Decomposition Theorems for Quasi-discrete Planar Domains

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Abstract

Let Ω be a domain in $\mathbb{R}^n$ and $D^\alpha u \geqslant 1$ on Ω for some smooth function $u : \mathbb{R}^n \to \mathbb{R}$ and multi-index α. Let E_s be the sublevel set at height s of u and consider the mulitilinear sublevel set operator

$$\Lambda_s^{\alpha,u}(f_1, \ldots, f_n) = \int_\Omega \chi_{E_s}(x) f_1(x_1) \ldots f_n(x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

where x_i denotes the i^{th} co-ordinate of $x \in \mathbb{R}^n$. It is natural to seek estimates of the form

$$|\Lambda_s^{\alpha,u}(f_1, \ldots, f_n)| \leqslant C s^\varepsilon \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}$$

for some $\varepsilon > 0$ and constant C independent of s, u and the f_i. Of course one must first decide which classes of domains Ω and functions u and what values of $p_1, \ldots, p_n$ to work with.

Motivated by recent work on such estimates, we ask what progress can be made in two dimensions by finding decompositions of domains Ω that have the $BC(m, n)$ property for some $m, n \in \mathbb{N}$, which says that the domain meets horizontal lines in at most m components and vertical lines in at most n. Estimates are easily established on $BC(1, 1)$ domains and so one is led to attempt to decompose $BC(m, n)$ domains, under appropriate further hypotheses, into a number of $BC(1, 1)$ domains which is bounded in terms of m and n.

For various reasons we choose to work in a quasi-discrete setting. We formulate this framework before stating and proving the principal results, and go on to discuss some of the issues that they raise and their possible applications.

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Chapter 1

Introduction

The starting point for our study is the following lemma due to van der Corput:

Lemma 1. *For each $k \in \mathbb{N}\backslash\{0\}$ there is a constant C_k such that if $\Phi : [a,b] \longrightarrow \mathbb{R}$ and $\Phi^{(k)}(t) \geqslant 1$ on $[a,b]$ (and furthermore Φ' is monotone if $k = 1$), then*

$$\left| \int_a^b e^{i\lambda\Phi(t)}\,\mathrm{d}t \right| \leqslant \frac{C_k}{\lambda^{1/k}}.$$

We can prove it using the *sublevel set estimate*:

Lemma 2. *For each $k \in \mathbb{N}\backslash\{0\}$ there is a constant D_k such that if $\Phi^{(k)} \geqslant 1$ on $[a,b]$, then*

$$\left|\{t \in [a,b] : |\Phi(t)| \leqslant s\}\right| \leqslant D_k s^{1/k}.$$

Proof. The case $k = 1$ follows easily from the Mean Value Theorem, giving $D_1 = 2$. Indeed, if there were some Φ with $\left|\{t \in [a,b] : |\Phi(t)| \leqslant s\}\right| > 2s$, then there would be a point $c \in (a,b)$ with $\Phi'(c) < 1$. Suppose inductively that the result holds for $k \leqslant r$. Assuming that $\Phi^{(r+1)}(t) \geqslant 1$ on $[a,b]$,

$$\begin{aligned} \left|\{t \in [a,b] : |\Phi(t)| \leqslant s\}\right| &\leqslant \left|\left\{t \in [a,b] : \left|\Phi^{(r)}(t)\right| \leqslant \alpha\right\}\right| \\ &\quad + \left|\left\{t \in [a,b] : \left|\Phi^{(r)}(t)\right| > \alpha,\ |\Phi(t)| \leqslant s\right\}\right| \\ &\leqslant D_1\alpha + 2D_r\left(\frac{s}{\alpha}\right)^{1/r}. \end{aligned}$$

(The final inequality above is seen by considering the function $\alpha^{-1}\Phi$.) Putting $\alpha = s^{\frac{1}{r+1}}$ gives the result for $k = r+1$ and the lemma is proved by induction. $\square$

Proof of Lemma 1. Firstly, we note that if $k \geqslant 2$, the hypothesis $\Phi^{(k)} \geqslant 1$ implies that the set $\{t \in [a,b] : |\Phi'(t)| \geqslant \alpha\}$ can be written as a disjoint union of $2k-2$ or fewer intervals (a_i, b_i) on each of which Φ' is monotonic and a set of measure zero. For, applying Rolle's Theorem $k-2$ times shows that Φ'' has at most $k-2$ zeroes, and so Φ' has at most $k-2$ turning points. Thus we can decompose $[a,b]$ into $k-1$ or fewer intervals on which Φ' is monotonic, and each of these is at worst split into two on passing to $\{t \in [a,b] : |\Phi'(t)| \geqslant \alpha\}$. Writing $N_k = 2k-2$, we have

$$\begin{aligned}
\left|\int_a^b e^{i\lambda\Phi}\right| &= \left|\int_{|\Phi'|<\alpha} e^{i\lambda\Phi} + \int_{|\Phi'|\geqslant\alpha} e^{i\lambda\Phi}\right| \\
&\leqslant D_{k-1}\alpha^{\frac{1}{k-1}} + \left|\frac{1}{i\lambda}\int_{|\Phi'|\geqslant\alpha} \frac{1}{\Phi'}(e^{i\lambda\Phi})'\right| \\
&\leqslant D_{k-1}\alpha^{\frac{1}{k-1}} + \frac{1}{\lambda}\sum_{i=1}^{N_k}\left(\left|\left[\frac{1}{\Phi'}\right]_{a_i}^{b_i}\right| + \int_{a_i}^{b_i}\left|\left(\frac{1}{\Phi'}\right)'\right|\right) \\
&\leqslant D_{k-1}\alpha^{\frac{1}{k-1}} + \frac{1}{\lambda}\left(\frac{N_k}{\alpha} + \sum_{i=1}^{N_k}\left|\int_{a_i}^{b_i}\left(\frac{1}{\Phi'}\right)'\right|\right) \\
&\leqslant D_{k-1}\alpha^{\frac{1}{k-1}} + \frac{2N_k}{\lambda\alpha}.
\end{aligned}$$

Now put $\alpha = \lambda^{-\frac{k-1}{k}}$. □

This method of proving the van der Corput lemma using the sublevel set estimate is taken from [2].

Remarks. These two results have some desirable properties:

- The constants C_k and D_k are independent of a and b.
- The estimates are sharp, as seen by putting in $\Phi(x) = x^k/k!$.
- The estimates scale, i.e. having them for a fixed a_0, b_0 implies them for all a, b.

It is natural to ask whether there are higher-dimensional analogues of these results, i.e. given some function u on $\Omega \subseteq \mathbb{R}^n$ and a multi-index α such that $D^\alpha u \geqslant 1$ on Ω, can we obtain estimates of the form

$$\left|\{x \in \Omega : |u(x)| \leqslant s\}\right| \leqslant C_\alpha s^\varepsilon \tag{1.1}$$

and

$$\left|\int_{\Omega} e^{i\lambda u(x)}\,\mathrm{d}x\right| \leqslant \frac{C_\alpha}{\lambda^\varepsilon} \tag{1.2}$$

for some $\varepsilon > 0$? And do they have nice properties like sharpness and scaling?

1.1 Choosing the domain Ω: two ideas

$BC(m)$ domains, type M functions and HV-convexity

The question we are immediately faced with is what kind of domain Ω should correspond to intervals on the line. The connected sets quickly suggest themselves, but must be rejected on account of the following example, which appears in [4]:

Example 3. *Let* $n = 2$, $\alpha = (0, 1)$ *and* $N \in \mathbb{N}$. *Define* $\Omega' = \left(0, \frac{1}{3}\right) \times (0, 1)$ *and for* $0 \leqslant j \leqslant N-1$, $\Omega_j = \left[\frac{1}{3}, 1\right) \times \left(\frac{j}{N}, \frac{j+1/2}{N}\right)$. *Let* $\Omega = \Omega' \cup \bigcup_{j=0}^{N-1} \Omega_j$. *(See Figure 1.1.) Choose a smooth* $\phi : [0,1] \longrightarrow \mathbb{R}$ *with* $\phi \geqslant 0$, $\phi \equiv 0$ *on* $\left[0, \frac{5}{12}\right]$ *and* $\phi \equiv 1$ *on* $\left[\frac{7}{12}, 1\right]$. *Define* u *by*

$$u(x,y) = \begin{cases} y & (x,y) \in \Omega' \\ y - \frac{j}{N}\phi(x) & (x,y) \in \Omega_j \end{cases}.$$

Then clearly $\frac{\partial u}{\partial y} \equiv 1$ *on* Ω, *while for* s *sufficiently small,*

$$\left|\{(x,y) \in \Omega : |u(x,y)| \leqslant s\}\right| \sim Ns.$$

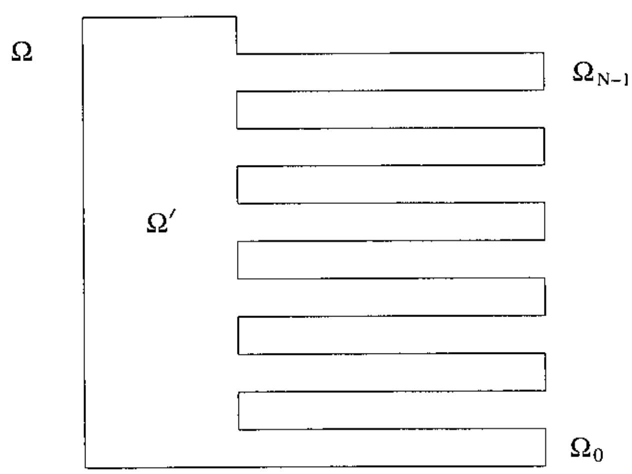

Figure 1.1: The set Ω in Example 3

What appears to have gone wrong is that the arbitrarily many "legs" of Ω allow some of the vertical cross-sections to have arbitrarily many components. Perhaps we could make some progress by putting a bound on the number of components of any axis-parallel cross-section. This train of thought leads to the following important definitions:

Definition 4.

- *A set $\Omega \subseteq \mathbb{R}^n$ is said to be $BC(m_1, \ldots, m_n)$ if for each i, any line L parallel to the i^{th} axis is cut into at most m_i pieces by Ω, i.e. $\Omega \cap L$ has at most m_i connected components. We abbreviate $BC(m, \ldots, m)$ to $BC(m)$. (The notation BC is motivated by the idea of having a* **b***ounded number of* **c***omponents of intersections with axis-parallel lines.)*

- *Let Ω be an open subset of $\mathbb{R}^n$. The function $p : \Omega \longrightarrow \mathbb{R}$ is said to be of* type M *if it has the property:*

  there exists N such that for all β with $|\beta| \leqslant M$ and all $s > 0$, the set $\{x \in \Omega : |D^\beta p(x)| \leqslant s\}$ is $BC(N)$.

  In other words, there is some N such that all sublevel sets of derivatives of order up to M of p are $BC(N)$.

- *The least such N is called the* type M constant *of p, and denoted $t_M(p)$.*

Observe that a polynomial $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ of degree d is type M for all M. Its type M constant $t_M(p)$ is bounded by a constant $C(M, n, d)$ depending on M, n and d but not on the coefficients of p.

We also introduce the notions of horizontal and vertical convexity, which are related to the $BC(1)$ property.

Definition 5. *The subset Ω of $\mathbb{R}^2$ is* horizontally convex *if whenever (x, y_0) and $(x', y_0) \in \Omega$ and $x < z < x'$, then $(z, y_0) \in \Omega$. Similarly, Ω is* vertically convex *if whenever $(x_0, y), (x_0, y') \in \Omega$ and $y < w < y'$, then $(x_0, w) \in \Omega$. We sometimes use the abbreviations 'H-convex' for 'horizontally convex'; 'V-convex'* *for 'vertically convex'; and 'HV-convex' for 'horizontally convex and vertically convex'.*

Note that the domain $\Omega \subseteq \mathbb{R}^2$ has the $BC(1)$ property if and only if it is both horizontally and vertically convex. Results involving these concepts will appear a bit further on, but first we sketch more of the background material.

Rectangles

Another possibility is to work with axis-parallel rectangular boxes. However, our next example shows that we cannot achieve estimates independent of the size of the box as in the one-dimensional case.

Example 6. *Let $u(x,y) = xy$ and A a square centred at the origin with side length $2a$. We have that*

$$\begin{aligned}\left|\{(x,y) \in A : |u(x,y)| \leqslant s\}\right| &= 4\Big(s + \int_{\frac{s}{a}}^{a} \frac{s}{t}\,\mathrm{d}t\Big)\\ &= 4s\Big(1 + \log\frac{1}{s} + 2\log a\Big)\end{aligned}$$

and by choosing $\log a$ large enough we see that there is no constant independent of a such that an estimate of the form (1.1) holds. (At this stage intuition suggests that we should be aiming for $\varepsilon < 1$ since the partial derivative in question is of order two.)

Furthermore, it is not difficult to show that oscillatory integral estimates are in general stronger than their sublevel set counterparts, and we can conclude that in this case we cannot achieve an estimate of the form (1.2) with a constant independent of the size of A either. As an instance of this principle, we have the following:

Example 7. *Let $Q = [0,1]^2$ and suppose there is some $\delta < 1$ such that*

$$|I(\lambda)| = \left|\int_Q e^{i\lambda\phi(x)}\mathrm{d}x\right| \leqslant \frac{C}{|\lambda|^{\delta}}.$$

Then we also have

$$|E_t| = \left|\{x \in Q : |\phi(x)| < t\}\right| \leqslant C' t^{\delta}.$$

To show this, choose a smooth, compactly-supported $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ that is identically 1 on $[-1,1]$. Note that we have $\chi_{E_t}(x) \leqslant \psi(\frac{\phi(x)}{t})$ pointwise on $\mathbb{R}^2$ since

$$x \in E_t \Rightarrow \left|\frac{\phi(x)}{t}\right| < 1 \Rightarrow \psi\Big(\frac{\phi(x)}{t}\Big) = 1.$$

Now

$$\begin{aligned}
|E(t)| = \int_Q \chi_{E_t} &\leqslant \int_Q \psi\Big(\frac{\phi(x)}{t}\Big)\,\mathrm{d}x \\
&= \int_Q \int_{-\infty}^{\infty} \hat{\psi}(y) e^{2\pi i y \phi(x)/t} \mathrm{d}y\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \hat{\psi}(y) \int_Q e^{i(2\pi y/t)\phi(x)} \mathrm{d}x\,\mathrm{d}y \quad \text{by Fubini} \\
&\leqslant C\Big(\frac{t}{2\pi}\Big)^{\delta} \int_{-\infty}^{\infty} \frac{|\hat{\psi}(y)|}{|y|^{\delta}}\,\mathrm{d}y \\
&= C' t^{\delta},
\end{aligned}$$

where $C' = (2\pi)^{-\delta} C \int_{-\infty}^{\infty} |\hat{\psi}(y)||y|^{-\delta}\mathrm{d}y$, which exists since $\delta < 1$.[1] Note that we could replace Q in this example by any subset of $\mathbb{R}^2$.

One solution to the problem highlighted in Example 6 would be to fix a box once and for all in which to work, say $Q = [0,1]^n$. For the phase function in Example 6, we then obtain the desired estimates on the oscillatory integral estimate for any $\varepsilon < 1$. However this is not the only weapon at our disposal. Notice that by inserting suitable functions $f(x)$ and $g(y)$ in said example, we obtain:

$$\begin{aligned}
\left|\int_R e^{i\lambda xy} f(x) g(y)\,\mathrm{d}x\,\mathrm{d}y\right| &\leqslant \left|\int_J \widehat{f\restriction_I}\Big(\frac{\lambda y}{2\pi}\Big) g(y)\,\mathrm{d}y\right| \\
&\leqslant \sqrt{\frac{2\pi}{\lambda}} \|f\|_2 \|g\|_2 \quad \text{by Plancherel and Cauchy-Schwartz}
\end{aligned}$$

for any rectangle $R = I \times J$ in $\mathbb{R}^2$, where I and J are intervals in $\mathbb{R}$.[2]

This connection with the Fourier Transform, as well as the frequent appear-

[1] Our definition of the Fourier Transform $\hat{f}$ of a suitable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is $\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{2\pi i x y} \mathrm{d}x$.

[2] We use the notation $f\restriction_A$ for the restriction of the function f to the subset A of its domain. A vertical line is perhaps a more common notation, but already has enough usage within this document!

ance in recent years of multilinear operators in harmonic analytic research,[3] supports the idea of inserting functions and aiming for estimates of the form

$$\left|\int\limits_{\Omega\cap\{|u(x)|\leqslant s\}} f_1(x_1)\dots f_n(x_n)\,\mathrm{d}x_1\dots\,\mathrm{d}x_n\right| \leqslant C_\alpha s^\varepsilon \|f_1\|_{p_1}\cdots\|f_n\|_{p_n} \tag{1.3}$$

$$\left|\int_\Omega e^{i\lambda u(x)} f_1(x_1)\dots f_n(x_n)\,\mathrm{d}x_1\dots\,\mathrm{d}x_n\right| \leqslant \frac{C_\alpha}{\lambda^\varepsilon} \|f_1\|_{p_1}\cdots\|f_n\|_{p_n} \tag{1.4}$$

where $D^\alpha u \geqslant 1$ on Ω. In most of the recent work done in this area, bilinear or multilinear approaches have been used. Usually this is in conjunction with a fixed box such as $[0,1]^n$, although some results extend to more general domains.

The calculations below with functions of an elementary nature soon reveal that the ideal ε we could wish for is $\varepsilon = \frac{1}{|\alpha|}$, with $\frac{1}{p_i} = 1 - \frac{\alpha_i}{|\alpha|}$. With this estimate in hand, we could obtain all possible others by interpolation with trivial estimates. To see how this ε comes about, suppose that we have an estimate of the form

$$\left|\int\limits_{Q\cap\{|u(x)|\leqslant s\}} \prod_{i=1}^n f_i(x_i)\,\mathrm{d}x_i\right| \leqslant Cs^\varepsilon \prod_{i=1}^n \|f_i\|_{p_i}$$

whenever $D^\alpha u \geqslant 1$ on Q. Putting $u = (x_1 + \dots + x_n)^{|\alpha|}$, $f_1 = \dots = f_n = \chi_{(0,1)}$ forces $\varepsilon \leqslant \frac{1}{|\alpha|}$, while putting $u = x_1^{\alpha_1}\dots x_n^{\alpha_n}$, $f_1 = \chi_{(0,s^{1/\alpha_1})}$, $f_2 = \dots = f_n = \chi_{(0,1)}$ entails that $\varepsilon \leqslant \frac{1}{\alpha_1 p_1'}$, where r' denotes the *conjugate* of an exponent $r \in [0,\infty]$, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$. Symmetrically, $\varepsilon \leqslant \frac{1}{\alpha_i p_i'}$ for $i = 2,\dots,n$. Hence we have $\varepsilon \leqslant \min\{\frac{1}{|\alpha|}, \frac{1}{\alpha_i p_i'}\}$. Considering the planes $\varepsilon = \frac{1}{|\alpha|}$, $\varepsilon = \frac{1}{\alpha_i p_i'}$ in $(\frac{1}{p_1},\dots,\frac{1}{p_n},\varepsilon)$-space shows that where they intersect, $\frac{1}{p_i} = 1 - \frac{\alpha_i}{|\alpha|}$. See [2].

1.2 Some results

So what results are known to date? Well, it is at least known that the estimates do hold for *some* $\varepsilon > 0$, although of course it may not be the optimal value of $\frac{1}{|\alpha|}$.

[3]We mention just a few examples out of the many possibilities. One of the most significant developments is the work of Lacey and Thiele on the bilinear Hilbert Transform ([8] and [9]), an offshoot of which, described in [10], is the shortest currently known proof of the boundedness of the Carleson operator. (The latter is the principal ingredient of proofs of the almost everywhere convergence of Fourier series of L^2 functions, originally established by Carleson.) Christ, in [5], has investigated trilinear operators, finding connections with important geometric and combinatorial results. Broad-ranging work by Grafakos and Torres on multilinear Calderón-Zygmund theory can be found in [6]. Other authors who have recently worked on multilinear operators include Kalton, Kenig, Stein, Tao and several more.

Theorem 8. *Suppose that $u : Q \longrightarrow \mathbb{R}$ is smooth, that $D^\alpha u \geqslant 1$ on Q, and that $p_1, \dots, p_n > 1$. Then there exist $\varepsilon > 0$ and $C > 0$ (depending only on α, n, $p_1, \dots, p_n$) such that*

$$\left| \int_{Q \cap \{|u(x)| \leqslant s\}} f_1(x_1) \dots f_n(x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n \right| \leqslant C s^\varepsilon \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

Under the further provision that α has two or more nonzero entries, at least one of which has value at least 2, there is an $\varepsilon' > 0$ and a $C' > 0$ (with the same dependencies) such that

$$\left| \int_Q e^{i\lambda u(x)} f_1(x_1) \dots f_n(x_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_n \right| \leqslant \frac{C'}{\lambda^{\varepsilon'}} \|f_1\|_{p_1} \cdots \|f_n\|_{p_n}.$$

We shall give a sketch proof of the first estimate, based on [3]. The proof goes by induction on the dimension, with base case $n = 2$, so first we must prove the result in this case.

Lemma 9. *Let $k \geqslant j \geqslant 1$ and let $p = \frac{k+1}{k}$, $q = \frac{j(k+1)}{j(k+1)-k}$. Then there exists a constant $C_{j,k}$ such that for any smooth u with $\frac{\partial^{j+k} u}{\partial x^j \partial y^k} \geqslant 1$ on Q, we have for $0 < s < 1$:*

$$\left| \int_{|u| \leqslant s} f(x) g(y) \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant \begin{cases} C_{j,k} s^{\frac{1}{j(k+1)}} \|f\|_p \|g\|_q & \text{if } j > 1 \\ C_{j,k} s^{\frac{1}{k+1}} (\log s^{-1})^{\frac{k}{k+1}} \|f\|_p \|g\|_q & \text{if } j = 1 \end{cases}.$$

Proof. First suppose that $j > 1$. Let $E = \{(x, y) \in Q : |u(x, y)| \leqslant s\}$ and for each y let $E^y = \{x \in \mathbb{R} : |u(x, y)| \leqslant s\}$. We suppress the dependence on s since s will remain fixed throughout this argument. It can be shown that for $y_0, y_1, \dots, y_k$

$$|E^{y_0} \cap E^{y_1} \cap \dots \cap E^{y_k}| \leqslant C'_{j,k} s^{\frac{1}{j}} \sum_{m=0}^{k} \prod_{l \neq m} |y_l - y_m|^{-\frac{1}{j}}. \tag{1.5}$$

(See [3] for details.) Now by Hölder's inequality,

$$\left| \int_{|u| \leqslant s} f(x) g(y) \, \mathrm{d}x \, \mathrm{d}y \right| \leqslant \|f\|_p \left\| \int \chi_E(x, y) g(y) \, \mathrm{d}y \right\|_{k+1}.$$

Denoting the second term on the right hand side by I, we have

$$\begin{aligned} I^{k+1} &= \int \chi_E(x, y_0) \dots \chi_E(x, y_k) \, g(y_0) \dots g(y_k) \, \mathrm{d}y_0 \dots \mathrm{d}y_k \, \mathrm{d}x \\ &= \int |E^{y_0} \cap \dots \cap E^{y_k}| \, g(y_0) \dots g(y_k) \, \mathrm{d}y_0 \dots \mathrm{d}y_k \\ &\leqslant C''_{j,k} s^{\frac{1}{j(k+1)}} \int \sum_{m=0}^{k} \prod_{l \neq m} |y_l - y_m|^{-\frac{1}{j}} g(y_0) \dots g(y_k) \, \mathrm{d}y_0 \dots \mathrm{d}y_k. \end{aligned}$$

Taking just the first term in the sum we have

$$\begin{aligned}\int \frac{g(y_0)\dots g(y_k)}{|y_1-y_0|^{1/j}\dots|y_k-y_0|^{1/j}}\,\mathrm{d}y_0\dots\mathrm{d}y_k &= \int g(y_0)\Big(I_{\frac{j-1}{j}}g\Big)^k(y_0)\,\mathrm{d}y_0\\ &\leqslant \|g\|_q\Big\|\Big(I_{\frac{j-1}{j}}g\Big)^k\Big\|_{\frac{j(k+1)}{k}}\\ &= \|g\|_q\Big\|I_{\frac{j-1}{j}}g\Big\|^k_{j(k+1)}\\ &\leqslant A_{j,k}\|g\|_q^{k+1},\end{aligned}$$

where I_β denotes fractional integration of order β. (A discussion of fractional integration can be found in Chapter 5 of [13]. The property we use is that $I_{\frac{j-1}{j}}$ maps L^q boundedly into $L^{j(k+1)}$.) By symmetry, the other terms in the summation obey the same estimate and we have done the case $j>1$.

If $j=1$ then we establish the desired estimate by once again establishing an appropriate estimate on the term I^{k+1} appearing in the above argument. This is achieved by multilinear interpolation with one copy of g in L^1 and the others in L^∞. (Consult [15] for the relevant theorem, which is a multilinear generalisation of the Marcinkiewicz interpolation theorem. The latter is discussed in [13], and a slightly stronger version of it is stated in the remarks on Lorentz spaces on page 18.) Thinking in $(1/p_1,\dots,1/p_{k+1})$ space, we can achieve all the estimates corresponding to the vertices $(0,\dots,0,1,0,\dots,0)$ with the 1 in the i^{th} place, for $i=1,\dots,k+1$. By interpolation we can get all the points on the intersection of the plane $x_1+\dots+x_{k+1}=1$ with the set $\{x\in\mathbb{R}^{k+1}: x_1,\dots,x_{k+1}>0\}$, and of course the point $\big(1/(k+1),\dots,1/(k+1)\big)$ lies in this intersection. The problem reduces to establishing that

$$\sup_{y_0}\int|E^{y_0}\cap\dots\cap E^{y_k}|\,\mathrm{d}y_1\dots\mathrm{d}y_k$$

exists. Using the estimate

$$|E^{y_0}\cap\dots\cap E^{y_k}|\leqslant C_k\min\Big\{1, s\sum_{m=0}^{k}\prod_{l\neq m}|y_l-y_m|^{-1}\Big\}$$

and k applications of the fact that for all $r\geqslant 0$, the function $\min\{1,\beta/t\log^r(t/\beta)\}$

has an $L^1[0,1]$ norm that is $O\big(\beta\log^{r+1}(\beta^{-1})\big)$, we have

$$\begin{aligned}
&\int \min\Big\{1, s\prod_{l\geqslant 1}|y_l-y_0|^{-1}\Big\}\mathrm{d}y_1\ldots\mathrm{d}y_k\\
&\leqslant A\int\min\Big\{1, s\prod_{l\geqslant 2}|y_l-y_0|^{-1}\log\Big(\frac{1}{s}\prod_{l\geqslant 2}|y_l-y_0|\Big)\Big\}\,\mathrm{d}y_2\ldots\mathrm{d}y_k\\
&\leqslant A'\int\min\Big\{1, s\prod_{l\geqslant 3}|y_l-y_0|^{-1}\log^2\Big(\frac{1}{s}\prod_{l\geqslant 3}|y_l-y_0|\Big)\Big\}\,\mathrm{d}y_3\ldots\mathrm{d}y_k\\
&\leqslant\ \ldots\\
&\leqslant D_k s\log^k(s^{-1}),\quad \text{where } A,\ A',\ldots \text{ and } D_k \text{ are constants.}
\end{aligned}$$

Again by symmetry the same estimate holds for the other terms in the sum and the $j=1$ case is completed. □

Thus we have proved Theorem 8 for the case $n=2$ with $p_1=\frac{k+1}{k}$, $p_2=\frac{j(k+1)}{j(k+1)-k}$, getting an exponent of $\frac{1}{j(k+1)}$ for $j\geqslant 1$ and $\frac{1}{k+1}-\eta$ for any $\eta\in\big(0,\frac{1}{k+1}\big)$ when $j=1$. By putting in $f\equiv g\equiv\chi_{(0,1)}$, the sublevel set estimate, i.e. the statement of the lemma without f and g, follows with the same exponents. Notice that this is equivalent to the case of $p=q=\infty$. There are trivial estimates with no s^ϵ decay when either p or q is 1. We can now interpolate to get an estimate with some power decay whenever $p,q>1$. (Although we expect a poor ϵ when either p or q is close to 1.) We have thus completed the $n=2$ case for arbitrary $p,q>1$.

For higher n we proceed by induction to establish the sublevel set estimate (equivalent to $p_1=\cdots=p_n=\infty$). Arguments similar to those in the $n=2$ case are used together with a higher order version of the Mean Value Theorem. Once again, details can be found in [3]. Then using interpolation with trivial estimates for some $p_i=1$, we obtain the first conclusion of Theorem 8.

In their very useful survey article, the authors of [4] draw attention to the fact that in certain cases of the problem in two dimensions, the methods outlined above can be used to obtain results for general HV-convex domains, sometimes with scale-invariance or the optimal exponents. Specifically, we have:

Theorem 10. *Let Ω be an HV-convex domain in $\mathbb{R}^2$ and α a multi-index such that $D^\alpha u\geqslant 1$ on Ω.*

- *If $\alpha = (1,1)$ and $1 \leqslant p < 2$, there is a constant C depending only on p such that*
$$\left|\int_{|u|\leqslant s} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant Cs^{1/p'}\|f\|_p\|g\|_p$$
and a constant C' depending only on Ω such that
$$\left|\int_{|u|\leqslant s} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant C's^{1/2}(\log s^{-1})^{1/2}\|f\|_2\|g\|_2.$$

- *If $\alpha = (j,k)$, $1 < j \leqslant k$ and p, q are as in Lemma 9, then there is a constant C depending only on α such that*
$$\left|\int_{|u|\leqslant s} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant Cs^{\frac{j}{k+1}}\|f\|_p\|g\|_q.$$

- *If $\alpha = (1,k)$, $\frac{1}{p_0} = \frac{k}{k+1}$, $\frac{1}{q_0} = \frac{1}{k+1}$ and $\left(\frac{1}{p},\frac{1}{q}\right) \neq \left(\frac{1}{p_0},\frac{1}{q_0}\right)$ is on the line segment between $\left(\frac{1}{p_0},\frac{1}{q_0}\right)$ and $(1,1)$, then there is a constant C depending only on k and p such that*
$$\left|\int_{|u|\leqslant s} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant Cs^{1/p'}\|f\|_p\|g\|_q$$
and a constant C' depending only on k and Ω such that
$$\left|\int_{|u|\leqslant s} f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant Cs^{\frac{1}{k+1}}(\log s^{-1})^{\frac{k}{k+1}}\|f\|_{p_0}\|g\|_{q_0}.$$

A combinatorial problem

Returning to the setting of $\Omega = Q$, if we insist on having estimates involving power decay with exponent $\frac{1}{|\alpha|}$, then even in the case $n = 2$, $\alpha = (1,1)$, $f = g = \chi_{(0,1)}$ the best established bound to date is $Cs^{1/2}(\log s^{-1})^{1/2}$, which is of course contained in Lemma 9. Here there is a connection with a combinatorial problem. A positive answer to the following question would allow us to remove the logarithm term in this particular case.

Question: Is there a constant $c_0 > 0$ such that for any set $E \subseteq Q = [0,1]^2$ with positive (Lebesgue) measure, there is an axis-parallel rectangle with corners in E that has area at least $c_0|E|^2$?

Suppose that we could answer the Question in the affirmative. Let u be our phase function with $\frac{\partial^2 u}{\partial x \partial y} \geqslant 1$ on Q, and let $E = \{(x,y) \in Q : |u(x,y)| \leqslant s\}$. Suppose we have a rectangle R with corners in E, labelled anticlockwise from the bottom-left as A, B, C, D. By Green's Theorem,

$$|R| \leqslant \int_R \frac{\partial^2 u}{\partial x \partial y}\, \mathrm{d}x\, \mathrm{d}y = \frac{1}{2}\int_{\partial R}\left(\frac{\partial u}{\partial y}\mathrm{d}y - \frac{\partial u}{\partial x}\mathrm{d}x\right) = u(A) + u(C) - u(B) - u(D).$$

Since $A, B, C, D \in E$, we have $|R| \leqslant 4s$. Therefore $|E| \leqslant \frac{2}{\sqrt{c_0}} s^{1/2}$. (Otherwise, by our assumption that the answer to the Question is yes, we could find a rectangle with corners in E of area greater than $c_0\big(2(s/c_0)^{1/2}\big)^2 = 4s$, a contradiction.)

It is clear that our Question could be posed for measures other than Lebesgue measure on $[0,1]$. For instance, adopting counting measure on $A = \{1, 2, \ldots, N\}$, the problem becomes that of determining the existence or otherwise of a $c_0 > 0$ such that whenever we have an M-element subset B of $A \times A$, there is a rectangle R with corners in B of area at least $c_0 M^2/N^2$. Even this is still unsolved, but we round off this subsection by returning to the Lebesgue measure case and establishing an upper bound for the c_0 appearing there.

We construct a series of sets $E_k \subseteq Q$ and see what we can infer from them regarding the constant c_0, supposing that it exists at all. Define $\Delta_1 = E_1$ as the open quadrilateral strip with corners at $(0,0)$, $(\delta, 0)$, $(1, 1-\delta)$ and $(1,1)$ for some small quantity δ. The biggest axis-parallel rectangle with corners in Δ_1 has area $\delta^2/4$. (Strictly speaking there is no 'biggest' such rectangle, but we can find rectangles with areas arbitrarily close to this value.) This gives us an initial bound of $c_0 \leqslant 1/4$.

It seems worth asking if we can add another similarly-shaped strip Δ_2 such that there are no larger axis-parallel rectangles with corners in $E_2 = \Delta_1 \cup \Delta_2$. If we consider making Δ_2 the strip of 'thickness' (defined as length of axis-parallel sides) $t_2 < \delta$, then by placing its bottom-left corner at $(2\delta, 0)$ we just avoid having squares of side δ with corners in E_2. By considering rectangles with left side in Δ_1 and right side in Δ_2, it becomes clear that we must have $2\delta t_2 \leqslant \delta^2/4$ and so the largest value t_2 may take is $\delta/8$. We select this value in order to maximise $|E_2|$. It is clear that there are now no axis-parallel rectangles with corners in E_2

and area greater than $\delta^2/4$. Figure 1.2 shows what is going on.

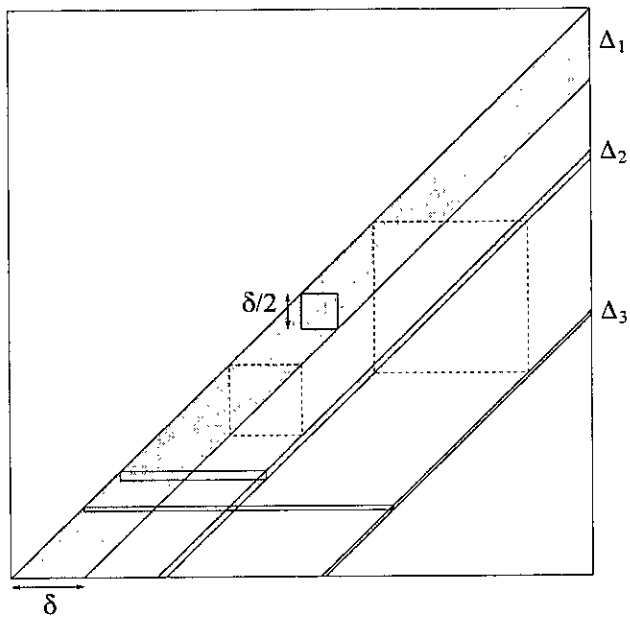

Figure 1.2: Constructing the Δ_i

We can repeat this argument to add a finite number k of strips Δ_i with bottom-left corners at $(d_i, 0)$ and thickness t_i. Using similar considerations, we seek values of d_i and t_i that preclude any axis-parallel rectangle with corners in E_k and area greater than $\delta^2/4$. We claim that this is achieved if we define the d_i and t_i by the recurrence relations:

$$\begin{aligned} & d_1 = 0, & & t_1 = \delta, \\ \text{and for } i \geqslant 1, \quad & d_{i+1} = 2(d_i + t_i) & \text{and} \quad & t_{i+1} = \frac{\delta^2}{4d_{i+1}}. \end{aligned} \tag{1.6}$$

To see this, consider an axis-parallel rectangle R with corners A, B, C and D (labelled anticlockwise from the bottom-left) in E_k for some minimal k. By minimality of k we have $B \in \Delta_k$. We also suppose that $k \geqslant 2$ to avoid trivial cases. Firstly, we claim that at least one of A or C must also be in Δ_k. For otherwise, the sides AB and BC would have to have length strictly greater than $d_{k-1} + t_{k-1}$, since $d_k = 2(d_{k-1} + t_{k-1})$. But A, C and D are in E_{k-1} and thus the sides AD and CD must have length strictly less than $d_{k-1} + t_{k-1}$, which is a contradiction.

Thus we know that at least one side of R has endpoints in Δ_k. Say it is BC. Clearly the side AB will be at its longest if it lies in Δ_1. In this case there is some $0 \leqslant a < t_k$ such that AB has length $d_k + a$. It is also clear by the geometry of E_k that BC must have length at most $t_k - a$. Therefore

$$\begin{aligned} |R| &\leqslant (d_k + a)(t_k - a) \\ &= d_k t_k - a(d_k - t_k) - a^2 \\ &\leqslant \frac{\delta^2}{4} \end{aligned}$$

since $d_k > t_k$ for $k \geqslant 2$.

Observe that all of our Δ_i, E_i, d_i and t_i depend on the quantity δ, which we have thus far left unspecified. In what follows, we shall be more rigorous and in particular more explicit about δ. Introducing $e_i = d_i/2\delta$ and $u_i = t_i/\delta$ (which are independent of δ by (1.6) and slightly easier to calculate with), we have the relations

$$e_{i+1} = 2e_i + u_i \quad \text{and} \quad u_{i+1} = \frac{1}{8e_{i+1}}$$

for $i \geqslant 1$, with of course $e_1 = 0$ and $u_1 = 1$. From these it is easily seen that the u_i can be defined more directly by

$$u_1 = 1, \quad u_2 = \frac{1}{8} \quad \text{and} \quad u_{i+1} = \frac{u_i}{2 + 8u_i^2} \text{ for } i \geqslant 2.$$

We claim that $c_0 \leqslant 4u^{-2}$, where $u = \sum_{i=1}^{\infty} u_i$. [4] For let $\varepsilon > 0$. Choose K such that $\left|\sum_{i=1}^{K} u_i - u\right| < \varepsilon/2$, and choose δ_1 such that for $\delta < \delta_1$ we can fit all of the K strips $\Delta_1, \ldots, \Delta_K$, as given by the construction procedure described above, into Q. From their defining relations (1.6), we can see that for each i, both d_i and t_i

[4] The sum converges by the Ratio Test: clearly $u_i > 0$ for all i and so we have for all i that

$$\frac{u_{i+1}}{u_i} = \frac{1}{2 + 8u_i^2} < \frac{1}{2}.$$

Now $u_i \to 0$ as $i \to \infty$ by comparison with 2^{-i}, and therefore

$$\frac{u_{i+1}}{u_i} \to \frac{1}{2}.$$

are $O(\delta)$. Thus we have for $\delta < \delta_1$ that

$$\begin{aligned} |E_K| &= \sum_{i=1}^{K} \left(t_i(1-d_i) - \frac{t_i^2}{2} \right) \\ &= \sum_{i=1}^{K} t_i - O(\delta^2) \\ &= \delta \sum_{i=1}^{K} u_i - O(\delta^2) \end{aligned}$$

where the $O(\delta^2)$ terms depends on K. Now choose $\delta < \delta_1$ such that the $O(\delta^2)$ term is less than $\delta\varepsilon/2$. Then

$$\begin{aligned} c_0 &\leqslant \frac{\delta^2}{4}|E_K|^{-2} \\ &= \frac{\delta^2}{4}\left(\delta \sum_{i=1}^{K} u_i - O(\delta^2)\right)^{-2} \\ &\leqslant \frac{\delta^2}{4}\left(\delta\left(u - \frac{\varepsilon}{2}\right) - \delta\frac{\varepsilon}{2}\right)^{-2} \\ &= \frac{1}{4}(u-\varepsilon)^{-2} \end{aligned}$$

and the claim is established.

Computing a few terms of the sum, we find that $c_0 \leqslant \frac{1}{6.167\cdots}$. (The second partial sum gives us $u \geqslant \frac{9}{8} \geqslant \sqrt{5/4}$ and so $c_0 < 1/5$, while the fourth partial sum tells us that $u \geqslant \frac{1053146031}{858106756} \geqslant \sqrt{6/4}$ and so $c_0 < 1/6$.) The methods here can in fact be tightened up to show that $c_0 \leqslant 1/8$. In order to keep this whole digression to a reasonable length, we defer this result to Appendix B.

Before ending this particular discussion, we mention Katz's paper [7], which settles the Question under further hypotheses. Note that we can re-phrase the Question as: does there exist a constant $C > 0$ such that if $E \subseteq [0,1]^2$ and all axis-parallel rectangles R with corners in E obey $|R| \leqslant \varepsilon^2$, then $|E| \leqslant C\varepsilon$? Katz proves that we can answer this re-phrasing of the Question affirmatively if we also stipulate that certain six-cornered figures whose corners lie in E must all have area at most ε^2.

Lorentz spaces

Estimates of the form 1.3 involving straight power decay of s with the optimal exponent $\varepsilon = 1/|\alpha|$ are possible when we restrict the class of phase functions involved. From now on we shall only discuss the sublevel set estimates, since, as we have already noted, their oscillatory integral counterparts are stronger statements and so pose additional difficulties. In the case of type M functions, we come encouragingly close with an estimate for when the f_i are in the Lorentz spaces $L^{p_i,1}$.

Before we state this estimate, we include a few basics on Lorentz spaces for the unfamiliar reader. The classical reference for this material is [14]. (These few paragraphs can be skipped by those in the know.) For a measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define its *distribution function* λ_f by

$$\lambda_f(t) = \big|\{x : |f(x) > t\}\big|, \quad t \in (0, \infty)$$

and the *decreasing rearrangement* f^* of f by

$$f^*(t) = \inf\{s : \lambda(s) \leqslant t\}, \quad t \in (0, \infty).$$

We also define the *averaged rearrangement* f^{**} of f by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(u)\,\mathrm{d}u, \quad t \in (0, \infty).$$

We introduce the quantity

$$\|f\|^*_{p,q} = \left(\frac{q}{p}\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

for $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, and

$$\|f\|^*_{p,q} = \sup_{t>0} t^{1/p} f^*(t)$$

for $q = \infty$, $1 \leqslant p \leqslant \infty$. We can now define the Lorentz spaces $L^{p,q}$ by

$$L^{p,q} = \{f : \|f\|^*_{p,q} < \infty\}.$$

In general, $\|\cdot\|^*_{p,q}$ is not a norm, but we can make $L^{p,q}$ into a Banach space by using the norm

$$\|f\|_{p,q} = \left(\frac{q}{p}\int_0^\infty \left(t^{1/p} f^{**}(t)\right)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}$$

for $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, and

$$\|f\|_{p,q} = \sup_{t>0} t^{1/p} f^{**}(t)$$

for $q = \infty$, $1 \leqslant p \leqslant \infty$. The quantity $\|\cdot\|^*_{p,q}$ is employed for its utility, and we have the relationship that for $1 < p \leqslant \infty$, if $f \in L^{p,q}$ then

$$\|f\|^*_{p,q} \leqslant \|f\|_{p,q} \leqslant \frac{p}{p-1}\|f\|^*_{p,q}.$$

The main Lorentz spaces as far as we are concerned are $L^{p,1}$, $L^{p,p} = L^p$ and $L^{p,\infty}$. Two important results are as follows:

Theorem 11. *If T is a linear operator that maps functions of the form $\sum_{i=1}^{k} c_i\chi_{E_i}$, where $|E_i| < \infty$, into a vector space B with order-preserving norm $\|\cdot\|$, and if $\|T\chi_E\| \leqslant C\|\chi_E\|^*_{r,1} = C|E|^{1/r}$ for some constant C independent of E, then*

$$\|Tf\| \leqslant C\|f\|^*_{r,1}$$

for all f in the domain of T.

The content is that when establishing the boundedness of a linear operator on $L^{p,1}$ spaces, it is enough to check it on characteristic functions of sets of finite measure.

For the second result, a strengthening of the well-known Marcinkiewicz interpolation theorem, we need the definition that the subadditive operator T is *restricted weak type* (r, p) if its domain D contains all functions of the form $\sum_{i=1}^{k} c_i\chi_{E_i}$, where $|E_i| < \infty$, is closed under truncation, and whenever $f \in D \cap L^{r,1}$ we have $\|Tf\|^*_{p,\infty} \leqslant K\|f\|^*_{r,1}$ for some K.

Theorem 12. *Suppose that T is a subadditive operator of restricted weak types (r_j, p_j), $j = 1, 2$, where $r_0 < r_1$ and $p_0 \neq p_1$. Let $1 \leqslant q \leqslant \infty$, $\theta \in (0, 1)$ and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

Then there is a constant A depending on θ such that

$$\|Tf\|^*_{p,q} \leqslant A\|f\|^*_{r,q}$$

for all $f \in \operatorname{dom}(T) \cap L^{r,q}$.

Having described the Lorentz spaces, we can now state and prove the estimate promised earlier.

Theorem 13. *Let $u : \Omega \longrightarrow \mathbb{R}$ be of type $M = |\alpha|$ with type M constant N. Then*

$$\left| \int_{\Omega\cap\{|u|\leqslant s\}\cap\{|D^\alpha u|\geqslant 1\}} \prod_{i=1}^n f_i(x_i)\,\mathrm{d}x_i \right| \leqslant Cs^{1/|\alpha|} \prod_{i=1}^n \|f_i\|_{p_i,1},$$

where $\frac{1}{p_i} = 1 - \frac{\alpha_i}{|\alpha|}$ and C depends only on α, N and n.

Proof. By Theorem 11, it is enough to establish the result on characteristic functions of sets, i.e. when $f_i \equiv \chi_{E_i}$ for all i. We proceed by induction on $|\alpha|$.

Firstly we treat the case of $\alpha = (1, 0, \dots, 0)$—so u is type 1. We want to show that

$$\int_{\Omega\cap\{|u|\leqslant s\}\cap\{|D^\alpha u|\geqslant 1\}} \prod \chi_{E_i}(x_i)\,\mathrm{d}x_i \leqslant Cs \prod |E_i|.$$

Thus it is enough to show that

$$\sup_{x'} \left| \left\{ x_1 : |u(x)| \leqslant s \text{ and } \left|\frac{\partial u}{\partial x_1}\right| \geqslant 1 \right\} \right| \leqslant Cs, \quad \text{where } x = (x_1, x').$$

But the set above consists of at most C_N intervals, for some constant C_N depending on the type 1 constant N of u. Applying the Mean Value Theorem to each of them, we get the desired estimate with $C = 2C_N$. By symmetry we get all the cases when $|\alpha| = 1$.

Now suppose the result holds for all multi-indices of size less than M and that $|\gamma| = M$ and $\gamma = \alpha + \beta$, where $0 < \alpha, \beta < \gamma$. We have

$$\begin{aligned}
\int_{\Omega\cap\{|u|\leqslant s\}\cap\{|D^\gamma u|\geqslant 1\}} \prod \chi_{E_i}(x_i)\,\mathrm{d}x_i \leqslant & \int_{\Omega\cap\{|u|\leqslant s\}\cap\{|D^\alpha u|\geqslant t\}} \prod \chi_{E_i}(x_i)\,\mathrm{d}x_i \\
& + \int_{\Omega\cap\{|D^\gamma u|\geqslant 1\}\cap\{|D^\alpha u|\leqslant t\}} \prod \chi_{E_i}(x_i)\,\mathrm{d}x_i \\
\leqslant & \; C\left(\frac{s}{t}\right)^{\frac{1}{|\alpha|}} \prod |E_i|^{1-\frac{\alpha_i}{|\alpha|}} \\
& + Ct^{1/|\beta|} \prod |E_i|^{1-\frac{\beta_i}{|\beta|}} \\
\leqslant & \; Cs^{1/|\gamma|} \prod |E_i|^{1-\frac{\gamma_i}{|\gamma|}}
\end{aligned}$$

by putting $t = \left(s^{|\beta|} \prod |E_i|^{\beta_i|\alpha| - \alpha_i|\beta|}\right)^{1/|\gamma|}$. The Theorem now follows by induction.

□

Refer to [1] for a discussion of Theorem 13. The question that immediately presents itself is whether we can improve the $L^{p,1}$ spaces appearing in the statement of Theorem 13 to L^p spaces, and this is one of the major motivations for our work.

1.3 A decompositional approach

A recent paper of Phong, Stein and Sturm [11] treats the case of polynomial u. They employ algebraic methods to provide a decomposition of the domains of integration appearing in certain multilinear sublevel set and oscillatory integral operators. Amongst other results, they achieve:

Theorem 14. *Let $\alpha \in \mathbb{N}^n \backslash \{0\}$ and $u \in \mathbb{R}[x_1, \ldots, x_n]$ a polynomial of degree d. For any $\Omega \subseteq \{x \in Q : |D^\alpha u(x)| > 1\}$ we have for all $s > 0$ that*

$$\left| \int_{\Omega \cap \{|u(x)| \leqslant s\}} \prod_{i=1}^n f_i(x_i)\, \mathrm{d}x_i \right| \leqslant C s^{1/|\alpha|} \log^{n-2}\left(2 + \frac{1}{s}\right) \prod_{i=1}^n \|f_i\|_{p_i}, \tag{1.7}$$

where $n \geqslant 2$, $\frac{1}{p_i} = 1 - \frac{\alpha_i}{|\alpha|}$ and C depends only on d and $|\alpha|$.

Notice that this establishes the desired estimate (1.3) with $\varepsilon = \frac{1}{|\alpha|}$ when $n = 2$ up to the degree of u. (The $n = 2$ case with $\Omega = [0, 1]^2$ had already been proved earlier, by completely different methods, for a wider class of functions by Carbery, Christ and Wright in [3].)

The proof of Theorem 14 works inductively, starting with the two-dimensional case. We give a sketch of how this base case is proved, following [11]. All of the results and proofs given in this section are due to Phong, Stein and Sturm, with only some minor presentational modifications made. Firstly, the following two elementary lemmas are needed.

Lemma 15. *Suppose that $\Omega \subseteq \mathbb{R}^2$ is open, $u(x, y)$ is smooth on Ω and for all $(x, y) \in \Omega$ that $|\partial_x^\alpha \partial_y^\beta u(x, y)| \geqslant 1$ and $|u(x, y)| \leqslant s$. Then for any axis-parallel* *rectangle $R \subseteq \Omega$ with sides of length l and h we have*

$$l^\alpha h^\beta \leqslant Cs,$$

where C depends only on α and β.

Proof. Firstly we observe that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is k times differentiable and we define $(\Delta_v f)(t) = f(t+v) - f(t)$, then we have

$$\begin{aligned}
&\int\limits_{x_0}^{x_0+v} \int\limits_{x_1}^{x_1+v} \cdots \int\limits_{x_{k-1}}^{x_{k-1}+v} f^{(k)}(t)\,\mathrm{d}t\,\mathrm{d}x_{k-1}\cdots\mathrm{d}x_1 \\
&= \left(\Delta_v^k f\right)(x_0) \\
&= \sum_{i=1}^{k}(-1)^i \binom{k}{i} f(x_0+iv).
\end{aligned} \tag{1.8}$$

Let (x_0, y_0) be the bottom left corner of R. We apply (1.8) to the function $f(t) = \partial_y^\beta u(t,y)$ with $v = l/\alpha$ and $k = \alpha$ to get

$$\left(\frac{l}{\alpha}\right)^\alpha \leqslant \sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\partial_y^\beta u(x_0+il/\alpha, y),$$

since $|\partial_x^\alpha \partial_y^\beta u(x,y)| \geqslant 1$ on R. Now for each i we apply (1.8) to the function $f(t) = u(x_0 + il/\alpha, t)$ with $v = h/\beta$ and $k = \beta$, getting

$$\begin{aligned}
&2^{\alpha+\beta}s \geqslant \left|\sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}(-1)^{i+j}\binom{\alpha}{i}\binom{\beta}{j}u\big(x_0+il/\alpha, y_0+jh/\beta\big)\right| \\
&= \int_{y_0}^{y_0+\frac{h}{\beta}}\int_{y_1}^{y_1+\frac{h}{\beta}}\cdots\int_{y_{k-1}}^{y_{k-1}+\frac{h}{\beta}}\sum_{i=0}^{\alpha}(-1)^i\binom{\alpha}{i}\partial_y^\beta u\big(x_0+il/\alpha, t\big)\,\mathrm{d}t\,\mathrm{d}y_{k-1}\cdots\mathrm{d}y_1 \\
&\geqslant \left(\frac{l}{\alpha}\right)^\alpha\left(\frac{h}{\beta}\right)^\beta
\end{aligned}$$

as claimed. $\square$

The second lemma is of key importance in our work. It says heuristically that domains in $\mathbb{R}^2$ with disjoint horizontal and vertical projections behave independently of one another with respect to the integral operators in question.

Lemma 16. *Let T be a bilinear operator given by*

$$T(f,g) = \int_{X\times Y} K(x,y)f(x)g(y)\,\mathrm{d}x\,\mathrm{d}y.$$

Assume that $\operatorname{supp} K \subseteq \bigcup_{k=1}^{\infty} I_k \times J_k$, *where* I_k, J_k *are measurable subsets of* X, Y *respectively such that* $|I_k \cap I_l| = |J_k \cap J_l| = 0$ *for* $k \neq l$. *Let* T_k *be the bilinear operator with kernel* $\chi_{I_k}(x)\chi_{J_k}(y)K(x,y)$. *If* p, q *are conjugate, let* $\|T_k\|$, $\|T\|$ *be the norms of* T_k, T *as bilinear operators on* $L^p(X) \times L^q(Y)$. *Then* $\|T\| \leqslant \sup_k \|T_k\|$.

Lemma 16 allows us to prove (14) when $u(x,y) = x^\alpha y^\beta$—the "model case".

Lemma 17. *Let* $(\alpha, \beta) \in \mathbb{N}^2$ *and* $u(x,y) = x^\alpha y^\beta$. *Then there is a* $C > 0$ *such that for all* $s > 0$ *we have*

$$\left| \int_{Q \cap \{|u(x)| \leqslant s\}} f(x)\, g(y)\, \mathrm{d}x\, \mathrm{d}y \right| \leqslant C s^{1/(\alpha+\beta)} \|f\|_p \|g\|_q,$$

where $p = \frac{\alpha+\beta}{\beta}$ *and* $q = \frac{\alpha+\beta}{\alpha}$.

Proof. Define $E_s = \{(x,y) \in Q : x^\alpha y^\beta < s\}$ and

$$W_s(f,g) = \int_{E_s} f(x)\, g(y)\, \mathrm{d}x\, \mathrm{d}y.$$

We want to show that the operator norm of W_s on $L^p[0,1] \times L^q[0,1]$ is bounded by $Cs^{1/(\alpha+\beta)}$ for some absolute constant C. Let N be the integer such that $2^{-N-1} < s \leqslant 2^{-N}$. We may as well suppose that $s = 2^{-N}$. Now define for each $i \geqslant 1$ and $k, l \geqslant 0$,

$$R_i(k,l) = \left\{(x,y) \in Q : 2^{-k-\frac{1}{2}} \leqslant x^\alpha \leqslant 2^{-k+\frac{1}{2}},\, 2^{-l-\frac{1}{2}} \leqslant y^\alpha \leqslant 2^{-l+\frac{1}{2}}\right\}.$$

Let $E_i = \bigcup_{k+l=N+i-1} R_i(k,l)$ for $i \geqslant 1$. Then we have that

$$W_s = \sum_i W_i = \sum_i \sum_{k+l=N+i-1} W_i(k,l),$$

where W_i and $W_i(k,l)$ are defined similarly to W_s but with E_i and $R_i(k,l)$ replacing E_s in the definition respectively. By the triangle inequality it is enough to show that $\|W_i\| \leqslant C' 2^{\frac{-N-i}{\alpha+\beta}}$ for some absolute C'. By Lemma 16 we have that

$$\|W_i\| \leqslant \sup_{k+l=N+i-1} \|W_i(k,l)\|.$$

But for any conjugate pair (r, r'), we have

$$|W_i(k,l)(f,g)| \leqslant \left(\int_{x^\alpha \sim 2^{-k}} 1.|f(x)|\,\mathrm{d}x\right)\left(\int_{y^\beta \sim 2^{-l}} 1.|g(y)|\,\mathrm{d}y\right) \tag{1.9}$$

$$\leqslant 2^{\left(-\frac{k}{\alpha r'}-\frac{l}{\beta r}\right)}\|f\|_r\|g\|_{r'} \tag{1.10}$$

by Hölder's Inequality. Putting $r = p$ and $r' = q$ we have $\|W_i(k,l)\|$ bounded by $C'2^{\frac{-N-i}{\alpha+\beta}}$ so long as $k + l = N + i - 1$, and the proof is finished. □

Next comes an important definition, which furnishes us with the building-blocks of the decomposition.

Definition 18. *The set $A \subseteq Q$ is a* curved trapezoid *if there exist $a < b$ and continuous monotonic $\phi, \psi : [a,b] \longrightarrow \mathbb{R}$ with $\phi(x) > \psi(x)$ on (a,b) such that*

$$A = \{(x,y) \in Q : a < x < b,\ \psi(x) < y < \phi(x)\}.$$

We now establish the case when $|\partial_x^\alpha \partial_y^\beta u| > 1$ and Ω is a curved trapezoid. There are two cases:

- The primary case is when ϕ is increasing and ψ decreasing (or *vice versa*). Here we may cut Ω along the line $y = c$ where $c = \frac{\phi(a)+\psi(a)}{2}$. Call the upper piece Ω'. (See the left-hand diagram in Figure 1.3.) We may also assume that $a = c = 0$. By Lemma 15, $x^\alpha y^\beta \leqslant Cs$ for all $x \in [0,b]$. Thus,

$$\{(x,y) \in \Omega' : |u(x,y)| \leqslant s\} \subseteq \{(x,y) \in \Omega' : x^\alpha y^\beta \leqslant Cs\}$$

  and so

$$\left|\int_{\Omega' \cap \{|u| \leqslant s\}} f(x)\,g(y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \left|\int_{\Omega' \cap \{x^\alpha y^\beta \leqslant Cs\}} f(x)\,g(y)\,\mathrm{d}x\,\mathrm{d}y\right|$$
$$\leqslant C' s^{\frac{1}{\alpha+\beta}}\|f\|_p\|g\|_q$$

  by applying the model case. The estimate for the lower piece is similar.

- In the secondary case, both ϕ and ψ are monotonic. Here we cut up A as shown in the right-hand diagram in Figure 1.3 and the estimate follows from the previous case and Lemma 16.

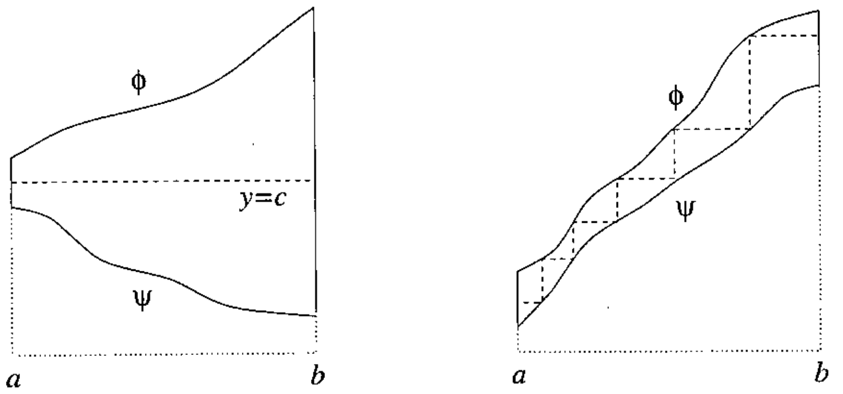

Figure 1.3: The two kinds of curved trapezoid

Finally, some algebraic geometry (Bézout's Theorem—refer to [12]) is used to show that any *algebraic domain* can be cut up into a controlled number of curved trapezoids. More precisely:

Definition 19. *A set $D \subseteq Q^n$ is called a* simple algebraic domain *of type (r, d, n) if there are $r' \leqslant r$ non-constant polynomials f_k of degree at most d such that*

$$D = \{x = (x_1, \ldots, x_n) \in Q^n : f_k(x) \geqslant A_k,\ k = 1, \ldots, r'\}.$$

We say that D is an algebraic domain *of type (r, d, n, w) if there are $w' \leqslant w$ simple algebraic domains of type (r, d, n) such that $D = D_1 \cup \cdots \cup D_{w'}$.*

The authors show that for any (r, d, w) there is an $M = M(r, d, w)$ such that for any algebraic domain D of type $(r, d, 2, w)$ one can find $M' \leqslant M$ curved trapezoids τ_i and a set Z of zero measure such that

$$D = \left(\bigsqcup_{i=1}^{M'} \tau_i \right) \sqcup Z,$$

where the square cups denote disjoint unions. Applying this result to our phase polynomial u in the sublevel set operator, and using the previous steps, the base case $n = 2$ of Theorem 14 is established.

1.4 Where now?

We pull together some of the ideas described in the previous sections. It is natural to seek the improvement of the estimate (Theorem 13) known for multilinear sublevel set operators involving type M functions from $L^{p,1}$ estimates to full L^p. To this end, one can ask whether appropriate decompositions can be found of $BC(m,n)$ domains (recall Definition 4) into curved trapezoids. As we saw in Section 1.3, curved trapezoids behave well with respect to the operators we are interested in, and so would seem like good building blocks on which to base such a decomposition. We also note that the continuity condition specified for curved trapezoids as defined in Section 1.3 was not used anywhere, and so it seems reasonable to drop this assumption in our considerations.

Another principle we can draw on is the orthogonality lemma (Lemma 16), also appearing in Section 1.3, which says roughly speaking that as far as our operators are concerned, domains that are the the union of several domains with mutually disjoint horizontal and vertical projections may be treated as just one of them. Thus we shall allow ourselves to perform decompositions "up to orthogonal families". As we shall soon see in the next chapter, one can easily decompose connected $BC(1)$ domains into curved trapezoids. We then notice that the connected components of a disconnected $BC(1)$ domain form an orthogonal family of connected $BC(1)$ domains. In view of Lemma 16, this leads us to the belief that $BC(1)$ domains, possibly disconnected, are suitable "atomic blocks" for any decomposition we might come up with.

The next objects to focus on would seem naturally enough to be $BC(2,1)$ domains. Progress in the continuous case seems fraught with technicalities, and it is at this stage that the idea of working in a quasi-discrete setting emerges. After formulating this notion in Chapter 2, progress is made and the relevant results are given in Chapter 3. Following on, we describe a method that allows us to decompose $BC(n,1)$ domains for any n, and that can be extended to decompose any $BC(m,n)$ domain provided the domain in question does not have any holes.

At some stage, we would expect that the derivative condition $D^\alpha u \geqslant 1$ should come into play. Indeed, considering the implications of this condition in the continuous setting when $\alpha = (1,1)$ leads us to formulate restrictions on how holes are allowed to be arranged within a $BC(m,n)$ domain. These restrictions turn out to be sufficient to allow a full decomposition of general $BC(m,n)$ domains to be found.

Once all the decomposition theorems have been given, we ask to what extent they can serve their original purpose, whether they have applications in other settings, and what questions are raised that might warrant future study.

Chapter 2

The Quasi-discrete Setting

In this chapter, we develop further some of the ideas mentioned in the Introduction to begin formulating and working on questions that arise naturally from the background material. The key concepts are $BC(m,n)$ domains, orthogonality and curved trapezoids.

Another idea, which is fundamental to the whole thesis, is that of working in an essentially discrete setting, based on the discrete two-dimensional plane, $\mathbb{Z}^2$. Here many of the barriers to progress found in $\mathbb{R}^2$ are removed, and one hopes that suitable approximation or limiting arguments can be found in order to wield the results proved in this chapter and the next back in a continuous context. The results from the discrete setting of course have value and interest in their own right, and this is also discussed later on.

We start by recalling a few definitions and introducing a new one.

Definition 20.

- *A* curved trapezoid *is a set of the form* $\{(x,y) \in \mathbb{R}^2 : a < x < b, f(x) < y < g(x)\}$, *where* f *and* g *are monotonic functions. (Notice that we have dropped the continuity assumption from the definition used in [11].)*

- *A set* $\Omega \subseteq \mathbb{R}^n$ *is said to be* $BC(m_1, \ldots, m_n)$ *if for each* i *and any line* L *parallel to the* i^{th} *axis,* $L \cap \Omega$ *has at most* m_i *connected components. We abbreviate* $BC(m, \ldots, m)$ *to* $BC(m)$.

- *Let* Ω *be an open subset of* $\mathbb{R}^n$. *The function* $p : \Omega \longrightarrow \mathbb{R}$ *is said to be of* type M *if it has the property:*

there exists N such that for all β with $|\beta| \leqslant M$ and all $s > 0$, the set $\{x \in \Omega : |D^\beta p(x)| \leqslant s\}$ is $BC(N)$.

The least such N is called the type M constant $t_M(p)$ *of p.*

- *The subset Ω of $\mathbb{R}^2$ is* horizontally convex *(H-convex) if whenever (x, y_0) and $(x', y_0) \in \Omega$ and $x < z < x'$, then $(z, y_0) \in \Omega$. Similarly, Ω is* vertically convex *(V-convex) if whenever $(x_0, y), (x_0, y') \in \Omega$ and $y < w < y'$, then $(x_0, w) \in \Omega$. We use 'HV-convex' to mean 'both horizontally and vertically convex'.*

Definition 21. *A family of sets $\{\Omega_i\}$ in $\mathbb{R}^n$ is said to be a $BC(m_1, \ldots, m_n)$ family if for all i, no line parallel to the i^{th} coordinate axis meets more than m_i of the Ω_i.*

Motivated by the methods of [11], as discussed in the Introduction, it seemed natural to ask whether one could decompose $BC(N)$ domains in $\mathbb{R}^2$ into a number of curved trapezoids bounded by a function of N. The simplest case, that of $BC(1)$ domains, readily yields to a straightforward decomposition without needing any further hypotheses.

A key result in [11] is Lemma 16, which was stated on page 21. What it says in effect is that in the context of their work, domains that are the the union of several domains of mutually disjoint horizontal and vertical projections may be treated as just one of them. In particular, since the connected components of a $BC(1)$ domain form a $BC(1)$ family, it is enough to consider connected $BC(1)$ domains.

Thus let Ω be a bounded connected subset of the (x, y)-plane with the $BC(1)$ property, i.e. that for all axis parallel lines L, the set $L \cap \Omega$ consists of a single component.

For $t \in \mathbb{R}$ let L_t denote the vertical line $x = t$. Clearly, for each t there exist $a(t)$ and $b(t)$ such that if $L_t \cap \Omega \neq \emptyset$ then $\pi_y(L_t \cap \Omega)$ is an interval with endpoints $a(t)$ and $b(t)$, where π_y denotes the usual projection onto the y-axis. It is easy to see that a and b can change monotonicity at most once, otherwise the

$BC(1)$ property would be violated. Up to symmetry, we have four cases of what Ω could look like, determined by whether either, both or neither of a, b change monotonicity. (See Figure 2.1.)

If both a and b are monotonic then we just have a curved trapezoid. If either one changes monotonicity, we make a vertical cut at the point where it does so and are left with at most three curved trapezoids. Thus we have proved:

Theorem 22. *A bounded, connected $BC(1)$ domain in $\mathbb{R}^2$ can be decomposed into three or fewer curved trapezoids.*

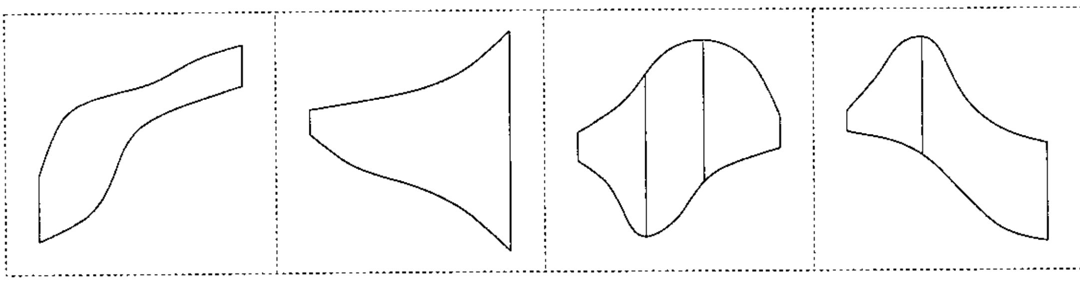

Figure 2.1: The possibilities in the $BC(1)$ case

The quasi-discrete setting

The next thing to investigate would seem to be $BC(2,1)$ domains, but the goal of decomposing $BC(2,1)$ domains in $\mathbb{R}^2$ has proved elusive thus far. However, one might consider approximating such domains by domains that are roughly speaking made up of squares aligned to some grid. It will suffice to work with unit squares and then scale. This discrete approach provides a more amenable setting for decompositions to take place, since many of the technicalities involved with $\mathbb{R}^2$ are done away with. However, first of all we must establish that the important notions of the BC properties and connectedness can be carried between the two settings safely.

Getting down to details, we think of $\mathbb{Z}^2$ as sitting inside $\mathbb{R}^2$ in the usual way. Furthermore, given a subset A of $\mathbb{Z}^2$, we can associate it with the subset of $\mathbb{R}^2$ that is the union of the unit squares centred at each point of A. This informal association ignores the question of what goes on at the boundaries of the squares, but given that these form a set of measure zero in the plane, this issue does not cause us any problems.

We define operators $\mathcal{S}$ and $\mathcal{T}$ that map between subsets of $\mathbb{Z}^2$ and subsets of $\mathbb{R}^2$, and give the exact meaning of the word 'quasi-discrete':

Definition 23.

- *For a subset U of $\mathbb{R}^2$, define $\mathcal{S}(U)$ as the set of points of $\mathbb{Z}^2$ lying in U, i.e. $U \cap \mathbb{Z}^2$.*
- *For a subset A of $\mathbb{Z}^2$, define $\mathcal{T}(A)$ as the interior of the union of the closed unit-side squares centred at each point of A. For aesthetic reasons we may write $\mathcal{T}(s)$ rather than $\mathcal{T}(\{s\})$ when s is a single point of $\mathbb{Z}^2$.*
- *We say that $\Omega \subseteq \mathbb{R}^2$ is a* quasi-discrete domain *if $\Omega = \mathcal{T}(A)$ for some $A \subseteq \mathbb{Z}^2$.*

A characterisation of the set $\mathcal{T}(A)$ for $A \subseteq \mathbb{Z}^2$ is as follows: for each $x \in \mathbb{R}^2$, let A_x be the set of elements of $\mathbb{Z}^2$ of shortest distance (with the usual metric) to x. Then A_x has either 1, 2 or 4 elements, depending on whether x lies on the inside, edge or corner of a unit square centred at a point of $\mathbb{Z}^2$. (See Figure 2.2.) We see that

$$x \in \mathcal{T}(A) \text{ if and only if } A_x \subseteq A. \tag{2.1}$$

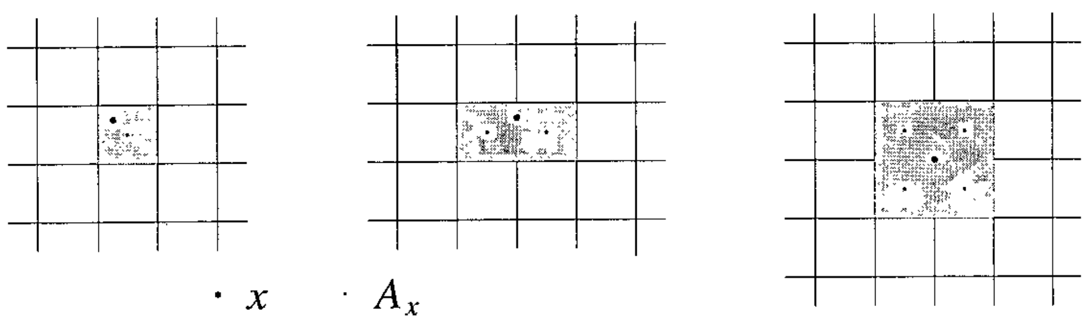

Figure 2.2: The possibilities for A_x

We also introduce a notion of $BC(m,n)$-ness and related ideas in $\mathbb{Z}^2$.

Definition 24. *For any $n \in \mathbb{N}$, let $\overline{n} = \{0, 1, \ldots, n\}$.*

- *We define an* interval *in $\mathbb{Z}$ as a set of the form $\overline{n}+m$ for some $n \in \mathbb{N}, m \in \mathbb{Z}$; or $\{n \in \mathbb{Z} : n \geqslant n_0\}$ for some $n_0 \in \mathbb{Z}$; or $\{n \in \mathbb{Z} : n \leqslant n_0\}$ for some $n_0 \in \mathbb{Z}$. We use round and square bracket notation in the same way as for intervals on the real line, for instance $(a, b] = \{x \in \mathbb{Z} : a < x \leqslant b\}$.*

- *We say that $A \subseteq \mathbb{Z}^2$ is a $BC(m, n)$* domain *if for each fixed y_0, the horizontal section $A \cap \{(x, y_0) : x \in \mathbb{Z}\}$ consists of at most m intervals and for each fixed x_0, the vertical section $A \cap \{(x_0, y) : y \in \mathbb{Z}\}$ consists of at most n intervals.*

- *We say that a family $\{A_i\}_{i \in I}$ of subsets of $\mathbb{Z}^2$ is $BC(m, n)$ if no line $\{(x, y_0) : x \in \mathbb{Z}\}$ meets more than m of the A_i and no line $\{(x_0, y) : y \in \mathbb{Z}\}$ meets more than n of them.*

- *We call $A \subseteq \mathbb{Z}^2$* horizontally convex *if whenever (x, y_0) and (x', y_0) are in A and $x < z < x'$, then $(z, y_0) \in A$. Vertical convexity is defined similarly.*

We observe the following about $\mathcal{S}$ and $\mathcal{T}$.

Lemma 25. *If $\Omega \subseteq \mathbb{R}^2$ is $BC(m, n)$ for some $m, n \in \mathbb{N}$, then $\mathcal{S}(\Omega)$ is also $BC(m, n)$. If $A \subseteq \mathbb{Z}^2$ is $BC(1)$ then $\mathcal{T}(A)$ is also $BC(1)$, i.e. $\mathcal{T}$ preserves HV-convexity.*

Proof. Let $\Omega \subseteq \mathbb{R}^2$ and suppose that for some $y_0 \in \mathbb{Z}$, the section $\mathcal{S}(\Omega) \cap \{(x, y_0) : x \in \mathbb{Z}\}$ has at least $m + 1$ intervals. Then there are $x_1 < z_1 < x_2 < \cdots < z_m < x_{m+1} \in \mathbb{Z}$ such that $(x_i, y_0) \in \mathcal{S}(\Omega)$ and $(z_i, y_0) \notin \mathcal{S}(\Omega)$ for all i. But $\mathcal{S}(\Omega) = \Omega \cap \mathbb{Z}^2$, and so $(x_i, y_0) \in \Omega$ and $(z_i, y_0) \notin \Omega$ for all i, which means that the cross-section of Ω at height y_0 must have at least $m + 1$ components.

Now let $A \subseteq \mathbb{Z}^2$ and suppose that $\mathcal{T}(A)$ is not HV-convex. Let's say that it fails H-convexity. Then we must have $x < z < x'$ and y_0 in $\mathbb{R}$ such that $(x, y_0), (x', y_0) \in \mathcal{T}(A)$ and $(z, y_0) \notin \mathcal{T}(A)$. By the characterisation of $\mathcal{T}$ given at 2.1, we have $A_{(x,y_0)}, A_{(x',y_0)} \subseteq A$ and $A_{(z,y_0)} \not\subseteq A$. This immediately tells us that A is not H-convex. $\square$

We note that $\mathcal{T}$ does not preserve $BC(m,n)$-ness generally. (See Figure 2.3.) However it is easily seen that things only go wrong on the boundaries of the unit squares centred at points of $\mathbb{Z}^2$.

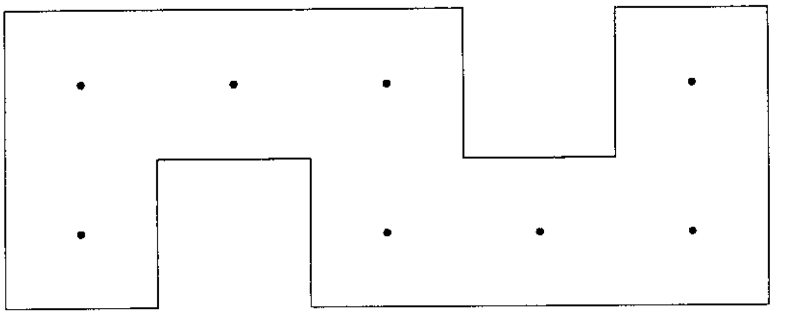

Figure 2.3: A $BC(2,1)$ domain A in $\mathbb{Z}^2$ such that $\mathcal{T}(A)$ is not $BC(2,1)$

Also needed is a formulation of path-connectedness for $\mathbb{Z}^2$:

Definition 26.

- *Let $p = (p_1, q_1)$ and $q = (q_1, q_2)$ be points of $A \subseteq \mathbb{Z}^2$. We say that p* abuts *q if $|p_1 - q_1| + |p_2 - q_2| = 1$. (Geometrically, p and q are non-diagonally adjacent.)*
- *A* path *in A from p to q is a function $\gamma : \overline{N} \longrightarrow A$ (for some $N \in \mathbb{N}$) such that $\gamma(0) = p$, $\gamma(N) = q$ and for each $i < N$, $\gamma(i)$ abuts $\gamma(i+1)$.*
- *We say that $A \subseteq \mathbb{Z}^2$ is path-connected if there exists a path between every two points of A.*

We would like this to be compatible with the usual path-connectedness in $\mathbb{R}^2$. We shall need:

Lemma 27. *The operator $\mathcal{T}$ has the following basic properties:*

1. *For all $A \subseteq B \subseteq \mathbb{Z}^2$, $\mathcal{T}(A) \subseteq \mathcal{T}(B)$.*
2. *For disjoint subsets A, B of $\mathbb{Z}^2$, $\mathcal{T}(A) \cap \mathcal{T}(B) = \emptyset$.*
3. *For all $A, B \subseteq \mathbb{Z}^2$, $\mathcal{T}(A \cup B) \supseteq \mathcal{T}(A) \cup \mathcal{T}(B)$.*

4. *For all disjoint $A, B \subseteq \mathbb{Z}^2$, $\mathcal{T}(A \cup B) = \mathcal{T}(A) \cup \mathcal{T}(B)$ if and only if no point of A abuts a point of B.*

Proof.

1. Let $A \subseteq B \subseteq \mathbb{Z}^2$. Then

$$\mathcal{T}(A) = \left(\bigcup_{x \in A} \overline{\mathcal{T}(x)} \right)^{\circ} \subseteq \left(\bigcup_{x \in B} \overline{\mathcal{T}(x)} \right)^{\circ} = \mathcal{T}(B).$$

2. Suppose not, and let $x \in \mathcal{T}(A) \cap \mathcal{T}(B)$, where $A \cap B = \emptyset$. Then, using the characterisation of $\mathcal{T}(\cdot)$ given at 2.1, $x \in \mathcal{T}(A)$ and $x \in \mathcal{T}(B)$ implies that $A_x \subseteq A$ and $A_x \subseteq B$, which is a contradiction.

3. Follows immediately from 1.

4. We prove the contrapositive of each implication.

   - First suppose that $a \in A$ abuts $b \in B$. Let p be the point midway between a and b. Clearly $p \notin \mathcal{T}(A)$ and $p \notin \mathcal{T}(B)$ but $p \in \mathcal{T}(A \cup B)$.

   - Suppose now that $\mathcal{T}(A \cup B) \neq \mathcal{T}(A) \cup \mathcal{T}(B)$. By 2, there is some $x \in \mathcal{T}(A \cup B) - \mathcal{T}(A) \cup \mathcal{T}(B)$. Consider the set A_x, which by 2.1 is a subset of $A \cup B$. Clearly A_x must have more than one element, that is to say x does not lie in the interior of any square, and so A_x must have either two or four elements.

     If A_x has two elements, s and t say, then by 2.1 it must be that $s \in A$, $t \in B$ or $s \in B$, $t \in A$. If A_x has four elements, say $A_x = \{s, t, u, v\} \subseteq A \cup B$, then by 2.1 it must be that at least one of s, t, u, v is not in A (hence is in B), and at least one of them is not in B (hence is in A). So in either case we have an element of A abutting an element of B.

□

Lemma 28. *$A \subseteq \mathbb{Z}^2$ is path-connected if and only if $\mathcal{T}(A)$ is path-connected.*

Proof. First suppose that A is path-connected and let $x, y \in \mathcal{T}(A)$. Then we can find closed unit-side squares S_x, S_y that contain x, y respectively and whose centres s_x, s_y lie in A. We can then "join the dots" to find a path between s_x and s_y, then join x to s_x and y to s_y with straight lines. So $\mathcal{T}(A)$ is path-connected.

Suppose conversely that $\mathcal{T}(A)$ is path-connected. We show by induction on $|A|$ that A is path-connected. The base case $|A| = 1$ is trivial. Assume that the result holds for $|A| \leqslant k$ and let $B \subseteq \mathbb{Z}^2$ with $|B| = k + 1$. Choose any point $s = (s_1, s_2) \in B$ and consider $\mathcal{T}(B \backslash \{s\})$. We claim that its connected components are all of the form $\mathcal{T}(B_i)$ for some $B_i \subseteq B \backslash \{s\}$, $i = 1, \ldots, n$. In fact, we have the following little result:

Claim. If $D \subseteq \mathbb{Z}^2$ and C is a connected component of $\mathcal{T}(D)$, then $C = \mathcal{T}(\mathcal{S}(C))$.

Proof. First suppose that $x \in C$. Then $A_x \subseteq D$ and so $\mathcal{T}(A_x) \subseteq \mathcal{T}(D)$. Since C is connected, $\mathcal{T}(A_x) \subseteq C$. In particular, $A_x \subseteq \mathcal{S}(C)$, and hence $x \in \mathcal{T}(\mathcal{S}(C))$.

Now suppose that $x \in \mathcal{T}(\mathcal{S}(C))$. Then $A_x \subseteq \mathcal{S}(C)$. So $A_x \subseteq C \subseteq \mathcal{T}(D)$ and $A_x \subseteq D$. Now the connected set $\mathcal{T}(A_x)$ is a subset of $\mathcal{T}(D)$ and hence $\mathcal{T}(A_x) \subseteq C$. □

Continuing with the proof of the lemma, we claim next that each B_i must abut s, otherwise the connectedness of $\mathcal{T}(B)$ is contradicted. For we have just seen that there is a finite number of connected components C_i of $\mathcal{T}(B \backslash \{s\})$, each of which is $\mathcal{T}(B_i)$ for some $B_i \subseteq B \backslash \{s\}$. By the inductive hypothesis, each B_i is path-connected. For simplicity, say that there are just two B_i—the argument is no deeper for more than two. Suppose for a contradiction that B_1 has no square abutting s. Clearly B_1 has no square abutting a square of B_2 either, otherwise the disconnectedness of $C_1 \cup C_2$ is contradicted. By part 3 of the above lemma,

$$\mathcal{T}(B) = \mathcal{T}(B_1 \cup B_2 \cup \{s\}) = \mathcal{T}(B_1) \cup \mathcal{T}(B_2 \cup \{s\}).$$

But this implies that $\mathcal{T}(B)$ is disconnected, and we have a contradiction.

Now, given $p, q \in B$, we can find paths in B from p to s and from q to s (by path-connectedness of the B_i and the fact that they abut s). Joining them together, we have a path from p to q. Hence B is path-connected and the result holds by induction.

□

Remember that open subsets of $\mathbb{R}^n$ are connected if and only if they are path-connected. From now on in the discrete situation, we drop the "path-" and just say that $A \subseteq \mathbb{Z}^2$ is connected if it is path-connected.

One final definition is needed for a concept of holes inside a subset A of $\mathbb{Z}^2$. In analogy with the definition in $\mathbb{R}^2$, we define the holes of $A \subseteq \mathbb{Z}^2$ to be the bounded connected components of the complement $\mathbb{Z}^2 \backslash A$. We note that it is not true that whenever H is a hole of $A \subseteq \mathbb{Z}^2$ then $\mathcal{T}(H)$ is a hole of $\mathcal{T}(A)$, though this does not cause us any difficulties. See Figure 2.4.

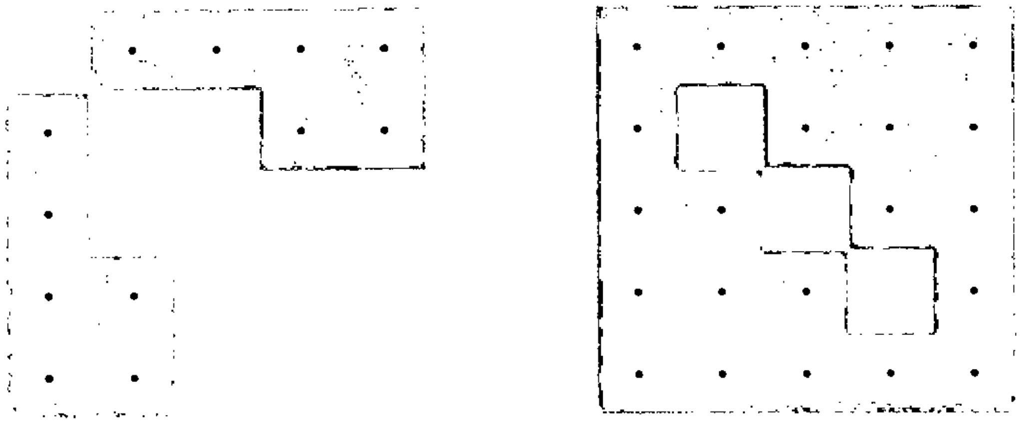

Figure 2.4: Examples in $\mathbb{Z}^2$: a disconnected $BC(2)$ domain and a connected $BC(2)$ domain with three holes

Chapter 3

The Main Decompositions

In this central chapter, our goal is to decompose bounded connected $BC(m,n)$ domains in $\mathbb{Z}^2$ into controlled numbers of $BC(1)$ domains. For the sake of clarity, we deal exclusively in the discrete setting ($\mathbb{Z}^2$, that is) and the quasi-discrete setting (sets $\mathcal{T}(A) \subseteq \mathbb{R}^2$ for $A \subseteq \mathbb{Z}^2$), with the results proved here being linked to the continuous case later. The order is as follows. Firstly two useful general results, Corollary 30 and Lemma 33, are given in Section 3.1. Then in Section 3.2 we treat $BC(2,1)$ domains, followed by $BC(n,1)$ domains for any $n \in \mathbb{N}$, which yields Theorems 37 and 40 respectively. Next come $BC(2,2)$ domains in Section 3.3, subdivided into the cases of those without any holes (Theorem 42) and those that do have holes (Theorem 47). Finally in Section 3.4 we consider $BC(3,3)$ domains (Theorem 52) and in doing so develop the final tools necessary for a decomposition of $BC(m,n)$ domains for any $m,n \in \mathbb{N}$, which we state as Theorem 55. The consequences for quasi-discrete domains are stated as Corollary 56.

3.1 Two general results

Before we go any further, we give two general results that are well-used in the subsequent domain decompositions. We recall that a family $\{A_i\}_{i \in I}$ of subsets of $\mathbb{Z}^2$ is called $BC(m,n)$ if no line $\{(x,y_0) : x \in \mathbb{Z}\}$ meets more than m of the A_i and no line $\{(x_0,y) : y \in \mathbb{Z}\}$ meets more than n of them, and that $A \subseteq \mathbb{Z}^2$ is connected if there is a path, i.e. a sequence of points each abutting the next, between every pair of points of A.

Theorem 29. *Let $\{A_j\}_{j=1}^{\infty}$ be a disjoint collection of bounded connected sets in $\mathbb{Z}^2$ that form a $BC(m,n)$ family. Then it is possible to sort the A_j into two (or fewer) $BC(m,n-1)$ families.*

Corollary 30. *A $BC(m,n)$ family as above may be split into 2^{m+n-2} or fewer $BC(1)$ families.*

Proof. Firstly, we note the following easily proved properties of intervals in $\mathbb{Z}$ and make a definition:

Lemma 31. *Let $I_1, I_2, \ldots, I_n$ be bounded intervals in $\mathbb{Z}$ such that $I_j \cap I_1 \neq \emptyset$ for $j = 2, \ldots, n$.*

1. *If $\cap_{j=2}^{n} I_j \neq \emptyset$ then $\cap_{j=1}^{n} I_j \neq \emptyset$.*
2. *If $I_j \cap [\sup I_1, \infty) \neq \emptyset$ for $j = 2, \ldots, n$ then $\cap_{j=1}^{n} I_j \neq \emptyset$.*

Definition 32. *We say that subsets $\{S_i\}_{i \in I}$ of $\mathbb{Z}^2$ have* x-overlap at x_0 *if $x_0 \in \bigcap_{i \in I} \pi_x S_i$; and that the S_i have* x-overlap *if $\bigcap_{i \in I} \pi_x S_i \neq \emptyset$.*

Now for the proof proper. Begin by choosing some $A_0 = A_{0,1}$. Let $\{A_{1,j}\}_j$ be the set of $A_j \neq A_0$ that have x-overlap with A_0 and for $i \geqslant 1$ let $\{A_{i+1,j}\}_j$ be the set of A_j not picked yet that have x-overlap with some $A_{i,j'}$. Also define for all i,

$$a_i = \inf \pi_x \Big(\bigcup_{l \leqslant i} \bigcup_{j} A_{l,j} \Big), \quad b_i = \sup \pi_x \Big(\bigcup_{l \leqslant i} \bigcup_{j} A_{l,j} \Big).$$

Evidently we have

$$\ldots \leqslant a_{i+1} \leqslant a_i \leqslant \ldots \leqslant a_0 < b_0 \leqslant \ldots \leqslant b_i \leqslant b_{i+1} \leqslant \cdots.$$

The result is soon derived from the following claim:

Claim.

1. For $i \geqslant 2$, none of the elements chosen at stage i have x-overlap with $[a_{i-2}, b_{i-2}]$, i.e.
$$\pi_x \Big(\bigcup_{j} A_{i,j} \Big) \cap [a_{i-2}, b_{i-2}] = \emptyset.$$

2. For $i = 0, 1, 2, \ldots$, there exist m_i, n_i, R_i, L_i such that

$$\pi_x\left(\bigcup_{l \leqslant i} \bigcup_j A_{l,j}\right) = \left[\inf \pi_x A_{m_i, L_i}, \sup \pi_x A_{n_i, R_i}\right].$$

3. For a fixed i, all $A_{i,j}$ have x-overlap with either $A_{m_{i-1}, L_{i-1}}$ or $A_{n_{i-1}, R_{i-1}}$.

4. For each i, $\{A_{i,j}\}_j$ is a $BC(m, n-1)$ family.

The proof of this claim goes by induction. The case $i = 0$ is trivial, so suppose inductively that the result holds for all $i \leqslant k$ and consider the case $i = k+1$. We begin by establishing 1. If $k = 0$ then it holds trivially, so suppose $k \geqslant 1$. But since $\pi_x(\bigcup_{l<k} \bigcup_j A_{l,j})$ is an interval, we must have that $\pi_x(\bigcup_j A_{k+1,j}) \cap [a_{k-1}, b_{k-1}] = \emptyset$, otherwise some $A_{k+1,j}$ would have been chosen earlier.

We now show 3. To see this, consider some $A_{k+1,j}$ (with $k \geqslant 1$, to avoid trivialities). Since it has no x-overlap with $[a_{k-1}, b_{k-1}]$, we must have that $\pi_x(A_{k+1,j})$ is included in $(-\infty, a_{k-1} - 1]$ or $[b_{k-1} + 1, \infty)$. Without loss, suppose it is the latter. Since $A_{k+1,j}$ must have x-overlap with some $A_{k,j'}$, it must have x-overlap with $[b_{k-1} + 1, b_k]$—in particular $b_k > b_{k-1}$. But because A_{n_k, R_k} has x-overlap with some $A_{l,j''}$ with $l \leqslant k-1$ and $b_k = \sup \pi_x A_{n_k, R_k}$, it is clear that

$$[b_{k-1} + 1, b_k] \subseteq \pi_x(A_{n_k, R_k}), \tag{3.1}$$

which establishes 3.

Now observe that at most $n-1$ of the $A_{k+1,j}$ may have x-overlap with $[b_k, \infty)$, which follows by applying part 2 of Lemma 31. Let $A_{n_{k+1}, R_{k+1}}$ be one such whose x-projection has greatest supremum; if there are none, just put $n_{k+1} = n_k$, $R_{k+1} = R_k$. Do the obvious corresponding thing to choose m_{k+1} and L_{k+1}. This establishes 2 for $i = k+1$.

Finally we tackle 4. Suppose for a contradiction that we had a collection of n of the $A_{k+1,j}$ that had x-overlap at the point x_0. By 1, (already proved for $i = k+1$), x_0 must lie in $(-\infty, a_{k-1} - 1]$ or $[b_{k-1} + 1, \infty)$. Without loss of generality, say $x_0 > b_{k-1}$. If $x_0 \in [b_{k-1} + 1, b_k]$ then by (3.1) we have $n+1$ of the A_j with x-overlap at x_0, which contradicts the $BC(m, n)$ hypothesis. Therefore

$x_0 > b_k$. But now, since each of the n sets $A_{k+1,j}$ in question has x-overlap with A_{n_k,R_k}, we can apply part 1 of Lemma 31 and arrive at a contradiction as before. The whole claim now follows by induction.

Continuing with the proof of the theorem, put

$$\mathcal{F}_1^1 = \{A_{i,j} : \text{ i even}, j \in \mathbb{N}\}, \quad \mathcal{F}_1^2 = \{A_{i,j} : \text{ i odd}, j \in \mathbb{N}\}.$$

By properties 2 and 4 of our claim, these are both $BC(m, n-1)$ families. Working inductively, begin the process again (if necessary) at some new A_0 hitherto unchosen and repeat, obtaining families $\mathcal{F}_2^1, \mathcal{F}_2^2, \mathcal{F}_3^1, \ldots$, until all the A_j have been used up. We see easily that the families

$$\mathcal{F}_1 = \bigcup_i \mathcal{F}_i^1 \qquad \text{and} \qquad \mathcal{F}_2 = \bigcup_i \mathcal{F}_i^2$$

satisfy the requirements of the theorem and we are done. □

The second general result is

Lemma 33. *The intersection of a $BC(m, m')$ domain with a $BC(n, n')$ domain is a $BC(m + n - 1, m' + n' - 1)$ domain, where $m, m', n, n' \geqslant 1$.*

Proof. It is enough to prove: if A, B are bounded nonempty subsets of $\mathbb{Z}$ having at most m and n connected components respectively, then $A \cap B$ has at most $m + n - 1$. This we do by induction on n.

Case $n = 1$. Let B be a single interval. We need to show that $A \cap B$ has at most m components. Well, writing $A = \bigcup_{j=1}^m I_j$ where the I_j are the connected components of A, we have $A \cap B = \bigcup_{j=1}^m I_j \cap B$. Since the intersection of two intervals is also an interval, we are done.

Case n=k+1. Suppose that the result holds whenever $n \leqslant k$ and let B have $k + 1$ connected components. Write $A = \bigcup_{j=1}^m I_j$ and $B = \bigcup_{l=1}^{k+1} J_l$, where the I_j and J_l are the connected components of A and B, ordered from left to right. Consider J_1, the leftmost component of B. If it meets none of the components of A, then we are reduced to a previous case. So suppose that J_1 meets $p \geqslant 1$ of the I_j. Now out of these p of the I_j, the remaining components of B can only meet the rightmost one, which must be I_{p+q} for some $q \geqslant 0$. Hence they can meet at most $m-(p-1)$ of all the I_j. Thus we have

$$A \cap B = \bigcup_{j=1}^{m} I_j \cap \left(J_1 \cup \bigcup_{l=2}^{k+1} J_l\right) = \left(\bigcup_{j=1}^{m} I_j \cap J_1\right) \cup \left(\bigcup_{j=1}^{m} I_j \cap \bigcup_{l=2}^{k+1} J_l\right).$$

Since by the inductive hypothesis $\bigcup_{j=1}^{m} I_j \cap J_1$ has at most p components and $\bigcup_{j=1}^{m} I_j \cap \bigcup_{l=2}^{k+1} J_l$ has at most $\big(m-(p-1)\big)+k-1$ components, $A \cap B$ has at most $m+(k+1)-1$ components. The result follows by induction. □

3.2 Decomposition of $BC(n,1)$ domains

$BC(2,1)$ domains

We begin the domain decompositions with the case of $BC(2,1)$ domains. This can be done using fairly elementary methods, and we obtain what seems like a good, low bound on the number of resulting $BC(1)$ domains. We then give a result for $BC(n,1)$ domains for any $n \in \mathbb{N}$, which requires more complicated methods and gives a bound that increases fairly quickly with n.

If $A \subseteq \mathbb{Z}^2$ is a bounded disconnected $BC(2,1)$ domain, Theorem 29 can be applied to the collection of connected components of A, separating them into two $BC(1)$ families. Bearing in mind Lemma 16 and the potential applications of what follows to integral operators, we now assume that A consists of just a single connected component.

So, let $A \subseteq \mathbb{Z}^2$ be a bounded connected $BC(2,1)$ domain. Take an injective path γ from bottom to top. (By this we mean: choose points $a=(a_1,a_2)$ and $b=(b_1,b_2)$ in A such that $a_2=\min \pi_y A$ and $b_2=\max \pi_y A$, and let γ be a path in A from a to b.) We can take γ to be x-monotonic (let's say increasing) by vertical convexity. We also choose γ to minimise the quantity $|D|$, where

$$D=\{t : \pi_y\gamma(t+1) < \pi_y\gamma(t)\}.$$

Claim. If $t_0 \in D$ then $\pi_y\gamma$ is increasing on $[t_0+1,t_1]$, where t_1 is defined to be $\inf\{t > t_0 : \pi_y\gamma(t) = \pi_y\gamma(t_0)+2\}$. That is to say that if γ dips by one square then it cannot dip again until it climbs to at least two squares above its original height.

Proof. Suppose not. There are three possibilities, namely that γ dips next at height $\pi_y t_0 - 1$, $\pi_y t_0$ or $\pi_y t_0 + 1$. These are sketched in Figure 3.1. Then for some height z (in fact we can take $z = \pi_y t_0$ or $\pi_y t_0 - 1$) there are at least 3 components in $\mathbb{N}$ of $\gamma^{-1}(\gamma^* \cap \pi_y^{-1}(z))$. But now $A \cap \pi_y^{-1}(z)$ must have at least 3 components or contradict the least dipping property of γ.

(The idea is that γ visits height z for three separated time periods, in between which it must have y-coordinates different from z. At these in-between stages it must be circumventing points of A^c, otherwise it could just carry on at height z with less dipping.) □

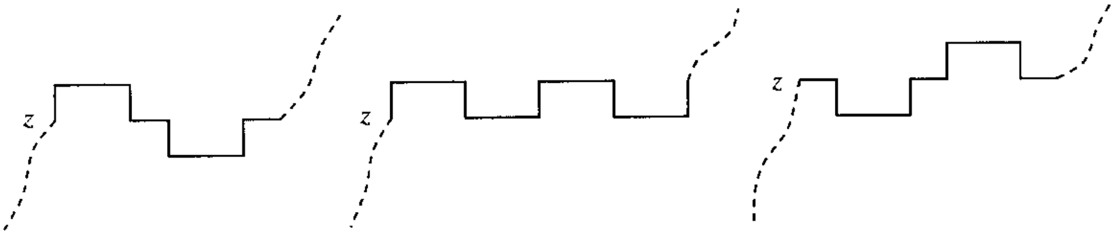

Figure 3.1: γ must climb a bit before it dips again

Hence γ is a disjoint union of y-increasing subpaths γ_j such that $\min \pi_y \gamma_{j+1} = \max \pi_y \gamma_j - 1$ and $\max \pi_y \gamma_j \geqslant \min \pi_y \gamma_j + 3$. At this point we need to introduce another object:

Definition 34. *Let $A \subseteq \mathbb{Z}^2$. We define, for $t \in \mathbb{Z}$, the* horizontal beams *at height t to be the connected components of the horizontal cross-section of A at height t. Vertical beams are defined similarly.*

Letting A_j be the union of horizontal beams of A that meet γ_j and $A' = \bigcup A_j$, we have:

$$\min \pi_y A_{j+2} = \max \pi_y A_{j+1} - 1 \geqslant \min \pi_y A_{j+1} + 2 = \max \pi_y A_j + 1$$

and so the families $\{A_{2j}\}$ and $\{A_{2j+1}\}$ are both vertically disjoint. (That is to say no A_{2j} has vertical overlap with any $A_{2j'}$ and no A_{2j+1} has vertical overlap with any $A_{2j'+1}$.)

Claim. These families are both horizontally disjoint too.

Proof. Let c be the x-coordinate of the rightmost point on the top horizontal beam of A_j. By the behaviour of γ, for any $d \geqslant \max \pi_y A_j$, the point $(c+1, d) \in A^c$. Since γ is x-increasing and $\pi_y A_{j+2} \geqslant \max \pi_y A_j + 1$, we have $\pi_x A_{j+2} > c$.

It remains to see that $c = \max \pi_x A_j$. If not, then there is a $p = (p_1, p_2)$ such that $p_1 = c + 1$ and $p_2 < \max \pi_y A_j - 1$. But then if γ were diverted through p parallel to the axes it would have less dipping, which is a contradiction. See Figure 3.2. □

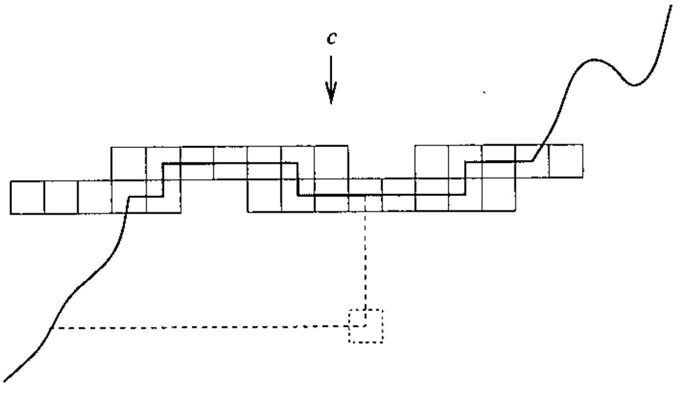

Figure 3.2: Showing that A_j and A_{j+2k} are horizontally disjoint

Recall that we defined $A' = \bigcup A_j$. The following property of A' holds:

Lemma 35. *A' is a $BC(2,1)$ domain.*

Proof. The horizontal part is clear. For vertical convexity, suppose that we have points p, q, r with $p_1 = q_1 = r_1$ and $p_2 < q_2 < r_2$ and such that $p, r \in A'$ but $q \in A_1^c$. By the $BC(2,1)$ property of A, $q \in A \backslash A'$. Without loss of generality, γ passes through a point s such that $s_1 < q_1$ and $s_2 = q_2$. Consider now the point $u = (t_1, r_2)$. If at height r_2, γ lies to the left of u then by the horizontal convexity of A', $u \in A'$, while if γ lies to the right of u then there must be a point z on γ with $z_1 = t_1$ and $r_2 > z_2 > t_2$. Similar reasoning shows that there is a point v with $v_1 = t_1$ and $v_2 < t_2$. Thus the vertical convexity of A is violated. (See Figure 3.3.) □

Similarly, we can show that each A_j is vertically convex. Since it is horizontally convex too (straight from the definition), it is $BC(1)$.

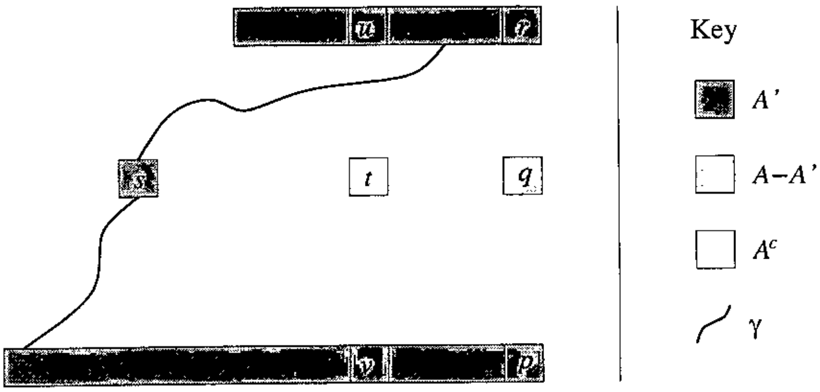

Figure 3.3: The situation in Lemma 35

Now consider $A \backslash A'$. Note that it is certainly H-convex. Also, since A' is V-convex, each vertical cross-section of $A \backslash A'$ consists of at most two intervals. Let A_L be the union of all the vertical intervals thus given that lie below a piece of A' together with any vertical intervals that contain no point of A' and put $A_U = A \backslash (A' \cup A_L)$. (See Figure 3.4 for the various possibilities.)

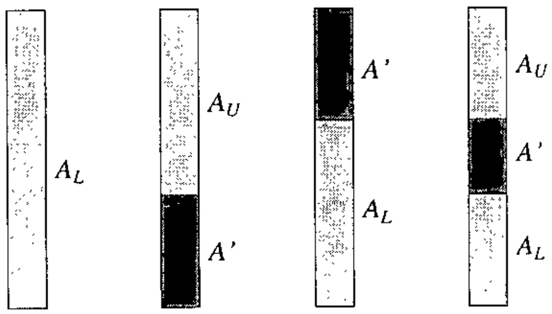

Figure 3.4: What A_L and A_U look like

Lemma 36. *A_L and A_U are $BC(1)$ domains.*

Proof. We treat only A_L, since the case of A_U is very similar. Vertical convexity is trivial. Suppose then that there are p, q, r with $p_1 < q_1 < r_1$ and $p_2 = q_2 = r_2$ and $p, r \in A_L$, $q \notin A_L$. By H-convexity of $A \backslash A'$, it must be that $q \in A_U$. Therefore, there must be a point $s \in A'$ with $s_1 = q_1$ and $s_2 < q_2$. Also, there is a $t \in A'$ such that (without loss of generality) $t_1 > r_1$ and $t_2 = r_2$. Note that the definitions of A_U and A_L imply that there can be no point $u \in A'$ with either $u_1 = q_1$, $u_2 > q_2$

or $u_1 = r_1$, $u_2 < r_2$. Furthermore, the H-convexity of $A \backslash A'$ implies that there is no $u \in A'$ with $q_1 < u_1 < r_1$ and $u_2 = q_2$. But this means there can be no path in A' between s and t, contradicting path-connectedness of A'. See Figure 3.5. $\square$

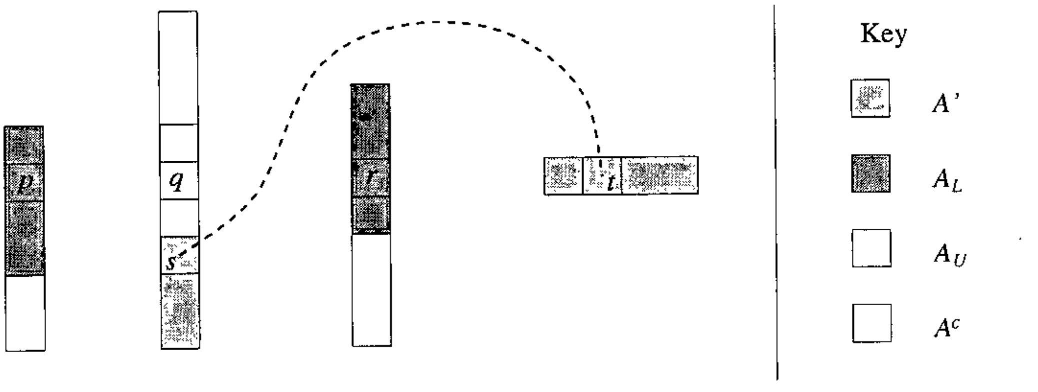

Figure 3.5: Illustration of Lemma 36

Thus we have shown that $A = \bigcup A_{2j} \cup \bigcup A_{2j+1} \cup A_L \cup A_U$, where the four sets on the right hand side are $BC(1)$ domains, as required. We sum things up as:

Theorem 37. *Let A be a bounded, connected $BC(2,1)$ domain in $\mathbb{Z}^2$. Then we can write A as the union of four or fewer $BC(1)$ domains.*

$BC(n, 1)$ domains

Having dealt with $BC(2,1)$ domains, we now give a decomposition of $BC(n,1)$ domains for any $n \in \mathbb{N}$. Let $A \subseteq \mathbb{Z}^2$ be a bounded $BC(n,1)$ domain. We consider the following algorithm, which is illustrated in Figure 3.6:

Algorithm.

- *Begin by choosing a path γ_1 in A that is monotonic in x and y and is maximal in y-length subject to this condition. (That is, $|\pi_y \gamma_1(\mathbb{N})|$ is maximal.) Define A_1 as the union of horizontal beams of A that meet γ_1.*

- *At the $k+1^{th}$ stage, choose a path γ_{k+1} in $A \backslash \bigcup_{j=1}^{k} A_j$ that is monotonic in x and y and maximal in y-length. Let A_{k+1} be the union of horizontal beams through γ_{k+1}.*

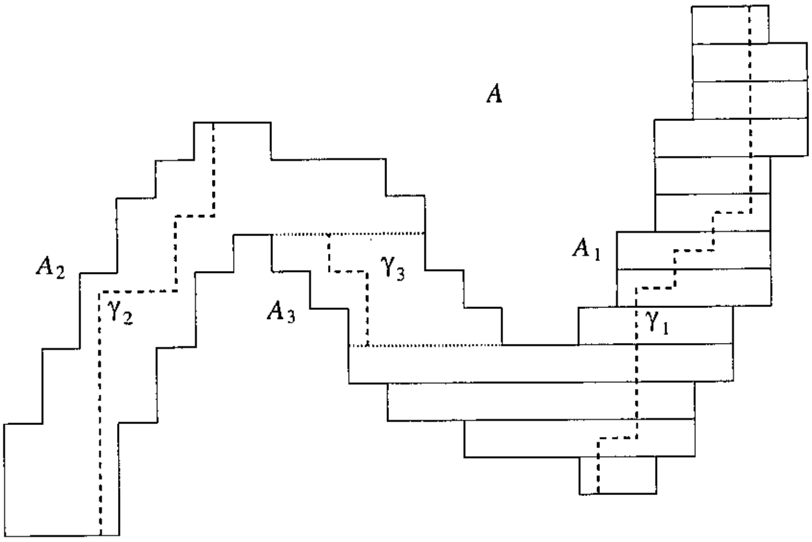

Figure 3.6: Illustration of the Algorithm at work

- *Repeat until we have exhausted the whole of* A, *i.e. we have* $A = \bigcup_{j=1}^{N} A_j$ *for some* N. *This must happen because* A *is finite.*

We note that the same reasoning as in the proof of Lemma 35 shows that A_k is $BC(1)$ for $k = 1, \ldots, N$.

We establish the following property of the Algorithm.

Lemma 38. *Let* A *be a bounded, connected* $BC(n, 1)$ *domain in* $\mathbb{Z}^2$ *and a vertical beam* L *of* A *be given. If we decompose* A *using the Algorithm, then no more than* $2^n - 1$ *of the resulting* A_i *can meet* L.

With this result in hand, we would then have that the A_i form a $BC(n, 2^n - 1)$ family. By Corollary 30, this would mean that the A_i can be sorted into 2^{2^n+n-3} or fewer orthogonal families.

Some notational set-up is required. Suppose that our bounded $A \subseteq \mathbb{Z}^2$ has been decomposed into A_i according to the Algorithm, and that we have nominated a vertical line L. Define $C_1^1 = L_1 = L$, and let A_1^1 be the first of the A_i to meet L_1. Now define inductively

$$L_i = L \cap A \Big\backslash \bigcup_{j=1}^{i-1} \bigcup_{l} A_l^j.$$

Note that L_i has at most 2^{i-1} components. (A trivial induction shows this.) Denote these by $C_1^i, \ldots, C_{2^{i-1}}^i$ and define A_j^i for $j = 1, 2, \ldots, 2^{i-1}$ as the first of the A_i to meet C_j^i.

Since A is bounded, we must have that eventually $L_M = \emptyset$ for some least M. The idea of the next result is that the higher M is, the greater the number of components some horizontal sections of A will be forced to have. Thus we shall find a bound on M and hence on the number of A_i meeting L.

Lemma 39. *For all k, either $L_k = \emptyset$ or L_k has up to 2^{k-1} connected components $C_1^k, \ldots, C_{2^{k-1}}^k$ such that for all $t \in \pi_y L_k$ the horizontal section*

$$\bigcup_{j=1}^{k-1} \bigcup_{l} A_l^j \cap \pi_y^{-1}(t)$$

has at least $k-1$ connected components.

Proof. The proof is by induction. The case of $k = 1$ is trivial—the union expression is empty.

Now suppose inductively that the result holds for k, and consider what happens with $k+1$. If $L_{k+1} = L \cap A \setminus \bigcup_{j=1}^{k} \bigcup_l A_l^j$ is empty then there is no more to prove, so suppose that L_{k+1} is nonempty. By our previous observation, L_{k+1} has (up to) 2^k components $C_1^{k+1}, \ldots, C_{2^k}^{k+1}$. Fix one of them, C_p^{k+1} say. Clearly $C_p^{k+1} \subseteq C_q^k$ for some q. By the inductive hypothesis,

$$\text{for all } t \in \pi_y C_q^k, \ \bigcup_{j=1}^{k-1} \bigcup_{l} A_l^j \cap \pi_y^{-1}(t) \text{ has at least } k-1 \text{ components.}$$

Recall that A_q^k is the first of the A_i to meet C_q^k. Now evidently either C_p^{k+1} lies above $A_q^k \cap C_q^k$ or below it. Suppose without loss of generality that it lies below. (The argument is very similar in the 'above' case.)

By the y-maximality of γ_q^k, we have that

$$\pi_y A_q^k = \pi_y \gamma_q^k(\mathbb{N}) \supseteq \pi_y C_q^k \supseteq \pi_y C_p^{k+1}.$$

So there is some point z of A_q^k at each height t of C_p^{k+1}. By the definition of the A_l^j's, z must be in a different connected component of the horizontal section of A

at height t from the points in $\bigcup_{j=1}^{k-1} \bigcup_l A_l^j \cap \pi_y^{-1}(t)$. Hence the horizontal section $\bigcup_{j=1}^{k} \bigcup_l A_l^j \cap \pi_y^{-1}(t)$ must have at least k components.

$\square$

Now let M be the smallest natural number such that $L_M = \emptyset$. Then $L_{M-1} \neq \emptyset$ and at each height t of L_{M-1} we have at least $M-2$ components in the horizontal section $\bigcup_{j=1}^{M-2} \bigcup_l A_l^j \cap \pi_y^{-1}(t)$. From the fact that L_M is empty, it follows that $C_i^{M-1} = L \cap A_i^{M-1}$ for all i. Hence (again by definition of the A_l^j's), for all $t \in \pi_y L_{M-1}$ the horizontal section $\bigcup_{j=1}^{M-1} \bigcup_l A_l^j \cap \pi_y^{-1}(t)$ must have at least $M-1$ components.

If A is $BC(n,1)$ then $M \leqslant n+1$, whence at most

$$2^{n-1} + 2^{n-2} + \cdots + 2 + 1 = 2^n - 1$$

of the A_i can meet L. For none of the A_l^j meeting L has an upper index of more than $M-1$, and there are no more than 2^{j-1} of the A_l^j for a fixed j.

Figures 3.8 and 3.7 show a sample decomposition which, incidentally, can be generalised to show that the bound in Lemma 38 is tight. Evidently, the choice of γ_k's in this example is slightly perverse—the obvious choice for γ_1 being the central vertical line—however there seemed to be no reasonable extra conditions that could be imposed during the choice of γ_k's.

We summarise the results of this section with a theorem.

Theorem 40. *Given a bounded, connected $BC(n,1)$ domain A in $\mathbb{Z}^2$, it is possible to decompose A into 2^{2^n+n-3} or fewer $BC(1)$ domains (each of which is an orthogonal family of connected $BC(1)$ domains).*

3.3 Decomposition of $BC(2,2)$ domains

The next move is to attempt a decomposition of $BC(2,2)$ domains. As in the $BC(2,1)$ case, we begin by reducing to the case of a single connected one, under the guidance of Lemma 16. Again, this is achieved by applying Theorem 29.

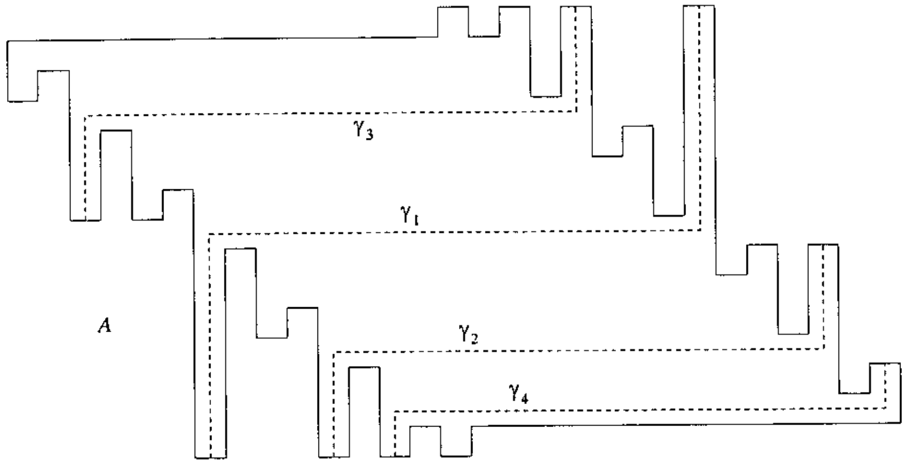

Figure 3.7: Our algorithm at work: A and the first few γ_i

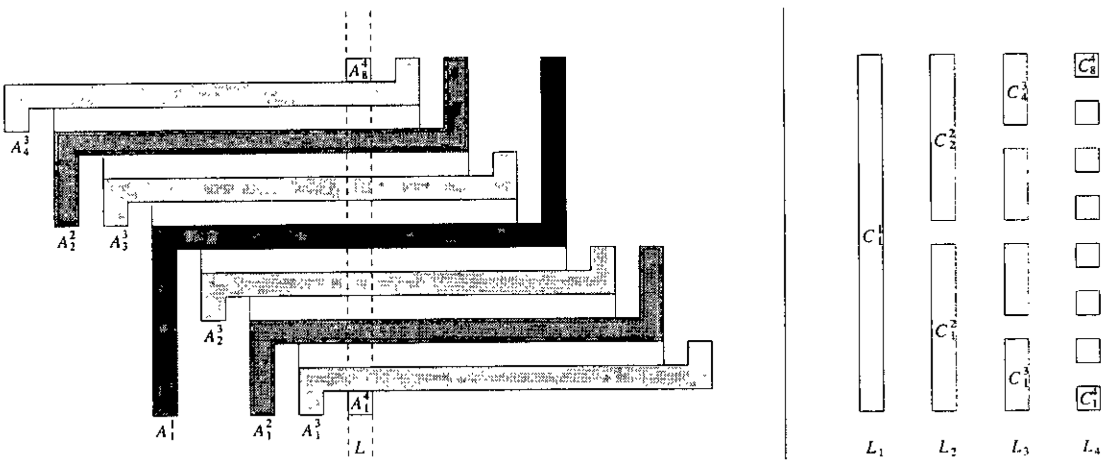

Figure 3.8: Our algorithm at work: the A_j^i and C_j^i

The main obstacle now is the fact that a $BC(2,2)$ domain may contain holes (by definition, bounded connected components of the complement), which frustrate attempts to use the methods of the previous section without further treatment.

Although it is possible to move forward by placing certain restrictions on the arrangement of holes, we postpone such discussion and deal meanwhile with the intermediate case of connected $BC(2,2)$ domains that do not have any holes.

$BC(2,2)$ domains without holes

Let $A \subseteq \mathbb{Z}^2$ be a bounded connected $BC(2,2)$ domain. Again we take an injective path γ from bottom to top. We can no longer take γ to be x-monotonic, but can still choose γ to minimise the quantity $|D| = |\{t : \pi_y\gamma(t+1) < \pi_y\gamma(t)\}|$.

Claim. It still holds that if $t_0 \in D$ then $\pi_y\gamma$ is increasing on $[t_0+1, t_1]$, where $t_1 = \inf\{t > t_0 : \pi_y\gamma(t) = \pi_y\gamma(t_0) + 2\}$. (That is, if γ dips by one square then it cannot dip again until it climbs to at least two squares above its original height.)

This is because the x-monotonicity of γ in Section 3.2 was not used in the proof of the corresponding claim on page 40 there. Thus, in the same way as before, we can write γ as the disjoint union of y-increasing subpaths γ_j, define A_j as the union of horizontal beams through γ_j, and put $A' = \bigcup A_j$. Once again, these objects have some useful properties.

Claim. $\{A_j\}$ is a $BC(2,4)$ family.

Proof. The horizontal considerations are trivial. For the vertical, we show that there can be no more than two A_j meeting any vertical beam of A. (In fact there is some j such that only A_j and A_{j-1} can meet the vertical beam.)

Suppose $k \geqslant 1$ and there are points $p \in A_j$ and $q \in A_{j+k}$ with $\pi_y p < \pi_y q$ and both on the same vertical beam of A. By definition there exist p', q' on γ in the same horizontal beam as p, q respectively. But since $k \geqslant 1$ there must be a dip on γ between p' and q', whereas the path of straight lines $p' \to p \to q \to q'$ lies in A and has no dips, contradicting the least dipping property of γ.

The only other possibility is $k = -1$, since for $k < -1$, $\pi_y A_j > \pi_y A_{j+k}$. $\square$

Notice that in the previous section we could decompose the A_j into two $BC(1)$ families by virtue of the x-monotonicity of γ. Now that we no longer have this property, the best we can do here is 16 $BC(1)$ families (invoking Corollary 30).

Lemma 41. *With $A' = \bigcup A_j$, we have that A' is a $BC(2,2)$ domain.*

Proof. Suppose not. Then we have points p, q, r, s, t lying on a vertical line (in increasing order of height), where $p, r, t \in A'$ and $q, s \notin A'$. Looking just at height $\pi_y r$ downwards, we see that if $q \notin A^c$ (i.e. $q \in A$!) then there exists by definition of A' a point $u \in A^c$ between q and γ at height $\pi_y q$. The fact that A has no holes then forces the existence of some point of A^c between p and r. So either way we have a point of A^c between p and r. The same argument applied to the upper part (height $\pi_y r$ upwards) yields a contradiction of the $BC(2,2)$-ness of A. $\square$

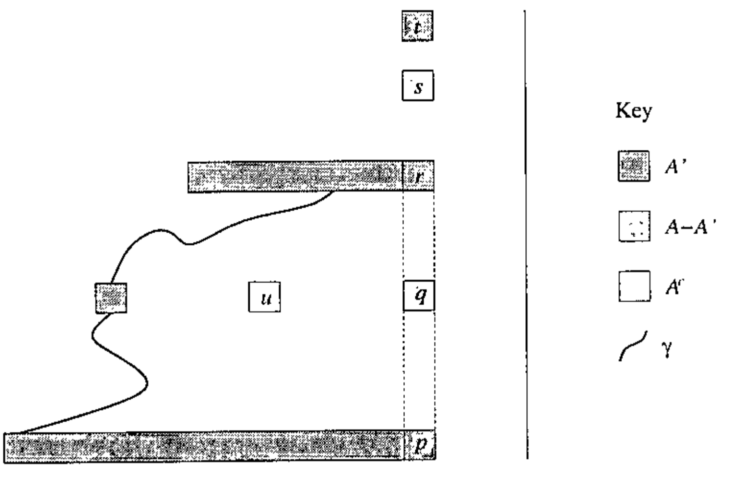

Figure 3.9: Showing that A' is $BC(2,2)$

Similarly we can show that each A_j is $BC(1,2)$.

It also follows easily that $A \backslash A_1$ is a $BC(1,4)$ domain. Applying Theorem 29 and Theorem 40, we can decompose $A \backslash A_1$ into a controlled number of $BC(1)$ domains. Drawing things together we have:

Theorem 42. *Let A be a bounded, connected $BC(2,2)$ domain in $\mathbb{Z}^2$ such that A has no holes in it. Then we can write A as the union of $2^{20} + 64$ or fewer $BC(1)$ domains.*

Proof. Recall that we split A into A', which could be split into 16 $BC(1)$ families of $BC(2,1)$ domains, and $A \backslash A'$, which was $BC(1,4)$ and possibly disconnected. By Theorem 37 we can split A' into 64 or fewer $BC(1)$ domains. By Corollary 30, we can split $A \backslash A'$ into eight or fewer orthogonal families of connected $BC(1,4)$ domains, each of which by Theorem 40 can be split into 2^{17} or fewer $BC(1)$ domains. □

$BC(2,2)$ domains with holes

As mentioned earlier, the appearance of holes in a $BC(2,2)$ (or higher) domain will generally scupper attempts to decompose them using the methods we have considered so far. Progress is however possible by imposing conditions on how the holes may be arranged. Specifically, we define the property $HP(m)$ of a subset A of $\mathbb{Z}^2$ as the negation of the statement: there exist holes $H_1, \ldots, H_m$ in A and $z_0 \in \mathbb{Z}$ such that one of the following holds:

- $\pi_x H_{i+1} > \pi_x H_i$ for all i and either $\pi_y H_{odd} < z_0 < \pi_y H_{even}$ or $\pi_y H_{odd} > z_0 > \pi_y H_{even}$.[1]
- $\pi_y H_{i+1} > \pi_y H_i$ for all i and either $\pi_x H_{odd} < z_0 < \pi_x H_{even}$ or $\pi_x H_{odd} > z_0 > \pi_x H_{even}$.

So roughly speaking, $HP(m)$ says that we cannot have m or more holes alternating about a horizontal or vertical line. (The HP notation comes from **h**ole **p**roperty.)

Why should this be a reasonable condition to impose? Well, let us imagine ourselves in the continuous setting for a moment. Suppose we have a type 1 function u with type constant 2 such that $\frac{\partial^2 u}{\partial x \partial y} > 0$ on Q. Thus all the sublevel sets of u and its first-order partial derivatives are $BC(2)$ domains. Let Ω be a sublevel set at level s. Suppose we had four holes in Ω arranged as in Figure 3.10. We claim that on the boundary ∂H_j of any given one of these holes, u is either identically s or identically $-s$. Evidently $|u| = s$ on ∂H_j. Suppose for a

[1] For subsets A and B of $\mathbb{R}$ or $\mathbb{Z}$, we say that $A > B$ if $a > b$ for all $a \in A$ and $b \in B$. If $B = \{z\}$ we just write it as z for ease.

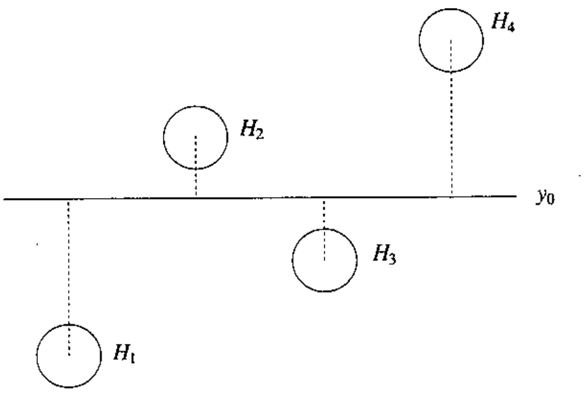

Figure 3.10: Here $HP(4)$ does not hold

contradiction that we had points p and q on ∂H_j with $u(p) = s$ and $u(q) = -s$. We can find a path γ from p to q in H_j. But then by the Intermediate Value Theorem, there would be a zero of u in H_j, which is a contradiction.

By Rolle's Theorem applied on appropriate horizontal lines, we could find in each hole a zero of $\frac{\partial u}{\partial x}$. Then $\frac{\partial^2 u}{\partial x \partial y} > 0$ tells us that we have x_1, x_2, x_3, x_4 and y_0 with $\frac{\partial u}{\partial x}(x_{odd}, y_0) > 0$ and $\frac{\partial u}{\partial x}(x_{even}, y_0) < 0$. Hence there exists s' such that the sublevel set of $\frac{\partial u}{\partial x}$ at level s' has at least 3 components on the horizontal section at y_0. Of course this contradicts our assumptions about u.

Returning to $\mathbb{Z}^2$, the next step is to see what imposing $HP(m)$ on a domain A can tell us about how all the holes in A are arranged. Some notational setup is required.

Definition 43. *Let $A \subseteq \mathbb{Z}^2$ and suppose that A has some finite number of holes. We call the set $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ of holes a* string *if there is a permutation σ on $\{1, 2, \ldots, n\}$ such that $\pi_x H_{\sigma(1)} < \pi_x H_{\sigma(2)} < \cdots < \pi_x H_{\sigma(n)}$ and either*

- *$\pi_y H_{\sigma(1)} < \pi_y H_{\sigma(n)}$ and whenever there are holes K_1, K_2 such that $\pi_x H_{\sigma(1)} < \pi_x K_1 < \pi_x K_2 < \pi_x H_{\sigma(n)}$ we have $\pi_y H_{\sigma(1)} < \pi_y K_1 < \pi_y K_2 < \pi_y H_{\sigma(n)}$ and whenever there are holes K_1, K_2 such that $\pi_y H_{\sigma(1)} < \pi_y K_1 < \pi_y K_2 < \pi_y H_{\sigma(n)}$ we have $\pi_x H_{\sigma(1)} < \pi_x K_1 < \pi_x K_2 < \pi_x H_{\sigma(n)}$; or*

- *$\pi_y H_{\sigma(1)} > \pi_y H_{\sigma(n)}$ and whenever there are holes K_1, K_2 such that $\pi_x H_{\sigma(1)} < \pi_x K_1 < \pi_x K_2 < \pi_x H_{\sigma(n)}$ we have $\pi_y H_{\sigma(1)} > \pi_y K_1 > \pi_y K_2 > \pi_y H_{\sigma(n)}$*

and whenever there are holes K_1, K_2 such that $\pi_y H_{\sigma(1)} > \pi_y K_1 > \pi_y K_2 > \pi_y H_{\sigma(n)}$ we have $\pi_x H_{\sigma(1)} < \pi_x K_1 < \pi_x K_2 < \pi_x H_{\sigma(n)}$.

(Note that this implies either $\pi_y H_{\sigma(1)} < \pi_y H_{\sigma(2)} < \cdots < \pi_y H_{\sigma(n)}$ or $\pi_y H_{\sigma(1)} > \pi_y H_{\sigma(2)} > \cdots > pi_y H_{\sigma(n)}$.) A string is called maximal *if it is not included in any strictly larger string.*

With each such set we associate a *hole diagram*, which represents the order in which any holes appear relative to the x- and y-coordinates. The diagrams consist simply of dots and diagonal lines, the dots representing one-element maximal strings and the lines representing longer maximal strings. It is easy to show that any arrangement of holes has a unique hole diagram. To do so, we can simply define the equivalence relation $\sim$ on the holes by saying that $H_1 \sim H_2$ if and only if $\{H_1, H_2\}$ is a string. Then the maximal strings are the equivalence classes. Figure 3.11 gives a couple of examples.

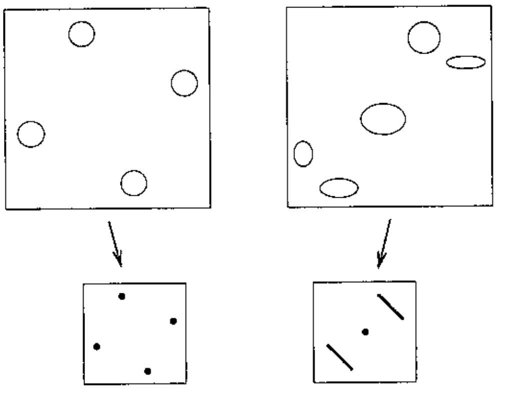

Figure 3.11: Two examples of hole diagrams

Lemma 44. *Let $A \subseteq \mathbb{Z}^2$ have property $HP(4)$. Then there are only eight possible hole diagrams for A, up to symmetry. They are shown in Figure 3.12.*

Proof. The proof is by induction on the number of holes. Let A be such a set. If it has no holes, then its hole diagram is just the first one shown in Figure 3.12. Suppose inductively that every such sublevel set with k holes has a hole diagram contained in Figure 3.12, modulo symmetry, and let A be a set with $k+1$ holes

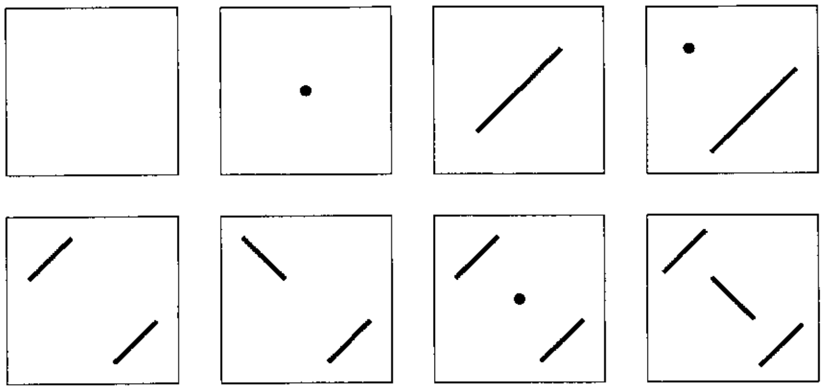

Figure 3.12: All the possible hole diagrams with $HP(4)$

and property $HP(4)$. Order the holes (arbitrarily) and imagine that the $k+1^{\text{th}}$ hole has been filled in. The resulting set, by the inductive hypothesis, has a hole diagram shown in Figure 3.12. We wish to see that adding in the $k+1^{\text{th}}$ hole leads either to one of the diagrams shown or an illegal arrangement.

Thus there are eight cases to consider, one for each of the eight diagrams. Let us just consider the last diagram, since the arguments here cover the other cases too. Remember exactly what the diagram means: we have $m+n+p$ holes

$$H_1, \dots, H_m, H_{m+1}, \dots, H_{m+n}, H_{m+n+1}, \dots, H_{m+n+p}$$

(where $m, n, p \geqslant 2$) such that $\pi_x(H_1) < \pi_x(H_2) < \cdots < \pi_x(H_{m+n+p})$ and

$$\pi_y(H_{m+n+1}) < \cdots < \pi_y(H_{m+n+p}) < \pi_y(H_{m+n}) < \cdots < \pi_y(H_{m+1}) < \\ < \pi_y(H_1) < \cdots < \pi_y(H_m).$$

A bit of renaming is useful here: let $H_1, \dots, H_6$ be the new names of the holes $H_1, H_m, H_{m+1}, H_{m+n}, H_{m+n+1}, H_{m+n+p}$, i.e. those at the ends of maximal strings as they appear from left to right. Now we divide our sublevel set into a number of areas, shown by dotted lines in Figure 3.13. We investigate what will happen if the $k+1^{\text{th}}$ hole (call it H^*) appears in each of these areas. (For instance, saying that H^* appears in area D means that $\pi_x(H_3) < \pi_x(H^*) < \pi_x(H_4)$ and $\pi_y(H^*) > \pi_y(H_2)$.) It will be seen that the results for all other areas follow by symmetry.

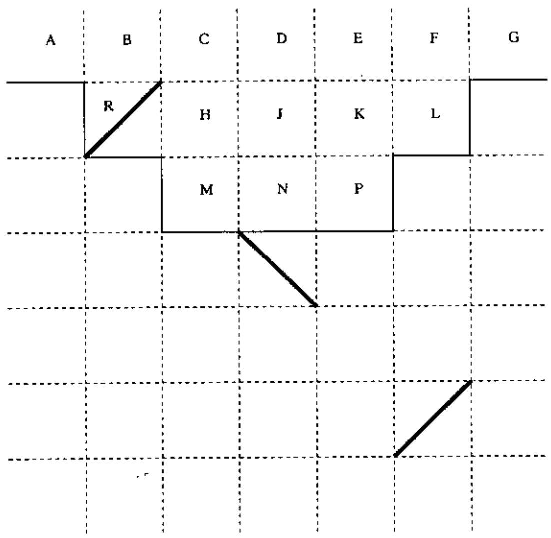

Figure 3.13: Places where H^* could appear

For the moment we consider only the areas A to P. (R requires special attention later.) We soon see that areas C and M are the only places where H^* can occur without creating an illegal arrangement, as the following table shows.

Region	Illegal Arrangement	Region	Illegal Arrangement
A	H^*, H_2, H_1, H_3	H	H_2, H^*, H_1, H_3
B	H^*, H_2, H_1, H_3	J	H_2, H^*, H_1, H_3
D	H_1, H_2, H_3, H^*	K	H_2, H^*, H_1, H_3
E	H_1, H_2, H_3, H^*	L	H_2, H^*, H_1, H_3
F	H_1, H_2, H_3, H^*	N	H_1, H^*, H_3, H_4
G	H_1, H_2, H_3, H^*	P	H_1, H^*, H_3, H_4

Finally we deal with the area R. Focus in on just the top-left maximal string, which we shall call $\mathcal{H}$. If $\mathcal{H} \cup \{H^*\}$ is also a maximal string, then there is nothing more to do. Otherwise, there must be some H^{**} such that $\pi_x(H^{**}) > \pi_x(H^*)$ and $\pi_y(H^{**}) < \pi_y(H^*)$ (or *vice versa*). Hence the holes H_1, H^*, H^{**}, H_2 form an illegal arrangement. The two cases are shown in Figure 3.14.

It is clear from Figure 3.13 that all other placements of the hole H^* are covered by symmetry. Furthermore, none of the arguments used to establish the result for the other seven diagrams (cf Figure 3.12) are any more difficult than those used above; therefore it is left to the diligent reader to verify the details.

□

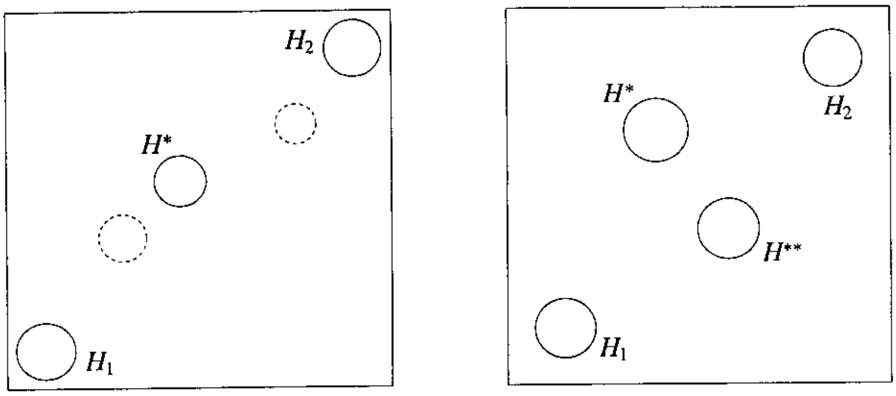

Figure 3.14: If H^* falls in the area R

Now that we have good control over how holes may appear in sets with $HP(4)$, we are not far from a decomposition of such sets into simpler objects. Shortly, we shall need to use Lemma 33 for the first time.

For the next step, we begin by looking at the case of $A \subseteq \mathbb{Z}^2$ whose holes appear as in the final diagram of Figure 3.12. (Cases corresponding to the other diagrams are all easier than this one.) We split up the sublevel set as shown in the left hand side of Figure 3.15, that is we choose $x_1 \in [\sup \pi_x H_2, \inf \pi_x H_3]$, $x_2 \in [\sup \pi_x H_4, \inf \pi_x H_5]$, $y_1 \in [\sup \pi_y H_6, \inf \pi_y H_4]$, $y_2 \in [\sup \pi_y H_3, \inf \pi_y H_1]$ and draw straight lines between the pairs of points (x_1, y_1) and $(x_1, 1)$; (x_1, y_1) and $(1, y_1)$; $(0, y_2)$ and (x_2, y_2); $(x_2, 0)$ and (x_2, y_2).

Denoting the two L-shaped areas by L_1 and L_2, we claim that and $A \cap L_2$ are $BC(2)$ domains *without holes.* The $BC(2)$ property comes simply by applying Lemma 33 with $m = m' = 2$ and $n = n' = 1$. The fact that $A \cap L_1$ and $A \cap L_2$ have no holes follows from the following lemma.

Lemma 45. *Let X be a $BC(m,n)$ domain with holes $\{H_i\}_{i \in I}$, and let Y be a $BC(1)$ domain that does not meet any of the holes of X. (So $Y \subseteq \left(\bigcup H_i\right)^c$.) Then $Z := X \cap Y$ is $BC(m,n)$ and has no holes.*

Proof. The fact that Z is $BC(m,n)$ follows immediately from Lemma 33. To prove that Z has no holes, suppose for a contradiction that it does have a hole, H say. By definition of holes, all points of H^c abutting H are in $Z = X \cap Y$. By considering the points either side of horizontal and vertical beams of H, we see that since Y is $BC(1)$, all points of H are also in Y. Also $H \subseteq Z^c = (X \cap Y)^c = X^c \cup Y^c$, which implies that $H \subseteq X^c$. Thus H is a hole in X. But we also have $H \subseteq Y \subseteq \left(\bigcup H_i\right)^c$, which is a clear contradiction. $\square$

Lemma 16 now suggests that regarding the three pieces on the diagonal, we need only consider one of them, which we now do. Concentrating on one of these diagonal pieces, D say, we decompose it in a similar manner. We divide it up as shown in the right hand side of Figure 3.15, that is surrounding each hole H by the rectangle $R = \pi_x H \times \pi_y H$, then linking these rectangles with a "staircase" that extends to the edges of D. A precise definition for the case of holes arranged as in Figure 3.15 appears in the following table.

$\pi_x s \in$	$\pi_y s \in$	$s \in$
$[\min \pi_x D, \min \pi_x H_1)$	$[\min \pi_y D, \max \pi_y H_1]$	K_1
	$(\max \pi_y H_1, \max \pi_y D]$	K_2
$[\min \pi_x H_i, \max \pi_x H_i]$	$[\min \pi_y D, \min \pi_y H_i)$	K_1
	$(\max \pi_y H_i, \max \pi_y D]$	K_2
$(\max \pi_x H_i, \min \pi_x H_{i+1})$	$[\min \pi_y D, \max \pi_y H_{i+1}]$	K_1
	$(\max \pi_y H_{i+1}, \max \pi_y D]$	K_2
$(\max \pi_x H_n, \max \pi_x D]$	$\pi_y D$	K_2

Thus we have $D = K_1 \sqcup K_2 \sqcup \bigsqcup_{i=1}^{n} R_i$. Intersecting with A and applying the following lemma together with Lemma 45, we find that $A \cap D$ can be decomposed into four (disconnected) $BC(1)$ domains along with the two $BC(2)$ domains without holes, $A \cap K_1$ and $A \cap K_2$.

Lemma 46. *Let A be a bounded connected $BC(m,n)$ domain and H a hole in it. Define $R = \pi_x H \times \pi_y H$. Then*

1. *$R \backslash H$ has at most $2(m+n-2)$ connected components, and*
2. *If C_i is one such, then $A \cap C_i$ is $BC(m-1, n-1)$.*

Proof.

1. Let ∂R denote the "boundary" of R, i.e. the set of points in R adjacent to a point in R^c. By definition of R and $BC(m-1, n-1)$-ness of H, it is

clear that $(\partial R)\backslash H$ has at most $2(m+n-2)$ connected components. Fix an arbitrary $z \in \mathbb{Z}^2\backslash R$. Given any $p \in R\backslash H$, we wish to see that p is in the same connected component of $R\backslash H$ as some $q \in \partial R$. Since $\mathbb{Z}^2\backslash H$ is connected, there is a path γ from p to z in $\mathbb{Z}^2\backslash H$. Let q be the first point on ∂R that γ reaches. Then the restriction of γ to $[0, \min \gamma^{-1} q]$ is a path in $R\backslash H$ from p to q. Therefore every element of $R\backslash H$ is indeed in the same connected component of $R\backslash H$ as some element of ∂R, whence the first part of the lemma is proved.

2. Suppose for contradiction that there is a horizontal section of $A \cap C_i$ having m connected components. Then there are $p_1, \ldots, p_m \in A \cap C_i$ and $q_1, \ldots, q_{m-1} \notin A \cap C_i$ all lying on the section and such that $\pi_x p_i < \pi_x q_i < \pi_x p_{i+1}$ for $i = 1, \ldots, m-1$. In fact we can in fact take the q_i to be in A^c. (For if $q_i \in A$ then it can't be in C_i. Since $p_i \in C_i$ and C_i is a connected component of $R\backslash H$, there must be a point of $H \subseteq A^c$ between p_i and q_i.) Since A is $BC(m,n)$, there can be no points of any hole to the left of p_1 or to the right of p_m. Also, there is a path in C_i from p_1 to p_m, and thus (by concatenating) a path from the left edge of R to its right edge that sits entirely in H^c. But there must also be a path in H from the bottom edge of R to its top edge. By the Jordan Curve Theorem the two paths must meet, which is a contradiction.

□

Thus we have proved:

Theorem 47. *Let $A \subseteq \mathbb{Z}^2$ be a bounded, connected, $BC(2,2)$ domain that has the $HP(4)$ property. Then A can be decomposed into orthogonal families of $BC(1)$ domains and $BC(2)$ domains without holes, whence we can use previous results to show that such a set may be decomposed into $2^{24} + 2^{10} + 4$ or fewer $BC(1)$ domains.*

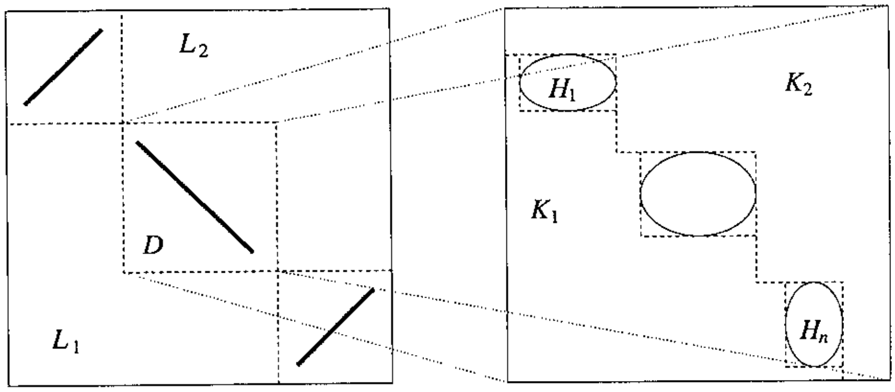

Figure 3.15: Final step in $BC(2)$ decomposition

3.4 $BC(3)$ and beyond

A few more techniques must be developed before we can achieve our goal of the decomposition of any $BC(m, n)$ domain. [2] The next case to consider is that of $BC(3, 2)$ domains. As in the treatment of $BC(2, 2)$ domains, we can classify the different possible layouts of holes and make use of this knowledge in our methods. However, when we move up to $BC(3, 3)$ domains, this method no longer seems feasible, due to a sudden growth in complexity. (Even in stepping from $BC(2, 2)$ to $BC(3, 2)$ we see a tripling in the number of possible hole diagrams.) So in the $BC(3, 3)$ case we give a new method of dealing with the holes, and it turns out that we can extend this to the general $BC(N)$ case. Once that is done, we shall have all the necessary machinery to perform a decomposition of any $BC(N)$ domain.

$BC(3, 2)$ domains

As in the $BC(2)$ case it seems best to consider first the case when there are no holes. At this point however, the methods used in decomposing $BC(2, 1)$ and $BC(2, 2)$ become less effective and new methods become desirable. In fact, we have already developed one, because the Algorithm described in Section 3.2

[2] Obviously it is enough to provide a decomposition of any $BC(N)$ domain and then putting $N = \max\{m, n\}$ covers $BC(m, n)$ domains. If $m \neq n$ we forfeit a tighter bound, but given the definition of type M functions, which involves $BC(N)$ and not $BC(m, n)$ domains, one feels that this is reasonable enough.

can be applied in this situation. It is possible to extend the methods there to decompose any $BC(m,n)$ domain without holes.

For suppose that $A \subseteq \mathbb{Z}^2$ is such a domain and suppose without loss of generality that $m < n$. Consider a vertical cross-section of A. It consists of at most n vertical beams. The proof of Lemma 38 shows that at most $2^m - 1$ of the A_i meet any given one of these n beams. Thus the A_i form a $BC(m, n(2^m - 1))$ family. We can also show, using the same ideas as those in the proof of Lemma 41, that each A_i is $BC(1,n)$. By Theorem 29 and Theorem 40, we can thus decompose A into $2^{2^n + n2^m + m - 5}$ or fewer $BC(1)$ domains.

A question that arises here is whether this bound is always smaller when $m < n$ than when $m > n$.

So in the present situation of $BC(3,2)$ domains, the Algorithm is used to give a $BC(2,9)$ family of $BC(1,3)$ domains, which can be split into 2^9 or fewer orthogonal families of $BC(1,3)$ domains. Applying our results on $BC(1,3)$ domains, we end up with 2^{17} or fewer $BC(1)$ domains.

We now attempt to tackle $BC(3,2)$ domains with holes. First we note that the holes must form a $BC(2,1)$ family, and use Theorem 29 to split it into two $BC(1)$ families. For the time being we choose just one of these families on which to concentrate.

We also impose the condition $HP(5,4)$, which says that we cannot have five holes alternating about a horizontal line, or four about a vertical line. A rigorous definition of $HP(m,n)$ for any $m,n \in \mathbb{N}$ comes in the same mould as that of the $HP(m)$ condition on page 51. Let P_1 be the statement that there exist holes $H_1, \ldots, H_m$ in A such that

- $\pi_x H_{i+1} > \pi_x H_i$ for all i and either $\pi_y H_{odd} < z_0 < \pi_y H_{even}$ or $\pi_y H_{odd} > z_0 > \pi_y H_{even}$

and let P_2 be the statement that there exist holes $H_1, \ldots, H_n$ in A such that

- $\pi_y H_{i+1} > \pi_y H_i$ for all i and either $\pi_x H_{odd} < z_0 < \pi_x H_{even}$ or $\pi_x H_{odd} > z_0 > \pi_x H_{even}$.

Then A is said to satisfy $HP(m,n)$ if it satisfies "neither P_1 nor P_2", that is $\neg(P_1 \vee P_2)$ in the notation of logic. See Figure 3.16.

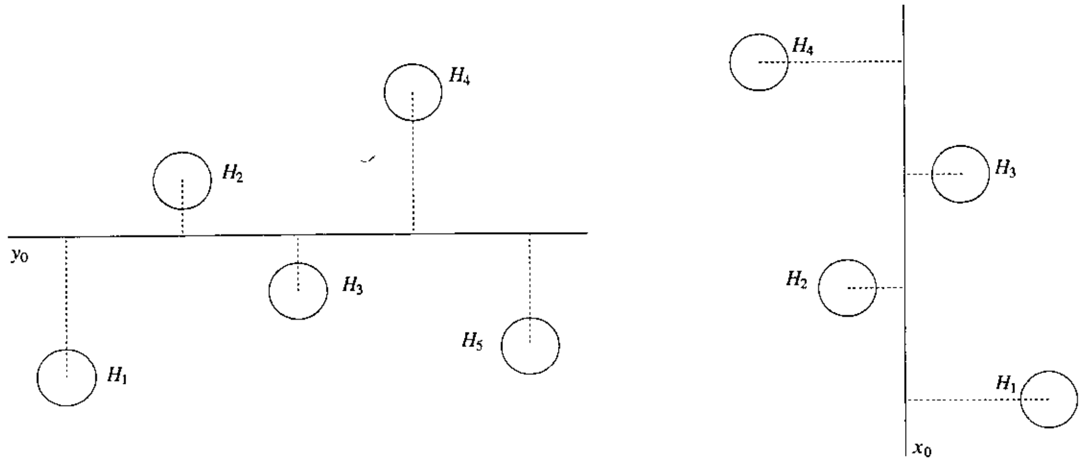

Figure 3.16: Illegal hole arrangements with $HP(5,4)$

Armed with these restrictions and using the same kind of reasoning as in Lemma 44 we find that the holes in this family must have a maximal string diagram that is, up to symmetry, a "subdiagram" of that shown in Figure 3.17. (Details are given in Appendix A. There are 27 cases to check, but the argument is no harder than in the case of $BC(2)$ domains with the $HP(2)$ condition.) If we now enumerate the holes as H_j^1 and place rectangles R_j^1 around them as before (i.e. so that $R_j^1 = \pi_x H_j^1 \times \pi_y H_j^1$) then it is easy to see that $Q \setminus \bigcup_j R_j^1$ can be divided up into four $BC(1)$ domains. Again, see Figure 3.17. Call these domains $A_1^1, \ldots, A_4^1$.

Repeat this procedure with the second of the two families of holes H_j^2 and call the resulting $BC(1)$ domains A_1^2, A_2^2, A_3^2 and A_4^2. We have

$$\begin{aligned} Q = Q \cap Q &= \left(\bigsqcup_{j=1}^{4} A_j^1 \sqcup \bigsqcup_j R_j^1 \right) \cap \left(\bigsqcup_{j=1}^{4} A_j^2 \sqcup \bigsqcup_j R_j^2 \right) \\ &= \left(\bigsqcup_{j,k=1}^{4} A_j^1 \cap A_k^2 \right) \sqcup \bigcup_{i,j} R_j^i. \end{aligned}$$

Intersecting with A on both sides, we have that $A \setminus \bigcup_{i,j} R_j^i = \bigsqcup_{j,k=1}^4 A_j^1 \cap A_k^2 \cap A$. By Lemmas 33 and 45, this is the union of at most sixteen $BC(3,2)$ domains without holes.

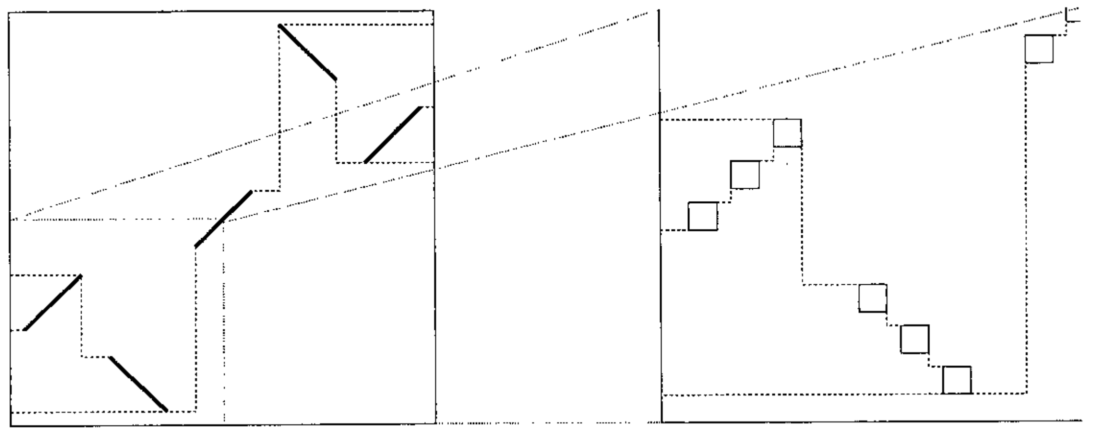

Figure 3.17: Decomposing $BC(3,2)$ domains

We are left with just the $A \cap R^i_j$ to deal with. To do so we just apply Lemma 46, which tells us that each of these can be decomposed into six $BC(2,1)$ domains. Bearing in mind that the R^i_j form two $BC(1)$ families, we have that $A \cap \bigcup_{i,j} R^i_j$ can be decomposed into twelve or fewer $BC(2,1)$ domains. Chasing the numbers back through previous results we have:

Theorem 48. *Let A be a bounded connected $BC(3,2)$ domain in $\mathbb{Z}^2$ that has property $HP(5,4)$. Then A can be decomposed into $2^{24} + 96$ or fewer $BC(1)$ domains.*

But wait a minute: this bound is smaller than the one obtained earlier for $BC(2)$ domains! This suggests that using the Algorithm of Section 3.2 should lower the bound there. Indeed this is the case. For $BC(2)$ domains without holes, the Algorithm gives a $BC(2,6)$ family of $BC(1,2)$ domains, whence $4.2^6 = 2^8$ $BC(1)$ domains. For $BC(2)$ domains with holes (under the conditions of Theorem 47) this translates into $4.2^{10} + 4 = 2^{12} + 4$ $BC(1)$ domains.

$BC(3,3)$ domains

In this subsection, we put the final piece in the jigsaw of techniques that will enable us to decompose a general square-type $BC(N)$ domain on which an appropriate HP condition holds. It is easier to follow what is going on in the $BC(3,3)$ case, which is why we do not immediately move to the general case.

Again the arrangements of holes in our domains entailed by the HP condition is crucially important.

Let A be a $BC(3)$ domain. If A has no holes, then we can use the Algorithm of Section 3.2 to divide A into a $BC(3, 22)$ family of connected $BC(1, 3)$ domain, each of which in turn we know how to decompose into 2^8 $BC(1)$ domains. Thus we can decompose A into 2^{30} or fewer $BC(1)$ domains.

So assume that A does have holes, and suppose further that $HP(5)$ holds on A. The holes form a $BC(2, 2)$ family, which we may split into four $BC(1)$ families. We treat these one by one before taking intersections and suchlike for an overall decomposition.

Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be one of these $BC(1)$ hole families, where they are enumerated left–right. For $i = 1, \ldots, k$ let $R_i = \pi_x H_i \times \pi_y H_i$. We define $L, U \subseteq Q$ as follows:

$\pi_x s \in$	and $\pi_y s \in$	$s \in$
$[\min \pi_x Q, \min \pi_x H_1)$	$[\min \pi_y Q, \min \pi_y H_1)$	L
	$[\min \pi_y H_1, \max \pi_y Q]$	U
$\pi_x H_i$	$[\min \pi_y Q, \min \pi_y H_i)$	L
	$(\max \pi_y H_i, \max \pi_y Q]$	U
$(\max \pi_x H_i, \min \pi_x H_{i+1})$	$[\min \pi_y Q, \min\{\max \pi_y H_i, \max \pi_y H_{i+1}\}]$	L
	$(\min\{\max \pi_y H_i, \max \pi_y H_{i+1}\}, \max \pi_y Q]$	U
$(\max \pi_y H_k, \max \pi_x Q]$	$[\min \pi_y Q, \min \pi_y H_k)$	L
	$[\min \pi_y H_k, \max \pi_y Q]$	U

Note that $Q = L \sqcup U \sqcup \bigsqcup_{i=1}^{k} R_i$. (This is all a lot easier to understand in a picture—see Figure 3.18.)

The aim now is to show that L is $BC(2, 1)$ and U is $BC(3, 1)$. The case of U turns out to be slightly more involved, so we shall deal with it first. Supposing for a contradiction that U were not $BC(3, 1)$, we would have $p_1, \ldots, p_4 \in U$ and $q_1, q_2, q_3 \in Q \backslash U$ such that $\pi_y p_i = y_0 = \pi_y q_i$ for all i and

$$\pi_x p_1 < \pi_x q_1 < \pi_x p_2 < \cdots < \pi_x q_3 < \pi_x p_4.$$

We also define $q_0 = (\min \pi_x Q - 1, y_0)$ and $q_4 = (\max \pi_x Q + 1, y_0)$.

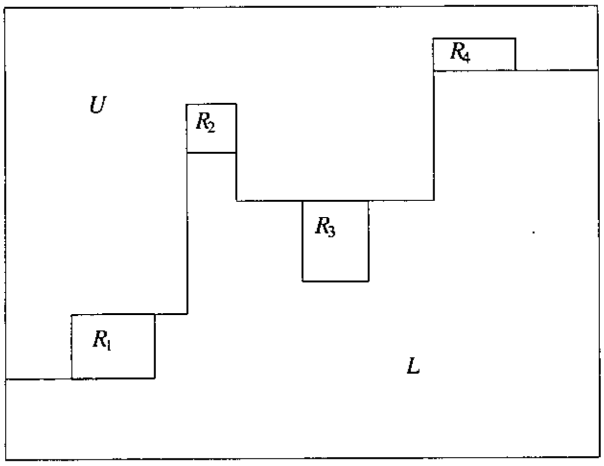

Figure 3.18: A picture of L and U

We define some subsets of Q as follows:

$$
\begin{aligned}
Q_0 &= [\min \pi_x Q, \pi_x p_1 - 1] \times [y_0, \max \pi_y Q] \\
P_i &= ([\pi_x q_{i-1} + 1, \pi_x q_i - 1] \times [\min \pi_y Q, y_0]) \backslash \{p_i\} \qquad i = 1, 2, 3, 4 \\
Q_i &= ([\pi_x p_i + 1, \pi_x p_{i+1} - 1] \times [y_0, \max \pi_x Q]) \backslash \{q_i\} \qquad i = 1, 2, 3 \\
Q_4 &= [\pi_x p_4 + 1, \min \pi_x Q] \times [y_0, \max \pi_x Q].
\end{aligned}
$$

We also define $P'_i = P_i \backslash \pi_y^{-1}(y_0)$ for $i = 1, 2, 3, 4$; and $Q'_i = Q_i \backslash \pi_y^{-1}(y_0)$ for $i = 0, \ldots, 4$, with the stipulation that $P_0 = P_5 = \emptyset$. See Figure 3.19. (The black vertical lines indicate squares that the rectangles R_j may not contain—unless they contain some q_i.)

The following lemma is the machinery that produces illegal hole arrangements to get the intended contradiction.

Lemma 49. *With the notation just introduced,*

1. *Suppose $i \leqslant 3$ ($i \geqslant 2$) and there is a rectangle R meeting $P_i \cap Q_i$ but not q_i. Then there is some $R' \subseteq Q'_i$ (P'_i).*
   Suppose that $i \leqslant 2$ ($i \geqslant 1$) and there is a rectangle R meeting $Q_i \cap P_{i+1}$ but not q_i. Then there is some $R' \subseteq P'_{i+1}$ (Q'_i).

2. *Suppose there is a rectangle R containing q_i where $i \leqslant 2$ ($i \geqslant 2$). Then there is some $R' \subseteq P'_{i+1}$ (P'_i).*

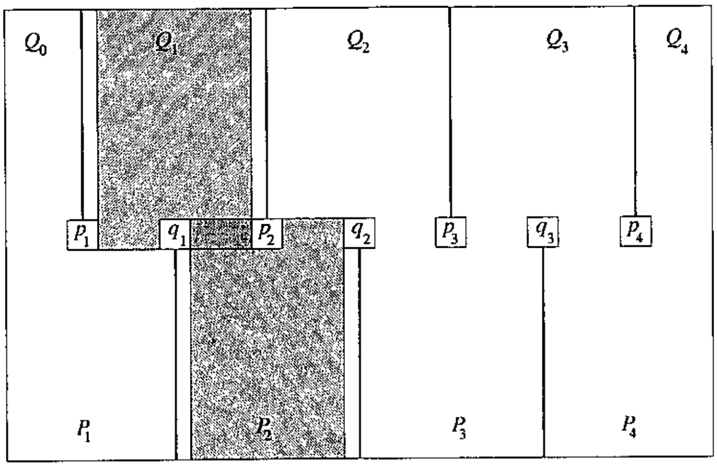

Figure 3.19: Defining the P_i, Q_i, p_i and q_i

3. Suppose $i \leqslant 3$ ($i \geqslant 1$) and there is a rectangle $R \subseteq Q_i'$ and that no R_j meets $Q_i \cap P_{i+1}$ ($Q_i \cap P_i$) or q_i. Then there is a rectangle $R' \subseteq P_{i+1}'$ (P_i').

Suppose that $i \leqslant 3$ ($i \geqslant 2$) and there is a rectangle $R \subseteq P_i'$ and that no R_j meets $P_i \cap Q_i$ ($P_i \cap Q_{i-1}$) or q_i (q_{i-1}). Then there is a rectangle $R' \subseteq Q_i'$ (Q_{i-1}').

Proof. We just prove the first statement under each number, as the others are very similar.

- Consider the rightmost rectangle meeting P_i, which is either R or some $R' \subseteq P_i'$. The next rectangle to the right of it is evidently (by definition of U, L etc.) in Q_i'.

- Same argument as above.

- Let R' be the rightmost hole in Q_i'. This case now divides into 2 subcases. Firstly we suppose that $\max \pi_x R' \geqslant \pi_x q_i$. Consider the next rectangle to the right. As before, it is evident that it lies in P_{i+1}'.

  Secondly, suppose that $\max \pi_x R' < \pi_x q_i$. Consider the next rectangle to the right. If it lay in P_i' then applying the first case to P_i' yields a contradiction. Hence it must lie in P_{i+1}'. (This is actually also a contradiction, meaning that $\max \pi_x R' < \pi_x q_i$ is impossible, but this redundancy does not appear in the similar case of $R' \subseteq P_i'$.)

□

Corollary 50. *If U is not $BC(3,1)$, then there are at least six holes alternating about a horizontal line.*

Proof. We have three cases to consider.

- First suppose that no rectangle meets the horizontal section at y_0. Pick a rectangle at random, which must therefore lie completely inside some P_i' or Q_i'. Using part 3 of the Lemma repeatedly, we find holes in $P_1', \ldots, P_4'$ and $Q_1', \ldots, Q_3'$ and are done.

- Suppose that there is some rectangle R that meets $P_i \cap Q_i$ but not q_i, where $i \leqslant 3$. Note that by $BC(1)$-ness of the hole family this can only happen once.

  If $i = 2$ or 3 then part 1 of the Lemma tells us that there are rectangles in P_i' and Q_i'. Thence part 3 gives that there are rectangles in $P_1', \ldots P_4'$ and $Q_1', \ldots, Q_3'$ in addition to R, so at least seven in total, alternating about y_0. (R does not take part here.)

  If $i = 1$, part 1 of the lemma gives us a rectangle in Q_1'. Then part 3 gives us holes in P_2', P_3', P_4' and Q_2', Q_3', and so at least seven in total, including R, that alternate about $\max \pi_y R$.

  Very similar arguments give the result when $R \subseteq P_4 \cup Q_4$ or $Q_i \cup P_{i+1}$.

- Suppose that there is some rectangle R containing a q_i. If $i = 2$ then by part 2 of the lemma we have rectangles in P_2' and P_3'. Thence by part 3 we have further rectangles in P_1', Q_1', Q_3' and P_4', and so at least seven altogether, including R, which alternate about $\min \pi_y R$.

  If $i = 1$ then we have a rectangle in P_2' by part 2 and ones in Q_2', P_3', Q_3', P_4' by part 3, so at least six in all, including R, that alternate about $\min \pi_y R$. The case $i = 3$ follows by symmetry.

□

However we know it is impossible to have more than five holes alternating about a line, whence U must be $BC(3,1)$ as claimed. The same kind of reasoning is used to show that L is $BC(2,1)$.

Now we already have a method to decompose $BC(n,1)$ domains into controlled numbers of $BC(1)$ domains, however it is possible here to improve the bound obtained by having a closer look at the domains L and U. We can exploit the fact that they are of a special form—not only $BC(n,1)$ for some n but they are unions of vertical beams all of whose top or bottom ends are at the same height. We give a more general result than necessary in the immediate context, since it can be used later on.

Lemma 51. *Let F be a bounded connected $BC(n,1)$ domain that is a union of vertical intervals all of whose lower endpoints are at the same height (which may as well be 0). Then F can be decomposed into $T(n)$ or fewer $BC(1)$ domains, where*

$$\begin{aligned} T(1) &= 1 \\ T(n) &= 1 + 2^{n-2}T(n-1) \qquad \textit{for } n > 1. \end{aligned} \tag{3.2}$$

(So $T(n) = O\big(2^{\frac{1}{2}n(n-1)}\big)$, a substantial improvement on 2^{2^n}.)

Proof. Choose a highest point p in F and let γ be the vertical line from p to $(\pi_x p, 0) =: q$. Take the union of horizontal beams through γ and call it A_1. By the usual arguments, A_1 is $BC(1)$, and it clearly contains the bottom line of F, i.e. the lowest point of every vertical beam of F. Hence every vertical beam of $F \backslash A_1$ is a subset of a vertical beam of F and contains the top point of that beam.

Evidently $F \backslash A_1$ may be disconnected and does not share the special form of F. However we claim that the connected components of $F \backslash A_1$ do have this form. So we wish to see that if C is a connected component of $F \backslash A_1$ then C is $BC(n-1,1)$ and is the union of vertical intervals all of whose lower endpoints are at the same height.

The $BC(n-1,1)$-ness of C is clear. To demonstrate its special form, suppose otherwise. Then there exist $a, b \in C$ with $a' := (a_1, a_2 - 1)$ and $b' := (b_1, b_2 - 1)$ in

A_1 and $a_2 \neq b_2$. We can assume that $a_1 < b_1$ and $a_2 < b_2$. Since C is connected and $F\backslash A_1$ is vertically convex, there is an x-monotonic path γ_1 from a to b.

Let z be the last square on γ_1 at height $b_2 - 1$. Between z and b' there must be a point $h \in F^c$. There must also be a point w on γ_1 with $w_1 = c_1, w_2 > c_2$. But every point from w down to $(w_1, 0)$ should be in F, and thus we have a contradiction. (See figure 3.20.)

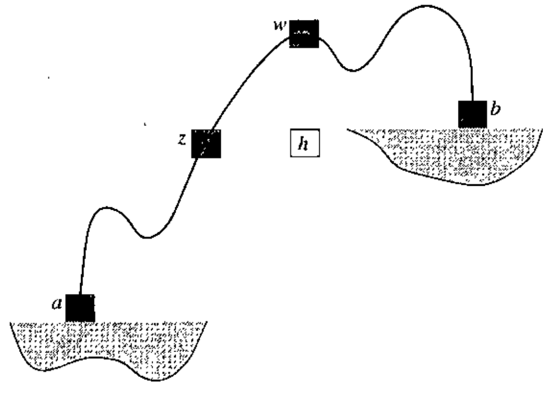

Figure 3.20: A more efficient decomposition of F

Now the connected components of $F\backslash A_1$ are a $BC(n-1,1)$ family, which can be split into 2^{n-2} or fewer $BC(1)$ families. This together with the claim just proven gives rise to the function $T(n)$ in the statement of the lemma, and we are done. (Of course, a symmetrical argument would work if we replaced 'lower' by 'upper' in the statement of the lemma.) □

In particular, the domains L and U can be decomposed respectively into two and five (or fewer) $BC(1)$ domains. If we name these seven domains A_j^1 for $j = 1, \ldots, 7$ then we have $Q = \bigsqcup_{j=1}^{7} A_j^1 \sqcup \bigsqcup_j R_j^1$ where the R_j^1 are the rectangles round the members of the $BC(1)$ family of holes under consideration. Repeating the whole procedure for the other three $BC(1)$ families of holes to obtain domains A_j^i and R_j^i for $i = 2, 3, 4$, we have that $Q = \bigsqcup_{j=1}^{7} A_j^i \sqcup \bigsqcup_j R_j^i$ for each $i = 1, 2, 3, 4$. Hence

$$Q = \left(\bigsqcup_{j,k,l,m} A_j^1 \cap A_k^2 \cap A_l^3 \cap A_m^4 \right) \sqcup \bigcup_{i,j} R_j^i.$$

Thus Q is the union of 7^4 $BC(1)$ domains and the R_j^i's. (Note that there may be some overlapping of the R_j^i's, but this does not present a problem. The only danger is worsening the bounds, but as we are not seeking best possible bounds, this does not concern us.) Taking the intersection with A, we find by Lemmas 33, 46 and 45 that A is the union of 7^4 $BC(3)$ domains without holes and 32 (possibly disconnected) $BC(2)$ domains.

At this stage, we might wish to apply the $BC(2)$ decomposition to complete the work in hand. The only barrier to this is that we are no longer assuming $HP(4)$—which was used in the $BC(2)$ decomposition—but the weaker condition $HP(5)$. However we are still able to proceed. Instead of ending up with four $BC(2)$ domains without holes and four $BC(1)$ domains, we can use the results of the preceding paragraphs to decompose each of the $BC(2)$ domains under consideration into $T(2) + T(3) = 7$ $BC(2)$ domains without holes and four $BC(1)$ domains. (Note that the holes in each $BC(2)$ domain form a $BC(1)$ family, since each has horizontal and vertical overlap with one of the R^i_j mentioned above.) Doing the arithmetic, we find that each holeless $BC(2)$ domain here can be decomposed into $2^8.4.7 + 4$ $BC(1)$ domains.

Referring back to Lemma 30 and the remarks at the start of this section, we have:

Theorem 52. *Let $A \subseteq \mathbb{Z}^2$ be a $BC(3)$ domain on which $HP(5)$ holds. Then A can be decomposed into $2^{34}.7^4 + 2^{17}.7 + 2^9$ or fewer $BC(1)$ domains.*

$BC(N)$ domains

Now let A be a general $BC(N)$ domain in $\mathbb{Z}^2$ on which $HP(N+2)$ holds, with $N > 3$. If A has no holes, we invoke the Algorithm of Section 3.2 to decompose it into a $BC\big(N, N.(2^N - 1)\big)$ family of $BC(1, N)$ domains and take things from there. So suppose that A does have holes. Enumerate them as H_j and let $R_j = \pi_x H_j \times \pi_y H_j$ as ever. The decomposition continues much along the lines of the previous section.

Firstly we divide the holes, which form a $BC(N-1)$ family, into 2^{2N-4} $BC(1)$ families, using Lemma 30. Then isolate one of these $BC(1)$ families (call it $\mathcal{H}$) and define the subsets L and U of Q exactly as in the last section: see Figure 3.18 and the preceeding table of definitions. Our aim is to show that L is $BC(r-1, 1)$

and U is $BC(r,1)$, where $r = \lceil \frac{N}{2} \rceil + 1$.[3] Again we consider only the slightly more complicated case of U. If it were not $BC(r,1)$, we would have $p_1, \ldots, p_{r+1} \in U$ and $q_1, \ldots, q_r \in Q \backslash U$ all at some height y_0 and such that

$$\pi_x p_1 < \pi_x q_1 < \pi_x p_2 < \cdots < \pi_x q_r < \pi_x p_{r+1}.$$

Define $q_0 = (\min \pi_x Q - 1, y_0)$, $q_{r+1} = (\max \pi_x Q + 1, y_0)$, and

$$\begin{aligned}
Q_0 &= [\min \pi_x Q, \pi_x p_1 - 1] \times [y_0, \max \pi_y Q] \\
P_i &= \big([\pi_x q_{i-1} + 1, \pi_x q_i - 1] \times [\min \pi_y Q, y_0]\big) \backslash \{p_i\} \qquad i = 1, \ldots, r+1 \\
Q_i &= \big([\pi_x p_i + 1, \pi_x p_{i+1} - 1] \times [y_0, \max \pi_x Q]\big) \backslash \{q_i\} \qquad i = 1, \ldots, r \\
Q_{r+1} &= [\pi_x p_4 + 1, \min \pi_x Q] \times [y_0, \max \pi_x Q].
\end{aligned}$$

We also define $P'_i = P_i \backslash \pi_y^{-1}(y_0)$ for $i = 1, \ldots, r+1$; and $Q'_i = Q_i \backslash \pi_y^{-1}(y_0)$ for $i = 0, \ldots, r$, with the stipulation that $P_0 = P_{r+2} = \emptyset$.

We now introduce a generalised version of Lemma 49:

Lemma 53.

1. *Suppose $i \leqslant r$ ($i \geqslant 2$) and there is a rectangle R meeting $P_i \cap Q_i$ but not q_i. Then there is some $R' \subseteq Q'_i$ (P'_i).*
   Suppose that $i \leqslant r - 1$ ($i \geqslant 1$) and there is a rectangle R meeting $Q_i \cap P_{i+1}$ but not q_i. Then there is some $R' \subseteq P'_{i+1}$ (Q'_i).

2. *Suppose there is a rectangle R containing q_i where $i \leqslant r - 1$ ($i \geqslant 2$). Then there is some $R' \subseteq P'_{i+1}$ (P'_i).*

3. *Suppose $i \leqslant r$ ($i \geqslant 1$) and there is a rectangle $R \subseteq Q'_i$ and that no R_j meets $Q_i \cap P_{i+1}$ ($Q_i \cap P_i$) or q_i. Then there is a rectangle $R' \subseteq P'_{i+1}$ (P'_i).*
   Suppose that $i \leqslant r$ ($i \geqslant 2$) and there is a rectangle $R \subseteq P'_i$ and that no R_j meets $P_i \cap Q_i$ ($P_i \cap Q_{i-1}$) or q_i (q_{i-1}). Then there is a rectangle $R' \subseteq Q'_i$ (Q'_{i-1}).

[3]The reason for r taking this form is that each new component appearing in a horizontal section of U or L forces two more holes to alternate about that section.

It is proved in exactly the same way as Lemma 49. As before, it has as a consequence:

Corollary 54. *If U is not $BC(r,1)$ then there are at least $2r$ holes alternating about a horizontal line.*

But $2r = N+2$ when N is even and $N+3$ when N is odd, and we know that we may only have at most $N+1$ holes alternating about a line. Thus we have a contradiction and U must be $BC(r,1)$. Similarly L is $BC(r,1)$.

From the previous section, we know that U and L can be decomposed respectively into $T(r)$ and $T(r-1)$ or fewer $BC(1)$ domains. Thus we have a decomposition of Q into $T(r)+T(r-1)$ or fewer $BC(1)$ domains along with the R_j associated to the holes in $\mathcal{H}$. Repeating the process for each of the remaining $BC(1)$ hole-families, we obtain similar decompositions. Taking intersections, we find that Q is the union of $\big(T(r)+T(r-1)\big)^{2^{2N-4}}$ or fewer $BC(1)$ domains together with the R_j. Then intersecting with A and applying Lemma 45 gives that A is the union of this same number of $BC(N)$ domains without holes and $\bigcup_j A \cap R_j$. The former we dealt with at the beginning of this section on page 69. The latter we can deal with using Lemma 46 and the same kind of considerations as in the $BC(3)$ case. Lemma 46 will leave us with $4N-4$ or fewer $BC(N-1)$ domains for each of the $A \cap R_j$. Although we cannot use a straightforward induction, we can decompose these into a bounded number of $BC(1)$ domains and $BC(N-2)$ domains using the $HP(N+2)$ condition, the results of the $BC(3)$ section and Lemma 46 once again. We repeat this procedure until we are left with nothing but $BC(1)$ domains, at each stage bearing in mind that we can only assume $HP(N+2)$ and none of the stronger HP conditions.

Summarising, we have:

Theorem 55. *Let $A \subseteq \mathbb{Z}^2$ be a $BC(N)$ domain ($N \geqslant 3$) on which the property $HP(N+2)$ holds. Then A can be decomposed into $B(N)$ or fewer $BC(1)$ domains, where B is a function from $\mathbb{N}$ to itself.*

Translating this into the quasi-discrete setting, we have

Corollary 56. *Let $\Omega = \mathcal{T}(A)$ be a quasi-discrete domain in $\mathbb{R}^2$. Suppose that A has the $BC(N)$ and $HP(N+2)$ properties. Then we can decompose Ω into $B(N)$ or fewer $BC(1)$ domains and a set of measure zero.*

Proof. By Theorem 55, we can decompose A into $B(N)$ or fewer $BC(1)$ domains, which we label $A_1, \ldots, A_K$. We have

$$\mathcal{T}(A) = \mathcal{T}\left(\bigsqcup_{i=1}^{K} A_i\right) = Z \sqcup \bigsqcup_{i=1}^{K} \mathcal{T}(A_i)$$

where Z is a set of measure zero consisting purely of points that are on the boundaries of squares. Since $\mathcal{T}$ preserves the $BC(1)$ property (by Lemma 25), the $\mathcal{T}(A_i)$ are all $BC(1)$ domains. $\square$

The final task for this section is to show how one can compute the bound function $B : \mathbb{N} \longrightarrow \mathbb{N}$. Firstly we need to define some more elementary functions from which B is built. We recall the function T that occurred in Lemma 51 defined by $T(1) = 1$ and $T(n+1) = 1 + 2^{n-2}T(n-1)$ for $n \geqslant 1$. Now given $N \in \mathbb{N}$, we define for $N \geqslant 3$,

$$\begin{aligned} r_N &= \lceil N/2 \rceil + 1, \\ S_N &= T(r_N) + T(r_N - 1), \quad \text{and} \\ U_N &= 2^{(N+1)(2^N+1)-6}, \end{aligned}$$

and for $N \geqslant 2$,

$$\begin{aligned} C_N &= 2^{2N-2} \quad \text{and} \\ D_N &= 4N - 4. \end{aligned}$$

Put $S_2 = 4$ and $U_2 = 2^8$. Note that S_N is the bound on the number of $BC(1)$ domains from the decomposition of the regions L and U defined on page 64; U_N is the bound on the number of $BC(1)$ domains from the decomposition of $BC(N)$ domains with no holes; C_N is the bound on the number of $BC(1)$ families produced from a $BC(N)$ family by Corollary 30; and D_N is the bound on the number of $BC(N-1)$ domains coming from Lemma 46. By inspecting how the numbers behave in the arguments of this chapter, we find that

$$B(N) = S_N^{C_{N-1}} U_N C_N + C_{N-1}^2 D_N \Big(S_N^{C_{N-2}} U_{N-1} C_{N-1} +$$
$$+ C_{N-2}^2 D_{N-1} \big(S_N^{C_{N-3}} U_{N-2} C_{N-2} + \cdots + C_2^2 D_3 (S_N U_2 C_2 + D_2) \ldots \big) \Big).$$

3.5 Issues arising

The results of this chapter raise questions in computational complexity. As we have seen, their proofs rely on various techniques of estimation and there are key algorithmic procedures underlying them all. None of the processes involved is claimed or conjectured to be efficient, and it would be an interesting undertaking to investigate if and how the bounds they give might be improved.

Firstly, let us consider Theorem 29 and Corollary 30, the first results given in this chapter. Recall that the Corollary tells us that we can arrange a $BC(m,n)$ family of sets into 2^{m+n-2} or fewer $BC(1)$ families. Certainly this bound is attained when $m = n = 2$, but we know of no $BC(3,2)$ family that requires a full eight $BC(1)$ families. Perhaps the bound here could be improved to something like $O(mn)$ or even $O(m+n)$, but we suspect that much subtler techniques would be required to establish such a bound. The main idea of the proof of Theorem 29 is at least very simple in concept.

Next, take the Algorithm in Section 3.2 given initially for the decomposition of $BC(n,1)$ domains and then extended at the outset of Section 3.4 to cover any $BC(n)$ domains without holes. The bound it gives on the number of $BC(1)$ domains produced is $2^{O(n2^n)}$. If this could be improved upon, then given the centrality of this process, it would give improved bounds for all but the most basic of our decompositions. Of course, the Algorithm makes use of Corollary 30, so an improvement of the latter would help matters here for free.

Finally, consider Lemma 51, which was used to decompose the area away from the holes in a general $BC(N)$ domain. It gave us an improved bound for a special kind of $BC(n,1)$ domain, but even the process used to establish this bound seemed rather inefficient, and one feels that further improvements are possible.

However, because of the rapid growth of the bound given by the Algorithm of

Section 3.2, it would seem to be the most obvious starting point for any efforts to streamline our methods and find better bounds.

Chapter 4

Applications and Discussion

Two possible uses for our results present themselves. One is in the original, continuous setting which motivated our work, and the other is in an entirely discrete setting. We devote a section of this chapter to each of them. The main results are Theorem 58 below and Theorem 63 on page 82. The reader may wish to refer back to Theorem 55 on page 71 for a reminder of the result we wish to apply. A review of Section 1.3 might also be useful for Section 4.1.

4.1 An application in the continuous setting

As mentioned above, the first application is in the context from which our results arose, namely of sublevel set operators on real-valued functions on $\mathbb{R}^2$. It extends what we know in a special case of the following theorem, taken from [4].

Theorem 57. *Let $\alpha \in \mathbb{N}^2$ be a multi-index and suppose that $D^\alpha u \geqslant 1$ on $[0,1]^2$. Suppose further that the intersection of any vertical line with any sublevel set of u has at most N connected components. Then there is a C depending on α and N such that*

$$\left| \int_{\{|u| \leqslant s\}} f_1(x_1) f_2(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \right| \leqslant C s^{1/|\alpha|} \|f_1\|_{p_1} \|f_2\|_{p_2},$$

where $\frac{1}{p_i} = 1 - \frac{\alpha_i}{\alpha}$.

The proof of this result can be found in [3], and since it uses completely different methods from those discussed in this document, we omit it here. Suffice it to say that the above theorem contains the $n = 2$ case of Theorem 14 and was proved before the latter.

The content of the result we are about to give is that in the scenario of Theorem 57, we can allow Ω to be any HV-convex set provided that u is a type 1 function with type 1 constant N and $\frac{\partial^2 u}{\partial x \partial y} > 0$ on $\pi_x\Omega \times \pi_y\Omega$. Without further ado, we have

Theorem 58. *Let $\Omega \subseteq Q_0$ be an HV-convex domain, where Q_0 is some closed, axis-parallel rectangle containing Ω. Let $u : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth type 1 function with type 1 constant N and α a multi-index such that $D^\alpha u \geqslant 1$ on Ω and $\frac{\partial^2 u}{\partial x \partial y} > 0$ on Q_0. Then there is a constant C depending only on N and α such that*

$$\left| \int_{\Omega \cap \{|u| \leqslant s\}} f_1(x_1)\, f_2(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \right| \leqslant C s^{1/|\alpha|} \|f_1\|_{p_1} \|f_2\|_{p_2},$$

where the p_i are as in Theorem 57 above.

Proof. Let E_s be the sublevel set of u at level s, i.e. $E_s = \{x \in Q_0 : |u(x)| \leqslant s\}$. The idea is that we approximate E_s by some $r\mathcal{T}(A) = \{rx : x \in \mathcal{T}(A)\}$, where $r > 0$ is a 'zoom factor' and $A \subseteq \mathbb{Z}^2$ obeys appropriate BC and HP conditions, apply Theorem 55 from Chapter 3, translate the resulting $BC(1)$ domains in $\mathbb{Z}^2$ into suitable domains in $\mathbb{R}^2$, and finally apply the methods of [11] as discussed in Section 1.3.

Firstly we note that the proof of Theorem 29 and Corollary 30 can be carried over to $\mathbb{R}^2$. (In fact, these results were first proved in $\mathbb{R}^2$ before noticing that they worked equally well in $\mathbb{Z}^2$!). By the Corollary and Lemma 16 we may assume that E_s is connected.

By the uniform continuity of u on Q_0, there is a $\delta > 0$ such that whenever $|x - x'| < \delta$, we have $|u(x) - u(x')| < s/2$. Let E be the set obtained from E_s by filling in any holes that do not contain a circle of radius δ. More precisely, denote the holes of E_s not containing a circle of radius δ by H_i, $i \in \mathbb{N}$, and put $E := E_s \cup \bigcup_{i \in \mathbb{N}} H_i$. So any point inside one of the H_i is within δ of a point of E_s, and hence $|u| < 3s/2$ on E. (We ignore the dependence of E on s, since it is irrelevant in the rest of the argument.) Note that E is connected because E_s is connected.

We know that there is some $\varepsilon > 0$ such that $\frac{\partial^2 u}{\partial x \partial y} \geqslant \varepsilon$ on Q_0. By uniform continuity once more, we choose $r < \delta/2\sqrt{2}$ such that whenever $|x - x'| < 2r$, we have $|u(x) - u(x')| < \delta^2\varepsilon/8$. Now we pick our approximating set $A \subseteq \mathbb{Z}^2$ by putting $A = \{z \in \mathbb{Z}^2 : r\overline{\mathcal{T}(z)} \cap E \neq \emptyset\}$. We note that $|u| < 2s$ on $r\overline{\mathcal{T}(A)}$, and that $E_s \subseteq E \subseteq r\mathcal{T}(A)$.

We claim that A satisfies $BC(2N-1)$ and $HP(N+2)$. We tackle the latter property first. Suppose for a contradiction that we had $N+2$ holes $H_1, \dots, H_{N+2}$ violating the condition. Without loss of generality, suppose that they alternate around a horizontal line L_0 and that H_1 is below the line.

By definition of E, all the holes of $\mathcal{T}(A)$ contain a circle of radius $\delta/2r$. Choose a height that is at least $\delta/2r$ down H_1 and consider the points s_1 and s_2 of $\mathbb{Z}^2$ immediately to the left and right of the horizontal cross-section of H_1 at that height.

Both of the sets $S_i := r\overline{\mathcal{T}(s_i)}$ must meet the boundary of E, on which $|u| = s$; let q_i be points where this happens. Since the q_i can be joined by a path lying entirely inside E_s^c except for the endpoints, we see that $u(q_1) = u(q_2)$. Let p_1 be the point midway down the right-hand edge of $r\overline{\mathcal{T}(s_1)}$ and p_2 the point midway down the left-hand edge of $r\overline{\mathcal{T}(s_2)}$, and let L_1 be the horizontal line joining p_1 and p_2. (See Figure 4.1.)

Now for $i = 1, 2$, $|p_i - q_i| < 2r$ and so $|u(p_i) - u(q_i)| < \delta^2\varepsilon/8$. Hence by the Mean Value Theorem, there must be a point c on L_1 such that

$$\begin{aligned} \left|\frac{\partial u}{\partial x}(c)\right| &= \frac{|u(p_2) - u(p_1)|}{|p_2 - p_1|} \\ &\leqslant \frac{|u(p_2) - u(q_2)| + |u(q_1) - u(p_1)|}{|p_2 - p_1|} \\ &\leqslant \frac{\delta\varepsilon}{4}. \end{aligned}$$

Therefore, since $\frac{\partial^2 u}{\partial x \partial y} \geqslant \varepsilon$ on Q_0 and the distance from c to L_0 is at least $\delta/2$, there is a point c_1 on rL_0 for which $\pi_x c_1 = \pi_x c$ and $\frac{\partial u}{\partial x}(c_1) \geqslant \delta\varepsilon/4$. We treat each of the holes $H_2, \dots, H_{N+2}$ in the same manner to obtain a sequence of points $c_1, \dots, c_{N+2}$

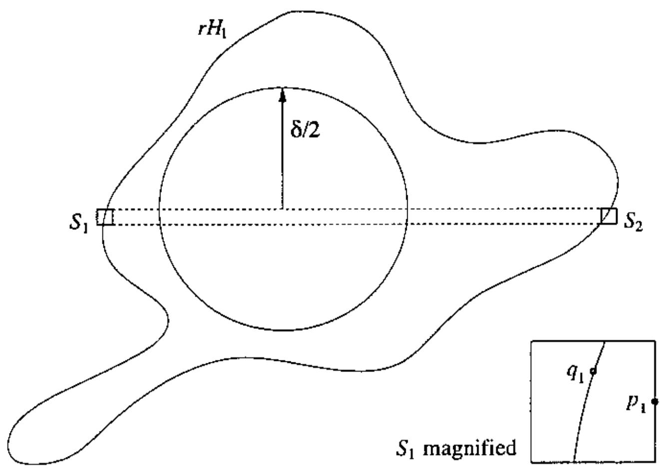

Figure 4.1: Analysing the holes of $r\mathcal{T}(A)$

on the horizontal line rL_0 such that $\pi_x(c_i) < \pi_x(c_{i+1})$ for all i, $\frac{\partial u}{\partial x}(c_{odd}) > 0$ and $\frac{\partial u}{\partial x}(c_{even}) < 0$. But this means that $\frac{\partial u}{\partial x}$ meets rL_0 in at least $N+1$ components, which contradicts the assumption that u is type 1 with type 1 constant N.

We now show that A is a $BC(2N-1)$ domain. Note that E is $BC(N)$, by its definition and the fact that E_s is $BC(N)$. Again, we suppose for a contradiction that A isn't $BC(2N-1)$, and that there are, without loss of generality, precisely $2N$ horizontal beams at the height y_0, named $C_1, \ldots, C_{2N}$ say. (More than $2N$ would make things easier, as we shall shortly see.) For each i, let $D_i = \overline{r\mathcal{T}(C_i)}$ and choose a point $x_i \in E \cap D_i$. By connectedness of E, each x_i must be joined by a path in E to the top or the bottom edge of $\overline{r\mathcal{T}(C_i)}$.

We claim that exactly N of the x_i can only be joined by such a path to the top edge of D_i and exactly N of the x_i can only be joined by such a path to the bottom edge of D_i. For if this were not the case, there would be (at least) $N+1$ of the x_i joined to the top of D_i or $N+1$ of the x_i joined to the bottom of D_i. Suppose without loss of generality that the former holds. Choose a height that lies between the height of the top edges of the D_i and the height of the highest x_i, and consider the horizontal cross section of E at that height. Since by definition E does not meet any of the gaps between the D_i, we can find a sequence of points $y_1, y_2, \ldots, y_{2N+1}$ such that for all i, $\pi_x y_i < \pi_x y_{i+1}$, $y_{2i+1} \in E$ and $y_{2i} \in E^c$. But this contradicts the fact that E is $BC(N)$.

This entails that there can be no path between any of the former category of x_i (those joined only to the top edge of D_i) and any of the latter category of x_i (those joined only to the bottom edge of D_i), which contradicts the connectedness of E. For any such path would necessarily meet both the bottom and the top of some fixed D_i. See Figure 4.2.

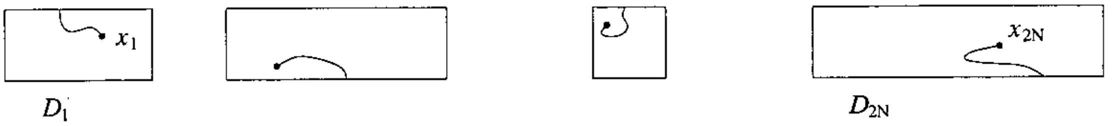

Figure 4.2: Showing that A is a $BC(2N-1)$ domain

Having established that A satisfies $HP(N+2)$ and $BC(2N-1)$, we can apply Theorem 55 of Chapter 3 to decompose it into $B(2N-1)$ or fewer $BC(1)$ domains. Say $A = \bigsqcup_{i=1}^{K} A_i$ where $K \leqslant B(2N-1)$. (Evidently with a bit of further analysis one could come up with a better bound, since A satisfies $HP(N+2)$, which is stronger than the $HP(2N+1)$ in the hypotheses of the Theorem.) Then we have

$$rT(A) \cap \Omega = Z \sqcup \bigsqcup_{i=1}^{K} rT(A_i) \cap \Omega$$

for some set Z of zero measure. By Lemmas 25 and 33, each $rT(A_i) \cap \Omega$ is a $BC(1)$ domain, and hence can be decomposed into three or fewer orthogonal families of curved trapezoids. Now the methods of [11] can be invoked to obtain the desired result. By Lemma 16, we may as well be dealing with $3K$ curved trapezoids Ω_i. On each of the Ω_i, we have $|u| \leqslant 2s$ and $D^\alpha u \geqslant 1$. Therefore

$$\begin{aligned}
\left| \int_{\Omega \cap \{|u| \leqslant s\}} f_1(x_1)\, f_2(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \right| &\leqslant \left| \int_{\Omega \cap rT(A)} f_1(x_1)\, f_2(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \right| \\
&\leqslant \sum_{i=1}^{3K} \left| \int_{\Omega_i} f_1(x_1)\, f_2(x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \right| \\
&\leqslant C s^{1/|\alpha|} \|f_1\|_{p_1} \|f_2\|_{p_2},
\end{aligned}$$

because the result holds on curved trapezoids, as shown in Section 1.3. $\square$

There is a need to be cautious here, since the arguments of [11] described in Section 1.3 are only stated for domains inside the unit square Q. The way to get around this is to analyse those arguments and observe that this restriction is unnecessary.

The principal limitation of this whole method is that it is very much anchored to the case of $\frac{\partial^2 u}{\partial x \partial y} > 0$. At present we do not know of any way to extend it to cover the case of having single-signed $D^\beta u$ for other multi-indices β. That said, the broad strategy of decomposing domains in $\mathbb{Z}^2$ rather than $\mathbb{R}^2$ certainly avoids a number of difficulties and the arguments involved are fairly elementary, as seen in Chapter 3. There are still technicalities to be faced when seeking suitable quasi-discrete approximations to general domains in $\mathbb{R}^2$ and in ensuring that relevant properties carry over. However it may be that the techniques of Chapter 3 will find uses separate from their original purpose of analysing sublevel set integral operators.

4.2 An application in the discrete setting

The second use for our results is to apply them in a purely discrete context. This would seem more elegant, but perhaps the potential knock-on applications are less obvious at the moment.

Recall that in motivating the $HP(N)$ conditions (see the subsection on page 51) we looked at the number of components of certain cross-sections of sublevel sets of the first partials of our phase function u. Firstly we need analogues of partial derivatives.

Definition 59. *Let f be a function from $\mathbb{Z}^2$ to $\mathbb{R}$. Define for all $(a,b) \in \mathbb{Z}^2$*

$$\begin{aligned} D^x f(a,b) &= f(a+1,b) - f(a,b) \\ D^y f(a,b) &= f(a,b+1) - f(a,b) \\ D^{xy} f(a,b) &= f(a+1,b+1) + f(a,b) - f(a+1,b) - f(a,b+1). \end{aligned}$$

Also, for $f : \mathbb{Z} \longrightarrow \mathbb{R}$ and all $a \in \mathbb{Z}$ define

$$Df(a) = f(a+1) - f(a).$$

Obviously in the discrete setting we have no intermediate value theorem. Note however that in the continuous case, the number of components of the cross-section $\{x \in \mathbb{R} : |f(x, y_0)| < s\}$ is equal to

$$\# \text{ components } \{x \in \mathbb{R} : f(x, y_0) \geqslant s\} \\ + \# \text{ components } \{x \in \mathbb{R} : f(x, y_0) \leqslant -s\} + \varepsilon,$$

where $\varepsilon \in \{-1, 0, 1\}$ and depends on f, s and y_0. Therefore we introduce

Definition 60. *For a function $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}$, define*

$$\begin{aligned} C_x(f, s, y_0) &:= \# \textit{ components } \{x \in \mathbb{Z} : f(x, y_0) \geqslant s\} \\ &\quad + \# \textit{ components } \{x \in \mathbb{Z} : f(x, y_0) \leqslant -s\} \\ C_y(f, s, x_0) &:= \# \textit{ components } \{y \in \mathbb{Z} : f(x_0, y) \geqslant s\} \\ &\quad + \# \textit{ components } \{y \in \mathbb{Z} : f(x_0, y) \leqslant -s\}. \end{aligned}$$

There now follows a Lemma that performs the rôle that Rolle's Theorem did in the continuous setting.

Lemma 61. *Let $f : \mathbb{Z} \longrightarrow \mathbb{R}$ and $a < b - 1$ be such that $|f(a)|, |f(b)| < s$ and $|f(c)| \geqslant s$ for all $c \in (a, b)$. Then there exist c^+ and c^- in $[a, b)$ such that $Df(c^+) > 0 > Df(c^-)$.*

The proof is trivial. As a consequence we have

Proposition 62. *Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{R}$ be such that $D^{xy}f > 0$ on $\mathbb{Z}^2$ and let $A = \{z \in \mathbb{Z}^2 : |f(z)| < s\}$ be a sublevel set of f. Suppose that there are holes $H_1, \ldots, H_m$ in A and $y_0 \in \mathbb{Z}$ such that for all i, $\pi_x(H_i) < \pi_x(H_{i+1})$, $\pi_y(H_{2i+1}) \leqslant y_0$ and $\pi_y(H_{2i}) > y_0$. Then there exist $x_1 < \cdots < x_m$ such that $D^x f(x_{2i+1}, y_0) > 0$ and $D^x f(x_{2i}, y_0) < 0$ for all i. Hence there is an s' such that $C_x(D^x f, s', y_0) \geqslant m$.*

Proof. For each i, choose $y_i \in \pi_y(H_i)$ and apply Lemma 61 to $f \restriction \pi_y^{-1}(y_i)$. In this way we find $z_i \in H_i$ such that for all i, $D^x f(z_{2i+1}) > 0$ and $D^x f(z_{2i}) < 0$. Now the assumption that $D^{xy} f > 0$ gives the existence of the x_i as required. □

It is now easy to prove

Theorem 63. *Let $A \subseteq \mathbb{Z}^2$ be bounded and $f : A \longrightarrow \mathbb{R}$. Suppose that $D^{xy} f > 0$ on A and that for all $x_0, y_0 \in \mathbb{Z}$, all $s \in \mathbb{R}$ and all $g \in \{f, D^x f, D^y f\}$,*

$$\left. \begin{array}{l} C_x(g, s, y_0) \\ C_y(g, s, x_0) \end{array} \right\} \leqslant N.$$

Then any sublevel set of f can be decomposed into C_N or fewer $BC(1)$ domains, where C_N is a constant depending only on N.

Proof. Let S be a sublevel set of f. The conditions imply that S is $BC(N+1)$. By the Proposition, S must have property $HP(N+3)$ (in fact $HP(N+1)$), and hence we can use Theorem 55 in Chapter 3 to decompose it as required. □

4.3 Discussion

We round off this chapter with a retrospective look at the thesis as a whole, and give some ideas for future lines of enquiry. The central question is whether our hole conditions $HP(m, n)$ are necessary to achieve decompositions of (approximations of) sublevel sets of type 1 functions.

Thinking back to the decompositions of Phong, Stein and Sturm ([11]), one notices that their decompositions, which apply to all algebraic domains, rely on nothing other than the polynomial nature of u. In particular, the mechanics of the decomposition do not require any derivative conditions. In contrast, our $HP(m, n)$ conditions are motivated by the condition $\frac{\partial^2 u}{\partial x \partial y} > 0$.

On the other hand, we have strong suspicions that examples can be produced showing that restrictions on the layout of holes are necessary for the type of decomposition we require, although we have not been able to complete all the details of such an example.

So are there any other conditions that could be imposed or deduced that would give us suitable control of hole arrangements? A condition fitting the bill would be that the number of holes in type 1 (or M) functions u is bounded by a constant C_N depending on $N = t_1(u)$. We show this would allow a decomposition considerably easier than that of the $BC(N)$ Domains subsection of Chapter 3 on page 69. The arguments closely mimic those found in the final four paragraphs of the subsection on $BC(3,3)$ domains, which begins on page 62. .

Leaving aside the process of transferring between continuous and discrete settings, suppose that $A \subseteq \mathbb{Z}^2$ be a $BC(N)$ domain that has at most C_N holes. Let H_1 be one of the holes and put $R_1 = \pi_x H_1 \times \pi_y H_1$ in the familiar way. Then $Q \backslash R_1$ (where $Q = \pi_x A \times \pi_y A$) can be divided into two $BC(1)$ domains as shown in Figure 4.3.

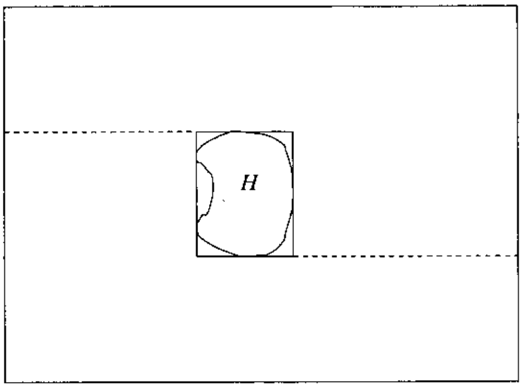

Figure 4.3: An easier decomposition when there are bounds on hole numbers

Call these two $BC(1)$ domains A_1^1 and A_2^1 and repeat this procedure for all of the other holes to get $A_1^2, A_2^2, \ldots, A_2^{C_N}$. Then we have

$$Q = \bigsqcup_{i_1,\ldots,i_{C_N}} \left(A_{i_1}^1 \cap A_{i_2}^2 \cap \cdots \cap A_{i_{C_N}}^{C_N} \right) \sqcup \bigcup_{i=1}^{C_N} R_i,$$

where for $j = 1, 2, \ldots, C_N$ each i_j is in $\{1, 2\}$. Thus Q is the union of 2^{C_N} or fewer $BC(1)$ domains together with the R_i by Lemma 33. Now taking intersections with A and applying Lemmas 33 and 45, we end up with 2^{C_N} or fewer $BC(N)$ domains without holes. The decomposition can now continue using the methods of the subsection on $BC(N)$ domains on page 69. The difference is that here we do not need to analyse and decompose the domains L and U that were introduced there, since in their place we have the much simpler $BC(1)$ domains.

Thus a bound of the type above on the number of holes would be very desirable, but unfortunately as yet we can see no grounds either for or against imposing such a condition.

We sum up with a list of a few questions that we believe would merit further investigation:

- Can the HV-convex domain Ω in Theorem 58 (page 76) be relaxed to a $BC(N)$ domain for any $N \in \mathbb{N}$? (Perhaps the restriction that Ω have no holes would be needed.)

- In the setting of Theorem 58, if one assumes that $D^{\beta}u > 0$ on $\pi_x\Omega \times \pi_y\Omega$ for some $\beta \neq (1,1)$ rather than $\frac{\partial^2 u}{\partial x \partial y} > 0$, can any useful extensions of or alternatives to the $HP(m,n)$ conditions be motivated or deduced?
Alternatively, is there some kind of inductive approach to treating the case of single-signed $D^{\beta}u$ for $\beta \neq (1,1)$ using the case $\frac{\partial^2 u}{\partial x \partial y} > 0$ as the base case for the induction?

- Is there any reason to believe or to deny that the number of holes in a sublevel set of a type 1 (or type M) function can be bounded by a constant depending only on $t_1(u)$?
Could such conditions controlling the holes be dropped altogether, or can one (as we suspect) produce examples showing a need for them?

- Can decompositional methods be used in dimensions higher than 2, or would inductive methods be more appropriate?

Appendix A

Hole Diagrams for Domains Satisfying the HP(5,4) Condition

Here we list all the possible hole diagrams (up to symmetry) for bounded domains in $\mathbb{Z}^2$ which obey the $HP(5,4)$ condition, an issue which was deferred from Section 3.4. We also give an indication of how to establish that these are indeed all the possible diagrams. A look at Section 3.4 and, more importantly, Section 3.3, might be helpful to the reader before proceeding with the present material. Familiarity with the concepts and notation there is assumed.

The diagrams, of which there are 27 in all, appear on pages 88 and 89. To establish that they constitute all the possible diagrams, one uses the same argument as in the $HP(4)$ case. Taking the diagrams in turn, one shows that adding another hole to any domain having that hole diagram has one of two consequences. Either it produces an illegal arrangement (i.e. one violating the $HP(5,4)$ condition), or it yields a domain whose diagram is on the list up to symmetry, possibly the same diagram as we started with. Since the hole diagram for any domain can be built up by introducing the holes one at a time, it follows that we have indeed listed all the possible hole diagrams for the situation in hand.

A representative example of the process is depicted in Figure A.1. We take the twelfth diagram on the list and record what happens if a new hole H^* is introduced in the different possible areas. As in the $HP(4)$ case, we give names to the holes at the ends of maximal strings. From left to right, we label them $H_1 \ldots H_5$, as shown in Figure A.1. In the table of Figure A.2, the second and fourth columns show either an illegal arrangement given rise to or the numbers of the new diagrams that can be produced.

The sharp-eyed reader will notice that two zones are missing from the left column of the table, namely $(3,2)$ and $(5,5)$. We can analyse these as in the $HP(4)$ case (see Figure 3.14 on page 56) and conclude that H^* is either absorbed into a maximal string, leaving us with the same diagram, or gives an illegal arrangement. Finally, note that the zone co-ordinates of the table work as for a matrix, i.e. row first then column.

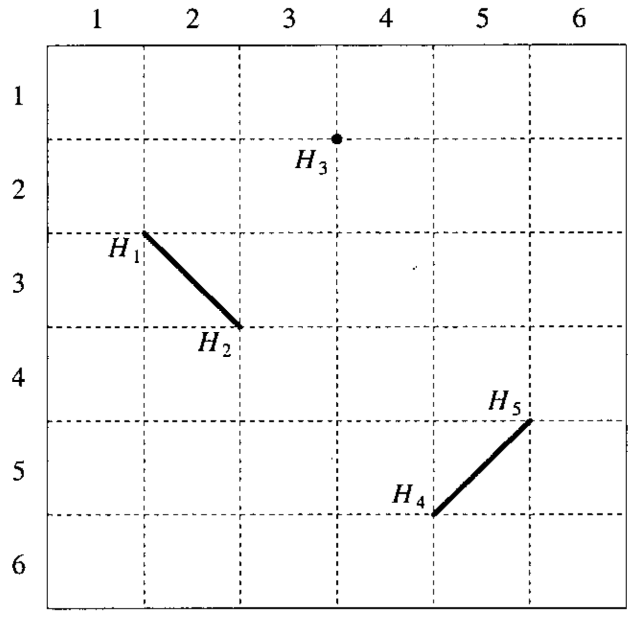

Figure A.1: A closer look at diagram 12

Of course, there are 26 other cases to check. In view of the fact that the material here does not constitute an important part of the thesis, they are omitted and left to the diligent (or masochistic) reader to verify. The most that the methods here can give is a mild improvement of the bounds in the decomposition of domains obeying the $HP(5,4)$ condition compared to the more general method described in Section 3.4.

Before giving the list of hole diagrams, we remark on an interesting phenomenon, which also appeared when enumerating the hole diagrams for sets with the $HP(4)$ condition. To wit, in both cases there is a 'maximum' diagram, mean-

Zone	Consequence	Zone	Consequence
1,1	H^*, H_3, H_1, H_4	3,1	H^*, H_1, H_2, H_3, H_4
1,2	"	4,1	"
1,3	"	5,1	"
1,4	19	6,1	"
2,3	"	3,3	H_3, H_1, H^*, H_2
1,5	H^*, H_2, H_5, H_4	4,2	H_3, H_1, H_2, H^*
2,5	"	5,2	"
3,5	"	6,2	"
1,6	"	4,4	13
2,6	"	4,5	22, 23
3,6	"	5,3	H_2, H_5, H^*, H_4
2,1	12	5,4	"
4,3	"	5,6	15, 24
4,6	"	6,3	H_2, H^*, H_3, H_4, H_5
6,4	"	6,5	H^*, H_4, H_5, H_2
2,2	H_1, H^*, H_2, H_3, H_4	6,6	"
2,4	H_3, H^*, H_2, H_4		
3,4	"		

Figure A.2: What happens if a hole is added to Diagram 12

ing a diagram from which all the others can be achieved by removing holes. We know of no reason *a priori* for this to be the case. Certainly the hole diagrams can be partially ordered by saying that $D_1 \preccurlyeq D_2$ if D_2 can be obtained from D_1 by adding holes. However, a quick inspection confirms that this ordering is not total. Thus while it would be reasonable to expect maximal diagrams, there would be no reason to expect a maximum one without further insights.

The diagrams appear on the next two pages.

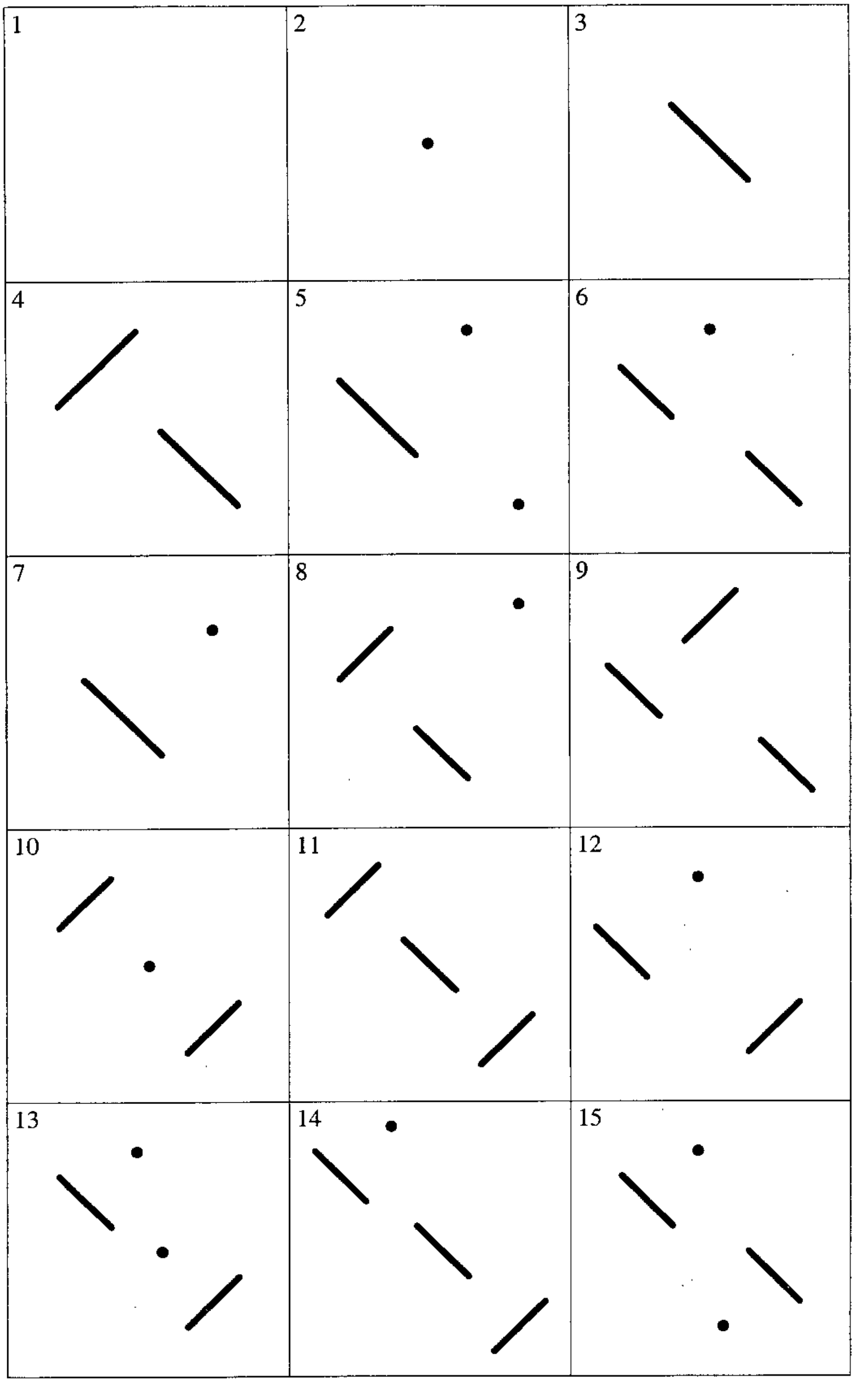

Figure A.3: The first 15 diagrams

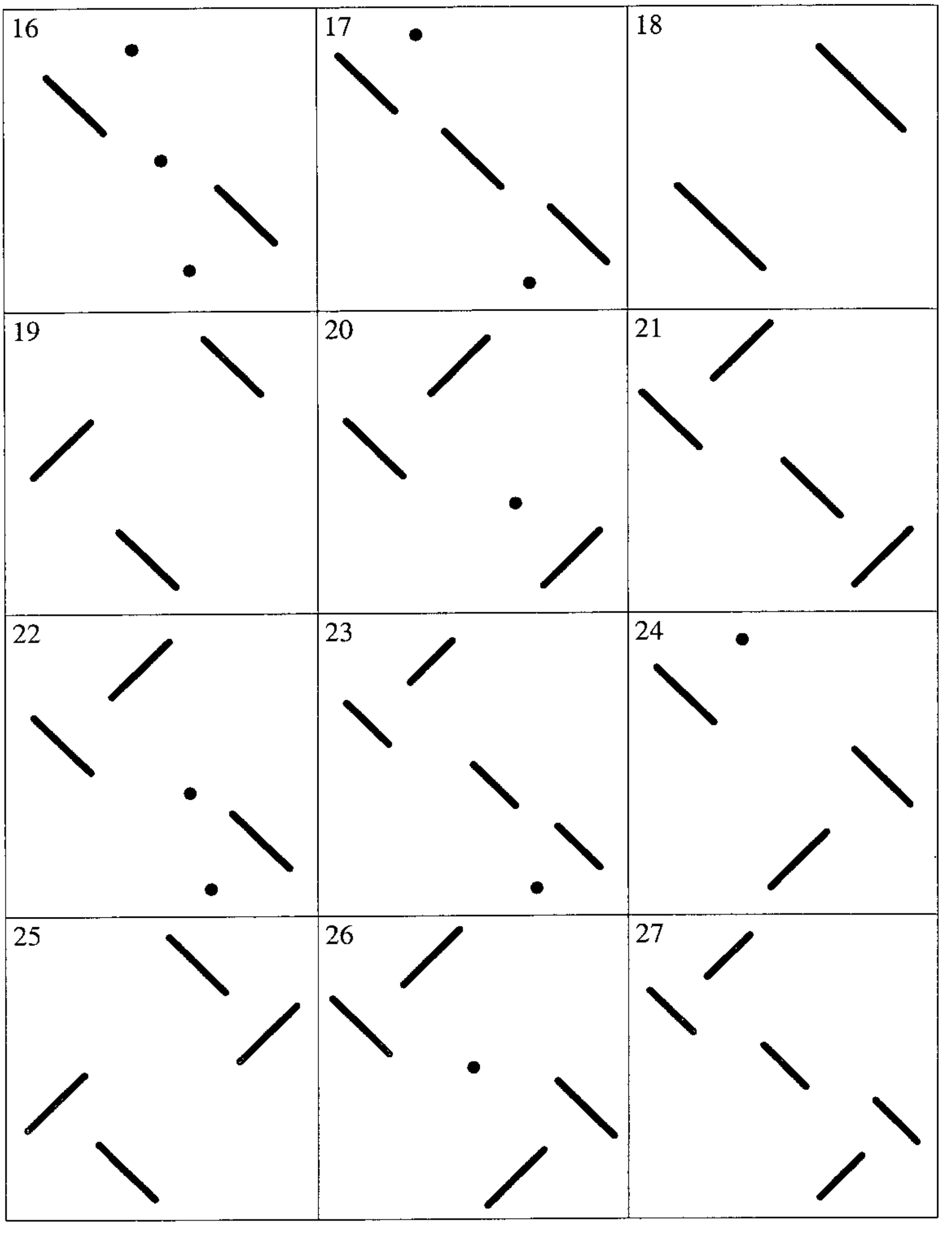

Figure A.4: The remaining 12 diagrams

Appendix B

More on the Rectangle Problem

As stated in Section 1.2, it is possible to tighten up the methods used in our estimate there of the constant c_0 in the Rectangle Problem. This realisation came out of a reduction of the scenario from a problem in $\mathbb{R}^2$ to one in $\mathbb{R}$. We describe this reformulation before giving the improved bound. As in Appendix A, familiarity with the notation and ideas of the relevant section is assumed.

Let E_K denote the union of K open strips $\Delta_1, \ldots, \Delta_K$ as in the previous discussion of the problem, but with the distances between them yet to be determined. Define F_K as the interior of the intersection of $\overline{E_K}$ with the x-axis, and view F_K as a subset of $\mathbb{R}$. Then F_K is the union of K disjoint open intervals, $I_j = (a_j, b_j)$ say.

Observe that there is an axis-parallel rectangle R with corners in E_K if and only if there exist $a \geqslant 0$ and $0 < y \leqslant x$ such that

$$\{a, a+x, a+y, a+x+y\} \subseteq F_K. \tag{B.1}$$

The values x and y correspond to the side lengths of R. (In particular, $x = y$ is allowed and represents a square.) One could describe the set B.1 as a smallest non-trivial two-dimensional arithmetic multiprogression.

We wish to find a good way of adding strips to $[0,1]^2$ while avoiding rectangles with corners in E_K and area greater than $(b_1 - a_1)^2/4$. In fact we can work at any scale we choose and then use the scaling and limiting arguments of the earlier discussion. The problem translates into finding a sequence of intervals in $\mathbb{R}$ while avoiding sets of the form B.1. The first technique which comes to mind is that of adopting a simple 'greedy' strategy of placing each interval I_j as close as possible to the previous one, thereby maximising its width.

For concreteness, let us define $I_1 = (0, 10000)$. We shall round off all our numbers to the nearest integer, since this loses little and simplifies things. Then we must avoid configurations like B.1 with $xy \geqslant 5000^2$. It is easily seen that we must have $I_2 = (20000, 21250)$ and $I_3 = (42500, 43088)$. (Thus far these are the same proportions as earlier.)

To analyse the situation further, we introduce for $m < n$ in $\mathbb{N}$

$$J_{m,n} = (a_n - b_m, b_n - a_m),$$

and $J_{m,m} = (0, b_m - a_m)$. These can be thought of as the sets of possible side lengths of rectangles where one end is in Δ_m and the other in Δ_n. So thus far we have:

$$\begin{array}{lll} J_{1,1} = (0, 10000) & J_{1,2} = (10000, 21250) & J_{1,3} = (32500, 43088) \\ & J_{2,2} = (0, 1250) & J_{2,3} = (21250, 23088) \\ & & J_{3,3} = (0, 588). \end{array}$$

We now claim that, supposing we have introduced K intervals I_j, sets of the form B.1 with $xy \geqslant 5000^2$ are avoided if and only if $J_{1,1}$ together with the $J_{m,n}$ for $m < n$ are pairwise disjoint and $|I_j| \leqslant 5000^2/a_j$. (The role of the second condition is clear: it corresponds to the long, thin rectangles with left vertices in Δ_1 and right vertices in Δ_j.) For suppose that there were some $z \in J_{m,n} \cap J_{m',n'}$, where $m < m'$. There are two cases to investigate: $m' \geqslant n$ and $m' < n$. (See Figure B.1.) In both cases it is easy to see that a set of the form B.1 is given rise to. In the first case we take $x = z$ and in the second $y = z$. It is also easily seen that $J_{m,n} > 10000$ for $m < n$, and so we have $xy > 10000^2$. The converse is similar.

The 'greedy' strategy entails that we introduce the I_j for $j \geqslant 4$ so that at each stage the left endpoint of $J_{j-1,j}$ is equal to the right endpoint of $J_{j-2,j-1}$. In other words, $a_j = 2b_{j-1} - a_{j-2}$. This works up to $j = 11$, but if we introduce I_{12} according to this rule then we run out of luck because $J_{1,7}$ and $J_{7,12}$ meet.

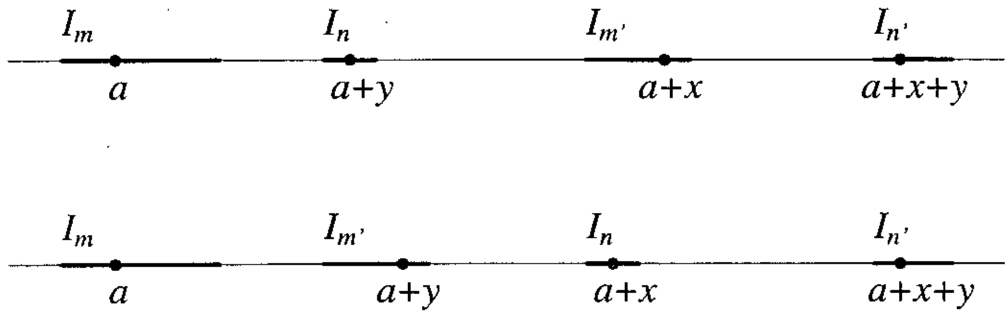

Figure B.1: Interpreting the $J_{m,n}$

The situation is rather intricate: each new I_j adds a further j of the $J_{m,n}$ and it becomes increasingly difficult to ensure that none of them overlap.

The most obvious way to seek a valid I_{12} is to replace a_{12} with $a_{12}+(b_7-a_1)-(a_{12}-b_7)=2b_7-a_1$, then redefine $b_{12}=5000^2/a_{12}$ and try again. In general, suppose we had found $I_1,\ldots,I_{N-1}$ with disjoint $J_{m,n}$ for $1 \leqslant m < n \leqslant N-1$. Our initial guess for a_N is $2b_{N-1}-a_{N-2}$ and for b_N is $5000^2/a_N$. if we find that $J_{m,n}$ and $J_{p,N}$ overlap (where $1 \leqslant m, p \leqslant N-1$ and $m < n$), we replace a_N by $b_n - a_m + b_p$, redefine b_N as $5000^2/a_N$ and see if this works. If not, we redefine I_N in a similar way and keep trying until we find one that works. At the very worst, we could end up with $a_N = 2b_{N-1}$, and so the process must end. Continuing with the example at hand, we arrive at $I_{12}=(290145, 290231)$.

This process is perfectly suited to being carried out by a computer. In particular, calculating the first 50 of the I_j when $I_1=(0,10000)$ and with integer rounding shows that the constant c_0 is at most about $5/42$. As stated, the number of computations is growing quite quickly from one stage to the next, while the gains (i.e. reduction in the upper bound for c_0) are diminishing, and it is unclear how much use such numerical methods could be in making progress on the problem. Further insight into the situation might well allow more significant gains than sheer number-crunching.

Out of interest, we list $I_4,\ldots,I_{11}$ in the present model, leaving the reader to verify that the corresponding $J_{m,n}$ are indeed disjoint for $m < n$.

$$(66176, 66553) \quad (90606, 90881) \quad (115586, 115802) \quad (140998, 141175)$$
$$(166764, 166913) \quad (192828, 192957) \quad (219150, 219264) \quad (245700, 245801).$$

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