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ALGEBRAIC POINCARÉ COBORDISM

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Introduction

The object of this paper is to give a reasonably leisurely account of the algebraic Poincaré cobordism theory of Ranicki [15], [16] and the further development due to Weiss [19], along with some of the applications to manifolds and vector bundles. It is a companion paper to Ranicki [17], which is an introduction to algebraic surgery using forms and formations.

Algebraic Poincaré cobordism is modelled on the bordism groups $\Omega_*(X)$ of maps $f : M \to X$ from manifolds to a space X. The Wall [18] surgery obstruction groups $L_*(A)$ of a ring with involution A were expressed in [15] as the cobordism groups of A-module chain complexes C with a quadratic Poincaré duality

$$\psi \ : \ H^{n-*}(C) \ \cong \ H_*(C) \ ,$$

and the surgery obstruction $\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(X)])$ of an n-dimensional normal map $(f, b) : M \to X$ was expressed as the cobordism class (C, ψ) of an n-dimensional f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex C such that

$$H_*(C) \ = \ \ker(f_* : H_*(\widetilde{M}) \to H_*(\widetilde{X}))$$

together with an n-dimensional quadratic Poincaré duality ψ. The passage from the bundle map $b : \nu_M \to \nu_X$ to ψ used an equivariant chain level version of the relationship established by Thom between the Wu classes of the normal bundle ν_M of a manifold M and the action of the Steenrod algebra on the Thom class of ν_M.

A chain bundle (C, γ) over a ring with involution A is an A-module chain complex C together with a Tate $\mathbb{Z}_2$-hypercohomology class $\gamma \in \widehat{H}^0(\mathbb{Z}_2; C^* \otimes C^*)$. The L-groups $L^*(C, \gamma)$ of [19] are the cobordism groups of symmetric Poincaré complexes over A with a chain bundle map to (C, γ), which are related to the quadratic L-groups by an exact sequence of abelian groups

$$\dots \to L_n(A) \to L^n(C, \gamma) \to Q_n(C, \gamma) \to L_{n-1}(A) \to \dots$$

with the Q-groups $Q_*(C, \gamma)$ defined purely homologically. The surgery obstruction groups $L_*(A)$ of [18] and the symmetric L-groups $L^*(A)$ of Mishchenko [13] are

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particular examples of the generalized L-groups $L^*(C,\gamma)$. The main novelty of this paper is an explicit formula obtained in §7 for the addition of elements in $Q_*(C,\gamma)$. The Wu classes $v_*(\nu)\in H^*(X;\mathbb{Z}_2)$ of a $(k-1)$-spherical fibration ν over a space X (e.g. the sphere bundle of a k-plane bundle) determine a chain bundle $(C(\widetilde{X}),\gamma(\nu))$ over $\mathbb{Z}[\pi_1(X)]$, with $\widetilde{X}$ the universal cover of X, and with a morphism

$$\pi_{n+k}(T(\nu))\to Q_n(C(\widetilde{X}),\gamma)\ .$$

For a k-plane bundle ν the morphism factors through the flexible signature map of [19]

$$\Omega_n(X,\nu)\ =\ \pi_{n+k}(T(\nu))\to L^n(C(\widetilde{X}),\gamma(\nu))\ .$$

with $\Omega_n(X,\nu)$ the bordism group of normal maps $(f:M\to X, b:\nu_M\to\nu)$ from n-dimensional manifolds.

In subsequent joint work with Frank Connolly a computation of $Q_*(C,\gamma)$ will be used to compute the Cappell Unil-groups in certain special cases.

The titles of the sections are

§1. **Rings with involution**

In §1 we show how an involution $a\mapsto\overline{a}$ on a ring A determines a duality involution functor

$$\text{(f.g. projective left } A\text{-modules)}\to\text{(f.g. projective left } A\text{-modules)}\ .$$

More generally, duality can be defined using an antistructure on A in the sense of Wall [18], and the L-theory results described in this paper all have versions for rings with antistructure.

Let A be an associative ring with 1, together with an involution

$$\bar{\ }\ :\ A\to A\ ;\ a\mapsto\overline{a}\ ,$$

that is a function satisfying

$$\overline{a+b}\ =\ \overline{a}+\overline{b}\ ,\ \overline{\overline{a}}\ =\ a\ ,\ \overline{ab}\ =\ \overline{b}.\overline{a}\ ,\ \overline{1}\ =\ 1\in A\ (a,b\in A)\ .$$

In the topological applications $A = \mathbb{Z}[\pi]$ is a group ring, for some group π equipped with a morphism $w : \pi \to \mathbb{Z}_2 = \{+1, -1\}$, and the involution is defined by

$$\bar{\ } : A \to A \ ; \ \sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g w(g) g^{-1} \ .$$

We take A-modules to be left A-modules, unless a right A-action is expressly specified. Given an A-module M there is defined a right A-module M^t with the same additive group and

$$M^t \times A \to M^t \ ; \ (x, a) \mapsto \overline{a} x \ .$$

The *dual* of an A-module M is the A-module with additive group

$$M^* \ = \ \mathrm{Hom}_A(M, A)$$

and A acting by

$$A \times M^* \to M^* \ ; \ (a, f) \mapsto (x \mapsto f(x).\overline{a}) \ .$$

The *dual* of an A-module morphism $f \in \mathrm{Hom}_A(M, N)$ is the A-module morphism

$$f^* \ : \ N^* \to M^* \ ; \ g \mapsto (x \mapsto g(f(x))) \ .$$

For a f.g. (finitely generated) projective A-module M the natural A-module isomorphism

$$M \to M^{**} \ ; \ x \mapsto (f \mapsto \overline{f(x)})$$

will be used to identify

$$M^{**} \ = \ M \ .$$

§2. **Chain complexes**

In order to adequately deal with the quadratic nature of the function $C \to C^t \otimes_A C$ sending an A-module chain complex C to the $\mathbb{Z}$-module chain complex $C^t \otimes_A C$ it is necessary to use the equivalence of Dold [7] and Kan [8] (cf. May [10, §22]) between positive $\mathbb{Z}$-module chain complexes and simplicial $\mathbb{Z}$-modules. This is now recalled, along with some other properties of chain complexes that we shall require.

An A-module chain complex

$$C \ : \ \dots \to C_{r+1} \xrightarrow{d} C_r \xrightarrow{d} C_{r-1} \to \dots \ (r \in \mathbb{Z})$$

is n-*dimensional* if each C_r $(0 \leq r \leq n)$ is a f.g. projective A-module and $C_r = 0$ for $r < 0$ and $r > n$. By an abuse of terminology chain complexes of the chain homotopy type of an n-dimensional chain complex will also be called n-dimensional.

The *suspension* of an A-module chain complex C is the A-module chain complex defined by

$$d_{SC} \ = \ d_C \ : \ SC_r \ = \ C_{r-1} \to SC_{r-1} \ = \ C_{r-2} \ .$$

If C is n-dimensional then SC is $(n+1)$-dimensional.

Given an A-module chain complex C let

$$C^r \ = \ (C_r)^* \ (r \in \mathbb{Z}) \ .$$

The *dual A-module* chain complex C^{-*} is defined by

$$d_{C^{-*}} \;=\; (d_C)^* \;:\; (C^{-*})_r \;=\; C^{-r} \to (C^{-*})_{r-1} \;=\; C^{-r+1}\ .$$

The *n-dual A-module chain complex* C^{n-*} is defined by

$$d_{C^{n-*}} \;=\; (-1)^r (d_C)^* \;:\; (C^{n-*})_r \;=\; C^{n-r} \to (C^{n-*})_{r-1} \;=\; C^{n-r+1}\ .$$

The n-fold suspension of the dual S^nC^{-*} is related to the n-dual C^{n-*} by the isomorphism

$$S^nC^{-*} \to C^{n-*}\ ;\ x \mapsto (-1)^{r(r-1)/2}x\ (x \in C^{n-r})\ .$$

In particular, C^{0-*} is isomorphic to (but not identical to) C^{-*}.

A *chain map up to sign* between A-module chain complexes

$$f \;:\; C \to D$$

is a collection of A-module morphisms

$$\{f \in \mathrm{Hom}_A(C_r, D_r) \,|\, r \in \mathbb{Z}\}$$

such that

$$d_D f \;=\; \pm f d_C \;:\; C_r \to D_{r-1}\ (r \in \mathbb{Z})\ .$$

If the sign is always + this is just a *chain map* $f : C \to D$, as usual.

Given A-module chain complexes C, D let $C^t \otimes_A D$, $\mathrm{Hom}_A(C, D)$ be the $\mathbb{Z}$-module chain complexes defined by

$$(C^t \otimes_A D)_n \;=\; \sum_{p+q=n} C_p \otimes_A D_q\ ,$$

$$d_{C^t\otimes_A D}(x \otimes y) \;=\; x \otimes d_D(y) + (-1)^q d_C(x) \otimes y\ ,$$

$$\mathrm{Hom}_A(C, D)_n \;=\; \sum_{q-p=n} \mathrm{Hom}_A(C_p, D_q)\ ,$$

$$d_{\mathrm{Hom}_A(C,D)}(f)(x) \;=\; d_D(f(x)) + (-1)^q f(d_C(x))\ .$$

A cycle $f \in \mathrm{Hom}_A(C, D)_n$ is a chain map up to sign $f : S^nC \to D$, and

$$H_n(\mathrm{Hom}_A(C, D)) \;=\; H_0(\mathrm{Hom}_A(S^nC, D))$$

is the $\mathbb{Z}$-module of chain homotopy classes of chain maps $S^nC \to D$.

For finite-dimensional C the *slant isomorphism* of $\mathbb{Z}$-module chain complexes

$$C^t \otimes_A D \to \mathrm{Hom}_A(C^{-*}, D)\ ;\ x \otimes y \mapsto (f \mapsto \overline{f(x)}\,.\,y)$$

will be used to identify

$$C^t \otimes_A D \;=\; \mathrm{Hom}_A(C^{-*}, D)\ .$$

A cycle

$$f \in (C^t \otimes_A D)_n \;=\; \mathrm{Hom}_A(C^{-*}, D)_n$$

is a chain map $f : C^{n-*} \to D$. Thus

$$H_n(C^t \otimes_A D) \;=\; H_n(\mathrm{Hom}_A(C^{-*}, D)) \;=\; H_0(\mathrm{Hom}_A(C^{n-*}, D))$$

is the $\mathbb{Z}$-module of chain homotopy classes of chain maps $C^{n-*} \to D$.

The *algebraic mapping cone* $C(f)$ of an A-module chain map $f : C \to D$ is the A-module chain complex defined as usual by

$$d_{C(f)} = \begin{pmatrix} d_D & (-1)^{r-1}f \\ 0 & d_C \end{pmatrix} : C(f)_r = D_r \oplus C_{r-1} \to C(f)_{r-1} = D_{r-1} \oplus C_{r-2} .$$

The relative homology A-modules

$$H_n(f) = H_n(C(f))$$

are such that there is defined an exact sequence

$$\dots \to H_n(C) \xrightarrow{f_*} H_n(D) \to H_n(f) \to H_{n-1}(C) \to \dots .$$

Let $C(\Delta^n)$ denote the cellular chain complex of the standard n-simplex Δ^n with the standard cell structure consisting of $\begin{pmatrix} n+1 \\ r+1 \end{pmatrix}$ r-cells $(0 \le r \le n)$.

Given a $\mathbb{Z}$-module chain complex C let $K(C)$ denote the simplicial $\mathbb{Z}$-module defined by the Dold-Kan construction, with one n-simplex for each chain map $C(\Delta^n) \to C$ and the evident face and degeneracy maps d_i, s_i, such that

$$\pi_n(K(C)) = H_n(C) \ (n \ge 0) .$$

Given a chain $y \in C_n$ and cycles $x_i \in \ker(d : C_{n-1} \to C_{n-2})$ $(0 \le i \le n)$ such that

$$dy = \sum_{i=0}^{n} (-1)^i x_i \in C_{n-1}$$

let $(y; x_0, \dots, x_n)$ denote the n-simplex of $K(C)$ defined by the chain map

$$f : C(\Delta^n) \to C$$

with

$$\begin{aligned} f &: C(\Delta^n)_n = \mathbb{Z} \to C_n \ ; \ 1 \mapsto y \ , \\ d_i f &: C(\Delta^{n-1})_{n-1} = \mathbb{Z} \to C_{n-1} \ ; \ 1 \mapsto x_i \ (0 \le i \le n) \ . \end{aligned}$$

The chain $x \in C_n$ is identified with the n-simplex $(x; dx, 0, \dots, 0) \in K(C)^{(n)}$.

Given $\mathbb{Z}$-module chain complexes C, D and a simplicial map

$$f : K(C) \to K(D)$$

(which need not preserve the $\mathbb{Z}$-module structure) there is defined a function

$$f : C_n \to D_n \ ; \ x \mapsto f(x) = f(x; dx, 0, \dots, 0)$$

such that

$$df(x) = f(dx) \in D_{n-1} \ (x \in C_n) \ .$$

In general, $f : C_n \to D_n$ is only additive on the level of homology, with

$$f(x + x') - f(x) - f(x') = d[x, x']_f \in D_n \ (x, x' \in \ker(d : C_n \to C_{n-1})$$

where the function

$$[\ ,\ ]_f : \ker(d : C_n \to C_{n-1}) \times \ker(d : C_n \to C_{n-1}) \to D_{n+1}$$

is defined by

$$[x, x']_f = f(0; x+x', x, -x', 0, \dots, 0) - f(0; x, x, 0, \dots, 0) - f(0; x', 0, -x', 0, \dots, 0) .$$

The induced functions

$$f_* \ : \ H_n(C) \to H_n(D) \ ; \ x \mapsto f(x)$$

are thus morphisms of abelian groups, which fit into an exact sequence

$$\dots \to H_{n+1}(f) \to H_n(C) \xrightarrow{f_*} H_n(D) \to H_n(f) \to H_{n-1}(C) \to \dots \ ,$$

with the relative group $H_n(f)$ $(= \pi_n(f))$ the set of equivalence classes of pairs $(x,y) \in C_n \times D_{n+1}$ such that

$$dx \ = \ 0 \in C_{n-1} \ , \ f(x) \ = \ dy \in D_n \ ,$$

subject to the equivalence relation

$$\begin{aligned}&(x,y) \sim (x',y') \text{ if there exist } (u,v) \in C_{n+1} \times D_{n+2} \text{ such that}\\ &\qquad x - x' \ = \ du \in C_n \ , \ y - y' \ = \ f(u;x,x',0,\dots,0) + dv \in D_{n+1} \ ,\end{aligned}$$

and addition by

$$(x,y) + (x',y') \ = \ (x+x', y+y'+[x,x']_f) \in H_n(f) \ .$$

If $f : K(C) \to K(D)$ does preserve the $\mathbb{Z}$-module structure (so that $[\ ,\ ]_f = 0$) then f is essentially just a chain map $f : C \to D$, and the relative homology groups $H_*(f)$ are just the homology groups $H_*(C(f))$ of the algebraic mapping cone $C(f)$, as usual.

§3. **Symmetric, quadratic and hyperquadratic structures**

An n-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\\ \text{hyperquadratic}\end{cases}$ structure on an A-module chain complex C is a cycle representing an element of the $\begin{cases}\mathbb{Z}_2\text{-hypercohomology}\\ \mathbb{Z}_2\text{-hyperhomology}\\ \text{Tate } \mathbb{Z}_2\text{-hypercohomology}\end{cases}$ group $\begin{cases}H^n(\mathbb{Z}_2; C^t \otimes_A C)\\ H_n(\mathbb{Z}_2; C^t \otimes_A C)\\ \widehat{H}^n(\mathbb{Z}_2; C^t \otimes_A C)\end{cases}$ in the sense of Cartan and Eilenberg [6].

Given an A-module chain complex C let the generator $T \in \mathbb{Z}_2$ act on $C^t \otimes_A C$ by the *transposition involution*

$$T \ : \ C^t \otimes_A C \to C^t \otimes_A C \ ; \ x \otimes y \ \mapsto \ (-1)^{pq} y \otimes x \ (x \in C_p, y \in C_q) \ .$$

For finite-dimensional C use the slant isomorphism to identify

$$C^t \otimes_A C \ = \ \mathrm{Hom}_A(C^{-*}, C) \ .$$

Under this identification the transposition involution corresponds to the duality involution on $\mathrm{Hom}_A(C^{-*}, C)$

$$T \ : \ \mathrm{Hom}_A(C^{-*}, C) \to \mathrm{Hom}_A(C^{-*}, C) \ ; \ \phi \mapsto (-1)^{pq}\phi^* \ (\phi \in \mathrm{Hom}_A(C^p, C_q)) \ .$$

A cycle $\phi \in \mathrm{Hom}_A(C^{-*}, C)_n = (C^t \otimes_A C)_n$ is a chain map $\phi : C^{n-*} \to C$, and $H_n(\mathrm{Hom}_A(C^{-*}, C))$ is the $\mathbb{Z}$-module of chain homotopy classes of A-module chain maps $C^{n-*} \to C$. Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$-module resolution of $\mathbb{Z}$

$$W \ : \ \cdots \to \mathbb{Z}[\mathbb{Z}_2] \overset{1-T}{\to} \mathbb{Z}[\mathbb{Z}_2] \overset{1+T}{\to} \mathbb{Z}[\mathbb{Z}_2] \overset{1-T}{\to} \mathbb{Z}[\mathbb{Z}_2] \to 0$$

and let $\widehat{W}$ be the complete resolution

$$\widehat{W} \ : \ \cdots \to \mathbb{Z}[\mathbb{Z}_2] \overset{1-T}{\to} \mathbb{Z}[\mathbb{Z}_2] \overset{1+T}{\to} \mathbb{Z}[\mathbb{Z}_2] \overset{1-T}{\to} \mathbb{Z}[\mathbb{Z}_2] \to \dots \ .$$

The $\begin{cases} \mathbb{Z}_2\text{-}\textit{hypercohomology} \\ \mathbb{Z}_2\text{-}\textit{hyperhomology} \\ \textit{Tate } \mathbb{Z}_2\text{-}\textit{hypercohomology} \end{cases}$ *groups* of a $\mathbb{Z}[\mathbb{Z}_2]$-module chain complex C are defined by

$$\begin{cases} H^n(\mathbb{Z}_2; C) \ = \ H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C)) \\ H_n(\mathbb{Z}_2; C) \ = \ H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C) \\ \widehat{H}^n(\mathbb{Z}_2; C) \ = \ H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C)) \ . \end{cases}$$

The evident short exact sequence of $\mathbb{Z}[\mathbb{Z}_2]$-module chain complexes

$$0 \to SW^{-*} \to \widehat{W} \to W \to 0$$

induces a long exact sequence of abelian groups

$$\cdots \to H_n(\mathbb{Z}_2; C) \overset{1+T}{\to} H^n(\mathbb{Z}_2; C) \overset{J}{\to} \widehat{H}^n(\mathbb{Z}_2; C) \overset{H}{\to} H_{n-1}(\mathbb{Z}_2; C) \to \dots \ .$$

An element $\begin{cases} \phi \in H^n(\mathbb{Z}_2; C) \\ \psi \in H_n(\mathbb{Z}_2; C) \\ \theta \in \widehat{H}^n(\mathbb{Z}_2; C) \end{cases}$ is represented by an n-cycle of $\begin{cases} \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C) \\ W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} C \\ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C) \end{cases}$

which is just a collection of chains of C $\begin{cases} \{\phi_s \in C_{n+s} \mid s \geq 0\} \\ \{\psi_s \in C_{n-s} \mid s \geq 0\} \\ \{\theta_s \in C_{n+s} \mid s \in \mathbb{Z}\} \end{cases}$ such that

$$\begin{cases} d_C(\phi_s) + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) \ = \ 0 \in C_{n+s-1} \ (s \geq 0, \phi_{-1} = 0) \\ d_C(\psi_s) + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1} T\psi_{s+1}) \ = \ 0 \in C_{n-s-1} \ (s \geq 0) \\ d_C(\theta_s) + (-1)^{n+s-1}(\theta_{s-1} + (-1)^s T\theta_{s-1}) \ = \ 0 \in C_{n+s-1} \ (s \in \mathbb{Z}) \end{cases}$$

with

$$1 + T \ : \ H_n(\mathbb{Z}_2; C) \to H^n(\mathbb{Z}_2; C) \ ;$$

$$\psi \ = \ \{\psi_s \mid s \geq 0\} \mapsto \{(1+T)\psi_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases} \} \ ,$$

$$J \ : \ H^n(\mathbb{Z}_2; C) \to \widehat{H}^n(\mathbb{Z}_2; C) \ ;$$

$$\phi \ = \ \{\phi_s \mid s \geq 0\} \mapsto \{J\phi_s = \begin{cases} \phi_s & \text{if } s \geq 0 \\ 0 & \text{if } s \leq -1 \end{cases} \} \ ,$$

$$H \ : \ \widehat{H}^n(\mathbb{Z}_2; C) \to H_{n-1}(\mathbb{Z}_2; C) \ ;$$

$$\theta \ = \ \{\theta_s \mid s \in \mathbb{Z}\} \mapsto H\theta \ = \ \{H\theta_s = \theta_{-s-1} \mid s \geq 0\} \ .$$

Given an A-module chain complex C use the action of $T \in \mathbb{Z}_2$ on $C^t \otimes_A C$ by the transposition involution to define the $\mathbb{Z}$-module chain complex

$$\begin{aligned} W^{\%}C &= \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C^t \otimes_A C) \\ W_{\%}C &= W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C^t \otimes_A C) \\ \widehat{W}^{\%}C &= \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C^t \otimes_A C) \ . \end{aligned}$$

We shall be mainly concerned with finite-dimensional C, using the slant isomorphism to identify

$$C^t \otimes_A C \ = \ \mathrm{Hom}_A(C^{-*}, C)$$

and

$$\begin{aligned} W^{\%}C &= \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathrm{Hom}_A(C^{-*}, C)) \\ W_{\%}C &= W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_A(C^{-*}, C) \\ \widehat{W}^{\%}C &= \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, \mathrm{Hom}_A(C^{-*}, C)) \ . \end{aligned}$$

An n-*dimensional* $\begin{cases} \textit{symmetric} \\ \textit{quadratic} \\ \textit{hyperquadratic} \end{cases}$ *structure* on a finite-dimensional A-module chain complex C is a cycle $\begin{cases} \phi \in (W^{\%}C)_n \\ \psi \in (W_{\%}C)_n \\ \theta \in (\widehat{W}^{\%}C)_n \ , \end{cases}$ which is just a collection of A-module morphisms

$$\begin{cases} \{\phi_s \in \mathrm{Hom}_A(C^{n-r+s}, C_r) \,|\, r \in \mathbb{Z}, s \geq 0\} \\ \{\psi_s \in \mathrm{Hom}_A(C^{n-r-s}, C_r) \,|\, r \in \mathbb{Z}, s \geq 0\} \\ \{\theta_s \in \mathrm{Hom}_A(C^{n-r+s}, C_r) \,|\, r \in \mathbb{Z}, s \in \mathbb{Z}\} \end{cases}$$

such that

$$\begin{cases} d\phi_s + (-1)^r \phi_s d^* + (-1)^{n+s-1}(\phi_{s-1} + (-1)^s T\phi_{s-1}) \ = \ 0 \\ \qquad\qquad : \ C^{n-r+s-1} \to C_r \ (s \geq 0, \phi_{-1} = 0) \\ d\psi_s + (-1)^r \psi_s d^* + (-1)^{n-s-1}(\psi_{s+1} + (-1)^{s+1} T\psi_{s+1}) \ = \ 0 \\ \qquad\qquad : \ C^{n-r-s-1} \to C_r \ (s \geq 0) \\ d\theta_s + (-1)^r \theta_s d^* + (-1)^{n+s-1}(\theta_{s-1} + (-1)^s T\theta_{s-1}) \ = \ 0 \\ \qquad\qquad : \ C^{n-r+s-1} \to C_r \ (s \in \mathbb{Z}) \ . \end{cases}$$

An *equivalence* $\begin{cases} \xi : \phi \to \phi' \\ \chi : \psi \to \psi' \\ \nu : \theta \to \theta' \end{cases}$ of n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$ structures on C is a chain $\begin{cases} \xi \in (W^{\%}C)_{n+1} \\ \chi \in (W_{\%}C)_{n+1} \\ \nu \in (\widehat{W}^{\%}C)_{n+1} \end{cases}$ such that

$$\begin{cases} \phi' - \phi \ = \ d(\xi) \in (W^{\%}C)_n \\ \psi' - \psi \ = \ d(\chi) \in (W_{\%}C)_n \\ \theta' - \theta \ = \ d(\nu) \in (\widehat{W}^{\%}C)_n \ . \end{cases}$$

The n-*dimensional* $\begin{cases} \textit{symmetric} \\ \textit{quadratic} \\ \textit{hyperquadratic} \end{cases}$ *structure group* $\begin{cases} Q^n(C) \\ Q_n(C) \\ \widehat{Q}^n(C) \end{cases}$ of a chain complex C is the abelian group of equivalence classes of n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$ structures on C, that is

$$\begin{cases} Q^n(C) \ = \ H^n(\mathbb{Z}_2; C^t \otimes_A C) \ = \ H_n(W^{\%}C) \\ Q_n(C) \ = \ H_n(\mathbb{Z}_2; C^t \otimes_A C) \ = \ H_n(W_{\%}C) \\ \widehat{Q}^n(C) \ = \ \widehat{H}^n(\mathbb{Z}_2; C^t \otimes_A C) \ = \ H_n(\widehat{W}^{\%}C) \ . \end{cases}$$

The Q-groups are related by a long exact sequence

$$\dots \to Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \to \dots$$

involving the morphisms induced in homology by the $\mathbb{Z}$-module chain maps

$$1+T \ : \ W_{\%}C \to W^{\%}C \ , \ J \ : \ W^{\%}C \to \widehat{W}^{\%}C \ , \ H \ : \ \widehat{W}^{\%}C \to S(W_{\%}C)$$

defined by

$$\begin{array}{l} 1+T \ : \ (W_{\%}C)_n \to (W^{\%}C)_n \ ; \\ \qquad \{\psi_s \in (C^t \otimes_A C)_{n-s} \,|\, s \geq 0\} \mapsto \{((1+T)\psi)_s = \begin{cases} (1+T)\psi_0 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases} \} \ , \\ J \ : \ (W^{\%}C)_n \to (\widehat{W}^{\%}C)_n \ ; \\ \qquad \{\phi_s \in (C^t \otimes_A C)_{n+s} \,|\, s \geq 0\} \mapsto \{(J\phi)_s = \begin{cases} \phi_s & \text{if } s \geq 0 \\ 0 & \text{if } s \leq -1 \end{cases} \} \ , \\ H \ : \ (\widehat{W}^{\%}C)_n \to (W_{\%}C)_{n-1} \ ; \\ \qquad \{\theta_s \in (C^t \otimes_A C)_{n+s} \,|\, s \in \mathbb{Z}\} \ \mapsto \ \{(H\theta)_s = \theta_{-s-1} \,|\, s \geq 0\} \ . \end{array}$$

An n-dimensional symmetric structure $\phi \in (W^{\%}C)_n$ is equivalent to the symmetrization $(1+T)\psi$ of an n-dimensional quadratic structure $\psi \in (W_{\%}C)_n$ if and only if the n-dimensional hyperquadratic structure $J(\phi) \in (\widehat{W}^{\%}C)_n$ is equivalent to 0. An A-module chain map $f : C \to D$ induces a $\mathbb{Z}[\mathbb{Z}_2]$-module chain map

$$f^t \otimes_A f \ : \ C^t \otimes_A C \to D^t \otimes_A D \ ; \ x \otimes y \mapsto f(x) \otimes f(y)$$

and hence $\mathbb{Z}$-module chain maps

$$\begin{array}{l} f^{\%} \ : \ W^{\%}C \to W^{\%}D \ , \\ f_{\%} \ : \ W_{\%}C \to W_{\%}D \ , \\ \widehat{f}^{\%} \ : \ \widehat{W}^{\%}C \to \widehat{W}^{\%}D \ . \end{array}$$

An A-module chain homotopy

$$g \ : \ f \simeq f' \ : \ C \to D$$

determines $\mathbb{Z}$-module chain homotopies

$$\begin{aligned}
(g;f,f')^{\%} &: f^{\%} \simeq f'^{\%} : W^{\%}C \to W^{\%}D \\
(g;f,f')_{\%} &: f_{\%} \simeq f'_{\%} : W_{\%}C \to W_{\%}D \\
\widehat{(g;f,f')}^{\%} &: \widehat{f}^{\%} \simeq \widehat{f}'^{\%} : \widehat{W}^{\%}C \to \widehat{W}^{\%}D
\end{aligned}$$

with

$$\begin{aligned}
(g;f,f')^{\%} \ : \ (W^{\%}C)_n \ &= \ \sum_{s=0}^{\infty}(C^t \otimes_A C)_{n+s} \\
&\to (W^{\%}D)_{n+1} \ = \ \sum_{s=0}^{\infty}\sum_q D^t_{n-q+s+1} \otimes_A D_q \ ; \\
\sum_{s=0}^{\infty}\phi_s \mapsto \sum_{s=0}^{\infty}&((f^t \otimes_A g + g^t \otimes_A f')(\phi_s) + (-1)^{q+s-1}(g^t \otimes_A g)(T\phi_{s-1}))
\end{aligned}$$

and similarly for $(g;f,f')_{\%}$, $\widehat{(g;f,f')}^{\%}$. Thus the induced morphisms in the Q-groups

$$\begin{aligned}
f^{\%} &: Q^n(C) \to Q^n(D) \\
f_{\%} &: Q_n(C) \to Q_n(D) \\
\widehat{f}^{\%} &: \widehat{Q}^n(C) \to \widehat{Q}^n(D)
\end{aligned}$$

depend only on the chain homotopy class of f, and are isomorphisms if f is a chain equivalence. For finite-dimensional C,D the slant isomorphisms are used to identify $f^t \otimes_A f : C^t \otimes_A C \to D^t \otimes_A D$ with

$$\mathrm{Hom}_A(f^*,f) \ : \ \mathrm{Hom}_A(C^{-*},C) \to \mathrm{Hom}_A(D^*,D) \ ; \ \theta \mapsto f\theta f^* \ ,$$

and similarly for $f^{\%}, f_{\%}, \widehat{f}^{\%}$ and $(g;f,f')^{\%}, (g;f,f')_{\%}, \widehat{(g;f,f')}^{\%}$.

Although all the Q-groups are chain homotopy invariant, only the hyperquadratic Q-groups $\widehat{Q}^*(C)$ are additive. The sum of A-module chain maps $f,g : C \to D$ is an A-module chain map $f+g : C \to D$ such that

$$\begin{aligned}
&(f+g)^{\%} - f^{\%} - g^{\%} \ : \ Q^n(C) \to H_n(C^t \otimes_A C) \xrightarrow{f^t \otimes_A g} H_n(D^t \otimes_A D) \to Q^n(D) \ , \\
&(f+g)_{\%} - f_{\%} - g_{\%} \ : \ Q_n(C) \to H_n(C^t \otimes_A C) \xrightarrow{f^t \otimes_A g} H_n(D^t \otimes_A D) \to Q_n(D) \ , \\
&\widehat{(f+g)}^{\%} - \widehat{f}^{\%} - \widehat{g}^{\%} \ = \ 0 \ : \ \widehat{Q}^n(C) \to \widehat{Q}^n(D)
\end{aligned}$$

with

$$\begin{aligned}
&Q^n(C) \to H_n(C^t \otimes_A C) \ ; \ \phi \ = \ \{\phi_s \,|\, s \geq 0\} \mapsto \phi_0 \ , \\
&Q_n(C) \to H_n(C^t \otimes_A C) \ ; \ \psi \ = \ \{\psi_s \,|\, s \geq 0\} \mapsto (1+T)\psi_0 \ , \\
&H_n(D^t \otimes_A D) \to Q^n(D) \ ; \ \theta \mapsto \{\phi_s = \begin{cases} (1+T)\theta & \text{if } s=0 \\ 0 & \text{if } s \geq 1 \end{cases}\} \ , \\
&H_n(D^t \otimes_A D) \to Q_n(D) \ ; \ \theta \mapsto \{\psi_s = \begin{cases} \theta & \text{if } s=0 \\ 0 & \text{if } s \geq 1 \end{cases}\} \ .
\end{aligned}$$

Given a finite-dimensional A-module chain complex C and $n \geq 0$ define the *n-fold suspension chain isomorphism*

$$\begin{aligned} S^n \ &: S^n(\widehat{W}^{\%}C) \to \widehat{W}^{\%}(S^nC) \ ; \\ &\theta \ = \ \{\theta_s \in \mathrm{Hom}_A(C^r, C_{m-r+s}) \,|\, s \in \mathbb{Z}\} \\ &\mapsto \ S^n\theta \ = \ \{(S^n\theta)_s = \theta_{s-n} \in \mathrm{Hom}_A(C^r, C_{m-n+r+s}) \,|\, s \in \mathbb{Z}\} \ . \end{aligned}$$

For any (finite-dimensional) A-module chain complexes C, D there is defined a simplicial map

$$I \ : \ K(\mathrm{Hom}_A(C,D)) \to K(\mathrm{Hom}_{\mathbb{Z}}(\widehat{W}^{\%}C, \widehat{W}^{\%}D))$$

sending a cycle $f \in \mathrm{Hom}_A(C,D)_n$ ($=$ a chain map up to sign $f : S^nC \to D$) to the $\mathbb{Z}$-module chain map up to sign

$$I(f) \ = \ \widehat{f}^{\%}S^n \ : \ S^n(\widehat{W}^{\%}C) \xrightarrow{S^n} \widehat{W}^{\%}S^nC \xrightarrow{\widehat{f}^{\%}} W^{\%}D \ .$$

An n-simplex $(g; f, f', 0, \dots, 0) \in K(\mathrm{Hom}_A(C,D))^{(n)}$ ($=$ an A-module chain homotopy up to sign $g : f \simeq f' : S^nC \to D$) is sent to the $\mathbb{Z}$-module chain homotopy up to sign

$$I(g; f, f') \ = \ (g; f, f')^{\%}S^n \ : \ I(f) \ \simeq \ I(f') \ : \ S^n(\widehat{W}^{\%}C) \to \widehat{W}^{\%}D \ .$$

The failure of I to be linear on chain maps up to sign $f : S^nC \to D$ is given by the chain homotopy up to sign

$$[f, f'] \ : \ (\widehat{f+f'})^{\%} \ \simeq \ \widehat{f}^{\%} + \widehat{f}'^{\%} \ : \ \widehat{W}^{\%}(S^nC) \to \widehat{W}^{\%}D$$

defined by

$$\begin{aligned} &[f, f'] \ : \ (S^n\widehat{W}^{\%}C)_m \to (\widehat{W}^{\%}D)_{m+n+1} \ ; \\ &\theta \ = \ \{\theta_s \in \mathrm{Hom}_A(C^r, C_{m-r+s}) \,|\, s \in \mathbb{Z}\} \\ &\mapsto [f, f']\theta \ = \ \{T^{n+1}f\theta_{s-n+1}f'^* \in \mathrm{Hom}_A(D^r, D_{m-n+r+s+1}) \,|\, s \in \mathbb{Z}\} \ . \end{aligned}$$

§4. Algebraic Wu classes

The algebraic Wu classes are the fundamental invariants of a duality structure on a chain complex C, which are obtained by an algebraic analogue of the Steenrod squares in the cohomology groups of a topological space. In the topological applications the algebraic Wu classes are closely related to the topological Wu classes, as explained in Ranicki [15].

Let S^rA ($r \in \mathbb{Z}$) denote the A-module chain complex

$$S^rA \ : \ \dots \to 0 \to A \to 0 \to \dots$$

concentrated in degree r. For any A-module chain complex C there are defined natural isomorphisms

$$H_0(\mathrm{Hom}_A(C, S^rA)) \to H^r(C) \ ; \ (f : C_r \to A) \mapsto f^*(1) \ .$$

An element $f \in H^r(C)$ is just a chain homotopy class of chain maps $f : C \to S^rA$. The Wu classes of a quadratic structure on C are the invariants of the equivalence class defined by sending an element $f \in H^r(C)$ to the induced equivalence class of quadratic structures on S^rA. The quadratic structure groups of the elementary complexes S^rA are identified with subquotients of the ground ring A.

An A-*group* M is an abelian group together with an A-action

$$A\times M\to M\ ;\ (a,x)\to ax$$

such that

$$a(x+y)\ =\ ax+ay\ ,\ a(bx)\ =\ (ab)x\ ,\ 1x\ =\ x\ (x,y\in M,a,b\in M)\ .$$

An A-module is an A-group M such that also

$$(a+b)x\ =\ ax+bx\in M\ .$$

An A-*morphism* of A-groups is a morphism of abelian groups

$$f\ :\ M\to N$$

such that

$$f(ax)\ =\ af(x)\in N\ (x\in M,a\in A)\ .$$

The set of A-group morphisms $f:M\to N$ defines an abelian group $\mathrm{Hom}_A(M,N)$, with addition by

$$(f+g)(x)\ =\ f(x)+g(x)\in N\ .$$

For A-modules M,N the A-morphisms $f:M\to N$ coincide with A-module morphisms.

For $\epsilon=\pm1$ let the generator $T\in\mathbb{Z}_2$ act on A by the ϵ-involution

$$T_\epsilon\ :\ A\to A\ ;\ a\mapsto\epsilon\overline{a}\ .$$

Define the $\begin{cases}\mathbb{Z}_2\text{-}\mathit{cohomology}\\ \mathbb{Z}_2\text{-}\mathit{homology}\\ \mathit{Tate}\ \mathbb{Z}_2\text{-}\mathit{cohomology}\end{cases}$ A-*groups* $\begin{cases}H^*(\mathbb{Z}_2;A,\epsilon)\\ H_*(\mathbb{Z}_2;A,\epsilon)\\ \widehat{H}^*(\mathbb{Z}_2;A,\epsilon)\end{cases}$ by

$$\begin{cases}H^r(\mathbb{Z}_2;A,\epsilon)\ =\ \begin{cases}\ker(1-T_\epsilon:A\to A) & \text{if } r=0\\ \widehat{H}^r(\mathbb{Z}_2;A,\epsilon) & \text{if } r\geq 1\\ 0 & \text{if } r<0\end{cases}\\ H_r(\mathbb{Z}_2;A,\epsilon)\ =\ \begin{cases}\mathrm{coker}(1-T_\epsilon:A\to A) & \text{if } r=0\\ \widehat{H}^{r+1}(\mathbb{Z}_2;A,\epsilon) & \text{if } r\geq 1\\ 0 & \text{if } r<0\end{cases}\\ \widehat{H}^r(\mathbb{Z}_2;A,\epsilon)\ =\ \ker(1-(-1)^rT_\epsilon:A\to A)/\mathrm{im}(1+(-1)^rT_\epsilon:A\to A)\ (r\in\mathbb{Z})\ .\end{cases}$$

The A-action

$$A\times\widehat{H}^r(\mathbb{Z}_2;A,\epsilon)\to\widehat{H}^r(\mathbb{Z}_2;A,\epsilon)\ ;\ (a,x)\mapsto ax\overline{a}$$

defines an A-module structure on $\widehat{H}^r(\mathbb{Z}_2;A,\epsilon)$. The A-actions

$$A\times H^0(\mathbb{Z}_2;A,\epsilon)\to H^0(\mathbb{Z}_2;A,\epsilon)\ ;\ (a,x)\mapsto ax\overline{a}$$
$$A\times H_0(\mathbb{Z}_2;A,\epsilon)\to H_0(\mathbb{Z}_2;A,\epsilon)\ ;\ (a,x)\mapsto ax\overline{a}$$

are not linear in A, and so do not define A-module structures.

For $\epsilon = +1$ the groups $\begin{cases} H^*(\mathbb{Z}_2; A, \epsilon) \\ H_*(\mathbb{Z}_2; A, \epsilon) \\ \widehat{H}^*(\mathbb{Z}TheA, \epsilon) \end{cases}$ are denoted by $\begin{cases} H^*(\mathbb{Z}_2; A) \\ H_*(\mathbb{Z}_2; A) \\ \widehat{H}^*(\mathbb{Z}_2; A). \end{cases}$

The natural $\mathbb{Z}$-module isomorphisms

$$\begin{aligned} &Q^n(S^rA) \to H^{2r-n}(\mathbb{Z}_2; A, (-1)^r) \ ; \ \phi \mapsto \phi_{2r-n}(1)(1) \\ &Q_n(S^rA) \to H_{n-2r}(\mathbb{Z}_2; A, (-1)^r) \ ; \ \psi \mapsto \psi_{n-2r}(1)(1) \\ &\widehat{Q}^n(S^rA) \to \widehat{H}^{r-n}(\mathbb{Z}_2; A, (-1)^r) \ ; \ \theta \mapsto \theta_{2r-n}(1)(1) \end{aligned}$$

will be used as identifications.

The *Wu classes* of a $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{hyperquadratic} \end{cases}$ structure $\begin{cases} \phi \in (W^{\%}C)_n \\ \psi \in (W_{\%}C)_n \\ \theta \in (\widehat{W}^{\%}C)_n \end{cases}$ are the invariants of the equivalence class of structures defined by the A-morphisms

$$\begin{cases} v_r(\phi) \ : \ H^{n-r}(C) \ = \ H_0(\mathrm{Hom}_A(C, S^{n-r}A)) \\ \qquad \to Q^n(S^{n-r}A) \ = \ H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) \ ; \ f \mapsto (f \otimes f)(\phi_{n-2r}) \\ v^r(\psi) \ : \ H^{n-r}(C) \ = \ H_0(\mathrm{Hom}_A(C, S^{n-r}A)) \\ \qquad \to Q_n(S^{n-r}A) \ = \ H_{2r-n}(\mathbb{Z}_2; A, (-1)^{n-r}) \ ; \ f \mapsto (f \otimes f)(\psi_{2r-n}) \\ \widehat{v}_r(\theta) \ : \ H^{n-r}(C) \ = \ H_0(\mathrm{Hom}_A(C, S^{n-r}A)) \\ \qquad \to \widehat{Q}^n(S^{n-r}A) \ = \ \widehat{H}^r(\mathbb{Z}_2; A) \ ; \ f \mapsto (f \otimes f)(\theta_{n-2r}) \ . \end{cases}$$

§5. **Algebraic Poincaré complexes**

An algebraic Poincaré complex is a chain complex with Poincaré duality, such as arises from a compact n-manifold or a normal map.

An *n-dimensional* $\begin{cases} \textit{symmetric} \\ \textit{quadratic} \end{cases}$ *(Poincaré) complex over A* $\begin{cases} (C, \phi) \\ (C, \psi) \end{cases}$ is an n-dimensional A-module chain complex C together with an n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ structure $\begin{cases} \phi \in (W^{\%}C) \\ \psi \in (W_{\%}C)_n \end{cases}$ (such that $\begin{cases} \phi_0 \\ (1+T)\psi_0 \end{cases} : C^{n-*} \to C$ is a chain equivalence).

An $(n+1)$*-dimensional* $\begin{cases} \textit{symmetric} \\ \textit{quadratic} \\ \textit{(symmetric, quadratic)} \end{cases}$ *(Poincaré) pair over A*

$$\left(f \ : \ C \to D \ , \begin{cases} (\delta\phi, \phi) \\ (\delta\psi, \psi) \\ (\delta\phi, \psi) \end{cases} \right)$$

consists of an n-dimensional A-module chain complex C, an $(n+1)$-dimensional A-module chain complex D, a chain map $f : C \to D$ and a cycle

$$\begin{cases} (\delta\phi, \phi) \in C(f^{\%} : W^{\%}C \to W^{\%}D)_{n+1} \ = \ (W^{\%}D)_{n+1} \oplus (W^{\%}C)_n \\ (\delta\psi, \psi) \in C(f_{\%} : W_{\%}C \to W_{\%}D)_{n+1} \ = \ (W_{\%}D)_{n+1} \oplus (W_{\%}C)_n \\ (\delta\phi, \psi) \in C((1+T)f_{\%} : W_{\%}C \to W^{\%}D)_{n+1} \ = \ (W^{\%}D)_{n+1} \oplus (W_{\%}C)_n \end{cases}$$

(such that the A-module chain map $D^{n+1-*} \to C(f)$ defined by

$$\begin{cases} (\delta\phi,\phi)_0 \;=\; \begin{pmatrix} \delta\phi_0 \\ \phi_0 f^* \end{pmatrix} \\ (1+T)(\delta\psi,\psi)_0 \;=\; \begin{pmatrix} (1+T)\delta\psi_0 \\ (1+T)\psi_0 f^* \end{pmatrix} \;:\; D^{n+1-r} \to C(f)_r \;=\; D_r \oplus C_{r-1} \\ (\delta\phi,(1+T)\psi)_0 \;=\; \begin{pmatrix} \delta\phi_0 \\ (1+T)\psi_0 f^* \end{pmatrix} \end{cases}$$

is a chain equivalence). The *boundary* of the pair is the n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \\ \text{quadratic} \end{cases}$ (Poincaré) complex $\begin{cases} (C,\phi) \\ (C,\psi) \\ (C,\psi). \end{cases}$

A *homotopy equivalence* of n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complexes

$$\begin{cases} (f,\chi) \;:\; (C,\phi) \to (C',\phi') \\ (f,\xi) \;:\; (C,\psi) \to (C',\psi') \end{cases}$$

is a chain equivalence $f : C \to C'$ together with an equivalence of $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ structures on C' $\begin{cases} \chi \;:\; f^{\%}(\phi) \to \phi' \\ \xi \;:\; f_{\%}(\psi) \to \psi' \;. \end{cases}$ There is a similar notion of homotopy equivalence for pairs.

An n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complex $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$ is *connected* if

$$\begin{cases} H_0(\phi_0 : C^{n-*} \to C) \;=\; 0 \\ H_0((1+T)\psi_0 : C^{n-*} \to C) \;=\; 0 \;. \end{cases}$$

It was shown in Ranicki [15] that there is a natural one-one correspondence between the homotopy equivalence classes of connected n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complexes over A and the homotopy equivalence classes of n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré pairs over A. A connected n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complex $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$ determines the n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré pair

$$(i_C : \partial C \to C^{n-*}, \begin{cases} (0,\partial\phi) \\ (0,\partial\psi) \end{cases})$$

defined by

$$i_C \;=\; (0 \;\; 1) \;:\; \partial C_r \;=\; C_{r+1} \oplus C^{n-r} \to C^{n-r}\ ,$$
$$d_{\partial C} \;=\; \begin{cases} \begin{pmatrix} d_C & (-1)^r\phi_0 \\ 0 & (-1)^r d_C^* \end{pmatrix} \;: \\ \begin{pmatrix} d_C & (-1)^r(1+T)\psi_0 \\ 0 & (-1)^r d_C^* \end{pmatrix} \;: \end{cases}$$
$$\partial C_r \;=\; C_{r+1} \oplus C^{n-r} \to \partial C_{r-1} \;=\; C_r \oplus C^{n-r+1}\ ,$$
$$\begin{cases} \partial\phi_0 \;=\; \begin{pmatrix} (-1)^{n-r-1}T\phi_1 & (-1)^{r(n-r-1)} \\ 1 & 0 \end{pmatrix} \;: \\ \partial\psi_0 \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;: \end{cases}$$
$$\partial C^{n-r-1} \;=\; C^{n-r} \oplus C_{r+1} \to \partial C_r \;=\; C_{r+1} \oplus C^{n-r}\ ,$$
$$\begin{cases} \partial\phi_s \;=\; \begin{pmatrix} (-1)^{n-r+s-1}T\phi_{s+1} & 0 \\ 0 & 0 \end{pmatrix} \;: \\ \qquad \partial C^{n-r+s-1} \;=\; C^{n-r+s} \oplus C_{r-s+1} \to \partial C_r \;=\; C_{r+1} \oplus C^{n-r}\ (s \geq 1)\ , \\ \partial\psi_s \;=\; \begin{pmatrix} (-1)^{n-r-s}T\psi_{s-1} & 0 \\ 0 & 0 \end{pmatrix} \;: \\ \qquad \partial C^{n-r-s-1} \;=\; C^{n-r-s} \oplus C_{r+s+1} \to \partial C_r \;=\; C_{r+1} \oplus C^{n-r}\ (s \geq 1)\ . \end{cases}$$

The $(n-1)$-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré complex

$$\begin{cases} \partial(C,\phi) \;=\; (\partial C, \partial\phi) \\ \partial(C,\psi) \;=\; (\partial C, \partial\psi) \end{cases}$$

is the *boundary* of the connected n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complex $\begin{cases} (C,\phi) \\ (C,\psi). \end{cases}$ The connected complex $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$ is a Poincaré complex if and only if the boundary $\begin{cases} \partial(C,\phi) \\ \partial(C,\psi) \end{cases}$ is contractible (= homotopy equivalent to 0). A Poincaré complex $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$ is the boundary of an $(n+1)$-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré pair $\begin{cases} (f : C \to D, (\delta\phi, \phi)) \\ (f : C \to D, (\delta\psi, \psi)) \end{cases}$ if and only if it is homotopy equivalent to the boundary of a connected $(n+1)$-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complex.

The n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré complexes $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$, $\begin{cases} (C',\phi') \\ (C',\psi') \end{cases}$ are *cobordant* if $\begin{cases} (C,\phi) \oplus (C', -\phi') \\ (C,\psi) \oplus (C', -\psi') \end{cases}$ is the boundary of an $(n+1)$-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré pair $\begin{cases} ((f\ f') : C \oplus C' \to D, (\delta\phi, \phi \oplus -\phi')) \\ ((f\ f') : C \oplus C' \to D, (\delta\psi, \psi \oplus -\psi'))\ . \end{cases}$ Homotopy equivalent Poincaré complexes are cobordant.

The $\begin{cases} \textit{symmetric} \\ \textit{quadratic} \end{cases}$ *L-groups* $\begin{cases} L^n(A) \\ L_n(A) \end{cases}$ $(n \geq 0)$ are the cobordism groups of n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré complexes over A. The quadratic L-groups $L_*(A)$ are 4-periodic, with isomorphisms

$$L_n(A) \to L_{n+4}(A) \ ; \ (C,\psi) \mapsto (S^2C,\psi) \ (n \geq 0) \ ,$$

and are just the surgery obstruction groups of Wall [18]. The symmetric L-groups $L^*(A)$ were introduced by Mishchenko [13]. The corresponding maps in the symmetric L-groups

$$L^n(A) \to L^{n+4}(A) \ ; \ (C,\phi) \mapsto (S^2C,\phi) \ (n \geq 0)$$

are not isomorphisms in general. The symmetric and quadratic L-groups are related by an exact sequence

$$\dots \to L_n(A) \overset{1+T}{\to} L^n(A) \to \widehat{L}^n(A) \to L_{n-1}(A) \to \dots$$

with

$$1+T \ : \ L_n(A) \to L^n(A) \ ; \ (C,\psi) \mapsto (C,(1+T)\psi)$$

and $\widehat{L}^n(A)$ the relative cobordism group of n-dimensional (symmetric, quadratic) Poincaré pairs over A. The relative L-groups $\widehat{L}^*(A)$ are 8-torsion, so that the symmetrization maps $1+T : L_n(A) \to L^n(A)$ are isomorphisms modulo 8-torsion. If $\widehat{H}^*(\mathbb{Z}_2;A) = 0$ (e.g. if $1/2 \in A$) then $\widehat{L}^*(A) = 0$ and the symmetrization maps are isomorphisms.

The *symmetric construction* of Ranicki [15] is the natural chain map

$$\phi_X \ = \ 1 \otimes \Delta \ : \ C(X) \to W^{\%}C(\widetilde{X}) \ = \ \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} C(\widetilde{X}))$$

induced by an Alexander-Whitney-Steenrod diagonal chain approximation Δ, for any space X and any regular cover $\widetilde{X}$, with π the group of covering translations. For $\widetilde{X} = X$ the mod 2 reduction of the composite

$$H_n(X) \xrightarrow{\phi_X} Q^n(C(X)) \xrightarrow{v_r} \mathrm{Hom}_{\mathbb{Z}}(H^{n-r}(X), Q^n(S^{n-r}\mathbb{Z}))$$

is given by the rth Steenrod square

$$v_r(\phi_X(x))(y) \ = \ \langle Sq^r(y), x\rangle \in \mathbb{Z}_2 \ .$$

The symmetric signature of Mishchenko [13] is defined for any n-dimensional geometric Poincaré complex X to be the symmetric Poincaré cobordism class

$$\sigma^*(X) \ = \ (C(\widetilde{X}), \phi_X([X])) \in L^n(\mathbb{Z}[\pi_1(X)]) \ .$$

The symmetric L-groups of $\mathbb{Z}$ are given by

$$L^n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} \text{ (signature)} & \text{if } n \equiv 0 (\mathrm{mod}\, 4) \\ \mathbb{Z}_2 \text{ (deRham invariant)} & \text{if } n \equiv 1 (\mathrm{mod}\, 4) \\ 0 & \text{if } n \equiv 2 (\mathrm{mod}\, 4) \\ 0 & \text{if } n \equiv 3 (\mathrm{mod}\, 4) \ . \end{cases}$$

The *quadratic construction* of [15] associates to any stable π-equivariant map $F : \Sigma^\infty \widetilde{X}_+ \to \Sigma^\infty \widetilde{Y}_+$ a natural chain map

$$\psi_F \ : \ C(X) \to W_\% C(\widetilde{Y}) \ = \ W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C(\widetilde{Y}) \otimes_{\mathbb{Z}[\pi]} C(\widetilde{Y}))$$

such that

$$(1+T)\psi_F \ = \ F^\% \phi_X - \phi_Y F_* \ : \ C(X) \to W^\% C(\widetilde{Y})$$

with $\widetilde{X}$ a regular cover of X with group of covering translations π, $\widetilde{X}_+ = \widetilde{X} \cup \{\text{pt.}\}$, and similarly for Y. For $\widetilde{X} = X$, $\widetilde{Y} = Y$, $\pi = \{1\}$ the mod 2 reduction of the composite

$$H_n(X) \xrightarrow{\psi_F} Q_n(C(Y)) \xrightarrow{v^r} \mathrm{Hom}_{\mathbb{Z}}(H^{n-r}(Y), Q_n(S^{n-r}\mathbb{Z}))$$

is given by the $(r+1)$th functional Steenrod square

$$v^r(\psi_F(x))(y) \ = \ \langle Sq^{r+1}_{(\Sigma^\infty y)F}(\Sigma^\infty \iota), \Sigma^\infty x \rangle \in \mathbb{Z}_2$$

with $\iota \in H^{n-r}(K(\mathbb{Z}_2, n-r); \mathbb{Z}_2) = \mathbb{Z}_2$ the generator.

The Wall [18] surgery obstruction of an n-dimensional normal map $(f, b) : M \to X$ was expressed in [15] as the quadratic Poincaré cobordism class

$$\sigma_*(f, b) \ = \ (C(f^!), e_\% \psi_F([X])) \in L_n(\mathbb{Z}[\pi_1(X)])$$

with $F : \Sigma^\infty \widetilde{X}_+ \to \Sigma^\infty \widetilde{M}_+$ a $\pi_1(X)$-equivariant S-dual of $T(\widetilde{b}) : T(\nu_{\widetilde{M}}) \to T(\nu_{\widetilde{X}})$ inducing the Umkehr chain map

$$f^! \ : \ C(\widetilde{X}) \ \simeq \ C(\widetilde{X})^{n-*} \xrightarrow{f^*} C(\widetilde{M})^{n-*} \ \simeq \ C(\widetilde{M})$$

and $e : C(\widetilde{M}) \to C(f^!)$ the inclusion in the algebraic mapping cone. The quadratic L-groups of $\mathbb{Z}$ are given by

$$L_n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} \ (\text{signature}/8) & \text{if } n \equiv 0 (\bmod 4) \\ 0 & \text{if } n \equiv 1 (\bmod 4) \\ \mathbb{Z}_2 \ (\text{Arf invariant}) & \text{if } n \equiv 2 (\bmod 4) \\ 0 & \text{if } n \equiv 3 (\bmod 4) \ . \end{cases}$$

§6. **Chain bundles**

A *bundle* over a finite-dimensional A-module chain complex C is a 0-dimensional hyperquadratic structure on C^{0-*}, that is a cycle

$$\gamma \in (\widehat{W}^\% C^{0-*})_0 \ ,$$

as defined by a collection of A-module morphisms

$$\{\gamma_s \in \mathrm{Hom}_A(C_{r-s}, C^{-r}) \,|\, r, s \in \mathbb{Z}\}$$

such that

$$(-1)^{r+1} d_C^* \gamma_s + (-1)^s \gamma_s d_C + (-1)^{s-1}(\gamma_{s-1} + (-1)^s T\gamma_{s-1}) \ = \ 0 \ : \ C_{r-s+1} \to C^{-r} \ .$$

An *equivalence* of bundles over C

$$\chi \ : \ \gamma \to \gamma'$$

is an equivalence of hyperquadratic structures, as defined by a collection of A-module morphisms

$$\{\chi_s \in \mathrm{Hom}_A(C_{r-s-1}, C^{-r}) \,|\, r, s \in \mathbb{Z}\}$$

such that

$$\gamma'_s - \gamma_s \;=\; (-1)^{r+1} d_C^* \chi_s + (-1)^s \chi_s d_C + (-1)^s (\chi_{s-1} + (-1)^s T\chi_{s-1}) \;:\; C_{r-s} \to C^{-r} \;.$$

Thus

$$\widehat{Q}^0(C^{0-*}) \;=\; H_0(\widehat{W}^{\%} C^{0-*})$$

is the abelian group of equivalence classes of bundles over C.

A *chain bundle* over A (C, γ) is a finite-dimensional A-module chain complex C together with a bundle $\gamma \in (\widehat{W}^{\%} C^{0-*})_0$.

Given a chain bundle (C, γ) over A and an A-module chain map $f : B \to C$ define the *pullback* chain bundle $(B, f^*\gamma)$ using the image of γ under the $\mathbb{Z}$-module chain map

$$\widehat{f^*} \;:\; \widehat{W}^{\%} C^{0-*} \to \widehat{W}^{\%} B^{0-*}$$

induced by the dual A-module chain map $f^* : C^{0-*} \to B^{0-*}$. The equivalence class of the pullback bundle $f^*\gamma$ depends only on the chain homotopy class of the chain map f, by the chain homotopy invariance of the Q-groups.

A *map* of chain bundles over A

$$(f, \chi) \;:\; (C, \gamma) \to (C', \gamma')$$

is a chain map $f : C \to C'$ together with an equivalence of bundles over C

$$\chi \;:\; \gamma \to f^*\gamma' \;.$$

The *composite* of chain bundle maps

$$(f, \chi) \;:\; (C, \gamma) \to (C', \gamma') \;,\; (f', \chi') \;:\; (C', \gamma') \to (C'', \gamma'')$$

is the chain bundle map

$$(f', \chi')(f, \chi) \;=\; (f'f, \chi + \widehat{f^*}^{\%}(\chi')) \;:\; (C, \gamma) \to (C'', \gamma'') \;.$$

A *homotopy of chain bundle maps*

$$(g, \eta) \;:\; (f, \chi) \;\simeq\; (f', \chi') \;:\; (C, \gamma) \to (C', \gamma')$$

is a chain homotopy

$$g \;:\; f \;\simeq\; f' \;:\; C \to C'$$

together with an equivalence of 1-dimensional hyperquadratic structures on C^{0-*}

$$\eta \;:\; \chi - \chi' + (g^*; f^*, f'^*)^{\%}(\gamma') \to 0 \;.$$

A map of chain bundles $(f, \chi) : (C, \gamma) \to (C', \gamma')$ is an *equivalence* if there exists a homotopy inverse. This happens precisely when $f : C \to C'$ is a chain equivalence, in which case any chain homotopy inverse

$$f' \;=\; f^{-1} \;:\; C' \to C$$

can be used to define a homotopy inverse

$$(f', \chi') \;:\; (C', \gamma') \to (C, \gamma) \;.$$

Given a chain bundle (B,β) over A and a finite-dimensional A-module chain complex C use the pullback construction to define abelian group morphisms

$$I_\beta \;:\; H_n(B^t\otimes_A C)\to \widehat{Q}^n(C)\;;\; f\mapsto \widehat{f}^{\%}(S^n\beta)\ ,$$

using the slant isomorphism

$$B^t\otimes_A C\to \mathrm{Hom}_A(B^{-*},C)\;;\; x\otimes y\mapsto (f\mapsto \overline{f(x)}\,.\,y)$$

to identify a cycle $f\in (B^t\otimes_A C)_n$ with a chain map $f: B^{n-*}\to C$. Weiss [19] developed an algebraic analogue of the representation theorem of Brown [3] to obtain for any ring with involution A the existence of a directed system $\{(B(r),\beta(r))\,|\,r\geq 0\}$ of chain bundles over A and chain bundle maps

$$(B(r),\beta(r))\to (B(r+1),\beta(r+1))$$

such that the abelian group morphisms

$$\varinjlim_r I_{\beta_r} \;:\; \varinjlim_r H_n(B(r)^t\otimes_A C)\to \widehat{Q}^n(C)$$

are isomorphisms for any finite-dimensional A-module chain complex C. In general, the direct limit A-module chain complex

$$B(\infty) \;=\; \varinjlim_r B(r)$$

is not finite-dimensional. As in [19] we shall ignore this inconvenience and treat $B(\infty)$ as if it were finite-dimensional, so that there is defined the *universal chain bundle* over A

$$(B(\infty),\beta(\infty)) \;=\; \varinjlim_r (B(r),\beta(r))\ ,$$

with the universal property that for any finite-dimensional A-module chain complex C the abelian group morphisms

$$I_{\beta(\infty)} \;:\; H_n(B(\infty)^t\otimes_A C)\to \widehat{Q}^n(C)$$

are isomorphisms. In particular, there are defined isomorphisms

$$H_0(\mathrm{Hom}_A(C,B(\infty)))\to \widehat{Q}^0(C^{0-*})\;;\; f\mapsto f^*(\beta(\infty))$$

for any finite-dimensional C. Thus every chain bundle (C,γ) has a classifying map

$$(f,\chi) \;:\; (C,\gamma)\to (B(\infty),\beta(\infty))$$

and the equivalence classes of bundles $\gamma\in(\widehat{W}^{\%}C^{0-*})_0$ over C are in one-one correspondence with the chain homotopy classes of chain maps $f: C\to B(\infty)$.

The *Wu classes* of a chain bundle (C,γ) are the Wu classes of γ, the A-module morphisms

$$\widehat{v}_r(\gamma) \;:\; H_r(C)\to \widehat{H}^r(\mathbb{Z}_2;A)\;;\; x\mapsto \langle \gamma_{-2r},x\otimes x\rangle\ .$$

The universal chain bundle $(B(\infty),\beta(\infty))$ is characterized by the property that the Wu classes define A-module isomorphisms

$$\widehat{v}_r(\gamma) \;:\; H_r(C)\to \widehat{H}^r(\mathbb{Z}_2;A)\ (r\geq 0)\ .$$

For example, if $A=\mathbb{Z}$ the chain bundle $(B(\infty),\beta(\infty))$ defined by

$$d_{B(\infty)} \;=\; \begin{cases} 2 & \text{if } r \text{ is odd} \\ 0 & \text{if } r \text{ is even} \end{cases} \;:\; B(\infty)_r \;=\; \mathbb{Z} \to B(\infty)_{r-1} \;=\; \mathbb{Z}\ ,$$
$$\beta(\infty)_s \;=\; \begin{cases} 1 & \text{if } 2r=s \\ 0 & \text{otherwise} \end{cases} \;:\; B(\infty)_{r-s} \;=\; \mathbb{Z} \to B(\infty)^{-r} \;=\; \mathbb{Z}$$

is universal, with the Wu classes defining isomorphisms

$$\widehat{v}_r(\beta(\infty)) \;:\; H_r(B(\infty)) \to \widehat{H}^r(\mathbb{Z}_2;\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}_2 & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd}\ . \end{cases}$$

The symmetric and quadratic constructions of Ranicki [15] were extended in Weiss [19] and Ranicki [16, 2.5] : a spherical fibration $\nu : X \to BG(k)$ determines a chain bundle $(C(\widetilde{X}),\gamma)$ over $\mathbb{Z}[\pi_1(X)]$, and there is defined a natural transformation of exact sequences from the certain exact sequence of Whitehead [20]

$$\begin{tikzcd}
\cdots \arrow[r] & \Gamma_{n+k+1}(T(\nu)) \arrow[r] \arrow[d] & \pi_{n+k}(T(\nu)) \arrow[r, "h"] \arrow[d] & \dot{H}_{n+k}(T(\nu)) \arrow[r] \arrow[d, "\phi_X U"] & \Gamma_{n+k}(T(\nu)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \widehat{Q}^{n+1}(C(\widetilde{X})) \arrow[r] & Q_n(C(\widetilde{X}),\gamma) \arrow[r] & Q^n(C(\widetilde{X})) \arrow[r, "J_\gamma"] & \widehat{Q}^n(C(\widetilde{X})) \arrow[r] & \cdots
\end{tikzcd}$$

with h the Hurewicz map and $U : \dot{H}_{n+k}(T(\nu)) \to H_n(X)$ the Thom isomorphism. The topological Wu classes of ν are the algebraic Wu classes of the induced chain bundle $(C(X;\mathbb{Z}_2), 1\otimes\gamma)$ over $\mathbb{Z}_2$

$$v_*(\nu) \;=\; \widehat{v}_*(1\otimes\gamma) \in H^*(X;\mathbb{Z}_2) \;=\; \mathrm{Hom}_{\mathbb{Z}_2}(H_*(X;\mathbb{Z}_2),\mathbb{Z}_2)\ .$$

§7. **Normal complexes**

An n-dimensional normal space (X,ν_X,ρ_X) (Quinn [14]) is a finite n-dimensional CW complex X together with a $(k-1)$-spherical fibration $\nu_X : X \to BG(k)$ and a map $\rho_X : S^{n+k} \to T(\nu_X)$ to the Thom space of ν_X. An n-dimensional geometric Poincaré complex X has a unique equivalence class of normal structures (ν_X,ρ_X), with ν_X the Spivak normal fibration and ρ_X representing the fundamental class $[X] \in H_n(X)$. A normal complex is the algebraic analogue of a normal space, consisting of a symmetric complex with normal chain bundle.

A *normal structure* (γ,θ) on an n-dimensional symmetric complex (C,ϕ) is a bundle $\gamma \in (\widehat{W}^{\%}C^{0-*})_0$ together with an equivalence of n-dimensional hyperquadratic structures on C

$$\theta \;:\; J(\phi) \to (\widehat{\phi}_0)^{\%}(S^n\gamma)\ ,$$

as defined by a chain $\theta \in (\widehat{W}^{\%}C)_{n+1}$ such that

$$J(\phi) - (\widehat{\phi}_0)^{\%}(S^n\gamma) \;=\; d\theta \in (\widehat{W}^{\%}C)_n\ .$$

The Wu classes of ϕ and γ are then related by a commutative diagram

$$\begin{array}{ccc} H^{n-r}(C) & \xrightarrow{\phi_0} & H_r(C) \\ {\scriptstyle v_r(\phi)}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{v}_r(\gamma)} \\ H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) & \xrightarrow{J} & \widehat{H}^r(Z_2; A) \ . \end{array}$$

An *equivalence* of n-dimensional normal structures on (C, ϕ)

$$(\chi, \eta) \ : \ (\gamma, \theta) \to (\gamma', \theta')$$

is an equivalence of bundles $\chi : \gamma \to \gamma'$ together with an equivalence of $(n+1)$-dimensional hyperquadratic structures on C

$$\eta \ : \ \theta - \theta' + (\widehat{\phi}_0)^{\%}(S^n\chi) \to 0 \ .$$

An n-dimensional symmetric Poincaré complex $(C, \phi \in (W^{\%}C)_n)$ has a unique equivalence class of normal structures (γ, θ), with the equivalence class of bundles $[\gamma] \in \widehat{Q}^0(C^{0-*})$ the image of the equivalence class of symmetric structures $[\phi] \in Q^n(C)$ under the composite

$$Q^n(C) \overset{J}{\to} \widehat{Q}^n(C) \overset{((\phi_0)^{\%})^{-1}}{\to} \widehat{Q}^n(C^{n-*}) \overset{(S^n)^{-1}}{\to} \widehat{Q}^0(C^{0-*}) \ .$$

If (γ, θ), (γ', θ') are two such normal structures on (C, ϕ) there exists an equivalence of bundles $\chi : \gamma \to \gamma'$. As $\phi_0 : C^{n-*} \to C$ is a chain equivalence the cycle

$$\theta - \theta' + (\phi_0)^{\%}(S^n\chi) \in (\widehat{W}^{\%}C)_{n+1}$$

is such that there exist a cycle $\lambda \in \widehat{W}^{\%}C^{n-*})_{n+1}$ and a chain $\mu \in (\widehat{W}^{\%}C)_{n+2}$ such that

$$\theta - \theta' + (\phi_0)^{\%}(S^n\chi) \ = \ (\phi_0)^{\%}(\lambda) + d\mu \in (\widehat{W}^{\%}C)_{n+1} \ .$$

There is now defined an equivalence of normal structures on (C, ϕ)

$$(\chi - (S^n)^{-1}(\lambda), \mu) \ : \ (\gamma, \theta) \to (\gamma', \theta') \ .$$

An *n-dimensional normal* (*Poincaré*) *complex* over A $(C, \phi, \gamma, \theta)$ is an n-dimensional symmetric (Poincaré) complex (C, ϕ) together with a normal structure (γ, θ). Symmetric Poincaré complexes are regarded as normal Poincaré complexes by choosing a normal structure in the unique equivalence class.

An n-dimensional normal complex $(C, \phi, \gamma, \theta)$ is *connected* if the n-dimensional symmetric complex (C, ϕ) is connected, that is

$$H_0(\phi_0 : C^{n-*} \to C) \ = \ 0 \ .$$

The correspondence described in §5 between the homotopy equivalence classes of connected n-dimensional $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ complexes and those of n-dimensional $\left\{\begin{array}{l}\text{symmetric}\\ \text{quadratic}\end{array}\right.$ Poincaré pairs has the following generalization to connected normal complexes and (symmetric, quadratic) Poincaré pairs.

A connected n-dimensional normal complex (C,ϕ,γ,θ) determines the n-dimensional (symmetric, quadratic) Poincaré pair

$$(i_C : \partial C \to C^{n-*}, (\delta\phi,\psi))$$

defined by

$$\begin{aligned}
i_C \ &= \ (0\ 1) \ : \ \partial C_r \ = \ C_{r+1} \oplus C^{n-r} \to C^{n-r} \ , \\
d_{\partial C} \ &= \ \begin{pmatrix} d_C & (-1)^r\phi_0 \\ 0 & (-1)^r d_C^* \end{pmatrix} \ : \\
&\quad \partial C_r \ = \ C_{r+1} \oplus C^{n-r} \to \partial C_{r-1} \ = \ C_r \oplus C^{n-r+1} \ , \\
\psi_0 \ &= \ \begin{pmatrix} \chi_0 & 0 \\ 1+\gamma_{-n}\phi_0^* & \gamma_{-n-1}^* \end{pmatrix} \ : \\
&\quad \partial C^r \ = \ C^{r+1} \oplus C_{n-r} \to \partial C_{n-r-1} \ = \ C_{n-r} \oplus C^{r+1} \ , \\
\psi_s \ &= \ \begin{pmatrix} \chi_{-s} & 0 \\ \gamma_{-n-s}\phi_0^* & \gamma_{-n-s-1}^* \end{pmatrix} \ : \\
&\quad \partial C^r \ = \ C^{r+1} \oplus C_{n-r} \to \partial C_{n-r-s-1} \ = \ C_{n-r-s} \oplus C^{r+s+1} \ (s \geq 1) \ , \\
\delta\phi_s \ &= \ \gamma_{-n-s} \ : \ C_r \to C^{n-r+s} \ (s \geq 0) \ .
\end{aligned}$$

The $(n-1)$-dimensional quadratic Poincaré complex

$$\partial(C,\phi,\gamma,\theta) \ = \ (\partial C,\psi)$$

is the *quadratic boundary* of the connected n-dimensional normal complex (C,ϕ,γ,θ). (Compare with the definition in §6 of the boundary $(n-1)$-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ Poincaré complex $\begin{cases} \partial(C,\phi) \\ \partial(C,\psi) \end{cases}$ of a connected n-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ complex $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$). Conversely, given an n-dimensional (symmetric, quadratic) Poincaré pair $(f : C \to D, (\delta\phi,\psi))$ there is defined a connected n-dimensional normal complex $(C(f),\phi,\gamma,\theta)$ with the symmetric structure

$$\phi_s \ = \ \begin{cases} \begin{pmatrix} \delta\phi_0 & 0 \\ (1+T)\psi_0 f^* & 0 \end{pmatrix} & \text{if } s = 0 \\ \begin{pmatrix} \delta\phi_1 & 0 \\ 0 & (1+T)\psi_0 \end{pmatrix} & \text{if } s = 1 \\ \begin{pmatrix} \delta\phi_s & 0 \\ 0 & 0 \end{pmatrix} & \text{if } s \geq 2 \end{cases}$$
$$: \ C(f)^r \ = \ D^r \oplus C^{r-1} \to C(f)_{n-r+s} \ = \ D_{n-r+s} \oplus C_{n-r+s-1} \ .$$

The normal structure (γ,χ) is determined up to equivalence by the Poincaré duality, with $\gamma \in \widehat{Q}^0(D^{-*})$ the image of $(\delta\phi/(1+T)\psi) \in Q^n(C(f))$ under the composite

$$Q^n(C(f)) \xrightarrow{((\delta\phi_0,(1+T)\psi_0)^{\%})^{-1}} Q^n(D^{n-*}) \xrightarrow{J} \widehat{Q}^n(D^{n-*}) \xrightarrow{S^{-n}} \widehat{Q}^0(D^{-*}) \ .$$

The composite isomorphism

$$\widehat{Q}^0(C(f)^{0-*}) \xrightarrow{S^n} \widehat{Q}^n(C(f)^{n-*}) \xrightarrow{(\delta\phi_0,(1+T)\psi_0)^{\%}} \widehat{Q}^n(D)$$

sends the equivalence class $[\gamma] \in \widehat{Q}^0(C(f)^{0-*})$ to the element $\alpha \in \widehat{Q}^n(D)$ represented by

$$\alpha_s = \begin{cases} \delta\phi_s & \text{if } s \geq 0 \\ f\psi_{-s-1}f^* & \text{if } s \leq -1 \end{cases} : D^r \to D_{n-r+s} .$$

There is thus a natural one-one correspondence between the homotopy equivalence classes of connected n-dimensional normal complexes over A and the homotopy equivalence classes of n-dimensional (symmetric, quadratic) Poincaré pairs over A. In §8 below this correspondence will be used to identify the cobordism group $\widehat{L}^n(A)$ of n-dimensional (symmetric, quadratic) Poincaré pairs over A with the cobordism group of n-dimensional normal complexes over A.

Let (B,β) be a chain bundle over A. A *normal* (B,β)*-structure* (γ,θ,f,χ) on an n-dimensional symmetric complex (C,ϕ) over A is a normal structure (γ,θ) on (C,ϕ) together with a chain bundle map

$$(f,\chi) : (C,\gamma) \to (B,\beta) .$$

There are also the corresponding relative notions of *normal* (B,β)-structure on symmetric and (symmetric, quadratic) pairs. For the universal chain bundle $(B(\infty),\beta(\infty))$ over A a normal $(B(\infty),\beta(\infty))$-structure (γ,θ,f,χ) on a symmetric complex (C,ϕ) is to all intents and purposes the same as a normal structure (γ,θ).

A normal (0,0)-structure $(\gamma,\theta,0,\chi)$ on an n-dimensional symmetric complex (C,ϕ) determines an equivalence to 0 of the hyperquadratic structure $J(\phi) \in (\widehat{W}^{\%}C)_n$

$$\xi = \theta + \phi_0^{\%}(S^n\chi) : J(\phi) \to 0 .$$

Such an equivalence $\xi : J(\phi) \to 0$ consists of a quadratic structure $\psi \in (W_{\%}C)_n$ and an equivalence of symmetric structures

$$\eta : (1+T)\psi \to \phi ,$$

with

$$\psi_s = \xi_{-s-1} \in \mathrm{Hom}_A(C^*,C)_{n-s} \ (s \geq 0) ,$$
$$\eta_s = \xi_s \in \mathrm{Hom}_A(C^*,C)_{n+s+1} \ (s \geq 0) .$$

Thus a normal (0,0)-structure on a symmetric complex (C,ϕ) is to all intents and purposes an equivalence of the symmetric structure ϕ to $(1+T)\psi$ for some quadratic structure ψ on C.

An *n-dimensional* (B,β)*-normal* (*Poincaré*) *complex* $(C,\phi,\gamma,\theta,f,\chi)$ is an n-dimensional symmetric (Poincaré) complex (C,ϕ) together with a normal (B,β)-structure $(\gamma,\theta,\phi,\chi)$.

In §8 below the cobordism group $\widehat{L}\langle B,\beta\rangle^n(A)$ of n-dimensional (B,β)-normal complexes over A will be identified with the twisted quadratic group $Q_n(B,\beta)$ (introduced by Weiss [19]) of equivalence classes of pairs (ϕ,θ) such that $(B,\phi,\beta,\theta,1,0)$ is an n-dimensional (B,β)-normal complex.

An *n-dimensional symmetric structure* (ϕ,θ) on a chain bundle (C,γ) is an n-dimensional symmetric structure $\phi\in(W^{\%}C)_n$ together with an equivalence of n-dimensional hyperquadratic structures on C

$$\theta\ :\ J(\phi)\to(\phi_0)^{\%}(S^n\gamma)\ ,$$

as defined by a chain $\theta\in(\widehat{W}^{\%}C)_{n+1}$ such that

$$J(\phi)-(\phi_0)^{\%}(S^n\gamma)\ =\ d(\theta)\in(\widehat{W}^{\%}C)_n\ .$$

Thus (C,ϕ) is an n-dimensional symmetric complex with normal structure (γ,θ).

An *equivalence* of n-dimensional symmetric structures on (C,γ)

$$(\xi,\eta)\ :\ (\phi,\theta)\to(\phi',\theta')$$

is defined by an equivalence of symmetric structures $\xi:\phi\to\phi'$ together with an equivalence of hyperquadratic structures on C

$$\eta\ :\ \theta-\theta'+J(\xi)+(\xi_0;\phi_0,\phi'_0)^{\%}(S^n\gamma)\to 0\ ,$$

as defined by chains $\xi\in(W^{\%}C)_{n+1}$, $\eta\in(\widehat{W}^{\%}C)_{n+2}$ such that

$$\begin{aligned}&\phi'-\phi\ =\ d(\xi)\in(W^{\%}C)_nC^{-*}\\&\theta'-\theta+J(\xi)+(\xi_0;\phi_0,\phi'_0)^{\%}(S^n\gamma)\ =\ d(\eta)\in(\widehat{W}^{\%}C)_{n+1}\ .\end{aligned}$$

The *twisted quadratic Q-group* $Q_n(C,\gamma)$ is the abelian group of equivalence classes of n-dimensional symmetric structures on a chain bundle (C,γ), with addition by

$$(\phi,\theta)+(\phi',\theta')\ =\ (\phi+\phi',\theta+\theta'+[\phi_0,\phi'_0](S^n\gamma))\in Q_n(C,\gamma)\ .$$

The twisted quadratic Q-groups $Q_*(C,\gamma)$ fit into an exact sequence of abelian groups

$$\dots\to\widehat{Q}^{n+1}(C)\xrightarrow{H_\gamma}Q_n(C,\gamma)\xrightarrow{N_\gamma}Q^n(C)\xrightarrow{J_\gamma}\widehat{Q}^n(C)\to\dots$$

with the morphisms

$$\begin{aligned}&H_\gamma\ :\ \widehat{Q}^{n+1}(C)\to Q_n(C,\gamma)\ ;\ \theta\mapsto(0,\theta)\ ,\\&N_\gamma\ :\ Q_n(C,\gamma)\to Q^n(C)\ ;\ (\phi,\theta)\mapsto\phi\ ,\\&J_\gamma\ :\ Q^n(C)\to\widehat{Q}^n(C)\ ;\ \phi\mapsto J(\phi)-(\phi_0)^{\%}(S^n\gamma)\end{aligned}$$

induced by simplicial maps. In the untwisted case $\gamma=0$ there is defined an isomorphism of exact sequences

$$\begin{tikzcd}\cdots \arrow[r] & \widehat{Q}^{n+1}(C) \arrow[r, "H"] \arrow[d, equal] & Q_n(C) \arrow[r, "1+T"] \arrow[d, "\cong"] & Q^n(C) \arrow[r, "J"] \arrow[d, equal] & \widehat{Q}^n(C) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \widehat{Q}^{n+1}(C) \arrow[r, "H_0"] & Q_n(C,0) \arrow[r, "N_0"] & Q^n(C) \arrow[r, "J_0"] & \widehat{Q}^n(C) \arrow[r] & \cdots\end{tikzcd}$$

with

$$Q_n(C)\to Q_n(C,0)\ ;\ \psi\mapsto((1+T)\psi,\theta)\ ,\ \theta_s\ =\ \begin{cases}\psi_{-s-1}&\text{if } s\le -1\\ 0&\text{if } s\ge 0.\end{cases}$$

The twisted quadratic groups $Q_*(C,\gamma)$ are covariant in (C,γ). Given a map of chain bundles $(f,\chi):(C,\gamma)\to(C',\gamma')$ and an n-dimensional symmetric structure (ϕ,θ) on (C,γ) define an n-dimensional symmetric structure on (C',γ')

$$(f,\chi)_\%(\phi,\theta) \;=\; (f^\%(\phi),\widehat{f}^\%(\theta)+(f\phi_0)^\%(S^n\chi))\ .$$

The resulting morphisms of the twisted quadratic Q-groups

$$(f,\chi)_\% \;:\; Q_n(C,\gamma)\to Q_n(C',\gamma')$$

depend only on the homotopy class of (f,χ). There is defined a morphism of exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{Q}^{n+1}(C) \arrow[r, "H_\gamma"] \arrow[d, "\widehat{f}^\%"'] & Q_n(C,\gamma) \arrow[r, "N_\gamma"] \arrow[d, "(f,\chi)_\%"'] & Q^n(C) \arrow[r, "J_\gamma"] \arrow[d, "f^\%"'] & \widehat{Q}^n(C) \arrow[r] \arrow[d, "\widehat{f}^\%"'] & \cdots \\
\cdots \arrow[r] & \widehat{Q}^{n+1}(C') \arrow[r, "H_{\gamma'}"] & Q_n(C',\gamma') \arrow[r, "N_{\gamma'}"] & Q^n(C') \arrow[r, "J_{\gamma'}"] & \widehat{Q}^n(C') \arrow[r] & \cdots
\end{tikzcd}$$

which is an isomorphism if (f,χ) is an equivalence.

The *characteristic element* of an n-dimensional (B,β)-normal complex $(C,\phi,\gamma,\theta,f,\chi)$ is defined by

$$(f,\chi)_\%(\phi,\theta)\in Q_n(B,\beta)\ .$$

In §8 the cobordism class of a (B,β)-normal complex will be identified with the characteristic element.

A *map of n-dimensional* $\left\{\begin{array}{l}\textit{normal}\\ (B,\beta)\textit{-normal}\end{array}\right.$ *complexes*

$$\left\{\begin{array}{l}(f,\xi,\chi,\eta)\;:\;(C,\phi,\gamma,\theta)\to(C',\phi',\gamma',\theta')\\ (f,\xi,\chi,\eta,h,\mu)\;:\;(C,\phi,\gamma,\theta,g,\lambda)\to(C',\phi',\gamma',\theta',g',\lambda')\end{array}\right.$$

consists of

(i) a chain map $f:C\to C'$,
(ii) an equivalence $\xi:f^\%(\phi)\to\phi'$ of n-dimensional symmetric structures on C',
(iii) an equivalence $\chi:\gamma\to f^*\gamma'$ of bundles on C,
(iv) an equivalence of $(n+1)$-dimensional hyperquadratic structures on C'

$$\eta\;:\;J(\xi)+\theta'-\widehat{f}^\%(\theta)+(\xi_0;f^\%\phi_0,\phi_0')^\%(S^n\gamma')+(f\phi_0)^\%(S^n\chi)\to 0\ ,$$

and in the (B,β)-normal case also

(v) a homotopy of bundle maps

$$(h,\mu)\;:\;(g,\lambda)\;\simeq\;(g',\lambda')(f,\chi)\;:\;(C,\gamma)\to(B,\beta)\ .$$

Note that $(C,\phi,\gamma,\theta,g,\lambda)$ and $(C',\phi',\gamma',\theta',g',\lambda')$ have the same characteristic element

$$(g,\lambda)_\%(\phi,\theta)\;=\;(g',\lambda')_\%(\phi',\theta')\in Q_n(B,\beta)\ .$$

It is convenient for computational purposes to describe the behaviour of the twisted quadratic groups under direct sum. The direct sum of chain bundles (C,γ), (C',γ') is the chain bundle

$$(C,\gamma)\oplus(C',\gamma') \;=\; (C\oplus C',\gamma\oplus\gamma')\ .$$

Let

$$\begin{array}{l} i \;=\; \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;:\; C\to C\oplus C'\ ,\ i' \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \;:\; C'\to C\oplus C'\ , \\ j \;=\; (\,1 \quad 0\,) \;:\; C\oplus C'\to C\ ,\ j' \;=\; (\,0 \quad 1\,) \;:\; C\oplus C'\to C'\ . \end{array}$$

The twisted quadratic groups of the direct sum are such that there is defined a long exact sequence

$$\begin{array}{l} \dots\to Q_n(C,\gamma)\oplus Q_n(C',\gamma') \xrightarrow{i_*} Q_n(C\oplus C',\gamma\oplus\gamma') \xrightarrow{j_*} H_n(C^t\otimes_A C') \\ \qquad \xrightarrow{k_*} Q_{n-1}(C,\gamma)\oplus Q_{n-1}(C',\gamma') \xrightarrow{i_*} Q_{n-1}(C\oplus C',\gamma\oplus\gamma')\to\dots \end{array}$$

with

$$\begin{array}{l} i_* \;=\; (\,i_\% \quad i'_\%\,) \;:\; Q_n(C,\gamma)\oplus Q_n(C',\gamma')\to Q_n(C\oplus C',\gamma\oplus\gamma')\ , \\ j_* \;:\; Q_n(C\oplus C',\gamma\oplus\gamma')\to H_n(C^t\otimes_A C')\ ;\ (\phi,\theta)\mapsto(j\otimes j')\phi_0\ , \\ k_* \;:\; H_n(C^t\otimes_A C')\to Q_{n-1}(C,\gamma)\oplus Q_{n-1}(C',\gamma')\ ; \\ \qquad (f:C^{n-*}\to C')\mapsto((0,\widehat{f}^{\%}(S^n\gamma')),(0,-\widehat{f^*}^{\%}(S^n\gamma))\ . \end{array}$$

For $\gamma=0$ and $\gamma'=0$ the long exact sequence collapses into split exact sequences of the untwisted quadratic Q-groups

$$0\to Q_n(C)\oplus Q_n(C')\to Q_n(C\oplus C')\to H_n(C^t\otimes_A C')\to 0\ .$$

§8. **Normal cobordism**

Given a k-plane vector bundle $\nu:X\to BO(k)$ over a space X let $\Omega_n(X,\nu)$ $(n\geq 0)$ denote the bordism groups of bundle maps

$$(f,b) \;:\; (M^n,\nu_M)\to(X,\nu)$$

with M^n a smooth closed n-manifold and $\nu_M:M\to BO(k)$ the normal bundle of an embedding $M^n\subset S^{n+k}$ (Lashof [9]). The Thom space of ν_M is given by

$$T(\nu_M) \;=\; E(\nu_M)/\partial E(\nu_M)$$

with $E(\nu_M)$ the tubular neighbourhood of M^n in S^{n+k}, so that there is defined a collapse map

$$\rho_M \;:\; S^{n+k}\to S^{n+k}/(S^{n+k}\backslash E(\nu_M)) \;=\; E(\nu_M)/\partial E(\nu_M) \;=\; T(\nu_M)\ .$$

The Pontrjagin-Thom isomorphism

$$\begin{array}{l} \Omega_n(X,\nu)\to\pi_{n+k}(T(\nu))\ ; \\ (f:M^n\to X,b:\nu_M\to\nu)\mapsto(T(b)(\rho_M):S^{n+k}\xrightarrow{\rho_M}T(\nu_M)\xrightarrow{T(b)}T(\nu)) \end{array}$$

has inverse

$$\begin{aligned}&\pi_{n+k}(T(\nu)) \to \Omega_n(X,\nu) ~;\\ &(\rho : S^{n+k} \to T(\nu)) \mapsto (f = \rho| : M^n = \rho^{-1}(X) \to X, b : \nu_M \to \nu) ~,\end{aligned}$$

using smooth transversality to choose a representative ρ transverse regular at the zero section $X \subset T(\nu)$.

Given a $(k-1)$-spherical fibration $\nu : X \to BG(k)$ over a space X let $\Omega^N_n(X,\nu)$ (resp. $\Omega^P_n(X,\nu)$) denote the bordism group of fibration maps

$$(f,b) ~:~ (M^n,\nu_M) \to (X,\nu)$$

with $(M^n, \nu_M : M \to BG(k), \rho_M : S^{n+k} \to T(\nu_M))$ an n-dimensional normal space (resp. geometric Poincaré complex with Spivak normal structure). According to the theory of Quinn [14] there is a geometric theory of transversality for normal spaces, so that by analogy with the Pontrjagin-Thom isomorphism for smooth bordism there is defined an isomorphism

$$\begin{aligned}&\Omega^N_n(X,\nu) \to \pi_{n+k}(T(\nu)) ~;\\ &(f : M^n \to X, b : \nu_M \to \nu) \mapsto (T(b)(\rho_M) : S^{n+k} \xrightarrow{\rho_M} T(\nu_M) \xrightarrow{T(b)} T(\nu)) ~,\end{aligned}$$

with inverse

$$\begin{aligned}&\pi_{n+k}(T(\nu)) \to \Omega^N_n(X,\nu) ~;\\ &(\rho : S^{n+k} \to T(\nu)) \mapsto (f = \rho| : M^n = \rho^{-1}(X) \to X, b : \nu_M \to \nu) ~.\end{aligned}$$

The geometric Poincaré and normal bordism groups for $n \geq 5$ are related by the Levitt-Jones-Quinn exact sequence

$$\dots \to L_n(\mathbb{Z}[\pi_1(X)]) \to \Omega^P_n(X,\nu) \to \Omega^N_n(X,\nu) \to L_{n-1}(\mathbb{Z}[\pi_1(X)]) \to \dots ~.$$

If $\nu : X \to BG(k)$ admits a TOP reduction $\widetilde{\nu} : X \to BTOP(k)$ the forgetful maps from manifold to normal space bordism $\Omega_n(X,\nu) \to \Omega^N_n(X,\nu)$ are isomorphisms, and

$$\Omega^P_n(X,\nu) ~=~ L_n(\mathbb{Z}[\pi_1(X)]) \oplus \Omega^N_n(X,\nu) ~.$$

A map of n-dimensional normal spaces

$$(f,b,c) : (M^n,\nu_M,\rho_M) \to (X^n,\nu_X,\rho_X)$$

is defined by a map of fibrations $(f,b) : (M,\nu_M) \to (X,\nu_X)$ together with a homotopy

$$c ~:~ T(b)\rho_M ~\simeq~ \rho_X ~:~ S^{n+k} \to T(\nu_X) ~.$$

The mapping cylinder of f

$$M(f) ~=~ M \times [0,1] \cup X/\{(x,1) = f(x) \,|\, x \in M\}$$

defines a cobordism $(M(f);M,X)$ of normal spaces, identifying

$$M ~=~ M \times \{0\} \subset M(f) ~.$$

If M^n and X^n are Poincaré complexes the corresponding element of the relative bordism group is just the surgery obstruction

$$(M(f);M \cup -X) ~=~ \sigma_*(f,b) \in \Omega^{N,P}_{n+1}(X,\nu_X) ~=~ L_n(\mathbb{Z}[\pi_1(X)]) ~.$$

Ignoring questions of finite-dimensionality (or assuming that X is a finite n-dimensional CW complex) it is therefore possible to define the inverse isomorphism to $\Omega^N_n(X,\nu) \to \pi_{n+k}(T(\nu))$ by

$$\pi_{n+k}(T(\nu)) \to \Omega^N_n(X,\nu)\ ;\ \rho \mapsto (X,\nu,\rho)\ ,$$

without an appeal to the transversality of normal spaces. The group $\pi_{n+k}(T(\nu))$ consists of the equivalence classes of normal structures $(\nu_X : X \to BG(k), \rho_X : S^{n+k} \to T(\nu_X))$ on X with $\nu_X = \nu$.

Following Weiss [19] we shall now identify the algebraic normal bordism groups $\widehat{L}\langle B,\beta\rangle^n(A)$ with the twisted quadratic groups $Q_n(B,\beta)$, the algebraic analogues of the homotopy groups of the Thom space $\pi_{n+k}(T(\nu))$.

A *cobordism* of n-dimensional normal complexes (C,ϕ,γ,θ), $(C',\phi',\gamma',\theta')$ is defined by an $(n+1)$-dimensional symmetric pair

$$((f\ f') : C\oplus C' \to D, (\delta\phi, \phi\oplus -\phi'))$$

together with bundle maps

$$(f,\zeta)\ :\ (C,\gamma) \to (D,\delta\gamma)\ ,\ (f',\zeta')\ :\ (C',\gamma') \to (D,\delta\gamma)$$

and an equivalence of hyperquadratic structures on D

$$\delta\theta\ :\ J(\delta\phi) - (\delta\phi_0; f\phi_0 f^*, f'\phi'_0 f'^*)^{\%}(S^n\delta\gamma) + f;(\phi'_0)^{\%}(S^n\zeta')) \to 0\ .$$

Similarly for the cobordism of (B,β)-normal complexes.

The *symmetric (B,β)-structure L-groups of A* $L\langle B,\beta\rangle^n(A)$ $(n\geq 0)$ of Weiss [19] are the cobordism groups of n-dimensional (B,β)-normal Poincaré complexes over A $(C,\phi,\gamma,\theta,f,\chi)$. For the $\begin{cases}\text{universal}\\ \text{zero}\end{cases}$ chain bundle $\begin{cases}(B(\infty),\beta(\infty))\\ (0,0)\end{cases}$ over A these are just the $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ L-groups

$$\begin{cases} L\langle B(\infty),\beta(\infty)\rangle^n(A)\ =\ L^n(A)\\ L\langle 0,0\rangle^n(A)\ =\ L_n(A)\ .\end{cases}$$

The *symmetric (B,β)-structure $\widehat{L}$-groups* $\widehat{L}\langle B,\beta\rangle^n(A)$ $(n\geq 0)$ are the cobordism groups of n-dimensional (B,β)-normal complexes over A. For the $\begin{cases}\text{universal}\\ \text{zero}\end{cases}$ chain bundle $\begin{cases}(B(\infty),\beta(\infty))\\ (0,0)\end{cases}$ over A these are just the

$$\begin{cases} \widehat{L}\langle B(\infty),\beta(\infty)\rangle^n(A)\ =\ \widehat{L}^n(A)\\ \widehat{L}\langle 0,0\rangle^n(A)\ =\ 0\ .\end{cases}$$

Algebraic surgery was used in Ranicki [15] to prove that every n-dimensional quadratic Poincaré complex (C,ψ) is cobordant to a highly-connected complex (C',ψ'), with

$$H_r(C')\ =\ 0\ (2r\leq n-2)\ .$$

The boundary of an n-dimensional normal complex (C,ϕ,γ,θ) is an $(n-1)$-dimensional quadratic Poincaré complex $(\partial C,\psi)$. Glueing on to (C,ϕ,γ,θ) the trace of the surgery making $(\partial C, \psi)$ highly-connected there is obtained an n-dimensional normal complex $(C', \phi', \gamma', \theta')$ which is cobordant to $(C, \phi, \gamma, \theta)$ and which has a highly-connected boundary, with

$$H_r(\partial C') \ = \ H_{r+1}(\phi'_0 : C'^{n-*} \to C') \ = \ 0 \ (2r \leq n-3) \ .$$

In particular, this shows that every normal complex is cobordant to a connected complex. Thus $\widehat{L}\langle B, \beta\rangle^n(A)$ is also the cobordism group of connected n-dimensional (B, β)-normal complexes over A. The one-one correspondence established in §7 between connected n-dimensional normal complexes and n-dimensional (symmetric, quadratic) Poincaré pairs generalizes to a one-one correspondence between connected n-dimensional (B, β)-normal complexes over A and n-dimensional (symmetric, quadratic) (B, β)-normal Poincaré pairs over A, for any chain bundle (B, β) over A. It follows that $\widehat{L}\langle B, \beta\rangle^n(A)$ can be identified with the cobordism group of n-dimensional (symmetric, quadratic) (B, β)-normal Poincaré pairs, and that there is defined an exact sequence

$$\dots \to L_n(A) \to L\langle B, \beta\rangle^n(A) \to \widehat{L}\langle B, \beta\rangle^n(A) \overset{\partial}{\to} L_{n-1}(A) \to \dots \ ,$$

with ∂ defined by the quadratic boundary

$$\partial \ : \ \widehat{L}\langle B, \beta\rangle^n(A) \to L_{n-1}(A) \ ; \ (C, \phi, \gamma, \theta, f, \chi) \mapsto \partial(C, \phi, \gamma, \theta) \ .$$

A map of n-dimensional normal complexes

$$(f, \xi, \chi, \eta) \ : \ (C, \phi, \gamma, \theta) \to (C', \phi', \gamma', \theta')$$

determines an $(n+1)$-dimensional symmetric pair $((f\ 1) : C \oplus C' \to C', (\xi, \phi \oplus -\phi'))$, bundle maps

$$(f, \chi) \ : \ (C, \gamma) \to (C', \gamma') \ , \ (1, 0) \ : \ (C', \gamma') \to (C', \gamma')$$

and an equivalence of hyperquadratic structures on C'

$$\eta : J(\xi) - (\xi_0; f\phi_0 f^*, \phi'_0)^{\%}(S^n\gamma') + f^{\%}(\theta - (\phi_0)^{\%}(S^n\gamma)) - \theta' \to 0 \ ,$$

defining a cobordism between $(C, \phi, \gamma, \theta)$ and $(C', \phi', \gamma', \theta')$ by analogy with the mapping cylinder construction of geometric normal bordisms. Similarly for maps of (B, β)-normal complexes. It follows that the abelian group morphisms

$$\begin{aligned} &\widehat{L}\langle B, \beta\rangle^n(A) \to Q_n(B, \beta) \ ; \ (C, \phi, \gamma, \theta, f, \chi) \mapsto (f, \chi)_{\%}(\phi, \theta) \ , \\ &Q_n(B, \beta) \to \widehat{L}\langle B, \beta\rangle^n(A) \ ; \ (\phi, \theta) \mapsto (B, \phi, \beta, \theta, 1, 0) \end{aligned}$$

are inverse isomorphisms.

For example, if $A = \mathbb{Z}$ and $(B(\infty), \beta(\infty))$ is the universal chain bundle over $\mathbb{Z}$ (as constructed at the end of §6) then

$$L\langle B(\infty), \beta(\infty)\rangle^n(\mathbb{Z}) \ = \ L^n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\bmod 4) \\ \mathbb{Z}_2 & \text{if } n \equiv 1 (\bmod 4) \\ 0 & \text{if } n \equiv 2,3 (\bmod 4)\ , \end{cases}$$

$$L_n(\mathbb{Z}) \ = \ \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 (\bmod 4) \\ \mathbb{Z}_2 & \text{if } n \equiv 2 (\bmod 4) \\ 0 & \text{if } n \equiv 1,3 (\bmod 4)\ , \end{cases}$$

$$\widehat{L}\langle B(\infty), \beta(\infty)\rangle^n(\mathbb{Z}) \ = \ Q_n(B(\infty), \beta(\infty)) \ = \ \begin{cases} \mathbb{Z}_8 & \text{if } n \equiv 0 (\bmod 4) \\ \mathbb{Z}_2 & \text{if } n \equiv 1,3 (\bmod 4) \\ 0 & \text{if } n \equiv 2 (\bmod 4)\ . \end{cases}$$

A spherical fibration $\nu : X \to BG(k)$ determines a chain bundle $(C(\widetilde{X}), \gamma)$ over $\mathbb{Z}[\pi_1(X)]$ ([15], [19]) and there is defined a natural transformation of exact sequences from the Levitt-Jones-Quinn Poincaré bordism sequence

$$\begin{tikzcd} \cdots \arrow[r] & L_n(\mathbb{Z}[\pi_1(X)]) \arrow[r] \arrow[d, equal] & \Omega^P_n(X,\nu) \arrow[r] \arrow[d] & \pi_{n+k}(T(\nu)) \arrow[r] \arrow[d] & L_{n-1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & L_n(\mathbb{Z}[\pi_1(X)]) \arrow[r] & L^n(C(\widetilde{X}),\gamma) \arrow[r] & Q_n(C(\widetilde{X}),\gamma) \arrow[r] & L_{n-1}(\mathbb{Z}[\pi_1(X)]) \arrow[r] & \cdots \end{tikzcd}$$

with $\Omega^P_n(X,\nu) \to L^n(C(\widetilde{X}),\gamma)$ a generalized symmetric signature map.

§9. **Normal Wu classes**

The Wu classes of the symmetric structure ϕ and the bundles β, γ in an n-dimensional (B,β)-normal complex $(C,\phi,\gamma,\theta,f,\chi)$ are related by a commutative diagram

$$\begin{tikzcd} H^{n-r}(C) \arrow[r, "\phi_0"] \arrow[d, "v_r(\phi)"'] & H_r(C) \arrow[r, "f_*"] \arrow[d, "\widehat{v}_r(\gamma)"'] & H_r(B) \arrow[dl, "\widehat{v}_r(\beta)"] \\ H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r}) \arrow[r, "J"] & \widehat{H}^r(\mathbb{Z}_2; A) & \end{tikzcd}$$

For any chain bundle (B,β) and any chain complex C we shall now define symmetric (B,β)-structure groups $Q\langle B,\beta\rangle^n(C)$ $(n \geq 0)$ to fit into an exact sequence

$$\cdots \to Q\langle B,\beta\rangle^n(C) \to Q^n(C) \oplus H_n(B^t \otimes_A C) \to \widehat{Q}^n(C) \to Q\langle B,\beta\rangle^{n-1}(C) \to \dots \ .$$

The Wu classes $v_r(\phi)$ of a symmetric complex (C,ϕ) will then be refined to the normal Wu classes of a (B,β)-normal complex $(C,\phi,\gamma,\theta,f,\chi)$

$$v_r \ = \ v_r(\phi,\gamma,\theta,f,\chi) \ : \ H^{n-r}(C) \to Q\langle B,\beta\rangle^n(S^{n-r}A)\ ,$$

with

$$v_r(\phi) \ : \ H^{n-r}(C) \xrightarrow{v_r} Q\langle B,\beta\rangle^n(S^{n-r}A) \to Q^n(S^{n-r}A) \ = \ H^{n-2r}(\mathbb{Z}_2; A, (-1)^{n-r})\ .$$

In §11 below the normal Wu classes will be used to define a $\mathbb{Z}_4$-valued quadratic function on $H^n(C)$ for a $2n$-dimensional symmetric Poincaré complex (C,ϕ) over $\mathbb{Z}_2$ with normal $(v_{n+1}=0)$-structure, as required to define the $\mathbb{Z}_8$-valued invariant of Brown [4].

Let (B,β) be a chain bundle over A, and let C be a finite-dimensional A-module chain complex. An *n-dimensional symmetric (B,β)-structure on C (ϕ,θ,f)* is defined by an n-dimensional symmetric structure $\phi\in(W^{\%}C)_n$ together with a chain $\theta\in(\widehat{W}^{\%}C)_{n+1}$ and a chain map $f:B^{n-*}\to C$ such that

$$J(\phi)-\widehat{f}^{\%}(S^n\beta) \;=\; d\theta\in(\widehat{W}^{\%}C)_n\ .$$

An n-dimensional (B,β)-normal structure $(\phi,\gamma,\theta,g,\chi)$ on C determines the n-dimensional symmetric (B,β)-structure $(\phi,\theta+(\phi_0)^{\%}(S^n\chi),\phi_0g^*)$ on C. Conversely, if $f^*:C^{n-*}\to B$ is a composite

$$f^*\ :\ C^{n-*}\ \xrightarrow{\ \phi_0\ }\ C\ \xrightarrow{\ g\ }\ B$$

(as is always the case up to chain homotopy if (C,ϕ) is a Poincaré complex) the symmetric (B,β)-structure (ϕ,θ,f) determines the n-dimensional (B,β)-normal structure $(\phi,g^*\gamma,\theta,g,0)$.

An *n-dimensional symmetric (B,β)-structure (Poincaré) complex over A* (C,ϕ,θ,f) is an n-dimensional A-module chain complex C together with an n-dimensional symmetric (B,β)-structure (ϕ,θ,f) (such that $\phi_0:C^{n-*}\to C$ is a chain equivalence). As for symmetric (Poincaré) pairs there is also the analogous notion of *symmetric (B,β)-structure (Poincaré) pair*. There is essentially no difference between symmetric (B,β)-structure Poincaré complexes and (B,β)-normal Poincaré complexes, so that the L-groups $L\langle B,\beta\rangle^n(A)$ $(n\geq 0)$ can also be regarded as the cobordism groups of n-dimensional symmetric (B,β)-structure Poincaré complexes over A.

An *equivalence* of n-dimensional symmetric (B,β)-structures on C

$$(\xi,\eta,g)\ :\ (\phi,\theta,f)\to(\phi',\theta',f')$$

is defined by an equivalence of symmetric structures $\xi:\phi\to\phi'$ together with a chain $\eta\in(\widehat{W}^{\%}C)_{n+2}$ and a chain homotopy $g:f\simeq f':B^{n-*}\to C$ such that

$$J(\xi)-(g;f,f')^{\%}(S^n\beta)-\theta'+\theta \;=\; d\eta\in(\widehat{W}^{\%}C)_{n+1}\ .$$

The *n-dimensional symmetric (B,β)-structure group of C $Q\langle B,\beta\rangle^n(C)$* is the abelian group of equivalence classes of n-dimensional symmetric (B,β)-symmetric structures on C, with addition by

$$(\phi,\theta,f)+(\phi',\theta',f') \;=\; (\phi+\phi',\theta+\theta'+[f,f'](S^n\beta),f+f')\in Q\langle B,\beta\rangle^n(C)\ .$$

There is also a more economical description of $Q\langle B,\beta\rangle^n(C)$ as the abelian group of equivalence classes of pairs (ψ,f) defined by an n-dimensional quadratic structure $\psi\in(W_{\%}C)_n$ and a chain map $f:B^{n-*}\to C$ such that

$$f_{\%}H(S^n\beta) \;=\; d\psi\in(W_{\%}C)_{n-1}\ ,$$

so that up to signs

$$f\beta_{-n-s-1}f^* \;=\; d\psi_s+\psi_sd^*+\psi_{s+1}+\psi^*_{s+1}\in \mathrm{Hom}_A(C^{-*},C)_{n-s}\ (s\geq 0)\ ,$$

subject to the equivalence relation

$$(\psi,f)\sim(\psi',f') \text{ if there exist a chain homotopy } g: f\simeq f': B^{n-*}\to C \text{ and an equivalence of quadratic structures } \chi\ :\ \psi'-\psi\to(g;f,f')_{\%}H(S^n\beta)\ ,$$

with addition by

$$(\psi,f)+(\psi',f')\ =\ (\psi+\psi'+H([f,f'](S^n\beta)),f+f')\ .$$

The pair (ψ,f) determines the triple (ϕ,θ,f) with

$$\phi_s\ =\ \begin{cases} f\beta_{s-n}f^* & \text{if } s\geq 1\\ f\beta_{-n}f^*+(1+T)\psi_0 & \text{if } s=0\ ,\end{cases}$$
$$\theta_s\ =\ \begin{cases} 0 & \text{if } s\geq 0\\ \psi_{-s-1} & \text{if } s\leq -1\ .\end{cases}$$

Conversely, a triple (ϕ,θ,f) determines the pair (ψ,f) with

$$\psi_s\ =\ \theta_{-s-1}\ (s\geq 0)\ .$$

Given an n-dimensional symmetric (B,β)-structure (ϕ,θ,f) on C, a chain bundle map $(g,\chi):(B,\beta)\to(B',\beta')$ and a chain map $h:C\to C'$ define the *pushforward* n-dimensional symmetric (B',β')-structure on C'

$$\langle g,\chi\rangle(h)^{\%}(\phi,\theta,f)\ =\ (h^{\%}(\phi),h^{\%}(\theta+S^n(\widehat{f}^{\%}\chi)),hfg^*)\ .$$

Thus the groups $Q\langle B,\beta\rangle^*(C)$ are covariant in both (B,β) and C, with pushforward abelian group morphisms

$$\begin{aligned}\langle g,\chi\rangle(h)^{\%}\ &:Q\langle B,\beta\rangle^n(C)\to Q\langle B',\beta'\rangle^n(C')\ ;\\ &(\phi,\theta,f)\mapsto(h^{\%}(\phi),\widehat{h}^{\%}(\theta+\widehat{f}^{\%}(S^n\chi)),hfg^*)\end{aligned}$$

depending only on the homotopy classes of (g,χ) and h.

An n-dimensional symmetric (B,β)-structure (ϕ,θ,f) on C determines an n-dimensional symmetric structure $\phi\in(W^{\%}C)_n$ on C, so that there is defined a forgetful map

$$s\ :\ Q\langle B,\beta\rangle^n(C)\to Q^n(C)\ ;\ (\phi,\theta,f)\mapsto\phi\ .$$

An n-dimensional quadratic structure $\psi\in(W_{\%}C)_n$ on C determines an n-dimensional symmetric (B,β)-structure $((1+T)\psi,\theta,0)$ on C for any (B,β), with

$$\theta_s\ =\ \begin{cases}\psi_{-s-1} & \text{if } s\leq -1\\ 0 & \text{if } s\geq 0\ .\end{cases}$$

Thus there are also defined forgetful maps

$$s\ :\ Q_n(C)\to Q\langle B,\beta\rangle^n(C)\ ;\ \psi\mapsto((1+T)\psi,\theta,0)\ ,$$

and $sr=1+T:Q_n(C)\to Q^n(C)$.

Let $P\langle B,\beta\rangle^n(C)$ be the abelian group of equivalence classes of n-dimensional symmetric (B,β)-structures (ϕ,θ,f) with $\phi=0$, to be denoted (θ,f), subject to the equivalence relation

$$(\theta,f)\sim(\theta',f') \text{ if there exists an equivalence of } (B,\beta)\text{-structures}$$

$$(0,\eta,g):(0,\theta,f)\to(0,\theta',f').$$

The symmetric (B,β)-structure groups $Q\langle B,\beta\rangle^*(C)$ and the groups $P\langle B,\beta\rangle^*(C)$ are related by a commutative braid of exact sequences of abelian groups

$$\begin{tikzcd}
Q_n(C) \arrow[dr] \arrow[rr, bend left, "1+T"] & & Q^n(C) \arrow[dr, "J"] \arrow[rr, bend left] & & P\langle B,\beta\rangle^{n-1}(C) \\
& Q\langle B,\beta\rangle^n(C) \arrow[ur] \arrow[dr] & & \widehat{Q}^n(C) \arrow[ur] \arrow[dr, "H"] & \\
P\langle B,\beta\rangle^n(C) \arrow[ur] \arrow[rr, bend right] & & H_n(B^t\otimes_A C) \arrow[ur] \arrow[rr, bend right] & & Q_{n-1}(C)
\end{tikzcd}$$

If (B,β) is the $\begin{cases}\text{universal}\\ \text{zero}\end{cases}$ chain bundle $\begin{cases}(B(\infty),\beta(\infty))\\ (0,0)\end{cases}$ the forgetful map

$$\begin{cases} Q\langle B(\infty),\beta(\infty)\rangle^n(C)\to Q^n(C)\ ;\ (\phi,\theta,f)\mapsto\phi\\ Q_n(C)\to Q\langle 0,0\rangle^n(C)\ ;\ \psi\mapsto((1+T)\psi,\theta)\end{cases}$$

is an isomorphism and

$$\begin{cases} P\langle B(\infty),\beta(\infty)\rangle^n(C)\ =\ 0\\ P\langle 0,0\rangle^n(C)\ =\ \widehat{Q}^{n+1}(C)\ .\end{cases}$$

The *Wu classes* of an n-dimensional symmetric (B,β)-structure (ϕ,θ,f) on C are the A-morphisms

$$\begin{aligned} v_r(\phi,\theta,f)\ &:H^{n-r}(C)\to Q\langle B,\beta\rangle^n(S^{n-r}A)\ ;\\ &(x:C\to S^{n-r}A)\mapsto\langle 1,0\rangle(x)^{\%}(\phi,\theta,f)\ .\end{aligned}$$

Now

$$Q\langle B,\beta\rangle^n(S^{n-r}A)\ =$$

$$\begin{cases} H_r(B) & \text{if } 2r<n\\ \{(a,b)\in A\oplus B_r \mid db=0\in B_{r-1}\}/{\sim} & \text{if } 2r=n\\ \{(a,b)\in A\oplus B_r \mid a+(-1)^r\overline{a}+\beta_{-2r}(b)(b)=0\in A, db=0\in B_{r-1}\}/\sim & \text{if } 2r>n\end{cases}$$

with the equivalence relation $\sim$ defined by

$$\begin{aligned}(a,b)\sim(a',b')\ \text{ if there exists } &(x,y)\in A\oplus B_{r+1} \text{ such that}\\ &a'-a=x+(-1)^{r+1}\overline{x}+\beta_{-2r-2}(y)(y)\\ &\qquad\qquad+\beta_{-2r-1}(y)(b)+\beta_{-2r-1}(b')(y)\in A,\\ &b'-b=dy\in B_r\end{aligned}$$

and addition by

$$(a,b)+(a',b')\ =\ (a+a'+\beta_{-2r}(b)(b'),b+b')\ .$$

The map to the symmetric Q-group is given by

$$Q\langle B,\beta\rangle^n(S^{n-r}A) \to Q^n(S^{n-r}A) \;=\; H^{n-2r}(\mathbb{Z}_2;A,(-1)^{n-r}) \;;$$
$$\begin{cases} b \mapsto \beta_{-2r}(b)(b) & \text{if } 2r<n \\ (a,b)\mapsto a+(-1)^r\overline{a}+\beta_{-2r}(b)(b) & \text{if } 2r=n \\ 0 & \text{if } 2r>n \;. \end{cases}$$

The Wu classes are given by

$$v_r(\phi,\theta,f) \;:\; H^{n-r}(C)\to Q\langle B,\beta\rangle^n(S^{n-r}A) \;;$$
$$z\mapsto \begin{cases} f^*(z) & \text{if } 2r<n \\ (\theta_{n-2r-1}(z)(z),f^*(z)) & \text{if } 2r\geq n \end{cases} \; (z\in C^{n-r}, d^*z=0) \;.$$

§10. **Forms**

In Ranicki [15] the even-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ L-groups $\begin{cases}L^{2n}(A)\\ L_{2n}(A)\end{cases}$ $(n\geq 0)$ were related to the Witt groups $\begin{cases}W^{(-1)^n}(A)\\ W_{(-1)^n}(A)\end{cases}$ of nonsingular $\begin{cases}(-1)^n\text{-symmetric}\\ (-1)^n\text{-quadratic}\end{cases}$ forms over A. In particular, it was shown that

$$\begin{cases} L^0(A) \;=\; W^{+1}(A) \\ L_{2n}(A) \;=\; W_{(-1)^n}(A) \end{cases} (n\geq 0) \;.$$

This relationship between L-groups and Witt groups will now be generalized to the even-dimensional symmetric (B,β)-structure L-groups $L\langle B,\beta\rangle^{2n}(A)$ and the Witt groups $W_{Q(n)}(A)$ of nonsingular $Q(n)$-quadratic forms over A, with (B,β) any chain bundle over A and

$$Q(n) \;=\; Q\langle B,\beta\rangle^{2n}(S^nA) \;.$$

Let $\epsilon=\pm 1$. An *ϵ-symmetric form over A* (M,λ) is a f.g. projective A-module M together with an element $\lambda\in \mathrm{Hom}_A(M,M^*)$ such that

$$\epsilon\lambda^* \;=\; \lambda \;:\; M\to M^* \;.$$

Equivalently, the form is defined by a pairing

$$\lambda \;:\; M\times M\to A \;;\; (x,y)\to \lambda(x,y) \;=\; \lambda(x)(y)$$

such that

$$\begin{aligned} \lambda(ax,by) \;&=\; b\lambda(x,y)\overline{a} \;, \\ \lambda(x+x',y) \;&=\; \lambda(x,y)+\lambda(x',y) \;, \\ \epsilon\overline{\lambda(y,x)} \;&=\; \lambda(x,y) \; (x,y\in M, a,b\in A) \;. \end{aligned}$$

Let $Q(\epsilon)$ be an A-group together with A-morphisms

$$\begin{aligned} r \;&:\; Q(\epsilon)\to H^0(\mathbb{Z}_2;A,\epsilon) \;=\; \{a\in A\,|\,\epsilon\overline{a}=a\} \;, \\ s \;&:\; H_0(\mathbb{Z}_2;A,\epsilon) \;=\; A/\{b-\epsilon\overline{b}\,|\,b\in A\}\to Q(\epsilon) \end{aligned}$$

such that

$$rs \;=\; 1+T_\epsilon \;:\; H_0(\mathbb{Z}_2;A,\epsilon)\to H^0(\mathbb{Z}_2;A,\epsilon) \;.$$

A $Q(\epsilon)$-*quadratic form over* A (M,λ,μ) is an ϵ-symmetric form (M,λ) together with an A-morphism $\mu : M \to Q(\epsilon)$ such that

$$\begin{aligned} r(\mu(x)) &= \lambda(x,x) \in H^0(\mathbb{Z}_2;A,\epsilon)\ , \\ \mu(x+y)-\mu(x)-\mu(y) &= s(\lambda(x,y)) \in Q(\epsilon)\ (x,y\in M)\ . \end{aligned}$$

There is an evident notion of *isomorphism* of $Q(\epsilon)$-quadratic forms.

A $Q(\epsilon)$-quadratic form (M,λ,μ) is *nonsingular* if $\lambda \in \mathrm{Hom}_A(M,M^*)$ is an isomorphism of A-modules.

A nonsingular $Q(\epsilon)$-quadratic form (M,λ,μ) is *hyperbolic* if there exists a direct summand $L\subset M$ such that

(i) the inclusion $j\in \mathrm{Hom}_A(L,M)$ fits into an exact sequence

$$0\to L \xrightarrow{j} M \xrightarrow{j^*\lambda} L^* \to 0\ ,$$

(ii) $\mu j \ =\ 0\ :\ L\to Q(\epsilon)$.

The $Q(\epsilon)$-*quadratic Witt group of* A $W_{Q(\epsilon)}(A)$ is the abelian group of equivalence classes of nonsingular $Q(\epsilon)$-quadratic forms (M,λ,μ), subject to the equivalence relation

$(M,\lambda,\mu)\sim(M',\lambda',\mu')$ if there exists an isomorphism
$(M,\lambda,\mu)\oplus(N,\nu,\rho)\to(M',\lambda',\mu')\oplus(N',\nu',\rho')$
for some hyperbolic $Q(\epsilon)$-quadratic forms
(N,ν,ρ), (N',ν',ρ').

For $Q(\epsilon)=H^0(\mathbb{Z}_2;A,\epsilon)$, $r=1$, $s=1+T_\epsilon$ a $Q(\epsilon)$-quadratic form (M,λ,μ) may be identified with the ϵ-symmetric form (M,λ), since λ determines μ by

$$\mu(x) \ =\ \lambda(x,x)\in H^0(\mathbb{Z}_2;A,\epsilon)\ (x\in M)\ .$$

The Witt group of ϵ-symmetric forms $W_{Q(\epsilon)}(A)$ is denoted by $W^\epsilon(A)$.

For $Q(\epsilon)=H_0(\mathbb{Z}_2;A,\epsilon)$, $r=1+T_\epsilon$, $s=1$ a $Q(\epsilon)$-quadratic form (M,λ,μ) is just a ϵ-quadratic form in the sense of Wall [18]. The Witt group of ϵ-quadratic forms $W_{Q(\epsilon)}(A)$ is denoted by $W_\epsilon(A)$.

For $Q(\epsilon)=\mathrm{im}(1+T_\epsilon : H_0(\mathbb{Z}_2;A,\epsilon)\to H^0(\mathbb{Z}_2;A,\epsilon))$, $r=$ projection, $s=$ injection a $Q(\epsilon)$-quadratic form (M,λ,μ) is just an ϵ-symmetric form (M,λ) for which there exists an ϵ-quadratic form $(M,\lambda,\mu : M\to H_0(\mathbb{Z}_2;A,\epsilon))$. Such an ϵ-symmetric form is *even*. The Witt group of even ϵ-symmetric forms $W_{Q(\epsilon)}(A)$ is denoted by $W\langle v_0\rangle^\epsilon(A)$.

For $\epsilon=+1$ $\begin{cases}\epsilon\text{-symmetric}\\ \epsilon\text{-quadratic}\end{cases}$ is abbreviated to $\begin{cases}\text{symmetric}\\ \text{quadratic}\ .\end{cases}$

A $2n$-dimensional $\begin{cases}\text{symmetric}\\ \text{quadratic}\end{cases}$ (Poincaré) complex over A $\begin{cases}(C,\phi)\\ (C,\psi)\end{cases}$ with $H^n(C)$ a f.g. projective A-module determines a (nonsingular) $\begin{cases}(-1)^n\text{-symmetric}\\ (-1)^n\text{-quadratic}\end{cases}$

form over A $\begin{cases} (H^n(C),\phi_0,v_n(\phi)) \\ (H^n(C),(1+T)\psi_0,v^n(\psi)) \end{cases}$ with

$$\begin{cases} v_n(\phi)\ :\ H^n(C)\to H^0(\mathbb{Z}_2;A,(-1)^n)\ ;\ x\mapsto \phi_0(x)(x) \\ v^n(\psi)\ :\ H^n(C)\to H_0(\mathbb{Z}_2;A,(-1)^n)\ ;\ x\mapsto \psi_0(x)(x)\ . \end{cases}$$

Conversely, a (nonsingular) $\begin{cases} (-1)^n\text{-symmetric} \\ (-1)^n\text{-quadratic} \end{cases}$ form $\begin{cases} (M,\lambda) \\ (M,\lambda,\mu) \end{cases}$ determines a $2n$-dimensional $\begin{cases} \text{symmetric} \\ \text{quadratic} \end{cases}$ (Poincaré) complex $\begin{cases} (C,\phi) \\ (C,\psi) \end{cases}$ such that

$$\begin{aligned} &\begin{cases} \phi_0 \\ (1+T)\psi_0 \end{cases} =\ \lambda\ :\ C^n\ =\ M\to C_n\ =\ M^*\ ,\ C_r\ =\ 0\ (r\neq n)\ , \\ &v^n(\psi)\ =\ \mu\ :\ H^n(C)\ =\ M\to H_0(\mathbb{Z}_2;A,(-1)^n)\ . \end{aligned}$$

The corresponding morphisms from the Witt groups to the L-groups

$$\begin{cases} W^{(-1)^n}(A)\to L^{2n}(A)\ ;\ (M,\lambda)\mapsto (C,\phi) \\ W_{(-1)^n}(A)\to L_{2n}(A)\ ;\ (M,\lambda,\mu)\mapsto (C,\psi) \end{cases}$$

were shown in Ranicki [15] to be isomorphisms for $n=0$ if A is any ring, and for all $n\geq 0$ if A is $\begin{cases} \text{a Dedekind} \\ \text{any} \end{cases}$ ring. For a Dedekind ring A the inverse isomorphism in symmetric L-theory is given by

$$L^{2n}(A)\to W^{(-1)^n}(A)\ ;\ (C,\phi)\mapsto (H^n(C)/(\text{torsion}),\phi_0)\ .$$

The inverse isomorphism in quadratic L-theory is given for any A by

$$\begin{aligned} &L_{2n}(A)\to W_{(-1)^n}(A)\ ; \\ &(C,\psi)\mapsto (\text{coker}(\begin{pmatrix} d^* & 0 \\ (1+T)\psi_0 & d \end{pmatrix} : C^{n-1}\oplus C_{n+2}\to C^n\oplus C_{n+1}),\begin{bmatrix} \psi_0 & d \\ 0 & 0 \end{bmatrix})\ . \end{aligned}$$

If A is a field this isomorphism can also be expressed as

$$(C,\psi)\mapsto (H^n(C),(1+T)\psi_0,v^n(\psi))$$

but this is not the case in general – see Milgram and Ranicki [12, p.406].

Given A-groups M,N and a symmetric bilinear pairing

$$\phi\ :\ N\times N\to M$$

such that

$$\phi(ay,ay')\ =\ a\phi(y,y')\in M\ (a\in A,y,y'\in N)$$

let $M\times_\phi N$ be the A-group of pairs $(x\in M,y\in N)$, with addition by

$$(x,y)+(x',y')\ =\ (x+x'+\phi(y,y'),y+y')\in M\times_\phi N$$

and A acting by

$$A\times(M\times_\phi N)\to M\times_\phi N\ ;\ (a,(x,y))\mapsto (ax,ay)\ .$$

There is then defined a short exact sequence of A-groups and A-morphisms

$$0\to M\to M\times_\phi N\to N\to 0$$

with

$$\begin{aligned} &M \to M \times_\phi N \ ; \ x \mapsto (x, 0) \\ &M \times_\phi N \to N \ ; \ (x, y) \mapsto y \ . \end{aligned}$$

Given a chain bundle (B, β) over A define the A-group

$$\begin{aligned} Q(n) \ &= \ Q\langle B, \beta\rangle^{2n}(S^n A) \\ &= \ \{(a, b) \in A \oplus B_n \,|\, db = 0 \in B_{n-1}\}/\!\sim \ , \end{aligned}$$

where

$$\begin{aligned} (a, b) \sim (a', b') \ \ &\text{if there exist } (x, y) \in A \oplus B_{n+1} \text{ such that} \\ a' - a \ &= \ x + (-1)^{n+1}\overline{x} + \beta_{-2n-2}(y)(y) \\ &\qquad\qquad + \beta_{-2n-1}(y)(b) + \beta_{-2n-1}(b')(y) \ , \\ b' - b \ &= \ dy \ , \end{aligned}$$

with addition by

$$(a, b) + (a', b') \ = \ (a + a' + \beta_{-2n}(b)(b'), b + b')$$

and A-action by

$$A \times Q(n) \to Q(n) \ ; \ (x, (a, b)) \mapsto x(a, b) \ = \ (xa\overline{x}, xb) \ .$$

The A-morphisms

$$\begin{aligned} r \ &: \ Q_{2n}(S^n A) \ = \ H_0(\mathbb{Z}_2; A, (-1)^n) \to Q(n) \ ; \ a \mapsto (a, 0) \ , \\ s \ &: \ Q(n) \to Q^{2n}(S^n A) \ = \ H^0(\mathbb{Z}_2; A, (-1)^n) \ ; \\ &\qquad\qquad (a, b) \mapsto a + (-1)^n\overline{a} + \beta_{-2n}(b)(b) \end{aligned}$$

are such that there is defined a commutative braid of exact sequences

$$\begin{tikzcd}[column sep=small]
H_{n+1}(B) \arrow[dr] \arrow[rr, bend left] & & Q_{2n}(S^n A) \arrow[dr, "r"] \arrow[rr, bend left, "1+T"] & & Q^{2n}(S^n A) \arrow[dr, "J"] \arrow[rr, bend left] & & 0 \\
& \widehat{Q}^{2n+1}(S^n A) \arrow[ur, "H"] \arrow[dr] & & Q(n) \arrow[ur, "s"] \arrow[dr] & & \widehat{Q}^{2n}(S^n A) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & P\langle B, \beta\rangle^{2n}(S^n A) \arrow[ur] \arrow[rr, bend right] & & H_n(B) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

with

$$Q(n) \to H_n(B) \ ; \ (a, b) \mapsto b \ .$$

If (B, β) is such that for all $y \in B_{n+1}$ there exists $x \in A$ such that

$$(\beta_{-2n-2} + \beta_{-2n-1}d)(y)(y) \ = \ x + (-1)^n\overline{x} \in A$$

(e.g. if $d = 0 : B_{n+1} \to B_n$ and $v_{n+1}(\beta) = 0 : H_{n+1}(B) \to \widehat{H}^{n+1}(\mathbb{Z}_2; A)$) then there is a natural identification of A-groups

$$Q(n) \;=\; Q_{2n}(S^n A) \times_{\beta_{-2n}} H_n(B)$$

with

$$\beta_{-2n} \;:\; H_n(B) \times H_n(B) \to Q_{2n}(S^n A)\ ;\ (b, b') \mapsto \beta_{-2n}(b)(b')\ .$$

For any chain bundle (B, β) and any $Q(n)$-quadratic form (M, λ, μ) there exist A-module morphisms

$$g \;:\; M \to B_n\ ,\ \psi \;:\; M \to M^*$$

such that

$$\begin{aligned} & dg \;=\; 0 \;:\; M \to B_{n-1}\ , \\ & \lambda - g^* \beta_{-2n} g \;=\; \psi + (-1)^n \psi^* \;:\; M \to M^*\ , \\ & \mu \;:\; M \to Q(n)\ ;\ x \mapsto (\psi(x)(x), g(x))\ . \end{aligned}$$

If (M, λ, μ) is a nonsingular form there is thus defined a $2n$-dimensional symmetric (B, β)-structure Poincaré complex (C, ϕ, θ, f) with

$$\begin{aligned} & \phi_0 \;=\; \lambda \;:\; C^n \;=\; M \to C_n \;=\; M^*\ ,\ C_r \;=\; 0\ (r \neq n)\ , \\ & \theta_{-1} \;=\; \psi \;:\; C^n \;=\; M \to C_n \;=\; M^*\ , \\ & f \;=\; g\lambda^{-1} \;:\; C_n \;=\; M^* \xrightarrow{\lambda^{-1}} M \xrightarrow{g} B_n\ , \\ & v_n(\phi, \theta, f) \;=\; \mu \;:\; H^n(C) \;=\; M \to Q(n)\ . \end{aligned}$$

The construction defines a morphism of abelian groups

$$W_{Q(n)}(A) \to L\langle B, \beta\rangle^{2n}(A)\ ;\ (M, \lambda, \mu) \mapsto (C, \phi, \theta, f)\ .$$

Conversely, if (C, ϕ, θ, f) is a $2n$-dimensional symmetric (B, β)-structure Poincaré complex such that $H^n(C)$ is a f.g. projective A-module there is defined a nonsingular $Q(n)$-quadratic form $(H^n(C), \phi_0, v_n(\phi, \theta, f))$, with

$$v_n(\phi, \theta, f) \;:\; H^n(C) \to Q(n)\ ;\ x \mapsto (\theta_{-1}(x)(x), f(x))\ .$$

It follows that for any ring A there is a natural identification of the 0-dimensional L-group with the Witt group

$$L\langle B, \beta\rangle^0(A) \;=\; W_{Q(0)}(A)\ .$$

For a field A the morphisms

$$W_{Q(n)}(A) \to L\langle B, \beta\rangle^{2n}(A)\ (n \geq 0)$$

are injections, which are split by

$$L\langle B, \beta\rangle^{2n}(A) \to W_{Q(n)}(A)\ ;\ (C, \phi, \theta, f) \mapsto (H^n(C), \phi_0, v_n(\phi, \theta, f))\ (n \geq 0)\ .$$

For any ring with involution A let $(B(\infty), \beta(\infty))$ be the universal chain bundle of Weiss [19] (cf. §6 above), with isomorphisms

$$\begin{aligned} & \widehat{v}_m(\beta(\infty)) \;:\; H_m(B(\infty)) \to \widehat{H}^m(\mathbb{Z}_2; A)\ , \\ & L\langle B(\infty), \beta(\infty)\rangle^m(A) \;\cong\; L^m(A) \end{aligned}$$

and an exact sequence

$$\cdots \to L_m(A) \overset{1+T}{\to} L^m(A) \to Q_m(B(\infty), \beta(\infty)) \overset{\partial}{\to} L_{m-1}(A) \to \ldots .$$

The cokernel of the symmetrization map in the Witt groups

$$\operatorname{coker}(1+T : L_0(A) \to L^0(A)) \;=\; \operatorname{im}(L^0(A) \to Q_0(B(\infty), \beta(\infty)))$$

was computed for noetherian A by Carlsson [5] in terms of 'Wu invariants' prior to the general theory of Weiss [19].

For $n \geq 0$ let $(B\langle n+1\rangle, \beta\langle n+1\rangle)$ be the $(v_{n+1}=0)$*-universal chain bundle over* A, characterized up to equivalence by the properties

(i) $\widehat{v}_r(\beta\langle n+1\rangle) : H_r(B\langle n+1\rangle) \to \widehat{H}^r(\mathbb{Z}_2; A)$ is an isomorphism for $r \neq n+1$,

(ii) $H_{n+1}(B\langle n+1\rangle) = 0$.

The $(v_{n+1}=0)$*-symmetric* L*-groups of* A are defined by

$$L\langle v_{n+1}\rangle^m(A) \;=\; L\langle B\langle n+1\rangle, \beta\langle n+1\rangle\rangle^m(A) \;(m \geq 0)\;.$$

Define the A-group

$$Q\langle v_{n+1}\rangle \;=\; Q\langle B\langle n+1\rangle, \beta\langle n+1\rangle\rangle^{2n}(S^nA)\;,$$

to fit into the commutative braid of exact sequences

$$\begin{tikzcd}[column sep=small]
0 \arrow[rr, bend left] \arrow[dr] & & Q_{2n}(S^nA) \arrow[rr, bend left, "1+T"] \arrow[dr, "r"] & & Q^{2n}(S^nA) \arrow[rr, bend left] \arrow[dr, "J"] & & 0 \\
& \widehat{Q}^{2n+1}(S^nA) \arrow[ur, "H"] \arrow[dr, "1"] & & Q\langle v_{n+1}\rangle \arrow[ur, "s"] \arrow[dr] & & \widehat{Q}^{2n}(S^nA) \arrow[ur] \arrow[dr] & \\
0 \arrow[rr, bend right] \arrow[ur] & & Q^{2n+1}(S^nA) \arrow[rr, bend right, "0"] \arrow[ur] & & \widehat{Q}^{2n}(S^nA) \arrow[rr, bend right] \arrow[ur, "1"] & & 0
\end{tikzcd}$$

In §11 below we shall make use of the surjections

$$L\langle v_{n+1}\rangle^{2n}(A) \to W_{Q\langle v_{n+1}\rangle}(A)\;;\; (C, \phi, \theta, f) \mapsto (H^n(C), \phi_0, v_n(\phi, \theta, f)) \;(n \geq 0)$$

defined for a field A.

§11. **An example**

As an illustration of the exact sequence of §8

$$\cdots \to L_n(A) \to L\langle B, \beta\rangle^n(A) \to Q_n(B, \beta) \overset{\partial}{\to} L_{n-1}(A) \to \ldots$$

we compute the Witt groups $L^0(A)$, $L_0(A)$, $L\langle v_1\rangle^0(A)$ for A a perfect field of characteristic 2, without appealing to the theorem of Arf [1] on the classification of quadratic forms over such A (cf. Example 2.14 of Ranicki [16]).

For any field $A \begin{cases} L^{2n}(A) \\ L_{2n}(A) \end{cases}$ is the Witt group of nonsingular $\begin{cases} (-1)^n\text{-symmetric} \\ (-1)^n\text{-quadratic} \end{cases}$ forms over A, and $\begin{cases} L^{2n+1}(A) = 0 \\ L_{2n+1}(A) = 0 \end{cases}$ $(n \geq 0)$ - see Ranicki [15] for details.

Let then A be a perfect field of characteristic 2, so that squaring defines an automorphism

$$A \to A \; ; \; a \mapsto a^2 \; .$$

Let A have the identity involution

$$^- \; : \; A \to A \; ; \; a \mapsto \overline{a} \; = \; a \; .$$

As an additive group

$$\widehat{H}^r(\mathbb{Z}_2; A) \; = \; A \; (r \in \mathbb{Z})$$

with A acting by

$$A \times \widehat{H}^r(\mathbb{Z}_2; A) \to \widehat{H}^r(\mathbb{Z}_2; A) \; ; \; (a, x) \mapsto a^2 x \; ,$$

and there is defined an isomorphism of A-modules

$$A \to \widehat{H}^r(\mathbb{Z}_2; A) \; ; \; a \mapsto a^2 \; .$$

The chain bundle over A $(B(\infty), \beta(\infty))$ defined by

$$\begin{aligned} & d_{B(\infty)} \; = \; 0 \; : \; B(\infty)_r \; = \; A \to B(\infty)_{r-1} \; = \; A \; , \\ & \beta(\infty)_s \; = \; \begin{cases} 1 \\ 0 \end{cases} : \; B(\infty)_r \; = \; A \to B(\infty)^{-r-s} \; = \; A \text{ if } \begin{cases} s = -2r \\ s \neq -2r \end{cases} \end{aligned}$$

is universal. The twisted quadratic groups of $(B(\infty), \beta(\infty))$ are given up to isomorphism by

$$\begin{aligned} Q_{2n}(B(\infty), \beta(\infty)) \; &= \; Q^\bullet(A) \; , \\ Q_{2n+1}(B(\infty), \beta(\infty)) \; &= \; Q_\bullet(A) \; , \end{aligned}$$

with the abelian groups $Q^\bullet(A)$, $Q_\bullet(A)$ defined by

$$\begin{aligned} Q^\bullet(A) \; &= \; \{a \in A \,|\, a + a^2 = 0\} \; = \; \mathbb{Z}_2 \; , \\ Q_\bullet(A) \; &= \; A/\{b + b^2 \,|\, b \in A\} \; , \end{aligned}$$

and isomorphisms defined by

$$\begin{aligned} & Q_{2n}(B(\infty), \beta(\infty)) \to Q^\bullet(A) \; ; \; (\phi, \theta) \mapsto \phi_0(1)(1) \; , \\ & \qquad \phi_0 \; : \; B(\infty)^n \; = \; A \to B(\infty)_n \; = \; A \; , \\ & Q_{2n+1}(B(\infty), \beta(\infty)) \to Q_\bullet(A) \; ; \; (\phi, \theta) \mapsto \theta_{-1}(1)(1) \; , \\ & \qquad \theta_{-1} \; : \; B(\infty)^{n+1} \; = \; A \to B(\infty)_{n+1} \; = \; A \; . \end{aligned}$$

A symmetric form over A (M, λ) is even if and only if

$$\lambda(x, x) \; = \; 0 \in A \; (x \in M) \; .$$

A nonsingular even symmetric form over A (M,λ) is hyperbolic, since for any $x \in M$ there exists $y \in M$ such that $\lambda(x)(y) = 1 \in A$, so that a hyperbolic summand may be split off (M,λ)

$$(M,\lambda) \;=\; (Ax \oplus Ay, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \oplus (M',\lambda') \;,$$

with

$$\mathrm{rank}_A M' \;=\; (\mathrm{rank}_A M) - 2 \;.$$

Thus

$$L\langle v_0\rangle^0(A) \;=\; W\langle v_0\rangle(A) \;=\; 0$$

and the symmetrization maps

$$1+T \;:\; L_{2n}(A) \;=\; L_0(A) \to L\langle v_0\rangle^0(A) \to L^{2n}(A)$$

are zero. It is now immediate from the exact sequence

$$\dots \to L_m(A) \overset{1+T}{\to} L^m(A) \to Q_m(B(\infty),\beta(\infty)) \overset{\partial}{\to} L_{m-1}(A) \to \dots$$

that

$$L^{2n}(A) \;=\; Q^\bullet(A) \;,\; L_{2n}(A) \;=\; Q_\bullet(A) \;.$$

In the symmetric case there is defined an isomorphism

$$L^{2n}(A) \to Q^\bullet(A) \;=\; \mathbb{Z}_2 \;;\; (C,\phi) \mapsto \phi_0(v)(v) \;=\; \mathrm{rank}_A H^n(C) \;,$$

sending a $2n$-dimensional symmetric Poincaré complex (C,ϕ) over A to the element $\phi_0(v)(v) \in Q^\bullet(A)$, with $v \in H^n(C)$ the unique cohomology class such that

$$\phi_0(x)(v) \;=\; \phi_0(x)(x) \in \widehat{H}^n(\mathbb{Z}_2;A) \;(x \in H^n(C)) \;.$$

The inverse isomorphism

$$Q^\bullet(A) \;=\; \mathbb{Z}_2 \to L^{2n}(A)$$

sends $1 \in Q^\bullet(A)$ to the $2n$-dimensional symmetric Poincaré complex (C,ϕ) defined by

$$\phi_0 \;=\; 1 \;:\; C^n \;=\; A \to C_n \;=\; A \;,\; C_r \;=\; 0 \;(r \neq 0) \;.$$

In the quadratic case there is defined an isomorphism

$$\partial \;:\; Q_\bullet(A) \to L_{2n}(A) \;;\; a \mapsto (C,\psi)$$

with (C,ψ) the $2n$-dimensional quadratic Poincaré complex over A given by

$$\psi_0 \;=\; \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \;:\; C^n \;=\; A \oplus A \to C_n \;=\; A \oplus A \;,\; C_r \;=\; 0 \;(r \neq n) \;.$$

The inverse isomorphism $L_{2n}(A) \to Q_\bullet(A)$ sends a $2n$-dimensional quadratic Poincaré complex (C,ψ) over A to the Arf invariant $c \in Q_\bullet(A)$ of the nonsingular quadratic form $(H^n(C),(1+T)\psi_0,v^n(\psi))$ over A, as defined by

$$c \;=\; \sum_{i=1}^{m} v^n(\psi)(x_{2i})v^n(\psi)(x_{2i+1}) \in Q_\bullet(A)$$

with $\{x_i \,|\, 1 \leq i \leq m\}$ any basis for $H^n(C)$ such that

$$(1+T)\psi_0(x_i,x_j) \;=\; \begin{cases} 1 & \text{if } (i,j) = (2r,2r+1) \text{ or } (2r+1,2r) \\ 0 & \text{otherwise} \;. \end{cases}$$

The chain bundle over A $(B\langle v_{n+1}\rangle, \beta\langle v_{n+1}\rangle)$ $(n \geq 0)$ defined by

$$B\langle v_{n+1}\rangle_r = \begin{cases} A & \text{if } r \neq n+1 \\ 0 & \text{if } r = n+1 \end{cases},$$
$$d = 0 : B\langle v_{n+1}\rangle_r \to B\langle v_{n+1}\rangle_{r-1},$$
$$\beta\langle v_{n+1}\rangle_s = \begin{cases} 1 \\ 0 \end{cases} : B\langle v_{n+1}\rangle_r \to B\langle v_{n+1}\rangle^{-r-s} \text{ if } \begin{cases} s = -2r, r \neq n+1 \\ \text{otherwise} \end{cases}$$

is $(v_{n+1} = 0)$-universal. Define symmetric bilinear pairings

$$\rho : A \times A \to A ; (a, b) \mapsto ab ,$$
$$\sigma : Q^\bullet(A) \times Q^\bullet(A) \to A ; (1, 1) \mapsto 1$$

such that

$$Q\langle B\langle v_{n+1}\rangle, \beta\langle v_{n+1}\rangle\rangle^{2n}(S^n A) = A \times_\rho A ,$$
$$Q_{2n}(B\langle v_{n+1}\rangle, \beta\langle v_{n+1}\rangle) = A \times_\sigma Q^\bullet(A) ,$$
$$Q_{2n+1}(B\langle v_{n+1}\rangle, \beta\langle v_{n+1}\rangle) = 0 .$$

Let

$$Q\langle v_1\rangle = Q\langle B\langle v_1\rangle, \beta\langle v_1\rangle\rangle^0(A)$$
$$= A \times_\rho A .$$

Given a nonsingular $Q\langle v_1\rangle$-quadratic form (M, λ, μ) over A there exist $v \in M$, $\psi \in \mathrm{Hom}_A(M, M^*)$ such that

$$\lambda(x, y) = \lambda(x, v)\lambda(y, v) + \psi(x)(y) + \psi(y)(x) \in A \ (x, y \in M)$$
$$\mu : M \to Q\langle v_1\rangle = A \times_\rho A ; x \mapsto (\psi(x)(x), \lambda(x, v)) .$$

The morphism

$$L\langle v_1\rangle^0(A) = W_{Q\langle v_1\rangle}(A) \to Q_0(B\langle v_1\rangle, \beta\langle v_1\rangle) ;$$
$$(M, \lambda : M \times M \to A, \mu : M \to Q\langle v_1\rangle) \mapsto \mu(v) = (\psi(v)(v), \lambda(v, v))$$

fits into a short exact sequence

$$0 \to L_0(A) \to L\langle v_1\rangle^0(A) \to Q_0(B\langle v_1\rangle, \beta\langle v_1\rangle) \to 0 .$$

The injection

$$L_{2n}(A) \to L\langle v_{n+1}\rangle^{2n}(A) \to W_{Q\langle v_1\rangle}(A) = L\langle v_1\rangle^0(A)$$

sends the cobordism class of a $2n$-dimensional quadratic Poincaré complex over A (C, ψ) to the Witt class of the nonsingular $Q\langle v_1\rangle$-quadratic form $(H^n(C), (1+T)\psi_0, iv^n(\psi))$, with i the canonical injection

$$i : H_0(\mathbb{Z}_2; A, (-1)^n) = A \to Q\langle v_1\rangle = A \times_\rho A ; a \mapsto (a, 0) .$$

In the special case $A = \mathbb{Z}_2$

$$L_{2n}(\mathbb{Z}_2) = Q_\bullet(\mathbb{Z}_2) = \mathbb{Z}_2 , L^{2n}(\mathbb{Z}_2) = Q^\bullet(\mathbb{Z}_2) = \mathbb{Z}_2 ,$$
$$Q\langle v_{n+1}\rangle = \mathbb{Z}_4 , Q_0(B\langle v_1\rangle, \beta\langle v_1\rangle) = \mathbb{Z}_4 ,$$

with

$$L\langle v_{n+1}\rangle^{2n}(\mathbb{Z}_2) = W_{\mathbb{Z}_4}(\mathbb{Z}_2) = \mathbb{Z}_8$$

the Witt group of nonsingular $\mathbb{Z}_4$-valued quadratic forms over $\mathbb{Z}_2$. See Weiss [19, §11] for the the algebraic Poincaré bordism interpretation of the work of Browder [2] and Brown [4] on the Kervaire invariant and its generalization, which applies also to the work of Milgram [11].

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