(x_i)_{i<n} \leq (y_j)_{j<m}$ iff there exists a function

$$f : \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, m-1\}$$

such that $\forall i < n\, x_i \leq y_{f(i)}$ and $x_i < y_{f(i)}$ whenever $\exists j \neq i\, f(i) = f(j)$. In general, this quasi-order is not a partial order. The *symmetric power* of $A$ is $A^{(\mathbb{N})}$, the quotient of $A^{<\omega}$ under the action of the permutation group of $\mathbb{N}$.

*Remark* 1.65. Let $A, A^{<\omega}, A^{(\mathbb{N})}$ as in the previous definition. The symmetric power $A^{(\mathbb{N})}$ is a partial order. If $A$ is well-founded (or set-like, or totally ordered), so are $A^{<\omega}$ and $A^{(\mathbb{N})}$.

**Lemma 1.66.** *Let $(A, <)$ be a set-like partial order. Let $x_0, \ldots, x_n \in A$. Let $\alpha_1, \ldots, \alpha_m$ be the set of lengths of $x_0, \ldots, x_n$, ordered by the greatest to the smallest. For $i = 1, \ldots, m$, let $k_i$ be the number of $j$ such that $\ell(x_j) = \alpha_i$. Then,*

$$\ell(x_0, \ldots, x_n) \leq k_1 \omega^{\alpha_1} + \cdots + k_m \omega^{\alpha_m},$$

*with equality holding if for each $i = 0 \ldots, m$ $\mathscr{P}(x_i)$ is a linearly ordered set.*

*Sketch of proof.* I will treat only the case in which $x_i$ are all ordinal numbers. The map $\psi : \mathbf{On}^{(\mathbb{N})} \to \mathbf{On}$ defined by

$$\psi(x_1, \ldots, x_n) = \omega^{x_1} + \cdots + \omega^{x_n}$$

is surjective (where $+$ is the natural sum of ordinals), by well known facts on the Cantor normal form of an ordinal. Moreover, $x < y$ implies $\psi(x) < \psi(y)$, and the conclusion follows. □

The symmetric power is a generalisation of bsym. See also [5] and [4] for other results on partial orders.

### 1.9.3 Ordered trees

**Definition 1.67.** Let $\leq$ be a (partial) order on a class $A$. Given a subclass $S \subseteq A$, $a \in A$ is an *upper bound* for $S$ iff $\forall x \in S\ a \geq x$.
$a$ is the *least upper bound* of $S$ iff $a$ is an upper bound for $S$ and $a \leq x$ for every $x$ upper bound for $S$.
The *greatest lower bound* is defined in a similar way.
If $\leq$ is linear, then a $S \subseteq A$ is a *convex* subclass iff

$$\forall x, y \in S\ \forall z \in A\ x \leq z \leq y \to z \in S.$$

**Definition 1.68 (Tree).** An ordered class $(A, \prec)$ is a *tree* iff

T1. $A$ is well-founded and set-like.

T2. $\forall a \in A$, $\mathscr{P}(a) := \{x \in A : x \prec a\}$ is linearly ordered.

T3. Every non-empty subclass of $A$ has a g.l.b.

A tree $A$ is a *binary tree* iff every $a \in A$ has at most 2 immediate successors. A structure $(A, \prec)$ is a *weak tree* if the set-like condition is dropped.

**Lemma 1.69.** *If $A$ is a weak tree, then*

1. *$A$ has a minimum, the* root *of the tree.*

2. *Every chain which has an upper bound has a l.u.b.*

*Proof.* 1. The g.l.b. of $A$ itself is the root of $A$.

2. Let $C \subset A$ be a bounded chain. The class of upper bounds of $C$ is non-empty, therefore it has a g.l.b. $a$, which is the l.u.b. of $C$.

□

**Definition 1.70 (Ordered tree).** A structure $(A,<,\prec)$ is an *ordered tree* iff

OT1. $(A,<)$ is a linear order.

OT2. $(A,\prec)$ is a tree.

OT3. For every $S \subseteq A$ $<$-convex subclass of $A$, the $\prec$-g.l.b. of $S$ is in $S$.

OT4. For every $a \in A$, $\mathscr{S}(a) := \{x \in A : a \preceq x\}$ is a $<$-convex subclass of $A$.

If in the above definition a tree is replaced by a weak tree, we would get a weak ordered tree.

In the following, convex will mean convex with respect to $<$. If $a \leq b \in A$ then $[a,b]$ is the convex subclass $\{x \in A : a \leq x \leq b\}$.

**Lemma 1.71.** *Every ordered tree is isomorphic in a unique way to a initial subtree of* **No**.

*Proof.* Let $(A,<,\prec)$ be a tree.

*Claim* 1. Suppose that $x,y,z \in A$, $x \preceq y$, $z \prec x$ and $z < x$. Then $z < y$. Similarly for $z > x$.

$\mathscr{S}(x)$ is convex, $z \notin \mathscr{S}(x)$ and $z < x$, therefore $z < \mathscr{S}(x)$. In particular, $z < y$.

I will define the isomorphism $\phi : A \to \mathbf{No}$ by induction. First, if 0 is the root of $A$, then $\phi(0) = 0$. Suppose that I have already defined $\phi$ on $\mathscr{P}(a)$, such that $\phi \restriction \mathscr{P}(a)$ is an ordered tree isomorphism. Then define

$$\phi(a) = \langle L \mid R \rangle,$$

where $L = \{\phi(x) : x \prec a \,\&\, x < a\}$ and $R = \{\phi(x) : x \prec a \,\&\, x > a\}$. By T1, $\phi(x)$ exists. I need to check that $\phi$ is an isomorphism, i.e. that if $x < y$ then $\phi(x) < \phi(y)$ and if $x \prec y$ then $\phi(x) \prec \phi(y)$.

If $x \prec y$, from the claim, the definition of $\phi$ and the cofinality theorem on **No** it follows that $\phi(x) \prec \phi(y)$.

If $x < y$, then by T3 the class $[x,y]$ has a $\prec$-g.l.b. $z$ and by OT3 $z \in [x,y]$. Without loss of generality, I can suppose that $x \leq z < y$. Then, by definition of $\phi(x)$, $\phi(x) \leq \phi(z)$, and, by definition of $\phi(y)$, $\phi(z) < \phi(y)$; therefore, $\phi(x) < \phi(y)$. □

**Corollary 1.72.** *For an ordered tree $(A,<,\prec)$, axiom* T2 *is a consequence of the other axioms. Moreover, the following statements are also true:*

- *$(A,\prec)$ is a binary tree.*

- $\forall \alpha \in \mathbf{On}$, *the class*

$$A(\alpha) := \{x \in A : \ell(x) < \alpha\}$$

*is a proper set.*

*Remark* 1.73. $(\mathbf{No}, <, \prec)$ can be defined as *the* maximal ordered tree, or equivalently as *the* ordered tree such that if $L < R$ are two sub*set*s of **No**, then the cut $(L \mid R)$ is non-empty.

An alternative definition of ordered trees, which is surprisingly simple, is the following.

**Definition 1.74 (Weak ordered tree).** An weak ordered tree is a triple $(A, <, f)$ such that:

OT1'. $(A, <)$ is a linearly ordered class.

OT2'. $f$ is a function from $\mathfrak{C}(A)$, the collection of non-empty $<$-convex subclasses of $A$, into $A$.

OT3'. For every $S \in \mathfrak{C}(A)$, $f(S) \in S$.

OT4'. For every $S, T \in \mathfrak{C}(A)$ such that $S \subseteq T$ and $f(T) \in S$ we have $f(S) = f(T)$.

Given a weak ordered tree (in the sense of 1.70) we obtain an weak ordered tree (in the sense of 1.74 ) defining $f(S)$ be the g.l.b. of $S$ for $S \in \mathfrak{C}(A)$. For the converse:

**Lemma 1.75.** *Let $(A, <, f)$ be a weak ordered tree. Define*

$$x \preceq y \text{ iff } \begin{cases} f([x,y]) = x & \text{if } x \leq y \text{ or} \\ f([y,x]) = x & \text{if } y < x. \end{cases}$$

*Then, $(A, <, \preceq)$ is a weak ordered tree.*

*Proof.*

*Claim* 1. $\preceq$ is a partial order.

Anti-symmetry and reflexivity are obvious. For transitivity, let $x \preceq y \preceq z$. Without loss of generality, $x \leq y$.

Suppose that $x \leq y \leq z$. By OT3', $a := f([x,z]) \in [x,y] \cup [y,z]$. If $a \in [x,y]$ then by OT4', $a = x$, i.e. $x \preceq z$. If $a \in [y,z]$, then $a = y$. Therefore $a \in [x,y]$ and so $a = x$.

Suppose that $x \leq z \leq y$. By OT4', $f([x,z]) = x$.

Suppose that $z \leq x < y$. This is impossible, because $f([z,y]) = y$, therefore, by OT4', $f([x,y]) = y$, so $x = y$.

*Claim* 2. For every $S \in \mathfrak{C}(A)$, $f(S)$ is the $\preceq$-minimum of $S$.

Let $a := f(S)$, and let $x \in S$. Without loss of generality, $a < x$. Then, $[a,x] \subseteq S$, therefore $a = f([a,x])$.

*Claim* 3. For every $a \in A$, $\mathscr{P}(a)$ is linearly ordered by $\preceq$.

Let $x \preceq a$ and $y \preceq a$. Without loss of generality, I can suppose that $x < a$. If $x \leq y < a$, then $x \preceq y$. If $y < x < a$, then $y \preceq x$. If $x < a < y$, let $b := f([x,y])$. If $b \in [x,a]$, then $b = x$, therefore $x \preceq y$. Otherwise, $y \preceq x$.

*Claim* 4. $\preceq$ is well-founded.

Suppose not. Let $x_0 \succeq x_1 \succeq x_2 \dots$ be an infinite sequence. Without loss of generality, after taking a subsequence, I can suppose that $(x_i)_{i \in \mathbb{N}}$ is an infinite $<$-descending sequence, i.e. $x_0 \geq x_1 \geq x_2 \dots$. Then, $f([x_0,x_i]) = x_i$ for every $i \in \mathbb{N}$.

Let $C := \bigcup_i [x_0,x_i]$. $C \in \mathfrak{C}(A)$, therefore $c := f(C)$ is defined, and $c \in C$. Therefore, $c \in [x_0,x_n]$ for some $n \in \mathbb{N}$ and so $c = x_n$. Therefore $x_i = x_n$ for every $i > n$.

*Claim* 5. For every $a \in A$, the class $\{x : a \preceq x\}$ is convex.

Let $a \preceq x_1$, $a \preceq x_2$ and $x_1 < y < x_2$. Without loss of generality, $a \leq y$. Then, $[a,y] \subset [a,x_2]$, therefore $f([a,y]) = a$.

*Claim* 6. Every nonempty subclass $S \subseteq A$ has a $\preceq$-g.l.b.

Let $T$ be the convex hull of $S$. I say that $a := f(T)$ is the g.l.b. of $S$. By claim 2, $a$ is a lower bound for $T$, and a fortiori for $S$. Let $y$ be a lower bound for $S$. $a \in T$, therefore there exist $x_1, x_2 \in S$ such that $x_1 \leq a \leq x_2$. $y \preceq x_1$ and $y \preceq x_2$, therefore, by the previous claim, $y \preceq a$. So, $a$ is the g.l.b. of $S$. □

*Example* 1.76. Let $(A,<)$ be the lexicographic sum $\mathbf{On} \oplus \mathbf{On}$, and let $\preceq$ coincide with $\leq$. Then $A$ is a weak ordered tree, but it is not set-like.

Of course, for proper sets, weak ordered tree and ordered tree are the same concept.

For a different account on the subject, see [7].

Lemma 1.75 can be generalised.

**Definition 1.77.** Let $A$ be a set. Let $\mathfrak{C}$ be a family of subsets of $A$ such that:

1. $A \in \mathfrak{C}$, $\emptyset \notin \mathfrak{C}$.
2. $\forall U \subseteq \mathfrak{C} \bigcap U \neq 0 \rightarrow (\bigcup U \in \mathfrak{C} \mathbin{\&} \bigcap U \in \mathfrak{C})$.

For $x_1, \dots, x_n \in A$, let $[x_1, \dots, x_n]$ be the l.u.b. of $\{x_1, \dots, x_n\}$, i.e.

$$[x_1, \dots, x_n] := \bigcap \{S \in \mathfrak{C} : x_1 \in S \mathbin{\&} \dots \mathbin{\&} x_n \in S\}$$

A $\mathfrak{C}$-tree is given by a function $f : \mathfrak{C} \rightarrow A$ such that

3. $\forall S \in \mathfrak{C}\ f(S) \in S$.

4. $\forall x \in A\ f([x]) = x$.

5. $\forall S \subseteq T \in \mathfrak{C}\ \big(f(T) \in S \rightarrow f(S) = f(T)\big)$.

*Example* 1.78. Let $(A, <, f)$ be an ordered tree (and a set). Let $\mathfrak{C} := \mathfrak{C}(A)$ be the set of non-empty convex subsets of $A$. Then, $(A, f)$ is a $\mathfrak{C}$-tree.

**Lemma 1.79.** *Every $\mathfrak{C}$-tree is a tree in a canonical way, with the definition $x \preceq y$ iff $f([x,y]) = x$.*

*Proof.*

*Claim* 1. $\preceq$ is a partial order.

Antisymmetry is obvious, reflexivity follows from axiom 4. For transitivity, let $x \preceq y \preceq z$. $[x,y] \cup [y,z] = [x,y,z] \in \mathfrak{C}$; let $c := f([x,y,z])$. Therefore, $c \in [x,y]$ or $c \in [y,z]$. If $c \in [y,z]$, $c = y$, so $c \in [x,y]$. If $c \in [x,y]$, $c = x$, therefore $x \preceq z$.

*Claim* 2. For every $S \in \mathfrak{C}$, $f(S)$ is the minimum of $S$.

Let $a := f(S)$, and let $x \in S$. Then, $[a,x] \subseteq S$, therefore $a = f([a,x])$.

*Claim* 3. For every $a \in A$, $\mathscr{P}(a)$ is linearly ordered by $\preceq$.

Let $x, y \in \mathscr{P}(a)$, $b := f([x,y,a])$. If $b \in [x,a]$, then $b = x$, so $x \preceq y$, otherwise $b \in [y,a]$, so $y \preceq x$.

*Claim* 4. $\preceq$ is well-founded.

Let $x_o \succeq x_1 \succeq x_2 \succeq \dots$. Therefore, $f([x_0,x_i]) = x_i$ for every $i \in \mathbb{N}$. Let $C := \bigcup_i [x_0,x_i]$; $C \in \mathfrak{C}$, so I can define $c := f(C)$. $c \in [x_0,x_n]$ for some $n \in \mathbb{N}$, so $c = x_n$, therefore $x_i = x_n$ for every $i \geq n$.

*Claim* 5. For every $a \in A$, the set $\mathscr{S}(a) := \{x \in A : a \preceq x\}$ is in $\mathfrak{C}$.

In fact, $\mathscr{S}(a) = \bigcup\{[a,x] : a \preceq x\}$.

*Claim* 6. Every non-empty subset $R \subseteq A$ has a g.l.b.

Let $T := \bigcap\{C \in \mathfrak{C} : R \subseteq C\}$. Then, $a := f(T)$ is the g.l.b. of $R$. In fact, $a$ is a lower bound of $T$, and a fortiori of $R$. Moreover, if $y$ is a lower bound for $R$, $R \subseteq \mathscr{S}(y)$ and $\mathscr{S}(y) \in \mathfrak{C}$, therefore $T \subseteq \mathscr{S}(y)$, so $y \preceq a$. □

*Example* 1.80. Let $(A, \preceq)$ be a tree (and a set). Define

$$\mathfrak{C} := \{\mathscr{S}(a) : a \in A\}, \quad f(\mathscr{S}(a)) := a.$$

Then $(A, f)$ is a $\mathfrak{C}$-tree.

**Lemma 1.81.** *Let $(A,<,\preceq)$ be an weak ordered tree, $(B,<)$ a linearly ordered class such that $(A,<)$ is a dense sub-order of $(B,<)$. Then there is a unique tree structure on $B$ such that $B$ is a weak ordered tree and $A$ is an initial subtree of $B$.*

*Proof.* I have to define $f:\mathfrak{C}(B)\to B$ satisfying definition 1.74, extending $f':\mathfrak{C}(A)\to A$. Let $S$ be a non-empty convex subclass of $B$. If $S=\{s\}$ is a singleton, $f(S):=s$. Otherwise, $T:=f(S)\cap A$ is a non-empty convex subset of $A$, and I define $f(S):=f'(T)$. The conclusion is now obvious. □

**Definition 1.82.** Let $(A,<)$ be a linearly ordered class. A Dedekind cut is a partition of $A$ into two non-empty subclasses $L,R$ such that $L<R$ and $L$ has no maximum. The *Dedekind completion* $A^{\mathscr{D}}$ of $(A,<)$ is the collection of all its Dedekind cuts with order defined by

$$(L,R)\leq(L',R')\leftrightarrow L\subseteq L',$$

and inclusion $\iota:A\to A^{\mathscr{D}}$ given by

$$\iota(a)=(\{x\in A:x<a\},\{x\in A:x\geq a\}).$$

$A$ is dense in its Dedekind completion, so lemma 1.81 applies to $A^{\mathscr{D}}$, if it is a class.

## 1.10 Structure on $\mathbf{No}^{\mathscr{D}}$

Let $\mathbf{No}^{\mathscr{D}}$ be the Dedekind completion of $\mathbf{No}$. I will define a tree structure on $\mathbf{No}^{\mathscr{D}}$ which extends the structure on $\mathbf{No}$ (I cannot use lemma 1.81 directly because $\mathbf{No}^{\mathscr{D}}$ is not a class[(9)]). If $x,y\in\mathbf{No}^{\mathscr{D}}$, $x\preceq y$ iff $x=y$ or $x,y\in\mathbf{No}$ and $x\preceq y$ or $x=\langle x^L\mid x^R\rangle\in\mathbf{No}$, $y=(L,R)\in\mathbf{No}^{\mathscr{D}}$ and $x^L\in L$, $x^R\in R$ for every $x^L,x^R$ canonical options of $x$.

With abuse of notation, given $L,R$ subclasses of $\mathbf{No}$, with $L<R$, I write $\langle L\mid R\rangle$ for the simplest $x\in\mathbf{No}^{\mathscr{D}}$ such that $L<x<R$ (if it exists). Every $x\in\mathbf{No}^{\mathscr{D}}$ is of the form $\langle L\mid R\rangle$, with $L=\{x'\in\mathbf{No}:x'<x\}$ and $R=\{x''\in\mathbf{No}:x''>x\}$.

Every element of $\mathbf{No}^{\mathscr{D}}$ has a sign expansion corresponding to it, possibly of length $\mathbf{On}$. But not every sign expansion of length at most $\mathbf{On}$ corresponds to an element of $\mathbf{No}^{\mathscr{D}}$.

For $x,y\in\mathbf{No}^{\mathscr{D}}$, define $x+y$ as

$$\langle\{x'+y':x'<x,y'<y\}\mid\{x''+y'':x''>x,y''>y\}\rangle.$$

(9)The proof that $\mathbf{No}^{\mathscr{D}}$ is not a class is a trivial modification of Cantor's diagonal argument showing that the set of real numbers is not countable.

*Remark* 1.83. $x+y$ is a well-defined element of $\mathbf{No}^{\mathscr{D}}$. Moreover, if $x,y \in \mathbf{No}$, then $x+y$ coincides with the usual sum.

$-x$ is defined as

$$-x = \langle \{ -x'' : x'' > x \} \mid \{ -x' : x' < x \} \rangle.$$

Every positive element of $\mathbf{No}^{\mathscr{D}}$ can be also represented in a unique way as $x = (\{ x' \in \mathbf{No} : 0 < x' < x \} \mid \{ x'' \in \mathbf{No} : x'' > x \})$. With this representation, we can define

$$xy = \langle \{ x'y' : 0 < x' < x, 0 < y' < y \} \mid \{ x''y'' : x'' > x, y'' > y \} \rangle.$$

Again, $xy$ is a well-defined element of $\mathbf{No}^{\mathscr{D}}$, which for $x,y \in \mathbf{No}$ coincides with the usual product.

Let

$$\begin{aligned} \eta &:= \langle \{ x \in \mathbf{No} : 0 < x \ll 1 \} \mid \{ x \in \mathbf{No} : 0 < x \;\&\; v(x) \geq 0 \} \rangle \\ &= \sup \{ x \in \mathbf{No} : 0 < x \ll 1 \}. \end{aligned}$$

Then, $\eta + \eta = \eta^2 = \eta$. Therefore, $\mathbf{No}^{\mathscr{D}}$ is not a ring. In particular, the sum is not associative[(10)].

*Remark* 1.84. Let $x > 0 \in \mathbf{No}$. Then

$$\eta x = \sup \{ y \in \mathbf{No} : 0 < y \ll x \} = \inf \{ y \in \mathbf{No} : y > 0 \;\&\; v(y) \geq v(x) \}$$

*Proof.* Let $x = \langle L \mid R \rangle$, where

$$L = \{ x^L \in \mathbf{No} : 0 < x^L < x \} \text{ and } R = \{ x^R \in \mathbf{No} : x^R > x \}.$$

Then,

$$\eta x = \langle \{ x^L \varepsilon : 0 < x^L < x \;\&\; 0 < \varepsilon \ll 1 \} \mid \{ x^R q : x^R > x \;\&\; q > 0 \;\&\; v(q) \geq 0 \} \rangle.$$

Let $0 < y \ll x$. Then $2y/x = \varepsilon \ll 1$, therefore

$$y = \varepsilon x^L,$$

with $x^L = x/2$, so $\sup \{ y \in \mathbf{No} : 0 < y \ll x \} \leq \eta x$.

On the other hand, if $y > 0$ and $v(y) \geq v(x)$, then $y \geq 2qx$ for some $q > 0 \in \mathbb{Q}$, therefore $y \geq qx^R$, with $x^R = 2x$, so $y > \eta x$. □

(10)
$$\begin{aligned} \eta + (\eta - \eta) &= \eta \\ (\eta + \eta) - \eta &= 0. \end{aligned}$$

I will introduce the notion of approximation associated to a surreal number.

**Definition 1.85.** Let $x \in \mathbf{No}$.

$$\Delta^L(x) := \sup\{\, \varepsilon \in \mathbf{No} : x \preceq x - \varepsilon \,\}$$
$$\Delta^R(x) := \sup\{\, \varepsilon \in \mathbf{No} : x \preceq x + \varepsilon \,\}$$
$$\Delta(x) := \max\{\, \Delta^L(x), \Delta^R(x) \,\}$$

Note that, in general, $\Delta^L, \Delta^R$ and $\Delta$ are not in $\mathbf{No}$, but in $\mathbf{No}^{\mathscr{D}}$.

*Remark* 1.86. Let $x, y \in \mathbf{No}$. If $|x - y| < \Delta(x)$, then $x \preceq y$.

Moreover,

$$\Delta^L(x) = \inf\{\, \varepsilon > 0 : x - \varepsilon \prec x \,\},$$

and similarly for $\Delta^R(x)$. Likewise,

$$\Delta(x) = \inf\{\, \varepsilon > 0 : x - \varepsilon \prec x \vee x + \varepsilon \prec x \,\}$$

In particular, if $y \prec x$ and $y < x$, then $x - y \geq \Delta^L(x)$.

*Example* 1.87.

- $\Delta(x) = \infty$ iff $x = 0$.
- The ring of omnific integers is the subclass of $\mathbf{No}$

  $$\mathbf{Oz} := \{x \in \mathbf{No} : \Delta(x) \geq 1\,\}.$$

  It is a subring of $\mathbf{No}$. Its elements are the surreal numbers with normal form

  $$\sum_{a \in \mathbf{No}} r_a \omega^a,$$

  with $r_a = 0$ for every $a < 0$, and $r_0 \in \mathbb{Z}$. Many properties of this ring are explained in [6] and [10].
- If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $\Delta(x) = \eta$.

**Lemma 1.88.** *For* $x, y \in \mathbf{No}$,

$$\Delta^L(x+y) \geq \min\{\, \Delta^L(x), \Delta^L(y) \,\}$$
$$\Delta^R(x+y) \geq \min\{\, \Delta^R(x), \Delta^R(y) \,\}$$
$$\Delta(x+y) \geq \min\{\, \Delta(x), \Delta(y) \,\}$$

*Proof.* It is enough to prove the first inequality.

$$(x+y)^L = \begin{cases} x+y^L \\ x^L+y. \end{cases}$$

Therefore

$$(x+y)-(x+y)^L = \begin{cases} x-x^L > \Delta^L(x) \\ y-y^L > \Delta^L(y) \end{cases} \geq \min\{\,\Delta^L(x), \Delta^L(y)\,\}.$$

Now apply remark 1.86 and the inverse cofinality theorem. □

**Corollary 1.89.** *If* $\Delta(b) < \Delta(a)$*, then* $\Delta(a+b) = \Delta(b)$.

**Lemma 1.90.** *For* $x,y \in \mathbf{No}$*,* $\Delta(xy) \geq \Delta(x)\Delta(y)$.

*Proof.* Let $x = \langle x^L \mid x^R \rangle$, $y = \langle y^L \mid y^R \rangle$ be any representations of $x,y$. Then, a typical option of $xy$ is of the form

$$(xy)^{\mathbf{o}} = xy - (x-x^{\mathbf{o}})(y-y^{\mathbf{o}}),$$

therefore $|(xy)^{\mathbf{o}} - xy| = |x-x^{\mathbf{o}}||y-y^{\mathbf{o}}|$. □

*Example* 1.91. In general, it is not true that $\Delta(xy) = \Delta(x)\Delta(y)$. For instance, take $x = 1/3, y = 3$.
Then, $\Delta(x) = \eta$, $\Delta(y) = 1$, $\Delta(xy) = \Delta(1) = 1 > \Delta(x)\Delta(y)$.

*Remark* 1.92. Let $a \in \mathbf{No}$, $x > 0 \in \mathbf{No}$. If $x \sim \omega^a$, then $\omega^a \preceq x$.

*Proof.* If $a = \langle a^L \mid a^R \rangle$ is any representation of $a$,

$$\omega^a = \langle q\omega^{a^L} \mid q\omega^{a^R} \rangle_{q>0\in\mathbb{Q}}.$$

$x \sim \omega^a$, therefore $q\omega^{a^L} < x < q\omega^{a^R}$, and the conclusion follows. □

**Lemma 1.93.** *Let* $r \in \mathbb{R}$*,* $a \in \mathbf{No}$*. Then,*

$$r\omega^a = \langle (r-\varepsilon)\omega^a \mid (r+\varepsilon)\omega^a \rangle_{\varepsilon>0\in\mathbb{Q}}.$$

*Proof.* Without loss of generality, $r > 0$. Let $a = \langle a^L \mid a^R \rangle$ be any representation of $a$. Then a typical option of $z := r\omega^a$ is of the form

$$z^{\mathbf{o}} := rq\omega^{a^{\mathbf{o}}} + (r\pm\varepsilon)\omega^a - (r\pm\varepsilon)q\omega^{a^{\mathbf{o}}}$$

For some $\varepsilon, q > 0 \in \mathbb{Q}$. If $a^{\mathbf{o}} = a^L < a$,

$$z^{\mathbf{o}} \simeq (r\pm\varepsilon)\omega^a$$

otherwise $a^{\mathbf{o}} = a^R > a$, and

$$z^{\mathbf{o}} \simeq \pm q\varepsilon\omega^{a^R} \gg \omega^a$$

Therefore,

$$z = \langle (r-\varepsilon)\omega^a \mid (r+\varepsilon)\omega^a \rangle_{\varepsilon>0\in\mathbb{Q}}.$$ □

**Lemma 1.94.** *Let $r \neq 0 \in \mathbb{R}$, $a \in \mathbf{No}$. Then, $\eta\,\omega^a \leq \Delta(r\omega^a) \leq \omega^a$.*

*Moreover, $\forall y \in \mathbf{No}\; y \simeq r\omega^a$ implies $r\omega^a \preceq y$.*

*Proof.* If $y \simeq r\omega^a$, then $\forall \varepsilon > 0 \in \mathbb{Q}$

$$(r-\varepsilon)\omega^a < y < (r+\varepsilon)\omega^a,$$

therefore, by lemma 1.93, $r\omega^a \preceq y$. This implies that $\Delta(r\omega^a) \geq \eta\,\omega^a$.

Without loss of generality, $r > 0$. If $r \leq 1$, $0$ is a canonical left option of $r\omega^a$, therefore $\Delta^L(r\omega^a) \leq \omega^a$.

If $r > 1$, let $n$ be the greatest natural number strictly less than $r$; $n < r \leq n+1$.

*Claim* 1. $n\omega^a \prec r\omega^a$.

The claim implies that $\Delta^L(r\omega^a) \leq \omega^a$. Let $r = \langle r^L \mid r^R \rangle$ be a representation of $r$ such that $r^L, r^R \in \mathbb{Q}$ are canonical options of $r$, $n \leq r^L < r$ and $r < r^R < r+1$. Let $a = \langle a^L \mid a^R \rangle$ be the canonical representation of $a$. $n = \langle n-1 \mid \rangle$, therefore a typical left option of $na$ is

$$(n\omega^a)^L = (n-1)\omega^a + q\omega^{a^L} \simeq (n-1)\omega^a,$$

while a typical right option is

$$(n\omega^a)^R = (n-1)\omega^a + q\omega^{a^R} \sim \omega^{a^R},$$

for some $q > 0 \in \mathbb{Q}$. On the other hand, by lemma 1.93, a typical left options of $r\omega^a$ is

$$(r\omega^a)^L = (r-\varepsilon)\omega^a > (n-1)\omega^a,$$

while a right option is

$$(r\omega^a)^R = (r+\varepsilon)\omega^a < \omega^{a^R}.$$

The conclusion follows easily. □

**Lemma 1.95.** *Let $x = \sum_{i<\alpha} r_i\omega^{a_i}$ be the normal form of $x \in \mathbf{No}$. Let $a \in \mathbf{No}$. For every $\gamma \leq \alpha$, define $x_\gamma := \sum_{i<\gamma} r_i\omega^{a_i}$. Then,*

1. *If $a_i > a\;\forall i < \alpha$, then $\Delta(x) > \omega^a$.*

2. *If* $a_i \geq a\ \forall i < \alpha$, *then* $\Delta(x) \geq \eta\,\omega^a$.

3. *If* $\alpha$ *is a limit ordinal, then*

$$x = \langle x_\gamma + (r_\gamma - \varepsilon)\omega^{a_\gamma} \mid x_\gamma + (r_\gamma + \varepsilon)\omega^{a_\gamma} \rangle_{\substack{\gamma<\alpha \\ 0<\varepsilon\in\mathbb{Q}}} .$$

*Moreover,* $\Delta(x) \geq \inf\{\,\omega^{a_i} : i < \alpha\,\}$.

4. *If* $\alpha = \beta + 1$ *and* $r_\beta t^\beta = z$, *then* $\Delta(x) = \Delta(z)$.

5. $\forall \gamma < \alpha\ x_\gamma \prec x$.

*Proof.* Induction on $\alpha$.

If $\alpha$ is a limit ordinal, then the first part of 3 is an immediate consequence of the definition of $\sum_{i<\alpha} r_i \omega^{a_i}$. Therefore,

$$|x - x^{\mathbf{o}}| \simeq \varepsilon\omega^{a_\gamma} > \begin{cases} \omega^a & \text{if } a_\gamma > a \\ \eta\,\omega^a & \text{if } a_\gamma \geq a. \end{cases}$$

and the first two points and the second part of 3 follow.

If $\alpha = \beta + 1$, let $y = x_\beta$, $z = \omega^{a_\beta} r_\beta$, i.e. $x = y + z$.

The case $\alpha = 1$ has already been proved. For $\alpha > 1$, by inductive hypothesis $\Delta(y) > \omega^{a_\beta}$ and $\Delta(z) \leq \omega^{a_\beta}$, therefore $\Delta(x) = \Delta(y+z) = \Delta(z) \geq \eta\,\omega^{a_\beta}$.

It remains to prove 5. However, $v(x - x_\gamma) \leq a_\gamma$ and, by point 1, $\Delta(x_\gamma) > \omega^{a_\gamma}$, therefore $x_\gamma \preceq x$. □

**Theorem 6.** *Let* $x = \sum_{i<\alpha} r_i \omega^{a_i}$ *be the normal form of* $x$.
*If* $\alpha = \beta + 1$, *then* $\Delta(x) = \Delta(r_\beta \omega^{a_\beta})$.
*If* $\alpha$ *is a limit ordinal, then* $\Delta(x) = \inf\{\,\omega^{a_i} : i < \alpha\,\}$.

*Proof.* The case $\alpha = \beta + 1$ follows from the previous lemma.

If $\alpha$ limit, then by the previous lemma $\Delta(x) \geq \inf\{\,\omega^{a_i} : i < \alpha\,\}$. Let $\gamma < \alpha$ and $y = \sum_{i\leq\gamma} r_i \omega^{a_i}$; $y \prec x$ and $|y - x| < \omega^{a_\gamma}$, therefore $\Delta(x) < \omega^{a_\gamma}$. □

# Chapter 2

# Integration

The integral of a recursive function is defined along the lines of Riemann integral on real numbers, and some of its properties are proved.

## 2.1 Definition

Given a recursively defined function $f(X) = \langle f^L \mid f^R \rangle$, I will try to define what the Riemann integral of $f$ is, knowing what the integrals of $f^{\mathbf{o}}$ are, for any $f^{\mathbf{o}}$ (left or right) option of $f$. I will write

$$\int_a^b f(t)\,\mathrm{d}t$$

for such an integral, or $\int_a^b f$ if the variable of integration is clear.

I will say what properties the function

$$\mathscr{I}(a,b,f) := \int_a^b f(t)\,\mathrm{d}t$$

should have. First, it should be "additive" in $(a,b)$, i.e.

$$\mathscr{I}(a,b,f) + \mathscr{I}(b,c,f) = \mathscr{I}(a,c,f)$$

for any $a,b,c$. This implies that I need only to define what $\mathscr{I}(0,a,f)$ is, and say

$$\mathscr{I}(a,b,f) := \mathscr{I}(0,b,f) - \mathscr{I}(0,a,f).$$

Second, it must be increasing in $f$: if $a < b$ and $f(t) < g(t)$ for all $t \in (a,b)$, then

$$\mathscr{I}(a,b,f) < \mathscr{I}(a,b,g).$$

These two properties are enough for our purpose: I will show that they define a function $\mathscr{I}(a,b,f)$, under some assumptions on how $f$ is defined, and that other natural properties of $\mathscr{I}$ (as, for instance, linearity in $f$) follow.

Simple case: the functions $f^{\mathbf{o}}$ do not depend on $x^{\mathbf{o}}$ and $f(x^{\mathbf{o}})$, but only on $x$. In that case, we know that

$$f^L(x) < f(x) < f^R(x).$$

So, it must be that

$$\int_a^b f^L(t)\,\mathrm{d}t < \int_a^b f(t)\,\mathrm{d}t < \int_a^b f^R(t)\,\mathrm{d}t$$

if $a < b$, and the other way round if $b < a$. Let's call $\mathcal{F}(x) = \int_0^x f(t)\,\mathrm{d}t$. In particular, if $a = x^L$ is a left option of $x$,

$$\mathcal{F}(x^L) + \int_{x^L}^x f^L(t)\,\mathrm{d}t < \mathcal{F}(x) < \mathcal{F}(x^L) + \int_{x^L}^x f^R(t)\,\mathrm{d}t,$$

and analogous formulae for $a = x^R$. So, I could write

$$\begin{aligned}\mathcal{F}(x) := \langle\, &\mathcal{F}(x^L) + \int_{x^L}^x f^L(t)\,\mathrm{d}t,\ \mathcal{F}(x^R) + \int_{x^R}^x f^R(t)\,\mathrm{d}t \mid \\ &\mathcal{F}(x^L) + \int_{x^L}^x f^R(t)\,\mathrm{d}t,\ \mathcal{F}(x^R) + \int_{x^R}^x f^L(t)\,\mathrm{d}t\,\rangle.\end{aligned} \tag{2.1}$$

The previous definition is sound, because we know already the value of $\int_a^b f^{\mathbf{o}}(t)\,\mathrm{d}t$ for any $a, b$, and, by induction on $x$, we know the value of $\mathcal{F}(x^{\mathbf{o}})$.

As a shorthand, I write

$$(\mathcal{F}(x))^{\mathbf{o}} = \mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^x f^{\mathbf{o}}(t)\,\mathrm{d}t.$$

However, this is not good enough: I must also split the interval $[x^L, x]$ (or $[x, x^R]$) into finitely many intervals $[k_i, k_{i+1}], i = 0, \dots, m-1$, choose a left (or right) option $f_i^{\mathbf{o}}$ for each $i$, and define:

$$(\mathcal{F}(x))^{\mathbf{o}} = \mathcal{F}(x^{\mathbf{o}}) + \sum_i \int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}}(t)\,\mathrm{d}t.$$

In the general case, I should write

$$(\mathcal{F}(x))^{\mathbf{o}} = \mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^x f^{\mathbf{o}}(t, t^{\mathbf{o}}, f(t^{\mathbf{o}}))\,\mathrm{d}t, \tag{2.2}$$

but it is not clear at all what the expression on the right means.

Let $a, b$ be two elements of **No**. I say that

$$P = (k_0, \dots, k_m)$$

is an *m-partition* of $(a, b)$ (and write $P[a, b]$) if $k_0 = a$, $k_m = b$ and $k_{i+1} > k_i$, for any $0 \le i < m$. I call $m$ the *length* of $P$. Given a partition $P[a, b] = (k_0, \dots, k_m)$ and $m$-tuple $g_P$ of functions $g_i(t, t^L, t^R)$, $i = 0, \dots, m-1$, I define

$$\int_a^b g_P(t, t^{\mathbf{o}})\,\mathrm{d}t := \sum_{i=0}^{m-1} \int_{k_i}^{k_{i+1}} g(t, k_i, k_{i+1})\,\mathrm{d}t$$

if the expression on the right makes sense (i.e. if I have already assigned a value to the various integrals). I say that $g_P$ associates to every interval $(k_i, k_{i+1})$ of $P$ a function $g_i$.

Now we have a candidate for the left side in (2.2): let $m$ be any natural number, let $P$ vary among all the possible $m$-partitions of $(x^L, x)$, let $f_P^L$ be an $m$-tuple of left options of $f$, then the expression

$$\mathcal{F}(x^L) + \int_{x^L}^{x} f_P^L(t, t^\mathbf{o}, f(t^\mathbf{o}))\, \mathrm{d}t \tag{2.3}$$

is a left option of $\mathcal{F}(x)$. Similarly, I can use an $m$-tuple of right options of $f$, or a partition of $(x, x^R)$, to obtain all the other options of $\mathcal{F}(x)$ (4 cases in total). (2.3) is a sound definition, because to compute it I only need to compute $\mathcal{F}(x^\mathbf{o})$ (which I know, by induction on $x$), and

$$\int_a^b f^\mathbf{o}(t, a, b, c, d)\, \mathrm{d}t$$

for any option $f^\mathbf{o}$ of $f$ and for any $a, b, c, d$ in **No**: so, I need only to suppose I know how to integrate such expressions.

Concluding, the recursive definition of $\mathcal{F}(x) := \int_0^x f(t)\, \mathrm{d}t$ is

$$\Big\langle\ \mathcal{F}(x^L) + \int_{x^L}^{x} f_P^L(t, t^\mathbf{o}, f(t^\mathbf{o})),\quad \mathcal{F}(x^R) - \int_{x}^{x^R} f_P^R(t, t^\mathbf{o}, f(t^\mathbf{o}))\ \Big| \\ \mathcal{F}(x^L) + \int_{x^L}^{x} f_P^R(t, t^\mathbf{o}, f(t^\mathbf{o})),\quad \mathcal{F}(x^R) - \int_{x}^{x^R} f_P^L(t, t^\mathbf{o}, f(t^\mathbf{o}))\ \Big\rangle,$$

where $P$ varies among the partitions of $(x^L, x)$ (or of $(x, x^R)$, according to the context).

## 2.2 Problems and examples

There are various difficulties with the previous definitions. Assume that $f$ is recursive over $\mathfrak{A}$.

I. I need to know that if $a < t < b$

$$f^L(t, a, b, f(a), f(b)) < f(t)$$

in order to be able to conclude that (2.3) is less than $\mathcal{F}(x)$, and so be able to use (2.3) as a left option of $\mathcal{F}(x)$.

II. I want the definition of $\mathcal{F}(x)$ to be uniform in $x$ and $f$, i.e. independent of the particular representations of $x$ and of $f$. Strictly correlated with this, I want that if $f < g$ and $a < b$, then $\int_a^b f < \int_a^b g$.

III. If I consider the number of options necessary to define $\mathcal{F}(x)$, I see that it is a proper class (at least one option for any possible partition of $(0,x)$): so, a priori there is no guarantee on the existence of $\mathcal{F}(x)$, even if I know that every right option of it is strictly greater than any left option. To solve this, I need some amount of saturation of the theory of $(\mathbf{No}, \mathfrak{A})$.

IV. What about other properties of the integral? How does it fit with already defined functions, such as polynomials, analytic functions, logarithm and exponential?

I will prove that the integral of a polynomial is equal to the formal one, and similarly for bounded analytic functions. I will also prove an integration by parts formula and the fundamental theorem of calculus. However, the integral of exp is not what we expect.

V. Finiteness theorems. Under some assumptions on $f$ (for instance, that it has a finite number of zeros), I will prove that $\mathcal{F}(x)$ has a finite number of zeros, that between any two zeros of $\mathcal{F}$ there is a zero of $f$, and that if $\mathcal{F}(a) < 0 < \mathcal{F}(b)$, then there is a zero of $\mathcal{F}$ in $(a,b)$.

The following two examples illustrate some of the computations one should perform and some of the difficulties one may encounter with the definition of the integral of an arbitrary function $f$ (recursively defined over some family).

- The integer part function.

$$[x] = \langle x-1 \mid x+1 \rangle$$

is a function defined in an "elementary" way, but surely it is not in a Hardy field.

If $x$ is a finite positive number, $[x]$ is the usual integer part. For negative finite argument $x$, the function $[x]$ behaves slightly differently from the usual integer part function: instead of returning the greatest integer below $x$, it will give the smallest integer above it For $x$ a generic surreal number, $[x]$ will return the sometimes the greatest omnific integer[(1)] below it, sometimes the smallest above it (the actual behaviour can be easily computed from the normal form of $x$). For instance, $[\omega - 1/2] = \omega$.

Let us compute $\mathcal{F}(x) := \int_0^x [t]\,\mathrm{d}t$ for some values of $x$. Suppose that $x \in \mathbb{N}$; so, $x = \langle x-1 \mid \rangle$. Then $\mathcal{F}(x) = \sum_{i=0}^{x} i = x(x-1)/2$. In fact,

$$\mathcal{F}(x) = \langle \mathcal{F}(x-1) + \int_{x-1}^{x} (t-1)\,\mathrm{d}t \mid \mathcal{F}(x-1) + \int_{x-1}^{x} (t+1)\,\mathrm{d}t \rangle,$$

[(1)]See [6] for the definition.

i.e.

$$\begin{aligned}\mathcal{F}(x)^{\mathbf{o}} &= \frac{(x-1)(x-2)}{2} + \left[\frac{t^2}{2} \pm t\right]_{x-1}^{x} \\ &= \frac{(x-1)(x-2)+x^2-(x-1)^2}{2} \pm 1 \\ &= \frac{x^2-3x+2+x^2-x^2+2x-1}{2} \pm 1 = \frac{x^2-x+1}{2} \pm 1 \\ &= \frac{x(x-1)}{2} + \frac{1}{2} \pm 1.\end{aligned}$$

Therefore,

$$\mathcal{F}(x) = \langle \frac{x(x-1)}{2} - \frac{1}{2} \mid \frac{x(x-1)}{2} + \frac{3}{2} \rangle = \frac{x(x-1)}{2}.$$

But

$$\begin{aligned}\mathcal{F}(\omega)^{\mathbf{o}} &= \mathcal{F}(n) + \int_n^{\omega} (t \pm 1)\,\mathrm{d}t \\ &= \mathcal{F}(n) + \left[\frac{t^2}{2} \pm t\right]_n^{\omega} = q + \frac{\omega^2}{2} \pm \omega\end{aligned}$$

where $n$ is any natural number and $q$ is some rational number (depending on $n$):

$$\mathcal{F}(\omega) = \langle \frac{\omega^2}{2} - \omega + q \mid \frac{\omega^2}{2} + \omega + q' \rangle_{n \in \mathbb{N}} = \frac{\omega^2}{2}.$$

Therefore, $\int_0^{\omega} [t]\,\mathrm{d}t = \int_0^{\omega} t\,\mathrm{d}t$.

• Let $\varepsilon$ be any positive surreal number., $y \in \mathbf{No}$. Let $g_y(t)$ the piecewise linear function that has values 1 in $t = y$, 0 outside the interval $(y - \varepsilon, y + \varepsilon)$.

Define

$$f(t) := \langle g_y(t) \mid \ \rangle_{y \in \mathbf{No}}.$$

Then $f(t) = 2$ for any $t \in \mathbf{No}$ (because $1 = g_t(t)$). So, I would expect that $\int_0^x f(t)\,\mathrm{d}t = 2x$. But,

$$\mathcal{F}(x)^{\mathbf{o}} - \mathcal{F}(x^{\mathbf{o}}) = \sum_i \int_{k_i}^{k_{i+1}} g_{y_i}(t)\,\mathrm{d}t$$

for some $k_0, \ldots, k_m$ and $y_0, \ldots, y_m$. If $\varepsilon$ is infinitesimal (for instance, $\varepsilon = 1/\omega$), the former sum is infinitesimal too, and it is then easy to check that $\mathcal{F}(x) \neq 2x$.

It is unclear whether in the last counterexample is essential that, in order to define $f$, I use a class of functions (instead of a set, as it should be); nevertheless, I think that I must impose some strong conditions on how $f$ is defined in order to have a useful definition of $\mathcal{F}$.

## 2.3 Conditions of integrability

Let $f = \langle f^L \mid f^R \rangle$ be a recursively defined function. I want to state some conditions under which the previous definitions make sense.

First, of course, I need that $f^\circ(x,a,b,c,d)$ is integrable for every $f^\circ$ option of $f$ and for every parameters $a,b,c,d \in \mathbf{No}$. Moreover, I need some kind of uniformity in the definition of $f$. To be more precise, the following conditions:

**Axiom 1.**

1a. For every $t' < t < t''$, for every $f^L, f^R$ options of $f$, one has

$$f^L(t,t',t'',f(t'),f(t'')) < f(t) < f^R(t,t',t'',f(t'),f(t''))$$

1b. For every $f^L$ there exists $f^{L'}$ such that for every $t_1' \leq t_2' < t < t_2'' \leq t_1''$

$$f^L(t,t_1',t_1'',f(t_1'),f(t_1'')) \leq f^{L'}(t,t_2',t_2'',f(t_2'),f(t_2''))$$

and analogous conditions for $f^R$ (that is, if I take $t_2$ which is a better approximation[(2)] of $t$ than $t_1$, I can obtain a better approximation of $f(t)$).

Axiom 1 is a (slightly) stronger version of the uniformity of the definition of $f$. In the rest of this thesis, when I will compute the integral of some recursively defined function, I will always assume that it satisfies this axiom.

In the second condition, it is often true that $f^{L'} = f^L$, but this usually does not simplify our task.

Besides, I assume to have a family $\mathfrak{A}$ of functions

$$g(x,\vec{y}) : \mathbf{No}^{n+1} \to \mathbf{No}$$

in $n+1$ variables ($n$ depends on $g$), with $x$, the first one, distinguished. Moreover, $g(x,\vec{c}) \in \mathfrak{A}$ for every $g \in \mathfrak{A}$ and $\vec{c} \in \mathbf{No}^n$; $\mathfrak{A}$ contains all the constants, the identity function, and $+$.

I also suppose that $\mathfrak{A}$ is obtained by an inductive process adding recursive functions over previous families, starting with only the constants, i.e.

$$\mathfrak{A} = \bigcup_{\beta<\alpha} \mathfrak{A}_\beta$$

(2)Given $t_1, t_2, t$ such that $t_1 \leq t_2 < t$ I say that $t_2$, as a left approximation of $t$, is better than $t_1$, and similarly for right ones.

where $\alpha$ is an ordinal, and $\mathfrak{A}_{\beta+1}$ is made only by functions recursive over $\mathfrak{A}_\beta$, and $\mathfrak{A}_\lambda$ is the union of the previous ones if $\lambda$ is a limit ordinal. In this case, I say that $\mathfrak{A}$ is *constructed inductively*. This will allow us to proceed by induction on the "complexity" of a function $g$ (namely, the smallest $\beta$ such that $g \in \mathfrak{A}_\beta$).

A consequence of it is that if $g \in \mathfrak{A}$ then every option of $g$ is also in $\mathfrak{A}$. This is weaker than being constructed inductively: see for instance the discussion in § 5.2.2 about the field of rational functions.

Finally, I assume that I have already defined somehow an integral for every function of one variable in $\mathfrak{A}$, satisfying

$$\int_a^b g + \int_b^c g = \int_a^c g$$

for every $a, b, c \in \mathbf{No}$, $g \in \mathfrak{A}$, and

$$\int_a^b g < \int_a^b h$$

if $a < b \in \mathbf{No}$, $g, h \in \mathfrak{A}$ and $g(t) < h(t)$ $\forall t \in (a,b)$ (and analogous condition if $g(t) \leq h(t)$). I call the first property *additivity* in the interval of integration,[(3)] the second *monotonicity* of the integral.

$f = \langle f^L \mid f^R \rangle$ will be a recursive over $\mathfrak{A}$ (and satisfying axioms 1). I can then define $\int f$ using formula (2.2). I will then suppose that $\int$ has some further properties on $\mathfrak{A}$, and prove some other property for $\int f$.

For shorthand, I will often write $f^{\mathbf{o}}(x, x^{\mathbf{o}})$ for $f^{\mathbf{o}}(x, x^L, x^R, f(x^L), f(x^R))$.

I wish to emphasise that the value of $\mathcal{F}(x)$ depends not only on the function $f$, but also on the following:

1. The value of $\int g$ as $g$ varies in $\mathfrak{A}$.

2. The recursive definition of $f$ in terms of functions in $\mathfrak{A}$.

Different definitions of the same function $f$ might give rise to different values of $\mathcal{F}$ (but I do not have actual examples of this phenomenon). Later, I will give some conditions under which this does non happen, i.e. the value of $\mathcal{F}$ will not depend on how $f$ is defined (but it will still depend on the value of $\int$ on $\mathfrak{A}$).

Therefore, by function we will usually mean application from **No** (or some set-bounded convex subclass of it) to **No**, together with some recursive definition for it (over some family $\mathfrak{A}$). The theory of integration we will build will consider only this kind of functions.

[(3)]In order to avoid confusion with the property $\int f + \int g = \int (f+g)$.

So, we have to prove that $\int f$ is well defined, and that $\int$ is monotone on $\mathfrak{A} \cup \{f\}$ (additivity of $\int$ in the interval of integration is immediate from the definition). But this does not come for free: we will need some further assumptions.

**Definition 2.1.** Let $f, g$ be recursive over $\mathfrak{A}$, $a < b \in \mathbf{No}$.

I say that $f < g$ provably in $(a, b)$ if there exists $P[a, b]$ and $f_P^R, g_P^L$ such that for each $i$ either

$$f(t) \leq g_i^L(t, k_i, k_{i+1}) \qquad \text{or}$$
$$f_i^R(t, k_i, k_{i+1}) \leq g(t)$$

in $(k_i, k_{i+1})$.

$f \leq g$ provably if

$$f^L(t, a', b') < g \qquad \text{and}$$
$$f < g^R(t, a', b')$$

in $(a', b')$ for each option $f^L, g^R$ and $a', b$, such that $a \leq a' < b' \leq b$.

$f = g$ provably if $f \leq g$ and $g \leq f$ provably.

**Axiom 2.** For every $h, g \in \mathfrak{A} \cup \{f\}$ either $h = g$ identically, or for every $a < b \in \mathbf{No}$ there exists a partition $P$ of $(a, b)$ such that on each interval $(k_i, k_{i+1})$ either $h(t) < g(t)$ or $h(t) > g(t)$, and provably so.

This axiom has two important consequences

1. If $f < g$, then there are witnesses for it (namely, $g_P^L$ or $f_P^R$).
2. A function either is identically zero, or has only finitely many zeros on any interval $(a, b)$.

With this hypothesis, I will prove the monotonicity of $\int$, and that $\int f$ is independent from the definition of $f$.

It will remain to prove that the integral of $f$ exists. As I said before, the number of options I use to define $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$ is too high (namely, a proper class): I need to cut it down if I want to be sure that $\mathcal{F}(x)$ exists.

**Axiom 3.** Let $\mathcal{L}$ be the first order language

$$\mathcal{L} := \left(+, <, f, g, \int_0^x g(t, y_1, y_2, z_1, z_2)\,\mathrm{d}t\right)_{g \in \mathfrak{A}}$$

Every subclass of $\mathbf{No}$ definable in $\mathcal{L}$ (with parameters) has a supremum in $\mathbf{No} \cup \{\pm\infty\}$. $f$ has at least a left and a right option.

If $g \in \mathfrak{A}$ and $g < f$ on some interval $(a, b)$, then there exists $\varepsilon > 0$ such that $g < f - \varepsilon$ on some subinterval of $(a, b)$.

*Remark* 2.2. For the last statement of axiom 3, is it enough that $f$ and $g$ are continuous at at least one point $x \in (a,b)$ and $g(x) < f(x)$.

I will also need that **No** is saturated (in the language $\mathscr{L}$), as will be clear from lemma 2.8.

Further properties of the integral will (usually) need additional hypothesis.

I repeat that in all this thesis, I will suppose that axiom 1 is true for every function $f$, and that the integral is monotone an additive on $\mathfrak{A}$. On the other hand, I will try to state explicitly which of the other axioms are used to prove each property.

## 2.4 Properties of the integral

The first, easy problem is to compute the integral of constant functions.

**Lemma 2.3.** *Let $a,b,c \in$ **No**. Then,*

$$\int_a^b c\,\mathrm{d}t = c(b-a).$$

*Proof.* Induction on $c,a,b$. First, I do induction on $c$, then on the cardinal product sym of $a,b$. Let $\langle a^L \mid a^R \rangle$, $\langle b^L \mid b^R \rangle$ and $\langle c^L \mid c^R \rangle$ be the canonical representations of $a,b,c$. Then, by definition of integral and inductive hypothesis,

$$\begin{aligned}\left((b-a)c\right)^{\circ} &= (b^{\circ}-a)c+(b-b^{\circ})c^{\circ},\ (b-a^{\circ})c+(a^{\circ}-a)c^{\circ}\\ &= \int_a^{b^{\circ}} c\,\mathrm{d}t + \int_{b^{\circ}}^{b} c^{\circ}\,\mathrm{d}t,\ \int_{a^{\circ}}^{b} c\,\mathrm{d}t + \int_a^{a^{\circ}} c^{\circ}\,\mathrm{d}t = \left(\int_a^b c\,\mathrm{d}t\right)^{\circ},\end{aligned}$$

proving that $(b-a)c \preceq \int_a^b c\,\mathrm{d}t$.

Conversely, let $b^L$ be a canonical left options of $b$, $P := (k_0,\dots,k_n)$ be a partition of $(b^L,b)$, $c_0^L,\dots,c_{n-1}^L$ be corresponding canonical left options of $c$.

$$\begin{aligned}\left(\int_a^b c\,\mathrm{d}t\right)^L &= \int_a^{b^L} c\,\mathrm{d}t + \sum_i \int_{k_i}^{k_{i+1}} c_i^L\,\mathrm{d}t\\ &= (b^L-a)c+\sum_i (k_{i+1}-k_i)c_i^L < (b^L-a)c+(b-b^L)c = (b-a)c.\end{aligned}$$

Similar inequalities can be proved taking right options of $b$ or of $c$, or taking options of $a$. This proves that $\int_a^b c\,\mathrm{d}t \preceq (b-a)c$. □

As usual, if $P,Q$ are partitions of $(a,b)$, I say that $P$ is a refinement of $Q$ if $Q$ is a subsequence of $P$. A basic lemma in the theory of Riemann integral on the reals is that I take a refinement of $P$, I obtain a better approximation of the integral. For this, we will need axiom 1.

**Lemma 2.4.** *Let $a < b \in \mathbf{No}$, let $P, Q$ be partitions of $(a,b)$, with $P$ a refinement of $Q$. Let $f_Q^L$ be a tuple of left options of $f$.*

*Then there exists $g_P^L$ tuple of left options of $f$ such that*

$$\int_a^b f_Q^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t \leq \int_a^b g_P^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t$$

*Proof.* W.l.o.g. I can suppose $Q = (k_1,\ldots,k_m)$, $P = Q \cup \{c\}$, $k_n < c < k_{n+1}$, $n < m$. I define $g_P = f_i$ on $(k_i, k_{i+1})$ if $i \neq n$. I apply axiom 1b to $f_n^L$ to obtain $f_n^{L'}$ and define $g_P = f_n^{L'}$ on both $(k_n, c)$ and $(c, k_{n+1})$.

$$\int_a^b f_Q^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t = \sum_i \int_{k_i}^{k_{i+1}} f_i^L(t,k_i,k_{i+1})\,\mathrm{d}t$$

$$= \sum_{i\neq n} \int_{k_i}^{k_{i+1}} f_i^L(t,k_i,k_{i+1})\,\mathrm{d}t + \int_{k_n}^{c} f_n^L(t,k_n,k_{n+1})\,\mathrm{d}t + \int_{c}^{k_{n+1}} f_n^L(t,k_n,k_{n+1})\,\mathrm{d}t$$

$$\leq \sum_{i\neq n} \int_{k_i}^{k_{i+1}} g_i^L(t,k_i,k_{i+1})\,\mathrm{d}t + \int_{k_n}^{c} g_n^L(t,k_n,c)\,\mathrm{d}t + \int_{c}^{k_{n+1}} g_n^L(t,c,k_{n+1})\,\mathrm{d}t$$

$$= \int_a^b g_P^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}})\,\mathrm{d}t \quad \square$$

Given $a < c < b$ and $P[a,b]$ such that $c \in P$, I define the restriction of $P$ to $[a,c]$ in the obvious way: if $P = (k_0,\ldots,k_n)$, its restriction is $(k_0,\ldots,k_m)$, where $k_m = c$. Given $g_P$ tuple of functions, I define the restriction of $g_P$ to $[a,c]$ in a similar way.

**Lemma 2.5.** *Let $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$, $a < b \in \mathbf{No}$, $P$ be a partition of $[a,b]$ and $f_P^L$ be a tuple of left options of $f$. Then,*

$$\int_a^b f_P^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t < \int_a^b f(t)\,\mathrm{d}t. \tag{2.4}$$

*Similar inequalities hold if $b > a$ or for right options $f_P^R$. Moreover, the definition of $\mathcal{F}$ is uniform.*

*Proof.* I will first prove (2.4). If $a \prec b$ or $b \prec a$, it follows from the very definition of $\int_a^b f(t)\,\mathrm{d}t$.

Otherwise, let $c := \langle a \mid b \rangle$. It follows that $c \prec a$ and $c \prec b$. Let $Q := P \cup \{c\}$, let $Q_1$ (let $Q_2$) be the restriction of $Q$ to $[a,c]$ (to $[c,b]$).

By lemma 2.4, there exists $g_Q^L$ tuple of left options of $f$ such that

$$\int_a^b f_P^L \leq \int_a^b g_Q^L = \int_a^c g_{Q_1}^L + \int_b^c g_{Q_2}^L < \int_a^c f + \int_c^b f = \int_a^b f$$

Choose a non-canonical representation $x = \langle y^L \mid y^R \rangle$ for $x$, and define $z^{\mathbf{o}} := \mathcal{F}(y^{\mathbf{o}}) + \int_{y^{\mathbf{o}}}^x f_P^{\mathbf{o}}$. I must prove that $z = \mathcal{F}(x)$. For every $y^L$ left option of $x$,

$$\mathcal{F}(y^L) + \int_{y^L}^x f_P^L < \mathcal{F}(y^L) + \int_{y_L}^x f = \mathcal{F}(x)$$

**Lemma 2.4.** *Let $a < b \in \mathbf{No}$, let $P, Q$ be partitions of $(a,b)$, with $P$ a refinement of $Q$. Let $f_Q^L$ be a tuple of left options of $f$.*

*Then there exists $g_P^L$ tuple of left options of $f$ such that*

$$\int_a^b f_Q^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t \leq \int_a^b g_P^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t$$

*Proof.* W.l.o.g. I can suppose $Q = (k_1,\dots,k_m)$, $P = Q \cup \{c\}$, $k_n < c < k_{n+1}$, $n < m$. I define $g_P = f_i$ on $(k_i, k_{i+1})$ if $i \neq n$. I apply axiom 1b to $f_n^L$ to obtain $f_n^{L'}$ and define $g_P = f_n^{L'}$ on both $(k_n, c)$ and $(c, k_{n+1})$.

$$\begin{aligned}
\int_a^b f_Q^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t &= \sum_i \int_{k_i}^{k_{i+1}} f_i^L(t,k_i,k_{i+1})\,\mathrm{d}t \\
&= \sum_{i\neq n} \int_{k_i}^{k_{i+1}} f_i^L(t,k_i,k_{i+1})\,\mathrm{d}t + \int_{k_n}^{c} f_n^L(t,k_n,k_{n+1})\,\mathrm{d}t + \int_{c}^{k_{n+1}} f_n^L(t,k_n,k_{n+1})\,\mathrm{d}t \\
&\leq \sum_{i\neq n} \int_{k_i}^{k_{i+1}} g_i^L(t,k_i,k_{i+1})\,\mathrm{d}t + \int_{k_n}^{c} g_n^L(t,k_n,c)\,\mathrm{d}t + \int_{c}^{k_{n+1}} g_n^L(t,c,k_{n+1})\,\mathrm{d}t \\
&= \int_a^b g_P^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}})\,\mathrm{d}t \qquad \square
\end{aligned}$$

Given $a < c < b$ and $P[a,b]$ such that $c \in P$, I define the restriction of $P$ to $[a,c]$ in the obvious way: if $P = (k_0,\dots,k_n)$, its restriction is $(k_0,\dots,k_m)$, where $k_m = c$. Given $g_P$ tuple of functions, I define the restriction of $g_P$ to $[a,c]$ in a similar way.

**Lemma 2.5.** *Let $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$, $a < b \in \mathbf{No}$, $P$ be a partition of $[a,b]$ and $f_P^L$ be a tuple of left options of $f$. Then,*

$$\int_a^b f_P^L(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t < \int_a^b f(t)\,\mathrm{d}t. \tag{2.4}$$

*Similar inequalities hold if $b > a$ or for right options $f_P^R$. Moreover, the definition of $\mathcal{F}$ is uniform.*

*Proof.* I will first prove (2.4). If $a \prec b$ or $b \prec a$, it follows from the very definition of $\int_a^b f(t)\,\mathrm{d}t$.

Otherwise, let $c := \langle a \mid b \rangle$. It follows that $c \prec a$ and $c \prec b$. Let $Q := P \cup \{c\}$, let $Q_1$ (let $Q_2$) be the restriction of $Q$ to $[a,c]$ (to $[c,b]$).

By lemma 2.4, there exists $g_Q^L$ tuple of left options of $f$ such that

$$\int_a^b f_P^L \leq \int_a^b g_Q^L = \int_a^c g_{Q_1}^L + \int_b^c g_{Q_2}^L < \int_a^c f + \int_c^b f = \int_a^b f$$

Choose a non-canonical representation $x = \langle y^L \mid y^R \rangle$ for $x$, and define $z^{\mathbf{o}} := \mathcal{F}(y^{\mathbf{o}}) + \int_{y^{\mathbf{o}}}^x f_P^{\mathbf{o}}$. I must prove that $z = \mathcal{F}(x)$. For every $y^L$ left option of $x$,

$$\mathcal{F}(y^L) + \int_{y^L}^x f_P^L < \mathcal{F}(y^L) + \int_{y_L}^x f = \mathcal{F}(x)$$

and similarly for right options of $f$ or of $x$. So, $z \preceq \mathcal{F}(x)$.

It remains to show that $\mathcal{F}(x) \preceq z$. But, by cofinality, for every $x^L$ canonical left option of $x$ there exists $y^L$ such that $x^L \leq y^L < x$. Therefore, given $P[x^L, x]$ and $f_P^L$, if $Q := P \cup \{y^L\}$, there exists $f_Q^L$ such that

$$\begin{aligned}\mathcal{F}(x)^L &= \mathcal{F}(x^L) + \int_{x^L}^{x} f_P^L \leq \mathcal{F}(x^L) + \int_{x^L}^{x} f_Q^L = \\ &= \mathcal{F}(x^L) + \int_{x^L}^{y^L} f_{Q_1}^L + \int_{y^L}^{x} f_{Q_2}^L \leq \mathcal{F}(y^L) + \int_{y^L}^{x} f_{Q_2}^L = z^L,\end{aligned}$$

where $Q_1, Q_2$ are the restrictions of $Q$ to $(x^L, y^L)$ and $(y^L, x)$. (I assumed for simplicity of notation that $x$ has no right options, but the general case is similar). □

The integral is linear in the function argument, provided that it is already linear on $\mathfrak{A}$, and that $\mathfrak{A}$ is constructed inductively.

**Lemma 2.6.** *Suppose that $\mathfrak{A}$ is also a vector space over* **No**, *and that $\int$ is linear on $\mathfrak{A}$. Let $f$, $g$ be recursive over $\mathfrak{A}$, and $a, b, \lambda \in$* **No**. *Then,*

$$\int_a^b f(t) + g(t)\,\mathrm{d}t = \int_a^b f(t)\,\mathrm{d}t + \int_a^b g(t)\,\mathrm{d}t \tag{2.5}$$

$$\int_a^b \lambda f(t)\,\mathrm{d}t = \lambda \int_a^b f(t)\,\mathrm{d}t \tag{2.6}$$

*Proof.* I will give all details of this proof, even if it is quite elementary. Both formulae are proved by induction on $f$, $g$, $a$, $b$, $\lambda$, i.e. I suppose to have proved the lemma for:

- $f^{\circ}(t,c), g(t), a', b', \lambda'$ for any $f^{\circ}$ option of $f$ and any $c, a', b', \lambda' \in \mathbf{No}$.
- The same as before with $f, g$ exchanged.
- $f, g, (a, b, \lambda)^{\circ}$ where I have already explained what I mean by the option of a tuple.

In both cases, without loss of generality I can suppose $a = 0$ and $b > 0$. I call $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$, $\mathcal{G}(x) := \int_0^x g(t)\,\mathrm{d}t$. Fix once for all $P = (k_0, \dots, k_n)$ a partition of $(a, b)$.

By definition of $+$,

$$(\mathcal{F}(x) + \mathcal{G}(x))^{\circ} = (\mathcal{F}(x))^{\circ} + \mathcal{G}(x), \mathcal{F}(x) + (\mathcal{G}(x))^{\circ}$$

I consider only the first kind of options (the others are similar).

Let $\mathcal{H}(x) := \int_0^x f(t) + g(t)\,\mathrm{d}t$.

$$\begin{aligned}
&\mathcal{G}(x) + \left(\mathcal{F}(x)\right)^{\mathbf{o}} = \\
&= \mathcal{G}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} g(t)\,\mathrm{d}t + \mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} f_P^{\mathbf{o}}(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t \\
&= \mathcal{G}(x^{\mathbf{o}}) + \sum_i \int_{k_i}^{k_{i+1}} g(t)\,\mathrm{d}t + \mathcal{F}(x^{\mathbf{o}}) + \sum_i \int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}}(t,k_i,k_{i+1})\,\mathrm{d}t \\
&= \mathcal{H}(x^{\mathbf{o}}) + \sum_i \int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}}(t,k_i,k_{i+1}) + g(t)\,\mathrm{d}t \\
&= \mathcal{H}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} (f+g)_P^{\mathbf{o}}(t,t^L,t^R)\,\mathrm{d}t = \left(\mathcal{H}(x)\right)^{\mathbf{o}}
\end{aligned}$$

where $(f+g)_P^{\mathbf{o}}(t,t',t'') := f_P^{\mathbf{o}}(t,t',t'',f(t'),f(t'')) + g(t)$, and I have used the definition of $\int$ and the additivity of $\int$ in the interval of integration for the first two identities, the inductive hypothesis for the third, and the definition of $\int$ for the last two. Therefore, every option of $\mathcal{F} + \mathcal{G}$ is an option of $\mathcal{H}$.

For the converse, let

$$(f+g)_i^{\mathbf{o}} = \begin{cases} f_i^{\mathbf{o}} + g & \text{or} \quad (2.7a) \\ f + g_i^{\mathbf{o}}. & \quad (2.7b) \end{cases}$$

Suppose, for instance, that (2.7a) is true, and that $f^{\mathbf{o}}$ is a left option of $f$. Then, by inductive hypothesis

$$\int_{k_i}^{k_{i+1}} (f+g)_i^{\mathbf{o}}(t,k_i,k_{i+1})\,\mathrm{d}t = \int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}}(t,k_i,k_{i+1})\,\mathrm{d}t + \int_{k_i}^{k_{i+1}} g(t)\,\mathrm{d}t.$$

Lemma 2.5 implies that

$$\int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}}(t,k_i,k_{i+1})\,\mathrm{d}t < \int_{k_i}^{k_{i+1}} f(t)\,\mathrm{d}t,$$

therefore

$$\sum_i \int_{k_i}^{k_{i+1}} (f+g)_i^{\mathbf{o}}(t,k_i,k_{i+1})\,\mathrm{d}t < \sum_i \int_{k_i}^{k_{i+1}} f(t)\,\mathrm{d}t \int_{k_i}^{k_{i+1}} g(t)\,\mathrm{d}t = \mathcal{F}(x) + \mathcal{G}(x).$$

I proceed in the same fashion for scalar multiplication.

Let $\mathcal{H}(\lambda,x) := \int_0^x \lambda f(t)\,\mathrm{d}t$.

$$\begin{aligned}
\left(\lambda\mathcal{F}(x)\right)^{\mathbf{o}} &= \lambda^{\mathbf{o}}\mathcal{F}(x) + (\lambda - \lambda^{\mathbf{o}})\mathcal{F}(x)^{\mathbf{o}} \\
&= \lambda^{\mathbf{o}}\mathcal{F}(x) + (\lambda - \lambda^{\mathbf{o}})\left(\mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} f_P^{\mathbf{o}}(t,t^{\mathbf{o}})\,\mathrm{d}t\right) \\
&= \mathcal{H}(\lambda^{\mathbf{o}},x) + \mathcal{H}(\lambda,x^{\mathbf{o}}) - \mathcal{H}(\lambda^{\mathbf{o}},x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} (\lambda - \lambda^{\mathbf{o}}) f_P^{\mathbf{o}}(t,t^{\mathbf{o}})\,\mathrm{d}t \\
&= \mathcal{H}(\lambda,x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} \left(\lambda^{\mathbf{o}} f(t) + (\lambda - \lambda^{\mathbf{o}}) f_P^{\mathbf{o}}(t,t^{\mathbf{o}})\right)\mathrm{d}t \\
&= \mathcal{H}(\lambda,x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} (\lambda f)_P^{\mathbf{o}}(t,t^{\mathbf{o}})\,\mathrm{d}t
\end{aligned}$$

where I have used the definition of product for the first identity, the definition of $\int$ for the second, the inductive hypothesis for the third and again the definition of the product for the last one, beside the formula (2.5).

Conversely, for every $i = 0, \dots, m-1$, choose $\lambda_i$ an option of $\lambda$. Then

$$\begin{aligned}(\mathcal{H}(\lambda,x))^{\mathbf{o}} &= \mathcal{H}(\lambda,x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} (\lambda f)_P^{\mathbf{o}} \\ &= \mathcal{H}(\lambda,x^{\mathbf{o}}) + \sum_i \int_{k_i}^{k_{i+1}} (\lambda_i^{\mathbf{o}} f + (\lambda - \lambda_i^{\mathbf{o}}) f_i^{\mathbf{o}})\end{aligned}$$

By inductive hypothesis, the previous is equal to

$$\mathcal{H}(\lambda,x^{\mathbf{o}}) + \sum_i ((\lambda - \lambda_i^{\mathbf{o}}) \int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}} + \lambda_i^{\mathbf{o}} \int_{k_i}^{k_{i+1}} f)$$

Again, if I suppose that $(\mathcal{H}(\lambda,x))^{\mathbf{o}}$ is a left option of $\mathcal{H}(\lambda,x)$, then, by lemma 2.5

$$(\lambda - \lambda_i^{\mathbf{o}}) \int_{k_i}^{k_{i+1}} f_i^{\mathbf{o}} + \lambda_i^{\mathbf{o}} \int_{k_i}^{k_{i+1}} f < \lambda \int_{k_i}^{k_{i+1}} f(t)\, \mathrm{d}t,$$

and the conclusion follows. □

Now, the fundamental lemma: the monotonicity of the integral. As I said before, for this I need that $\mathfrak{A}$ is constructed inductively, and that axiom 2 is true.

**Lemma 2.7.** *Let $a < b \in \mathbf{No}$, $f,g$ be recursive over $\mathfrak{A}$. If $f(t) \leq g(t)$ provably in $(a,b)$ then*

$$\int_a^b f(t)\, \mathrm{d}t \leq \int_a^b g(t)\, \mathrm{d}t.$$

*If moreover the inequality in the hypothesis is strict, then it is strict also in the conclusion. If instead I have $=$ in the hypothesis, I have $=$ also in the conclusion.*

*Proof.* Again, I proceed by induction on $f,g,a,b$. Let $c = \langle a \mid b \rangle$ If $a \not\preceq b$ and $b \not\preceq a$ then $c \prec a$ and $c \prec b$, so the conclusion follows by induction. Otherwise, w.l.o.g. $a \prec b$.

I will treat the case of $<$ first. Simple case: $f(x) = g^L(x,a,b,g(a),g(b))$ for some $g^L$ left option of $g$, or $g(x) = f^R(x,a,b,f(a),f(b))$. Then the conclusion follows from lemma 2.5 and the $f = g$ case of the inductive hypothesis.

In general, let $P[a,b], f_P^R, g_P^L$ as in the definition 2.1. Then I can use the inductive hypothesis, obtaining for each $i < m$

$$\int_{k_i}^{k_{i+1}} f(t) < \int_{k_i}^{k_{i+1}} g(t),$$

and the conclusion follows.

Now I will treat the case $f = g$. I must show $\left(\int_a^b f\right)^L < \int_a^b g$ (and similarly for right options and for $f, g$ exchanged). By definitions,

$$\left(\int_a^b f(t)\,\mathrm{d}t\right)^L = \begin{cases} \int_a^{b^L} f + \int_{b^L}^b f_P^L, & (2.8a) \\ \int_a^{b^R} f + \int_{b^R}^b f_P^R, & (2.8b) \\ \int_{a^L}^b f - \int_{a^L}^a f_P^R, & (2.8c) \\ \int_{a^R}^b f - \int_{a^R}^a f_P^L. & (2.8d) \end{cases}$$

Consider the left option (2.8a). There are two cases: either $b^L \le a$ or $b^L > a$.

If $a \le b^L < b$, then $f_i^L(t, k_i, k_{i+1}) < g(t)$ for $t \in (k_i, k_{i+1})$ for each $i$, so I can apply the inductive hypothesis, and obtain $\int_{b^L}^b f_P^L < \int_{b^L}^b g$.

If $a > b^L$, then the conclusion becomes

$$\int_{b^L}^b f_P^L < \int_a^b g + \int_{b^L}^a f.$$

Let $Q = P \cup \{a\}$, let $f_Q^{L'}$ as in lemma 2.4. Then, if $Q_1$ (if $Q_2$) is the restriction of $Q$ to $[a, b]$ (to $[b^L, a]$),

$$\int_{b^L}^b f_P^L \le \int_{b^L}^b f_Q^{L'} = \int_a^b f_{Q_1}^{L\,\prime} + \int_{b^L}^a f_{Q_2}^{L\,\prime} < \int_a^b g + \int_{b^L}^a f.$$

where, again, I have used the inductive hypothesis for the last inequality.

Consider now the left option (2.8b). This is the step where I use the fact that $f = g$ (and not merely that $f \le g$). Then,

$$\int_b^{b^R} f^R > \int_b^{b^R} g$$

by induction on $f$, and

$$\int_a^{b^R} f = \int_a^{b^R} g$$

by induction on $b$, and the conclusion follows.

The case (2.8c) is treated in a similar way to (2.8b), and (2.8d) to (2.8a).

It remains to prove the case $f \le g$. But then, by axion 2, either $f = g$, and I have just discussed it, or I can find $P[a, b]$ such that $f(t) < g(t)$ in each interval $(k_i, k_{i+1})$, and the conclusion follows from the $<$ case. □

For the remainder of this thesis, I will need that the integral is monotone and that every function in $\mathfrak{A} \cup \{f\}$ is either constant, or has only finitely many zeros. As I have shown above, these are consequences of axiom 2.

I will now consider the problem of good definition of integral.

Consider a simple example: $f(x) = x^2$. Remember that

$$(x^2)^{\circ} = x^{\circ} - (x - x^L)^{\alpha}(x - x^R)^{\beta},\ 0 \le \alpha \le 2,\ \alpha + \beta = 2,$$

$\mathcal{F}(0) = 0$ and $1 = \langle 0 \,|\, \rangle$.

$$(\mathcal{F}(1))^{\circ} = \sum_i \int_{k_i}^{k_{i+1}} t^2 - (t - k_i)^{\alpha_i}(t - k_{i+1})^{\beta_i}\, dt.$$

I claim that I can take $k_i$ rational for every $i$. The reason is the following.

Fix $P[0,1] = (k_0, \dots, k_m)$ and $\alpha_0, \dots, \alpha_{m-1}$ and call $z_P$ the resulting option of $\mathcal{F}(1)$.

If I refine $P$ by adding a single point that differs from each $k_i$ by a non infinitesimal amount, I obtain a partition $Q$ such that the corresponding approximation $z_Q$ is better than $z_P$ by a non infinitesimal amount $\rho$.

For each point $h_i \in Q$ I substitute it by a rational number $h_i'$: in this way, I obtain a partition $Q'$ and an approximation $z_{Q'}$ of $\mathcal{F}(1)$ which is slightly worse than $z_Q$ by a certain amount $\varepsilon$. However, choosing $h_i$ suitably, I can make this $\varepsilon$ smaller than any fixed positive real, and in particular smaller than $\rho$. So, $z_{Q'}$ is better than $z_P$, and I can discard $z_P$ altogether. Therefore, $\mathcal{F}(1)$ is well defined.

Of course, I could have proved this using Tarski's theorem(4), but, for an arbitrary function $f$ I do not have such a theorem.

For the rest of this section, I will suppose that axiom 3 is true.

**Lemma 2.8.** *Let $\mathbb{K} \subseteq$ **No** be an initial elementary substructure(5) of* **No**, *let $x \in \mathbb{K}$, and suppose that $\mathbb{K}$ is $\alpha$-saturated, where $\alpha$ is a cardinal number greater or equal to $\ell(x)$.*

*Then $\mathcal{F}(x) := \int_0^x f(t)$ is defined, and is in $\mathbb{K}$.*

*Proof.* Simple case: $x = 1$, $f$ has only one left option $f^L$ and one right option $f^R$.

Let $P[0,1]$ be an $m$-partition of $[0,1]$, and let

$$g^L(P) := g^L(k_0, \dots, k_m) := \int_0^1 f_P^L(t, t^{\circ}, f(t^{\circ}))\, dt$$

and similar for $g^R$.

I have to prove that there exists $z \in \mathbb{K}$ such that $g^L(P) < z < g^R(P')$ for every $P, P'[a,b]$. These conditions induce a type $T(z)$ (in this case, without parameters), given by formulae

$$\psi_m^L(z) := \forall k_0 = 0 < k_1 < \dots < k_m = 1\ \left(g^L(k_0, \dots, k_m) < z\right),$$

(4) The theory of real closed fields is complete and model complete in the language of ordered rings.

(5) In the language $\mathscr{L}$, defined in axion 3.

and similarly for $\psi^R_M(z)$.

First, I will prove that $T(z)$ is consistent. Suppose not. Then, for some $m \in \mathbb{N}$,

$$\mathbf{No} \models \forall z \neg\left(\psi^L_m(z) \,\&\, \psi^R_m(z)\right).$$

Let

$$z^L = \sup\left\{g^L(k_0,\ldots,k_m) : k_i \in \mathbf{No}\right\},$$

and similarly for $z^R$. By axiom 3, $z^L$ and $z^R$ are in **No**. By $\mathbb{K} \preceq \mathbf{No}$, $z^L$ and $z^R$ are in $\mathbb{K}$. We know already $z^L \leq z^R$. By inconsistency, it must be $z^L = z^R = g^L(P)$ for some $P[0,1]$ (or $z^L = z^R = g^R(P)$). Note that if axiom 3 were not true, it could happen that $z^L = z^R \in \mathbf{No}^{\mathscr{D}} \setminus \mathbf{No}$, and in that case $\mathcal{F}(1)$ would not be defined.

But $f^L_P(t,k_i,k_{i+1}) < f(t)$ on $(k_i,k_{i+1})$ for each $i$, so there is an interval $(a',b') \subseteq (0,1)$ and $\varepsilon > 0$ such that $f^L_P < f - \varepsilon$. Again by elementary equivalence, I can take $\varepsilon, a', b' \in \mathbb{K}$. Therefore,

$$\int_0^1 f^L_P < \int_0^1 f^R_Q(t) - \varepsilon(b'-a')$$

for every $Q[0,1]$ and $z^R - z^L \geq \varepsilon(b'-a')$, which is a contradiction.

So, $T(z)$ is consistent, and, because $\mathbb{K}$ is saturated (over the empty set), it has a realisation in $\mathbb{K}$. The simplest such realisation is $\mathcal{F}(1)$.

For a generic $x \in \mathbb{K}$ and arbitrary $f$, I proceed similarly, using saturation of $\mathbb{K}$ over the parameters $x^{\mathbf{o}}$ and $\mathcal{F}(x^{\mathbf{o}})$, as $x^{\mathbf{o}}$ varies among the options of $x$, and induction on $x$. The fact that $f$ has at least one right option ensures that $z^L < +\infty$ (and similarly for left options). □

For the proof previous lemma, it is not necessary that *every* $\mathscr{L}$-definable $S \subset \mathbf{No}$ has a supremum: it suffices that subclasses definable using existential formulae only have it.

For other properties of the integral, I will need further assumptions.

**Axiom 4 (Intermediate value).** Let $f\colon \mathbf{No} \to \mathbf{No}$, $a < b \in \mathbf{No}$. Suppose $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = 0$.

In the rest of this section, I will assume that axioms 1, 2 and 4 (beside axiom 1 and the consequences of axiom 2 mentioned above). Moreover, I define

$$\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t.$$

**Lemma 2.9 (Rolle).** *Let $a < b \in \mathbf{No}$. Suppose $\mathcal{F}(a) = \mathcal{F}(b)$. Then there exists $c \in (a,b)$ such that $f(c) = 0$.*

*Proof.* Suppose not. Then either $f(t) > 0$ or $f(t) < 0$ in $(a,b)$. But $\mathcal{F}(b) - \mathcal{F}(a) = \int_a^b f(t)\,dt$, so, by monotonicity, $\mathcal{F}(b) > \mathcal{F}(a)$ in the first case, and $\mathcal{F}(b) < \mathcal{F}(a)$ in the second. □

**Corollary 2.10.** *Let $a < b \in$ **No**. $\mathcal{F}$ has only finitely many zeros in $(a,b)$.*

**Lemma 2.11.** *Let $a < b \in$ **No**. $\mathcal{F}$ can change sign only finitely many times in $(a,b)$.*

*Proof.* As in the proof of lemma (2.9), between two sign changes of $\mathcal{F}$ there must be a zero of $f$: i.e. if $c < d < e$ and $\mathcal{F}(c), \mathcal{F}(e) < 0 < \mathcal{F}(d)$ then there exists $f \in (c,e)$ such that $f(f) = 0$. But by axiom 2, each $f \in \mathfrak{A}$ can have only finitely many zeros in $(a,b)$. □

The following theorem needs also axiom 3.

**Theorem 7 (Intermediate value).** *Let $\mathcal{F}(x) := \int_0^x f(t)$. Let $a,b,d \in$ **No** such that $a < b$ and $\mathcal{F}(a) < d < \mathcal{F}(b)$. Then there exists $c \in$ **No** such that $a < c < b$ and $\mathcal{F}(c) = d$.*

*Wrong proof.* Let's do the case $d = 0$. By the lemma 2.11, w.l.o.g. I can suppose that there exists $\zeta \in \mathbf{No}^{\mathcal{D}}$ such that $\mathcal{F}(t) < 0$ in $[a,\zeta)$ and $\mathcal{F}(t) > 0$ in $(\zeta, b]$. I must prove that $\zeta \in$ **No**. I will give an inductive definition of $\zeta$.

Let us solve the equation (in $c$) $\mathcal{F}(c) = 0$, $a < c < b$. It is necessary and sufficient that $\mathcal{F}(c)^L < 0 < \mathcal{F}(c)^R$, i.e. I have to solve inequalities of the kind:

$$\mathcal{F}(c^L) + \int_{c^L}^{c} f_P^L(t, t^\circ, f(t^\circ))\,dt < 0, \text{ i.e.}$$
$$\mathcal{F}(c^L) + \sum_i f_i^L(t, k_i, k_{i+1})\,dt < 0$$

I will give the options of $c$. First, I must say $a < c < b$, i.e. $a$ is among the left options, and $b$ among the right ones.

Suppose that I have already found some options $c^L, c^R$ such that $a \le c^L < \zeta < c^R \le b$. Fix $P[c^L, c^R]$, and suppose $k_n < \zeta < k_{n+1}$. Let

$$g(x) = \mathcal{F}(c^L) + \int_{c^L}^{x} f_P^L(t, t^\circ)\,dt :=$$
$$\mathcal{F}(c^L) + \sum_{i<n} \int_{k_i}^{k_{i+1}} f_i^L(t, k_i, k_{i+1})\,dt + \int_{k_n}^{x} f_n^L(t, k_n, k_{n+1})\,dt$$

where $x \in (k_n, k_{n+1})$.

$g(x) < \mathcal{F}(x)$, and $\mathcal{F}(x) < 0$ in $(a,\zeta)$, so $g(t)$ has only finitely many zeros in $(k_n, k_{n+1})$.

Let $c^{L'}$ be the rightmost zero of $g$ before $\zeta$, and let $c^{R'}$ be the leftmost zero after $\zeta$ (if they exists, otherwise use $a$ or $b$ respectively). Then, by induction on $f$, $g(t)$ does not change sign in $(c^{L'}, c^{R'})$. So $g(t) < 0$ in $(c^{L'}, c^{R'})$.

Add $c^{L'}$ to the left options of $c$ and $c^{R'}$ to the right ones. It follows that, at the end of this process, $\mathcal{F}(c) = 0$. The problem is that I am adding an option for every partition $P[a,b]$, and there is a proper class of partitions, therefore I cannot be sure that $c \in \mathbf{No}$. $\square$

I will give now the correct proof, using axiom 3.

*Correct proof.* By lemma 2.11, I can suppose that there exists $\zeta \in \mathbf{No}^{\mathscr{D}}$, the Dedekind completion of $\mathbf{No}$, such that $\mathcal{F}(t) < d$ in $[a, \zeta)$ and $\mathcal{F}(t) > d$ in $(\zeta, b]$.

I recall that $\mathcal{F}$ is not in the language $\mathscr{L}$, and that the cut $\zeta$ in general is not definable in $\mathscr{L}$. I must prove that $\zeta \in \mathbf{No}$. I will prove the lemma by induction on $f$ and $d$.

I need to give the options of $c$. First of all, $a < c < b$, so $a$ is a left option, $b$ a right one. Solving the equation $\mathcal{F}(c) = d$ is equivalent to solving the inequalities

$$\mathcal{F}(c)^L < d \qquad\qquad \mathcal{F}(c)^R > d \tag{2.9a}$$

$$\mathcal{F}(c) < d^R \qquad\qquad \mathcal{F}(c) > d^L. \tag{2.9b}$$

Consider (2.9b), for instance $\mathcal{F}(c) < d^R$. $\mathcal{F}(c) < d < d^R$ in $[a, \zeta)$, therefore $\mathcal{F}(c) = d^R$ has at most finitely many solutions. Let $c^L$ (let $c^R$) be the rightmost (the leftmost) solution smaller (greater) than $\zeta$, if it exists. By inductive hypothesis, $\mathcal{F}(x) - d^R$ does not change sign in $(c^L, c^R)$, therefore $\mathcal{F}(x) < d^R$ in $(c^L, c^R)$, and I can add them as left and right options of $c$.

Consider now (2.9a), for instance $\mathcal{F}(c)^L < d$. Suppose that I have already found some options $c^L, c^R \in \mathbb{K}$, with $a \le c^L < \zeta < c^R \le b$. Fix $m \in \mathbb{N}$, an $m$-tuple of left options $\vec{f}^L$ and a left option $c^L$. Let $\vec{k} = (k_0, \dots, k_m)$ be a partition of $[a,b]$, and, say, $k_n \le c^L < k_{n+1}$ and $k_l \le x < k_{l+1}$. Define

$$\begin{aligned} g^L_{\vec{f}^L}(x, \vec{k}, c^L) &:= \mathcal{F}(c^L) + \int_{c^L}^{x} \vec{f}^L(t, t^{\mathbf{o}}, f(t^{\mathbf{o}}))\,\mathrm{d}t = \\ &= \sum_{n<i<l} \int_{k_i}^{k_{i+1}} f_i^L(t, k_i, k_{i+1})\,\mathrm{d}t + \\ &+ \int_{c^L}^{k_{n+1}} f_n^L(t, k_n, k_{n+1})\,\mathrm{d}t + \int_{k_l}^{x} f_l^L(t, k_l, k_{l+1})\,\mathrm{d}t. \end{aligned} \tag{2.10}$$

I give analogous definitions for right options $c^R$ of $c$ or if $\vec{f}^R$ is an $m$-tuple of right options. I want now to add some extra options to $c$, ensuring that $g^L_{\vec{f}^L}(c, \vec{k}, c^L) < d$ is true (and similarly with $c^R$ or $\vec{f}^R$).

Notwithstanding its complicated aspect, $g^L(x) = d$ is an equation simpler than $\mathcal{F}(x) = d$, therefore the conclusion of the lemma holds for it. It follows that if $c^{R'}$ is the leftmost solution after $\zeta$ of such equation, $g^L(x) < d$ in the interval $(c^L, c^{R'})$. In fact, $g^L(x) < \mathcal{F}(x)$ in $(c^L, b)$ and $\mathcal{F}(x) < d$ in $(c^L, \zeta)$. $c^{R'}$ depends on the "old" option $c^L$, on the $m$-partition $\vec{k}$, beside the $m$-tuple of options $\vec{f}^L$.

Now, apparently, I can add $c^{R'}$ to the options of $c$, and at the end of this process obtain $\mathcal{F}(c) = d$.

Unfortunately, in this way of defining $c$ I use a whole class of options (instead of a proper set), and $c$ may not exist in **No**. I'll do something better. Let

$$h^R_{\vec{f}^L}(c^L, \vec{k}) = \inf\left\{ x \in \mathbf{No} : c^L < x < b \ \&\ \exists k_0, \dots, k_m g^L_{\vec{f}^L}(x, \vec{k}, c^L) \geq 0 \right\}.$$

By axiom 3, $h^R$ exists. Now, I can take $h^R$ as new right options of $c$. $h^R$ depends only on the "old" option $c^L$, besides $m$ and the chosen $m$-tuple of options $\vec{f}^{\mathbf{o}}$. Therefore, once I fix the "old" option $c^L$, I am adding only a proper set of "new" options (one for every $m \in \mathbb{N}$ and every possible choice of $\vec{f}^{\mathbf{o}}$), and now I can say that at the end of the process $c \in \mathbf{On}$. □

**Lemma 2.12 (O-minimality).** *Let $\mathfrak{A}$ be a family of functions, constructed inductively. Suppose that axiom 1 is true for every $f$ in $\mathfrak{A}$ and that the structure on* **No** *induced by $\mathfrak{A}$ is o-minimal. Then for every $f, g \in \mathfrak{A}$ if $f < g$ (on an interval $(a,b)$) then $f < g$ provably.*

*Proof.* In fact, for every $x \in (a,b)$ there exists $f^R_x$ right option of $f$ or $g^L_x$ left option of $g$ and $x^L, x^R$ options of $x$ such that either

$$f^R_x(x, x^R, x^L, f(x^L), f(x^R)) \leq g(x) \text{ or } f(x) \leq g^L_x(x, x^R, x^L, g(x^L), g(x^R)).$$

Let

$$V_x = \left\{ t \in (a,b) : \exists t', t''\ t' < t < t'' \ \&\ f^R_x(t, t', t'', f(t'), f(t'')) \leq g(t) \right\},$$

and similarly $U_x$ for $g^L_x$ and $f$. $V_x$ depends only on $f^R_x$, not on $x$ itself, so the class of $V_x$s and $U_x$s is actually a proper set. By o-minimality, every $V_x$ and $U_x$ is a finite union of intervals with extremes in **No**, therefore, by lemma 1.51, there exist $x_1, \dots, x_n$ such that

$$\mathbf{No} = V_{x_1} \cup \dots \cup V_{x_n} \cup U_{x_1} \cup \dots \cup U_{x_n}.$$

The conclusion easily follows (via axiom 1). □

Axioms 3 and the saturation of **No** (sufficient for the existence of the integral) are also immediate consequences of o-minimality. If $f$ is continous,[6] axiom 4 is

[6] I recall that every function in an o-minimal structure is piece-wise continous.

also true. Therefore, if I know that the hypothesis of the previous lemma holds, the only hypothesis in this section that needs to be checked is the second consequence of axiom 2 (i.e. that a function is either constant, or has finitely many zeros). For instance, it is true for analytic functions.

## 2.5 Integral of partial functions

Let $A$ be a set-bounded convex subclass of **No**,[7] and let $f : A \to \mathbf{No}$ be a function. Given $f$ is recursive (over some family of function $\mathfrak{A}$), I want to define its integral.

Let $a$ be the simplest element of $A$, and let $x \in A$. The options of $\int_a^x f(t)\,\mathrm{d}t$ are of the form

$$\int_a^{x^\mathbf{o}} f(t)\,\mathrm{d}t + \int_{x^\mathbf{o}}^x f^\mathbf{o}(t, t^\mathbf{o}, f(t^\mathbf{o}))\,\mathrm{d}t,$$

where $x^\mathbf{o}$ varies among the canonical options of $x$ *with respect to* $A$.

Let $x, y \in A$. Then,

$$\int_x^y f := \int_a^y f - \int_a^x f.$$

Note that if $A = \mathbf{No}$, the definition given here coincides with the one in § 2.1.

*Example* 2.13. $f(x) = 1/x$. The domain of $f$ can be partitioned into two set-bounded convex subclasses:

$$\operatorname{dom} f = (-\infty, 0) \sqcup (0, +\infty).$$

This allows us to define $\int_1^x 1/t\,\mathrm{d}t$ for $x > 0$. In § 5.1 we will see that it is equal to $\log x$.

The propositions, proved in this and the following chapter for total functions, hold, with the same proof, for funtions with domain a set-bounded subclass of **No**.

However, the value of $\int f$ may depend on the choice of the interval of definition of $f$; namely, if $B \subset A$, then the value of $\int f$ computed with repect to $A$ might be different from the one computed w.r.t. $B$. For instance, one can easily construct example of this phenomenon using $f(x) = \exp x$ (see § 5.2).

## 2.6 Concluding remarks

S. Norton and M. Kruskal have already defined an integration on **No** that is similar to what has been defined here.[8] I do not know enough about their work to tell how much it differs from the treatment presented in this thesis.

[7] see § 1.4 for the definition.

[8] See the note on pag. 38 of [6]. The story, as I have understood it, is that Norton gave a definition of an integral, that produces the desired result for the function $1/t$, but lacks some of the other "good" properties of an integral. Kruskal later improved his definition. However, as far as I know, none of this has been published.

Let $\mathfrak{A}$ be a family of functions (containing all the constants, the identity function and $+$), and $\int$ be an integral over it. As I said before, if $f$ is recursively defined over $\mathfrak{A}$, I will suppose that axiom 1 holds. Moreover, from what I have proved, it is reasonable to suppose that the integral is monotone and additive on the interval of integration, and this will always be the case in the following chapters. If $\mathfrak{A}$ is a vector space, I will also suppose that $\int$ is linear.

On the other hand, I will not make uses of the other axioms, unless explicitly specified.

These properties alone are enough to prove the following theorem.

**Theorem 8.** *Suppose that $f : \mathbf{No} \to \mathbf{No}$ is continuous at the point $a \in \mathbf{No}$. Then,*

$$\lim_{x \to 0} \frac{\int_a^{a+x} f(t)\,\mathrm{d}t}{x} = f(a).$$

*Proof.* Without loss of generality, we can suppose $a = f(a) = 0$. Let $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$. By definition, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(t)| < \varepsilon$ for every $t$ such that $|t| < \delta$. Therefore, by monotonicity, in the interval $[0, \delta)$,

$$\int_0^x -\varepsilon\,\mathrm{d}t \leq \mathcal{F}(x) \leq \int_0^x \varepsilon\,\mathrm{d}t,$$

and similarly in the interval $(-\delta, 0]$. We use lemma 2.3 to compute the integral of constant functions. So, $|\mathcal{F}(x)| < \varepsilon|x|$ for $|x| < \delta$, implying that $|\frac{\mathcal{F}(x)}{x}| < \varepsilon$ for $0 \neq |x| < \delta$. □

The previous theorem is the analogue of the fundamental theorem of calculus. However, in this context it is much less powerful than for the reals, because **No**, like every other non Archimedean ordered field, is totally disconnected, therefore we do not have uniqueness of the primitive of a function.

# Chapter 3

# Integration by parts

In this chapter I will prove the integration by parts formula. It is a natural extension of the arguments of previous chapter; however, the hypothesis will be simpler and the computations more involved.

## 3.1 Definition of multiple integral

Given a function $f = \langle f^L \mid f^R \rangle$, what is the meaning of $\int\int f(t)\,\mathrm{d}t$ ?

The reason of this question is that, if $f$ is in the family $\mathfrak{A}$, $\int f$ needs not to be recursive over $\mathfrak{A}$ according to definition in chapter 1.3, because I used a whole class of functions to define it.

Consider $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$.

$$\left(\mathcal{F}(x)\right)^{\mathbf{o}} = \mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} f^{\mathbf{o}}(t, k_i, k_{i+1})\,\mathrm{d}t = \mathcal{F}(x^{\mathbf{o}}) + \sum_i \int_{k_i}^{ki+1} f^{\mathbf{o}}(t, k_i, k_{i+1})\,\mathrm{d}t.$$

If I define $\mathcal{F}^{\mathbf{o}}(t, t^{\mathbf{o}}, \mathcal{F}(t^{\mathbf{o}}))$ accordingly to the previous expression, i.e.

$$\mathcal{F}^{\mathbf{o}}(t, t^{\mathbf{o}}, \mathcal{F}(t^{\mathbf{o}})) = \mathcal{F}(t^{\mathbf{o}}) + \int_{t^{\mathbf{o}}}^{t} f^{\mathbf{o}}(s, s^{\mathbf{o}}, f(s^{\mathbf{o}}))\,\mathrm{d}s,$$

what is the meaning of

$$\int_{x^{\mathbf{o}}}^{x} \mathcal{F}^{\mathbf{o}}(t, t^{\mathbf{o}}, \mathcal{F}(t^{\mathbf{o}}))\,\mathrm{d}t\ ? \tag{3.1}$$

Given $P$ an $m$-partition of $(x^{\mathbf{o}}, x)$, $\int_{x^{\mathbf{o}}}^{x} \mathcal{F}_P(t^{\mathbf{o}})$ has the usual meaning:

$$\int_{x^{\mathbf{o}}}^{x} \mathcal{F}_P(t^{\mathbf{o}}) = \begin{cases} \sum\limits_{0 \le i < m} \mathcal{F}(k_i)(k_{i+1} - k_i) & \text{or} \\ \sum\limits_{0 \le i < m} \mathcal{F}(k_{i+1})(k_{i+1} - k_i) & \end{cases}$$

Given $y \in (k_i, k_{i+1})$ and $f_P^{\mathbf{o}}$ an $m$-tuple of options of $f$, let

$$h_i(y) := \int_{k_i}^{y} f_i^{\mathbf{o}}(t, k_i, k_{i+1}, f(k_i), f(k_{i+1}))\,\mathrm{d}t, \tag{3.2}$$

for every $0 \leq i < m$. $\hbar_i$ is the integral of the function $f_i^{\mathbf{o}}$, which is simpler than $f$, therefore I can suppose that $\int_{k_i}^{k_{i+1}} \hbar_i(t)\,\mathrm{d}t$ has already been defined. Then define

$$\Big(\int_0^x \mathcal{F}(t)\,\mathrm{d}t\Big)^{\mathbf{o}} := \int_0^{x^{\mathbf{o}}} \mathcal{F}(t)\,\mathrm{d}t + \int_{x^{\mathbf{o}}}^x \mathcal{F}_P(t^{\mathbf{o}})\,\mathrm{d}t + \sum_i \int_{k_i}^{k_{i+1}} \hbar_i(t,k_i,k_{i+1})\,\mathrm{d}t$$
$$= \int_0^{x^{\mathbf{o}}} \mathcal{F}(t)\,\mathrm{d}t + \int_{x^{\mathbf{o}}}^x \mathcal{F}_P(t^{\mathbf{o}})\,\mathrm{d}t + \sum_i \int_{k_i}^{k_{i+1}} \int_{k_i}^t f_i^{\mathbf{o}}(s,k_i,k_{i+1})\,\mathrm{d}s\,\mathrm{d}t.$$

Analogous definitions work for other kinds of double integrals, such as $\int g(t)\mathcal{F}(t)\,\mathrm{d}t$.

## 3.2 Integration by parts

I will prove the formula of integration by parts. I have to suppose that $\mathfrak{A}$ is a family of functions inductively constructed, that $f,g$ are recursive over it, that $\int$ is defined, additive, monotone and linear on the **No**-vector space generated by $\mathfrak{A} \cup \{f,g\}$. I start with $f = \langle f^L \mid f^R \rangle$, $g = \langle g^L \mid g^R \rangle$, and define

$$\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t \quad \mathcal{G}(x) := \int_0^x g(t)\,\mathrm{d}t$$

The proof is done by induction over $f,g$ and over the extremes of integration: in particular, I will suppose that the integration by part formula is true for $(f^{\mathbf{o}},g)$ where $f^{\mathbf{o}}$ is any option of $f$, and similarly for $(f,g^{\mathbf{o}})$.

**Theorem 9.** *Let $f,g$: **No** $\to$ **No**, $a,b \in$ **No**. Then $(\mathcal{F}\mathcal{G})' = f\mathcal{G} + \mathcal{F}g$, i.e.*

$$\mathcal{F}(b)\mathcal{G}(b) - \mathcal{F}(a)\mathcal{G}(a) = \int_a^b \big(f(t)\mathcal{G}(t) + \mathcal{F}(t)g(t)\big)\,\mathrm{d}t \tag{3.3}$$

*Proof.* Suppose I have proved (3.3) for $a = 0$. Then,

$$\mathcal{F}(b)\mathcal{G}(b) - \mathcal{F}(a)\mathcal{G}(a) =$$
$$= \int_0^a f(t)\mathcal{G}(t)dt + \int_0^a \mathcal{F}(t)g(t)\,\mathrm{d}t - \int_0^b f(t)\mathcal{G}(t)dt + \int_0^b \mathcal{F}(t)g(t)\,\mathrm{d}t$$
$$= \int_a^b \big(f(t)\mathcal{G}(t) + \mathcal{F}(t)g(t)\big)\,\mathrm{d}t. \tag{3.4}$$

So, I have to prove that

$$\mathcal{F}(x)\mathcal{G}(x) = \int_0^x f(t)\mathcal{G}(t)dt + \int_0^x \mathcal{F}(t)g(t)\,\mathrm{d}t. \tag{3.5}$$

$$
\begin{aligned}
\big(\mathcal{F}(x)\mathcal{G}(x)\big)^{\mathbf{o}} = \mathcal{F}(x)^{\mathbf{o}}\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(x)^{\mathbf{o}} - \mathcal{F}(x)^{\mathbf{o}}\mathcal{G}(x)^{\mathbf{o}} =& \\
= \big[\mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} f^{\mathbf{o}}(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t\big]\,\mathcal{G}(x) +& \\
+ \big[\mathcal{G}(x^{\mathbf{o}\prime}) + \int_{x^{\mathbf{o}\prime}}^{x} g^{\mathbf{o}}(t,t^{\mathbf{o}},g(t^{\mathbf{o}}))\,\mathrm{d}t\big]\,\mathcal{F}(x) -& \\
- \big[\mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} f^{\mathbf{o}}(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t\big]\big[\mathcal{G}(x^{\mathbf{o}\prime}) + \int_{x^{\mathbf{o}\prime}}^{x} g^{\mathbf{o}}(t,t^{\mathbf{o}},g(t^{\mathbf{o}}))\,\mathrm{d}t\big]& \\
= \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}\prime}) - \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x^{\mathbf{o}\prime}) +& \\
+ \big(\mathcal{G}(x) - \mathcal{G}(x^{\mathbf{o}\prime})\big)\int_{x^{\mathbf{o}}}^{x} f^{\mathbf{o}}(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t + \big(\mathcal{F}(x) - \mathcal{F}(x^{\mathbf{o}})\big)\int_{x^{\mathbf{o}\prime}}^{x} g^{\mathbf{o}}(t,t^{\mathbf{o}},g(t^{\mathbf{o}}))\,\mathrm{d}t -& \\
- \int_{x^{\mathbf{o}}}^{x} f^{\mathbf{o}}(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t \int_{x^{\mathbf{o}\prime}}^{x} g^{\mathbf{o}}(t,t^{\mathbf{o}},g(t^{\mathbf{o}}))\,\mathrm{d}t,& \quad (3.6)
\end{aligned}
$$

where $x^{\mathbf{o}}, x^{\mathbf{o}\prime}$ are options of $x$. Fix $x^{\mathbf{o}}, x^{\mathbf{o}\prime}$ and partitions $P[x^{\mathbf{o}},x] = (h_0,\dots,h_m)$ and $Q[x^{\mathbf{o}\prime},x] = (k_0,\dots,k_{m'})$, fix $f_P^{\mathbf{o}}$ and $g_Q^{\mathbf{o}}$ tuples of options of $f$ and $g$. If $t \in [h_i,h_{i+1})$ and $t' \in [k_i,k_{i+1})$, define

$$
\begin{aligned}
h(t) &:= f_i^{\mathbf{o}}(t,h_i,h_{i+1},f(h_i),f(h_{i+1})) \qquad (3.7)\\
k(t') &:= g_i^{\mathbf{o}}(t',k_i,k_{i+1},g(k_i),g(k_{i+1})); \qquad (3.8)
\end{aligned}
$$

$h, k$ are defined on $[x^{\mathbf{o}},x]$ and $[x^{\mathbf{o}\prime},x]$ respectively. Let $\mathcal{H}(y) = \int_{x^{\mathbf{o}}}^{y} h(t)\,\mathrm{d}t$ and $\mathcal{K}(y) = \int_{x^{\mathbf{o}}}^{y} k(t)\,\mathrm{d}t$. (3.6) becomes

$$
\begin{aligned}
\mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}\prime}) - \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x^{\mathbf{o}\prime}) +& \\
+ \big(\mathcal{G}(x) - \mathcal{G}(x^{\mathbf{o}\prime})\big)\mathcal{H}(x) + \big(\mathcal{F}(x) - \mathcal{F}(x^{\mathbf{o}})\big)\mathcal{K}(x) - \mathcal{H}(x)\mathcal{K}(x)& \\
= \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}\prime}) - \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x^{\mathbf{o}\prime}) +& \\
+ \mathcal{G}(x)\mathcal{H}(x) - \mathcal{G}(x^{\mathbf{o}\prime})\mathcal{H}(x) + \mathcal{K}(x)\mathcal{F}(x) - \mathcal{K}(x)\mathcal{F}(x^{\mathbf{o}}) - \mathcal{K}(x)\mathcal{H}(x).& \quad (3.9)
\end{aligned}
$$

Now I use induction on $f, g$ and $x$: I can apply (3.3) to various products in (3.9), obtaining

$$
\begin{aligned}
\mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}\prime}) - \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x^{\mathbf{o}\prime}) -& \\
- \mathcal{G}(x^{\mathbf{o}\prime})\mathcal{H}(x) - \mathcal{K}(x)\mathcal{F}(x^{\mathbf{o}}) - \mathcal{K}(x)\mathcal{H}(x) +& \\
+ \int_{x^{\mathbf{o}}}^{x} (g\mathcal{H} + h\mathcal{G}) + \int_{x^{\mathbf{o}\prime}}^{x} (\mathcal{K}f + k\mathcal{F})& \quad (3.10)
\end{aligned}
$$

Suppose now that $x^{\mathbf{o}}$ and $x^{\mathbf{o}\prime}$ are both left options of $x$, $f^{\mathbf{o}}$ is a left option of $f$ and $g^{\mathbf{o}}$ is a left option of $g$ (there are 16 cases in total). I have to prove that (3.10) is strictly less than

$$
\int_0^x (f\mathcal{G} + \mathcal{F}g)(t)\,\mathrm{d}t.
$$

W.l.o.g. I can suppose that $x^{\mathbf{o}} = x^{\mathbf{o}\prime} = a$. Then, I apply again (3.5) to $\mathcal{K}(x)\mathcal{H}(x)$ and (3.10) becomes

$$\begin{aligned}\mathcal{F}(a)\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(a) - \mathcal{F}(a)\mathcal{G}(a) + \\ + \int_a^x \big(g(t)\mathcal{H}(t) + (\mathcal{G}(t) - \mathcal{G}(a))h(t) + \\ + f(t)\mathcal{K}(t) + (\mathcal{F}(t) - \mathcal{F}(a))k(t) - k(t)\mathcal{H}(t) - \mathcal{K}(t)h(t)\big)\,\mathrm{d}t \quad (3.11)\end{aligned}$$

However, $\mathcal{F}(t) - \mathcal{F}(a) - \mathcal{H}(t) > 0$ and $f(t) > h(t)$ for $t \in (a,x)$, and similarly for $g$, $k$, therefore (3.11) is strictly less than

$$\begin{aligned}&\mathcal{F}(a)\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(a) - \mathcal{F}(a)\mathcal{G}(a) + \\ &+ \int_a^x \big(g(t)\mathcal{H}(t) + (\mathcal{G}(t) - \mathcal{G}(a) - \mathcal{K}(t))f(t) + f(t)\mathcal{K}(t) + (\mathcal{F}(t) - \mathcal{F}(a) - \mathcal{H}(t))g(t)\big)\,\mathrm{d}t \\ &= \mathcal{F}(a)\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(a) - \mathcal{F}(a)\mathcal{G}(a) + \\ &+ \int_a^x g(t)\mathcal{F}(t) + \mathcal{G}(t)f(t)\,\mathrm{d}t + \mathcal{G}(a)(\mathcal{F}(a) - \mathcal{F}(x)) + \mathcal{F}(a)(\mathcal{G}(a) - \mathcal{G}(x)) \\ &= \int_a^x (g\mathcal{F} + \mathcal{G}f)\,\mathrm{d}t + \mathcal{F}(a)\mathcal{G}(a) = \int_0^x (g\mathcal{F} + \mathcal{G}f)\,\mathrm{d}t, \quad (3.12)\end{aligned}$$

where I have used once again the inductive hypothesis in the last line.

The same kind computations works in the other cases, as long as $x^{\mathbf{o}}$ and $x^{\mathbf{o}\prime}$ are on the same side of $x$ (i.e. both left options or both right options). It remains to treat the cases when they are on opposite sides. W.l.o.g., I can suppose $x^{\mathbf{o}} < x < x^{\mathbf{o}\prime}$.

It is better to treat in uniformly all cases. I re-start from (3.9), this time without assuming that $x^{\mathbf{o}} = x^{\mathbf{o}\prime}$. To increase readability, I write $a = x^{\mathbf{o}}$, $b = x^{\mathbf{o}\prime}$. Let us call $\Lambda := \mathcal{F}(a)\mathcal{G}(x) + \mathcal{F}(x)\mathcal{G}(b) - \mathcal{F}(a)\mathcal{G}(b) - \int_0^x(\mathcal{F}g + \mathcal{G}f)$.

*Claim* 3.1. $\Lambda = \int_a^x \big((\mathcal{G}(b) - \mathcal{G}(t))f(t) + (\mathcal{F}(a) - \mathcal{F}(t))g(t)\big)\,\mathrm{d}t$.

*Proof of claim.*

$$\begin{aligned}\Lambda &= \mathcal{F}(a)\mathcal{G}(a) + \int_a^x \mathcal{F}(a)g(t)\,\mathrm{d}t + \mathcal{F}(a)\mathcal{G}(b) + \\ &\quad + \int_a^x \mathcal{G}(b)f(t)\,\mathrm{d}t - \mathcal{F}(a)\mathcal{G}(b) - \int_0^x \mathcal{F}g + \mathcal{G}f \\ &= \mathcal{F}(a)\mathcal{G}(a) + \int_a^x \mathcal{F}(a)g(t) + \mathcal{G}(b)f(t)\,\mathrm{d}t - \int_0^x (\mathcal{F}g + \mathcal{G}f) \\ &= \int_0^a (\mathcal{F}g + \mathcal{G}f) + \int_a^x \mathcal{F}(a)g(t) + \mathcal{G}(b)f(t)\,\mathrm{d}t - \int_0^x (\mathcal{F}g + \mathcal{G}f) \\ &= \int_a^x \big((\mathcal{F}(a) - \mathcal{F}(t))g(t) + (\mathcal{G}(b) - \mathcal{G}(t))f(t)\big)\,\mathrm{d}t, \quad (3.13)\end{aligned}$$

where I have used the inductive hypothesis on to compute $\mathcal{F}(a)\mathcal{G}(a)$. □

I am interested in the sign of the expression (3.9) minus $\int_0^x(\mathcal{F}g+\mathcal{G}f)$, which is equal to

$$\begin{aligned}
\Lambda + &\mathcal{G}(x)\mathcal{H}(x) - \mathcal{G}(b)\mathcal{H}(x) + \mathcal{K}(x)\mathcal{F}(x) - \mathcal{K}(x)\mathcal{F}(a) - \mathcal{K}(x)\mathcal{H}(x) \\
&= \int_a^x((\mathcal{G}(b)-\mathcal{G})f + (\mathcal{F}(a)-\mathcal{F})g + \mathcal{G}h + g\mathcal{H} - \mathcal{G}(b)h) + \\
&\qquad + \int_b^x(\mathcal{K}f + k\mathcal{F} - \mathcal{F}(a)k) - \mathcal{K}(x)\mathcal{H}(x) \\
&= \int_a^x((\mathcal{G}(b)-\mathcal{G})f + (\mathcal{F}(a)-\mathcal{F}+\mathcal{H})g + (\mathcal{G}-\mathcal{G}(b))h) + \\
&\qquad + \int_b^x(\mathcal{K}f + k\mathcal{F} - \mathcal{F}(a)k) - \mathcal{K}(x)\mathcal{H}(x) = \\
= \int_a^x((\mathcal{G}(b)-\mathcal{G})(f-h) + (\mathcal{F}(a)-\mathcal{F}-\mathcal{H})g) &+ \int_b^x(\mathcal{K}f + k\mathcal{F} - \mathcal{F}(a)k) - \mathcal{K}(x)\mathcal{H}(x),
\end{aligned} \tag{3.14}$$

where I have used the inductive hypothesis to compute $\mathcal{H}(x)\mathcal{G}(x)$ and $\mathcal{K}(x)\mathcal{F}(x)$. Let $I$ the smallest interval containing $a, b$ and $x$. Extend $h, k$ to all $I$ by choosing some options of $f, g$. Suppose now again that $a = x^{\mathbf{o}}$ and $b = x^{\mathbf{o}'}$ are both left options of $x$ and that $h, k$ are left options of $f, g$ respectively. Then, $h < f$, $k < g$, $\mathcal{K}(t) + \mathcal{G}(b) < \mathcal{G}(t)$ and $\mathcal{H}(t) + \mathcal{F}(a) < \mathcal{F}(t)$ on $I$. Therefore, (3.14) is strictly less than

$$\begin{aligned}
&\int_a^x(-\mathcal{K}(f-h) + (\mathcal{F}(a)-\mathcal{F}-\mathcal{H})g) + \int_b^x(\mathcal{K}f + k\mathcal{F} - \mathcal{F}(a)k) - \mathcal{K}(x)\mathcal{H}(x) \\
&\quad = \int_b^a \mathcal{K}f + \int_a^x(\mathcal{K}h + (\mathcal{F}(a)-\mathcal{F}+\mathcal{H})g) + \int_b^x(\mathcal{F}-\mathcal{F}(a))k - \mathcal{K}(x)\mathcal{H}(x) \\
&\qquad = \int_b^a(\mathcal{K}f + \mathcal{F}k) - \int_b^x \mathcal{F}(a)k + \\
&\qquad \int_a^x(\mathcal{K}h + (\mathcal{F}(a)-\mathcal{F}+\mathcal{H})g + \mathcal{F}k) - \mathcal{K}(x)\mathcal{H}(x)
\end{aligned} \tag{3.15}$$

which is is strictly less than

$$\begin{aligned}
&\mathcal{F}(a)\mathcal{K}(a) - \mathcal{F}(a)\mathcal{K}(x) - \mathcal{K}(x)\mathcal{H}(x) + \int_a^x(\mathcal{K}h + (\mathcal{F}(a)-\mathcal{F}+\mathcal{H})k + \mathcal{F}k) \\
&\quad = \mathcal{F}(a)\mathcal{K}(a) - \mathcal{F}(a)\mathcal{K}(x) - \mathcal{K}(x)\mathcal{H}(x) + \int_a^x(\mathcal{K}h + \mathcal{H}k + \mathcal{F}(a)k) = \\
&\qquad - \mathcal{K}(x)\mathcal{H}(x) + \int_a^x(\mathcal{K}h + \mathcal{H}k) = 0
\end{aligned} \tag{3.16}$$

where I have used again induction to compute $\mathcal{H}(x)\mathcal{K}(x)$. The other cases (i.e. $a > x$ or $b > x$ or $h > f$ or $k > g$) are similar.

On the other hand, w.l.o.g.

$$\left(\int_0^x f(t)\mathcal{G}(t)dt + \int_0^x \mathcal{F}(t)g(t)\,\mathrm{d}t\right)^{\mathbf{o}} =$$

$$= \int_0^{x^{\mathbf{o}}} f\mathcal{G} + \int_{x^{\mathbf{o}}}^x (f\mathcal{G})^{\mathbf{o}} + \int_0^x \mathcal{F}g$$

$$= \int_0^{x^{\mathbf{o}}} (f\mathcal{G} + \mathcal{F}g) + \int_{x^{\mathbf{o}}}^x (f^{\mathbf{o}}\mathcal{G} + f\mathcal{G}^{\mathbf{o}} - f^{\mathbf{o}}\mathcal{G}^{\mathbf{o}}) + \int_0^x \mathcal{F}g \quad (3.17)$$

By induction on $x$,

$$\int_0^{x^{\mathbf{o}}} (f\mathcal{G} + \mathcal{F}g) = \mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x^{\mathbf{o}}).$$

As before, fix a partition $P[x^{\mathbf{o}},x]$ and tuples $f_P^{\mathbf{o}}$, $g_P^{\mathbf{o}}$ of options of $f,g$. Let $h(t) = f_i^{\mathbf{o}}(t,k_i,k_{i+1},f(k_i),f(k_{i+1}))$ for $t \in [k_i,k_{i+1})$, and similarly for $k$. Let $\mathcal{K}(y) = \int_{x^{\mathbf{o}}}^y k(t)\,\mathrm{d}t$ and similarly for $\mathcal{H}(y)$. Then, (3.17) becomes

$$\mathcal{F}(x^{\mathbf{o}})\mathcal{G}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^x (\mathcal{F}g + h\mathcal{G} + (f-h)(\mathcal{G}(x^{\mathbf{o}}) + \mathcal{K}))$$

$$= \mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^x (\mathcal{F}g + f\mathcal{K} + h(\mathcal{G} - \mathcal{K} - \mathcal{G}(x^{\mathbf{o}}))) \quad (3.18)$$

Suppose now for simplicity that $x^{\mathbf{o}} < x$ is a left option of $x$, and that $h,k$ are left options of $f,g$ respectively. I have to prove that (3.18) is strictly less than $\mathcal{F}(x)\mathcal{G}(x)$.

By inductive hypothesis,

$$\int_{x^{\mathbf{o}}}^x h\mathcal{G} = \mathcal{H}(x)\mathcal{G}(x) - \int_{x^{\mathbf{o}}}^x \mathcal{H}g,$$

and similarly for $f\mathcal{K}$ and $h\mathcal{K}$. So, (3.18) is equal to

$$\mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}}) + \mathcal{F}(x)\mathcal{K}(x) + \mathcal{H}(x)\mathcal{G}(x) - \mathcal{H}(x)\mathcal{K}(x) + \int_{x^{\mathbf{o}}}^x (\mathcal{F}g - \mathcal{F}k - \mathcal{H}g + \mathcal{H}k - h\mathcal{G}(x^{\mathbf{o}}))$$

$$= \mathcal{F}(x)\mathcal{G}(x^{\mathbf{o}}) + \mathcal{F}(x)\mathcal{K}(x) + \mathcal{H}(x)\mathcal{G}(x) - \mathcal{H}(x)\mathcal{G}(x^{\mathbf{o}}) - \mathcal{H}(x)\mathcal{K}(x) + \int_{x^{\mathbf{o}}}^x (\mathcal{F} - \mathcal{H})(g - k)$$

$$= \mathcal{F}(x)(\mathcal{G}(x^{\mathbf{o}}) + \mathcal{K}(x)) + \mathcal{H}(x)(\mathcal{G}(x) - \mathcal{G}(x^{\mathbf{o}}) - \mathcal{K}(x)) + \int_{x^{\mathbf{o}}}^x (\mathcal{F} - \mathcal{H})(g - k)$$

$$= \mathcal{F}(x)\mathcal{G}(x) - (\mathcal{F}(x) - \mathcal{F}(x^{\mathbf{o}}) - \mathcal{H}(x))(\mathcal{G}(x) - \mathcal{G}(x^{\mathbf{o}}) - \mathcal{K}(x)) + \int_{x^{\mathbf{o}}}^x (\mathcal{F} - \mathcal{H} - \mathcal{F}(x^{\mathbf{o}}))(g - k) \quad (3.19)$$

Call

$$m(y) := f(y) - h(y) \qquad \mathcal{M}(y) := \int_{x^{\mathbf{o}}}^x m(t)\,\mathrm{d}t = \mathcal{F}(x) - \mathcal{F}(x^{\mathbf{o}}) - \mathcal{H}(x)$$

$$n(y) := g(y) - k(y) \qquad \mathcal{N}(y) := \int_{x^{\mathbf{o}}}^x n(t)\,\mathrm{d}t = \mathcal{G}(x) - \mathcal{G}(x^{\mathbf{o}}) - \mathcal{K}(x).$$

I have to prove that (3.19) is strictly less than $\mathcal{F}(x)\mathcal{G}(x)$. This is equivalent to

$$\mathcal{M}(x)\mathcal{N}(x) - \int_{x^o}^{x} \mathcal{M}(t)n(t)\,dt > 0.$$

i.e.

$$\int_{x^o}^{x} (\mathcal{M}(x) - \mathcal{M}(t))n(t)\,dt > 0 \tag{3.20}$$

But $m > 0$ and $n > 0$ in the interval $(x^o, x)$; therefore, $\mathcal{M}(x) > \mathcal{M}(t)$ and the conclusion follows. □

### 3.2.1 Error checking

There are some methods to easily check whether there is any mistake in the algebraic manipulations like the ones in the previous proof; they do not guarantee the correctness of the computations, however they can detect many errors.

The first one is dimensional argument: if $t$ is of dimension $[t]$, $f$ and $h$ of dimension $[f]$, $g$ and $k$ of dimension $[g]$, then $\mathcal{F}\mathcal{G}$ has dimension $[fgt^2]$ and can only be added to or compared with quantities of the same dimension; moreover the dimension must be preserved by algebraic manipulations. For instance, if I start with $\mathcal{F}\mathcal{G}$ I cannot end with an expression containing $h\mathcal{G}$ among its summands.

The second method is more specific to the surreal numbers. Take an option of $xy$: $x^o y + xy^o - x^o y^o$. If I substitute $x$ instead of $x^o$ and anything instead of $y$, I obtain the product $xy$ itself. Same thing can be said for the sum $x+y$. Therefore, if I start from a composition of sums and products (i.e. a polynomial) $p(x,y)$, and consider an option $p(x,y)^o = q(x,y,x^o,y^o)$, I must have that $q(x,y,x,z) = p(x,y) = q(x,y,z,y)$. And this must remain true after any algebraic manipulation of $q$.

For the integral, something even stronger can be said: if I consider $\mathcal{F}(x) = \int_0^x f(t)\,dt$ and I take an option $\mathcal{F}(x)^o = g(x, x^o, f^o)$ then not only $g(x,x,f^o) = \mathcal{F}(x)$, but also $g(x,y,f) = \mathcal{F}(x)$.

But what happens after applying some theorem, like theorem 9, to an option? I can still use the previous trick, bearing in mind that I will obtain an identity only if theorem 9 is true. For instance, consider the expression (3.10), which has been obtained from an option of $\mathcal{F}(x)\mathcal{G}(x)$. If, say, I substitute $h = f$ (and therefore $\mathcal{H} = \mathcal{F} - \mathcal{F}(a)$), I obtain the expression

$$\begin{aligned}
&\mathcal{F}(a)\mathcal{G}(a) + \mathcal{F}(x)\mathcal{G}(a) - \mathcal{F}(a)\mathcal{G}(a) + \\
&+ \int_a^x \left(g(\mathcal{F} - \mathcal{F}(a)) + (\mathcal{G} - \mathcal{G}(a)f) + f\mathcal{K} + (\mathcal{F} - \mathcal{F}(a))k - k(\mathcal{F} - \mathcal{F}(a)) - \mathcal{K}f\right) \\
&= \mathcal{F}(a)\mathcal{G}(a) + \mathcal{F}(x)\mathcal{G}(a) - \mathcal{F}(a)\mathcal{G}(a) + \int_a^x \left(g(\mathcal{F} - \mathcal{F}(a)) + (\mathcal{G} - \mathcal{G}(a)f)\right),
\end{aligned} \tag{3.21}$$

which, integrating by parts, is equal to $\mathcal{F}(x)\mathcal{G}(x)$: this independently from the value of $a$ and of $k$.

What if I have applied instead some inequality, for instance $h < f$ ? If the expression is an option $z = \lambda^{\mathbf{o}}$ and at the end the new expression $z'$ satisfies $z \leq z' \leq \lambda$, then I can apply the same trick to check the validity of the manipulation. On the other hand, if instead I obtain $z' < z < \lambda$, often this trick fails: but usually I do not need to find a $z'$ which is a worse approximation of $\lambda$ than $z$ in the first place.

## 3.3 Concluding remarks

It is now natural to ask about other formulae known for the Riemann integral over the reals. For instance, one may wonder whether about the validity on **No** of the formula for composite functions corresponding to

$$(\mathcal{G} \circ \mathcal{F})' = (g \circ \mathcal{F})f.$$

While it is true for $f$ and $g$ both polynomials, we will see that in the general case it fails even for the simplest kind of functions; for example,

$$f(x) = x + c, \quad c \in \mathbf{No}.$$

# Chapter 4

# Polynomials and analytic functions

In this chapter I will give recursive definitions for polynomials in $\mathbf{No}[x]$.
I will also give a recursive definition for germs of analytic functions in $\mathbb{R}[[x]]$, for infinitesimal values of $x \in \mathbf{No}$.

Moreover, I will compute their integral, and prove that it is equal to the "formal" integral.

Finally, I will give generalisations of some closure theorems from polynomials to recursively definable functions.

## 4.1 Polynomials

Let $p(X) \in \mathbf{No}[X]$ be a polynomial, $x \in \mathbf{No}$. I want to give an explicit inductive formula for $p(x)$.

If we write

$$p(X) = \sum_{i=0}^{n} a_i X^i$$

then, by definition of sum,

$$p(x) = \langle \sum_{\substack{0 \le i \le n \\ i \ne m}} a_i x^i + (a_m x^m)^L \mid \sum_{\substack{0 \le i \le n \\ i \ne m}} a_i x^i + (a_m x^m)^R \rangle_{0 \le m \le n}$$

or, more concisely,

$$p(x)^{\mathbf{o}} = \big(\sum_{\substack{0 \le i \le n \\ i \ne m}} a_i x^i\big) + (a_m x^m)^{\mathbf{o}} \tag{4.1}$$

It remains to treat the case where $p(X)$ is a monomial, i.e. $p(X) = aX^n$. By definition of product,

$$(xy)^{\mathbf{o}} = x^{\mathbf{o}} y + x y^{\mathbf{o}} - x^{\mathbf{o}} y^{\mathbf{o}},$$

i.e.

$$xy - (xy)^{\mathbf{o}} = (x - x^{\mathbf{o}})(y - y^{\mathbf{o}}).$$

It follows that for any $x_1, \dots, x_n \in \mathbf{No}$

$$x_1 \cdots x_n - (x_1 \cdots x_n)^{\mathbf{o}} = (x_1 - x_1^{\mathbf{o}}) \cdots (x_n - x_n^{\mathbf{o}})$$

In particular,

$$(ax^n)^{\mathbf{o}} = ax^n - (a - a^{\mathbf{o}})(x - x_1^{\mathbf{o}}) \cdots (x - x_n^{\mathbf{o}})$$

where $x_1^{\mathbf{o}}, \dots, x_n^{\mathbf{o}}$ are options of $x$. By cofinality, I can choose among $x_1^{\mathbf{o}}, \dots, x_n^{\mathbf{o}}$ the 'best' left option $x^L$ (i.e. the greatest) and the 'best' right option (i.e. the smallest), and say that

$$(ax^n)^{\mathbf{o}} = ax^n - (a - a^{\mathbf{o}})(x - x^L)^{\alpha}(x - x^R)^{\beta} \tag{4.2}$$

where $0 \leq \alpha \leq n$ and $\alpha + \beta = n$. Clearly, (4.2) is a left option if and only if $\beta$ is even and $a^{\mathbf{o}} < a$, or $\beta$ is odd and $a^{\mathbf{o}} > a$.

Putting 4.1 and (4.2) together, we obtain:

$$p(x)^{\mathbf{o}} = p(x) - (a_m - a_m^{\mathbf{o}})(x - x^L)^{\alpha}(x - x^R)^{\beta} \tag{4.3}$$

where $0 \leq m \leq n$ and $\alpha + \beta = m$. Moreover I can always take $\alpha = 0, 1, n-1$ or $n$.

## 4.2 Analytic functions

Let $(a_i)_{i \in \mathbb{N}}$ be a sequence of real numbers. Let $x \in \mathbf{No}$ be an infinitesimal (positive) surreal number. Then it is possible to give a meaning to the expression

$$f(x) = \sum_i a_i x^i,$$

using the fact that $\mathbf{No}$ can be identified canonically with $\mathbb{R}((\mathbf{No}))$, the generalised power series field with real coefficients and surreal exponents. It is not obvious that $f$ can be defined recursively, and that the corresponding integral coincides with the integration term-by-term.

I will give the recursive formula of $f(x)$ (for $x > 0$). The proof will be postponed.

If $f$ is a polynomial, we know already how to define it. Otherwise, given $x \in \mathbf{No}$ infinitesimal, given $x^{\mathbf{o}}$ an option of $x$ which is infinitesimal too, we can consider the Taylor expansion of $f$ at $x^{\mathbf{o}}$, truncated at the $n^{\text{th}}$ term for any $n \in \mathbb{N}$,

$$p_n(x, x^{\mathbf{o}}) := \sum_{i \leq n} \frac{f^{(i)}(x^{\mathbf{o}})}{i!}(x - x^{\mathbf{o}})^i \tag{4.4}$$

and say that $f(x)$ is more or less $p_n(x, x^{\mathbf{o}})$. This means that if $\varepsilon$ is any positive real then

$$p_n(x, x^{\mathbf{o}}) - \varepsilon(x - x^{\mathbf{o}})^n < f(x) < p_n(x, x^{\mathbf{o}}) + \varepsilon(x - x^{\mathbf{o}})^n \tag{4.5}$$

if $x^{\mathbf{o}}$ is a left option, and similarly for right options. This, because

$$\left(f(t) - p_n(t,t_0)\right)/(t-t_0)^n$$

is infinitesimal if $t$ and $t_0$ are both infinitesimal (because all the coefficients of $f$ are *real* numbers). It suffices to define $f(x)$ using the formula (4.5), taking into consideration that $f^{(n)}(0) = a_n/n!$, and letting $\varepsilon$ vary among all possible positive real (or rational) numbers. The only difficulty is that I need to define $f, f', f'', \ldots$ all at the same time (because to compute $p_n$ I need them), but this is not a problem.

Note that to define $f(x)$ I use only the values of $f^{(n)}(x^{\mathbf{o}})$ where $x^{\mathbf{o}}$ is an infinitesimal option of $x$: for instance, the only infinitesimal option of $c = \frac{1}{\omega}$ is 0, so I can compute $f(c)$ directly from the Taylor expansion in 0:

$$f(c) = \langle \sum_{i \leq n} a_i c^i - \varepsilon c^n \mid \sum_{i \leq n} a_i c^i + \varepsilon c^n \rangle$$

Moreover, the definition of $f(x)$ is uniform.

Concluding, I have the formula

$$\left(f(x)\right)^{\mathbf{o}} = p_n(x, x^{\mathbf{o}}) \pm \varepsilon (x - x^{\mathbf{o}})^n. \tag{4.6}$$

### 4.2.1 Justification of the definition for analytic functions

Let

$$f(X) = \sum_{i \in \mathbb{N}} a_i X^i$$

be a power series with real coefficients, $x > 0 \in \mathbf{No}$ be an infinitesimal surreal number, $x = \sum_{j<\alpha} r_j \omega^{c_j}$ be its normal form. I will prove that (4.6) defines $f(x)$, using induction on $x$.

If $f$ is a polynomial, the conclusion is a consequence of formula (4.3).

Otherwise, let $z$ be the surreal number defined by (4.6). (4.5) implies that $z \preceq f(x)$. Conversely,

$$f(x) = \sum_{\substack{i \in \mathbb{N} \\ j_1, \ldots, j_i < \alpha}} s_{i, j_i, \ldots, j_i} \omega^{c_{j_1} + \cdots + c_{j_i}} = \sum_{k < \delta} t_k \omega^{d_k},$$

for some $\delta \in \mathbf{On}$, $s_{i,\bar{\jmath}}, t_k \in \mathbb{R}$, $d_k \in \mathbf{No}$.

If $\delta$ is a limit ordinal, then

$$f(x)^{\mathbf{o}} = \sum_{k \leq \beta} t_k \omega^{d_k} \pm \varepsilon \omega^{d_\beta}$$

for some $\beta < \delta$ and $\varepsilon > 0 \in \mathbf{No}$. Therefore, $|f(x) - f(x)^{\mathbf{o}}| \simeq \varepsilon \omega^{d_\beta}$. Moreover,

$$d_\beta = c_{j_1} + \cdots + c_{j_i} \geq i c_\lambda$$

for some $i \in \mathbb{N}$, $j_1, \dots, j_i < \alpha$ and $c_\lambda := \min\left\{c_{j_1}, \dots, c_{j_i}\right\}$. Therefore,

$$|f(x) - f(x)^{\mathbf{o}}| \geq \varepsilon\omega^{ic_\lambda}$$

for some $i \in \mathbb{N}$, $0 < \varepsilon \in \mathbb{R}$, $\lambda < \alpha$.

If $\delta = \gamma + 1$,

$$|f(x) - f(x)^{\mathbf{o}}| = |t_\gamma - t_\gamma^{\mathbf{o}}||\omega^{d_\gamma} - \left(\omega^{d_\gamma}\right)^{\mathbf{o}}| \geq \varepsilon\omega^{d_\gamma} \geq \varepsilon\omega^{ic_\lambda}$$

for some $\varepsilon > 0 \in \mathbb{R}$, $i \in \mathbb{N}$, $\lambda < \alpha$.

Using (4.6), I obtain that for every $n \in \mathbb{N}$, $\varepsilon > 0 \in \mathbb{R}$ there exist $z^R$ and $z^L$ options of $z$ such that

$$|z^R - z^L| \leq \varepsilon|x - x^{\mathbf{o}}|^n.$$

If $\alpha$ is a limit ordinal, I can suppose $|x - x^{\mathbf{o}}| < \varepsilon\omega^{c_\lambda}$. If $\alpha = \gamma + 1$, I can suppose $|x - x^{\mathbf{o}}| \leq s\omega^{c_\gamma}$ for some $s > 0 \in \mathbb{R}$.

In both cases,

$$\Delta(f(x)) \geq \inf\left\{\varepsilon\omega^{ic_\lambda} : i \in \mathbb{N}, \varepsilon > 0 \in \mathbb{Q}, \lambda < \alpha\right\}.$$

So, for our representation $\langle z^L \mid z^R \rangle$ of $z$,

$$\inf\left\{|z^R - z^L| \leq \Delta(f(x))\right\}.$$

Moreover, by equation (4.5), $z^L < f(x) < z^R$, therefore, by remark 1.86, $z = f(x)$.

## 4.3 Integral of polynomials

The integral of a polynomial is what we expect:

**Theorem 10.** *Let* $p(x) = \sum_{i=0}^{n} a_i x^i$ *be a polynomial in one variable with coefficients in* **No**. *Then,*

$$\int_0^x p(t)\,\mathrm{d}t = \sum_{i=0}^{n} \frac{a_i}{i+1} x^{i+1}.$$

*Proof.* I call the expression on the right the formal integral of $p$. By linearity of $\int$, it is enough to prove the lemma for $p(x) = x^n$. I will use induction on $n$, the degree of the polynomial, and on $x$.

Let $\mathcal{F}(x) = \int_0^x t^n\,\mathrm{d}t$. I want to prove $\mathcal{F}(x) = x^{n+1}/(n+1)$. As usual, for some $P[x^{\mathbf{o}}, x] = (k_0, \dots, k_n)$,

$$\begin{aligned}\mathcal{F}(x)^{\mathbf{o}} &= \mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} \left(t^n\right)^{\mathbf{o}}\mathrm{d}t \\ &= \frac{(x^{\mathbf{o}})^{n+1}}{n+1} + \sum_i \int_{k_i}^{k_{i+1}} (t^n - (t - k_i)^\alpha (t - k_{i+i})^\beta)\,\mathrm{d}t,\end{aligned}$$

where I have used the formula (4.2) and the induction on $x$.

We know that **No** is an ordered domain, so $x^n$ satisfies all universal formulae true in $\mathbb{R}$ (of course, we know more than that: but this is enough for our purpose); in particular, $y := x^{n+1}/(n+1)$ is in the correct cut in order to be the integral of $x^n$. I.e. if $\mathcal{F}(x^{\circ}) = (x^{\circ})^{n+1}/(n+1)$, for every $x^{\circ}$ option of $x$, then $\mathcal{F}(x) \preceq y$.

It remains to show that $y \preceq \mathcal{F}(x)$. To prove it, I can use the cofinality theorem. Thus, it is enough to prove that $\mathcal{F}(x)^{\circ} - y$ is small, i.e. for every $y^{\circ} \prec y$ there exists $\mathcal{F}(x)^{\circ}$ such that $|y - y^{\circ}| \geq |y - \mathcal{F}(x)^{\circ}|$ and $\mathcal{F}(x)^{\circ}$ is on the same side of $y$ as $y^{\circ}$.

The integrand is a polynomial in $t$ of degree less than $n$, so I can apply the inductive hypothesis, and say that its integral is equal to the formal integral.

I use the following formula (which can be proved in many different ways: for instance, integrating by parts):

$$\int_a^b (t-a)^{\alpha}(t-b)^{\beta}\,\mathrm{d}t = \frac{(-1)^{\beta}(b-a)^{n+1}}{(n+1)\binom{n}{\alpha}} \tag{4.7}$$

where $n := \alpha + \beta$, and $\int$ is the formal integral of polynomials.

In the following, I will suppose $x^{\circ} < x$ (the other case is similar). Apply (4.7) to obtain:

$$\mathcal{F}(x)^{\circ} = y + \sum_i \frac{(-1)^{\beta+1}}{(n+1)\binom{n}{\alpha}} \delta_i^{n+1}, \tag{4.8}$$

where $\delta_i := k_{i+1} - k_i$.

Call $\Delta := |x - x^{\circ}|$. Choose the $m$-partition such that $\delta_i = \delta = \Delta/m$ for every $i$. Then

$$|\mathcal{F}(x)^{\circ} - \mathcal{F}(x)| = \frac{\Delta^{n+1}}{m^n(n+1)\binom{n}{\alpha}} = q\Delta^{n+1}$$

where $q > 0$ is a rational number that can be chosen as small as we want, simply taking a smaller $\delta$, i.e. refining the partition.

On the other hand, by (4.2), putting $a = 1/(n+1)$

$$|y - y^{\circ}| \geq |a - a^{\circ}||x - x^{\circ}|^n \tag{4.9}$$

where $x^{\circ}$ (where $a^{\circ}$) is a canonical option of $x$ (of $a$)[(1)]. But $\varepsilon = |a - a^{\circ}|$ is positive real (because $a$ is real), so we can find a partition of $(x^{\circ}, x)$ such that $0 < q < a$, and the theorem follows.

I should also prove that $y^{\circ}$ can be found to lie on the same side of $y$ as $\mathcal{F}(x)^{\circ}$, but this follows from the fact that we need only to change the parity of $\beta$ to switch side of $y$, as is obvious from (4.8). □

[(1)]This means that for every $y^{\circ} \prec y$ we can find $a^{\circ} \prec a$ and $x^{\circ} \prec x$ such that (4.9) holds.

### 4.3.1 Example

As an example, let us compute directly

$$c := \int_0^\omega x + 1 \,\mathrm{d}x.$$

I will assume that I have already computed $\int_a^b x \,\mathrm{d}x$ for arbitrary $a, b$, and $\int_0^n x + 1 \,\mathrm{d}x$ for $n \in \mathbb{N}$.

I have to prove that $c = \frac{\omega^2}{2} + \omega$. **No** is a real closed field, therefore $c \preceq \frac{\omega^2}{2} + \omega$. Conversely, a right option of $c$ is

$$\begin{aligned} c^R &:= \int_0^n x + 1 \,\mathrm{d}x + \int_n^\omega x^R + 1 \,\mathrm{d}x \\ &= \frac{n^2}{2} + n + \sum_{0 \le i < m} \int_{k_i}^{k_{i+1}} k_{i+1} + 1 \,\mathrm{d}x \\ &= \frac{n^2}{2} + n + \sum_{0 \le i < m} (k_{i+1} - k_i)(k_{i+1} + 1). \end{aligned}$$

with $n \in \mathbb{N}$ and $(k_0, \dots, k_m)$ a partition of $(n, \omega)$. Define $\Delta := \omega - n$, and $\delta := \Delta/m$. Choose $k_{i+1} - k_i = \delta$, i.e. $k_i = n + i\delta$. Then,

$$\begin{aligned} c^R &= \frac{n^2}{2} + n + \delta \sum_{0 \le i < m} (n + \delta(i+1) + 1) \\ &= \frac{n^2}{2} + n + m\delta(n + \delta + 1) + \delta^2\left(\frac{m^2 - m}{2}\right) \\ &= \delta^2\left(\frac{m^2 + m}{2}\right) + \delta(mn + 1) + \frac{n^2}{2} + n \\ &= \frac{\Delta^2}{2}\left(1 + \frac{1}{m}\right) + O(\Delta) \\ &= \frac{\omega^2}{2}\left(1 + \frac{1}{m}\right) + O(\omega). \end{aligned}$$

where $y = O(z)$ means that $v(y) \le v(z)$. Therefore, by cofinality, we can choose $c^R = \frac{\omega^2}{2}(1 + \frac{1}{m})$.

A left option of $c$ is

$$\begin{aligned} c^L &:= \frac{n^2}{2} + n + \sum_{0 \le i < m} \int_n^\omega (x+1)^L \,\mathrm{d}x \\ &= \frac{n^2}{2} + n + \sum_{i \in I} \int_{k_i}^{k_{i+1}} k_i + 1 \,\mathrm{d}x + \sum_{i \in J} \int_{k_i}^{k_{i+1}} x \,\mathrm{d}x \\ &= \frac{n^2}{2} + n + \sum_{i \in I} (k_i + 1)(k_{i+1} - k_i) + \sum_{i \in J} \frac{k_{i+1}^2 - k_i^2}{2}. \end{aligned}$$

where $I \sqcup J = \{0, 1, \dots, m\}$. Choose $k_i$ as above. Then,

$$c^L = \cdots = \frac{n^2}{2} + n + \frac{\Delta^2}{2} + n\Delta + |I|(1 - \frac{\delta}{2}).$$

The best left options are obtained by setting $|I| = 0$ (this is what we expected, because for $i \in I$ we chose $x^L + 1$ as a left option of $x + 1$, while for $i \in J$ we chose $x$, and the latter should be a better approximation of $x + 1$). Therefore, $c^L = \frac{\omega^2}{2} + n^2$.

In conclusion,

$$\langle \frac{\omega^2}{2} + n^2 \mid \frac{\omega^2}{2}(1 + \frac{1}{m}) \rangle \preceq c$$

and one can check that the expression on the left is equal to $\frac{\omega^2}{2} + \omega$.

## 4.4 Integral of analytic functions

Let

$$f(X) = \sum_{i \in \mathbb{N}} a_i X^i,$$

$a_i \in \mathbb{R}$, be an analytic function (defined for $X$ infinitesimal). The formal integral of $f$ is

$$\mathcal{G}(X) := \sum_{i \in \mathbb{N}} \frac{a_i X^{i+1}}{i+1}$$

**Theorem 11.** *For $f, \mathcal{G}$ as before, the integral $f$ is equal to $\mathcal{G}$, i.e. for $x$ infinitesimal,*

$$\int_0^x f(t)\,\mathrm{d}t = \mathcal{G}(x).$$

*Proof.* The proof is by induction on $x$.

If $f$ is a polynomial, the conclusion follows from theorem 10.

Otherwise, call $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$.

$$\mathcal{F}(x)^{\mathbf{o}} = \mathcal{F}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^x f^{\mathbf{o}}(t, t^{\mathbf{o}})\,\mathrm{d}t.$$

By (4.6), $f^{\mathbf{o}} = p_n(t, t^{\mathbf{o}}) \pm \varepsilon(t - t^{\mathbf{o}})^n$, where $p_n$ is the Taylor series expansion of $f$ at $x^{\mathbf{o}}$. By definition, $\mathcal{G}^{(i+1)} = f^{(i)}$, therefore

$$\mathcal{G}(x)^{\mathbf{o}} = q_n(x, x^{\mathbf{o}}) \pm \varepsilon(x - x^{\mathbf{o}})^n,$$

where

$$q_n(x, x^{\mathbf{o}}) := \sum_{0 \le i \le n} \frac{\mathcal{G}^{(i)}(x^{\mathbf{o}})(x - x^{\mathbf{o}})^i}{i!} = \mathcal{G}(x^{\mathbf{o}}) + \sum_{0 \le i \le n-1} \frac{f^{(i)}(x^{\mathbf{o}})(x - x^{\mathbf{o}})^{i+1}}{(i+1)!}$$

is the Taylor series expansion of $\mathcal{G}$ at $x^{\mathbf{o}}$.

By inductive hypothesis,

$$\mathcal{F}(x)^{\mathbf{o}} = \mathcal{G}(x^{\mathbf{o}}) + \int_{x^{\mathbf{o}}}^{x} p_{n-1}(t,t^{\mathbf{o}}) \pm \varepsilon (t-t^{\mathbf{o}})^{n-1}\, \mathrm{d}t$$

(I have supposed, for simplicity, that the partition of $(x^{\mathbf{o}},x)$ has length 1). But the integrand is a polynomial in $t$, therefore I can apply theorem 10, and obtain that the previous is equal to

$$\mathcal{G}(x^{\mathbf{o}}) + \sum_{0\leq i\leq n-1} \frac{f^{(i)}(x^{\mathbf{o}})(x-x^{\mathbf{o}})^{i+1}}{(i+1)!} \pm \varepsilon (x-x^{\mathbf{o}})^{n} = \left(\mathcal{G}(x)\right)^{\mathbf{o}},$$

so $\left(\mathcal{F}(x)\right)^{\mathbf{o}} = \left(\mathcal{G}(x)\right)^{\mathbf{o}}$. In particular, $\mathcal{G}(x) \preceq \mathcal{F}(x)$.

If the partition of $(x^{\mathbf{o}},x)$ has length greater than 1, I can apply the elementary equivalence of **No** with $\mathbb{R}$ in the language $\mathscr{L}_{an}$ to obtain the every option of $\mathcal{F}(x)$ is also an option of $\mathcal{G}(x)$, obtaining $\mathcal{F}(x) \preceq \mathcal{G}(x)$. □

## 4.5 Real closure

I will give a proof of the fact that **No** is a real closed field, starting from the knowledge that it is an ordered domain. First, I recall the definition of real closed field.

**Definition 4.1.** An ordered field $\mathbb{K}$ is real closed iff

1. Every positive element has a square root.

2. Every polynomial of odd degree has a root.

**Theorem 12.** *Let $\mathbb{K}$ be an ordered domain. The following are equivalent:*

- *$\mathbb{K}$ is a real closed field.*

- *$\mathbb{K}$ is elementarily equivalent to $\mathbb{R}$ in the language of ordered rings $(0,1,+,\cdot,<)$.*

- *$\mathbb{K}$ is a field and every polynomial $p(x) \in \mathbb{K}[x]$ satisfies the intermediate value property:*

$$\forall a < b \in \mathbb{K}\ p(a) < 0 < p(b) \rightarrow \exists c \in \mathbb{K}\ a < c < b \ \&\ p(c) = 0.$$

- *$\mathbb{K}$ is maximal, i.e. every ordered domain containing $\mathbb{K}$ and algebraic over it coincides with $\mathbb{K}$ itself.*

- *$\mathbb{K}[i]$ is an algebraically closed field, where $i = \sqrt{-1}$.*

The real closure of **No** is a proper class definable by a certain formula. The next lemma shows that it coincides with **No** itself.

**Lemma 4.2.** *Let* $\mathbb{K}$ *be a real closed field containing* **No**. *Let* $p(x) \in \mathbf{No}[x]$ *be a polynomial with a root* $\zeta \in \mathbb{K}$. *Then* $\zeta \in \mathbf{No}$.

*Proof.* Order the polynomials in $\mathbf{No}[x]$ using the lexicographic order induced by $\preceq$, with monomials of higher degree more important than monomials of lower degree. This gives a well-founded partial order on $\mathbf{No}[x]$. I will prove the lemma using induction on $p$.

First, I can suppose there are $l < r \in \mathbf{No} \cup \{\pm\infty\}$ such that $\zeta$ is the only root of $p(x)$ in $\mathbb{K}$ in the interval $(l, r)$. In fact, between two roots of $p$ there is always a root (in $\mathbb{K}$) of its derivative $p'$, and $p'$ is simpler than $p$, therefore its roots are in **No**.

I will define $c = \langle c^L \mid c^R \rangle$ such that $p(c) = 0$ and $c \in (l, r)$. I will give the options of $c$. First of all, I want $l < c < r$, so $l$ is a left option, $r$ a right one.

In order to have $p(c) = 0$ it is necessary and sufficient to have

$$p(c)^L < 0 < p(c)^R$$

for every left and right option of $p(c)$. Let

$$p(x) = \sum_{i=0}^{n} a_i x^i. \tag{4.10}$$

By formula (4.3),

$$p(c)^{\mathbf{o}} = p(c) - (a_m - a_m^{\mathbf{o}})(c - c^L)^{\alpha}(c - c^R)^{\beta}$$

Fix $a_m^{\mathbf{o}} \prec a_m$. Suppose I have already found some options $c^L$ and $c^R$ of $c$: I want to give some other options ensuring that (4.10) is true. Let

$$q(x) := p(x) - (a_m - a_m^{\mathbf{o}})(x - c^L)^{\alpha}(x - c^R)^{\beta}.$$

$q(x) \in \mathbf{No}[x]$ is strictly simpler than $p(x)$, because the coefficients of degree greater than $m$ are unchanged, while the $m$-coefficient is $a_m^{\mathbf{o}}$, which is strictly simpler than $a_m$. Therefore, I can apply induction, and say that all its roots are in **No**. Suppose, for instance, that $p^{\mathbf{o}} = p^L$ is a left option of $p$. Let $c^{R'}$ be the leftmost greater than $\zeta$, or $c^R$ if there is none. Then $q(x) < p(x)$ in $(c^L, c^R)$, $p(x) < 0$ in $(c^L, \zeta)$ and $q(x)$ does not change sign in $(\zeta, c^{R'})$ (because $\mathbb{K}$ is real closed), so $q(x) < 0$ in $(c^L, c^{R'})$.

Consequently, if I add $c^{R'}$ to the right options of $c$, I ensure that $q(c) < 0$, namely $p(c)^L < 0$.

A minor problem: the lexicographic order on **No**$[x]$ induced by $\preceq$ is not set-like: therefore, it seems that I might be giving a proper class of options for $c$. But when I do the inductive step I do not take an arbitrary polynomial simpler than $p$, but one which is an option of $p$, and this ensures that I never add more than a set of options for $c$. □

*Example* 4.3. Conway's proof that **No** is a field follows from applying the previous proof to the polynomial $ax-1$.

*Proof.* Let $p(x) = ax - 1$. Then, $q(x)$ in the previous proof is in one of the following forms:

$$q(x) = \begin{cases} (ax-1)-(a-a^{\mathbf{o}})(x-c^{\mathbf{o}}) = a^{\mathbf{o}}x+(a-a^{\mathbf{o}})c^{\mathbf{o}}-1 & (4.11) \\ ax. & (4.12) \end{cases}$$

(4.12) yields the left option 0 for $c$. (4.11) produces the option

$$c^{\mathbf{o}\prime} = \frac{1+(a^{\mathbf{o}}-a)c^{\mathbf{o}}}{a^{\mathbf{o}}}$$

if $a^{\mathbf{o}} \neq 0$. If $a^{\mathbf{o}} = 0$, (4.11) becomes the constant function $ac^{\mathbf{o}}-1$, which gives no options for $c$. □

*Example* 4.4. Clive Bach's algorithm for finding $\sqrt{a}$ in [6] is the application of the proof to $x^2 - a$.

*Proof.* Let $p(x) = x^2 - a$. Then,

$$q(x)^L = \begin{cases} x^2 - a^R & (4.13) \\ x^2 - a - (x-c^{\mathbf{o}})^2 = 2xc^{\mathbf{o}} - (c^{\mathbf{o}})^2 - a & (4.14) \end{cases}$$

and

$$q(x)^R = \begin{cases} x^2 - a^L & (4.15) \\ x^2 - a - (x-c^L)(x-c^R) = (c^L+c^R)x - c^Lc^R - a. & (4.16) \end{cases}$$

(4.13) and (4.15) yield the options $\sqrt{a^R}$ and $\sqrt{a^L}$ respectively. (4.14) and (4.16) give respectively

$$c^{R\prime} = \frac{a+(c^{\mathbf{o}})^2}{2c^{\mathbf{o}}} \tag{4.17}$$

and

$$c^{L\prime} = \frac{a+c^Lc^R}{c^L+c^R}, \tag{4.18}$$

where none of the denominators can be 0. Instead of (4.17), Bach uses

$$\frac{a+c^Lc^{L^*}}{c^L+c^{L^*}} \text{ or } \frac{a+c^Rc^{R^*}}{c^R+c^{R^*}},$$

where $c^L, c^{L^*}$ are "old" left options (and $c^R, c^{R^*}$ "old" right options) of $c$, but I can always take the best among $c^L, c^{L^*}$ instead. □

*Example* 4.5. Other polynomials do not yield to such simple algorithms. For instance, to solve the polynomial $x^3 - a$, I need to solve first polynomials of the kind

$$x^3 - (x - c^L)^\alpha (x - c^R)^\beta - a,$$

where $\alpha + \beta = 3$, beside of course polynomials of type

$$x^3 - a^{\mathbf{o}}.$$

*Remark* 4.6. Let $S$ be an initial subset of $\mathbf{No}$. Let $L < R$ be subsets of $S$, and let $x = \langle L \mid R \rangle$. Then, every ancestor of $x$ is in $S$.

*Proof.* Let $z \prec x$. Without loss of generality, $z < x$. By the inverse cofinality theorem, there exists $y \in L$ such that $z \leq y < x$. Therefore $z \preceq y$, but $y \in S$ and $S$ is initial, so $z \in S$. □

**Lemma 4.7.** *Let $S, R$ be initial subsets of* $\mathbf{No}$. *Then the sets*

$$S + R := \{x + y : x \in S, y \in R\} \text{ and } -S := \{-x : x \in S\}$$

*are initial subsets of* $\mathbf{No}$.

*If $S, R$ are also additive subgroups of* $\mathbf{No}$, *then $\langle SR \rangle$, the additive subgroup generated by $SR := \{xy : x \in S, y \in R\}$, is an initial subset of* $\mathbf{No}$.

*The additive subgroup and the subring of* $\mathbf{No}$ *generated by $S$ are initial subsets of* $\mathbf{No}$.

*Proof.* The fact that $-S$ is initial is obvious.

Suppose for contradiction that $S+R$ is not initial. Let $(x,y)$ the simplest element of in the cardinal product $S \times R$ such that $\exists z \prec x+y\, z \notin S+R$. Without loss of generality, $z < x+y$. Let $x = \langle x^L \mid x^R \rangle$ and $y = \langle y^L \mid y^R \rangle$ be their canonical representations. Then $x+y = \langle x^L + y, x + y^L \mid x^R + y, x + y^R \rangle$. By the inverse cofinality theorem, there exist $x^L$ (or $y^L$) such that $t := x^L + y$ (or $t := x + y^L$) satisfies $z \leq t < x+y$. Therefore, $z \preceq t$. But, by minimality of $(x,y)$, every ancestor of $t$ is in $S+R$, and in particular $z \in S+R$, a contradiction.

Suppose that $S, R$ are subgroups of $\mathbf{No}$, and that, for contradiction, $\langle SR \rangle$ is not initial. An element of $w \in \langle SR \rangle$ is of the form $w = w_1 + \cdots + w_n$ for some $n \in \mathbb{N}$, $w_i \in SR$ $i = 1, \ldots, n$.

Consider again the order $\prec$ on the cardinal product $S \times R$. Let

$$G := (S \times R)^{(\mathbb{N})}$$

be the symmetric power of $S \times R$ with the induced order, defined in 1.64. There is a surjective map $\psi : G \to \langle SR \rangle$

$$\Psi((x_1,y_1),\dots,(x_n,y_n)) := x_1y_1 + \dots + x_ny_n.$$

Let $g := ((x_1,y_1),(x_2,y_2),\dots) \in G$ minimal such that $w := \psi(g)$ has an ancestor not in $\langle SR \rangle$. Let $n \in \mathbb{N}$ be the cardinality of the support of $(x_1y_1, x_2y_2, \dots)$. Without loss of generality, I can suppose $x_iy_i \neq 0$ if $i \leq n$, while $x_i = y_i = 0$ if $i > n$.

If $n > 1$, let $a := x_1y_1$, $b := x_2y_2 + \dots + x_ny_n$. $w = a + b$. Let $A := \{t \in \mathbf{No} : t \preceq a\}$, $B := \{t \in \mathbf{No} : t \preceq b\}$. By definition, $A, B$ are initial subsets of $\mathbf{No}$, therefore $A + B$ is an initial subset of $\mathbf{No}$. Moreover, the minimality of $g$ implies that $A, B \subseteq \langle SR \rangle$, so $A + B \subseteq \langle SR \rangle$. $w \in A + B$ and $A + B$ is initial, so all ancestors of $w$ are in $A + B \subseteq \langle SR \rangle$, a contradiction.

If $n = 1$, $g = ((x,y))$, i.e. $w = xy$; let $x = \langle x^L \mid x^R \rangle$ and $y = \langle y^L \mid y^R \rangle$ be their canonical representations. Without loss of generality, $z < xy$. A left option of $xy$ is $t := xy^L + x^Ly - x^Ly^L$ or $t := xy^R + x^Ry - x^Ry^R$. By the inverse cofinality theorem there exists $t$ such that $z \leq t < xy$, so $z \preceq t$. But $((x^{\mathbf{o}},y),(x,y^{\mathbf{o}}),(x^{\mathbf{o}},y^{\mathbf{o}}))$ is strictly simpler than $(x,y)$, therefore all the ancestors of $t$ are in $\langle SR \rangle$, a contradiction.

Finally, the additive subgroup generated by $S$ is the union of initial subsets of $\mathbf{No}$, so it is an initial subset of $\mathbf{No}$, and similarly for the subring. □

It is not true in general that if $S, R$ are initial subgroups of $\mathbf{No}$, then $SR$ is an initial subclass of $\mathbf{No}$. For instance, take $S = R$ to be the subgroup generated by $\mathbb{Z}$ and $\omega$. Then, $\omega^2 + \omega = \omega(\omega + 1) \in SR$, but $\omega^2 + 1 \notin SR$.

**Corollary 4.8.** *Let $\mathbb{K}$ be an initial subring of* $\mathbf{No}$. *Let $L < R$ be subsets of $\mathbb{K}$, and let $c := \langle L \mid R \rangle$. Then, $\mathbb{K}[c]$ is also an initial subring of* $\mathbf{No}$.

*Proof.* $\mathbb{K} \cup \{c\}$ is an initial subset of $\mathbf{No}$, therefore the ring generated by it is initial too. □

Given $\mathbb{K} \subseteq \mathbf{No}$ a subring of $\mathbf{No}$, its real closure is the class of all surreal numbers that are algebraic over $\mathbb{K}$.

**Lemma 4.9.** *Let $\mathbb{K}$ be an initial subring of* $\mathbf{No}$. *Let $\overline{\mathbb{K}} \subset \mathbf{No}$ be its real closure. Then, $\overline{\mathbb{K}}$ is an initial subfield of* $\mathbf{No}$.

*Proof.* Let $\mathbb{F}$ be the union of all initial subsets of $\overline{\mathbb{K}}$; it is obviously initial, and by lemma 4.7 it is a subring of $\overline{\mathbb{K}}$. I want to prove that $\overline{\mathbb{K}} = \mathbb{F}$.

Following the proof of 4.2, I introduce on $\mathbb{F}[x]$ the lexicographic order induced by $\prec$. Let $p \in \mathbb{F}[x]$, let $c \in \overline{\mathbb{K}}$ be a root of $p(x)$. I have to prove that $c \in \mathbb{F}$. Suppose that I have already proved it for every polynomial simpler than $p(x)$. I want to find $L < R \in \mathbb{F}$ such that $c = \langle L \mid R \rangle$; then, by remark 4.6, $c \in \mathbb{F}$. Let us compute $L, R$ using the procedure of lemma 4.2. First, I put in them the roots of the derivative $p'(x)$, which is simpler than $p(x)$: therefore, all these roots are in $\mathbb{F}$. Then, given $c^L, c^R$ "old" options of $c$ (which I can suppose are in $\mathbb{F}$), I construct "new" options $d$ as roots of the polynomial

$$q(x) := p(x) - (a_m - a_m^{\mathbf{o}})(x - c^L)^{\alpha}(x - c^R)^{\beta},$$

where $a_m$ is the $m^{th}$ coefficient of $p(x)$ and $a_m^{\mathbf{o}} \prec a_m$. Therefore, $q(x)$ is simpler than $p(x)$. Moreover, its coefficients are in $\mathbb{F}$, because $a_m \in \mathbb{F}$, $a_m^{\mathbf{o}}$ is simpler than $a_m$, and therefore in $\mathbb{F}$, $c^L$ and $c^R$ are in $\mathbb{F}$. Therefore, $d$ is a root of a polynomial in $\mathbb{F}[x]$ simpler than $p(x)$, so $d \in \mathbb{F}$. □

**Corollary 4.10.** *If $S$ is an initial subset of* **No**, *then the smallest real closed field containing it is an initial subfield of* **No**.

*Proof.* $\mathbb{Q}$ is an initial subset of **No**, therefore $S \cup \mathbb{Q}$ is initial too, so, by lemma 4.7, the ring generated by it is also initial, and its real closure is initial by the previous lemma. □

The statements and the proofs of the previous lemmas work also for initial classes instead of sets.

If instead of considering *all* polynomials with coefficients in $\mathbb{K}$, I consider only the polynomial up to a fixed degree $n \in \mathbb{N}$, the lemma and the corollary are still true (with the same proof). For instance, if I take $n = 1$, I can say that the smallest field containing an initial subset of **No** is initial.

The following theorem was proved with different methods in [7]

**Theorem 13.** *Let $\mathbb{K}$ be a real closed field and a proper set. Then, $\mathbb{K}$ is isomorphic to an initial subfield of* **No**.

*Proof.* If $\mathbb{K} = \mathbb{Q}$, it is true.

If $\mathbb{F}$ real closed and an initial subfield of **No** and $\mathbb{K}$ is (isomorphic to) the real closure of $\mathbb{F}(a)$ for some $a$ transcendental over $\mathbb{F}$, let $(L, R)$ be the cut over $\mathbb{F}$ of $a$. If $c \in (L \mid R)$, then $\mathbb{F}(c)$ is isomorphic to $\mathbb{F}(a)$, and the real closure of $\mathbb{F}(c)$ is isomorphic to $\mathbb{K}$. If I take $c = \langle L \mid R \rangle$, then $\mathbb{F} \cup \{c\}$ is an initial subset of **No**, and the conclusion follows by corollary 4.10.

In general, let $(c_\beta)_{\beta<\alpha}$ be a transcendence basis of $\mathbb{K}$ over $\mathbb{Q}$, let $\mathbb{K}_0$ be the real closure of $\mathbb{Q}$, and for $0 < \beta < \alpha$ let

$$\mathbb{K}_\beta := \begin{cases} \text{the real closure of } \mathbb{K}_\gamma(c_\gamma) & \text{if } \beta = \gamma + 1 \\ \bigcup_{\gamma<\beta} \mathbb{K}_\gamma & \text{if } \beta \text{ is a limit ordinal.} \end{cases}$$

By the previous case and induction on $\beta$, for every $\beta < \alpha$ $\mathbb{K}_\beta$ is isomorphic to an initial subfield of **No**, and the conclusion follows. □

It is not true that every ordered field (which is also a set) is isomorphic to an initial subfield of **No**. For instance, take $\mathbb{K} := \mathbb{Q}(\sqrt{2} + 1/\omega) \subset \mathbf{No}$. Suppose, for contradiction, that there exists an isomorphism of ordered fields $\psi$ between $\mathbb{K}$ and an initial subfield of **No**. Let $z = \psi(\sqrt{2} + 1/\omega)$. Then, $\sqrt{2} \prec z$, but $\sqrt{2} \notin \psi(\mathbb{K})$.

*Conjecture* 4.11. Let $G$ be an initial ordered subgroup of **No**, and let $\mathbb{K}$ be an ordered field. Assume that $\mathbb{K}$, with the natural valuation, is an Henselian and with value group $G$. Then, $\mathbb{K}$ is isomorphic to an initial ordered subfield of **No**.

Note that the value group of every initial subfield of **No** is also initial.

The main problem with the proof of lemma 4.2 is that I have to know in advance the existence of a real closed field $\mathbb{K}$ embedding **No**: it is used twice, once to assure the existence of a root $\zeta$ of $p$ 'somewhere', and second to assure that a polynomial $q$ does not change sign between two consecutive roots. Of course, this is not a problem for polynomials, but becomes an issue for other kinds of functions.

Let me rephrase the lemma using the following

**Definition 4.12.** I say that a function $f\colon \mathbf{No} \to \mathbf{No}$ satisfies the intermediate value property at $d \in \mathbf{No}$ iff for all $a < b \in \mathbf{No}$ such that $f(a) < d < f(b)$ there exists $c \in (a, b)$ such that $f(c) = d$.

$f$ satisfies the I.V.P. iff it satisfies the I.V.P. at every $d \in \mathbf{No}$.

I will now prove that every polynomial satisfies the I.V.P: while this is not very interesting for polynomials, it is for other kind of recursively defined function.

*Proof.* It suffices to prove the case $d = 0$. Let

$$p(x) = \sum_{i=0}^{n} a_i x^i.$$

Again, I do induction on $p$.

I can suppose that $\zeta \in \mathbf{No}^{\mathscr{D}}$, the Dedekind completion of **No**, is such that $p(x) < 0$ in $[a, \zeta)$ and $p(x) > 0$ in $(\zeta, b]$. I will give options for $\zeta$.

First, $a, b$ are left and right options. Then, if $q(x)$ and $c^{R'}$ as in the proof of lemma 4.9, I can apply the inductive hypothesis, and conclude that $q(x)$ does not change sign in the interval $(\zeta, c^{R'})$. The fact that $p(c) = 0$ follows trivially. □

The following is a weak form of axiom 2

**Axiom 5.** Let $g : \mathbf{No} \to \mathbf{No}$. Either $g$ is constant or $\forall c \in \mathbf{No}\ \forall a' < a'' \in \mathbf{No}\ \exists m \in \mathbb{N}$ $\exists a_0, \ldots, a_m \in \mathbf{No}^{\mathscr{D}}$ such that $a' = a_0 < a_1 < \cdots < a_m = a''$ and for $i = 0, \ldots, m-1$ $g\restriction(a_i, a_{i+1}) < c$ or $g\restriction(a_i, a_{i+1}) > c$.

Now I can adapt this proof to recursive functions.

**Lemma 4.13.** *Suppose that $\mathfrak{A}$ is a family of functions satisfying axiom 5, and such that every function in it satisfies the I.V.P. at 0. Suppose that $f$ is uniformly recursive over $\mathfrak{A}$. Then, $f$ satisfies the I.V.P. at 0.*

*Proof.* Proceed as in the previous proof. Let $c^L, c^R$ be "old" options of $c$. Let

$$g(x) := f^{\mathbf{o}}(x, c^L, c^R, f(c^L), f(c^R))$$
$$\zeta := \sup(\{x \in [a,b] : f(x) < 0\} \cup \{b\})$$

for some $f^{\mathbf{o}}$ option of $f$. By axiom 5, $g$ has only finitely many zeros in $(c^L, c^R)$, therefore I can take $c^{L'}$ the rightmost before $\zeta$ and $c^{R'}$ the leftmost after $\zeta$, and by hypothesis g does not change sign in the interval $I = (c^{L'}, c^{R'})$.

Say that $f^{\mathbf{o}} = f^L$. Then, by uniformity, $c \in (c^L, c^R) \Rightarrow g(c) < f(c)$. Moreover, $f(c) < 0$ if $c < \zeta$, therefore $g(x) < 0$ in $I$. The rest of the proof is the same as before. □

*Example* 4.14. Consider the integer part function $[x] - 1/2$. Why cannot I apply the previous proof to it? I.e. take the equation $[x] = 1/2$. We know that $[-1] = -1 < 1/2$ and $[2] = 2 > 1/2$. So, if $[x]$ did satisfy the I.V.P. at $1/2$, I would find $x \in (-1, 2)$ such that $[x] = 1/2$.

*Fake proof.* This is equivalent to solving:

$$x > -1 \qquad x < 2 \tag{4.19a}$$
$$[x] > 0 \qquad [x] < 1 \tag{4.19b}$$
$$x - 1 < \frac{1}{2} \qquad x + 1 > \frac{1}{2} \tag{4.19c}$$

(4.19c) plus (4.19a) produce

$$\frac{-1}{2} < x < \frac{3}{2}$$

Let us solve (4.19b) using the method in the previous proof: I have to find the maximum of the set

$$\{x \in \mathbf{No} : -1 < x < 2 \;\&\; [x] = 0\}.$$

However *this set has no maximum*: $[x] = 0$ on $(-1, 1)$, but $[1] = 1$. □

I can say something more.

**Definition 4.15.** Let $n \geq 0 \in \mathbb{N}$. A function $g : \mathbf{No}^{n+1} \to \mathbf{No}$ satisfies the sup property iff for every $\vec{b} \in \mathbf{No}^n$, $d' < a'' \in \mathbf{No} \cup \{\pm\infty\}$, $c \in \mathbf{No}$ the infimum and the supremum of the class

$$\left\{x \in \mathbf{No} : d' < x < a'' \;\&\; g(x,\vec{b}) \leq c\right\}$$

are in $\mathbf{No} \cup \{\pm\infty\}$, and the same with $\geq$ instead of $\leq$.

**Theorem 14.** *Let $\mathfrak{A}$ be a family of functions, such that every $g \in \mathfrak{A}$ satisfies the* sup *property. Let $f$ be a function uniformly recursive over $\mathfrak{A}$ and satisfying axiom 5.*

*Then, $f$ satisfies the* sup *property.*

*Proof.* Let $a', a'', c \in \mathbf{No}$. Let

$$\zeta := \sup\left\{x \in \mathbf{No} : a' < x < a'' \;\&\; f(x) \leq c\right\} \cup \{a''\} \in \mathbf{No}^{\mathscr{D}}.$$

By axiom 5 w.l.o.g. I can suppose that $f(x) < c$ in the interval $[a', \zeta)$, and $f(x) > c$ in the interval $(\zeta, a'']$.

I will prove that $\zeta \in \mathbf{No}$ by induction on $c$.

I will construct a $d \in \mathbf{No}$ "as near as possible" to $\zeta$; I will give the options of such a $d$. First, $a' < \zeta < a''$, therefore $a'$ is a left option, $a''$ a right one.

Given a representation $c = \langle c^L \mid c^R \rangle$ and given $x = \langle x^L \mid x^R \rangle \in \mathbf{No}$, $f(x) \leq c$ is equivalent to

$$f(x) < c^R \text{ and} \tag{4.20}$$
$$f^L(x, x^{\mathbf{o}}, f(x^{\mathbf{o}})) < c \tag{4.21}$$

for every $x^{\mathbf{o}}$ option of $x$, $c^L$, $c^R$ options of $c$ and $f^L, f^R$ options of $f$.

Let $d^{R'}$ be the infimum of the class

$$\left\{x \in \mathbf{No} : f(x) \geq c^R \;\&\; a' < x < a''\right\} \cup \{a'\}.$$

By inductive hypothesis $d^{R'} \in \mathbf{No}$. Moreover, $\forall x < \zeta\ f(x) < c < c^R$, therefore $d^{R'} \geq \zeta$. If $d^{R'} = \zeta$, I have proved the conclusion; otherwise, add $d^{R'}$ to the right options of $d$.

Let $d^{L'}$ be the supremum of the class

$$\left\{x \in \mathbf{No} : f(x) \leq c^L \;\&\; a' < x < a''\right\} \cup \{a''\}.$$

Again, $d^{L'} \in \mathbf{No}$. Moreover, $\forall x > \zeta\ f(x) \geq c > c^L$, therefore $d^{L'} \leq \zeta$. If $d^{L'} \neq \zeta$, add $d^{L'}$ to the left options of $d$.

Fix $d^L, d^R$ "old" options of $d$. Let $d^{R''}$ be the infimum of the class

$$\left\{x \in \mathbf{No} : f^L(x, d^L, d^R, f(d^L), f(d^R)) \geq c \;\&\; d^L < x < d^R\right\} \cup \{d^R\}.$$

By hypothesis $d^{R''} \in \mathbf{No}$. Moreover, by uniformity

$$\forall x \in (d^L, d^R)\, f^L(x, d^{\mathbf{o}}, f(d^{\mathbf{o}})) < f(x)$$

and $\forall x < \zeta\ f(x) < c$, therefore $d^{R''} \geq \zeta$. Again, if $d^{R''} = \zeta$ I have proved that $\zeta \in \mathbf{No}$, otherwise $d^{R''}$ is a "new" right option of $d$.

Proceed similarly for

$$d^{L''} = \sup\{x \in \mathbf{No} : f^R(x, d^L, d^R, f(d^L), f(d^R)) \leq c \;\&\; d^L < x < d^R\} \cup \{d^L\}.$$

If the construction was not broken by $\zeta$ being equal to some of the "new" options of $d$, I obtain in this way a $d \in \mathbf{No}$ such that $d \preceq \zeta$ (with the simplicity relation on $\mathbf{No}^{\mathcal{D}}$ induced by the one on $\mathbf{No}$). Moreover, $f(d) = c$ and $d' < d < d''$, therefore $d = \zeta$.

□

Let $f : \mathbf{No} \to \mathbf{No}$ be as in the previous theorem.

**Corollary 4.16.** *$f$ can be extended in a unique way to $f : \mathbf{No}^{\mathcal{D}} \to \mathbf{No}^{\mathcal{D}}$ such that $f$ continuous at every $\zeta \in \mathbf{No}^{\mathcal{D}} \setminus \mathbf{No}$;*

$$f(\zeta) := \limsup_{\substack{x \to \zeta^- \\ x \in \mathbf{No}}} f(x) = \lim_{\substack{x \to \zeta \\ x \in \mathbf{No}}} f(x)$$

**Corollary 4.17.** *If moreover $f$ is continuous, then $f$ satisfies the intermediate value property.*

*Example* 4.18. Let $f(x) = \omega^x$. It is not continuous at any $x \in \mathbf{No}$. Nevertheless, $f$ can be extended to $\mathbf{No}^{\mathcal{D}}$. For instance,

$$\omega^\eta = \sup\{\omega^x : v(x) < 1\} = \inf\{\omega^x : v(x) \geq 1\} = \inf\left\{\omega, \omega^{1/2}, \omega^{1/4}, \dots\right\}.$$

The sign expansion of $\omega^\eta$ is given by the sign expansion of $\omega^{1/\omega}$ followed by infinitely many pluses; $\omega^\eta = \omega^{1/\omega} : +\infty$.

*Example* 4.19. Let $f(x) = x - [x]$. On the interval $(1/2, 3/2)$, $f$ satisfies the hypothesis of theorem 14. Let us see how the proof works for $c = 1/2$. Let

$$\zeta = \sup\left\{x \in \mathbf{No} : f(x) > \frac{1}{2} \;\&\; \frac{1}{2} < x < \frac{3}{2}\right\}.$$

I must show that $\zeta \in \mathbf{No}$. $f(x) = \langle x^L - [x], -1 \mid x^R - [x], 1 \rangle$. $f(x) = 1/2$ iff

$$x^L - [x] < \frac{1}{2} < x^R - [x] \text{ and}$$
$$0 < f(x) < 1.$$

However, $f(x) = 1$ in the interval $(1/2, 3/2)$ iff $x = 1$ and $\zeta = 1$.

I can also generalise lemma 4.7.

**Definition 4.20.** Given $S \subseteq \mathbf{No}$ and a function $g : \mathbf{No}^n \to \mathbf{No}$, I say that $S$ is closed under $g$ iff $g(S^n) \subseteq S$. Given a family of functions $\mathfrak{A}$, I say that $S$ is closed under $\mathfrak{A}$ iff it is closed under every $g \in \mathfrak{A}$. The closure of $S$ under $\mathfrak{A}$ is $S^{\mathfrak{A}}$, the intersection of all subclasses of **No** closed under $\mathfrak{A}$ and containing $S$.

Note that if both $S$ and $\mathfrak{A}$ are sets, then $S^{\mathfrak{A}}$ is also a set.

**Theorem 15.** *Suppose that $\mathfrak{A}$ is a family of functions and that $f : \mathbf{No}^n \to \mathbf{No}$ is recursive over it.*(2)

*Suppose that for every $S \subset \mathbf{No}$ initial subset of* **No**, *$S^{\mathfrak{A}}$ is also initial. Then for every $R \subset \mathbf{No}$ initial subset of* **No**, *$R^{\mathfrak{A} \cup \{f\}}$ is also initial.*

*Proof.* Let $T$ be the maximal initial subset of **No** contained in $R^{\mathfrak{A} \cup \{f\}}$. By hypothesis, $T$ is closed under $\mathfrak{A}$. I have to prove that $T$ is also closed under $f$. Given $\vec{a} \in T^n$, an option of $f(\vec{a})$ is of the form

$$g(\vec{a}, \vec{a}^{\mathbf{o}}, f(\vec{a}^{\mathbf{o}})),$$

where $g \in \mathfrak{A}$ and $\vec{a}^{\mathbf{o}}$ is a vector of options of $\vec{a}$, i.e.

$$\vec{a}^{\mathbf{o}} = (a_1, \dots, a_{i-1}, a_i^{\mathbf{o}}, a_{i+1}, \dots, a_n),$$

with $a_i^{\mathbf{o}}$ a standard option of $a_i$ for every $i = 0 \dots, n$. Hence, $\vec{a}^{\mathbf{o}} \prec \vec{a}$ in the order of $\mathbf{No}^n$ induced by $\preceq$ via *bnd*. Therefore, by induction I can suppose to have already proved $f(\vec{a}^{\mathbf{o}}) \in T$. It follows that $g(\vec{a}, \vec{a}^{\mathbf{o}}, f(\vec{a}^{\mathbf{o}})) \in T$. Then, every options of $f(\vec{a})$ is in $T$, so, by remark 4.6, $f(\vec{a}) \in T$. □

**Definition 4.21.** Let $S \subset \mathbf{No}$, and $g : \mathbf{No}^{n+1} \to \mathbf{No}$. I say that $S$ is closed under solutions of $g$ iff for every $\vec{a} \in S^n$ and every $c \in S$, every isolated zero of $h(x) := g(x, \vec{a}) - c$ (in **No**) is in $S$. The closure of $S$ under solutions of $g$ is the smallest subclass of **No** closed under $g$ and under solutions of $g$.

**Lemma 4.22.** *Let $\mathfrak{A}$ be a family of functions satisfying axion 5 and the I.V.P. Let $f : \mathbf{No} \to \mathbf{No}$ be non-constant and uniformly recursive over $\mathfrak{A}$, satisfying axiom 5 and the I.V.P.. Suppose that for every $S$ initial subset of* **No**, *the closure of $S$ under solutions of $\mathfrak{A}$ and under $f$ is also initial. Then $R$, the closure of $S$ under solutions of $\mathfrak{A} \cup \{f\}$, is initial.*

(2) not necessarily uniformly

*Proof.* Usual procedure. Let $T$ be the maximal initial subset of $R$. By hypothesis, $T$ is closed under $f$ and under solutions of $\mathfrak{A}$. I have to prove that $T$ is also closed under solutions of $f$. Let $c \in T$, $a$ be a zero of $f(x) - c$. By axiom 5, there are only finitely many zeros of $f(x) - c$. I will prove the lemma by induction on $c$.

I will give $L < R \subseteq T$ such that $a = \langle L \mid R \rangle$, implying that $a \in T$. Let $f = \langle f^L \mid f^R \rangle$. Then, $f(a) = c$ iff

$$f^L(a, a^\circ, f(a^\circ)) < c < f^R(a, a^\circ, f(a^\circ)) \tag{1}$$

$$c^L < f(a) < c^R. \tag{2}$$

Let $a^L, a^R \in T$ be "old" options of $a$.

1. Let $a^{L'} \in \mathbf{No}$ be the the rightmost zero before $a$ of

   $$g(x) := f^L(x, a^\circ, f(a^\circ)) - c,$$

   and let $a^{R'}$ the leftmost one after it. By the I.V.P. on $\mathfrak{A}$, $g(x)$ does not change sign in $(a^{L'}, a^{R'})$, and $g(a) < f(a) - c = 0$, therefore $g(x) < 0$ in $(a^{L'}, a^{R'})$. Moreover, by hypothesis on $T$, $a^{L'}, a^{R'} \in T$. Add $a^{L'}, a^{R'}$ to the options of $a$. Do the same for $f^R$.

2. Let $a^{L'}$ be the rightmost zero before $a$ of

   $$g(x) := f(x) - c^L,$$

   and let $a^{R'}$ be the leftmost after it. By the I.V.P. on $f$, $g(x)$ does not change sign in the interval $(a^{L'}, a^{R'})$, and, by induction on $a$, $a^{L'}, a^{R'} \in T$. Again, add $a^{L'}, a^{R'}$ to the options of $a$.

At the end of the process, we obtain $a' \in T$ such that $f(a') = c$. But we cannot be sure that $a' = a$. However, if for instance $a' > a$, we can restart the whole algorithm adding $a'$ to the right options of $a$. Because $f(x) = c$ has only finitely many solutions, the process must terminate at $a$. □

*Question* 4.23. Let $f : \mathbf{No}^{n+1} \to \mathbf{No}$ be uniformly recursive over a family of functions $\mathfrak{A}$. Suppose that for every $\vec{a} \in \mathbf{No}^n$ $\exists! c \in \mathbf{No}$ $f(c, a) = 0$. Call $h : \mathbf{No}^n \to \mathbf{No}$ the function such that $f(h(\vec{x}), \vec{x}) = 0$. Under which hypothesis (on $f$ and on $\mathfrak{A}$) can we prove that $h$ is uniformly recursive over $\mathfrak{A} \cup \{f\}$? A related question: is it possible to generalise lemma 4.22 to functions of many variables?

Using techniques similar to those employed here, one can prove that the concatenation function $x : y$ is *not* uniformly recursive over any family $\mathfrak{A}$ satisfying axiom 5 and the I.V.P. In particular, it is not uniformly recursive over the family of polynomial functions.(3)

(3) See also [13] and remark 1.20.

# Chapter 5

# Other functions

I will prove that the integral of $1/x$ is $\log x$. On the other hand, I will show that in general $\int_0^a \exp t \,\mathrm{d}t \neq \exp a - 1$.

## 5.1 Logarithm

A variation of the following lemma is attributed to M. Kruskal in [6].

**Lemma 5.1.** *Let* $x > 0 \in \mathbf{No}$. *Then,*

$$\left(\frac{1}{x}\right)^{\circ} = \frac{1-(1-\frac{x}{x^L})^{\alpha}(1-\frac{x}{x^R})^{\beta}}{x}, \tag{5.1}$$

*where* $x^L$ *and* $x^R$ *are positive options of* $x$, *and* $\left(1/x\right)^{\circ}$ *is a left option iff* $\alpha$ *is even.*

Note that, after cancellation of the denominator with the numerator, (5.1) is a polynomial in $x$.

*Proof.* I already know that $\mathbf{No}$ is an ordered field (lemma 4.2). Call $y$ the number defined by (5.1).

First, I prove that $y$ is a number: it is sufficient to prove that $xy^L < 1 < xy^R$, i.e.

$$1-(1-\frac{x}{x^L})^{\alpha}(1-\frac{x}{x^R})^{\beta} < 1$$

iff $\alpha$ is even (and greater iff $\alpha$ is odd), namely

$$(x^L-x)^{\alpha}(x^R-x)^{\beta} \gtrless 0$$

iff $\alpha$ is even (odd), which is obvious.

Then, I want to prove that $xy = 1$. First, I prove that $1 \preceq xy$, i.e. that $0 < xy$. But if I take $\alpha = \beta = 0$, I obtain $y > 0$, which implies $xy > 0$.

Then, I prove that $xy \preceq 1$, i.e. that $(xy)^L < 1 < (xy)^R$.

$$(xy)^{\mathbf{o}} = x^{\mathbf{o}}y + (x - x^{\mathbf{o}})y^{\mathbf{o}} = x^{\mathbf{o}}y + (x - x^{\mathbf{o}})\frac{1 - (1 - \frac{x}{x^L})^{\alpha}(1 - \frac{x}{x^R})^{\beta}}{x}.$$

Suppose for instance that $\alpha$ is even and $x^{\mathbf{o}} < x$. Then, I have to prove that the previous expression is $< 1$, i.e.

$$x^{\mathbf{o}}y < \frac{x - (x - x^{\mathbf{o}}) + (x - x^{\mathbf{o}})(1 - \frac{x}{x^L})^{\alpha}(1 - \frac{x}{x^R})^{\beta}}{x} \text{ i.e.}$$

$$y < 1 - (1 - \frac{x}{x^{\mathbf{o}}})(1 - \frac{x}{x^L})^{\alpha}(1 - \frac{x}{x^R})^{\beta}.$$

The lemma is a consequence of the following

*Claim* 2. If $x_1^{\mathbf{o}}, \dots, x_n^{\mathbf{o}}$ is a tuple of options of $x$, then

$$y > \frac{\prod_i (1 - (1 - \frac{x}{x_i^{\mathbf{o}}}))}{x} \tag{5.2}$$

iff $\alpha$, the number of left options, is even, and less otherwise.

In fact, let $x^R$ and $x^L$ be the best left and right approximations of $x$ among $x_1^{\mathbf{o}}, \dots, x_n^{\mathbf{o}}$. It is then easy to see that if $\alpha$ is even, then (5.1) is greater or equal than (5.2) (and less or equal if $\alpha$ is odd). □

I will now consider the integral of $1/x$. In [10], Gonshor defines the logarithm, and proves that it is the compositional inverse of exp.

**Definition 5.2 (Logarithm).** Let $z > 0 \in \mathbf{No}$. The definition of $\log z$ is the following.

Suppose that $z = \omega^a$, $a = \langle a^L \mid a^R \rangle \in \mathbf{No}$. Then,

$$\log z = \langle \log(\omega^{a^L}) + n, \log(\omega^{a^R}) - \omega^{\frac{a^R - a}{n}} \mid \log(\omega^{a^R}) - n, \log(\omega^{a^L}) + \omega^{\frac{a - a^L}{n}} \rangle_{n \in \mathbb{N}}. \tag{5.3}$$

If $z \in \mathbb{R}$, $\log z$ coincides with the logarithm for real numbers.

If $z = 1 + \varepsilon$, where $\varepsilon \in \mathbf{No}$ is infinitesimal, $\log z$ is defined by the power series expansion of log at 1.

Every $z > 0 \in \mathbf{No}$ can be written in a unique way as $z = xry$, where $x = \omega^a$, $r > 0 \in \mathbb{R}$ and $y = 1 + \varepsilon$, with $\varepsilon \in \mathbf{No}$ infinitesimal.

$$\log z := \log x + \log r + \log y.$$

Gonshor proves various properties of $\log z$, in particular the following:

**Lemma 5.3.**

- log *satisfies the functional equation* $\log(xy) = \log x + \log y$.
- $\log : \mathbf{No}^+ \to \mathbf{No}$ *is surjective.*

- $\log x < x^{1/n}$ *for every* $n \in \mathbb{N}$ *and for every* $x$ *infinite.*
- *For* $x = ry$, *where* $r > 0 \in \mathbb{R}$ *and* $y = 1 + \varepsilon$, $\varepsilon \in \mathbf{No}$ *infinitesimal,* $\log x$ *coincides with the corresponding analytic function.*
- $x > y$ *implies* $\log x > \log y$.

By [19] (see also [18]), the previous lemma is enough to prove the following.

**Corollary 5.4.** $\mathbb{R}$ *is an elementary substructure of* $\mathbf{No}$ *in the language* $\mathscr{L}_{an}(\log)$.

I will prove that $\log x$ coincides with the integral of $1/t$. Before I need the following technical tool.

**Lemma 5.5.** *Let* $0 < a < b \in \mathbf{No}$, $c > 0 \in \mathbf{No}$, $P = (k_0, \dots, k_m)$ *be a partition of* $(a, b)$, $(\alpha_1 \dots, \alpha_m)$, $(\beta_1, \dots, \beta_m)$ *tuples of natural numbers.*

*Define* $cP := (ck_0, \dots, ck_m)$ *partition of* $(ca, cb)$, *and*

$$\int_a^b \left(\frac{1}{t}\right)_P^{\mathbf{o}} \mathrm{d}t := \sum_i \int_{k_i}^{k_{i+1}} \frac{1 - (1 - \frac{t}{k_i})^\alpha (1 - \frac{t}{k_{i+1}})^\beta}{t} \mathrm{d}t$$

*and similarly for* $\int_{ca}^{cb} (1/t)_{cP}^{\mathbf{o}} \mathrm{d}t$. *Then,*

$$\int_a^b \left(\frac{1}{t}\right)_P^{\mathbf{o}} \mathrm{d}t = \int_{ca}^{cb} \left(\frac{1}{t}\right)_{cP}^{\mathbf{o}} \mathrm{d}t.$$

*Proof.* It is enough to prove the lemma for $m = 1$. Then, $\int_a^b (1/t)_P^{\mathbf{o}} \mathrm{d}t$ is the integral

$$\int_a^b \frac{1 - (1 - \frac{t}{a})^\alpha (1 - \frac{t}{b})^\beta}{t} \mathrm{d}t.$$

But the integrand is a polynomial in $t$, and for polynomials the integral is equal to the formal integral. Therefore I can apply the change of variable $s = ct$, obtaining

$$\int_a^b \left(\frac{1}{t}\right)_P^{\mathbf{o}} \mathrm{d}t = \int_{ca}^{cb} \frac{1 - (1 - \frac{cs}{a})^\alpha (1 - \frac{cs}{b})^\beta}{cs} c \,\mathrm{d}s = \int_{ca}^{cb} \left(\frac{1}{s}\right)_{cP}^{\mathbf{o}} \mathrm{d}s. \qquad \square$$

**Theorem 16.** *Let* $x > 0 \in \mathbf{No}$. *Then,*

$$\log x = \int_1^x \frac{1}{t} \mathrm{d}t.$$

*Proof.* Call $f(x) := \int_1^x 1/t \,\mathrm{d}t$. I will prove that $f(x) = \log x$ by induction on $x$. Note that taking $\alpha = \beta = 0$ in (5.1), I obtain 0 as a left option of $1/x$, taking $\alpha = 0, \beta = 1$, I obtain $1/x^R$ as a left option, and $\alpha = 1, \beta = 0$ yields $1/x^L$ as a right option.

By inductive hypothesis,

$$f(x)^L = \log(x^L) + \int_{x^L}^x \left(\frac{1}{t}\right)^L \mathrm{d}t,$$

(and similarly with $x^R$ or $(1/t)^R$). The integrand is a polynomial in $t$, therefore previous expression, by corollary 5.4, is less than $\log x$, so I have proved that $f(x) \preceq \log(x)$.

The other direction $\log x \preceq f(x)$ is more difficult.

First, suppose that $x = \omega^a$, $a \in \mathbf{No}$. Then, $x^{\mathbf{o}} = r\omega^{a^{\mathbf{o}}}$, where $r > 0 \in \mathbb{R}$. Consider

$$\int_{x^L}^{x} \left(\frac{1}{t}\right)^{\mathbf{o}} \mathrm{d}t = \sum_{0 \leq i < m} \int_{k_i}^{k_{i+1}} \frac{1 - (1 - \frac{t}{k_i})^{\alpha_i}(1 - \frac{t}{k_{i+1}})^{\beta_i}}{t} \mathrm{d}t.$$

Take $\alpha_i = 0$, $\beta_i = 1$ for every $i$ to obtain the left option of $f(x)$

$$\log(x^L) + \sum_i \int_{k_i}^{k_{i+1}} \frac{1}{k_{i+1}} \mathrm{d}t = \log(x^L) + m - \sum_i \frac{k_i}{k_{i+1}}.$$

Now take

$$k_i = x^L \left(\frac{x}{x^L}\right)^{\frac{i}{m}},$$

i.e.

$$\frac{k_i}{k_{i+1}} = \left(\frac{x}{x^L}\right)^{-1/m},$$

yielding the left option

$$\log(x^L) + m - m\left(\frac{x}{x^L}\right)^{-1/m}.$$

If $x = \omega^a$ and $x^L = r\omega^{a^L}$, it follows that $\left(\frac{x}{x^L}\right)^{-1/m}$ is infinitesimal, therefore for every $n \in \mathbb{N}$

$$f(x) > \log(x^L) + n.$$

On the other hand, taking $\alpha = 1$, $\beta = 0$, $k_i$ as before, I obtain the right option

$$\log(x^L) + m\left(\frac{x}{x^L}\right)^{1/m} - m,$$

which implies

$$f(x) < \log(x^L) + \left(\frac{x}{x^L}\right)^{1/n}.$$

The other two kind of options in 5.3 are obtained taking $x^R$ instead of $x^L$.

Now suppose that $x$ is log-bounded, i.e. that $x = r + \varepsilon$, where $r > 0 \in \mathbb{R}$, and $\varepsilon \in \mathbf{No}$ is infinitesimal. It is easy to see that $\log r = f(r)$. Moreover, $\frac{1}{x}$ is analytic in a neighbourhood of $r$, therefore its integral coincides with the formal integral as a power series, which is equal to $\log x$.

Suppose that $z = xry$, where $x, r, y$ are as in the definition 5.2. To conclude the lemma it remains to prove that

$$f(z) = f(x) + f(r) + f(y) = \log z.$$

I will prove it by induction on $x, r, y$. First, suppose that $y = 1$. If $r = 1$ I have already proved it. Otherwise,

$$z^L = (xr)^L = \omega^a r^L + s(r - r^L)\omega^{a^L},$$

and similarly for right options, where $s > 0 \in \mathbb{R}$ and $r^L \in \mathbb{R}$ is some positive option of $r$ (positive because we are working in the domain $\mathbf{No}^{>0}$). Note that

$$z^L = r^L \omega^a \left(1 + (s(\frac{r^L}{r} - 1)\omega^{a^L - a})\right) = xr^L y',$$

where $y' - 1$ is infinitesimal, so I can apply the inductive hypothesis to $z^L$. Therefore, by lemma 5.5

$$\begin{aligned} f(z)^L &= f(z^L) + \int_{z^L}^{z} \left(\frac{1}{t}\right)^L \mathrm{d}t = \log(xr^L y) + \int_{xr^L y'}^{z} \left(\frac{1}{t}\right)^L \mathrm{d}t \\ &= \log(xr) + \log(\frac{r^L}{r} y') + \int_{\frac{r^L}{r} y'}^{1} \left(\frac{1}{t}\right)^L \mathrm{d}t. \end{aligned} \tag{5.4}$$

Note that $w := \frac{r^L}{r} y' < 1$ is log-bounded, therefore $\log w = f(w)$, and

$$f(z)^L = \log(xr) + \int_w^1 \left(\frac{1}{t}\right)^L - \left(\frac{1}{t}\right) \mathrm{d}t < \log(xr).$$

Similar reasoning for other options.

If $y \neq 1$, then

$$z^{\mathbf{o}} = xr(y^{\mathbf{o}}),$$

where $y^{\mathbf{o}} = 1 + \varepsilon^{\mathbf{o}}$. Again, I can apply the inductive hypothesis to $z^{\mathbf{o}}$ and lemma 5.5, obtaining

$$f(z)^{\mathbf{o}} = \log(xr) + \log(y^{\mathbf{o}}) + \int_{xry^{\mathbf{o}}}^{xry} \left(\frac{1}{t}\right)^{\mathbf{o}} \mathrm{d}t = \log(xry) + \log(y^{\mathbf{o}}/y) + \int_{y^{\mathbf{o}}/y}^{1} \left(\frac{1}{t}\right)^{\mathbf{o}} \mathrm{d}t.$$

But $\frac{y^{\mathbf{o}}}{y} - 1$ is infinitesimal, therefore $\log(y^{\mathbf{o}}/y) = f(y^{\mathbf{o}}/y)$, and I can conclude as before. □

## 5.2 Exponential

### 5.2.1 Translation invariance

**Lemma 5.6.** *Suppose that $\mathbb{K}$ is a field, $f\colon (\mathbb{K}, +) \to (\mathbb{K}, \cdot)$ is a homomorphism of groups with $f(1) \neq 1$, and $\int_x^y g(t)\,\mathrm{d}t$ (defined on some family of functions containing $f$) is a functional satisfying*

1. *For every* $a, b, c \in \mathbb{K}$
$$\int_a^b g + \int_b^c g = \int_a^c g.$$

2. *For every* $\lambda \in \mathbb{K}$,
$$\int_a^b \lambda g = \lambda \int_a^b g.$$

3. *For every* $c \in \mathbb{K}$
$$\int_a^b g(t+c)\,\mathrm{d}t = \int_{a+c}^{b+c} g(t)\,\mathrm{d}t.$$

4. $\int_0^1 f(t)\,\mathrm{d}t = f(1) - 1.$

*Call* $\mathcal{F}(x) := \int_0^x f(t)\,\mathrm{d}t$. *Then,* $\mathcal{F}(x) = f(x) - 1$ *for every* $x \in \mathbb{K}$.

Note that $\mathbb{K}$ is not assumed to be an ordered field. I call property 3 translation invariance of $\int$.

*Proof.* Call $e := f(1)$.
$$\int_0^x f(t+1)\,\mathrm{d}t = \int_0^x f(1)f(t)\,\mathrm{d}t = e\mathcal{F}(x).$$

On the other hand, by translation invariance of $\int$,
$$\begin{aligned}\int_0^x f(t+1)\,\mathrm{d}t = \int_1^{x+1} f(t)\,\mathrm{d}t &= \mathcal{F}(x+1) - \mathcal{F}(1) = \mathcal{F}(x+1) - e + 1\\ &= \mathcal{F}(x) + \int_x^{x+1} f(t)\,\mathrm{d}t - e + 1 = \mathcal{F}(x) + f(x)(e-1) - e + 1\end{aligned}$$

Therefore,
$$e\mathcal{F}(x) = \mathcal{F}(x) + f(x)(e-1) - e + 1,$$

so
$$\mathcal{F}(x)(e-1) = (f(x)-1)(e-1).$$
□

If we apply the previous result to $\mathbb{K} = \mathbf{No}$, $f = \exp$ and $\int$ the integral on $\mathbf{No}$, we see that if $\int$ is translation invariant, then $\mathcal{F}(x) := \int_0^x \exp t\,\mathrm{d}t = \exp x - 1$. But this is not true: we shall see that $\mathcal{F}(\omega) = \exp(\omega)$. Therefore, $\int$ is *not* translation invariant.

### 5.2.2 Integral of exp

The exponential function exp has been defined in [10]. It is the inverse of the logarithm log. I recalled its recursive definition in section 1.5.

Let $\mathfrak{A}$ be the field of rational functions on **No**.

By lemma 5.1, $1/x$ is recursive over polynomials functions. However, the definition of rational functions is not recursive. For instance, one can check that the options of $1/x^2$ are at least as much complicated as the function $1/x^2$ itself.(1) This means that the partial order on the field of rational functions induced by the relationship "$f$ is an option of $g$" is not well-founded, implying that this field is *not* an inductively constructed family. However, even in this case, the definition of $\int$ makes sense, not as a definition, but as a requirement: we are imposing that $\int_0^x f$ is the simplest element in a certain convex set.

The integral on $\mathfrak{A}$ given by the formal integral(2) satisfies the previous requirement.

The exponential function exp is (uniformly) recursive over $\mathfrak{A}$, therefore I can define $\mathcal{F}(x) := \int_0^x \exp t \, dt$, and see if $\mathcal{F}(x) = \exp x - 1$. It is easy to see that if $x \in \mathbb{R}$, $\mathcal{F}(x) = \exp x - 1$. The function $\exp x$ is analytic, therefore the answer is affirmative for $x$ bounded.

On the other hand, we will prove(3) that $\mathcal{F}(\omega) = \exp \omega$.

*Proof.* I recall that $\exp \omega = \omega^\omega$, the simplest surreal number greater than $\omega^n$ for every natural $n$.

For $n \in \mathbb{N}$, let

$$[x]_n := \sum_{i=0}^{n} \frac{x^i}{i!}$$

be the Taylor expansion of $\exp x$. $[x]_n$ is a polynomial, therefore we know how to compute its integral:

$$\int_0^x [t]_n \, dt = [x]_{n+1} - 1.$$

Moreover, $[x]_n$ is a left option of $\exp x$, therefore $\mathcal{F}(x) > [x]_{n+1} - 1$, in particular $\mathcal{F}(\omega) > \omega^n$ for all $n \in \mathbb{N}$, so $\omega^\omega \preceq \mathcal{F}(\omega)$. To obtain the conclusion, it is enough to check that every right option of $\mathcal{F}(\omega)$ is greater than $\omega^\omega$.

---

(1)This does not mean that someone else might not find a true recursive definition for rational functions. Only that the definition we can extract from lemma 5.1 is not recursive.

(2)The formal integral of a rational function is a combination of rational functions, log and arctan. Moreover, arctan is definable in $\mathscr{L}_{an}$. Therefore, it is defined on **No**.

(3)Thanks to prof. O. Costin for having pointed this out.

*Claim* 1. Let $n \in \mathbb{N}$, $a < b \in \mathbf{No}$ such that $a$ is finite and $b$ is infinite. Then,

$$\int_a^b \frac{\exp b}{[b-t]_n}\,\mathrm{d}t - (\exp b - \exp a)$$

is positive infinite.

It is enough to prove that

$$\int_a^b \frac{1}{[b-t]_n}\,\mathrm{d}t - 1 + \exp(a-b)$$

is positive non infinitesimal. By hypothesis, the integral is a function in $a, b$ definable in $\mathscr{L}_{an}(\exp)$. Therefore, I can make the change of variable $t' = b - t$. Call $c := b - a$; $c$ is infinite, so $\exp(-c)$ is infinitesimal, and the claim becomes

$$v\Big(\int_0^c \frac{1}{[t]_n}\,\mathrm{d}t - 1\Big) \geq 0.$$

Moreover, $\mathbb{R}$ is an elementary $\mathscr{L}_{an}(\exp)$-substructure of $\mathbf{No}$, therefore it is enough to prove in $\mathbb{R}$ that

$$\int_0^{+\infty} \frac{1}{[t]_n}\,\mathrm{d}t > 1.$$

But for $t > 0$, $1/[t]_n > \exp(-t)$, so

$$\int_0^{+\infty} \frac{1}{[t]_n}\,\mathrm{d}t > \int_0^{+\infty} \exp(-t)\,\mathrm{d}t = 1.$$

The only right options of $\mathcal{F}(\omega)$ are of the form

$$z^R := \mathcal{F}(q) + \int_n^{\omega} (\exp t)_P^R\,\mathrm{d}t,$$

for some $q \in \mathbb{N}$ and $P = (k_0, \dots, k_m)$ partition of $(q, \omega)$. Let $\bar{\imath} < m$ be such that $k_{\bar{\imath}}$ is finite, while $k_{\bar{\imath}+1}$ is infinite. Call $a = k_{\bar{\imath}}$, $b = k_{\bar{\imath}+1}$. The right option $(\exp t)^R$ of $\exp t$ in the interval $(a, b)$ are of the form either $\exp b/[b-t]_n$ or $\exp a/[a-t]_{2n+1}$, with $[a-t]_{2n+1}$ positive. But the second case cannot happen, because $[a-t]_{2n+1} < 0$ for any infinite $t \in (a, b)$, therefore we have only right options of the first kind. Let

$$\Delta := \int_a^b \frac{\exp b}{[b-t]_{n+1}}\,\mathrm{d}t - (\exp b - \exp a).$$

By the claim, $\Delta$ is positive infinite. Therefore, for every $r \in \mathbb{R}$,

$$z^R > (\exp q - 1) + (r + \exp b - \exp a) + \sum_{i \neq \bar{\imath}} \int_{k_i}^{k_{i+1}} \exp_i^R(t)\,\mathrm{d}t \geq$$
$$\geq r + \exp q - 1 + \sum_i (\exp k_{i+1} - \exp k_i) = r + \exp \omega - 1.$$

This proves that every right option $z^R$ is greater than $\omega^\omega$. □

A similar phenomenon happens at every infinite power of $\omega$.

If $a > 0 \in \mathbf{No}$, then $\mathcal{F}(\omega^a) = \exp(\omega^a)$. Loosely speaking, 1 is too small w.r.t. $\omega^a$, and the approximation with which exp has been defined is not good enough to detect it. More precisely, $\Delta(\mathcal{F}(\omega^a))$ is infinite.

### 5.2.3 Recursive definition of exp

Let $\mathfrak{A}$ be the family of functions definable over $\mathbf{No}$ (with parameters) in the language $\mathscr{L}_{an}$. Obviously, exp is recursive over $\mathfrak{A}$, and $f(x) := 1 + \exp x$ is recursive over $\mathfrak{A} \cup \{\exp\}$. But $f(x)$ is not recursive over $\mathfrak{A}$. I will sketch the proof of this fact.

Instead of $\mathfrak{A}$, consider $\mathfrak{A}'$, the family of functions definable in $\mathscr{L}_{an}$ without using any parameter. It is easy enough to define recursively over $\mathfrak{A}'$ a function $g(x)$ which coincides with $f(x)$ for $x$ bounded. Take any such recursive $g$. Suppose, for contradiction, that $f = g$ and consider $a := g(\omega)$. Let $T$ be the type of $f(\omega) = \omega^\omega + 1$ over $\mathbb{R}$ in the language $\mathscr{L}_{an}$. If $x \in \mathbb{R}$, $g(x) = 1 + \exp x \in \mathbb{R}$. Therefore, $a$ would be equal to the simplest surreal number in $T$. But the simplest element in $T$ is $\omega^\omega$, so $g(\omega) \neq f(\omega)$.

For the general case, in the recursive definition of $f$ only a set $\mathfrak{A}''$ of elements from $\mathfrak{A}$ can be involved, i.e. only a set $S$ of parameters from $\mathbf{No}$ can be involved. Let $\mathbb{K}$ be an initial elementary $\mathscr{L}_{an}$-substructure of $\mathbf{No}$ containing $S$ and which is set. Let $g$ be any function recursive over $\mathfrak{A}''$, such that $g(x) = f(x)$ for every $x \in \mathbb{K}$. Let $c := \langle \mathbb{K} \,|\, \rangle$ the simplest surreal number greater than $\mathbb{K}$. As before, $g(c) \neq f(c)$.

### 5.2.4 Other exponential functions

Let $c = \sum_{i<\alpha} r_i \omega^{a_i}$. Then, $c$ is *purely infinite* iff $a_i > 0$ for all $i$. Theorem 5 gives the value of $\exp c$ for $c$ purely infinite. For $c$ finite, we can use $e(x)$ the power series expansion of $\exp x$ to compute $\exp c$. In general, every $c \in \mathbf{No}$ can be expressed uniquely as a sum $c = c' + c''$, with $c'$ purely infinite and $c''$ finite. Therefore, $\exp c = (\exp c')e(c'')$.

Consider now the function $2^x := \exp(x \log 2)$. If $x = x' + x''$ is the decomposition of $x$ into purely infinite and finite part, define $k(x) := (\exp x')2^{x''}$. Finally, define $f(x) := k(x/\log 2)$.

It is easy to see that $f$ satisfies the following properties:

- $f(x+y) = f(x)f(y)$.
- $f(x) = e(x)$ for $x$ finite.
- $f : \mathbf{No} \to \mathbf{No}^{>0}$ is surjective.
- $f(x) > x^n$ for $x$ large enough.

Therefore, $\mathbf{No}_{an}(f)$ is elementary equivalent to $\mathbf{No}_{an}(\exp)$, where $\mathbf{No}_{an}$ is the canonical $\mathscr{L}_{an}$-structure on $\mathbf{No}$. However, $f \neq \exp$.

Of course, all the previous construction can be done taking any real number instead of 2 to define $f$, obtaining a whole family of different "exponential functions".

### 5.2.5 An alternative definition of integral

Define

$$\left(\fint_a^b f(t)\,\mathrm{d}t\right)^{\mathbf{o}} = \fint_a^b f^{\mathbf{o}}(t,t^{\mathbf{o}},f(t^{\mathbf{o}}))\,\mathrm{d}t.$$

With this definition, the integral is linear, monotone, invariant under translations and satisfies the integration by part formula. However, it is not additive, i.e. it does not satisfy

$$\fint_a^b f + \fint_b^c f = \fint_a^c f.$$

Note that every option of $\fint$ is an option also of $\int$, therefore this integral is worse than the one I have used until now.

# Conclusion

The central themes of this thesis are the functions on the Surreal Numbers uniformly recursive (over some family of functions) and their integral.

In chapter 4, we saw that being recursive has some non-trivial consequences, such as the sup property.[4]

The guiding idea is to define the integral for functions definable in $T_{an}(\exp)$, in such a way that the resulting structure on **No** is an elementary extension of the corresponding structure on the reals.

The definition given here of integral for such functions, mimicking the Riemann integral for real functions, is satisfactory for polynomials and restricted analytic functions. Moreover, it has many of the properties we expect, as shown in chapters 2–4. I recall the ones I deem the most important: monotonicity, additivity, linearity and integration by parts.

However, one of the fundamental properties of the integral, namely the translation invariance, is missing in general, and this gives problems when we try to integrate the exponential.

Finally, here are some of the open problems and unsolved questions.

- A "natural" definition of integral that gives the right answer for the exponential function; it is the main open problem of this thesis.

- The "implicit function theorem" for recursive functions, already mentioned in question 4.23. Given a recursive function $f(x,y)$ such that $\forall x\ \exists y\ f(x,y)=0$, is it possible to find a function $h(x)$, recursive over a suitable family, such that $f(x,h(x)) = 0$ for all $x$? For instance, the function $\sqrt{x}$ is recursive over the rational functions;[5] I would like a general theorem ensuring that functions obtained as solution of certain equations are recursive.

- A simpler form of the axioms in the second chapter. While axiom 1 seems a natural choice, the other axioms are less convincing.

(4) See definition 4.15.

(5) See example 4.4

- Given a family of functions $\mathfrak{A}$, which kind of functions $f$ can be recursively defined over it? The results in chapter 4 give some properties for $f$, while §5.2.3 and the concatenation function produce some counterexamples. Can we give some more conditions on recursive functions?

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