X

UNIQUENESS THEOREMS FOR A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS

by

DAVID L. COLTON

Thesis presented for the Degree of Doctor of Philosophy of the University of Edinburgh in the Faculty of Science

October, 1967.

ACKNOWLEDGEMENT

I would like to thank my supervisor, Professor Arthur Erdélyi, for his advice and guidance during the course of my research. I am also indebted to my office mate Jet Wimp for several helpful discussions.

I gratefully acknowledge financial support received through a Junior Fellowship from the University of Edinburgh.

CONTENTS

INTRODUCTION

Of fundamental importance in physics are problems whose mathematical formulation requires at least three dimensions. Since in many ways one and two dimensional problems are easier to handle, one of the major efforts of mathematicians has been to reduce three dimensional problems to those of lower dimensions. Fourier analysis, separation of variables, integral transforms, and the introduction of various kinds of axial symmetry are some of the more familiar methods that have been devised with this aim in mind. This thesis is concerned with the study of the two dimensional equations that result when Fourier analysis is applied to the three dimensional Helmholtz or reduced wave equation.

The Helmholtz Equation

We begin our study by considering the three dimensional Helmholtz equation in cylindrical coordinates ρ, ϕ, z :

$$\Delta\Phi + k^2\Phi \equiv \frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} + k^2\Phi = 0 \qquad (0.1)$$

The following uniqueness theorem was first stated by Sommerfeld[22] and later extended and refined by Magnus[18], Rellich[20] and Levine[16].

<u>Theorem 0.1</u> : Let F be the exterior of a finite three dimensional closed surface ∂D (∂D denotes the boundary of a simply connected domain D). ∂D satisfies such regularity

conditions as to insure the validity of the following case of Green's identity:

$$\int_{G_a} (\bar{u}\Delta u - u\Delta\bar{u})\,dV = \int_{\Sigma(a,0)} \left(\bar{u}\frac{\partial u}{\partial n} - u\frac{\partial \bar{u}}{\partial n}\right) dS$$

where $\Sigma(a, 0)$ denotes the sphere of arbitrary radius a centred at the origin and containing ∂D in its interior, G_a is the domain bounded by $\Sigma(a, 0)$ and ∂D, u is a twice continuously differentiable function in G_a, vanishing on ∂D, (with $\bar{u}$ as its complex conjugate) and $\frac{\partial}{\partial n}$ indicates differentiation in the direction of the exterior normal of $\Sigma(a, 0)$. Then $u(\rho, \phi, z) \equiv 0$ is the only continuous function in $\bar{F} = F \cup \partial D$ such that

a) $u \in C^{(2)}(F) \cap C^{(1)}(\bar{F})$ ($C^{(n)}(F)$ denote the space of functions defined in F which are n times continuously differentiable in F).

b) $\Delta u + k^2 u = 0$ in F.

c) $u = 0$ on ∂D.

d) $\lim_{R\to\infty} R(\frac{\partial u}{\partial R} - iku) = 0$ uniformly along all rays from the origin, where $R = (\rho^2 + z^2)^{1/2}$.

e) $u = O(R^{-1})$ as $R \to \infty$.

Condition d) is known as the "radiation condition" and e) is known as the "finiteness condition". A bounded, simply connected domain D that admits the application of the Gauss integral theorem(12)* will be called a <u>normal domain</u>. The conditions imposed on ∂D in theorem 0.1 are equivalent to requiring that the domain D be normal(12)*. Rellich showed that conditions d) and e) could be replaced by

* p. 4-5.

$$\text{d}') \quad \lim_{R\to\infty} \int_{\Sigma(R,0)} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS = 0 .$$

See (12)* for more information. Notice that theorem 0.1 holds only for u defined in the exterior of a finite closed surface ∂D. For the situation in which there are infinite boundaries very little is known. See (16) for a survey of what has been done up until 1964 in this area of research.

We now expand solutions $\Phi(\rho, \phi, z)$ of (0.1) in a Fourier series in ϕ, that is,

$$\Phi(\rho,\phi,z) = \Phi_0(\rho,z) + \sum_{n=1}^{\infty} \Phi_n^c(\rho,z)\cos n\phi + \sum_{n=1}^{\infty} \Phi_n^s(\rho,z)\sin n\phi \tag{0.2}$$

Here
$$\begin{aligned} \Phi_n^c(\rho,z) &= \frac{1}{\pi}\int_0^{2\pi} \Phi(\rho,\phi',z)\cos n\phi'\,d\phi' \quad , n\geq 1 \\ \Phi_n^s(\rho,z) &= \frac{1}{\pi}\int_0^{2\pi} \Phi(\rho,\phi',z)\sin n\phi'\,d\phi' \quad , n\geq 1 \end{aligned} \tag{0.3}$$

and Φ_n^s and Φ_n^c satisfy the equation

$$\frac{\partial^2 \Phi_n}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \Phi_n}{\partial \rho} + \frac{\partial^2 \Phi_n}{\partial z^2} - \frac{n^2}{\rho^2}\Phi_n + k^2\Phi_n = 0 \tag{0.4}$$

We introduce the change of variables $\rho^n U_n(\rho, z) = \Phi_n(\rho, z)$ and (0.3) becomes

$$L_{n+1/2}(U_n) \equiv \frac{\partial^2 U_n}{\partial \rho^2} + \frac{2n+1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U_n}{\partial z^2} + k^2 U_n = 0 \tag{0.5}$$

In the above analysis $2n+1$ must be a positive, odd integer.

* p. 107-114.

However if we begin our study of equation (0.1) by expanding $\Phi(\rho, \phi, z)$ over a wedge shaped domain $0 \le \phi \le 2\pi\alpha$ (α is a real number) instead of $0 \le \phi \le 2\pi$, then it is seen that $2n+1$ can assume arbitrary positive real values depending on the choice of α.

For the situation in which the solution $\Phi(\rho, \phi, z)$ of (0.1) is considered to be defined and continuous in the exterior of a bounded domain, only solutions u of $L_{n+1/2}(u) = 0$ which are bounded along the ρ axis for $\rho > \rho_0$ (where ρ_0 is some positive number) are permitted. However if we consider unbounded domains (which arise for example if we wish to examine the field resulting from the diffraction of a plane electromagnetic wave by an infinite conducting cylinder whose axis is the line $\rho = 0$) solutions u of $L_{n+1/2}(u) = 0$ no longer need necessarily be bounded along $\rho = 0$ and we need to examine singular solutions of $L_{n+1/2}(u) = 0$. From the analytic theory of this equation (to be developed presently) it is seen that this is essentially the study of $L_{1/2-n}(u) = 0$. Hence investigation of questions of uniqueness for $L_{n+1/2}(u) = 0$ for $n < 0$ may provide clues as to sufficient criteria for uniqueness of solutions of the Helmholtz equation in unbounded domains.

Because of its intimate connection with the three dimensional Helmholtz equation, considerable attention has been devoted in recent years to the study of boundary value problems for equation (0.5) (see the survey article by Heins[11]). As of yet, however, questions of uniqueness of solutions has not been examined. This is due basically to the fact that for $n \ne -1/2$ the coefficient of $\frac{\partial u}{\partial \rho}$ in (0.5) is unbounded in any region containing a segment of the axis $\rho = 0$, and the mathematical study of boundary value problems involving such "singular" partial differential equations is of only very recent origin. It is the purpose of this thesis to examine questions of uniqueness for equation (0.5) where n is an arbitrary real number. To the author's knowledge this is the first time uniqueness theorems have been obtained for an exterior boundary value problem involving a singular partial differential equation.

Singular Partial Differential Equations

Ordinary linear differential equations with rational coefficients, such as Bessel's and Legendre's equation, made an early appearance in analysis and today their study forms an integral part of the theory of ordinary differential equations. A similar theory for singular partial differential equations (i.e. partial differential equations with rational coefficients) is still only partially developed even though mathematicians of the calibre of Appell, Goursat and Picard devoted their energies towards it. For the state of the art as of 1951, see the survey article by Erdelyi[4]. Recently considerable attention has been given to a class of singular partial differential equations involving Bessel's operator

$$D_\nu(u) \equiv \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} \tag{0.6}$$

since such equations arise frequently in many areas of mathematical physics. For more information on this important class of equations, see the survey article by Weinstein[25].

Singular partial differential equations are also of interest as a source of "improperly posed" problems, i.e. boundary value problems in which either existence, uniqueness, or continuous dependence upon the boundary data fails to hold. See[19] for some results along this line. The following quotation from ([2] p. 230) should indicate why the mathematician and physicist are interested in such seemingly pathological cases: "The stipulation about existence, uniqueness, and stability of solutions dominates classical mathematical physics. They are deeply inherent in the ideal of a unique, complete, and stable determination of physical events by appropriate conditions at the boundaries, at infinity, at times $t = 0$, or in the past. Laplace's vision of the possibility of calculating the whole future of the physical world from complete data of the present state is an extreme expression of this attitude. However, this rational ideal of casual-mathematical determination was gradually eroded by confrontation with physical reality. Nonlinear phenomena, quantum theory, and the advent of powerful numerical methods have shown that "properly posed" problems are by far not the only ones which appropriately reflect real phenomena. So far, unfortunately, little mathematical progress has been made in the important task of solving or even identifying and formulating such problems which are not "properly posed" but still are important and motivated by realistic situations."

Although questions of existence and uniqueness have played a fundamental and basic role in the study of partial differential equations whose coefficients are twice continuously differentiable, it is only very recently that a similar study has been initiated for the singular case[21, 19].

We now explain briefly the terminology used in the work that will follow. Consider the following partial differential equation of elliptic type in normal form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a(x,y)\frac{\partial u}{\partial y} + b(x,y)\frac{\partial u}{\partial x} + c(x,y)\,u = 0 \tag{0.7}$$

and assume that a, b and c are rational functions of x and y. Let $\phi(x, y)$ be an irreducible factor of one of the denominators. The algebraic curve $C = \phi(x, y) = 0$ is a <u>singular curve</u> of equation (0.7). A point (a, b) of C is called a <u>nonsingular point</u> of C if either $\phi_x(a, b) \neq 0$ or $\phi_y(a, b) \neq 0$ and it is called a general point of C if it is nonsingular and if C is the only singular curve of (0.7) which passes through that point. If there is a fundamental system of solutions $u_1(x, y)$, $u_2(x, y)$ which can be represented in the canonical form

$$\begin{aligned} u_1(x,y) &= \phi^{\rho_1} S_1(x-a, y-b) \\ u_2(x,y) &= \phi^{\rho_2} S_2(x-a, y-b) \end{aligned} \tag{0.8}$$

(or in special cases in a similar form containing terms in log ϕ, where the $S(x-a, y-b)$ are power series convergent in some neighbourhood of (a, b); and if such a representation holds in the neighbourhood of each general point (a, b) of C, with the ρ_i independent of a, b, x, y then $\phi(x\ y) = 0$ may be called a <u>singular curve of the regular type</u>.

From this point on we will consider equation (0.5) in the form

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{0.9}$$

where (x,y) are cartesian coordinates and ν is an arbitrary real number which can be either positive or negative.

The Analytic Theory of $L_\nu(u) = 0$.

In either of the half planes $y > 0$ and $y < 0$, (0.9) is an elliptic partial differential equation with analytic coefficients, and hence every twice continuously differentiable solution is an analytic function of x and y in each such half plane(2)*. The line $y = 0$, which will be called the axis is a singular curve of the regular type with exponents $\rho_1 = 0$, $\rho_2 = 1 - 2\nu$ (see equation (0.8)). Consequently, there are solutions of (0.9) which are regular on (some portion of) the axis. It is seen from the differnetial equation that $\partial u/\partial y = 0$ on the axis for such regular solutions. For $2\nu \neq -1,-2,-3$ each regular solution can be continued across the axis as an even function of y i.e. for $2\nu \neq -1,-2,-3..$ a regular solution is an analytic function of x and y^2 in some domain D that is symmetric with respect to the axis $y = 0$. if $2\nu = -1,-2,-3,..$ the assumption that u is an even function of y will be part of the definition of regularity, viz.

Def. 0.1 : A solution $u(x, y)$ of $L_\nu(u) = 0$ will be called regular if it is an analytic function of x and y^2 in some region which is symmetric with respect to the axis $y = 0$.

Theorem 0.2 : (The Correspondence Principle): Let $u^{(2\nu)}$ be a solution of $L_\nu(u) = 0$ in a region D not intersecting with $y = 0$. Then $u^+ = y^{2\nu-1} u^{(2\nu)}$ is a solution of $L_{1-\nu}(u^+) = 0$ and vice versa.

Proof: (25)

From theorem 0.2 we see that for $\nu \neq \frac{1}{2}$ a fundamental system

* p. 502.

of solutions of $L_\nu(u) = 0$ about a point $(a, 0)$ consists of a regular solution $u_1(x, y)$ of $L_\nu(u) = 0$ and a solution of the form $u_2(x, y) = y^{1-2\nu} u^+(x, y)$ where $u^+(x, y)$ is a regular solution of $L_{1-\nu}(u) = 0$. This holds even for 2ν equal to an integer (provided $2\nu \neq 1$) where we might have expected logarithmic solutions. Hence the study of singular solutions of $L_\nu(u) = 0$ for $\nu > 1/2$ is essentially the study of regular solutions of $L_\nu(u) = 0$ for $\nu < 1/2$. Note that the case when $2\nu = -1, -3, -5, \ldots$ is exceptional in the sense that any solution is an even function of y and hence regular.

__Theorem 0.3__: (The Identification Principle): Let $2\nu \neq -1, -3, -5, \ldots$ and let $u_1(x, y)$ and $u_2(x, y)$ be two regular solutions of $L_\nu(u) = 0$ in a region D which is symmetric with respect to the axis $y = 0$. If $u_1(x, y) = u_2(x, y)$ along some segment of the axis contained in D, then $u_1(x, y) \equiv u_2(x, y)$ throughout D.

__Proof__: (24, 25)

__Theorem 0.4__: (The Recursion Formula): Let $u^{(2\nu)}(x,y)$ be a solution of $L_\nu(u) = 0$ in a region D not intersecting with $y = 0$. Then $u^+(x,y) = \frac{1}{y} \frac{\partial u^{(2\nu)}(x,y)}{\partial y}$ is a solution of $L_{\nu+1}(u) = 0$ in D.

__Outline of Results__

Much of the analysis in this work is based on our first showing that any regular solution $u(x, y)$ of $L_\nu(u) = 0$ can be expanded in a Bessel-Jacobi series and then extracting properties of $u(x, y)$ from such an expansion. The use of such special function expansions can be considered analogous to the use of power series and Laurent expansions in the study of analytic functions of a complex variable.

The case $\nu = 0$ is always excluded since in this case $L_\nu(u) = 0$ reduces to the classical two dimensional Helmholtz equation in cartesian coordinates.

In chapters one and two we consider the following uniqueness question: Let $F = R_2 - \overline{D}$ where D is a normal domain symmetric with respect to the axis $y = 0$ and R_2 is the x-y plane. Let $u(x,y)$ be a regular solution of $L_\nu(u) = 0$ in $\overline{F}$ and suppose that $u(x,y) = 0$ on ∂D. Then under what conditions can we state that $u(x,y) = 0$ in F? Examples are given to show that $u(x,y) = 0$ on ∂D alone does not imply $u = 0$ in F. The following results are obtained:

1) Assume $\nu > -\frac{1}{2}$. If $\lim\limits_{r\to\infty} \int_0^\pi r^{2\nu+1} \sin^{2\nu}\theta \left|\frac{\partial u}{\partial r} - iku\right|^2 d\theta = 0$ (where $x = r\cos\theta$, $y = r\sin\theta$) then $u(x,y) = 0$. It is shown that if $u(x, y)$ is regular for $r > a$ the above radiation condition is equivalent to there existing a rectangle T (independent of r) enclosing $[-1, +1]$ in the complex $\xi = \cos\theta$ plane such that for each fixed $r > a$ $u(x, y) = \tilde{u}(r, \xi)$ is an analytic function of ξ in T and $\lim\limits_{r\to\infty} r^{\nu+\frac{1}{2}}\left(\frac{\partial\tilde{u}}{\partial r} - ik\tilde{u}\right) = 0$ uniformly for ξ contained in T.

2) Assume $\nu < -\frac{1}{2}$, $2\nu \neq -1, -3, -5, \ldots$. If there exists a rectangle T as described in 1) and if $\lim\limits_{r\to\infty} r^{\nu+\frac{1}{2}}\left(\frac{\partial\tilde{u}}{\partial r} - ik\tilde{u}\right) = 0$ uniformly for ξ contained in T then $u(x,y) = \tilde{u}(r,\xi) = 0$.

3) It is shown that for $\nu = -1, -2, -3, \ldots$ a solution to the exterior Dirichlet problem satisfying the conditions in 2) does not in general exist for given analytic boundary data even though

if a solution does exist it is unique. This is of added interest since it is often tacitly assumed or hoped that elliptic boundary value problems are well posed if uniqueness holds[7]. Such misguided hopes are encouraged by the fact that Perron's method yields both existence and uniqueness theorems provided one can derive a maximum modulus principle[2] p. 342.

In chapter three regular solutions of $L_\nu(u) = 0$ for the case $2\nu = -1, -3, -5, \ldots$ are decomposed into a function of the form $y^{1-2\nu}\, u^+(x,y)$ where $u^+(x,y)$ is a regular solution of $L_{1-\nu}(u) = 0$ and a function of the form $Q_1(x,y)e^{ikx} + Q_2(x,y)e^{-ikx}$ where $Q_1(x,y)$ and $Q_2(x,y)$ are polynomials in x and y^2. Hence the study of $L_\nu(u) = 0$ for such values of ν is reduced to examining $L_\nu(u) = 0$ for $2\nu \geq 3$.

In chapter four solutions $u(x,y)$ of $L_\nu(u) = 0$ which are regular in the whole plane are considered for $\nu < -\frac{1}{2}$, $2\nu \neq -1, -3, -5, \ldots$. Conditions are established to insure that if $\lim_{r\to\infty} r^{\nu+\frac{1}{2}}\, u(x,y) = 0$ pointwise for $\theta \in [0, \pi]$ $(x = r\cos\theta,$ $y = r\sin\theta)$ then $\psi(x,y) = 0$. It is noted that for $\nu > -\frac{1}{2}$ uniqueness theorems are easily obtainable by direct generalization of results for the classical Helmholtz equation.

Concluding Remarks

Previous work on the equation $L_\nu(u) = 0$ has been done by Henrici[13] and Gilbert and Howard[8], each of whom considered solutions of $L_\nu(u) = 0$ for $\nu > 0$ which are regular in some region containing the origin.

The author believes that results similar to those obtained in this thesis can be derived without serious difficulty for the following equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{x}\frac{\partial u}{\partial x} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{0.10}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{0.11}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\mu}{x}\frac{\partial u}{\partial x} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + \left\{k^2 + e^2/r\right\} u = 0 \tag{0.12}$$

Here μ, ν, k, e are real valued constants.

CHAPTER I

In this chapter we will investigate uniqueness questions for

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{1.1}$$

where ν and k are real numbers, $\nu > -\frac{1}{2}$, $k > 0$. More specifically we ask the following: Given a normal domain (see introduction) D symmetric with respect to the axis $y = 0$, $F = R_2 - \bar{D}$ (R_2 is two dimensional Euclidean space), u a regular solution of (1.1) (see introduction) such that $u = 0$ on ∂D, under what additional conditions is $u \equiv 0$ in F ? The fact that some additional condition is needed can be seen by the following example:

<u>Example 1.1</u>: Let $x = r\cos\theta$, $y = r\sin\theta$, $D = \{(x,y) \mid r \le a,\ a > 0\}$ and let $u(x,y) = \tilde{u}(r,\theta) = \frac{r^{-\nu}}{H_\nu^{(1)}(ka)} H_\nu^{(1)}(kr) - \frac{r^{-\nu}}{H_\nu^{(2)}(ka)} H_\nu^{(2)}(kr)$ where H denotes Hankel's function. On ∂D $\tilde{u}(a,\theta) = 0$ but $\tilde{u}(r,\theta) \not\equiv 0$ in $F = R_2 - \bar{D}$. $\tilde{u}(r,\theta)$ is a solution of $L_\nu(u) = 0$ which is independent of θ.

For $\nu = 0$ the following radiation condition assures uniqueness ([12] p. 108)

$$\lim_{r\to\infty} \int_0^{2\pi} r \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d\theta = 0 \tag{1.2}$$

In the more general case of equation (1.1) it will be shown that (1.2) must be replaced by

$$\lim_{r\to\infty} \int_0^{\pi} r^{2\nu+1} \sin^{2\nu}\theta \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d\theta = 0 \tag{1.3}$$

Observe that for $\nu = 0$ (1.3) reduces to (1.2). Henceforth in this chapter (1.3) will be referred to as the radiation condition.

Condition (1.3) is implied by stronger radiation conditions which resemble those first derived by Sommerfeld (see Introduction). If $\nu \geq 0$ we have

$$\left| \int_0^\pi r^{2\nu+1} \sin^{2\nu}\theta \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d\theta \right| \leq \pi \max_{0 \leq \theta \leq \pi} r^{2\nu+1} \sin^{2\nu}\theta \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2$$

and hence (1.3) holds if

$$\lim_{r \to \infty} r^{\nu + 1/2} \sin^\nu\theta \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right| = 0 \tag{1.4}$$

uniformly for $0 \leq \theta \leq \pi$. If $-\frac{1}{2} < \nu \leq 0$ we have

$$\int_0^\pi r^{2\nu+1} \sin^{2\nu}\theta \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d\theta \leq \max_{0 \leq \theta \leq \pi} r^{2\nu+1} \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 \int_0^\pi \sin^{2\nu}\theta \, d\theta$$

and hence (1.3) holds if

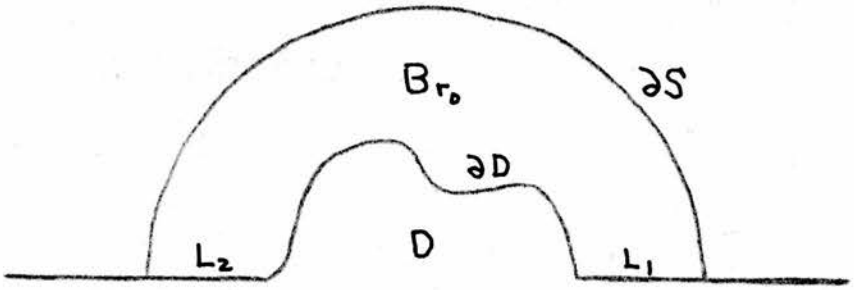
$$\lim_{r \to \infty} r^{\nu + 1/2} \left| \frac{\partial \tilde{u}}{\partial r} - iku \right| = 0 \tag{1.5}$$

uniformly for $0 \leq \theta \leq \pi$. For $\nu \geq 0$ condition (1.5) implies that condition (1.4) holds also.

Statement of Main Theorem and Outline of Proof.

Theorem 1.1: Let $F = R_2 - \bar{D}$ where D is a normal domain symmetric with respect to the axis $y = 0$. Let $u(x,y)$ be a regular solution of $L_\nu(u) = 0$ in $\bar{F}$. Assume $\nu > -\frac{1}{2}$, $k > 0$, and let the radiation condition (1.3) hold. Then if $u(x,y) = 0$ on ∂D, $u(x,y) \equiv 0$ in F .

<u>Outline of Proof</u>: Since all regular solutions of (1.1) for $\nu > -\frac{1}{2}$ are even functions of y (see Introduction) it is only necessary to examine solutions u of $L_\nu(u) = 0$ in the half plane $y \geq 0$. For r_0 sufficiently large let B_{r_0} be the region bounded by that part of ∂D lying in $y \geq 0$, portions of the axis L_1 and L_2, and the semicircle $\partial S = \left\{(x,y) \middle| r = r_0, y \geq 0\right\}$. See Figure 1.1 below.



<u>Figure 1.1</u>

First a representation for $u(x,y) = \tilde{u}(r,\theta)$ valid for $r \geq r_0$ will be obtained. Next it will be shown that if

$$\lim_{r_0 \to \infty} \int_{\partial S} y^{2\nu} |u(x,y)|^2 dS = 0 \tag{1.6}$$

then $u(x,y) \equiv 0$ in F. Finally Green's formula[5] will be applied to $u(x,y)$ in the region B_{r_0} and it will be shown that (1.6) is implied by the radiation condition.

<u>Preliminary Theorems</u>

Assume $\nu > -\frac{1}{2}$.

<u>Theorem 1.2</u>: Let $u(x,y)$ be a regular solution of $L_\nu(u) = 0$ for $r \geq r_0$ where $x = r \cos \theta$, $y = r \sin \theta$. Then if u satisfies the radiation condition (1.3), u can be expressed for $r \geq r_0$ / $\theta \in (0, \pi)$ as

$$u(x,y) = \tilde{u}(r,\theta) = r^{-\nu} \sum_{n=0}^{\infty} a_n H^{(1)}_{\nu+n}(kr)\, C_n^{\nu}(\cos\theta)$$

where $H^{(1)}_{\nu+n}$ is a Hankel function, C_n^{ν} a Gegenbauer polynomial and a_n, $n = 0,1,\ldots.$ are independent of r and θ.

Proof: For fixed $r \geq r_o$, $\tilde{u}(r,\theta) \in C^{(1)}([0,\pi])$ and hence the Fourier cosine series of $\sin^{\nu}\theta\, \tilde{u}(r,\theta)$ converges uniformly to $\sin^{\nu}\theta\, \tilde{u}(r,\theta)$ in $\varepsilon \leq \theta \leq \pi - \varepsilon$ where ε is a fixed positive number, $0 < \varepsilon < \pi$ ([26] p. 368, 410). Now since Gegenbauer polynomials are constant multiples of Jacobi polynomials ([3] VII p. 174) by the equiconvergence theorem for Jacobi series ([23] p. 239) it is seen that for $\theta \in (0,\pi)$ we have

$$\tilde{u}(r,\theta) = \sum_{n=0}^{\infty} h_n C_n^{\nu}(\cos\theta) \int_0^{\pi} \tilde{u}(r,\varphi) \sin^{2\nu}\varphi\, C_n^{\nu}(\cos\varphi)\, d\varphi$$

where C_n^{ν} denotes the Gegenbauer polynomial ([3] Vol. II, p. 174) and

$$h_n = \frac{2(n+\nu)\, n!\, \Gamma(\nu)}{\pi^{1/2}\, (2\nu)_n\, \Gamma(\nu+1/2)} \tag{1.7}$$

In polar coordinates equation (1.1) becomes

$$\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{2\nu+1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2} + \frac{2\nu}{r^2 \tan\theta}\frac{\partial \tilde{u}}{\partial \theta} + k^2 \tilde{u} = 0$$

Now let $v_n(r) = \displaystyle\int_0^{\pi} \tilde{u}(r,\varphi) \sin^{2\nu}\varphi\, C_n^{\nu}(\cos\varphi)\, d\varphi$

$$0 = \int_0^{\pi} C_n^{\nu}(\cos\varphi)\sin^{2\nu}\varphi \left\{ \frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{2\nu+1}{r}\frac{\partial \tilde{u}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{u}}{\partial \varphi^2} + \frac{2\nu}{r^2\tan\varphi}\frac{\partial \tilde{u}}{\partial \varphi} + k^2\tilde{u} \right\} d\varphi$$

$$= v_n'' + \frac{2\nu+1}{r} v_n' + k^2 v_n + \frac{1}{r^2}\int_0^{\pi} C_n^{\nu}(\cos\varphi)\sin^{2\nu}\varphi \left\{ \frac{2\nu}{\tan\varphi}\tilde{u}_{\varphi} + \tilde{u}_{\varphi\varphi} \right\} d\varphi$$

where $u_{\phi} = \frac{\partial u}{\partial \phi}$. Now consider the integral in the last term, setting $f(\phi) = C_n^{\nu}(\cos\phi)$ for notational convenience,

$$\int_0^\pi f(\varphi)\sin^{2\nu}\varphi\left[\frac{2\nu}{\tan\varphi}\tilde{u}_\varphi + \tilde{u}_{\varphi\varphi}\right]d\varphi = \int_0^\pi f(\varphi)\frac{\partial}{\partial\varphi}\left[\sin^{2\nu}\varphi\,\tilde{u}_\varphi\right]d\varphi$$

Now $\frac{\partial u}{\partial \phi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \phi}$ and $x = r\cos\phi$, $y = r\sin\phi$.

Hence $\frac{\partial u}{\partial \phi} = -y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = O(y)$ as $y \to 0$ by Weinstein's recursion formula (see Introduction.). Therefore

$$\int_0^\pi f(\varphi)\frac{\partial}{\partial\varphi}\left[\sin^{2\nu}\varphi\,\tilde{u}_\varphi\right]d\varphi = -\int_0^\pi f'(\varphi)\sin^{2\nu}\varphi\,\tilde{u}_\varphi\,d\varphi + f(\varphi)\sin^{2\nu}\varphi\,\tilde{u}_\varphi\Big|_0^\pi$$

The last term in the above line vanishes from the above discussion and therefore integrating by parts again we have

$$\int_0^\pi f(\varphi)\frac{\partial}{\partial\varphi}\left[\sin^{2\nu}\varphi\,\tilde{u}_\varphi\right]d\varphi = \int_0^\pi\left\{f''(\varphi)\sin^{2\nu}\varphi + 2\nu f'(\varphi)\sin^{2\nu-1}\varphi\cos\varphi\right\}\tilde{u}(r,\varphi)\,d\varphi$$

$$- f'(\varphi)\sin^{2\nu}\varphi\;\tilde{u}(r,\varphi)\Big|_0^\pi$$

By our notation $f'(\varphi) = \frac{d}{d\varphi}C_n^\nu(\cos\varphi) = -\sin\varphi\,\frac{d\,C_n^\nu(\cos\varphi)}{d(\cos\varphi)} = O(\sin\varphi)$

Hence the last term in the above line vanishes and from Gegenbauer's differential equation

$$(1-x^2)y'' - (2\nu+1)xy' + n(n+2\nu)y = 0 \tag{1.8}$$

we have

$$\int_0^\pi f(\varphi)\frac{\partial}{\partial\varphi}\left[\sin^{2\nu}\varphi\,\tilde{u}_\varphi\right]d\varphi = -n(n+2\nu)\int_0^\pi C_n^\nu(\cos\varphi)\sin^{2\nu}\varphi\,\tilde{u}(r,\varphi)\,d\varphi$$

$$= -n(n+2\nu)\,V_n(r)$$

Hence $v_n(r)$ satisfies

$$v'' + \frac{2\nu+1}{r} v' + \left(k^2 - \frac{n(n+2\nu)}{r^2}\right) v = 0 \tag{1.9}$$

i.e.

$$v_n(r) = a_n r^{-\nu} H^{(1)}_{\nu+n}(kr) + b_n r^{-\nu} H^{(2)}_{\nu+n}(kr)$$

where $H^{(1)}_{n+\nu}$, $H^{(2)}_{n+\nu}$ denote Hankel functions of the first and second kind respectively. Now we have, using the radiation condition and Schwartz's lemma, ?

$$\lim_{r\to\infty}\left| r^{\nu+1/2}\left(\frac{dv_n}{dr} - ikv_n\right)\right|^2 = \lim_{r\to\infty}\left|\int_0^\pi \left\{ r^{\nu+1/2} \sin^{2\nu}\varphi \left(\frac{\partial \tilde{u}}{\partial r} - ik\tilde{u}\right) C_n^\nu(\cos\varphi)\, d\varphi\right|^2$$

$$\leq \lim_{r\to\infty} \int_0^\pi r^{2\nu+1} \sin^{2\nu}\varphi \left|\frac{\partial \tilde{u}}{\partial r} - ik\tilde{u}\right|^2 d\varphi \int_0^\pi \sin^{2\nu}\varphi \left[C_n^\nu(\cos\varphi)\right]^2 d\varphi$$

$$= 0$$

From [3](Vol. II, pp. 79, 85, 31) we have

$$\frac{d}{dr}\left[r^{-\nu} H^{(1)}_\nu(kr)\right] = -kr^{-\nu} H^{(1)}_{\nu+1}(kr)$$

$$\frac{d}{dr}\left[r^{-\nu} H^{(2)}_\nu(kr)\right] = -kr^{-\nu} H^{(2)}_{\nu+1}(kr) \tag{1.10}$$

$$H^{(1)}_\nu(kr) = \sqrt{\frac{2}{k\pi r}}\, e^{i(kr - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} + O(r^{-3/2}), \quad r\to\infty$$

$$H^{(2)}_\nu(kr) = \sqrt{\frac{2}{k\pi r}}\, e^{-i(kr - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)} + O(r^{-3/2}), \quad r\to\infty \tag{1.11}$$

Hence for large r we have

$$r^{\nu+1/2}\left(\frac{dv_n}{dr} - ikv_n\right) = a_n\left[O\left(\frac{1}{r}\right)\right] + b_n\left[-2ik\sqrt{\frac{2}{\pi k}}\, e^{-i(kr - \frac{1}{2}(\nu+n+1)\pi - \frac{1}{4}\pi)} + O(r^{-1})\right]$$

Since this last expression must tend to zero as r tends to infinity, conclude that $b_n = 0$. Hence our theorem is proved.

The next theorem we are going to prove can be considered as an analogue of Rellich's lemma[12] (p. 109) for the classical Helmholtz equation.

Theorem 1.3 (Rellich): Let $V(x) \in C^2$ for $|x| > r_o$ ($r_o > 0$, $x = (x_1, \ldots, x_n)$) be a (complex valued) solution of $\Delta_n V + k^2 V = 0$ $\left(\Delta_n \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)$ with $k > 0$. Let

$$\lim_{r\to\infty} \int_{|x| = r} |V|^2 \, dS = 0$$

where dS denotes an element of surface area on the sphere $|x| = r$. Then $V(x) \equiv 0$.

Theorem 1.4: Assume $u(x,y) = \tilde{u}(r,\theta)$ is a regular solution of $L_\nu(u) = 0$ in F and the radiation condition (1.3) holds. Let $\nu > -\frac{1}{2}$. If

$$\lim_{r\to\infty} r^{2\nu+1} \int_0^\pi \sin^{2\nu}\phi \, |\tilde{u}(r,\phi)|^2 \, d\phi = 0$$

then $u(x,y) \equiv 0$.

Proof: $u(x,y) = \tilde{u}(r,\theta) = r^{-\nu} \sum_{n=0}^{\infty} a_n H^{(1)}_{\nu+n}(kr)\, C^{\nu}_n(\cos\theta)$ from theorem 1.2. Parseval's relation for series of Gegenbauer polynomials gives us

$$\sum_{n=0}^{\infty} h_n^{-1} \left| r^{-\nu} a_n H^{(1)}_{\nu+n}(kr) \right|^2 = \int_0^\pi \sin^{2\nu}\varphi \, |\tilde{u}(r,\varphi)|^2 d\varphi$$

where h_n is defined by (1.7). Hence

$$r^{2\nu+1}\int_0^\pi \sin^{2\nu}\varphi\,|\tilde{u}(r,\varphi)|^2 d\varphi = r\sum_{n=0}^\infty h_n^{-1}\,|a_n H_{\nu+n}^{(1)}(kr)|^2$$

Therefore $\forall \epsilon > 0, \exists r_0$, such that for $r > r_0$

$$r\,h_n^{-1}|a_n H_{n+\nu}^{(1)}(kr)|^2 < \epsilon \tag{1.12}$$

But from (1.11) we have

$$H_{\nu+n}^{(1)}(kr) = \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-(\nu+n)\pi/2-\pi/4)} + O(r^{-3/2}),\ r\to\infty$$

Hence conclude $|a_n| < \sqrt{\epsilon\, h_n}\sqrt{\frac{\pi k}{2}}$ and since ϵ was arbitrary $a_n = 0$, $n = 0,1,2,\ldots$, which implies $u \equiv 0$ for $r > r_0$. By analyticity of $u(x,y)$, $u(x,y) \equiv 0$.

We are now in a position to prove theorem 1.1.

Proof of Theorem 1.1

It is convenient to consider two separate cases, that for $\nu > 0$ and $-\frac{1}{2} < \nu < 0$. For $\nu \geq 0$ Green's formula can be applied directly. Green's formula for equation (1.1) is [5]

$$\int_{\partial B_{r_0}} y^{2\nu}\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)ds = \int_{B_{r_0}} y^{2\nu}\left(g\,L(f) - f\,L(g)\right)dx\,dy \tag{1.13}$$

(See Figure 1.1) where $f,g \in C^{(2)}(\overline{B}_{r_0})$, $\frac{\partial}{\partial n}$ indicates differentiation in the direction of the exterior normal of ∂B_{r_0}, $L(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y}$ and ∂B_{r_0} is traversed in a counterclockwise direction. Now if f and g are regular solutions of $L_y(u) = 0$, then $\frac{\partial f}{\partial n} = \frac{\partial g}{\partial n} = 0$ along the axis $y = 0$ since f and g are even functions of y. Let $f = g = 0$ on ∂D. Equation (1.13) now becomes

$$\int_{\partial S} y^{2\nu} \left(g \frac{\partial f}{\partial r} - f \frac{\partial g}{\partial r} \right) ds = \int_{B_{r_0}} y^{2\nu} \left(g L(f) - f L(g) \right) dx\, dy \tag{1.14}$$

For $-\frac{1}{2} < \nu < 0$ Green's formula is valid only if the path of integration does not contain points of the axis $y = 0$ since in this case $y^{2\nu}$ in (1.13) becomes infinite. We have however the following

$$\int_{\partial \Sigma} y^{2\nu} \left(g \frac{\partial f}{\partial n} - f \frac{\partial g}{\partial n} \right) ds = \int_{\Sigma} y^{2\nu} \left(g L(f) - f L(g) \right) dx\, dy \tag{1.15}$$

where $\partial \Sigma = L_1 + C_1 + L_2 + C_2$ where C_2 lies on ∂D (see Figure 1.2 below).

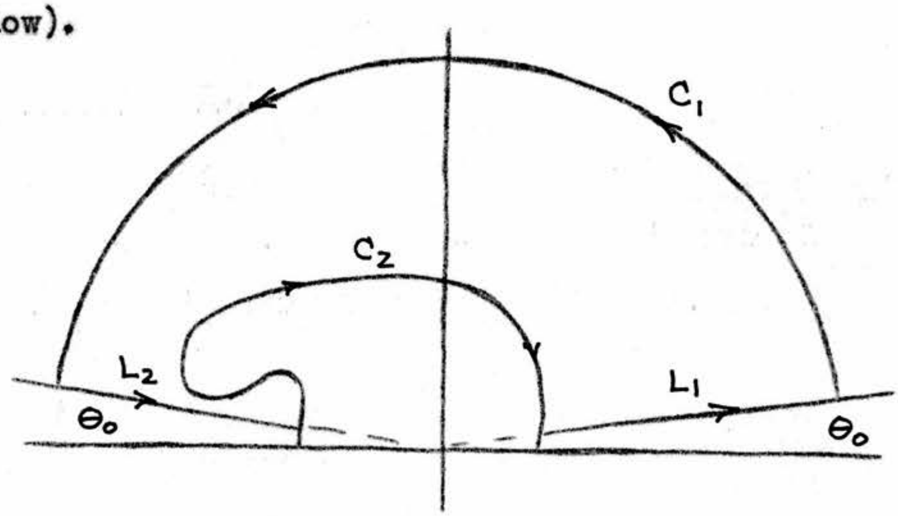


Figure 1.2

Now in (1.15) assume f, g are bounded for $y \geq 0$ and $\frac{\partial f}{\partial y}$, $\frac{\partial g}{\partial y} = O(y)$ as $y \to 0$. Now let $\theta_0 \to 0$ in Figure 1.2 and conclude that formula (1.13) is valid in this case also since the integrals along L_1, L_2 vanish in the limit. Now note that Weinstein's recursion formula (see Introduction) implies that for regular solutions u of $L_\nu(u) = 0$ we have $\frac{\partial u}{\partial y}$, $\frac{\partial \bar{u}}{\partial y} = O(y)$ as $y \to 0$. Hence conclude formula (1.14) is valid for $f = u$, $g = \bar{u}$ / where $u \in C^{(2)}(\overline{B_{r_0}})$, $L_\nu(u) = 0$. Let

$$R(r_0) = \int_{\partial S} y^{2\nu} \left|\frac{\partial u}{\partial r} - iku\right|^2 dS = \int_{\partial S} y^{2\nu} \left(\frac{\partial \bar{u}}{\partial r} - ik\bar{u}\right)\left(\frac{\partial u}{\partial r} - iku\right) dS$$

$$= \int_{\partial S} y^{2\nu} \left|\frac{\partial u}{\partial r}\right|^2 dS + k^2 \int_{\partial S} y^{2\nu} |u|^2 dS + ik \int_{\partial S} y^{2\nu} \left(\bar{u}\frac{\partial u}{\partial r} - u\frac{\partial \bar{u}}{\partial r}\right) dS$$

Now apply (1.14) with $f = u$, $g = \bar{u}$ and arrive at

$$R(r_0) = \int_{\partial S} y^{2\nu} \left|\frac{\partial u}{\partial r}\right|^2 dS + k^2 \int_{\partial S} y^{2\nu} |u|^2 dS \tag{1.16}$$

Since $\lim_{r_0 \to \infty} R(r_0) = 0$ and both terms in (1.16) are positive conclude that

$$\lim_{r_0 \to \infty} \int_{\partial S} y^{2\nu} |u(x,y)|^2 \, dS = 0 \quad .$$

By theorem 1.4 this implies $u(x,y) \equiv 0$.

CHAPTER II

In this chapter we will investigate uniqueness questions for

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{2.1}$$

where $\nu < -\frac{1}{2}$, $k > 0$. More specifically we ask the following: Given a normal domain (see introduction) D symmetric with respect to the axis $y = 0$, F the exterior of $\bar{D}$, u a regular solution (see introduction) of (2.1) in $\overline{F}$ such that $u = 0$ on ∂D, under what additional condition is $u \equiv 0$ in F? Here it will always be assumed that 2ν is not a negative odd integer since in this case all solutions of (2.1) are regular and up to a function of the form $y^{1-2\nu} u^+$ where u^+ is a regular solution of $L_{1-\nu}\, u = 0$ the general solution can be explicitly expressed in terms of $1-2\nu$ arbitrary constants (see chapter three).

For $\nu > -\frac{1}{2}$ uniqueness theorems for (2.1) depend on expanding $u(x,y) = \tilde{u}(r, \cos\theta)$ in a Gegenbauer series for fixed r (see chapter one. In chapter one $\tilde{u}$ denoted u as a function of r and θ. Due to the fact that u is an even function of y, u can also be expressed as a function of r and $\cos\theta$ and in this chapter $\tilde{u}$ denotes this latter function):

$$u(x,y) = \tilde{u}(r, \cos\theta) = \sum_{n=0}^{\infty} a_n(r)\, C_n^\nu(\cos\theta) \tag{2.2}$$

the $a_n(r)$ are then determined from the following representation formula

$$a_n(r) = h_n^{-1} \int_{-1}^{1} (1-\cos^2\theta)^{\nu-1/2}\, C_n^\nu(\cos\theta)\, \tilde{u}(r,\cos\theta)\, d(\cos\theta) \tag{2.3}$$

where h_n is a normalization constant depending only on n. It was shown in chapter one that for $\nu > -\frac{1}{2}$ the following radiation condition assures uniqueness:

$$\lim_{r \to \infty} \int_{-1}^{1} r^{2\nu+1} (1-\cos^2\theta)^{\nu-1/2} \left| \frac{\partial \tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d(\cos\theta) = 0 \tag{2.4}$$

where $x = r \cos \theta$, $y = r \sin \theta$.

For $\nu < -\frac{1}{2}$ the methods outlined above cannot be applied immediately since the weight function $(1 - \cos^2\theta)^{\nu-\frac{1}{2}}$ in (2.3) and (2.4) no longer is integrable over $[-1, +1]$. This can be dealt with by considering contour integrals instead of real integrals. In this sense our work bears some resemblance to that of Mackie(17). Much of the forthcoming analysis is devoted to considering equations similar to (2.2) and (2.3) when $\xi = \cos\theta$ is complex valued. It will be shown that if $u(x,y)$ is a regular solution of $L_\nu(u) = 0$ for $r > a$ then for fixed $r > a$ $\tilde{u}(r,\xi)$ can be extended to an analytic function of ξ in some rectangle in the complex ξ-plane enclosing the points ± 1. For $\nu < -\frac{1}{2}$ $(2\nu \neq -1, -3, -5, \ldots)$ the following conditions are shown to ensure uniqueness:

1) There exists a rectangle T, independent of r, enclosing $[-1, +1]$ in the complex ξ-plane such that for each fixed $r > a$ $\tilde{u}(r, \xi)$ is an analytic function of ξ in T.

2) $\lim_{r \to \infty} r^{\nu+\frac{1}{2}} \left(\frac{\partial \tilde{u}}{\partial r} - ik\tilde{u}\right) = 0$ uniformly in T . (2.5)

Finally for $\nu > -\frac{1}{2}$ it is shown that condition (2.4) is equivalent to conditions 1) and 2) above.

Statement of Main Theorem and Outline of Proof

Theorem 2.1: Assume $\nu < -\frac{1}{2}$, $2\nu \neq -1, -3, -5, \ldots$. Let $F = R_2 - \overline{D}$/ where D is a normal domain symmetric with respect to the axis $y = 0$. Let $u(x,y) = \tilde{u}(r, \cos\theta)$ be a regular solution of $L_\nu(u) = 0$ in $\overline{F}$ and hence regular for $r > a$ where a is large enough. Suppose that

1) there exists a rectangle T, independent of r, enclosing $[-1, +1]$ in the complex ξ plane such that for each fixed $r > a$ $\tilde{u}(r,\xi)$ $(\xi = \cos\theta)$ is an analytic function of ξ in T.

2) $\lim\limits_{r \to \infty} r^{\nu+\frac{1}{2}} \left(\frac{\partial\tilde{u}}{\partial r} - ik\tilde{u}\right) = 0$ uniformly for ξ in T.

Then if $u(x, y) = 0$ on ∂D, $u(x,y) \equiv 0$ in F.

Outline of Proof: As pointed out in chapter one it is only necessary to examine solutions u of $L_\nu(u) = 0$ in the half plane $y \geq 0$. An expansion theorem for $u(x,y)$ in terms of a Bessel-Jacobi series valid for $r > a$ where a is sufficiently large will be obtained. Using this series an asymptotic expansion for $\tilde{u}(r, \xi)$ and $\frac{\partial\tilde{u}(r,\xi)}{\partial r}$ will be obtained for large r. By the use of Green's formula it will be shown that if $u = 0$ on ∂D then for $0 < \theta_0 < \pi$

$$\int_{\theta_0}^{\pi-\theta_0} r^{2\nu+1} \sin^{2\nu}\theta \left(\tilde{u}\frac{\partial\overline{\tilde{u}}}{\partial r} - \overline{\tilde{u}}\frac{\partial\tilde{u}}{\partial r}\right) d\theta = 0 \tag{2.6}$$

for r large enough where $\overline{\tilde{u}}$ denotes the complex conjugate of $\tilde{u}$. Finally by using (2.6) coupled with the above mentioned asymptotic expansion we will show that $u(x, y) \equiv 0$.

Preliminary Theorems

Theorem 2.2: Suppose $-\infty < \nu < +\infty$, $-\infty < k < \infty$ and let $u(x,y)$ be a regular solution of $L_\nu(u) = 0$ for $r_1 < r < r_2$ where $r = +\sqrt{x^2+y^2}$. Then for every closed interval $[r_3, r_4]$ contained in (r_1, r_2) there exists a neighbourhood T of $[-1, +1]$ in the complex $\xi = \cos\theta$ plane such that $u(x,y) = \tilde{u}(r,\xi)$ is an analytic function of ξ for $r \in [r_3, r_4]$, $\xi \in T$.

Proof: Since $u(x,y)$ is regular for $r_1 < r < r_2$, $u(x,y)$ is an analytic function of x and y^2 in this annulus (see introduction). Since we have

$$x = r\cos\theta = r\xi$$

$$y^2 = r^2\sin^2\theta = r^2(1-\xi^2) \tag{2.7}$$

we can conclude that for $r_1 < r < r_2$, $\xi \in [-1, +1]$, $\tilde{u}(r,\xi)$ is an analytic function of r and ξ since an analytic function of an analytic function is analytic ((1) p. 136). Hence for each fixed $\xi_0 \in [-1, +1]$, $r_1 < r_0 < r_2$, we can write

$$\tilde{u}(r,\xi) = \sum_{m,n} a_{mn}(r-r_0)^m(\xi-\xi_0)^n \tag{2.8}$$

for $|r-r_0| < \delta(ro, \xi_0)$, $|\xi - \xi_0| < \eta(r_0\ \xi_0)$. Hence about each point $\xi_0 \in [-1, +1]$ there exists a disc of radius $\eta(r_0, \xi_0)$ such that $\tilde{u}(r, \xi)$ is analytic in thisdisc for $|r-r_0| < \delta(r_0, \xi_0)$. By the Heine Borel theorem we can pick out a finite number of these discs which cover $[-1, +1]$. Let these discs have their centres at the points $\xi_1, \ldots, \xi_k$. For each ξ_k we have associated with it the positive number $\delta(r_o, \xi_k)$. Let

$$\delta(r_0) = \min \{ \delta(r_0, \xi_\ell) : 0 \leq \ell \leq k \} \tag{2.9}$$

then for each fixed r, $|r - r_o| < \delta(r_o)$, $\tilde{u}(r,\xi)$ can be continued analytically into this finite union of discs which cover $[-1, +1]$. Now consider the closed interval $[r_3, r_4]$ contained in (r_1, r_2). For each fixed r_o in $[r_3, r_4]$ there exists a $\delta(r_o)$ such that for $|r-r_o| < \delta(r_o)$ $\tilde{u}(r,\xi)$ can be continued analytically into some region in the ξ plane enclosing $[-1, +1]$. By the Heine- Borel theorem there exists a finite number of these intervals $|r-r_j| < \delta(r_j)$, $0 \leq j \leq m$, which cover $[r_3, r_4]$. Let T_j be the region into which $\tilde{u}(r,\xi)$ can be continued analytically for $|r-r_j| < \delta(r_j)$ and let $T = \bigcap_{j=1}^{m} T_j$. This T satisfies the requirements for T in the theorem and hence we are done.

Szego([23] p. 238) has shown that if $f(x)$ is an analytic function on the closed segment $[-1, +1]$ then $f(x)$ can be expanded in a Gegenbauer series which is convergent in the interior of the greatest ellipse with foci at ± 1 in which $f(x)$ is regular. There it is assumed that the index ν of the Gegenbauer polynomials $C_n^\nu(x)$ is greater than $-\frac{1}{2}$. Since the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are well defined for $\alpha, \beta \neq 0, -1, -2, \ldots,$ and

$$C_n^\nu(x) = \frac{\Gamma(\nu + 1/2)\, \Gamma(n+2\nu)}{\Gamma(2\nu)\, \Gamma(n+\nu+1/2)} P_n^{(\nu-1/2,\, \nu-1/2)}(x) \tag{2.10}$$

the question arises as to the possibility of expanding an arbitrary analytic function in a Jacobi series of equal negative non-integer indices. In view of theorem 2.2 and recalling the analysis of chapter one, this is the case of interest in the study of regular solutions of $L_\nu(u) = 0$ for $\nu < -\frac{1}{2}$, $2\nu \neq -1, -3, -5, \dots$. The following theorem shows that the above mentioned expansion is in general impossible for $\nu = -1, -2, -3, \dots$.

<u>Theorem 2.3</u>: Let $\nu = -1,-2,-3,\dots$ and $f(x)$ be a polynomial of degree -2ν. Then it is not possible to expand $f(x)$ in a Jacobi series

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x)$$

where the series converges in some region containing $[-1, +1]$.

<u>**Proof**</u>: From ([23] p. 61) we have

$$P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x) = \binom{n+\nu-\frac{1}{2}}{n} {}_2F_1\left(-n, n+2\nu ; \nu+\tfrac{1}{2} ; \tfrac{1-x}{2}\right) \tag{2.11}$$

where ${}_2F_1$ denotes the hypergeometric function. Hence if ν is a negative integer $P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x)$ is of degree n for $0 \le n \le -\nu$ and $-2\nu+1 \le n$ whereas $P_n^{(\nu-\frac{1}{2},--\frac{1}{2})}(x)$ is of degree $-n-2\nu$, and hence of degree strictly less than n, for $-\nu+1 \le n \le -2\nu$. Since $f(x)$ is a polynomial of degree -2ν it is therefore impossible to expand $f(x)$ in a Jacobi series with a finite number of terms. Now suppose it were possible to expand $f(x)$ in an infinite series of Jacobi polynomials, i.e.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x) \tag{2.12}$$

where there does not exist an N such that, for $n > N$, $a_n = 0$ and the series (2.12) converges in some region containing $[-1, +1]$. Let C be a simple closed curve lying in this region and enclosing $[-1, +1]$. From ([23], p. 245) we have for $\alpha > -\frac{1}{2}$

$$\frac{1}{\pi i}\int_C (y^2-1)^{\alpha-1/2}\, Q_n^{(\alpha-1/2,\alpha-1/2)}(y)\, P_m^{(\alpha-1/2,\alpha-1/2)}(y)\, dy = \delta_{mn}\, g_n^\alpha$$

$$g_n^\alpha = \frac{2^{2\alpha-1}}{n+\alpha}\,\frac{\Gamma^2(n+\alpha+1/2)}{\Gamma(n+1)\,\Gamma(n+2\alpha)} \tag{2.13}$$

where $Q_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(y)$ denotes a Jacobi function of the second kind. If we expand $f(x)$ in a Jacobi series for some (fixed) $\alpha > -\frac{1}{2}$ and apply (2.13) conclude (since $f(x)$ is a polynomial of degree -2ν) that for $n > -2\nu$

$$\int_C f(y)(y^2-1)^{\alpha-1/2}\, Q_n^{(\alpha-1/2,\alpha-1/2)}(y)\, dy = 0 \tag{2.14}$$

By analytic continuation with respect to α (2.13) and (2.14) holds true for $\alpha -\frac{1}{2}$ not equal to a negative integer, in particular for $\alpha = \nu$. Now note that due to the asymptotic expansion of the $P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x)$ ([23], p. 195) the series (2.12) converges uniformly in every compact subset of its ellipse of convergence and hence termwise integration is permissible. (2.13) and (2.14) now imply that in the series (2.12) $a_n = 0$ for $n > -2\nu$ which is a contradiction.

We next derive an expansion theorem for solutions $u(x,y)$ of

$L_\nu(u) = 0$ which satisfy conditions 1) and 2) of theorem 2.1. This could have been accomplished by extending the expansion theorem of Szego as mentioned on page 27 to the case when $\nu < -\frac{1}{2}$, $2\nu \neq 1, -2, -$ ... . Then by using the orthogonality property (2.13) the general expansion theorem for $u(x,y)$ could have been derived using the method of theorem 1.2 since $Q_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x)$ satisfies the same ordinary differential equation as $C_n^\nu(x)$. However the following approach allows us to handle the case ν equal to a negative integer (which by theorem 2.3 is excluded in the method outlined above) and thereby bring to light the improperly posed problems which arise in this situation.

__Theorem 2.4:__ Let $u(x,y) = \tilde{u}(r, \cos\theta)$ be a regular solution of $L_\nu(u) = 0$ for $r > a > 0$ and $2\nu \neq -1, -3, -5, \ldots$. Suppose further that $u(x,y)$ satisfies hypotheses 1) and 2) of theorem 2.1. Then for any $b > a$ and $r \geq b > a$ we have

$$u(x,y) = \tilde{u}(r, \cos\theta) = r^{-\nu} \sum_{n=0}^{\infty} a_n \frac{H_{\nu+n}^{(1)}(kr)}{H_{\nu+n}^{(1)}(kb)} P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\cos\theta)$$

__Proof:__ Let m be an integer greater than or equal to $-2\nu+1$. Since by hypothesis for each fixed $r > a$ $\tilde{u}$ is an analytic function of $\xi = \cos\theta$ in T, so is $\frac{\partial^m \tilde{u}}{\partial \xi^m}$. Now expand $\frac{\partial^m \tilde{u}}{\partial \xi^m}$ in a Jacobi series of index $m + \nu - \frac{1}{2}$ for fixed $r = b$ ((23), p. 238),

$$\frac{\partial^m \tilde{u}}{\partial \xi^m} = \sum_{n=m}^{\infty} a_{n-m} P_{n-m}^{(m+\nu-\frac{1}{2},\, m+\nu-\frac{1}{2})}(\xi) \tag{2.15}$$

From ([23], p. 238) the above series converges uniformly in some ellipse containing $[-1, +1]$ and in view of the fact that $|P_n^{(\alpha,\beta)}(\xi)|^{1/n} > 1$ for n large enough and $\xi \notin [-1, +1]$ ([23] p. 195) we can use the root test to conclude that $\overline{\lim}_{n\to\infty} |a_{n-m}|^{1/n} < 1$. This implies the weaker estimate $|a_{n-m}| = O(n^{-p})$ for any fixed integer p. We now use the following relationship between the Jacobi polynomials ([3], Vol. II, p. 170):

$$2^m D^m P_n^{(\alpha,\alpha)}(x) = (n+2\alpha+1)_m \, P_{n-m}^{(\alpha+m,\alpha+m)}(x) \;; \quad m = 1, \cdots, n \tag{2.16}$$

where $D = \frac{d}{dx}$. Hence for $n \geq m$

$$\underbrace{\int_1^\xi \cdots \int_1^\xi}_{m \text{ times}} P_{n-m}^{(m+\nu-1/2,\, m+\nu-1/2)}(\xi)\, d\xi = 2^{-m}(n+2\nu)_m \left[P_n^{(\nu-1/2,\, \nu-1/2)}(\xi) + f_n(\xi) \right] \tag{2.17}$$

where $f_n(\xi)$ is a polynomial of degree at most $m-1$. (Define

$$\gamma_k = 2^{k-m}(n+2\nu)_{m-k} \, P_{n-m+k}^{(\nu-1/2+m-k,\, \nu-1/2+m-k)}(1)$$

$$I_k(\xi) = \underbrace{\int_1^\xi \cdots \int_1^\xi}_{k \text{ times}} d\xi \quad ; \qquad I_0(\xi) = 1$$

By (2.16) we have

$$2^{-m}(n+2\nu)_m \int_1^\xi P_{n-m}^{(m+\nu-1/2,\, m+\nu-1/2)}(\xi)\, d\xi = \int_1^\xi D^m P_n^{(\nu-1/2,\, \nu-1/2)}(\xi)\, d\xi$$

$$= D^{m-1} P_n^{(\nu-1/2,\, \nu-1/2)}(\xi) - \gamma_1$$

Repeating this operation m times we have formula (2.17) where

$$f_n(\xi) = -\sum_{\ell=0}^{m-1} \gamma_{m-\ell}\, I_\ell(\xi)$$

).

Since from ([3], VII, p. 169) we have

$$P_n^{(\alpha,\alpha)}(1) = \frac{(\alpha+1)_n}{n!} = O(n^{\alpha}) \tag{2.18}$$

and $(n+2\nu)_{m-k} = O(n^{m-k})$ we can conclude from the above representation of $f_n(\xi)$ that if we express $f_n(\xi)$ as $f_n(\xi) = \sum_{k=0}^{m-1} \beta_k \xi^k$ then $\beta_k = O(n^{\nu-\frac{1}{2}})$ for $0 \le k \le m-1$. Therefore since $a_{n-m} = O(n^{-p})$ for any integer p it is possible to integrate (2.15) termwise m times and then rearrange the series to get

$$\tilde{u}(r,\xi) = \sum_{n=m}^{\infty} b_n(r) P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi) + h(r,\xi) = S(r,\xi) + h(r,\xi) \tag{2.19}$$

where $h(r,\xi)$ is a polynomial of degree at most $m-1$ in ξ with coefficients depending on r and where $b_n(r) = O(n^{-p})$ for any integer p. Note that $S(r,\xi)$ is an analytic function of ξ (for fixed r) in the rectangle T since $\tilde{u}(r,\xi)$ and $h(r,\xi)$ both have this property. Now for $\alpha - \frac{1}{2}$ not equal to a negative integer by reasoning exactly as in the derivation of equation (2.14), we have for $n \ge m$

$$\int_C h(r,\xi)(\xi^2-1)^{\alpha-\frac{1}{2}} Q_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\xi)\, d\xi = 0 \tag{2.20}$$

where C is a smooth curve surrounding $[-1, +1]$ and lying within the rectangle T. Setting $\alpha = \nu$ in (2.20) we have from (2.13) and (2.19) that

$$b_n(r) = (\pi i \; g_n^{\nu})^{-1} \int_C \tilde{u}(r,\xi)(\xi^2-1)^{\nu-\frac{1}{2}} Q_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\, d\xi$$

where g_n^ν is defined in (2.13). Since the curve C can be chosen independent of r we can apply the methods of theorem 1.2 and conclude from the partial differential equation and the radiation condition that

$$b_n(r) = b_n \frac{H^{(1)}_{\nu+n}(kr)}{H^{(1)}_{\nu+n}(kb)} \qquad r \geq b > a$$

where b_n is independent of r. Since $S(r,\xi)$ is an analytic function of ξ for $r = b$ and $b_n(b) = b_n$ conclude that $b_n = O(n^{-p})$ for any integer p just as we concluded that $a_{n-m} = O(n^{-p})$ in (2.15). For real ξ $|P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)| = O(n^{-\frac{1}{2}})$ uniformly for $\xi \in [-1, +1]$ ([23], p. 164) and

$$\left| \frac{H^{(1)}_{\nu+n}(kr)}{H^{(1)}_{\nu+n}(kb)} \right| = O\left(\frac{b}{r}\right)^n$$

uniformly for r on compact subsets of $[b, \infty)$ [10]. Hence for each fixed r_o, $b < r_o < \infty$, $S(r,\xi)$ is dominated over the interval $[b, r_o]$ by

$$\sum_{n=m}^{\infty} n^{-p}\left(\frac{b}{r}\right)^n \qquad b \leq r \leq r_o, \ p \text{ any integer}$$

This series converges uniformly for $b \leq r \leq r_o$. Similar arguments show that the series obtained by applying $\frac{\partial}{\partial\theta}$, $\frac{\partial^2}{\partial\theta^2}$, $\frac{\partial}{\partial r}$, $\frac{\partial^2}{\partial r^2}$ termwise to $S(r, \xi)$ converge uniformly. Hence the operator L_ν can be applied to $S(r, \xi)$ termwise with the result that $L_\nu(S) = 0$ for $r \geq b$, $\theta \in [0, 2\pi]$ since r_o can be taken arbitrarily large. By methods identical to theorem 2.5 (which follows this theorem) it is seen that $S(r,\xi)$ satisfies conditions 1) and 2) of theorem 2.1. Hence $h(r,\xi)$ is a polynomial in ξ of degree at most m-1 with coefficients depending on r which satisfies $L_\nu(h) = 0$ for $r \geq b$, $\theta \in [0, 2\pi]$ and also the radiation condition[25]. Our proof will be complete if we show that $h(r,\xi)$ is of the form

$$h(r,\xi) = r^{-\nu} \sum_{n=0}^{m-1} c_n \frac{H^{(1)}_{\nu+n}(kr)}{H^{(1)}_{\nu+n}(kb)} P_n^{(\nu-1/2,\nu-1/2)}(\xi)$$

For ν not equal to a negative integer this is immediate since we can expand $h(r,\xi)$ in a Jacobi series of a finite number of terms and then use (2.13) and the methods of theorem (1.2). Therefore consider the case in which ν equals a negative integer. In this situation we set $m = -2\nu+1$ and go to the partial differential equation.

The polynomial $h(r,\xi)$ can be written in the form

$$h(r,\xi) = \sum_{n=0}^{-2\nu} c_n(r) \cos n\theta \tag{2.21}$$

Applying L_ν to $h(r,\xi)$ we have

$$0 = \sum_{n=0}^{-2\nu} \left[c_n'' + \frac{2\nu+1}{r} c_n' + \left(k^2 - \frac{n^2}{r^2}\right) c_n \right] \cos n\theta - c_n \frac{2\nu n}{r^2 \tan\theta} \sin n\theta \tag{2.22}$$

Now $\dfrac{\sin n\theta}{\tan\theta} = \dfrac{\sin n\theta}{\sin\theta}\cos\theta = U_{n-1}(\cos\theta)\, T_1(\cos\theta)$ where T_n, U_n denote Tchebichef polynomials of the first and second kind respectively ([3], Vol. II, p. 183). We have

$$U_{n-1}T_1 = \frac{1}{2}\left[U_n + U_{n-2}\right]$$

$$\frac{1}{2}U_{2n} = -\frac{1}{2} + \sum_{m=0}^{n} T_{2m} \qquad ; \qquad \frac{1}{2}U_{2n-1} = \sum_{m=0}^{n-1} T_{2m+1}$$

Hence

$$0 = \sum_{n=0}^{-2\nu}\left[c_n'' + \frac{2\nu+1}{r}c_n' + \left(k^2 - \frac{n^2}{r^2}\right)c_n\right]\cos n\theta$$

$$-c_n\frac{2\nu n}{r^2}\begin{cases} -1 + 2\sum_{m=0}^{\frac{n-2}{2}}\cos 2m\theta + \cos 2n\theta & \text{if } n \text{ is even} \\ 2\sum_{m=0}^{\frac{n-3}{2}}\cos(2m+1)\theta + \cos 2n\theta & \text{if } n \text{ is odd} \end{cases} \tag{2.23}$$

Therefore by the orthogonality properties of $\cos n\theta$ and the fact that $h(r, \xi)$ (and hence $c_n(r)$ in (2.21)) satisfies the radiation condition (2.5). Conclude

$$c_n(r) = \sum_{j=n}^{-2\nu} d_j\, r^{-\nu} H_{\nu+j}^{(1)}(kr) \tag{2.24}$$

Substituting (2.24) into (2.21) and rearranging we have

$$h(r,\xi) = r^{-\nu}\sum_{n=0}^{-2\nu} H_{\nu+n}^{(1)}(kr)\, g_n(\xi)$$

where $g_n(\xi)$ is a polynomial of degree at most -2ν in ξ. Now using the fact that the $H_{\nu+n}^{(1)}(kr)$ are linearly independent for $0 \le n \le -\nu$ whereas $H_{-\nu-n}^{(1)}(kr) = e^{(\nu+n)\pi i}\, H_{\nu+n}^{(1)}(kr)$ and applying the partial differential equation conclude that $g_n(\xi)$ must satisfy Jacobi's equation

$$(1-\xi^2)y'' - (2\nu+1)\xi y' + n(n+2\nu)y = 0 \tag{2.25}$$

i.e. $$h(r,\xi) = r^{-\nu}\sum_{n=0}^{-2\nu} a_n\, H_{\nu+n}^{(1)}(kr)\, P_n^{(\nu-\frac{1}{2},\,\nu-\frac{1}{2})}(\xi)$$

Here the a_n are not fully determined since the preceding analysis shows that it is in fact a linear combination of a_n and $a_{-n-2\nu}$ which is uniquely determined. This concludes the proof.

Corollary 2.1: Let $f(\xi)$ be an analytic function of $\xi = \cos\theta$ for $\xi \in [-1, +1]$ and let ν be a negative integer. Then in general no solution $u(x,y)$ of $L_\nu(u) = 0$ regular for $r \geq 1$ exists which satisfies hypotheses 1) and 2) of theorem 2.1 and assumes the values $f(\xi)$ on $r = 1$.

Proof: By theorem 2.4 $u(x,y) = \tilde{u}(r, \xi) = r^{-\nu} \sum_{n=0}^{\infty} a_n \frac{H^{(1)}_{\nu+n}(kr)}{H^{(1)}_{\nu+n}(k)}$ $P_n^{(\nu-\frac12, \nu-\frac12)}(\xi)$ for $r \geq 1$ since $\tilde{u}(r, \xi)$ is regular for $r \geq 1$. Hence $f(\xi) = \sum_{n=0}^{\infty} a_n P_n^{(\nu-\frac12, \nu-\frac12)}(\xi)$ where convergence is uniform in some ellipse containing $[-1, +1]$. By theorem 2.3 there exist analytic functions $f(\xi)$ for which no such expansion exists.

The following theorem is a generalization of a result due to Karp[15] who considered the case $\nu = 0$, i.e. the classical two dimensional Helmholtz equation. The hypothesis of the theorem is implied by the hypothesis of theorem 2.4.

Theorem 2.5: Assume $-\infty < \nu < +\infty$, $2\nu \neq -1, -3, -5 \ldots$ Let $u(x,y)$ satisfy $L_\nu(u) = 0$ for $r > a$ and have the expansion

$$u(x,y) = \tilde{u}(r, \xi) = r^{-\nu} \sum_{n=0}^{\infty} a_n \frac{H^{(1)}_{\nu+n}(kr)}{H^{(1)}_{\nu+n}(kb)} P_n^{(\nu-\frac12, \nu-\frac12)}(\xi), \quad r \geq b > a$$

then there exists a convergent expansion in the form

$$u(x,y) = \tilde{u}(r,\xi) = r^{-\nu} H_\nu^{(1)}(kr) \sum_{n=0}^{\infty} \frac{F_n(\xi)}{r^n} + r^{-\nu} H_{\nu+1}^{(1)}(kr) \sum_{n=0}^{\infty} \frac{G_n(\xi)}{r^n}$$

where the series converges uniformly and absolutely for $r \geq b > a$, $\xi = \cos\theta$ contained in some rectangle T in the complex ξ plane enclosing $[-1, +1]$. $F_0(\xi)$ and $G_0(\xi)$ determine $u(x,y)$ uniquely and are analytic functions of ξ in the above mentioned rectangle.

__Proof:__ It is known([3] Vol. II, p. 34) that

$$H_{\nu+n}^{(1)}(kr) = H_\nu^{(1)}(kr)\, R_{n,\nu}(kr) - H_{\nu-1}(kr)\, R_{n-1,\nu+1}(kr) \tag{2.26}$$

where $R_{n,\nu}(kr)$, $R_{n-1,\nu+1}(kr)$ are the Lommel polynomials defined as

$$R_{n,\nu}(r) = \sum_{k=0}^{[n/2]} \frac{(-1)^n (n-k)!\, \Gamma(\nu+n-k)}{k!\,(n-2k)!\,\Gamma(\nu+k)} \left(\tfrac{1}{2} r\right)^{-n+2k} \tag{2.27}$$

Consider now the series

$$E(\omega,\xi) = \sum_{n=0}^{\infty} \frac{a_n P_n^{(\nu-1/2,\,\nu-1/2)}(\xi)}{H_{\nu+n}^{(1)}(kb)} R_{n,\nu}(\omega) \tag{2.28}$$

$$Q(\omega,\xi) = \sum_{n=0}^{\infty} \frac{a_n P_n^{(\nu-1/2,\,\nu-1/2)}(\xi)}{H_{\nu+n}^{(1)}(kb)} R_{n-1,\nu+1}(\omega) \tag{2.29}$$

which result when (2.26) is substituted into the Bessel-Jacobi representation for $u(x,y)$ given in the statement of the theorem. Here ω is a complex variable. Replace ω by $kbe^{i\alpha}$ where $0 \leq \alpha \leq 2\pi$. Then for example we can rewrite (2.28) as

$$E\left(kbe^{i\varphi}, \xi\right) = \sum_{n=0}^{\infty} \frac{a_n \; P_n^{(\nu-1/2,\, \nu-1/2)}(\xi) \frac{1}{\Gamma(\nu+n)} \left(\frac{kb}{2} e^{i\varphi}\right)^{n+\nu}}{\frac{1}{\Gamma(\nu+n)} \left(\frac{kb}{2} e^{i\varphi}\right)^{n+\nu} H_{\nu+n}^{(1)}(kb)} R_{n,\nu}\left(kbe^{i\varphi}\right) \tag{2.30}$$

(if ν is a negative integer, for $n \le -\nu$ omit the factor $\frac{1}{\Gamma(\nu+n)}$ which appears in the numerator and denomenator of each term in the above series). Using the formula ([3] V. II, p. 35),[15]

$$\lim_{n\to\infty} \frac{(z/2)^{\nu+n} R_{n,\nu}(z)}{\Gamma(\nu+n)} = \left(\frac{z}{2}\right) J_\nu(z) \tag{2.31}$$

$$\lim_{n\to\infty} \frac{\left(\frac{kb}{2}\right)^{n+\nu}}{\Gamma(\nu+n)} H_{\nu+n}^{(1)}(kb) = -\frac{i}{\pi} \tag{2.32}$$

and recalling that for $r = b$ $\tilde{u}(r,\xi)$ is an analytic function of ξ in some rectangle enclosing $[-1, +1]$ (theorem 2.2) and hence $\tilde{u}(b,\xi) = b^{-\nu} \sum_{n=0}^{\infty} a_n P_n^{(\nu-1/2,\, \nu-1/2)}$ converges absolutely uniformly on compact subsets of this rectangle, we can conclude that the right hand side of (2,28) is uniformly and absolutely convergent for $|\omega| = kb$ and ξ contained in some rectangle enclosing $[-1, +1]$. Now the series (2.28) is a series of polynomials in $\frac{1}{\omega}$ which, as we have just seen, converges uniformly and absolutely on the circle $\left|\frac{1}{\omega}\right| = \frac{1}{kb}$ and hence define an analytic function for $\left|\frac{1}{\omega}\right| < \frac{1}{kb}$ ([26] p. 95, 97). A similar result is seen to hold for equation (2.29). Hence

$$u(x,y) = \tilde{u}(r,\xi) = r^{-\nu} H_\nu^{(1)}(kr)\, E\left(\tfrac{1}{r}, \xi\right) + r^{-\nu} H_{\nu+1}^{(1)}(kr)\, Q\left(\tfrac{1}{r}, \xi\right) \tag{2.23}$$

where E and Q are analytic functions of $\frac{1}{r}$ and are regular in the interior of the circle $\left|\frac{1}{r}\right| = \frac{1}{b}$ in the complex $\frac{1}{r}$ plane. The first part of the theorem follows from this statement. If F_o and G_o are known, F_n and G_n can be computed recursively by substituting the series into the differential equation (see[15]). It is also easily verified that

$$F_o(\xi) = \sum_{n=0}^{\infty} \frac{a_{2n}(-1)^n}{H^{(1)}_{2n+\nu}(kb)} P_{2n}^{(\nu-1/2,\nu-1/2)}(\xi) \tag{2.34}$$

$$G_o(\xi) = \sum_{n=0}^{\infty} \frac{a_{2n+1}(-1)^n}{H^{(1)}_{2n+\nu+1}(kb)} P_{2n+1}^{(\nu-1/2,\nu-1/2)}(\xi) \tag{2.35}$$

which are seen to be analytic in the same rectangle $\tilde{u}(b, \xi)$ is analytic since they are the uniform limit of analytic functions in this rectangle.

The above theorem implies the following important corollary.

Corollary 2.2: Let $u(x,y) = \tilde{u}(r, \cos\theta)$ satisfy the conditions of theorem 2.1. Then $\lim\limits_{r\to\infty} \sqrt{\frac{\pi k}{2}}\, r^{\nu+1/2}\, \tilde{u}(r, \cos\theta) \exp\left\{-ikr + \frac{i\pi}{2} + \frac{i\nu\pi}{2}\right\}$ $= f_o(\cos\theta)$ exists uniformly for $\xi = \cos\theta$ contained in some rectangle in the complex ξ plane enclosing $[-1, +1]$, where

$$F_o(\xi) = \tfrac{1}{2}\left[f_o(\xi) + f_o(-\xi)\right]$$

$$-iG_o(\xi) = \tfrac{1}{2}\left[f_o(\xi) - f_o(-\xi)\right]$$

Proof: From theorems 2.4 and 2.5 and the asymptotic expansion for Hankel functions (see theorem 1.2) we have

$f_o(\xi) = F_o(\xi) - iG_o(\xi)$. From (2.34) and 2.35 it is seen that $F_0(\xi) = F_0(-\xi)$ and $G_0(\xi) = -G_0(-\xi)$. The corollary follows from these facts.

<u>Theorem 2.6</u>: Let $\nu < -\frac{1}{2}$. If $u(x,y) = \tilde{u}(r, \cos\theta)$ is a regular solution of $L_\nu(u) = 0$ in the exterior $\bar{F}$ of a normal domain D symmetric with respect to the axis $y = 0$ and $u(x,y) = 0$ on ∂D then for $0 < \theta_0 < \pi$ and r sufficiently large

$$\int_{\theta_0}^{\pi-\theta_0} r^{2\nu+1} \sin^{2\nu}\theta \left(\tilde{u}\frac{\partial\bar{\tilde{u}}}{\partial r} - \bar{\tilde{u}}\frac{\partial\tilde{u}}{\partial r}\right) d\theta = 0$$

where $\bar{\tilde{u}}$ denotes the complex conjugate of $\tilde{u}$.

<u>Proof</u>: For $y > 0$ Green's formula[5] is

$$\int_{\partial\Sigma_1} y^{2\nu}\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)ds = \iint_{\Sigma_1} y^{2\nu}\left[gL(f) - fL(g)\right]dx\,dy \qquad (2.36)$$

where $L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y}$, $\partial\Sigma_1$ is a sufficiently smooth simple closed curve in the half plane $y > 0$ traversed counterclockwise and bounding a closed domain Σ_1 $f, g \in C^{(2)}(\bar{\Sigma}_1)$. For $y < 0$ we can write the identity as

$$\int_{\partial\Sigma_2} (-y)^{2\nu}\left(g\frac{\partial f}{\partial n} - f\frac{\partial g}{\partial n}\right)ds = \iint_{\Sigma_2} (-y)^{2\nu}\left[gL(f) - fL(g)\right]dx\,dy \qquad (2.37)$$

where $\partial\Sigma_2$ is traversed counterclockwise. Now consider

$$\int_{\partial\Sigma_1} y^{2\nu}\left(u\frac{\partial\bar{u}}{\partial n} - \bar{u}\frac{\partial u}{\partial n}\right)ds + \int_{\partial\Sigma_2} (-y)^{2\nu}\left(u\frac{\partial\bar{u}}{\partial n} - \bar{u}\frac{\partial u}{\partial n}\right)ds \qquad (2.38)$$

where

$\partial\Sigma_1 = L_1 + C_1 + L_2 + C_2$ and $\partial\Sigma_2 = L_3 + C_3 + L_4 + C_4$ (see Figure 2.1. L_1, L_2, L_3, L_4 are line segments connecting the circle containing C_1, C_4 with ∂D. C_2 and C_3 lie on ∂D.)

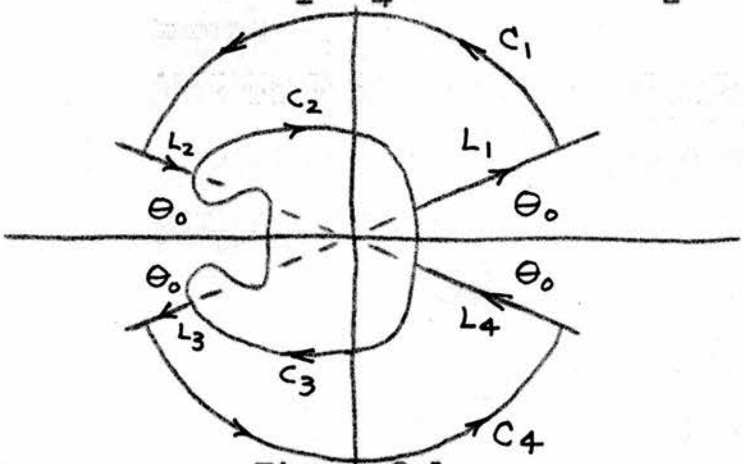


Figure 2.1

By hypothesis both u and $\bar{u}$ vanish on C_2 and C_3 (since C_2 abd C_3 lie on ∂D) and since u and $\bar{u}$ satisfy $L_\nu(u) = 0$ the double integrals in (2.36) and (2.37) taken over Σ_1, Σ_2 respectively vanish if we set $g=u$, $f=\bar{u}$. Observing that 1) u is an even function of y, 2) $\frac{\partial u}{\partial\theta}$ is an odd function of θ, 3) along L_1 and L_3 $\frac{\partial u}{\partial n} = -\frac{1}{r}\frac{\partial u}{\partial\theta}$; along L_2 and L_4 $\frac{\partial u}{\partial n} = \frac{1}{r}\frac{\partial u}{\partial\theta}$, ~~4) the integrals along L_1, L_4 and L_2, L_3 in (2.38) are traversed in opposite directions~~ conclude that the contribution along the straight line paths in (2.38) cancel each other. Hence since $u = \bar{u} = 0$ along C_2 and C_3 we have

$$0 = \int_{C_1} y^{2\nu}\left(u\frac{\partial\bar{u}}{\partial n} - \bar{u}\frac{\partial u}{\partial n}\right)ds + \int_{C_4}(-y)^{2\nu}\left(u\frac{\partial\bar{u}}{\partial n} - \bar{u}\frac{\partial u}{\partial n}\right)ds$$

$$= 2\int_{\theta_0}^{\pi-\theta_0} r^{2\nu+1}\sin^{2\nu}\theta\left(\tilde{u}\frac{\partial\tilde{\bar{u}}}{\partial r} - \tilde{\bar{u}}\frac{\partial\tilde{u}}{\partial r}\right)d\theta$$

This proves our theorem.

This concludes the preliminary results. We now prove the main theorem of this chapter viz. theorem 2.1.

Proof of theorem 2.1 : If $\tilde{u}$ satisfies the hypothesis of theorem 2.1 then by theorem 2.4 the hypothesis of theorem 2.5 is also satisfied. The series representation for $\tilde{u}$ derived in theorem 2.5 converges uniformly and absolutely for $r \geq b > a$ and $\xi = \cos\theta$ contained in some rectangle in the complex ξ plane enclosing $[-1, +1]$. Hence by this theorem and corollary 2.2 we can write

$$u(x,y) = \tilde{u}(r,\xi) = \sqrt{\frac{2}{\pi k}}\, r^{-\nu-1/2} e^{i(kr - \frac{\nu\pi}{2} - \pi/4)} \left[f_0(\xi) + O(\tfrac{1}{r}) \right] \tag{2.39}$$

$$\frac{\partial \tilde{u}(r,\xi)}{\partial r} = -(\nu + 1/2)\, \frac{\tilde{u}(r,\xi)}{r} + ik\,\tilde{u}(r,\xi)$$

$$= ik\, r^{-\nu-1/2} e^{i(kr - \frac{\nu\pi}{2} - \pi/4)} \left[f_0(\xi) + O(\tfrac{1}{r}) \right] \tag{2.40}$$

From theorem 2.6 we have for r sufficiently large

$$0 = \int_{\theta_0}^{\pi-\theta_0} r^{2\nu+1} \sin^{2\nu}\theta \left(\tilde{u} \frac{\partial \bar{\tilde{u}}}{\partial r} - \bar{\tilde{u}} \frac{\partial \tilde{u}}{\partial r} \right) d\theta = 2ik \int_{\theta_0}^{\pi-\theta_0} |f_0(\cos\theta)|^2 d\theta + O(\tfrac{1}{r})$$

Letting r tend to infinity we have therefore

$$\int_{\theta_0}^{\pi-\theta_0} |f_0(\cos\theta)|^2 d\theta = 0$$

Hence since $f_0(\cos\theta)$ is continuous, $f_0(\cos\theta) \equiv 0$ for $\theta_0 \leq \theta \leq \pi - \theta_0$. Since θ_0 can be chosen arbitrarily close to zero, conclude that $f_o(\cos\theta) \equiv 0$ for $0 \leq \theta \leq \pi$ by continuity of $f_o(\cos\theta)$, and hence $f_o(\cos\theta) \equiv 0$ for all θ, $0 \leq \theta \leq 2\pi$. Theorem 2.5 and corollary 2.2 now imply that $u(x,y) = \tilde{u}(r, \cos\theta) \equiv 0$. The theorem is proved.

At first glance conditions 1) and 2) of theorem 2.1 which treats the case $\nu < -\frac{1}{2}$, $2\nu \neq -1, -3, -5, \ldots$ seems much more restrictive than condition (2.4) for the case $\nu > -\frac{1}{2}$. The following theorem shows that the class of solutions considered in each case is essentially the same.

<u>Theorem 2.7</u>: Suppose $\nu > -\frac{1}{2}$ and $u(x,y) = \tilde{u}(r, \cos\theta)$ is a solution of $L_\nu(u) = 0$ regular for $r \geq b > a$. Then the following two conditions are equivalent:

1) $$\lim_{r\to\infty} \int_0^\pi r^{2\nu+1} \sin^{2\nu}\theta \left| \frac{\partial\tilde{u}}{\partial r} - ik\tilde{u} \right|^2 d\theta = 0 .$$

2) There exists a rectangle T (independent of r) enclosing $[-1, +1]$ in the complex $\xi = \cos\theta$ plane such that for each fixed $r > a$ $u(x,y) = \tilde{u}(r, \xi)$ is an analytic function of ξ in T and $\lim_{r\to\infty} r^{\nu+\frac{1}{2}} \left| \frac{\partial\tilde{u}}{\partial r} - ik\tilde{u} \right| = 0$ uniformly for ξ contained in T.

<u>Proof</u>: The fact that 2) implies 1) was shown in Chapter One where the limit in 2) was required to hold only for $\xi \varepsilon [-1, +1]$. If 1) holds then for $r \geq b > a$ we have by theorem 1.2 (it is easily shown using theorem 2.2 that the interval $(0,\pi)$ in the theorem can be replaced by $[0,\pi]$).

$$u(x,y) = \tilde{u}(r,\xi) = r^{-\nu} \sum_{n=0}^{\infty} a_n \frac{H^{(1)}_{\nu+n}(kr)}{H^{(1)}_{\nu+n}(kb)} C_n^\nu(\cos\theta) \tag{2.41}$$

By theorem 2.5 we see that there exists a rectangle T as defined in condition 2) and by methods identical to colollary 2.2. we have $\frac{\partial \bar{u}}{\partial r} = ik\bar{u} + O\left(\frac{1}{r}\right)$ uniformly as r tends to infinity for ξ contained in T. Conclude that 2) holds and hence the theorem is proved.

CHAPTER III

It is the purpose of this chapter to investigate solutions of the equation

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{3.1}$$

where 2ν is a negative odd integer and $k > 0$. If solutions of the classical Helmholtz equation in three dimensions which are regular in some exterior domain are expanded in a Fourier series, then the Fourier coefficients Φ are a function of ρ and z (Helmholtz's equation is assumed to be expressed in cylindrical coordinates ρ, ϕ, z) and satisfy $L_{n+\frac{1}{2}}(\overset{\rho^{-n}\Phi}{\cancel{\psi}}) = 0$ if we set $\rho = y$, $z = x$ in (3.1). The study of singular solutions of the last mentioned equation is essentially the study of equation (3.1) for the values of ν indicated above, viz. 2ν a negative odd integer. The reader is referred to the introduction for more detailed information. The main characteristic which distinguishes this case from other real values of ν is that it is now possible for a regular solution $u(x,y)$ of $L_\nu(u) = 0$ to vanish along the axis $y = 0$ without having $u(x,y) \equiv 0$. The following theorem essentially reduces the study of equation (3.1) for 2ν a negative odd integer to the study of $L_{1-\nu}(u) = 0$. This extends the results previously obtained by Hyman for the situation in which $k = 0$[14].

<u>Theorem 3.1</u>: Suppose $2\nu = -1, -3, -5, \ldots$ and $u(x,y)$ is a solution of $L_\nu(u) = 0$ regular in some domain D containing a segment of the axis $y = 0$. Then $u(x,y)$ is of the form

$$u(x,y) = Q_1(x,y)e^{ikx} + Q_2(x,y)e^{-ikx} + y^{1-2\nu}u^+(x,y)$$

where $Q_1(x,y)$, $Q_2(x,y)$ are polynomials of degree at most $-\frac{1}{2}(2\nu+1)$ in x and $-(\nu+\frac{1}{2})$ in y^2 and $u^+(x,y)$ is a solution of $L_{1-\nu}(u) = 0$ regular in D.

Proof: A fundamental system of solutions of equation (3.1) can be written in the following canonical form

$$u_1(x,y) = \sum_{m,n} d_{mn}(x-a)^n y^{2m} \quad ; \quad u_2(x,y) = y^{1-2\nu} \sum_{m,n} c_{mn}(x-a)^n y^{2m} \tag{3.2}$$

where the power series converge in some neighbourhood of (a,0) (see introduction). By Weinstein's correspondence principle (see introduction) $u_2(x,y)$ is of the form $y^{1-2\nu}u^+(x,y)$ where $L_{1-\nu}(u^+) = 0$. Now consider $u_1(x,y)$. Since the series representation of $u_1(x,y)$ is absolutely convergent on compact subsets contained in its region of convergence, we can rearrange this series on such sets to obtain

$$u_1(x,y) = \sum_{n=0}^{\infty} y^{2n} g_n(x) \tag{3.3}$$

Using the method of Frobenius(6) we have

$$4n(n+\nu-\tfrac{1}{2})g_n = -(D^2+k^2)g_{n-1} \tag{3.4}$$

where $D \equiv \frac{d}{dx}$. The recurrence relation (3.4) tells us that $(D^2 + k^2)g_{-\nu-1/2} = 0$ and by repeated application of this formula we can conclude that $(D^2 + k^2)^{-\nu+\frac{1}{2}} g_0(x) = 0$.

Since $g_0(x) = u(x,0)$ by (3.3), we have

$$(D^2+k^2)^{-\nu+1/2} u(x,0) = 0 \tag{3.5}$$

$$\text{i.e} \quad u(x,0) = p_1(x) e^{ikx} + p_2(x) e^{-ikx} \tag{3.6}$$

where $p_1(x)$ and $p_2(x)$ are polynomials of degree at most $-\nu-\frac{1}{2}$. Again from (3.4) it is seen that $g_n(x)$ for $n > -\nu + \frac{1}{2}$ is uniquely dependent on $g_{-\nu-1/2}(x)$ which can be arbitrarily chosen. Therefore

$$u_1(x,y) = \sum_{n=0}^{-\nu-1/2} y^{2n} \frac{(-1)^n (D^2+k^2)^n u(x,0)}{4^n n! (\nu+1/2)_n} + \sum_{n=-\nu+1/2}^{\infty} y^{2n} g_n(x) \tag{3.7}$$

In view of (3.6) this equation can be rewritten as

$$u_1(x,y) = Q_1(x,y) e^{ikx} + Q_2(x,y) e^{-ikx} + y^{1-2\nu} \sum_{m,n} f_{mn} y^{2m} (x-a)^n \tag{3.8}$$

where Q_1 and Q_2 are functions described in the statement of the theorem and f_{mn} are constants. By (3.4), (3.5) and (3.7) it is seen that $\omega(x,y) = Q_1(x,y)e^{ikx} + Q_2(x,y)e^{-ikx}$ satisfies $L_\nu(\omega) = 0$ and since $L_\nu(u) = 0$ we can conclude that the third term on the right hand side of (3.8) is of the form $y^{1-2\nu} u^+(x,y)$ where $L_{1-\nu}(u^+) = 0$. (See arguments at the beginning of this proof). This proves the theorem.

CHAPTER IV

It is the purpose of this chapter to investigate uniqueness questions for solutions regular in the whole space of the equation

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + k^2 u = 0 \tag{4.1}$$

where $\nu < -\frac{1}{2}$, $k > 0$. For $\nu > -\frac{1}{2}$ uniqueness theorems follow readily from the analysis of the Bessel-Gegenbauer series representation of $u(x,y)$ (see chapter one) similar to the investigation of the Fourier-Bessel series for the classical Helmholtz equation. The reader is referred to the papers of Hartman and Wilcox[9, 10] for details. Hence in this chapter we concern ourselves solely with the case $\nu < -\frac{1}{2}$. For such values of ν our main interest is in deriving results analogous to the following theorem of Magnus for the classical Helmholtz equation. Again let us stress the point that for $\nu > -\frac{1}{2}$ such results are easily obtainable. In the following $x = r\cos\theta$, $y = r\sin\theta$.

<u>Theorem 4.1</u> (Magnus). Let $u(x,y)$ be a solution regular in the whole plane of $L_0(u) = 0$ having the Fourier-Bessel expansion

$$u(x,y) = \sum_{n=0}^{\infty} a_n J_n(kr)\cos n\theta \quad .$$

Suppose that

$$\frac{1}{r}\iint_{\Delta(0,r)} |u(x,y)|^2\, dx\, dy \le C \qquad \text{for all } r$$

(or equivalently $\sum_{n=0}^{\infty} |a_n|^2 \le C$) where C is a positive constant and $\Delta(0,r)$ denotes a disc of radius r centred at the origin in the (x,y) plane. If

$$\lim_{r\to\infty} r^{1/2}\, u(x,y) = 0$$

for every (fixed) $\theta \in [0, \pi]$ then $u(x,y) \equiv 0$.

<u>Proof</u>: See[9]

The reason that problems of uniqueness for (4.1) are quite difficult for $\nu < -\frac{1}{2}$ and $\nu > -\frac{1}{2}$ is due basically to the fact that the condition of regularity in a neighbourhood of the axis $y = 0$ serves essentially as a boundary condition and the strength of this condition is seriously weakened as we pass into the range $\nu < -\frac{1}{2}$. For example if $\nu > -\frac{1}{2}$ $(\nu \neq 0)$ then any twice continuously differentiable solution $u(x,y)$ of $L_\nu(u) = 0$ in a domain D containing a portion of the axis is also regular in D whereas for $\nu < -\frac{1}{2}$ this is not the case. (This is due to the fact that the exponents of the singular line $y = 0$ are 0 and $1-2\nu$ (see introduction).) The well known methods of attacking (4.1) viz. integral operators and Bessel-Gegenbauer expansions[8, 13] run into difficulties for $\nu < -\frac{1}{2}$ and recourse often has to be made to properties of solutions $u(x,y)$ of $L_\nu(u) = 0$ in the complex y or $\xi = \cos\theta$ plane[17]; Chapter two of this thesis) In chapter two of this thesis strong use was made of the fact that $u(x,y) = \tilde{u}(r,\xi)$ (where $\xi = \cos\theta$) is an analytic function of ξ (for each fixed r) in some rectangle in the complex ξ plane enclosing $[-1, +1]$. In general the size of this rectangle will of course depend on r. With these facts in mind we proceed to determine sufficient conditions on solutions $u(x,y)$ of $L_\nu(u) = 0$ which are regular in the whole plane to ensure that $u(x,y) \equiv 0$.

<u>Theorem 4.2</u>: Assume $\nu < -\frac{1}{2}$, $2\nu \neq -1, -3, -5, \ldots$ and let $u(x,y) = \tilde{u}(r, \xi)$ be a solution of $L_\nu(u) = 0$ which is regular in the whole real x,y plane. Suppose there exists a rectangle T in the complex $\xi = \cos\theta$ plane enclosing $[-1, +1]$ such that $\tilde{u}(r,\xi)$ is analytic in T for all fixed r, $0 \le r < \infty$, i.e. T is independent of r. If

1) $\sup\left\{ r^{\nu+\frac{1}{2}} |\tilde{u}(r,\xi)| : \delta \le r < \infty,\ \xi \in T \right\} \le M(\delta) < \infty$

   where δ and M are positive constants, δ arbitrary but greater than zero.

2) $\lim\limits_{r\to\infty} r^{\nu+\frac{1}{2}} u(x,y) = 0$ pointwise for $\theta \in [0, \pi]$

   where $x = r\cos\theta$, $y = r\sin\theta$

then $u(x,y) \equiv 0$.

<u>Proof</u>: Proceeding exactly as in the proof of theorem 2.4, except that instead of employing the radiation condition (2.5) to determine the coefficients of the Jacobi expansion for $\tilde{u}(r,\xi)$ we use the fact that $\tilde{u}(r,\xi)$ is regular at $r = 0$, we have

$$u(x,y) = \tilde{u}(r,\xi) = r^{-\nu} \sum_{n=0}^{\infty} a_n J_{\nu+n}(kr)\, P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi) \tag{4.2}$$

where

$$\pi i\, a_n g_n^\nu r^{-\nu} J_{\nu+n}(kr) = \int_C (\xi^2-1)^{\nu-\frac{1}{2}} \tilde{u}(r,\xi)\, Q_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(\xi)\, d\xi \tag{4.3}$$

formula (4.3) holding for $n \ge -2\nu$ if ν equals a negative integer and for all n if $2\nu \neq -1, -2, -3, \ldots$. Here C is an ellipse lying in T with foci at ± 1, $Q_n^{(\nu-\frac{1}{2}, \nu-\frac{1}{2})}(\xi)$ is a Jacobi function of the second kind and g_n^ν is given by

$$g_n^\nu = \frac{2^{2\nu-1}}{n+\nu} \frac{\Gamma^2(n+\nu+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(n+2\nu)} \tag{4.4}$$

Note that due to the hypothesis of the theorem we can use the same ellipse C for all values of r, $0 \leq r < \infty$, i.e. C is independent of r. We first examine the behaviour of $Q_n^{(\nu-\frac{1}{2}, \nu-\frac{1}{2})}(\xi)$ along the ellipse C. From ([3], V.II, p. 170) we have

$$Q_n^{(\alpha,\alpha)}(\xi) = \frac{2^{2\alpha-1}\,\Gamma^2(n+\alpha+1)}{\Gamma(2n+2\alpha+2)\left(\frac{\xi-1}{2}\right)^{n+1}(\xi^2-1)^{2\alpha}}\ {}_2F_1\left(n+1, n+\alpha+1; 2n+2\alpha+2; 2(1-\xi)^{-1}\right) \tag{4.5}$$

where ${}_2F_1$ denotes the hypergeometric series. From ([3] V.1, p.77) we have

$$\left(\frac{\xi-1}{2}\right)^{-a-\lambda} {}_2F_1\left(a+\lambda, a-c+1+\lambda; a-b+1+2\lambda; 2(1-\xi)^{-1}\right) =$$

$$= \frac{2^{a+b}\,\Gamma(a-b+2\lambda+1)\,\Gamma(\frac{1}{2})\,\lambda^{-\frac{1}{2}}}{\Gamma(a-c+\lambda+1)\,\Gamma(c-b+\lambda)} e^{-(a+\lambda)\omega}(1-e^{-\omega})^{-c+\frac{1}{2}}(1+e^{-\omega})^{c-a-b-\frac{1}{2}}\left[1+O(\lambda^{-1})\right] \tag{4.6}$$

as $\lambda \to \infty$

where $|\arg \lambda| \leq \pi - \delta$, $\delta > 0$, $\xi \pm (\xi^2-1)^{\frac{1}{2}} = e^{\pm\omega}$, $\xi \notin [-1, +1]$. Setting $a = 1$, $c = -\alpha+1$, $b = -2\alpha$ in (4.6) we have

$$Q_n^{(\alpha,\alpha)}(\xi) = \Gamma(\tfrac{1}{2})\,(\xi^2-1)^{-2\alpha}\,n^{-\frac{1}{2}}\,e^{-(1+n)\omega}(1-e^{-2\omega})^{\alpha-\frac{1}{2}}\left[1+O(\tfrac{1}{n})\right] \tag{4.7}$$

which implies

$$\frac{|\,Q^{(\alpha,\alpha)}(\xi)\,|^{1/n}}{e^{-\omega}} = \frac{|\,Q_n^{(\alpha,\alpha)}(\xi)\,|^{1/n}}{|\,\xi-(\xi^2-1)^{1/2}|} = 1+o(1) \quad \text{as } n\to\infty$$

which is valid for all real non integer α and for $\xi \notin [-1, +1]$. Here $(\xi^2 - 1)^{1/2}$ is positive for $\xi > 1$.

Now from (4.3) and the hypothesis of the theorem we have for $n \geq -2\nu$

$$|\,\pi g_n^{\nu} a_n r^{1/2} J_{\nu+n}(kr)\,| \leq M \max_{\xi\in C} |\,Q_n^{(\nu-1/2,\nu-1/2)}(\xi)\,| \tag{4.9}$$

where M is independent of n, r, θ. Since the curve C is bounded away from $[-1, +1]$ for $\xi\varepsilon C$ we have $|\xi-(\xi^2-1)^{1/2}| \leq \rho < 1$ where ρ is a positive constant independent of ξ. For ε small enough we can find a γ such that $\rho(1+\varepsilon) \leq \gamma < 1$. The right hand side of (4.8) is less than $1+\varepsilon$ for $n > n_o(\varepsilon)$ and hence for $n>n_o$ $\xi\varepsilon C$ the inequality $|Q_n^{(\alpha,\alpha)}(\xi)| < \gamma^n$ holds. Here n_o does not depend on r since the ellipse C does not depend on r. Hence we have

$$|\,\pi g_n^{\nu} a_n r^{1/2} J_{\nu+n}(kr)\,| \leq M\gamma^n \quad \text{for } n>n_o \tag{4.10}$$

Now consider the sequence of integers n, $n > n_o$, and let r_{nk} denote the k^{th} local maximum of $J_{\nu+n}(kr)$. In view of the asymptotic behaviour of Bessel's function ([3] V. II, p. 85)

$$(2\pi r)^{1/2} J_\mu(r) = 2\cos(r - \tfrac{1}{2}\mu\pi - \pi/4) + o(1) \quad \text{as } r\to\infty \tag{4.11}$$

we have from(4.10) by letting r run through the values r_{nk} $k = 1,2,3,\ldots\ldots$, n fixed, that for each fixed integer $n > n_o$

$$|\,g_n^{\nu} a_n \sqrt{2\pi k}\,| \leq M\gamma^n \tag{4.12}$$

Since (4.4) implies that $|g_n^\nu| = O(\frac{1}{n})$ and M and γ are independent of n we can conclude from (4.12) that

$$\overline{\lim_{n\to\infty}} |a_n|^{1/n} < 1 \tag{4.13}$$

We now make use of Lommel's inequality[9]

$$|J_\mu(r)| \leq 1 \quad for \ r, \mu \geq 0 \tag{4.14}$$

and the uniform bound on Jacobi's polynomials ([23], p. 164)

$$|P_n^{(\nu-1/2,\nu-1/2)}(x)| = O(n^{-1/2}) \ ; \ x \in [-1,+1], \ 2\nu \neq -1,-3,-5,\dots \tag{4.15}$$

We have using (4.11)

$$\left| \sqrt{2\pi k}\ r^{\nu+1/2}\tilde{u}(r,\xi) - 2\sum_{n=0}^{\infty} a_n \cos(kr - \tfrac{\pi}{2}(\nu+n) - \pi/4)\, P_n^{(\nu-1/2,\nu-1/2)}(\xi) \right|$$

$$\leq \sum_{n=0}^{\infty} |a_n P_n^{(\nu-1/2,\nu-1/2)}(\xi)|\, |(2\pi kr)^{1/2} J_{\nu+n}(kr) - 2\cos(kr - \pi/2(\nu+n) - \pi/4)| \tag{4.16}$$

By (4.14) the factor of $|a_n P_n^{(\nu-1/2,\nu-1/2)}(\xi)|$ in (4.16) is uniformly bounded for $n > [\nu+1]$ and $r \geq 0$ and hence by (4.13) and (4.15) the series is uniformly convergent for $\xi \in [-1, +1]$. Since by (4.11) each term tends to zero as r tends to infinity we can conclude that for $\xi \in [-1, +1]$

$$\lim_{r\to\infty} \left| \sqrt{2\pi k}\ r^{\nu+1/2}\tilde{u}(r,\xi) - 2\sum_{k=0}^{\infty} a_n \cos(kr - \pi/2(\nu+n) - \pi/4)\, P_n^{(\nu-1/2,\nu-1/2)}(\xi) \right| = 0 \tag{4.17}$$

and hence by the hypothesis of the theorem

$$\lim_{r\to\infty} \sum_{n=0}^{\infty} a_n \cos(kr - \pi/2(\nu+n) - \pi/4)\, P_n^{(\nu-1/2,\nu-1/2)}(\xi) = 0 \tag{4.18}$$

Formula (4.18) can be written as

$$\lim_{r\to\infty} \Big\{ \cos\left(kr - \frac{\pi\nu}{2} - \pi/4\right) \sum_{n=0}^{\infty} a_{2n}(-1)^n P_{2n}^{(\nu-1/2,\nu-1/2)}(\xi) +$$

$$+ \sin\left(kr - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) \sum_{n=0}^{\infty} a_{2n+1}(-1)^n P_{2n+1}^{(\nu-1/2,\nu-1/2)}(\xi) \Big\} = 0 \tag{4.19}$$

Formula (4.19) implies that for $\xi \in [-1, +1]$

$$\sum_{n=0}^{\infty} a_{2n}(-1)^n P_{2n}^{(\nu-1/2,\nu-1/2)}(\xi) = \sum_{n=0}^{\infty} a_{2n+1}(-1)^n P_{2n+1}^{(\nu-1/2,\nu-1/2)}(\xi) = 0 \tag{4.20}$$

From ([23], p. 195) we have for non-negative integer α and $\xi \notin [-1, +1]$

$$\frac{|P_n^{(\alpha,\alpha)}(\xi)|}{|\xi + (\xi^2-1)^{1/2}|} = 1 + o(1) \quad \text{as } n \to \infty \tag{4.21}$$

Hence in view of (4.13) and (4.15) each of the series in (4.20) is an analytic function of ξ in some rectangle enclosing $[-1, +1]$ since each of the series converges uniformly in some such rectangle and the uniform limit of analytic functions is analytic. Since each of the series is zero along $[-1, +1]$, by the identity principle they are equal to zero in the above mentioned rectangle.

Now for $n \quad 0$, $2\nu \neq -1, -3, -5, \ldots$ we have from formula (2.13) that

$$\pi i \int_{C_1} (\xi^2-1)^{\nu-1/2} Q_n^{(\nu-1/2,\nu-1/2)}(\xi)\, P_m^{(\nu-1/2,\nu-1/2)}(\xi)\, d\xi = \delta_{mn}\, g_n^\nu \tag{4.22}$$

where g_n^ν is given by (4.4) and C_1 is a simple closed curve lying in a rectangle in which both series in (4.20) converge uniformly. Hence $a_n = 0$ for $n \geq 0$ if $\nu \neq -1, -2, -3, \ldots.$ and for $n \geq -2\nu$ if $\nu = -1, -2, -3, \ldots.$ If the first case holds, by (4.2) we are done. Therefore assume $\nu = -1, -2, -3\ldots$ when by the above analysis we have

$$\sum_{n=0}^{-\nu} a_{2n} (-1)^n P_{2n}^{(\nu-1/2,\nu-1/2)}(\xi) = \sum_{n=0}^{-\nu-1} a_{2n+1} (-1)^n P_{2n+1}^{(\nu-1/2,\nu-1/2)}(\xi) = 0 \tag{4.23}$$

From (2.11) we have $P_n^{(\nu-1/2,\nu-1/2)}(\xi) = h_n P_{-n-2\nu}^{(\nu-1/2,\nu-1/2)}(\xi)$ where h_n is a normalization factor and also that for $0 \leq n \leq -\nu$ $P_n^{(\nu-1/2,\nu-1/2)}(\xi)$ is a polynomial of degree n in ξ. Hence if (4.23) vanishes we must have for $0 \leq n \leq -\nu$

$$a_n + (-1)^{n+\nu} h_n^{-1} a_{-n-2\nu} = 0 \tag{4.24}$$

Since $J_{\nu+n}(kr) = (-1)^{\nu+n} J_{-\nu-n}(kr)$ we have from (4.2) (since $a_n = 0$ for $n \geq -2\nu$) that $u(x,y) \equiv 0$ and the theorem is proved.

<u>Example 4.1</u>: The rectangle T in the hypothesis of theorem 4.2 cannot be replaced by the line segment $[-1, +1]$ since in this case $u(x,y) = e^{ikx}$ satisfies all the conditions of the theorem, but $u(x,y) \not\equiv 0$.

REFERENCES

1. H. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables (Editions Scientifiques Hermann, Paris, 1963).
2. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II (Wiley, New York, 1962).
3. A. Erdélyi, Higher Transcendental Functions, Vols. I and II (McGraw Hill, New York, 1953).
4. __________, The Analytic Theory of Systems of Partial Differential Equations, Bull. Amer. Math. Soc. 57 (1951).
5. __________, Singularities of Generalized Axially Symmetric Potentials, Comm. Pure and App. Math. 9 (1956).
6. __________, Axially Symmetric Potentials and Fractional Integration, J. Soc. Ind. App. Math. 13 (1965).
7. J. Ericksen, Nonexistence Theorems in Linearized Elastostatics, J. of Diff. Eqns 1 (1965).
8. R. Gilbert and H. Howard, On Solutions of the Generalized Axially Symmetric Wave Equation Represented by Bergman Operators, Proc. Lond. Math. Soc. 15 (1965).
9. P. Hartman, On Solutions of $\Delta V + V = 0$ in an Exterior Region, Math. Zeit. 71 (1959).
10. P. Hartman and C. Wilcox, On Solutions of the Helmholtz Equation in Exterior Domains, Math. Zeit. 75 (1961).
11. A. Heine, Axially-Symmetric Boundary Value Problems, Bull. Amer. Math. Soc. 71 (1965).
12. C. Hellwig, Partial Differential Equations (Blaisdell, New York, 1960).
13. P. Henrici, Zur Funktionentheorie der Wellengleichung, Comm. Math. Helv. 27 (1953).
14. M. Hyman, Concerning Analytic Solutions of the Generalized Potential Equation, Proc. Konnikl. Akad. Weten. Amsterdam 57 (1954).
15. S. Karp, A Convergent 'Farfield' Expansion for Two Dimensional Radiation Functions, Comm. Pure and App. Math. 14 (1961).

REFERENCES (Contd.)

16. L. Levine, A Uniqueness Theorem for the Reduced Wave Equation, Comm. Pure and App. Math. 17 (1964).

17. A. Mackie, Contour Integral Solutions of a Class of Differential Equations, J. of Rat. Mech. Anal. 4 (1955).

18. W. Magnus, Über Eindeutigkeitsfragen bei einer Randwertaufgabe von $\Delta u + k^2 u = 0$, Jber. Deutsch. Math. Verein. 52 (1943).

19. S. Parter, On the Existence and Uniqueness of Symmetric Axially Symmetric Potentials, Arch. Rat. Mech. and Anal. 20 (1965).

20. R. Rellich, Über das Asymptotische Verhalten der Lösungen von $\Delta u + u = 0$ in imendichen Gebieten, Jber. Deutsch. Math.-Verein 53 (1943).

21. M. Schecter, On the Dirichlet Problem for Second Order Elliptic Equations with Coefficients Singular at the Boundary, Comm. Pure and App. Math. 13 (1960).

22. A. Sommerfeld, Die Greenehe Function der Schwingungsgleichung, Jber. Deutsch. Math.-Verein 21 (1912).

23. G. Szego, Orthogonal Polynomials (A.M.S. New York, 1948).

24. A. Weinstein, Generalized Axially Symmetric Potential Theory, Bull. Amer. Math. Soc. 59 (1953).

25. ____________, Singular Partial Differential Equations and Their Applications, Proceedings of the Symposium on Fluid Dynamics and Applied Mathematics (Gordon and Breach, New York, 1961).

26. E. Titchmarsh, The Theory of Functions (Oxford University Press, London, 1939).