DISPERSION RELATIONS FOR SCATTERING BY SPIN-DEPENDENT POTENTIALS

A Thesis
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by

MUHAMMAD JAMEEL
B.Sc. (Hons.); M.Sc. (Karachi)

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PREFACE

I am indebted to Dr. D.J. Candlin for suggesting the field of inquiry, and to Dr. P.W. Higgs for his invaluable guidance, encouragement and help throughout the work leading to the preparation of this manuscript. It is also a pleasure to express my grateful appreciation of the hospitality extended to me at the Tait Institute by Professor N. Kemmer, F.R.S., and other members of the Department.

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The material presented in this dissertation is asserted to be original except where some reference has been cited.

CONTENTS

CHAPTER I

INTRODUCTION

The understanding of the two-nucleon interaction is the central problem in the theory of nuclear forces. Whether specifically many-body forces turn out to be important or not, the investigation of the two-body interaction is bound to give us some insight into the mechanism responsible for the strongest binding known to man.

The earliest attempts to unravel the mystery of these forces have been devoted to the search for a reliable 2-nucleon potential. This was only natural because experience had taught us that potentials furnished a very useful way of describing interaction between particles. Moreover, this approach received an early impetus when the Yukawa potential successfully accounted for the "periphery" of the nucleonic interaction. Even now, there are good reasons to believe that at least for low energies, the Schrödinger equation together with a "correct" potential should tell us a lot about nucleon-nucleon scattering. This is all the more plausible in the case of nucleons because, by virtue of their greater mass, their velocity remains essentially non-relativistic even up to pion production

threshold. Of course, when production processes become energetically possible, the potential picture is expected to lose much of its significance anyway.

However, in spite of the early successes of a Yukawa's meson theory, it became increasingly clear from the analysis of experimental data that a static, central potential was inadequate as a realistic model. If the internucleon force could at all be described in terms of a potential, it has to be spin- and velocity-dependent as well. This meant the introduction of terms to take account of tensor and spin-orbit forces etc., and this continues to be done in an ad hoc sort of manner. The result is the flowering of a wide variety of forms of 2-nucleon potentials, each with some success to its credit. (See, e.g., the review by R.J.N. Phillips[18]).

So far, the only attempts, in theory, to go beyond pure guesswork (or phenomenology, as it is politely called) have been along two essentially different directions. The first type of computations, which were initiated by Yukawa and have since been assiduously followed up by many physicists (e.g. Taketani et al.) may be collectively classed as "meson-theoretical calculations". The main idea behind these efforts if the belief (perhaps quite true) that nuclear forces are mediated by a meson cloud and, naturally, the closer together the nucleons are, the larger the number of pions

taking part. This rather loose inverse relationahip between the range of a force and the corresponding mass of the intermediate state is now widely accepted among elementary particle physicists. However, the quantitative realisation of the Taketani programme takes the form of computing one perturbation graph after another in the framework of field theory. The one pion exchange graph, of course, yielded the famous Yukawa potential which, as expected, was very successful in explaining the long range effect of the nucleon force. Higher order effects have not had the same success in the "medium" range, with the consequent flourishing of phenomenology. Apart from the increasing technical difficulty of calculating higher order graphs, it is very probable that the perturbation series is divergent in which case the whole project is essentially illusory, and any occasional agreements with experiments must be regarded as fortuitous. There have been many variations from straight expansion in the coupling strength or the number of intermediate particles. A comprehensive survey is given in the review of Moravcsik & Noyes[16].

A more hopeful approach has appeared in recent years, following Mandelstam's proposal of a double dispersion relations for the scattering amplitude in a two-particles-in, two-particles-out process. It was initiated by Charap & Fubini[4], who indicated how one can obtain a useful

"potential", starting from the Mandelstam representation. But he could do it reliably only for the case of scalar particles, mainly because no one had derived a Mandelstam representation, or even a single variable dispersion relation, for two particles of spin $\frac{1}{2}$ interacting through a general spin and velocity-dependent potential. Up to now, there have appeared only two papers on this subject and both have touched the problem only partially. Hamilton[9] has proved dispersion relations which are similar to ~~what~~ those Khuri[12] obtained for central potentials, considering only an additional tensor term. The other attempt has been by Buslayev[3] who considers a particle scattered by a spin-orbit potential.

The present work is intended to fill this gap; we obtain dispersion relations (in energy, for fixed momentum transfer) for two spin $\frac{1}{2}$ particles interacting through a complete potential subject only to reasonable physical requirements. The relations obtained turn out to be substantially different in form from those derived or postulated as yet; and the methods used have the virtue of being generalisable to the scattering between systems of arbitrary spin, as indicated in the last chapter. The derivation of these dispersion relations is also a step forward in the fulfilment of the Charap-Fubini programme of deducing a realistic two-nucleon potential.

We will now give a brief outline of this manuscript.

In Chapter II, we discuss the form of the potential for a two-nucleon system, and then go on to derive some properties of the resolvent of the Hamiltonian. We also have a brief look at the spectral decomposition of the total Hamiltonian. Most of the matter contained in this chapter is of a preliminary character, and no originality is claimed.

Chapter III forms the bulk of the thesis in which we obtain the analytic properties and asymptotic behaviour of the Hamiltonian Green's function in the complex energy plane. We start by deriving an integral equation for the Green's function incorporating the outgoing boundary condition. This equation does not have a bounded or square integrable kernel so that the usual methods of solution do not apply. However, it is seen that the kernel is only "weakly" singular (to use the terminology of Mikhlin[15]), and Fredholm's theorems can still be applied. This is the central point of the argument, which permits us to infer that the Green's function is analytic except on the spectrum of the Hamiltonian. We then proceed to investigate the spectrum of the total Hamiltonian, and the approach is in part borrowed from Povzner[19]. The chapter concludes with a detailed investigation of the Green's function at high energy.

The analyticity and asymptotics obtained in Chapter III help us in writing down an explicit form for the scattering

amplitude in Chapter IV. The amplitude is treated in detail, and split into five parts with different spin-invariants, as e.g. in Goldberger, Nambu & Oehme[8]. Each coefficient separately obeys a dispersion relation which is then written down. We close the chapter with a few physical remarks of interest.

Chapter V contains a generalisation of the above results to systems with arbitrarily high spins. It gives only a sketch of the arguments without going into much cumbersome detail.

Summary of Notation

The notation used in the manuscript is summarised here for convenience and easy of reference.

$\underline{r}$, $\underline{k}$ = vectors in the 3-space

$\hat{r}$ = unit vector in the same direction as $\underline{r}$.

d^3r = volume element

$\tilde{dr}$ = surface element

Ω = the entire 3-dimensional space.

K_ρ = sphere with radius ρ and centre origin.

Σ_ρ = surface of the sphere K_ρ

$\frac{d}{dn_s}$ = derivative along the outward drawn normal to some surface at the point $\underline{s}$.

D_x = domain of definition of the operator x.

CHAPTER II

THE HAMILTONIAN AND ITS RESOLVENT

The Schrödinger equation for a system of two interacting nucleons* may be written as

$$(H_0 + V)\,|\,\rangle = E\,|\,\rangle \,. \qquad (1)$$

where E is the total energy and $|\,\rangle$ the state vector of the system; H_0 is the kinetic energy operator of the two particles and V is the interaction between them. Without loss of generality, we may go over to the centre-of-mass system and transcribe the abstract equation (1) in the position representation, thus:

$$\left(-\nabla_r^2 + V(\underline{\sigma}_1, \underline{\sigma}_2 ; \underline{r})\right) u(\underline{\sigma}_1, \underline{\sigma}_2 ; \underline{r}) = E\, u(\underline{\sigma}_1, \underline{\sigma}_2 ; \underline{r}) \qquad (2)$$

where $\underline{\sigma}_1$ and $\underline{\sigma}_2$ are the Pauli spin matrices for the two nucleons and $\underline{r}$ is the distance between them. (We have chosen units such that $\frac{1}{2}\hbar = 1$, $2M = 1$ where M is the mass of the nucleon.) In what follows, we will drop the suffix from ∇_r^2 when there is no ambiguity; we will also frequently omit to exhibit explicitly the functional dependence of the potential V and the wave-function u on spin.

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* In this work, the only relevant property of nucleons is that they have spin $\frac{1}{2}$; consequently, most of the results up to Chapter IV are true for any spin $\frac{1}{2}$ particle.

1. The Form of the Potential

The equation (2) describes the relative motion of the two particles in terms of a force field derived from the inter-nucleon potential $V(\sigma_1, \sigma_2; \underline{r})$. Let us now try to find the most general permissible form of V, subject to reasonable physical requirements. Such investigations have been carried out by many physicists, among them Eisenbud & Wigner[5], Okubo & Marshak[14] and Nishijima[17]. Following them, we list below the conditions that must be satisfied by a non-relativistic two-particle potential in order that it be a physically useful concept. These are

1) translational invariance, i.e. V depends only on the difference of space coordinates.
2) Hermiticity.
3) Rotational invariance.
4) Galilean invariance; this requires that V can depend only on the relative velocity (Or relative momentum) of the two nucleons.
5) Invariance under space inversion.
6) Invariance under time reversal.
7) Symmetry, i.e. V should be invariant under the interchange of the two particles.

In addition to these seven requirements, which are rather obvious on general physical grounds, we will use

another one the reasons for which are not so compelling, viz. we will require that

8) the potential V should not contain any terms higher than linear in the relative momentum of the two particles.

According to Nishijima[17], the solution of the Schrödinger equation is no longer uniquely determined by the usual boundary conditions if the potential contains second or higher powers of the relative momentum. This can be seen by making the replacement $\underline{p} \to i\nabla$ in the Schrödinger equation, which no longer remains definitely elliptic.

The most general form of the potential, compatible with the above requirements, can be shown to be*

$$V(\underline{r}) = V_1(r) + V_2(r)\,\underline{\sigma}_1 \cdot \underline{\sigma}_2 + V_3(r)\,\frac{(\underline{\sigma}_1 \cdot \underline{r})(\underline{\sigma}_2 \cdot \underline{r})}{r^2} + i\,V_4(r)\,(\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{r} \wedge \underline{\nabla}\,, \qquad (3)$$

where r denotes the magnitude $|\underline{r}|$ of the vector $\underline{r}$, and the coefficients $V_i(r)$ $(i = 1,2,3,4)$ are all real-valued. Note that the usual tensor and spin-orbit forces are included, but terms containing e.g. $(\underline{\sigma}_1 \cdot \nabla)(\underline{\sigma}_2 \cdot \nabla)$ have been excluded by the eighth requirement. This point turns out to be a crucial one in our discussion (as will be pointed out at the appropriate place) and it is nice that such terms can be barred on physical grounds.

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* See for e.g. Eisenbud & Wigner[5]. A much more general problem of the same nature is also investigated in Chapter V of this manuscript.

It is perhaps relevant to remark here that we have left out of account any iso-spin considerations. In fact it turns out that, on the basis of charge-independence, the form (3) is simply multiplied by I or $\underline{\tau}_1 . \underline{\tau}_2$ where $\underline{\tau}_1$ and $\underline{\tau}_2$ are the usual iso-spin matrices for the two nucleons. This does not lead to any difficulty in the discussions that follow and so we have decided to drop it. Since charge independence appears to be valid in strong-interactions, we have not lost any generality by this simplification.

As to the coefficients $V_i(r)$, we will not need to assume any particular functional form for them; they are only required to satisfy certain general "good-behaviour" conditions, viz. that

$|V_1(r)|$, $|V_2(r)|$, $|V_3(r)|$ and $|rV_4(r)|$ are each $<$ some $c(r)$ where $c(r)$ is a positive function of r such that:

$$\left.\begin{aligned} &\text{i)} \quad \int_0^\infty r\, c(r)\, dr < \infty \\ &\text{ii)} \quad c(r) = O\left(\frac{1}{r^{3+\epsilon}}\right) \\ &\text{iii)} \quad \left|\frac{\partial c(r)}{\partial \underline{r}}\right| = O\left(\frac{1}{r^{4+\epsilon}}\right) \end{aligned}\right\} \qquad (4)$$

$\frac{\partial}{\partial \underline{r}}$ denote differentiation along any direction.

The Hamiltonian is now defined, as usual, by

$$H = -\nabla_r^2 + V(\underline{r}) , \tag{5}$$

where ∇^2 is just the Laplacian operator and $V(\underline{r})$ is given by equation (3).

Let us say a word or two about the domain of definition $\mathcal{D}_H$ of this operator. The operator H is certainly well-defined over the space C_2 of all "finite" functions i.e. the space of all functions $u(\underline{r})$, $\underline{r} \in \Omega$, (Ω is the entire 3-dimensional space) which are twice continuously differentiable, and which also vanish outside a finite sphere in Ω. This last restriction is placed mainly for simplicity in proofs and may, in fact, be relaxed considerably.

We will, therefore, take C_2 as the domain of definition of the Hamiltonian H. Then, by the properties of the Laplacian and the construction of $V(\underline{r})$, it is clear that H is a linear, Hermitian symmetric operator in the linear vector space C_2. This, however, does not imply that H is a self-adjoint operator in the strict mathematical sense of the term. A rigorous proof of "self-adjointness" will require a much more thorough investigation (as, e.g., in Kato[10]), and we will not attempt it here. Moreover, we will not require this stringent property in this work except when (as in Theorem 2 of this chapter) we use the existence of a spectral family of the operator H. Since the possibility of a spectral resolution of H is intimately connected with the completeness of the energy eigenstates

which is usually assumed extensively in quantum mechanics, this lack of rigour here is not a serious or unique drawback. With this in mind, we will henceforth use the terms "self-adjoint" and "Hermitian" synonymously and may refer to H as a self-adjoint operator.

2. The Resolvent of H.

The resolvent of an operator H is defined as

$$R_\lambda = (H - \lambda I)^{-1}. \tag{6}$$

where λ is any number, in general complex. The investigation of the resolvent of an operator is very important in the study of eigenvalue problems. In fact, as we shall see later on, R_λ is closely related to the Green's function for the total Hamiltonian.

The main result of this section will be the proof of an important theorem about the kernel of the operator R_λ, but, before we can do that, we will have to prove two auxiliary lemmas. The proofs of these follow Povzner[19] very closely, and are given here only for completeness and for establishing some notations.

Lemma 1. For a given ρ, there exist two functions $G_{1,\rho}(\underline{r}, \underline{s})$ and $G_{2,\rho}(\underline{r}, \underline{s})$, defined for $\underline{r} \in K_\rho$, $\underline{s} \in K_{\rho+2}$, such that

$$\int_{K_{\rho+2}} |G_{i,\rho}(\underline{r},\underline{s})|^2 d^3s < M_\rho \quad (i=1,2\,;\ \underline{r} \in K_\rho) \tag{7}$$

where M_ρ is some constant depending only on ρ, and the functions G_i are such that for $u \in \mathcal{D}_H$ and $u^x \equiv Hu$ we have

$$u(\underline{r}) = \int_{K_{\rho+2}} G_{1,\rho}(\underline{r},\underline{s})\, u^x(\underline{s})\, d^3s + \int_{K_{\rho+2}} G_{2,\rho}(\underline{r},\underline{s})\, u(\underline{s})\, d^3s. \quad (\underline{r} \in K_\rho) \tag{8}$$

Proof:

Let $g_\tau(\underline{r}, \underline{s})$ be the Green's function for the Dirichlet's problem for Laplace equation inside the sphere K_τ. Using the known properties of g_τ, and applying Green's formula, we get

$$u(\underline{r}) = -\int_{K_\tau} g_\tau(\underline{r},\underline{s})\, \nabla^2 u(\underline{s})\, d^3s + \int_{\Sigma_\tau} \left\{ \frac{d}{dn_s} g_\tau(\underline{r},\underline{s}) \right\} u(\underline{s})\, d^3s. \tag{9}$$

where $\underline{r}, \underline{s} \in K_\tau$ and $\frac{d}{dn_s}$ denotes differentiation along the outward drawn normal at a variable point $\underline{s}$ of the surface Σ_τ.

Since (9) is true for any τ, we can integrate over $\rho+1 \leq \tau \leq \rho+2$, and obtain

$$u(\underline{r}) = \int_{\rho+1}^{\rho+2} d\tau \int_{K_\tau} g_\tau(\underline{r},\underline{s}) \left[u^x(\underline{s}) - V(\underline{s}) u(\underline{s}) \right] d^3s$$

$$+ \int_{\rho+1}^{\rho+2} d\tau \int_{\Sigma_\tau} \frac{d}{dn_s} g_\tau(\underline{r},\underline{s}) \, u(\underline{s}) \, \tilde{d}\underline{s} \, . \qquad (10)$$

where we have used the fact that $H = -\nabla^2 + V$.

Let us now construct the function $g_1(\underline{r}, \underline{s})$ in the following way.

For $|\underline{s}| = \tau$, $\rho+1 \leq \tau \leq \rho+2$, and $\underline{r} \in K_\rho$, we define

$$g_1(\underline{r},\underline{s}) = \frac{d}{dn_s} g_\tau(\underline{r},\underline{s}) \qquad (11)$$

and for $|\underline{s}| < \rho+1$, $\underline{r} \in K_\rho$, we take $g_1(\underline{r}, \underline{s}) = 0$.

With this definition, we can write the second integral on the right hand side of (10) as an integral over a complete sphere.

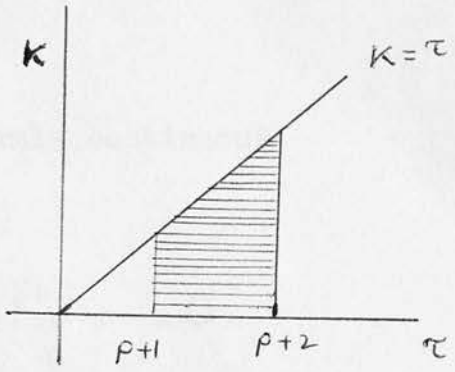
$$\int_{K_{\rho+2}} g_1(\underline{r},\underline{s}) \, u(\underline{s}) \, d^3s \, .$$

As for the first integral, it is transformed into spherical polar coordinates as

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_{\rho+1}^{\rho+2} d\tau \int_0^{\tau} k^2 dk \, g_\tau(\underline{r},\underline{s}) \left[u^x(\underline{s}) - V(\underline{s}) u(\underline{s}) \right]$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\rho+1} k^2 dk \left\{ \int_{\rho+1}^{\rho+2} g_\tau(\underline{r},\underline{s}) \, d\tau \right\} \left[u^x(\underline{s}) - V(\underline{s}) u(\underline{s}) \right]$$

$$+ \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \int_{\rho+1}^{\rho+2} k^2 dk \left\{ \int_K^{\rho+2} g_\tau(\underline{r},\underline{s}) \, d\tau \right\} \left[u^x(\underline{s}) - V(\underline{s}) u(\underline{s}) \right] .$$



area of integration

Defining $g_2(\underline{r}, \underline{s}) = \int_{\rho+1}^{\rho+2} g_\tau(\underline{r}, \underline{s})\, d\tau$, for $|\underline{s}| \leq \rho+1$

and $= \int_{|\underline{s}|}^{\rho+2} g_\tau(\underline{r}, \underline{s})\, d\tau$, for $\rho+1 < |\underline{s}| \leq \rho+2$, (12)

we observe that the above integral can be written as

$$\int_{K_{\rho+2}} g_2(\underline{r}, \underline{s}) \left[u^x(\underline{s}) - V(\underline{s})\, u(\underline{s}) \right] d^3s$$

Here, we encounter a small difficulty due to the spin- and velocity-dependence of the potential. This is because our $V(\underline{s})$ contains a differential operator, acting on $u(\underline{s})$. In order to set $u(\underline{s})$ free, we use integration by parts, and due to the finiteness of u and the boundary condition on g_τ , the surface term disappears so that we get

$$\int_{K_{\rho+2}} g_2(\underline{r}, \underline{s})\, u^x(\underline{s})\, d^3s + \int_{K_{\rho+2}} V(\underline{s})\, g_2(\underline{r}, \underline{s})\, u(\underline{s})\, d^3s .$$

Now, the desired result (8) can be obtained simply by setting

$$\left.\begin{aligned} G_{1,\rho}(\underline{r}, \underline{s}) &= g_2(\underline{r}, \underline{s}) \\ G_{2,\rho}(\underline{r}, \underline{s}) &= g_1(\underline{r}, \underline{s}) + V(\underline{s})\, g_2(\underline{r}, \underline{s}) . \end{aligned}\right\} \qquad (13)$$

Q.E.D.

Lemma 2. The functions G_i are uniformly continuous in root mean square over $\underline{s}$:

$$\lim_{\underline{r}_1 \to \underline{r}} \int_{K_{\rho+2}} \left| G_{i,\rho}(\underline{r},\underline{s}) - G_{i,\rho}(\underline{r}_1,\underline{s}) \right|^2 d^3s = 0 .$$

$$(i = 1, 2 \; ; \; \underline{r} \in K_\rho) . \qquad (14)$$

and the limiting process is uniform.

The proof of this lemma follows immediately from the definition of the functions G_i and the properties of g_ρ.

Now, we are in a position to prove the following:

Theorem II - 1: The resolvent R_λ of the Hamiltonian H is an integral operator, whose kernel $G(\underline{r}, \underline{s}; \lambda)$ is of the Carleman type, i.e. there exists a positive $M(\underline{r}, \lambda)$, depending only on $\underline{r}$ and λ, such that

$$\int_\Omega \left| G(\underline{r}, \underline{s} ; \lambda) \right|^2 d^3s < M(\underline{r}, \lambda) . \qquad (15)$$

Proof: Let us define

$$\left.\begin{aligned} G'_{i,\rho}(\underline{r},\underline{s}) &= G_{i,\rho}(\underline{r},\underline{s}) \qquad \text{for} \quad \underline{s} \in K_{\rho+2} \\ &= 0 \qquad\qquad\quad \text{for} \quad \underline{s} \notin K_{\rho+2} \end{aligned}\right\} \underline{r} \in K_\rho .$$

The relation (8) can then be written as

$$u(\underline{r}) = \int_\Omega G'_{1,\rho}(\underline{r},\underline{s})\, u^x(\underline{s})\, d^3s + \int_\Omega G'_{2,\rho}(\underline{r},\underline{s})\, u(\underline{s})\, d^3s . \qquad (8')$$

We now apply the resolvent operator R_λ to some element

$f \varepsilon L^2$, and get $R_\lambda(f) = u$ (say). By definition $R_\lambda = (H - \lambda I)^{-1}$ so that $f = H u - \lambda u$ i.e. $u \varepsilon \mathcal{D}_H$. Also $Hu = f + \lambda u$. Substituting this expression for $u^x (\equiv Hu)$ in (8'), we get ($\underline{r} \varepsilon K_\rho$) :

$$R_\lambda(f) = \int_\Omega G'_{1,\rho}(\underline{r},\underline{s}) \left[f(\underline{s}) + \lambda R_\lambda(f) \right] d^3s$$

$$+ \int_\Omega G'_{2,\rho}(\underline{r},\underline{s}) R_\lambda(f) \, d^3s. \qquad (16)$$

Since the functions $G'_{i,\rho}$ are also square integrable over $\underline{s}$, the above integrals can be viewed as scalar products in the space L^2, e.g. the second integral may be interpreted as $(G', R_\lambda f)$. We can then transfer the operation of R_λ from f to G' (considered as an L^2 function of $\underline{s}$) and reinterpret:

$$R_\lambda(f) = \int_\Omega G'_{1,\rho}(\underline{r},\underline{s}) f(\underline{s}) \, d^3s + \int_\Omega \lambda \, \overline{R_{\bar\lambda}\left(\bar G'_{1,\rho}(\underline{r},\underline{s})\right)} f(\underline{s}) \, d^3s$$

$$+ \int_\Omega \overline{R_{\bar\lambda}\left(\bar G'_{2,\rho}(\underline{r},\underline{s})\right)} f(\underline{s}) d^3s \qquad (\underline{r} \in K_\rho) \quad (17)$$

Let us consider the bounds on the different terms.

Since $\left\| R_{\bar\lambda}\left(\bar G'_{i,\rho}(\underline{r},\underline{s})\right) \right\| \leq \left\| R_{\bar\lambda} \right\| \left\| G'_{i,\rho}(\underline{r},\underline{s}) \right\|$

(with the norms defined in the usual way) it follows from lemma 1, and the definition of R_λ that

$$\left\| R_{\bar{\lambda}} \left(\bar{G}'_{i,\rho}(\underline{r},\underline{s}) \right) \right\| \leq \frac{1}{|\mathrm{Im}\,\lambda|} M_\rho \,. \qquad (18)$$

Let us now define

$$G_\rho(\underline{r},\underline{s};\lambda) \equiv G'_{1,\rho}(\underline{r},\underline{s}) + \lambda \overline{R_{\bar{\lambda}}\left(\bar{G}'_{1,\rho}(\underline{r},\underline{s})\right)} + \overline{R_{\bar{\lambda}}\left(G'_{2,\rho}(\underline{r},\underline{s})\right)} \qquad (\underline{r} \in K_\rho) \qquad (19)$$

so that we can write (17) neatly as

$$R_\lambda(f) = \int_\Omega G_\rho(\underline{r},\underline{s};\lambda) f(\underline{s}) \, d^3s \,. \qquad (\underline{r} \in K_\rho) \qquad (20)$$

and we have the norm

$$\left\| G_\rho(\underline{r},\underline{s};\lambda) \right\|_{\underline{s}} \leq M_\rho + \frac{|\lambda| M_\rho + M_\rho}{|\mathrm{Im}\,\lambda|} \leq \frac{M_\rho(2|\lambda|+1)}{|\mathrm{Im}\,\lambda|} \,. \qquad (21)$$

If $\underline{r} \in K_\tau$ and $\rho_1, \rho_2 > \tau$, then it follows from equation (20) that

$$\int_\Omega G_{\rho_1}(\underline{r},\underline{s};\lambda) f(\underline{s}) \, d^3s = \int_\Omega G_{\rho_2}(\underline{r},\underline{s};\lambda) f(\underline{s}) \, d^3s \,.$$

and, because f is arbitrary, we can deduce that

$$G_{\rho_1}(\underline{r},\underline{s};\lambda) = G_{\rho_2}(\underline{r},\underline{s};\lambda)$$

i.e., for $\rho > |\underline{r}|$, the function G_ρ does not depend on ρ.

Hence, we may take $G_\rho(\underline{r}, \underline{s}; \lambda) = G(\underline{r}, \underline{s}; \lambda)$ for $\rho > |\underline{r}|$, and obtain

$$R_\lambda(f) = \int_\Omega G(\underline{r}, \underline{s}; \lambda) f(\underline{s}) d^3s . \qquad (\underline{r} \in \Omega) . \tag{22}$$

where G also satisfies (15) by virtue of the relation (21). This concludes the proof of the theorem.

We will now study some of the more obvious properties of the kernel $G(\underline{r}, \underline{s}; \lambda)$.

1) If we assume that the state vectors $|\underline{s}\rangle$, parameterised by the position vector $\underline{s}$, form a complete set in some Hilbert space, then it follows immediately from equation (22) that $G(\underline{r}, \underline{s}; \lambda)$ are just the matrix elements of the resolvent R_λ in terms of that basic set. This set is in fact just the familiar "coordinate representation" widely used in quantum mechanics, and almost exclusively used in this work.

2) Symmetry: From the relation $(R_\lambda f, g) = (f, R_{\bar\lambda} g)$ it follows right away that

$$G(\underline{r}, \underline{s}; \lambda) = \overline{G(\underline{s}, \underline{r}; \bar\lambda)} . \tag{23}$$

3) Let us start with the identity

$$(H - \lambda I) R_\lambda f = f .$$

Writing it in the coordinate representation, we have

$$\int_{\Omega} (-\nabla_r^2 + V(\underline{r}) - \lambda)\, G(\underline{r}, \underline{s}\,;\lambda)\, f(\underline{s})\, d^3s = f(\underline{r})$$

This relation is equivalent to the formal equation (since f is arbitrary)

$$(-\nabla_r^2 + V(\underline{r}) - \lambda)\, G(\underline{r}, \underline{s}\,;\lambda) = \delta^3(\underline{r} - \underline{s}) \qquad (24)$$

which tells us that $G(\underline{r}, \underline{s}; \lambda)$ is the Green's function for the full Schrödinger equation, a result which we had anticipated by the notation. A large portion of our work will be devoted to the study of this "total Hamiltonian Green's function" for the appropriate spin-dependent potential (3). This is because any information about G leads to some knowledge of the scattering solutions of the Schrödinger equation which, in turn, give us the scattering amplitudes.

Incidentally, it follows that, being a Green's function, G has the property of being symmetric in $\underline{r}$, $\underline{s}$, viz.

$$G(\underline{r}, \underline{s}\,;\lambda) = G(\underline{s}, \underline{r}\,;\lambda). \qquad (23a)$$

Combining (23) and (23a), we get

$$G(\underline{r}, \underline{s}\,;\bar{\lambda}) = \overline{G(\underline{r}, \underline{s}\,;\lambda)} \qquad (23b)$$

3. The Spectral Family of H.

Since the Hamiltonian H is a self-adjoint transformation, it possesses a "spectral family, i.e. there exists, a family of projection operators $\{\mathcal{E}_\mu\}$ such that the operator H admits of the representation

$$H = \int_{-\infty}^{\infty} \mu \, d\mathcal{E}_\mu \,. \tag{25}$$

where the integral is understood in the sense of an operator-valued Stieltje's integral. $\{\mathcal{E}_\mu\}$ is also known as the "resolution of the identity" of the operator H.

We will now state without proof some properties of $\mathcal{E}_\mu$.

Theorem II - 2. The projection, $\mathcal{E}_\mu$, is generated by some spectral function $\theta(\underline{r}, \underline{s}; \mu)$ i.e.

$$(\mathcal{E}_\beta - \mathcal{E}_\alpha) f = \int_{\Omega} \left[\theta(\underline{r}, \underline{s}; \beta) - \theta(\underline{r}, \underline{s}; \alpha)\right] f(\underline{s}) \, d^3s. \tag{26}$$

and the function θ has the following properties:

1. $\Delta_{\alpha,\beta}\, \theta(\underline{r}, \underline{s}; \mu) \equiv \theta(\underline{r}, \underline{s}; \beta) - \theta(\underline{r}, \underline{s}; \alpha)$, is continuous in the variables $\underline{r}$ and $\underline{s}$ for any α, β.

2. $\theta(\underline{r}, \underline{s}; \mu)$ has bounded variation on μ in every finite interval (α, β); i.e. there exists a constant $M_{\rho,n}$, depending only on ρ and n, such that the total variation of $\theta(\underline{r}, \underline{s}; \mu)$ over the interval (α, β) does

not exceed $M_{\rho,n}$ for $\underline{r}, \underline{s} \in K_\rho$ and $|\alpha|, |\beta| < n$.

3. $\theta(\underline{r}, \underline{s}; \mu) = \overline{\theta(\underline{s}, \underline{r}; \mu)}$.

4. There exists a number $N_{\rho,n}$ such that

$$\int_\Omega |\theta(\underline{r}, \underline{s}; \mu)|^2 d^3s \leq N_{\rho,n} \quad \text{for } \underline{r} \in K_\rho \text{ and } |\mu| < n.$$

5. $$\int_\Omega \Delta_{\alpha,\beta} \theta(\underline{r}, \underline{t}; \mu) \Delta_{\alpha_1,\beta_1} \theta(\underline{t}, \underline{s}; \mu) d^3t = \Delta_{\alpha',\beta'} \theta(\underline{r}, \underline{s}; \mu).$$

where (α', β') is the overlap of the intervals (α, β) and (α_1, β_1).

The proof of this theorem may be found in the work of Povzner[19]. The results are quite well-known anyway.

CHAPTER III

THE GREEN'S FUNCTION

In this chapter, we will study the Green's function of the total Hamiltonian in greater detail with a view to finding its behaviour in the complex energy plane. As before, we denote by $G(\underline{r}, \underline{s}; \lambda^2)$ the kernel of the resolvent operator $(H - \lambda^2 I)^{-1}$ and introduce further the notation

$$\tilde{G}(\underline{r}, \underline{s}; \lambda) \equiv G(\underline{r}, \underline{s}; \lambda^2)$$

Then, by equation (24) of Chapter II, we have

$$\left(-\nabla_q^2 + V(\underline{q}) - \lambda^2\right) \tilde{G}(\underline{q}, \underline{s}; \lambda) = \delta^3(\underline{q} - \underline{s}) \qquad (1)$$

1. The Integral Equation

In order to study the analytic properties of G, it is more convenient to transform the equation (1), with appropriate boundary conditions (for obtaining outgoing waves only), into an integral equation. This can be done by standard procedure, introducing the function

$$\tilde{G}_0(\underline{r}, \underline{q}; \lambda) \equiv \frac{e^{i\lambda|\underline{r} - \underline{q}|}}{4\pi|\underline{r} - \underline{q}|}$$ which is the Green's

function for the "free" Hamiltonian, i.e. it satisfies

the equation

$$(-\nabla_q^2 - \lambda^2)\,\tilde{G}_0(\underline{r},\underline{q};\lambda) = \delta^3(\underline{r}-\underline{q}) \qquad (2)$$

With the help of (1) and (2), and the Green's formula, we obtain

$$\tilde{G}(\underline{r},\underline{s};\lambda) = \tilde{G}_0(\underline{r},\underline{s};\lambda) - \int_{\Omega} \tilde{G}_0(\underline{r},\underline{q};\lambda)\, V(\underline{q})\, \tilde{G}(\underline{q},\underline{s};\lambda) \qquad (3)$$

where Ω is the entire 3-dimensional space. (The boundary condition has been incorporated in (3) by choosing $(+ i\lambda)$ in the exponential of G_0 instead of $(- i\lambda)$). Note that equation (3) is actually a 4 x 4 matrix equation because V involves the 2 x 2 Pauli spin matrice of the two particles. However, wewill continue to consider (3) formally as a single equation. This is justified because the $\underline{\sigma}$-matrices do not involve $\underline{r}$ or E. We also observe that since $V(\underline{q})$ contains a differential operator, equation (3) is in fact an integro-differential equation. We will now proceed to reduce it to a pure integral equation.

From the form (II-3) of $V(\underline{q})$, we see that the relevant term in (4) is

$$i\int_{\Omega} \tilde{G}_0(\underline{r},\underline{q};\lambda)\, V_4(q)\, (\underline{\sigma}_1 + \underline{\sigma}_2)\cdot \underline{q} \wedge \nabla_q\, \tilde{G}(\underline{q},\underline{s};\lambda) \qquad (4)$$

Here ∇_q is operating on $\tilde{G}$, and our problem is to set $\tilde{G}$ free. The obvious solution is, of course, to transfer the operation of ∇_q from $\tilde{G}$ to $\tilde{G}_0$ through integration by parts. The integral (5) can be written as

$$i\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{I} \qquad , \text{ where}$$

$$\underline{I} = \int_{\Omega} \tilde{G}_0(\underline{r},\underline{q};\lambda)\, V_4(q)\, \underline{q}\wedge\nabla_q\, \tilde{G}(\underline{q},\underline{s};\lambda)\, d^3q .$$

Introducing Cartesian coordinates, we can easily integrate by parts. Then, observing that the surface terms vanish because of the boundary conditions, we obtain

$$\underline{I} = -\int_{\Omega} \underline{q}\wedge\nabla_q \left\{ \tilde{G}_0(\underline{r},\underline{q};\lambda) V_4(q) \right\} \tilde{G}(\underline{q},\underline{s};\lambda)\, d^3q .$$

where $\underline{q}\wedge\nabla_q$ now operates only on the terms within the brackets. We can simplify it even further; in fact, since the coefficient $V_4(q)$ is a function of the scalar $|\underline{q}|$ only, its "level surfaces" are concentric spheres about the origin. The gradient $\nabla V_4(q)$ is therefore in the direction of $\underline{q}$, and so

$$\underline{q}\wedge\nabla V_4(q) = 0$$

Thus, the expression (4) reduces to

$$-i\int_{\Omega} (\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{q}\wedge\nabla_q \left\{ \tilde{G}_0(\underline{r},\underline{q};\lambda) \right\} V_4(q)\, \tilde{G}(\underline{q},\underline{s};\lambda)\, d^3q . \qquad \text{(4a)}$$

We can now calculate $\nabla_q \tilde{G}_o$ explicitly, and we have

$$-i \int_{\Omega} \tilde{G}_o(\underline{r}, \underline{q}; \lambda)(\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{q} \wedge \hat{\underline{q}}_r \left(i\lambda - \frac{1}{|\underline{r} - \underline{q}|}\right) \cdot V_4(q)\, \tilde{G}(\underline{q}, \underline{s}; \lambda)\, d^3q \,.$$

[$\hat{\underline{q}}_r \equiv$ unit vector in the direction $(\underline{q} - \underline{r})$.] (4b)

Taking into account all the terms of the potential, the equation (4) can be written as

$$\tilde{G}(\underline{r}, \underline{s}; \lambda) = \tilde{G}_o(\underline{r}, \underline{s}; \lambda) - \int_{\Omega} K(\underline{r}, \underline{q}; \lambda)\, \tilde{G}(\underline{q}, \underline{s}; \lambda)\, d^3q \,. \tag{5}$$

where the kernel K is expressed as

$$K(\underline{r}, \underline{q}; \lambda) = \tilde{G}_o(\underline{r}, \underline{q}; \lambda)\, U(\underline{r}, \underline{q}; \lambda) \tag{5a}$$

$$U(\underline{r}, \underline{q}; \lambda) = V_1(q) + \underline{\sigma}_1 \cdot \underline{\sigma}_2\, V_2(q) + \frac{(\underline{\sigma}_1 \cdot \underline{q})(\underline{\sigma}_2 \cdot \underline{q})}{q^2} V_3(q)$$

$$+ i(\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{q} \wedge \hat{\underline{q}}_r \left(\frac{1}{|\underline{r} - \underline{q}|} - i\lambda\right) V_4(q) \tag{5b}$$

The above is the fundamental integral equation whose solutions will be studied in this work.

There are two points of interest to be noted here. Firstly, thatequation (5) is formally similar to the basic integral equation of the usual "spin-less" theory of

Schrödinger scattering, if we identify $U(\underline{r}, \underline{q}; \lambda)$ as some sort of "effective potential". Thus the replacement of $V(q)$ (a function of one scalar quantity) by $U(\underline{r}, \underline{q}; \lambda)$ (function of two vectors and a scalar) is the chief complication caused by the inclusion of potentials depending on spin and velocity. This circumstance necessitates a departure from the usual methods of proving the analyticity of G.

The second, and more important, point is that although equation (5) looks like a Fredholm equation, the conventional Fredholm theory cannot be applied here because the kernel $K(\underline{r}, \underline{q}; \lambda)$ is neither continuous nor square integrable over $\underline{r}$ and $\underline{q}$. Fortunately, it turns out that this equation belongs to a more general class of integral equations -- known as "weakly singular" equations -- to which some of the results of Fredholm theory can be extended.

Definition: An integral equation

$$\varphi(r) = f(r) + \gamma \int_{\omega_n} K(r,s)\, \varphi(s)\, d^n s \,, \qquad (6)$$

where the integral is taken over a bounded region ω_n in an n-dimensional space, and r, s are vectors in this space, is said to have a weak singularity if its kernel has the form

$$K(r,s) = \frac{H(r,s)}{|r-s|^{\alpha}} \,.$$

where $0 < \alpha < n$, and $H(\underline{r}, \underline{s})$ is bounded for all $\underline{r}$, $\underline{s}$.

Thus, a restricted sort of singularity is allowed in the kernel. The vital point about such equations is that, after a finite number of iterations (depending on α and n), the kernel becomes bounded. This makes it possible to generalise most of Fredholm's theory to include this type of equations (e.g. Mikhlin[15]). In particular, what is relevant to our problem is the fact that the Fredholm's alternative is also valid for equations with weakly singular kernels. We may recall here with advantage that, the Fredholm alternative says that the parameter Υ in equation (6) is either a regular value or a characteristic value of the equation. (We will not reproduce the proofs of these results as they are readily accessible in Mikhlin's book).

Let us now look at our basic integral equation (5). The maximum power of $|\underline{r} - \underline{q}|$ in the denominator is 2 (one each from $\tilde{G}_o$ and U, 5a, b) whereas the dimensionality of space is 3, so that (5) belongs to the class of weakly singular integral equations. It will now be apparent why the condition (8) on the potential (Chapter II), which excluded quadratic and higher powers of derivatives, is so essential to our programme. For each power of the differential operator, operating on the free Green's function $\tilde{G}_o$ in the integral equation, will introduce an extra $|\underline{r} - \underline{q}|$ in the denominator, and we simply cannot accommodate any more. A potential containing $(\underline{\sigma}_1.\nabla)(\underline{\sigma}_2.\nabla)$, for instance, will contribute a term

$\frac{1}{|\underline{r} - \underline{q}|^3}$ to the equation (5) which will no longer remain only weakly singular. The Fredholm theory will then break down completely because it will take an infinite number of iterations to make the kernel bounded, and we will be in real trouble. However, for the equation (5), as it stands now, the Fredholm alternative is valid and so we have the following.

Theorem III - 1. The equation (5) has a unique solution if and only if the corresponding homogeneous equation

$$\psi(\underline{r};\lambda) = -\frac{1}{4\pi}\int_{\Omega} U(\underline{r},\underline{q};\lambda)\frac{e^{i\lambda|\underline{r}-\underline{q}|}}{|\underline{r}-\underline{q}|}\psi(\underline{q};\lambda)\,d^3q \qquad (7)$$

has no bounded solution.

If the equation (7) is solvable for some values of λ, and has a finite number of linearly independent solutions $\psi_1, \ldots, \psi_n$, then for the solvability of equation (5), it is necessary and sufficient that

$$\int_{\Omega} \tilde{G}_0(\underline{r},\underline{s};\lambda)\,\psi_i(\underline{r};\lambda)\,d^3r = 0 \qquad (i = 1, 2, \ldots n).$$

Thus, we are naturally led to the study of the solvability of the homogeneous equation (7). This can be done in a way similar to that of Povzner[19], but we give the proof for the sake of completeness.

Theorem III - 2. The equation (7) does not possess a solution for $\text{Im}\,\lambda^2 \neq 0$; and for $\text{Im}\,\lambda^2 = 0$, it has a solution if and only if λ^2 is an eigenvalue of the Hamiltonian H.

Proof: First of all, we note that the equation (7) can be recast into the original form (i.e. before the integration by parts carried out in the beginning of this chapter). In fact, equation (7) is equivalent to

$$\psi(\underline{r};\lambda) = -\frac{1}{4\pi}\int_{\Omega}\frac{e^{i\lambda|\underline{r}-\underline{q}|}}{|\underline{r}-\underline{q}|}V(\underline{q})\psi(\underline{q};\lambda)\,d^3q \cdot \quad (7a)$$

Operating on both sides with $(-\nabla_r^2 - \lambda^2)$, we find then that ψ also obeys

$$H\psi = \lambda^2\psi \ .$$

But this is impossible for $\text{Im}\,\lambda^2 \neq 0$ because H is a self-adjoint operator, and so has only real eigenvalues. Thus, we have proved that equation (7) does not have a non-trivial solution for $\text{Im}\,\lambda^2 \neq 0$.

For real λ^2, a non-trivial eigenfunction is possible, but the same argument tells us that, in that case, λ^2 must be an eigenvalue of H. This shows the <u>necessity</u> of λ^2 belonging to the spectrum of H in order that equation (7) has a non-trivial solution. Let us now prove sufficiency.

This is easily done and consists simply in transforming

the differential equation $(-\nabla_{\mathbf{r}}^2 + V(\underline{r}) - \lambda^2)\psi(\mathbf{r}) = 0$ into an integral equation. Using the fact that

$$\left(-\nabla_{\underline{r}}^2 - \lambda^2\right) \frac{e^{i\lambda|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} = \delta^3(\underline{r}-\underline{s})$$

and applying Green's formula to $\psi(\mathbf{r})$ and $\dfrac{e^{i\lambda|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|}$ we obtain the following integral equation for ψ:

$$\psi(\underline{r}) = -\int \frac{e^{i\lambda|\underline{r}-\underline{q}|}}{4\pi|\underline{r}-\underline{q}|} V(\underline{q})\,\psi(\underline{q})\,d^3q .$$

so that every eigenfunction of H also furnishes a solution of (7a) and hence also, of equation (7).

Q.E.D.

The above theorem ensures a unique solution of the integral equation (5) for complex λ^2 and for real λ^2 not belonging to the eigenvalue spectrum of H. Let us introduce, for brevity, the notation

$$T_\lambda \psi \equiv -\frac{1}{4\pi} \int_\Omega U(\underline{r},\underline{q};\lambda) \frac{e^{i\lambda|\underline{r}-\underline{q}|}}{|\underline{r}-\underline{q}|} \psi(\underline{q};\lambda)\,d^3q \qquad (8)$$

so that the integral equation (5) can be written as

$$\tilde{G} = \tilde{G}_o + T_\lambda \tilde{G} . \qquad (9)$$

Then, we have shown, in effect, that the operator $(I - T_\lambda)$

where I is the identity, possesses an inverse for $\mathrm{Im}\,\lambda^2 \neq 0$, and for $\mathrm{Im}\,\lambda^2 = 0$ if only λ^2 is not an eigenvalue of H.

We will now discuss the continuity properties of $T_\lambda \psi$ as a function of the parameter λ.

Theorem III - 3. The family of operators T_λ (for $\mathrm{Im}\,\lambda \geq 0$) depend continuously on λ.

Proof: Let us choose ρ such that, for $\underline{r} \in \Omega - K_\rho$ i.e. the point $\underline{r}$ outside the sphere K_ρ, $\| T_\lambda \psi \|$ does not exceed $\frac{\varepsilon}{2} \| \psi \|$. (Here $\| \ \|$ denote the upper bound of the absolute value, and ε is a given positive number, however small). This is always possible because

$$| T_\lambda \psi | = \left| \frac{1}{4\pi} \int_\Omega \frac{e^{i\lambda |\underline{r} - \underline{q}|}}{|\underline{r} - \underline{q}|} U(\underline{r}, \underline{q}; \lambda) \psi(\underline{q}) \, d^3q \right|$$

$$\leq \frac{1}{4\pi} \int_\Omega \frac{e^{-\mathrm{Im}\lambda |\underline{r} - \underline{q}|}}{|\underline{r} - \underline{q}|} \cdot |U(\underline{r}, \underline{q}; \lambda)| \cdot |\psi(\underline{q})| \, d^3q$$

$$\leq \frac{1}{4\pi} \cdot \| \psi \| \int_\Omega \frac{e^{-\mathrm{Im}\lambda |\underline{r} - \underline{q}|}}{|\underline{r} - \underline{q}|} \cdot |U(\underline{r}, \underline{q}; \lambda)| \, d^3q$$

By taking ρ sufficiently large, the above integral can be made as small as we please for $\underline{r}$ outside K_ρ. In particular, we can choose ρ such that

$$| T_\lambda \psi | < \frac{\varepsilon}{2} \| \psi \| \qquad \text{for} \quad \underline{r} \in \Omega - K_\rho .$$

On the other hand, for $\underline{r} \in K_\rho$, one may obviously choose λ and μ so close that

$$\frac{1}{4\pi} \int \frac{\left| e^{i\lambda|\underline{r}-\underline{q}|} - e^{i\mu|\underline{r}-\underline{q}|} \right|}{|\underline{r}-\underline{q}|} \cdot |U(\underline{r},\underline{q};\lambda)| \, d^3q$$

does not exceed $\frac{1}{2}\varepsilon$. Thus, the theorem is proved.

It is now straightforward to prove the continuity of $\tilde{G}(\underline{r}, \underline{s}; \lambda)$ in the variable λ. Let (α,β) be an interval on the real axis which does not include any point belonging to the discrete spectrum of the Hamiltonian. Then, by theorems 2 and 3 of this chapter, the equation

$$\mathcal{G}(\underline{r},\underline{s};\lambda) = \frac{e^{i\lambda|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} - \frac{1}{4\pi} \int_{\Omega} \frac{e^{i\lambda|\underline{r}-\underline{q}|}}{|\underline{r}-\underline{q}|} U(\underline{r},\underline{q};\lambda) \cdot \mathcal{G}(\underline{q},\underline{s};\lambda) \, d^3q. \qquad (10)$$

has a unique solution for $\alpha \leqslant \mathrm{Re}\ \lambda^2 \leqslant \beta$, $\mathrm{Im}\ \lambda \geqslant 0$. Also, in this region, $\mathcal{G}(\underline{r}, \underline{s}; \lambda)$ coincides with our Green's function, $\tilde{G}(\underline{r}, \underline{s}; \lambda)$ for $\mathrm{Im}\ \lambda^2 \neq 0$. Further, by Theorem 4, the operator T_λ is continuous in $\lambda(\mathrm{Im}\ \lambda \geqslant 0)$; and since, for every point in the above region, the operator $I - T_\lambda$ has an inverse, therefore the family of operators $\tilde{T}_\lambda \equiv (I - T_\lambda)^{-1}$ is also continuous in λ. But

$$\mathcal{G}(\underline{r},\underline{s};\lambda) - \mathcal{G}(\underline{r},\underline{s};\mu) = \tilde{T}_\lambda \frac{e^{i\lambda|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} - \tilde{T}_\mu \frac{e^{i\mu|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|}$$

and so

$$\left| \mathcal{G}(\underline{r},\underline{s};\lambda) - \mathcal{G}(\underline{r},\underline{s};\mu) \right|$$

$$= \left| \tilde{T}_\lambda \frac{e^{i\lambda|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} - \tilde{T}_\lambda \frac{e^{i\mu|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} + \tilde{T}_\lambda \frac{e^{i\mu|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} - \tilde{T}_\mu \frac{e^{i\mu|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} \right|$$

$$\leq \left\| \tilde{T}_\lambda \right\| \cdot \left| \frac{e^{i\lambda|\underline{r}-\underline{s}|} - e^{i\mu|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} \right|$$

$$+ \left\| \tilde{T}_\lambda - \tilde{T}_\mu \right\| \cdot \left| \frac{e^{i\mu|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} \right| .$$

whence follows the continuity of $G(\underline{r}, \underline{s}; \lambda)$ in λ. We can summarise these results in the form of:

Theorem III - 4. Let (α,β) be an interval on the real axis, not containing any point of the discrete spectrum of H. In the region $\alpha \leq \mathrm{Re}\,\lambda^2 \leq \beta$, $\mathrm{Im}\,\lambda \geq 0$, the Green's function $\tilde{G}(\underline{r}, \underline{s}; \lambda) \equiv G(\underline{r}, \underline{s}; \lambda^2)$ tends to a definite limit as λ^2 tends to some point $\varepsilon(\alpha,\beta)$. Further, this convergence is easily seen to be uniform when $\underline{r}$, $\underline{s}$ vary in some finite sphere in Ω.

Before inferring the analytic properties of $\tilde{G}(\underline{r}, \underline{s}; \lambda)$

in detail, it is necessary to investigate the eigenvalue spectrum of the Hamiltonian.

2. The Spectrum of H

First, we will discuss the location of the continuous spectrum, and start with a known result from functional analysis:-

$$\left((-\mathcal{E}_\mu + \mathcal{E}_\nu) f, f\right) = \lim_{\delta \to 0} \frac{1}{\pi} \int_{\mu+i\delta}^{\nu+i\delta} dz \; \mathrm{Im}(R_z f, f) \qquad (11)$$

where f is a finite real function, and the other symbols have the usual meaning. This result is valid only if (μ, ν) does not contain any discrete eigenvalue of H. Then, using (26) of Chapter II, we obtain

$$\int_\Omega \int_\Omega \Delta_{\mu,\nu}\, \theta(\underline{r}, \underline{s}; \eta)\, f(\underline{r})\, f(\underline{s})\, d^3r\, d^3s$$

$$= \lim_{\delta \to 0} \frac{1}{\pi} \int_{\mu+i\delta}^{\nu+i\delta} dz \int_\Omega \int_\Omega \mathrm{Im}\, \tilde{G}(\underline{r}, \underline{s}; \sqrt{z})\, f(\underline{r}) f(\underline{s})\, d^3r\, d^3s. \qquad (12)$$

The integrals on the right hand side are over finite regions due to the finiteness of the function f. By theorem 4, the limiting process is also uniform and so we can interchange the order of integration and limiting, and get

$$\int_\Omega \int_\Omega \Delta_{\mu,\nu}\,\theta(\underline{r},\underline{s};\eta)\, f(\underline{r}) f(\underline{s})\, d^3r\, d^3s$$

$$= \frac{1}{\pi} \int_\mu^\nu d\eta \int_\Omega \int_\Omega \mathrm{Im}\, \tilde{G}(\underline{r},\underline{s};\sqrt{\eta})\, f(\underline{r}) f(\underline{s})\, d^3r\, d^3s .$$

Since f is arbitrary, we have in fact

$$\Delta_{\mu,\nu}\,\theta(\underline{r},\underline{s};\eta) = \frac{1}{\pi} \int_\mu^\nu d\eta\, \mathrm{Im}\, \tilde{G}(\underline{r},\underline{s};\sqrt{\eta}) . \qquad (13)$$

From this formula, we can deduce almost immediately:

Theorem III - 5. The operator H does not have a negative continuous spectrum.

For negative η , $G(\underline{r}, \underline{s}; \sqrt{\eta})$ is a solution of the integral equation

$$\tilde{G}(\underline{r},\underline{s};\sqrt{\eta}) = \frac{e^{-\sqrt{|\eta|}\,|\underline{r}-\underline{s}|}}{4\pi|\underline{r}-\underline{s}|} - \frac{1}{4\pi} \int_\Omega \frac{e^{-\sqrt{|\eta|}\,|\underline{r}-\underline{q}|}}{|\underline{r}-\underline{q}|}\, U(\underline{r},\underline{q})\, \tilde{G}(\underline{q},\underline{s};\sqrt{\eta})\, d^3q$$

whence it follows that

$$\mathrm{Im}\, \tilde{G}(\underline{r},\underline{s};\sqrt{\eta}) = 0 \qquad \text{for } \eta \text{ real and } < 0 \qquad (14)$$

From (13) and (14) it follows that

$$\Delta_{\mu,\nu}\,\theta(\underline{r},\underline{s};\eta) = 0 \qquad \text{for } \mu, \nu < 0 \qquad (15)$$

i.e. it has no eigenvalues in the interval (μ, ν), Since it is true for any negative interval (μ, ν), which does not

contain any discrete eigenvalue, the assertion of the theorem is proved.

Thus, we have shown that the continuous spectrum of the Hamiltonian can only lie along the positive real axis (there cannot be any complex eigenvalues because H is self-adjoint).

The problem of the discrete spectrum is much more difficult. It has been known for some time that if the potential V is spherically symmetric, then the Hamiltonian does not have an L^2-eigenfunction belonging to a positive eigenvalue. Attempts to generalise this result have been only partially successful. Kato[11] claims to have obtained the same result for a general class of non-symmetric potentials, but even he does not take the spin-orbit term into account. On physical grounds, of course, it is reasonable to expect that the Hamiltonian will not have a positive discrete spectrum -- a "bound" state with positive total energy will be a monstrosity indeed! But, from the mathematical standpoint, the question still remains open.

Fortunately, we can make more definite pronouncements about the discrete spectrum on the negative real axis. In fact, we will prove below that H has only a <u>finite</u> number of negative eigenvalues belonging to L^2-eigenfunctions. In the proofs below, we will often implicitly assume the

truth of the following two obvious statements:-

a) If A is a self-adjoint operator, defined on L^2, its expectation value

$$(\psi, A\psi) = \int_{\Omega} \psi A\psi d^3r, \quad \psi \varepsilon L^2, \quad \text{is always real.}$$

b) For A to have a negative (discrete) eigenvalue, it is necessary and sufficient that there exist a $\psi \varepsilon L^2$ such that $(\psi, A\psi) < 0$.

Next we will prove a useful lemma.

Lemma: Let there be two self-adjoint operators $H_1 = -\nabla^2 + V_1(\underline{r})$ and $H_2 = -\nabla^2 + V_2(\underline{r})$, such that $(\psi, V_1\psi) < (\psi, V_2\psi)$ for an arbitrary $\psi \varepsilon L^2$. Then the operator H_2 does not have more (discrete) negative eigenvalues than H_1.

Proof: Suppose the lemma is false and let H_1 have n and H_2 $(n + 1)$ negative eigenvalues. Let us assume for simplicity that they are not degenerate, and call their eigenfunctions as $\psi_i^{(1)}$ and $\psi_i^{(2)}$ respectively. The ψ_i's are, of course, L^2 and we can normalise them to unity : $(\psi, \psi) = 1$.

Since the eigenfunctions belonging to different eigenvalues are mutually orthogonal, the eigenfunctions of H_2 span an $(n+1)$-dimensional sub-space of the space L^2. Hence, given any n functions from L^2, we can find a $\psi \varepsilon L^2$ which is orthogonal to all of them and for which $(\psi, H_2\psi) < \lambda_{n+1} < 0$ where λ_{n+1} is the algebraically largest eigenvalue of H_2. Taking the eigenfunction $\psi_i^{(1)}$ of H_1 as a particular set of n L^2-functions, we obtain the result that there exists a $\psi \varepsilon L^2$ such that $(\psi, \psi_i^{(1)}) = 0$ $(i = 1,2,\ldots, n)$ and

$(\psi, H_2\psi) \leq \lambda_{n+1} < 0.$

However, by the condition of the lemma, $(\psi, H_1\psi) < (\psi, H_2\psi)$. Therefore, this $\psi \varepsilon L^2$ is such that $(\psi, \psi_i^{(1)}) = 0$ $(i = 1,2,\ldots, n)$ and $(\psi, H_1\psi) < 0$, which implies that H_1 has at least $(n + 1)$ negative eigenvalues, contrary to our assumption. Hence the lemma. Actually the result of this lemma is quite well-known in one form or another. This particular statement is best suited to our purpose.

It is apparent now that we have to look for a suitable simple potential with which our $V(\underline{r})$ can be compared. Due to the well-behaviour requirements imposed on the coefficients $V_i(r)$ in Chapter II, we can always find a positive number α such that for any $\psi \varepsilon L^2$,

$$\left|(\psi, V(\underline{r})\psi)\right| < \alpha\,(\psi, c(r)\psi) + \left|(\psi, iV_4(r)(\underline{\sigma}_1+\underline{\sigma}_2)\cdot \underline{r}\wedge\underline{\nabla}\,\psi)\right|, \qquad (16)$$

where $c(r)$ is a positive, spherically symmetric function. Note that it is again the spin-orbit term which is obstinately non-conformist. However, it turns out that it is easily calculated explicitly.

$$(\psi, iV_4(r)(\underline{\sigma}_1+\underline{\sigma}_2)\cdot \underline{r}\wedge\underline{\nabla}\,\psi) = i\int \bar{\psi}(\underline{r})V_4(r)(\underline{\sigma}_1+\underline{\sigma}_2)\cdot \underline{r}\wedge\underline{\nabla}_r\psi(\underline{r})\,d^3r.$$

Since the expectation value is real, we can write

$$2\left(\psi, iV_4(r)(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{r}\wedge\underline{\nabla}\,\psi\right) = i\int_{\Omega}\bar{\psi}(\underline{r})\,V_4(r)\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{r}\wedge\underline{\nabla}\psi(\underline{r})\,d^3r$$

$$+\int_{\Omega}\bar{\psi}(\underline{r})\{V_4(r)\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot(\underline{r}\wedge\overleftarrow{\underline{\nabla}})\}\psi(\underline{r})\,d^3r.$$

Let us integrate the first integral by parts in the same way as we obtained the integral equation for Green's function. Since V_4 and ψ also vanish at spatial infinity ($\psi\varepsilon L^2$), we can throw away the boundary term and get:

$$\int_{\Omega}\bar{\psi}\,V_4\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{r}\wedge\underline{\nabla}\psi\,d^3r = -\int_{\Omega}\{\bar{\psi}\,V_4\}(\underline{\sigma}_1+\underline{\sigma}_2)\cdot(\underline{r}\wedge\overleftarrow{\nabla})\,\psi\,d^3r.$$

where ∇ now acts on the product $\psi(\underline{r})V_4(r)$. But V_4 depends only on the scalar r and so $\nabla V_4(r)$ is parallel to $\underline{r}$ so that $\underline{r}\wedge\nabla V_4(r) = 0$. Then, we get

$$\int_{\Omega}\bar{\psi}\,V_4\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{r}\wedge\nabla\psi\,d^3r = -\int\bar{\psi}\left\{V_4\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{r}\wedge\overleftarrow{\nabla}\right\}\psi\,d^3r$$

This leads to

$$\left(\psi, iV_4(r)\,(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\underline{r}\wedge\nabla\,\psi\right) = 0 \qquad \text{for any } \psi\varepsilon L^2.$$

Now (16) reduces to

$$\left|\left(\psi, V(\underline{r})\psi\right)\right| < \alpha\left(\psi, c(r)\psi\right)$$

$$\therefore \left(\psi, V(\underline{r})\psi\right) > -\alpha\left(\psi, c(r)\psi\right) \qquad \text{any } \psi\varepsilon L^2 \qquad (16a)$$

$((\psi, c(r)\psi)$ is positive definite because $c(r)$ is a positive function).

Hence, by the above lemma, our Hamiltonian $-\nabla^2 + V(\underline{r})$ does not have more negative (discrete) eigenvalues than a similar Hamiltonian with potential $-\alpha c(r)$.

Let us now put the potential $-\alpha c(r)$ in the Schrödinger equation and separate the variables. The solution can in the usual notation be written as:

$$\psi_{n\ell m}(\underline{r}) = R_{n\ell}(r)\, Y_{\ell m}(\theta, \varphi)$$

where the function $\phi(r) = rR_{n\ell}(r)$ satisfies the radial equation

$$L_\ell(\phi) \equiv \left[-\frac{d^2}{dr^2} - \alpha c(r) + \frac{\ell(\ell+1)}{r^2} \right] \phi(r) = 0$$

Owing to the condition $c(r) = O\left(\frac{1}{r^{2+\varepsilon}}\right)$ (this condition is actually weaker than that imposed on our potential), it is easy to see that, for sufficiently large ℓ, $-\alpha c(r) + \frac{\ell(\ell+1)}{r^2} > 0$.

Then

$$(\phi, L_\ell \phi) = \int \bar{\phi} \left(-\frac{d^2}{dr^2}\right) \phi \, dr + \int \bar{\phi} \left\{ -\alpha c(r) + \frac{\ell(\ell+1)}{r^2} \right\} \phi \, dr$$

$$= \int \left| \frac{d\phi}{dr} \right|^2 dr + \int \bar{\phi} \left\{ -\alpha c(r) + \frac{\ell(\ell+1)}{r^2} \right\} \phi \, dr$$

$$> 0$$

(after integrating the 1st integral by parts and ignoring the surface term).

i.e. $L_\ell(\phi)$ does not have a negative eigenvalue, for sufficiently large ℓ. For smaller ℓ, $L_\ell(\phi)$ can have only a finite number of negative eigenvalues, and each eigenvalue will have only a finite multiplicity due to the angular functions $Y_{\ell m}(\theta,\phi)$. Thus, the negative discrete spectrum of the operator $-\nabla^2 - \alpha c(r)$ is finite, and hence also of our Hamiltonian (because of condition 16(a)).

We have thus established the following facts about the spectrum of H:

(1) There is no complex spectrum.

(2) There is no negative, continuous spectrum.

(3) There is probably no positive, discrete spectrum, but we have not been able to prove this.

(4) The negative discrete spectrum is finite.

3. Analytic Properties of G.

On the basis of the above information, and the other theorems proved previously, we can now make the following statements about the behaviour of the Green's function $\tilde{G}(\underline{r}, \underline{s}; \lambda) \equiv G(\underline{r}, \underline{s}; \lambda^2)$ in the finite λ^2 (or energy — E) plane:-

i) The function $G(\underline{r}, \underline{s}; E)$ is analytic in the entire complex E-plane, cut along the positive real axis, with the exception of a finite number of points E_j on the negative real axis where it has simple poles. The residues at these poles must be real because, according to formula (14), $G(\underline{r}, \underline{s}; E)$ does not have an imaginary part for $E < 0$.

ii) The limit

$$\lim_{\epsilon \to 0+} G(\underline{r}, \underline{s}; E + i\epsilon)$$

exists for all $E > 0$, with the possible exception of a finite number of points $E_i > 0$, for which the homogeneous equation

$$w = G_0 U w$$

may have non-trivial solutions. These points E_i belong to the positive discrete spectrum of Hamiltonian, if any.

iii) It has the symmetry property

$$G(\underline{r}, \underline{s}; E + i\epsilon) = \overline{G(\underline{r}, \underline{s}; E - i\epsilon)} \qquad (17)$$

so that the discontinuity across the cut is given by

$$\lim_{\epsilon \to 0+} 2i \, \mathrm{Im}\, G(\underline{r}, \underline{s}; E + i\epsilon) .$$

The vanishing of the imaginary part for negative E

is thus another expression of the absence of a negative continuous spectrum, which would give rise to a cut.

The above properties are all we would require to know about the behaviour of $G(\underline{r}, \underline{s}; E)$ in the finite E-plane. Let us now study the function for large $|E|$.

4. Asymptotics of G

We want to know the behaviour of $G(\underline{r}, \underline{s}; E)$ at high energy. We start with the equation

$$\left(-\nabla_r^2 + V(\underline{r}) - E\right) G(\underline{r}, \underline{s}; E) = \delta^3(\underline{r} - \underline{s}), \tag{1}$$

with

$$V(\underline{r}) = V_1(r) + V_2(r)\, \underline{\sigma}_1 \cdot \underline{\sigma}_2 + V_3(r)\, \frac{(\underline{\sigma}_1 \cdot \underline{r})(\underline{\sigma}_2 \cdot \underline{r})}{r^2} + i\, V_4(r)\, (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{r} \wedge \nabla$$

$$= \tilde{V}(\underline{r}) + i\, V_4(r)\, \underline{\sigma} \cdot \underline{r} \wedge \nabla \qquad (\underline{\sigma} = \underline{\sigma}_1 + \underline{\sigma}_2) \tag{18}$$

where we have lumped all gradient-free terms into $\tilde{V}(\underline{r})$ because they can all be treated in the same way.

Let us write

$$G(\underline{r}, \underline{s}; E) = \frac{e^{i\phi(\underline{r})}}{4\pi |\underline{r} - \underline{s}|}. \tag{19}$$

This is always possible and the form is, of course, suggested by the structure of the Green's function for the "free" Hamiltonian. The function ϕ will also depend on E and possibly also on $\underline{s}$, but we have suppressed these variables for convenience.

Since G is a matrix in the spin-space of the two particles, so must ϕ be. This implies that ϕ will not in general commute with $\nabla\phi$ or $\nabla^2\phi$, which introduces some complications as will be seen later.

Let us now substitute (19) into (1), to obtain:

$$-(\nabla^2 e^{i\phi})\frac{1}{4\pi|\underline{r}-\underline{s}|} + 2(\nabla e^{i\phi})\cdot\frac{(\underline{r}-\underline{s})}{4\pi|\underline{r}-\underline{s}|^3}$$

$$+ e^{i\phi}\,\delta^3(\underline{r}-\underline{s}) + \tilde{V}(\underline{r})\frac{e^{i\phi}}{4\pi|\underline{r}-\underline{s}|}$$

$$+ iV_4(r)\frac{\underline{\sigma}\wedge\underline{r}\cdot(\nabla e^{i\phi})}{4\pi|\underline{r}-\underline{s}|} - iV_4(r)\frac{\underline{\sigma}\wedge\underline{r}\cdot(\underline{r}-\underline{s})}{4\pi|\underline{r}-\underline{s}|^3}e^{i\phi}$$

$$-E\frac{e^{i\phi}}{4\pi|\underline{r}-\underline{s}|} = \delta^3(\underline{r}-\underline{s}) \qquad (20)$$

The δ-function singularity is taken care of if

$$\phi(\underline{r})\Big|_{\underline{r}=\underline{s}} = 0\,, \qquad (21)$$

and we require, in addition, that

$$-\left(\nabla^2 e^{i\phi}\right)\frac{1}{4\pi|\underline{r}-\underline{s}|}+\left(iV_4(r)\,\underline{\sigma}\wedge\underline{r}+\frac{2(\underline{r}-\underline{s})}{|\underline{r}-\underline{s}|^2}\right).\frac{(\nabla e^{i\phi})}{4\pi|\underline{r}-\underline{s}|}$$

$$+\tilde{V}(\underline{r})\frac{e^{i\phi}}{4\pi|\underline{r}-\underline{s}|}+iV_4(r)\frac{\underline{\sigma}\wedge\underline{r}.\underline{s}}{4\pi|\underline{r}-\underline{s}|^3}e^{i\phi}-E\frac{e^{i\phi}}{4\pi|\underline{r}-\underline{s}|}=0.$$

Multiplying from the right by $-4\pi\,|\underline{r}-\underline{s}|\,e^{-i\phi}$, we get

$$\left(\nabla^2 e^{i\phi}\right)e^{-i\phi}-\left(iV_4(r)\,\underline{\sigma}\wedge\underline{r}+\frac{2(\underline{r}-\underline{s})}{|\underline{r}-\underline{s}|^2}\right).\left(\nabla e^{i\phi}\right)e^{-i\phi}$$

$$-\tilde{V}(\underline{r})-iV_4(r)\frac{\underline{\sigma}\wedge\underline{r}.\underline{s}}{|\underline{r}-\underline{s}|^2}+E=0. \qquad \text{——— (22)}$$

Now, it can be verified by direct computation that

$$(\nabla e^{i\phi})\,e^{-i\phi}=i\nabla\phi+\sum_{r=1}^{\infty}\frac{[i\phi,[i\phi,\cdots[i\phi,i\nabla\phi]]\cdots]_r}{(r+1)!} \qquad \text{——— (23)}$$

where $[\ldots[\ ]\ldots]_r$ denotes the r-fold multiple commutator. Also:

$$\left(\nabla^2 e^{i\phi}\right)e^{-i\phi}=i\nabla^2\phi-(\nabla\phi)^2+\sum_{r=1}^{\infty}\frac{[i\phi,[i\phi,\ldots[i\phi,i\nabla^2\phi]]\cdots]}{(r+1)!}$$

$$+\frac{1}{3!}\left\{\left[i\phi,-(\nabla\phi)^2\right]+2i\nabla\phi.\left[i\phi,i\nabla\phi\right]\right\}+\cdots\cdots \text{(24)}$$

Substituting (23) and (24) in (22), we obtain

$$E - \tilde{V}(\underline{r}) - i V_4(r) \frac{\underline{\sigma} \wedge \underline{r} \cdot \underline{s}}{|\underline{r} - \underline{s}|^2}$$

$$- i \left(i V_4(r) \underline{\sigma} \wedge \underline{r} + \frac{2(\underline{r} - \underline{s})}{|\underline{r} - \underline{s}|^2} \right) \cdot \nabla \phi$$

$$+ i \nabla^2 \phi - (\nabla \phi)^2 \qquad + \text{terms involving commutators}$$

of ϕ with $\nabla\phi$, $(\nabla\phi)^2$ and $\nabla^2\phi$ etc. $= 0$ (25)

To find the asymptotic solution, we must expand ϕ in decreasing powers of E^β where β is some positive number. Further, if the series is to make any sense at all as $|E| \to \infty$ it must not contain too high powers of E; call the highest power E^α. Then, we can write the asymptotic series for ϕ as:

$$\phi = E^\alpha \sum_{j=0}^{\infty} \phi_j E^{-j\beta}, \qquad \text{with } j \text{ integral}; \ \beta > 0 \qquad (26)$$

where α and β are yet to be determined.

If we put the series (26) in (25), we see that the presence of the infinite series of commutators will generate terms containing larger and larger power of E^α, so that the equation (25) will become meaningless as $E \to \infty$ if α is positive. This seems to indicate that in the series (26), we can start at best with E^0. However,

in that case, the commutators will not yield any positive power of E; and, in fact, no other term will have a positive power of E except the term E^1 itself. Since, after substitution of (26), the equation (25) must be an identity in E (at least for large E), the above situation is impossible. The only way out of this dilemma is to conclude that

i) the series (26) does begin with a positive power of E (i.e. $\alpha > 0$)

ii) and that the first term ϕ_0 is diagonal so that it commutes with everything else.

It now remains to determine the values of α and β. It is easy to see that the term cancelling E^1 must come from $(\nabla\phi)^2$; if any other term yields one containing E^1, then $(\nabla\phi)^2$ will provide a higher power of E which will have no balancing term. This consideration fixed $\alpha = \frac{1}{2}$, and a little reflection will convince one that β must also be equal to $\frac{1}{2}$. Thus, the asymptotic series is

$$\phi = E^{\frac{1}{2}}\phi_0 + \sum_{j=0}^{\infty} \phi_{j+1} E^{-\frac{1}{2}j}, \quad \text{with } \phi_0 \text{ diagonal.} \qquad (26(a))$$

Since higher terms will vanish as $E \to \infty$, we need consider only ϕ_0 and ϕ_1. Substituting $\phi = E^{\frac{1}{2}}\phi_0 + \phi_1$, in (25) and remembering that ϕ_0 is diagonal, we obtain

$$E - \tilde{V}(\underline{r}) - i V_4(r) \frac{\underline{\sigma} \wedge \underline{r} \cdot \underline{s}}{|\underline{r} - \underline{s}|^2} + i E^{\frac{1}{2}} \nabla^2 \phi_0 + i \nabla^2 \phi_1$$

$$- i \left(i V_4(r) \underline{\sigma} \wedge \underline{r} + \frac{2(\underline{r} - \underline{s})}{|\underline{r} - \underline{s}|^2} \right) \cdot \left(E^{\frac{1}{2}} \nabla \phi_0 + \nabla \phi_1 \right)$$

$$- \left(E^{\frac{1}{2}} \nabla \phi_0 + \nabla \phi_1 \right) \cdot \left(E^{\frac{1}{2}} \nabla \phi_0 + \nabla \phi_1 \right) + \text{terms involving}$$

commutators of ϕ_1 with $\nabla \phi_1$ etc. + terms having ϕ_2, ϕ_3 etc. $= 0$. (27)

The important point to note here is that the commutators and terms higher than ϕ_1 will NOT contain positive powers of E, so that we can equate the coefficients of E and $E^{\frac{1}{2}}$ in (27) to zero, without taking any account of commutators. We obtain then the following relation:-

From coefficient of E :-

$$1 - (\nabla \phi_0)^2 = 0. \qquad (28)$$

From coefficient of $E^{\frac{1}{2}}$:-

$$i \nabla^2 \phi_0 - 2 \nabla \phi_0 \cdot \nabla \phi_1 + \left\{ V_4(r) \underline{\sigma} \wedge \underline{r} - \frac{2i(\underline{r} - \underline{s})}{|\underline{r} - \underline{s}|^2} \right\} \cdot \nabla \phi_0 = 0. \qquad (29)$$

We will solve these equations successively for ϕ_0 and ϕ_1. The equation (28) is a very convenient one for obtaining ϕ_0,

but, unfortunately, there is too much freedom in its solution. However, we observe that this equation does not involve any potentials so that its solution will also be independent of the potential. In particular, we can take $V = 0$, in which case the solution is very well-known. In fact, the "free" Green's function, with out-going boundary condition is just $\frac{e^{i\sqrt{E}\,|\underline{r}-\underline{s}|}}{4\pi\,|\underline{r}-\underline{s}|}$ so that, in our notation

$$\phi = \sqrt{E}\,|\underline{r}-\underline{s}|\,. \qquad \text{(Zero potential case)} \qquad (30)$$

Comparing it with the expansion (26(a)), we see that

$$\phi_0 = |\underline{r}-\underline{s}| \;;\; \phi_1, \phi_2, \cdots \text{etc.} = 0\,.$$

But the equations for ϕ_1 etc. involve the potential so that we are justified in identifying only the first term ϕ_0 with the corresponding term in the "free" Green's function. Thus we obtain uniquely

$$\phi_0 = +\,|\underline{r}-\underline{s}|\,. \qquad (31)$$

It is easily verified that it satisfies equation (28) as well as (21); it also verifies our conclusion that the first term ϕ_0 does not involve any spin-matrices.

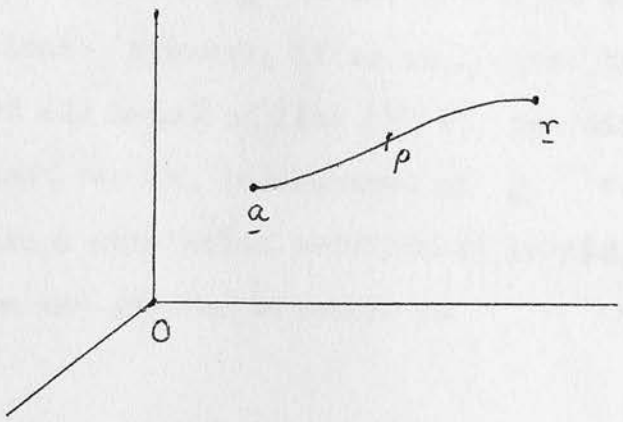
Using (31), the equation (29) reduces to

$$\left[\nabla\phi_1 - \tfrac{1}{2} V_4(r)\, \underline{\sigma} \wedge \underline{r} \right] \cdot (\underline{r} - \underline{s}) = 0. \qquad (32)$$

This is equivalent to writing

$$\nabla\phi_1 - \tfrac{1}{2} V_4(r)\, \underline{\sigma} \wedge \underline{r} = (\underline{r} - \underline{s}) \wedge \underline{F}(\underline{r}) , \qquad (32(a))$$

where $\underline{F}$ is at the moment quite arbitrary. This equation



can be solved in a standard way by taking a line integral from some constant point $\underline{a}$ to the point $\underline{r}$. The integral is independent of the choice of the path because $\nabla\phi_1$ is obviously irrotational ($\nabla \wedge \nabla\phi_1 \equiv 0$). We can, therefore, take the path to be a straight line from $\underline{a}$ to $\underline{r}$.

Thus $$\phi_1(\underline{r}) = \tfrac{1}{2} \int_{\underline{a}}^{\underline{r}} V_4(|\underline{\rho}|)\, \underline{\sigma} \wedge \underline{\rho} \cdot d\underline{\rho} + \int_{\underline{a}}^{\underline{r}} (\underline{\rho} - \underline{s}) \wedge \underline{F} \cdot d\underline{\rho} \qquad (33)$$

Also : $$\underline{\rho} = \tfrac{1}{2}(1-k)\,\underline{a} + \tfrac{1}{2}(1+k)\,\underline{r} \quad , \text{ with } -1 \le k \le 1.$$

$$\therefore d\underline{\rho} = \tfrac{1}{2}(\underline{r} - \underline{a})\, dk .$$

Then, the integral becomes

$$\phi_1(\underline{r}) = -\frac{1}{4}\int_{-1}^{1} dK\ \underline{\sigma}.\underline{r}\wedge\underline{a}\ V_4\left\{\left|\tfrac{1}{2}(1+K)\underline{r}+\tfrac{1}{2}(1-K)\underline{a}\right|\right\}$$

$$-\frac{1}{2}\int_{-1}^{1} dK\ \underline{F}(K;\underline{r},\underline{s}).\left\{\underline{r}\wedge\underline{a}+\underline{s}\wedge(\underline{r}-\underline{a})\right\}. \qquad \text{——— (34)}$$

Taking different values of $\underline{a}$ in the above, we will get different solutions. However, it is well-known that any two solutions of any equation like (32(a)) can differ only by a constant vector, independent of $\underline{r}$. We can, therefore, obtain a particular solution by taking $\underline{a} = \underline{s}$, and then the general solution is

$$\phi_1(\underline{r}) = -\frac{1}{4}\int_{-1}^{1} dK\ \underline{\sigma}.\underline{r}\wedge\underline{s}\ V_4\left\{\left|\tfrac{1}{2}(1+K)\underline{r}+\tfrac{1}{2}(1-K)\underline{s}\right|\right\} + \text{const. vector}$$

Now, using the condition (21), viz. $\phi(\underline{r})\Big|_{\underline{r}=\underline{s}} = 0$, it is easy to see that this constant vector must vanish so that

$$\phi_1(\underline{r}) = -\frac{1}{4}\,\underline{\sigma}.\underline{r}\wedge\underline{s}\int_{-1}^{1} dK\ V_4\left\{\left|\tfrac{1}{2}(1+K)\underline{r}+\tfrac{1}{2}(1-K)\underline{s}\right|\right\}. \qquad \text{——— (35)}$$

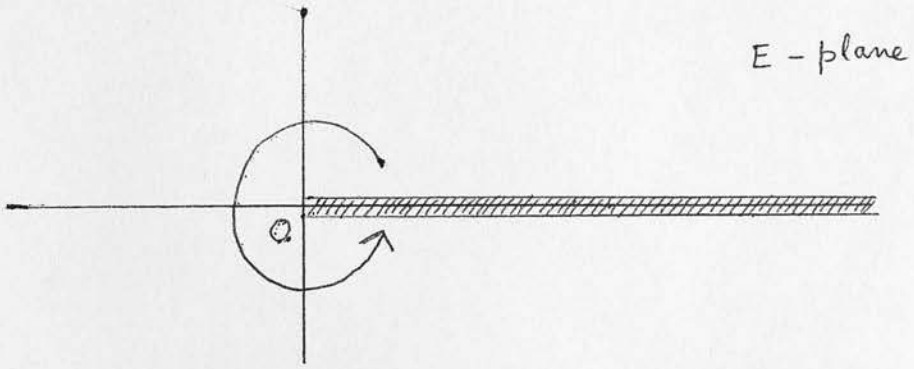
Combining equations (19), (26(a)), (31) and (35), we obtain the following asymptotic estimation for G :-

$$G(\underline{r}, \underline{s}\,; E) \sim \frac{e^{i\sqrt{E}\,|\underline{r}-\underline{s}|}}{4\pi\,|\underline{r}-\underline{s}|} \times$$

$$\times\; e^{-\frac{1}{4}(\sigma_1+\sigma_2),\, \underline{r}\wedge\underline{s} \int_{-1}^{1} dk\, V_4\left(|\frac{1}{2}(1+k)\underline{r} + \frac{1}{2}(1-k)\underline{s}|\right)} + O\left(\frac{1}{\sqrt{E}}\right)$$

$$\sim \frac{e^{i\sqrt{E}\,|\underline{r}-\underline{s}|}}{4\pi\,|\underline{r}-\underline{s}|}\left(1 + O\left(\frac{1}{\sqrt{E}}\right)\right). \qquad (36)$$

i.e. at large energy, $G(\underline{r}, \underline{s}; E)$ behaves essentially as the Green's function for the free Hamiltonian. A similar result was obtained by Faddeyev[6], using a different technique, in the absence of non-central forces.



Now, the cut in the energy-plane is along the positive real axis, so we can define $\sqrt{E}$ as follows:

Take $E = Re^{i\theta}$; $0 \leq \theta < 2\pi$, $0 < R$.

and $\sqrt{E} = |R^{\frac{1}{2}}|\, e^{i\phi}$; $0 \leq \phi < \pi$.

so that the whole E-plane is mapped onto the upper half of the $\sqrt{E}$-plane. Therefore, in formula (36), $\sqrt{E}$ always has

a non-negative imaginary part. This ensures that, for large complex E, $G(\underline{r}, \underline{s}; E)$ vanishes very rapidly; to be precise

$$\left| G(\underline{r}, \underline{s}; E) \right| \sim \frac{e^{-\operatorname{Im}\sqrt{E}\,|\underline{r}-\underline{s}|}}{|\underline{r}-\underline{s}|} \quad \text{as} \quad |E| \to \infty. \tag{37}$$

and $\operatorname{Im}\sqrt{E}$ is always ≥ 0.

This completes our study of the Green's function.

CHAPTER IV

THE SCATTERING AMPLITUDE

In this chapter, we will write down dispersion relations for the elastic scattering amplitude of two nucleons interacting through the potential $V(\underline{r})$ as given by II - (3). We will use the Green's function $G(\underline{r}, \underline{s}; E)$ whose properties have been investigated in the last chapter.

Following the usual convention, we write an eigenstate ψ_o of the free Hamiltonian, for two particles of spin $\frac{1}{2}$, as:-

$$\psi_o = e^{i\underline{k}\cdot\underline{r}} \chi_\sigma^\nu \quad , \quad |\underline{k}|^2 = E \, , \qquad (1)$$

where σ is the total spin $(\sigma_1 + \sigma_2)$ and ν the component in a chosen direction. Two particles of spin $\frac{1}{2}$ can form either a triplet state $(\uparrow\uparrow)$ of multiplicity 3, or a singlet state $(\uparrow\downarrow)$ of multiplicity one. In the notation essentially of Blatt & Weisskopf[1], we can write

$$\chi_\sigma^\nu = \mathcal{Y}_{\sigma\,\sigma_1\sigma_2}^{\nu}$$

$$= \sum_{\nu_1=-\sigma_1}^{+\sigma_1} \sum_{\nu_2=-\sigma_2}^{+\sigma_2} C_{\sigma_1\sigma_2}(\sigma, \nu\,;\, \nu_1, \nu_2)\, Y_{\sigma_1\nu_1}(1)\, Y_{\sigma_2\nu_2}(2) \qquad (2)$$

where the Y's are simultaneous eigenstates of spin + one component of spin for each particle, and the C's are appropriate Clebsch-Gordon coefficients. Thus, for a state with two spin $\frac{1}{2}$ particles $(\underline{\sigma} = 0, 1)$, χ_σ^ν is a 4-component column vector

$$\chi_\sigma^\nu = \begin{pmatrix} \mathcal{Y}^0_{0\,\frac{1}{2}\,\frac{1}{2}} \\ \mathcal{Y}^1_{1\,\frac{1}{2}\,\frac{1}{2}} \\ \mathcal{Y}^0_{1\,\frac{1}{2}\,\frac{1}{2}} \\ \mathcal{Y}^{-1}_{1\,\frac{1}{2}\,\frac{1}{2}} \end{pmatrix} \qquad (3)$$

in the product spinor space of the two particles. The χ's are also orthonormal,

$$\text{i.e.} \quad \bar{\chi}_{\sigma'}^{\nu'} \chi_\sigma^\nu = \delta_{\sigma\sigma'} \delta_{\nu\nu'} . \qquad (4)$$

We can now write down the outgoing scattered waves with the help of our Green's function $G(\underline{r}, \underline{s}; E)$:

$$\psi_+ = \psi_0 + \int G(\underline{r}, \underline{s}; E)\, V(\underline{s})\, \psi_0(\underline{s})\, d^3s \qquad E = |\underline{k}|^2$$

$$= e^{i\underline{k}\cdot\underline{r}} \chi_\sigma^\nu + \int G(\underline{r}, \underline{s}; E) \Big[V_1(s) + V_2(s)\, \underline{\sigma}_1 \cdot \underline{\sigma}_2 + V_3(s) \frac{(\underline{\sigma}_1 \cdot \underline{s})(\underline{\sigma}_2 \cdot \underline{s})}{s^2} + i\, V_4(s)\, (\underline{\sigma}_1 + \underline{\sigma}_2) \cdot \underline{s} \wedge \nabla_s \Big] e^{i\underline{k}\cdot\underline{s}}\, d^3s\, \chi_\sigma^\nu$$

$$= e^{i\underline{k}\cdot\underline{r}} \chi_\sigma^\nu + \int G(\underline{r}, \underline{s}; E) \Big[\tilde{V}(\underline{s}) - \sqrt{E}\, V_4(s)\, \underline{\sigma} \cdot \underline{s} \wedge \hat{\underline{k}} \Big] e^{i\underline{k}\cdot\underline{s}} d^3s\, \chi_\sigma^\nu .$$

——— (5)

where $\tilde{V}$ includes all terms except the spin-orbit one.

Let us now use the well-known expression for the scattering amplitude in potential theory, viz.

$$T = (\psi_0 , V\psi_+)$$

$$= (V\psi_0 , \psi_+) \qquad \text{since } V \text{ is Hermitian.}$$

In detail:

$$T = \chi_{\sigma'}^{\nu'} \Big[\int_{\Omega} d^3r \left\{ \tilde{V}(\underline{r}) + \sqrt{E}\, V_4(r)\, \underline{\sigma}\cdot\underline{r}\wedge\hat{\underline{k}}' \right\} e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}$$

$$+ \int_{\Omega} d^3r\, d^3s \left\{ \tilde{V}(\underline{r}) + \sqrt{E}\, V_4(r)\, \underline{\sigma}\cdot\underline{r}\wedge\hat{\underline{k}}' \right\} e^{-i\underline{k}'\cdot\underline{r}} \times$$

$$\times\, G(\underline{r},\underline{s};E) \left\{ \tilde{V}(\underline{s}) - \sqrt{E}\, V_4(s)\, \underline{\sigma}\cdot\underline{s}\wedge\hat{\underline{k}} \right\} e^{i\underline{k}\cdot\underline{s}} \Big] \chi_{\sigma}^{\nu} \cdot$$

$$\left(|\underline{k}|^2 = |\underline{k}'|^2 = E \right) . \qquad (6)$$

The amplitude is a function of energy ($\underline{k}^2 = \underline{k}'^2 = E$) and the scattering angle $\cos^{-1}(\hat{\underline{k}}\cdot\hat{\underline{k}}')$, in addition to invariant combinations of the spin variables and the unit vectors $\hat{\underline{k}}$, $\hat{\underline{k}}'$. We go over into the more usual variables of energy and momentum transfer:

$$\underline{\tau} \equiv \underline{k} - \underline{k}' \quad , \quad \underline{n} \equiv \underline{k} + \underline{k}' = \sqrt{4E - \tau^2}\, \hat{\underline{n}} \, .$$

Then, we have

$$T_{\sigma'\nu',\,\sigma\nu}(E,\tau) = \bar{\chi}_{\sigma'}^{\nu'} \Big[\int d^3r \left\{ \tilde{V}(\underline{r}) + \sqrt{E}\, V_4(r)\, \underline{\sigma}\cdot\underline{r}\wedge(\hat{\underline{n}} - \underline{\tau}) \right\} e^{i\underline{\tau}\cdot\underline{r}}$$

$$+ \int d^3r\, d^3s \left\{ \tilde{V}(\underline{r}) + \sqrt{E}\, V_4(r)\, \underline{\sigma}\cdot\underline{r}\wedge(\hat{\underline{n}} - \underline{\tau}) \right\} e^{\frac{1}{2}i\underline{\tau}\cdot(\underline{r}+\underline{s})}\, G(\underline{r},\underline{s};E) \times$$

$$\times\, e^{-i\sqrt{E-\frac{\tau^2}{4}}\,\hat{\underline{n}}\cdot(\underline{r}-\underline{s})} \left\{ \tilde{V}(\underline{s}) - \sqrt{E}\, V_4(s)\, \underline{\sigma}\cdot\underline{s}\wedge(\hat{\underline{n}} + \underline{\tau}) \right\} \Big] \chi_{\sigma}^{\nu} \cdot$$

$$\qquad (7)$$

The analytic properties of T can be read off from (7) if the integrals are convergent.

Since all the $V_i(r) = O\left(\frac{1}{r^{3+\varepsilon}}\right)$ $[i=1,2,3]$ and $r V_4(r) = O\left(\frac{1}{r^{3+\varepsilon}}\right)$ and τ is real, the first integral is uniformly convergent for all finite E.

The second integral, however, gives trouble. Because of the reality of τ , $\left| e^{\frac{1}{2} i \underline{\tau} \cdot (\underline{r} + \underline{s})} \right| = 1$, and so the only important terms are

$$G(\underline{r}, \underline{s}; E)\, e^{- i \sqrt{E - \tau^2/4}\ \ \hat{\underline{n}} \cdot (\underline{r} - \underline{s})}$$

From the estimation III - (37) for the Green's function, we have the bound

$$\left| G(\underline{r}, \underline{s}; E)\, e^{- i \sqrt{E - \frac{\tau^2}{4}}\ \hat{\underline{n}} \cdot (\underline{r} - \underline{s})} \right| \leq e^{\left(- \mathrm{Im}\sqrt{E} + \mathrm{Im}\sqrt{E - \tau^2/4}\right) |\underline{r} - \underline{s}|}$$

In the complex E-plane, cut along the positive real axix, Im $\sqrt{E}$ is always positive, and for $\sqrt{E - \tau^2/4}$ the worst case occurs when the imaginary part is positive. Now, it can be proved (cf. e.g. Bogoliubov[2]) that, since τ^2 is positive,

$$\mathrm{Im}\sqrt{E - \frac{\tau^2}{4}} \quad > \quad \mathrm{Im}\sqrt{E}$$

so that the second integral in equation (7) diverges badly. This disaster is already encountered in Khuri's work and is not related to the introduction of spin-dependent terms in

the potential. It can be averted only by imposing more stringent conditions on the potential. In fact, it is necessary to require not only that the $V_i(r) = O(\frac{1}{r^{3+\varepsilon}})$ but that

$$V_i(r)\,,\ r V_4(r) \sim \frac{e^{-\alpha r}}{r^{3+\varepsilon}} \qquad \text{where } \alpha > 0 \quad (i = 1,2,3.) \tag{8}$$

Then, the integrand in the second integral on the right hand of (7) is bounded:

$$\left|\text{integrand}\right| \leq \left|(1+\sqrt{E})^2\right| \frac{e^{-\alpha r}}{r^{3+\varepsilon}} \cdot \frac{e^{-\alpha s}}{s^{3+\varepsilon}}\, e^{\left(-\operatorname{Im}\sqrt{E} + \operatorname{Im}\sqrt{E-\tau^2/4}\right)|\underline{r}-\underline{s}|}$$

It can also be seen easily that

$$\operatorname{Im}\sqrt{E-\tau^2/4} < \operatorname{Im}\sqrt{E} + \frac{\tau}{2} \qquad \text{so that}$$

$$\left|\text{integrand}\right| < \left|(1+\sqrt{E})^2\right| \frac{e^{-\alpha r}}{r^{3+\varepsilon}} \cdot \frac{e^{-\alpha s}}{s^{3+\varepsilon}} \cdot e^{\frac{\tau}{2}|\underline{r}-\underline{s}|}$$

$$< \left|(1+\sqrt{E})^2\right| \frac{e^{-\alpha r}}{r^{3+\varepsilon}} \cdot \frac{e^{-\alpha s}}{s^{3+\varepsilon}}\, e^{\frac{1}{2}\tau(r+s)}$$

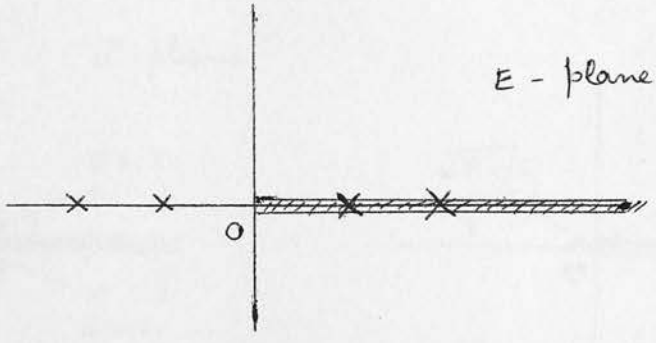
$$\leq \left|(1+\sqrt{E})^2\right| \frac{1}{r^{3+\varepsilon}} \cdot \frac{1}{s^{3+\varepsilon}} \quad , \text{ if } \tau \leq 2\alpha$$

Hence, for finite E and for $\tau \leq 2\alpha$, the second integral is also uniformly convergent in $\underline{r}$ and $\underline{s}$. The integral,

therefore, has the same analytic properties in the finite complex energy plane, as the integrand.

Apart from the Green's function, which has already been studied previously the other terms in the integrand contribute branch points at $E = 0$ and $E = \frac{\tau^2}{4}$. However, the E-plane is already cut along the positive real axis due to the singularities of G so that there is essentially no additional singularity in this sheet.

Hence $T_{\sigma'\nu', \sigma\nu}(E, \tau)$, for real τ, has the following behaviour in the finite complex E-plane:

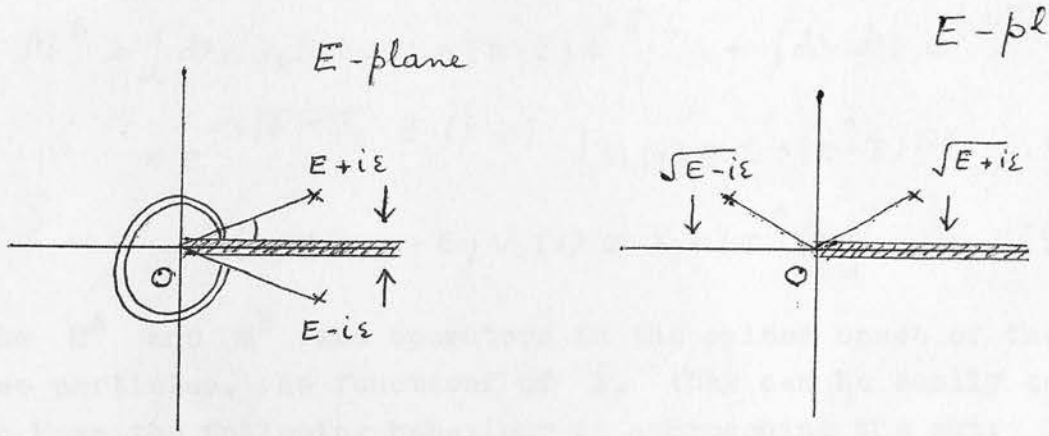


T is meromorphic in the whole complex plane, cut along the positive real axis. The only other singularities are poles, lying on the real axis, corresponding to the discrete spectrum of the Hamiltonian. On the negative axis, the poles are finite in number; but on the positive axis, we will have to <u>assume</u> that there are only a finite number of poles, if any at all. This assumption is very plausible, because for a potential not containing the spin orbit term it has been proved (Kato[11]) that there

is no positive discrete spectrum.

It now remains to investigate the behaviour of T as we approach the cut from above and from below. This is necessary in order that we can express the discontinuity across the cut as an imaginary part.

We know already that $G(E - i\varepsilon) = \overline{G(E + i\varepsilon)}$. The only other energy-dependent factors appearing in the expression (7) for T are $\sqrt{E}$ and $\sqrt{E - \frac{\tau^2}{4}}$, whose behaviour will be clear from the diagrams below.



It is obvious that

$$\lim_{\epsilon \to 0+} \sqrt{E+i\varepsilon} = - \lim_{\epsilon \to 0+} \sqrt{E-i\varepsilon} \qquad \text{for } E > 0 .$$

A similar argument can be applied to $\sqrt{E - \frac{\tau^2}{4}}$ although, in this case, the discontinuity starts only at $\frac{\tau^2}{4}$, i.e. further down the positive axis.

If we now look at equation (7), it will be apparent that we can decompose T into two parts which behave slightly differently at the cut. In fact, we can write:

$$T_{\sigma'\nu',\sigma\nu}(E,\tau;\hat{n},\hat{\tau}) = \bar{\chi}^{\nu'}_{\sigma'}\left[M^A(E,\tau;\hat{n},\hat{\tau}) + \sqrt{E}\, M^B(E,\tau;\hat{n},\hat{\tau})\right]\chi^{\nu}_{\sigma} \quad (9)$$

where

$$M^A = \int d^3r\, \tilde{V}(r)\, e^{i\underline{\tau}\cdot\underline{r}} + \int d^3r\, d^3s\, e^{\frac{1}{2}i\underline{\tau}\cdot(\underline{r}+\underline{s})} \times$$
$$\times\, e^{-i\sqrt{E-\tau^2/4}\,\hat{n}\cdot(\underline{r}-\underline{s})} \left\{\tilde{V}(\underline{r})\, G(\underline{r},\underline{s};E)\,\tilde{V}(\underline{s}) - \right.$$
$$\left. - E\, V_4(r)\,\underline{\sigma}\cdot\underline{r}\wedge(\hat{n}-\hat{\tau})\, G(\underline{r},\underline{s};E)\, V_4(s)\,\underline{\sigma}\cdot\underline{s}\wedge(\hat{n}+\hat{\tau})\right\}. \quad (9a)$$

and

$$M^B = \int d^3r\, V_4(r)\,\underline{\sigma}\cdot\underline{r}\wedge(\hat{n}-\hat{\tau})\, e^{i\underline{\tau}\cdot\underline{r}} + \int d^3r\, d^3s\, e^{\frac{1}{2}i\underline{\tau}\cdot(\underline{r}+\underline{s})} \times$$
$$\times\, e^{-i\sqrt{E-\tau^2/4}\,\hat{n}\cdot(\underline{r}-\underline{s})} \left\{V_4(r)\,\underline{\sigma}\cdot\underline{r}\wedge(\hat{n}-\hat{\tau})\, G(\underline{r},\underline{s};E)\,\tilde{V}(\underline{s}) - \right.$$
$$\left. - \tilde{V}(\underline{r})\, G(\underline{r},\underline{s};E)\, V_4(s)\,\underline{\sigma}\cdot\underline{s}\wedge(\hat{\tau}+\hat{n})\right\}. \quad (9b)$$

The M^A and M^B are operators in the spinor space of the two particles. As functionf of E, they can be easily seen to have the following behaviour in approaching the cut:

$$\lim_{\epsilon\to 0+} M^{A,B}(E-i\epsilon) = \lim_{\epsilon\to 0+} \overline{M^{A,B}(E+i\epsilon)} \qquad (E>0). \quad (10)$$

It is worth noting here that this convenient property holds for M^A and M^B separately, but <u>not</u> for the total amplitude. Hence the necessity for the decomposition (9).

Let us now look at the behaviour of M^A and M^B for large $|E|$. Observing that, asymptotically, $e^{-i\sqrt{E-\tau^2/4}\,\hat{n}\cdot(r-s)}$ will be compensated by $G(\underline{r}, \underline{s}; E)$ by virtue of the estimation III - (37), it is obvious from inspection that, for large energy, M^B approaches a constant, whereas M^A increases not faster than E. Hence we require two subtractions for M^A and one for M^B. This is in contrast with the case of central potentials (e.g. Khuri's work) where it turns out that the scattering amplitude does not need any subtractions if only we separate off the first Born term.

We have now obtained all the information we require in order to write down integral representations for M^A and M^B.

However, it will be more convenient to "round off" all the $\underline{\sigma}$-matrices and find dispersion relations for their coefficients only. By arguments, based on physical considerations (similar to those used in Chapter II for obtaining the potential), it can be shown that we can express M^A and M^B as a sum of five amplitudes with different spin invariants. We choose the invariants used by Goldberger, Nambu and Ohme[8], and write

$$M^A = M^A_\alpha + M^A_\beta (\underline{\sigma}_1\cdot\hat{n})(\underline{\sigma}_2\cdot\hat{n}) + M^A_\gamma (\underline{\sigma}_1\cdot\hat{\tau})(\underline{\sigma}_2\cdot\hat{\tau}) + M^A_\delta (\underline{\sigma}_1\cdot\hat{n}\wedge\hat{\tau})(\underline{\sigma}_2\cdot\hat{n}\wedge\hat{\tau}) + M^A_\epsilon (\underline{\sigma}_1+\underline{\sigma}_2)\cdot\hat{n}\wedge\hat{\tau}. \quad (11)$$

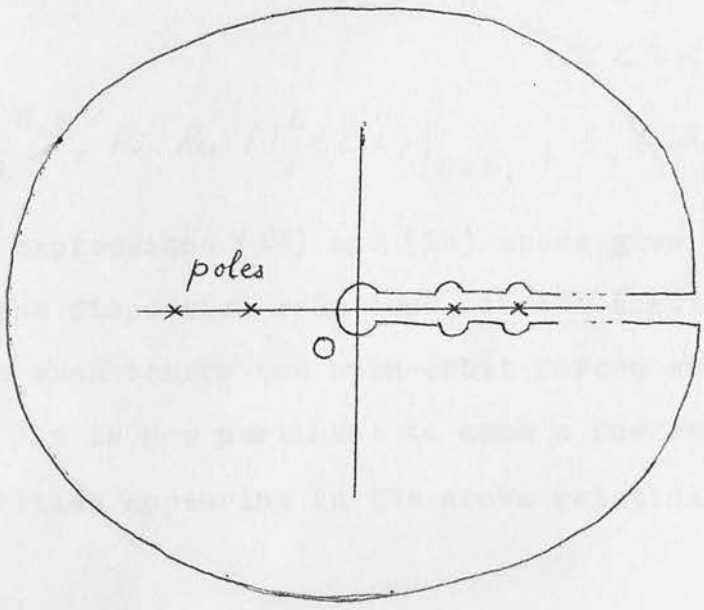
and similarly for M^B.

The coefficients $M^A_\alpha, \ldots M^B_\alpha, \ldots$ are now functions only of the scalars E and τ. The scattering amplitude can then be written as:

$$T_{\sigma'\nu',\sigma\nu}(E,\tau;\hat{n},\hat{\tau}) = \bar{\chi}^{\nu'}_{\sigma'}\Big[\left(M^A_\alpha + \sqrt{E}\,M^B_\alpha\right) + \left(M^A_\beta + \sqrt{E}\,M^B_\beta\right)\underline{\sigma}_1\cdot\hat{n}\;\underline{\sigma}_2\cdot\hat{n} + \left(M^A_\gamma + \sqrt{E}\,M^B_\gamma\right)\underline{\sigma}_1\cdot\hat{\tau}\;\underline{\sigma}_2\cdot\hat{\tau} + \left(M^A_\delta + \sqrt{E}\,M^B_\delta\right)\underline{\sigma}_1\cdot\hat{\underline{n}}\wedge\hat{\underline{\tau}}\;\underline{\sigma}_2\cdot\hat{\underline{n}}\wedge\hat{\underline{\tau}} + \left(M^A_\epsilon + \sqrt{E}\,M^B_\epsilon\right)(\underline{\sigma}_1+\underline{\sigma}_2)\cdot\hat{\underline{n}}\wedge\hat{\underline{\tau}}\Big]\chi^{\nu}_{\sigma}\,. \tag{12}$$

Further, since the spin coefficients do not involve energy, the analytic properties and asymptotic behaviour of the coefficient $M^A_\alpha \ldots M^B_\alpha \ldots$ are the same as those of their respective parents M^A and M^B.

We will now write the dispersion relations for a typical M^A_ζ and M^B_ζ (ζ can be α, β, γ, δ, ε). The contour of integration in the E-plane is shown below.



$$\mathrm{Re}\, M^{A}_{\xi}(E,\tau) = \frac{1}{\pi} P \int_{0}^{\infty} \frac{E-E_a}{E'-E_a} \cdot \frac{E-E_b}{E'-E_b} \frac{\mathrm{Im}\, M^{A}_{\xi}(E',\tau)\, dE'}{E'-E}$$

$$+ \frac{E-E_b}{E_a-E_b} \mathrm{Re}\, M^{A}_{\xi}(E_a,\tau) + \frac{E-E_a}{E_b-E_a} \mathrm{Re}\, M^{A}_{\xi}(E_b,\tau)$$

$$+ \sum_{j} \frac{E-E_b}{E_j-E_b} \cdot \frac{E-E_a}{E_j-E_a} \cdot \frac{R_j^{A,\xi}}{E-E_j} \qquad (\tau < 2\alpha) \qquad (13)$$

where $R_j^{A,\xi} = \mathrm{Re.\ Res.}\, M^{A}_{\xi}(E,\tau)\big|_{E=E_j}$; $E_j \in$ discrete spectrum of

We have included the possibility of having poles on the positive real axis as well (due to the positive spectrum of H), and the subtractions are carried out at positive energies E_a, E_b.

Similarly,

$$\mathrm{Re}\, M^{B}_{\xi}(E,\tau) = \frac{1}{\pi} P \int_{0}^{\infty} \frac{E-E_a}{E'-E_a} \frac{\mathrm{Im}\, M^{B}_{\xi}(E',\tau)\, dE'}{E'-E}$$

$$+ \mathrm{Re}\, M^{B}_{\xi}(E_a,\tau) + \sum_{j} \frac{E-E_a}{E_j-E_a} \frac{R_j^{B,\xi}}{E-E_j} \cdot \qquad (\tau < 2\alpha) \qquad (14)$$

where $R_j^{B,\xi} = \mathrm{Re.\ Res.}\, M^{B}_{\xi}(E,\tau)\big|_{E=E_j}$; $E_j \in$ discrete spectrum of H.

The expressions (13) and (14) above give the correct form of the dispersion relations for the scattering amplitude when tensor and spin-orbit forces are also present. It is now pertinent to make a few remarks about the quantities appearing in the above relations.

1) Continuation into the Unphysical Region. In the above integrals, the functions are integrated over all positive energies. However, not all the positive values can occur in physical scattering, which is characterised by $|\cos\theta| \leq 1$ where θ is the scattering angle. In fact, it is easy to see that for a finite momentum transfer τ, we must have $E \geq \tau^2/4$ for physical scattering. We require, therefore a prescription for continuing Im M_ℓ to the "unphysical" region $0 \leq E < \tau^2/4$. The usual method is to expand $M(E,\tau)$ (dropping super- and sub-scripts for the time being) into partial waves:

$$M(E,\tau) = \sum_{\ell=0}^{\infty} (2\ell+1) M_\ell(E) P_\ell\left(1 - \frac{\tau^2}{2E}\right) . \qquad (15)$$

The entire $\cos\theta$-dependence is thus transferred to the Legendre polynomials, and we can make an explicit continuation. However, it is not at all evident that the series (15) is convergent in the region $\tau^2 > 4E$, which is precisely the range in which we are interested. Klein and Zemach[13] have carried out a careful investigation of this problem in the case of central potentials, and have shown that the series is in fact convergent in the region $E < \tau^2/4$, provided the potential has an asymptotic exponential decay $\sim e^{-ar}$ where $a > |\frac{\tau}{2}|$, which is precisely the restriction we had to place on the potential in order to obtain any dispersion relation at all. Essentially,

their method consists in studying an expression equivalent to our 9(a) and 9(b) for real energy E and complex momentum transfer τ (or equivalently complex angle of scattering). They then show that the imaginary part of the amplitude (which is all we actually require) is an analytic function of τ in a certain region of the complex plane. This result is then utilised to prove the convergence of the partial-wave expansion inside an ellipse, of semi-major axis $1 + \frac{2a^2}{E}$, in the complex $\cos\theta$-plane. This region is easily seen to include the range on the real axis, in which we are interested, provided $|\tau| < 2a$. We can carry out an analysis in exactly the same way as Klein & Zemach to show that the expansion (15) is valid in the entire interval $E \geqslant 0$, with essentially no more restrictions on the potential (or the momentum transfer) than those imposed in, for instance, Khuri's work[12].

2) Calculation of the Residue

The calculation of the residue R_j is quite simple if we use the following expression for the Green's function:

$$G(\underline{r}, \underline{s}; E + i\varepsilon) = \langle \underline{r} | \frac{1}{H - E - i\varepsilon} | \underline{s} \rangle .$$

Introducing complete sets of energy eigenstates, we will get

$$G(\underline{r},\underline{s};E) = \sum_{E',E''} \frac{\langle \underline{r}|E'\rangle\langle E'|E''\rangle\langle E''|\underline{s}\rangle}{E'-E-i\varepsilon}$$

when the $\sum$ is a symbolic summation which includes integrations etc. Using the orthogonality of the eigenstates, we can write

$$G(\underline{r},\underline{s};E+i\varepsilon) = \sum_{E'} \frac{\bar{\psi}(\underline{r},E')\psi(\underline{s},E')}{E'-E-i\varepsilon} + \phi(\underline{r},\underline{s};E)$$

where the summation is now over discrete eigenvalues only. The residue at a particular eigenvalue is, therefore,

$$\text{Res. } G(\underline{r},\underline{s};E)\Big|_{E=E_j} = \sum_{k} \bar{\psi}_k(\underline{r},E_j)\psi_k(\underline{s},E_j) \qquad (16)$$

where the $\sum_k$ takes account of the degeneracy of the eigenvalue E_j.

Since the poles of the scattering amplitude are only due to the poles of the Green's function, this method applies to the calculation of all the R_j's.

In neutron-proton scattering, for instance, the only pole comes from the deuteron bound state. (This, of course, is an additional physical information). The residue at this pole can thus be computed from the deuteron wave-function.

(3) The Question of Subtractions

As mentioned earlier, dispersion relations for the scattering amplitude from central potential require no subtractions at all. On the other hand, our amplitude requires two subtractions. The necessity for these subtractions comes essentially from the presence of the spin-orbit potential. If $V_4(r)$ were zero, we could also obtain a dispersion relation without any subtractions, even in the presence of other spin-dependent terms. Mathematically, the spin-orbit term is distinguished from others by virtue of containing a differential operator so that every time it operates on an exponential of the form $e^{i\underline{k}\cdot\underline{r}}$, it brings down a power of $\sqrt{E}$; these powers of $E^{\frac{1}{2}}$ accumulate and worsen the asymptotic behaviour. But, from the physical point of view, it is rather difficult to understand why the presence of a spin-orbit coupling should make the scattering amplitude diverge linearly at high energies. Of course, it is doubtful whether potential theory has any physical content at all at infinitely high energies.

CHAPTER V

GENERALISATION TO HIGHER SPIN

We will now look for a generalisation of the previous results to scattering between two systems of arbitrary spin without, however, going into full details. As a first step, we will deduce the most general potential permissible for such a system, subject to the same sort of physical requirements as laid down in Chapter II. For the spin $\frac{1}{2}$ case, the problem is much simpler because we need to consider only those terms which are linear in the σ-matrices, all higher terms being reducible to a linear combination of the σ's. In the case of higher spin, this is no longer true and we are naturally led to investigate the question: what is the highest rank of irreducible tensor that we can construct from the spin matrices of a particle of spin s ?

In quantum mechanics, the spin of a system is represented by three matrices, say M_i $(i = 1,2,3)$, which, when sandwiched between appropriate spinors, form a vector in the three-dimensional configuration space. It is usually convenient to consider the matrix-system M_i <u>itself</u> as a vector in the ordinary space. The dimensionality of the matrices M_i is

related to the magnitude of the spin; a system with spin s can have $(2s + 1)$ orientations with respect to any given direction in space so that the state of this system can be written as a column vector with $(2s + 1)$ components and, therefore, the matrices M_i operate in a $(2s + 1)$-dimensional space. They are also known (see, e.g. Schiff[21]) to satisfy the following relations

$$M_i M_j - M_j M_i = i \epsilon_{ijk} M_k \qquad (1)$$

$$(i, j, k = 1, 2, 3)$$

and $$M^2 \equiv M_i M_i = s(s+1)\, I\,. \qquad (2)$$

where summation over repeated indices is understood, ε_{ijk} is the usual totally antisymmetric tensor of the 3rd rank in 3 dimensions, and I is a $(2s + 1)$-dimensional unit matrix.

Now, tensors $T_{i_1, i_2 \ldots, i_r}$ of any rank r may be classified into irreducible sets under permutations of the indices. When we form the tensor from the matrices M_i, as for instance $T_{i_1 i_2 \ldots, i_r} = M_{i_i} M_{i_2} \cdots M_{i_r}$ (each i_r can take on the values 1, 2, 3), we need consider only totally symmetric combinations, since those which are antisymmetric in any pair of indices reduce to a lower rank on account of the commutation rules (1). Thus, our problem now is to determine the number of

independent components of an irreducible, totally symmetric tensor of rank r in a three-dimensional space.

It is known that a completely symmetric tensor of rank r in a space of dimensions ν has ${}^{r+\nu-1}C_r$ independent components. In our case $\nu = 3$, so that the number is ${}^{r+2}C_r$. However, a general <u>irreducible</u> totally symmetric tensor $T_{i_1 \cdots i_r}$ has the additional property of vanishing trace:

$$T_{i_1 i_2 \cdots i_{r-2} jj} = 0 .$$

which gives some more restrictions. In fact, since r indices can be contracted in ${}^{r}C_2$ ways, there are as many relations among the components of T. The number of independent components in an irreducible totally symmetric tensor of rank r in 3 dimensions is, therefore, only

$$ {}^{r+2}C_r - {}^{r}C_2 = {}^{r+2}C_2 - {}^{r}C_2 = 2r+1 . \qquad (3)$$

Starting with the lowest ranks (scalar unit matrix, vector M_i), we must include tensors of sufficiently high rank such that we obtain $(2s + 1)^2$ independent components in all, which will then provide us with one complete basic set of $(2s + 1) \times (2s + 1)$ square matrices. Before we can proceed any further, we must prove that these linearly independent matrices can be built up

starting from the lowest rank and going to successively higher rank; in other words, that we must not skip any particular rank. This is easy to see for suppose, at a certain rank p, $T^{(p)}_{i_1,\ldots i_p}$ becomes reducible; then its components satisfy some additional equation. But the only irreducible tensor equation, which contains M's multiplied p times and no more, is just

$$T^{(p)}_{i_1 i_2 \ldots i_p} = 0$$

Multiplication of this equation by further M factors, followed by symmetrisation and subtraction of traces, leads to the more general result

$$T^{(q)}_{i_1 i_2 \ldots i_q} = 0 \qquad \text{where} \quad q \geq p$$

i.e. once a tensor T becomes reducible, all higher rank tensors formed from the M-matrices also reduce automatically.

Hence, the highest rank n of irreducible, totally symmetric tensor, made from the M_i's is such that

$$\sum_{r=0}^{n} (2r+1) = (2s+1)^2. \qquad (4)$$

This equation yields n = 2s, which means that we need consider tensors T, constructed from M_i, only up to the rank 2s where s is the spin of the particle being considered.

(Parenthetical Remark: Since we obtain a complete set of

linearly independent $(2s+1)\times(2s+1)$ matrices by proceeding only up to rank $2s$, the higher ranks must be reducible. By an argument, similar to one used above, we have then

$$T^{(2s+1)}_{i_1 i_2 \cdots i_{2s+1}} = 0 . \qquad (5)$$

This equation gives us a general auxiliary relation for spin s matrices, like those we are already familiar with in the spin $\frac{1}{2}$ and spin 1 case. In fact, these relations can be easily derived from (5), together with relation (2):-

Spin 0 : We have $M_i = 0$ trivially.

Spin $\frac{1}{2}$: $T^{(2)}_{ij} = 0$, i.e. $M_i M_j + M_j M_i - \frac{2}{3} M_K M_K \delta_{ij} = 0$

or $\sigma_i \sigma_j + \sigma_j \sigma_i = \frac{1}{2} \delta_{ij} I$.

Spin 1 : $T^{(3)}_{ijK} = 0$

i.e. Symm.$(M_i M_j M_k)$ - Trace $= 0$.

which gives (writing the M_i for spin 1 as β's):

$$\beta_i\beta_j\beta_K + \beta_i\beta_K\beta_j + \beta_j\beta_i\beta_K + \beta_j\beta_K\beta_i + \beta_K\beta_i\beta_j + \beta_K\beta_j\beta_i$$

$$= \frac{2}{5}\{\beta_a\beta_a(\beta_i\delta_{jK} + \beta_j\delta_{iK} + \beta_K\delta_{ij}) + \beta_a(\beta_i\delta_{jK} + \beta_j\delta_{iK} + \beta_K\delta_{ij})\beta_a$$

$$+ (\beta_i\delta_{jK} + \beta_j\delta_{iK} + \beta_K\delta_{ij})\beta_a\beta_a\}$$

$$= 2(\beta_i\delta_{jK} + \beta_j\delta_{Ki} + \beta_K\delta_{ij}) ,$$

using (1) and (2).

Further use of commutation relations will bring this equation in the familiar Duffin-Kemmer form:

$$\beta_i \beta_j \beta_k + \beta_k \beta_j \beta_i = \beta_i \delta_{jk} + \beta_k \delta_{ij} .$$

And, similarly, for higher spins).

Now, we are ready to investigate the permissible form of the potential between two particles or system, one with spin s and the other with spin s'. For this potential to have a physical sense, it must again satisfy the same requirements that were enumerated in the second chapter, with one exception -- it need not be symmetric under the exchange of the two particles. The conditions imposed on the potential are then

1) translational invariance
2) Hermiticity
3) rotational invariance
4) Galilean invariance
5) invariance under space inversion
6) invariance under time reversal
7) linearity in relative momentum.

As before, conditions (1) and (4) require that the potential should depend only on the relative position $\underline{r}$ and the relative momentum $\underline{p}$. Also, by (3), (5) and (6), it must be a proper scalar under rotation and reflections. Further, it should be Hermitian and contain $\underline{p}$ only

linearly. With these requirements, we proceed to construct the potential.

Let us recollect the basic "bricks" available to us, from which the potential has to be built. These are the following:-

the matrix-vector : $M_i^{(1)}$ which can be repeated up to 2s times.

the matrix-vector : $M_i^{(2)}$ " " " " 2s' "

the vector : $\underline{r}$, which can be repeated any number of times.

the vector : $\underline{p}$, which can occur only once.

(Also, without loss of generality, we can take $s \leqslant s'$).

We will start the construction of the permissible invariants by studying the term which does not involve any spin-matrices, viz. the coefficient of the unit matrix. This must be a scalar function of $\underline{r}$ and $\underline{p}$, and, therefore, can depend only on r^2, p^2, $\underline{r}.\underline{p}$, or $(\underline{r} \wedge \underline{p})^2$. Of these, p^2 and $(\underline{r} \wedge \underline{p})^2$ are immediately excluded by the linearity condition (7). The term $\underline{r}.\underline{p}$ changes sign under time-reversal, and can only occur bilinearly, which is again forbidden by the condition (7). We are thus left with just r^2, so that the first term is simply a function $V(r)$ of the scalar $r = |\underline{r}|$. Further, such a function satisfies all the requirements (1) through (7) so that it can occur as a multiplication factor with all other terms as well, and will in fact govern the "shape" of the potential.

Let us now proceed to investigate the permissible invariants which can be constructed from $\underline{M}^{(1)}$, $\underline{M}^{(2)}$, $\underline{r}$ and $\underline{p}$. First of all, we observe that we can construct irreducible totally symmetric tensors of orders up to $2s$ from the matrices $M^{(1)}$ and $2s'$ from the matrices $M^{(2)}$. Calling these $T^{(1)}$ and $T^{(2)}$ respectively, we have at our disposal the following tensorial sets:

From $M^{(1)}$

$T^{(1)}_{(\alpha_1)}$ where the rank $a_1 = 1, 2, \ldots 2s$.

From $M^{(2)}$

$T^{(2)}_{(\alpha_2)}$ where the rank $a_2 = 1, 2, \ldots, 2s'$

From the tensors $T^{(1)}$ and $T^{(2)}$, we can form direct products which are in general reducible. In fact we can express the direct product of $T^{(1)}_{(\alpha_1)}$ and $T^{(2)}_{(\alpha_2)}$ as a linear combination of irreducible direct products with appropriate Clebsch-Gordon coefficients. Let us denote these irreducible direct products of $T^{(1)}_{(\alpha_1)}$ and $T^{(2)}_{(\alpha_2)}$ by $T^{(1,2)}_{(\alpha : \alpha_1, \alpha_2)}$ where the ranks α can take on the values $|\alpha_1 - \alpha_2|$, $|\alpha_1 - \alpha_2| + 1$, $\ldots$ $|\alpha_1 + \alpha_2|$.

We thus need to consider only the tensors

$T^{(1,2)}_{(\alpha : \alpha_1, \alpha_2)}$ where $a_1 = 0, 1, \ldots, 2s$

$a_2 = 0, 1, \ldots, 2s'$

$\alpha = |a_1 - a_2|, |a_1 - a_2| + 1, \ldots$

$\ldots |a_1 + a_2|$.

In particular

$T^{(1,2)}_{(\alpha : \alpha_1, 0)}$ and $T^{(1,2)}_{(\alpha : 0, \alpha_2)}$ are just the

single $T^{(1)}_{(\alpha_1)}$ and $T^{(2)}_{(\alpha_2)}$ respectively, and $T^{(1,2)}_{(0:0,0)}$ is the unit matrix.

Now we can classify the scalars as follows:-

Class I (those which do not involve r, p):-

These will be constructed from the spin matrices alone by suitable contractions. As a matter of fact, this class will just consist of the tensorial sets $T^{(1,2)}$ of rank zero, viz.

$$T^{(1,2)}_{(0:\beta,\beta)} \qquad \text{where } \beta = 1, 2, \ldots, 2s.$$

Such invariants also satisfy the other requirements, in particular those related to space and time reflection, and are, therefore, allowed.

Class II (those which involve M's and r-factors):-

For any α (≥ 1), we can make up a scalar from $T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}$ by contracting it with α factors r. The properties of such a combinstion under space and time reversal are easily obtained from the following "parity" table:

	Space inversion	Time Reversal
matrices (M)	even (+1)	odd (-1)
r	odd (-1)	even (+1)
p	odd (-1)	odd (-1)

$\therefore$ space parity $= (-1)^{a}$

time parity $= (-1)^{a_1 + a_2}$.

Hence, the allowed invariants are $T^{(1,2)}_{(\alpha : \alpha_1, \alpha_2)}\, r_{(a)}$

where both a and $(a_1 + a_2)$ are even. Examples of such invariants are:

$M^{(1)}_{i_1} M^{(1)}_{i_2} r_{i_1} r_{i_2}$,

$M^{(1)}_{i} M^{(2)}_{j} r_i r_j$,

$M^{(1)}_{i_1} M^{(2)}_{i_1} M^{(2)}_{i_2} M^{(2)}_{i_3} r_{i_2} r_{i_3}$, etc.

} properly symmetrized and trace subtracted

Class III (these involve M_i, r_i and single p_i)

To form a scalar with a single $\underline{p}$, we need a vector made up of the $\underline{M}^{(1)}$, $\underline{M}^{(2)}$ and $\underline{r}$. We can get a vector from $T^{(1,2)}_{(\alpha : \alpha_1, \alpha_2)}$ by adjoining $(\alpha - 1)$, α, or $(\alpha + 1)$ factors $\underline{r}$. (Using the Kronecker δ and the ε). We can, therefore, subdivide this class into three sub-classes.

III(a) : This contains terms of the type $T^{(1,2)}_{(\alpha : \alpha_1, \alpha_2)}\, r_{(\alpha-1)}\, p$.

Here space parity $= (1)\ (-1)^{a-1}\ (-1) = (-1)^{a}$

time parity $= (-1)^{a_1 + a_2}\ (+1)\ (-1) = (-1)^{a_1 + a_2 + 1}$.

Hence, we require : a even, $(a_1 + a_2)$ odd.

Example:

$(\epsilon_{ijk} M^{(1)}_{i} M^{(2)}_{j} M^{(2)}_{l}) r_k p_l$ (properly symmetrised between k, l)

III(b) : This class consists of terms like

$$\epsilon_{(\ )}\ T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}\ r_{(\alpha)}\ p$$

Here, space parity = $(1)\ (-1)^{\alpha}\ (-1) = (-1)^{\alpha+1}$,

and time parity = $(-1)^{\alpha_1+\alpha_2}\ (+1)(-1) = (-1)^{\alpha_1+\alpha_2+1}$,

which requires : α odd, $(\alpha_1 + \alpha_2)$ odd.

Example:

$$\varepsilon_{ijk}\ M_i^{(1)}\ r_j\ p_k \qquad (\alpha = 1,\ \alpha_1 + \alpha_2 = 1).$$

(This is the usual spin-orbit term).

III(c) : Here, we have invariants of the form

$$T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}\ r_{(\alpha+1)}\ p$$

Therefore, space parity = $(+1)(-1)^{\alpha+1}\ (-1) = (-1)^{\alpha+2}$

time parity = $(-1)^{\alpha_1+\alpha_2}\ (+1)(-1) = (-1)^{\alpha_1+\alpha_2+1}$

Hence the restrictions : α even, $(\alpha_1 + \alpha_2)$ odd.

Example:

$$(\varepsilon_{ijk}\ M_i^{(1)}\ M_j^{(2)}\ M_\ell^{(2)})\ r_k\ r_\ell\ r_m\ p_m \quad \text{(properly symmetrised etc.)}$$

This completes the enumeration of invariants. We can summarise our results in the following way:-

Class I:

$$T^{(1,2)}_{(0:\beta,\beta)} \quad : \text{ where } \beta = 0,1,2,\ldots,2s.$$

(this includes the unit matrix $T^{(1,2)}_{(0:0,0)}$).

Class II:

$$T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}\ r_{(\alpha)} \quad : \text{ with } \alpha \text{ even, } (\alpha_1+\alpha_2) \text{ even.}$$

Class III:

III(a): $T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}\, r_{(\alpha-1)}\, p$: with α even, $(\alpha_1 + \alpha_2)$ odd.

III(b): $\varepsilon_{(\ldots)}\, T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}\, r_{(\alpha)}\, p$: with α odd, $(\alpha_1 + \alpha_2)$ odd.

III(c): $T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}\, r_{(\alpha+1)}\, p$: with α even, $(\alpha_1 + \alpha_2)$ odd.

Let us now introduce the notation A_i, B_i, C_i for a general invariant of the classes I, II and III respectively. Recalling that each such invariant can be multiplied by an arbitrary function of $r = |\underline{r}|$, we can write the required potential as:

$$V = V_i^A(r)\, A_i + V_i^B(r)\, B_i + V_i^C(r)\, C_i \qquad (6)$$

where summation over i is implied.

The expression (6) gives the most general permissible potential between two systems of spin s and s' respectively, subject to the physical requirements (1) - (7) enumerated earlier. It will be useful, for instance, in studying the interaction between two nuclei of arbitrary spin, or indeed whenever the potential between any two particles may be spin-dependent.

If the two particles are identical, then $s = s'$, and we also have to take into account the symmetry under the exchange of the two particles. In the above formalism, this is quite simple. Class I invariants are already

symmetric; as for the Classes II and III, we have to symmetrise them between the indices (1) and (2). This is accomplished simply by replacing everywhere $T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)}$ by the symmetric combination $T^{(1,2)}_{(\alpha:\alpha_1,\alpha_2)} + T^{(1,2)}_{(\alpha:\alpha_2,\alpha_1)}$ This will in general reduce the number of possible terms by a factor of 2.

Let us now put the potential (6) into a two-particle Schrödinger equation as before. Since we have restricted the potential to be at most linear in the relative momentum, we will have, in the coordinate representation ($\underline{p} \longrightarrow i\underline{\nabla}$), only the first power of the differential operator ∇. The integral equation for the Green's function of this equation will be a complicated matrix equation, but will again have only a weakly singular kernel, so that the whole discussion in Chapter III will go through mutatis mutandis. We can, therefore, deduce the analytic properties and asymptotic behaviour of the Green's function in complete analogy with the spin $\frac{1}{2}$ case.

In writing down the scattering amplitude, we will have to investigate the problem in greater detail, in particular the "crossing" property, i.e. the approach to the cut from above and from below. This, and the subsequent decomposition into invariant amplitudes, will require some effort; but presumably there will be no essential difficulty in obtaining the dispersion relations for the scattering amplitude.

Such relations may prove of value in the study of scattering of heavy nuclei in the non-relativistic region of energy.

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