FRACTIONAL CALCULUS
FRACTIONAL POWERS OF OPERATORS
and
MELLIN MULTIPLIER TRANSFORMS

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Doctor of Science

The University of Edinburgh

1994

CONTENTS

ABSTRACT

We shall present a theory of fractional calculus for generalised functions on $(0, \infty)$ and use this theory as a basis for extensions to some related areas.

In the first section, appropriate spaces of test-functions and generalised functions on $(0, \infty)$ are introduced and the properties of operators of fractional calculus obtained relative to these spaces. Applications are given to hypergeometric integral equations, Hankel transforms and dual integral equations of Titchmarsh type.

In the second section, the Mellin transform is used to define fractional powers of a very general class of operators. These definitions include standard operators as special cases. Of particular interest are powers of differential operators of Bessel or hyper-Bessel type which are related to integral operators with special functions, notably G-functions, as kernels.

In the third section, we examine operators whose Mellin multipliers involve products and/or quotients of Γ-functions. There is a detailed study of the range and invertibility of such operators in weighted L^p-spaces and in appropriate spaces of smooth functions. The Laplace and Stieltjes transforms give two particular examples.

In the final section, we show how our theory of fractional calculus on $(0, \infty)$ can be used to develop a corresponding theory on $\mathbf{R}^n$ in the presence of radial symmetry. In this framework the mapping properties of multidimensional fractional integrals and Riesz potentials are obtained very precisely.

DECLARATION

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DECLARATION

Joint Authorship of Papers

W. LAMB gained his Ph.D. under my supervision. Paper 13 contains a substantial contribution from myself.

W. J. SPRATT gained his Ph.D. under my supervision. Papers 17, 18, 19 and 20 present and extend material from the thesis. I made a very substantial contribution and produced the published versions. Paper 20 is almost entirely my own.

V. S. KIRYAKOVA visited me at the University of Strathclyde. Papers 15 and 16 resulted from our collaboration. I had a major input to the production of these papers, with Professor Kiryakova preparing the final versions.

B. S. RUBIN visited me at the University of Strathclyde. Papers 23 and 24 represent collaborative work to which both of us made very substantial contributions. I produced the versions submitted.

All papers other than those indicated above are my own work.

INTRODUCTION

We shall present a theory of fractional integration for certain spaces of generalised functions on the half-line $(0, \infty)$ and discuss some applications of this theory, as well as several related topics.

In the first instance, the theory is presented in an elementary and direct fashion. Subsequently we show how this approach can be streamlined by use of the Mellin transform. This leads to detailed examination of a class of Mellin multiplier transforms, which can be thought of as analogues for $(0, \infty)$ of pseudo-differential operators on the whole real line. Although the majority of our results relate to $(0, \infty)$, we also include some recent work which extends the one-dimensional theory to fractional integrals and Riesz potentials of functions defined on n-dimensional space.

"Fractional Calculus" is the (slightly unfortunate) name which has been given to the study of operators involving derivatives or integrals of order α, where α is not necessarily a positive integer. Some of these ideas are as old as calculus itself. For example, in 1695 Leibniz and L'Hôpital were in correspondence about a possible interpretation of a derivative of order 1/2. We might expect fractional derivatives to be inverses of fractional integrals and since this is so we shall concentrate on fractional integrals. Fractional integration has been studied in various settings including functions defined on $(0, \infty)$, $\mathbf{R}$, orthants in $\mathbf{R}^n$, domains in the complex plane, light cones, homogeneous spaces and Orlicz spaces. The results obtained have led to many developments. We might mention, as one instance, how results of Hardy and Littlewood obtained in the 1920's have provided the basis for areas of modern harmonic analysis, such as weighted norm inequalities. For a selection of ideas related to fractional integrals, see [14] and [26].

The last 50 years have seen the development of an area of mathematics sometimes called "generalised integral transformations" in which classical integral transforms are extended to appropriate spaces of generalised functions or distributions (in the sense of Schwartz). Possibly the best known example is the extension of the Fourier transform to tempered distributions. The aim is to enlarge the class of problems which can be solved by use of the relevant integral transform. In certain cases, the need for a distributional approach arises naturally. An illustration is afforded by the dual integral equations studied in [6], where the change in behaviour at $x = 1$ suggests that there may be a rôle for the δ-distribution concentrated at 1 in studying behaviour which may be too "wild" for a classical treatment.

In studying a particular generalised integral transformation, authors typically introduce a space of test-functions (and corresponding generalised functions) which are tailor-made for that one transformation. The resulting spaces vary considerably, depending on the kernel of the integral transformation. A difficulty arises when we require to apply a sequence of operators, rather than just a single operator, in order to solve a problem. Then, all the operators involved should have good mapping properties relative to the same space (or scale of spaces). Our work makes use of a family of spaces relative to which the mapping properties of many operators can be described completely.

Our spaces $F_{p,\mu}$ of test-functions and the corresponding spaces $F'_{p,\mu}$ of generalised functions were originally conceived for the study of operators of fractional calculus called the Erdélyi-Kober operators, to which the Riemann-Liouville and Weyl fractional integrals are simply related; see [1], [4]. However, the spaces are amenable to application of the Hankel transform [5] and by combining the two sets of results we can completely solve the dual integral equations in [6]. Much more can be done. The properties of a large class of Mellin multiplier transforms can be described in great detail; see [17], [18], [19]. The contrast with the corresponding classical theory in versions of L^p with power weights is remarkable.

Use of the Mellin transform leads to another extension of fractional calculus. We can define arbitrary α^{th} powers, T^α, of operators T satisfying an appropriate functional relation involving the Mellin transform. The Riemann-Liouville and Weyl fractional integrals and fractional derivatives are particular cases. However, we can obtain powers of much more complicated operators such as Bessel-type differential operators [7]. The theory is distinct from the spectral approach for powers of an operator T mapping a space into itself. Our operators map one space into another different space in our $F_{p,\mu}$ (or $F'_{p,\mu}$) family. Paradoxically perhaps, the theory is often simpler than in the spectral approach, a notable example being the validity of the index law $(T^\alpha)^\beta = T^{\alpha\beta}$. We compare and contrast the two approaches in [13].

Fractional powers of an operator form a semigroup under composition and our work consequently impinges on the theory of semigroups of operators. The Hille-Yosida theorem is the fundamental result for strongly continuous semigroups. When an operator fails to satisfy the hypotheses of this theorem, the operator does not generate a strongly continuous semigroup. In an attempt to overcome this, the concept of an α-integrated semigroup has recently been introduced and has led to some interesting results. Although this is not our main concern here, some of the ideas are surveyed in [14].

Having outlined the interplay between fractional calculus, fractional powers of operators, generalised functions and Mellin multiplier transforms, we shall now comment briefly on the papers involved in this presentation.

We may group them under the following headings.

1. Fractional Calculus in $F_{p,\mu}$ and $F'_{p,\mu}$ with Applications
2. Fractional Powers of Operators via the Mellin Transform
3. Range and Invertibility of Mellin Multiplier Transforms
4. Radial Spherical Fourier Transforms.

We consider each in turn.

1. Fractional Calculus in $F_{p,\mu}$ and $F'_{p,\mu}$ with Applications.

The relevant papers are [1], [2], [3], [4], [5], [6]. The material in these papers can be found gathered together in [25], although the latter also contains various extensions (involving very weak restrictions on parameters). The spaces $F_{p,\mu}$ (of test-functions) and $F'_{p,\mu}$ (of generalised functions) are introduced and simple properties obtained. These spaces are used in all subsequent papers. However, it should be noted that in later papers, such as [19], a slight adjustment is made to the definition (essentially a change of notation) which makes conditions on the parameters more aesthetically pleasing and independent of p. For example, the condition $\mathrm{Re}(\mu_1 - \mu_2) = 1/p_1 - 1/p_2$ in [3, Corollary 2.3] becomes $\mathrm{Re}\ \mu_1 = \mathrm{Re}\ \mu_2$ in the revised notation.

In [1] the basic theory of fractional calculus is developed with the Erdélyi-Kober operators $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$ as the starting point. These are homogeneous operators mapping $F_{p,\mu}$ into itself under conditions of great generality. Analytic continuation with respect to α, which is possible when dealing with smooth test-functions, leads to definitions of I_m^α and K_m^α for all complex α. For Re $\alpha > 0$ we have the usual Riemann-Liouville and Weyl fractional integrals while for Re $\alpha < 0$ we can recover fractional derivatives. In [4] we show how to extend the previous definitions to allow a much wider range of parameter values. A direct approach is adopted and concrete forms of the extended operators are obtained. Subsequently the various definitions can be harmonised via multipliers. Thus, all versions of the operator $I_m^{\eta,\alpha}$ correspond to the same Mellin multiplier, namely $\Gamma(\eta+1-s/m)\Big/\Gamma(\eta+\alpha+1-s/m)$. See, for example , [22]. Extension to $F'_{p,\mu}$ is routine on using adjoints.

The first application of the theory, to hypergeometric integral equations, is presented in [2] and extended in [25, Chapter 4]. The simplicity of our results is in marked contrast to the highly technical conditions in the two fundamental papers of E.R. Love cited in [2]. This arises because Love imposes very mild local integrability conditions on his functions, whereas we can imbed classical functions from weighted L^p spaces into appropriate $F'_{p,\mu}$ spaces wherein all the operators are easily handled under very general conditions.

The properties of the Hankel transform are presented in [5] and extended in [25, Chapter 5]. The mapping properties of this transform relative to $L^p(0,\infty)$ are not particularly good unless $p = 2$, and there are major difficulties for $p > 2$. In contrast, our $F_{p,\mu}$ theory is simple, elegant and valid for all $p \geq 1$. We go on to establish six fundamental results, connecting the Erdélyi-Kober operators and Hankel transforms, which are valid under very mild restrictions on the parameters. As in the classical case, these are the main tools for the study in [6] of dual integral equations of Titchmarsh type. As previously noted, the rôle of the distribution δ_1 emerges naturally and leads to the resolution of questions of existence and uniqueness (or non-uniqueness) of classical solutions of the dual integral equations.

2. Fractional Powers of Operators via the Mellin Transform

The relevant papers are [7], [8], [9], [10], [11], [12], [13], [14], [15] and [16]. Much of this material is summarised in [26, pages 99-139]. The first of these papers forms the transition between the direct approach of earlier papers and the subsequent extensive use of the Mellin transform. In [7], we consider a class of differential operators T, sometimes called Hyper-Bessel operators. By relating

$D \equiv d/dx$ to $D_m \equiv d/dx^m$ for a suitably chosen m, we can obtain expressions for $T^n(n = 1, 2, ...)$ which extend easily to T^α. Convenient expressions involving the Mellin transform are to be noted. Similar much more elaborate expressions appear in [10]. By choosing particular operators T we recover results of Love [8] and Buschman [9] which were obtained some years ago by other means.

The Mellin transform expressions in [7] serve as motivation for the theory in [10], [11], [12]. The presence here of a complex number γ with Re $\gamma \neq 0$ is crucial for the definition of T^α. It is this number γ which ensures that T maps from one $F_{p,\mu}$ space into a different one. As previously noted, this makes life easier rather than harder. In [13] we compare and contrast our theory with the spectral method for defining fractional powers of an operator mapping a space into itself. This and other related matters are also discussed in [14].

In [15] and [16] we return to ideas related to [7]. We study one particular aspect of the solution of equations involving Hyper-Bessel operators. On this occasion the analysis is conducted in certain spaces of continuous functions but an $F_{p,\mu}$ theory can also be developed.

3. Range and Invertibility of Mellin Multiplier Transforms

The relevant papers are [17], [18], [19], [20], [21] and [22].

By a Mellin multiplier transform we mean an operator T, acting on suitable functions ϕ, such that $\mathcal{M}(T\phi) = h.\mathcal{M}\phi$, where $\mathcal{M}$ denotes the Mellin transform and h is a suitable function called the (Mellin) multiplier of T. Such operators can be related to pseudo-differential operators under a change of variable which turns $\mathcal{M}$ into $\mathcal{F}$, the Fourier transform. Interest attaches to deriving mapping properties of T from analytic properties of h. P.G. Rooney has developed an extensive theory in power weighted L^p spaces for operators T whose multiplier h involves products and/or quotients of Γ-functions. In [17] we study a simple example of such an operator, characterise its range and, by renorming, turn the operator into a homeomorphism. The operator is related to the Laplace transform and we obtain real inversion formulae of Widder-Post type. In [18] we carry out a similar investigation for an operator related to the Stieltjes transform. It is already clear that in a L^p setting things become very complicated. However, subspaces of $F_{p,\mu}$ spaces appear as ranges and it is therefore natural to look at such operators wholly in an $F_{p,\mu}$ setting. In [19] a complete theory is developed which again displays elegance and simplicity, the contrast with

L^p being as notable as the contrast already mentioned in connection with [2]. A corollary of our theory is the emergence of a whole family of spaces $F_{p,\mu,r}$, each of which is invariant under the action of the Erdélyi-Kober operators and suitable for fractional calculus. There is great scope for further development, including details of the distributional theory or further studies of multipliers similar to [20]. We content ourselves in [21] with one simple illustration. A summary can be found in [22].

4. Radial Spherical Fourier Transforms

In [23] and [24] we bring together work of B.S. Rubin on so-called radial spherical Fourier transforms and our results for $F_{p,\mu}, F'_{p,\mu}$ on $(0,\infty)$. It is natural to ask how to generalise the definition of our spaces from one dimension to n dimensions. In [23] we offer one possibility, motivated by consideration of the expansion in terms of spherical harmonics of functions defined on $\mathbf{R}^n - \{0\}$. These spaces provide a framework for the study in [24] of multi-dimensional fractional integrals and Riesz potentials. For smooth functions we can relate these to one-dimensional fractional integrals acting on individual spherical harmonics, the latter belonging to our original $F_{p,\mu}$ space under appropriate conditions. The corresponding distributional theory gives desirable mapping properties of the potentials under very general conditions, which again contrasts sharply with what happens in weighted versions of $L^p(\mathbf{R}^n)$. There is scope for a variety of future investigations, including applications to boundary-value problems and the study of other radial spherical Fourier transforms via Mellin transforms acting on the individual spherical harmonics.

In conclusion, we may say that the spaces $F_{p,\mu}$ and $F'_{p,\mu}$ on $(0,\infty)$ provide an excellent setting for the study of many differential and integral operators. Our original foray into fractional calculus spawned the study of fractional powers of operators and studies of the range and invertibility of certain classes of Mellin multiplier transforms. These studies suggest various future investigations. We have already mentioned other radial spherical Fourier transforms on $\mathbf{R}^n$. Even on $(0,\infty)$, there is plenty scope. One possibility is the extension from power weighted spaces to spaces with more general weights, such as those satisfying the Muckenhoupt A_p condition.

It can justifiably be claimed that the classical notion of a fractional integral has connections with many topics in analysis, some of which are discussed in [14], [26] and a few of which have been treated in detail in the accompanying books and papers.

LIST OF PAPERS AND BOOKS

1. "A Theory of Fractional Integration for Generalised Functions". SIAM J. Math. Anal., 6(1975), 583-599.

2. "Solution of Hypergeometric Integral Equations involving Generalised Functions". Proc. Edinburgh Math. Soc. (2), 19(1975), 265-285.

3. "A note on the spaces $F'_{p,\mu}$". Proc. Royal Soc. Edinburgh, 77A (1977), 39-47.

4. "A Theory of Fractional Integration for Generalised Functions II". Proc. Royal Soc. Edinburgh, 77A (1977) , 335-349.

5. "The Hankel Transform of Some Classes of Generalised Functions and Connections with Fractional Integration". Proc. Royal Soc. Edinburgh, 81A (1978), 95-117.

6. "Solution of Dual Integral Equations of Titchmarsh Type using Generalised Functions". Proc. Royal Soc. Edinburgh, 83A (1979), 263-281.

7. "Fractional Powers of a Class of Ordinary Differential Operators". Proc. London Math. Soc. (3), 45 (1982), 519-546.

8. "A Note on the Index Laws of Fractional Calculus". J. Austral. Math. Soc., Series A, 34 (1983), 356-363.

9. "On an Index Law and a Result of Buschman". Proc. Royal Soc. Edinburgh, 96A (1984), 231-247.

10. "Fractional Powers of a Class of Mellin Multiplier Transforms I". Applicable Analysis, 21 (1986), 89-127.

11. "Fractional Powers of a Class of Mellin Multiplier Transforms II". Applicable Analysis, 21 (1986), 129-149.

12. "Fractional Powers of a Class of Mellin Multiplier Transforms III". Applicable Analysis, 21 (1986), 151-173.

13. (With W. Lamb) "On Relating Two Approaches to Fractional Calculus". J. Math. Anal. Appl., 132 (1988), 590-610.

14. "Fractional Integrals and Semigroups"
in *Semigroups of Linear and Nonlinear Operations and Applications* (edited by G. R. Goldstein and J. A. Goldstein), Kluwer, Dordrecht, 1993.

15. (With V. S. Kiryakova)
"On Solving Hyper-Bessel Differential Equations by means of Meijer's G-Functions II: The Nonhomogeneous Case".
Strathclyde Mathematics Research Report, 1992/20.

16. (With V. S. Kiryakova)
"Explicit Solution of the Non-Homogeneous Hyper-Bessel Differential Equation".
Comptes Rendus de l'Acad. Bulg. des Sciences, 46 (1993), 23-26.

17. (With W. J. Spratt)
"On the Range and Invertibility of a Class of Mellin Multiplier Transforms I".
J. Math. Anal. Appl., 156 (1991), 568-587.

18. (With W. J. Spratt)
"On the Range and Invertibility of a Class of Mellin Multiplier Transforms II".
Strathclyde Mathematics Research Report, 1988/7.

19. (With W. J. Spratt)
"On the Range and Invertibility of a Class of Mellin Multiplier Transforms III".
Canad. J. Math., 43 (1991), 1323-1338.

20. (With W. J. Spratt)
"A Class of Mellin Multipliers".
Canad. Math. Bull., 35 (1992), 252-260.

21. "Boundedness of Mellin Multiplier Transforms on L^p-Spaces with Power Weights".
Strathclyde Mathematics Research Report, 1993/20.

22. "Connections between Fractional Calculus and some Mellin Multiplier Transforms"
in *Univalent Functions, Fractional Calculus and their Applications* (edited by H. M. Srivastava and S. Owa), Ellis Horwood, 1989.

23. (With B. S. Rubin)
"Multidimensional Fractional Integrals of Distributions I".
Strathclyde Mathematics Research Report, 1994/2.

24. (With B. S. Rubin)
"Multidimensional Fractional Integrals of Distributions II".
Strathclyde Mathematics Research Report, 1994/3.

25. *Fractional Calculus and Integral Transforms of Generalised Functions*
Research Notes in Mathematics No. 31, Pitman, London, 1979.

26. (With G. F. Roach, Editor)
Fractional Calculus
Research Notes in Mathematics No. 138, Pitman, London, 1985.

1

FRACTIONAL CALCULUS IN

$F_{p,\mu}$ and $F'_{p,\mu}$

WITH APPLICATIONS

CONTENTS

Paper 1

SIAM J. MATH. ANAL.
Vol. 6, No. 3, May 1975

A THEORY OF FRACTIONAL INTEGRATION FOR GENERALIZED FUNCTIONS*

ADAM C. McBRIDE†

Abstract. In this paper, we develop a theory of fractional integration for certain classes of generalized functions and give one simple application.

First, we introduce the appropriate spaces of testing-functions and generalized functions and state some of their basic properties. Next, we discuss the various operators of fractional integration including the Riemann–Liouville and Weyl fractional integrals and the Erdélyi–Kober operators. Use of analytic continuation enables us to obtain a precise description of the mapping properties of these operators relative to the testing-function spaces. We extend the operators to the generalized functions using adjoints and deduce the corresponding mapping properties using standard theorems. Finally, we solve a differential equation involving generalized functions using the previous theory.

The theory is much more general than that developed in Erdélyi and McBride [6].

1.1. Conventions. We begin by making certain conventions which will be adhered to throughout. Generalized functions will be denoted by letters such as f, g, etc., while testing-functions will be denoted by Greek letters such as ϕ, ψ, etc. The value assigned to a testing-function ϕ by a generalized function f will be denoted by (f, ϕ).

Our testing-functions will be complex-valued infinitely differentiable functions on the open interval $(0, \infty)$. The space of all such functions will be denoted by C^∞. For each p, $1 \leqq p < \infty$, L_p is the set of (measurable) functions ϕ for which

$$|\phi|_p = \left(\int_0^\infty |\phi(x)|^p \, dx\right)^{1/p} < \infty.$$

L^p will denote the set of equivalence classes of such functions which differ on a set of measure zero. L_∞ will denote the space of (measurable) functions ϕ for which

$$|\phi|_\infty = \text{essential supremum of } \phi \text{ over } (0, \infty)$$

is finite. L^∞ is the corresponding space of equivalence classes. The numbers p and q will always be related by

$$\frac{1}{p} + \frac{1}{q} = 1$$

and unless otherwise stated, $1 \leqq p \leqq \infty$.

For any $x \in (0, \infty)$ and complex number μ, x^μ means $\exp(\mu \log x)$ where $\log x$ is real.

Where any term is not defined explicitly, we shall use the terminology of Zemanian [16].

* Received by the editors March 26, 1973, and in revised form March 12, 1974.

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1.2. Introduction. We shall be concerned with the following operators of fractional integration:

$$I_{x^m}^{\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_0^x (x^m - u^m)^{\alpha-1}u^{m-1}\phi(u)\,du, \tag{1.1}$$

$$K_{x^m}^{\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_x^\infty (u^m - x^m)^{\alpha-1}u^{m-1}\phi(u)\,du, \tag{1.2}$$

$$I_{x^m}^{\eta,\alpha}\phi(x) = x^{-m\eta-m\alpha}I_{x^m}^{\alpha}x^{m\eta}\phi(x), \tag{1.3}$$

$$K_{x^m}^{\eta,\alpha}\phi(x) = x^{m\eta}K_{x^m}^{\alpha}x^{-m\eta-m\alpha}\phi(x). \tag{1.4}$$

Here $m > 0$ is real, $\operatorname{Re}\alpha > 0$, η is a suitably restricted complex number and ϕ is defined on $(0, \infty)$. When $m = 1$, we obtain $I_x^{\alpha}\phi$ and $K_x^{\alpha}\phi$, which are respectively the Riemann–Liouville and Weyl integrals of order α of ϕ, while $I_x^{\eta,\alpha}$ and $K_x^{\eta,\alpha}$ are the Erdélyi–Kober operators [9].

Such operators arise in many situations, notably in connection with certain ordinary and partial differential equations (see, for instance, [3], [4] and [11]), integral transforms ([2], [10] and [14]) and dual and triple integral equations ([1] and [7]). On the other hand, the theory of generalized functions, or distributions, has led to great advances in the theory of differential equations ([8], [15]) and elsewhere. In this paper, we combine these two methods in developing a theory of fractional integration for a class of generalized functions.

It is possible to develop a theory for I_x^{α} and K_x^{α} based on the concept of the convolution of distributions [8], but this cannot be extended to the more general operators above. Instead, we pursue an approach based on adjoint operators. In [6], a space $\mathscr{I}$ of testing-functions was introduced such that (under suitable restrictions on the parameters) $K_{x^m}^{\eta,\alpha}$ is an automorphism of $\mathscr{I}$ and $I_{x^m}^{\eta,\alpha}$ is an automorphism of the generalized function space $\mathscr{I}'$. In this paper, we introduce classes $F'_{p,\mu}$ of generalized functions, relative to which the mapping properties of all four operators above can be obtained. The theory is much more general than that in [6] and also more flexible, since other operations such as differentiation and multiplication by arbitrary powers of x are easily handled.

In § 2, we study the spaces $F_{p,\mu}$ proceeding via the spaces $F_p \equiv F_{p,0}$. Certain simple operators are also discussed. The results are then extended to $F'_{p,\mu}$, and, in addition, we obtain a structure theorem for $F'_{p,\mu}$ in the case $p < \infty$.

Section 3 is devoted to a detailed study of the operators of fractional integration on $F_{p,\mu}$ and $F'_{p,\mu}$. It appears easier to obtain results for $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$ first, deducing properties of $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$, rather than to proceed in the opposite direction. The whole theory depends on the work of Kober in [9].

As indicated above, we would expect to obtain applications of the theory to generalized integral transforms (notably the Hankel transform) and to integral equations. These we hope to discuss in future papers, and we refer the interested reader to the author's thesis [13]. Here we content ourselves with just one application. In § 4, we discuss relations between fractional integration and the operator

$$L_\nu \equiv \frac{d^2}{dx^2} + \frac{2\nu + 1}{x}\frac{d}{dx}. \tag{1.5}$$

Formulas are given for the solution of

$$L_\nu f = g,$$

where f and g are generalized functions. Again, the results are much more general than those in [6].

2.1. The testing-function spaces F_p. For each p, $1 \leqq p \leqq \infty$, we define F_p by

$$F_p = \left\{\phi : \phi \in C^\infty \text{ and } x^k \frac{d^k\phi}{dx^k} \in L_p \ (k = 0, 1, 2, \cdots)\right\}. \tag{2.1}$$

With the usual pointwise operations of addition and scalar multiplication, F_p becomes a complex linear space. For $\phi \in F_p$, $k = 0, 1, 2, \cdots$, define γ_k^p by

$$\gamma_k^p(\phi) = \left| x^k \frac{d^k\phi}{dx^k} \right|_p. \tag{2.2}$$

The collection

$$M_p = \{\gamma_k^p : k = 0, 1, 2, \cdots\} \tag{2.3}$$

is a countable multinorm and, with the topology generated by M_p, F_p becomes a countably multinormed space. We define convergent sequences and Cauchy (or fundamental) sequences as in [16, § 1.6]. As usual, every convergent sequence is a fundamental sequence, but the converse is also true, i.e., F_p is complete.

THEOREM 2.1. *For $1 \leqq p \leqq \infty$, F_p is a complete countably multinormed space* (*and hence a Fréchet space*).

Proof. Define an operator δ on F_p by

$$(\delta\phi)(x) = x\frac{d\phi}{dx}, \qquad \delta \equiv x\frac{d}{dx}. \tag{2.4}$$

Since $x^k(d^k\phi/dx^k) \in L_p$, $k = 0, 1, 2, \cdots$, $\Leftrightarrow \delta^k\phi \in L_p$, $k = 0, 1, 2, \cdots$, we may rewrite (2.1) as

$$F_p = \{\phi : \phi \in C^\infty \text{ and } \delta^k\phi \in L_p (k = 0, 1, 2, \cdots)\}. \tag{2.5}$$

The proof is completed by an argument analogous to that in [16, pp. 253–4] using Hölder's inequality rather than Schwarz's inequality at the appropriate stage.

It can be shown similarly that F_p is a testing-function space in the sense of [16, p. 39], and we will call the elements of F_p *testing-functions.*

We conclude this section with an easy lemma which will be used frequently.

LEMMA 2.2. *$\phi \in F_p \Rightarrow x^{1/p}\phi(x)$ is bounded on* $(0, \infty)$, $1 \leqq p \leqq \infty$.

Proof. It is sufficient to consider the case when $\phi(x)$ is real-valued. Suppose first that $1 \leqq p < \infty$. Choose a, b with $0 < a < b < \infty$. Integrating by parts, we have

$$\int_a^b x\phi'(x)\{\phi(x)\}^{p-1}\,dx = \frac{1}{p}[x\{\phi(x)\}^p]_a^b - \frac{1}{p}\int_a^b \{\phi(x)\}^p\,dx.$$

Now $\phi \in F_p \Rightarrow x\phi'(x) \in L_p$. Also $\{\phi(x)\}^{p-1} \in L_q$ so, by Hölder's inequality, the left-hand side is bounded as $a \to 0+$ or $b \to \infty$. Since the same is true of the integral on the right, the result follows in this case.

The case $p = \infty$ is trivial since then $x^{1/p}\phi(x) = \phi(x)$ is essentially bounded and continuous and hence bounded on $(0, \infty)$.

2.2. The generalized function spaces F'_p. A functional f on F_p is (sequentially) continuous if, whenever ϕ_n converges to ϕ in the topology of F_p, $(f, \phi_n) \to (f, \phi)$ as $n \to \infty$. F'_p will denote the complex linear space of continuous linear functionals on F_p with the usual operations of addition and scalar multiplication. We assign to F'_p the topology of weak (or pointwise) convergence. From Theorem 2.1 and also [16, Thm. 1.8-3] we have the following.

THEOREM 2.3. *F'_p is complete,* $1 \leqq p \leqq \infty$.

Any function $f \in L_q$ generates an element $\tilde{f} \in F'_p$ by means of the formula

$$(2.6) \qquad (\tilde{f}, \phi) = \int_0^\infty f(x)\phi(x)\,dx, \qquad \phi \in F_p.$$

Generalized functions with an integral representation of this form will be called *regular*; those with no such representation will be called *singular*. An example of a singular element of F'_p is provided by $\delta_a, a > 0$, defined by

$$(\delta_a, \phi) = \phi(a), \qquad \phi \in F_p.$$

We shall use regular functionals to motivate the definition of various operators on F'_p in the sequel.

It is interesting to compare the spaces F'_p with other spaces of generalized functions, in particular with $\mathscr{D}'$, the distributions on $(0, \infty)$, and $\mathscr{E}'$, the distributions on $(0, \infty)$ with compact support. For the theory of $\mathscr{D}(=\mathscr{D}(0, \infty))$, $\mathscr{D}'$, $\mathscr{E}(=\mathscr{E}(0, \infty))$ and $\mathscr{E}'$, see [16]. It is clear that for each p, $1 \leqq p \leqq \infty$,

$$\mathscr{D} \subset F_p \subset \mathscr{E},$$

both inclusions being strict. Further, since $\mathscr{D}$ is dense in $\mathscr{E}$, [16, p. 37], F_p is dense in $\mathscr{E}$. Also it can be shown that if $1 \leqq p < \infty$, $\mathscr{D}$ is dense in F_p; the proof, which is rather intricate, is omitted. However, $\mathscr{D}$ is *not* dense in F_∞; for instance, we cannot approximate a (nonzero) constant function in the F_∞-topology by functions with compact support.

Now suppose $1 \leqq p < \infty$. Let $\{\phi_n\}$ converge to ϕ in $\mathscr{D}$ (i.e., in the topology of $\mathscr{D}$). The supports of ϕ and ϕ_n, $n = 1, 2, \cdots$, are all contained in some closed interval $[a, b]$ with $0 < a < b < \infty$, so that

$$\begin{aligned}\gamma_k^p(\phi_n - \phi) &= \left\{\int_a^b \left| x^k \frac{d^k}{dx^k}(\phi_n - \phi)\right|^p dx\right\}^{1/p} \\ &\leqq b^k(b-a)^{1/p} \sup_{a \leqq x \leqq b} \left|\frac{d^k}{dx^k}(\phi_n - \phi)\right| \\ &\to 0 \qquad \text{as } n \to \infty,\end{aligned}$$

by definition of convergence in $\mathscr{D}$. Hence

$$\text{convergence in } \mathscr{D} \Rightarrow \text{convergence in } F_p, \qquad 1 \leqq p < \infty,$$

and hence, $F'_p \subset \mathscr{D}'$. We can also show that

$$\text{convergence in } \mathscr{D} \Rightarrow \text{convergence in } F_\infty,$$

but since $\mathscr{D}$ is not dense in F_∞, we cannot deduce that $F'_\infty \subset \mathscr{D}'$. In the other direction, however, we can show that for $1 \leqq p \leqq \infty$, $\mathscr{E}' \subset F'_p$. In summary, we have Theorem 2.4.

THEOREM 2.4. *$\mathscr{E}' \subset F'_p$, $1 \leqq p \leqq \infty$, and $F'_p \subset \mathscr{D}'$, $1 \leqq p < \infty$.*

After we have defined generalized differentiation below, we will be able to prove a structure theorem for the elements of F'_p, $p < \infty$.

2.3. The spaces $F_{p,\mu}$ and $F'_{p,\mu}$. In order to be able to consider certain operations such as multiplication by arbitrary powers of x and differentiation, we must introduce generalizations of the spaces F_p and F'_p. For any complex number μ and $1 \leqq p \leqq \infty$, we define $F_{p,\mu}$ by

$$F_{p,\mu} = \{\phi : x^{-\mu}\phi(x) \in F_p\}. \tag{2.7}$$

$F_{p,\mu}$ is given the topology generated by the multinorm

$$M_{p,\mu} = \{\gamma_k^{p,\mu} : k = 0, 1, 2, \cdots\}, \tag{2.8}$$

where, for $\phi \in F_{p,\mu}$,

$$\gamma_k^{p,\mu}(\phi) = \gamma_k^p(x^{-\mu}\phi), \tag{2.9}$$

where γ_k^p is given by (2.2). It follows that the mapping $\phi \to x^\mu \phi$ is an isomorphism of F_p onto $F_{p,\mu}$. From Theorems 2.1 and 2.3 we immediately have Theorem 2.5.

THEOREM 2.5. *For each complex number μ and $1 \leqq p \leqq \infty$, $F_{p,\mu}$ is a Fréchet space and $F'_{p,\mu}$ is complete.*

Note in passing that we will continue to write

$$F_p \equiv F_{p,0}. \tag{2.10}$$

For each complex number λ, we define the operator x^λ on $F_{p,\mu}$ by

$$(x^\lambda\phi)(x) = x^\lambda\phi(x), \qquad 0 < x < \infty. \tag{2.11}$$

No confusion should arise from using the same symbol for the function x^λ and the operation of multiplying by this function. We define δ' on $F_{p,\mu}$ by

$$(\delta'\phi)(x) = \frac{d}{dx}(x\phi), \tag{2.12}$$

while for $m > 0$ we shall write

$$D_m \equiv \frac{d}{dx^m}, \qquad D_1 = D.$$

Note that

$$\delta' = \delta + I,$$

where I is the identity operator and δ is defined by (2.4). It is easy to prove the next theorem.

THEOREM 2.6. *Let λ, μ be complex numbers and $1 \leqq p \leqq \infty$.*

(i) *x^λ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+\lambda}$ with inverse $x^{-\lambda}$.*

(ii) *δ, δ' are continuous linear mappings of $F_{p,\mu}$ into itself.*

(iii) *D_m is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-m}$.*

To define the corresponding operators on $F'_{p,\mu}$, we use adjoint operators. For $f \in F'_{p,\mu}$ we define $x^\lambda f$, δf, $\delta' f$ and $D_m f$ by

$$(x^\lambda f, \phi) = (f, x^\lambda \phi), \qquad \phi \in F_{p,\mu-\lambda}, \tag{2.13}$$

$$(\delta f, \phi) = (f, -\delta'\phi), \qquad \phi \in F_{p,\mu}, \tag{2.14}$$

$$(\delta' f, \phi) = (f, -\delta\phi), \qquad \phi \in F_{p,\mu}, \tag{2.15}$$

$$(D_m f, \phi) = \left(f, -\frac{1}{m} D x^{-m+1}\phi\right), \qquad \phi \in F_{p,\mu+m}. \tag{2.16}$$

The motivation for (2.14)–(2.16) is supplied by taking f to be a regular functional, $\phi \in \mathscr{D}$ and integrating by parts. Using Theorem 2.6 and [16, Thm. 1.10-1] we immediately obtain the following.

THEOREM 2.7. *Let λ, μ be complex numbers and $1 \leqq p \leqq \infty$.*

(i) *x^λ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$ with inverse $x^{-\lambda}$.*

(ii) *δ, δ' are continuous linear mappings of $F'_{p,\mu}$ into itself.*

(iii) *D_m is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+m}$.*

We conclude this section with the following structure theorem.

THEOREM 2.8. *Let μ be any complex number and $1 \leqq p < \infty$. Any $f \in F'_{p,\mu}$ is of the form*

$$f = x^{-\mu} \sum_{k=0}^{r} x^k D^k \tilde{h}_k, \tag{2.17}$$

where r is a positive integer, $h_k \in L_q$, $k = 0, 1, \cdots, r$, and $\tilde{h}_k$ is defined as in (2.6).

Proof. The proof is analogous to a number of proofs in the literature and is omitted. (See, for instance, [15, pp. 272–274].)

3.1. The operators $I_{x^m}^{\eta;\alpha}$ on $F_{p,\mu}$. We are now ready to discuss the mapping properties of the operators (1.1)–(1.4) of fractional integration. In this section we study $I_{x^m}^{\alpha}$ and $I_{x^m}^{\eta,\alpha}$ and the corresponding results for $K_{x^m}^{\alpha}$ and $K_{x^m}^{\eta,\alpha}$ are given in the next section.

It is convenient to begin with $I_{x^m}^{\eta,\alpha}$ since (under suitable conditions) it maps L_p into L_p whereas $I_{x^m}^{\alpha}$ does not. The mapping properties of $I_{x^m}^{\alpha}$ can be derived using the relation

$$I_{x^m}^{\alpha}\phi(x) = x^{m\alpha}I_{x^m}^{0,\alpha}\phi(x). \tag{3.1}$$

This approach has the slight disadvantage that some of the more obvious results such as

$$I_{x^m}^{\alpha}\phi = \frac{d}{dx^m}I_{x^m}^{\alpha+1}\phi \tag{3.2}$$

appear much later than usual.

LEMMA 3.1. *For* $1 \leqq p \leqq \infty$, $\operatorname{Re}\alpha > 0$, $I_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* L_p *into* L_p *provided* $m \operatorname{Re}\eta + m > 1/p$.

Proof. The result for $m = 1$ is proved by Kober in [9, Thm. 2]. The general result follows by a simple change of variable.

This leads to Theorem 3.2.

THEOREM 3.2. *For* $1 \leqq p \leqq \infty$, $\operatorname{Re}\alpha > 0$, $I_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu}$ *provided* $\operatorname{Re}(m\eta + \mu) + m > 1/p$.

Proof. Suppose first that $\mu = 0$. Since F_p is a subspace of L_p, Lemma 3.1 shows that $I_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of F_p into L_p. From (1.1) and (1.3),

$$I_{x^m}^{\eta,\alpha}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_0^1 (1 - t^m)^{\alpha-1}t^{m\eta+m-1}\phi(xt)\,dt. \tag{3.3}$$

Differentiating under the integral sign in (3.3) gives

$$\delta I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,\alpha}\,\delta\phi,$$

from which it follows by induction that for $k = 0, 1, 2, \cdots$,

$$x^k\frac{d^k}{dx^k}I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,\alpha}x^k\frac{d^k\phi}{dx^k}. \tag{3.4}$$

By Lemma 3.1, for some number M (depending only on η, α and m),

$$\gamma_k^p(I_{x^m}^{\eta,\alpha}\phi) \leqq M\gamma_k^p(\phi),$$

and the theorem is proved for $\mu = 0$.

The general result follows from the previous case using the relation

$$I_{x^m}^{\eta,\alpha}\phi(x) = x^{\mu}I_{x^m}^{\eta+(\mu/m),\alpha}x^{-\mu}\phi, \qquad \phi \in F_{p,\mu}$$

and Theorem 2.6 (i).

We shall, in fact, prove much more about $I_{x^m}^{\eta,\alpha}$ shortly. One result we shall need is

$$I_{x^m}^{\eta+\alpha,\beta}I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,\alpha+\beta}\phi, \qquad \phi \in F_{p,\mu}, \tag{3.5}$$

valid provided $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}(m\eta + \mu) + m > 1/p$. Theorem 3.2 involves restrictions on η and α. That the restriction $\operatorname{Re}(m\eta + \mu) + m > 1/p$ is necessary is seen by taking $p = \infty$, $\phi(x) = x^\mu$, whence

$$I_{xm}^{\eta;\alpha}\phi(x) = \frac{\Gamma(\eta + (\mu/m) + 1)}{\Gamma(\alpha + \eta + (\mu/m) + 1)} x^\mu$$

which belongs to $F_{\infty,\mu}$ provided $\operatorname{Re}(\eta + (\mu/m) + 1) > 0$. On the other hand, we now proceed to remove the restriction $\operatorname{Re}\alpha > 0$ using analytic continuation. We make the following definition.

DEFINITION. Let V_1, V_2 be two countably multinormed spaces. Suppose that to each α in some domain D of the complex plane there corresponds a continuous linear mapping T_α from V_1 to V_2. We shall say that T_α is *analytic with respect to α in D* if there exists a continuous linear mapping $\partial T_\alpha/\partial\alpha$ of V_1 into V_2 such that, for each fixed $\phi \in V_1$,

$$\frac{1}{h}[T_{\alpha+h}\phi - T_\alpha\phi] - \frac{\partial T_\alpha}{\partial\alpha}\phi$$

converges to zero in the topology of V_2 as the (complex) increment $h \to 0$ in any manner.

It is easy to show that if $f(\alpha)$ is an analytic function of α in D (in the usual sense) and T_α is analytic in D (in the sense of the above definition), then the operator $f(\alpha)T_\alpha$ is analytic in D.

THEOREM 3.3. *On $F_{p,\mu}$, $I_{xm}^{\eta;\alpha}$ is analytic with respect to α for* $\operatorname{Re}\alpha > 0$, *provided that* $\operatorname{Re}(m\eta + \mu) + m > 1/p$.

Proof. See [13].

Notes. 1. It is clear that, under the hypotheses of Theorem 3.3, $I_{xm}^{\eta;\alpha}\phi(x)$ is, for each fixed x, an analytic function of α in the usual sense for $\operatorname{Re}\alpha > 0$.

2. A similar argument shows that, for each fixed α with $\operatorname{Re}\alpha > 0$, $I_{xm}^{\eta;\alpha}$ is analytic on $F_{p,\mu}$ with respect to η in the half-plane $\operatorname{Re}\eta > (1/m)((1/p) - m - \operatorname{Re}\mu)$.

We shall be concerned with analytic continuation with respect to α. We require the following lemma which can be proved by straightforward differentiation.

LEMMA 3.4. *Let* $\operatorname{Re}\alpha > 0$, $\phi \in F_{p,\mu}$, $\operatorname{Re}(m\eta + \mu) + m > 1/p$. *Then*

$$\delta I_{xm}^{\eta;\alpha+1}\phi = I_{xm}^{\eta;\alpha+1}\,\delta\phi = mI_{xm}^{\eta;\alpha}\phi - (m\eta + m\alpha + m)I_{xm}^{\eta;\alpha+1}\phi.$$

Rearranging the result of Lemma 3.4 gives

$$mI_{xm}^{\eta;\alpha}\phi = (m\eta + m\alpha + m)I_{xm}^{\eta;\alpha+1}\phi + I_{xm}^{\eta;\alpha+1}\,\delta\phi. \tag{3.6}$$

By Theorem 3.3 and the remark following our definition above, the right-hand side is analytic with respect to α for $\operatorname{Re}\alpha > -1$. We use (3.6) to continue $I_{xm}^{\eta;\alpha}$ analytically, in the first instance to $-1 < \operatorname{Re}\alpha \leqq 0$ and hence, step by step, to the whole complex α-plane.

Still assuming $\operatorname{Re}(m\eta + \mu) + m > 1/p$, we may put $\alpha = 0$ in (3.6) to obtain

$$I_{xm}^{\eta;0}\phi = \phi. \tag{3.7}$$

We can now prove our first main result.

THEOREM 3.5. *Let* $\operatorname{Re}(m\eta + \mu) + m > 1/p$, $1 \leqq p \leqq \infty$.

(i) *For any complex number α, $I_{xm}^{\eta;\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself.*

(ii) *For fixed* η, $I_{x^m}^{\eta,\alpha}$ *is entire with respect to* α *on* $F_{p,\mu}$.

(iii) *If, in addition,* $\operatorname{Re}(m\eta + m\alpha + \mu) + m > 1/p$, $I_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}$ *and*

$$(I_{x^m}^{\eta,\alpha})^{-1} = I_{x^m}^{\eta+\alpha,-\alpha}.$$

Proof. Parts (i) and (ii) follow using Theorems 3.2 and 3.3 along with sufficiently many applications of formula (3.6). We now prove (iii).

By analytic continuation, (3.5) is valid provided only $\operatorname{Re}(m\eta + \mu) + m > 1/p$ and $\operatorname{Re}(m\eta + m\alpha + \mu) + m > 1/p$. (This second condition was redundant before with $\operatorname{Re}\alpha > 0$). In this case for $\phi \in F_{p,\mu}$,

$$I_{x^m}^{\eta+\alpha,-\alpha} I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,0}\phi = \phi$$

by (3.7) and

$$I_{x^m}^{\eta,\alpha} I_{x^m}^{\eta+\alpha,-\alpha}\phi = I_{x^m}^{\eta+\alpha,0}\phi = \phi$$

by (3.7). The result follows.

Finally in this section, we state the mapping properties of $I_{x^m}^{\alpha}$ For $\operatorname{Re}\alpha > 0$, we have, from (1.1) and (1.3),

$$I_{x^m}^{\alpha}\phi(x) = x^{m\alpha} I_{x^m}^{0,\alpha}\phi(x). \tag{3.8}$$

We use (3.8) to define $I_{x^m}^{\alpha}$ for all α, this definition coinciding with (1.1) for $\operatorname{Re}\alpha > 0$. By Theorems 2.6 (i) and 3.2, $I_{x^m}^{\alpha}$ is a contintinuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$ provided $\operatorname{Re}\mu + m > 1/p$. In this case also we can prove

$$I_{x^m}^{\alpha}\phi(x) = \frac{d}{dx^m} I_{x^m}^{\alpha+1}\phi(x). \tag{3.9}$$

If we had developed the theory of $I_{x^m}^{\alpha}$ without proceeding via $I_{x^m}^{\eta,\alpha}$, we would use (3.9) to continue $I_{x^m}^{\alpha}$ analytically from $\operatorname{Re}\alpha > 0$ to the whole complex α-plane.

Still assuming $\operatorname{Re}\mu + m > 1/p$, we have from (3.7) and (3.8) that, for $\phi \in F_{p,\mu}$,

$$I_{x^m}^{0}\,\phi = \phi.$$

It follows from (3.9) that, for $n = 0, 1, 2, \cdots$,

$$I_{x^m}^{-n}\phi = \left(\frac{d}{dx^m}\right)^n \phi, \tag{3.10}$$

as might be expected.

It can also be proved using (3.5) and (3.8) that for $\phi \in F_{p,\mu}$, $(1/p) - m - \operatorname{Re}\mu < \min(0, m\operatorname{Re}\alpha, m\operatorname{Re}\beta)$,

$$I_{x^m}^{\alpha}\, I_{x^m}^{\beta}\phi = I_{x^m}^{\alpha+\beta}\phi = I_{x^m}^{\beta} I_{x^m}^{\alpha}\phi. \tag{3.11}$$

This leads to Theorem 3.6.

THEOREM 3.6. $I_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$ *provided* $\operatorname{Re}\mu + m > 1/p$. $I_{x^m}^{0}$ *is the identity operator. If, in addition,* $\operatorname{Re}(\mu + m\alpha) + m > 1/p$, $I_{x^m}^{\alpha}$ *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and*

$$(I_{x^m}^{\alpha})^{-1} = I_{x^m}^{-\alpha}.$$

Equation (3.11) enables us to write down an explicit expression for $I_{x^m}^{\alpha}$ for any α; if $\phi \in F_{p,\mu}$, $\operatorname{Re}\mu + m > 1/p$, $\operatorname{Re}\alpha + n > 0$, then

$$I_{x^m}^{\alpha}\phi(x) = \frac{m}{\Gamma(\alpha + n)}\left(\frac{d}{dx^m}\right)^n \int_0^x (x^m - u^m)^{\alpha+n-1}u^{m-1}\phi(u)\,du. \tag{3.12}$$

We mention also the second index law for the operators $I_{x^m}^{\alpha}$; if $\phi \in F_{p,\mu}$, $-\operatorname{Re}\mu - m + (1/p) < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$, then

$$x^{m\alpha}I_{x^m}^{\beta}x^{m\gamma}\phi(x) = I_{x^m}^{-\gamma}x^{-m\beta}I_{x^m}^{-\alpha}\phi(x). \tag{3.13}$$

We shall not prove (3.13) here, but defer the proof to a subsequent paper where the result arises naturally in connection with hypergeometric integral equations. Equations (3.11) and (3.13) have been studied in the case $m = 1$ by Love [12] for ordinary functions and by Erdélyi [5] for a class of generalized functions.

3.2. The operators $K_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$. We now consider the operators $K_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$. For $\operatorname{Re}\alpha > 0$, we have from (1.2) and (1.4) that

$$\begin{aligned} K_{x^m}^{\eta,\alpha}\phi(x) &= x^{m\eta}K_{x^m}^{\alpha}x^{-m\eta-m\alpha}\phi(x) \\ &= \frac{m}{\Gamma(\alpha)}\int_1^{\infty}(t^m - 1)^{\alpha-1}t^{-m\eta-m\alpha+m-1}\phi(xt)\,dt. \end{aligned} \tag{3.14}$$

We obtain the properties of $K_{x^m}^{\eta,\alpha}$ using arguments similar to those for $I_{x^m}^{\eta,\alpha}$. We shall mention only the salient points.

THEOREM 3.7. *Let* $\operatorname{Re}(m\eta - \mu) > -1/p$, $1 \leqq p \leqq \infty$.

(i) *For any complex number* α, $K_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into itself.*

(ii) *For fixed* η, $K_{x^m}^{\eta,\alpha}$ *is entire with respect to* α *on* $F_{p,\mu}$.

(iii) *If, in addition,* $\operatorname{Re}(m\eta + m\alpha - \mu) > -1/p$, $K_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}$ *and*

$$(K_{x^m}^{\eta,\alpha})^{-1} = K_{x^m}^{\eta+\alpha,-\alpha}.$$

Proof. (i) For $\operatorname{Re}\alpha > 0$, the result follows using a result of Kober [9] and differentiating under the integral sign in (3.14). We extend the definition of $K_{x^m}^{\eta,\alpha}$ to $\operatorname{Re}\alpha \leqq 0$ using the formula

$$mK_{x^m}^{\eta,\alpha}\phi(x) = (m\eta + m\alpha)K_{x^m}^{\eta,\alpha+1}\phi(x) - K_{x^m}^{\eta,\alpha+1}\,\delta\phi(x), \tag{3.15}$$

which is an analogue of (3.6) and is valid for $\phi \in F_{p,\mu}$ if $\operatorname{Re}(m\eta - \mu) > -1/p$. Use of (3.15) completes the proof of (i).

As regards (ii) and (iii), we proceed as for $I_{x^m}^{\eta,\alpha}$ using (3.15) and the additional results

$$K_{x^m}^{\eta,0}\phi = \phi, \tag{3.16}$$

valid for $\phi \in F_{p,\mu}$, $\operatorname{Re}(m\eta - \mu) > -1/p$, and

$$K_{x^m}^{\eta,\alpha}K_{x^m}^{\eta+\alpha,\beta}\phi = K_{x^m}^{\eta,\alpha+\beta}\phi, \tag{3.17}$$

valid when $\phi \in F_{p,\mu}$, $\operatorname{Re}(m\eta - \mu) > -1/p$ and $\operatorname{Re}(m\eta + m\alpha - \mu) > -1/p$.

To obtain the results for K_{xm}^{α}, we note first that for $\operatorname{Re}\alpha > 0$,

$$K_{xm}^{\alpha}\phi = K_{xm}^{0;\alpha}x^{m\alpha}\phi \tag{3.18}$$

from (1.2) and (1.4). We use (3.18) to define K_{xm}^{α} for all α. Using Theorem 3.7 gives the following.

THEOREM 3.8. *If* $\operatorname{Re}(\mu + m\alpha) < 1/p$, K_{xm}^{α} *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$. *If also* $\operatorname{Re}\mu < 1/p$, K_{xm}^{α} *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and*

$$(K_{xm}^{\alpha})^{-1} = K_{xm}^{-\alpha}.$$

K_{xm}^{0} *is the identity operator on* $F_{p,\mu}$ *if* $\operatorname{Re}\mu < 1/p$.

Using (3.15), we can show that, for $\operatorname{Re}(\mu + m\alpha) < 1/p$,

$$K_{xm}^{\alpha}\phi(x) = -K_{xm}^{\alpha+1}\frac{d}{dx^m}\phi(x), \qquad \phi \in F_{p,\mu}, \tag{3.19}$$

from which it follows by induction that if $\operatorname{Re}\mu - mn < 1/p, n = 0, 1, 2, \cdots$,

$$K_{xm}^{-n}\phi = \left(-\frac{d}{dx^m}\right)^n\phi. \tag{3.20}$$

The first index law for the operators K_{xm}^{α} is

$$K_{xm}^{\alpha}K_{xm}^{\beta}\phi = K_{xm}^{\alpha+\beta}\phi = K_{xm}^{\beta}K_{xm}^{\alpha}\phi, \tag{3.21}$$

valid when $(1/p) - \operatorname{Re}\mu > \max(m\operatorname{Re}\alpha, m\operatorname{Re}\beta, m\operatorname{Re}(\alpha+\beta))$. The second index law states that for $\phi \in F_{p,\mu}$, $\operatorname{Re}\mu - (1/p) < \min(0, m\operatorname{Re}\gamma), \alpha + \beta + \gamma = 0$,

$$x^{m\gamma}K_{xm}^{\beta}x^{m\alpha}\phi = K_{xm}^{-\alpha}x^{-m\beta}K_{xm}^{-\gamma}\phi. \tag{3.22}$$

For discussion of (3.21) and (3.22) we again refer the reader to [5] and [12].

3.3. The action of $I_{xm}^{\eta,\alpha}$ and $K_{xm}^{\eta,\alpha}$ on $F'_{p,\mu}$. We are now ready to develop the theory of fractional integration on the spaces $F'_{p,\mu}$ of generalized functions. As usual, our definitions are motivated by considering regular functionals.

Let $f \in F'_{p,\mu}$. From adjoint considerations we are led to define $I_{xm}^{\eta,\alpha}f$, for $\operatorname{Re}\alpha > 0$, by

$$(I_{xm}^{\eta,\alpha}f, \phi) = (f, K_{xm}^{\eta+1-(1/m),\alpha}\phi), \tag{3.23}$$

where $\phi \in F_{p,\mu}$. However, the right-hand side is meaningful provided only $\operatorname{Re}(m\eta - \mu) + m > 1/q$ by Theorem 3.7; in this case, we can remove the restriction $\operatorname{Re}\alpha > 0$ and use (3.23) to define $I_{xm}^{\eta,\alpha}f$ for all complex α.

THEOREM 3.9. *Let* $1 \leqq p \leqq \infty$ *and let* α *be any complex number.*

(i) $I_{xm}^{\eta,\alpha}$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu}$ *provided that* $\operatorname{Re}(m\eta - \mu) + m > 1/q$.

(ii) *If, in addition,* $\operatorname{Re}(m\eta + m\alpha - \mu) + m > 1/q$, $I_{xm}^{\eta,\alpha}$ *is an automorphism of* $F'_{p,\mu}$ *and*

$$(I_{xm}^{\eta,\alpha})^{-1} = I_{xm}^{\eta+\alpha,-\alpha}.$$

Proof. By Theorem 3.7 (i), $K_{xm}^{\eta+1-(1/m),\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided $\operatorname{Re} m(\eta + 1 - (1/m)) - \mu > -1/p$, i.e., $\operatorname{Re}(m\eta - \mu)$

$+ m > 1/q$. Part (i) now follows by [16, Thm. 1.10-1]. Part (ii) follows similarly using Theorem 3.7 (iii) in conjunction with [16, Thm. 1.10-2].

Using (3.17) and (3.23) we see that if $f \in F'_{p,\mu}$,

$$I_{x^m}^{\eta+\alpha,\beta} I_{x^m}^{\eta,\alpha} f = I_{x^m}^{\eta,\alpha+\beta} f \tag{3.24}$$

provided $\operatorname{Re}(m\eta - \mu) + m > 1/q$, $\operatorname{Re}(m\eta + m\alpha - \mu) + m > 1/q$, while from (3.16),

$$I_{x^m}^{\eta,0} f = f \tag{3.25}$$

provided $\operatorname{Re}(m\eta - \mu) + m > 1/q$. Equations (3.24) and (3.25) are analogous to (3.5) and (3.7), respectively.

It is now clear that to obtain results for $F'_{p,\mu}$ from the corresponding results for $F_{p,\mu}$ (e.g., to obtain Theorem 3.9 from Theorem 3.5) we interchange μ and $-\mu$, p and q in the restrictions on the parameters. This trend, which continues below, is to be expected from consideration of Hölder's inequality. If $\phi \in F_{p,\mu}$, $\int_0^\infty f(x)\phi(x)\,dx$ will converge if $f(x) = x^{-\mu} g(x)$ with $g \in L_q$ and, in particular, if $f \in F_{q,-\mu}$.

We note in passing that for fixed $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu}$ and $\operatorname{Re}(m\eta - \mu) + m > 1/q$,

$$(I_{x^m}^{\eta,\alpha} f, \phi)$$

is an entire function of α by virtue of Theorem 3.7 (ii). However, this will not be needed here.

Proceeding as before, we are led to define $K_{x^m}^{\eta,\alpha}$ for any α and any $f \in F'_{p,\mu}$ by

$$(K_{x^m}^{\eta,\alpha} f, \phi) = (f, I_{x^m}^{\eta-1+(1/m),\alpha} \phi), \qquad \phi \in F_{p,\mu}. \tag{3.26}$$

Using Theorem 3.5 we obtain the following result analogous to Theorem 3.7.

THEOREM 3.10. *For $1 \leqq p \leqq \infty$ and any complex α, $K_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself provided* $\operatorname{Re}(m\eta + \mu) > -1/q$. *If, in addition,* $\operatorname{Re}(m\eta + m\alpha + \mu) > -1/q$, *$K_{x^m}^{\eta,\alpha}$ is an automorphism of $F'_{p,\mu}$ and*

$$(K_{x^m}^{\eta,\alpha})^{-1} = K_{x^m}^{\eta+\alpha,-\alpha}.$$

For $f \in F'_{p,\mu}$, we have analogues of (3.16) and (3.17).

$$K_{x^m}^{\eta,0} f = f \tag{3.27}$$

for $\operatorname{Re}(m\eta + \mu) > -1/q$; if in addition, $\operatorname{Re}(m\eta + m\alpha + \mu) > -1/q$,

$$K_{x^m}^{\eta,\alpha} K_{x^m}^{\eta+\alpha,\beta} f = K_{x^m}^{\eta,\alpha+\beta} f. \tag{3.28}$$

Finally we discuss the properties of $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$ on $F'_{p,\mu}$.

Let $f \in F'_{p,\mu}$. As before, from adjoint considerations, we are led to define $I_{x^m}^{\alpha} f$ for any complex α by

$$(I_{x^m}^{\alpha} f, \phi) = (f, x^{m-1} K_{x^m}^{\alpha} x^{-m+1} \phi). \tag{3.29}$$

The right-hand side is meaningful provided only that $\phi \in F_{p,\mu-m\alpha}$ and $m - \operatorname{Re}\mu > 1/q$ by Theorem 3.8. Similarly, for $\phi \in F_{p,\mu-m\alpha}$, $f \in F'_{p,\mu}$, we define $K_{x^m}^{\alpha} f$ by

$$(K_{x^m}^{\alpha} f, \phi) = (f, x^{m-1} I_{x^m}^{\alpha} x^{-m+1} \phi). \tag{3.30}$$

Use of Theorems 3.6 and 3.8 proves the next theorem.

THEOREM 3.11. (i) *$I^{\alpha}_{x^m}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$ provided $m - \operatorname{Re}\mu > 1/q$. $I^0_{x^m}$ is the identity operator. If, in addition, $m + \operatorname{Re}(m\alpha - \mu) > 1/q$, $I^{\alpha}_{x^m}$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and*

$$(I^{\alpha}_{x^m})^{-1} = I^{-\alpha}_{x^m}.$$

(ii) *$K^{\alpha}_{x^m}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$ provided $\operatorname{Re}(m\alpha - \mu) < 1/q$. If, in addition, $-\operatorname{Re}\mu < 1/q$, $K^{\alpha}_{x^m}$ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and*

$$(K^{\alpha}_{x^m})^{-1} = K^{-\alpha}_{x^m}.$$

$K^0_{x^m}$ is the identity operator on $F'_{p,\mu}$ if $-\operatorname{Re}\mu < 1/q$.

For $f \in F'_{p,\mu}$, we have the following index laws analogous to (3.11), (3.13), (3.21) and (3.22).

$$I^{\alpha}_{x^m} I^{\beta}_{x^m} f = I^{\alpha+\beta}_{x^m} f = I^{\beta}_{x^m} I^{\alpha}_{x^m} f \tag{3.31}$$

provided $(1/q) - m + \operatorname{Re}\mu < \min(0, m\operatorname{Re}\alpha, m\operatorname{Re}\beta)$.

$$x^{m\alpha} I^{\beta}_{x^m} x^{m\gamma} f = I^{-\gamma}_{x^m} x^{-m\beta} I^{-\alpha}_{x^m} f \tag{3.32}$$

provided $(1/q) - m + \operatorname{Re}\mu < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$.

$$K^{\alpha}_{x^m} K^{\beta}_{x^m} f = K^{\alpha+\beta}_{x^m} f = K^{\beta}_{x^m} K^{\alpha}_{x^m} f \tag{3.33}$$

provided $(1/q) + \operatorname{Re}\mu > \max(m\operatorname{Re}\alpha, m\operatorname{Re}\beta, m\operatorname{Re}(\alpha+\beta))$.

$$x^{m\gamma} K^{\beta}_{x^m} x^{m\alpha} f = K^{-\alpha}_{x^m} x^{-m\beta} K^{-\gamma}_{x^m} f \tag{3.34}$$

provided $-(1/q) - \operatorname{Re}\mu < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$.

4.1. The operators L_v. For any suitable function ϕ and any complex number v, we define the differential operator L_v by

$$(L_v\phi)(x) = \frac{d^2\phi}{dx^2} + \frac{2v+1}{x}\frac{d\phi}{dx}. \tag{4.1}$$

In this section, we consider connections between L_v and operators of fractional integration which have been discussed for ordinary functions by Erdélyi in [3], and one of which has been established for the class $\mathscr{F}'$ of generalized functions by Erdélyi and McBride in [6].

It is immediate from Theorem 2.6 that for all complex numbers μ and v and for $1 \leqq p \leqq \infty$, L_v is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-2}$. For $f \in F'_{p,\mu}$, we define $L_v f$ by

$$(L_v f, \phi) = (f, xL_{-v}x^{-1}\phi), \qquad \phi \in F_{p,\mu+2}. \tag{4.2}$$

The motivation for (4.2) is supplied by taking f to be a regular functional generated by a C^2 function, taking $\phi \in \mathscr{D}$ and integrating by parts. Using [16, Thm. 1.10-1], we immediately deduce Theorem 4.1.

THEOREM 4.1. *For any complex numbers μ and v and for $1 \leqq p \leqq \infty$, L_v is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+2}$.*

As regards connections with fractional integration, we have Theorem 4.2.

THEOREM 4.2. *Let* $\phi \in F_{p,\mu}$, $1 \leqq p \leqq \infty$.

(i) *If* $\mathrm{Re}\,(2\nu + \mu) > 1/p$,

$$I_{x^2}^{\nu,\alpha} L_\nu \phi = L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} \phi. \tag{4.3}$$

(ii) *If* $\mathrm{Re}\,(2\nu - \mu) > -1/p$,

$$L_{-\nu} K_{x^2}^{\nu,\alpha} \phi = K_{x^2}^{\nu,\alpha} L_{-\nu-\alpha} \phi. \tag{4.4}$$

Proof. To prove (i) we can proceed as in [6, § 6], or use (3.15). The proof of (ii) is similar, so we shall omit the details.

(4.3) and (4.4) give perhaps the neatest relations between L_ν and fractional integration operators on $F_{p,\mu}$. We now give the corresponding results for $F'_{p,\mu}$.

THEOREM 4.3. *Let* $f \in F'_{p,\mu}$, $1 \leqq p \leqq \infty$.

(i) *If* $\mathrm{Re}\,(2\nu - \mu) > 1/q$,

$$I_{x^2}^{\nu,\alpha} L_\nu f = L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} f. \tag{4.5}$$

(ii) If $\mathrm{Re}\,(2\nu + \mu) > -1/q$,

$$L_{-\nu} K_{x^2}^{\nu,\alpha} f = K_{x^2}^{\nu,\alpha} L_{-\nu-\alpha} f. \tag{4.6}$$

Proof. (i) For $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu+2}$, (3.23) and (4.2) give

$$\begin{aligned}(I_{x^2}^{\nu,\alpha} L_\nu f, \phi) &= (f, xL_{-\nu} x^{-1} K_{x^2}^{\nu+(1/2),\alpha} \phi)\\ &= (f, xL_{-\nu} K_{x^2}^{\nu,\alpha} x^{-1} \phi)\end{aligned}$$

and similarly

$$(I_{x^2}^{\nu,\alpha} L_{\nu+\alpha} f, \phi) = (f, xK_{x^2}^{\nu,\alpha} L_{-\nu-\alpha} x^{-1} \phi).$$

The result now follows from Theorem 4.2 (ii) with μ, ϕ replaced by $\mu + 1$ and $x^{-1}\phi$, respectively. Part (ii) follows similarly from Theorem 4.2 (i).

4.2. Solution of $L_\nu \phi = \psi$. Suppose $\psi \in F_{p,\mu-2}$ is given. We wish to find $\phi \in F_{p,\mu}$ such that $L_\nu \phi = \psi$, i.e.,

$$\begin{aligned}&\frac{d^2\phi}{dx^2} + \frac{2\nu+1}{x}\frac{d\phi}{dx} = \psi\\ &\Rightarrow \frac{d}{dx}\left(x^{2\nu+1}\frac{d\phi}{dx}\right) = x^{2\nu+1}\psi.\end{aligned} \tag{4.7}$$

The problem then reduces to inverting $D = d/dx$, and for this we fall back on Theorems 3.6 and 3.8, which tell us that D is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ provided $\mathrm{Re}\,\mu \neq 1/p$ and

$$D^{-1} = \begin{cases} I_x^1, & \mathrm{Re}\,\mu > 1/p,\\ -K_x^1, & \mathrm{Re}\,\mu < 1/p.\end{cases}$$

As regards the case $\mathrm{Re}\,\mu = 1/p$, take $p = \infty$ so that $\mu = 0$. Then $D\phi = 0$ for *every* constant function $\phi \in F_\infty$, and clearly D is not invertible in this case. Using (4.7) we easily obtain the following theorem.

THEOREM 4.4. *For each* $\psi \in F_{p,\mu-2}$, *the equation* $L_\nu\phi = \psi$ *has a unique solution* $\phi \in F_{p,\mu}$ *provided* $\mathrm{Re}\,(2\nu + \mu) \neq 1/p$ *and* $\mathrm{Re}\,\mu \neq 1/p$.

(i) *If* $\mathrm{Re}\,(2\nu + \mu) > 1/p$, $\mathrm{Re}\,\mu > 1/p$,

$$\phi = I_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1}\psi.$$

(ii) *If* $\mathrm{Re}\,(2\nu + \mu) > 1/p$, $\mathrm{Re}\,\mu < 1/p$,

$$\phi = -K_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1}\psi.$$

(iii) *If* $\mathrm{Re}\,(2\nu + \mu) < 1/p$, $\mathrm{Re}\,\mu > 1/p$,

$$\phi = -I_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1}\psi.$$

(iv) *If* $\mathrm{Re}\,(2\nu + \mu) < 1/p$, $\mathrm{Re}\,\mu < 1/p$,

$$\phi = K_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1}\psi.$$

The results are particularly simple when $\nu = -\frac{1}{2}$ (in which case (ii) is redundant) and $\nu = 0$ (when (ii) and (iii) are redundant). We can use these special cases in conjunction with Theorem 4.2 to derive alternative expressions for the solution ϕ in the general case. For instance, for the case $\nu = -\frac{1}{2}$, we obtain Theorem 4.5.

THEOREM 4.5. *If* $\mathrm{Re}\,(2\nu + \mu) \neq 1/p$ and $\mathrm{Re}\,\mu \neq 1/p$, *the* (*unique*) *solution* $\phi \in F_{p,\mu}$ *of* $L_\nu\phi = \psi$, $\psi \in F_{p,\mu-2}$, *is given as follows*:

(i) *If* $\mathrm{Re}\,(2\nu + \mu) > 1/p$, $\mathrm{Re}\,(-1 + \mu) > 1/p$

$$\phi = I_{x^2}^{-1/2,\nu+1/2} I_x^2 I_{x^2}^{\nu,-\nu-1/2}\psi,$$

(ii) *If* $\mathrm{Re}\,(2\nu + \mu) < 1/p$, $\mathrm{Re}\,(-1 + \mu) < 1/p$

$$\phi = -K_{x^2}^{1/2,-\nu-1/2} I_x^1 K_x^1 K_{x^2}^{-\nu,\nu+1/2}\psi, \qquad 1/p < \mathrm{Re}\,\mu < (1/p) + 1,$$

$$\phi = K_{x^2}^{1/2,-\nu-1/2} K_x^2 K_{x^2}^{-\nu,\nu+1/2}\psi, \qquad \mathrm{Re}\,\mu < 1/p.$$

Proof.

(i) Since $\mathrm{Re}\,(2\nu + \mu) > 1/p$, we may take $\alpha = -\nu - \frac{1}{2}$ in (4.3) to get

$$I_{x^2}^{\nu,-\nu-1/2} L_\nu\phi = L_{-1/2} I_{x^2}^{\nu,-\nu-1/2}\phi$$

$$\Rightarrow \psi = L_\nu\phi = I_{x^2}^{-1/2,\nu+1/2} L_{-1/2} I_{x^2}^{\nu,-\nu-1/2}\phi$$

using Theorem 3.5 (since $\mathrm{Re}\,\mu + 1 > 1/p$). Provided $(L_{-1/2})^{-1}$ exists, we may now invert obtaining

$$\phi = L_\nu^{-1}\psi = I_x^{-1/2,\nu+1/2}(L_{-1/2})^{-1} I_{x^2}^{\nu,-\nu-1/2}\psi,$$

and we use Theorem 4.4 (i) to substitute for $(L_{-1/2})^{-1}$.

Part (ii) is proved similarly using (4.4).

There are other possible expressions for ϕ, but we shall not list them here.

4.3. Solution of $L_\nu f = g$. Suppose now that $g \in F'_{p,\mu+2}$ is given. We have to find $f \in F'_{p,\mu}$ such that $L_\nu f = g$. To obtain the solution, we can either imitate the methods of § 4.2 or else take adjoints.

THEOREM 4.6. *For each* $g \in F'_{p,\mu+2}$, *the equation* $L_\nu f = g$ *has a unique solution* $f \in F'_{p,\mu}$ *provided* $\mathrm{Re}(2\nu - \mu) \neq 1/q$ *and* $\mathrm{Re}\,\mu \neq 1/q$.

(i) *If* $\mathrm{Re}(2\nu - \mu) > 1/q$, $-\mathrm{Re}\,\mu > 1/q$,

$$f = I_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1} g.$$

(ii) *If* $\mathrm{Re}(2\nu - \mu) > 1/q$, $-\mathrm{Re}\,\mu < 1/q$,

$$f = -K_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1} g.$$

(iii) *If* $\mathrm{Re}(2\nu - \mu) < 1/q$, $-\mathrm{Re}\,\mu > 1/q$,

$$f = -I_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1} g.$$

(iv) *If* $\mathrm{Re}(2\nu - \mu) < 1/q$, $-\mathrm{Re}\,\mu < 1/q$,

$$f = K_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1} g.$$

As an illustration consider (i). Under the given conditions, $xL_{-\nu}x^{-1}$ is invertible on $F_{p,\mu+2}$, and from Theorem 4.4 (iv), if $\psi \in F_{p,\mu}$,

$$xL_{-\nu}x^{-1}\phi = \psi \Leftrightarrow \phi = xK_x^1 x^{2\nu-1} K_x^1 x^{-2\nu}\psi.$$

Hence taking adjoints, where $g \in F'_{p,\mu+2}$, we see that

$$L_\nu f = g \Leftrightarrow f = x^{-2\nu} I_x^1 x^{2\nu-1} I_x^1 x g.$$

This is a perfectly acceptable expression for the solution, but to obtain the form in (i) we use the index laws (3.31) and (3.32). Indeed,

$$\begin{aligned} f &= x^{-2\nu} I_x^1 x^{2\nu-1} I_x^1 x g = I_x^{1-2\nu} x^{-1} I_x^{2\nu} I_x^1 x g \\ &= I_x^1 (I_x^{-2\nu} x^{-1} I_x^{2\nu+1}) x g = I_x^1 (x^{-2\nu-1} I_x^1 x^{2\nu}) x g, \end{aligned}$$

from which (i) follows; the above steps are all valid under the given conditions. Parts (ii)–(iv) are similar.

Again other equivalent solution formulas can be obtained if required via (4.5) and (4.6).

Acknowledgments. The contents of this paper form part of the author's Ph.D. thesis written at the University of Edinburgh under the supervision of Professor A. Erdélyi.

I would like to record my sincere thanks to Professor Erdélyi for his help and encouragement and for many stimulating discussions.

Finally, I would like to thank the referee for several helpful suggestions.

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Paper 2

Reprinted from PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY
Vol. 19 (Series II), Part 3, March 1975

SOLUTION OF HYPERGEOMETRIC INTEGRAL EQUATIONS INVOLVING GENERALISED FUNCTIONS

by ADAM C. McBRIDE

(Received 10th December 1973)

1. Introduction

In a previous paper (9), we introduced the spaces $F_{p,\mu}$ of testing-functions and the corresponding spaces $F'_{p,\mu}$ of generalised functions. For $1 \leqq p \leqq \infty$, $F_p(= F_{p,0})$ is the linear space of all complex-valued measurable functions ϕ defined on $(0, \infty)$ which are infinitely differentiable on $(0, \infty)$ and for which $x^k \dfrac{d^k\phi}{dx^k} \in L^p(0, \infty)$ for each $k = 0, 1, 2, \ldots$. In symbols,

$$F_p = \left\{\phi \in C^\infty(0, \infty) : x^k \frac{d^k\phi}{dx^k} \in L^p(0, \infty) \text{ for } k = 0, 1, 2, \ldots\right\}. \tag{1.1}$$

F_p is equipped with the topology generated by the semi-norms $\{\gamma_k^p\}_{k=0}^\infty$ where, for $\phi \in F_p$,

$$\gamma_k^p(\phi) = \left\| x^k \frac{d^k\phi}{dx^k} \right\|_p \quad (k = 0, 1, 2, \ldots) \tag{1.2}$$

(and $\| \ \|_p$ denotes the usual $L^p(0, \infty)$ norm). Then, for any complex number μ, and $1 \leqq p \leqq \infty$, we define $F_{p,\mu}$ by

$$F_{p,\mu} = \{\phi : x^{-\mu}\phi(x) \in F_p\}. \tag{1.3}$$

$F_{p,\mu}$ is equipped with the topology generated by the semi-norms $\{\gamma_k^{p,\mu}\}_{k=0}^\infty$ where, for $\phi \in F_{p,\mu}$,

$$\gamma_k^{p,\mu}(\phi) = \gamma_k^p(x^{-\mu}\phi) \quad (k = 0, 1, 2, \ldots) \tag{1.4}$$

with γ_k^p as in (1.2). Finally, $F'_{p,\mu}$ is the linear space of continuous linear functionals on $F_{p,\mu}$; it is equipped with the topology of weak (or pointwise) convergence.

The spaces $F_{p,\mu}$ and $F'_{p,\mu}$ are amenable to the study of various operators of fractional integration and, in (9), we investigated the mapping properties of $I_{x^m}^\alpha$ and $K_{x^m}^\alpha$, the Riemann-Liouville and Weyl fractional integrals respectively, as well as the Erdélyi-Kober operators $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$. We also gave a simple application to differential operators. In this paper, we are going to turn our attention to integral operators.

We shall be concerned with four operators $H_i(a, b;\ c;\ m)$ $(i = 1, 2, 3, 4)$

typical of which is $H_1(a, b;\ c;\ m)$, defined for Re $c>0$ and suitable functions ϕ by

$$H_1(a, b;\ c;\ m)\phi(x) = \int_0^x \frac{(x^m-t^m)^{c-1}}{\Gamma(c)} F\left(a, b;\ c;\ 1-\frac{x^m}{t^m}\right) mt^{m-1}\phi(t)dt. \quad (1.5)$$

Here $m>0$ is real, a, b and c are complex and $F(a, b;\ c;\ z) \equiv {}_2F_1(a, b;\ c;\ z)$ is the Gauss hypergeometric function. For $m = 1$, the operators $H_i(a, b;\ c;\ m)$ have been discussed at length for classical functions by Love in (5) and (6). Love's results unified the work of many authors who had earlier treated particular cases; see (5) and (6) for references. Here we work with generalised functions in $F'_{p,\mu}$ rather than classical functions. With tools such as analytic continuation available, it is not surprising that the restrictions on the parameters involved are not so numerous as in Love's work. On the other hand, we end up with a generalised solution which may or may not correspond to a classical solution. However, we shall give one result to indicate how, in a particular case, we can recover a classical solution.

In Section 2, we gather together a few facts about $F(a, b;\ c;\ z)$ which we require in the sequel. In Section 3, we develop the properties of $H_1(a, b;\ c;\ m)$ on $F_{p,\mu}$ by establishing a connection with the operators $I_{x^m}^{\eta;\alpha}$. Analytic continuation and results in (9) enable us to extend the definition of $H_1(a, b;\ c;\ m)$ to values of c with Re $c \leqq 0$ (although in this case we will no longer be able to use the integral representation (1.5)). In Section 4, we introduce the other three operators $H_i(a, b;\ c;\ m)$ $(i = 2, 3, 4)$ on $F_{p\mu}$ and obtain connections between them. $H_3(a, b;\ c;\ m)$ and $H_4(a, b;\ c;\ m)$ are the adjoints of $H_2(a, b;\ c;\ m)$ and $H_1(a, b;\ c;\ m)$ respectively and moreover we find that

$$[H_2(a, b;\ c;\ m)]^{-1} = H_1(-a, -b;\ -c;\ m)$$

a fact which does not emerge clearly in (5).

The use of adjoint operators enables us to define the operators

$$H_i(a, b;\ c;\ m) \quad (i = 1, 2, 3, 4)$$

on $F'_{p,\mu}$ and in Section 5 we obtain their mapping properties as well as giving formulae for the solution f of

$$H_i(a, b;\ c;\ m)f = g,$$

where g is a given generalised function. Finally, in Section 6, we compare and contrast our results with those of Love and also discuss the problem of finding classical solutions.

In the course of our travels, we establish on $F_{p,\mu}$ and $F'_{p,\mu}$ the second index laws for the operators $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$; if $\alpha+\beta+\gamma = 0$, then

$$x^{m\alpha}I_{x^m}^{\beta}x^{m\gamma}f = I_{x^m}^{-\gamma}x^{-m\beta}I_{x^m}^{-\alpha}f$$
$$x^{m\gamma}K_{x^m}^{\beta}x^{m\alpha}f = K_{x^m}^{-\alpha}x^{-m\beta}K_{x^m}^{-\gamma}f$$

are valid under appropriate conditions. These arise naturally in the discussions; they have been discussed for classical functions by Love (7) and for a class of generalised functions by Erdélyi (1).

Throughout, we shall use the notation and terminology of (**9**). In particular, we shall assume that $m>0$, that $1 \leqq p \leqq \infty$ (unless the contrary is stated) and that p, q are connected by $1/p+1/q = 1$.

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We shall denote Gauss' hypergeometric function by $F(a, b;\ c;\ z)$. Thus for complex numbers a, b and c with $c \neq 0, -1, -2, \ldots$, and for $|z|<1$,

$$F(a, b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \tag{2.1}$$

where, for example,

$$(a)_0 = 1,$$
$$(a)_n = a(a+1)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a) \quad (n \geqq 1).$$

For brevity we shall write

$$F^*(a, b;\ c;\ z) = F(a, b;\ c;\ z)/\Gamma(c).$$

Thus, for $|z| < 1$ and for any complex numbers a, b and c

$$F^*(a, b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{\Gamma(c+n)} \frac{z^n}{n!} \tag{2.2}$$

The restriction $c \neq 0, -1, -2, \ldots$ is no longer necessary since the reciprocal of the gamma function is an entire function. $F^*(a, b;\ c;\ z)$ as defined by (2.2) is an entire function of a, b and c, and an analytic function of z for $|z|<1$.

We shall require values of $F(a, b;\ c;\ z)$ for z on the negative real axis. We therefore extend $F^*(a, b;\ c;\ z)$ to the half-plane $\mathrm{Re}\, z<\frac{1}{2}$ using one of Kummer's relations ((**2**), p. 105)

$$F^*(a, b;\ c;\ z) = (1-z)^{-a}F^*\left(a, c-b;\ c; \frac{z}{z-1}\right) \tag{2.3}$$

using the principal branch of $(1-z)^{-a}$. The extended function is an entire function of a, b and c, and an analytic function of z for $\mathrm{Re}\, z<\frac{1}{2}$. Also, by (2.2) and analytic continuation,

$$F^*(a, b;\ c;\ z) = F^*(b, a;\ c;\ z) \tag{2.4}$$

for $\mathrm{Re}\, z<\frac{1}{2}$.

To discuss the operator $H_1(a, b;\ c;\ m)$ we require the following result.

Lemma 2.1. *Let a, b and c be complex numbers and let $\delta>0$. Then there exists a constant M, independent of v, such that, for $0<v<1$, the four expressions*

$$|F^*(a, b;\ c;\ 1-1/v^m)|, \quad \left|\frac{\partial}{\partial a} F^*(a, b;\ c;\ 1-1/v^m)\right|$$

$$\left|\frac{\partial}{\partial b} F^*(a, b;\ c;\ 1-1/v^m)\right|, \quad \left|\frac{\partial}{\partial c} F^*(a, b;\ c;\ 1-1/v^m)\right|$$

are all less than or equal to $Mv^{\min(m\,\mathrm{Re}\, a,\, m\,\mathrm{Re}\, b)-\delta}$.

Proof. The proof of this result for $m = 1$ is given in (**8**). The general result then follows easily.

3

We now proceed to the discussion of $H_1(a, b; c; m)$ on $F_{p,\mu}$. We recall that for Re $c>0$, $m>0$ and suitable functions ϕ, $H_1(a, b; c; m)\phi$ is defined, for $0<x<\infty$, by

$$H_1(a, b; c; m)\phi(x) = \int_0^x (x^m - t^m)^{c-1} F^*(a, b; c; 1 - x^m/t^m) m t^{m-1} \phi(t) dt. \quad (3.1)$$

To begin with, we note

Lemma 3.1. *Let* Re $c>0$, $-\mathrm{Re}\,\mu - m + 1/p < \min(m\,\mathrm{Re}\,a, m\,\mathrm{Re}\,b)$, $\phi \in F_{p,\mu}$. *Then*

(i) *the integral* (3.1) *for* $H_1(a, b; c; m)\phi(x)$ *exists and defines a continuous function of* x *on* $(0, \infty)$,

(ii) *for each fixed* $x \in (0, \infty)$, $H_1(a, b; c; m)\phi(x)$ *is an analytic function of the (single) variables* a, b, c *in the regions* $-\mathrm{Re}\,\mu - m + 1/p < \min(m\,\mathrm{Re}\,a, m\,\mathrm{Re}\,b)$ *and* Re $c>0$.

Proof. For $x \in (0, \infty)$ we have, from (3.1),

$$H_1(a, b; c; m)\phi(x) = x^{mc} \int_0^1 (1 - v^m)^{c-1} F^*(a, b; c; 1 - 1/v^m) m v^{m-1} \phi(xv) dv. \quad (3.2)$$

By Lemma 2.1 above and Lemma 2.2 of (9), for any given $\delta>0$, there exists M independent of $v \in (0, 1)$ such that

$$| (1 - v^m)^{c-1} F^*(a, b; c; 1 - 1/v^m) m v^{m-1} \phi(xv) |$$
$$\leqq M(1 - v^m)^{\mathrm{Re}\,c - 1} v^{\min(m\,\mathrm{Re}\,a,\, m\,\mathrm{Re}\,b) - \delta} v^{m-1} (xv)^{\mathrm{Re}\,\mu - 1/p}$$

for $v \in (0, 1)$. Under the given conditions on the parameters, the right-hand side of this inequality is an integrable function of v over $(0, 1)$ provided δ is chosen sufficiently close to 0. Hence the integral on the right of (3.2) converges uniformly on compact subsets of $(0, \infty)$ and (i) follows. (ii) follows similarly using Lemma 2.1 since, under the given conditions, we may differentiate under the integral sign in (3.2) with respect to a, b or c.

The main use of Lemma 3.1 is in resolving a minor technical detail below. The information it gives turns out to be relatively little as we shall see later.

To obtain a full description of the mapping properties of $H_1(a, b; c; m)$ on $F_{p,\mu}$ we proceed to establish a connection with fractional integrals.

Lemma 3.2. *Let*

Re $\alpha>0$, Re $\beta>0$, $-\mathrm{Re}\,\mu - m + 1/p < \min(m\,\mathrm{Re}\,\xi, m\,\mathrm{Re}\,\eta)$, $\phi \in F_{p,\mu}$.

Then, for $x>0$,

$$I_{x^m}^{\eta,\alpha} I_{x^m}^{\xi,\beta} \phi(x) = x^{-m\eta - m\alpha} H_1(\xi + \beta - \eta, \beta; \alpha + \beta; m) x^{m\eta - m\beta} \phi(x). \quad (3.3)$$

Proof.

$$I_{x^m}^{\eta,\alpha} I_{x^m}^{\xi,\beta}\phi(x)$$
$$=\frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)}\int_0^x (x^m-u^m)^{\alpha-1}u^{m\eta+m-1}du\,\frac{mu^{-m\xi-m\beta}}{\Gamma(\beta)}\int_0^u (u^m-t^m)^{\beta-1}t^{m\xi+m-1}\phi(t)dt.$$

By Lemma 2.2 of (**9**), there is a constant M: $|\phi(t)| \leqq Mt^{\text{Re}\,\mu-1/p}(0<t<\infty)$. It then follows easily that the repeated integral is absolutely convergent under the given conditions on the parameters. By Fubini's Theorem we may justifiably invert the order of integration to obtain

$$\frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x t^{m\xi+m-1}\phi(t)dt\int_t^x (x^m-u^m)^{\alpha-1}(u^m-t^m)^{\beta-1}u^{m\eta-m\xi-m\beta}mu^{m-1}du.$$

Under the substitution $w = (u^m-t^m)/(x^m-t^m)$ the inner integral becomes

$$(x^m-t^m)^{\alpha+\beta-1}t^{m\eta-m\xi-m\beta}\int_0^1 (1-w)^{\alpha-1}w^{\beta-1}[1-w(1-x^m/t^m)]^{\eta-\xi-\beta}dw$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}(x^m-t^m)^{\alpha+\beta-1}t^{m\eta-m\xi-m\beta}F(\xi+\beta-\eta,\ \beta;\ \alpha+\beta;\ 1-x^m/t^m)$$

using Euler's Integral, formula (10) on p. 59 of (**2**). Finally, therefore,

$$I_{x^m}^{\eta,\alpha} I_{x^m}^{\xi,\beta}\phi(x)$$
$$=\frac{x^{-m\eta-m\alpha}}{\Gamma(\alpha+\beta)}\int_0^x (x^m-t^m)^{\alpha+\beta-1}F(\xi+\beta-\eta,\ \beta;\ \alpha+\beta;\ 1-x^m/t^m)mt^{m-1}t^{m\eta-m\beta}\phi(t)dt$$
$$= x^{-m\eta-m\alpha}H_1(\xi+\beta-\eta,\ \beta;\ \alpha+\beta;\ m)x^{m\eta-m\beta}\phi(x) \quad \text{as required.}$$

This completes the proof.

Corollary 3.3. *Let*

$$\text{Re}\,c>\text{Re}\,b>0,\ -\text{Re}\,\mu-m+1/p<\min\,(m\,\text{Re}\,a,\ m\,\text{Re}\,b),\ \phi\in F_{p,\mu}.$$

Then for $x>0$,
$$H_1(a,\ b;\ c;\ m)\phi(x) = I_{x^m}^{c-b}x^{-ma}I_{x^m}^{b}x^{ma}\phi(x). \tag{3.4}$$

Proof. In Lemma 3.2, we take $\alpha = c-b$, $\beta = b$, $\xi = \eta+a-b$, and replace μ and $\phi(x)$ by $\mu-m\eta+mb$ and $x^{-m\eta+mb}\phi(x)$. The conditions in Corollary 3.3 then imply that the conditions of Lemma 3.2 are satisfied so that (after a slight rearrangement)
$$H_1(a,\ b;\ c;\ m)\phi(x) = x^{m\eta+mc-mb}I_{x^m}^{\eta,c-b}I_{x^m}^{\eta+a-b,b}x^{-m\eta+mb}\phi(x) \tag{3.5}$$
for $x>0$. The free parameter η disappears when we rewrite (3.5) in terms of the inhomogeneous operators $I_{x^m}^{\alpha}$ and (3.4) follows almost immediately.

In (**9**), we extended the operators $I_{x^m}^{\eta,\alpha}$ and $I_{x^m}^{\alpha}$ on $F_{p,\mu}$ using analytic continuation to values of α with $\text{Re}\,\alpha \leqq 0$. The right-hand side of (3.4) therefore has a meaning even if the condition $\text{Re}\,c>\text{Re}\,b>0$ is removed. We must however retain the restriction $-\text{Re}\,\mu-m+1/p<\min\,(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$; for details, we refer the reader to Section 3 of (9). We now use (3.4) to extend the definition of $H_1(a, b;\ c;\ m)$ on $F_{p,\mu}$ removing the restriction $\text{Re}\,c>0$.

Definition 3.4. For $\phi \in F_{p,\mu}$ with $-\text{Re}\,\mu-m+1/p<\min\,(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$, we define $H_1(a, b;\ c;\ m)\phi$ by

$$H_1(a, b;\ c;\ m)\phi(x) = I_{x^m}^{c-b}x^{-ma}I_{x^m}^{b}x^{ma}\phi(x) \quad (x>0).$$

Note. We must be a little careful here and check that this new definition coincides with the original where both make sense, namely where the above conditions are satisfied and $\text{Re}\,c>0$. Certainly this is the case if $\text{Re}\,c>\text{Re}\,b>0$ by Corollary 3.3. However, under the given conditions both sides are analytic functions of b (using Lemma 3.1 for the left-hand side). The principle of analytical continuation then gives the desired result for $\text{Re}\,c>0$.

With Definition 3.4 available, we can now use results in (9) to obtain the mapping properties of $H_1(a, b;\ c;\ m)$.

Theorem 3.5. *If* $-\text{Re}\,\mu-m+1/p<\min\,(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$, $H_1(a, b;\ c;\ m)$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+mc}$.

If, in addition, $-\text{Re}\,\mu-m+1/p<\min\,(m\,\text{Re}\,c,\ m\,\text{Re}\,(a+b))$, $H_1(a, b;\ c;\ m)$ *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+mc}$ *and, for* $\psi \in F_{p,\mu+mc}$,

$$[H_1(a, b;\ c;\ m)]^{-1}\psi(x) = x^{-ma}I_{x^m}^{-b}x^{ma}I_{x^m}^{b-c}\psi(x). \tag{3.6}$$

Notice that, in the above theorem, " isomorphism " is used in the sense of Zemanian (**10**, p. 27).

Proof. Let $\phi \in F_{p,\mu}$. Then $x^{ma}\phi \in F_{p,\mu+ma}$ (Theorem 2.6 (i) of (9)). Since $\text{Re}\,(\mu+ma)+m>1/p$, we can apply Theorem 3.6 of (9) with μ and ϕ replaced by $\mu+ma$ and $x^{ma}\phi$ respectively to deduce that $I_{x^m}^{b}x^{ma}\phi \in F_{p,\mu+ma+mb}$. Then

$$x^{-ma}I_{x^m}^{b}x^{ma}\phi \in F_{p,\mu+mb}$$

(Theorem 2.6 (i) of (9)). Finally, since $\text{Re}\,(\mu+mb)+m>1/p$, we can again apply Theorem 3.6 of (9) with μ and ϕ replaced by $\mu+mb$ and $x^{-ma}I_{x^m}^{b}x^{ma}\phi$ respectively to obtain $I_{x^m}^{c-b}x^{-ma}I_{x^m}^{b}x^{ma}\phi \in F_{p,\mu+mc}$. Further, the theorems quoted above also show that $H_1(a, b;\ c;\ m)$, being the composition of four continuous linear mappings, is itself a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$. The second part of the theorem is proved similarly; the extra conditions are needed to ensure the invertibility of $I_{x^m}^{b}$ and $I_{x^m}^{c-b}$ (see Theorem 3.6 of (9) again). We note that it is also possible to prove the theorem by using (3.5) above instead of Theorem 3.5 of (9).

One interesting consequence is

Corollary 3.6. *If*

$\psi \in F_{p,\mu+mc}$ *and* $-\text{Re}\,\mu-m+1/p<\min\,(m\,\text{Re}\,a,\ m\,\text{Re}\,b,\ m\,\text{Re}\,c,\ m\,\text{Re}\,(a+b))$, *then for* $x>0$,

$$[H_1(a, b;\ c;\ m)]^{-1}\psi(x) = x^{-ma}H_1(-a, b-c;\ -c;\ m)x^{ma}\psi(x). \tag{3.7}$$

Proof. This follows immediately from (3.6) and Definition 3.4; in the latter a, b, c, μ and $\phi(x)$ have to be replaced by $-a$, $b-c$, $-c$, $\mu+ma+mc$ and $x^{ma}\phi(x)$ respectively.

Next we remark that if $\phi \in F_{p,\mu}$ and $-\operatorname{Re}\mu-m+1/p<min\,(m \operatorname{Re} a, m \operatorname{Re} b)$, then

$$H_1(a,\, b;\; c;\; m)\phi = H_1(b,\, a;\; c;\; m)\phi. \tag{3.8}$$

Indeed, this follows for Re c>0 from (2.4) and (3.1) and then in general by analytic continuation. It is hardly surprising that the restrictions on the parameters are symmetric in a and b. However, the right-hand side in Definition 3.4 is not symmetric in a and b. We exploit this lack of symmetry to establish the second index law for the operators $I^{\alpha}_{x^m}$ on the spaces $F_{p,\mu}$. There are various ways of stating this of which we choose the following (see (3.13) of (9)).

Theorem 3.7. *If $\alpha+\beta+\gamma = 0$ and if $\phi \in F_{p,\mu}$ where*

$$-\operatorname{Re}\mu-m+1/p<\min\,(0,\, m \operatorname{Re}\gamma)$$

then, for $x>0$,

$$x^{m\alpha}I^{\beta}_{x^m}x^{m\gamma}\phi(x) = I^{-\gamma}_{x^m}x^{-m\beta}I^{-\alpha}_{x^m}\phi(x). \tag{3.9}$$

Proof. Under the given conditions we apply (3.8) with a, b, c, μ and $\phi(x)$ replaced by $-\alpha$, β, β, $\mu-m\beta$ and $x^{-m\beta}\phi(x)$ respectively. To do this we need

$$-\operatorname{Re}\,(\mu-m\beta)-m+1/p<\min\,(-m \operatorname{Re}\alpha,\, m \operatorname{Re}\beta)$$

or

$$-\operatorname{Re}\mu-m+1/p<\min\,(-m \operatorname{Re}\,(\alpha+\beta),\, 0) = \min\,(m \operatorname{Re}\gamma,\, 0)$$

and this is the case by hypothesis. Hence (3.8) gives

$$H_1(-\alpha,\, \beta;\; \beta;\; m)(x^{-m\beta}\phi) = H_1(\beta,\, -\alpha;\; \beta;\; m)(x^{-m\beta}\phi).$$

Using Definition 3.4 then gives

$$I^{0}_{x^m}x^{m\alpha}I^{\beta}_{x^m}x^{-m\alpha}x^{-m\beta}\phi = I^{\beta+\alpha}_{x^m}x^{-m\beta}I^{-\alpha}_{x^m}x^{m\beta}x^{-m\beta}\phi. \tag{3.10}$$

Theorem 3.6 shows that under the given circumstances, $\phi \in F_{p,\mu}$ implies that $x^{m\alpha}I^{\beta}_{x^m}x^{-m\alpha-m\beta}\phi \in F_{p,\mu}$ and, in addition, $I^{0}_{x^m}$ is the identity operator on $F_{p,\mu}$. Hence, putting $\gamma = -\alpha-\beta$ in (3.10) gives (3.9) as required.

4

We now introduce three more integral operators related to $H_1(a,\, b;\; c;\; m)$. For any complex numbers a and b, $\operatorname{Re} c>0$ ($m>0$ as usual) and suitable functions ϕ, we define $H_2(a,\, b;\; c;\; m)\phi$ by

$$H_2(a,\, b;\; c;\; m)\phi(x) = \int_0^x (x^m-t^m)^{c-1}F^*(a,\, b;\; c;\; 1-t^m/x^m)mt^{m-1}\phi(t)dt, \tag{4.1}$$

where $x>0$. Proceeding as in (5, p. 195), we deduce that, for $x>0$,

$$H_2(a,\, b;\; c;\; m)\phi(x) = x^{ma}H_1(a,\, c-b;\; c;\; m)x^{-ma}\phi(x) \tag{4.2}$$

whenever either side exists. In particular, if we apply Theorem 3.5 with b, μ and $\phi(x)$ replaced by $c-b$, $\mu-ma$ and $x^{-ma}\phi(x)$ respectively we find that

$$\phi \in F_{p,\mu} \Rightarrow H_2(a, b; c; m)\phi \in F_{p,\mu+mc}$$

provided that $-\text{Re}(\mu-ma)-m+1/p<\min(m \text{ Re } a, m \text{ Re }(c-b))$ and $\text{Re } c>0$. However, the right-hand side of (4.2) is meaningful even without the restriction $\text{Re } c>0$. We can therefore use (4.2) to extend the definition of $H_2(a, b; c; m)$ on $F_{p,\mu}$.

Definition 4.1. For $\phi \in F_{p,\mu}$ with $-\text{Re }\mu-m+1/p<\min(0, m \text{ Re }(c-a-b))$, we define $H_2(a, b; c; m)\phi$ by

$$H_2(a, b; c; m)\phi(x) = x^{ma}H_1(a, c-b; c; m)x^{-ma}\phi(x) \quad (x>0).$$

In view of the preceding remarks, the definition is meaningful and agrees with (4.1) when, in addition, $\text{Re } c>0$.

Using Theorem 3.5, we can easily prove

Theorem 4.2. *If* $-\text{Re }\mu-m+1/p<\min(0, m \text{ Re }(c-a-b))$, *then*

$$H_2(a, b; c; m)$$

is a continuous linear mapping of $F_{p,\mu}$ *into* $F_{p,\mu+mc}$ *and, for* $\phi \in F_{p,\mu}$,

$$H_2(a, b; c; m)\phi(x) = x^{ma}I_{x^m}^{b}x^{-ma}I_{x^m}^{c-b}\phi(x) \quad (x>0). \tag{4.3}$$

If, in addition,

$$-\text{Re }\mu-m+1/p<\min(m \text{ Re }(c-a), m \text{ Re }(c-b)),$$

then $H_2(a, b; c; m)$ *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+mc}$ *and, for any* $\psi \in F_{p,\mu+mc}$, *the equation*

$$H_2(a, b; c; m)\phi = \psi \tag{4.4}$$

has a unique solution $\phi \in F_{p,\mu}$ *given by*

$$\phi(x) = I_{x^m}^{b-c}x^{ma}I_{x^m}^{-b}x^{-ma}\psi(x) \quad (x>0). \tag{4.5}$$

This leads to

Corollary 4.3. *Let*

$-\text{Re }\mu-m+1/p<\min(0, m \text{ Re }(c-a-b), m \text{ Re }(c-a), m\text{Re }(c-b))$, $\phi \in F_{p,\mu}$ *and* $\psi \in F_{p,\mu+mc}$. *Then*

$$H_2(a, b; c; m)\phi = \psi \Leftrightarrow \phi = H_1(-a, -b; -c; m)\psi.$$

Proof. Since by hypothesis,

$$-\text{Re }(\mu+mc)-m+1/p$$
$$<\min(m \text{ Re }(-c), m \text{ Re }(-a-b), m \text{ Re }(-a), m \text{ Re }(-b)),$$

we can apply Definition 3.4 with a, b, c, μ and $\phi(x)$ replaced by $-a$, $-b$, $-c$, $\mu+mc$ and $\psi(x)$ respectively to obtain

$$H_1(-a, -b; -c; m)\psi(x) = I_{x^m}^{b-c}x^{ma}I_{x^m}^{-b}x^{-ma}\psi(x) \quad (x>0).$$

The result follows at once from Theorem 4.2 and in particular (4.5).

Our other two operators are the adjoints of $H_1(a, b; c; m)$ and $H_2(a, b; c; m)$ and reduce in the case $m = 1$ to those studied by Love in (6). For complex numbers a and b, $\operatorname{Re} c > 0$ and suitable functions ϕ, we define $H_3(a, b; c; m)\phi$ and $H_4(a, b; c; m)\phi$ on $(0, \infty)$ by

$$H_3(a, b; c; m)\phi(x) = mx^{m-1} \int_x^\infty (t^m - x^m)^{c-1} F^*(a, b; c; 1 - x^m/t^m)\phi(t)dt \quad (4.6)$$

$$H_4(a, b; c; m)\phi(x) = mx^{m-1} \int_x^\infty (t^m - x^m)^{c-1} F^*(a, b; c; 1 - t^m/x^m)\phi(t)dt. \quad (4.7)$$

As we might expect, it is possible to express these operators in terms of operators of the form $K_{x^m}^{\eta,\alpha}$ or $K_{x^m}^{\alpha}$. Indeed, proceeding as in Lemma 3.2, we find that for $x > 0$,

$$K_{x^m}^{\xi,\beta} K_{x^m}^{\eta,\alpha}\phi(x) = x^{m\xi - m + 1} H_3(\xi + \beta - \eta, \alpha; \alpha + \beta; m) x^{-m\xi - m\beta - m\alpha + m - 1}\phi(x) \quad (4.8)$$

provided $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, $\phi \in F_{p,\mu}$ and $\operatorname{Re} \mu - 1/p < \min(m \operatorname{Re} \xi,\ m \operatorname{Re} \eta)$. It follows, as before, that

$$H_3(a, b; c; m)\phi(x)$$
$$= x^{-m\eta - ma - mb + mc + m - 1} K_{x^m}^{a+b-c+\eta,\, c-b} K_{x^m}^{\eta,\, b} x^{m\eta + ma + mb - m + 1}\phi(x),$$

or

$$H_3(a, b; c; m)\phi(x) = x^{m-1} K_{x^m}^{c-b} x^{-ma} K_{x^m}^{b} x^{ma - m + 1}\phi(x) \quad (4.9)$$

provided

$\operatorname{Re} c > \operatorname{Re} b > 0$, $\phi \in F_{p,\mu}$ and $\operatorname{Re} \mu - m + 1/q < \min(-m \operatorname{Re} c,\ -m \operatorname{Re}(a+b))$. As before, we can use (4.9) to extend the definition of $H_3(a, b; c; m)$ on $F_{p,\mu}$ removing the restriction $\operatorname{Re} c > \operatorname{Re} b > 0$.

Definition 4.4. For $\phi \in F_{p,\mu}$ and

$$\operatorname{Re} \mu - m + 1/q < \min(-m \operatorname{Re} c,\ -m \operatorname{Re}(a+b)),$$

define $H_3(a, b; c; m)\phi$ by

$$H_3(a, b; c; m)\phi(x) = x^{m-1} K_{x^m}^{c-b} x^{-ma} K_{x^m}^{b} x^{ma - m + 1}\phi(x) \quad (x > 0).$$

By analytic continuation, this definition coincides with (4.6) when in addition $\operatorname{Re} c > 0$. Using the mapping properties of $K_{x^m}^{\eta,\alpha}$ or $K_{x^m}^{\alpha}$ derived in Theorems 3.7 and 3.8 of (9), we immediately obtain

Theorem 4.5. *If*

$$\operatorname{Re} \mu - m + 1/q < \min(-m \operatorname{Re} c,\ -m \operatorname{Re}(a+b)),\ H_3(a, b; c; m)$$

is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$. If, in addition,

$$\operatorname{Re} \mu - m + 1/q < \min(-m \operatorname{Re} a,\ -m \operatorname{Re} b),\ H_3(a, b; c; m)$$

is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+mc}$ and, for any $\psi \in F_{p,\mu+mc}$, the equation

$$H_3(a, b; c; m)\phi = \psi \quad (4.10)$$

has a unique solution $\phi \in F_{p,\mu}$ *given by*

$$\phi(x) = x^{-ma+m-1} K_{xm}^{-b} x^{ma} K_{xm}^{b-c} x^{-m+1} \psi(x) \quad (x>0). \tag{4.11}$$

As regards $H_4(a, b;\ c;\ m)$, we may use (4.6) and (4.7) and proceed as in ((6), pp. 1073-4) to show that for Re $c>0$,

$$H_4(a,\ b;\ c;\ m)\phi(x) = x^{ma} H_3(a,\ c-b;\ c;\ m) x^{-ma}\phi(x) \tag{4.12}$$

whenever either side exists. In particular, from Theorem 4.5, (4.12) is valid if $\phi \in F_{p,\mu}$, Re $\mu - m + 1/q < \min(m \operatorname{Re}(a-c),\ m \operatorname{Re}(b-c))$ and Re $c>0$. The right-hand side is meaningful even without the restriction Re $c>0$ and we can use (4.12) to extend the definition of $H_4(a, b;\ c;\ m)$ on $F_{p,\mu}$.

Definition 4.6. For $\phi \in F_{p,\mu}$ and

$$\operatorname{Re} \mu - m + 1/q < \min(m \operatorname{Re}(a-c),\ m \operatorname{Re}(b-c)),$$

define $H_4(a, b;\ c;\ m)\phi$ on $(0, \infty)$ by

$$H_4(a,\ b;\ c;\ m)\phi(x) = x^{ma} H_3(a,\ c-b;\ c;\ m) x^{-ma}\phi(x).$$

The definition agrees with (4.7) when also Re $c>0$.

Using Definition 4.4 and Theorem 4.5 we obtain

Theorem 4.7. *If*

$$\operatorname{Re} \mu - m + 1/q < \min(m \operatorname{Re}(a-c),\ m \operatorname{Re}(b-c)),\ H_4(a,\ b;\ c;\ m)$$

is a continuous linear mapping of $F_{p,\mu}$ *into* $F_{p,\mu+mc}$ *and, for* $\phi \in F_{p,\mu}$,

$$H_4(a,\ b;\ c;\ m)\phi(x) = x^{ma+m-1} K_{xm}^{b} x^{-ma} K_{xm}^{c-b} x^{-m+1} \phi(x) \quad (x>0). \tag{4.13}$$

If, in addition, Re $\mu - m + 1/q < \min(0,\ m \operatorname{Re}(a+b-c))$, $H_4(a,\ b;\ c;\ m)$ *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+mc}$ *and, for any* $\psi \in F_{p,\mu+mc}$, *the equation*

$$H_4(a,\ b;\ c;\ m)\phi = \psi \tag{4.14}$$

has a unique solution $\phi \in F_{p,\mu}$ *given by*

$$\phi(x) = x^{m-1} K_{xm}^{b-c} x^{ma} K_{xm}^{-b} x^{-ma-m+1} \psi(x) \quad (x>0). \tag{4.15}$$

Comparing (4.15) with Definition 4.4 produces the following analogue of Corollary 4.3.

Corollary 4.8. *Let*

Re $\mu - m + 1/q < \min(0,\ m \operatorname{Re}(a+b-c),\ m \operatorname{Re}(a-c),\ m \operatorname{Re}(b-c))$, $\phi \in F_{p,\mu}$ *and* $\psi \in F_{p,\mu+mc}$. *Then*

$$H_4(a,\ b;\ c;\ m)\phi = \psi \Leftrightarrow \phi = H_3(-a,\ -b;\ -c;\ m)\psi.$$

To conclude this section, we state the second index law for the operators K_{xm}^{α} analogous to Theorem 3.7.

Theorem 4.9. *If* $\alpha+\beta+\gamma = 0$ *and if* $\phi \in F_{p,\mu}$ *where*

$$\operatorname{Re} \mu - 1/p < \min(0,\ m \operatorname{Re} \gamma),$$

then, for $x>0$,

$$x^{m\gamma}K_{x^m}^{\beta}x^{m\alpha}\phi(x) = K_{x^m}^{-\alpha}x^{-m\beta}K_{x^m}^{-\gamma}\phi(x). \tag{4.16}$$

This can be proved by observing that under appropriate conditions

$$H_4(a, b;\ c;\ m)\phi = H_4(b, a;\ c;\ m)\phi \tag{4.17}$$

and proceeding as in Theorem 3.7 making use of Theorem 3.8 of (9).

5

We are now going to discuss the operators $H_i(a, b;\ c;\ m)$ $(i = 1, 2, 3, 4)$ relative to the spaces $F'_{p,\mu}$.

We consider first $H_1(a, b;\ c;\ m)$ and to motivate our definition we deal with regular functionals. We require the following

Definition 5.1. For each complex number μ and $1 \leqq p \leqq \infty$, we define L_μ^p by

$$L_\mu^p = \{f : x^{-\mu}f(x) \in L^p(0, \infty)\}.$$

We can turn L_μ^p into a Banach space by introducing the norm $\|\ \|_{p,\mu}$ defined by

$$\| f \|_{p,\mu} = \| x^{-\mu}f(x) \|_p \quad (f \in L_\mu^p)$$

where $\|\ \|_p$ denotes the usual norm on $L^p(0, \infty)$.

If $f \in L_\mu^p$ and $\operatorname{Re} c>0$ we can define $H_1(a, b;\ c;\ m)f$ using (1.5), with ϕ replaced by f, under appropriate conditions on a, b and μ. Indeed we can prove

Lemma 5.2. *If*

$\operatorname{Re} c>0$ *and* $-\operatorname{Re}\mu-m+1/p<\min(m\operatorname{Re} a,\ m\operatorname{Re} b)$, $H_1(a, b;\ c;\ m)$ (*as given by* (1.5)) *is a continuous linear mapping of* L_μ^p *into* $L_{\mu+mc}^p$.

Proof. Using (1.5) and putting $t = xv$ we have (see (3.2)) for $x>0$,

$$H_1(a, b;\ c;\ m)f(x) = x^{mc}\int_0^1 (1-v^m)^{c-1}F^*(a, b;\ c;\ 1-1/v^m)mv^{m-1}f(xv)dv.$$

For simplicity write $d = \min(m\operatorname{Re} a,\ m\operatorname{Re} b)$ and choose $\delta>0$ such that $-\operatorname{Re}\mu-m+1/p<d-\delta$. Applying Lemma 2.1, there exists a constant M such that

$$\begin{aligned}&| x^{-\mu-mc}H_1(a, b;\ c;\ m)f(x) |\\ &\qquad \leqq M\int_0^1 (1-v^m)^{\operatorname{Re} c-1}v^{d-\delta+\operatorname{Re}\mu}mv^{m-1}\,| (xv)^{-\mu}f(xv) |\,dv\\ &\qquad = M\Gamma(c)I_{x^m}^{1/m(d-\delta+\operatorname{Re}\mu),\ \operatorname{Re} c}(| x^{-\mu}f(x) |)\end{aligned}$$

(see (3.3) of (9)) $= g(x)$ say. Then

$$| x^{-\mu-mc}H_1(a, b;\ c;\ m)f(x) | \leqq g(x). \tag{5.1}$$

Now since $(d-\delta+\text{Re}\,\mu)+m>1/p$, we can apply an extension of a result of Kober (4, Theorem 2 (i)) to deduce that, since $|x^{-\mu}f(x)|\in L^p(0,\infty)$, $g\in L^p(0,\infty)$ and there is a constant K independent of f such that

$$\| g(x)\|_p \leqq K\| x^{-\mu}f(x)\|_p. \tag{5.2}$$

Kober's result deals with the case $m = 1$, $\mu = 0$. The general case follows easily by a simple change of variable; compare Lemma 3.1 of (9). Now (5.1), (5.2) give

$$\| x^{-\mu-mc}H_1(a, b;\ c;\ m)f(x)\|_p \leqq K\| x^{-\mu}f(x)\|_p$$

$$\Rightarrow \| H_1(a, b;\ c;\ m)f(x)\|_{p,\mu+mc} \leqq K\| f(x)\|_{p,\mu}$$

and the result follows.

Now let $f\in L^q_{-\mu}$. Then f generates a regular functional, τf say, in the space $F'_{p,\mu}$ according to the formula

$$(\tau f, \phi) = \int_0^\infty f(x)\phi(x)dx \quad (\phi\in F_{p,\mu}) \tag{5.3}$$

the integral on the right being absolutely convergent by Hölder's inequality. If $\text{Re}\,c>0$ and $\text{Re}\,\mu-m+1/q<\min(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$, Lemma 5.2 shows that $H_1(a, b;\ c;\ m)f$ exists and belongs to $L^q_{-\mu+mc}$. Hence $H_1(a, b;\ c;\ m)f$ generates a regular functional $\tau H_1(a, b;\ c;\ m)f\in F'_{p,\mu-mc}$ according to the formula

$$(\tau H_1(a, b;\ c;\ m)f, \phi) = \int_0^\infty H_1(a, b;\ c;\ m)f(x)\phi(x)dx \quad (\phi\in F_{p,\mu-mc}). \tag{5.4}$$

It seems reasonable that we should define $H_1(a, b;\ c;\ m)$ on the space $F'_{p,\mu}$ in such a way that if the above conditions are satisfied,

$$H_1(a, b;\ c;\ m)\tau f = \tau H_1(a, b;\ c;\ m)f. \tag{5.5}$$

Note. No confusion should arise from using the same symbol $H_1(a, b;\ c;\ m)$ for the operator on $F'_{p,\mu}$ as well as the operator on $F_{p,\mu}$.

(5.5) implies that if $\text{Re}\,c>0$, $\text{Re}\,\mu-m+1/q<\min(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$, $H_1(a, b;\ c;\ m)$ maps regular functionals in $F'_{p,\mu}$ into regular functionals in $F'_{p,\mu-mc}$. Further if $\phi\in F_{p,\mu-mc}$, we have

$$(H_1(a, b;\ c;\ m)\tau f, \phi) = (\tau H_1(a, b;\ c;\ m)f, \phi) \qquad \text{by (5.5)}$$

$$= \int_0^\infty H_1(a, b;\ c;\ m)f(x)\phi(x)dx \qquad \text{by (5.4)}$$

$$= \int_0^\infty \left(\int_0^x (x^m-t^m)^{c-1}F^*(a, b;\ c;\ 1-x^m/t^m)mt^{m-1}f(t)dt\right)\phi(x)dx \qquad \text{by (1.5)}$$

$$= \int_0^\infty f(t)\left(mt^{m-1} \int_t^\infty (x^m - t^m)^{c-1} F^*(a, b;\ c;\ 1 - x^m/t^m)\phi(x)dx \right) dt$$

$$= \int_0^\infty f(t) H_4(a, b;\ c;\ m)\phi(t)dt \qquad \text{by (4.7)}$$

or, using (5.3),

$$(H_1(a, b;\ c;\ m)\tau f, \phi) = (\tau f, H_4(a, b;\ c;\ m)\phi). \tag{5.6}$$

The inversion of the order of integration above is justified since the integrals involved are absolutely convergent by Lemma 5.2, Theorem 4.7 and Hölder's inequality. The derivation of (5.6) required Re $c > 0$ in order that we could use (1.5) to obtain $H_1(a, b;\ c;\ m)f$. If Re $c \leqq 0$, the integral in (1.5) will not exist for an arbitrary element $f \in L^q_{-\mu}$, nor can we use a formula such as Definition 3.4 unless f has some additional differentiability. Nevertheless, by Theorem 4.7, the right-hand side of (5.6) is still meaningful provided only $\phi \in F_{p, \mu - mc}$ and Re $\mu - m + 1/q < \min(m \text{ Re } a,\ m \text{ Re } b)$. We can even go further and replace τf by any functional in $F'_{p,\mu}$, regular or not. Hence we are led to the following definition.

Definition 5.3. For $f \in F'_{p,\mu}$ and Re $\mu - m + 1/q < \min(m \text{ Re } a,\ m \text{ Re } b)$, we define $H_1(a, b;\ c;\ m)f$ as the member of $F'_{p,\mu-mc}$ such that

$$(H_1(a, b;\ c;\ m)f, \phi) = (f, H_4(a, b;\ c;\ m)\phi) \quad (\phi \in F_{p,\mu-mc}).$$

Remark 5.4. No confusion should arise from the use of "f" to denote an arbitrary generalised function as well as a classical function generating the regular functional τf. Indeed it is often convenient to identify a classical function with the functional it generates, although we shall not do so in this paper.

Since Theorem 4.7 details the mapping properties of $H_4(a, b;\ c;\ m)$ on the spaces $F_{p,\mu}$, we can use standard theorems on adjoints (e.g. Theorems 1.10-1 and 1.10-2 in (**10**)) to obtain properties of $H_1(a, b;\ c;\ m)$ on the spaces $F'_{p,\mu}$.

Theorem 5.5. *If* Re $\mu - m + 1/q < \min(m \text{ Re } a, m \text{ Re } b)$, $H_1(a, b;\ c;\ m)$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu-mc}$ *and, for* $f \in F'_{p,\mu}$,

$$H_1(a, b;\ c;\ m)f = I_{x^m}^{c-b} x^{-ma} I_{x^m}^{b} x^{ma} f. \tag{5.7}$$

If, in addition, Re $\mu - m + 1/q < \min(m \text{ Re } c, m \text{ Re } (a+b))$, $H_1(a, b;\ c;\ m)$ *is an isomorphism of* $F'_{p,\mu}$ *onto* $F'_{p,\mu-mc}$ *and, for any* $g \in F'_{p,\mu-mc}$, *the equation*

$$H_1(a, b;\ c;\ m)f = g \tag{5.8}$$

has a unique solution $f \in F'_{p,\mu}$ *given by*

$$f = x^{-ma} I_{x^m}^{-b} x^{ma} I_{x^m}^{b-c} g. \tag{5.9}$$

Proof. By Theorem 4.7, $H_4(a, b;\ c;\ m)$ is a continuous linear mapping of $F_{p,\mu-mc}$ into $F_{p,\mu}$ under the given conditions; the first statement follows from Theorem 1.10-1 of (**10**). To establish (5.7), let $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu-mc}$.

Then

$$\begin{aligned}
(H_1(a, b;\ c;\ m)f, \phi) &= (f, H_4(a, b;\ c;\ m)\phi) && \text{by Definition 5.3}\\
&= (f, x^{ma+m-1}K_{x^m}^{b}x^{-ma}K_{x^m}^{c-b}x^{-m+1}\phi) && \text{by (4.13)}\\
&= (x^{ma}f, x^{m-1}K_{x^m}^{b}x^{-m+1}x^{-ma}x^{m-1}K_{x^m}^{c-b}x^{-m+1}\phi) && \text{by (2.13) of (9)}\\
&= (I_{x^m}^{b}x^{ma}f, x^{-ma}x^{m-1}K_{x^m}^{c-b}x^{-m+1}\phi) && \text{by (3.29) of (9)}\\
&= (x^{-ma}I_{x^m}^{b}x^{ma}f, x^{m-1}K_{x^m}^{c-b}x^{-m+1}\phi) && \text{by (2.13) of (9)}\\
&= (I_{x^m}^{c-b}x^{-ma}I_{x^m}^{b}x^{ma}f, \phi) && \text{by (3.29) of (9)}
\end{aligned}$$

from which (5.7) follows. The two applications of (3.29) of (9) above are valid under the given conditions. The remainder of Theorem 5.5 can be proved similarly.

If we compare Theorems 3.5 and 5.5, we see that the restrictions on the parameters in one are obtained from those in the other by interchanging p and q, μ and $-\mu$. This continues the trend we first mentioned after Theorem 3.9 of (9).

We can handle the other operators similarly. We mention the salient facts and omit proofs.

Consideration of regular functionals leads as before to

Definition 5.6. For $f \in F'_{p,\mu}$, we define $H_i(a, b;\ c;\ m)f\ (i = 2, 3, 4)$ to be the elements of $F'_{p,\mu-mc}$ such that, for all $\phi \in F_{p,\mu-mc}$,

$$(H_2(a, b;\ c;\ m)f, \phi) = (f, H_3(a, b;\ c;\ m)\phi) \tag{5.10}$$

$$(H_3(a, b;\ c;\ m)f, \phi) = (f, H_2(a, b;\ c;\ m)\phi) \tag{5.11}$$

$$(H_4(a, b;\ c;\ m)f, \phi) = (f, H_1(a, b;\ c;\ m)\phi) \tag{5.12}$$

(whenever the right-hand sides are meaningful).

Theorem 5.7. *If* $\operatorname{Re}\mu - m + 1/q < \min(0, m\operatorname{Re}(c-a-b))$, $H_2(a, b;\ c;\ m)$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu-mc}$ *and, for* $f \in F'_{p,\mu}$,

$$H_2(a, b;\ c;\ m)f = x^{ma}I_{x^m}^{b}x^{-ma}I_{x^m}^{c-b}f. \tag{5.13}$$

If, in addition, $\operatorname{Re}\mu - m + 1/q < \min(m\operatorname{Re}(c-a), m\operatorname{Re}(c-b))$, $H_2(a, b;\ c;\ m)$ *is an isomorphism of* $F'_{p,\mu}$ *onto* $F'_{p,\mu-mc}$ *and, for each* $g \in F'_{p,\mu-mc}$, *the equation*

$$H_2(a, b;\ c;\ m)f = g \tag{5.14}$$

has a unique solution $f \in F'_{p,\mu}$ *given by*

$$f = H_1(-a, -b;\ -c;\ m)g = I_{x^m}^{b-c}x^{ma}I_{x^m}^{-b}x^{-ma}g. \tag{5.15}$$

Theorem 5.8. *If*

$$-\operatorname{Re}\mu - m + 1/p < \min(-m\operatorname{Re}c, -m\operatorname{Re}(a+b)),\ H_3(a, b;\ c;\ m)$$

is a continuous linear mapping of $F'_{p,\mu}$ *into* $F'_{p,\mu-mc}$ *and, for* $f \in F'_{p,\mu}$,

$$H_3(a, b;\ c;\ m)f = x^{m-1}K_{x^m}^{c-b}x^{-ma}K_{x^m}^{b}x^{ma-m+1}f. \tag{5.16}$$

If, in addition, $-\mathrm{Re}\,\mu-m+1/p<\min(-m\,\mathrm{Re}\,a,\ -m\,\mathrm{Re}\,b)$, $H_3(a, b;\ c;\ m)$ *is an isomorphism of* $F'_{p,\mu}$ *onto* $F'_{p,\mu-mc}$ *and, for each* $g\in F'_{p,\mu-mc}$, *the equation*

$$H_3(a,\ b;\ c;\ m)f=g \tag{5.17}$$

has a unique solution $f\in F'_{p,\mu}$ *given by*

$$f=x^{-ma+m-1}K_{xm}^{-b}x^{ma}K_{xm}^{b-c}x^{-m+1}g. \tag{5.18}$$

Theorem 5.9. *If*

$$-\mathrm{Re}\,\mu-m+1/p<\min(m\,\mathrm{Re}\,(a-c),\ m\,\mathrm{Re}\,(b-c)),\ H_4(a,\ b;\ c;\ m)$$

is a continuous linear mapping of $F'_{p,\mu}$ *into* $F'_{p,\mu-mc}$ *and, for* $f\in F'_{p,\mu}$,

$$H_4(a,\ b;\ c;\ m)f=x^{ma+m-1}K_{xm}^{b}x^{-ma}K_{xm}^{c-b}x^{-m+1}f. \tag{5.19}$$

If, in addition, $-\mathrm{Re}\,\mu-m+1/p<\min(0,\ m\,\mathrm{Re}\,(a+b-c))$, $H_4(a, b;\ c;\ m)$ *is an isomorphism of* $F'_{p,\mu}$ *onto* $F'_{p,\mu-mc}$ *and, for each* $g\in F'_{p,\mu-mc}$, *the equation*

$$H_4(a,\ b;\ c;\ m)f=g \tag{5.20}$$

has a unique solution $f\in F'_{p,\mu}$ *given by*

$$f=H_3(-a,\ -b;\ -c;\ m)g=x^{m-1}K_{xm}^{b-c}x^{ma}K_{xm}^{-b}x^{-ma-m+1}g. \tag{5.21}$$

Finally, we state the second index laws for I_{xm}^{α} and K_{xm}^{α} on $F'_{p,\mu}$.

Theorem 5.10. *Let* $f\in F'_{p,\mu}$ *and* $\alpha+\beta+\gamma=0$.

(i) *If* $\mathrm{Re}\,\mu-m+1/q<\min(0,\ m\,\mathrm{Re}\,\gamma)$, *then*

$$x^{m\alpha}I_{xm}^{\beta}x^{m\gamma}f=I_{xm}^{-\gamma}x^{-m\beta}I_{xm}^{-\alpha}f. \tag{5.22}$$

(ii) *If* $-\mathrm{Re}\,\mu-1/q<\min(0,\ m\,\mathrm{Re}\,\gamma)$, *then*

$$x^{m\gamma}K_{xm}^{\beta}x^{m\alpha}f=K_{xm}^{-\alpha}x^{-m\beta}K_{xm}^{-\gamma}f. \tag{5.23}$$

The basis of Theorem 5.10 is the symmetry of the $H_i(a, b;\ c;\ m)f$ between a and b which is inherited from the symmetry of $H_i(a,\ b;\ c;\ m)\phi$ as exemplified by (3.8) and (4.17).

6

In this section we compare and contrast our results with those of Love in (5) and (6). Since Love's results are stated for $m=1$, we shall also take $m=1$ in the remainder of the paper. Corresponding results for general m are easily obtained by simple changes of variables.

As usual we shall focus attention on $H_1(a, b;\ c;\ 1)$. We shall have occasion to consider $H_1(a, b;\ c;\ 1)f$ where f is either a classical function or a generalised function. If f is a classical function (for example in L_μ^p) we must assume that $\mathrm{Re}\,c>0$ and define $H_1(a,\ b;\ c;\ 1)f$ by (1.5) with ϕ replaced by f and $m=1$; that is

$$H_1(a,\ b;\ c;\ 1)f(x)=\int_0^x(x-t)^{c-1}F^*(a,\ b;\ c;\ 1-x/t)f(t)dt\quad(x>0) \tag{6.1}$$

with appropriate conditions on a and b to guarantee the existence of the integral.

If f is a generalised function, we define $H_1(a, b;\ c;\ 1)f$ by Definition 5.3 and use (5.7), with $m = 1$.

Working with certain classes of locally integrable classical functions (defined below), Love has to impose considerable restrictions on the parameters and proofs are different for different ranges of the parameters. On the other hand, in our results for generalised functions, the restrictions are less numerous and there is no need to split up proofs into different cases. This is hardly surprising when we consider the powerful tools, such as analytic continuation, which are available for generalised functions but not for locally integrable functions.

In (5), Love gives six solution formulae, valid under appropriate sets of conditions (the sets not being disjoint) for the equation

$$H_1(a, b;\ c;\ 1)f = g, \tag{6.2}$$

where f, g are classical functions satisfying an appropriate local integrability condition. Three of these formulae are

$$f(x) = x^{-a} I_x^{-b} x^a I_x^{b-c} g(x) \tag{6.3}$$

$$f(x) = x^{-b} I_x^{b-c} x^{c-a} I_x^{-b} x^{a+b-c} g(x) \tag{6.4}$$

$$f(x) = x^{-a} \frac{d^n}{dx^n} \{x^a H_2(-a, n-b;\ n-c;\ 1) g(x)\} \tag{6.5}$$

(where n is a positive integer, $n - \text{Re}\, c > 0$ and $H_2(-a, n-b;\ n-c;\ 1)g(x)$ is defined using the analogue of (4.1) with m = 1). The other three are obtained by interchanging a and b and using the fact that

$$H_1(b, a;\ c;\ 1)f = H_1(a, b;\ c;\ 1)f$$

(compare (3.8)).

As a contrast, we consider (6.2) where f, g are generalised functions and suppose that the conditions of Theorem 5.5 are satisfied with $m = 1$, i.e.

$$\text{Re}\, \mu - 1/p < \min\,(\text{Re}\, a, \text{Re}\, b, \text{Re}\, c, \text{Re}\,(a+b)),\ g \in F'_{p,\mu-c}.$$

Theorem 5.5 ensures that the right-hand side of (6.3) is the unique solution in $F'_{p,\mu}$. However, (6.4) and (6.5) are also valid under the same conditions and are merely alternative ways of writing (6.3). Indeed, if we apply Theorem 5.10 (i) with α, β, γ, μ and f replaced by $c-a$, $-b$, $a+b-c$, $\mu-c$ and g (which is permissible under the given conditions)

$$\begin{aligned} & x^{-b} I_x^{b-c} x^{c-a} I_x^{-b} x^{a+b-c} g \\ & = x^{-b} I_x^{b-c} I_x^{-a-b+c} x^b I_x^{a-c} g \\ & = x^{-b} I_x^{-a} x^b I_x^{a-c} g \qquad \text{using (3.31) of (9);} \end{aligned}$$

we have (6.3) with a and b interchanged. As regards (6.5), suppose n is a non-negative integer; then we may apply Theorem 5.7 with a, b, c, μ and f replaced by $-a, n-b, n-c, \mu-c$ and g to deduce that

$$x^{-a}\frac{d^n}{dx^n}\{x^a H_2(-a, n-b;\ n-c;\ 1)g$$

$$= x^{-a}\frac{d^n}{dx^n}x^a x^{-a} I_x^{n-b} x^a I_x^{b-c} g$$

$$= x^{-a} I_x^{-n} I_x^{n-b} x^a I_x^{b-c} g \qquad \text{(compare (3.10) of (9))}$$

$$= x^{-a} I_x^{-b} x^a I_x^{b-c} g \qquad \text{using (3.31) of (9)}$$

and we have (6.3) again.

We now indicate how our results involving generalised functions can be used to obtain results concerning classical solutions of (6.2); thus given $g(x)$ defined for $x \in (0, \infty)$ we try to find $f(x)$ such that

$$\int_0^x (x-t)^{c-1} F^*(a, b;\ c;\ 1-x/t) f(t)dt = g(x), \tag{6.6}$$

where equality is to hold almost everywhere on $(0, \infty)$. (Here $\operatorname{Re} c > 0$ as indicated earlier.) We shall identify functions differing only on a set of measure zero on $(0, \infty)$. In (5), Love discusses (6.6) relative to spaces Q_r. (We use r rather than q to avoid confusion with $1/p+1/q = 1$.) By definition, for each real r,

$$Q_r = \{f : x^r f(x) \in L^1(0, X) \text{ for each } X \in (0, \infty)\}. \tag{6.7}$$

We shall discuss (6.6) relative to the spaces L_μ^p (see Definition 5.1). Given $f \in L_\mu^p$, we again denote by τf the element of $F'_{q,-\mu}$ defined by (5.3). We can regard $f \to \tau f$ as an imbedding of L_μ^p into $F'_{q,-\mu}$. We shall use τ to denote any such imbedding and will not show the dependence on p, μ explicitly; this should not cause confusion.

It is natural to try to compare the spaces Q_r and L_μ^p. Since local integrability does not imply integrability over $(0, \infty)$, there is no hope of any inclusion of the form $Q_r \subset L_\mu^p$. However, in the other direction we have

Lemma 6.1. *$L_\mu^p \subset Q_r$ provided $-\operatorname{Re}\mu - 1/q < r$.*

Proof. Let $f \in L_\mu^p$ so that $\int_0^X |x^{-\mu} f(x)|^p dx < \infty$ for any $X \in (0, \infty)$. Then, by Hölder's inequality, if $1 < p < \infty$,

$$\int_0^X |x^r f(x)|\, dx = \int_0^X x^{r+\operatorname{Re}\mu} |x^{-\mu} f(x)|\, dx$$
$$\leq \left\{\int_0^X x^{(r+\operatorname{Re}\mu)q} dx\right\}^{1/q} \left\{\int_0^X |x^{-\mu} f(x)|^p dx\right\}^{1/p} < \infty$$

since $(r+\operatorname{Re}\mu)q > -1$ by hypothesis. The proof is similar if $p = 1$ or ∞. The result follows.

We know from Lemma 5.2 that if (6.6) is to have a solution f in L^p_μ, g must belong to $L^p_{\mu+c}$. We could regard this as a very rough analogue of Theorem 7, p. 185 of (5); however, our condition on g is an integrability condition whereas Love's condition, namely $I_x^{-c}g \in Q_r$, is more a differentiability condition. We can prove theorems corresponding to other results in (5). For example, compare (5, Corollary 1, p. 179) and

Theorem 6.2. *If* $g \in L^p_{\mu+c}$ *where*

$$-\operatorname{Re}\mu - 1/q < \min(\operatorname{Re} a, \operatorname{Re} b, \operatorname{Re} c, \operatorname{Re}(a+b)) \text{ and } \operatorname{Re} c > 0,$$

(6.6) *has at most one solution* $f \in L^p_\mu$.

Proof. It is instructive to give two proofs.

(i) Choose r such that $-\operatorname{Re}\mu - 1/q < r < \min(\operatorname{Re} a, \operatorname{Re} b, \operatorname{Re} c, \operatorname{Re}(a+b))$. Then also $-\operatorname{Re}(\mu+c) - 1/q < r < \min(\operatorname{Re} a, \operatorname{Re} b, \operatorname{Re} c, \operatorname{Re}(a+b))$ since $\operatorname{Re} c > 0$. Thus $g \in L^p_{\mu+c} \Rightarrow g \in Q_r$ by Lemma 6.1. By Corollary 1 on p. 179 of (5), (6.6) has at most one solution $f \in Q_r$. However, under the given conditions, $L^p_\mu \subset Q_r$ by Lemma 6.1 again; the result follows.

(ii) g generates a regular functional $\tau g \in F'_{q,-\mu-c}$ via (5.3). Thus applying τ to (6.6) gives

$$\tau H_1(a, b; c; 1)f = \tau g \tag{6.8}$$

in $F'_{q,-\mu-c}$. Also any solution f in L^p_μ of (6.6) generates $\tau f \in F'_{q,-\mu}$ and under the given conditions, (5.5) is valid and gives

$$\tau H_1(a, b; c; 1)f = H_1(a, b; c; 1)\tau f. \tag{6.9}$$

On the left-hand side $H_1(a, b; c; 1)$ is defined via (6.6) while on the right-hand side we use Definition 5.3. From (6.8) and (6.9) we seek $f \in L^p_\mu$ satisfying

$$H_1(a, b; c; 1)\tau_f = \tau g. \tag{6.10}$$

Now, under the given conditions, we may apply Theorem 5.5 with p, μ replaced by q and $-\mu$ to deduce that the equation

$$H_1(a, b; c; 1)h = \tau g \tag{6.11}$$

has a unique solution $h \in F'_{q,-\mu}$ given by

$$h = x^{-a}I_x^{-b}x^a I_x^{b-c}\tau g. \tag{6.12}$$

Thus (6.6) will have either no solution in L^p_μ or exactly one solution in L^p_μ (since we are identifying functions which differ only on a set of measure zero), depending on whether there exists $f \in L^p_\mu$ such that $h = \tau f$, that is, depending on whether h, as given by (6.12), is a regular functional.

We might call h, as given by (6.12), a generalised solution of (6.6). The second proof of Theorem 6.2 has therefore established

Corollary 6.3. *If* $g \in L^p_{\mu+c}$ *with*

$$-\text{Re}\,\mu - 1/q < \min(\text{Re}\,a, \text{Re}\,b, \text{Re}\,c, \text{Re}\,(a+b)) \text{ and } \text{Re}\,c > 0,$$

(6.6) *has a unique generalised solution in* $F'_{q,-\mu}$ *given by*

$$h = x^{-a} I_x^{-b} x^a I_x^{b-c} \tau g.$$

If we are interested in classical rather than generalised solutions we must put further restrictions on our parameters and on g. We then obtain results analogous to Theorems 8-12 in (5). Selecting just one at random, we shall prove an analogue of Theorem 11. Our proof requires the following lemma.

Lemma 6.4. *Let* $f \in L^p_\mu$.

(i) *For any complex numbers* μ *and* η,

$$\tau x^\eta f = x^\eta \tau f. \tag{6.13}$$

(ii) *If* $\text{Re}\,\alpha > 0$ *and* $\text{Re}\,\mu > -1/q$,

$$\tau I_x^\alpha f = I_x^\alpha \tau f. \tag{6.14}$$

Proof. We prove (ii); (i) can be proved similarly using (2.13) of (9).

Since $\text{Re}\,\alpha > 0$ and $\text{Re}\,\mu > -1/q$, $I_x^\alpha f$ is defined by

$$I_x^\alpha f(x) = \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt \quad (x > 0). \tag{6.15}$$

Under the given conditions we may apply Theorem 2.1 (i) of (4) (with $f(x)$, ζ replaced by $x^{-\mu}f(x)$ and μ) to deduce that $I_x^\alpha f \in L^p_{\mu+\alpha}$ so that $\tau I_x^\alpha f \in F'_{q,-\mu-\alpha}$. On the other hand, $f \in L^p_\mu \Rightarrow \tau f \in F'_{q,-\mu}$ and we may apply Theorem 3.11 (with p, μ replaced by q, $-\mu$) to deduce that $I_x^\alpha \tau f \in F'_{q,-\mu-\alpha}$ also. (Here I_x^α is interpreted in the sense of (3.29) of (9)). Thus both sides of (6.14) exist and belong to $F'_{q,-\mu-\alpha}$. To prove equality let $\phi \in F_{q,-\mu-\alpha}$. Then

$$
\begin{aligned}
(I_x^\alpha \tau f, \phi) &= (\tau f, K_x^\alpha \phi) && \text{by (3.29) of (9)}\\
&= \int_0^\infty f(x) K_x^\alpha \phi(x)dx && \text{by (5.3)}\\
&= \int_0^\infty f(x)\left(\int_x^\infty \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)}\phi(t)dt\right)dx &&\\
&= \int_0^\infty \phi(t)dt \int_0^t \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x)dx &&\\
&= \int_0^\infty I_x^\alpha f(x)\phi(x)dx && \text{by (6.15)}\\
\Rightarrow (I_x^\alpha \tau f, \phi) &= (\tau I_x^\alpha f, \phi) && \text{by (5.3).}
\end{aligned}
$$

The inversion of the order of integration in the fourth line above is justified since the repeated integrals involved are both absolutely convergent by Hölder's inequality. This completes the proof.

We can now prove

Theorem 6.5. *If* $\operatorname{Re} b<0$, $\operatorname{Re} c>0$, $-\operatorname{Re}(\mu+b)-1/q<\min(0, \operatorname{Re} a)$ *and* $I_x^{b-c}g\in L_{\mu+b}^p$ *(that is, there exists* $G\in L_{\mu+b}^p$ *such that* $g=I_x^{c-b}G$*), then* (6.6) *has a unique solution* $f\in L_\mu^p$ *given by*

$$f(x)=x^{-a}I_x^{-b}x^aG(x).$$

Proof. Under the given conditions, we certainly have $\operatorname{Re} c>0$ and also $-\operatorname{Re}\mu-1/q<\min(\operatorname{Re} a, \operatorname{Re} b, \operatorname{Re} c, \operatorname{Re}(a+b))$. Further, since $\operatorname{Re}(c-b)>0$ we may apply Theorem 2.1 (i) of (4) (with $f(x)$, ζ and α replaced by $x^{-\mu-b}G(x)$, $\mu+b$ and $c-b$) to prove that $G\in L_{\mu+b}^p\Rightarrow g\in L_{\mu+c}^p$. Hence, by Corollary 6.3, (6.6) has a unique generalised solution $h\in F'_{q,-\mu}$ given by

$$\begin{aligned} h &= x^{-a}I_x^{-b}x^aI_x^{b-c}\tau g \\ &= x^{-a}I_x^{-b}x^aI_x^{b-c}\tau I_x^{c-b}G \\ &= x^{-a}I_x^{-b}x^aI_x^{b-c}I_x^{c-b}\tau G && \text{by (6.14)} \\ &= x^{-a}I_x^{-b}x^a\tau G && \text{using Theorem 3.11(i) of (9)} \\ &= x^{-a}I_x^{-b}\tau x^aG && \text{by (6.13)} \\ &= x^{-a}\tau I_x^{-b}x^aG. && \text{by (6.14)} \end{aligned}$$

Thus, by (6.13) we have

$$h=\tau x^{-a}I_x^{-b}x^aG. \tag{6.16}$$

The first application of (6.14) above is valid since $\operatorname{Re}(c-b)>0$ and

$$\operatorname{Re}(\mu+b)>-1/q$$

and the second is valid since $\operatorname{Re}(-b)>0$ and $\operatorname{Re}(\mu+b+a)>-1/q$. From (6.16) we have $h=\tau f$ where

$$f=x^{-a}I_x^{-b}x^aG.$$

By examining the Proof of Theorem 6.2, we see that $f\in L_\mu^p$ is a solution of (6.6) and since τ is a 1-1 mapping of L_μ^p into $F'_{q,-\mu}$, uniqueness of h gives uniqueness of f. The result follows.

Here we mention that in (6), Love discusses the operators $H_3(a, b;\ c;\ 1)$ and $H_4(a, b;\ c;\ 1)$ relative to the spaces R_r; for each real r

$$R_r=\{f: x^rf(x)\in L^1(X, \infty) \text{ for each } X>0\}.$$

(Compare with (6.7).) Analogously to Lemma 6.1 we find that $L_\mu^p\subset R_r$ provided that $\operatorname{Re}\mu+1/q<-r$. We can then proceed as above to prove theorems on classical solutions analogous to those in (6). Details are similar to those above and are omitted.

Lemma 5.2 showed that the integrability condition $g\in L_{\mu+c}^p$ was necessary for (6.6) to have a solution $f\in L_\mu^p$. On the other hand, the condition $I_x^{b-c}g\in L_{\mu+b}^p$ in Theorem 6.5 states that g has a fractional derivative of order $c-b$ (belonging to $L_{\mu+b}^p$). This is characteristic in the sense that any sufficient condition for a classical solution (in L^p_μ) of (6.6) (or the corresponding equations for the other operators) seems to require of g a certain degree of differentiability. For further discussions in this direction we refer the reader to Higgins (3).

In conclusion, we remark that results for special cases such as Jacobi polynomials and Legendre functions can be obtained by the appropriate choices of a, b, c and m.

Acknowledgements

This paper is based on part of the author's Ph.D. thesis written at the University of Edinburgh under the supervision of Professor A. Erdélyi. It is a pleasure to record my thanks to Professor Erdélyi for his help and guidance and to the Carnegie Trust for the Universities of Scotland for the award of a Scholarship.

I am also grateful to the referee for his many useful comments on the original draft of the paper.

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Paper 3

Proceedings of the Royal Society of Edinburgh, 77A, 39–47, 1976

* A note on the spaces $F_{p,\mu}$

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(MS received 14 April 1976. Read 25 October 1976)

Synopsis

This note is concerned with the spaces $F'_{p,\mu}$ of generalised functions introduced in a previous paper. A necessary and sufficient condition for an inclusion of the form

$$F'_{p_1,\mu_1} \subseteq F'_{p_2,\mu_2}$$

to hold is established. The case $p = \infty$ leads to consideration of a class $G'_{\infty,\mu}$ whose simple properties are noted. Some consequences of relevance to fractional integrals and Hankel transforms are indicated.

1. Introduction

The purpose of this short note is to consider a question connected with the spaces $F_{p,\mu}$ of testing-functions and the corresponding spaces $F'_{p,\mu}$ of generalised functions which were introduced in [3]. The answer to this question is relevant to a discussion of the Hankel transform of elements of $F'_{p,\mu}$, a topic which we intend to explore in a future paper.

For convenience we recall briefly the necessary definitions. We consider complex-valued infinitely differentiable functions on $(0, \infty)$. For $1 \leq p \leq \infty$, let

$$F_p = \{\phi \in C^\infty(0, \infty) : x^k d^k\phi/dx^k \in L^p(0, \infty) \quad \text{for } k = 0, 1, 2, \ldots\}. \tag{1.1}$$

We equip F_p with the topology generated by the semi-norms γ_k^p $(k = 0, 1, 2, \ldots)$ defined by

$$\gamma_k^p(\phi) = \| x^k d^k\phi/dx^k \|_p. \tag{1.2}$$

Then for any complex number μ,

$$F_{p,\mu} = \{\phi : x^{-\mu}\phi(x) \in F_p\} \tag{1.3}$$

with the topology generated by the semi-norms $\gamma_k^{p,\mu}$ $(k = 0, 1, 2, \ldots)$ given by

$$\gamma_k^{p,\mu}(\phi) = \gamma_k^p(x^{-\mu}\phi). \tag{1.4}$$

$F'_{p,\mu}$ is the space of continuous linear functionals on $F_{p,\mu}$ equipped with the topology of pointwise (weak) convergence.

The question we shall consider is as follows: 'Are there any inclusions of the form

$$F_{p_1,\mu_1} \subseteq F_{p_2,\mu_2} \text{ or } F'_{p_1,\mu_1} \subseteq F'_{p_2,\mu_2}$$

apart from the trivial ones when $p_1 = p_2$ and $\operatorname{Re}\mu_1 = \operatorname{Re}\mu_2$?'

The answer is affirmative. In section 2 we prove that, if $1 \leq p_1 \leq p_2 \leq \infty$ and

* This paper was assisted in publication by a grant from the Carnegie Trust for the Universities of Scotland.

μ_1, μ_2 are complex numbers, then

$$F_{p_1,\mu_1} \subseteq F_{p_2,\mu_2} \tag{1.5}$$

if and only if $\operatorname{Re}(\mu_1-\mu_2) = 1/p_1 - 1/p_2$. Furthermore, the inclusion is strict when $p_1 < p_2$. We can deduce easily that

$$F'_{p_1,\mu_1} \supseteq F'_{p_2,\mu_2}$$

in the case $1 \leqq p_1 \leqq p_2 < \infty$. For the case $p = \infty$, we are led to consider subspaces $G_{\infty,\mu}$ of $F_{\infty,\mu}$ and the corresponding dual spaces $G'_{\infty,\mu}$. These spaces are studied in section 3 and we show in particular that

$$F'_{p_1,\mu_1} \supseteq G'_{\infty,\mu_2}$$

provided $1 \leqq p_1 < \infty$ and $\operatorname{Re}(\mu_1-\mu_2) = 1/p_1$.

In section 4, we contrast these results with the situation for the spaces L^p_μ where, for $1 \leqq p \leqq \infty$ and any complex number μ,

$$L^p_\mu = \{f\colon x^{-\mu}f(x) \in L^p(0,\infty)\}. \tag{1.6}$$

The analogue of (1.5), namely $L^{p_1}_{\mu_1} \subseteq L^{p_2}_{\mu_2}$ is false. We also explore some of the consequences of (1.5) for various results on fractional integrals established by Kober [1] as well as theorems on the Fourier or Hankel transforms derived from those in [6].

2

Let $1 \leqq p_1 \leqq p_2 \leqq \infty$ and let μ_1, μ_2 be complex numbers. In this section we first show that a necessary and sufficient condition for

$$F_{p_1,\mu_1} \subseteq F_{p_2,\mu_2} \tag{2.1}$$

to hold is that $\operatorname{Re}(\mu_1-\mu_2) = 1/p_1 - 1/p_2$. To do this, we proceed by means of a number of simple steps and, where convenient, we use p, r instead of p_1, p_2 for typographical convenience.

First we recall the definition of the spaces $\mathscr{D}_{L^p}$ as defined in [5, p. 199]. For $1 \leqq p \leqq \infty$,

$$\mathscr{D}_{L^p} = \{\phi\colon D^k\phi \in L^p(-\infty,\infty),\ k = 0, 1, 2, \ldots\} \tag{2.2}$$

and the topology on $\mathscr{D}_{L^p}$ is that generated by the semi-norms

$$\nu^p_k(\phi) = \| D^k\phi \|_{L^p(-\infty,\infty)} \quad (k = 0, 1, 2, \ldots). \tag{2.3}$$

We have the following lemma:

LEMMA 2.1. *If* $1 \leqq p < r \leqq \infty$, *then* $\mathscr{D}_{L^p} \subset \mathscr{D}_{L^r}$, *the inclusion being strict. Further, the identity mapping from* $\mathscr{D}_{L^p}$ *into* $\mathscr{D}_{L^r}$ *is continuous.*

Proof. Most of the details required appear in [5, pp. 199–200]. We merely note that to establish the strict inclusion, we may consider

$$\psi(x) = \omega(x)x^a, \tag{2.4}$$

where $-p^{-1} < a < -r^{-1}$ and $\omega(x)$ is an infinitely differentiable function on $(-\infty, \infty)$ such that

$$\omega(x) = \begin{cases} 0 & x < 1 \\ 1 & x > 2. \end{cases}$$

Then $\psi \in \mathscr{D}_{L^r}$ but $\psi \notin \mathscr{D}_{L^p}$.

The relevance of this result to our investigations is indicated by the next lemma.

LEMMA 2.2. *Let $1 \leqq p \leqq \infty$ and let μ be a complex number. Then the operator $T_{p,\mu}$ defined by*

$$(T_{p,\mu}\phi)(x) = e^{(1/p-\mu)x}\phi(e^x) \quad (\phi \in F_{p,\mu}) \tag{2.5}$$

is a homeomorphism of $F_{p,\mu}$ onto $\mathscr{D}_{L^p}$.

[Note: The operator $T_{p,\mu}$ is the same as the operator $C_{1-\mu p,p}$ used by Rooney in [4] when $p<\infty$.]

Proof. It is obvious that $T_{p,\mu}$ is $1-1$. As regards continuity, we first note, as in [4], that if $\phi \in F_{p,\mu}$

$$\| T_{p,\mu}\phi \|_{L^p(-\infty,\infty)} = \gamma_0^{p,\mu}(\phi), \tag{2.6}$$

where $\gamma_k^{p,\mu}$ is defined in (1.4). Also by induction

$$D^k T_{p,\mu}\phi = T_{p,\mu}[(1/p-\mu)I+\delta]^k\phi \quad (\phi \in F_{p,\mu}) \tag{2.7}$$

($k = 0, 1, 2, \ldots$) where I denotes the identity mapping on $F_{p,\mu}$ and $\delta \equiv xd/dx$. Combining (2.6 and 2.7), we can show after a little routine algebra that

$$\nu_k^p(T_{p,\mu}\phi) \leqq \sum_{i=0}^{k} C_i\gamma_i^{p,\mu}(\phi)$$

for certain constants C_i independent of ϕ. It follows that $T_{p,\mu}$ is a continuous mapping of $F_{p,\mu}$ into $\mathscr{D}_{L^p}$. That $T_{p,\mu}$ maps $F_{p,\mu}$ onto $\mathscr{D}_{L^p}$ is easily seen and

$$(T_{p,\mu}^{-1})\psi(x) = x^{\mu-1/p}\psi(\log x) \quad (\psi \in \mathscr{D}_{L^p}). \tag{2.8}$$

The continuity of $T_{p,\mu}^{-1}$ can be proved directly but follows immediately from the Open Mapping Theorem for Fréchet spaces [7, Theorem 17.1]. This completes the proof of Lemma 2.2.

We now combine the results of Lemmas 2.1 and 2.2.

COROLLARY 2.3. *Let $1 \leqq p_1 \leqq p_2 \leqq \infty$. Then, if* $\operatorname{Re}(\mu_1-\mu_2) = 1/p_1-1/p_2$

$$F_{p_1,\mu_1} \subseteqq F_{p_2,\mu_2}$$

and the identity mapping from F_{p_1,μ_1} into F_{p_2,μ_2} is continuous. Also if $p_1<p_2$, the inclusion is strict.

Proof. First we remark that if μ is any complex number and $1 \leqq p \leqq \infty$, then $F_{p,\mu} = F_{p,\operatorname{Re}\mu}$; this is a consequence of the fact that for $0<x<\infty$, $x^{i(\operatorname{Im}\mu)}$ is infinitely differentiable and $|x^{i(\operatorname{Im}\mu)}| = 1$. Thus it is sufficient to prove the result when μ_1 and μ_2 are real. The case $p_1 = p_2$ is now a triviality; so we consider only the case $1 \leqq p_1<p_2 \leqq \infty$, $\mu_1-\mu_2 = 1/p_1-1/p_2$. By Lemma 2.1, $\mathscr{D}_{L^{p_1}} \subset \mathscr{D}_{L^{p_2}}$. But

$$1/p_1-\mu_1 = 1/p_2-\mu_2$$

so that $T_{p_1,\mu_1} = T_{p_2,\mu_2}$

$$\Rightarrow T_{p_1,\mu_1}^{-1}(\mathscr{D}_{L^{p_1}}) \subset T_{p_2,\mu_2}^{-1}(\mathscr{D}_{L^{p_2}})$$

$$\Rightarrow F_{p_1,\mu_1} \subset F_{p_2,\mu_2}$$

by Lemma 2.2. The strict inclusion and continuity of the embedding follow from the homeomorphic properties of $T_{p_1,\mu_1} = T_{p_2,\mu_2}$ and Lemma 2.1.

For future reference we write down a function ϕ in F_{p_2,μ_2} which does not belong to F_{p_1,μ_1} when $p_1<p_2$ and $\operatorname{Re}(u_1-\mu_2) = 1/p_1-1/p_2$.

Using $T^{-1}_{p_1, \mathrm{Re}\,\mu_1} = T^{-1}_{p_2, \mathrm{Re}\,\mu_2}$ as given by (2.8) and ψ as in (2.4), we can choose ϕ to be given by

$$\phi(x) = x^{\mu_2 - 1/p_2}\omega(\log x)(\log x)^a,$$

where $-1/p_1 < a < -1/p_2$. (Here again we use the fact that $F_{p_1,\mu_1} = F_{p_1, \mathrm{Re}\,\mu_1}$ etc.) Alternatively we could take

$$\phi(x) = x^{\mu_2 - 1/p_2}\lambda(x)(\log x)^a, \tag{2.9}$$

where $-1/p_1 < a < -1/p_2$ and λ is an infinitely differentiable function on $(0, \infty)$ such that

$$\lambda(x) = \begin{cases} 0 & x < C_1 \\ 1 & x > C_2 \end{cases} \tag{2.10}$$

for positive constants C_1, C_2 with $C_1 < C_2$.

In the other direction we have the following result.

LEMMA 2.4. *Let* $1 \leqq p_1 \leqq p_2 \leqq \infty$. *Then if* $\mathrm{Re}\,(\mu_1 - \mu_2) \neq 1/p_1 - 1/p_2$, the inclusion (2.1) *is false.*

Proof. Again we may assume without loss of generality that μ_1, μ_2 are real.

(i) $1/p_1 - \mu_1 < 1/p_2 - \mu_2$. Let $b = (1/p_2 - \mu_2) - (1/p_1 - \mu_1)$; then $b > 0$. Let

$$\phi(x) = \lambda(x)x^{\mu_1 - 1/p_1 - b/2}$$

with λ as in (2.10). Then, since $x^{-\mu_1}\phi(x) = \lambda(x)x^{-1/p_1 - b/2}$, $\phi \in F_{p_1,\mu_1}$. On the other hand

$$x^{-\mu_2}\phi(x) = \lambda(x)x^{-\mu_2 + \mu_1 - 1/p_1 - b/2} = \lambda(x)x^{-1/p_2 + b/2}$$

and this function does not belong to L^{p_2} since $b > 0$.

So, in this case, (2.1) is false.

(ii) $1/p_1 - \mu_1 > 1/p_2 - \mu_2$. Again we let $b = (1/p_2 - \mu_2) - (1/p_1 - \mu_1) < 0$ and choose $\phi(x) = \lambda(1/x)x^{\mu_1 - 1/p_1 - b/2}$ with λ as in (2.10). Again it is easy to check that $\phi \in F_{p_1,\mu_1}$ but $\phi \notin F_{p_2,\mu_2}$.

This completes the proof of Lemma 2.4.

We collect our results together in the following theorem.

THEOREM 2.5. *Let* $1 \leqq p_1 \leqq p_2 \leqq \infty$ *and let* μ_1, μ_2 *be complex numbers.*

(i) $F_{p_1,\mu_1} \subseteq F_{p_2,\mu_2}$ if and only if $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1 - 1/p_2$.

(ii) If $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1 - 1/p_2$, then the identity mapping from F_{p_1,μ_1} into F_{p_2,μ_2} is continuous.

(iii) If $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1 - 1/p_2$ and $p_1 < p_2$, then F_{p_1,μ_1} is a proper subspace of F_{p_2,μ_2}.

We mentioned in [3] that $\mathscr{D}(= \mathscr{D}(0, \infty))$ is dense in $F_{p,\mu}$ for any complex μ and for $1 \leqq p < \infty$. Hence F_{p_1,μ_1} is dense in F_{p_2,μ_2} when $1 \leqq p_1 \leqq p_2 < \infty$ and

$$\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1 - 1/p_2.$$

This enables us to prove the next result.

THEOREM 2.6. *Let* $1 \leqq p_1 \leqq p_2 < \infty$ *and let* μ_1, μ_2 *be complex numbers. Then* $F'_{p_1,\mu_1} \supseteq F'_{p_2,\mu_2}$ *if and only if* $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1 - 1/p_2$. *Further, if* $p_1 < p_2$, *then* $F'_{p_1,\mu_1} \supset F'_{p_2,\mu_2}$ *(strict inclusion).*

Proof. If $\operatorname{Re}(\mu_1-\mu_2) = 1/p_1-1/p_2$, then $F'_{p_1,\mu_1} \supseteq F'_{p_2,\mu_2}$ by virtue of our previous remark together with [8, Theorem 1.8–2].

For the converse we again assume without loss of generality that μ_1 and μ_2 are real. Let $b = (1/p_2-\mu_2)-(1/p_1-\mu_1)>0$. (The case $b<0$ is similar.) Also write

$$q_i = p_i/(p_i-1) \quad (i = 1, 2).$$

Then $1<q_2 \leqq q_1 \leqq \infty$ and $b = (1/q_1-\mu_2)-(1/q_2-\mu_1)$. Let

$$f(x) = \lambda(x)x^{-\mu_2-1/q_2-b/2}$$

with λ as in (2.10). Then as in Lemma 2.4, $x^{\mu_1+\mu_2}f(x) \in F_{q_2,\mu_1}$ but $x^{\mu_1+\mu_2}f(x) \notin F_{q_1,\mu_2}$

$$\Rightarrow f \in F_{q_2,-\mu_2} \text{ (but } f \notin F_{q_1,-\mu_1}).$$

By Hölder's inequality, f generates an element, $\tilde{f}$ say, of F'_{p_2,μ_2} by means of the formula

$$(\tilde{f}, \phi) = \int_0^\infty f(x)\phi(x)dx. \quad (\phi \in F_{p_2,\mu_2}). \tag{2.11}$$

On the other hand, consider the function

$$\phi(x) = \lambda(x)x^{\mu_1-1/p_1-b/4}. \tag{2.12}$$

This function $\phi \in F_{p_1,\mu_1}$ since $b>0$, but for x sufficiently large

$$f(x)\phi(x) = x^{-\mu_2-1/q_2-b/2+\mu_1-1/p_1-b/4} = x^{-1+b/4}$$

and since $-1+b/4>-1$, $\displaystyle\int_0^\infty f(x)\phi(x)dx$ does not exist. Thus we cannot use (2.11) to extend $\tilde{f}$ to an element of F'_{p_1,μ_1}. Indeed, there is no such extension. To see this, let α be an infinitely differentiable function such that $0 \leqq \alpha(x) \leqq 1$ for all $x \in (0, \infty)$ and

$$\alpha(x) = \begin{cases} 1 & 0<x<1 \\ 0 & x>2. \end{cases}$$

Then let $\phi_n(x) = \alpha(x/n)\phi(x)$ with ϕ as in (2.12). It can be shown that ϕ_n converges to ϕ in the topology of F_{p_1,μ_1}. [For a similar calculation, see **2**, pp. 63–4.] Thus if $\tilde{f}_E$ denotes any extension of $\tilde{f}$ to an element of F'_{p_1,μ_1} we would have

$$(\tilde{f}_E, \phi) = \lim_{n\to\infty} (\tilde{f}_E, \phi_n).$$

But, since $\phi_n \in F_{p_2,\mu_2}$,

$$(\tilde{f}_E, \phi_n) = (\tilde{f}, \phi_n) = \int_0^\infty f(x)\phi_n(x)dx \to \infty \text{ as } n\to\infty$$

as above and we have a contradiction. Hence $F'_{p_1,\mu_1} \not\supseteq F'_{p_2,\mu_2}$ in this case.

Finally, the strict inclusion in the theorem requires similar arguments to the above and the proof is therefore omitted.

This completes the proof of Theorem 2.6.

Since $\mathscr{D}$ is not dense in $F_{\infty,\mu}$ for any complex μ we must exclude the case $p = \infty$ in Theorem 2.6. To plug the resulting gap we are led to consider a certain subspace, $G_{\infty,\mu}$ say, of $F_{\infty,\mu}$ and the corresponding dual space $G'_{\infty,\mu}$. This investigation is carried out in the next section.

3

We define the space G_∞ by

$$G_\infty = \left\{\phi \in F_\infty : \begin{array}{l}\text{for each non-negative integer } k, \\ x^k d^k\phi/dx^k \to 0 \text{ as } x\to 0+ \text{ and as } x\to\infty\end{array}\right\} \tag{3.1}$$

and equip G_∞ with the topology induced by that on F_∞ (that is, the topology generated by the semi-norms γ_k^∞ ($k = 0, 1, 2, \ldots$) defined in (1.2)). Then, for each complex number μ, we define $G_{\infty,\mu}$ by

$$G_{\infty,\mu} = \{\phi \colon x^{-\mu}\phi(x) \in G_\infty\} \tag{3.2}$$

and equip $G_{\infty,\mu}$ with the topology induced by that on $F_{\infty,\mu}$ (that is, the topology generated by the semi-norms $\gamma_k^{\infty,\mu}$ ($k = 0, 1, 2, \ldots$) defined in (1.4)).

Using the operator $T_{\infty,\mu}$ defined in (2.5), it is easy to check that $G_{\infty,\mu}$ is homeomorphic to the space $(\mathscr{B})$ defined in [5, p. 199] so that the properties of $G_{\infty,\mu}$ follow almost at once from those of $(\mathscr{B})$ in [5]. We list those of interest to us in the form of a theorem.

THEOREM 3.1. *Let* $1 \leqq p_1 < \infty$ *and let* μ_1, μ_2 *be complex numbers.*

(i) $F_{p_1,\mu_1} \subseteqq G_{\infty,\mu_2}$ *if and only if* $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1$.

(ii) *If* $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1$, *then* F_{p_1,μ_1} *is a proper subspace of* G_{∞,μ_2} *and the identity mapping from* F_{p_1,μ_1} *into* G_{∞,μ_2} *is continuous.*

(iii) *For any complex number* μ, $\mathscr{D}(= \mathscr{D}(0, \infty))$ *is dense in* $G_{\infty,\mu}$.

Proof. (i) We may assume again that μ_1, μ_2 are real. From [5], p. 200, $\mathscr{D}_{L^{p_1}} \subset (\mathscr{B})$. Thus if $\mathrm{Re}\,(\mu_1 - \mu_2) = 1/p_1$, $T_{p_1,\mu_1}^{-1}(\mathscr{D}_{L^{p_1}}) \subset T_{\infty,\mu_2}^{-1}(\mathscr{B})$. By Lemma 2.2 and our remark above, $F_{p_1,\mu_1} \subset G_{\infty,\mu_2}$ as required. The converse follows at once from Lemma 2.4.

(ii) This follows as in Corollary 2.3. For the strict inclusion, we can use the function ϕ in (2.9) with $p_2 = \infty$ and $-1/p_1 < a < 0$.

(iii) By [5, p. 199] $\mathscr{D}(-\infty, \infty)$ is dense in $(\mathscr{B})$. Our result follows on using the homeomorphism $T_{\infty,\mu}^{-1}$ from $(\mathscr{B})$ onto $G_{\infty,\mu}$ which also maps $\mathscr{D}(-\infty, \infty)$ onto $\mathscr{D}(0, \infty)$.

The corresponding result for generalised functions is contained in the next theorem.

THEOREM 3.2. *Let* $1 \leqq p_1 \leqq p_2 < \infty$ *and let* μ_1, μ_2, μ_3 *be complex numbers such that*

$$\mathrm{Re}\,\mu_1 - 1/p_1 = \mathrm{Re}\,\mu_2 - 1/p_2 = \mathrm{Re}\,\mu_3.$$

Then

$$G'_{\infty,\mu_3} \subseteqq F'_{p_2,\mu_2} \subseteqq F'_{p_1,\mu_1} \subseteqq \mathscr{D}'(= \mathscr{D}'(0, \infty)).$$

Proof. Under the given conditions we have the chain of inclusions

$$\mathscr{D} \subset F_{p_1,\mu_1} \subseteqq F_{p_2,\mu_2} \subset G_{\infty,\mu_3}$$

each inclusion in the chain being dense and each embedding continuous (by Corollary 2.3 and Theorem 3.1). The result therefore follows from [8, Theorem 1.8–2].

Theorems in [3] concerning the spaces $F'_{\infty,\mu}$ give rise to corresponding results for $G'_{\infty,\mu}$. As an example we consider again the Erdélyi-Kober operators $I_{x^m}^{\eta,\alpha}$, $K_{x^m}^{\eta,\alpha}$; for the relevant definitions see [3].

THEOREM 3.3 (i) *If* $\mathrm{Re}\,(m\eta + \mu) + m > 0$, then $I_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $G_{\infty,\mu}$ *into itself; if also* $\mathrm{Re}\,(m\eta + m\alpha + \mu) + m > 0$, *then* $I_{x^m}^{\eta,\alpha}$ *is an automorphism of* $G_{\infty,\mu}$.

(ii) *If* $\mathrm{Re}\,(m\eta-\mu)>0$, *then* $K_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $G_{\infty,\mu}$ *into itself; if also* $\mathrm{Re}\,(m\eta+m\alpha-\mu)>0$, *then* $K_{x^m}^{\eta,\alpha}$ *is an automorphism of* $G_{\infty,\mu}$.

Proof. We prove (i), (ii) being similar. For (i), it only remains in view of [3, Theorem 3.5], to show that if $\phi\in G_{\infty,\mu}$,

$$x^k d^k/dx^k(x^{-\mu}I_{x^m}^{\eta,\alpha}\phi)\to 0 \text{ as } x\to 0 \text{ and as } x\to+\infty, \text{ for } k=0,1,2,\ldots.$$

Now if $\mathrm{Re}\,\alpha>0$,

$$|x^{-\mu}I_{x^m}^{\eta,\alpha}\phi(x)| = m\left|[\Gamma(\alpha)]^{-1}\int_0^1 (1-t^m)^{\alpha-1}t^{m\mu+m-1+\mu}(xt)^{-\mu}\phi(xt)dt\right|$$
$$\leqq m\sup_{0<v<x}|v^{-\mu}\phi(v)|\,|\Gamma(\alpha)|^{-1}\int_0^1 (1-t^m)^{\mathrm{Re}\,\alpha-1}t^{\mathrm{Re}\,(m\eta+\mu)+m-1}dt.$$

The integral on the right is finite under the given conditions and since $v^{-\mu}\phi(v)\to 0$ as $v\to 0+$, it follows that $x^{-\mu}I_{x^m}^{\eta,\alpha}\phi(x)\to 0$ as $x\to 0+$.

Now let $\{x_n\}_{n=1}^{\infty}$ be any sequence of positive real numbers tending to ∞.

Let $f_n(t)=(1-t^m)^{\alpha-1}t^{m\eta+m-1+\mu}(x_nt)^{-\mu}\phi(x_nt)$ $(0<t<1)$. Then for each fixed t, $x_nt\to\infty$ as $n\to\infty\Rightarrow(x_nt)^{-\mu}\phi(x_nt)\to 0$ as $n\to\infty$ (since $\phi\in G_{\infty,\mu}$), i.e. f_n converges pointwise to 0. Also $|f_n(t)|\leqq(1-t^m)^{\mathrm{Re}\,\alpha-1}t^{\mathrm{Re}\,(m\eta+\mu)+m-1}\gamma_0^{\infty,\mu}(\phi)$ so that the right-hand side is integrable over (0, 1) under the given conditions ($\mathrm{Re}\,\alpha>0$ still). By Lebesgue's Dominated Convergence Theorem, $\int_0^1 f_n(t)dt\to 0$ as $n\to\infty$; that is, $(x_n)^{-\mu}(I_{x^m}^{\eta,\alpha}\phi)(x_n)\to 0$ as $n\to\infty$ for any sequence $\{x_n\}$ with $x_n\to\infty$. Thus $x^{-\mu}I_{x^m}^{\eta,\alpha}\phi(x)\to 0$ as $n\to\infty$. Also, for $k=1,2,\ldots$ we may differentiate under the integral sign to obtain

$$x^k d^k/dx^k(x^{-\mu}I_{x^m}^{\eta,\alpha}\phi)=x^{-\mu}\sum_{i=0}^{k}C_iI_{x^m}^{\eta,\alpha}(x^id^i\phi/dx^i)$$

for certain constants $C_i(i=0,\ldots,k)$ and since $x^id^i\phi/dx^i\to 0$ as $x\to 0+$ and as $x\to\infty$ we may apply the previous reasoning to obtain the required result for $\mathrm{Re}\,\alpha>0$. The case $\mathrm{Re}\,\alpha\leqq 0$ is now easily handled using [3, formula (3.6)] and observing that δ is a continuous linear mapping of $G_{\infty,\mu}$ into itself. This completes the proof of Theorem 3.3.

The corresponding results for $G'_{\infty,\mu}$ follow immediately from [8, Theorems 1.10–1 and 1.10–2]. Thus, as regards the theory of fractional integration, there seems nothing to choose between $F'_{\infty,\mu}$ and $G'_{\infty,\mu}$. However, in certain cases, it will be more advantageous to use $G'_{\infty,\mu}$ as we shall see in our work on the Hankel transform.

4

Let us consider now the spaces L_μ^p as defined in (1.6). Such weighted L^p spaces have been considered, for instance, in [4]; our space L_μ^p is Rooney's space $L_{1-\mu p,p}$.

It might be thought, as a result of Theorem 2.5, that there might be an inclusion of the form

$$L_{\mu_1}^{p_1}\subseteqq L_{\mu_2}^{p_2} \tag{4.1}$$

when $1\leqq p_1\leqq p_2\leqq\infty$ and $\mathrm{Re}\,(\mu_1-\mu_2)=1/p_1-1/p_2$. However, this is soon seen to be false by considering the operator $T_{p,1/p}$ (see (2.5)) defined on $L_{1/p}^p$ by

$$(T_{p,1/p}\phi)(x)=\phi(e^x)\quad(-\infty<x<\infty).$$

For $1 \leqq p \leqq \infty$, $T_{p,1/p}$ is an isometric isomorphism of $L^p_{1/p}$ onto $L^p(-\infty, \infty)$ so that $L^{p_1}_{1/p_1} \subseteq L^{p_2}_{1/p_2}$ is equivalent to $L^{p_1}(-\infty, \infty) \subseteq L^{p_2}(-\infty, \infty)$ which is false. The falsity of (4.1) in general then follows by considering the mapping

$$\phi \to x^{\mathrm{Re}\,\mu_1 - 1/p_1}\phi = x^{\mathrm{Re}\,\mu_2 - 1/p_2}\phi.$$

Let us examine two well-known results in the light of our findings above. First we consider the Erdélyi-Kober operator $I_x^{\eta,\alpha}$ of fractional integration defined for $\mathrm{Re}\,\alpha > 0$ and suitable functions ϕ by

$$I_x^{\eta,\alpha}\phi(x) = [\Gamma(\alpha)]^{-1}x^{-\eta-\alpha}\int_0^x (x-t)^{\alpha-1}t^\eta\phi(t)dt.$$

In [1, Theorem 2], Kober proved that, if $\mathrm{Re}\,\eta > 1/p - 1$, then $I_x^{\eta,\alpha}$ is a continuous linear mapping of $L^p(=L^p_0)$ into $L^r_{1/r-1/p}$ under the following additional alternative hypotheses:

(i) $1 \leqq p \leqq \infty$, $p = r$

(ii) $1 < p < r < \infty$, $1/p - 1/r \leqq \mathrm{Re}\,\alpha \leqq 1/p$

(iii) $1 \leqq p \leqq r \leqq \infty$, $\mathrm{Re}\,\alpha > 1/p$.

In view of the falsity of (4.1), the results obtained under (i, ii, iii) are independent; for instance we cannot immediately deduce that under (ii) from that under (i). However, when we study $I_x^{\eta,\alpha}$ relative to $F_{p,\mu}$, the situation is completely different. In [3, Theorem 3.2] with $m = 1$, we proved that $I_x^{\eta,\alpha}$ was a continuous linear mapping of F_p into F_p for $\mathrm{Re}\,\eta > 1/p - 1$ (and $\mathrm{Re}\,\alpha > 0$). In view of Theorem 2.5, it follows that $I_x^{\eta,\alpha}$ is a continuous linear mapping of F_p into $F_{r,1/r-1/p}$ whenever conditions (ii or iii) are satisfied so that all the results may be subsumed in just one. Furthermore, in cases (ii and iii), if $p < r$, $I_x^{\eta,\alpha}$ cannot possibly map F_p *onto* $F_{r,1/r-1/p}$ because of the strict inclusion in Theorem 2.5. Thus the largest candidate for the range of $I_x^{\eta,\alpha}$ on F_p is F_p itself and we saw in [3, Theorem 3.5] that if $\mathrm{Re}\,\eta > 1/p - 1$ (and $\mathrm{Re}\,(\eta+\alpha) > 1/p - 1$), then $I_x^{\eta,\alpha}$ does indeed map F_p onto F_p.

Similar considerations apply to the Fourier sine and cosine transforms on F_p. Theorem 74 of [6] shows that if $1 < p \leqq 2$, the Fourier transform is a continuous linear mapping of $L^p(-\infty, \infty)$ into $L^q(-\infty, \infty)$ $(q = p(p-1)^{-1})$. It follows easily that $H_{-\frac{1}{2}}$ (the Fourier cosine transform) is a continuous linear mapping of $L^p(0, \infty)$ into $L^q(0, \infty)$ for $1 < p \leqq 2$ (as is the Fourier sine transform). We could use this to prove that $H_{-\frac{1}{2}}$ is a continuous linear mapping of F_p into F_q under the same conditions. On the other hand, if we use [6, Theorem 80], we can show that $H_{-\frac{1}{2}}$ is a continuous linear mapping of F_p into $F_{p,2/p-1}$ and this result is stronger than the other since for $1 < p \leqq 2$,

$$F_{p,2/p-1} \subseteq F_q$$

by Theorem 2.5, with strict inclusion for $1 < p < 2$. Thus $H_{-\frac{1}{2}}$ cannot map F_p onto F_q for $1 < p \leqq 2$. In fact, we shall prove in a subsequent paper that $H_{-\frac{1}{2}}$ maps F_p onto $F_{p,2/p-1}$ not merely for $1 < p \leqq 2$ but for $1 < p < \infty$. This result is, in turn, a special case of a much more general result to the effect that H_ν, the Hankel transform of order ν, maps $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ under very general conditions on p, μ and ν.

We should mention that results concerning the properties of fractional integrals and Hankel transforms in the spaces L^p_μ $(\mu \neq 0)$ have been obtained by several authors, e.g. Flett [9] and Rooney [10]. Comments similar to the above apply to these; suffice to say that they are in accord with our results under the appropriate restrictions on the parameters.

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(*Issued* 15 *June* 1977)

Proceedings of the Royal Society of Edinburgh, 77A, 335–349, 1977

A theory of fractional integration for generalised functions II*

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(MS received 14 April 1976. Read 25 October 1976)

SYNOPSIS

In a previous paper [2], a theory of fractional integration was developed for certain spaces $F_{p,\mu}$ of generalised functions. In this paper we extend this theory by relaxing some of the restrictions on the various parameters involved. In particular we show how a generalised Erdélyi-Kober operator can be defined on $F'_{p,\mu}$ for $1 \leqq p \leqq \infty$ and for all complex numbers μ except for those lying on a countable number of lines of the form $\operatorname{Re}\mu$ = constant in the complex μ-plane. Mapping properties of these generalised operators are obtained and several applications mentioned.

1. INTRODUCTION

In a previous paper [2], we introduced the spaces $F_{p,\mu}$ of testing-functions and the corresponding spaces $F'_{p,\mu}$ of generalised functions and showed that they were well suited to the study of operators of fractional integration. For instance, the operator $I_{x^m}^{\eta,\alpha}$ defined for $\operatorname{Re}\alpha>0$ by

$$I_{x^m}^{\eta,\alpha}\phi(x) = [\Gamma(\alpha)]^{-1}mx^{-m\eta-m\alpha}\int_0^x (x^m-u^m)^{\alpha-1}u^{m\eta+m-1}\phi(u)du \tag{1.1}$$

is a continuous linear mapping of $F_{p,\mu}$ into itself if $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$ [2, Theorem 3.2]. We extended the definition of $I_{x^m}^{\eta,\alpha}$ to values of α with $\operatorname{Re}\alpha \leqq 0$ using analytic continuation and found that under certain conditions $I_{x^m}^{\eta,\alpha}$ was an automorphism of $F_{p,\mu}$ and also of $F'_{p,\mu}$. It is the purpose of this paper to show that these results hold under still more general conditions. For instance, as regards $I_{x^m}^{\eta,\alpha}$, we show how the condition $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$ mentioned above can be replaced by $\operatorname{Re}(m\eta+\mu)+m \neq p^{-1}-ml$ $(l = 0, 1, 2, \ldots)$ with similar generalisations for the other operators studied.

The functions in $F_{p,\mu}$ are complex-valued infinitely differentiable functions defined on $(0, \infty)$. As in [2], for $1 \leqq p<\infty$, $F_p(= F_{p,0})$ is defined by

$$F_p = \{\phi \in C^\infty(0, \infty): x^kD^k\phi \in L^p(0, \infty) \text{ for } k = 0, 1, 2, \ldots\}, \tag{1.2}$$

where $D \equiv d/dx$. For the case $p = \infty$, it is convenient to modify the definition in [2]. Thus, in this paper,

$$F_\infty = \{\phi \in C^\infty(0, \infty): \text{for each non-negative integer } k,\ x^kD^k\phi\to 0 \text{ as } x\to 0+ \text{ and as } x\to\infty\}. \tag{1.3}$$

This is the space denoted by G_∞ in [4] and is a subspace of F_∞ as defined in [2].

* This paper was assisted in publication by a grant from the Carnegie Trust for the Universities of Scotland.

However, we shall not use the latter in this paper and for typographical convenience we shall write F_∞ rather than G_∞ for the space defined by (1.3). This should not cause any confusion. For $1 \leqq p \leqq \infty$, we equip F_p with the topology generated by the seminorms $\gamma_k^p (k = 0, 1, 2, \ldots)$ defined by

$$\gamma_k^p(\phi) = \| x^k D^k \phi \|_p \quad (\phi \in F_p), \tag{1.4}$$

where $\| \ \|_p$ denotes the $L^p(0, \infty)$ norm.

For any complex number μ, and $1 \leqq p \leqq \infty$,

$$F_{p,\mu} = \{\phi: x^{-\mu}\phi(x) \in F_p\}. \tag{1.5}$$

$F_{p,\mu}$ is equipped with the topology generated by the seminorms $\gamma_k^{p,\mu}(k = 0, 1, 2, \ldots)$ given by

$$\gamma_k^{p,\mu}(\phi) = \gamma_k^p(x^{-\mu}\phi) \quad (\phi \in F_{p,\mu}). \tag{1.6}$$

Finally, $F'_{p,\mu}$ is the space of continuous linear functionals on $F_{p,\mu}$ equipped with the topology of pointwise convergence.

The basic properties of the spaces $F_{p,\mu}$ and $F'_{p,\mu}$ were developed in [2, 4]. In particular we recall that

$$C_0^\infty(0, \infty) \text{ is dense in } F_{p,\mu} \text{ for } 1 \leqq p \leqq \infty \text{ and any } \mu, \tag{1.7}$$

where

$$C_0^\infty(0, \infty) = \{\phi \in C^\infty(0, \infty): \phi \text{ has compact support in } (0, \infty)\}. \tag{1.8}$$

(1.7) is crucial to our development and justifies the modified definition of $F_{p,\mu}$ in the case $p = \infty$.

The extension which we are going to discuss makes extensive use of the invertibility of the differentiation operator $D \equiv d/dx$. In section 2, we investigate the behaviour of D on $F_{p,\mu}$. Indeed, in [2] we saw that D is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ if $1 \leqq p \leqq \infty$, $\operatorname{Re}\mu \neq p^{-1}$ (the case $p = \infty$ needing minor modification in view of our modified definition). We show that in the case $\operatorname{Re}\mu = p^{-1}$, D does not map $F_{p,\mu}$ onto $F_{p,\mu-1}$ and hence is not a homeomorphism.

In section 3, the extension process is carried out in detail for the operator $I_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$. The theory for $K_{x^m}^{\eta,\alpha}$ is outlined and results for $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$ follow easily. The corresponding theory for $F'_{p,\mu}$ is then obtained using adjoint considerations. We find that results in [2, 4] continue to hold under our more general conditions.

We should mention that the definition of $I_x^{\eta,\alpha}$, as given by (1.1) for $m = 1$, has been extended, relative to the spaces $L^p(0, \infty)$, in [7]. We give an approach that develops naturally from earlier work and find that the results of Erdélyi emerge as particular cases of our results. Similar comments apply to other operators.

Finally, in section 4, we give three applications of our results. Firstly we give conditions under which the equation

$$I_{x^m}^{\eta,\alpha} f = g \tag{1.9}$$

with $I_{x^m}^{\eta,\alpha}$ as in (1.1) has a classical solution $f \in L_\mu^p$ when $g \in L_\mu^p$ is given. Here

$$L_\mu^p = \{f: x^{-\mu}f(x) \in L^p(0, \infty)\}. \tag{1.10}$$

Secondly we indicate how results on the hypergeometric operators $H_i(a, b; c; m)$ $(i = 1, 2, 3, 4)$ introduced in [3] remain true under more general conditions. Lastly we mention the extended validity of results in [2] connected with the differential operator L_ν.

Throughout the paper we adhere to the conventions used in [2] except that we use 'homeomorphism' rather than 'isomorphism' in accordance with common usage. We mention in particular that $1 \leqq p \leqq \infty$ unless the contrary is specifically stated. Also p, q are always related by $p^{-1}+q^{-1} = 1$.

2. In this section we consider the invertibility of the differentiation operator $D \equiv d/dx$ on $F_{p,\mu}$. From [2, Theorem 2.6] we see that D is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-1}$ for $1 \leqq p \leqq \infty$ and for any complex number μ. Further, from [2, Theorems 3.6 and 3.8], D is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ if $\operatorname{Re}\mu \neq p^{-1}$, $1 \leqq p<\infty$ and

$$(D^{-1}\phi)(x) = \begin{cases} \displaystyle\int_0^x \phi(t)dt & \operatorname{Re}\mu>p^{-1} \\ \displaystyle -\int_x^\infty \phi(t)dt & \operatorname{Re}\mu<p^{-1} \end{cases}$$

for $\phi \in F_{p,\mu-1}$. The corresponding result for $\operatorname{Re}\mu \neq 0, p = \infty$ is also true and follows easily by a simple modification of the proof in [2] using [4, Theorem 3.3 (i)] rather than [2, Theorem 3.5].

It remains to consider the exceptional case $\operatorname{Re}\mu = p^{-1}$ and here the situation is different.

THEOREM 2.1. *If* $1 \leqq p \leqq \infty$ *and* $\operatorname{Re}\mu = p^{-1}$, *then* D *is not a homeomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu-1}$.

Proof. First we consider the case $1 \leqq p<\infty$, $\operatorname{Re}\mu = p^{-1}$. Let

$$\phi(x) = \lambda(x)(\log x)^a \quad (0<x<\infty),$$

where $-p^{-1}<a<0$ and $\lambda \in C^\infty(0, \infty)$ is such that

$$\lambda(x) = \begin{cases} 0 & 0<x\leqq 2 \\ 1 & x \geqq e. \end{cases}$$

As in [2], we shall write

$$\delta \equiv xD \equiv xd/dx. \tag{2.1}$$

We next define ψ on $(0, \infty)$ by $\psi(x) = (\delta\phi)(x)$. Then for $k = 0, 1, 2, \ldots$,

$$(\delta^k\psi)(x) = (\delta^{k+1}\phi)(x)$$

so that

$$(\delta^k\psi)(x) = \begin{cases} 0 & 0<x\leqq 2 \\ a(a-1)\ldots(a-k)(\log x)^{a-k-1} & x \geqq e \end{cases} \tag{2.2}$$

for $k = 0, 1, 2, \ldots$. Since, for $\operatorname{Re}\mu = p^{-1}$,

$$\int_e^\infty |\, x^{-\mu}(\log x)^{a-k-1}\,|^p\, dx = \int_1^\infty u^{(a-k-1)p} du<\infty$$

$(k = 0, 1, 2, \ldots)$, it follows easily that $\psi \in F_{p,\mu}$. Thus if we write

$$\chi(x) = x^{-1}\psi(x),$$

$\chi \in F_{p,\mu-1}$ and $\chi(x) = D\phi(x)$. However, $\phi \notin F_{p,\mu}$; for

$$\int_e^\infty |\, x^{-\mu}\phi(x)|^p\, dx = \int_e^\infty x^{-1}(\log x)^{ap} dx = \int_1^\infty u^{ap} du$$

diverges since $ap > -1$. Further, suppose there were some other function $\phi_1 \in F_{p,\mu}$ such that $D\phi_1 = \chi$. Then

$$\phi_1(x) = \phi(x) + c,$$

where c is a constant. Now the analysis in [2, Lemma 2.2] shows that $\phi_1(x) \to 0$ as $x \to \infty$; also, since $a < 0$, $\phi(x) \to 0$ as $x \to \infty$. Hence $c = 0$, $\phi_1 = \phi$ and we have a contradiction since $\phi_1 \in F_{p,\mu}$, $\phi \notin F_{p,\mu}$. Thus χ does not belong to the range of D on $F_{p,\mu}$ and the result is proved in this case.

Now consider the case $p = \infty$, $\mathrm{Re}\,\mu = 0$ and let

$$\phi(x) = \lambda(x)(\log x)^a$$

with λ as above and $0 < a < 1$. Certainly $\phi \notin F_{\infty,\mu}$. But if $\psi = \delta\phi$ it follows easily from (2.2) that $\psi \in F_{\infty,\mu}$ so that χ defined by $\chi(x) = x^{-1}\psi(x)$ belongs to $F_{\infty,\mu-1}$ and $\chi = D\phi$. Proceeding as above, we can show that there is no function $\phi_1 \in F_{\infty,\mu}$ such that $\chi = D\phi_1$. Hence D does not map $F_{\infty,\mu}$ onto $F_{\infty,\mu-1}$ and so is not a homeomorphism.

This completes the proof of Theorem 2.1.

One important consequence of Theorem 2.1 is that it is impossible to extend the operator $I_{x^m}^{\eta,\alpha}$ to the whole of certain spaces $F_{p,\mu}$ as we shall see in the next section. We also mention the following corollary.

COROLLARY 2.2. *δ (defined by (2.1)) is an automorphism of $F_{p,\mu}$ if and only if* $\mathrm{Re}\,\mu \neq p^{-1}$.

Proof. This follows immediately from Theorem 2.1 as well as [2, Theorem 2.6 (i)].

3. We proved in [2, Theorem 3.5] that $I_{x^m}^{\eta,\alpha}$, defined by (1.1) for $\mathrm{Re}\,\alpha > 0$ and extended as in [2] to $\mathrm{Re}\,\alpha \leq 0$, was a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$ provided that $\mathrm{Re}\,(m\eta+\mu)+m > p^{-1}$. Further, for fixed η satisfying this inequality, $I_{x^m}^{\eta,\alpha}$ was entire with respect to α on $F_{p,\mu}$; in particular for each fixed ϕ in $F_{p,\mu}$, and each fixed $x \in (0, \infty)$, $(I_{x^m}^{\eta,\alpha}\phi)(x)$ was an entire function of α. We now consider how to relax the condition $\mathrm{Re}\,(m\eta+\mu)+m > p^{-1}$.

First suppose that $\mathrm{Re}\,\mu > p^{-1}$ and $\mathrm{Re}\,\alpha > 0$. Then, if $\phi \in F_{p,\mu}$,

$$I_{x^m}^{-1,\alpha+1}\delta\phi = mI_{x^m}^{0,\alpha}\phi, \tag{3.1}$$

as is easily seen on integrating the left-hand side by parts. The restriction $\mathrm{Re}\,\alpha > 0$ can be dropped using analytic continuation so that (3.1) holds for $\phi \in F_{p,\mu}$ provided only that $\mathrm{Re}\,\mu > p^{-1}$. In this situation, δ is invertible by Corollary 2.2 so that we can rewrite (3.1) in the form

$$I_{x^m}^{-1,\alpha}\phi = mI_{x^m}^{0,\alpha-1}\delta^{-1}\phi \quad (\phi \in F_{p,\mu}) \tag{3.2}$$

when $\mathrm{Re}\,\mu > p^{-1}$. However, the right-hand side of (3.2) is meaningful provided $\mathrm{Re}\,\mu + m > p^{-1}$, $\mathrm{Re}\,\mu \neq p^{-1}$ by [2, Theorem 3.5] and Corollary 2.2 above. We can therefore use (3.2) to extend the definition of $I_{x^m}^{-1,\alpha}$ to spaces $F_{p,\mu}$ with $\mathrm{Re}\,\mu > p^{-1} - m$, $\mathrm{Re}\,\mu \neq p^{-1}$. Now, for any η and α

$$I_{x^m}^{\eta,\alpha} = x^{-m\eta-m}I_{x^m}^{-1,\alpha}x^{m\eta+m}\phi \tag{3.3}$$

and it follows that we can use (3.2 and 3.3) to extend the definition of $I_{x^m}^{\eta,\alpha}\phi$ to spaces $F_{p,\mu}$ satisfying the conditions

$$\mathrm{Re}\,(m\eta+\mu)+m > p^{-1} - m, \quad \mathrm{Re}\,(m\eta+\mu)+m \neq p^{-1}.$$

The formula used for the extension can be written in the form

$$I_{x^m}^{\eta,\alpha}\phi(x) = I_{x^m}^{\eta+1,\alpha-1}x^{-m(\eta+1)}(D_m)^{-1}x^{m\eta}\phi, \tag{3.4}$$

where

$$D_m \equiv d/dx^m \equiv m^{-1}x^{-m}\delta.$$

Expressions for $(D_m)^{-1}$ for the cases

$$\operatorname{Re}(m\eta+\mu)+m>p^{-1} \quad \text{and} \quad p^{-1}-m<\operatorname{Re}(m\eta+\mu)+m<p^{-1}$$

can be obtained as in [2]. The right-hand side of (3.4) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$ under the above conditions.

We can now repeat the process and extend the definition of $I_{x^m}^{\eta,\alpha}$ to all spaces $F_{p,\mu}$ except those for which $\operatorname{Re}(m\eta+\mu)+m = \dfrac{1}{p}-ml$ for some non-negative integer l. For $1 \leqq p \leqq \infty$, $m>0$ and any complex number μ we shall define the set $A_{p,\mu,m}$ of complex numbers by

$$A_{p,\mu,m} = \{\eta\colon \operatorname{Re}(m\eta+\mu)+m \neq p^{-1}-ml, \quad l=0,1,2,\ldots\}. \tag{3.5}$$

Definition 3.1. For $\eta \in A_{p,\mu,m}$ and any complex number α, we define $I_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$ by

$$I_{x^m}^{\eta,\alpha}\phi(x) = I_{x^m}^{\eta+k,\alpha-k}x^{-m(\eta+k)}(D_m)^{-k}x^{m\eta}\phi(x), \tag{3.6}$$

where k is a non-negative integer such that $\operatorname{Re}(m\eta+\mu)+m>p^{-1}-mk$.

A few comments are in order concerning this definition.

Notes

1. The operator on the right-hand side of (3.6) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$ under the given conditions, with $I_{x^m}^{\eta+k,\alpha-k}$ being defined as in [2].
2. Our extended definition agrees with the previous definition in [2] in the case $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$; this follows by k applications of (3.4) which are valid under these circumstances.
3. We must check that the definition is independent of the choice of the non-negative integer k. Assume also that $\operatorname{Re}(m\eta+\mu)+m>p^{-1}-ml$, where l is another non-negative integer and that $l>k$ (without loss of generality). Then using results in [2], we have

$$\begin{aligned}
&I_{x^m}^{\eta+k,\alpha-k}x^{-m(\eta+k)}(D_m)^{-k}x^{m\eta}\phi \\
&= I_{x^m}^{\eta+l,d-l}I_{x^m}^{\eta+k,l-k}x^{-m(\eta+k)}(D_m)^{-k}x^{m\eta}\phi \\
&= I_{x^m}^{\eta+l,\alpha-l}x^{-m(\eta+l)}I_{x^m}^{l-k}(D_m)^{l-k}(D_m)^{-l}x^{m\eta}\phi \\
&= I_{x^m}^{\eta+l,\alpha-l}x^{-m(\eta+l)}(D_m)^{-l}x^{m\eta}\phi
\end{aligned}$$

for all $\phi \in F_{p,\mu}$ and we have established the independence of the definition of k.
4. Suppose $\operatorname{Re}(m\eta+\mu)+m<p^{-1}$ and that k is the smallest positive integer for which $\operatorname{Re}(m\eta+\mu)+m>p^{-1}-mk$; in other words,

$$p^{-1}-mk<\operatorname{Re}(m\eta+\mu)+m<p^{-1}-m(k-1).$$

Then, using results in [2], (3.6) takes on the simple form

$$I_{x^m}^{\eta,\alpha}\phi = (-1)^k I_{x^m}^{\eta+k,\alpha-k}K_{x^m}^{-\eta-k,k}\phi, \tag{3.7}$$

where both the operators on the right-hand side are defined as in [2]. It is sometimes convenient to use this particular expression for $I_{x^m}^{\eta,\alpha}$ as we shall see below.

We see that when $\mu = 0$ and $m = 1$, (3.7) and the conditions for its validity agree exactly with the corresponding statements in [7], although we have arrived at them by a different approach [see 7, p. 295, formula (5)]. Also the same exceptional strips $\operatorname{Re}\eta = -\dfrac{1}{q} - l(l = 0, 1, 2, \ldots)$ emerge clearly as the complement, in the complex plane, of $A_{p,0,1}$.

5. Using the analyticity of the Erdélyi-Kober operators discussed in [2], we can show that if p, μ, m, α are fixed, $I_{x^m}^{\eta,\alpha}$ is analytic on $F_{p,\mu}$ for $\eta \in A_{p,\mu,m}$. In particular, if ϕ is a fixed function in $F_{p,\mu}$ and $x \in (0, \infty)$ is fixed, then $I_{x^m}^{\eta,\alpha}\phi(x)$ is an analytic function of η in $A_{p,\mu,m}$.

In view of [2, Theorem 3.5], we might hope that, under appropriate conditions, $I_{x^m}^{\eta,\alpha}$ would be an automorphism of $F_{p,\mu}$ onto itself with $(I_{x^m}^{\eta,\alpha})^{-1} = I_{x^m}^{\eta+\alpha,-\alpha}$, and this is indeed the case. However, we cannot deduce the result by analytic continuation with respect to η since $A_{p,\mu,m}$ is not connected. Instead we proceed as follows.

Let $\phi \in C_0^\infty(0, \infty)$. Then by straightforward differentiation, for $\operatorname{Re}\alpha < 0$,

$$I_{x^m}^{-1+\alpha,-\alpha}\phi = x^m D_m I_{x^m}^{-1+\alpha,1-\alpha}\phi$$

and this result can be extended to all complex values of α by analytic continuation. Next, for any η,

$$\begin{aligned} I_{x^m}^{\eta+\alpha,-\alpha}\phi &= x^{-m(\eta+1)} I_{x^m}^{-1+\alpha,-\alpha} x^{m(\eta+1)}\phi \\ &= x^{-m\eta} D_m x^{m(\eta+1)} I_{x^m}^{\eta+\alpha,-\alpha+1}\phi. \end{aligned}$$

and repeating the process k times ($k = 0, 1, 2, \ldots$) gives, by induction,

$$I_{x^m}^{\eta+\alpha,-\alpha}\phi = x^{-m\eta}(D_m)^k x^{m(\eta+k)} I_{x^m}^{\eta+\alpha,-\alpha+k}\phi \tag{3.8}$$

for $\phi \in C_0^\infty(0, \infty)$. Further, this result remains true if we interpret $I_{x^m}^{\eta+\alpha,-\alpha}\phi$ and $I_{x^m}^{\eta+\alpha,-\alpha+k}\phi$ in the sense of Definition 3.1 using (3.7).

Now let $\phi \in F_{p,\mu}$ and assume that $\eta + \alpha \in A_{p,\mu,m}$. Then since both sides of (3.8) are continuous linear mappings of $F_{p,\mu}$ into itself and since $C_0^\infty(0, \infty)$ is dense in $F_{p,\mu}$, (3.8) will hold for all $\phi \in F_{p,\mu}$ under the stated conditions. On the other hand from (3.6), we find that if $\eta \in A_{p,\mu,m}$ and $\eta + \alpha \in A_{p,\mu,m}$, then

$$(I_{x^m}^{\eta,\alpha})^{-1}\phi = x^{-m\eta}(D_m)^k x^{m(\eta+k)} I_{x^m}^{\eta+\alpha,-\alpha+k}\phi \tag{3.9}$$

for $\phi \in F_{p,\mu}$ and with k as in Definition 3.1; in inverting $I_{x^m}^{\eta+k,\alpha-k}$ on $F_{p,\mu}$ we may use [2, Theorem 3.5 (iii)] under the given conditions. Comparison of (3.8 and 3.9) gives our required result which we state in the following theorem.

THEOREM 3.2. *Let* $1 \leqq p \leqq \infty$, $m > 0$ *and let* η, α *be any complex numbers.*

(i) *If* $\eta \in A_{p,\mu,m}$, *then* $I_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into itself and for fixed* α *and* m *is analytic on* $F_{p,\mu}$ *for* $\eta \in A_{p,\mu,m}$.

(ii) *If also* $\eta + \alpha \in A_{p,\mu,m}$, *then* $I_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}$ *and* $(I_{x^m}^{\eta,\alpha})^{-1} = I_{x^m}^{\eta+\alpha,-\alpha}$.

Proof. The results follow from Note 1 following Definition 3.1 and formulae (3.8 and 3.9).

Of course if $\eta \notin A_{p,\mu,m}$, the process used above breaks down because D_m is not invertible at some stage and it is therefore not possible to extend $I_{x^m}^{\eta,\alpha}$ to the whole of

$F_{p,\mu}$ in this case (although $I_{x^m}^{\eta,\alpha}$ would still be meaningful for some elements of $F_{p,\mu}$ such as those in $C_0^\infty(0, \infty)$).

An entirely analogous process can be applied to extend the definition of $K_{x^m}^{\eta,\alpha}$ to a wider class of spaces $F_{p,\mu}$ than in [2]. We merely outline the salient points.

Starting from the formula

$$\delta K_{x^m}^{0,\alpha}\phi = -mK_{x^m}^{1,\alpha-1}\phi$$

analogous to (3.1) and valid for $\phi \in F_{p,\mu}$ with $\operatorname{Re}(-\mu) > -p^{-1}$ we obtain

$$K_{x^m}^{0,\alpha}\phi = -m\delta^{-1}K_{x^m}^{1,\alpha-1}\phi$$

and hence, if $\phi \in F_{p,\mu}$ with $\operatorname{Re}(m\eta-\mu) > -p^{-1}$,

$$K_{x^m}^{\eta,\alpha}\phi = -x^{m\eta}(D_m)^{-1}x^{-m(\eta+1)}K_{x^m}^{\eta+1,\alpha-1}\phi,$$

which is essentially an adjoint version of (3.4). The right-hand side is meaningful provided $\operatorname{Re}(m\eta-\mu) > -p^{-1}-m$, $\operatorname{Re}(m\eta-\mu) \neq -p^{-1}$ and by repeating the process we can define $K_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$ if $\eta \in A'_{p,\mu,m}$ where

$$A'_{p,\mu,m} = \{\eta\colon \operatorname{Re}(m\eta-\mu) \neq -p^{-1}-ml,\ l = 0, 1, 2, \ldots\}. \tag{3.10}$$

We notice in passing that

$$\eta \in A_{p,\mu,m} \text{ if and only if } \eta+1-m^{-1} \in A'_{q,-\mu,m}, \tag{3.11}$$

which accords with $K_{x^m}^{\eta+1-1/m,\alpha}$ being the formal adjoint of $I_{x^m}^{\eta,\alpha}$ in the sense of [2, formula 3.23].

The extension of $K_{x^m}^{\eta,\alpha}$ is afforded by the following definition.

Definition 3.3. For $\eta \in A'_{p,\mu,m}$ and any complex number α, we define $K_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$ by

$$K_{x^m}^{\eta,\alpha}\phi = (-1)^k x^{m\eta}(D_m)^{-k}x^{-m(\eta+k)}K_{x^m}^{\eta+k,\alpha-k}\phi, \tag{3.12}$$

where k is a non-negative integer such that $\operatorname{Re}(m\eta-\mu) > -p^{-1}-mk$.

Comments analogous to those in Notes 1–5 following Definition 3.1 apply here. For instance, if $\operatorname{Re}(m\eta-\mu) < -p^{-1}$ and k is the smallest positive integer for which we can apply Definition 3.3, so that

$$-p^{-1}-mk < \operatorname{Re}(m\eta-\mu) < -p^{-1}-m(k-1),$$

then (3.12) takes on the simple form

$$K_{x^m}^{\eta,\alpha}\phi = (-1)^k I_{x^m}^{-\eta-k,k}K_{x^m}^{\eta+k,\alpha-k}\phi, \tag{3.13}$$

which is essentially an adjoint version of (3.7). As regards invertibility, we need

$$K_{x^m}^{\eta+\alpha,-\alpha}\phi = (-1)^k K_{x^m}^{\eta+\alpha,-\alpha+k}x^{m(\eta+k)}(D_m)^k x^{-m\eta}\phi \quad (\phi \in F_{p,\mu}),$$

which is the adjoint version of (3.8), valid for $\eta+\alpha \in A'_{p,\mu,m}$. The following theorem then follows easily.

THEOREM 3.4. *Let* $1 \leqq p \leqq \infty$, $m > 0$ *and let* μ, α *be complex numbers.*

(i) *If* $\eta \in A'_{p,\mu,m}$, *then* $K_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu}$ *and for fixed* α, m *is analytic on* $F_{p,\mu}$ *for* $\eta \in A'_{p,\mu,m}$.

(ii) *If also* $\eta+\alpha \in A'_{p,\mu,m}$, $K_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}$ *and*

$$(K_{x^m}^{\eta,\alpha})^{-1} = K_{x^m}^{\eta+\alpha,-\alpha}.$$

Results proved in [2] remain true for our extended operators. In each case the technique is to use (3.6) together with the corresponding results in [2].

For instance, from [2, formula 3.6] we have, for $\phi \in F_{p,\mu}$,

$$mI_{x^m}^{\eta,\alpha}\phi = (m\eta + m\alpha + m)I_{x^m}^{\eta,\alpha+1}\phi + I_{x^m}^{\eta,\alpha+1}\delta\phi \tag{3.14}$$

provided $\eta \in A_{p,\mu,m}$ while if $\eta \in A'_{p,\mu,m}$, [2, (3.15)] gives

$$mK_{x^m}^{\eta,\alpha}\phi = (m\eta + m\alpha)K_{x^m}^{\eta,\alpha+1}\phi - K_{x^m}^{\eta,\alpha+1}\delta\phi. \tag{3.15}$$

One consequence of (3.14 and 3.15) is that

$$\begin{aligned} \delta I_x^{-1,1}\phi &= I_x^{-1,0}\phi = \phi \\ -\delta K_x^{0,1}\phi &= K_x^{0,0}\phi = \phi \end{aligned} \tag{3.16}$$

valid provided $-1 \in A_{p,\mu,1}$ and $0 \in A'_{p,\mu,1}$ respectively.

It remains to consider the operators $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$ on $F_{p,\mu}$. As usual for any α we define the operators by

$$I_{x^m}^{\alpha}\phi = x^{m\alpha}I_{x^m}^{0,\alpha}\phi$$

$$K_{x^m}^{\alpha}\phi = K_{x^m}^{0,\alpha}x^{m\alpha}\phi.$$

The following results are almost immediate from Theorems 3.2 and 3.4.

THEOREM 3.5. (i) *If* $0 \in A_{p,\mu,m}$, *then* $I_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$.

(ii) *If also* $0 \in A_{p,\mu+m\alpha,m}$, *then* $I_{x^m}^{\alpha}$ *is a homeomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and* $(I_{x^m}^{\alpha})^{-1} = I_{x^m}^{-\alpha}$.

Proof. The only statement which needs comment is that $(I_{x^m}^{\alpha})^{-1} = I_{x^m}^{-\alpha}$.

This follows from Theorem 3.2 (ii) and the fact that for $\psi \in F_{p,\mu+m\alpha}$

$$(x^{m\alpha}I_{x^m}^{0,\alpha})^{-1}\psi = I_{x^m}^{\alpha,-\alpha}x^{-m\alpha}\psi = x^{-m\alpha}I_{x^m}^{0,-\alpha}\psi.$$

THEOREM 3.6. (i) *If* $0 \in A'_{p,\mu+m\alpha,m}$, *then* $K_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$.

(ii) *If also* $0 \in A'_{p,\mu,m}$, *then* $K_{x^m}^{\alpha}$ *is a homeomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and* $(K_{x^m}^{\alpha})^{-1} = K_{x^m}^{-\alpha}$.

The proof is omitted.

As regards the wider validity of the index laws we have the following results.

THEOREM 3.7. (i) *If* $\{0, \alpha, \beta\} \subseteq A_{p,\mu,m}$, *then, for* $\phi \in F_{p,\mu}$,

$$I_{x^m}^{\alpha}I_{x^m}^{\beta}\phi = I_{x^m}^{\alpha+\beta}\phi = I_{x^m}^{\beta}I_{x^m}^{\alpha}\phi.$$

(ii) *If* $\{0, \gamma\} \subseteq A_{p,\mu,m}$, *and* $\alpha+\beta+\gamma = 0$, *then, for* $\phi \in F_{p,\mu}$,

$$x^{m\alpha}I_{x^m}^{\beta}x^{m\gamma}\phi = I_{x^m}^{-\gamma}x^{-m\beta}I_{x^m}^{-\alpha}\phi.$$

Proof. These results can be established by applying to [2, formulae (3.11 and 3.13)] the technique described after Theorem 3.4.

THEOREM 3.8. (i) *If* $\{-\alpha, -\beta, -\alpha-\beta\} \subseteq A'_{p,\mu,m}$, *then, for* $\phi \in F_{p,\mu}$,

$$K^{\alpha}_{x^m} K^{\beta}_{x^m} \phi = K^{\alpha+\beta}_{x^m} \phi = K^{\beta}_{x^m} K^{\alpha}_{x^m} \phi.$$

(ii) *If* $\{0, \gamma\} \subseteq A'_{p,\mu,m}$, *then, for* $\phi \in F_{p,\mu}$,

$$x^{m\gamma} K^{\beta}_{x^m} x^{m\alpha} \phi = K^{-\alpha}_{x^m} x^{-m\beta} K^{-\gamma}_{x^m} \phi.$$

The proof, which uses [2, formulae (3.21 and 3.22)], is omitted. Also relevant here is [3, Theorems 3.7 and 4.9].

To define our operators on $F'_{p,\mu}$ we proceed via adjoint operators as in [2]. Thus for $f \in F'_{p,\mu}$, and appropriate values of η, we define $I^{\eta,\alpha}_{x^m} f$ by

$$(I^{\eta,\alpha}_{x^m} f, \phi) = (f, K^{\eta+1-1/m,\alpha}_{x^m} \phi) \quad (\phi \in F_{p,\mu}) \tag{3.17}$$

[see 3, (3.23)]. The operator $K^{\eta+1-1/m,\alpha}_{x^m}$ on the right is to be interpreted in the sense of Definition 3.3. Thus by Theorem 3.4, the right-hand side of (3.17) is meaningful provided $\eta+1-m^{-1} \in A'_{p,\mu,m}$ which is equivalent to $\eta \in A_{q,-\mu,m}$ by (3.11).

THEOREM 3.9. (i) *If* $\eta \in A_{q,-\mu,m}$, *then* $I^{\eta,\alpha}_{x^m}$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu}$ *and for each* $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu}$, $(I^{\eta,\alpha}_{x^m} f, \phi)$ *is an analytic function of* η *in* $A_{q,-\mu,m}$.

(ii) *If also* $\eta+\alpha \in A_{q,-\mu,m}$, *then* $I^{\eta,\alpha}_{x^m}$ *is a automorphism of* $F'_{p,\mu}$ *and*

$$(I^{\eta,\alpha}_{x^m})^{-1} = I^{\eta+\alpha,-\alpha}_{x^m}.$$

Proof. The results follow immediately from Theorem 3.4 together with [6, Theorems 1.10–1 and 1.10–2].

We notice again that the condition on η is obtained from that in Theorem 3.2 by interchanging p and q, μ and $-\mu$.

Next, for $f \in F'_{p,\mu}$, we define $K^{\eta,\alpha}_{x^m} f$ by

$$(K^{\eta,\alpha}_{x^m} f, \phi) = (f, I^{\eta-1+1/m,\alpha}_{x^m} \phi) \quad (\phi \in F_{p,\mu}) \tag{3.18}$$

[see 2 (3.26)]. The right-hand side is meaningful provided $\eta \in A'_{q,-\mu,m}$ by (3.11) and we have the following results.

THEOREM 3.10. (i) *If* $\eta \in A'_{q,-\mu,m}$, *then* $K^{\eta,\alpha}_{x^m}$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu}$ *and for each* $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu}$, $(K^{\eta,\alpha}_{x^m} f, \phi)$ *is an analytic function of* η *in* $A'_{q,-\mu,m}$.

(ii) *If also* $\eta+\alpha \in A'_{q,-\mu,m}$, *then* $K^{\eta,\alpha}_{x^m}$ *is an automorphism of* $F'_{p,\mu}$ *and*

$$(K^{\eta,\alpha}_{x^m})^{-1} = K^{\eta+\alpha,-\alpha}_{x^m}.$$

Proof. Immediate from Theorem 3.2 above and [6, Theorems 1.10–1 and 1.10–2].

For $f \in F'_{p,\mu}$, we define $I^{\alpha}_{x^m} f$ and $K^{\alpha}_{x^m} f$ by

$$(I^{\alpha}_{x^m} f, \phi) = (f, x^{m-1} K^{\alpha}_{x^m} x^{-m+1} \phi) \quad (\phi \in F_{p,\mu-m\alpha}) \tag{3.19}$$

$$(K^{\alpha}_{x^m} f, \phi) = (f, x^{m-1} I^{\alpha}_{x^m} x^{-m+1} \phi) \quad (\phi \in F_{p,\mu-m\alpha}). \tag{3.20}$$

We gather together the properties of these operators in the following two theorems.

THEOREM 3.11. (i) *If* $0 \in A_{q,-\mu,m}$, *then* $I_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu-m\alpha}$.

(ii) *If also* $0 \in A_{q,\ -\mu+m\alpha,m}$, *then* $I_{x^m}^{\alpha}$ *is a homeomorphism of* $F'_{p,\mu}$ *onto* $F'_{p,\mu-m\alpha}$ *and*

$$(I_{x^m}^{\alpha})^{-1} = I_{x^m}^{-\alpha}.$$

(iii) *If* $\{0, \alpha, \beta\} \subseteq A_{q,-\mu,m}$, *then for* $f \in F'_{p,\mu}$,

$$I_{x^m}^{\alpha} I_{x^m}^{\beta} f = I_{x^m}^{\alpha+\beta} f = I_{x^m}^{\beta} I_{x^m}^{\alpha} f.$$

(iv) *If* $\{0, \gamma\} \subseteq A_{q,-\mu,m}$, *then for* $f \in F'_{p,\mu}$,

$$x^{m\alpha} I_{x^m}^{\beta} x^{m\gamma} f = I_{x^m}^{-\gamma} x^{-m\beta} I_{x^m}^{-\alpha} f.$$

Proof. (i), (ii) follow from Theorem 3.6 above and [6, Theorems 1.10–1 and 1.10–2]. (iii), (iv) are proved using (3.19) together with Theorem 3.8.

THEOREM 3.12. (i) *If* $0 \in A'_{q,-\mu+m\alpha,m}$, *then* $K_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu-m\alpha}$.

(ii) *If also* $0 \in A'_{q,-\mu,m}$, *then* $K_{x^m}^{\alpha}$ *is a homeomorphism of* $F'_{p,\mu}$ *onto* $F'_{p,\mu-m\alpha}$ *and*

$$(K_{x^m}^{\alpha})^{-1} = K_{x^m}^{-\alpha}.$$

(iii) *If* $\{-\alpha, -\beta, -\alpha-\beta\} \subseteq A'_{q,-\mu,m}$, *then for* $f \in F'_{p,\mu}$,

$$K_{x^m}^{\alpha} K_{x^m}^{\beta} f = K_{x^m}^{\alpha+\beta} f = K_{x^m}^{\beta} K_{x^m}^{\alpha} f.$$

(iv) *If* $\{0, \gamma\} \subseteq A'_{q,-\mu,m}$, *then for* $f \in F'_{p,\mu}$,

$$x^{m\gamma} K_{x^m}^{\beta} x^{m\alpha} f = K_{x^m}^{-\alpha} x^{-m\beta} K_{x^m}^{-\gamma} f.$$

Proof. (i), (ii) require Theorem 3.5 and [6, Theorems 1.10–1 and 1.10–2]. (iii), (iv) are proved using (3.20) together with Theorem 3.7.

We conclude this section with a few results concerning the composition of the operators $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$. For instance, we have the following theorem on commutativity.

THEOREM 3.13. *Let* $\phi \in F_{p,\mu}$, $f \in F'_{p,\mu}$, *let* α, β *be complex numbers and let* m, n *be positive real numbers. Then*

(i) $I_{x^m}^{\eta,\alpha} I_{x^n}^{\xi,\beta} \phi = I_{x^n}^{\xi,\beta} I_{x^m}^{\eta,\alpha} \phi$ *if* $\eta \in A_{p,\mu,m}$, $\xi \in A_{p,\mu,n}$.

(ii) $K_{x^m}^{\eta,\alpha} K_{x^n}^{\xi,\beta} \phi = K_{x^n}^{\xi,\beta} K_{x^m}^{\eta,\alpha} \phi$ *if* $\eta \in A'_{p,\mu,m}$, $\xi \in A'_{p,\mu,n}$.

(iii) $I_{x^m}^{\eta,\alpha} K_{x^n}^{\xi,\beta} \phi = K_{x^n}^{\xi,\beta} I_{x^m}^{\eta,\alpha} \phi$ *if* $\eta \in A_{p,\mu,m}$, $\xi \in A'_{p,\mu,n}$.

(iv) $I_{x^m}^{\eta,\alpha} I_{x^n}^{\xi,\beta} f = I_{x^n}^{\xi,\beta} I_{x^m}^{\eta,\alpha} f$ *if* $\eta \in A_{q,-\mu,m}$, $\xi \in A_{q,-\mu,n}$.

(v) $K_{x^m}^{\eta,\alpha} K_{x^n}^{\xi,\beta} f = K_{x^n}^{\xi,\beta} K_{x^m}^{\eta,\alpha} f$ *if* $\eta \in A'_{q,-\mu,m}$, $\xi \in A'_{q,-\mu,n}$.

(vi) $I_{x^m}^{\eta,\alpha} K_{x^n}^{\xi,\beta} f = K_{x^n}^{\xi,\beta} I_{x^m}^{\eta,\alpha} f$ *if* $\eta \in A_{q,-\mu,m}$, $\xi \in A'_{q,-\mu,n}$.

Proof. (i) First suppose $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$, $\operatorname{Re}(m\eta+\mu)+m > p^{-1}$ and $\operatorname{Re}(n\xi+\mu)+n > p^{-1}$. Then both sides are defined by absolutely convergent repeated integrals and the result follows in this case on using Fubini's theorem to invert the order of integration.

The restrictions $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$ can be removed by using (3.14) or analytic continuation. The general result now follows from the previous case together with (3.7); the details are similar to those in (iii) below:

(i) is proved similarly.

(ii) if $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$, $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$ and $\operatorname{Re}(n\xi-\mu)>-p^{-1}$ both sides are given by absolutely convergent repeated integrals and we can use Fubini's theorem. Again the restrictions $\operatorname{Re}\alpha>0$, $\operatorname{Re}\beta>0$ can be removed using analytic continuation. Suppose now that we choose a positive integer k such that

$$p^{-1}-mk<\operatorname{Re}(m\eta+\mu)+m<p^{-1}-m(k-1)$$

and assume still that $\operatorname{Re}(n\xi-\mu)>-p^{-1}$. Then

$$\begin{aligned} I_{x^m}^{\eta,\alpha}K_{x^n}^{\xi,\beta}\phi &= (-1)^k I_{x^m}^{\eta+k,\alpha-k}K_{x^m}^{-\eta-k,k}K_{x^n}^{\xi,\beta}\phi \quad \text{by (3.7)}\\ &= (-1)^k I_{x^m}^{\eta+k,\alpha-k}K_{x^n}^{\xi,\beta}K_{x^m}^{-\eta-k,k}\phi \quad \text{by (ii)}\\ &= K_{x^n}^{\xi,\beta}(-1)^k I_{x^m}^{\eta+k,\alpha-k}K_{x^m}^{-\eta-k,k}\phi \quad \text{by the previous case}\\ &= K_{x^n}^{\xi,\beta}I_{x^m}^{\eta,\alpha}\phi \quad \text{by (3.7) again.}\end{aligned}$$

We now have (iii) for $\eta\in A_{p,\mu,m}$, $\operatorname{Re}(n\xi-\mu)>-p^{-1}$. Finally, we can replace the restriction $\operatorname{Re}(n\xi-\mu)>-p^{-1}$ by $\xi\in A'_{p,\mu,n}$ using (3.13) and (i).

(iv)–(vi) follow immediately from (i)–(iii) on taking adjoints.

This completes the proof of Theorem 3.13.

An interesting situation arises when we consider the generalisations of [**2**, formulae (3.5, 3.17, 3.24, 3.28)].

For instance, consider the formula

$$I_{x^m}^{\eta+\alpha,\beta}I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,\alpha+\beta}\phi \quad (\phi\in F_{p,\mu}). \tag{3.21}$$

Our considerations above show that the left-hand side defines a continuous linear mapping of $F_{p,\mu}$ into itself provided $\eta\in A_{p,\mu,m}$ and $\eta+\alpha\in A_{p,\mu,m}$ while the right-hand side defines a continuous linear mapping of $F_{p,\mu}$ into itself provided only that $\eta\in A_{p,\mu,m}$. This seems to indicate that it should be possible to remove the restriction $\eta+\alpha\in A_{p,\mu,m}$ or, in other words, to remove the singularities corresponding to $\eta+\alpha\notin A_{p,\mu,m}$. To see how this can be done let us, for the moment, write

$$T_{x^m}^{\eta,\alpha,\beta}\phi = I_{x^m}^{\eta+\alpha,\beta}I_{x^m}^{\eta,\alpha}\phi \tag{3.22}$$

for $\phi\in F_{p,\mu}$, where $\eta\in A_{p,\mu,m}$, $\eta+\alpha\in A_{p,\mu,m}$. If we make use of (3.6) and an analogue of (3.8) we obtain

$$T_{x^m}^{\eta,\alpha,\beta}\phi = I_{x^m}^{\eta+\alpha+k,\beta-k}x^{-m(\eta+\alpha+k)}(D_m)^{-k}x^{m(\eta+\alpha)}x^{-m(\eta+\alpha)}(D_m)^k x^{m(\eta+\alpha+k)}I_{x^m}^{\eta,\alpha+k}\phi$$

or

$$T_{x^m}^{\eta,\alpha,\beta}\phi = T_{x^m}^{\eta,\alpha+k,\beta-k}\phi, \tag{3.23}$$

where k is a positive integer such that $\operatorname{Re}(m\eta+m\alpha+\mu)+m>p^{-1}-mk$. In this case $(\eta+\alpha+k)\in A_{p,\mu,m}$ so that the right-hand side defines a continuous linear mapping of $F_{p,\mu}$ into itself. Thus we can use (3.23) to extend the definition of $T_{x^m}^{\eta,\alpha,\beta}$ to values of η, α with $\alpha+\alpha\notin A_{p,\mu,m}$. As usual this extension is independent of the integer k satisfying $\eta+\alpha+k\in A_{p,\mu,m}$. The extended operator is analytic for $\eta\in A_{p,\mu,m}$ and we can use (3.23) together with [2, formula (3.5)] to show that $T_{x^m}^{\eta;\alpha,\beta}$ coincides with $I_{x^m}^{\eta;\alpha+\beta}$ on $F_{p,\mu}$ when $\eta \in A_{p,\mu,m}$ so that (3.21) holds provided only that $\eta \in A_{p,\mu,m}$.

A similar process can be carried through for $K_{x^m}^{\eta;\alpha}K_{x^m}^{\eta+\alpha,\beta}$ on $F_{p,\mu}$ and by taking adjoints we can obtain corresponding results on $F'_{p,\mu}$. We state the results without proof in the following theorem.

THEOREM 3.14. *Let $\phi \in F_{p,\mu}$, $f \in F'_{p,\mu}$. Then*

(i) $I_{x^m}^{\eta+\alpha,\beta}I_{x^m}^{\eta;\alpha}\phi = I_{x^m}^{\eta;\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu,m}$

(ii) $K_{x^m}^{\eta;\alpha}K_{x^m}^{\eta+\alpha,\beta}\phi = K_{x^m}^{\eta;\alpha+\beta}\phi$ *if* $\eta \in A'_{p,\mu,m}$

(iii) $I_{x^m}^{\eta+\alpha,\beta}I_{x^m}^{\eta;\alpha}f = I_{x^m}^{\eta;\alpha+\beta}f$ *if* $\eta \in A_{q,-\mu,m}$

(iv) $K_{x^m}^{\eta;\alpha}K_{x^m}^{\eta+\alpha,\beta}f = K_{x^m}^{\eta;\alpha+\beta}f$ *if* $\eta \in A'_{q,-\mu,m}$.

We remark that in spite of Theorem 3.14 (i), for instance, we cannot remove the condition $\eta+\alpha \in A_{p,\mu,m}$ if we want $I_{x^m}^{\eta;\alpha}$ to be an automorphism of $F_{p,\mu}$ (Theorem 3.2 (ii)). The situation is analogous to considering $z^{-1}.z = 1$ at $z = 0$. Theorem 3.14 gives us one instance of the removal of singularities. We shall encounter a number of other examples in work on the Hankel transform in a later paper.

4. As a first application of the results of section 3, let us study the equation

$$[\Gamma(\alpha)]^{-1}mx^{-m\eta-m\alpha}\int_0^x (x^m-u^m)^{\alpha-1}u^{m\eta+m-1}f(u)du = g(x) \tag{4.1}$$

relative to the spaces L_μ^p where, for $1 \leqq p \leqq \infty$ and any complex number μ,

$$L_\mu^p = \{f\colon x^{-\mu}f(x) \in L^p(0, \infty)\}. \tag{4.2}$$

In (4.1) we assume $\operatorname{Re}\alpha>0$ and that η is a suitably restricted complex number. Indeed from [1, Theorem 2], the left-hand side defines a continuous linear mapping of L_μ^p into L_μ^p provided $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$. Hence a necessary condition for (4.1) to have a solution f in L_μ^p is that the given function $g \in L_\mu^p$. However, this is not a sufficient condition, as is well known. Nevertheless, we shall show how, under very general conditions, (4.1) has, for a given $g \in L_\mu^p$, a unique 'generalised solution' in a sense to be made precise. Further, the generalised solution is generated by the unique L_μ^p solution whenever the latter exists. (In L_μ^p we identify functions which differ only on a set of measure zero and work with the resulting equivalence classes.)

Let $g \in L_\mu^p$. We wish to know whether there exists a function $f \in L_\mu^p$ satisfying

$$I_{x^m}^{\eta;\alpha}f = g,$$

where $\operatorname{Re}\alpha>0$ and $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$. Since $g \in L_\mu^p$, Hölder's inequality shows that g generates an element, τg say, of $F'_{q,-\mu}$ via the formula

$$(\tau g, \phi) = \int_0^\infty g(x)\phi(x)dx \quad (\phi \in F_{q,-\mu}). \tag{4.3}$$

We can regard $g \to \tau g$ as an embedding of L_μ^p into $F'_{q,-\mu}$. As in [3], we shall use τ for any such embedding and will not exhibit the dependence on p, μ explicitly.

Consider the equation

$$I_{x^m}^{\eta;\alpha}h = \tau g \tag{4.4}$$

in $F'_{q,-\mu}$, where $I_{x^m}^{\eta,\alpha}$ is interpreted in the sense of (3.17) above. Since $\operatorname{Re}(m\eta+\mu)+m>p^{-1}$, we know that $\eta\in A_{p,\mu,m}$. Hence provided only that $\eta+\alpha\in A_{p,\mu,m}$, (4.4) has a unique solution $h\in F'_{q,-\mu}$ by Theorem 3.9 and

$$h = I_{x^m}^{\eta+\alpha,\,-\alpha}\tau g. \tag{4.5}$$

It is this functional h which we might refer to as the *generalised solution* of (4.1). If (4.1) has a classical solution $f\in L_\mu^p$, then since $\operatorname{Re}\alpha>0$, $\tau I_{x^m}^{\eta,\alpha}f = I_{x^m}^{\eta,\alpha}\tau f$; indeed, this was the motivation for [2, formula (3.23)]. Hence applying τ to (4.1) gives $I_{x^m}^{\eta,\alpha}\tau f = \tau g$, so that

$$h = I_{x^m}^{\eta+\alpha,\,-\alpha}I_{x^m}^{\eta,\alpha}\tau f = \tau f$$

in this case, as mentioned above. On the other hand, it is not always possible to find an $f\in L_\mu^p$ which will generate h and then of course there is no solution in L_μ^p. To see more clearly the crux of the matter, we write

$$h = x^{-m\eta}(D_m)^k x^{m(\eta+k)}I_{x^m}^{\eta+\alpha,\,-\alpha+k}\tau g \tag{4.6}$$

using the analogue of (3.8) for generalised functions which is valid under the given conditions. If the integer k is such that $-\operatorname{Re}\alpha+k>0$, (4.6) gives

$$h = x^{-m\eta}(D_m)^k\tau x^{m(\eta+k)}I_{x^m}^{\eta+\alpha,\,-\alpha+k}g \tag{4.7}$$

since $\operatorname{Re}(m\eta+mk+\mu)+m>p^{-1}$. Thus a classical solution f exists if and only if $(D_m)^k x^{m(\eta+k)}I_{x^m}^{\eta+\alpha,\,-\alpha+k}g$ exists as a classical function. If this latter condition holds

$$f = x^{-m\eta}(D_m)^k x^{m(\eta+k)}I_{x^m}^{\eta+\alpha,\,-\alpha+k}g \tag{4.8}$$

a formula similar to several in [5, § 2.4]. If the above condition does not hold, the process stops at (4.7). We can sum up our conclusions in the following theorem.

THEOREM 4.1. *Let $g\in L_\mu^p$ where* $\operatorname{Re}(m\eta+\mu)+m>\dfrac{1}{p}$, *and let* $\operatorname{Re}\alpha>0$. *If $\eta+\alpha\in A_{p,\mu,m}$ then* (4.1) *has a unique generalised solution h given by* (4.5). *If k is a positive integer such that* $-\operatorname{Re}\alpha+k>0$ *and if $(D_m)^k x^{m(\eta+k)}I_{x^m}^{\eta+\alpha,\,-\alpha+k}g$ exists in the classical sense, then* (4.1) *has a unique solution $f\in L_\mu^p$ given by* (4.8).

Results for special cases of (4.1) including Abel-type integral equations can be obtained by appropriate choices of η, α and m.

As a second application of our results we consider again the hypergeometric operators $H_i(a, b; c; m)$ $(i = 1, 2, 3, 4)$ which were discussed in [3]. As in [3, Definition 3.4], we define $H_1(a, b; c; m)$ on $F_{p,\mu}$ by

$$H_1(a, b; c; m)\phi = I_{x^m}^{c-b}x^{-ma}I_{x^m}^{b}x^{ma}\phi(x) \tag{4.9}$$

In [3] we saw that $H_1(a, b; c; m)$ was a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$ provided $-\operatorname{Re}\mu-m+p^{-1}<\min(m\operatorname{Re}a,\ m\operatorname{Re}b)$. But if we now interpret the operators $I_{x^m}^{c-b}$ and $I_{x^m}^{b}$ on the right-hand side in the sense of section 3 we find that the right-hand side of (4.9) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$ provided only that $0\in A_{p,\mu+ma,m}$ and $0\in A_{p,\mu+mb,m}$ (from Theorem 3.5 (i)) or equivalently that $\{a, b\}\subseteq A_{p,\mu,m}$. In fact the following theorem is easily proved.

THEOREM 4.2. (i) *If* $\{a, b\} \subseteq A_{p,\mu,m}$, *then* $H_1(a, b; c; m)$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+mc}$.

(ii) *If also* $\{c, a+b\} \subseteq A_{p,\mu,m}$, *then* $H_1(a, b; c; m)$ *is a homeomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+mc}$ *and*

$$[H_1(a, b; c; m)]^{-1} = x^{-ma}H_1(-a, b-c; -c; m)x^{ma}.$$

Proof. The result follows on applying to [3, Theorem 3.5 and Corollary 3.6] the technique stated after Theorem 3.4 above.

Similar considerations apply to the other three operators on $F_{p,\mu}$. We summarise the results in the form of a table. In the column headed 'c.l.m.' we give the conditions under which the various operators are continuous linear mappings from $F_{p,\mu}$ into $F_{p,\mu+mc}$ while the column headed 'homeo' gives the conditions under which they are homeomorphisms.

TABLE 4.1

$H_i(a, b; c; m)$: $F_{p,\mu} \to F_{p,\mu+mc}$

	c.l.m.	homeo
$H_1(a, b; c; m)$	$\{a, b\} \subseteq A_{p,\mu,m}$	$\{a, b, c, a+b\} \subseteq A_{p,\mu,m}$
$H_2(a, b; c; m)$	$\{0, c-a-b\} \subseteq A_{p,\mu,m}$	$\{0, c-a, c-b, c-a-b\} \subseteq A_{p,\mu,m}$
$H_3(a, b; c; m)$	$\{-c, -a-b\} \subseteq A_{q,-\mu,m}$	$\{-a, -b, -c, -a-b\} \subseteq A_{q,-\mu,m}$
$H_4(a, b; c; m)$	$\{a-c, b-c\} \subseteq A_{q,-\mu,m}$	$\{0, a-c, b-c, a+b-c\} \subseteq A_{q,-\mu,m}$

All the results in [3] such as

$$[H_1(a, b; c; m)]^{-1} = H_2(-a, -b; -c; m)$$

remain true under the appropriate more general conditions.

The operators are defined on $F'_{p,\mu}$ using adjoints. Thus for $f \in F'_{p,\mu}$, $H_i(a, b; c; m)f$ is defined by

$$(H_i(a, b; c; m)f, \phi) = (f, H_{5-i}(a, b; c; m)\phi) \quad (\phi \in F_{p,\mu-mc}) \quad \text{for} \quad i = 1, 2, 3, 4.$$

Using [6, Theorems 1.10–1 and 1.10–2] and Table 4.1 we easily obtain the following table for the operators on $F'_{p,\mu}$.

TABLE 4.2

$H_i(a, b; c; m)$: $F'_{p,\mu} \to F'_{p,\mu-mc}$

	c.l.m.	homeo
$H_1(a, b; c; m)$	$\{a, b\} \subseteq A_{q,-\mu,m}$	$\{a, b, c, a+b\} \subseteq A_{q,-\mu,m}$
$H_2(a, b; c; m)$	$\{0, c-a-b\} \subseteq A_{q,-\mu,m}$	$\{0, c-a, c-b, c-a-b\} \subseteq A_{q,-\mu,m}$
$H_3(a, b; c; m)$	$\{-c, -a-b\} \subseteq A_{p,\mu,m}$	$\{-a, -b, -c, -a-b\} \subseteq A_{p,\mu,m}$
$H_4(a, b; c; m)$	$\{a-c, b-c\} \subseteq A_{p,\mu,m}$	$\{0, a-c, b-c, a+b-c\} \subseteq A_{p,\mu,m}$

As a third application we consider again the operator L_ν defined for each complex number ν by

$$L_\nu \equiv \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}.$$

This operator can be applied meaningfully to elements in $F'_{p,\mu}$ and we can prove the following generalisation of [2, Theorem 4.3].

THEOREM 4.3. (i) *If* $\nu \in A_{q, -\mu-2, 2}$, *then for* $f \in F'_{p,\mu}$,

$$I_{x^2}^{\nu, \alpha} L_\nu f = L_{\nu+\alpha} I_{x^2}^{\nu, \alpha} f.$$

(ii) *If* $\nu \in A'_{q, -\mu, 2}$, *then for* $f \in F'_{p,\mu}$,

$$L_{-\nu} K_{x^2}^{\nu, \alpha} f = K_{x^2}^{\nu, \alpha} L_{-\nu-\alpha} f.$$

Proof. The results are obtained by applying to [2, Theorem 4.2] the technique described after Theorem 3.4 and then taking adjoints.

We note also that

$$A_{q, -\mu-2, 2} \subseteq A_{q, -\mu, 2} \quad \text{and} \quad A'_{q, -\mu, 2} \subseteq A'_{q, -\mu-2, 2}.$$

A fourth important application of our results arises in connection with the theory of the Hankel transform for elements of $F'_{p,\mu}$. This turns out to be a rather extensive investigation which we intend to pursue in a later paper.

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(*Issued 29 September* 1977)

Paper 5

Proceedings of the Royal Society of Edinburgh, **81A**, 95–117, 1978.

The Hankel transform of some classes of generalized functions and connections with fractional integration*†

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(*Communicated by the late Professor A. Erdélyi*)

(MS received 18 February 1977. Revised MS received 5 January 1978. Read 9 January 1978)

Synopsis

In previous papers [**11**, **12**, **13**], certain spaces $F'_{p,\mu}$ of generalized functions were studied from the point of view of fractional calculus. In this paper, we show how a Hankel transform H_ν of order ν can be defined on $F'_{p,\mu}$ for all complex numbers ν except for those lying on a countable number of lines of the form Re ν = constant in the complex ν-plane. The mapping properties of H_ν on $F'_{p,\mu}$ are obtained. Various connections between H_ν (or modifications of H_ν) and operators of fractional integration are examined.

§1. Introduction

In this paper we are concerned again with the spaces $F_{p,\mu}$ of testing-functions and the corresponding spaces $F'_{p,\mu}$ of generalized functions as defined in [**12**]. It seems convenient to recall immediately the basic definitions.

Our testing-functions are complex-valued infinitely differentiable functions defined on $(0, \infty)$. For $1 \leqq p < \infty$,

$$F_p = \{\phi \in C^\infty(0, \infty) \colon x^k d^k\phi/dx^k \in L^p(0, \infty) \text{ for } k = 0, 1, 2, \ldots\} \tag{1.1}$$

while

$$F_\infty = \left\{ \phi \in C^\infty(0, \infty) \colon \begin{array}{l} \text{for each } k = 0, 1, 2, \ldots, \\ x^k d^k\phi/dx^k \to 0 \text{ as } x \to 0+ \text{ and } x \to \infty \end{array} \right\} \tag{1.2}$$

For $1 \leqq p \leqq \infty$, F_p is equipped with the topology generated by the semi-norms γ_k^p $(k = 0, 1, 2, \ldots)$ defined by

$$\gamma_k^p(\phi) = \|x^k d^k\phi/dx^k\|_p \quad (\phi \in F_p) \tag{1.3}$$

where $\| \ \|_p$ denotes the norm on $L^p(0, \infty)$. Next, for each complex number μ,

$$F_{p,\mu} = \{\phi \colon x^{-\mu}\phi(x) \in F_p\}. \tag{1.4}$$

* This paper was assisted in publication by a grant from the Carnegie Trust for the Universities of Scotland.

† This paper is dedicated to the memory of my great friend and former supervisor, Professor Arthur Erdélyi, without whose help this work would never have existed.

$F_{p,\mu}$ is equipped with the topology generated by the semi-norms $\gamma_k^{p,\mu}$ ($k=0, 1, 2, \ldots$) defined by

$$\gamma_k^{p,\mu}(\phi)=\gamma_k^p(x^{-\mu}\phi) \quad (\phi\in F_{p,\mu}) \tag{1.5}$$

with γ_k^p as in (1.3). Finally, $F'_{p,\mu}$ is the space of continuous linear functionals on $F_{p,\mu}$ equipped with the topology of weak (or pointwise) convergence. We mention again that the space $F_{\infty,\mu}$ as defined above differs from that in **[11]** and corresponds to the space $G_{\infty,\mu}$ in **[13]**.

The purpose of this paper is twofold:

(i) to define the Hankel transform H_ν on the spaces $F'_{p,\mu}$ and to obtain its mapping properties relative to them.

(ii) to investigate some of the connections between the Hankel transform and fractional integration and establish the conditions under which they are valid in $F'_{p,\mu}$.

We hope to carry this a stage further in a future paper whose aim will be

(iii) to apply some of the results obtained to solve certain dual or triple integral equations involving elements of the $F'_{p,\mu}$ spaces.

Various authors have defined the Hankel transform for various classes of distributions. One particularly notable theory appeared in a series of papers by Zemanian which form the basis of **[23]**, Ch. 5] and which has been extended in **[8, 9** and **3]**. Before proceeding, therefore, we must attempt to justify our alternative theory.

As regards the behaviour of the Hankel transform itself, the situation is as follows. In Zemanian's theory, for each (real) ν, H_ν is an automorphism of $\mathscr{H}'_\nu$ [**23**, Theorem 5.10-1] although its behaviour relative to $\mathscr{H}'_{\nu_0}$ ($\nu\neq\nu_0$) is not immediately obvious; the definition of the spaces $\mathscr{H}'_\nu$ will be recalled in Section 2 below. In our theory we shall find below that H_ν is an homeomorphism of $F'_{p,\mu}$ onto $F'_{p,2/p-1-\mu}$ for all but certain exceptional values; for fixed p, the set of exceptional values of (μ, ν) is of measure zero as a subset of two-dimensional complex space.

It is no surprise that the Hankel transform is well behaved relative to the spaces $F'_{p,\mu}$. We have developed a theory of fractional integration in these spaces in two previous papers [**11** and **12**] and it is well known that there are many connections between fractional integration and the Hankel transform; a few of these results appear in [**4**, **6**, **7**, **14**, **17** and **19**]. In a sense, knowledge of the behaviour of the Erdélyi–Kober operators is equivalent to knowledge of the behaviour of the Hankel transform. To discuss these results for generalized functions we must find spaces relative to which the properties of all the operators involved can be found. Since the behaviour of the Erdélyi–Kober operators on $\mathscr{H}'_\nu$ is not obvious, we can regard this as part of the *raison d'être* of our theory. We compare and contrast the two theories at more length in Section 2.

At this point we should mention that a similar investigation to ours has been carried out in a recent paper by Braaksma and Schuitman [**1**]. They use Mellin transforms to study the Hankel transform on certain spaces $T(a, b)$ which are essentially countable unions of spaces of the form $F_{\infty,-c}$. Indeed for real a, b and c, $T(a, b)=\bigcup_{a<c<b} F_{\infty,-c}$. Not surprisingly, our results are in accord with theirs for the case $p=\infty$. Nevertheless, their use of countable-union spaces leads to the restrictions on the parameters taking on a more complicated form. Since the

mapping properties of H_ν can be ascertained on each $F_{\infty,-c}$, it seems simpler to work with them. The results for $T(a, b)$ can then be deduced easily if required. Furthermore, our methods, which do not involve Mellin transforms, handle all values of p in the range $1 \leqq p \leqq \infty$ simultaneously and this is essential for the L^p theory of Hankel transforms; the theory in **[1]** deals only with the case $p = \infty$. We may therefore regard our theory as incorporating that of Braaksma and Schuitman. (A similar comparison can be drawn between the results on fractional integrals in §5 of **[1]** and our theory in **[11, 12]**.)

Meanwhile in Section 3 we begin our investigations into the behaviour of H_ν on $F_{p,\mu}$. We find that if $-\text{Re}\,\nu - \frac{3}{2} + p^{-1}$, $< \text{Re}\,\mu < p^{-1}$ we may use the integral representation

$$(H_\nu \phi)(x) = \int_0^\infty \sqrt{(xt)} J_\nu(xt) \phi(t)\, dt \quad (\phi \in F_{p,\mu}) \tag{1.6}$$

and thus obtain a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$. We give a proof which may not be the simplest but makes use of some results on fractional integrals. Further we would expect that under appropriate conditions H_ν would be invertible with $H_\nu = H_\nu^{-1}$. This is indeed the case and follows from the classical Hankel inversion theorem if also $\text{Re}\ \nu > -\frac{1}{2}$. For $\text{Re}\ \nu \leqq -\frac{1}{2}$ we fall back on analytic continuation and prove that under the conditions above H_ν is analytic on $F_{p,\mu}$ in the sense that for each fixed $\phi \in F_{p,\mu}$

$$[H_{\nu+h}\phi - H_\nu \phi]/h$$

converges to a limit in the topology of $F_{p,2/p-1-\mu}$ as $h \to 0$ in any manner. This might seem to be an example of using a sledgehammer to crack a nut and other methods are possible. However the analyticity of H_ν is of great importance in later sections. Indeed analytic continuation is one of the two main tools in this paper; the other is the fact that $C_0^\infty(0, \infty)$ is dense in $F_{p,\mu}$ for $1 \leqq p \leqq \infty$ and any complex μ.

In Section 4 we pause to consider how the results of Section 3 are in accord with other standard results in the literature. In particular, we indicate how various special cases can be deduced from results in **[20]** on the Fourier transform and also indicate connections with results from **[18]** on the extendability of operators on weighted L^p spaces.

In Section 5 we show how by redefining H_ν we can relax the condition $-\text{Re}\ \nu - \frac{3}{2} + p^{-1} < \text{Re}\,\mu < p^{-1}$ very considerably. At each stage our new definition coincides with the previous definition whenever both are meaningful. We first remove the restriction $\text{Re}\,\mu < p^{-1}$, which of course necessitates leaving behind the simple integral representation (1.6). This operation is quite painless but when we turn to the second restriction $-\text{Re}\ \nu - \frac{3}{2} + p^{-1} < \text{Re}\,\mu$ difficulties arise. Our extended definition requires the invertibility of certain differential operators and we know from **[12**, Section 2] that $D \equiv d/dx$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ if and only if $\text{Re}\,\mu \neq p^{-1}$. It is this restriction which gives rise to the exceptional values mentioned above. We shall be able to define H_ν on $F_{p,\mu}$ provided that

$$\text{Re}\,(\mu + \nu) \neq -\tfrac{3}{2} + p^{-1} - 2l \text{ where } l = 0, 1, 2, \ldots.$$

The mapping properties of the extended operator H_ν are easily deduced from those in Section 3 using results in [**11**] and we have $H_\nu = H_\nu^{-1}$ under appropriate conditions.

In Section 6 we consider a number of operators closely related to H_ν particularly the modified Hankel operator $S^{\eta,\alpha}$ used extensively in [**19**]. The definitions of these operators on $F_{p,\mu}$ are extended using the extended definition of H_ν and their mapping properties obtained.

We come at last in Section 7 to the definition of H_ν on $F'_{p,\mu}$. From consideration of Parseval's equation, we are led to define H_ν for our generalized functions as the adjoint of the (extended) operator H_ν on the testing-functions. The properties of H_ν on $F'_{p,\mu}$ are quickly obtained from those in Section 5 using standard theorems on adjoints; indeed most of the hard work goes into Sections 3 and 5. This completes the first part of our project.

We then turn our attention to the sundry connections between the Erdélyi–Kober operators and the Hankel transform or modifications of it. The conditions necessary for their validity on $F_{p,\mu}$ are established in Section 8 and the corresponding results on $F'_{p,\mu}$ follow readily in Section 9. We come across a number of instances of "removable singularities" analogous to [**12**, Theorem 3.14], where the composition of two operators can be interpreted in the sense of analytic continuation on a space where one or other of the individual operators is not defined.

As indicated above we defer our main application of the theory to dual integral equations until a later paper.

We shall adhere to the conventions adopted in our previous papers. In particular we mention again that $1 \leqq p \leqq \infty$ unless the contrary is explicitly stated and that p and q are always connected by the relation $p^{-1}+q^{-1}=1$. Also, $C_0^\infty(0,\infty)$ will denote the set of infinitely differentiable functions on $(0,\infty)$ whose support is a compact subset of $(0,\infty)$.

§2

In this section, we compare and contrast our theory with that developed in [**23**].

First it is convenient to recall the definition of the spaces $\mathscr{H}_\nu$ introduced in [**23**]. (We use ν here to avoid confusion with $F_{p,\mu}$.) For each complex number ν, $\mathscr{H}_\nu$ is the complex linear space of infinitely differentiable functions ϕ such that, for every pair of non-negative integers m and k,

$$\gamma^\nu_{m,k}(\phi) = \sup_{0<x<\infty} |x^m (x^{-1}D)^k x^{-\nu-\frac{1}{2}}\phi(x)| < \infty \tag{2.1}$$

$\mathscr{H}_\nu$ is equipped with the topology generated by the semi-norms $\gamma^\nu_{m,k}$ ($m,k=0$, 1, 2, . . .).

It is natural to ask whether there are any connections between these spaces and our spaces $F_{p,\mu}$. It is fairly easy to see that there are no inclusions of the form $F_{p,\mu} \subseteq \mathscr{H}_\nu$. Indeed, consider the function ϕ defined by

$$\phi(x) = \lambda(x)x^{\mu-1/p-1}$$

where $1 \leqq p \leqq \infty$, μ is any complex number and λ is an infinitely differentiable function on $(0, \infty)$ such that

$$\lambda(x) = \begin{cases} 0 & 0<x<1 \\ 1 & x>2 \end{cases}$$

Then $\phi \in F_{p,\mu}$. On the other hand, since ϕ is not of rapid descent as $x \to \infty$, $\phi \notin \mathcal{H}_\nu$ by [**23**, Lemma 5.2-1].

In the opposite direction we state the following theorem.

THEOREM 2.1. *Let* $1 \leq p \leq \infty$ *and let* μ, ν *be complex numbers. Then*
(i) $\mathcal{H}_\nu \subseteq F_{p,\mu}$ *if and only if* $\operatorname{Re} \nu > \operatorname{Re} \mu - \frac{1}{2} - p^{-1}$.
(ii) *If* $\operatorname{Re} \nu > \operatorname{Re} \mu - \frac{1}{2} - p^{-1}$, $F'_{p,\mu} \subseteq \mathcal{H}'_\nu$.

The proof involving some routine analysis is omitted.

Theorem 2.1 effectively says that when $\mathcal{H}'_\nu$ and $F'_{p,\mu}$ are comparable, the former contains "more" generalized functions than the latter. Thus although both theories treat spaces of distributions, relative to which the Hankel transform is "well-behaved", that in [**23**] includes our theory in a certain sense, under the above conditions. On the other hand we shall also define H_ν on $F'_{p,\mu}$ for values of μ, ν and p which do not satisfy the condition $\operatorname{Re} \nu > \operatorname{Re} \mu - \frac{1}{2} - 1/p$. Further, in solving all but the simplest problems, it is necessary to use a sequence of different operators. Therefore, if we are solving problems in a space of distributions, the mapping properties of all the operators, relative to the given space, must be known.

To illustrate these remarks in the context of the Hankel transform let us consider for a moment mixed boundary value problems such as those which arise frequently in potential theory and which are discussed comprehensively in [**19**]. One important method of solving the dual integral equations which arise involves the use of the Hankel transform (or certain modifications of it) in conjunction with the Erdélyi–Kober operators of fractional integration [see **19**, Ch. 2]. To solve the corresponding problems for distributions entails using spaces of generalized functions which are not only amenable to the Hankel transform but also to the various fractional integration operators. The spaces $F'_{p,\mu}$ fill the bill nicely and we shall explore some of the theory in the rest of this paper.

But how do the spaces $\mathcal{H}'_\nu$ behave as regards fractional calculus? Let us consider the simple differentiation operator D. It follows easily from [**23**, Lemma 5.3-3 (ii)] that D is a continuous linear mapping of $\mathcal{H}_{\nu+1}$ into $\mathcal{H}_\nu$ for any complex ν. On the other hand, from part (i) of the same lemma, D maps $\mathcal{H}_{-\frac{1}{2}}$ *onto* $\mathcal{H}_{+\frac{1}{2}}$ rather than $\mathcal{H}_{-\frac{1}{2}}$ and the range of D on $\mathcal{H}_\nu$ for arbitrary ν does not seem to be readily obtainable. Thus the precise behaviour of even this simple operator relative to $\mathcal{H}_\nu$ or $\mathcal{H}'_\nu$ requires a fairly extensive investigation. By contrast, from [**12**, Theorem 2.1], D is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ except when $\operatorname{Re} \mu = p^{-1}$.

With these few remarks as a prologue, we now proceed to our theory.

§3

For $\phi \in F_{p,\mu}$, we define $H_\nu \phi$, the Hankel transform of order ν of ϕ, by

$$(H_\nu \phi)(x) = \int_0^\infty \sqrt{(xt)} J_\nu(xt) \phi(t)\, dt \quad (0<x<\infty) \tag{3.1}$$

provided that the integral exists. If ϕ were only known to belong to $L^p(0,\infty)$ for instance, there would be no guarantee that the integral would exist even for almost all $x\in(0,\infty)$ and we would have to define $H_\nu\phi$ using mean convergence (for instance, convergence in the $L^q(0,\infty)$ norm). Fortunately the differentiability of the functions in F_p and $F_{p,\mu}$ obviates this difficulty and enables us to use (3.1) under fairly general conditions which we now indicate.

LEMMA 3.1. *Let* $\phi\in F_{p,\mu}$ *with* $-\mathrm{Re}\ \nu-\frac{3}{2}+p^{-1}<\mathrm{Re}\ \mu<p^{-1}$. *Then* $H_\nu\phi$, *as defined by* (3.1), *is infinitely differentiable on* $(0,\infty)$ *and for* $k=0, 1, 2, \ldots$

$$\delta^k H_\nu\phi=(-1)^k H_\nu(\delta+1)^k\phi \quad (\delta\equiv x\, d/dx). \tag{3.2}$$

Proof. First recall from [**11**, Lemma 2.2] that if $\phi\in F_{p,\mu}$, $t^{1/p-\mathrm{Re}\mu}\phi(t)$ is bounded on $(0,\infty)$. Also, by [**5**, p. 11, formula (50)],

$$z^{\nu+1}J_\nu(z)=d/dz(z^{\nu+1}J_{\nu+1}(z))$$

for any z, ν. Using these facts and integration by parts, we find that

$$(H_\nu\phi)(x)=-\int_0^\infty (xt)^{-\frac{1}{2}}J_{\nu+1}(xt)\{(\delta\phi)(t)-(\nu+\tfrac{1}{2})\phi(t)\}\, dt. \tag{3.3}$$

Since $\phi\in F_{p,\mu}\Rightarrow\delta\phi-(\nu+\frac{1}{2})\phi\in F_{p,\mu}$, the integral on the R.H.S. is absolutely convergent and uniformly convergent when x is restricted to any compact subset of $(0,\infty)$. Hence $H_\nu\phi$ is continuous on $(0,\infty)$.

If $\psi=\delta\phi-(\nu+\frac{1}{2})\phi$, (3.3) can be written in the form

$$(H_\nu\phi)(x)=-\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)\psi(u/x)\, du/x. \tag{3.4}$$

We may differentiate under the integral sign in (3.4) to obtain

$$\begin{aligned} d/dx[(H_\nu\phi)(x)]&=x^{-1}\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)[(u/x)\psi'(u/x)+\psi(u/x)]\, du/x\\ &=x^{-1}\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)[(\delta+1)\psi](u/x)\, du/x.\end{aligned}$$

Since $(\delta+1)\psi\in F_{p,\mu}$, the integral again converges absolutely and uniformly on any compact subset of $(0,\infty)$ under the given restrictions on the parameters. Hence $H_\nu\phi$ is differentiable on $(0,\infty)$ and

$$(\delta H_\nu\phi)(x)=\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)[(\delta+1)\psi](u/x)\, du/x.$$

By induction it follows that $H_\nu\phi$ is infinitely differentiable on $(0,\infty)$ and for $k=0$, $1, 2, \ldots,$

$$(\delta^k H_\nu\phi)(x)=(-1)^{k-1}\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)[(\delta+1)^k\psi](u/x)\, du/x \tag{3.5}$$

Now $(\delta+1)^k\psi=(\delta+1)^k[\delta-(\nu+\frac{1}{2})]\phi=[\delta-(\nu+\frac{1}{2})]\ (\delta+1)^k\phi$ since our operators are polynomials in δ. Hence from (3.5)

$$(\delta^k H_\nu\phi)(x)=(-1)^{k-1}\int_0^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)[(\delta-(\nu+\tfrac{1}{2}))(\delta+1)^k\phi](u/x)\,du/x$$
$$=(-1)^{k-1}\int_0^\infty (xt)^{-\frac{1}{2}}J_{\nu+1}(xt)[(\delta-(\nu+\tfrac{1}{2}))(\delta+1)^k\phi](t)\,dt$$
$$=(-1)^k[H_\nu(\delta+1)^k\phi](x)$$

from (3.3) with ϕ replaced by $(\delta+1)^k\phi$. This completes the proof.

We can now proceed to investigate the mapping properties of H_ν on the $F_{p,\mu}$ spaces. It is probably easiest to split the process up into a number of smaller stages. First we have

LEMMA 3.2. *If* $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu<p^{-1}$, *then* H_ν *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,2/p-1-\mu}$.

Proof. We give a proof which makes use of properties of the Erdélyi–Kober operators of fractional integration. Again we write $\psi(t)=(\delta\phi)(t)-(\nu+\frac{1}{2})\phi(t)$. From (3.4) we have

$$x^{-1}(H_\nu\phi)(1/x)=-\int_0^1 u^{-\frac{1}{2}}J_{\nu+1}(u)\psi(ux)\,du-\int_1^\infty u^{-\frac{1}{2}}J_{\nu+1}(u)\psi(ux)\,du.$$

Using properties of $J_\nu(u)$, we can find constants C_1 and C_2 such that

$$|x^{-1}(H_\nu\phi)(1/x)|\leqq C_1\int_0^1 u^{\operatorname{Re}\nu+\frac{1}{2}}|\psi(ux)|\,du+C_2\int_1^\infty u^{-1}|\psi(ux)|\,du$$
$$\Rightarrow|x^{-\mu-1}H_\nu\phi(1/x)|\leqq C_1\int_0^1 u^{\operatorname{Re}\nu+\frac{1}{2}+\operatorname{Re}\mu}|(ux)^{-\mu}\psi(ux)|\,du$$
$$+C_2\int_1^\infty u^{-1+\operatorname{Re}\mu}|(ux)^{-\mu}\psi(ux)|\,du$$
$$=C_1 I_x^{\operatorname{Re}\nu+\frac{1}{2}+\operatorname{Re}\mu,1}|x^{-\mu}\psi(x)|+C_2K_x^{-\operatorname{Re}\mu,1}|x^{-\mu}\psi(x)|.$$

By hypothesis, $|x^{-\mu}\psi(x)|\in L_p$. Since we have $\operatorname{Re}(\nu+\frac{1}{2}+\mu)>-q^{-1}$ and $-\operatorname{Re}\mu>-p^{-1}$ we may apply **[11**, Lemma 3.1] to deduce that there exist constants C_3, C_4 (independent of ϕ) such that

$$\|x^{-\mu-1}H_\nu\phi(1/x)\|_p\leqq C_3\,\|x^{-\mu}\psi\|_p+C_4\,\|x^{-\mu}\psi\|_p.$$

Using the change of variable $t=1/x$ we find that

$$\|x^{-\mu-1}H_\nu\phi(1/x)\|_p=\|t^{\mu+1-2/p}H_\nu\phi(t)\|_p$$

and therefore we can find a constant C_5 (independent of ϕ) such that

$$\|x^{\mu+1-2/p}H_\nu\phi(x)\|_p\leqq C_5\,\|x^{-\mu}\psi(x)\|_p.$$

Similarly we see from (3.4) and (3.5) that for $k=0,\,1,\,2,\dots$

$$\|x^{\mu+1-2/p}\delta^k H_\nu\phi\|_p\leqq C_5\,\|x^{-\mu}(\delta+1)^k\psi\|_p.$$

It now follows that for each $k=0,\,1,\,2,\dots$ there exist constants $C_k^0,\dots\,C_k^k$

(independent of ϕ) such that

$$\|x^k d^k/dx^k(x^{\mu+1-2/p}H_\nu\phi)\|_p \leqq \sum_{l=0}^{k} C_k^l \|x^l d^l/dx^l(x^{-\mu}\psi)\|_p.$$

Finally substituting for ψ, we deduce that there exist constants $D_k^0, \ldots D_k^{k+1}$ (independent of ϕ) such that

$$\|x^k d^k/dx^k(x^{\mu+1-2/p}H_\nu\phi)\|_p \leqq \sum_{l=0}^{k+1} D_k^l \|x^l d^l/dx^l(x^{-\mu}\phi)\|_p$$

or

$$\gamma_k^{p,2/p-1-\mu}(H_\nu\phi) \leqq \sum_{l=0}^{k+1} D_k^l \gamma_l^{p,\mu}(\phi). \tag{3.6}$$

Thus $H_\nu\phi \in F_{p,2/p-1-\mu}$ and H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$. This completes the proof.

It is natural to ask whether H_ν maps $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ under appropriate conditions. It turns out that the answer is affirmative. It is convenient to give a proof making use of the principle of analytic continuation and it is this that we consider next.

Let $\phi \in F_{p,\mu}$ and suppose $-\mathrm{Re}\ \nu - \frac{3}{2} + p^{-1} < \mathrm{Re}\ \mu < p^{-1}$ as in Lemma 3.2. Then $H_\nu\phi$ exists and, for fixed $\phi \in F_{p,\mu}$ and for fixed $x \in (0, \infty)$, we can consider

$$(H_\nu\phi)(x) = \int_0^\infty \sqrt{(xt)} J_\nu(xt)\phi(t)\, dt$$

as a function of ν in the half-plane $\mathrm{Re}\ \nu > -\mathrm{Re}\ \mu - \frac{3}{2} + p^{-1}$. Since, for each $y \in (0, \infty)$, $J_\nu(y)$ is an entire function of ν [**5**, p. 5], we would expect $(H_\nu\phi)(x)$ to be an analytic function of ν in the half-plane indicated with

$$\partial/\partial\nu[H_\nu\phi(x)] = \int_0^\infty \sqrt{(xt)}\partial/\partial\nu[J_\nu(xt)]\phi(t)\, dt.$$

This is a consequence of

THEOREM 3.3 *Let* $-\mathrm{Re}\ \nu - \frac{3}{2} + p^{-1} < \mathrm{Re}\ \mu < p^{-1}$ *and let* $\phi \in F_{p,\mu}$. *Then, as the complex increment h tends to* 0 *(in any manner),* $h^{-1}[H_{\nu+h}\phi - H_\nu\phi]$ *converges in the topology of* $F_{p,2/p-1-\mu}$ *to the function* $[\partial H_\nu/\partial\nu]\phi$ *defined by*

$$[\partial H_\nu/\partial\nu]\phi(x) = \int_0^\infty \sqrt{(xt)}\partial/\partial\nu[J_\nu(xt)]\phi(t)\, dt \quad (0 < x < \infty). \tag{3.7}$$

Proof. Let $\phi \in C_0^\infty(0, \infty)$ and let $x \in (0, \infty)$ be fixed. The function

$$(H_\nu\phi)(x) = \int_0^\infty \sqrt{(xt)} J_\nu(xt)\phi(t)\, dt$$

is analytic in ν on $S_\mu = \{\nu: -\mathrm{Re}\ \nu - \frac{3}{2} + p^{-1} < \mathrm{Re}\ \mu < p^{-1}\}$ and for $k = 1, 2, \ldots$

$$\partial^k/\partial\nu^k[H_\nu\phi(x)] = \int_0^\infty \sqrt{(xt)}\partial^k/\partial\nu^k[J_\nu(xt)]\phi(t)\, dt,$$

differentiation under the integral sign being justified by uniform convergence.

Hence, in this case, $\partial/\partial\nu[H_\nu\phi(x)]=([\partial H_\nu/\partial\nu]\phi)(x)$ where the operator $\partial H_\nu/\partial\nu$ is defined by (3.7).

For fixed ν in the strip S_μ we can choose $\varepsilon>0$ such that $|h|<\varepsilon\Rightarrow\nu+h\in S_\mu$. For such h we may use the Cauchy integral formula to obtain

$$[h^{-1}(H_{\nu+h}\phi-H_\nu\phi)-(\partial H_\nu/\partial\nu)\phi](x)=\frac{h}{2\pi i}\int_C\frac{H_s\phi(x)}{(s-\nu-h)(s-\nu)^2}ds$$

where C is any closed contour in S_μ surrounding ν and $\nu+h$. By (3.6), there exist constants D_k^l, independent of $\phi\in F_{p,\mu}$, such that

$$\gamma_k^{p,2/p-1-\mu}(H_s\phi)\leqq\sum_{l=0}^{k+1}D_k^l\gamma_l^{p,\mu}(\phi)\quad(\phi\in F_{p,\mu})$$

for $s\in S_\mu$ and examination of the proof of Lemma 3.2 shows that these constants can be chosen to be independent of $s\in C$. If $1\leqq p<\infty$, the quantity

$$\gamma_k^{p,\mu}\left(\frac{h}{2\pi i}\int_C\frac{H_s\phi(x)}{(s-\nu-h)(s-\nu)^2}ds\right)$$

involves a double integral which converges absolutely. We may invert the order of integration by Fubini's theorem and, using our previous remark, we can find constants C_k^l, independent of h satisfying $|h|<\varepsilon$ and independent of $\phi\in F_{p,\mu}$, such that

$$\gamma_k^{p,2/p-1-\mu}(h^{-1}[H_{\nu+h}\phi-H_\nu\phi]-(\partial H_\nu/\partial\nu)\phi)\leqq|h|\sum_{l=0}^{k+1}C_k^l\gamma_l^{p,\mu}(\phi).\tag{3.8}$$

A similar result follows somewhat more easily in the case $p=\infty$.

We know that H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ when $\nu\in S_\mu$. A calculation, similar to the above, using the Cauchy integral formula shows the same is true of $\partial H_\nu/\partial\nu$. (An alternative would be to derive the asymptotic expansion of $(\partial J_\nu/\partial\nu)(x)$; [see **10**].) This, together with the fact that $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$, shows that (3.8) holds for all $\phi\in F_{p,\mu}$. Hence, as $h\to0$ in any manner.

$$\gamma_k^{p,2/p-1-\mu}(h^{-1}[H_{\nu+h}\phi-H_\nu\phi]-(\partial H_\nu/\partial\nu)\phi)\to0$$

and the theorem is proved.

Remarks 3.4

1. Theorem 3.3 contains the weaker result that if Re $\mu<p^{-1}$, $(H_\nu\phi)(x)$ is analytic in the half-plane Re $\nu>-\frac{3}{2}+p^{-1}-$Re μ for fixed $x\in(0,\infty)$ and fixed $\phi\in F_{p,\mu}$.
2. We can prove similarly that $\partial^kH_\nu/\partial\nu^k$ exists as the kth order Fréchet derivative of H_ν under the above conditions [see **16**, pp. 205–210].

For Re $\nu>-\frac{1}{2}$, the Hankel inversion theorem [**22**, p. 456] shows that $H_\nu=H_\nu^{-1}$ under the conditions of Theorem 3.3. To remove the restriction Re $\nu>-\frac{1}{2}$, we need

LEMMA 3.5. *Let* $-q^{-1}<$Re $\mu<p^{-1}$. *Then for fixed* $\phi\in F_{p,\mu}$ *and fixed* $x\in(0,\infty)$, $H_\nu H_\nu\phi(x)$ *is analytic in* ν *on the region* Re $\nu>$max $(-$Re $\mu-\frac{3}{2}+p^{-1}$, Re $\mu-\frac{3}{2}+q^{-1})$.

Proof. Writing $T_{\nu,h}=h^{-1}[H_{\nu+h}\phi-H_\nu\phi]-(\partial H_\nu/\partial\nu)\phi$, we can write

$$\begin{aligned}h^{-1}[H_{\nu+h}H_{\nu+h}\phi-H_\nu H_\nu\phi]&=h(T_{\nu,h}T_{\nu,h}\phi+T_{\nu,h}\partial H_\nu/\partial\nu\phi+\partial H_\nu/\partial\nu T_{\nu,h}\phi\\&\quad+\partial H_\nu/\partial\nu\partial H_\nu/\partial\nu\phi)+H_\nu h^{-1}(H_{\nu+h}\phi-H_\nu\phi)\\&\quad+h^{-1}(H_{\nu+h}-H_\nu)H_\nu\phi.\end{aligned}$$

Taking $|h|$ small enough, we can use arguments similar to those in Theorem 3.3 to show that each term converges in the topology of $F_{p,\mu}$ as $h\to0$ if

$$\max(-\operatorname{Re}\nu-\tfrac{3}{2}+p^{-1},-q^{-1})<\operatorname{Re}\mu<\min(p^{-1},\operatorname{Re}\nu+\tfrac{3}{2}-q^{-1}).$$

The required result then follows.

We notice that the inequalities in the hypotheses of Lemma 3.5 imply that $\operatorname{Re}\nu>-1$ now and not $\operatorname{Re}\nu>-\frac{3}{2}$ as previously. It might appear that we have expended a great deal of effort in travelling the short distance from $-\frac{1}{2}$ to -1. However, the results on analyticity will be useful later. Furthermore we can now obtain a complete description of the mapping properties of H_ν on $F_{p,\mu}$ by means of

LEMMA 3.6. *Let* $\phi\in F_{p,\mu}$ *with* $\max(-\operatorname{Re}\nu-\frac{3}{2}+p^{-1},-q^{-1})<\operatorname{Re}\mu<\min(p^{-1},\operatorname{Re}\nu+\frac{3}{2}-q^{-1})$.
Then

$$H_\nu H_\nu\phi=\phi. \tag{3.9}$$

Proof. First assume also that $\operatorname{Re}\nu>-\frac{1}{2}$.

If $\phi\in C_0^\infty(0,\infty)$, then certainly $\phi\in L^1(0,\infty)$ and, since $\operatorname{Re}\nu>-\frac{1}{2}$, (3.9) follows by the Hankel inversion theorem, [**22** p. 456]. (Although Watson only considers real values of ν, the result is easily modified for complex ν.)

Next, let $\phi\in F_{p,\mu}$ be arbitrary. By [**12**, (1.7)], $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$ and hence there is a sequence $\{\phi_n\}_{n=1}^\infty$ of elements in $C_0^\infty(0,\infty)$ such that ϕ_n converges to ϕ in $F_{p,\mu}$ as $n\to\infty$. By the previous case, for $n=1,2,\ldots,$

$$H_\nu H_\nu\phi_n=\phi_n \tag{3.10}$$

But, as mentioned above, under the given conditions, $H_\nu H_\nu$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu}$. (3.9) follows on letting $n\to\infty$ in (3.10).

We have now proved that under the extra hypothesis $\operatorname{Re}\nu>-\frac{1}{2}$

$$H_\nu H_\nu\phi(x)=\phi(x)$$

for each fixed $\phi\in F_{p,\mu}$ and each fixed $x\in(0,\infty)$. However the left-hand side is an analytic function of ν in the half-plane $\operatorname{Re}\nu>\max(-\operatorname{Re}\mu-\frac{3}{2}+p^{-1},\operatorname{Re}\mu-\frac{3}{2}+q^{-1})$ using Lemma 3.5. By the principle of analytic continuation, the restriction $\operatorname{Re}\nu>-\frac{1}{2}$ can be removed. This completes the proof of Lemma 3.6.

We can summarize our conclusions in

THEOREM 3.7. H_ν *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,2/p-1-\mu}$ *provided* $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu<p^{-1}$. *If also* $-q^{-1}<\operatorname{Re}\mu<\operatorname{Re}\nu+\frac{3}{2}-q^{-1}$, *then* H_ν *is a homeomorphism of* $F_{p,\mu}$ *onto* $F_{p,2/p-1-\mu}$ *and* $H_\nu^{-1}=H_\nu$.

Proof. This follows immediately from Lemmas 3.2 and 3.6.

There are other ways of arriving at Theorem 3.7 but the method we have chosen both makes use of some earlier results and paves the way for using analytic continuation as a tool in later sections.

§4

Before attempting to relax the conditions in Theorem 3.7, we consider briefly how certain special cases can be derived from results in the literature.

Some facts about $H_{-\frac{1}{2}}$ and $H_{\frac{1}{2}}$, the Fourier cosine and sine transforms can be obtained from results in [**20**]. For instance, [**20**, Theorem 74] can be used to show that $H_{-\frac{1}{2}}$ is a continuous linear mapping of $L^p(0,\infty)$ into $L^q(0,\infty)$ when $1<p\leqq 2$. For $\phi\in F_p$, the definition of $H_{-\frac{1}{2}}\phi$ using convergence in the L^q norm coincides with (3.1). From this it follows fairly easily that $H_{-\frac{1}{2}}$ is a continuous linear mapping of F_p into F_q when $1<p\leqq 2$.

To get a sharper result, we could use [**20**, Theorem 80] to prove in an analogous way that for $1<p\leqq 2$, $H_{-\frac{1}{2}}$ maps F_p continuously into $F_{p,2/p-1}$. As we remarked in [**13**], if $1<p\leqq 2$

$$F_{p,2/p-1}\subseteq F_q$$

the inclusion map being continuous and for $1<p<2$, the inclusion is strict [**13**, Theorem 2.5]. Thus for $1<p<2$, $H_{-\frac{1}{2}}$ cannot map F_p *onto* F_q. That it maps F_p onto $F_{p,2/p-1}$ can be seen using [**20**, Theorem 83] in conjunction with properties of the Riemann–Liouville fractional integral $I_x^{2/p-1}$ developed in [**11**] and the Open Mapping Theorem for Fréchet spaces [**21**, Theorem 17.1]. We also note that conditions under which $H_{-\frac{1}{2}}$ maps $F_{p,2/p-1}$ into F_p can be derived from [**20**, Theorem 79].

Other special cases can be derived from results in [**14** and **15**] concerning operators of the form

$$\mathscr{J}_\nu\phi(x)=\lim_{n\to\infty}(q)\int_0^n (xt)^{\frac{1}{2}-\nu}J_{\nu-\frac{1}{2}}(xt)\phi(t)\,dt \quad (\phi\in L^p) \tag{4.1}$$

where $\lim(q)$ denotes convergence in the L^q norm.

Finally we should mention work of Rooney on the extendability of the Hankel transform relative to weighted L^p spaces. For $1\leqq p\leqq\infty$ and any complex μ we shall write

$$L^p_\mu=\{f\colon x^{-\mu}f(x)\in L^p(0,\infty)\} \tag{4.2}$$

so that L^p_μ is the space $L_{1-\mu p,p}$ in [**17** and **18**]. Transcribing results in [**18**], we find that H_ν can be extended to a continuous linear mapping from L^p_μ into $L^r_{1/p+1/r-1-\mu}$ provided that $1<p\leqq r<\infty$ and $\max(1/p,\ 1-1/r)\leqq 1/p-\mathrm{Re}\ \mu<\mathrm{Re}\ \nu+\frac{3}{2}$. If we take $p=r$, results for $F_{p,\mu}$ follow fairly easily although the restrictions on the parameters are more stringent than those in Theorem 3.3 above.

§5

We now consider whether it is possible to extend the definition of H_ν to spaces $F_{p,\mu}$ which do not satisfy the conditions $-\mathrm{Re}\ \nu-\frac{3}{2}+p^{-1}<\mathrm{Re}\ \mu<p^{-1}$ in Lemma

3.2 and Theorem 3.7. First we remove the restriction $\operatorname{Re}\mu<p^{-1}$.

Let $\phi\in F_{p,\mu}$ with $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu<p^{-1}$. Then from (3.3), we have

$$(H_\nu\phi)(x)=-x^{-1}H_{\nu+1}N_\nu\phi \tag{5.1}$$

where, as in [**23**, p. 135], N_ν is the differential operator defined by

$$N_\nu\phi=x^{\nu+\frac{1}{2}}\,d/dx(x^{-\nu-\frac{1}{2}}\phi)=d\phi/dx-(\nu+\tfrac{1}{2})x^{-1}\phi. \tag{5.2}$$

Now the right-hand side of (5.1) exists provided only that

$$-\operatorname{Re}(\nu+1)-\tfrac{3}{2}+p^{-1}<\operatorname{Re}\mu-1<p^{-1}$$

or

$$-\operatorname{Re}\nu-\tfrac{3}{2}+p^{-1}<\operatorname{Re}\mu<p^{-1}+1$$

We can therefore use (5.1) to *define* H_ν on $F_{p,\mu}$ subject to the last condition. It is clear from [**11**, Theorem 2.6] and Lemma 3.2 above that H_ν, as so defined, is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$. Furthermore if $\phi\in F_{p,\mu}$ where $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu<p^{-1}$, $H_\nu\phi$ as defined by (5.1) agrees with $H_\nu\phi$ as defined by (3.1), in view of (3.3). Also, it is easy to show, using Theorem 3.3 and (5.2) that if $\operatorname{Re}\mu<p^{-1}+1$, H_ν is analytic in ν in the half-plane $\operatorname{Re}\nu>-\operatorname{Re}\mu-\frac{3}{2}+p^{-1}$ with Fréchet derivative $\partial H_\nu/\partial\nu$ given by

$$[\partial H_\nu/\partial\nu]\phi=-x^{-1}\{\partial H_{\nu+1}/\partial\nu N_\nu\phi-H_{\nu+1}x^{-1}\phi\}\quad(\phi\in F_{p,\mu})$$

Having weakened the condition $\operatorname{Re}\mu<p^{-1}$ to $\operatorname{Re}\mu<p^{-1}+1$, we can now repeat the process to define H_ν on $F_{p,\mu}$ subject only to the condition $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu$. (5.1) motivates the following definition.

DEFINITION 5.1. Let $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu$. For $\phi\in F_{p,\mu}$, we define $H_\nu\phi$ by

$$(H_\nu\phi)(x)=(-1)^k x^{-k}H_{\nu+k}N_{\nu+k-1}\cdots N_\nu\phi(x) \tag{5.3}$$

where k is any non-negative integer such that $\operatorname{Re}\mu<p^{-1}+k$. [Compare **23**, p. 163, formula (2).]

Before we proceed, there are a number of points to be resolved concerning this definition.

Notes

1. Firstly we note that since $-\operatorname{Re}(\nu+k)-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu-k<p^{-1}$, the right-hand side of (5.3) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ by [**11**, Theorem 2.6] and Lemma 3.2 above.

2. To justify the symbol H_ν on the left-hand side of (5.3), we have to show that our new definition agrees with that in (3.1) wherever both make sense, that is, for functions $\phi\in F_{p,\mu}$ with $-\operatorname{Re}\nu-\frac{3}{2}+p^{-1}<\operatorname{Re}\mu<p^{-1}$. However, this is easily checked using k applications of (5.1) which are valid under the given hypotheses.

3. We must also check that our definition is independent of our choice of non-negative integer k satisfying $\operatorname{Re}\mu<p^{-1}+k$. To this end, suppose that l is another non-negative integer such that $\operatorname{Re}\mu<p^{-1}+l$. Also we may take $k<l$

without loss of generality. Then for $\phi \in F_{p,\mu}$ and $-\text{Re}\ \nu-\frac{3}{2}+p^{-1}<\text{Re}\ \mu$, we have

$$\begin{aligned}(-1)^l x^{-l} H_{\nu+l} N_{\nu+l-1} \dots N_\nu \phi \\ &= (-1)^k x^{-k}(-1)^{l-k} x^{-(l-k)} H_{\nu+l} N_{\nu+l-1} \dots N_{\nu+k}(N_{\nu+k-1} \dots N_\nu \phi) \\ &= (-1)^k x^{-k} H_{\nu+k}(N_{\nu+k-1} \dots N_\nu \phi)\end{aligned}$$

applying (5.1) $(l-k)$ times with ϕ replaced by $N_{\nu+k-1} \dots N_\nu \phi$. This is valid since $-\text{Re}\ (\nu+k)-\frac{3}{2}+p^{-1}<\text{Re}\ \mu-k<p^{-1}+(l-k)$. The required result follows.

From results in §3, we can easily obtain the mapping properties of our extended operator.

THEOREM 5.2

(i) *If* $-\text{Re}\ \nu-\frac{3}{2}+p^{-1}<\text{Re}\ \mu$, *then* H_ν *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,2/p-1-\mu}$.

(ii) H_ν *is analytic in* ν *in the half-plane* $\text{Re}\ \nu>-\text{Re}\ \mu-\frac{3}{2}+p^{-1}$.

(iii) *If* $-\text{Re}\ \nu-\frac{3}{2}+p^{-1}<\text{Re}\ \mu<\text{Re}\ \nu+\frac{3}{2}-q^{-1}$, *then* H_ν *is a homeomorphism of* $F_{p,\mu}$ *onto* $F_{p,2/p-1-\mu}$ *and* $H_\nu=H_\nu^{-1}$.

Proof. (i) and (ii) follow almost immediately. The proof of (iii) is similar to that of Lemma 3.6 and is therefore omitted.

To relax the restrictions in Theorem 5.2 further, we have to leave behind (5.1) of which maximum use has been made. As in [**23**], we define the operator M_ν by

$$M_\nu \phi = x^{-\nu-\frac{1}{2}}\, d/dx(x^{\nu+\frac{1}{2}}\phi). \tag{5.4}$$

Proceeding as in [**23**, Lemma 5.4-1(7)] we can prove that

$$H_\nu M_\nu \phi = x H_{\nu+1}\phi \tag{5.5}$$

for $\phi \in C_0^\infty(0,\infty)$ and, by continuity, (5.5) is true for $\phi \in F_{p,\mu}$ provided that $-\text{Re}\ \nu-\frac{3}{2}+1/p<\text{Re}\ \mu-1$. However the right-hand side of (5.5) is meaningful if $-\text{Re}\ \nu-\frac{3}{2}+1/p<\text{Re}\ \mu+1$. To exploit this, we would like to write

$$H_\nu \phi = x H_{\nu+1} M_\nu^{-1}\phi \tag{5.6}$$

under appropriate conditions. In [**11**], we saw that D is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ if and only if $\text{Re}\ \mu \neq p^{-1}$ and, for $\psi \in F_{p,\mu-1}$,

$$D^{-1}\psi = \begin{cases} \displaystyle\int_0^x \psi(t)\,dt & \text{Re}\ \mu>p^{-1} \\ \displaystyle -\int_x^\infty \psi(t)\,dt & \text{Re}\ \mu<p^{-1} \end{cases}$$

It follows easily that M_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ if and only if $\text{Re}\,(\mu+\nu+\frac{1}{2})\neq p^{-1}$. (If $\text{Re}\,(\mu+\nu+\frac{1}{2})=p^{-1}$, M_ν^{-1} will not be defined on the whole of $F_{p,\mu-1}$.) Thus the right-hand side of (5.6) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ provided that

$$\text{Re}\,(\mu+\nu)>-2-\tfrac{3}{2}+p^{-1}, \quad \text{Re}\,(\mu+\nu)\neq -\tfrac{3}{2}+p^{-1}. \tag{5.7}$$

For such values of ν we can use (5.6) to *define* H_ν on $F_{p,\mu}$. Because of (5.5), the new definition agrees with the previous (extended) definition in the case $\text{Re}\,(\mu+\nu)>-\frac{3}{2}+p^{-1}$. By repeating the process, we can extend the definition of H_ν to the set of complex numbers

$$\Omega_{p,\mu}=\{\nu\colon \mathrm{Re}\,(\mu+\nu)\neq -\tfrac{3}{2}+p^{-1}-2l \text{ for } l=0,1,2,\ldots\} \tag{5.8}$$

DEFINITION 5.3 Let $\nu\in\Omega_{p,\mu}$ and let k be a non-negative integer such that $\mathrm{Re}\,(\mu+\nu)>-\frac{3}{2}+p^{-1}-2k$. We define H_ν on $F_{p,\mu}$ $(1\leqq p\leqq\infty)$ by

$$(H_\nu\phi)(x)=x^k H_{\nu+k}M_{\nu+k-1}^{-1}\ldots M_\nu^{-1}\phi \quad (\phi\in F_{p,\mu}) \tag{5.9}$$

where $H_{\nu+k}$ is defined using (5.3).

Notes. Comments similar to those after Definition 5.1 are in order here.

1. Since $-\mathrm{Re}\,(\nu+k)-\frac{3}{2}+p^{-1}<\mathrm{Re}\ \mu+k$ and $\nu\in\Omega_{p,\mu}$, the right-hand side of (5.9) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ by Theorem 5.2 in conjunction with [**11**, Theorem 3.2].

2. By k applications of (5.6), our new definition agrees with that of Definition 5.1 when $\mathrm{Re}\,(\mu+\nu)>-\frac{3}{2}+p^{-2}$.

3. We must check that our definition is independent of the choice of the non-negative integer satisfying $\mathrm{Re}\,(\mu+\nu)>-\frac{3}{2}+p^{-1}-2k$. The details are similar to those in Note 3 following Definition 5.1 and are omitted.

We would expect H_ν to be a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$, with $H_\nu=H_\nu^{-1}$, under appropriate conditions and this is indeed the case. However, since $\Omega_{p,\mu}$ is not connected, it is not merely a case of applying analytic continuation in conjunction with Theorem 5.2 (iii). Instead we use the following Lemma.

LEMMA 5.4. *Let $1\leqq p\leqq\infty$ and let $\nu\in\Omega_{p,\mu}$. Then for $\phi\in F_{p,\mu}$,*

$$H_\nu\phi=M_\nu H_{\nu+1}x^{-1}\phi \tag{5.10}$$

Proof. $\nu\in\Omega_{p,\mu}\Rightarrow\nu+1\in\Omega_{p,\mu-1}$ so that $H_\nu\phi$, $H_{\nu+1}x^{-1}\phi$ are both defined. First assume $\mathrm{Re}\,(\mu+\nu)>-\frac{3}{2}+1/p$ and let $\phi\in C_0^\infty(0,\infty)$. The method of [**23**, Lemma 5.4-1(6)] (with ϕ replaced by $x^{-1}\phi$) establishes (5.10) in this case and the result for a general $\phi\in F_{p,\mu}$ follows since $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$.

The proof for a general value of $\nu\in\Omega_{p,\mu}$ now follows from the previous case on using (5.9).

We are now ready to give a precise description of the behaviour of H_ν on $F_{p,\mu}$.

THEOREM 5.5 *Let $1\leqq p\leqq\infty$ and let μ, ν be complex numbers.*

(i) *If $\nu\in\Omega_{p,\mu}$, H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$.*

(ii) *For fixed p and μ, H_ν is analytic on $F_{p,\mu}$ for $\nu\in\Omega_{p,\mu}$.*

(iii) *If $\nu\in\Omega_{p,\mu}\cap\Omega_{p,2/p-1-\mu}$, H_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ and*

$$H_\nu^{-1}=H_\nu.$$

Proof. We consider only (iii).

Let $\nu\in\Omega_{p,\mu}\cap\Omega_{p,2/p-1-\mu}$ and let k, l be non-negative integers such that $\mathrm{Re}\,(\mu+\nu)>-\frac{3}{2}+p^{-1}-2k$ and $\mathrm{Re}\,(2p-1-\mu+\nu)>-\frac{3}{2}+p^{-1}=2l$. Then if $\phi\in F_{p,\mu}$ and $\mathrm{Re}\,(2/p-1-\mu+\nu)>-\frac{3}{2}+p^{-1}$ (so that we may take $l=0$)

$$(H_\nu\phi)(x)=x^k H_{\nu+k}M_{\nu+k-1}^{-1}\ldots M_\nu^{-1}\phi \tag{5.11}$$

from (5.9). But under the given conditions $H_{\nu+k}$ is a homeomorphism of $F_{p,\mu+k}$ onto $F_{p,2/p-1-(\mu+k)}$ by Theorem 5.2(iii). Hence by (5.11), H_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ and, for $\psi \in F_{p,2/p-1-\mu}$,

$$H_\nu^{-1}\psi = M_\nu \ldots M_{\nu+k-1} H_{\nu+k} x^{-k}\psi.$$

But, by an easy induction based on (5.10), the right-hand side is simply $H_\nu\psi$ under the given conditions and we have the required result. The case $l>0$ follows easily from the previous case.

This completes the proof of Theorem 5.5.

We mention one consequence of Theorem 5.5.

COROLLARY 5.6. *Let* $-\mathrm{Re}\ \nu - \frac{3}{2} + p^{-1} < \mathrm{Re}\ \mu < p^{-1}$ *and let* $\nu \in \Omega_{p,2/p-1-\mu}$. *Then the equation*

$$\int_0^\infty \sqrt{(xt)}J_\nu(xt)\phi(t)\,dt = \psi(x)$$

has, for each $\psi \in F_{p,2/p-1-\mu}$, *a unique solution* $\phi \in F_{p,\mu}$.

Proof. If $-\mathrm{Re}\ \nu - \frac{3}{2} + p^{-1} < \mathrm{Re}\ \mu < p^{-1}$, then $\nu \in \Omega_{p,\mu}$ and for $\phi \in F_{p,\mu}$, $H_\nu\phi$ is given by (3.1). The result follows at once from Theorem 5.5 (iii).

§6

Various operators related to H_ν can now be extended easily. For instance, (4.1) indicates that the operator $\mathcal{J}_\nu$ of Okikiolu can be extended to other values of ν via the relation

$$\mathcal{J}_\nu\phi = x^{-\nu} H_{\nu-\frac{1}{2}} x^{-\nu}\phi. \tag{6.1}$$

Similar considerations apply to operators involving the composition of two Hankel transforms as studied in [**17**]. However we shall only look in detail at one important modification of H_ν.

For complex numbers η and α, we define the modified Hankel transform $S^{\eta,\alpha}$ by

$$S^{\eta,\alpha}\phi(x) = 2^\alpha x^{-\alpha}\int_0^\infty t^{1-\alpha}J_{2\eta+\alpha}(xt)\phi(t)\,dt \tag{6.2}$$

under appropriate conditions. The operator $S^{\eta,\alpha}$ was introduced in [**7**] and used extensively in [**19**] in connection with dual integral equations. By Lemma 3.1, if $\phi \in F_{p,\mu}$, the integral in (6.2) converges for all $x \in (0,\infty)$ provided that $-\mathrm{Re}\,(2\eta+\alpha) - \frac{3}{2} + p^{-1} < \mathrm{Re}\ \mu + \frac{1}{2} - \mathrm{Re}\ \alpha < p^{-1}$ and in that case

$$S^{\eta,\alpha}\phi = 2^\alpha x^{-\frac{1}{2}-\alpha} H_{2\eta+\alpha} x^{\frac{1}{2}-\alpha}\phi. \tag{6.3}$$

We can use this to extend $S^{\eta,\alpha}$ to those values of η,α such that $\mathrm{Re}\,(2\eta+\alpha+\frac{1}{2}-\alpha+\mu) \neq -\frac{3}{2}+p^{-1}-2l$ $(l=0, 1, 2, \ldots)$. Accordingly, let

$$A_{p,\mu} = \{\eta : \mathrm{Re}\,(2\eta+\mu)+2 \neq p^{-1}-2l \quad (l=0, 1, 2, \ldots)\} \tag{6.4}$$

(6.4) is the case $m=2$ of [**12**, formula 3.5].

DEFINITION 6.1. For any complex number α and for $\eta \in A_{p,\mu}$, define $S^{\eta,\alpha}$ on $F_{p,\mu}$

by

$$S^{\eta,\alpha}\phi(x)=2^{\alpha}x^{-\frac{1}{2}-\alpha}H_{2\eta+\alpha}x^{\frac{1}{2}-\alpha}\phi(x)$$

where $H_{2\eta+\alpha}$ is defined using Definition 5.3.

To discuss the invertibility of $S^{\eta,\alpha}$, let

$$A'_{p,\mu}=\{\eta:\mathrm{Re}\,(2\eta-\mu)\neq -p^{-1}-2l, \quad (l=0,1,2,\ldots)\} \tag{6.5}$$

This is the case $p=2$ in [**12**, formula 3.10]. Then we have the following.

THEOREM 6.2

(i) *If α is any complex number and $\eta\in A_{p,\mu}$, then $S^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-2-\mu}$.*

(ii) *If also $\eta+\alpha\in A'_{p,\mu}$, then $S^{\eta,\alpha}$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-2-\mu}$ and $(S^{\eta,\alpha})^{-1}=S^{\eta+\alpha,-\alpha}$.*

Proof. The results follow in a routine manner from Theorem 5.5 and the details are omitted.

§7

We are now ready to define our generalized Hankel transform on the spaces $F'_{p,\mu}$. Our definition is motivated by considering regular functionals.

Let $f\in L^p_\mu$ (where L^p_μ is defined by (4.2)). By Hölder's inequality, f generates a functional, τf say, on $F_{q,-\mu}(p^{-1}+q^{-1}=1)$ according to the formula

$$(\tau f,\phi)=\int_0^\infty f(x)\phi(x)\,dx \quad (\phi\in F_{q,-\mu}). \tag{7.1}$$

The mapping $f\to\tau f$ provides an imbedding of L^p_μ into $F'_{q,-\mu}$. We shall use τ to denote any such imbedding and omit any explicit mention of p or μ.

Suppose now that $H_\nu f$ exists in the classical sense. As we have seen above, we may expect $H_\nu f$ to belong to $L^p_{2/p-1-\mu}$ under appropriate conditions. In this case, $H_\nu f$ generates an element $\tau H_\nu f$ of $F'_{q,-2/p+1+\mu}=F'_{q,2/q-1+\mu}$. For our definition to be reasonable we must ensure that

$$H_\nu\tau f=\tau H_\nu f \tag{7.2}$$

for such functions f, where H_ν on the left-hand side is our generalized Hankel transform and H_ν on the right is the classical Hankel transform.

If $\phi\in F_{q,2/q-1+\mu}$, (7.2) implies that

$$(H_\nu\tau f,\phi)=(\tau H_\nu f,\phi)=\int_0^\infty H_\nu f(x)\phi(x)\,dx=\int_0^\infty f(x)H_\nu\phi(x)\,dx$$

by Parseval's formula. The validity of this step can be checked by first taking $\phi\in C_0^\infty(0,\infty)$ and f with compact support and then using continuity and Hölder's inequality. In particular, Parseval's equality is valid for each of the special cases mentioned in §4. We are therefore led to the equation

$$(H_\nu\tau f,\phi)=(\tau f,H_\nu\phi) \quad (\phi\in F_{q,2/q-1+\mu}). \tag{7.3}$$

Although we arrived at (7.3) by considering regular functionals only, it is natural to use (7.3) to define $H_\nu g$ for any $g\in F'_{q,-\mu}$, regular or not. We shall continue to use f rather than g for an arbitrary element of $F'_{p,\mu}$. Thus we require that

$$(H_\nu f, \phi) = (f, H_\nu \phi) \tag{7.4}$$

for $f \in F'_{q,-\mu}$, $\phi \in F_{q,2/q-1+\mu}$ under appropriate conditions. Further, although our derivation above used an integral representation for $H_\nu \phi$, the right-hand side of (7.4) is meaningful under much wider conditions if we use Definition 5.3. Finally, therefore, we arrive at the following definition.

DEFINITION 7.1. Let $1 \leqq p \leqq \infty$ and let μ be any complex number. For $\nu \in \Omega_{q,-\mu}$ we define H_ν on $F'_{p,\mu}$ by

$$(H_\nu f, \phi) = (f, H_\nu \phi)$$

for $f \in F'_{p,\mu}$, $\phi \in F_{p,2/p-1-\mu}$ where $H_\nu \phi$ is defined as in Definition 5.3.

Notes

1. Since $\Omega_{q,-\mu} = \Omega_{p,2/p-1-\mu}$, the right-hand side is meaningful and H_ν defines a continuous linear mapping of $F_{p,2/p-1-\mu}$ into $F_{p,\mu}$.

2. By our motivation above, our definition of a generalized Hankel transform extends the classical transform for functions in L^p_μ in that, when both exist, they agree in the sense of (7.2). In particular, our theory is consistent with Rooney's theory of extendability of the classical Hankel transform in [**18**] but allows us to relax the restrictions further by passing to generalized functions.

The mapping properties of H_ν on $F'_{p,\mu}$ are easily obtained from those of H_ν on $F_{p,2/p-1-\mu}$.

THEOREM 7.2 (i) *If $\nu \in \Omega_{q,-\mu}$, then H_ν is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,2/p-1-\mu}$.*

(ii) *If also $\nu \in \Omega_{p,\mu}$, then H_ν is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,2/p-1-\mu}$ and $H_\nu^{-1} = H_\nu$.*

(iii) *For each fixed $f \in F'_{p,\mu}$ and each fixed $\phi \in F_{p,2/p-1-\mu}$, $(H_\nu f, \phi)$ is an analytic function of ν in $\Omega_{q,-\mu}$.*

Proof. (i) As was stated in Note 1 following Definition 7.1, H_ν is a continuous linear mapping of $F_{p,2/p-1-\mu}$ into $F_{p,\mu}$ if $\nu \in \Omega_{q,-\mu}$. The result follows at once from [**23**, Theorem 1.10-1].

(ii) is proved similarly using Theorem 5.5 (iii) and [**23**, Theorem 1.10-2] while (iii) can be proved using Theorem 5.5(ii).

Analogues of various formulae in §5 hold for our generalized Hankel transform. For instance if $\nu \in \Omega_{q,-\mu}$, the result (cf. (5.1))

$$H_\nu f = -x^{-1} H_{\nu+1} N_\nu f \quad (f \in F'_{p,\mu})$$

can be established by taking adjoints in (5.10) and using the fact that the adjoint of M_ν is $-N_\nu$. Similar reasoning applies to other formulae in [**23**, p. 143].

To define $S^{\eta,\alpha}$ on $F'_{p,\mu}$, we have formally

$$\begin{aligned}(S^{\eta,\alpha} f, \phi) &= (2^\alpha x^{-\frac{1}{2}-\alpha} H_{2\eta+\alpha} x^{\frac{1}{2}-\alpha} f, \phi)\\ &= (f, 2^\alpha x^{\frac{1}{2}-\alpha} H_{2\eta+\alpha} x^{-\frac{1}{2}-\alpha} \phi)\\ &= (f, x S^{\eta,\alpha} x^{-1} \phi).\end{aligned}$$

Hence for $f\in F'_{p,\mu}$, we define $S^{\eta,\alpha}f$ by

$$(S^{\eta,\alpha}f,\phi)=(f,xS^{\eta,\alpha}x^{-1}\phi)\quad(\phi\in F_{p,2/p-\mu})\tag{7.5}$$

By Theorem 6.2, the right-hand side is meaningful if $\eta\in A_{q,-\mu}$ and we have the following result which we state without proof.

THEOREM 7.3. (i) *If α is any complex number and $\eta\in A_{q,-\mu}$, then $S^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,2/p-\mu}$.*

(ii) If also $\eta+\alpha\in A'_{q,-\mu}$, then $S^{\eta,\alpha}$ is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,2/p-\mu}$ and $(S^{\eta,\alpha})^{-1}=S^{\eta+\alpha,-\alpha}$.

§8

We now consider connections between the Hankel transform and operators of fractional integration in $F'_{p,\mu}$. For Re $\alpha>0$, $m>0$ and $\phi\in F_{p,\mu}$ we define $I_{x^m}^{\eta;\alpha}\phi$ and $K_{x^m}^{\eta;\alpha}\phi$ on $(0,\infty)$ by

$$(I_{x^m}^{\eta;\alpha}\phi)(x)=[\Gamma(\alpha)]^{-1}mx^{-m\eta-m\alpha}\int_0^x(x^m-u^m)^{\alpha-1}u^{m\eta+m-1}\phi(u)\,du$$

$$(K_{x^m}^{\eta;\alpha}\phi)(x)=[\Gamma(\alpha)]^{-1}mx^{m\eta}\int_x^\infty(u^m-x^m)^{\alpha-1}u^{-m\eta-m\alpha+m-1}\phi(u)\,du$$

provided that Re $(m\eta+\mu)+m>p^{-1}$ and Re $(m\eta-\mu)>-p^{-1}$ respectively. The restriction Re $\alpha>0$ was removed by analytic continuation in **[11]** and an extension to more general values of η was discussed in **[12]** where the mapping properties of the operators can be found.

By making simple changes of variables in results from **[6]**, we find that for $\phi\in L^2$, Re $\alpha>0$ and Re $\nu>-1$,

$$I_{x^2}^{\nu/2-\frac14,\alpha}H_\nu\phi=H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac14,\alpha}\phi.\tag{8.1}$$

We consider the conditions under which (8.1) holds for $\phi\in F_{p,\mu}$. From results in **[12**, §3] and Theorem 5.5 above, the left-hand side maps $F_{p,\mu}$ continuously into $F_{p,2/p-1-\mu}$ if $\nu\in\Omega_{p,\mu}\cap\Omega_{q,-\mu}$. The corresponding conditions for the right-hand side are $\nu\in\Omega_{p,\mu}$ and $\nu+2\alpha\in\Omega_{p,\mu}$. The fact that these two sets of conditions are not the same indicates that further investigation is called for.

If k is a non-negative integer then by **[12**, (3.6)] we have formally

$$I_{x^2}^{\nu/2-\frac14,\alpha}=I_{x^2}^{\nu/2-\frac14+k,\alpha-k}x^{-\nu+\frac12-2k}(\tfrac12x^{-1}D)^{-k}x^{\nu-\frac12}\tag{8.2}$$

while if we use (5.10) we obtain formally

$$H_\nu=M_\nu M_{\nu+1}\dots M_{\nu+k-1}H_{\nu+k}x^{-k}=x^{-\nu+\frac12}(x^{-1}D)^kx^{\nu+k-\frac12}H_{\nu+k}x^{-k}\tag{8.3}$$

Combining (8.2) and (8.3) we obtain

$$I_{x^2}^{\nu/2-\frac14,\alpha}H_\nu\phi=2^kI_{x^2}^{\nu/2-\frac14+k,\alpha-k}x^{-k}H_{\nu+k}x^{-k}\phi$$

or

$$I_{x^2}^{\nu/2-\frac14,\alpha}H_\nu\phi=2^kx^{-k}I_{x^2}^{(\nu+k)/2-\frac14,\alpha-k}H_{\nu+k}x^{-k}\phi\tag{8.4}$$

Now the calculations above are valid for $\phi\in F_{p,\mu}$ provided that $\nu\in\Omega_{p,\mu}$ and

$\nu \in \Omega_{q,-\mu}$. On the other hand, the right-hand side is meaningful provided only that $\nu \in \Omega_{p,\mu}$ and $\nu+2k \in \Omega_{q,-\mu}$. This enables us to remove the restriction $\nu \in \Omega_{q,-\mu}$, while retaining the restriction $\nu \in \Omega_{p,\mu}$. Namely, if $\nu \notin \Omega_{q,-\mu}$ we can choose a non-negative integer k such that $\nu+2k \in \Omega_{q,-\mu}$ and we use (8.4) to define $I_{x^2}^{\nu/2-\frac{1}{4},\alpha} H_\nu$ on $F_{p,\mu}$ in this case. We should mention the following points.

(i) The definition of the extended operator is independent of the non-negative integer k satisfying the condition $\nu+2k \in \Omega_{q,-\mu}$.

(ii) The extended operator $I_{x^2}^{\nu/2-\frac{1}{4},\alpha} H_\nu$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ for any complex α and $\nu \in \Omega_{p,\mu}$.

(iii) The extended operator is analytic in ν on $F_{p,\mu}$ for $\nu \in \Omega_{p,\mu}$.

We have therefore seen that in a sense the singularities corresponding to $\nu \in \Omega_{p,\mu} - \Omega_{q,-\mu}$ are removable.

Similar considerations apply to the operator $H_{\nu+2\alpha} I_{x^2}^{\nu/2-\frac{1}{4},\alpha}$ on the right-hand side of (8.1). We find that if $\nu \in \Omega_{p,\mu}$, $\nu+2\alpha \in \Omega_{p,\mu}$ and $\phi \in F_{p,\mu}$

$$H_{\nu+2\alpha} I_{x^2}^{\nu/2-\frac{1}{4},\alpha} \phi = 2^{-k} x^k H_{\nu-k+2(\alpha+k)} I_{x^2}^{(\nu-k)/2-\frac{1}{4},\alpha+k} x^k \phi. \tag{8.5}$$

However, the right-hand side is meaningful provided only that $\nu \in \Omega_{p,\mu}$ and $(\nu+2\alpha)+2k \in \Omega_{p,\mu}$. We can use (8.5) to extend $H_{\nu+2\alpha} I_{x^2}^{\nu/2-\frac{1}{4},\alpha}$ to $F_{p,\mu}$ provided only $\nu \in \Omega_{p,\mu}$ and the extended operator is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$.

We have now established that, for $\phi \in F_{p,\mu}$, both sides of (8.1) define elements of $F_{p,2/p-1-\mu}$ provided only that $\nu \in \Omega_{p,\mu}$. It remains to establish equality. We state our result as a theorem.

THEOREM 8.1. *If $\phi \in F_{p,\mu}$ and $\nu \in \Omega_{p,\mu}$, then* (8.1) *holds in the usual sense if also $\nu \in \Omega_{q,-\mu}$ and $\nu+2\alpha \in \Omega_{p,\mu}$ and, otherwise, in the sense that the analytic continuations of both sides are equal.*

Proof. By the continuity of the (extended) operators, it is sufficient to establish the result for $\phi \in C_0^\infty(0,\infty)$ (regarded as an element of $F_{p,\mu}$). Using Kober's result together with analytic continuation, (8.1) holds provided that

$$\operatorname{Re}(\mu+\nu) > -\tfrac{3}{2}+p^{-1}, \quad \operatorname{Re}(\mu+\nu+2\alpha) > -\tfrac{3}{2}+p^{-1}, \quad \operatorname{Re}(-\mu+\nu) > -\tfrac{1}{2}-p^{-1}.$$

Now, let k be a non-negative integer such that $\operatorname{Re}(\mu+\nu) > -\frac{3}{2}+p^{-1}-2k$ and $\operatorname{Re}(\mu+\nu+2\alpha) > -\frac{3}{2}+p^{-1}-2k$. Then, if $\phi \in F_{p,\mu}$,

$$\begin{aligned} I_{x^2}^{\nu/2-\frac{1}{4},\alpha} H_\nu \phi &= I_{x^2}^{\nu/2-\frac{1}{4},\alpha} x^k H_{\nu+k} x^{-\nu-k+\frac{1}{2}} (x^{-1}D)^{-k} x^{\nu-\frac{1}{2}} \phi \\ &= x^k I_{x^2}^{(\nu+k)/2-\frac{1}{4},\alpha} H_{\nu+k} x^{-\nu-k+\frac{1}{2}} (x^{-1}D)^{-k} x^{\nu-\frac{1}{2}} \phi \\ &= x^k H_{\nu+k+2\alpha} I_{x^2}^{(\nu+k)/2-\frac{1}{4},\alpha} x^{-\nu-k+\frac{1}{2}} (x^{-1}D)^{-k} x^{\nu-\frac{1}{2}} \phi \end{aligned}$$

by Kober's result with μ, ν replaced by $\mu+k$, $\nu+k$ respectively. Also

$$\begin{aligned} H_{\nu+2\alpha} I_{x^2}^{\nu/2-\frac{1}{4},\alpha} \phi &= 2^{-k} x^k H_{\nu+2\alpha+k} I_{x^2}^{(\nu-k)/2-\frac{1}{4},\alpha+k} x^k \phi \\ &= x^k H_{\nu+2\alpha+k} I_{x^2}^{(\nu+k)/2-\frac{1}{4},\alpha} x^{-\nu-k+\frac{1}{2}} (x^{-1}D)^{-k} x^{\nu-\frac{1}{2}} \phi \end{aligned}$$

by (8.5) above and [**12**, Definition 3.1]. Hence our result holds for $\nu \in \Omega_{p,\mu}$ $\nu+2\alpha \in \Omega_{p,\mu}$ and $\operatorname{Re}(-\mu+\nu) > -\frac{1}{2}+p^{-1}$.

Finally, using similar considerations, the restriction $\operatorname{Re}(-\mu+\nu) > -\frac{1}{2}+p^{-1}$ can be replaced by the condition $\nu \in \Omega_{q,-\mu}$; the details are omitted.

This completes the proof of Theorem 8.1.

(8.1) gives a simple connection between operators of the form $I_{x^2}^{\eta,\alpha}$ and H_ν. There is a similar connection between the operators $K_{x^2}^{\eta,\alpha}$ and H_ν which we state in the following theorem.

THEOREM 8.2. *If $\phi \in F_{p,\mu}$ and $\nu \in \Omega_{p,\mu}$, then, in a sense analogous to that of Theorem 8.1,*

$$H_\nu K_{x^2}^{\nu/2+\frac{1}{4},\alpha}\phi = K_{x^2}^{\nu/2+\frac{1}{4},\alpha} H_{\nu+2\alpha}\phi. \tag{8.6}$$

Proof. The proof is similar to that of Theorem 8.1 and is omitted.

We next establish relations between $I_{x^2}^{\eta,\alpha}$, $K_{x^2}^{\eta,\alpha}$ and the modified Hankel operator $S^{\eta,\alpha}$. These are well-known and are given in [**19**]. We merely establish the range of values for which they are valid and indicate how this may be deduced from our results above. We give six results from [**19**, p. 274, formulae 13A–18A].

THEOREM 8.3. *Let $\phi \in F_{p,\mu}$. Then*

(i) $I_{x^2}^{\eta+\alpha,\beta} S^{\eta,\alpha}\phi = S^{\eta,\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu}$, $\eta+\alpha \in A'_{p,\mu}$.

(ii) $K_{x^2}^{\eta,\alpha} S^{\eta+\alpha,\beta}\phi = S^{\eta,\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu}$, $\eta+\alpha \in A_{p,\mu}$.

(iii) $S^{\eta+\alpha,\beta} S^{\eta,\alpha}\phi = I_{x^2}^{\eta,\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu}$, $\eta+\alpha \in A'_{p,\mu}$.

(iv) $S^{\eta+\alpha,\beta} I_{x^2}^{\eta,\alpha}\phi = S^{\eta,\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu}$, $\eta+\alpha \in A_{p,\mu}$.

(v) $S^{\eta,\alpha} K_{x^2}^{\eta+\alpha,\beta}\phi = S^{\eta,\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu}$, $\eta+\alpha \in A'_{p,\mu}$.

(vi) $S^{\eta,\alpha} S^{\eta+\alpha,\beta}\phi = K_{x^2}^{\eta,\alpha+\beta}\phi$ *if* $\eta \in A_{p,\mu}$, $\eta+\alpha \in A_{p,\mu}$.

The right-hand sides provide analytic continuations of the left-hand sides for $\eta \in A_{p,\mu}$ in (i)–(v) *and $\eta \in A'_{p,\mu}$ in* (vi).

Proof. We consider (i); the other parts are similar.

If $\eta \in A_{p,\mu}$, the right-hand side of (i) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-2-\mu}$. The same is true of the left-hand side, the potential singularities for $\eta+\alpha \notin A_{p,2/p-2-\mu}$ $(=A'_{p,\mu})$ being removable. To see this, we rewrite the left-hand side in the form

$$2^\alpha x^{-\alpha-\frac{1}{2}} I_{x^2}^{\eta+\alpha/2-\frac{1}{4},\beta} H_{2\eta+\alpha} x^{\frac{1}{2}-\alpha}\phi$$

and we are in the situation discussed above for (8.1) with $\nu = 2\eta+\alpha$ and μ replaced by $\mu+\frac{1}{2}-\alpha$.

For a verification of equality we can proceed as in [**19**], taking $\phi \in C_0^\infty(0,\infty)$ and using analytic continuation with respect to ν, as well as techniques similar to those above.

Results obtained in [**17**] concerning the products of Hankel transforms can be used to derive the following theorem.

THEOREM 8.4. *If $\nu \in \Omega_{p,\mu}$ and $\nu+\gamma \in \Omega_{p,2/p-1-\mu}$, then for $\phi \in F_{p,\mu}$,*

$$H_{\nu+\gamma} H_\nu \phi = K_{x^2}^{(\nu+\gamma)/2+\frac{1}{4},-\gamma/2} I_{x^2}^{\nu/2-\frac{1}{4},\gamma/2}\phi$$

$$H_{\nu+\gamma} H_\nu \phi = I_{x^2}^{\nu/2-\frac{1}{4},\gamma/2} K_{x^2}^{(\nu+\gamma)/2+\frac{1}{4},-\gamma/2}\phi$$

Proof. The details are similar in spirit to those in Theorem 8.1 and are omitted.

As a special case, we can obtain properties of

$$H_+ = H_{\frac{1}{2}}H_{-\frac{1}{2}} \quad \text{and} \quad H_- = H_{-\frac{1}{2}}H_{\frac{1}{2}},$$

the Hilbert transforms of odd and even functions. Another corollary ties together formulae (8.1) and (8.6) above.

COROLLARY 8.5. *If $\phi \in F_{p,\mu}$ and $\nu \in \Omega_{p,\mu}$, then*

$$I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu\phi = H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac{1}{4},\alpha}\ \phi = K_{x^2}^{\nu/2+\frac{1}{4},\alpha}H_{\nu+2\alpha}\phi = H_\nu K_{x^2}^{\nu/2+\frac{1}{4},\alpha}\phi$$

where equality holds in the sense of Theorem 8.1.

Proof. The result follows formally on putting $\gamma = 2\alpha$ in Theorem 8.4 and using Theorems 8.1 and 8.2. The removal of singularities is carried out as in Theorem 8.1.

Finally in this section we mention a result involving three Hankel transforms. This is a special case of a result for Watson transforms which has been proved in [**2**] using Mellin transforms for functions in $L^p(0,\infty)$.

THEOREM 8.6. *If $\alpha \in \Omega_{p,\mu}$, $\beta \in \Omega_{q,-\mu}$, $\gamma \in \Omega_{p,\mu}$, then, for $\phi \in F_{p,\mu}$,*

$$H_\alpha H_\beta H_\gamma \phi = H_\gamma H_\beta H_\alpha \phi.$$

Proof. We merely mention that an alternative to the use of Mellin transforms is provided by Theorem 8.4 and Corollary 8.5 above together with [**12**, Theorems 3.13 and 3.14].

§9

As usual it is an easy matter to obtain the corresponding results on $F'_{p,\mu}$ by using adjoint operators. For instance, let us consider again the operator $I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu$. Using [**12**, (3.17)] and Definition 7.1 above we have formally

$$(I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu f,\ \phi) = (f,\ H_\nu K_{x^2}^{\nu/2+\frac{1}{4},\alpha}\phi) \tag{9.1}$$

where $f \in F'_{p,\mu}$, $\phi \in F_{p,2/p-1-\mu}$. Now, in the first instance, the right-hand side will be meaningful provided that $\nu/2+\frac{1}{4} \in A'_{p,2/p-1-\mu}$ and $\nu \in \Omega_{p,2/p-1-\mu}$ (by [**12**, Theorem 3.10] and Theorem 5.5 above) or equivalently $\nu \in \Omega_{p,\mu}$ and $\nu \in \Omega_{q,-\mu}$. However we may remove the restriction $\nu \in \Omega_{p,\mu}$ and consequently the right-hand side is meaningful provided only that $\nu \in \Omega_{q,-\mu}$, if we interpret it in the sense of its analytic continuation. We can then use (9.1) to define an operator, which we again denote by $I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu$ on $F'_{p,\mu}$ for any complex α and $\nu \in \Omega_{q,-\mu}$. This operator is the analytic continuation to the whole of $\Omega_{q,-\mu}$ of the operator formed by the composition of $I_{x^2}^{\nu/2-\frac{1}{4},\alpha}$ and H_ν.

Similar considerations apply to the other operators appearing in Corollary 8.5 and we have the following theorem.

THEOREM 9.1. *If $f \in F'_{p,\mu}$ and $\nu \in \Omega_{q,-\mu}$ then*

$$I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu f = H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac{1}{4},\alpha}f = K_{x^2}^{\nu/2+\frac{1}{4},\alpha}H_{\nu+2\alpha}f = H_\nu K_{x^2}^{\nu/2+\frac{1}{4},\alpha}f \tag{9.2}$$

where if appropriate the operators are interpreted in the sense of their analytic continuations.

Proof. The result follows immediately from Corollary 8.5 which states in effect that the operators are formally self-adjoint.

Theorem 9.1 contains many classical results as special cases. For instance we give the following corollary.

COROLLARY. 9.2. *Let* $1<p\leq 2$, $\operatorname{Re}\alpha>0$, $\operatorname{Re}(\nu+\mu)>-\frac{3}{2}+p^{-1}$, $\operatorname{Re}(\nu-\mu)>-\frac{1}{2}-p^{-1}$. *Then, for* $g\in L^p_\mu$,

$$I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu g=H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac{1}{4},\alpha}g=K_{x^2}^{\nu/2+\frac{1}{4},\alpha}H_{\nu+2\alpha}g=H_\nu K_{x^2}^{\nu/2+\frac{1}{4},\alpha}g$$

Proof. Using Rooney's result quoted in Section 4 together with a simple extension of [6, Theorem 2], we see that, under the given conditions, all four expressions define functions in $L^p_{2/p-1-\mu}$ and therefore generate elements in $F'_{q,-2/p+1+\mu}$. Using the notation introduced in (7.1) we have

$$\begin{aligned}\tau I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu g&=I_{x^2}^{\nu/2-\frac{1}{4},\alpha}\tau H_\nu g\quad\text{since}\quad \operatorname{Re}(\nu-\mu)>-\tfrac{1}{2}-p^{-1},\quad \operatorname{Re}\alpha>0\\&=I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu\tau g\quad\text{(see Note 2 following Definition 7.1)}\\&=H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac{1}{4},\alpha}\tau g\quad\text{(by Theorem 9.1 applied to }\tau g\in F'_{q,-\mu})\\&=H_{\nu+2\alpha}\tau I_{x^2}^{\nu/2-\frac{1}{4},\alpha}g=\tau H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac{1}{4},\alpha}g\quad\text{as before}\end{aligned}$$

Hence $I_{x^2}^{\nu/2-\frac{1}{4},\alpha}H_\nu g=H_{\nu+2\alpha}I_{x^2}^{\nu/2-\frac{1}{4},\alpha}g$ (a.e.). The other equalities are proved similarly.

The analogues of Theorem 8.4 and 8.6 are obvious and we state them without proof.

THEOREM 9.3. *If* $\nu\in\Omega_{q,-\mu}$ *and* $\nu+\gamma\in\Omega_{p,\mu}$ *then, for* $f\in F'_{p,\mu}$,

$$H_{\nu+\gamma}H_\nu f=K_{x^2}^{(\nu+\gamma)/2+\frac{1}{4},-\gamma/2}I_{x^2}^{\nu/2-\frac{1}{4},\gamma/2}f=I_{x^2}^{\nu/2-\frac{1}{4},\gamma/2}K_{x^2}^{(\nu+\gamma)/2+\frac{1}{4},-\gamma/2}f$$

THEOREM 9.4. *If* $\alpha\in\Omega_{q,-\mu}$, $\beta\in\Omega_{p,\mu}$ *and* $\gamma\in\Omega_{q,-\mu}$, *then, for* $f\in F'_{p,\mu}$,

$$H_\alpha H_\beta H_\gamma f=H_\gamma H_\beta H_\alpha f.$$

We mention that various classical results can be deduced from Theorem 9.4 by using the method of proof of Corollary 9.2.

Finally we state the results for the modified Hankel operators $S^{\eta,\alpha}$ on $F'_{p,\mu}$.

THEOREM 9.5. *Let* $f\in F'_{p,\mu}$. *Then*

(i) $I_{x^2}^{\eta+\alpha,\beta}S^{\eta,\alpha}f=S^{\eta,\alpha+\beta}f$ *if* $\eta\in A_{q,-\mu}$, $\eta+\alpha\in A'_{q,-\mu}$.

(ii) $K_{x^2}^{\eta,\alpha}S^{\eta+\alpha,\beta}f=S^{\eta,\alpha+\beta}f$ *if* $\eta\in A_{q,-\mu}$, $\eta+\alpha\in A_{q,-\mu}$.

(iii) $S^{\eta+\alpha,\beta}S^{\eta,\alpha}f=I_{x^2}^{\eta,\alpha+\beta}f$ *if* $\eta\in A_{q,-\mu}$, $\eta+\alpha\in A'_{q,-\mu}$.

(iv) $S^{\eta+\alpha,\beta}I_{x^2}^{\eta,\alpha}f=S^{\eta,\alpha+\beta}f$ *if* $\eta\in A_{q,-\mu}$, $\eta+\alpha\in A_{q,-\mu}$.

(v) $S^{\eta,\alpha}K_{x^2}^{\eta+\alpha,\beta}f=S^{\eta,\alpha+\beta}f$ *if* $\eta\in A_{q,-\mu}$, $\eta+\alpha\in A'_{q,-\mu}$.

(vi) $S^{\eta,\alpha}S^{\eta+\alpha,\beta}f=K_{x^2}^{\eta,\alpha+\beta}f$ *if* $\eta\in A'_{q,-\mu}$, $\eta+\alpha\in A_{q,-\mu}$.

Furthermore, the right-hand sides provide analytic continuations of the corresponding left-hand sides for $\eta\in A_{q,-\mu}$ *for* (i)–(v) *and* $\eta\in A'_{q,-\mu}$ *for* (vi).

Proof. We give a quick proof of (i); the others are similar.

Using [**12**, (3.17)] and (7.5) above we have, for $f \in F'_{p,\mu}$ and $\phi \in F_{p,2/p-\mu}$,

$$(I_{x^2}^{\eta+\alpha,\beta} S^{\eta,\alpha} f, \phi) = (f, x S^{\eta,\alpha} x^{-1} K_{x^2}^{\eta+\alpha+\frac{1}{2},\beta} \phi)$$
$$= (f, x S^{\eta,\alpha} K_{x^2}^{\eta+\alpha,\beta} y^{-1} \phi).$$

Now the right-hand side is meaningful provided $\eta + \alpha \in A'_{p,2/p-1-\mu}$ and $\eta \in A_{p,2/p-1-\mu}$ or equivalently $\eta + \alpha \in A'_{q,-\mu}$, $\eta \in A_{q,-\mu}$. Applying Theorem 8.3(v) with p, μ and ϕ replaced by q, $-\mu$ and $x^{-1}\phi$ respectively

$$(I_{x^2}^{\eta+\alpha,\beta} S^{\eta,\alpha} f, \phi) = (f, x S^{\eta,\alpha+\beta} x^{-1} \phi) = (S^{\eta,\alpha+\beta} f, \phi)$$

using (7.5) again. The result follows.

We hope in a future paper to apply these results to the solution of certain dual or triple integral equations involving elements of $F'_{p,\mu}$, which are generalizations of classical equations whose solutions can be obtained using the operators above; [see, for instance, **19**].

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(*Issued* 12 *September* 1978)

2

FRACTIONAL POWERS OF OPERATORS VIA THE MELLIN TRANSFORM

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Paper 6

Solution of dual integral equations of Titchmarsh type using generalised functions*

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(MS received 5 September 1978. Revised MS received 11 November 1978. Read 30 April 1979)

SYNOPSIS

The theories of fractional calculus and of the Hankel transform developed in [**5**, **6** and **8**] for the spaces $F'_{p,\mu}$ of generalised functions are used to study distributional analogues of dual integral equations of Titchmarsh type. These are shown to have infinitely many solutions in $F'_{p,\mu}$ under very general conditions on the parameters involved. These results are used to study the corresponding classical problem in weighted L^p spaces. Existence and uniqueness of classical solutions are investigated and examples given of both uniqueness and non-uniqueness for the classical problem.

§1

The mathematical formulation of many problems in mathematical physics produces a pair of dual integral equations of the form

$$\begin{aligned} S^{\nu_1/2-\alpha_1,2\alpha_1}f &= g_1 \quad \text{on} \quad (0,1) \\ S^{\nu_2/2-\alpha_2,2\alpha_2}f &= g_2 \quad \text{on} \quad (1,\infty). \end{aligned} \tag{1.1}$$

Here g_1, g_2 are known functions on $(0, 1)$ and $(1, \infty)$ respectively, ν_1, ν_2, α_1, α_2 are complex numbers and $S^{\eta,\alpha}$ is an operator defined, for appropriate complex numbers η and α and appropriate functions f (defined almost everywhere on $(0,\infty)$), by

$$S^{\eta,\alpha}f(x) = 2^{\alpha}x^{-\alpha}\int_0^{\infty} t^{1-\alpha}J_{2\eta+\alpha}(xt)f(t)\,dt \quad (0<x<\infty). \tag{1.2}$$

These so-called dual integral equations of Titchmarsh type are discussed fully, within a classical framework, in [**11**].

The basic questions connected with (1.1) are those of existence and uniqueness, that is, whether a function f, defined almost everywhere on $(0,\infty)$ and satisfying (1.1), exists and, if so, whether it is unique. The purpose of the present paper is to discuss these basic questions by examining an analogue of (1.1) for certain classes of generalised functions. Such an exercise seems justified on at least three counts.

(i) It is conceivable that singularities could arise at the point $x = 1$ and that generalised functions are required to handle rigorously the δ-type distributions which could then occur.

(ii) A space of testing-functions, all of which are infinitely differentiable, provides a setting in which all the formal analysis associated with (1.1) can be made rigorous without much difficulty.

* This paper was assisted in publication by a grant from the Carnegie Trust for the Universities of Scotland.

(iii) A study of the operator $S^{\eta,\alpha}$ relative to, say, weighted L^p spaces, leads to difficulties concerning its range which can be overcome to some extent by imbedding the weighted L^p spaces in appropriate spaces of generalised functions and extending $S^{\eta,\alpha}$ suitably.

A similar investigation to ours has been carried out recently by Walton [**12**]. Hence some justification for our work is required. In [**12**], Walton uses the spaces $\mathcal{H}'_\nu$ introduced by Zemanian in [**13**]. However, as we mentioned in [**8**], the properties of even simple differential or integral operators are not easily described in full relative to $\mathcal{H}'_\nu$; only certain special combinations of operators can be inverted with precision. By contrast, we have available in [**5** and **6**] a complete fractional calculus for our spaces $F'_{p,\mu}$ and in [**8**] a complete description of the behaviour of $S^{\eta,\alpha}$ relative to these spaces. This allows us to avoid some of the weighty analysis in [**12**] involving Bessel functions and to proceed, instead, along the lines of [**11**, §4.2]. Furthermore, we are able to study (1.1) under much more general conditions on the parameters ν_1, ν_2, α_1 and α_2 than does Walton because of the extension processes carried out in [**6** and **8**].

Mention must also be made of another similar investigation carried out by Braaksma and Schuitman [**1**]. They study certain dual integral equations in spaces closely related to the case $p=\infty$ in our theory and extend the operators involved to ranges of the parameters analogous to those in [**6** and **8**]. However, our theory deals with all values of p in the range $1\leqq p\leqq\infty$ whereas the spaces in [**1**] do not seem suitable to the development of a theory in weighted L^p spaces for $p<\infty$. Our theory can thus be regarded as an extension of that in [**1**] which also contains some results on regularity similar to those of Walton.

To get a precise formulation of our problem, it is convenient to recall the definition of the spaces $F'_{p,\mu}$ here. For $1\leqq p<\infty$,

$$F_p=\{\phi\in C^\infty(0,\infty)\colon x^k\, d^k\phi/dx^k\in L^p(0,\infty) \text{ for } k=0,1,2,\ldots\}$$

while

$$F_\infty=\{\phi\in C^\infty(0,\infty)\colon x^k\, d^k\phi/dx^k\to 0 \text{ as } x\to 0+ \text{ and as } x\to\infty \text{ for } k=0,1,2,\ldots\}.$$

For $1\leqq p\leqq\infty$ and any complex number μ,

$$F_{p,\mu}=\{\phi\colon x^{-\mu}\phi(x)\in F_p\}.$$

$F_{p,\mu}$ is given the topology generated by the semi-norms $\gamma_k^{p,\mu}$ $(k=0,1,2,\ldots)$ defined by

$$\gamma_k^{p,\mu}(\phi)=\|x^k\, d^k/dx^k(x^{-\mu}\phi)\|_p \quad (\phi\in F_{p,\mu}).$$

Finally, $F'_{p,\mu}$ is the space of continuous linear functionals on $F_{p,\mu}$, equipped with the topology of pointwise convergence.

As in [**8**, formula (6.4)], we define the set $A_{p,\mu}$ of complex numbers by

$$A_{p,\mu}=\{\eta\colon \operatorname{Re}(2\eta+\mu)+2\neq 1/p-2l \quad (l=0,1,2,\ldots)\}.$$

Then from [**8**, Theorem 7.3] we see that, if $\eta\in A_{q,-\mu}(^1/p+{}^1/q=1)$ and α is complex, then an operator $S^{\eta,\alpha}$ can be defined on $F'_{p,\mu}$, which is an extension of $S^{\eta,\alpha}$ as defined by (1.2); further, $S^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,2/p-\mu}$. With this in mind, we arrive at the following problem.

PROBLEM 1.1. Let $g_1, g_2 \in F'_{p,2/p-\mu}$ and let ν_1, ν_2, α_1 and α_2 be suitably restricted complex numbers. Find $f \in F'_{p,\mu}$ such that

$$S^{\nu_1/2-\alpha_1,2\alpha_1}f = g_1 \quad \text{on } (0,1)$$

$$S^{\nu_2/2-\alpha_2,2\alpha_2}f = g_2 \quad \text{on } (1,\infty)$$

in the sense of distributions.

This means that, for all $\phi_1, \phi_2 \in F_{p,2/p-\mu}$ with supports in $(0, 1)$, $(1, \infty)$ respectively,

$$(S^{\nu_1/2-\alpha_1,2\alpha_1}f, \phi_1) = (g_1, \phi_1)$$

$$(S^{\nu_2/2-\alpha_2,2\alpha_2}f, \phi_2) = (g_2, \phi_2).$$

In view of the above motivation, we will obviously require $\nu_i/2-\alpha_i \in A_{q,-\mu}$ for $i=1,2$ if the problem is to be meaningful. The other conditions which are relevant appear naturally in due course.

The plan of campaign is as follows. In §2, we deal with a small technical lemma which is required in §3 to show the existence of a solution of Problem 1.1 under very general conditions on the parameters. The uniqueness investigation turns out to be much more interesting than existence and in §4, we find that Problem 1.1 always has infinitely many solutions $f \in F'_{p,\mu}$ under the conditions of §3. However, in §5, when we apply our results to obtain information about the corresponding classical problem, as given by (1.1) and (1.2), we discover that we sometimes have a unique classical solution and sometimes infinitely many classical solutions. We deal in detail with the simplest case and give specific examples of uniqueness and non-uniqueness. Finally, in §6, we describe briefly what happens in a more complicated situation when the values of the parameters are such that one or more of the operators $S^{\eta,\alpha}$ occurring can no longer be defined by the simple integral (1.2) and must be interpreted in an extended sense. We do not go into any great detail but give two examples which indicate how the theory can be applied to the electrified disc [**11**, Ch. 3] and a crack problem in elasticity [**11**, §4.5].

Throughout this paper we will make frequent reference to [**6** and **8**] and will use their notation. In particular, unless the contrary is stated, we assume that $1 \leqq p \leqq \infty$ and that μ is any complex number. For such p and μ,

$$L^p_\mu = \{f\colon x^{-\mu}f(x) \in L^p(0,\infty)\}.$$

p and q will always be connected by the relation $1/p+1/q=1$ and any function $f \in L^q_{-\mu}$ generates a regular functional $\tau f \in F'_{p,\mu}$ by means of the prescription

$$(\tau f, \phi) = \int_0^\infty f(x)\phi(x)\,dx. \tag{1.3}$$

Occasionally we shall write

$$\tau f = \tilde{f}. \tag{1.4}$$

§2

At an early stage in our investigation we are faced with the following technicality.

PROBLEM 2.1. For $1 \leqq p \leqq \infty$ and for any complex number μ, let $f_1, f_2 \in F'_{p,\mu}$. Does there exist $f \in F'_{p,\mu}$ such that (in the sense of distributions)

$$f = f_1 \quad \text{on } (0, 1); \quad f = f_2 \quad \text{on } (1, \infty)?$$

The difficulty for us is that standard results such as those in [4, p. 144 or 10, p. 27] cannot be applied as they stand since $(0, \infty) \neq (0, 1) \cup (1, \infty)$. Instead we fall back on a structure theorem for $F'_{p,\mu}$ which we recall briefly.

THEOREM 2.2. *Let $1 \leqq p \leqq \infty$ and let μ be any complex number. Then any $g \in F'_{p,\mu}$ can be written in the form*

$$g = \sum_{k=0}^{n} \delta^k \tilde{g}_k$$

where n is a non-negative integer, $g_k \in L^q_{-\mu}$ $(k = 0, 1, \ldots, n)$, $\tilde{g}_k$ is defined via (1.3) *and* (1.4) *and δ is interpreted in the sense of* [**5**, *formula* (2.14)].

Proof. This can be deduced from results in [**10**, pp. 199–201] by using the homeomorphism $T_{p,\mu}$ defined in [**7**, Lemma 2.2]. The details are routine and are omitted. (See also [**5**].)

We can now deal easily with Problem 2.1.

THEOREM 2.3. *Problem* 2.1 *has infinitely many solutions $f \in F'_{p,\mu}$, any two of which differ by a distribution of the form*

$$\sum_{k=0}^{m} a_k \, \delta^k \, \delta_1$$

where $a_0, a_1, \ldots, a_m$ are constants and $\delta_1 \in F'_{p,\mu}$ is defined by

$$(\delta_1, \phi) = \phi(1) \quad (\phi \in F_{p,\mu}). \tag{2.1}$$

Proof. Let f_1, f_2 be as in Problem 2.1. By Theorem 2.2, for some positive integer n,

$$f_i = \sum_{k=0}^{n} \delta^k \tilde{f}_k^{(i)}$$

where $f_k^{(i)} \in L^q_{-\mu}$ $(i = 1, 2;\ k = 0, 1, \ldots, n)$. Then a solution of Problem 2.1 is given by

$$f = \sum_{k=0}^{n} \delta^k \tilde{f}_k$$

where, for $k = 0, 1, \ldots, n$, $f_k \in L^q_{-\mu}$ is such that almost everywhere

$$f_k(x) = \begin{cases} f_k^{(1)}(x) & 0 < x \leqq 1 \\ f_k^{(2)}(x) & x > 1. \end{cases}$$

The proof is completed by using [10, p. 100] and a little algebra.

§3

It is convenient at this stage to introduce four non-negative integers k_1, k_2, l_1, l_2 defined relative to $\nu_1, \nu_2, \alpha_1, \alpha_2$ as follows.

DEFINITION 3.1. (i) Let $\nu_i/2+\alpha_i \in A'_{q,-\mu}$ $(i=1,2)$. If $\mathrm{Re}\,(\nu_i+2\alpha_i+\mu)>-{}^1/q$, then $k_i=0$. Otherwise, k_i is the unique positive integer such that

$$\mathrm{Re}\,(\nu_i+2\alpha_i+\mu)+2(k_i-1)<-{}^1/q<\mathrm{Re}\,(\nu_i+2\alpha_i+\mu)+2k_i.$$

(ii) Let $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i=1,2)$. If $\mathrm{Re}\,(\nu_i-2\alpha_i-\mu)+2>{}^1/q$, then $1_i=0$. Otherwise, 1_i is the unique positive integer such that

$$\mathrm{Re}\,(\nu_i-2\alpha_i-\mu)+21_i<{}^1/q<\mathrm{Re}\,(\nu_i-2\alpha_i-\mu)+21_i+2.$$

Remark 3.2. The sets $A_{q,-\mu}$ and $A'_{q,-\mu}$ are defined as in [**8**, formulae (6.4) and (6.5)]. The restrictions $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ arise naturally from the operators involved. The integers k_2, 1_2 play a role in our initial existence theorem while k_1, 1_1 are needed in the uniqueness investigation in the next section. The integers k_1, k_2, 1_1 and 1_2 retain the above meaning for the rest of the paper. Similarly, for the rest of the paper, λ will be the complex number

$$\lambda=(\nu_1+\nu_2)/2-(\alpha_1-\alpha_2) \tag{3.1}$$

with ν_1, ν_2, α_1, α_2 as in Problem 1.1. (3.1) is suggested by the theory in [**11**, §4.2] and, indeed, our proofs are essentially generalisations of these results.

Equipped with the above notation, we are now ready to prove the following existence theorem.

THEOREM 3.3. *Let $g_i \in F'_{p,2/p-\mu}$, $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ for $i=1,2$. Then Problem 1.1 has a solution $f \in F'_{p,\mu}$.*

Proof. By using the hypotheses and the facts that $A'_{q,-\mu}=A_{q,\mu-2/p}$, $A_{q,-\mu}=A'_{q,\mu-2/p}$, we can deduce from [**6**, Theorems 3.9 and 3.10] that

$$I_{x^2}^{\nu_1/2+\alpha_1,\lambda+k_2-\nu_1}K_{x^2}^{\nu_2/2-\alpha_2,1_2}g_1,\; K_{x^2}^{\nu_1/2-\alpha_1,\nu_2-\lambda+1_2}I_{x^2}^{\nu_2/2+\alpha_2,k_2}g_2$$

exist as elements of $F'_{p,2/p-\mu}$. Hence, by Theorem 2.3, there exists a functional $h \in F'_{p,2/p-\mu}$ such that

$$h=\begin{cases} I_{x^2}^{\nu_1/2+\alpha_1,\lambda+k_2-\nu_1}K_{x^2}^{\nu_2/2-\alpha_2,1_2}g_1 & \text{on } (0,1)\\ K_{x^2}^{\nu_1/2-\alpha_1,\nu_2-\lambda+1_2}I_{x^2}^{\nu_2/2+\alpha_2,k_2}g_2 & \text{on } (1,\infty). \end{cases} \tag{3.2}$$

Indeed, we saw that there are infinitely many such functionals h but any one will do here. Next, we define H by

$$H=K_{x^2}^{\nu_2/2-\alpha_2+1_2,-1_2}h. \tag{3.3}$$

Since $\mathrm{Re}\,(\nu_2-2\alpha_2+21_2-1+2/p-\mu)+2>{}^1/p$, $H \in F'_{p,2/p-\mu}$ also. Finally, we define f by

$$f=S^{\nu_2/2+\alpha_2+k_2,\nu_1-2\alpha_1-\lambda-k_2}H. \tag{3.4}$$

By [**8**, Theorem 7.3], $f \in F'_{p,\mu}$. We will show that f is a solution of Problem 1.1. By [**6**, Theorem 3.14 and **8**, Theorem 9.5],

$$\begin{aligned} S^{\nu_1/2-\alpha_1,2\alpha_1}f &= I_{x^2}^{\lambda-\nu_1/2+\alpha_1,\nu_1-\lambda}S^{\nu_1/2-\alpha_1,\lambda-\nu_1+2\alpha_1}f\\ &= I_{x^2}^{\lambda-\nu_1/2+\alpha_1,\nu_1-\lambda}I_{x^2}^{\lambda-\nu_1/2+\alpha_1+k_2,-k_2}H\\ &= I_{x^2}^{\lambda-\nu_1/2+\alpha_1+k_2,\nu_1-\lambda-k_2}K_{x^2}^{\nu_2/2-\alpha_2+1_2,-1_2}h. \end{aligned}$$

If $\phi \in F_{p,2/p-\mu}$ has support in $(0, 1)$, then since $\operatorname{Re}(\nu_2+2\alpha_2+\mu)+2k_2 > -1/q$, so does $I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,-1_2} K_{x^2}^{\lambda-\nu_1/2+\alpha_1+k_2+1/2,\nu_1-\lambda-k_2}\phi$ (and therefore the K operator is defined as in [**5**]). Thus, for such functions ϕ,

$$\begin{aligned}(S^{\nu_1/2-\alpha_1,2\alpha_1}f, \phi) &= (h, I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,-1_2} K_{x^2}^{\lambda-\nu_1/2+\alpha_1+k_2+1/2,\nu_1-\lambda-k_2}\phi)\\ &= (I_{x^2}^{\nu_1/2+\alpha_1,\lambda+k_2-\nu_1} K_{x^2}^{\nu_2/2-\alpha_2,1_2} g_1, I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,-1_2}\\ &\quad \times K_{x^2}^{\lambda-\nu_1/2+\alpha_1+k_2+1/2,\nu_1-\lambda-k_2}\phi)\\ &= (I_{x^2}^{\lambda-\nu_1/2+\alpha_1+k_2,\nu_1-\lambda-k_2} I_{x^2}^{\nu_1/2+\alpha_1,\lambda+k_2-\nu_1} K_{x^2}^{\nu_2/2-\alpha_2+1_2,-1_2}\\ &\quad \times K_{x^2}^{\nu_2/2-\alpha_2,1_2} g_1, \phi)\\ &= (g_1, \phi)\end{aligned}$$

where we have used [**6**, Theorems 3.9, 3.10 and 3.13] and (3.2) above. Thus

$$S^{\nu_1/2-\alpha_1,2\alpha_1}f = g_1 \quad \text{on } (0, 1). \tag{3.5}$$

Similarly, we have

$$\begin{aligned}S^{\nu_2/2-\alpha_2,2\alpha_2}f &= K_{x^2}^{\nu_2/2-\alpha_2,\lambda-\nu_2} S^{\lambda-\nu_2/2-\alpha_2,2\alpha_2+\nu_2-\lambda}f\\ &= K_{x^2}^{\nu_2/2-\alpha_2,\lambda-\nu_2} I_{x^2}^{\nu_2/2+\alpha_2+k_2,-k_2} H\\ &= K_{x^2}^{\nu_2/2-\alpha_2+1_2,\lambda-\nu_2-1_2} I_{x^2}^{\nu_2/2+\alpha_2+k_2,-k_2} h\end{aligned}$$

using [**8**, Theorem 3.13]. If $\phi \in F_{p,2/p-\mu}$ has support in $(1, \infty)$, then so does $K_{x^2}^{\nu_2/2+\alpha_2+k_2+1/2,-k_2} I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,\lambda-\nu_2-1_2}\phi$ because $\operatorname{Re}(\nu_2-2\alpha_2-\mu)+21_2+2 > 1/q$ and the I operator can be defined as in [**5**]. Thus, for such functions ϕ,

$$\begin{aligned}(S^{\nu_2/2-\alpha_2,2\alpha_2}f, \phi) &= (h, K_{x^2}^{\nu_2/2+\alpha_2+k_2+1/2,-k_2} I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,\lambda-\nu_2-1_2}\phi)\\ &= (K_{x^2}^{\nu_1/2-\alpha_1,\nu_2-\lambda+1_2} I_{x^2}^{\nu_2/2+\alpha_2,k_2} g_2, K_{x^2}^{\nu_2/2+\alpha_2+k_2+1/2,-k_2}\\ &\quad \times I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,\lambda-\nu_2-1_2}\phi)\\ &= (g_2, \phi)\end{aligned}$$

as in the previous case. Hence

$$S^{\nu_2/2-\alpha_2,2\alpha_2}f = g_2 \quad \text{on } (1, \infty). \tag{3.6}$$

In view of (3.5) and (3.6), this completes the proof.

Remark 3.4. (i) A careful examination of the proof shows the need for the various restrictions on the parameters. Also k_2 and 1_2 play a vital role.

(ii) The fact that we chose any admissible h and found a solution suggests that we have non-uniqueness. This is indeed the case as we shall see in the next section.

§4

We now turn to the question of uniqueness for Problem 1.1. To answer this question it is clearly necessary to decide whether there exists a non-zero element $f \in F'_{p,\mu}$ satisfying the homogeneous Problem 1.1.

$$\begin{aligned}S^{\nu_1/2-\alpha_1,2\alpha_1}f &= 0 \quad \text{on } (0, 1)\\ S^{\nu_2/2-\alpha_2,2\alpha_2}f &= 0 \quad \text{on } (1, \infty).\end{aligned} \tag{4.1}$$

We will show that under the hypotheses of Theorem 3.3, such an f exists and hence Problem 1.1 will have infinitely many solutions $f \in F'_{p,\mu}$.

As a first step we prove the following lemma.

LEMMA 4.1. *If* $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i=1,2)$ *and* $f \in F'_{p,\mu}$ *satisfies* (4.1) *then* f *must be of the form*

$$f = S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2+1_1}K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1}\left(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1\right) \tag{4.2}$$

where r *is a non-negative integer,* $a_0, \ldots, a_r$ *are constants,* δ' *is defined by* [**5**, *formula* (2.15)] *and* δ_1 *by* (2.1) *above.*

Proof. First we note that by [**6**, Definition 3.3], $K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1}$ is invertible on $F'_{p,2/p-\mu}$ and $(K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1} = I_{x^2}^{\nu_1/2+\alpha_1+k_1,-k_1}$. Hence by [**6**, Theorems 3.13 and 3.14 and **8**, Theorem 9.5],

$$\begin{aligned}(K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1}S^{\nu_1/2-\alpha_1+1_1,\lambda-\nu_1+2\alpha_1-1_1}f \\ &= I_{x^2}^{\nu_1/2+\alpha_1+k_1,-k_1}I_{x^2}^{\nu_1/2+\alpha_1,\lambda-\nu_1}S^{\nu_1/2-\alpha_1+1_1,2\alpha_1-1_1}f \\ &= I_{x^2}^{\nu_1/2+\alpha_1+k_1,\lambda-\nu_1-k_1}K_{x^2}^{\nu_1/2-\alpha_1+1_1,-1_1}S^{\nu_1/2-\alpha_1,2\alpha_1}f.\end{aligned}$$

If $\phi \in F_{p,2/p-\mu}$ has support in $(0,1)$, then, since $\operatorname{Re}(\nu_1+2\alpha_1+\mu)+2k_1 > -{}^1/q$, so does $I_{x^2}^{\nu_1/2-\alpha_1+1_1-1/2,-1_1}K_{x^2}^{\nu_1/2+\alpha_1+k_1+1/2,\lambda-\nu_1-k_1}\phi$. Hence, for such ϕ,

$$\begin{aligned}((K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1}S^{\nu_1/2-\alpha_1+1_1,\lambda-\nu_1+2\alpha_1-1_1}f,\ \phi) \\ &= (S^{\nu_1/2-\alpha_1,2\alpha_1}f,\ I_{x^2}^{\nu_1/2-\alpha_1+1_1-1/2,-1_1}K_{x^2}^{\nu_1/2+\alpha_1+k_1+1/2,\lambda-\nu_1-k_1}\phi) \\ &= 0 \quad \text{by (4.1)}.\end{aligned}$$

On the other hand, we may write

$$\begin{aligned}(K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1}S^{\nu_1/2-\alpha_1+1_1,\lambda-\nu_1+2\alpha_1-1_1}f \\ &= I_{x^2}^{\nu_1/2+\alpha_1+k_1,-k_1}K_{x^2}^{\nu_1/2-\alpha_1+1_1,\nu_2-\lambda-1_1}S^{\nu_2/2-\alpha_2,2\alpha_2}f\end{aligned}$$

and by a similar argument we can show that, since $\operatorname{Re}(\nu_1-2\alpha_1-\mu)+21_1+2 > {}^1/q$,

$$((K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1}S^{\nu_1/2-\alpha_1+1_1,\lambda-\nu_1+2\alpha_1-1_1}f,\ \phi) = 0$$

for all $\phi \in F_{p,2/p-\mu}$ with support contained in $(1,\infty)$. Hence, by [**10**, p. 100]

$$(K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1}S^{\nu_1/2-\alpha_1+1_1,\lambda-\nu_1+2\alpha_1-1_1}f = \sum_{s=0}^{r} b_s D^s\,\delta_1 \tag{4.3}$$

where $b_0, \ldots, b_r$ are constants and D denotes generalised differentiation. However, proceeding formally, we see that $D^s\,\delta_1$ is a linear combination of $\delta_1, (\delta')\,\delta_1, \ldots, (\delta')^s\,\delta_1$ and conversely $(\delta')^s\,\delta_1$ is a linear combination of $\delta_1, D\,\delta_1, \ldots, D^s\,\delta_1$ $(s=0,1,\ldots,r)$. Hence (4.3) can be written in the form

$$(K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1})^{-1}S^{\nu_1/2-\alpha_1+1_1,\lambda-\nu_1+2\alpha_1-1_1}f = \sum_{s=0}^{r} a_s(\delta')^s\,\delta_1 \tag{4.4}$$

where $a_0, \ldots, a_r$ are constants. The result then follows from (4.4) on using [**8**, Theorem 7.3].

Remark 4.2. (i) We observe that this time it is the integers k_1 and l_1 which play the vital role.

(ii) (4.4) is more convenient than (4.3) since in each term $((\delta')^s\,\delta_1, \phi)$, where $\phi \in F_{p,2/p-\mu}$, we may regard δ_1 as an element of $F'_{p,2/p-\mu}$ while in the term $(D^s\delta_1, \phi)$ we would have to regard δ_1 as an element of the space $F'_{p,2/p-\mu-s}$, which varies as s varies.

Having identified the form of possible solutions of the homogeneous Problem 1.1, we now have to decide which expressions of the form (4.2) do give solutions. As the calculations are a little involved, we will proceed in a number of stages.

LEMMA 4.3. *Let $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ for $i=1, 2$. The following are necessary and sufficient conditions on $a_0, \ldots, a_r$ for f (as defined by* (4.2)*) to satisfy $S^{\nu_1/2-\alpha_1, 2\alpha_1}f=0$ on $(0, 1)$.*

(i) *If $k_1=k_2=0$, r and $a_0, \ldots, a_r$ are arbitrary.*

(ii) *If $k_1=0$, $k_2>0$, $a_0, \ldots, a_r$ are such that*

$$\sum_{s=0}^{r} a_s(-1)^s(\nu_2+2\alpha_2+1+2h_2)^s=0 \quad (h_2=0, 1, \ldots, k_2-1). \tag{4.5}$$

(iii) *If $k_1>0$, $k_2=0$, $a_0, \ldots, a_r$ are such that*

$$\sum_{s=0}^{r} a_s(-1)^s(\nu_1+2\alpha_1+1+2h_1)^s=0 \quad (h_1=0, 1, \ldots, k_1-1). \tag{4.6}$$

(iv) *If $k_1>0$, $k_2>0$, $a_0, \ldots, a_r$ are such that both* (4.5) *and* (4.6) *hold.*

Proof. With f as in (4.2),

$$\begin{aligned} S^{\nu_1/2-\alpha_1, 2\alpha_1}f &= K_x^{\nu_1/2-\alpha_1, l_1} S^{\nu_1/2-\alpha_1+l_1, 2\alpha_1-l_1} S^{\nu_2/2+\alpha_2, \lambda-\nu_2-2\alpha_2+l_1} \\ &\quad \times K_{x^2}^{-\nu_1/2-\alpha_1-k_1, k_1}\Big(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1\Big) \\ &= K_{x^2}^{-\nu_1/2-\alpha_2, l_1} I_{x^2}^{\nu_2/2+\alpha_2, \nu_1-\lambda} K_{x^2}^{-\nu_1/2-\alpha_1-k_1, l_1}\Big(\sum_{x=0}^{r} a_s(\delta')^s\,\delta_1\Big) \\ &= (-1)^{l_1} I_{x^2}^{-\nu\ /2+\alpha_1-l_1, l_1}(-1)^{k_2} I_{x^2}^{\nu_2/2+\alpha_2+k_2, \nu_1-\lambda-k_2} \\ &\quad \times K_{x^2}^{-\nu_2/2-\alpha_2-k_2, k_2} K_{x^2}^{-\nu_1/2-\alpha_1-k_1, k_1}\Big(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1\Big). \end{aligned}$$

Here we have used [6, Definition 3.3 and 8, Theorem 9.5]. Now let $\phi \in F_{p,2/p-\mu}$ have support in $(0, 1)$. Then

$$(S^{\nu_1/2-\alpha_1, 2\alpha_1}f, \phi) = \Big(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1, I_{x^2}^{-\nu_1/2-\alpha_1-k_1-1/2, k_1} I_{x^2}^{-\nu_2/2-\alpha_2-k_2-1/2, k_2}\psi\Big)$$

where

$$\psi = (-1)^{k_2+l_1} K_{x^2}^{\nu_2/2+\alpha_2+k_2+1/2, \nu_1-\lambda-k_2} K_{x^2}^{-\nu_1/2+\alpha_1-l_1+1/2, l_1}\phi.$$

Since $\mathrm{Re}\,(\nu_1-2\alpha_1-\mu)+2l_1<1/q$ and $\mathrm{Re}\,(\nu_2+2\alpha_2+\mu)+2k_2>-1/q$, ψ also has support in $(0, 1)$. There are four possibilities.

(i) $k_1 = k_2 = 0$. In this case

$$(S^{\nu_1/2-\alpha_1,2\alpha_1}f, \phi) = \left(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1, \psi\right) = 0$$

for all choices of r and $a_0, \ldots, a_r$.

(ii) $k_1 = 0$, $k_2 > 0$. In this case $(I_{x^2}^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi)(x) = x^{\nu_2+2\alpha_2+1}P(x^2)$ for $x \geqq 1$ where P is a polynomial of degree at most k_2-1. Since $\operatorname{Re}(\nu_2+2\alpha_2+1+2k_2-2+\mu-{}^2/p) < -{}^1/p$, it follows that every such polynomial can arise from suitable functions $\phi \in F_{p,2/p-\mu}$ with support in $(0, 1)$. By taking $P(t) = t^{h_2}$ $(h_2 = 0, 1, \ldots, k_2-1)$ we see that $S^{\nu_1/2-\alpha_1,2\alpha_1}f = 0$ on $(0, 1)$ if and only if

$$\sum_{s=0}^{r} (a_s(\delta')^s\,\delta_1, x^{\nu_2+2\alpha_2+1+2h_2}) = 0$$

for $h_2 = 0, 1, \ldots, k_2-1$ (where, strictly speaking, in the last bracket we are dealing with a function in $F_{p,2/p-\mu}$ which equals $x^{\nu_2+2\alpha_2+1+2h_2}$ for $x \geqq 1$). (4.5) now follows easily via [5, formula (2.15)].

(iii) $k_1 > 0$, $k_2 = 0$. This case is similar to (ii).

(iv) $k_1 > 0$, $k_2 > 0$. As in (ii), $I_{x^2}^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi(x) = x^{\nu_2+2\alpha_2+1}P(x^2)$ for $x \geqq 1$ where P is a polynomial of degree $\leqq k_2-1$. If $\operatorname{Re}(\nu_1+2\alpha_1)+2k_1 \neq \operatorname{Re}(\nu_2+2\alpha_2)+2k_2$, we can show after a fairly routine calculation that, for $x \geqq 1$,

$$I_{x^2}^{-\nu_1/2-\alpha_1-k_1-1/2,k_1}I_{x^2}^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi(x) = x^{\nu_1+2\alpha_1+1}Q_1(x^2) + x^{\nu_2+2\alpha_2+1}Q_2(x^2)$$

where Q_1 and Q_2 are polynomials of degrees at most k_1-1 and k_2-1 respectively. Then, by proceeding as in (ii), (iii) we find conditions on $a_0, \ldots, a_r$ as stated. In the case $\operatorname{Re}(\nu_1+2\alpha_1)+2k_1 = \operatorname{Re}(\nu_2+2\alpha_2)+2k_2$, there is an added complication and we have to add on an extra term of the form $x^{\nu_1+2\alpha_1+2k_1-1}Q_3({}^1/x^2)\log({}^1/x^2)$ where Q_3 is a polynomial of degree at most $\min(k_1-1, k_2-1)$. However, as $\log 1 = 0$, the problem again reduces to evaluating polynomials at $x = 1$ and the proof goes through. We omit the full details which are rather tedious.

This completes the proof of Lemma 4.3.

Remark 4.4. The conditions in (ii)–(iv) above put constraints on r. For instance in (4.5), we require $a_0, \ldots, a_r$ to be such that the numbers $\nu_2+2\alpha_2+1+2h_2$ $(h_2 = 0, 1, \ldots, k_2-1)$ are solutions of $\sum_{s=0}^{r} a_s(-1)^s z^s = 0$ which requires $r \geqq k_2$. Likewise (4.6) implies that $r \geqq k_1$ while case (iv) produces k_1+k_2 constraints on $a_0, \ldots, a_r$.

The situation as regards $(1, \infty)$ is very similar and the relevant facts are as follows.

LEMMA 4.5. *Let $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ for $i = 1, 2$. The following are necessary and sufficient conditions on $a_0, \ldots, a_r$ for f (as defined by (4.2)) to satisfy $S^{\nu_2/2-\alpha_2,2\alpha_2}f = 0$ on $(1, \infty)$.*

(i) *If $1_1 = 1_2 = 0$, r and $a_0, \ldots, a_r$ are arbitrary.*

(ii) *If $1_1 = 0, 1_2 > 0$,*

$$\sum_{s=0}^{r} a_s(\nu_2-2\alpha_2+1+2j_2)^s = 0 \quad \textit{for} \quad j_2 = 0, 1, \ldots, 1_2-1. \tag{4.7}$$

(iii) *If* $1_1>0, 1_2=0$,

$$\sum_{s=0}^{r} a_s(\nu_1-2\alpha_1+1+2j_1)^s=0 \quad \textit{for} \quad j_1=0, 1, \ldots, 1_1-1. \tag{4.8}$$

(iv) *If* $1_1>0, 1_2>0, a_0, \ldots, a_r$ *such that both* (4.7) *and* (4.8) *hold.*

Proof. The details are entirely similar to those in Lemma 4.3. We merely mention that the crux is to use [**8**, Theorem 9.5] to write

$$(S^{\nu_2/2-\alpha_2,2\alpha_2}f, \phi)=\left(\sum_{s=0}^{r} a_s(\delta')^s\, \delta_1,\, K_{x^2}^{-\nu_2/2+\alpha_2-1_2+1/2,1_2}K_{x^2}^{-\nu_2/2+\alpha_1-1_1+1/2,1_1}\psi\right)$$

where

$$\psi=(-1)^{1_1+1_2}I_{x^2}^{\nu_2/2-\alpha_2+1_2-1/2,\lambda-\nu_2-1_2}I_{x^2}^{-\nu_1/2-\alpha_1-k_1-1/2,k_1}\phi.$$

Comments analogous to Remark 4.4 apply here.

Putting together the results of Lemmas 4.3 and 4.5 leads to

THEOREM 4.6. *Let* $\nu_i/2+\alpha_i\in A'_{q,-\mu}$, $\nu_i/2-\alpha_i\in A_{q,-\mu}$ *for* $i=1,2$. *Then there are infinitely many elements* $f\in F'_{p,\mu}$ *satisfying* (4.1).

Proof. There are 16 cases to consider depending on whether each of k_1, k_2, 1_1 and 1_2 is zero or positive. For instance, if $k_1>0$, $k_2>0$, $1_1>0$, $1_2>0$, f, as given by (4.2), is a solution of (4.1) if and only if

$$\sum_{s=0}^{r}(-1)^s a_s(\nu_1+2\alpha_1+1+2h_1)^s=0 \quad (h_1=0, 1, \ldots, k_1-1)$$

$$\sum_{s=0}^{r}(-1)^s a_s(\nu_2+2\alpha_2+1+2h_2)^s=0 \quad (h_2=0, 1, \ldots, k_2-1)$$

$$\sum_{s=0}^{r} a_s(\nu_1-2\alpha_1+1+2j_1)^s=0 \quad (j_1=0, 1, \ldots; 1_1-1)$$

and

$$\sum_{s=0}^{r} a_s(\nu_2-2\alpha_2+1+2j_2)^s=0 \quad (j_2=0, 1, \ldots, 1_2-1),$$

the first two conditions coming from case (iv) in Lemma 4.3 and the second two from case (iv) in Lemma 4.5. We have a system of $k_1+k_2+1_1+1_2$ homogeneous equations in the variables $a_0, \ldots, a_r$ and the system has infinitely many solutions if r is such that $r+1>k_1+k_2+1_1+1_2$. The other cases are similar and the details are omitted.

As an immediate consequence of Theorem 4.6, we obtain our non-uniqueness theorem for Problem 1.1 which we can state as follows.

THEOREM 4.7. *Let* $g_i\in F'_{p,2/p-\mu}$, $\nu_i/2+\alpha_i\in A'_{q,-\mu}$ *and* $\nu_i/2-\alpha_i\in A_{q,-\mu}$ *for* $i=1,2$. *Then Problem* 1.1 *has infinitely many solutions* $f\in F'_{p,\mu}$, *each of which is of the form*

$$S^{\nu_2/2+\alpha_2+k_2,\nu_1-2\alpha_1-\lambda-k_2}H+S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2+1_1}K_{x^2}^{-\nu_1/2-\alpha_1-k_1,k_1}\left(\sum_{s=0}^{r} a_s(\delta')^s\, \delta_1\right)$$

and

$$\operatorname{Re}(-\nu_1-2\alpha_1-2k_1-\mu)+2>{}^1/q \quad [\text{by } \mathbf{5}, \text{ Lemma 3.1}].$$

The latter holds by definition of k_1 and the former will hold if $k_2=0$ and (5.10) holds. It is fairly clear that there are various possibilities and we shall not go through them all. We content ourselves with just one.

THEOREM 6.1. *Let* $k_1>0$, $k_2=1_1=1_2=0$ *and* $\operatorname{Re}(\nu_2+2\alpha_2+\mu)\geqq {}^1/2-{}^1/q+\operatorname{Re}\lambda$ *and let* (5.3) *hold. Then the homogeneous Problem* 5.1 *has no non-trivial solution in* $L^q_{-\mu}$.

Proof. This is similar to that of Theorem 5.6(ii) and is omitted.

Next suppose that $k_1=1_1=1_2=0$ but $k_2>0$. Again we have to look at

$$(S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}(\delta')^s\,\delta_1,\phi)=((\delta')^s\,\delta_1, xS^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}x^{-1}\phi)$$

for $\phi\in F_{p,\mu}$. Using [**6**, Definition 3.3 and **8**, Theorem 8.3],

$$xS^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}x^{-1}\phi=(-1)^{k_2}xI_{x^2}^{-\nu_2/2-\alpha_2-k_2,k_2}S^{\nu_2/2+\alpha_2+k_2,\lambda-\nu_2-2\alpha_2-k_2}x^{-1}\phi. \tag{6.5}$$

An argument similar to the above then yields as possible candidates

$$f(x)=\sum_{s=0}^{r}c_s x^{-\lambda+\nu_2+2\alpha_2+s}I_{x^2}^{\rho_s,k_2}(J_{\lambda-s}(x)) \tag{6.6}$$

where $\rho_s=-2k_2-\lambda+s$ and now we are back to the previous situation.

If $1_1=1_2=0$, and $k_1>0$, $k_2>0$, possible solutions in $L^q_{-\mu}$ of the homogeneous Problem 5.1 would take the form

$$f(x)=\sum_{s=0}^{r}c_s x^{-\lambda+\nu_2+2\alpha_2+s}I_{x^2}^{\gamma_s,k_1}I_{x^2}^{\rho_s,k_2}J_{\lambda-s}(x) \tag{6.7}$$

where $c_0,\ldots,c_r$ are suitably restricted constants and γ_s, ρ_s are similar to the above but again we will not pursue the matter.

Remark 6.2. Expressions such as (6.4) and (6.6) can be written in terms of ${}_1F_2$ hypergeometric functions. This follows from [**3**, p. 195, formula (65)] and a simple change of variable. Similarly (6.7) can be written in terms of hypergeometric functions since the result of applying an Erdélyi–Kober operator to a ${}_1F_2$ function is a ${}_2F_3$ function under appropriate conditions [**3**, p. 200, formula (95)].

We mentioned after Example 5.10 that we would consider some applications to physical problems and it is in connection with these that the cases $1_1>0$ or $1_2>0$ (or both) arise fairly naturally.

EXAMPLE 6.3. Let $\nu_1=0$, $\nu_2=0$, $\alpha_1=1$, $\alpha_2={}^1/2$. Then Problem 5.1 arises in connection with the electrostatic potential due to a charged disc [**11,** §3.1]. In this case $\lambda=-{}^1/2$. If we try to treat this problem within the previous framework with $1_1=1_2=0$, we would be given $g_i\in L^q_{2/q-2+\mu}$ $(i=1,2)$ and would seek $f\in L^q_{-\mu}$ where, by (5.3),

$$\max({}^1/p,{}^1/q)\leqq {}^1/q+\operatorname{Re}\mu-{}^1/2+2<{}^3/2$$

and

$$\max({}^1/p, {}^1/q) \leqq {}^1/q + \mathrm{Re}\,\mu - {}^1/2 + 1 < {}^3/2 \Rightarrow -{}^1/2 + \max({}^1/p, {}^1/q) \leqq {}^1/q + \mathrm{Re}\,\mu < 0.$$

However, as $\max({}^1/p, {}^1/q) \geqq {}^1/2$, no such values of q and μ exist. Thus, to study the problem of the electrified disc in $L^q_{-\mu}$ we must use values of q and μ which require $l_1 > 0$, $l_2 > 0$ or both.

The condition $l_i > 0$ $(i = 1, 2)$ means that $\mathrm{Re}(\nu_i - 2\alpha_i - \mu) + 2 < {}^1/q$ and hence the conditions demanded in Lemma 5.2 are not satisfied by $S^{\nu_i/2-\alpha_i, 2\alpha_i}$ on $L^q_{-\mu}$ and we may not interpret the operator in the sense of Rooney's extension.

However we can generalise our operators in the manner suggested by (6.5). Namely, if $l_i > 0$ $(i = 1, 2)$, define $S^{\nu_i/2-\alpha_i, 2\alpha_i}$ on $L^q_{-\mu}$ by

$$S^{\nu_i/2-\alpha_i, 2\alpha_i} f = (-1)^{l_i} I_{x^2}^{-\nu_i/2+\alpha_i-l_i, l_i} S^{\nu_i/2-\alpha_i+l_i, 2\alpha_i-l_i} f. \tag{6.8}$$

By Lemma 5.2 and [**5**, Lemma 3.1], the right-hand side defines a continuous linear mapping of $L^q_{-\mu}$ into $L^q_{2/q-2+\mu}$ provided that $1 < p < \infty$,

$$\max({}^1/p, {}^1/q) \leqq {}^1/q + \mathrm{Re}(\mu - {}^1/2 + 2\alpha_i - l_i) < \mathrm{Re}\,\nu_i + l_i + {}^3/2$$

and $\mathrm{Re}(-\nu_i + 2\alpha_i - 2l_i + {}^2/q - 2 + \mu) + 2 > {}^1/q$, the last two restrictions being automatically satisfied by choice of l_i. Further, using appropriate parts of [**8**, Theorem 8.3] it can be shown that (5.4) still holds for this more general operator. We may refer to (6.8) as the extension of $S^{\nu_i/2-\alpha_i, 2\alpha_i}$ in the sense that, for spaces $L^q_{-\mu}$ with $\mathrm{Re}(\nu_i - 2\alpha_i - \mu) + 2 > {}^1/q$, $l_i = 0$ and we recover Rooney's extended operator.

Again we will not embark on a major investigation but merely give two examples of what happens in the case of the homogeneous Problem 5.1.

EXAMPLE 6.4. We consider again the problem of the disc introduced in Example 6.3. The homogeneous problem corresponds to the disc being uncharged and we would expect f, which can be regarded as a measure of the potential, to be zero; see [**11**, Ch. 3]. With $\nu_1 = 0$, $\nu_2 = 0$, $\alpha_1 = 1$, $\alpha_2 = {}^1/2$ as before, we now choose $k_1 = k_2 = l_2 = 0$, $l_1 = 1$. To study the problem in $L^q_{-\mu}$ we require values of q and μ such that $(1 < q < \infty)$,

$$\max({}^1/p, {}^1/q) \leqq {}^1/q + \mathrm{Re}\,\mu + {}^1/2 < {}^3/2, \quad \mathrm{Re}(\mu + {}^1/q) > 0$$
$$\Rightarrow \max({}^1/p - {}^1/2, {}^1/q - {}^1/2, 0) < \mathrm{Re}\,\mu + {}^1/q < 1.$$

Such values of μ and q can be found and for these values we have to try to find a non-trivial solution $f \in L^q_{-\mu}$ such that

$$S^{-1,2} f = 0 \quad \text{on } (0, 1); \qquad S^{-1/2,1} f = 0 \quad \text{on } (1, \infty)$$

where $S^{-1,2}$ is interpreted via (6.8) and $S^{-1/2,1}$ as in Lemma 5.2. Since (5.4) still holds, the argument proceeds as before (as $k_1 = k_2 = 0$) and we obtain, from (4.2), possible solutions of the form

$$f(x) = \sum_{s=0}^{r} c_s x^{1/2+s} J_{1/2-s}(x) \tag{6.9}$$

where $c_0, \ldots, c_r$ are restricted according to Lemma 4.5. For the right-hand side of (6.9) to belong to $L^q_{-\mu}$, we need in particular $\mathrm{Re}\,\mu + {}^1/q < -r$, contradicting

Re $\mu+{}^1/q>0$, unless $c_0,\ldots,c_r$ are all zero. Hence, we have no non-trivial solution in $L^q_{-\mu}$ as anticipated. This does not, in itself, rule out the existence of say, a locally integrable solution f. This requires a theory relative to a space of such functions which might not need the extension (6.8) but would have other difficulties; [see **12**]. An existence investigation for the charged disc can be carried out using the above results and ideas similar to those in Theorem 5.11.

EXAMPLE 6.5. For $\nu_1=-{}^1/2$, $\nu_2=-{}^1/2$, $\alpha_1=-{}^1/4$, $\alpha_2={}^1/4$, Problem 5.1 arises in connection with the stress distribution across a Griffith crack; see [**11**, §4.5 and **12**]. Again no spaces $L^q_{-\mu}$ can be used which correspond to $k_1=k_2=1_1=1_2=0$. However if we take $k_1=k_2=1_1=0$ and $1_2=1$, we need values of μ, q such that $\max({}^1/p+1, {}^1/q+1, 1)<\text{Re}\,\mu+{}^1/q<2$, $1<q<\infty$, which can always be achieved. Since $\lambda=0$, (5.9) gives the prospective solution

$$f(x)=\sum_{s=0}^{r} c_s x^s J_{-s}(x) \tag{6.10}$$

of the homogeneous Problem 5.1. However the right-hand side of (6.10) only belongs to $L^q_{-\mu}$ if $c_0=\cdots=c_r=0$ and again there is no non-trivial solution in $L^q_{-\mu}$. Comments similar to those at the end of the previous example apply here also.

Perhaps enough has now been said to justify the use of generalised functions in studying dual equations of Titchmarsh type, not merely for its own sake but also for the light it sheds on existence and uniqueness questions for the corresponding classical equations. It remains to be seen how such methods can be extended to more general dual integral equations or to triple integral equations such as those discussed in [**11**, §6.2].

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(*Issued* 21 *September* 1979)

Paper 7

Offprint from

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

Third Series *Volume XLV* *November 1982*

FRACTIONAL POWERS OF A CLASS OF ORDINARY DIFFERENTIAL OPERATORS

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ADAM C. McBRIDE

[Received 21 November 1980]

1.

In this paper, we shall be concerned with the class of ordinary differential operators of the form

$$L = x^{a_1}Dx^{a_2}Dx^{a_3} \ldots x^{a_n}Dx^{a_{n+1}} \tag{1.1}$$

where n is a positive integer, $a_1, \ldots, a_{n+1}$ are complex numbers, and $D \equiv d/dx$. L will act on classes of functions defined on $(0, \infty)$ so that, in (1.1), $x > 0$ and x^{a_k} will stand for $\exp(a_k \log x)$ where $\log x$ is the principal branch of the logarithm ($k = 1, \ldots, n+1$). For technical reasons, we shall assume that

$$a = \sum_{k=1}^{n+1} a_k \quad \text{is real.} \tag{1.2}$$

(The reason for this assumption is explained in Remark 3.10(i).) Further, we shall write

$$m = |a-n|. \tag{1.3}$$

Many different particular cases of (1.1) turn up in practice and, indeed, the impetus for this investigation was a paper by Sprinkhuizen-Kuyper [**16**] which dealt with the class of operators of the form

$$D^2 + \nu x^{-1}D \tag{1.4}$$

corresponding to the values

$$n = 2, \quad a_1 = -1, \quad a_2 = 2-\nu, \quad a_3 = \nu - 1, \tag{1.5}$$

or, equivalently,

$$n = 2, \quad a_1 = -\nu, \quad a_2 = \nu, \quad a_3 = 0. \tag{1.6}$$

In the case where $a = n$, it is clear that, for any complex number λ, $Lx^\lambda = c_\lambda x^\lambda$ for some constant c_λ depending on λ. It is to be expected, therefore, that if φ belongs to a certain space X of functions, then so does $L\varphi$. The theory of fractional powers of various classes of operators mapping a Banach space X into itself has been extensively developed in, for instance, [**1**, **4**, and **5**]. However, if X is a space of testing-functions and X' the corresponding space of generalized functions, X is not, in general, a Banach space but, rather, a countably multinormed space or the inductive limit of a family of such spaces [**17**, Chapter 2]. In [**6**], Lamb has developed a theory of fractional powers of operators from a Fréchet space X into itself and has extended these results to generalized functions. Therefore, we shall not consider the case where $a = n$, or $m = 0$, any further in this paper.

When $a \neq n$, considerations similar to the above indicate that, if φ belongs to a certain space of functions, $L\varphi$ will, in general, belong to a different space so that the theories mentioned for the case where $m = 0$ do not apply. Nevertheless, and perhaps

Proc. London Math. Soc. (3), 45 (1982), 519–546.

paradoxically, a theory can be developed using more elementary ideas and dispensing with the more abstract apparatus of resolvents, Dunford integrals and so on. The object is to obtain an expression for positive integral powers, L^r, of L which remains meaningful when r is replaced by a general complex number α. The resulting expression can then be taken as the definition of L^α, the αth power of L. There is nothing new in this approach. For instance, in the development of fractional calculus [15], one can take the r-fold repeated integral operator I^r defined by

$$(I^r\varphi)(x) = [\Gamma(r)]^{-1}\int_0^x (x-t)^{r-1}\varphi(t)\,dt$$

as the motivation for the definition of the Riemann–Liouville fractional integral operator I^α defined for $\operatorname{Re}\alpha > 0$ by

$$(I^\alpha\varphi)(x) = [\Gamma(\alpha)]^{-1}\int_0^x (x-t)^{\alpha-1}\varphi(t)\,dt.$$

Indeed, it is fractional calculus which lies at the heart of the matter here. The key step is to obtain an expression for L^r which involves derivatives, not with respect to x but with respect to x^m, where m is given by (1.3). (This is obviously a non-starter when $m = 0$.) We then define L^α in terms of fractional integrals or derivatives with respect to x^m. An extensive theory of such fractional calculus has been developed in [7, 9, and 10] and gathered together in [11]. We shall rely heavily on this theory in this paper. As regards applications, the cases where $m = 1$ and $m = 2$ figure prominently in [11], $m = 2$ being intimately connected with Hankel transforms [11, Chapter 6] and generalized axially symmetric potential theory [11, § 3.6]. However, a general positive value of m may have seemed a trivial or worthless generalization. We shall show in this paper how such general values of m emerge in a reasonably natural way.

In § 2, we recall for convenience the definitions of the spaces $F_{p,\mu}$ of testing-functions and the corresponding spaces $F'_{p,\mu}$ of generalized functions. We also recall the appropriate operators of fractional calculus and list such of their properties as are necessary to make the paper reasonably self-contained. In § 3, we obtain the required expressions for L^r referred to above, as well as expressions for positive integral powers of the formal adjoint,

$$L' = (-1)^n x^{a_{n+1}} D x^{a_n} \ldots x^{a_2} D x^{a_1}, \tag{1.7}$$

of L and of the related operators

$$M = (-1)^n L, \quad M' = (-1)^n L'. \tag{1.8}$$

It soon becomes clear (and it is hardly surprising) that the cases in which $a < n$ and $a > n$ produce different expressions so that the two cases have to be handled separately in the subsequent theory. The results in § 3 provide the motivation for the definitions of fractional powers of L, L', M, M' which are given in § 4 along with the mapping properties relative to the $F_{p,\mu}$ spaces.

It is to be expected that, if $\operatorname{Re}\alpha < 0$, then L^α will be an integral operator rather than a differential operator and it is of interest to identify the kernels of this and other operators. Therefore, in § 5, we relate the definitions of our fractional powers to results involving the Mellin transform, using which it is an easy matter, in § 6, to discover that the kernels can be expressed in terms of Meijer's G-function.

In § 7, we extend our definitions of fractional powers to the spaces $F'_{p,\mu}$. As usual, this involves the use of adjoint operators, and mapping properties in $F'_{p,\mu}$ are easily obtained from the results in §4 by standard methods. In §8, we use the results in $F'_{p,\mu}$ to obtain existence and uniqueness results for classical and weak solutions of certain integral equations involving the G-function. Finally, in §9, we examine our results for the case where $n = 2$ in more detail relating them to our earlier work on operators involving the ${}_2F_1$ hypergeometric function [**8**] and [**11**, Chapter 4] and reconciling our fractional powers with the operators discussed in [**11**, §3.6] and [**16**].

Throughout, we shall use the terminology, notation, and conventions of [**11**]. In particular, μ and α will denote general complex numbers, m will be real and positive, $1 \leqslant p \leqslant \infty$ (unless the contrary is explicitly stated), while p and q are related by the equation $1/p + 1/q = 1$.

2.

In this section we recall for convenience the definitions of the spaces in which we shall be working and the properties of some relevant operators relative to these spaces. Fuller details and proofs can be found in [**11**, Chapters 2 and 3] or in [**7**, **9**, **10**].

DEFINITION 2.1. (i) For $1 \leqslant p < \infty$,

$$F_p = \{\varphi \in C^\infty(0,\infty) \colon x^k d^k\varphi/dx^k \in L^p(0,\infty) \text{ for } k = 0,1,2,\dots\} \tag{2.1}$$

while

$$F_\infty = \{\varphi \in C^\infty(0,\infty) \colon x^k d^k\varphi/dx^k \to 0 \text{ as } x \to 0+ \text{ and as } x \to \infty \text{ for } k = 0,1,2,\dots\}. \tag{2.2}$$

(ii) For $1 \leqslant p \leqslant \infty$ and any complex number μ,

$$F_{p,\mu} = \{\varphi \colon x^{-\mu}\varphi(x) \in F_p\}. \tag{2.3}$$

(iii) For $k = 0,1,2,\dots$, we define $\gamma_k^{p,\mu}$ on $F_{p,\mu}$ by

$$\gamma_k^{p,\mu}(\varphi) = \| x^k d^k(x^{-\mu}\varphi(x))/dx^k \|_p \tag{2.4}$$

where $\|\ \ \|_p$ denotes the usual norm on $L^p(0,\infty)$. For each k, $\gamma_k^{p,\mu}$ is a semi-norm on $F_{p,\mu}$ and $\gamma_0^{p,\mu}$ is a norm. With the topology generated by $\{\gamma_k^{p,\mu}\}_{k=0}^\infty$ [**17**, p. 9], $F_{p,\mu}$ is a Fréchet space.

THEOREM 2.2. *For any complex number λ, the mapping x^λ defined by*

$$(x^\lambda\varphi)(x) = x^\lambda\varphi(x) \quad (0 < x < \infty) \tag{2.5}$$

is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu+\lambda}$ with inverse $x^{-\lambda}$.

NOTATION 2.3. For any positive real number m, we shall write

$$D_m \equiv d/dx^m = m^{-1}x^{1-m}d/dx, \tag{2.6}$$

$$D \equiv D_1 \quad \text{and} \quad \delta \equiv xD. \tag{2.7}$$

THEOREM 2.4. (i) *D_m is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m}$ and is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-m}$ if and only if* $\operatorname{Re}\mu \neq 1/p$.

(ii) *δ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ and is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ if and only if* $\operatorname{Re}\mu \neq 1/p$.

Theorem 2.4 describes the behaviour of differentiation operators on the spaces $F_{p,\mu}$. As regards integration, we shall require the following definitions.

DEFINITION 2.5. We define the sets $A_{p,\mu,m}$ and $A'_{p,\mu,m}$ of complex numbers by

$$A_{p,\mu,m} = \{\eta\colon \operatorname{Re}(m\eta+\mu)+m \neq 1/p-ml\ (l = 0,1,2,\ldots)\}, \tag{2.8}$$

$$A'_{p,\mu,m} = \{\eta\colon \operatorname{Re}(m\eta-\mu) \neq -1/p-ml\ (l = 0,1,2,\ldots)\}. \tag{2.9}$$

DEFINITION 2.6. (i) Let $\operatorname{Re}(m\eta+\mu)+m > 1/p$, and $\varphi \in F_{p,\mu}$. For $\operatorname{Re}\alpha > 0$, we define $I_m^{\eta,\alpha}\varphi$ by

$$I_m^{\eta,\alpha}\varphi(x) = [\Gamma(\alpha)]^{-1}x^{-m\eta-m\alpha}\int_0^x (x^m-u^m)^{\alpha-1}u^{m\eta}\varphi(u)\,d(u^m). \tag{2.10}$$

The definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by means of the formula

$$I_m^{\eta,\alpha}\varphi = (\eta+\alpha+1)I_m^{\eta,\alpha+1}\varphi+m^{-1}I_m^{\eta,\alpha+1}\delta\varphi. \tag{2.11}$$

(ii) Let $\operatorname{Re}(m\eta-\mu) > -1/p$, and $\varphi \in F_{p,\mu}$. For $\operatorname{Re}\alpha > 0$, we define $K_m^{\eta,\alpha}\varphi$ by

$$K_m^{\eta,\alpha}\varphi(x) = [\Gamma(\alpha)]^{-1}x^{m\eta}\int_x^\infty (u^m-x^m)^{\alpha-1}u^{-m\eta-m\alpha}\varphi(u)\,d(u^m). \tag{2.12}$$

The definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by means of the formula

$$K_m^{\eta,\alpha}\varphi = (\eta+\alpha)K_m^{\eta,\alpha+1}\varphi-m^{-1}K_m^{\eta,\alpha+1}\delta\varphi. \tag{2.13}$$

(iii) Let $\eta \in A_{p,\mu,m}$, $\varphi \in F_{p,\mu}$, and let α be any complex number. For $\operatorname{Re}(m\eta+\mu)+m > 1/p$, we define $I_m^{\eta,\alpha}\varphi$ as in (i). Otherwise, if k is the unique positive integer such that

$$1/p-mk < \operatorname{Re}(m\eta+\mu)+m < 1/p-m(k-1),$$

then

$$I_m^{\eta,\alpha}\varphi = (-1)^k I_m^{\eta+k,\alpha-k}K_m^{-\eta-k,k}\varphi \tag{2.14}$$

where $I_m^{\eta+k,\alpha-k}$ and $K_m^{-\eta-k,k}$ are defined as in (i) and (ii) respectively.

(iv) Let $\eta \in A'_{p,\mu,m}$, $\varphi \in F_{p,\mu}$, and let α be any complex number. For $\operatorname{Re}(m\eta-\mu) > -1/p$, we define $K_m^{\eta,\alpha}$ as in (ii). Otherwise, if k is the unique positive integer such that

$$-1/p-mk < \operatorname{Re}(m\eta-\mu) < -1/p-m(k-1),$$

then

$$K_m^{\eta,\alpha}\varphi = (-1)^k K_m^{\eta+k,\alpha-k}I_m^{-\eta-k,k}\varphi \tag{2.15}$$

where $K_m^{\eta+k,\alpha-k}$ and $I_m^{-\eta-k,k}$ are defined as in (ii) and (i) respectively.

(v) For $0 \in A_{p,\mu,m}$, $\varphi \in F_{p,\mu}$, and any complex number α, we define $I_m^\alpha\varphi$ by

$$I_m^\alpha\varphi = x^{m\alpha}I_m^{0,\alpha}\varphi \tag{2.16}$$

where $I_m^{0,\alpha}$ is defined as in (iii) and $x^{m\alpha}$ as in (2.5).

(vi) For $\varphi \in F_{p,\mu}$ and any complex number α such that $-\alpha \in A'_{p,\mu,m}$, we define $K_m^\alpha\varphi$ by

$$K_m^\alpha\varphi = K_m^{0,\alpha}x^{m\alpha}\varphi \tag{2.17}$$

where $K_m^{0,\alpha}$ is defined as in (iv) and $x^{m\alpha}$ as in (2.5).

We now list the mapping properties of these operators.

THEOREM 2.7. (i) *If* $\eta \in A_{p,\mu,m}$, *then* $I_m^{\eta,\alpha}$ *is a continuous linear mapping from* $F_{p,\mu}$ *into itself. If also* $\eta+\alpha \in A_{p,\mu,m}$, then $I_m^{\eta,\alpha}$ *is a homeomorphism from* $F_{p,\mu}$ *onto itself and*

$$(I_m^{\eta,\alpha})^{-1} = I_m^{\eta+\alpha,-\alpha}. \tag{2.18}$$

(ii) *If* $\eta \in A'_{p,\mu,m}$, *then* $K_m^{\eta,\alpha}$ *is a continuous linear mapping from* $F_{p,\mu}$ *into itself. If also* $\eta+\alpha \in A'_{p,\mu,m}$, *then* $K_m^{\eta,\alpha}$ *is a homeomorphism from* $F_{p,\mu}$ *onto itself and*

$$(K_m^{\eta,\alpha})^{-1} = K_m^{\eta+\alpha,-\alpha}. \tag{2.19}$$

(iii) *If* $0 \in A_{p,\mu,m}$, *then* I_m^{α} *is a continuous linear mapping from* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$. *If also* $\alpha \in A_{p,\mu,m}$, *then* I_m^{α} *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and*

$$(I_m^{\alpha})^{-1} = I_m^{-\alpha}. \tag{2.20}$$

(iv) *If* $-\alpha \in A'_{p,\mu,m}$, *then* K_m^{α} *is a continuous linear mapping from* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$. *If also* $0 \in A'_{p,\mu,m}$, *then* K_m^{α} *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and*

$$(K_m^{\alpha})^{-1} = K_m^{-\alpha}. \tag{2.21}$$

(v) *If* $0 \in A_{p,\mu,m}$, *then* I_m^0 *is the identity operator on* $F_{p,\mu}$ *and for* $n = 1,2,\ldots,$

$$I_m^{-n} = (D_m)^n \quad \text{on } F_{p,\mu}. \tag{2.22}$$

If $0 \in A'_{p,\mu,m}$, *then* K_m^0 *is the identity operator on* $F_{p,\mu}$. *If* $n \in A'_{p,\mu,m}$ $(n = 1,2,\ldots)$, *then*

$$K_m^{-n} = (-D_m)^n \quad \text{on } F_{p,\mu}. \tag{2.23}$$

Next we list a number of results which we shall need later.

THEOREM 2.8. *Let* ξ, η, λ, μ, α, *and* β *be complex numbers and let* $\varphi \in F_{p,\mu}$.
(i) *If* $\eta \in A_{p,\mu,m}$, *then*

$$x^{\lambda} I_m^{\eta,\alpha}\varphi = I_m^{\eta-\lambda/m,\alpha} x^{\lambda}\varphi. \tag{2.24}$$

(ii) *If* $\eta \in A'_{p,\mu,m}$, *then*

$$x^{\lambda} K_m^{\eta,\alpha}\varphi = K_m^{\eta+\lambda/m,\alpha} x^{\lambda}\varphi. \tag{2.25}$$

(iii) *If* $\xi \in A_{p,\mu,m}$ *and* $\eta \in A_{p,\mu,m}$, *then*

$$I_m^{\eta,\alpha} I_m^{\xi,\beta}\varphi = I_m^{\xi,\beta} I_m^{\eta,\alpha}\varphi. \tag{2.26}$$

(iv) *If* $\xi \in A'_{p,\mu,m}$ *and* $\eta \in A'_{p,\mu,m}$, *then*

$$K_m^{\eta,\alpha} K_m^{\xi,\beta}\varphi = K_m^{\xi,\beta} K_m^{\eta,\alpha}\varphi. \tag{2.27}$$

(v) *If* $\xi \in A_{p,\mu,m}$ *and* $\eta \in A'_{p,\mu,m}$, *then*

$$K_m^{\eta,\alpha} I_m^{\xi,\beta}\varphi = I_m^{\xi,\beta} K_m^{\eta,\alpha}\varphi. \tag{2.28}$$

(vi) *If* $\eta \in A_{p,\mu,m}$ *and* $\eta+\alpha \in A_{p,\mu,m}$, *then*

$$I_m^{\eta,\alpha} I_m^{\eta+\alpha,\beta}\varphi = I_m^{\eta,\alpha+\beta}\varphi. \tag{2.29}$$

(vii) *If* $\eta \in A'_{p,\mu,m}$ *and* $\eta+\alpha \in A'_{p,\mu,m}$, *then*

$$K_m^{\eta,\alpha} K_m^{\eta+\alpha,\beta}\varphi = K_m^{\eta,\alpha+\beta}\varphi. \tag{2.30}$$

(viii) *If* $\{0,\alpha,\beta\} \subseteq A_{p,\mu,m}$, *then*

$$I_m^{\alpha} I_m^{\beta}\varphi = I_m^{\alpha+\beta}\varphi = I_m^{\beta} I_m^{\alpha}\varphi. \tag{2.31}$$

(ix) *If* $\{-\alpha, -\beta, -\alpha-\beta\} \subseteq A'_{p,\mu,m}$, *then*

$$K_m^\alpha K_m^\beta \varphi = K_m^{\alpha+\beta}\varphi = K_m^\beta K_m^\alpha \varphi. \tag{2.32}$$

DEFINITION 2.9. We let $F'_{p,\mu}$ denote the space of continuous linear functionals on $F_{p,\mu}$, and equip it with the topology of pointwise convergence.

DEFINITION 2.10. Let $f \in F'_{p,\mu}$.

(i) For $\eta \in A_{q,-\mu,m}$, we define $I_m^{\eta,\alpha} f \in F'_{p,\mu}$ by

$$(I_m^{\eta,\alpha} f, \varphi) = (f, K_m^{\eta+1-1/m,\alpha}\varphi) \quad (\varphi \in F_{p,\mu}).$$

(ii) For $\eta \in A'_{q,-\mu,m}$, we define $K_m^{\eta,\alpha} f \in F'_{p,\mu}$ by

$$(K_m^{\eta,\alpha} f, \varphi) = (f, I_m^{\eta-1+1/m,\alpha}\varphi) \quad (\varphi \in F_{p,\mu}).$$

(iii) For $0 \in A_{q,-\mu,m}$, we define $I_m^\alpha f \in F'_{p,\mu-m\alpha}$ by

$$(I_m^\alpha f, \varphi) = (f, x^{m-1} K_m^\alpha x^{-m+1}\varphi) \quad (\varphi \in F_{p,\mu-m\alpha}).$$

(iv) For $-\alpha \in A'_{q,-\mu,m}$, we define $K_m^\alpha f \in F'_{p,\mu-m\alpha}$ by

$$(K_m^\alpha f, \varphi) = (f, x^{m-1} I_m^\alpha x^{-m+1}\varphi) \quad (\varphi \in F_{p,\mu-m\alpha}).$$

REMARK 2.11. The mapping properties of the operators $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$, I_m^α, and K_m^α on $F'_{p,\mu}$ are obtained from those on $F_{p,\mu}$ by using standard results on adjoints [17, Theorems 1.10–1 and 1.10–2]. Analogues of (2.24)–(2.32) can likewise be obtained by replacing f by φ and by interchanging p with q and μ with $-\mu$ in the conditions on the parameters.

3.

We now begin the development of our theory by examining the differential operator L given by (1.1) in more detail. It is an immediate consequence of Theorems 2.2 and 2.4 (i) that L is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+a-n}$, where a is given by (1.2). If $a = n$, then L is a continuous linear mapping from $F_{p,\mu}$ into itself. However, as indicated in § 1, we shall be interested in the case where $a \neq n$ or $m > 0$, where m is given by (1.3).

Firstly, we rewrite L, using Notation 2.3, in the form

$$L = x^{a_1-1}\delta x^{a_2-1}\delta \ldots x^{a_n-1}\delta x^{a_{n+1}}. \tag{3.1}$$

L will be invertible as a mapping from $F_{p,\mu}$ onto $F_{p,\mu+a-n}$ provided that each operator δ in (3.1) can be inverted. From Theorem 2.4 (ii), we see that this is the case if and only if $\operatorname{Re}(\mu+a-n-\sum_{i=1}^k a_i+k) \neq 1/p$ $(k = 1, \ldots, n)$. These conditions may be written as

$$\operatorname{Re}\left(\sum_{i=k+1}^{n+1} a_i + k - n + \mu\right) \neq 1/p \quad (k = 1, \ldots, n).$$

Thus an important role will be played by the following numbers.

NOTATION 3.1. With the notation of (1.1) and (1.3), we shall write

$$b_k = \left(\sum_{i=k+1}^{n+1} a_i + k - n\right)\Big/ m \quad (k = 1, \ldots, n). \tag{3.2}$$

THEOREM 3.2. *L is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+a-n}$ for all complex numbers $a_1,\dots,a_{n+1}$ and is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu+a-n}$ if and only if* $\operatorname{Re}(mb_k+\mu) \neq 1/p\ (k = 1,\dots,n)$.

Proof. This is immediate from the preamble.

By making use of the numbers b_k $(k = 1,\dots,n)$ and m, we can obtain the most convenient form of the operator L for our purposes.

LEMMA 3.3. *With b_k $(k = 1,\dots,n)$ as in* (3.2),

$$L = m^n x^{a-n} \prod_{k=1}^{n} x^{m-mb_k} D_m x^{mb_k} \tag{3.3}$$

or $L = m^n x^{a-n} T_1 T_2 \dots T_n$ where

$$T_k = x^{m-mb_k} D_m x^{mb_k} \quad (k = 1,\dots,n). \tag{3.4}$$

(Since, as is easily checked, the operators $T_1,\dots,T_n$ commute, the product in (3.3) *is unambiguous.)*

Proof. The result follows easily from (2.6) and the identities

$$mb_1 = a - a_1 + 1 - n, \quad mb_n = a_{n+1},$$
$$mb_k + 1 - mb_{k+1} = a_{k+1} \quad (k = 1,\dots,n-1).$$

Lemma 3.3 leads to an examination of an operator of the form

$$T = x^{m-mv} D_m x^{mv} \tag{3.5}$$

where v is a complex number.

LEMMA 3.4. *Let $\varphi \in F_{p,\mu}$.*
(i) *If $v \in A_{p,\mu,m}$, then $T\varphi = I_m^{v,-1}\varphi$, as given by Definition* 2.6 (iii).
(ii) *If $1 - v \in A'_{p,\mu,m}$, then $T\varphi = -K_m^{1-v,-1}\varphi$, as given by Definition* 2.6 (iv).

Proof. These results follow fairly easily from Definition 2.6 and Theorem 2.7 (v).

To indicate how these facts can be used, we next give expressions for positive integral powers of L, L', M, and M', dealing first with the case where $a < n$.

LEMMA 3.5. *Let r be a positive integer, $a < n$, $\varphi \in F_{p,\mu}$, and $b_k \in A_{p,\mu,m}$ $(k = 1,\dots,n)$. Then*

$$L^r \varphi = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k,-r} \varphi. \tag{3.6}$$

Proof. We note first that the I operators commute, by Theorem 2.8 (iii), so that the product on the right-hand side is unambiguous. The result is true for $r = 1$ by Lemma 3.4 and the fact that $m = n - a$ in this case. Assume the result is true for a certain value of r. Then from (3.6), (2.16), and (2.22),

$$L^r \varphi = m^{nr} x^{-mr} \prod_{k=1}^{n} x^{-mb_k+mr} (D_m)^r x^{mb_k} \varphi$$

$$= m^{nr} \left(\prod_{k=1}^{n} x^{-mb_k} (D_m)^r x^{mb_k+mr} \right) x^{-mr} \varphi. \tag{3.7}$$

Bearing in mind that the operators in the products commute with each other, we can use (3.7) and (3.3) (with φ replaced by $L^r\varphi$) to obtain

$$\begin{aligned} L^{r+1}\varphi &= LL^r\varphi \\ &= m^{n(r+1)}x^{-m}\left(\prod_{k=1}^{n} x^{m-mb_k}(D_m)^{r+1}x^{mb_k+mr}\right)x^{-mr}\varphi \\ &= m^{n(r+1)}x^{-m(r+1)}\left(\prod_{k=1}^{n} x^{m(r+1)-mb_k}(D_m)^{r+1}x^{mb_k}\right)\varphi \\ &= m^{n(r+1)}x^{-m(r+1)}\prod_{k=1}^{n} I_m^{b_k,-(r+1)}\varphi. \end{aligned}$$

Hence (3.6) is true for all positive integers r by induction.

REMARK 3.6. In the proof of Lemma 3.5, it is tempting to use the expression for L derived from (3.6) to calculate $LL^r\varphi$. However, by Theorem 3.2, $L^r\varphi \in F_{p,\mu-mr}$, and the fact that $b_k \in A_{p,\mu,m}$ does not imply that $b_k \in A_{p,\mu-mr,m}$. Hence this procedure is invalid and a slightly longer argument is required. A similar remark applies to the proofs of the corresponding results for $(L')^r$, M^r, and $(M')^r$.

LEMMA 3.7. *Let r be a positive integer and let $a < n$.*
(i) *If $b_k \in A_{p,\mu,m}$ $(k = 1,\dots,n)$ and $\varphi \in F_{q,-\mu+mr}$, then*

$$(L')^r\varphi = m^{nr}\prod_{k=1}^{n} K_m^{b_k+1-1/m,-r}x^{-mr}\varphi. \tag{3.8}$$

(ii) *If $-b_k \in A'_{q,-\mu,m}$ $(k = 1,\dots,n)$ and $\varphi \in F_{q,-\mu+mr}$, then*

$$M^r\varphi = m^{nr}\prod_{k=1}^{n} K_m^{-b_k,-r}x^{-mr}\varphi. \tag{3.9}$$

(iii) *If $-b_k \in A'_{q,-\mu,m}$ $(k = 1,\dots,n)$ and $\varphi \in F_{p,\mu}$, then*

$$(M')^r\varphi = m^{nr}x^{-mr}\prod_{k=1}^{n} I_m^{-b_k-1+1/m,-r}\varphi. \tag{3.10}$$

Proof. The details which are similar to those in Lemma 3.5 are omitted.

REMARK 3.8. We note that (i) can be obtained from (ii) and vice versa. L' is obtained formally from M on replacing a_k by a_{n+2-k} $(k = 1,\dots,n+1)$, from which it follows that b_k must be replaced by the quantity

$$\begin{aligned} b'_k &= \left(\sum_{i=k+1}^{n+1} a_{n+2-i}+k-n\right)\bigg/ m \\ &= \left(a-\sum_{i=n-k+2}^{n+1} a_i+k-n\right)\bigg/ m \\ &= (a-(mb_{n-k+1}-(n-k+1)+n)+k-n)/m \\ &= (a-n+1)/m-b_{n-k+1} \\ &= -b_{n-k+1}-1+1/m, \end{aligned}$$

where we have used (3.2) and the fact that $m = n-a$ in this case. Equation (3.8) follows from (3.9) and vice versa since the operators in the products commute. Similarly, (iii) can be derived from Lemma 3.5.

As regards the case where $a > n$, we can prove the following results.

LEMMA 3.9. *Let r be a positive integer and let $m = a-n > 0$.*
(i) *If $\varphi \in F_{q,-\mu-mr}$ and $b_k - 1 \in A_{q,-\mu,m}$ $(k = 1,\ldots,n)$, then*

$$L^r\varphi = m^{nr} \prod_{k=1}^{n} I_m^{b_k-1,-r} x^{mr}\varphi. \tag{3.11}$$

(ii) *If $\varphi \in F_{p,\mu}$ and $b_k - 1 \in A_{q,-\mu,m}$ $(k = 1,\ldots,n)$, then*

$$(L')^r\varphi = m^{nr}x^{mr} \prod_{k=1}^{n} K_m^{b_k-1/m,-r}\varphi. \tag{3.12}$$

(iii) *If $\varphi \in F_{p,\mu}$ and $1 - b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then*

$$M^r\varphi = m^{nr}x^{mr} \prod_{k=1}^{n} K_m^{1-b_k,-r}\varphi. \tag{3.13}$$

(iv) *If $\varphi \in F_{q,-\mu-mr}$ and $1 - b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then*

$$(M')^r\varphi = m^{nr} \prod_{k=1}^{n} I_m^{-b_k+1/m,-r} x^{mr}\varphi. \tag{3.14}$$

Proof. Again the details are similar to those in Lemma 3.5 and are omitted.

REMARK 3.10. (i) All the work prior to Lemma 3.5 goes through even if the number a is allowed to be complex. However, if $\operatorname{Im} a \neq 0$, the proofs of Lemmas 3.5, 3.7, and 3.9 fail because the term x^{a-n} presents difficulty. The restriction, however, is not a major one when we observe that $F_{p,\mu} = F_{p,\operatorname{Re}\mu}$ for all complex numbers μ.

(ii) Formulae (3.6) and (3.8)–(3.14) can be used as the motivation for the definition of fractional powers of L, L', M, and M' in the two cases $a < n$ and $a > n$. It is already clear that these two cases will require separate treatment. In the sequel, we shall concentrate on L and L' in the case where $a < n$ and on M and M' in the case where $a > n$ and omit discussion of the others.

(iii) In stating the results in Lemmas 3.5, 3.7, and 3.9, we have assumed that certain conditions hold for $k = 1,\ldots,n$. This will continue in what follows and we shall ignore other situations (for example, $b_k \in A_{p,\mu,m}$ for some values of k and $1 - b_k \in A'_{p,\mu,m}$ for other values of k) which might present further difficulties.

4.

With the motivation supplied by §3, we can now give our definitions of fractional powers of L and M. We shall give details for L and L' in the case where $a < n$ and then simply state the corresponding results for M and M' when $a > n$.

DEFINITION 4.1. Let $m = n - a > 0$, let α be any complex number, and let $b_k \in A_{p,\mu,m}$ for $k = 1,\ldots,n$. Then we define L^α on $F_{p,\mu}$ by

$$L^\alpha\varphi = m^{n\alpha}x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha}\varphi \quad (\varphi \in F_{p,\mu}). \tag{4.1}$$

NOTES. 1. Equation (4.1) is obtained by formally replacing r by α in (3.6).

2. We emphasize that α may be *any* complex number.

3. The operators $I_m^{b_k,-\alpha}$ commute, by Theorem 2.8 (iii), so that the product on the right-hand side is unambiguous.

As a first step we can state the following result.

THEOREM 4.2. *If $a < n$ and $b_k \in A_{p,\mu,m}$ for $k = 1,\dots,n$, then L^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m\alpha}$.*

Proof. For each k, $I_m^{b_k,-\alpha}$ is a continuous linear mapping from $F_{p,\mu}$ into itself by Theorem 2.7 (i). The result follows easily from Theorem 2.2.

To obtain further information we require two results generalizing properties of indices in ordinary algebra.

LEMMA 4.3. *If $a < n$ and $b_k \in A_{p,\mu,m}$ for $k = 1,\dots,n$, then L^0 is the identity operator on $F_{p,\mu}$.*

Proof. This is almost immediate from Theorem 2.7 (v), Definition 2.6 (v), and Theorem 2.8 (i).

THEOREM 4.4 (First index law). *Let $a < n$, α and β be any complex numbers, $\varphi \in F_{p,\mu}$, and $b_k \in A_{p,\mu,m}$ for $k = 1,\dots,n$.*

(i) *If $b_k - \beta \in A_{p,\mu,m}$ for $k = 1,\dots,n$, then*

$$L^\alpha L^\beta \varphi = L^{\alpha+\beta}\varphi. \tag{4.2}$$

(ii) *If $b_k - \alpha \in A_{p,\mu,m}$ for $k = 1,\dots,n$, then*

$$L^\beta L^\alpha \varphi = L^{\alpha+\beta}\varphi. \tag{4.3}$$

(iii) *If $\{b_k - \alpha, b_k - \beta\} \subseteq A_{p,\mu,m}$ for $k = 1,\dots,n$, then*

$$L^\alpha L^\beta \varphi = L^{\alpha+\beta}\varphi = L^\beta L^\alpha \varphi. \tag{4.4}$$

Proof. It is clearly sufficient to prove (i). Under the stated hypotheses, we may use Theorem 2.8 (i), (iii), and (vi) to obtain

$$\begin{aligned} L^\alpha L^\beta \varphi &= m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha} m^{n\beta} x^{-m\beta} \prod_{k=1}^{n} I_m^{b_k,-\beta}\varphi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k-\beta,-\alpha} I_m^{b_k,-\beta}\varphi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k,-(\alpha+\beta)}\varphi \\ &= L^{\alpha+\beta}\varphi. \end{aligned}$$

This completes the proof.

REMARK 4.5. It is interesting to note that in (i), the right-hand side is meaningful without the extra condition $b_k - \beta \in A_{p,\mu,m}$ $(k = 1,\dots,n)$. Thus we could use the right-hand side to provide an analytic continuation of the left-hand side to values such that

$b_k-\beta \notin A_{p,\mu,m}$ for at least one value of k. We have commented on this before in [10] and [11, p. 72] in connection with the first index laws for I_m^α and K_m^α, quoted in Theorem 2.8 (viii) and (ix). We can think of this as removing 'removable singularities' by analogy with $z^{-1}z = 1$ at $z = 0$. We could therefore combine all three parts of Theorem 4.4 into one.

COROLLARY 4.6. *If $a < n$, α and β are complex numbers, $\varphi \in F_{p,\mu}$, and $b_k \in A_{p,\mu,m}$ for $k = 1,\ldots,n$, then*

$$L^\alpha L^\beta \varphi = L^{\alpha+\beta}\varphi = L^\beta L^\alpha \varphi, \tag{4.5}$$

the first and third members being interpreted in terms of the appropriate analytic continuations, where necessary.

We shall concentrate on the cases where analytic continuation is not necessary. For instance, we can use Theorem 4.4 to obtain further information about L^α.

THEOREM 4.7. *If $a < n$ and $\{b_k, b_k-\alpha\} \subseteq A_{p,\mu,m}$ for $k = 1,\ldots,n$, then L^α is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-m\alpha}$ with inverse $L^{-\alpha}$.*

Proof. Let $\varphi \in F_{p,\mu}$ and $\psi \in F_{p,\mu-m\alpha}$. Under the given conditions, Lemma 4.3 and Theorem 4.4 give

$$L^{-\alpha}L^\alpha\varphi = L^0\varphi = \varphi, \quad L^\alpha L^{-\alpha}\psi = L^0\psi = \psi.$$

The result follows at once.

COROLLARY 4.8. *If $a < n$ and $b_k - 1 \in A_{p,\mu,m}$ for $k = 1,\ldots,n$, then L is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-m}$ and the inverse of L is L^{-1} as given on $F_{p,\mu-m}$ by (4.1) with $\alpha = -1$.*

Proof. Since if $b_k - 1 \in A_{p,\mu,m}$ then $b_k \in A_{p,\mu,m}$, as can be seen from (2.8), the result follows from Theorem 4.7.

For future reference we record the salient facts about L'.

DEFINITION 4.9. Let $m = n-a > 0$, let α be any complex number, and let $b_k \in A_{p,\mu,m}$ $(k = 1,\ldots,n)$. Then we define $(L')^\alpha$ on $F_{q,-\mu+m\alpha}$ by

$$(L')^\alpha\varphi = m^{n\alpha}\prod_{k=1}^{n} K_m^{b_k+1-1/m,\,-\alpha}x^{-m\alpha}\varphi \quad (\varphi \in F_{q,-\mu+m\alpha}). \tag{4.6}$$

THEOREM 4.10. *Let $m = n-a > 0$ and let α, β be any complex numbers.*

(i) *If $b_k \in A_{p,\mu,m}$ $(k = 1,\ldots,n)$, then $(L')^\alpha$ is a continuous linear mapping from $F_{q,-\mu+m\alpha}$ into $F_{q,-\mu}$. If also $b_k - \alpha \in A_{p,\mu,m}$ $(k = 1,\ldots,n)$, then $(L')^\alpha$ is a homeomorphism from $F_{q,-\mu+m\alpha}$ onto $F_{q,-\mu}$ and $[(L')^\alpha]^{-1} = (L')^{-\alpha}$.*

(ii) *If $b_k \in A_{p,\mu,m}$ $(k = 1,\ldots,n)$, then $(L')^0$ is the identity operator on $F_{q,-\mu}$.*

(iii) *Let $\varphi \in F_{q,-\mu+m\alpha+m\beta}$. If $\{b_k, b_k-\beta\} \subseteq A_{p,\mu,m}$ for $k = 1,\ldots,n$, then*

$$(L')^\beta(L')^\alpha\varphi = (L')^{\alpha+\beta}\varphi. \tag{4.7}$$

If $\{b_k, b_k-\alpha, b_k-\beta\} \subseteq A_{p,\mu,m}$ for $k = 1,\ldots,n$, then

$$(L')^\alpha(L')^\beta\varphi = (L')^{\alpha+\beta}\varphi = (L')^\beta(L')^\alpha\varphi. \tag{4.8}$$

Proof. The details are similar to those for L and are therefore omitted. Once again we observe that in (4.7) the right-hand side provides an analytic continuation of the left to cover the cases where $b_k-\beta \notin A_{p,\mu,m}$. Likewise (4.8) can be regarded as holding subject only to the conditions $b_k \in A_{p,\mu,m}$ $(k = 1,\ldots,n)$ provided that the first and third members are interpreted in terms of their analytic continuations.

We now turn to the case where $a > n$ and consider fractional powers of M and M'.

DEFINITION 4.11. Let $m = a-n > 0$, let α be any complex number, and let $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$. Then we define M^α on $F_{p,\mu}$ by

$$M^\alpha\varphi = m^{n\alpha}x^{m\alpha}\prod_{k=1}^{n} K_m^{1-b_k,-\alpha}\varphi \quad (\varphi \in F_{p,\mu}). \tag{4.9}$$

The motivation for Definition 4.11 comes from Lemma 3.9(iii).

THEOREM 4.12. *Let $m = a-n > 0$ and let α,β be any complex numbers.*

(i) *If $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then M^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$. If also $1-b_k-\alpha \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$ then M^α is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu+m\alpha}$ and $(M^\alpha)^{-1} = M^{-\alpha}$.*

(ii) *If $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then L^0 is the identity mapping on $F_{p,\mu}$.*

(iii) *Let $\varphi \in F_{p,\mu}$. If $\{1-b_k, 1-b_k-\beta\} \subseteq A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then*

$$M^\alpha M^\beta\varphi = M^{\alpha+\beta}\varphi. \tag{4.10}$$

If $\{1-b_k, 1-b_k-\alpha, 1-b_k-\beta\} \subseteq A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then

$$M^\alpha M^\beta\varphi = M^{\alpha+\beta}\varphi = M^\beta M^\alpha\varphi. \tag{4.11}$$

Equations (4.10) *and* (4.11) *hold if $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$ provided that the appropriate terms are regarded in terms of their analytic continuations.*

Proof. The proof of this theorem is omitted.

DEFINITION 4.13. Let $m = a-n > 0$, let α be any complex number, and let $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$. Then we define $(M')^\alpha$ on $F_{q,-\mu-m\alpha}$ by

$$(M')^\alpha\varphi = m^{n\alpha}\prod_{k=1}^{n} I_m^{-b_k+1/m,-\alpha}x^{m\alpha}\varphi \quad (\varphi \in F_{q,-\mu-m\alpha}). \tag{4.12}$$

The motivation for Definition 4.13 comes from Lemma 3.9(iv).

THEOREM 4.14. *Let $m = a-n > 0$ and let α,β be any complex numbers.*

(i) *If $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then $(M')^\alpha$ is a continuous linear mapping from $F_{q,-\mu-m\alpha}$ into $F_{q,-\mu}$. If also $1-b_k-\alpha \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then $(M')^\alpha$ is a homeomorphism from $F_{q,-\mu-m\alpha}$ onto $F_{q,-\mu}$ and $[(M')^\alpha]^{-1} = (M')^{-\alpha}$.*

(ii) *If $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then $(M')^0$ is the identity mapping on $F_{q,-\mu}$.*

(iii) *Let $\varphi \in F_{q,-\mu-m\alpha-m\beta}$. If $\{1-b_k, 1-b_k-\beta\} \subseteq A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then*

$$(M')^\beta(M')^\alpha\varphi = (M')^{\alpha+\beta}\varphi. \tag{4.13}$$

If $\{1-b_k, 1-b_k-\alpha, 1-b_k-\beta\} \subseteq A'_{p,\mu,m}$ $(k = 1,\ldots,n)$, then

$$(M')^\alpha(M')^\beta\varphi = (M')^{\alpha+\beta}\varphi = (M')^\beta(M')^\alpha\varphi. \tag{4.14}$$

Equations (4.13) *and* (4.14) *hold if* $1-b_k \in A'_{p,\mu,m}$ $(k = 1,\ldots,n)$ *provided that the appropriate terms are interpreted in terms of their analytic continuations.*

Proof. The proof of Theorem 4.14 is also omitted.

5.

In the case where $\operatorname{Re}\alpha < 0$, we might expect L^α, M^α etc. to be integral rather than differential operators and it is of interest to try to identify the kernels of these integral operators. Expressions such as (4.1) and (4.9) do not yield the kernels without a great deal of effort involving special functions. However, use of the Mellin transform helps considerably. In this section, we show how this approach leads us to the same definitions of fractional powers as in §4. Previously, any value of p in the range $1 \leqslant p \leqslant \infty$ could be considered. In the Mellin transform approach the restriction $1 \leqslant p \leqslant 2$ enters but, by way of compensation, we shall be able to identify the kernels easily in §6.

The Mellin transform, $\mathcal{M}\varphi$, of a function φ is defined formally by

$$\mathcal{M}\varphi(s) = \int_0^\infty x^{s-1}\varphi(x)\,dx, \tag{5.1}$$

where s is a complex number. If $\varphi \in C_0^\infty(0,\infty)$, (5.1) is meaningful for any s. However, if $\varphi \in F_{p,\mu}$, we must restrict s to the line $\operatorname{Re}s = 1/p - \operatorname{Re}\mu$.

THEOREM 5.1. *For* $1 \leqslant p \leqslant 2$ *and any complex number* μ, *let* Φ *be defined on* $(-\infty,\infty)$ *by*

$$\Phi(t) = \mathcal{M}\varphi(1/p - \operatorname{Re}\mu + it) = \int_0^\infty x^{1/p - \operatorname{Re}\mu - 1 + it}\varphi(x)\,dx.$$

Then the mapping $\varphi \to \Phi$ *is a continuous linear mapping from* $F_{p,\mu}$ *into* $L^q(-\infty,\infty)$.

Proof. This is a consequence of [**13**, Lemma 2.3] once we observe that our space $F_{p,\mu}$ is continuously imbedded in Rooney's space $L_{1-p\operatorname{Re}\mu,\,p}$.

REMARK 5.2. From now on, in dealing with $\mathcal{M}\varphi(s)$ for $\varphi \in F_{p,\mu}$, it will be implicit that $\operatorname{Re}s = 1/p - \operatorname{Re}\mu$. In view of Theorem 5.1, to establish results on $F_{p,\mu}$ involving $\mathcal{M}$, it will be sufficient to take $\varphi \in C_0^\infty(0,\infty)$ and use the fact that $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$ [**11**, Corollary 2.7] in conjunction with the continuity of $\mathcal{M}$. However, the restriction $1 \leqslant p \leqslant 2$ mentioned above becomes operative immediately in this approach.

The following result is easily proved.

LEMMA 5.3. *Let* $1 \leqslant p \leqslant 2$, *let* $\varphi \in F_{p,\mu}$, *let* λ *be any complex number (and let* $\operatorname{Re}s = 1/p - \operatorname{Re}\mu$). *Then*

(i) $\mathcal{M}(x^\lambda\varphi)(s-\lambda) = \mathcal{M}\varphi(s)$, (5.2)

(ii) $\mathcal{M}(D\varphi)(s+1) = -s\mathcal{M}\varphi(s)$. (5.3)

Proof. (i) is trivial but we should perhaps note that both sides are well-defined in the spirit of Remark 5.2 since

$$\operatorname{Re} s = 1/p - \operatorname{Re}\mu \;\Rightarrow\; \operatorname{Re}(s-\lambda) = 1/p - \operatorname{Re}(\mu+\lambda).$$

(ii) follows at once by integration by parts for $\varphi \in C_0^\infty(0,\infty)$ and hence for all $\varphi \in F_{p,\mu}$ by continuity.

Of more substance is the effect of $\mathscr{M}$ on $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$.

THEOREM 5.4. *Let* $1 \leqslant p \leqslant 2$, *let* $\varphi \in F_{p,\mu}$, *and let* α *be any complex number.*
(i) *If* $\eta \in A_{p,\mu,m}$, *then*

$$\mathscr{M}(I_m^{\eta,\alpha}\varphi)(s) = \frac{\Gamma(\eta+1-s/m)}{\Gamma(\eta+\alpha+1-s/m)}\mathscr{M}\varphi(s). \tag{5.4}$$

(ii) $\eta \in A'_{p,\mu,m}$, *then*

$$\mathscr{M}(K_m^{\eta,\alpha}\varphi)(s) = \frac{\Gamma(\eta+s/m)}{\Gamma(\eta+\alpha+s/m)}\mathscr{M}\varphi(s). \tag{5.5}$$

Proof. We consider only (i), the proof of (ii) being similar. We deal with the various forms of $I_m^{\eta,\alpha}$ as given in Definition 2.6(i), (iii). The validity of (5.4) for the case where $\operatorname{Re}(m\eta+\mu)+m > 1/p$ and $\operatorname{Re}\alpha > 0$ is established, in slightly different notation, in [12, Corollary 4.1]. For $\operatorname{Re}(m\eta+\mu)+m > 1/p$ and $-1 < \operatorname{Re}\alpha \leqslant 0$, we use (2.11), (5.2), (5.3), and the previous case to obtain

$$\mathscr{M}(I_m^{\eta,\alpha}\varphi)(s) = \left[(\eta+\alpha+1)\frac{\Gamma(\eta+1-s/m)}{\Gamma(\eta+\alpha+2-s/m)} - \frac{s}{m}\frac{\Gamma(\eta+1-s/m)}{\Gamma(\eta+\alpha+2-s/m)}\right]\mathscr{M}\varphi(s)$$

which reduces to (5.4) via $\Gamma(z+1) = z\Gamma(z)$. Proceeding step-by-step, we can thus establish (5.4) for $\operatorname{Re}(m\eta+\mu)+m > 1/p$ and likewise (5.5) for $\operatorname{Re}(m\eta-\mu) > -1/p$. We now use these results along with (2.14) and the formula $\Gamma(z+1) = z\Gamma(z)$, the latter being applied $2k$ times, where k is as in (2.14). The details, which are routine, are omitted. This completes the proof.

REMARK 5.5. The expression (5.4) shows clearly the relevance of the set $A_{p,\mu,m}$. For (5.4) to be meaningful, we require that $\operatorname{Re}(\eta+1-s/m)$ should not be zero or a negative integer. Bearing in mind that $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$, we see easily that this condition is equivalent to $\eta \in A_{p,\mu,m}$. Similar remarks apply to (5.5). Theorem 5.4 also makes the extended definitions of $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ in Definition 2.6(iii) and (iv) appear perfectly natural.

We can now obtain expressions for the Mellin transforms of our fractional powers.

THEOREM 5.6. *Let* $1 \leqslant p \leqslant 2$, $\varphi \in F_{p,\mu}$, $m = n-a > 0$, *and*

$$b_k \in A_{p,\mu,m} \quad (k = 1,\ldots,n).$$

Then

$$\mathscr{M}(L^\alpha\varphi)(s+m\alpha) = m^{n\alpha}\prod_{k=1}^{n}\frac{\Gamma(b_k+1-s/m)}{\Gamma(b_k-\alpha+1-s/m)}\mathscr{M}\varphi(s). \tag{5.6}$$

Proof. By Theorem 4.2, if $\varphi \in F_{p,\mu}$ then $L^\alpha\varphi \in F_{p,\mu-m\alpha}$ and, in keeping with Remark 5.2, $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$ implies $\operatorname{Re}(s+m\alpha) = 1/p - \operatorname{Re}(\mu - m\alpha)$. Thus both sides of (5.6) are meaningful and equality follows from (4.1), (5.2), and (5.4).

REMARK 5.7. As mentioned above we could use the Mellin transform method to arrive at (4.1) in the case where $1 \leqslant p \leqslant 2$. The first step would be to establish (5.6) for the case where $\alpha = r$, a positive integer, and this is easily done via (3.3), (5.2), and (5.3). With this motivation, we would then use (5.6) as the definition of L^α which we could regard as being similar to a multiplier transform as discussed by Rooney in **[13]**, although here we are mapping one space into a different space. For $1 \leqslant p \leqslant 2$, $\mathscr{M}$ is one-to-one and we can deduce via (5.4) that (5.6) is equivalent to (4.1) in this case. Similar comments apply to the other cases.

Analogues of (5.6) for $(L')^\alpha$, M^α, and $(M')^\alpha$ can easily be obtained. We shall list just one of the results for use below.

THEOREM 5.8. *Let* $1 \leqslant p \leqslant 2$, $\varphi \in F_{p,\mu}$, $m = a - n > 0$, *and*

$$1 - b_k \in A'_{p,\mu,m} \quad (k = 1, \ldots, n).$$

Then

$$\mathscr{M}(M^\alpha\varphi)(s - m\alpha) = m^{n\alpha} \prod_{k=1}^{n} \frac{\Gamma(1 - b_k + s/m)}{\Gamma(1 - b_k - \alpha + s/m)} \mathscr{M}\varphi(s). \tag{5.7}$$

Proof. The result follows easily from (4.9), (5.2), and (5.5).

6.

Armed with (5.6) and (5.7) as alternatives to (4.1) and (4.9), at least when $1 \leqslant p \leqslant 2$, we set about identifying the kernels of L^α and M^α in the case where $\operatorname{Re}\alpha < 0$.

First, we need a technical lemma.

LEMMA 6.1. *Let K be a function defined (almost everywhere) on $(0, \infty)$ and such that*

$$\int_0^\infty x^{(1/p - \operatorname{Re}\mu)/m - 1} |K(x)| \, dx < \infty. \tag{6.1}$$

Let T be the integral transform defined, for suitable functions φ, by

$$T\varphi(x) = \int_0^\infty K(x^m/t^m)\, \varphi(t)\, dt/t$$

where $m > 0$. Then, for $1 \leqslant p \leqslant \infty$ and any complex number μ,

(i) *T is a continuous linear mapping from L^p_μ into itself, where*

$$L^p_\mu = \{f\colon x^{-\mu} f(x) \in L^p(0, \infty)\} \tag{6.2}$$

is equipped with the norm

$$\| f \|_{p,\mu} = \| x^{-\mu} f(x) \|_p, \tag{6.3}$$

(ii) *T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$,*

(iii) *for $\varphi \in C_0^\infty(0, \infty)$ and $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$,*

$$\mathscr{M}(T\varphi)(s) = m^{-1} \mathscr{M}K(s/m) \mathscr{M}\varphi(s).$$

Proof. For $x > 0$, let $K_1(x) = K(x^m)$. Then, by (6.1),

$$\int_0^\infty x^{-\mathrm{Re}\,\mu - 1/q} |K_1(x)|\, dx = m^{-1} \int_0^\infty t^{(1/p - \mathrm{Re}\,\mu)/m - 1} |K(t)|\, dt < \infty$$

and

$$T\varphi(x) = \int_0^\infty K_1(x/t)\phi(t)\, dt/t.$$

(i) follows easily by a well-known result on Mellin convolutions [12, Lemma 3.1]. We may also write

$$T\varphi(x) = \int_0^\infty K_1(v)\varphi(x/v)\, dv/v.$$

When $\varphi \in F_{p,\mu}$, we may differentiate under the integral sign, subject to (6.1), and obtain $\delta T\varphi = T\delta\varphi$ where δ is defined by (2.7). Since $\varphi \in F_{p,\mu}$ implies $\delta\varphi \in L^p_\mu$, (ii) now follows in a routine manner. (iii) can be proved by inverting the order of integration on the left-hand side which, again, can be justified subject to (6.1). This completes the proof.

In view of Lemma 6.1, we seek, in the case of L, a function whose Mellin transform is of the form

$$\prod_{k=1}^n \Gamma(b_k + 1 - s) \Big/ \prod_{k=1}^n \Gamma(b_k - \alpha + 1 - s).$$

Formula (14) of [3, p. 337] leads us to Meijer's G-function, properties of which can be found in, for instance, [2, Chapter V].

DEFINITION 6.2. Let n be a positive integer, let $m > 0$, let $v_1, \ldots, v_n$ be complex numbers, and let γ be a complex number with $\mathrm{Re}\,\gamma > 0$. We define $G_1(v_1, \ldots, v_n; \gamma; m)\varphi$ for suitable functions φ by

$$(G_1(v_1, \ldots, v_n; \gamma; m)\varphi)(x) = x^{-m} m^{-n\gamma} \int_0^x G^{n,0}_{n,n}\left(\frac{t^m}{x^m} \,\middle|\, \begin{matrix} v_1 + \gamma, \ldots, v_n + \gamma \\ v_1, \ldots, v_n \end{matrix}\right) \varphi(t)\, d(t^m). \tag{6.4}$$

REMARK 6.3. We note that

$$G^{n,0}_{n,n}\left(u \,\middle|\, \begin{matrix} v_1 + \gamma, \ldots, v_n + \gamma \\ v_1, \ldots, v_n \end{matrix}\right) = 0 \quad \text{for } u > 1, \tag{6.5}$$

and that the condition $\mathrm{Re}\gamma > 0$ ensures that

$$u^{-1} G^{n,0}_{n,n}\left(u \,\middle|\, \begin{matrix} v_1 + \gamma, \ldots, v_n + \gamma \\ v_1, \ldots, v_n \end{matrix}\right)$$

is absolutely integrable at $u = 1$. These facts can be deduced from, for instance, [2, p. 208, formulae (5) and (6)].

LEMMA 6.4. *Let the hypotheses in Definition* 6.2 *be satisfied. If* $\mathrm{Re}(mv_k + \mu) + m > 1/p$ *for* $k = 1, \ldots, n$, *then* $G_1(v_1, \ldots, v_n; \gamma; m)$ *is a continuous linear mapping from* $F_{p,\mu}$ *into* $F_{p,\mu}$.

Proof. For $\varphi \in F_{p,\mu}$, we have

$$G_1(\nu_1,\ldots,\nu_n;\gamma;m)\varphi(x) = m^{-n\gamma+1}\int_0^\infty K(x^m/t^m)\varphi(t)\,dt/t$$

where

$$K(u) = u^{-1}G_{n,n}^{n,0}\left(u^{-1}\,\middle|\,\begin{matrix}\nu_1+\gamma,\ldots,\nu_n+\gamma\\ \nu_1,\ldots,\nu_n\end{matrix}\right)\quad (u>0).$$

To apply Lemma 6.1, we recall (6.5) and consider

$$\int_0^\infty u^{(1/p-\operatorname{Re}\mu)/m-1}\,|K(u)|\,du$$
$$= \int_0^1 t^{(\operatorname{Re}\mu-1/p)/m+1}\left|t^{-1}G_{n,n}^{n,0}\left(t\,\middle|\,\begin{matrix}\nu_1+\gamma,\ldots,\nu_n+\gamma\\ \nu_1,\ldots,\nu_n\end{matrix}\right)\right|dt. \tag{6.6}$$

At the origin,

$$G_{n,n}^{n,0}\left(t\,\middle|\,\begin{matrix}\nu_1+\gamma,\ldots,\nu_n+\gamma\\ \nu_1,\ldots,\nu_n\end{matrix}\right) = O(|t|^\nu)$$

where $\nu = \max(\operatorname{Re}\nu_1,\ldots,\operatorname{Re}\nu_n)$ by [**2**, p. 212]. By hypothesis, $\operatorname{Re}(m\nu+\mu)+m > 1/p$ and this ensures the convergence of (6.6) at $t=0$, while convergence at $t=1$ is catered for by Remark 6.3. Thus (6.6) is finite and the result now follows easily from Lemma 6.1.

LEMMA 6.5. *Under the hypotheses of Lemma* 6.4, *we have*

$$\mathcal{M}(x^{m\gamma}G_1(\nu_1,\ldots,\nu_n;\gamma;m)\varphi)(s-m\gamma) = m^{-n\gamma}\frac{\prod_{k=1}^n\Gamma(\nu_k+1-s/m)}{\prod_{k=1}^n\Gamma(\nu_k+\gamma+1-s/m)}\mathcal{M}\varphi(s) \tag{6.7}$$

for $\varphi \in C_0^\infty(0,\infty)$ *and* $\operatorname{Re}s = 1/p-\operatorname{Re}\mu$.

Proof. By [**2**, p. 209] and (6.5) we can write (6.4) in the form

$$m^{-n\gamma+1}\int_0^\infty G_{n,n}^{0,n}\left(\frac{x^m}{t^m}\,\middle|\,\begin{matrix}-\nu_1,\ldots,-\nu_n\\ -\nu_1-\gamma,\ldots,-\nu_n-\gamma\end{matrix}\right)\varphi(t)\,dt/t. \tag{6.8}$$

The required result follows when we use Lemma 6.1 (ii) in conjunction with [**3**, p. 337, formula (14)].

We have now reached the stage of relating L^α to the G-function.

THEOREM 6.6. *If* $m = n-a > 0$, $\operatorname{Re}(mb_k+\mu)+m > 1/p$ $(k=1,\ldots,n)$, *and* $\operatorname{Re}\alpha < 0$, *then*

$$L^\alpha\varphi = x^{-m\alpha}G_1(b_1,\ldots,b_n;-\alpha;m)\varphi\quad(\varphi\in F_{p,\mu}). \tag{6.9}$$

Proof. Under the given conditions, both sides of (6.9) define functions in $F_{p,\mu-m\alpha}$ by Theorem 4.2 and Lemma 6.4. They are equal when $\varphi \in C_0^\infty(0,\infty)$ and $1 \leqslant p \leqslant 2$ in view of Theorem 5.6 and Lemma 6.5. Using the continuity of the operators, established in Theorem 4.2 and Lemma 6.4, together with the fact that $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$ [**11**, Corollary 2.7], we can establish (6.9) in the case where $1 \leqslant p \leqslant 2$. In the case where $p > 2$, the above proof for $p = 2$ will hold for $\varphi \in C_0^\infty(0,\infty)$ and

$\operatorname{Re}(mb_k+\mu)+m > \frac{1}{2}\ (k = 1,\dots,n)$. We can relax $\frac{1}{2}$ to $1/p$ in the latter inequalities using analytic continuation with respect to $b_1,\dots,b_n$; analyticity of the left-hand side follows from [11, Theorem 3.31] and of the right-hand side from the analyticity of the G-function. We omit the details. Finally, continuity and density complete the proof as before.

We now state, without proofs, results for other fractional powers.

DEFINITION 6.7. In the notation of Definition 6.2, we define $G_2(v_1,\dots,v_n;\gamma;m)\varphi$ for suitable functions φ by

$$
\begin{aligned}
&(G_2(v_1,\dots,v_n;\gamma;m)\varphi)(x)\\
&\quad = x^{-m}m^{-n\gamma}\int_x^\infty G^{0,n}_{n,n}\left(\frac{t^m}{x^m}\,\middle|\, \begin{matrix} -v_1,\dots,-v_n \\ -v_1-\gamma,\dots,-v_n-\gamma\end{matrix}\right)\varphi(t)\,d(t^m)\\
&\quad = m^{-n\gamma+1}\int_x^\infty G^{0,n}_{n,n}\left(\frac{t^m}{x^m}\,\middle|\, \begin{matrix} 1-v_1,\dots,1-v_n \\ 1-v_1-\gamma,\dots,1-v_n-\gamma\end{matrix}\right)\varphi(t)\,dt/t. \qquad (6.10)
\end{aligned}
$$

THEOREM 6.8. *If* $m = n-a > 0$, $\operatorname{Re}(mb_k+\mu)+m > 1/p\ (k = 1,\dots,n)$ *and* $\operatorname{Re}\alpha < 0$, *then for* $\varphi \in F_{q,-\mu+m\alpha}$,

$$(L')^\alpha\varphi = G_2(b_1+1-1/m,\dots,b_n+1-1/m;\,-\alpha;m)x^{-m\alpha}\varphi. \qquad (6.11)$$

THEOREM 6.9. *Let* $m = a-n > 0$, $\operatorname{Re}(mb_k+\mu)-m < 1/p\ (k = 1,\dots,n)$, *and* $\operatorname{Re}\alpha < 0$.
(i) *For* $\varphi \in F_{p,\mu}$,

$$M^\alpha\varphi = x^{m\alpha}G_2(1-b_1,\dots,1-b_n;\,-\alpha;m)\varphi. \qquad (6.12)$$

(ii) *For* $\varphi \in F_{q,-\mu-m\alpha}$,

$$(M')^\alpha\varphi = G_1(-b_1+1/m,\dots,-b_n+1/m;\,-\alpha;m)x^{m\alpha}\varphi. \qquad (6.13)$$

REMARK 6.10. (i) In Theorems 6.6, 6.8, and 6.9, we have dealt with the simplest restrictions on the parameters $b_1,\dots,b_n$. For instance, if, in Theorem 6.6, $b_k \in A_{p,\mu,m}$ but $\operatorname{Re}(mb_k+\mu)+m < 1/p$ for some k, the proof breaks down and the investigation becomes more complicated. Here, we do not try to identify the kernels of our fractional powers in these more complicated cases. Some indication of what is involved can be obtained from (2.14) and (2.15). Likewise, for $\operatorname{Re}\alpha > 0$, the operator becomes an integro-differential operator while $\operatorname{Re}\alpha = 0\ (\alpha \neq 0)$ presents its own problems. We shall not pursue this further either.

(ii) In the case where $n = 2$, the G-functions appearing in (6.4) and (6.10) become hypergeometric functions of type ${}_2F_1$. In § 9, we shall examine this case in more detail and reconcile our results in this paper with those obtained in earlier work [8] and [11, Chapter 4].

7.

We now discuss the extension of L^α and M^α to the classes $F'_{p,\mu}$ of generalized functions.

We consider first the case where $a < n$. The motivation for the definitions of the extended operators comes, as usual, from consideration of regular functionals. If $g \in F_{q,-\mu}$, then g generates an element $\tilde{g} \in F'_{p,\mu}$ by means of the formula

$$(\tilde{g}, \varphi) = \int_0^\infty g(x)\varphi(x)\,dx \quad (\varphi \in F_{p,\mu}). \tag{7.1}$$

By Theorem 3.2, if r is a positive integer, $L^r g \in F_{q,-\mu-mr}$ and generates an element $(L^r g)^\sim \in F'_{p,\mu+mr}$ in an analogous way. The extended operator $(L^r)^\sim$, say, should be such that

$$(L^r)^\sim \tilde{g} = (L^r g)^\sim \tag{7.2}$$

as an equality in $F'_{p,\mu+mr}$. Thus, if $\varphi \in C_0^\infty(0,\infty)$,

$$\begin{aligned}((L^r)^\sim \tilde{g}, \varphi) = ((L^r g)^\sim, \varphi) &= \int_0^\infty (L^r g)(x)\varphi(x)\,dx \\ &= \int_0^\infty g(x)((L')^r\varphi)(x)\,dx = (\tilde{g}, (L')^r\varphi)\end{aligned} \tag{7.3}$$

on integrating by parts. Since $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu+mr}$ [**11**, Corollary 2.7], (7.3) will hold for all $\varphi \in F_{p,\mu+mr}$. This suggests that for any $f \in F'_{p,\mu}$, regular or not, we should define $(L^r)^\sim f$ by

$$((L^r)^\sim f, \varphi) = (f, (L')^r\varphi) \quad (\varphi \in F_{p,\mu+mr}).$$

This in turn suggests the definition of the extension of L^α to $F'_{p,\mu}$. We shall denote this extension by L^α rather than $(L^\alpha)^\sim$, since no confusion should arise.

DEFINITION 7.1. Let $a < n$, let α be any complex number, and let

$$b_k \in A_{q,-\mu,m} \quad (k = 1, \ldots, n).$$

If $f \in F'_{p,\mu}$, we define $L^\alpha f \in F'_{p,\mu+m\alpha}$ by

$$(L^\alpha f, \varphi) = (f, (L')^\alpha\varphi) \quad (\varphi \in F_{p,\mu+m\alpha}) \tag{7.4}$$

where $(L')^\alpha\varphi$ is defined by (4.6).

NOTE. By Theorem 4.10 (i), $(L')^\alpha\varphi$ exists as an element of $F_{p,\mu}$ under the given conditions so that the right-hand side of (7.4) is meaningful. Equation (7.4) therefore defines $L^\alpha f$ as a linear functional on $F_{p,\mu+m\alpha}$. That $L^\alpha f$ is also continuous is a consequence of the definition of convergence in $F'_{p,\mu+m\alpha}$.

We can easily obtain the properties of L^α on $F'_{p,\mu}$ from those of $(L')^\alpha$ on $F_{p,\mu+m\alpha}$.

THEOREM 7.2. *Let $a < n$ and let α and β be any complex numbers.*

(i) *If $b_k \in A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then L^α is a continuous linear mapping from $F'_{p,\mu}$ into $F'_{p,\mu+m\alpha}$. If also $b_k - \alpha \in A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then L^α is a homeomorphism from $F'_{p,\mu}$ onto $F'_{p,\mu+m\alpha}$ with $(L^\alpha)^{-1} = L^{-\alpha}$.*

(ii) *If $b_k \in A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then L^0 is the identity operator on $F'_{p,\mu}$.*

(iii) *Let $f \in F'_{p,\mu}$. If $\{b_k, b_k - \beta\} \subseteq A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then*

$$L^\alpha L^\beta f = L^{\alpha+\beta} f. \tag{7.5}$$

If $\{b_k, b_k - \alpha, b_k - \beta\} \subseteq A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then

$$L^\alpha L^\beta f = L^{\alpha+\beta} f = L^\beta L^\alpha f. \tag{7.6}$$

Proof. These results follow by applying standard results on adjoint operators [17, Theorems 1.10–1 and 1.10–2] in conjunction with Theorem 4.10. Consider, for instance, (iii). Under the given conditions, $L^\beta f \in F'_{p,\mu+m\beta}$ and $L^\alpha(L^\beta f) \in F'_{p,\mu+m\alpha+m\beta}$. Likewise, $L^{\alpha+\beta} f \in F'_{p,\mu+m\alpha+m\beta}$ so that both sides of (7.5) define elements of the same space. To prove equality, let $\varphi \in F_{p,\mu+m\alpha+m\beta}$. Then

$$\begin{aligned}(L^\alpha L^\beta f, \varphi) &= (L^\beta f, (L')^\alpha \varphi) = (f, (L')^\beta (L')^\alpha \varphi)\\ &= (f, (L')^{\alpha+\beta}\varphi) = (L^{\alpha+\beta} f, \varphi)\end{aligned}$$

where we have used (7.4) and Theorem 4.10(iii). This gives equality. Equation (7.6) and parts (i) and (ii) are established similarly.

The theory for $(L')^\alpha$ in the case where $a < n$ can now be stated briefly, the motivation for the definition being similar to that above.

DEFINITION 7.3. Let $a < n$, let α be any complex number, and let

$$b_k \in A_{q,-\mu,m} \quad (k = 1, \ldots, n).$$

If $f \in F'_{q,-\mu-m\alpha}$, we define $(L')^\alpha f \in F'_{q,-\mu}$ by

$$((L')^\alpha f, \varphi) = (f, L^\alpha \varphi) \quad (\varphi \in F_{q,-\mu}).$$

THEOREM 7.4. *Let $a < n$ and let α, β be any complex numbers.*

(i) *If $b_k \in A_{q,-\mu,m} (k = 1, \ldots, n)$, then $(L')^\alpha$ is a continuous linear mapping from $F'_{q,-\mu-m\alpha}$ into $F'_{q,-\mu}$. If also $b_k - \alpha \in A_{q,-\mu,m} (k = 1, \ldots, n)$, then $(L')^\alpha$ is a homeomorphism from $F'_{q,-\mu-m\alpha}$ onto $F'_{q,-\mu}$ with $[(L')^\alpha]^{-1} = (L')^{-\alpha}$.*

(ii) *If $b_k \in A_{q,-\mu,m} (k = 1, \ldots, n)$, then $(L')^0$ is the identity operator on $F'_{q,-\mu}$.*

(iii) *Let $f \in F'_{q,-\mu-m\alpha-m\beta}$. If $\{b_k, b_k - \beta\} \subseteq A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then*

$$(L')^\beta (L')^\alpha f = (L')^{\alpha+\beta} f. \tag{7.7}$$

If $\{b_k, b_k - \alpha, b_k - \beta\} \subseteq A_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then

$$(L')^\alpha (L')^\beta f = (L')^{\alpha+\beta} f = (L')^\beta (L')^\alpha f. \tag{7.8}$$

Proof. The details, which are similar to those in Theorem 7.2, are omitted.

Finally, we deal with the case where $a > n$. Again proofs are omitted.

DEFINITION 7.5. Let $a > n$, let α be any complex number, and let

$$1 - b_k \in A'_{q,-\mu,m} \quad (k = 1, \ldots, n).$$

If $f \in F'_{p,\mu}$, we define $M^\alpha f \in F'_{p,\mu-m\alpha}$ by

$$(M^\alpha f, \varphi) = (f, (M')^\alpha \varphi) \quad (\varphi \in F_{p,\mu-m\alpha}).$$

THEOREM 7.6. *Let $a > n$ and let α, β be any complex numbers.*

(i) *If $1 - b_k \in A'_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then M^α is a continuous linear mapping from $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$. If also $1 - b_k - \alpha \in A'_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then M^α is a homeomorphism from $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and $(M^\alpha)^{-1} = M^{-\alpha}$.*

(ii) *If $1 - b_k \in A'_{q,-\mu,m} (k = 1, \ldots, n)$, then M^0 is the identity operator on $F'_{p,\mu}$.*

(iii) *Let $f \in F'_{p,\mu}$. If $\{1 - b_k, 1 - b_k - \beta\} \subseteq A'_{q,-\mu,m}$ $(k = 1, \ldots, n)$, then*

$$M^\alpha M^\beta f = M^{\alpha+\beta} f. \tag{7.9}$$

If $\{1-b_k, 1-b_k-\alpha, 1-b_k-\beta\} \subseteq A'_{q,-\mu,m}$ $(k = 1, \dots, n)$, *then*

$$M^\alpha M^\beta f = M^{\alpha+\beta} f = M^\beta M^\alpha f. \tag{7.10}$$

DEFINITION 7.7. Let $a > n$, let α be any complex number, and let

$$1-b_k \in A'_{q,-\mu,m} \quad (k = 1, \dots, n).$$

If $f \in F'_{q,-\mu+m\alpha}$, we define $(M')^\alpha f \in F'_{q,-\mu}$ by

$$((M')^\alpha f, \varphi) = (f, M^\alpha \varphi) \quad (\varphi \in F_{q,-\mu}).$$

THEOREM 7.8. *Let $a > n$ and let α, β be any complex numbers.*

(i) *If* $1-b_k \in A'_{q,-\mu,m}$ $(k = 1, \dots, n)$, *then* $(M')^\alpha$ *is a continuous linear mapping from* $F'_{q,-\mu+m\alpha}$ *into* $F'_{q,-\mu}$. *If also* $1-b_k-\alpha \in A'_{q,-\mu,m}$ $(k = 1, \dots, n)$, *then* $(M')^\alpha$ *is a homeomorphism from* $F'_{q-\mu+m\alpha}$ *onto* $F'_{q,-\mu}$ *and* $[(M')^\alpha]^{-1} = (M')^{-\alpha}$.

(ii) *If* $1-b_k \in A'_{q,-\mu,m}$ $(k = 1, \dots, n)$, *then* $(M')^0$ *is the identity operator on* $F'_{q,-\mu}$.

(iii) *Let* $f \in F'_{q,-\mu+m\alpha+m\beta}$. *If* $\{1-b_k, 1-b_k-\beta\} \subseteq A'_{q,-\mu,m}$ $(k = 1, \dots, n)$, *then*

$$(M')^\beta (M')^\alpha f = (M')^{\alpha+\beta} f. \tag{7.11}$$

If $\{1-b_k, 1-b_k-\alpha, 1-b_k-\beta\} \subseteq A'_{q,-\mu,m}$ $(k = 1, \dots, n)$, *then*

$$(M')^\alpha (M')^\beta f = (M')^{\alpha+\beta} f = (M')^\beta (M')^\alpha f. \tag{7.12}$$

REMARK 7.9. Once again, the equations (7.5)–(7.12) hold under weaker conditions provided that we interpret appropriate expressions in terms of analytic continuation. For instance, let us consider (7.5). Suppose that $b_k \in A_{q,-\mu,m}$ $(k = 1, \dots, n)$ and let $f \in F'_{p,\mu}$ and $\varphi \in F_{p,\mu+m\alpha+m\beta}$. Under the additional assumption $b_k - \beta \in A_{q,-\mu,m}$ $(k = 1, \dots, n)$ we have, from (4.7),

$$(L^\alpha L^\beta f, \varphi) = (f, (L')^\beta (L')^\alpha \varphi) = (f, (L')^{\beta+\alpha} \varphi).$$

However, the last expression is meaningful without the additional assumption and we can use it to extend the definition of $L^\alpha L^\beta f$. In this way, we may say that (7.5) holds provided only that $b_k \in A_{q,-\mu,m}$ $(k = 1, \dots, n)$. Similar considerations apply to (7.6)–(7.12).

8.

In this section we show how our theory in §7 can be applied to obtain information about integral transforms involving the G-function relative to the spaces L^p_μ defined by (6.1) and (6.2). Instead of starting with L and reading off $a, b_1, \dots, b_n$, in this section we start with complex numbers $b_1, \dots, b_n$, from which we calculate $a, a_1, \dots, a_n$ using (3.2) and (1.2) and then form L, L', M, and M'.

Assume first that $m = n - a > 0$, $\operatorname{Re} \gamma > 0$, and that

$$\operatorname{Re}(mb_k + \mu) + m > 1/p \quad (k = 1, \dots, n).$$

We can consider the operator $G_1(b_1, \dots, b_n; \gamma; m)$ defined via (6.4). By Lemma 6.1 (i), this operator is a continuous linear mapping from L^p_μ into L^p_μ.

Suppose now that $g \in L^p_\mu$ is given and that we seek $f \in L^p_\mu$ such that

$$G_1(b_1,\ldots,b_n;\gamma;m)f = g \quad \text{a.e. on } (0,\infty). \tag{8.1}$$

To answer the questions of existence and uniqueness, we use our distributional theory. Firstly, (8.1) is equivalent to

$$x^{m\gamma}G_1(b_1,\ldots,b_n;\gamma;m)f = x^{m\gamma}g \quad \text{a.e. on } (0,\infty). \tag{8.2}$$

The functions f and $x^{m\gamma}g$ generate functionals $\tilde{f} \in F'_{q,-\mu}$ and $(x^{m\gamma}g)^{\sim} \in F'_{q,-\mu-m\gamma}$ respectively in the manner of (7.1). If $\varphi \in C_0^\infty(0,\infty)$, we then obtain

$$((x^{m\gamma}g)^{\sim},\varphi) = \int_0^\infty x^{m\gamma}G_1(b_1,\ldots,b_n;\gamma;m)f(x)\varphi(x)\,dx.$$

We now substitute for $G_1(b_1,\ldots,b_n;\gamma;m)f$ via (6.4) and interchange the order of integration (Fubini's theorem supplying the justification) to obtain, after routine calculation,

$$\begin{aligned}((x^{m\gamma}g)^{\sim},\varphi) &= \int_0^\infty f(x)G_2(b_1+1-1/m,\ldots,b_n+1-1/m;\gamma;m)x^{m\gamma}\varphi(x)\,dx\\ &= (\tilde{f},(L')^{-\gamma}\varphi)\\ &= (L^{-\gamma}\tilde{f},\varphi)\end{aligned}$$

where we have used (6.11) and (7.4) and regarded φ as a member of $F_{q,-\mu-m\gamma}$. Thus any classical solution $f \in L^p_\mu$ of (8.1) satisfies the equation

$$(L^{-\gamma}\tilde{f},\varphi) = ((x^{m\gamma}g)^{\sim},\varphi) \quad (\varphi \in C_0^\infty(0,\infty)).$$

Since $C_0^\infty(0,\infty)$ is dense in $F_{q,-\mu-m\gamma}$, we deduce that $h = \tilde{f}$ satisfies

$$L^{-\gamma}h = (x^{m\gamma}g)^{\sim}, \tag{8.3}$$

where equality holds in the sense of $F'_{q,-\mu-m\gamma}$.

DEFINITION 8.1. Let $m = n-a > 0$, $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(mb_k+\mu)+m > 1/p$ $(k = 1,\ldots,n)$, and $g \in L^p_\mu$. A functional $h \in F'_{q,-\mu}$ satisfying (8.3) will be called a *weak solution* of (8.1) or (8.2).

The discussion so far shows that, if $f \in L^p_\mu$ is a classical solution of (8.1), then $\tilde{f}$ is a weak solution of (8.1). However, it is possible to have a weak solution when no classical solution exists.

THEOREM 8.2. *Let* $m = n-a > 0$, $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(mb_k+\mu)+m > 1/p$ $(k = 1,\ldots,n)$ *and* $g \in L^p_\mu$. *Then* (8.1) *has a unique weak solution* $h \in F'_{q,-\mu}$ *given by*

$$h = L^\gamma((x^{m\gamma}g)^{\sim}). \tag{8.4}$$

Proof. For each k, $\operatorname{Re}(mb_k+\mu+m\gamma)+m > 1/p$ since $\operatorname{Re}\gamma > 0$, so that $\{b_k, b_k+\gamma\} \subseteq A_{p,\mu,m}$. By Theorem 7.2, $L^{-\gamma}$ is a homeomorphism from $F'_{q,-\mu}$ onto $F'_{q,-\mu-m\gamma}$ under the given conditions, with inverse L^γ. Equation (8.4) therefore follows at once from (8.3).

COROLLARY 8.3. *Under the hypotheses of Theorem* 8.2, *equation* (8.1) *has at most one solution* $f \in L^p_\mu$.

Proof. If f_1, f_2 were two solutions in L^p_μ of (8.1), $\tilde{f}_1$ and $\tilde{f}_2$ would be weak solutions of (8.1). By Theorem 8.2, $\tilde{f}_1 = \tilde{f}_2$ so that $f_1 = f_2$ as elements of L^p_μ.

To illustrate the possibility of no classical solution, we consider the following example.

EXAMPLE 8.4. Let $m = n-a > 0$, $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(mb_k+\mu)+m > 1/p$ $(k = 1,\ldots,n)$, and let δ_1 be the element in $F'_{q,-\mu}$ defined by

$$(\delta_1, \psi) = \psi(1) \quad (\psi \in F_{q,-\mu}). \tag{8.5}$$

For $\varphi \in F_{q,-\mu-m\gamma}$, (7.4), (6.11), and (6.10) give

$$\begin{aligned}(L^{-\gamma}\delta_1, \varphi) &= (\delta_1, (L')^{-\gamma}\varphi) = ((L')^{-\gamma}\varphi)(1)\\ &= m^{-n\gamma+1}\int_1^\infty G^{0,n}_{n,n}\left(t^m \,\middle|\, \begin{matrix}-b_1+1/m,\ldots,-b_n+1/m\\ -b_1+1/m-\gamma,\ldots,-b_n+1/m-\gamma\end{matrix}\right)t^{m\gamma}\varphi(t)\,dt/t.\end{aligned} \tag{8.6}$$

Using properties of the G-function [2, p. 213], we may write (8.6) in the form $((x^{m\gamma}g)^{\sim}, \varphi)$ where

$$g(x) = m^{-n\gamma+1}x^{-1}G^{n,0}_{n,n}\left(x^{-m} \,\middle|\, \begin{matrix}b_1+1-1/m+\gamma,\ldots,b_n+1-1/m+\gamma\\ b_1+1-1/m,\ldots,b_n+1-1/m\end{matrix}\right) \tag{8.7}$$

for $x > 0$, g being identically zero on $(0, 1)$. Proceeding as in the proof of Lemma 6.4, we can show that $g \in L^p_\mu$ under the given conditions. Thus, with g as in (8.7), (8.1) has the weak solution $h = \delta_1$ but, as δ_1 is a singular distribution, there can be no classical solution $f \in L^p_\mu$ of (8.1).

An indication of what governs the existence or non-existence of classical solutions of (8.1) can be obtained from the following purely formal analysis, which can be justified for functions in $F_{p,\mu}$, rather than L^p_μ, under appropriate conditions:

$$\begin{aligned}&G_1(b_1,\ldots,b_n;\gamma;m)f = g\\ \Rightarrow\ & L^{-\gamma}f = x^{m\gamma}G_1(b_1,\ldots,b_n;\gamma;m)f = x^{m\gamma}g \quad \text{(by (6.9))}\\ \Rightarrow\ & m^{-n\gamma}x^{m\gamma}\prod_{k=1}^{n} I_m^{b_k,\gamma}f = x^{m\gamma}g \quad \text{(by (4.1))}\\ \Rightarrow\ & f = m^{n\gamma}\prod_{k=1}^{n} I_m^{b_k+\gamma,-\gamma}g \quad \text{(by (2.18))}.\end{aligned}$$

For the last expression to make sense, it has to be possible to work out n 'fractional derivatives' of order γ and this is where the restriction on g comes in. However, we shall not pursue this matter any further here.

A similar collection of results can be stated briefly in the case where $m = a-n > 0$. In this case we may consider the equation

$$G_2(1-b_1,\ldots,1-b_n;\gamma;m)f = g \tag{8.8}$$

and the corresponding equation

$$M^{-\gamma}h = (x^{-m\gamma}g)^{\sim} \tag{8.9}$$

for weak solutions h of (8.8). Under appropriate conditions, a classical solution of (8.8) generates a weak solution $h = \tilde{f}$. We can again state a uniqueness theorem.

THEOREM 8.5. *Let* $m = a-n > 0$, $\operatorname{Re}\gamma > 0$, $\operatorname{Re}(mb_k+\mu)-m < 1/p$ $(k = 1,\dots,n)$ *and* $g \in L^p_\mu$. *Then* (8.8) *has a unique weak solution* $h \in F'_{q,-\mu}$ *given by*

$$h = M^\gamma((x^{-m\gamma}g)\tilde{\ }). \tag{8.10}$$

Proof. This follows easily from Theorem 7.6.

COROLLARY 8.6. *Under the hypotheses of Theorem* 8.5, *equation* (8.1) *has at most one solution* $f \in L^p_\mu$.

Proof. This is almost immediate from Theorem 8.5.

Once again, by considering $M^{-\gamma}\delta_1$, with δ_1 as in (8.5), it is easy to construct an example where there is no classical solution $f \in L^p_\mu$, and a criterion involving fractional derivatives for $g \in L^p_\mu$ to belong to the range of $G_2(1-b_1,\dots,1-b_n;\gamma;m)$ on L^p_μ could be derived if desired.

REMARK 8.7. Finally in this section, we should mention that Rooney [14] has conducted a comprehensive investigation into the behaviour of integral transforms involving the G-function on the spaces L^p_μ. In connection with our remark on ranges above, it must be mentioned that Rooney obtains a complete characterization of the range of his operators, the range being that of a certain fractional integral in the case corresponding to our operators given by (6.4) and (6.10). Rooney makes extensive use of the Mellin transform and multipliers and his results apply to the values $1 < p < \infty$ and $m = 1$ (although the extension to general positive m could easily be obtained).

9.

In this final section we look in a little more detail at the case where $n = 2$ in order to see how the results tie in with some of our earlier work [8] and also with the work of Sprinkhuizen-Kuyper [16].

We are thus concerned with

$$L = x^{a_1}Dx^{a_2}Dx^{a_3} \tag{9.1}$$

where $a = a_1+a_2+a_3$ is real and $a \neq 2$. From (3.2) we obtain

$$b_1 = (a_2+a_3-1)/m, \quad b_2 = a_3/m. \tag{9.2}$$

Dealing first with the case where $m = 2-a > 0$, we have

$$L^\alpha = m^{2\alpha}x^{-m\alpha}I_m^{b_1,-\alpha}I_m^{b_2,-\alpha}. \tag{9.3}$$

The form of the right-hand side indicates that, in this case, L^α is related to the hypergeometric operators discussed in [8] and [11, Chapter 4].

THEOREM 9.1. *If* $m = 2-a > 0$, $\{b_1,b_2\} \subseteq A_{p,\mu,m}$, *and* $\varphi \in F_{p,\mu}$, *then*

$$L^\alpha\varphi = m^{2\alpha}x^{-mb_1}H_1(b_2-b_1-\alpha,\ -\alpha;\ -2\alpha;m)x^{mb_1+m\alpha}\varphi, \tag{9.4}$$

where $H_1(b_2-b_1-\alpha, -\alpha; -2\alpha;m)$ *is as in* [11, Definition 4.6]. *In particular, if*

$\operatorname{Re}\alpha < 0$ *and* $\operatorname{Re}(mb_k+\mu)+m > 1/p$ $(k = 1,2)$, *then*

$$L^\alpha\varphi(x) = [\Gamma(-2\alpha)]^{-1}m^{2\alpha}x^{-mb_1}\int_0^x (x^m-t^m)^{-2\alpha-1}$$
$$\times {}_2F_1(b_2-b_1-\alpha,\ -\alpha;\ -2\alpha;1-x^m/t^m)t^{mb_1+m\alpha}\varphi(t)\,d(t^m), \tag{9.5}$$

where ${}_2F_1$ *denotes the Gauss hypergeometric function.*

Proof. From [**11**, Definition 4.6], when $\{b_1,b_2\} \subseteq A_{p,\mu,m}$ and $\varphi \in F_{p,\mu}$,

$$m^{2\alpha}x^{-mb_1}H_1(b_2-b_1-\alpha,\ -\alpha;\ -2\alpha;m)x^{mb_1+m\alpha}\varphi$$
$$= m^{2\alpha}x^{-mb_1}I_m^{-\alpha}x^{m(b_1-b_2+\alpha)}I_m^{-\alpha}x^{m(b_2-b_1-\alpha)}x^{mb_1+m\alpha}\varphi,$$

which readily reduces to $L^\alpha\varphi$ via (9.3), (2.16), and (2.24). Equation (9.5) is obtained from [**11**, Definition 4.2]. This completes the proof.

REMARK 9.2. We could also have arrived at (9.5) via Theorem 6.6. Under the given conditions

$$L^\alpha\varphi(x) = x^{-m\alpha-m}m^{2\alpha}\int_0^x G^{2,0}_{2,2}\left(\frac{t^m}{x^m}\middle|\begin{matrix}b_1-\alpha,b_2-\alpha\\ b_1,b_2\end{matrix}\right)\varphi(t)\,d(t^m).$$

To show that this gives (9.5) we have to show that, for $0 < u < 1$,

$$G^{2,0}_{2,2}\left(u\middle|\begin{matrix}b_1-\alpha,b_2-\alpha\\ b_1,b_2\end{matrix}\right) = (1-u)^{-2\alpha-1}u^{b_2}\,{}_2F_1(b_2-b_1-\alpha,\ -\alpha;\ -2\alpha;1-u)/\Gamma(-2\alpha). \tag{9.6}$$

If we extend the right-hand side to have the value zero for $u \geqslant 1$, the resulting function has as its Mellin transform

$$\Gamma(b_1+s)\Gamma(b_2+s)/\Gamma(b_1-\alpha+s)\Gamma(b_2-\alpha+s),$$

as can be checked from, for instance, [**3**, p. 337, formula (10)]. As this is also the Mellin transform of

$$G^{2,0}_{2,2}\left(u\middle|\begin{matrix}b_1-\alpha,b_2-\alpha\\ b_1,b_2\end{matrix}\right)$$

and this function is identically zero for $u > 1$ by (6.5), we have established (9.6).

For another illustration we consider M^α in the case where $a > 2$. By Definition 4.11 with $n = 2$, $m = a-2$, and $\varphi \in F_{p,\mu}$, we have

$$M^\alpha\varphi = m^{2\alpha}x^{m\alpha}K_m^{1-b_1,-\alpha}K_m^{1-b_2,-\alpha}\varphi$$

and we can express M^α in terms of another hypergeometric operator.

THEOREM 9.3. *If* $m = a-2 > 0$, $\{1-b_1, 1-b_2\} \subseteq A'_{p,\mu,m}$, *and* $\varphi \in F_{p,\mu}$, *then*

$$M^\alpha\varphi = m^{2\alpha}x^{m\alpha-mb_1+1}H_3(b_2-b_1-\alpha,\ -\alpha;\ -2\alpha;m)x^{mb_1+2m\alpha-1}\varphi, \tag{9.7}$$

where $H_3(b_2-b_1-\alpha, -\alpha; -2\alpha; m)$ *is as in* [**11**, p. 94]. *In particular, if* $\operatorname{Re}\alpha < 0$ *and* $\operatorname{Re}(mb_k+\mu)-m < 1/p$ $(k = 1,2)$, *then*

$$M^\alpha\varphi(x) = [\Gamma(-2\alpha)]^{-1}m^{2\alpha}x^{m\alpha-mb_1+m}\int_x^\infty (t^m-x^m)^{-2\alpha-1}$$
$$\times {}_2F_1(b_2-b_1-\alpha,\ -\alpha;\ -2\alpha;1-x^m/t^m)t^{mb_1+2m\alpha-m}\varphi(t)\,d(t^m). \tag{9.8}$$

Proof. This is similar to that of Theorem 9.1 and is omitted.

It is possible to study the other six cases, namely $(L')^\alpha$, M^α, $(M')^\alpha$ for $a < 2$ and L^α, $(L')^\alpha$, and $(M')^\alpha$ for $a > 2$ in a similar way but we omit the details. Instead we examine a particular case of Theorem 9.1.

EXAMPLE 9.4. Let $a_1 = -1$, $a_2 = 2-v$, and $a_3 = v-1$ so that $a = 0$ and $m = 2$. These are the values in (1.5) for which L becomes the operator $D^2 + vx^{-1}D$ considered by Sprinkhuizen-Kuyper [16]. In this case $b_1 = \frac{1}{2}(a_2+a_3+1-2) = 0$ and $b_2 = \frac{1}{2}a_3 = \frac{1}{2}(v-1)$. Then, under the conditions of Theorem 9.1,

$$\begin{aligned} L^\alpha\varphi(x) &= [\Gamma(-2\alpha)]^{-1}2^{2\alpha}\int_0^x (x^2-t^2)^{-2\alpha-1} \\ &\quad \times {}_2F_1(\tfrac{1}{2}v-\tfrac{1}{2}-\alpha, -\alpha; -2\alpha; 1-x^2/t^2)t^{2\alpha}\varphi(t)\,d(t^2) \\ &= [\Gamma(-2\alpha)]^{-1}\int_0^x \{(x^2-t^2)/2t\}^{-2\alpha-1} \\ &\quad \times {}_2F_1(-\alpha+\tfrac{1}{2}(v-1), -\alpha; -2\alpha; 1-x^2/t^2)\varphi(t)\,dt. \end{aligned} \tag{9.9}$$

This last expression turns up in [16, Theorem 2.5] (with α replaced by $-\alpha$). The only difference is that the result in [16] is concerned with functions defined on $[1,\infty)$ so that the lower limit in the integral is 1 and it is assumed that $x > 1$. It should be mentioned that the values given in (1.6) give rise to the same result, as the only effect is to interchange b_1 and b_2.

REMARK 9.5. Once again, it would be possible to obtain explicit expressions for $(L')^\alpha$, M^α, and $(M')^\alpha$ for the particular operator L in Example 9.4. Expressions similar to those in [16] appear with minor differences in the limits of integration because of the different domains of the functions in our work and in [16]. Generalized functions in the spaces $F'_{p,\mu}$ can also be handled as particular cases of the theory in § 7 and the results compared with the distributional results in [16]. We omit the details. However, it seems fair to say that our theory is in agreement with that of Sprinkhuizen-Kuyper but can handle a much larger class of operators.

In view of Example 9.4, it seems appropriate to conclude by looking at another situation where such an operator occurs. For each complex number η, we shall write

$$L_\eta = D^2 + (2\eta+1)x^{-1}D. \tag{9.10}$$

On replacing v by $2\eta+1$ in Example 9.4, we have $b_1 = 0$ and $b_2 = \eta$ so that, from (4.1),

$$L_\eta^\alpha = 2^{2\alpha}x^{-2\alpha}I_2^{0,-\alpha}I_2^{\eta,-\alpha}$$

on $F_{p,\mu}$ provided that $\{0,\eta\} \subseteq A_{p,\mu,2}$. As mentioned in [11, § 3.6] L_η plays an important role in generalized axially symmetric potential theory and elsewhere. In [11, Theorem 3.59], we proved that, under appropriate conditions,

$$I_2^{\eta,\beta}L_\eta = L_{\eta+\beta}I_2^{\eta,\beta}.$$

Now we offer the following extension of this result.

THEOREM 9.6. *Let* $\{0, \eta, \eta-\alpha, \eta+\beta\} \subseteq A_{p,\mu,2}$.
(i) *If* $\varphi \in F_{p,\mu}$, *then*

$$I_2^{\eta,\beta}(L_\eta)^\alpha \varphi = (L_{\eta+\beta})^\alpha I_2^{\eta,\beta}\varphi. \tag{9.11}$$

(ii) *If* $\varphi \in F_{q,-\mu+2\alpha}$, *then*

$$(L'_\eta)^\alpha K_2^{\eta+\frac{1}{2},\beta}\varphi = K_2^{\eta+\frac{1}{2},\beta}(L'_{\eta+\beta})^\alpha \varphi, \tag{9.12}$$

where L'_η *and* $L'_{\eta+\alpha}$ *denote the formal adjoints of* L_η *and* $L_{\eta+\alpha}$ *respectively.*

Proof. We prove (i), the proof of (ii) being similar. For $\varphi \in F_{p,\mu}$, we have

$$\begin{aligned}
I_2^{\eta,\beta}(L_\eta)^\alpha\varphi &= I_2^{\eta,\beta}2^{2\alpha}x^{-2\alpha}I_2^{0,-\alpha}I_2^{\eta,-\alpha}\varphi && \text{(by (4.1))}\\
&= 2^{2\alpha}x^{-2\alpha}I_2^{\eta-\alpha,\beta}I_2^{\eta,-\alpha}I_2^{0,-\alpha}\varphi && \text{(by (2.24),(2.26))}\\
&= 2^{2\alpha}x^{-2\alpha}I_2^{\eta,\beta-\alpha}I_2^{0,-\alpha}\varphi && \text{(by (2.29))}\\
&= 2^{2\alpha}x^{-2\alpha}I_2^{0,-\alpha}I_2^{\eta+\beta,-\alpha}I_2^{\eta,\beta}\varphi && \text{(by (2.26),(2.29))}\\
&= (L_{\eta+\beta})^\alpha I_2^{\eta,\beta}\varphi && \text{(by (4.1)).}
\end{aligned}$$

This completes the proof.

REMARK 9.7. Once again, the conditions $\eta-\alpha \in A_{p,\mu,m}$ and $\eta+\beta \in A_{p,\mu,m}$ can be removed if we interpret the left-hand and right-hand sides respectively of (9.11) and (9.12) in terms of their analytic continuations. Thus only $\{0,\eta\} \in A_{p,\mu,m}$ is crucial. Similar remarks apply to the corresponding theorem for generalized functions.

THEOREM 9.8. *Let* $\{0, \eta, \eta-\alpha, \eta+\beta\} \subseteq A_{q,-\mu,2}$.
(i) *If* $f \in F'_{p,\mu}$, *then*

$$I_2^{\eta,\beta}(L_\eta)^\alpha f = (L_{\eta+\beta})^\alpha I_2^{\eta,\beta} f.$$

(ii) *If* $f \in F'_{q,-\mu-2\alpha}$, *then*

$$(L'_\eta)^\alpha K_2^{\eta+\frac{1}{2},\beta} f = K_2^{\eta+\frac{1}{2},\beta}(L'_{\eta+\beta})^\alpha f.$$

Proof. These results follow from Definitions 7.1 and 7.3 and Theorem 9.6.

In conclusion, we may say that, by expressing a differential operator in an alternative form, it is possible to discover in a natural way how operators of the form $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$, with an appropriate value of m, can be used to study problems associated with the differential operator. Other instances of this will be discussed elsewhere.

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TYPESET AND PRINTED IN GREAT BRITAIN
BY JOHN WRIGHT AND SONS (PRINTING) LTD
AT THE STONEBRIDGE PRESS, BRISTOL

Paper 8

J. Austral. Math. Soc. (Series A) 34 (1983), 356-363

A NOTE ON THE INDEX LAWS OF FRACTIONAL CALCULUS

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(Received 16 January 1981)

Communicated by E. Strzelecki

Abstract

Two index laws for fractional integrals and derivatives, which have been extensively studied by E. R. Love, are shown to be special cases of an index law for general powers of certain differential operators, by means of the theory developed in a previous paper. Discussion of the two index laws, which are rather different in appearance, can thus be unified.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 26 A 33.

1

In [2], Love discussed two index laws for fractional integrals and derivatives and gave detailed conditions for their validity. These laws were also discussed in a distributional setting by Erdélyi [1] and later by McBride; see, for instance, [3, Chapter 3]. In order to state the index laws, we shall work, for convenience, with functions in the class $C^\infty(0,\infty)$ of smooth, complex-valued functions defined on $(0,\infty)$. Throughout, m will denote a positive real number.

Let α be a complex number and let ϕ ($\in C^\infty(0,\infty)$) be a suitably restricted function.

(i) For $\operatorname{Re}\alpha > 0$, we define $I_m^\alpha\phi$ by

$$(1.1)\qquad (I_m^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x^m - t^m)^{\alpha-1}\phi(t)\,d(t^m)$$

where $d(t^m) = mt^{m-1}\,dt$. The definition is extended step-by-step to the region $\operatorname{Re}\alpha \leqslant 0$ by repeated application of the formula

$$I_m^\alpha \phi = D_m I_m^{\alpha+1}\phi \tag{1.2}$$

where

$$D_m \equiv \frac{d}{dx^m} \equiv m^{-1}x^{1-m}\frac{d}{dx} \equiv m^{-1}x^{1-m}D. \tag{1.3}$$

(ii) For $\operatorname{Re}\alpha > 0$, we define $K_m^\alpha\phi$ by

$$(K_m^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t^m - x^m)^{\alpha-1}\phi(t)\,d(t^m). \tag{1.4}$$

The definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by repeated application of the formula

$$K_m^\alpha\phi = (-D_m)K_m^{\alpha+1}\phi. \tag{1.5}$$

We can now state the two index laws referred to above.

(i) (First Index Law). For any complex numbers α and β,

$$I_m^\alpha I_m^\beta \phi = I_m^{\alpha+\beta}\phi = I_m^\beta I_m^\alpha\phi, \tag{1.6}$$

$$K_m^\alpha K_m^\beta \phi = K_m^{\alpha+\beta}\phi = K_m^\beta K_m^\alpha\phi. \tag{1.7}$$

(ii) (Second Index Law). For complex numbers α, β and γ such that $\alpha+\beta+\gamma = 0$,

$$x^{m\alpha} I_m^\beta x^{m\gamma}\phi = I_m^{-\gamma} x^{-m\beta} I_m^{-\alpha}\phi, \tag{1.8}$$

$$x^{m\alpha} K_m^\beta x^{m\gamma}\phi = K_m^{-\gamma} x^{-m\beta} K_m^{-\alpha}\phi. \tag{1.9}$$

(Throughout we shall use x^λ to denote the operation of multiplying a function of the variable x by x^λ.)

The first index law is very familiar but the second index law is much less familiar and seems, in the first instance, rather strange and unexpected. The object of this note is to point out that both laws can be brought under the same umbrella. In a recent paper [4], we have shown how it is possible to define general powers of an ordinary differential operator

$$L \equiv x^{a_1}Dx^{a_2}Dx^{a_3}\cdots x^{a_n}Dx^{a_{n+1}} \qquad \left(D \equiv \frac{d}{dx}\right) \tag{1.10}$$

of order n, as well as powers of the related operators

$$L' = (-1)^n x^{a_{n+1}}Dx^{a_n}\cdots x^{a_3}Dx^{a_2}Dx^{a_1}, \tag{1.11}$$

$$M = (-1)^n L \quad \text{and} \quad M' = (-1)^n L'. \tag{1.12}$$

This was done under the assumption that the complex numbers $a_1, \ldots, a_{n+1}$ were such that

$$a = \sum_{i=1}^{n+1} a_i \quad \text{is real} \tag{1.13}$$

and

$$m = |a - n| > 0. \tag{1.14}$$

The powers satisfied a "first index law" so that, for instance,

$$L^{\alpha}L^{\beta}\phi = L^{\alpha+\beta}\phi = L^{\beta}L^{\alpha}\phi, \tag{1.15}$$

$$(L')^{\alpha}(L')^{\beta}\phi = (L')^{\alpha+\beta}\phi = (L')^{\beta}(L')^{\alpha}\phi, \tag{1.16}$$

under appropriate conditions. The two cases $a < n$ and $a > n$ produced different expressions for the general powers. We shall show that in the case $a < n$, (1.6) and (1.7) lead to (1.15) and (1.16) and, conversely, by choosing a suitable L, that (1.15) and (1.16) contain (1.6) and (1.7) so that, in a sense, (1.15) and (1.16) are equivalent to (1.6) and (1.7) in this case. More interestingly perhaps, in the case $a > n$, analogues of (1.15) and (1.16) for M and M' are equivalent to (1.8) and (1.9) so that the *first* index law for M and M' gives rise to the *second* index law for I_m^{α} and K_m^{α}.

2

In what follows, ϕ will be a function in $C^{\infty}(0, \infty)$, such that all the subsequent formal analysis is valid. For instance, we may choose ϕ to be an element of the space $F_{p,\mu}$ defined in [3, Chapter 2]. Precise conditions under which the various steps can be justified within the framework of the $F_{p,\mu}$ spaces can be found in [3] and will not be detailed here.

For $m > 0$, $\operatorname{Re}\alpha > 0$ and suitable complex numbers η, we define the Erdélyi-Kober operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ by

$$I_m^{\eta,\alpha}\phi = x^{-m\eta-m\alpha}I_m^{\alpha}x^{m\eta}\phi, \tag{2.1}$$

$$K_m^{\eta,\alpha}\phi = x^{m\eta}K_m^{\alpha}x^{-m\eta-m\alpha}\phi, \tag{2.2}$$

where I_m^{α} and K_m^{α} are as in (1.1)–(1.5). Thus, for $\operatorname{Re}\alpha > 0$ and suitable η,

$$(I_m^{\eta,\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u^m)^{\alpha-1}u^{m\eta}\phi(xu)\,d(u^m), \tag{2.3}$$

$$(K_m^{\eta,\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_1^{\infty} (u^m-1)^{\alpha-1}u^{-m\eta-m\alpha}\phi(xu)\,d(u^m), \tag{2.4}$$

while, for any complex α,

$$I_m^{\eta,\alpha}\phi = (\eta+\alpha+1)I_m^{\eta,\alpha+1}\phi + m^{-1}I_m^{\eta,\alpha+1}\delta\phi, \tag{2.5}$$

$$K_m^{\eta,\alpha}\phi = (\eta+\alpha)K_m^{\eta,\alpha+1}\phi - m^{-1}K_m^{\eta,\alpha+1}\delta\phi \tag{2.6}$$

[3, formulae (3.14) and (3.18)], where

$$\delta \equiv x\frac{d}{dx}. \tag{2.7}$$

We can prove that

$$I_m^{\eta,\alpha} I_m^{\xi,\beta}\phi = I_m^{\xi,\beta} I_m^{\eta,\alpha}\phi \tag{2.8}$$

under appropriate conditions. For Re $\alpha > 0$, Re $\beta > 0$, (2.8) follows on using (2.3) and interchanging the order of integration; the restriction on α and β can then be removed by repeated application of (2.5) together with the fact that δ commutes with each I operator. Also, by (1.6) and (2.1),

$$I_m^{\eta-\gamma,-\alpha} I_m^{\eta,-\gamma}\phi = I_m^{\eta,-\alpha-\gamma}\phi. \tag{2.9}$$

Hence by (2.8) and (2.9), for suitable functions ψ,

$$I_m^{\eta,-\gamma} I_m^{\eta-\gamma,-\alpha}\psi = I_m^{\eta,-\alpha-\gamma}\psi.$$

(2.1) then gives

$$x^{-m\eta+m\gamma} I_m^{-\gamma} x^{m\eta} x^{-m\eta+m\gamma+m\alpha} I_m^{-\alpha} x^{m\eta-m\gamma}\psi = x^{-m\eta+m\alpha+m\gamma} I_m^{-\alpha-\gamma} x^{m\eta}\psi. \tag{2.10}$$

If we write $\phi(x) = x^{m\eta-m\gamma}\psi(x)$ and $\beta = -\alpha - \gamma$ so that $\alpha + \beta + \gamma = 0$, (2.10) becomes (1.8). Thus we may say that the first index law for I_m^α together with the commutativity of the Erdélyi-Kober operators leads to the second index law for I_m^α. Similarly, we can show that (1.7) and the result

$$K_m^{\eta,\alpha} K_m^{\xi,\beta}\phi = K_m^{\xi,\beta} K_m^{\eta,\alpha}\phi \tag{2.11}$$

lead to (1.9). This gives us one way of viewing the second index laws for I_m^α and K_m^α.

3

Now we show how the two index laws for I_m^α and K_m^α are related to the first index law for general powers of the operators L, L', M and M' defined by (1.10)–(1.12). As indicated above, the two cases $a < n$ and $a > n$ need separate treatment. We shall consider L and L' for $a < n$ and M and M' for $a > n$.

The method used in [4] relied on rewriting the operator L, defined by (1.10), in the equivalent form

$$L = m^n x^{a-n} \prod_{k=1}^{n} x^{m-mb_k} D_m x^{mb_k} \tag{3.1}$$

where

$$b_k = \frac{1}{m}\left(\sum_{i=k+1}^{n+1} a_i + k - n\right) \qquad (k = 1,\ldots,n). \tag{3.2}$$

In the case $a < n$, induction shows that, for $r = 1, 2, \ldots$,

$$L^r = m^{nr} x^{-mr} \prod_{k=1}^{n} x^{mr-mb_k} (D_m)^r x^{mb_k}$$

and, since $(D_m)^r = I_m^{-r}$ under appropriate conditions, we can use (2.1) to write

$$L^r = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k,-r}$$

which in turn leads to the definition of L^α, for any complex number α, as the operator

$$L^\alpha = m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha}. \tag{3.3}$$

(The product on the right-hand side is unambiguous in view of (2.8).) (2.1) and (2.8) give

$$\begin{aligned} L^\alpha L^\beta \phi &= m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha} m^{n\beta} x^{-m\beta} \prod_{k=1}^{n} I_m^{b_k,-\beta} \phi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k-\beta,-\alpha} \prod_{k=1}^{n} I_m^{b_k,-\beta} \phi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k-\beta,-\alpha} I_m^{b_k,-\beta} \phi \end{aligned} \tag{3.4}$$

while

$$L^{\alpha+\beta} \phi = m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k,-(\alpha+\beta)} \phi. \tag{3.5}$$

That the right-hand sides of (3.4) and (3.5) are equal is a consequence of (2.9), which in turn is a consequence of (1.6). Thus we may say that

$$L^\alpha L^\beta = L^{\alpha+\beta} \tag{3.6}$$

is a consequence of the first index law for I_m^α in this case. Conversely, we may regard (1.6) as a special case of (3.6) corresponding to

$$L = mD_m \equiv x^{1-m} D. \tag{3.7}$$

In the notation of (1.10), $n = 1$, $a_1 = 1 - m$, $a_2 = 0$, $b_1 = 0$ so that

$$L^{-\alpha} = m^{-\alpha} x^{m\alpha} I_m^{0,\alpha} = m^{-\alpha} I_m^\alpha.$$

Thus $L^{-\alpha} L^{-\beta} = L^{-(\alpha+\beta)} \Rightarrow I_m^\alpha I_m^\beta = I_m^{\alpha+\beta}$ as required.

In a similar fashion, for $a < n$, we define $(L')^\alpha$ by

$$(L')^\alpha \phi = m^{n\alpha} \prod_{k=1}^{n} K_m^{b_k+1-1/m,-\alpha} x^{-m\alpha} \phi. \tag{3.8}$$

The index law

$$(3.9) \qquad (L')^{\alpha}(L')^{\beta} = (L')^{\alpha+\beta}$$

is a consequence of (1.7). Conversely, with L as in (3.7), we find that

$$(L')^{-\alpha} = m^{-\alpha}x^{m-1}K_m^{\alpha}x^{1-m}$$

so that (1.7) is a special case of (3.9).

We now consider the case $a > n$. (1.12) and (3.1) give

$$M = m^n x^m \prod_{k=1}^{n} x^{m-mb_k}(-D_m)x^{mb_k}.$$

By induction, we obtain, for $r = 1, 2, \ldots$,

$$M^r = m^{nr}x^{mr} \prod_{k=1}^{n} x^{m-mb_k}(-D_m)^r x^{mb_k+m(r-1)}$$

and, since $(-D_m)^r = K_m^{-r}$ under appropriate conditions, (2.1) gives

$$M^r = m^{nr}x^{mr} \prod_{k=1}^{n} K_m^{1-b_k,-r}.$$

(The inductive step requires the result $(-D_m)^r x^{m(r+1)}(-D_m) = x^m(-D_m)^{r+1}x^{mr}$, a special case of (1.9) which can be established by Leibnitz' formula.) This suggests that we define M^{α}, for any complex number α, to be the operator

$$(3.10) \qquad M^{\alpha} = m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\alpha}.$$

From (2.2) and (2.11), we obtain

$$(3.11) \qquad \begin{aligned} M^{\alpha}M^{\beta}\phi &= m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\alpha} m^{n\beta}x^{m\beta} \prod_{k=1}^{n} K_m^{1-b_k,-\beta}\phi \\ &= m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k-\beta,-\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\beta}\phi \\ &= m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k-\beta,-\alpha}K_m^{1-b_k,-\beta}\phi \end{aligned}$$

while

$$(3.12) \qquad M^{\alpha+\beta}\phi = m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k,-(\alpha+\beta)}\phi.$$

The right-hand sides of (3.11) and (3.12) are equal provided that

$$K_m^{1-b_k-\beta,-\alpha}K_m^{1-b_k,-\beta}\phi = K_m^{1-b_k,-(\alpha+\beta)}\phi,$$

which, by (2.2), is equivalent to

$$(3.13) \qquad x^{m-mb_k-m\beta}K_m^{-\alpha}x^{-m+mb_k+m\beta+m\alpha}x^{m-mb_k}K_m^{-\beta}x^{-m+mb_k+m\beta}\phi$$
$$= x^{m-mb_k}K_m^{-(\alpha+\beta)}x^{-m+mb_k+m\alpha+m\beta}\phi, \quad \text{or}$$
$$K_m^{-\alpha}x^{m(\alpha+\beta)}K_m^{-\beta}\psi = x^{m\beta}K_m^{-(\alpha+\beta)}x^{m\alpha}\psi$$

where $\psi(x) = x^{-m+mb_k+m\beta}\phi(x)$. (3.13) is simply (1.9) with α, β, γ and ϕ replaced by β, $-(\alpha+\beta)$, α and ψ respectively. Thus the equation

$$(3.14) \qquad M^\alpha M^\beta = M^{\alpha+\beta}$$

is a consequence of the second index law for K_m^α. Conversely, (1.9) is a special case of (3.14), corresponding to

$$(3.15) \qquad L = xDx^m.$$

In the notation of (1.10), $n = 1$, $a_1 = 1$, $a_2 = m$, $b_1 = 1$ so that

$$M^\alpha = m^\alpha x^{m\alpha}K_m^{0,-\alpha} = m^\alpha x^{m\alpha}K_m^{-\alpha}x^{m\alpha}.$$

If $\alpha+\beta+\gamma = 0$, then

$$M^{\gamma+\alpha}\psi = M^\gamma M^\alpha\psi$$
$$\Rightarrow m^{\gamma+\alpha}x^{m(\gamma+\alpha)}K_m^{-(\gamma+\alpha)}x^{m(\gamma+\alpha)}\psi = m^\gamma x^{m\gamma}K_m^{-\gamma}x^{m\gamma}m^\alpha x^{m\alpha}K_m^{-\alpha}x^{m\alpha}\psi$$
$$\Rightarrow x^{m\alpha}K_m^{\beta}x^{m\gamma}\phi = K_m^{-\gamma}x^{-m\beta}K_m^{-\alpha}\phi$$

where $\phi(x) = x^{m\alpha}\psi(x)$. This gives (1.9).

Similarly, for $a > n$, we define $(M')^\alpha$ by

$$(3.16) \qquad (M')^\alpha\phi = m^{n\alpha}\prod_{k=1}^{n} I_m^{-b_k+1/m,-\alpha}x^{m\alpha}\phi.$$

The index law

$$(3.17) \qquad (M')^\alpha(M')^\beta = (M')^{\alpha+\beta}$$

is a consequence of (1.8). Conversely, with L as in (3.15),

$$(M')^\alpha = m^\alpha x^{m\alpha+m-1}I_m^{-\alpha}x^{m\alpha-m+1}$$

and, as above, we can show that (1.8) is a special case of (3.17).

Finally, we mention that the results are valid in the setting of distribution theory, for instance in the spaces $F'_{p,\mu}$ introduced in [3, Chapter 2].

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[1] A. Erdélyi, 'Fractional integrals of generalised functions,' *J. Austral. Math. Soc.* **14** (1972), 30–37.

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J. Austral. Math. Soc. (Series A) 34 (1983), 356-363

A NOTE ON THE INDEX LAWS OF FRACTIONAL CALCULUS

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(Received 16 January 1981)

Communicated by E. Strzelecki

Abstract

Two index laws for fractional integrals and derivatives, which have been extensively studied by E. R. Love, are shown to be special cases of an index law for general powers of certain differential operators, by means of the theory developed in a previous paper. Discussion of the two index laws, which are rather different in appearance, can thus be unified.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 26 A 33.

1

In [2], Love discussed two index laws for fractional integrals and derivatives and gave detailed conditions for their validity. These laws were also discussed in a distributional setting by Erdélyi [1] and later by McBride; see, for instance, [3, Chapter 3]. In order to state the index laws, we shall work, for convenience, with functions in the class $C^\infty(0, \infty)$ of smooth, complex-valued functions defined on $(0, \infty)$. Throughout, m will denote a positive real number.

Let α be a complex number and let ϕ ($\in C^\infty(0, \infty)$) be a suitably restricted function.

(i) For Re $\alpha > 0$, we define $I_m^\alpha \phi$ by

$$(1.1) \qquad (I_m^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x^m - t^m)^{\alpha-1} \phi(t)\, d(t^m)$$

where $d(t^m) = mt^{m-1}\,dt$. The definition is extended step-by-step to the region $\operatorname{Re}\alpha \leqslant 0$ by repeated application of the formula

$$I_m^\alpha\phi = D_m I_m^{\alpha+1}\phi \tag{1.2}$$

where

$$D_m \equiv \frac{d}{dx^m} \equiv m^{-1}x^{1-m}\frac{d}{dx} \equiv m^{-1}x^{1-m}D. \tag{1.3}$$

(ii) For $\operatorname{Re}\alpha > 0$, we define $K_m^\alpha\phi$ by

$$(K_m^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t^m - x^m)^{\alpha-1}\phi(t)\,d(t^m). \tag{1.4}$$

The definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by repeated application of the formula

$$K_m^\alpha\phi = (-D_m)K_m^{\alpha+1}\phi. \tag{1.5}$$

We can now state the two index laws referred to above.

(i) (First Index Law). For any complex numbers α and β,

$$I_m^\alpha I_m^\beta\phi = I_m^{\alpha+\beta}\phi = I_m^\beta I_m^\alpha\phi, \tag{1.6}$$

$$K_m^\alpha K_m^\beta\phi = K_m^{\alpha+\beta}\phi = K_m^\beta K_m^\alpha\phi. \tag{1.7}$$

(ii) (Second Index Law). For complex numbers α, β and γ such that $\alpha + \beta + \gamma = 0$,

$$x^{m\alpha}I_m^\beta x^{m\gamma}\phi = I_m^{-\gamma}x^{-m\beta}I_m^{-\alpha}\phi, \tag{1.8}$$

$$x^{m\alpha}K_m^\beta x^{m\gamma}\phi = K_m^{-\gamma}x^{-m\beta}K_m^{-\alpha}\phi. \tag{1.9}$$

(Throughout we shall use x^λ to denote the operation of multiplying a function of the variable x by x^λ.)

The first index law is very familiar but the second index law is much less familiar and seems, in the first instance, rather strange and unexpected. The object of this note is to point out that both laws can be brought under the same umbrella. In a recent paper [4], we have shown how it is possible to define general powers of an ordinary differential operator

$$L \equiv x^{a_1}Dx^{a_2}Dx^{a_3}\cdots x^{a_n}Dx^{a_{n+1}} \qquad \left(D \equiv \frac{d}{dx}\right) \tag{1.10}$$

of order n, as well as powers of the related operators

$$L' = (-1)^n x^{a_{n+1}}Dx^{a_n}\cdots x^{a_3}Dx^{a_2}Dx^{a_1}, \tag{1.11}$$

$$M = (-1)^n L \quad \text{and} \quad M' = (-1)^n L'. \tag{1.12}$$

This was done under the assumption that the complex numbers $a_1, \ldots, a_{n+1}$ were such that

$$a = \sum_{i=1}^{n+1} a_i \quad \text{is real} \tag{1.13}$$

and

$$m = |a - n| > 0. \tag{1.14}$$

The powers satisfied a "first index law" so that, for instance,

$$L^{\alpha}L^{\beta}\phi = L^{\alpha+\beta}\phi = L^{\beta}L^{\alpha}\phi, \tag{1.15}$$

$$(L')^{\alpha}(L')^{\beta}\phi = (L')^{\alpha+\beta}\phi = (L')^{\beta}(L')^{\alpha}\phi, \tag{1.16}$$

under appropriate conditions. The two cases $a < n$ and $a > n$ produced different expressions for the general powers. We shall show that in the case $a < n$, (1.6) and (1.7) lead to (1.15) and (1.16) and, conversely, by choosing a suitable L, that (1.15) and (1.16) contain (1.6) and (1.7) so that, in a sense, (1.15) and (1.16) are equivalent to (1.6) and (1.7) in this case. More interestingly perhaps, in the case $a > n$, analogues of (1.15) and (1.16) for M and M' are equivalent to (1.8) and (1.9) so that the *first* index law for M and M' gives rise to the *second* index law for I_m^{α} and K_m^{α}.

2

In what follows, ϕ will be a function in $C^{\infty}(0, \infty)$, such that all the subsequent formal analysis is valid. For instance, we may choose ϕ to be an element of the space $F_{p,\mu}$ defined in [3, Chapter 2]. Precise conditions under which the various steps can be justified within the framework of the $F_{p,\mu}$ spaces can be found in [3] and will not be detailed here.

For $m > 0$, $\operatorname{Re}\alpha > 0$ and suitable complex numbers η, we define the Erdélyi-Kober operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ by

$$I_m^{\eta,\alpha}\phi = x^{-m\eta-m\alpha}I_m^{\alpha}x^{m\eta}\phi, \tag{2.1}$$

$$K_m^{\eta,\alpha}\phi = x^{m\eta}K_m^{\alpha}x^{-m\eta-m\alpha}\phi, \tag{2.2}$$

where I_m^{α} and K_m^{α} are as in (1.1)–(1.5). Thus, for $\operatorname{Re}\alpha > 0$ and suitable η,

$$(I_m^{\eta,\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u^m)^{\alpha-1}u^{m\eta}\phi(xu)\,d(u^m), \tag{2.3}$$

$$(K_m^{\eta,\alpha}\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_1^{\infty} (u^m-1)^{\alpha-1}u^{-m\eta-m\alpha}\phi(xu)\,d(u^m), \tag{2.4}$$

while, for any complex α,

$$I_m^{\eta,\alpha}\phi = (\eta+\alpha+1)I_m^{\eta,\alpha+1}\phi + m^{-1}I_m^{\eta,\alpha+1}\delta\phi, \tag{2.5}$$

$$K_m^{\eta,\alpha}\phi = (\eta+\alpha)K_m^{\eta,\alpha+1}\phi - m^{-1}K_m^{\eta,\alpha+1}\delta\phi \tag{2.6}$$

[3, formulae (3.14) and (3.18)], where

$$\delta \equiv x\frac{d}{dx}. \tag{2.7}$$

We can prove that

$$I_m^{\eta,\alpha} I_m^{\xi,\beta}\phi = I_m^{\xi,\beta} I_m^{\eta,\alpha}\phi \tag{2.8}$$

under appropriate conditions. For $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, (2.8) follows on using (2.3) and interchanging the order of integration; the restriction on α and β can then be removed by repeated application of (2.5) together with the fact that δ commutes with each I operator. Also, by (1.6) and (2.1),

$$I_m^{\eta-\gamma,-\alpha} I_m^{\eta,-\gamma}\phi = I_m^{\eta,-\alpha-\gamma}\phi. \tag{2.9}$$

Hence by (2.8) and (2.9), for suitable functions ψ,

$$I_m^{\eta,-\gamma} I_m^{\eta-\gamma,-\alpha}\psi = I_m^{\eta,-\alpha-\gamma}\psi.$$

(2.1) then gives

$$x^{-m\eta+m\gamma} I_m^{-\gamma} x^{m\eta} x^{-m\eta+m\gamma+m\alpha} I_m^{-\alpha} x^{m\eta-m\gamma}\psi = x^{-m\eta+m\alpha+m\gamma} I_m^{-\alpha-\gamma} x^{m\eta}\psi. \tag{2.10}$$

If we write $\phi(x) = x^{m\eta-m\gamma}\psi(x)$ and $\beta = -\alpha-\gamma$ so that $\alpha+\beta+\gamma = 0$, (2.10) becomes (1.8). Thus we may say that the first index law for I_m^α together with the commutativity of the Erdélyi-Kober operators leads to the second index law for I_m^α. Similarly, we can show that (1.7) and the result

$$K_m^{\eta,\alpha} K_m^{\xi,\beta}\phi = K_m^{\xi,\beta} K_m^{\eta,\alpha}\phi \tag{2.11}$$

lead to (1.9). This gives us one way of viewing the second index laws for I_m^α and K_m^α.

3

Now we show how the two index laws for I_m^α and K_m^α are related to the first index law for general powers of the operators L, L', M and M' defined by (1.10)–(1.12). As indicated above, the two cases $a < n$ and $a > n$ need separate treatment. We shall consider L and L' for $a < n$ and M and M' for $a > n$.

The method used in [4] relied on rewriting the operator L, defined by (1.10), in the equivalent form

$$L = m^n x^{a-n} \prod_{k=1}^{n} x^{m-mb_k} D_m x^{mb_k} \tag{3.1}$$

where

$$b_k = \frac{1}{m}\left(\sum_{i=k+1}^{n+1} a_i + k - n\right) \qquad (k = 1,\ldots,n). \tag{3.2}$$

In the case $a < n$, induction shows that, for $r = 1, 2, \ldots,$

$$L^r = m^{nr} x^{-mr} \prod_{k=1}^{n} x^{mr-mb_k} (D_m)^r x^{mb_k}$$

and, since $(D_m)^r = I_m^{-r}$ under appropriate conditions, we can use (2.1) to write

$$L^r = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k,-r}$$

which in turn leads to the definition of L^α, for any complex number α, as the operator

$$L^\alpha = m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha}. \tag{3.3}$$

(The product on the right-hand side is unambiguous in view of (2.8).) (2.1) and (2.8) give

$$\begin{aligned} L^\alpha L^\beta \phi &= m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k,-\alpha} m^{n\beta} x^{-m\beta} \prod_{k=1}^{n} I_m^{b_k,-\beta} \phi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k-\beta,-\alpha} \prod_{k=1}^{n} I_m^{b_k,-\beta} \phi \\ &= m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k-\beta,-\alpha} I_m^{b_k,-\beta} \phi \end{aligned} \tag{3.4}$$

while

$$L^{\alpha+\beta} \phi = m^{n(\alpha+\beta)} x^{-m(\alpha+\beta)} \prod_{k=1}^{n} I_m^{b_k,-(\alpha+\beta)} \phi. \tag{3.5}$$

That the right-hand sides of (3.4) and (3.5) are equal is a consequence of (2.9), which in turn is a consequence of (1.6). Thus we may say that

$$L^\alpha L^\beta = L^{\alpha+\beta} \tag{3.6}$$

is a consequence of the first index law for I_m^α in this case. Conversely, we may regard (1.6) as a special case of (3.6) corresponding to

$$L = mD_m \equiv x^{1-m} D. \tag{3.7}$$

In the notation of (1.10), $n = 1$, $a_1 = 1 - m$, $a_2 = 0$, $b_1 = 0$ so that

$$L^{-\alpha} = m^{-\alpha} x^{m\alpha} I_m^{0,\alpha} = m^{-\alpha} I_m^\alpha.$$

Thus $L^{-\alpha} L^{-\beta} = L^{-(\alpha+\beta)} \Rightarrow I_m^\alpha I_m^\beta = I_m^{\alpha+\beta}$ as required.

In a similar fashion, for $a < n$, we define $(L')^\alpha$ by

$$(L')^\alpha \phi = m^{n\alpha} \prod_{k=1}^{n} K_m^{b_k+1-1/m,-\alpha} x^{-m\alpha} \phi. \tag{3.8}$$

The index law

$$(L')^{\alpha}(L')^{\beta} = (L')^{\alpha+\beta} \tag{3.9}$$

is a consequence of (1.7). Conversely, with L as in (3.7), we find that

$$(L')^{-\alpha} = m^{-\alpha}x^{m-1}K_m^{\alpha}x^{1-m}$$

so that (1.7) is a special case of (3.9).

We now consider the case $a > n$. (1.12) and (3.1) give

$$M = m^n x^m \prod_{k=1}^{n} x^{m-mb_k}(-D_m)x^{mb_k}.$$

By induction, we obtain, for $r = 1, 2, \ldots,$

$$M^r = m^{nr}x^{mr} \prod_{k=1}^{n} x^{m-mb_k}(-D_m)^r x^{mb_k+m(r-1)}$$

and, since $(-D_m)^r = K_m^{-r}$ under appropriate conditions, (2.1) gives

$$M^r = m^{nr}x^{mr} \prod_{k=1}^{n} K_m^{1-b_k,-r}.$$

(The inductive step requires the result $(-D_m)^r x^{m(r+1)}(-D_m) = x^m(-D_m)^{r+1}x^{mr}$, a special case of (1.9) which can be established by Leibnitz' formula.) This suggests that we define M^{α}, for any complex number α, to be the operator

$$M^{\alpha} = m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\alpha}. \tag{3.10}$$

From (2.2) and (2.11), we obtain

$$\begin{aligned} M^{\alpha}M^{\beta}\phi &= m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\alpha} m^{n\beta}x^{m\beta} \prod_{k=1}^{n} K_m^{1-b_k,-\beta}\phi \\ &= m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k-\beta,-\alpha} \prod_{k=1}^{n} K_m^{1-b_k,-\beta}\phi \\ &= m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k-\beta,-\alpha}K_m^{1-b_k,-\beta}\phi \end{aligned} \tag{3.11}$$

while

$$M^{\alpha+\beta}\phi = m^{n(\alpha+\beta)}x^{m(\alpha+\beta)} \prod_{k=1}^{n} K_m^{1-b_k,-(\alpha+\beta)}\phi. \tag{3.12}$$

The right-hand sides of (3.11) and (3.12) are equal provided that

$$K_m^{1-b_k-\beta,-\alpha}K_m^{1-b_k,-\beta}\phi = K_m^{1-b_k,-(\alpha+\beta)}\phi,$$

which, by (2.2), is equivalent to

$$(3.13) \qquad x^{m-mb_k-m\beta}K_m^{-\alpha}x^{-m+mb_k+m\beta+m\alpha}x^{m-mb_k}K_m^{-\beta}x^{-m+mb_k+m\beta}\phi$$
$$= x^{m-mb_k}K_m^{-(\alpha+\beta)}x^{-m+mb_k+m\alpha+m\beta}\phi, \quad \text{or}$$
$$K_m^{-\alpha}x^{m(\alpha+\beta)}K_m^{-\beta}\psi = x^{m\beta}K_m^{-(\alpha+\beta)}x^{m\alpha}\psi$$

where $\psi(x) = x^{-m+mb_k+m\beta}\phi(x)$. (3.13) is simply (1.9) with α, β, γ and ϕ replaced by β, $-(\alpha+\beta)$, α and ψ respectively. Thus the equation

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is a consequence of the second index law for K_m^α. Conversely, (1.9) is a special case of (3.14), corresponding to

$$(3.15) \qquad L = xDx^m.$$

In the notation of (1.10), $n = 1$, $a_1 = 1$, $a_2 = m$, $b_1 = 1$ so that

$$M^\alpha = m^\alpha x^{m\alpha}K_m^{0,-\alpha} = m^\alpha x^{m\alpha}K_m^{-\alpha}x^{m\alpha}.$$

If $\alpha + \beta + \gamma = 0$, then

$$M^{\gamma+\alpha}\psi = M^\gamma M^\alpha\psi$$
$$\Rightarrow m^{\gamma+\alpha}x^{m(\gamma+\alpha)}K_m^{-(\gamma+\alpha)}x^{m(\gamma+\alpha)}\psi = m^\gamma x^{m\gamma}K_m^{-\gamma}x^{m\gamma}m^\alpha x^{m\alpha}K_m^{-\alpha}x^{m\alpha}\psi$$
$$\Rightarrow x^{m\alpha}K_m^\beta x^{m\gamma}\phi = K_m^{-\gamma}x^{-m\beta}K_m^{-\alpha}\phi$$

where $\phi(x) = x^{m\alpha}\psi(x)$. This gives (1.9).

Similarly, for $a > n$, we define $(M')^\alpha$ by

$$(3.16) \qquad (M')^\alpha\phi = m^{n\alpha}\prod_{k=1}^{n} I_m^{-b_k+1/m,-\alpha}x^{m\alpha}\phi.$$

The index law

$$(3.17) \qquad (M')^\alpha(M')^\beta = (M')^{\alpha+\beta}$$

is a consequence of (1.8). Conversely, with L as in (3.15),

$$(M')^\alpha = m^\alpha x^{m\alpha+m-1}I_m^{-\alpha}x^{m\alpha-m+1}$$

and, as above, we can show that (1.8) is a special case of (3.17).

Finally, we mention that the results are valid in the setting of distribution theory, for instance in the spaces $F'_{p,\mu}$ introduced in [3, Chapter 2].

References

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Proceedings of the Royal Society of Edinburgh, **96A**, 231–247, 1984

Paper 9

On an index law and a result of Buschman

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(MS received 3 October 1983)

Synopsis

A result for the Erdélyi-Kober operators, mentioned briefly by Buschman, is discussed together with a second related result. The results are proved rigorously by means of an index law for powers of certain differential operators and are shown to be valid under conditions of great generality. Mellin multipliers are used and it is shown that, in a certain sense, the index law approach is equivalent to, but independent of, the duplication formula for the gamma function. Various statements can be made concerning fractional integrals and derivatives which produce, as special cases, simple instances of the chain rule for differentiation and changes of variables in integrals.

1.

The simplest version of the chain rule for differentiation gives rise to results such as

$$(D_m\phi)(x) = m^{-1}x^{1-m}(D\phi)(x) \tag{1.1}$$

or, in operational form,

$$D_m = m^{-1}x^{1-m}D. \tag{1.2}$$

Here m is real and positive, x^{1-m} in (1.2) stands for the operation of multiplying a function of the variable x by x^{1-m} and

$$D_m \equiv d/dx^m, \quad D \equiv D_1 \equiv d/dx. \tag{1.3}$$

It is also an easy matter to express $D_m^n \equiv (D_m)^n$ in terms of D by repeated use of (1.2) or to obtain D_{nm} in terms of D_m for $n = 1, 2, \ldots$. It is interesting to contemplate what might happen in the cases when positive integral powers of derivatives are replaced by fractional derivatives or fractional integrals. Is it possible to relate D_{nm}^{α} ($\operatorname{Re}\alpha > 0$, $n = 1, 2, \ldots$) to fractional powers of D_m or to relate fractional integrals with respect to x^{nm} to fractional integrals with respect to x^m? The present note considers two formulae which, when written in an appropriate form, answer this question in the affirmative.

One of the formulae was mentioned briefly by Buschman in [**1**]; see, in particular, [**1**, (3.8)]. However, it does not seem to have attracted a great deal of attention since. Analysis in [**1**] is formal and makes use of the Mellin transform and properties of the gamma function, notably the duplication formula. Buschman says, "The Mellin transformation not only is suggestive of these identities, but it also provides an indirect method of proof of them for certain restricted ranges of the parameters". One purpose of this paper is to present a rigorous analysis of

Buschman's result and of a second (essentially adjoint) result. However, another purpose is to show how the results are related to ideas developed by the author in other papers [**5**] and [**6**]. Like Buschman, we shall make extensive use of the Mellin transform but we shall not use the duplication formula. Instead, we obtain our results as a particular case of earlier work on index laws for fractional powers of a class of ordinary differential operators.

Some details from earlier papers will be recalled from time to time in order to make the present paper reasonably self-contained. For instance, rigour will be supplied by studying the various operators in the context of the $F_{p,\mu}$ spaces, whose definition will be recalled below. (These spaces were originally intended as spaces of testing functions which generated corresponding spaces $F'_{p,\mu}$ of distributions; however, we shall not be working distributionally in this paper.) Within this context, our main tools will be

(i) the so-called Erdélyi-Kober operators, whose properties are developed extensively in [**3**, Chap. 3] but which will be treated slightly differently here;

(ii) an explicit expression in terms of Erdélyi-Kober operators for fractional powers of certain ordinary differential operators, as discussed in [**4**];

(iii) an index law of the form $(T^\alpha)^\beta = T^{\alpha\beta}$ as established in [**5**] and [**6**] for classes of Mellin multiplier transforms T.

We would like to suggest that our approach is of intrinsic interest since it gives a worthwhile application of earlier theory, provides rigorous proofs under conditions of great generality and shows that, in some sense, a particular case of an index law for operators turns out to be equivalent to the duplication formula for the gamma function.

2.

We first introduce the spaces $F_{p,\mu}$ mentioned above.

DEFINITION 2.1. Let μ be any complex number.

(i) For $1 \leqq p < \infty$, we define the set $F_{p,\mu}$ of functions by

$$F_{p,\mu} = \{\phi \in C^\infty(0,\infty) \colon x^k D^k(x^{-\mu}\phi) \in L^p(0,\infty) \text{ for } k = 0, 1, 2, \ldots\} \tag{2.1}$$

while

$$F_{\infty,\mu} = \{\phi \in C^\infty(0,\infty) \colon x^k D^k(x^{-\mu}\phi) \to 0 \text{ as } x \to 0+ \text{ and as } x \to \infty \text{ for } k = 0, 1, 2, \ldots\}. \tag{2.2}$$

(ii) For $1 \leqq p \leqq \infty$ and $k = 0, 1, 2, \ldots$, we define $\gamma_k^{p,\mu}$ on $F_{p,\mu}$ by

$$\gamma_k^{p,\mu}(\phi) = \|x^k D^k(x^{-\mu}\phi)\|_p \tag{2.3}$$

where $\| \ \|_p$ denotes the usual $L^p(0,\infty)$ norm.

Remark 2.2. For each fixed p and μ, $F_{p,\mu}$ is a vector space and a Fréchet space with respect to the topology generated by the separating collection $\{\gamma_k^{p,\mu}\}_{k=0}^\infty$ of seminorms. Analytic and topological properties of these spaces are described in detail in [3, Chap. 2].

Next we state the behaviour of the Mellin transform on $F_{p,\mu}$.

Formally, the Mellin transform $\mathcal{M}$ is defined by

$$(\mathcal{M}\phi)(s) = \int_0^\infty x^{s-1}\phi(x)\,dx \tag{2.4}$$

where s is a complex variable and the integral is interpreted in an appropriate sense.

THEOREM 2.3. *For $\phi \in F_{p,\mu}$, $(\mathcal{M}\phi)(s)$ exists provided that $1 \leqq p \leqq 2$ and*

$$\operatorname{Re} s = 1/p - \operatorname{Re} \mu. \tag{2.5}$$

In that case, if $s = \sigma + it$ (where $\sigma = 1/p - \operatorname{Re}\mu$) and $(\mathcal{M}\phi)(\sigma+it)$ is regarded as a function of $t \in (-\infty, \infty)$, then $\mathcal{M}$ is a continuous linear mapping from $F_{p,\mu}$ into $L^q(-\infty,\infty)$, where $q = p/(p-1)$.

Proof. This appears as [**4**, Lemma 2.3].

Remark 2.4.

(i) For $p = 1$, the integral in (2.4) converges absolutely under the hypotheses of Theorem 2.3, while for $1 < p \leqq 2$, $(\mathcal{M}\phi)(s) \equiv (\mathcal{M}\phi)(\sigma+it)$ exists as a limit in the $L^q(-\infty,\infty)$ mean.

(ii) From now on, whenever we consider $(\mathcal{M}\phi)(s)$ for $\phi \in F_{p,\mu}$ $(1 \leqq p \leqq 2)$, it will always be the case that s and μ are related by (2.5).

We now introduce the idea of $F_{p,\mu}$ multipliers for which the following notation is helpful.

Notation 2.5.

(i) Ω will denote a region in the complex plane which is the union of a finite or countably infinite collection of disjoint strips, each of which has one of the forms

$$\{s: a < \operatorname{Re} s < b\}, \quad \{s: \operatorname{Re} s < c\} \quad \text{or} \quad \{s: \operatorname{Re} s > d\}$$

where a, b, c and d are real numbers. (There can be at most one strip of each of the second and third forms.)

(ii) Let Ω be as in (i). For each fixed p in the range $1 \leqq p \leqq \infty$, we let

$$\Omega_p = \{\mu : 1/p - \mu \in \Omega\}. \tag{2.6}$$

(2.6) ensures that, when s and μ are related by (2.5) and p is fixed, $s \in \Omega$ if and only if $\mu \in \Omega_p$.

DEFINITION 2.6. Let g be a complex-valued function analytic in a region Ω (of the type in Notation 2.5(i)). We shall say that g is an $F_{p,\mu}$ multiplier if there exists a (unique) linear transformation R (depending on g) such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, R is a continuous linear transformation from $F_{p,\mu}$ into $F_{p,\mu}$,

(ii) for $1 < p \leqq 2$, $\mu \in \Omega_p$ and $\phi \in F_{p,\mu}$,

$$(\mathcal{M}(R\phi))(s) = g(s)(\mathcal{M}\phi)(s). \tag{2.7}$$

We may then refer to R as a Mellin multiplier transform.

To illustrate these ideas, we give an example of an operator which will play a rôle later.

EXAMPLE 2.7. We define the operator δ on $F_{p,\mu}$ by

$$(\delta\phi)(x) = x\, d\phi/dx \quad \text{or} \quad \delta \equiv xD. \tag{2.8}$$

From [**3**, Corollary 2.14], we see that for $1 \leqq p \leqq \infty$ and any complex number μ, δ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$. If, further, $\operatorname{Re}\mu \neq 1/p$, then δ is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ with

$$(\delta^{-1}\psi)(x) = \begin{cases} \displaystyle\int_0^x \psi(t)\, dt/t & (\operatorname{Re}\mu > 1/p) \\ \displaystyle -\int_x^\infty \psi(t)\, dt/t & (\operatorname{Re}\mu < 1/p) \end{cases} \tag{2.9}$$

for $\psi \in F_{p,\mu}$, (2.9) being derived from [**3**, Theorem 2.13]. To relate δ to the Mellin transform, we let $\phi \in C_0^\infty(0, \infty)$ and use integration by parts to show that

$$(\mathcal{M}(\delta\phi))(s) = -s(\mathcal{M}\phi)(s). \tag{2.10}$$

Since $C_0^\infty(0, \infty)$ is dense in $F_{p,\mu}$ for all p and μ [**3**, Corollary 2.7] and $\delta, \mathcal{M}$ are continuous on $F_{p,\mu}$ for $1 < p \leqq 2$ (Theorem 2.3 above), (2.10) holds for $\phi \in F_{p,\mu}$ if $1 < p \leqq 2$, μ is any complex number and $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$ (Remark 2.4(ii)). On comparing (2.10) with (2.7), we see that the function $g(s) = -s$, with domain equal to the whole complex plane, is an $F_{p,\mu}$ multiplier and the corresponding operator R is δ. If $\operatorname{Re}\mu \neq 1/p$, then $\operatorname{Re} s \neq 0$ by (2.5) and, in particular, $s \neq 0$. With $\phi = \delta^{-1}\psi$, as given by (2.9), we can rewrite (2.10) in the form

$$(\mathcal{M}(\delta^{-1}\psi))(s) = -s^{-1}(\mathcal{M}\psi)(s). \tag{2.11}$$

By comparison with (2.7), we see that the function g defined by

$$g(s) = -s^{-1}\ (s \in \Omega) \quad \text{where} \quad \Omega = \{s : \operatorname{Re} s \neq 0\}$$

is an $F_{p,\mu}$ multiplier and the corresponding operator R is δ^{-1}. That the form of R is different on the two components of $\Omega_p = \{\mu : \operatorname{Re}\mu \neq 1/p\}$ is typical.

In [**5**] and [**6**], we considered the case in which the function $g(s)$ in (2.7) is of the form

$$g(s) = h(s-\gamma)/h(s) \quad (s \in \Omega) \tag{2.12}$$

and were more interested in the operator $T = x^\gamma R$ than in R itself. Here γ could be any complex number but since $\gamma = 0$ produces the trivial case $T = R = I$, the identity operator, we are primarily concerned with $\gamma \neq 0$. It is not hard to see that if R maps $F_{p,\mu}$ into itself, then $x^\gamma R$ maps $F_{p,\mu}$ into $F_{p,\mu+\gamma}$. If $1/p - \operatorname{Re}\mu = \operatorname{Re} s$, then $1/p - \operatorname{Re}(\mu+\gamma) = \operatorname{Re}(s-\gamma)$ so that we should compare $(\mathcal{M}(T\phi))(s-\gamma)$ with $(\mathcal{M}\phi)(s)$. With this motivation, we are led to the following definition.

DEFINITION 2.8. We shall say that T belongs to the class $\mathcal{F}$ if there is a triple (h, Ω, γ) such that

(i) Ω is a region of the type described in Notation 2.5(i), γ is a complex number, h is a function such that the function g defined on Ω by (2.12) is an $F_{p,\mu}$ multiplier;

(ii) for $1<p<\infty$ and $\mu\in\Omega_p$, T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+\gamma}$;

(iii) for $1<p\leqq 2$, $\mu\in\Omega_p$ and $\phi\in F_{p,\mu}$,

$$(\mathcal{M}(T\phi))(s-\gamma)=\frac{h(s-\gamma)}{h(s)}(\mathcal{M}\phi)(s) \quad (s\in\Omega). \tag{2.13}$$

We say that the triple (h, Ω, γ) generates T and sometimes write $T\equiv T(h, \Omega, \gamma)$.

Formally, (2.13) shows that for $n=1, 2, 3, \ldots,$

$$(\mathcal{M}(T^n\phi))(s-n\gamma)=\frac{h(s-n\gamma)}{h(s)}(\mathcal{M}\phi)(s)$$

and we use this as the motivation for the definition of T^α as the unique operator in $\mathcal{F}$ generated by the triple $(h, \Omega, \alpha\gamma)$, so that for $1<p\leqq 2$, $\mu\in\Omega_p$ and $\phi\in F_{p,\mu}$,

$$(\mathcal{M}(T^\alpha\phi))(s-\alpha\gamma)=\frac{h(s-\alpha\gamma)}{h(s)}(\mathcal{M}\phi)(s). \tag{2.14}$$

However, this is only valid if the complex number α is such that the function

$$g_\alpha(s)=h(s-\alpha\gamma)/h(s)$$

is an $F_{p,\mu}$ multiplier. Accordingly, we make the following precise definition.

DEFINITION 2.9. Let $T\equiv T(h, \Omega, \gamma)\in\mathcal{F}$ and let $A\equiv A(h, \Omega, \gamma)$ be given by

$$A=\{\alpha\colon\ (h, \Omega, \alpha\gamma) \text{ generates an element of } \mathcal{F}\}. \tag{2.15}$$

(This is the set denoted by A_F in **[6]** and differs from the set A in **[5]**!) Then for $\alpha\in A$, we define T^α to be the unique operator in $\mathcal{F}$ generated by $(h, \Omega, \alpha\gamma)$. In particular, when $\alpha\in A$,

(i) for $1<p<\infty$ and $\mu\in\Omega_p$, T^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+\alpha\gamma}$,

(ii) for $1<p\leqq 2$, $\mu\in\Omega_p$ and $\phi\in F_{p,\mu}$, (2.14) holds.

Again we give two examples relevant to what follows.

EXAMPLE 2.10.

(i) Let λ be any complex number and let $T=x^\lambda$, i.e. $(T\phi)(x)=x^\lambda\phi(x)$. Formally, $(\mathcal{M}(T\phi))(s-\lambda)=(\mathcal{M}\phi)(s)$ and by arguing as in Example 2.7, we find that $T\in\mathcal{F}$ and is generated by the triple (h, Ω, γ) where $\Omega=\mathbb{C}$ (the complex plane), $h(s)\equiv 1$ $(s\in\Omega)$ and $\gamma=\lambda$. In this case, $A=\mathbb{C}$ also and (2.14) shows that $T^\alpha=(x^\lambda)^\alpha=x^{\alpha\lambda}$, as expected.

(ii) For $m>0$, let $T=D_m$, as defined by (1.3). In the case $m=1$, we can modify Example 2.7 to show that for $1<p\leqq 2$, any complex number μ and $\phi\in F_{p,\mu}$,

$$(\mathcal{M}(D\phi))(s+1)=-s(\mathcal{M}\phi)(s).$$

(1.2) then leads to

$$(\mathcal{M}(D_m\phi))(s+m)=(\mathcal{M}(m^{-1}x^{1-m}D\phi))(s+m)=m^{-1}(\mathcal{M}(D\phi))(s+m+1-m)$$

or

$$(\mathcal{M}(D_m\phi))(s+m) = -(s/m)(\mathcal{M}\phi)(s). \tag{2.16}$$

Formally, $-(s/m) = \Gamma(1-s/m)/\Gamma(-s/m) = h(s+m)/h(s)$ where $h(s) = [\Gamma(1-s/m)]^{-1}$. The quotient of gamma functions has removable singularities at $s = km$ ($k = 1, 2, 3, \ldots$) and $-(s/m)$ serves as an analytic continuation. With that understanding, we could use the triple

$$\Omega = \mathbb{C}, \quad h(s) = [\Gamma(1-s/m)]^{-1}, \quad \gamma = -m$$

to generate D_m. However, we then experience difficulty with $g_\alpha(s) = h(s+\alpha m)/h(s)$ which takes the form $\Gamma(1-s/m)/\Gamma(1-\alpha-s/m)$. The singularities are no longer removable in general. Accordingly, we shall use the triple (h, Ω, γ) given by

$$\Omega = \{s\colon \operatorname{Re} s \neq km \text{ for } k = 1, 2, \ldots\}, \quad h(s) = [\Gamma(1-s/m)]^{-1} \quad (s \in \Omega), \ \gamma = m. \tag{2.17}$$

By arguments similar to those in [**6**], it can be shown that the set $A \equiv A(h, \Omega, \gamma) = \mathbb{C}$. Thus, for any complex number α, we may define $D_m^\alpha \equiv (D_m)^\alpha$ as a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m\alpha}$ for $1 < p < \infty$ and $\mu \in \Omega_p$ which is such that

$$(\mathcal{M}(D_m^\alpha\phi))(s+m\alpha) = \frac{\Gamma(1-s/m)}{\Gamma(1-\alpha-s/m)}(\mathcal{M}\phi)(s) \tag{2.18}$$

when $1 < p \leqq 2$, $\mu \in \Omega_p$ and $\phi \in F_{p,\mu}$. Again as in [**6**], we mention that if $\operatorname{Re} \alpha > 0$ then $D_m^{-\alpha}$ is an integral rather than a differential operator and we may write

$$D_m^{-\alpha} = I_m^\alpha \quad (\operatorname{Re} \alpha > 0) \tag{2.19}$$

where for $1 < p < \infty$, $\operatorname{Re} \mu + m > 1/p$ and $\phi \in F_{p,\mu}$,

$$(I_m^\alpha\phi)(x) = [\Gamma(\alpha)]^{-1}\int_0^x (x^m - t^m)^{\alpha-1}\phi(t)\, d(t^m) \tag{2.20}$$

with $d(t^m) = mt^{m-1}\, dt$. The form of I_m^α is more elaborate for other values of $\mu \in \Omega_p$ and we shall not state it explicitly here.

Since $T^\alpha \in \mathcal{F}$ by construction, it is possible to define powers of T^α under appropriate circumstances. In particular, we quote the following index law which we shall use later.

THEOREM 2.11. *Let $T \in \mathcal{F}$, $\{\alpha, \alpha\beta\} \subseteq A$, $1 < p < \infty$ and $\mu \in \Omega_p$. Then, as operators on $F_{p,\mu}$,*

$$(T^\alpha)^\beta = T^{\alpha\beta}. \tag{2.21}$$

Proof. This is established in [**6**, Theorem 4.9].

3.

Having outlined the theory of powers of operators in the class $\mathcal{F}$, we now turn our attention to a certain subclass of $\mathcal{F}$, consisting of ordinary differential operators of a particular kind. A typical operator is of the form

$$T = x^{a_1}Dx^{a_2}Dx^{a_3} \cdots x^{a_n}Dx^{a_{n+1}} \tag{3.1}$$

where n is a positive integer, $a_1, \dots, a_{n+1}$ are complex numbers,

$$a = \sum_{k=1}^{n+1} a_k \text{ is real,} \tag{3.2}$$

$$m = n - a > 0 \tag{3.3}$$

and $D = d/dx$ as usual. The domain of T is to be thought of as one of the $F_{p,\mu}$ spaces.

LEMMA 3.1. *For $1 \leqq p \leqq \infty$ and any complex number μ, T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m}$.*

Proof. From Example 2.10, T is a composition of $2n+1$ continuous mappings and maps $F_{p,\mu}$ into $F_{p,\mu+a_1+\dots+a_{n+1}-n} = F_{p,\mu+a-n} = F_{p,\mu-m}$ by (3.2) and (3.3).

In trying to apply the theory in §2, we must try to find a suitable triple (h, Ω, γ) to generate T. Definition 2.8 entails that T maps $F_{p,\mu}$ into $F_{p,\mu+\gamma}$. Comparison with Lemma 3.1 shows that $\gamma = -m$. Finding h is not so easy. It might be thought that, since each of the $2n+1$ operators in (3.1) is a member of $\mathscr{F}$ and is generated by an appropriate triple, it should be possible to cobble together a triple generating T from the individual triples described in Example 2.10. Certainly, we can form a product of functions of s but rewriting this product in the form $h(s+m)/h(s)$ is another matter. In [**4**], we solved the problem by rewriting each D as

$$D = mx^{m-1}D_m \tag{3.4}$$

with m as in (3.3). Although (3.4), being a rearrangement of (1.2), is valid for any $m > 0$, it is essential to use the particular value of m in (3.3); it seems that no other value will help our cause here. It is then possible to identify h and Ω so that our triple to generate T is complete. Formally, we obtain

$$(\mathscr{M}(T\phi))(s+m) = m^n \prod_{k=1}^{n} \frac{\Gamma(b_k + 1 - s/m)}{\Gamma(b_k - s/m)} (\mathscr{M}\phi)(s) \tag{3.5}$$

where for $k = 1, \dots, n$,

$$b_k = \left(\sum_{i=k+1}^{n+1} a_i + k - n \right) \Big/ m. \tag{3.6}$$

Formula (3.5) is obtained by putting $\alpha = 1$ in [**4**, formula (5.6)]. However, we shall supply a rigorous proof based solely on Example 2.10. To be rigorous, we must state the conditions on s or μ which ensure the validity of (3.5). Each factor in the product is reminiscent of the function $\Gamma(1-s/m)/\Gamma(-s/m)$ which appeared in Example 2.10(ii) and, by similar arguments, we require that $\operatorname{Re}(b_k + 1 - s/m)$ is not equal to zero or a negative integer for $k = 1, \dots, n$. Thus we let

$$\Omega = \{s : \operatorname{Re}(b_k + 1 - s/m) \neq 0, -1, -2, \dots \text{ for } k = 1, \dots, n\}. \tag{3.7}$$

On using (2.5), we see that the condition on s in (3.7) becomes $\operatorname{Re}(mb_k + \mu) + m \neq 1/p - ml$ for $k = 1, \dots, n$ and $l = 0, 1, 2, \dots$. In [**3**] and [**4**], we made extensive use of the sets $A_{p,\mu,m}$ of complex numbers defined by

$$A_{p,\mu,m} = \{\eta : \operatorname{Re}(m\eta + \mu) + m \neq 1/p - ml \text{ for } l = 0, 1, 2, \dots\} \tag{3.8}$$

and we shall use this notation, rather than Ω_p, in what follows. The restriction above becomes $b_k \in A_{p,\mu,m}$ $(k=1,\ldots,n)$. Accordingly, we make the following statement.

THEOREM 3.2. *Let* $1<p<\infty$, *let* μ *be any complex number and let* $b_k \in A_{p,\mu,m}$ $(k=1,\ldots,n)$. *Then* (3.5) *holds for* $\phi \in F_{p,\mu}$ *and* $s \in \Omega$ *(defined by* (3.7)*).*

Proof. We first note that by (1.2) and Example 2.10,

$$(\mathcal{M}(D_m\psi))(s-mb_k+m)=m^{-1}(\mathcal{M}(D\psi))(s-mb_k+1)=m^{-1}(mb_k-s)(\mathcal{M}\psi)(s-mb_k)$$

or

$$(\mathcal{M}(D_m\psi))(s-mb_k+m)=\frac{\Gamma(b_k+1-s/m)}{\Gamma(b_k-s/m)}(\mathcal{M}\psi)(s-mb_k) \tag{3.9}$$

whenever $\psi \in F_{p,\mu+mb_k}$, $s\in\Omega$ and $k=1,\ldots,n$. We note also that from (3.6),

$$mb_1=1-a_1-m,\quad mb_n=a_{n+1},\quad mb_{k+1}-mb_k=1-a_{k+1}\quad (k=1,\ldots,n-1). \tag{3.10}$$

By repeated use of (3.9) (with various choices for ψ) and (3.10), we obtain

$$\begin{aligned}
&(\mathcal{M}(T\phi))(s+m)\\
&=m^n(\mathcal{M}(x^{a_1+m-1}D_mx^{a_2+m-1}D_m\cdots x^{a_n+m-1}D_mx^{a_{n+1}}\phi))(s+m)\\
&=m^n(\mathcal{M}(x^{-mb_1}D_mx^{mb_1-mb_2+m}D_m\cdots x^{mb_{n-1}-mb_n+m}D_mx^{mb_n}\phi))(s+m)\\
&=m^n(\mathcal{M}(D_mx^{mb_1-mb_2+m}D_m\cdots x^{mb_{n-1}-mb_n+m}D_mx^{mb_n}\phi))(s-mb_1+m)\\
&=m^n\frac{\Gamma(b_1+1-s/m)}{\Gamma(b_1-s/m)}(\mathcal{M}(x^{mb_1-mb_2+m}D_m\cdots x^{mb_{n-1}-mb_n+m}D_mx^{mb_n}\phi))(s-mb_1)\\
&=m^n\frac{\Gamma(b_1+1-s/m)}{\Gamma(b_1-s/m)}\\
&\quad\times(\mathcal{M}(D_mx^{mb_2-mb_3+m}D_m\cdots x^{mb_{n-1}-mb_n+m}D_mx^{mb_n}\phi))(s-mb_2+m)\\
&=m^n\frac{\Gamma(b_1+1-s/m)}{\Gamma(b_1-s/m)}\frac{\Gamma(b_2+1-s/m)}{\Gamma(b_2-s/m)}\\
&\quad\times(\mathcal{M}(x^{mb_2-mb_3+m}D_m\cdots x^{mb_{n-1}-mb_n+m}D_mx^{mb_n}\phi))(s-mb_2)
\end{aligned}$$

and so on, (3.5) following after n repetitions.

COROLLARY 3.3. *The operator* T *defined by* (3.1) *belongs to* $\mathcal{F}$ *and is generated by the triple* (h,Ω,γ) *where* Ω *is given by* (3.7), $\gamma=-m$ *and*

$$h(s)=m^{ns/m}\left[\prod_{k=1}^{n}\Gamma(b_k+1-s/m)\right]^{-1}\quad(s\in\Omega). \tag{3.11}$$

Proof. This follows immediately from (3.5) above.

The next stage is to obtain the set A defined in (2.15) and this requires examination of the function

$$g_\alpha(s)=\frac{h(s+m\alpha)}{h(s)}=m^{n\alpha}\prod_{k=1}^{n}\frac{\Gamma(b_k+1-s/m)}{\Gamma(b_k+1-\alpha-s/m)}.$$

As in Example 2.10(ii), it transpires that $A=\mathbb{C}$, since, by arguments similar to those in [**6**, §5], each factor in the product is an $F_{p,\mu}$ multiplier for every complex number α and the same therefore applies to $g_\alpha(s)$. We can now use Definition 2.9 to define T^α.

THEOREM 3.4. *For the operator T given by (3.1) and for any complex number α, T^α is the unique operator in $\mathcal{F}$ generated by the triple (h, Ω, γ) where h is given by (3.11), Ω by (3.7) and $\gamma=-m\alpha$. In particular, if $b_k \in A_{p,\mu,m}$ $(k=1,\ldots,n)$, then*

(i) *for $1<p<\infty$, T^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m\alpha}$,*

(ii) *for $1<p\leqq 2$ and $\phi \in F_{p,\mu}$,*

$$(\mathcal{M}(T^\alpha\phi))(s+m\alpha)=m^{n\alpha}\prod_{k=1}^{n}\frac{\Gamma(b_k+1-s/m)}{\Gamma(b_k+1-\alpha-s/m)}(\mathcal{M}\phi)(s) \quad (s\in\Omega). \quad (3.12)$$

Proof. This is immediate from Definition 2.9 and Corollary 3.3. We mention that (3.12) also appears as [**4**, (5.6)] but has been derived here by an alternative approach.

4.

Formula (3.12) focuses attention on a function of the form $\Gamma(\eta+1-s/m)/\Gamma(\eta+1+\alpha-s/m)$ for complex numbers η and α. As indicated above, this function defines an $F_{p,\mu}$ multiplier under appropriate conditions, the corresponding operator being the Erdélyi-Kober operator $I_m^{\eta,\alpha}$. In order to obtain a concrete representation of $I_m^{\eta,\alpha}$, it is convenient to introduce the related Erdélyi-Kober operator $K_m^{\eta,\alpha}$ with the rather similar multiplier $\Gamma(\eta+s/m)/\Gamma(\eta+\alpha+s/m)$. To avoid trouble, we must ensure that $\eta+s/m$ is not a non-positive integer. Use of (2.5) leads to the following analogue of (3.8).

DEFINITION 4.1. For $1\leqq p\leqq\infty$, $m>0$ and any complex number μ, we define the set $A'_{p,\mu,m}$ of complex numbers by

$$A'_{p,\mu,m}=\{\eta\colon \operatorname{Re}(m\eta-\mu)\neq -1/p-ml \ (l=0,1,2,\ldots)\}. \quad (4.1)$$

In the case $1<p<\infty$, we find, as in the case of $I_m^{\eta,\alpha}$, that $\Gamma(\eta+s/m)/\Gamma(\eta+\alpha+s/m)$ is an $F_{p,\mu}$ multiplier provided that $\eta\in A'_{p,\mu,m}$ and α is any complex number. Accordingly, the following definition is meaningful.

DEFINITION 4.2.

(i) For $1<p<\infty$, $m>0$, $\eta\in A_{p,\mu,m}$ and any complex number α, $I_m^{\eta,\alpha}$ is the unique continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ such that, if $1<p\leqq 2$,

$$(\mathcal{M}(I_m^{\eta,\alpha}\phi))(s)=\frac{\Gamma(\eta+1-s/m)}{\Gamma(\eta+1+\alpha-s/m)}(\mathcal{M}\phi)(s) \quad (\phi\in F_{p,\mu}). \quad (4.2)$$

(ii) For $1<p<\infty$, $m>0$, $\eta\in A'_{p,\mu,m}$ and any complex number α, $K_m^{\eta,\alpha}$ is the unique continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ such that, if $1<p\leqq 2$,

$$(\mathcal{M}(K_m^{\eta,\alpha}\phi))(s)=\frac{\Gamma(\eta+s/m)}{\Gamma(\eta+\alpha+s/m)}(\mathcal{M}\phi)(s) \quad (\phi\in F_{p,\mu}). \quad (4.3)$$

By using results in Example 2.10, notably (2.20), and various results in [**3**, Chap. 3], we can obtain the following descriptions of $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$.

THEOREM 4.3. *Let* $1<p<\infty$, $m>0$, μ *and* α *be any complex numbers and* $\phi \in F_{p,\mu}$.

(i) *Let* $\operatorname{Re}(m\eta+\mu)+m>1/p$. *Then, if* $\operatorname{Re}\alpha>0$,

$$I_m^{\eta,\alpha} = x^{-m\eta-m\alpha} I_m^{\alpha} x^{m\eta} \tag{4.4}$$

as operators on $F_{p,\mu}$, *where* I_m^{α} *is given by* (2.20). *The restriction* $\operatorname{Re}\alpha>0$ *is removed by successive applications of the formula*

$$I_m^{\eta,\alpha}\phi = (\eta+\alpha+1) I_m^{\eta,\alpha+1}\phi + m^{-1} I_m^{\eta,\alpha+1}\,\delta\phi \tag{4.5}$$

where δ *is given by* (2.8). *In particular, for* $\operatorname{Re}\alpha<0$,

$$I_m^{\eta,\alpha} = x^{-m\eta-m\alpha} D_m^{-\alpha} x^{m\eta} \tag{4.6}$$

where $D_m^{-\alpha}$ *is given by* (2.18) (*with* α *replaced by* $-\alpha$).

(ii) *Let* $\operatorname{Re}(m\eta-\mu)>-1/p$. *Then, if* $\operatorname{Re}\alpha>0$,

$$K_m^{\eta,\alpha} = x^{m\eta} K_m^{\alpha} x^{-m\eta-m\alpha} \tag{4.7}$$

where, for $\psi \in F_{p,\mu}$,

$$(K_m^{\alpha}\psi)(x) = [\Gamma(\alpha)]^{-1} \int_x^{\infty} (t^m - x^m)^{\alpha-1} \psi(t)\, d(t^m) \quad (x>0). \tag{4.8}$$

The restriction $\operatorname{Re}\alpha>0$ *is removed by successive applications of the formula*

$$K_m^{\eta,\alpha}\phi = (\eta+\alpha) K_m^{\eta,\alpha+1}\phi - m^{-1} K_m^{\eta,\alpha+1}\delta\phi. \tag{4.9}$$

(iii) *Let* $\eta \in A_{p,\mu,m}$ *and* α *be any complex number. For* $\operatorname{Re}(m\eta+\mu)+m>1/p$, *we define* $I_m^{\eta,\alpha}$ *as in* (i). *Otherwise, if* k *is the unique positive integer such that* $1/p-mk<\operatorname{Re}(m\eta+\mu)+m<1/p-m(k-1)$, *then*

$$I_m^{\eta,\alpha} = (-1)^k I_m^{\eta+k,\alpha-k} K_m^{-\eta-k,k} \tag{4.10}$$

where $I_m^{\eta+k,\alpha-k}$ *and* $K_m^{-\eta-k,k}$ *are defined as in* (i) *and* (ii) *respectively.*

(iv) Let $\eta \in A'_{p,\mu,m}$ *and* α *be any complex number. For* $\operatorname{Re}(m\eta-\mu)>-1/p$, $K_m^{\eta,\alpha}$ *is as in* (ii). *Otherwise, if* k *is the unique positive integer such that* $-1/p-mk<\operatorname{Re}(m\eta-\mu)<-1/p-m(k-1)$, *then*

$$K_m^{\eta,\alpha} = (-1)^k K_m^{\eta+k,\alpha-k} I_m^{-\eta-k,k} \tag{4.11}$$

where $K_m^{\eta+k,\alpha-k}$ *and* $I_m^{-\eta-k,k}$ *are defined as in* (ii) *and* (i) *respectively.*

Proof. The details are omitted.

Remark 4.4.

(i) The expressions (4.4)–(4.11) can be used to obtain explicit expressions for $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ as integral, differential or integro-differential operators, depending on the values of η and α but we shall not require these here, except in the simplest cases such as (4.4) or (4.7).

(ii) Once the explicit expressions referred to in (i) have been obtained, it can be checked that the operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ define continuous linear mappings from

$F_{p,\mu}$ into $F_{p,\mu}$ in the cases $p=1$ and $p=\infty$ too (subject to the other restrictions in Definition 4.2). Again, details can be found in [**3**, Chap. 3] and we shall use these facts below when we establish relationships between the Erdélyi-Kober operators.

We can now rewrite the expression for T^α in Theorem 3.4 in terms of Erdélyi-Kober operators.

THEOREM 4.5. *Let T be given by* (3.1), *α be any complex number, $1<p<\infty$ and $b_k \in A_{p,\mu,m}$ $(k=1,\ldots,n)$. Then, as operators on $F_{p,\mu}$,*

$$T^\alpha = m^{n\alpha}x^{-m\alpha}\prod_{k=1}^{n} I_m^{b_k,-\alpha}. \tag{4.12}$$

Proof. First we note that the operators $I_m^{b_k,-\alpha}$ $(k=1,\ldots,n)$ commute, since their multipliers do. Thus, the product on the right-hand side is unambiguous. Then, if $1<p\leqq 2$ and $\phi\in F_{p,\mu}$,

$$\begin{aligned}\left(\mathcal{M}\left(m^{n\alpha}x^{-m\alpha}\prod_{k=1}^{n} I_m^{b_k,-\alpha}\phi\right)\right)(s+m\alpha) &= \left(\mathcal{M}\left(m^{n\alpha}\prod_{k=1}^{n} I_m^{b_k,-\alpha}\phi\right)\right)(s)\\ &= m^{n\alpha}\prod_{k=1}^{n}\frac{\Gamma(b_k+1-s/m)}{\Gamma(b_k+1-\alpha-s/m)}(\mathcal{M}\phi)(s), \quad \text{by (4.2).}\end{aligned}$$

The result follows in this case, by (3.12) and (4.2). For the case $2<p<\infty$, we choose $\phi\in C_0^\infty(0,\infty)$ (regarded as a subset of $F_{p,\mu}$) and use continuity and density in the standard way to complete the proof.

Remark 4.6

(i) We mention that similar results involving operators of the form $K_m^{\eta,\alpha}$ arise in the calculation of powers of T', the formal adjoint of T, as well as in the study of operators of the form (3.1) for which $a>n$ (in contrast to (3.3)). Some instances can be found in [**4**] but, again, we need not quote them here. Instead, we shall indicate briefly in §5 how results for the operators $I_m^{\eta,\alpha}$ can easily be converted into results for $K_m^{\eta,\alpha}$.

(ii) Formula (4.12) and related results turn up in [**4**]. The purpose of proving (4.12) again here is to show that the theory in [**4**] can be derived, from a different starting point, by using the more general theory in [**5**] and [**6**].

5.

We are now ready to use the index law (2.21) in the case where α is replaced by a positive integer r and T is a differential operator of the type discussed above. It will be sufficient to restrict attention to an operator T of the form

$$T = x^{a_1}Dx^{a_2};\quad a_1+a_2=a \text{ is real};\quad m=1-a>0. \tag{5.1}$$

As a preliminary, we must examine T^r which can be written down explicitly without recourse to Definition 2.9. We obtain formally

$$T^r = x^{c_1}Dx^{c_2}D\cdots x^{c_r}Dx^{c_{r+1}}$$

where the numbers c_k $(k=1,\ldots,r+1)$ are related to a_1 and a_2 by

$$c_1=a_1,\quad c_{r+1}=a_2,\quad c_k=a_1+a_2\quad (k=2,\ldots,r). \tag{5.2}$$

From (5.2), $\sum_{k=1}^{r+1} c_k=r(a_1+a_2)=ra$, while the order of T^r is r and $r-ra=r(1-a)=rm>0$. We see that T^r is of the same form as the operator in (3.1) with n replaced by r, the numbers $a_1,\ldots,a_{n+1}$ by $c_1,\ldots,c_{r+1}$, a by ra and m by rm. By analogy with (3.6), we define the numbers d_k $(k=1,\ldots,r)$ by

$$d_k=\left(\sum_{i=k+1}^{r+1} c_i+k-r\right)\Big/rm. \tag{5.3}$$

For T as defined by (5.1), (3.6) produces the single number $b_1=a_2/m$ and a routine calculation shows that

$$d_k=(b_1-r+k)/r\quad (k=1,\ldots,r). \tag{5.4}$$

In order that an analogue of (4.12) should hold as an operator equation on $F_{p,\mu}$, we must ensure that $d_k\in A_{p,\mu,rm}$ $(k=1,\ldots,r)$ which means that

$$\operatorname{Re}\{rm(b_1-r+k)/r+\mu\}+rm\neq 1/p-rml$$

or

$$\operatorname{Re}(mb_1+\mu)+m\neq 1/p-m(rl+k-1) \tag{5.5}$$

for $k=1,\ldots,r$ and $l=0,1,2,\ldots$ Since $rl+k-1$ is a non-negative integer, (5.5) will be satisfied provided that $b_1\in A_{p,\mu,m}$. We can then summarise our discussion in a theorem.

THEOREM 5.1. *Let T be as in* (5.1), *let β be a complex number, r any positive integer, $1<p<\infty$ and $b_1\in A_{p,\mu,m}$. Then, as operators on $F_{p,\mu}$,*

$$(T^r)^\beta=(rm)^{r\beta}x^{-rm\beta}\prod_{k=1}^{r} I_{rm}^{(b_1-r+k)/r,\,-\beta}. \tag{5.6}$$

Proof. This follows from the preamble on replacing T, a, m, n, α and $\{b_1,\ldots,b_n\}$ in (4.12) by T^r, ra, rm, r, β and $\{(b_1-r+k)/r\colon k=1,\ldots,r\}$.

On the other hand, if we apply (4.12) to the operator T in (5.1) and replace α by $r\beta$, we obtain at once

$$T^{r\beta}=m^{r\beta}x^{-mr\beta}I_m^{b_1,-r\beta} \tag{5.7}$$

under the hypotheses in Theorem 5.1. We can now apply (2.21), observing that in the conditions stated in Theorem 2.11, we may take $A=\mathbb{C}$, while $\mu\in\Omega_p$ becomes $b_1\in A_{p,\mu,m}$. We deduce from (5.6) and (5.7) that, as operators on $F_{p,\mu}$,

$$I_m^{b_1,-r\beta}=r^{r\beta}\prod_{k=1}^{r} I_{rm}^{(b_1-r+k)/r,\,-\beta}. \tag{5.8}$$

Remark 5.2. An interesting point arises at this stage. Theorem 2.11 involves the restriction $1<p<\infty$ so that, in the first instance, (5.8) holds for this range of values of p (and subject to the other conditions in Theorem 5.1). However, the operators appearing on both sides of (5.8) remain continuous linear mappings from $F_{p,\mu}$ into itself even if $p=1$ or $p=\infty$, as is shown in [3, Chap. 3]. Since they agree when applied to functions in $C_0^\infty(0,\infty)$ (regarded as a subset of $F_{2,\mu}$ for suitable μ) and since $C_0^\infty(0,\infty)$ is dense in $F_{1,\mu}$ and $F_{\infty,\mu}$ for any μ [**3**, Corollary 2.7], we deduce that (5.8) continues to hold if $p=1$ or $p=\infty$.

THEOREM 5.3. *Let* $1\leqq p\leqq\infty$, μ *and* α *be arbitrary complex numbers,* $m>0$, $\eta\in A_{p,\mu,m}$ *and r any positive integer. Then, as operators on* $F_{p,\mu}$,

$$I_m^{\eta,\alpha}=r^{-\alpha}\prod_{k=1}^{r} I_{rm}^{(\eta-r+k)/r,\alpha/r}. \tag{5.9}$$

Proof. This follows from Remark 5.2 and (5.8) on putting $b_1=\eta$, $\beta=-\alpha/r$.

Remark 5.4. If we put $m=1$ in (5.9), we obtain the result by Buschman [**1**, (3.8)] referred to earlier, in slightly different notation. We would emphasise that, in our approach, (5.9) arose naturally as an application of the index law but now that we have discovered the appropriate formula, we can check it independently by means of the duplication formula

$$\prod_{k=1}^{r}\Gamma(z+(k-1)/r)=(2\pi)^{(r-1)/2}r^{1/2-rz}\Gamma(rz)\quad(r=2,3,4,\ldots) \tag{5.10}$$

for the gamma function due to Gauss and Legendre [**2**, p. 4]. By (4.1), the right-hand side of (5.9) is a Mellin multiplier transform corresponding to the multiplier

$$g(s)=r^{-\alpha}\prod_{k=1}^{r}\frac{\Gamma((\eta+k-s/m)/r)}{\Gamma((\eta+\alpha+k-s/m)/r)}.$$

By applying (5.10) to the numerator and denominator on the right-hand side, with z replaced by $(\eta+1-s/m)/r$ and $(\eta+\alpha+1-s/m)/r$ respectively, we obtain

$$\begin{aligned} g(s)&=\frac{r^{-\alpha}(2\pi)^{(r-1)/2}r^{1/2-\eta-1+s/m}\Gamma(\eta+1-s/m)}{(2\pi)^{(r-1)/2}r^{1/2-\eta-\alpha-1+s/m}\Gamma(\eta+\alpha+1-s/m)}\\ &=\frac{\Gamma(\eta+1-s/m)}{\Gamma(\eta+\alpha+1-s/m)}\end{aligned}$$

which shows that $g(s)$ is also the multiplier for $I_m^{\eta,\alpha}$ and (5.9) follows in the case where $1<p<\infty$ (with $p=1$, $p=\infty$ being dealt with as before).

EXAMPLE 5.5. In certain cases, (5.9) can be proved from scratch by using results involving special functions. As an illustration, we consider the case $r=2$, $\operatorname{Re}\alpha>0$, $\operatorname{Re}(m\eta+\mu)+m>1/p$ $(1\leqq p\leqq\infty)$. Then all the operators in (5.9) are integral operators of the form described in Theorem 4.3(i). Also [**3**, Lemma 4.4] shows that, for $\phi\in F_{p,\mu}$,

$$\begin{aligned}&(I_{2m}^{(\eta-1)/2,\alpha/2}I_{2m}^{\eta/2,\alpha/2}\phi)(x)\\ &\quad=[\Gamma(\alpha)]^{-1}x^{-m(\eta+\alpha-1)}\int_0^x(x^{2m}-t^{2m})^{\alpha-1}{}_2F_1((\alpha+1)/2,\alpha/2;\alpha;1-x^{2m}/t^{2m})\\ &\quad\quad\times t^{m(\eta-\alpha-1)}(2m)t^{2m-1}\phi(t)\,dt.\end{aligned}$$

But, from [**2**, p. 101, (6)], we see that, under appropriate conditions,

$${}_2F_1(a+1/2,a;2a;z)=(1-z)^{-\frac12}[\tfrac12+(1-z)^{\frac12}/2]^{1-2a}.$$

Accordingly, we obtain

$$\begin{aligned}(I_{2m}^{(\eta-1)/2,\alpha/2}I_{2m}^{\eta/2,\alpha/2}\phi)(x) &= [\Gamma(\alpha)]^{-1}x^{-m(\eta+\alpha-1)}\int_0^x (x^{2m}-t^{2m})^{\alpha-1}(t^m/x^m) \\ &\quad\times[(t^m+x^m)/t^m]^{1-\alpha}2^{\alpha-1}t^{m(\eta-\alpha)+m-1}(2m)\phi(t)\,dt \\ &= 2^{\alpha}[\Gamma(\alpha)]^{-1}x^{-m(\eta+\alpha)}\int_0^x (x^m-t^m)^{\alpha-1}t^{m\eta}\phi(t)mt^{m-1}\,dt \\ &= 2^{\alpha}(I_m^{\eta,\alpha}\phi)(x) \quad \text{by (4.4).}\end{aligned}$$

(5.9) then follows at once. It seems probable that other cases can also be handled by means of properties of generalised hypergeometric functions or the G-function but we shall not elaborate here.

It is no surprise that there is an analogue of (5.9) for the Erdélyi-Kober operator defined by (4.3).

THEOREM 5.6. *Let* $1 \leqq p \leqq \infty$, μ *and* α *be arbitrary complex numbers,* $m > 0$, $\eta \in A'_{p,\mu,m}$ *and* r *be any positive integer. Then, as operators on* $F_{p,\mu}$,

$$K_m^{\eta,\alpha} = r^{-\alpha}\prod_{k=1}^{r} K_{rm}^{(\eta-1+k)/r,\,\alpha/r}. \tag{5.11}$$

Proof. This can be dealt with in various ways. To see that (5.11) is formally correct, we observe that the operator $K_m^{\eta,\alpha}$ is the formal adjoint of $I_m^{\eta-1+1/m,\alpha}$ while $K_{rm}^{(\eta-1+k)/r,\,\alpha/r}$ is the formal adjoint of $I_{rm}^{(\eta-1+k)/r-1+1/rm,\,\alpha/r}$. Since the I operators commute, the right-hand side of (5.11) is the formal adjoint of

$$r^{-\alpha}\prod_{k=1}^{r} I_{rm}^{(\eta-1+k)/r-1+1/rm,\,\alpha/r} = r^{-\alpha}\prod_{k=1}^{r} I_{rm}^{(\eta-1+1/m-r+k)/r,\,\alpha/r}.$$

The condition $\eta \in A'_{p,\mu,m}$ is equivalent to $\eta-1+1/m \in A_{q,-\mu,m}$ $(q = p/(p-1))$. Thus (5.11) is obtained formally by replacing η by $\eta-1+1/m$ in (5.9) and taking adjoints. Alternatively, it is possible to use (4.3), (5.10) and the method of Remark 5.4. A third possibility is to make use of results in [**4**]. We omit the details.

Remark 5.7.

(i) Formula (5.11) which is the second formula referred to earlier does not seem to be present in the literature.

(ii) We would emphasise that (5.9) and (5.11) have been proved rigorously and would draw attention to the very general conditions on the parameters under which the results are valid.

6.

In this final section, we indicate how formulae (5.9) and (5.11) lead to relations between fractional integrals and derivatives of the type mentioned in §1. We shall concentrate on (5.9) since (5.11) is similar.

THEOREM 6.1. *Let* $1 \leqq p \leqq \infty$, $m > 0$, μ *be any complex number,* $\eta \in A_{p,\mu,m}$ *and* r *be any positive integer.*

(i) *If* $\operatorname{Re}\alpha>0$, *then as operators from* $F_{p,\mu+m\eta}$ *into* $F_{p,\mu+m\eta+m\alpha}$,

$$I_m^\alpha = r^{-\alpha}x^{m(\alpha+r)}(x^{-m(1+\alpha)}I_{rm}^{\alpha/r})^r \tag{6.1}$$

where each I operator is defined by an appropriate version of (2.20).

(ii) *If* $\operatorname{Re}\beta>0$, *then as operators from* $F_{p,\mu+m\eta}$ *into* $F_{p,\mu+m\eta-m\beta}$,

$$D_m^\beta = r^{\beta}x^{m(-\beta+r)}(x^{-m(1-\beta)}D_{rm}^{\beta/r})^r \tag{6.2}$$

where each D operator is interpreted as in Example 2.10(ii).

Proof. Both parts can be handled simultaneously by means of multipliers since (2.18) is valid for any complex α and $D_m^\beta = I_m^{-\beta}$, $D_{rm}^{\beta/r} = I_{rm}^{-\beta/r}$ by (2.19). This approach again uses (5.10). We omit the details but instead derive (6.1) formally from (5.9). If $\psi\in F_{p,\mu+m\eta}$, then $\psi(x)=x^{m\eta}\phi(x)$ where $\phi\in F_{p,\mu}$ so that, by (4.4) and (5.9),

$$\begin{aligned}
x^{m(\alpha+r)}([x^{-m(1+\alpha)}I_{rm}^{\alpha/r}]^r\psi)(x) &= x^{m(\alpha+r)}[x^{-m(1+\alpha)}I_{rm}^{\alpha/r}]^{r-1}[x^{-m(1+\alpha)}I_{rm}^{\alpha/r}\,x^{m\eta}\phi(x)]\\
&= x^{m(\alpha+r)}[x^{-m(1+\alpha)}I_{rm}^{\alpha/r}]^{r-1}[x^{m(\eta-1)}I_{rm}^{\eta/r,\alpha/r}\phi(x)]\\
&= x^{m(\alpha+r)}[x^{-m(1+\alpha)}I_{rm}^{\alpha/r}]^{r-2}[x^{m(\eta-2)}I_{rm}^{(\eta-1)/r,\,\alpha/r}I_{rm}^{\eta/r,\,\alpha/r}\phi(x)]\\
&= \cdots\\
&= x^{m(\alpha+r)}\,.\,x^{m(\eta-r)}\left(\prod_{k=1}^{r} I_{rm}^{(\eta-r+k)/r,\alpha/r}\phi\right)(x)\\
&= x^{m(\eta+\alpha)}r^{\alpha}(I_m^{\eta,\alpha}\phi)(x)\\
&= r^{\alpha}x^{m(\eta+\alpha)}\,.\,x^{-m\eta-m\alpha}I_m^{\alpha}x^{m\eta}\phi(x)\\
&= r^{\alpha}I_m^{\alpha}\psi(x).
\end{aligned}$$

(6.1) then follows at once.

Remark 6.2. Although (6.1) and (6.2) have been proved under appropriate conditions within the $F_{p,\mu}$ structure, they are applicable to functions which do not belong to any $F_{p,\mu}$ class. In the case of derivatives, this arises because differentiability is a local property while in the case of integrals we can argue as follows. The validity of (6.1) entails checking that when the operators on both sides are applied to a suitable function ϕ and the resulting functions are evaluated at each fixed $x\in(0,\infty)$, we always get the same value in both cases. When x is fixed, the only values of ϕ which are relevant are those attained on the interval $(0, x)$. Accordingly, we can replace ϕ by an equivalent function ϕ_x defined by

$$\phi_x(t)=\begin{cases}\phi(t) & t\in(0,x)\\ 0 & t\in(x+1,\infty).\end{cases} \tag{6.3}$$

Provided that $\phi_x\in F_{p,\mu}$ for some p and μ, we are in business and this can often be arranged by means of smooth cut-off functions. A case in point is when ϕ has the form of a power of the variable, say $\phi(t)=t^\lambda$ for some λ. This function ϕ does not belong to any $F_{p,\mu}$ space but the function ϕ_x defined via (6.3) is an element of $F_{p,\mu}$ provided that $1\leqq p\leqq\infty$ and $\operatorname{Re}(\lambda-\mu)>-1/p$.

EXAMPLE 6.3. With the last point in Remark 6.2 in mind, we give a simple illustration of (6.2) for the case $\beta=4$. If we take $r=2$ and apply the right-hand side to the function $x^{m\lambda}$ (m being inserted for convenience) we obtain

$$\begin{aligned}2^4x^{m(-2)}x^{3m}D^2_{2m}x^{3m}D^2_{2m}x^{m\lambda} &= 2^4x^mD^2_{2m}x^{3m}D^2_{2m}x^{(2m)(\lambda/2)}\\ &= 2^4x^mD^2_{2m}x^{3m}(\lambda/2)(\lambda/2-1)x^{(2m)(\lambda/2-2)}\\ &= 2^2\lambda(\lambda-2)x^mD^2_{2m}x^{(2m)(\lambda/2-1/2)}\\ &= 2^2\lambda(\lambda-2)x^m(\lambda/2-1/2)(\lambda/2-3/2)x^{(2m)(\lambda/2-5/2)}\\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)x^{m(\lambda-4)}.\end{aligned}$$

On the other hand, with $r=4$, we get

$$\begin{aligned}4^4(x^{3m}D_{4m})^4x^{m\lambda} &= 4^4(x^{3m}D_{4m})^3x^{3m}D_{4m}x^{(4m)(\lambda/4)}\\ &= 4^4(x^{3m}D_{4m})^3x^{3m}(\lambda/4)x^{(4m)(\lambda/4-1)}\\ &= 4^3\lambda(x^{3m}D_{4m})^2x^{3m}D_{4m}x^{(4m)(\lambda-1)/4}\\ &= 4^3\lambda(x^{3m}D_{4m})^2x^{3m}((\lambda-1)/4)x^{(4m)(\lambda-5)/4}\\ &= 4^2\lambda(\lambda-1)(x^{3m}D_{4m})x^{3m}D_{4m}x^{(4m)(\lambda-2)/4}\\ &= 4^2\lambda(\lambda-1)x^{3m}D_{4m}x^{3m}((\lambda-2)/4)x^{(4m)(\lambda-6)/4}\\ &= 4\lambda(\lambda-1)(\lambda-2)x^{3m}D_{4m}x^{(4m)(\lambda-3)/4}\\ &= 4\lambda(\lambda-1)(\lambda-2)x^{3m}((\lambda-3)/4)x^{(4m)(\lambda-7)/4}\\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)x^{m(\lambda-4)}.\end{aligned}$$

In both cases, we obtain $D^4_mx^{m\lambda}$, as expected. These verifications are hardly novel. The novelty of (6.1) and (6.2) resides in the formulae obtained when α and β are non-integral.

We shall not pursue the ramifications of (6.1) and (6.2) here but merely make the following concluding remarks.

Remark 6.4.

(i) It is possible that some familiar results can be obtained by applying (6.1) and (6.2) to some special functions whose behaviour at 0 and ∞ is such that they do not belong to any $F_{p,\mu}$ space but can be handled in the spirit of Remark 6.2.

(ii) (6.1) can be recast in the form

$$x^{-m(1+\alpha)}I^{\alpha/r}_{rm} = (r^\alpha x^{-m(\alpha+r)}I^\alpha_m)^{1/r} \tag{6.4}$$

where the right-hand side is interpreted as an r-th root in an appropriate sense. Similar comments apply to (6.2).

(iii) There is an analogue of (6.1) for the operator K^α_m, defined by (4.8), which takes the form

$$K^\alpha_m = r^{-\alpha}x^{m(\alpha+r)}(x^{-m(1+\alpha)}K^{\alpha/r}_{rm})^r \tag{6.5}$$

for $\operatorname{Re}\alpha>0$. Conditions for the validity of (6.5) within the framework of the $F_{p,\mu}$ spaces can be derived from Theorem 5.6. The result can be extended by the use of cut-off functions since all activity is concentrated on an interval of the form (x,∞), in contrast to the interval $(0,x)$ in Remark 6.2. For an analogue of (6.2), we offer

$$(-D_m)^\beta = r^\beta x^{m(r-\beta)}(x^{m(\beta-1)}(-D_{rm})^{\beta/r})^r \tag{6.6}$$

for $\operatorname{Re}\beta>0$. Formulae (6.1), (6.2), (6.5) and (6.6) are the results promised in §1.

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(*Issued* 13 *August* 1984)

Paper 10

Applicable Analysis, 1986, Vol. 21, pp. 89–127
0003-6811/86/2102-0089 $18.50/0

Printed in Great Britain

Fractional Powers of a Class of Mellin Multiplier Transforms Part I

Communicated by G. F. ROACH

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AMS (MOS): 47G05, 26A33

Abstract The paper is concerned with a class of Mellin multiplier transforms, mapping one weighted $L^p(0,\infty)$ space into another, whose symbols are of a particular form. An expression is easily obtained for positive integral powers of such operators and this forms the basis of an extension to fractional powers. A rigorous framework for the analysis is described. Analogues of the index laws of ordinary algebra are established under stated conditions. Connections between powers of an operator and of its adjoint are explored. The theory is illustrated by means of simple integral operators. These examples serve to indicate some of the limitations of a classical setting and provide the stimulus for a distributional treatment which will be presented elsewhere.

(*Received for Publication June 26, 1985*)

1. There is currently considerable interest in what are sometimes called Mellin multiplier transforms. From the theoretical point of view, they are closely related to pseudo-differential operators which have an extensive literature. However, their importance in relation to practical problems is also well established. One notable example is the work of Costabel, Stephan and Wendland who, in [2] and elsewhere, have investigated the role played by Mellin multiplier transforms in the boundary integral equation approach to the solution of boundary value problems for partial differential equations (such as the bi-Laplacian) in polygonal plane domains or domains with corners.

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1. There is currently considerable interest in what are sometimes called Mellin multiplier transforms. From the theoretical point of view, they are closely related to pseudo-differential operators which have an extensive literature. However, their importance in relation to practical problems is also well established. One notable example is the work of Costabel, Stephan and Wendland who, in [2] and elsewhere, have investigated the role played by Mellin multiplier transforms in the boundary integral equation approach to the solution of boundary value problems for partial differential equations (such as the bi-Laplacian) in polygonal plane domains or domains with corners.

Another area, which has come to be known as fractional calculus, has developed extensively over the last fifty years. The use of operators involving derivatives and integrals of fractional order often provides an elegant formalism for solving problems which might otherwise involve heavy manipulations with special functions. Examples can be found in [10], particularly in the article by Sneddon where the connections with mixed boundary value problems in potential theory are discussed.

In this paper, the first of a series, we bring together strands from these two areas by presenting a method, based on Mellin multiplier transforms, of defining fractional powers of a certain class of operators. The class includes as particular cases operators which occur in many problems in applied mathematics. We hope to incorporate the useful aspects of the two separate areas and, by working distributionally in a later paper, to provide a full answer to questions of existence and uniqueness. Meanwhile, we introduce a little more detail in order to describe the motivation for our theory.

If X is a non-empty set and $T:X \to X$ is an operator whose domain is the whole of X, there is no difficulty in defining the operators $T^n (n = 1,2,\ldots)$ by repeated composition. In recent years, much interest has centred on trying to define T^α for (real or complex) values of α other than positive integers, such powers being referred to as fractional powers of T. Significant progress was made by, amongst others, Balakrishnan [1] and Komatsu [4] within the framework of closed linear operators in Banach spaces. These investigations relied heavily on the apparatus of spectral theory, Dunford integrals, symbolic calculus and so on. The object was to find a suitable expression for $T^n (n = 1,2,\ldots)$ which continued to be meaningful when n was replaced by a complex number α subject to some restriction such as $\operatorname{Re} \alpha>0$. With this definition of fractional powers of T, it was possible to show that the index laws

$$T^\alpha T^\beta = T^\beta T^\alpha = T^{\alpha+\beta} \tag{1.1}$$

$$(T^\alpha)^\beta = T^{\alpha\beta} \tag{1.2}$$

held under appropriate conditions. Typically, (1.2) was established under the rather severe restrictions α real, $0<\alpha<1$, Re $\delta>0$; see, for instance, [12].

When the elements of X are themselves functions, other approaches are sometimes possible. Suppose, for example, that $X = L^2(-\infty,\infty)$, and that T is a bounded linear mapping from $L^2(-\infty,\infty)$ into itself which has the form of a Fourier multiplier transform with symbol g. This means that, for $f \in L^2(-\infty,\infty)$,

$$(F(Tf))\ (\xi) = g(\xi)\ (Ff)\ (\xi) \qquad (-\infty<\xi<\infty) \tag{1.3}$$

where F is the Fourier transform. Since F is invertible on $L^2(-\infty,\infty)$, it follows from (1.3) that the equation

$$T = F^{-1}gF \tag{1.4}$$

holds in the sense of operators on $L^2(-\infty,\infty)$. Proceeding formally from (1.4), we have

$$T^n = F^{-1}g^nF \qquad (n = 1,2,\ldots)$$

which suggests that we might try to define T^α by

$$T^\alpha = F^{-1}g^\alpha F \tag{1.5}$$

provided that some sense can be attached to the function g^α (via Riemann surfaces, perhaps). Such a programme can be contemplated when $X = L^p(-\infty,\infty)$ for any p in the range $1<p<\infty$ since a precise characterisation can be given [11, Chapter 4] of a class of symbols g which generate bounded linear mappings from $L^p(-\infty,\infty)$ into itself which are of the form (1.4) (with an appropriate interpretation). In theory it would seem possible to check whether or not g^α belongs to this class of symbols. However, problems may well arise because of the existence of branch points, with a complete breakdown if $g(\xi) = 0$ for some real ξ. Thus, with this approach, which belongs to the realm of pseudodifferential operators, there is no guarantee of success.

By simple changes of variables, results for the Fourier transform of functions defined on $(-\infty,\infty)$ can be converted into results for the Mellin transform of functions defined on $(0,\infty)$. Formally the Mellin transform, $M\phi$, of a function ϕ is defined by

$$(M\phi)(s) = \int_0^\infty x^{s-1}\,\phi(x)\,dx. \tag{1.6}$$

We shall spell out below conditions on the function ϕ and the complex variable s which ensure the existence of $(M\phi)(s)$. Meanwhile, we simply remark that there is a class of symbols g which give rise to bounded linear operators from $L^p(0,\infty)$ into itself $(1<p<\infty)$ which have the form

$$T = M^{-1}gM \tag{1.7}$$

so that, for $\phi \in L^p(0,\infty)$,

$$(M(T\phi))(s) = g(s)\;(M\phi)(s) \tag{1.8}$$

under suitable conditions. An operator T of the form (1.7) is sometimes called a Mellin multiplier transform. Such operators have been investigated in detail by Rooney in [8] and [9] and also figure in the work of Lewis and Parenti [5] on a class of pseudo-differential operators. As before, (1.7) suggests the formula

$$T^\alpha = M^{-1}g^\alpha M \tag{1.9}$$

for general powers of T. However, similar problems arise with (1.9) to those encountered with (1.5), and the method may fail. In this paper, we shall avoid such problems by considering operators T which satisfy a modified version of (1.8).

Suppose firstly that the symbol g is of the form

$$g(s) = \frac{h(s-\gamma)}{h(s)} \tag{1.10}$$

where γ is a complex number and h is a function defined on an appropriate region of the complex plane. Suppose also that the operator T satisfies the relation

$$(\mathcal{M}(T\phi))(s-\gamma) = g(s)\ (\mathcal{M}\phi)(s) \tag{1.11}$$

or, equivalently,

$$(\mathcal{M}(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)}\ (\mathcal{M}\phi)(s). \tag{1.12}$$

From (1.6) we see that the left-hand side of (1.12) can be written as $(\mathcal{M}(x^{-\gamma}T\phi(x)))(s)$ so that, by comparison with (1.7), we are considering operators of the form

$$T = x^{\gamma}\mathcal{M}^{-1}g\mathcal{M} \tag{1.13}$$

where x^{γ} denotes the operation of multiplying a function of x by x^{γ} and g is of the form (1.10). When $\gamma = 0$, $g(s) \equiv 1$ and T reduces to the identity operator on the appropriate space. Needless to say, we shall be more concerned with the case $\gamma \neq 0$. The presence of the weight factor naturally leads to the use of weighted $L^p(0,\infty)$ spaces and, indeed, these are used extensively by Rooney in [8] and [9]. However, whereas operators of the form (1.7) map a weighted $L^p(0,\infty)$ space into itself, operators of the form (1.13) will map one such space into a different space in general. It might be thought that this would complicate matters considerably but, perhaps paradoxically, the reverse is the case.

To give an inkling of what lies ahead, we present a little formal analysis. By two applications of (1.12), we obtain

$$\begin{aligned}(\mathcal{M}(T^2\phi))(s-2\gamma) &= (\mathcal{M}(T(T\phi))\ )\ ((s-\gamma)-\gamma)\\ &= \frac{h((s-\gamma)-\gamma)}{h(s-\gamma)}\ (\mathcal{M}(T\phi))(s-\gamma)\\ &= \frac{h(s-2\gamma)}{h(s-\gamma)}\ \frac{h(s-\gamma)}{h(s)}\ (\mathcal{M}\phi)(s)\\ &= \frac{h(s-2\gamma)}{h(s)}\ (\mathcal{M}\phi)(s).\end{aligned}$$

An easy induction argument shows that, formally, for $n = 1,2,\ldots,$

$$(M(T^n\phi))(s-n\gamma) = \frac{h(s-n\gamma)}{h(s)} (M\phi)(s).$$

This in turn suggests that the operator T^α should be defined in such a way that

$$(M(T^\alpha\phi))(s-\alpha\gamma) = \frac{h(s-\alpha\gamma)}{h(s)} (M\phi)(s) \tag{1.14}$$

or, equivalently,

$$T^\alpha = x^{\alpha\gamma} M^{-1} g_\alpha M \tag{1.15}$$

where the symbol g_α is given by

$$g_\alpha(s) = \frac{h(s-\alpha\gamma)}{h(s)} . \tag{1.16}$$

There is now no difficulty over branch points. Instead we have to check whether or not g_α belongs to an appropriate class of symbols. More importantly, some justification has to be supplied for the formal analysis and this gives rise to a number of interesting points. It is the intention of this paper to provide a rigorous discussion of these matters within the classical setting of weighted $L^p(0,\infty)$ spaces. It is proposed to extend the theory to certain classes of generalised functions in a subsequent paper.

In §2, we review a number of results for Mellin multipliers in weighted $L^p(0,\infty)$ spaces and describe the particular class of multipliers which will concern us here. The standard results refer to spaces for which $1 < p < \infty$ but we shall indicate briefly our attitude to the cases $p = 1$ and $p = \infty$. In §3, we develop the theory of powers of operators in a certain class L and, in particular, we state conditions under which the index laws (1.1) and (1.2) are valid. In §4 we show that if an operator T belongs to the class L, then so does its formal adjoint T'. This is of some interest in its own right but is also an essential preparation for the distributional sequel. Finally, in §5, we illustrate the results of the preceding sections in the case of certain simple operators, reproducing, in particular, some well-known facts in fractional calculus. Other more elaborate examples could have been given but these have been deferred to a later paper.

2. In this section we shall collect together some notation and standard results which will be used extensively later. We begin with a few details concerning weighted L^p spaces.

Definition 2.1

For $1 \leq p \leq \infty$ and any complex number μ,

$$L^p_\mu = \{f : x^{-\mu}f(x) \in L^p(0,\infty)\}. \tag{2.1}$$

Thus, L^p_μ is the space of (equivalence classes of) complex-valued measurable functions f defined (almost everywhere) on $(0,\infty)$ such that the quantity

$$\| f \|_{p,\mu} \equiv \| x^{-\mu}f(x) \|_{L^p(0,\infty)} \tag{2.2}$$

is finite, where $\| \ \|_{L^p(0,\infty)}$ denotes the usual $L^p(0,\infty)$ norm.

Remark 2.2

(i) Here, and elsewhere, for $0 < x < \infty$ and any complex number λ, x^λ will mean $\exp(\lambda \log x)$ where log denotes the principal value of the logarithm.

(ii) The spaces $(L^p_\mu, \| \ \|_{p,\mu})$ are Banach spaces. For μ real and $1 \leq p < \infty$, they correspond to Rooney's spaces in [8] and [9] except for a slight change of notation, which means that our space $(L^p_\mu, \| \ \|_{p,\mu})$ is Rooney's space $(L_{1-\mu p,p}, \| \ \|_{1-\mu p,p})$. However, it is convenient to allow complex values of μ and the case $p = \infty$ can also be included without any difficulty.

(iii) It is easy to see that the mapping x^λ defined on L^p_μ by

$$(x^\lambda\phi)(x) = x^\lambda\phi(x) \qquad (0 < x < \infty) \tag{2.3}$$

is a homeomorphism from L^p_μ onto $L^p_{\mu+\lambda}$ with inverse $x^{-\lambda}$, for any complex numbers μ and λ and for $1 \leqslant p \leqslant \infty$. (There should be no confusion because of the use of the symbol x^λ to denote the function x^λ as well as the operation of multiplying a function of x by x^λ.)

Next we consider the behaviour of the Mellin transform on the spaces L^p_μ.

<u>Lemma 2.3</u>

For $\phi \in L^p_\mu$, $(M\phi)(s)$, as defined by (1.6), exists provided that $1 \leqslant p \leqslant 2$ and

$$\text{Re } s = 1/p - \text{Re } \mu. \tag{2.4}$$

In that case, if $s = \sigma + it$ and $(M\phi)(\sigma + it)$ is regarded as a function of t on $(-\infty,\infty)$ then M is a continuous linear mapping from L^p_μ into $L^q(-\infty,\infty)$, where $q = p/(p-1)$.

<u>Proof:-</u> The result follows immediately from [8, Lemma 4.1] via the change of notation mentioned in Remark 2.2 (ii).

<u>Remark 2.4</u>

(i) From now on, whenever we consider $(M\phi)(s)$ for $\phi \in L^p_\mu$ $(1 \leqslant p \leqslant 2)$, it will always be the case that s and μ are related by (2.4).

(ii) For $p = 1$, the integral defining $(M\phi)(s)$ converges absolutely under the hypotheses of Lemma 2.3, while for $1 < p \leqslant 2$, $(M\phi)(s) \equiv (M\phi)(\sigma + it)$ has to be interpreted as a limit in the $L^q(-\infty,\infty)$ mean.

We must now make clear what we shall mean by an L^p_μ multiplier. Here we are influenced by particular examples, some of which will be discussed later.

Notation 2.5

(i) Throughout the paper, Ω will denote a domain in the complex plane which is the union of a finite or countably infinite collection of disjoint, open strips parallel to the imaginary axis. In other words, Ω can be written as the union of a finite or countably infinite collection of disjoint strips, each of which has one of the forms

$\{s : a < \operatorname{Re} s < b\}$, $\{s : \operatorname{Re} s < c\}$ or $\{s : \operatorname{Re} s > d\}$

where a, b, c and d are real numbers. (There can be at most one strip of each of the second and third forms.)

(ii) Let Ω be as in (i). For each fixed p in the range $1 \leqslant p \leqslant \infty$, we let

$$\Omega_p = \{\mu : 1/p - \mu \in \Omega\}. \tag{2.5}$$

(2.5) ensures that, when s and μ are related by (2.4) and p is fixed, $s \in \Omega$ if and only if $\mu \in \Omega_p$.

Definition 2.6

Let g be a complex-valued function analytic on a domain Ω (of the type in Notation 2.5(i)). We shall say that g is an L^p_μ multiplier if there exists a (unique) linear transformation R (depending on g) such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, R is a bounded linear transformation from L^p_μ into L^p_μ

(ii) for $1 < p \leqslant 2$, $\mu \in \Omega_p$ and $\phi \in L^p_\mu$,

$$(\mathcal{M}(R\phi))\ (s) = g(s)\ (\mathcal{M}\phi)(s). \tag{2.6}$$

We shall then say that g is the symbol of R and refer to R as a (Mellin) multiplier transform.

Remark 2.7

(i) The motivation for Definition 2.6 is to be found in the work of Rooney [9] which in turn is derived from corresponding results for the Fourier transform described by Stein [11]. However, we differ in our approach in some respects. For instance, we are working in the domain Ω whereas Rooney works in a single strip. Rooney's theory could be applied to each strip in Ω individually so that the difference might appear minor. In one sense this is so, but in another sense, a new facet enters since the form of T on L^p_μ may vary as $1/p - \mu$ moves from one component of Ω to another. We shall give an example in §5.

(ii) A comment is in order regarding our attitude to the values $p = 1$ and $p = \infty$. Rooney considers the range $1 < p < \infty$ because then he can identify a large class of L^p_μ multipliers [9, Theorem 1]. However, in this paper, we shall not be attempting to characterise possible symbols g. We adopt the view that, by some means or another, it is possible to decide whether or not a given function g is an L^p_μ multiplier and take it from there. Many of our proofs boil down to using the fact that

$$C_o^\infty(0,\infty) = \{\phi : \phi \text{ is infinitely differentiable and has compact support in } (0,\infty)\} \tag{2.7}$$

is dense in L^p_μ for $1 < p < \infty$ and any complex number μ, together with the continuity of the Mellin transform on L^p_μ for $1 < p \leq 2$ (Remark 2.4(ii)). But these properties of density and continuity hold also when $p = 1$. Thus, in the case when R is a bounded linear transformation from L^1_μ into L^1_μ, we may extend (2.6) to include $p = 1$. In contrast $C_o^\infty(0,\infty)$ is not dense in L^∞_μ for any choice of μ so that our method of proof is doomed to failure in the case $p = \infty$ (although when we work distributionally, as we shall do in a later paper, we are back in business). To summarise, we shall examine L^p_μ multipliers in the sense of Definition 2.6 with the attitude that, if the operator R also maps L^1_μ into L^1_μ or L^∞_μ into L^∞_μ, then that is a bonus. Again, we shall see examples in §5.

As indicated in §1, we are particularly interested in the case where $g(s) = h(s-\gamma)/h(s)$. The following definition is suggested.

Definition 2.8

H will denote the set of all ordered triples (h,Ω,γ) such that

(i) Ω is a domain of the type considered in Notation 2.5(i)

(ii) γ is a complex number

(iii) h is a complex-valued function such that h is analytic on $\{s-\gamma : s \in \Omega\}$ and $1/h$ is analytic on Ω

(iv) the function g defined on Ω by $g(s) = h(s-\gamma)/h(s)$ is an L^p_μ multiplier for $1 < p < \infty$ and $\mu \in \Omega_p$, in the sense of Definition 2.6.

Remark 2.9

(i) The conditions above are convenient for our purposes but could be made less restrictive.

(ii) As mentioned in §1, we can allow $\gamma = 0$ in which case the corresponding operator R is simply the identity operator on the appropriate L^p_μ space. In useful applications, we usually have $\operatorname{Re} \gamma \neq 0$. In this case, it is not the operator R which matters to us but rather the operator $T = x^\gamma R$, with x^γ defined via (2.3). We therefore state the following result.

Theorem 2.10

A triple $(h,\Omega,\gamma) \in H$ generates a (unique) linear transformation T such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, $T \in B(L^p_\mu, L^p_{\mu+\gamma})$

(ii) for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in L^p_\mu$,

$$(M(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)} (M\phi)(s). \tag{2.8}$$

Proof:- The result follows immediately on taking $T = x^\gamma R$ with R as in Definition 2.6 since

$$(M(x^{\gamma}R\phi))(s-\gamma) = (M(R\phi))(s)$$

for $\phi \in L^p_\mu$, $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$.

3. We now have a description of the type of operators whose powers we shall shortly construct. Theorem 2.10 suggests that the following definition is appropriate.

Definition 3.1

L will denote the class of all operators T generated by triples $(h,\Omega,\gamma) \in H$ in the sense of Theorem 2.10. More precisely, $T \in L$ if there exists a triple $(h,\Omega,\gamma) \in H$ (as in Definition 2.8) such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, $T \in B(L^p_\mu, L^p_{\mu+\gamma})$

(ii) for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in L^p_\mu$,

$$(M(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)}(M\phi)(s).$$

(Recall that (2.4) and (2.5) apply.)

We propose to define powers of operators T in the class L. However, we must first deal with a rather important matter, namely, that if $T \in L$ then T can be generated by infinitely many triples in H. Trivially, if T is generated by $(h,\Omega,\gamma) \in H$, then T is also generated by the triple $(ch,\Omega,\gamma) \in H$ where c is any non-zero constant. We mention another less trivial instance to which we shall refer again later.

Lemma 3.2

Let $T \in L$ be generated by the triple $(h,\Omega,\gamma) \in H$ where $\gamma \neq 0$. For each integer k, define h_k on the domain of h by

$$h_k(s) = \exp(2\pi kis/\gamma)\, h(s) \tag{3.1}$$

(so that $h_0 \equiv h$). Then, for each integer k, the triple (h_k,Ω,γ) also belongs to H and generates T.

Proof:- That $(h_k,\Omega,\gamma) \in H$ is immediate from Definition 2.8 since the functions $\exp(2\pi kis/\gamma)$ and $\exp(-2\pi kis/\gamma)$ are entire and h_k inherits the analyticity properties of h. That (h_k,Ω,γ) generates T follows since

$$\frac{h_k(s-\gamma)}{h_k(s)} = \frac{\exp(2\pi kis/\gamma)\ \exp(-2\pi ki)\ h(s-\gamma)}{\exp(2\pi kis/\gamma)\ h(s)} = \frac{h(s-\gamma)}{h(s)} .$$

Remark 3.3

As regards a representation for T, any one of the functions h_k is as good as any other. However, we shall see shortly that the choice of representation is crucial as regards defining general powers of T. Accordingly, we must declare that in the theory which follows, whenever $T \in L$, it is to be understood that we are using a fixed representation $(h,\Omega,\gamma) \in H$ for T and that the definitions and theorems are stated in terms of that fixed representation. For emphasis, we shall often write $T \equiv T(h,\Omega,\gamma)$.

Now we are ready to define powers of an operator $T \in L$. The motivation is supplied by (1.14) which suggests that if T is generated by (h,Ω,γ), then T^α should be generated by $(h,\Omega,\alpha\gamma)$. To make sure that the latter triple is in the class H we make an extra assumption.

Definition 3.4

For any triple $(h,\Omega,\gamma) \in H$, we shall define the set A of complex numbers by

$$A \equiv A(h,\Omega,\gamma) = \{\alpha\colon\ (h,\Omega,\alpha\gamma) \in H\}. \tag{3.2}$$

The alternative notation $A(h,\Omega,\gamma)$ indicates that A depends on the triple (h,Ω,γ). However, we shall usually just write A on the understanding that it is constructed from the fixed representation for T which is under discussion. (See Remark 3.3.)

Definition 3.5

Let $T \equiv T(h,\Omega,\gamma) \in L$ and let $\alpha \in A$. We define T^{α} to be the operator in L generated by the triple $(h,\Omega,\alpha\gamma)$. More precisely, T^{α} is the linear operator such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, $T^{\alpha} \in B(L^p_{\mu}, L^p_{\mu+\alpha\gamma})$

(ii) for $1 < p \leqslant 2$, $\mu \in \Omega_p$, and $\phi \in L^p_{\mu}$,

$$(M(T^{\alpha}\phi))(s-\alpha\gamma) = \frac{h(s-\alpha\gamma)}{h(s)} (M\phi)(s). \tag{3.3}$$

Remark 3.6

(i) Definition 3.5 is meaningful in view of Definition 3.4.

(ii) The actual composition of A will vary with T, as noted above. We shall not attempt to characterise the possible sets A that can arise but simply remark that for some choices of T, A is the whole complex plane while for other operators T conditions such as Re $\alpha \geqslant 0$ are typical. We shall give examples below. Meanwhile, we proceed with the theory on the assumption that, for a given operator $T \equiv T(h,\Omega,\gamma)$, the set $A(h,\Omega,\gamma)$ can be calculated by some means or another, so that admissible values of α are known.

We shall dispose first of a triviality.

Lemma 3.7

Let $T \equiv T(h,\Omega,\gamma) \in L$. Then $0 \in A$ and for $1 < p < \infty$, $\mu \in \Omega_p$, T^{0} is the identity operator on L^p_{μ}.

Proof:- This follows on putting $\alpha = 0$ in (3.3) and using the fact that M is one-to-one on L^p_{μ} for $1 < p \leqslant 2$, $\mu \in \Omega_p$.

Next we shall state conditions under which the first index law (1.1) is satisfied and give a rigorous (as opposed to formal) proof.

Theorem 3.8

Let $T \equiv T(h,\Omega,\gamma) \in L$, $1 < p < \infty$, $\{\alpha,\beta,\alpha+\beta\} \subseteq A$ and $\{\mu, \mu+\beta\gamma\} \subseteq \Omega_p$. Then, as operators on L^p_μ,

$$T^\alpha T^\beta = T^{\alpha+\beta}. \tag{3.4}$$

Proof:- Since $\beta \in A$ and $\mu \in \Omega_p$, T^β exists as an element of $B(L^p_\mu, L^p_{\mu+\beta\gamma})$ by Definition 3.5(i). Similarly, since $\alpha \in A$ and $\mu+\beta\gamma \in \Omega_p$, $T^\alpha \in B(L^p_{\mu+\beta\gamma}, L^p_{\mu+\beta\gamma+\alpha\gamma})$ and since $\alpha+\beta \in A$, $\mu \in \Omega_p$, we have $T^{\alpha+\beta} \in B(L^p_\mu, L^p_{\mu+(\alpha+\beta)\gamma})$. Hence both sides of (3.4) define elements of $B(L^p_\mu, L^p_{\mu+\alpha\gamma+\beta\gamma})$ under the given conditions. To prove equality, it is therefore sufficient to prove that

$$T^\alpha T^\beta \phi = T^{\alpha+\beta}\phi$$

for $\phi \in C_o^\infty(0,\infty)$, since the latter set is dense in L^p_μ. To do this, we regard ϕ as an element of L^2_μ with μ chosen so that $\{\mu,\mu+\beta\gamma\} \subseteq \Omega_2$. Then, for Re $s = 1/2 -$ Re μ,

$$\begin{aligned}(M(T^\alpha T^\beta\phi))(s - (\alpha+\beta)\gamma) &= (M(T^\alpha(T^\beta\phi)))((s-\beta\gamma) - \alpha\gamma)\\ &= \frac{h(s-\beta\gamma-\alpha\gamma)}{h(s-\beta\gamma)}\,(M(T^\beta\phi))(s-\beta\gamma)\\ &= \frac{h(s-\beta\gamma-\alpha\gamma)}{h(s-\beta\gamma)}\,\frac{h(s-\beta\gamma)}{h(s)}\,(M\phi)(s)\\ &= \frac{h(s-(\alpha+\beta)\gamma)}{h(s)}\,(M\phi)(s)\end{aligned}$$

by two applications of (3.3) which are valid under the given conditions. But, by (3.3) again, the right-hand side is $(M(T^{\alpha+\beta}\phi))(s-(\alpha+\beta)\gamma)$ so that

$$(M(T^\alpha T^\beta\phi))(s-(\alpha+\beta)\gamma) = (M(T^{\alpha+\beta}\phi))(s-(\alpha+\beta)\gamma)$$

and, since M is one-to-one on $L^2_{\mu+(\alpha+\beta)\gamma}$, we obtain $T^\alpha T^\beta\phi = T^{\alpha+\beta}\phi$. In view of our earlier remark, this completes the proof.

The conditions under which $T^{\beta}T^{\alpha} = T^{\alpha+\beta}$ are obtained from those in Theorem 3.8 by interchanging α and β. By taking both results together, we can obtain the following corollary.

<u>Corollary 3.9</u>
Let $T \equiv T(h,\Omega,\gamma) \in L$, $\{\alpha, -\alpha\} \subseteq A$, $1 < p < \infty$ and $\{\mu, \mu+\alpha\gamma\} \subseteq \Omega_p$. Then T^{α} is a homeomorphism from L^p_{μ} onto $L^p_{\mu+\alpha\gamma}$ and, as operators on $L^p_{\mu+\alpha\gamma}$,

$$(T^{\alpha})^{-1} = T^{-\alpha}. \tag{3.5}$$

<u>Proof</u>:- By (3.4) with α,β replaced by $-\alpha$, α respectively we have, on using Lemma 3.7,

$$T^{-\alpha}T^{\alpha} = T^{o} = \text{identity operator on } L^p_{\mu}.$$

On the other hand, if we use (3.4) with β replaced by $-\alpha$ and μ replaced by $\mu+\alpha\gamma$, we obtain

$$T^{\alpha}T^{-\alpha} = T^{o} = \text{identity operator on } L^p_{\mu+\alpha\gamma}.$$

The result now follows easily.

<u>Remark 3.10</u>
Corollary 3.9 may seem very satisfactory but it must be emphasised that the restriction $\{\alpha, -\alpha\} \subseteq A$ is a very severe one and is not satisfied in many particular cases. We shall illustrate this in §5 for the case of simple integral operators. The failure of Corollary 3.9 supplies the motivation for the extension of the theory to appropriate classes of generalised functions, as hinted earlier. We shall pursue this extension in a subsequent paper.

We come now to the precise formulation of the second index law (1.2).

<u>Theorem 3.11</u>
Let $T \equiv T(h,\Omega,\gamma) \in L$, $\{\alpha, \alpha\beta\} \subseteq A$, $1 < p < \infty$ and $\mu \in \Omega_p$. Then, as

operators on L^p_μ,

$$(T^\alpha)^\beta = T^{\alpha\beta}. \tag{3.6}$$

Proof:- Since $\alpha\beta \in A$ and $\mu \in \Omega_p$, we know that the right-hand side of (3.6) exists as an element of $B(L^p_\mu, L^p_{\mu+(\alpha\beta)\gamma})$ by Definition 3.5(i). Similarly, since $\alpha \in A$ and $\mu \in \Omega_p$, T^α exists as an operator in $B(L^p_\mu, L^p_{\mu+\alpha\gamma})$ of the class L. Since, by assumption, the triple $(h,\Omega, \beta(\alpha\gamma)) = (h,\Omega, (\alpha\beta)\gamma) \in H$ (as $\alpha\beta \in A$), we can construct $(T^\alpha)^\beta \in L$ according to Definition 3.5 and the resulting operator belongs to $B(L^p_\mu, L^p_{\mu+\alpha\beta\gamma})$ also. It remains to prove equality in (3.6) and, as in the proof of Theorem 3.8, it is sufficient to prove that

$$(T^\alpha)^\beta\phi = T^{\alpha\beta}\phi$$

for $\phi \in C_o^\infty(0,\infty)$, where ϕ is regarded as an element of L^2_μ and $\mu \in \Omega_2$. Then, for Re s = 1/2 - Re μ, repeated use of (3.3) gives

$$\begin{aligned}(M((T^\alpha)^\beta\phi))(s-\alpha\beta\gamma) &= (M((T^\alpha)^\beta\phi))(s-\beta(\alpha\gamma)) \\ &= \frac{h(s-\beta(\alpha\gamma))}{h(s-\alpha\gamma)} \quad (M(T^\alpha\phi))(s-\alpha\gamma) \\ &= \frac{h(s-(\alpha\beta)\gamma)}{h(s-\alpha\gamma)} \quad \frac{h(s-\alpha\gamma)}{h(s)} \ (M\phi)(s) \\ &= \frac{h(s-\alpha\beta\gamma)}{h(s)} \quad (M\phi)(s) \\ &= (M(T^{\alpha\beta}\phi))(s-\alpha\beta\gamma)\end{aligned}$$

each step being valid under the given conditions. The proof is now completed as in Theorem 3.8.

Remark 3.12

At this stage it is interesting to examine the restrictions on the parameters in Theorems 3.8 and 3.11. We consider first Theorem 3.8 and, in particular, the restrictions on μ therein. The right-hand side of (3.4) is meaningful provided that $\mu \in \Omega_p$ (and $\alpha+\beta \in A$). The extra restriction $\mu+\beta\gamma \in \Omega_p$ was needed in order that the operators T^α and T^β appearing on the left-hand side of (3.4) should exist individually on the appropriate spaces. The proof of Theorem 3.8 shows that, if $\mu+\beta\gamma \notin \Omega_p$ or equivalently $s-\beta\gamma \notin \Omega$, the combination $T^\alpha T^\beta$ can still be regarded as existing on L^p_μ in the sense that singularities caused by the factor $h(s-\beta\gamma)$ cancel out. We could then regard $T^{\alpha+\beta}$ as giving a continuation of $T^\alpha T^\beta$ to a wider range of L^p_μ spaces. It could even be regarded as a sort of analytic continuation in a rather weak sense. The proof shows that, if p, μ, ϕ and s are fixed with $1 < p \leq 2$, $\mu \in \Omega_p$, $\phi \in L^p_\mu$ and $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$, then the functions

$$f_1(\alpha,\beta) = (M(T^\alpha T^\beta\phi))(s-\alpha\gamma-\beta\gamma), \quad f_2(\alpha,\beta) = (M(T^{\alpha+\beta}\phi))(s-\alpha\gamma-\beta\gamma)$$

are analytic functions of α and β in the region

$$\{(\alpha,\beta) : \alpha \in A,\ \beta \in A,\ \alpha+\beta \in A\}$$

provided that $\mu+\beta\gamma \in \Omega_p$, both being equal to

$$\frac{h(s-\alpha\gamma-\beta\gamma)}{h(s)} (M\phi)(s).$$

However, f_2 remains analytic without the extra restriction $\mu+\beta\gamma \in \Omega_p$ and provides an analytic continuation of f_1 to the larger region. It would appear that any stronger version of analytic continuation would require more precise information about h or a concrete expression for T^α (and T^β, $T^{\alpha+\beta}$) obtained by other means. We shall return to this in §5.

Just as the restriction $\mu+\beta\gamma \in \Omega_p$ in Theorem 3.8 can be removed in the sense just described, so can the restriction $\mu+\alpha\gamma \in \Omega_p$ in Corollary 3.9. Then we find that the conditions required for (3.5) to hold are precisely those required for (3.6) to hold in the case $\beta = -1$. Thus Corollary 3.9 can also be deduced from Theorem 3.11 and the conditions in the two approaches can be reconciled.

Finally in this section we return to the matter mentioned in Remark 3.3. We indicated that we could use any of the functions h_k defined in (3.1) to construct powers of T. We consider now how these are related.

Definition 3.13

Let $T \equiv T(h,\Omega,\gamma) \in L$ and let $\gamma \neq 0$. For any integer k and for $\alpha \in A$, let $[T^\alpha]_k$ denote the α^{th} power of T calculated by means of the triple (h_k,Ω,γ) where h_k is given by (3.1). This means, in particular, that if $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in L^p_\mu$, then

$$(M([T^\alpha]_k\phi))(s-\alpha\gamma) = \frac{h_k(s-\alpha\gamma)}{h_k(s)} \quad (M\phi)(s) \tag{3.7}$$

by (3.3). $[T^\alpha]_o$ is simply T^α again.

Theorem 3.14

Let $T \equiv T(h,\Omega,\gamma) \in L$ and let $\gamma \neq 0$. Then for any integer k and $\mu \in \Omega_p$

$$[T^\alpha]_k = \exp(-2\pi ki\alpha)\ [T^\alpha]_o = \exp(-2\pi ki\alpha)T^\alpha \tag{3.8}$$

as operators in $B(L^p_\mu, L^p_{\mu+\alpha\gamma})$.

Proof:- Let $\phi \in C_o^\infty(0,\infty)$ and regard ϕ as an element of L^2_μ. Then for $\mu \in \Omega_2$ and Re s = 1/2 - Re μ, (3.7) gives

$$\begin{aligned}(M([T^\alpha]_k\phi))(s-\alpha\gamma) &= \frac{\exp(2\pi ki(s-\alpha\gamma)/\gamma)}{\exp(2\pi kis/\gamma)} \ \frac{h(s-\alpha\gamma)}{h(s)}\ (M\phi)(s)\\ &= \exp(-2\pi ki\alpha)\ \frac{h(s-\alpha\gamma)}{h(s)} \quad (M\phi)(s)\\ &= (M(\exp(-2\pi ki\alpha)T^\alpha)\phi)(s-\alpha\gamma)\end{aligned}$$

by (3.3). The proof of (3.8) is now completed by using density and continuity in the usual way.

Remark 3.15

We see from Theorem 3.14 that the number of distinct operators $[T^{\alpha}]_k$ obtained as k varies over the set of all integers is determined by the behaviour of $\exp(-2\pi ki\alpha)$. Obviously, this is tantamount to elementary considerations of complex roots of unity. In particular, if $\alpha = 1/n$ where n is a positive integer, we obtain n distinct "n^{th} roots of T", namely, the operators $[T^{1/n}]_k$ for $k = 0,1,\ldots,n-1$. Again, we shall consider an example touching on this matter in §5.

4. We now turn our attention to powers of the adjoint of an operator in the class L. This is an interesting investigation in its own right but the results will have an important role to play in the extension of the theory in §3 to generalised functions in a subsequent paper. We require a few preliminaries.

Remark 4.1

Once again, we shall be concerned principally with values of p in the range $1 < p < \infty$, although the ideas can be extended to $p = 1$ or $p = \infty$ in certain cases. As usual, for $1 \leqslant p \leqslant \infty$, we let

$$q = p/(p-1) \quad \text{so that} \quad 1/p + 1/q = 1 \tag{4.1}$$

with the understanding that if $p = 1$ then $q = \infty$ and that if $p = \infty$ then $q = 1$.

Definition 4.2

Let $1 < p < \infty$, let μ_1 and μ_2 be complex numbers and let $T \in B(L^p_{\mu_1}, L^p_{\mu_2})$. Then we define T', the formal adjoint of T, to be the (unique) linear operator in $B(L^q_{-\mu_2}, L^q_{-\mu_1})$ such that

$$\int_0^{\infty} (T\phi)(x)\ \psi(x)dx = \int_0^{\infty} \phi(x)(T'\psi)(x)dx \tag{4.2}$$

for all $\phi \in L^p_{\mu_1}$, $\psi \in L^q_{-\mu_2}$.

Remark 4.3

That T' is well-defined and belongs to $B(L^q_{-\mu_2}, L^q_{-\mu_1})$ is a standard result from duality theory based on Hölder's inequality. Indeed, the result is much more general allowing different values of p in the two spaces and giving information on norms but we shall not need these extra facts here. We shall be concerned with the case when $T \in L$. We might hope that then $T' \in L$ also and that, under appropriate conditions, we might have

$$(T^{\alpha})' = (T')^{\alpha}. \tag{4.3}$$

We shall show that our expectations are realised. One of our main tools is Parseval's formula for the Mellin transform which we shall state in a version appropriate to $L^2(0,\infty) = L^2_o$.

Theorem 4.4

For any functions ϕ_1 and ϕ_2 in L^2_o,

$$\int_{-\infty}^{\infty} (M\phi_1)(1/2+it)(M\phi_2)(1/2-it)dt = 2\pi \int_0^{\infty} \phi_1(x)\phi_2(x)dx. \tag{4.4}$$

Proof:- This can be derived by making simple changes of variable in the corresponding version of Parseval's formula for the Fourier transform on $L^2(-\infty,\infty)$.

We can now state our first main result. In the statement, we shall repeat for convenience some known facts about T in order to show the interplay between T and T'.

Theorem 4.5

If $T \equiv T(h,\Omega,\gamma) \in L$, then $T' \in L$ and is generated by the triple (h', Ω', γ') where

$$\Omega' = \{s' : 1+\gamma-s' \in \Omega\},\ \gamma' = \gamma,\ h'(s') = [h(1-s')]^{-1}\ (s' \in \Omega'). \tag{4.5}$$

In particular, let $\mu \in \Omega_p$ where $1 < p < \infty$. Then

(i) $T \in B(L^p_\mu, L^p_{\mu+\gamma})$ and $T' \in B(L^q_{-\mu-\gamma}, L^q_{-\mu})$

(ii) for $1 < p \leq 2$, $\phi \in L^p_\mu$,

$$(M(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)} (M\phi)(s) \qquad (s \in \Omega)$$

(iii) for $1 < q \leq 2$, $\psi \in L^q_{-\mu-\gamma}$,

$$(M(T'\psi))(s'-\gamma) = \frac{h'(s'-\gamma)}{h'(s')} (M\psi)(s') \qquad (s' \in \Omega'). \tag{4.6}$$

(The use of h' should not cause confusion since we shall not be concerned with the derivative of h.)

Proof:- As regards (i) the statement concerning T is immediate from Definition 3.1(i) and that for T' then follows from Remark 4.3. (ii) is repeated from Definition 3.1(ii). It remains to deal with (iii). We shall first show that the expression $h'(s'-\gamma)/h'(s')$ can serve as a multiplier under the given conditions, calling the resulting multiplier transform $\hat{T}$. Then we shall show that $\hat{T} = T'$ and the theorem will be proved.

To simplify the presentation, we shall introduce two conventions. Firstly, we shall assume that μ and $\mu+\gamma$ are real. There is no loss of generality since for any complex number ν and $1 \leq p \leq \infty$, the spaces L^p_ν and $L^p_{\mathrm{Re}\,\nu}$ are identical. Secondly, the variables s and s' will always be related by the formula

$$s+s' = 1+\gamma. \tag{4.7}$$

Taken together, these two conventions and (2.4), (2.5) entail that if $\mu \in \Omega_p$, then $\mathrm{Re}\, s = 1/p - \mu$ and $\mathrm{Re}\, s' = 1/q + \mu + \gamma$. Also, by (4.5) and (4.7), for $s \in \Omega$,

$$\frac{h'(s'-\gamma)}{h'(s')} = \frac{h(1-s')}{h(1-s'+\gamma)} = \frac{h(s-\gamma)}{h(s)}. \tag{4.8}$$

Consider the operator U defined by

$$(U\phi)(x) = x^{-1}\phi(x^{-1}) \qquad (0 < x < \infty).$$

It is easy to check that, for any μ (real or complex) and $1 < p < \infty$, U is a homeomorphism from L^p_μ onto $L^p_{2/p-1-\mu}$ with inverse $U^{-1} = U$. Also for $1 < p \leq 2$, $\mu \in \Omega_p$, we have

$$(M(U\phi))(1-s) = (M\phi)(s) \qquad (s \in \Omega). \tag{4.9}$$

Define the operator $\hat{T}$ by

$$\hat{T} = x^\gamma UTUx^\gamma$$

where x^γ is defined via (2.3). The condition $\mu \in \Omega_p$ is equivalent to $1/p-\mu \in \Omega$ or $1/q-(2/q-1+\mu) \in \Omega$ so that in this case $T \in B(L^q_{2/q-1+\mu}, L^q_{2/q-1+\mu+\gamma})$ by Definition 3.1(i). Hence $\hat{T} \in B(L^q_{-\mu-\gamma}, L^q_{-\mu})$ since T is the composition of five bounded linear operators. Further, for $1 < q \leq 2$, $\mu \in \Omega_p$, $\psi \in L^q_{-\mu-\gamma}$ and $\operatorname{Re} s' = 1/q+\mu+\gamma$,

$$\begin{aligned}
(M(\hat{T}\psi))(s'-\gamma) &= (M(x^\gamma UTUx^\gamma\psi))(s'-\gamma) \\
&= (M(UTUx^\gamma\psi))(s') \\
&= (M(TUx^\gamma\psi))(1-s') && \text{by (4.9)} \\
&= (M(TUx^\gamma\psi))(s-\gamma) && \text{by (4.7)} \\
&= \frac{h(s-\gamma)}{h(s)}\,(M(Ux^\gamma\psi))(s) && \text{by (2.8)} \\
&= \frac{h'(s'-\gamma)}{h'(s')}\,(M(Ux^\gamma\psi))(s) && \text{by (4.8)} \\
&= \frac{h'(s'-\gamma)}{h'(s')}\,(M(x^\gamma\psi))(1-s) && \text{by (4.9)} \\
&= \frac{h'(s'-\gamma)}{h'(s')}\,(M\psi)(1-s+\gamma) \\
&= \frac{h'(s'-\gamma)}{h'(s')}\,(M\psi)(s') && \text{by (4.7).}
\end{aligned}$$

We have therefore obtained the multiplier operator $\hat{T}$ which we sought.

For $\mu \in \Omega_p$, both T' and $\hat{T}$ are elements of $B(L^q_{-\mu-\gamma}, L^q_{-\mu})$. Thus, to prove that $\hat{T} = T'$, it is sufficient to prove that $\hat{T}\psi = T'\psi$ for $\psi \in C_o^\infty(0,\infty)$. As before we regard ψ as an element of $L^2_{-\mu-\gamma}$ where $\mu \in \Omega_2$. Then both $\hat{T}\psi$ and $T'\psi$ are elements of $L^2_{-\mu}$. Let $\phi \in L^2_\mu$ so that $T\phi \in L^2_{\mu+\gamma}$. By applying Theorem 4.4 to both sides of (4.2) and cancelling the factors of 2π we get

$$\int_{-\infty}^{\infty} (M(T\phi))(1/2-\mu-\gamma+it)(M\psi)(1/2+\mu+\gamma-it)dt$$

$$=\int_{-\infty}^{\infty} (M\phi)(1/2-\mu+it)(M(T'\psi))(1/2+\mu-it)dt.$$

By (2.8) with $s = 1/2-\mu+it$, the left-hand side can be written as

$$\int_{-\infty}^{\infty} \frac{h(1/2-\mu-\gamma+it)}{h(1/2-\mu+it)} (M\phi)(1/2-\mu+it)(M\psi)(1/2+\mu+\gamma-it)dt.$$

It follows that

$$\int_{-\infty}^{\infty} (M\phi)(1/2-\mu+it)\left\{\frac{h(1/2-\mu-\gamma+it)}{h(1/2-\mu+it)} (M\psi)(1/2+\mu+\gamma-it) - \right.$$

$$\left. - (M(T'\psi))(1/2+\mu-it)\right\}dt = 0.$$

Since the range of M on L^2_μ is the whole of $L^2(-\infty,\infty)$, we must have

$$(M(T'\psi))(1/2+\mu-it) = \frac{h(1/2-\mu-\gamma+it)}{h(1/2-\mu+it)} (M\psi)(1/2+\mu+\gamma-it).$$

Then with $s = 1/2-\mu+it$, we have $s' = 1/2+\mu+\gamma-it$ by (4.7) so that, by (4.5),

$$(M(T'\psi))(s'-\gamma) = \frac{h'(s'-\gamma)}{h'(s')} (M\psi)(s').$$

The right-hand side is $(M(\hat{T}\psi))(s'-\gamma)$ from above. Since M is one-to-one on $L^2_{-\mu}$, we deduce that $\hat{T}\psi = T'\psi$ for all $\psi \in C_o^\infty(0,\infty)$. In view of our previous remarks, this completes the proof of the theorem.

Remark 4.6

As mentioned in §3, the operator T' in Theorem 4.5 can be generated by triples other than (h', Ω', γ'). However, we can regard (4.5) as the most natural choice of triple to generate T' and, from now on, whenever T is generated by (h,Ω,γ) , we shall always use the triple (4.5) to generate T'.

Since $T' \in L$, we can construct $(T')^\alpha$ under appropriate conditions by using Definition 3.5 and the powers of T' will satisfy index laws in the usual way. We shall state the relevant results, mainly to draw attention to the conditions under which they are valid.

Theorem 4.7

Let $T \equiv T(h,\Omega,\gamma) \in L$, $1 < p < \infty$, $\mu \in \Omega_p$ and $\alpha \in A$ (as given by (3.2)). Then $(T')^\alpha \in B(L^q_{-\mu-\alpha\gamma}, L^q_{-\mu})$ and is such that, for $1 < q \leqslant 2$ and $\psi \in L^q_{-\mu-\alpha\gamma}$

$$(M((T')^\alpha\psi))(s'-\alpha\gamma) = \frac{h'(s'-\alpha\gamma)}{h'(s')} (M\psi)(s') \qquad (s' \in \Omega'). \quad (4.10)$$

In particular, $(T')^0$ is the identity operator on $L^q_{-\mu}$.

Proof:- This follows in a fairly routine manner by the use of (4.6) and Definition 3.5. We merely mention that, since $\alpha \in A$, $h(s-\alpha\gamma)/h(s)$ is an L^p_μ multiplier and hence $h'(s'-\alpha\gamma)/h'(s')$ is an $L^q_{-\mu-\alpha\gamma}$ multiplier by considerations similar to those in Theorem 4.5.

Theorem 4.8

Let $T \equiv T(h,\Omega,\gamma) \in L$, $1 < p < \infty$.

(i) If $\{\alpha,\beta, \alpha+\beta\} \subseteq A$ and $\{\mu, \mu+\beta\gamma\} \subseteq \Omega_p$, then, as operators on $L^q_{-\mu-\alpha\gamma-\beta\gamma}$,

$$(T')^\beta(T')^\alpha = (T')^{\alpha+\beta}. \qquad (4.11)$$

(ii) If $\{\alpha, -\alpha\} \subseteq A$ and $\{\mu, \mu+\alpha\gamma\} \subseteq \Omega_p$, then $(T')^{\alpha}$ is a homeomorphism from $L^q_{-\mu-\alpha\gamma}$ onto $L^q_{-\mu}$ and

$$[(T')^{\alpha}]^{-1} = (T')^{-\alpha}. \tag{4.12}$$

Proof:- (i) is similar to Theorem 3.8 and the proof is omitted. The reversal of order on the left-hand side of (4.11) by comparison with (3.4) is to be expected. (ii) follows immediately from (i).

Theorem 4.9

Let $T \equiv T(h,\Omega,\gamma) \in L$, $1 < p < \infty$, $\{\alpha, \alpha\beta\} \subseteq A$ and $\mu \in \Omega_p$. Then $((T')^{\alpha})^{\beta} \in B(L^q_{-\mu-\alpha\beta\gamma}, L^q_{-\mu})$ and

$$((T')^{\alpha})^{\beta} = (T')^{\alpha\beta}. \tag{4.13}$$

Proof:- This is similar to that of Theorem 3.11 and is omitted.

Comments analogous to those in Remarks 3.10 and 3.12 apply to the index laws for T' but we shall not spell out the details. Instead we shall deal with the formula (4.3).

Theorem 4.10

Let $T \equiv T(h,\Omega,\gamma) \in L$, $1 < p < \infty$, $\mu \in \Omega_p$ and $\alpha \in A$. Then, as operators on $L^q_{-\mu-\alpha\gamma}$,

$$(T^{\alpha})' = (T')^{\alpha}. \tag{4.3}$$

Proof:- The hypotheses ensure that both operators exist and belong to $B(L^q_{-\mu-\alpha\gamma}, L^q_{-\mu})$ by Definition 3.5, Theorem 4.5 and Theorem 4.7. For $1 < q \lesssim 2$ and $\psi \in L^q_{-\mu-\alpha\gamma}$, the method used in the proof of Theorem 4.5 (with γ replaced by $\alpha\gamma$) shows that

$$(M((T^{\alpha})'\psi))(s'-\alpha\gamma) = \frac{h'(s'-\alpha\gamma)}{h'(s')} (M\psi)(s') \qquad (s' \in \Omega').$$

Comparison with (4.10) reveals that

$$(M((T^{\alpha})'\psi))(s'-\alpha\gamma) = (M((T')^{\alpha}\psi))(s'-\alpha\gamma)$$

from which (4.3) follows in the usual way.

For future use, we note the more explicit form of (4.3) which guarantees that if $\phi \in L^p_\mu$, $\psi \in L^q_{-\mu-\alpha\gamma}$ and the conditions of Theorem 4.10 are satisfied, then

$$\int_0^\infty (T^\alpha\phi)(x)\psi(x)dx = \int_0^\infty \phi(x)((T')^\alpha\psi)(x)dx. \tag{4.14}$$

5. In this final section we give a few examples to illustrate the theory just developed. We start with a rather trivial case.

Example 5.1

Let λ be a fixed complex number and take T to be the operator x^λ as defined by (2.3). Then, as mentioned in Remark 2.2(iii), $x^\lambda \in B(L^p_\mu, L^p_{\mu+\lambda})$ for any complex number μ and $1 \leqslant p \leqslant \infty$. Also, for $1 < p \leqslant 2$ (and $p = 1$ too), $\phi \in L^p_\mu$ and $\mathrm{Re}\, s = 1/p - \mathrm{Re}\, \mu$,

$$(M(x^\lambda\phi))(s-\lambda) = (M\phi)(s).$$

By comparison with (2.8), we see that x^λ can be generated by means of the triple $(h,\Omega,\gamma) \in H$ where Ω is the entire complex plane, $h(s) \equiv 1$ $(s \in \Omega)$ and $\gamma = \lambda$. Also, the set A corresponding to (h,Ω,γ) is the entire complex plane. Hence $(x^\lambda)^\alpha$ can be defined as an operator on L^p_μ for any complex number α and (3.3) shows that, as operators,

$$(x^\lambda)^\alpha = x^{\alpha\lambda}$$

as we might expect. (We have calculated $[T^\alpha]_0$ in the notation of Definition 3.13.) This example illustrates one of the possibilities mentioned in Remark 3.6(ii). Since $T' = T$, it is not surprising that all the results in §§3,4 hold in a trivial way.

The next two examples are much more typical. The operators involved are simple enough to allow us to obtain concrete representations for appropriate powers but at the same time are sufficiently non-trivial to provide worthwhile illustrations of our theory.

Example 5.2

Let $T = I_1$ where, for suitable functions ϕ,

$$(I_1\phi)(x) = \int_0^x \phi(t)dt \qquad (0 < x < \infty). \tag{5.1}$$

(The suffix 1, apart from the menial task of helping to prevent confusion with the identity operator, also serves to indicate that we are dealing with the case m = 1 of the operator $I_m (m > 0)$ defined by

$$(I_m\phi)(x) = \int_0^x \phi(t)d(t^m) = \int_0^x \phi(t)mt^{m-1}dt$$

which can be treated similarly.) We shall obtain a triple (h,Ω,γ) generating T by working from first principles and shall see incidentally how (2.4) and (2.5) arise naturally.

Firstly we mention that $I_1 \in B(L^p_\mu, L^p_{\mu+1})$ provided that $1 \leq p \leq \infty$ and $\text{Re}\,\mu + 1 > 1/p$. This is an easy consequence of many results in the literature on Hardy kernels such as [3, Theorem 2]. It would seem that, for this choice of T, $\gamma = 1$ and $\Omega_p = \{\mu: \text{Re}\,\mu > 1/p-1\}$ for $1 < p < \infty$.

To obtain an appropriate relationship involving the Mellin transform and the restriction on the variable s, we shall make use of the case p = 1, which happens to be available for this T. Accordingly, let $\phi \in L^1_\mu$ with $\text{Re}\,\mu > 0$. From (2.4) we take $\text{Re}\,s = 1 - \text{Re}\,\mu$ in calculating $(M\phi)(s)$, while for the Mellin transform of $I_1\phi \in L^1_{\mu+1}$, the appropriate variable is s-1. We obtain

$$|(M(I_1\phi))(s-1)| = \left|\int_0^\infty x^{s-2}\left(\int_0^x \phi(t)dt\right)dx\right| \leq \int_0^\infty x^{-1-\text{Re}\,\mu}\left(\int_0^x |\phi(t)|dt\right)dx.$$

Since $|\phi| \in L^1_\mu$, $I_1(|\phi|) \in L^1_{\mu+1}$ as above, so that the repeated integral is absolutely convergent. By Fubini's Theorem, we may invert the order of integration to obtain

$$(M(I_1\phi))(s-1) = \int_0^\infty \phi(t) \; (\int_t^\infty x^{s-2}dx)dt.$$

For convergence of the inner integral we require $\mathrm{Re}(s-2) < -1$ or $\mathrm{Re}\ s < 1$. In this case,

$$(M(I_1\phi))(s-1) = -(s-1)^{-1} \int_0^\infty t^{s-1}\phi(t)dt$$

or

$$(M(I_1\phi))(s-1) = (1-s)^{-1}(M\phi)(s) \tag{5.2}$$

for $\phi \in L_\mu^1$ and $\mathrm{Re}\ s < 1$. For $\phi \in L_\mu^p$ with $1 < p \leqslant 2$, $\mathrm{Re}\ \mu+1 > 1/p$, (5.2) is still valid for $\mathrm{Re}\ s < 1$. Indeed, we can handle $\phi \in C_o^\infty(0,\infty)$ as in the case $p = 1$ and then use density and continuity in the usual way. We see that $\Omega = \{s: \mathrm{Re}\ s < 1\}$ while (2.4), (2.5) guarantee that this is in accord with the choice of Ω_p above. It remains to find a function h such that $h(s-1)/h(s) = (1-s)^{-1}$ for $\mathrm{Re}\ s < 1$. From various possibilities we shall opt for $h(s) = [\Gamma(1-s)]^{-1}$. We can then check that h satisfies the conditions in Definition 2.8, (iv) being automatic since we started with T which was known to be bounded. Hence $(h,\Omega,\gamma) \in H$ where

$$\Omega = \{s: \mathrm{Re}\ s < 1\},\ \gamma = 1,\ h(s) = [\Gamma(1-s)]^{-1}\ (s \in \Omega) \tag{5.3}$$

and this triple generates $T = I_1$.

With respect to the triple (h,Ω,γ) in (5.3), the α^{th} power of I_1, which we shall denote by I_1^α, must satisfy

$$(M(I_1^\alpha\phi))(s-\alpha) = \frac{h(s-\alpha)}{h(s)}\ (M\phi)(s) = \frac{\Gamma(1-s)}{\Gamma(1-s+\alpha)}\ (M\phi)(s) \tag{5.4}$$

by (3.3). To find the set A of admissible values of α, we must examine the symbol $\Gamma(1-s)/\Gamma(1-s+\alpha)$. In the notation of [8, Theorem 5.1], our symbol may be written as $m_1(c,1,c,1+\alpha)$ where c is at our disposal. Suppose that p and μ are fixed with $1 < p < \infty$ and

Re $\mu+1 > 1/p$. We can choose c so that $1/p + \text{Re}(c-\mu)$ is not equal to zero or a negative integer. Then [8, Theorem 5.1] guarantees that $\Gamma(1-s)/\Gamma(1-s+\alpha)$ is an L^p_μ multiplier provided that Re $\alpha \geqslant 0$. The case Re $\alpha = 0$ raises some interesting points but we shall not discuss them here, apart from the usual observation that I_1^0 is the identity operator. For the case Re $\alpha > 0$, Theorem 2.10 shows that $I_1^\alpha \in B(L^p_\mu, L^p_{\mu+\alpha})$ provided that $1 < p < \infty$, $\mu \in \Omega_p$ and that I_1^α satisfies (5.4) when $1 < p \leqslant 2$ and $\phi \in L^p_\mu$. It is not hard to check that this operator is the usual Riemann-Liouville fractional integral operator of order α defined by

$$(I_1^\alpha\phi)(x) = [\Gamma(\alpha)]^{-1} \int_0^x (x-t)^{\alpha-1}\phi(t)dt. \tag{5.5}$$

One way is to use Kober's work [3], which gives the extra information that $I_1^\alpha \in B(L^p_\mu, L^p_{\mu+\alpha})$ for $p = 1$ or $p = \infty$, Re $\mu+1 > 1/p$ also.

When Re $\alpha < 0$, the symbol $\Gamma(1-s)/\Gamma(1-s+\alpha)$ remains well-defined for Re $s < 1$ since the reciprocal of the gamma function is entire. However, Rooney's result quoted above no longer applies and this is no surprise since we have passed from the realms of integration into differentiation. For instance, when $\alpha = -1$, the symbol collapses to $-s$ and, for $\phi \in C_0^\infty(0,\infty)$, integration by parts gives

$$(M(D\phi))(s+1) = -s(M\phi)(s).$$

So, as we might expect, the case $\alpha = -1$ corresponds in some sense to $D = d/dx$, which is not defined on the whole of L^p_μ. Thus, within the framework of the L^p_μ spaces, Re $\alpha < 0$ presents problems and the admissible set A is not the whole complex plane. We shall leave further consideration of this to a subsequent paper but remark that it gives a good motivation for a distributional treatment.

Example 5.3

Let $T = K_1$ where, for suitable functions ϕ,

$$(K_1\phi)(x) = \int_x^\infty \phi(t)\,dt \qquad (0 < x < \infty). \tag{5.6}$$

By arguing as in Example 5.2, we can show that K_1 is generated by the triple $(h', \Omega', \gamma') \in H$ where

$$\Omega' = \{s' : \text{Re } s' > 1\}, \quad \gamma' = 1, \quad h'(s') = \Gamma(s') \quad (s' \in \Omega'). \tag{5.7}$$

(The reason for the use of dashes will become apparent in Example 5.5.) The analogue of (5.4) is

$$(M(K_1^\alpha\phi))(s'-\alpha) = \frac{h'(s'-\alpha)}{h'(s')} = \frac{\Gamma(s'-\alpha)}{\Gamma(s')}(M\phi)(s') \tag{5.8}$$

and [8, Theorem 5.1] guarantees that $\Gamma(s'-\alpha)/\Gamma(s')$ is an L_μ^p multiplier for Re $\alpha \geqslant 0$. For Re $\alpha > 0$, we have the concrete representation

$$(K_1^\alpha\phi)(x) = [\Gamma(\alpha)]^{-1} \int_x^\infty (t-x)^{\alpha-1}\,\phi(t)\,dt \tag{5.9}$$

revealing K_1^α as the usual Weyl fractional integral operator of order α. From [3], it follows that, in this case, $K_1^\alpha \in B(L_\mu^p, L_{\mu+\alpha}^p)$ for $1 \leqslant p \leqslant \infty$, Re$(\mu+\alpha) < 1/p$. When Re $\alpha < 0$, we experience the same difficulties as with I_1.

Example 5.4

We now examine the conditions under which the index laws hold for I_1^α and K_1^α, restricting attention to Re $\alpha > 0$ for the reasons outlined above. This will illustrate the ideas in Remarks 3.10 and 3.12. Theorem 3.8 guarantees that for $1 < p < \infty$, Re $\alpha > 0$, Re $\beta > 0$ and $\mu \in \Omega_p$

$$I_1^\alpha I_1^\beta = I_1^{\alpha+\beta} (= I_1^\beta I_1^\alpha) \tag{5.10}$$

as operators on L_μ^p. Kober's results show that the result remains true for $p = 1$ and $p = \infty$. Here the condition $\mathrm{Re}(\alpha+\beta) > 0$ is a consequence of $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$ and the extra condition $\mu+\beta\gamma \in \Omega_p$ in the theorem is also redundant since, with $\gamma = 1$,

$$\mu \in \Omega_p,\ \mathrm{Re}\,\beta > 0 \Rightarrow \mathrm{Re}\,\mu > 1/p - 1,\ \mathrm{Re}\,\beta > 0 \Rightarrow \mathrm{Re}(\mu+\beta) > 1/p - 1$$

$$\Rightarrow \mu+\beta \in \Omega_p.$$

Thus both sides of (5.10) exist under identical conditions and no analytic continuation is needed in the sense of Remark 3.12. By way of a contrast, consider

$$K_1^{\alpha}K_1^{\beta} = K_1^{\alpha+\beta} \tag{5.11}$$

as an operator equation in L_μ^p for $1 < p < \infty$, $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$. Drawing on our results in Example 5.3, we see that the right-hand side of (5.11) is an element of $B(L_\mu^p, L_{\mu+\alpha+\beta}^p)$ provided that $\mathrm{Re}(\mu+\alpha+\beta) < 1/p$ while the left-hand side is an element of the same space provided that $\mathrm{Re}(\mu+\beta) < 1/p$ and $\mathrm{Re}((\mu+\beta) + \alpha) < 1/p$. This time the extra condition $\mathrm{Re}(\mu+\beta) < 1/p$ is not a consequence of the others. However, for almost all $x \in (0,\infty)$, $(K_1^{\alpha}K_1^{\beta}\phi)(x)$ and $(K_1^{\alpha+\beta}\phi)(x)$ are analytic in α and β when both inequalities are satisfied and it is in this sense that we can regard the right-hand side as providing an analytic continuation of the left-hand side when the restriction $\mathrm{Re}(\mu+\beta) < 1/p$ is removed. This version of analytic continuation makes use of the concrete representation (5.9) which is available to us.

We mention briefly that Corollary 3.9 fails miserably when $T = I_1$ or $T = K_1$ because of the lack of differentiability of functions in L_μ^p. As regards Theorem 3.11 we simply remark that the restriction $\mathrm{Re}(\alpha\beta) > 0$ is not a consequence of $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$.

Example 5.5

Next we shall use I_1 and K_1 to illustrate the ideas in §4. The results obtained are familiar and can be derived by other means. However, it is useful to show how they fit into our theory. We generate I_1 via the triple (h,Ω,γ) in (5.3) and proceed to find $(I_1)'$. As in the proof of Theorem 4.5, we let s and s' be the Mellin transform variables appropriate to L^p_μ and $L^q_{-\mu-1}$ respectively so that $\mathrm{Re}\, s = 1/p - \mathrm{Re}\,\mu$, $\mathrm{Re}\, s' = 1/q + \mathrm{Re}\,\mu+1$ and $\mathrm{Re}(s+s') = 1/p+1/q+1 = 2$ by analogy with (4.7), since $\gamma = 1$ here. (4.5) then shows that $\gamma' = 1$ and

$$h'(s') = [h(1-s')]^{-1} = [\{\Gamma(1-(1-s'))\}^{-1}]^{-1} = \Gamma(s').$$

Also, if $s \in \Omega$, $\mathrm{Re}\, s < 1$ so that $\mathrm{Re}\, s' = 2-\mathrm{Re}\, s > 1$, giving $\Omega' = \{s': \mathrm{Re}\, s' > 1\}$. We have the precise triple (h', Ω', γ') in (5.7) which generates K_1. Thus $(I_1)' = K_1$. We can then invoke Theorem 4.10 to deduce that for $\mathrm{Re}\,\alpha > 0$, $1 < p < \infty$ and $\mu \in \Omega_p$, $(I_1^\alpha)' = K_1^\alpha$ as mappings from $L^q_{-\mu-\alpha}$ into $L^q_{-\mu}$ and (5.8) is satisfied for $1 < q \leq 2$, $\phi \in L^q_{-\mu-\alpha}$, $s' \in \Omega'$. In Example 5.3, we mentioned that $K_1^\alpha \in B(L^p_\mu, L^p_{\mu+\alpha})$ for $1 \leq p \leq \infty$ and $\mathrm{Re}(\mu+\alpha) < 1/p$. The latter condition, when applied to K_1^α as a mapping from $L^q_{-\mu-\alpha}$ into $L^q_{-\mu}$, becomes $\mathrm{Re}(-\mu) < 1/q$ or $\mathrm{Re}\,\mu+1/q > 0$. This in turn becomes $\mathrm{Re}\, s' > 1$ or $s' \in \Omega'$, so that the various conditions are all in accord. Equation (4.11) has the form

$$\int_0^\infty (I_1^\alpha\phi)(x)\psi(x)dx = \int_0^\infty \phi(x)(K_1^\alpha\psi)(x)dx \tag{5.12}$$

for $\mathrm{Re}\,\alpha > 0$, $1 < p < \infty$, $\mu \in \Omega_p$, $\phi \in L^p_\mu$ and $\psi \in L^q_{-\mu-\alpha}$. (5.12) is sometimes referred to as "fractional integration by parts" and appears in the paper by Love and Young [6]. It can, of course, be obtained from (5.5) and (5.9) on writing the left-hand side as a repeated integral and inverting the order of integration. This approach, which is justified by a combination of Hölder's inequality and Fubini's theorem, shows that the cases $p = 1$ and $p = \infty$ can be brought into the fold.

Example 5.6

In our examples so far, we have started with T, found a generating triple (h,Ω,γ) and used the triple to find the set A of admissible values of α for which T^α could then be defined. However, we could start with the triple and work backwards to find T. To illustrate this point, we stay with the quantities that turned up in our earlier examples and make a few modifications.

Consider first the symbol $\Gamma(1-s)/\Gamma(1-s+\alpha)$ appearing in (5.4). In Example 5.2, the restriction Re $s < 1$ entered because of the definition of I_1 in (5.1). However, the symbol is meaningful for any complex number $s \neq 1,2,\ldots$ (and any complex number α). Further, Rooney's result [8, Theorem 5.1] used previously shows that we still obtain an L^p_μ multiplier in this more general situation for Re $\alpha > 0$. If

$$\Omega = \{s: \text{Re } s \neq k \text{ for } k = 1,2,\ldots\},\ h(s) = [\Gamma(1-s)]^{-1}\ (s \in \Omega),\quad \gamma = 1 \tag{5.13}$$

the triple $(h,\Omega,\gamma) \in H$ and generates an operator $T \in L$. This operator T will be an element of $B(L^p_\mu, L^p_{\mu+1})$ for $1 < p < \infty$ and $\mu \in \Omega_p$, where

$$\Omega_p = \{\mu: \text{Re } \mu \neq 1/p - k \text{ for } k = 1,2,\ldots\}. \tag{5.14}$$

For values of μ such that Re $\mu > 1/p - 1$, T is simply given by (5.1), as before. For other values of $\mu \in \Omega_p$, the form of T is more complicated. Nevertheless we can still calculate T^α for Re $\alpha \geqslant 0$ and obtain an element of L. Concentrating on Re $\alpha > 0$, we can say that for Re $\mu > 1/p - 1$, T^α is given by (5.5) while for $k < 1/p - \text{Re } \mu < k+1$ $(k = 1,2,\ldots)$, we obtain

$$T^\alpha = (-1)^k\, I_1^\alpha\, D^k\, K_1^k \tag{5.15}$$

where I_1^α is given by (5.5), K_1^k by (5.9) with α replaced by k and D^k denotes differentiation k times. We shall not give a proof of

(5.15) here but similar situations will arise in our subsequent distributional approach where differentiation can be handled in a more routine fashion. Since T^α is an extended form of I_1^α we shall use the stop-gap notation I_E^α for this operator.

A similar programme can be carried out by starting with the symbol $\Gamma(s'-\alpha)/\Gamma(s')$ in (5.8). We can use the triple (h', Ω', γ') where

$$\Omega = \{s': \mathrm{Re}(s'-\alpha) \neq -k \text{ for } k=0,1,2,\ldots\},\ h'(s') = \Gamma(s')(s'\in\Omega'),$$
$$\gamma' = 1 \tag{5.16}$$

to generate an extended version K_E^α of K_1^α ($\mathrm{Re}\,\alpha > 0$) as an operator in $B(L^q_{-\mu-\alpha}, L^q_{-\mu})$. Since in that case $\mathrm{Re}(s'-\alpha) = 1/q + \mathrm{Re}\,\mu$, we see that for $\mathrm{Re}\,\mu > -1/q$ or, equivalently, $\mathrm{Re}\,\mu > 1/p - 1$, $K_E^\alpha = K_1^\alpha$ as given by (5.9) while for $k < 1/p - \mathrm{Re}\,\mu < k+1$,

$$K_E^\alpha = (-1)^k\, K_1^\alpha\, D^k\, I_1^k \tag{5.17}$$

with an appropriate interpretation.

Finally, we mention that the arguments in Example 5.5 carry over to the extended operators so that for $\mathrm{Re}\,\alpha > 0$, $1 < p < \infty$, $\mu \in \Omega_p$,

$$(I_E^\alpha)' = K_E^\alpha \tag{5.18}$$

as operators in $B(L^q_{-\mu-\alpha}, L^q_{-\mu})$.

<u>Example 5.7</u>

We pursue some of the ideas in Example 5.6 a little further by considering the triple

$$\Omega = \{s: \mathrm{Re}\, s \neq k \text{ for all integers } k\},\ h(s) = \Gamma(s)\ (s\in\Omega),\ \gamma = 1.$$

By comparison with (5.16) we have replaced s' by s in order to discuss an operator in $B(L^p_\mu, L^p_{\mu+1})$ rather than $B(L^q_{-\mu-1}, L^q_{-\mu})$, altered Ω but retained the same γ and the same defining expression for h. By arguing as in Example 5.6, we see that $(h,\Omega,\gamma) \in H$ generates an operator which coincides with K_E, the extended version of K_1, on the spaces L^p_μ to be considered. Thus, for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in L^p_\mu$,

$$(M(K_E\phi))(s-1) = \frac{\Gamma(s-1)}{\Gamma(s)} (M\phi)(s) = (s-1)^{-1}(M\phi)(s). \tag{5.19}$$

However, we can rewrite (5.19) in the form

$$(M(K_E\phi))(s-1) = -(1-s)^{-1}(M\phi)(s) = \frac{\hat{h}(s-1)}{\hat{h}(s)} (M\phi)(s) \tag{5.20}$$

where

$$\hat{h}(s) = [\Gamma(1-s)]^{-1} \exp(i\pi s) \quad (s \in \Omega). \tag{5.21}$$

Thus the triple $(\hat{h},\Omega,\gamma)$ also generates K_E on the spaces under discussion. We have here a variant of the ideas in Lemma 3.2. As in Definition 3.13 we can calculate α^{th} powers of K_E with respect to both triples (h,Ω,γ) and $(\hat{h},\Omega,\gamma)$, under appropriate conditions. The first triple produces K_E^α as described in Example 5.6, the operator being a member of $B(L^p_\mu, L^p_{\mu+\alpha})$ provided that $1/p-\text{Re}(\mu+\alpha) \neq k$ (k any integer) and $1 < p < \infty$, $\text{Re}\,\alpha > 0$ as usual. The second triple produces an operator $T \in L$ where

$$(M(T\phi))(s-\alpha) = \frac{\hat{h}(s-\alpha)}{\hat{h}(s)} (M\phi)(s) = \exp(-i\pi\alpha) \frac{\Gamma(1-s)}{\Gamma(1-s+\alpha)} (M\phi)(s) \tag{5.22}$$

for $1 < p \leq 2$, $\mu \in \Omega_p$, $\phi \in L^p_\mu$ and $1/p - \text{Re}(\mu+\alpha) \neq k$ (k any integer). On comparing (5.22) with (5.4) and our results in Example 5.6, we can identify T as the operator $\exp(-i\pi\alpha)I_E^\alpha$ under the appropriate conditions. Thus we have two different α^{th} powers of K_E, namely K_E^α and $\exp(-i\pi\alpha)I_E^\alpha$ which bear a more curious relationship to each other than that exemplified by (3.8). However, we shall not pursue this relationship any further here.

Remark 5.8

(i) The discussions in Examples 5.6 and 5.7 bear out the point made in Remark 2.7(i).

(ii) We have chosen simple examples to illustrate the theory but it should be emphasised that many more elaborate operators can also be studied. For instance, in [7], various Volterra integral operators involving Meijer's G-function or, in particular cases, the ${}_2F_1$ hypergeometric function arose as negative fractional powers of certain ordinary differential operators. Although [7] was geared to certain spaces of testing functions and distributions, the integral operators can be studied in L^p_μ spaces even although the differential operators could not. In each case the integral operators can be written as the composition of operators each of which is of the form x^λ (as in (2.3)), I_1^α (as in (5.5)) or K_1^α (as in (5.9)) or simple modifications of them. Since we can handle each of the three types individually it might be thought that the composition could be handled also in a straightforward fashion. Unfortunately, this may not be so since, in general, if T_1 and T_2 belong to the class L and Re $\alpha > 0$,

$$(T_1T_2)^\alpha \neq T_1^\alpha T_2^\alpha \ .$$

Indeed T_1T_2 need not belong to L so that our theory cannot be applied until some reformulation is carried out. We shall not present any details here but merely sign off with one particular case in which progress can be made.

Example 5.9

Let $1 < p < \infty$ and let μ, a and b be complex numbers such that Re$(\mu+a) + 1 > 1/p$ and Re$(\mu+b) + 1 > 1/p$. Consider T defined on L^p_μ by

$$(T\phi)(x) = x^{-a}\, I_1\, x^{a-b-1}\, I_1\, x^b\phi(x).$$

By Examples 5.1 and 5.2, $T \in B(L^p_\mu, L^p_{\mu+1})$. For Re $\alpha > 0$ we find that under the same conditions

$$(T^\alpha\phi)(x) = x^{-a} I_1^\alpha\, x^{a-b-\alpha} I_1^\alpha x^b \phi(x) \tag{5.23}$$

where I_1^α is as in (5.5). The expression (5.23) may alternatively be written in the form

$$(T^\alpha\phi)(x) = [\Gamma(2\alpha)]^{-1} x^{-a} \int_0^x (x-t)^{2\alpha-1} {}_2F_1(b-a+\alpha,\alpha;2\alpha;1-x/t)\, t^{a-\alpha}\phi(t)\,dt. \tag{5.24}$$

These facts can be found in [7, formulae (9.3) and (9.5)]. By using (5.4) we discover that the appropriate triple for T is (h,Ω,γ) where $h(s) = [\Gamma(1-s+a)\Gamma(1-s+b)]^{-1}$, $\Omega = \{s: \text{Re } s < \min(\text{Re } a+1, \text{Re } b+1)\}$, $\gamma = 1$. Theorem 3.11 then shows that if Re $\beta > 0$ and Re $\alpha\beta > 0$, then a β^{th} power of the integral operator in (5.24) can be obtained by replacing α by $\alpha\beta$ throughout.

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Paper 11

Applicable Analysis, 1986, Vol. 21, pp. 129–149
0003-6811/86/2102-0129 $18.50/0

Fraction Powers of a Class of Mellin Multiplier Transforms Part II

Communicated by G. F. Roach

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Abstract In a previous paper we developed a theory of fractional powers for a class of Mellin multiplier transforms within the framework of weighted $L^p(0,\infty)$ spaces. In the present paper, we extend these ideas to a wider class of operators acting on certain spaces of testing-functions. Fractional powers can be defined and index laws established under conditions of great generality. Results involving adjoints are stated in preparation for a distributional treatment in a subsequent paper.
(*Received for Publication May 2, 1985*)

§1 In a previous paper [2] we developed a theory of fractional (that is, not necessarily integer) powers of a class $\mathcal{L}$ of operators mapping one space of weighted L^p functions on $(0,\infty)$ into another such space. These operators were characterised by a certain type of functional relation involving the Mellin transform $\mathcal{M}$. Formally,

$$(\mathcal{M}\phi)(s) = \int_0^\infty x^{s-1}\,\phi(x)\,dx. \qquad (1.1)$$

The Mellin transforms of a function ϕ and $T\phi$, its image under an operator T of the class $\mathcal{L}$, were such that

$$(\mathcal{M}(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)}\,(\mathcal{M}\phi)(s) \qquad (1.2)$$

for some (non-zero) complex number γ and some suitable function h. The form of (1.2) suggests that an α^{th} power of T, which we denote by T^α, should be such that

$$(\mathcal{M}(T^{\alpha}\phi))(s-\alpha\gamma) = \frac{h(s-\alpha\gamma)}{h(s)}(\mathcal{M}\phi)(s) . \tag{1.3}$$

[2] was devoted to the development of a rigorous theory based on these simple ideas. Extensive use was made of multipliers for the Mellin transform. In particular, for (1.3) to be meaningful we had to restrict α to lie in a certain admissible set A, namely, the set of all complex numbers α for which the function $h(s-\alpha\gamma)/h(s)$ was a multiplier.

The theory in [2] is fine as far as it goes but in many cases it does not go far enough. Suppose that we wish to solve the equation

$$Tf = g \tag{1.4}$$

where g is a given classical function and f is to be found. A classical f will exist provided that g is in the range of T and, if T were one-to-one, we would write $f = T^{-1}g$ as usual. However, the range of T will not in general be the entire weighted L^p space which is serving as the codomain. For instance, if T is a Volterra integral operator, the range will consist of functions which are differentiable almost everywhere and the inverse operator will be a differential operator which will be unbounded. Thus, within the context of [2], $\alpha = -1$ will not belong to A. More generally, if $\alpha \in A$, the operator T^{α} will be bounded but will not in general be a homeomorphism so that questions of existence and uniqueness arise in connection with solutions f of the equation

$$T^{\alpha}f = g \tag{1.5}$$

where g is given. In trying to make sure that (1.5) has a solution, we can either restrict the class of possible functions g on the right-hand side or enlarge the class of objects f which we are prepared to accept as solutions. Since the first alternative is somewhat defeatist, the second is to be preferred. The aim is then to imbed the original domain and codomain of T in larger spaces X and Y, respectively, with the functions f and g in (1.4), say, becoming $\tilde{f} \in X$ and $\tilde{g} \in Y$, to extend T to a mapping $\tilde{T}$ from X into Y, to try to choose X and Y in such a way that $\tilde{T}$ becomes a homeomorphism and to ensure that

$$Tf = g \Rightarrow \tilde{T}\tilde{f} = \tilde{g} . \tag{1.6}$$

We are then assured of finding a generalised solution $F \in X$ of the equation

$$\tilde{T}F = G$$

for every $G \in Y$. This solution F could be regarded as a generalised solution of (1.4). A classical solution f of (1.4) may not exist, in which case F is the best we can do, but if (1.4) does have a classical solution f, we can recover it from F in view of (1.6). Similar comments apply to (1.5).

Our aim in this paper and its sequel [3] is to extend the theory developed in [2] along the lines suggested in the previous paragraph. In our case, the spaces X and Y will be spaces of generalised functions so that the generalised solutions will be distributions. Those distributional solutions which are regular will provide classical solutions, when the latter exist. The method relies heavily on adjoint operators. Thus, the extended operator $\tilde{T}$ above turns out to be the adjoint of the operator T' (the L^p adjoint of T) acting on appropriate spaces of testing-functions. Since these testing-functions are smooth, analysis becomes easy yet rigorous and the mapping properties of T' improve greatly by comparison with the weighted L^p spaces. It turns out that if T is in the class $\mathcal{J}$ then so is T' and that if $\alpha \in A$, then $(T')^{\alpha}$ can be defined as a continuous linear mapping, not merely between two weighted L^p spaces but also between two testing-function spaces. Furthermore, if we work with testing-functions, $(T')^\alpha$ may exist for values of α other than those in A; that is, we obtain an admissible set A_F at least as large as A and often very much larger. We can then define an extended version of T^α as the adjoint of $(T')^\alpha$ for all $\alpha \in A_F$ and this leads to a rapid solution of (1.5) in terms of generalised functions. It cannot be over-emphasised that, by working distributionally, we obtain enormous advantages in many cases, as we hope to make clear by means of examples in [3].

In §2, we introduce the weighted L^p spaces L^p_μ and the testing-function spaces $F_{p,\mu}$ along with some standard notation and simple properties. The chief results in §3 show that every L^p_μ (Mellin) multiplier is also an $F_{p,\mu}$ multiplier but that in the $F_{p,\mu}$ structure additional multipliers, corresponding to differential operators, become available. In §4, we introduce the class $\mathfrak{F}$ of operators which are generated by $F_{p,\mu}$ multipliers in the same way as the class $\mathfrak{L}$ is generated by the L^p_μ multipliers, via (1.2). The admissible set A_F of an operator $T \in \mathfrak{F}$ is introduced (as mentioned above) and T^α is defined for $\alpha \in A_F$. A comment on choice of generating triples leads to the observation that we can associate n distinct n^{th} roots with $T \in \mathfrak{F}$ under appropriate conditions. The index laws

$$T^\alpha T^\beta = T^{\alpha+\beta} \quad ; \quad (T^\alpha)^\beta = T^{\alpha\beta}$$

are established and conditions for the invertibility of T^α derived. Next, we show that, if $T \in \mathfrak{F}$, then $T' \in \mathfrak{F}$ where T' is the formal (L^p-structure) adjoint of T. This long section concludes with properties of powers of T', including the observation that, for $\alpha \in A_F$,

$$(T^\alpha)' = (T')^\alpha \ .$$

The preparation for [3] is then complete.

§2 It is convenient to begin by recalling some material from [2].

Definition 2.1 For $1 \leq p \leq \infty$ and any complex number μ,

$$L_\mu^p = \{f : x^{-\mu} f(x) \in L^p(0,\infty)\}. \tag{2.1}$$

Thus, L_μ^p is the space of (equivalence classes of) complex-valued measurable functions f defined (almost everywhere) on $(0,\infty)$ such that the quantity

$$\|f\|_{p,\mu} = \|x^{-\mu} f(x)\|_{L^p(0,\infty)} \tag{2.2}$$

is finite, where $\| \ \|_{L^p(0,\infty)}$ denotes the usual $L^p(0,\infty)$ norm. For each p and μ, $(L_\mu^p, \| \ \|_{p,\mu})$ is a Banach space.

We summarise, without proof, the behaviour of the Mellin transform on the spaces L_μ^p.

Lemma 2.2 For $1 \leq p \leq 2$ and any complex number μ, the Mellin transform $(\mathfrak{m}\phi)(s)$ of a function $\phi \in L_\mu^p$ exists provided that

$$\operatorname{Re} s = 1/p - \operatorname{Re} \mu. \tag{2.3}$$

For $p = 1$, the integral in (1.1) defining $(\mathfrak{m}\phi)(s)$ converges absolutely. For $1 < p \leq 2$, and $s = \sigma + it$ ($\sigma = 1/p - \operatorname{Re} \mu$), the integral in (1.1), regarded as a function of t on $(-\infty,\infty)$, converges in the $L^q(-\infty,\infty)$ norm, where $1/p + 1/q = 1$. Further, $\mathfrak{m}$ is a continuous linear mapping from L_μ^p into $L^q(-\infty,\infty)$.

Remark 2.3 From now on, whenever we consider $(\mathfrak{m}\phi)(s)$ for $\phi \in L_\mu^p$ $(1 \leq p \leq 2)$, s and μ will always be related by (2.3).

Notation 2.4

(i) Ω will denote a domain in the complex plane which is the union of a finite or countably infinite collection of disjoint, open strips parallel to the imaginary axis, that is, strips having one or other of the forms

$\{s : a < \mathrm{Re}\, s < b\}$, $\{s : \mathrm{Re}\, s < c\}$ or $\{s : \mathrm{Re}\, s > d\}$

where a, b, c and d are real numbers. (There can be at most one strip of each of the second or third forms.)

(ii) For any domain Ω as in (i) and for any p in the range $1 \leq p \leq \infty$, we let

$$\Omega_p = \{\mu : 1/p - \mu \in \Omega\} \quad . \tag{2.4}$$

Remark 2.5 (2.3) and (2.4) together ensure that, for fixed p,

$$s \in \Omega \quad \text{if and only if } \mu \in \Omega_p . \tag{2.5}$$

We adopt the following definition of an L^p_μ multiplier.

Definition 2.6 Let g be a complex-valued function analytic on a domain Ω (as in Notation 2.4(i)). We shall say that g is an L^p_μ multiplier if there exists a (unique) linear transformation R (depending on g) such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, R is a bounded linear transformation from L^p_μ into L^p_μ, that is, $R \in B(L^p_\mu)$

(ii) for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in L^p_\mu$,

$$(\mathfrak{M}(R\phi))(s) = g(s)\ (\mathfrak{M}\phi)(s) . \tag{2.6}$$

Next, we summarise the basic properties of the $F_{p,\mu}$ spaces required below. Full details can be found in [1, Chapter 2].

Definition 2.7 Let μ be any complex number.

(i) For $1 \leq p < \infty$,

$$F_{p,\mu} = \{\phi \in C^\infty(0,\infty) : x^k d^k (x^{-\mu}\phi(x))/dx^k \in L^p(0,\infty) \text{ for } k=0,1,2,\ldots\} \tag{2.7}$$

while

$$F_{\infty,\mu}=\{\phi \in C^{\infty}(0,\infty): x^k d^k(x^{-\mu}\phi(x))/dx^k \to 0 \text{ as } x\to 0+ \text{ and as } x\to\infty \tag{2.8}$$

for $k = 0,1,2,\ldots\}$.

(ii) For $1 \leq p \leq \infty$ and $k = 0,1,2,\ldots$, we define $\gamma_k^{p,\mu}$ on $F_{p,\mu}$ by

$$\gamma_k^{p,\mu}(\phi) = \|x^k d^k(x^{-\mu}\phi(x))/dx^k\|_{L^p(0,\infty)} \quad . \tag{2.9}$$

For fixed p and μ, $\gamma_k^{p,\mu}$ is a seminorm on $F_{p,\mu}$ for $k = 1,2,\ldots$, while $\gamma_0^{p,\mu}$ is a norm. Accordingly, $F_{p,\mu}$ is equipped with the topology corresponding to the countable multinorm $\{\gamma_k^{p,\mu}\}_{k=0}^{\infty}$ in the sense of [5, pp. 7-13].

<u>Theorem 2.8</u> For $1 \leq p \leq \infty$ and any complex number μ,

(i) $F_{p,\mu}$ is complete (and hence a Fréchet space)

(ii) $C_0^{\infty}(0,\infty)$ is dense in $F_{p,\mu}$.

Since our aim is to exploit the differentiability of the functions in $F_{p,\mu}$, we shall make extensive use of the operators D and δ defined by

$$(D\phi)(x) = d\phi/dx \quad , \quad (\delta\phi)(x) = x\, d\phi/dx. \tag{2.10}$$

<u>Theorem 2.9</u> Let μ be any complex number.

(i) For $1 \leq p \leq \infty$, D is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-1}$. Further, D is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-1}$ if and only if Re $\mu \neq 1/p$ and, for $\psi \in F_{p,\mu-1}$,

$$(D^{-1}\psi)(x) = \begin{cases} \displaystyle\int_0^x \psi(t)\,dt & \text{Re } \mu > 1/p \\ \displaystyle -\int_x^{\infty} \psi(t)\,dt & \text{Re } \mu < 1/p \ . \end{cases} \tag{2.11}$$

(ii) For $1 \leq p \leq 2$, $\phi \in F_{p,\mu}$ and Re $s = 1/p -$ Re μ,

$$(\mathfrak{M}(D\phi))(s+1) = -s(\mathfrak{M}\phi)(s) \quad . \tag{2.12}$$

<u>Corollary 2.10</u> Let μ be any complex number.

(i) For $1 \leq p \leq \infty$, δ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$. Further, δ is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ if and only if $\operatorname{Re}\mu \neq 1/p$ and, for $\psi \in F_{p,\mu}$,

$$(\delta^{-1}\psi)(x) = \begin{cases} \int_0^x t^{-1}\psi(t)\,dt & \operatorname{Re}\mu > 1/p \\ -\int_x^\infty t^{-1}\psi(t)\,dt & \operatorname{Re}\mu < 1/p \end{cases} . \tag{2.13}$$

(ii) For $1 \leq p \leq 2$, $\phi \in F_{p,\mu}$ and $\operatorname{Re} s = 1/p - \operatorname{Re}\mu$,

$$(\mathfrak{M}(\delta\phi))(s) = -s\,(\mathfrak{M}\phi)(s). \tag{2.14}$$

Remark 2.11 (2.12) and (2.14) are not proved explicitly in [1]. However for $\phi \in C_0^\infty(0,\infty)$, they can be established via integration by parts. The proof for any $\phi \in F_{p,\mu}$ then uses the density of $C_0^\infty(0,\infty)$ in $F_{p,\mu}$ (Theorem 2.8(ii)) and the continuity of $\mathfrak{M}$ as a mapping from L_μ^p into $L^q(-\infty,\infty)$ (Lemma 2.2). Since a similar but more complicated proof is given below, we omit further details.

§3 To start the extension of the theory from L_μ^p to $F_{p,\mu}$, we make the following definition, analogous to Definition 2.6.

Definition 3.1 Let g be a complex-valued function analytic on a domain Ω (as in Notation 2.4(i)). We shall say that g is an $F_{p,\mu}$ multiplier if there exists a (unique) linear mapping R (depending on g) such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, R is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$

(ii) for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in F_{p,\mu}$,

$$(\mathfrak{M}(R\phi))(s) = g(s)\,(\mathfrak{M}\phi)(s). \tag{3.1}$$

Remark 3.2 Our attitude to the cases $p = 1$ and $p = \infty$ is the same as that in [2, Remark 2.7(ii)]. It may happen that R is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ for either or both of the cases $p = 1$ and $p = \infty$, whereupon (3.1) will be valid for $p = 1$, if appropriate. In the main, however, we shall be concerned with $1 < p < \infty$ where classes of multipliers have been obtained relative to L^p_μ, for example in [4].

Our immediate aim is to demonstrate that the class of $F_{p,\mu}$ multipliers is strictly larger than the class of L^p_μ multipliers.

Theorem 3.3 Every L^p_μ multiplier is an $F_{p,\mu}$ multiplier.

Proof:- In the previous notation, let g be an L^p_μ multiplier so that g is the symbol of an operator $Q \in B(L^p_\mu)$ for $1 < p < \infty$ and $\mu \in \Omega_p$. To obtain the required operator R involves a rather tedious argument which establishes the appropriate differentiability.

Let $\phi \in C_o^\infty(0,\infty)$ and regard ϕ as an element of L^2_μ with μ chosen so that $\mu \in \Omega_2$ and, in addition, $1/2 - \text{Re}\,\mu \neq 0$. Since $\delta\phi \in L^2_\mu$, (2.14) and (3.1) give

$$(\mathfrak{M}(Q(\delta\phi)))(s) = g(s)\ (\mathfrak{M}(\delta\phi))(s) = -sg(s)(\mathfrak{M}\phi)(s) = -s(\mathfrak{M}(Q\phi))(s).$$

By (2.3), $\text{Re}\, s = 1/2 - \text{Re}\,\mu \neq 0$, so that we may write

$$(\mathfrak{M}(Q\phi))(s) = -s^{-1}(\mathfrak{M}(Q(\delta\phi)))(s). \tag{3.2}$$

Again, $\text{Re}\,\mu \neq 1/2$ ensures that $-s^{-1}$ is the symbol of the operator on L^2_μ defined formally by the appropriate part of (2.13); this can be proved directly or by observing that, for $\text{Re}\,\mu > 1/2$,

$\delta(\int_0^x t^{-1}\psi(t)dt) = \psi(x)$ for any $\psi \in L^2_\mu$ (not merely in $F_{p,\mu}$) with a similar result for $\text{Re}\,\mu < 1/2$. Hence (3.2) can be written in the form

$$(\mathfrak{M}(Q\phi))(s) = (\mathfrak{M}(\delta^{-1}Q\delta\phi))(s). \tag{3.3}$$

[Note that on the right-hand side of (3.3) we are using δ^{-1} to stand for the appropriate version of the integral operator in (2.13) and we are not supposing that $Q\delta\phi$ is differentiable, since that is what we are trying to prove!] Since $\mathfrak{M}$ is one-to-one on L^2_μ, we obtain

$$Q\phi = \delta^{-1}Q\delta\phi \qquad \text{a.e. on } (0,\infty). \tag{3.4}$$

We wish to turn equality a.e. into equality everywhere on $(0,\infty)$.

Define $R : C_o^\infty(0,\infty) \subseteq L^2_\mu \to L^2_\mu$ by

$$(R\phi)(x) = \begin{cases} \int_0^x t^{-1}\,(Q\delta\phi)(t)\,dt & \text{Re}\,\mu > 1/2 \\ -\int_x^\infty t^{-1}(Q\delta\phi)(t)\,dt & \text{Re}\,\mu < 1/2 \end{cases}.$$

The form of R is suggested by (2.13) and (3.4). First note that $(R\phi)(x)$ exists for all $x \in (0,\infty)$ since $Q\delta\phi \in L^2_\mu$ so that, for $\text{Re}\,\mu > 1/2$,

$$\left|\int_0^x t^{-1}(Q\delta\phi)(t)dt\right| \le \left(\int_0^x |t^{-\mu}(Q\delta\phi)(t)|^2 dt\right)^{\frac{1}{2}}\left(\int_0^x |t^{\mu-1}|^2 dt\right)^{\frac{1}{2}} < \infty$$

by the Cauchy-Schwarz inequality, with a similar argument for $\text{Re}\,\mu < 1/2$. By considering integrals with limits x and x + h, we can easily show that $R\phi$ is continuous on $(0,\infty)$. This in turn enables us to prove that $R\phi$ is differentiable on $(0,\infty)$. Indeed, for any $\psi \in C_o^\infty(0,\infty)$, $R\psi = Q\psi$ a.e. In particular, for $\psi = \delta\phi$ and $\text{Re}\,\mu > 1/2$,

$$(R\phi)(x) = \int_0^x t^{-1}(R\delta\phi)(t)\,dt \qquad (x > 0).$$

Thus, for $x > 0$ and $x + h > 0$,

$$\left|h^{-1}\{(R\phi)(x+h) - (R\phi)(x)\} - x^{-1}(R\delta\phi)(x)\right|$$
$$= \left|h^{-1}\int_x^{x+h} t^{-1}(R\delta\phi)(t)dt - x^{-1}(R\delta\phi)(x)\right|$$

tends to zero as $h \to 0$ by the mean-value theorem for integrals. A similar argument applies when $\operatorname{Re}\mu < 1/2$. In either case $R\phi$ is differentiable and $(D(R\phi))(x) = x^{-1}(R\delta\phi)(x)$ or

$$\delta R\phi = R\delta\phi \qquad (\phi \in C_o^\infty(0,\infty)).$$

A simple induction argument proves that $R\phi \in C^\infty(0,\infty)$ and that, for $k = 0,1,2,\ldots,$

$$\delta^k R\phi = R\,\delta^k\phi \qquad (\phi \in C_o^\infty(0,\infty)).$$

Equivalently, since $x^k D^k\phi = (\delta(\delta-1)\ldots(\delta-k+1)\phi)(x)$, for $k = 0,1,2,\ldots,$

$$x^k(D^k R\phi)(x) = (R(x^k D^k\phi))(x) \qquad (\phi \in C_o^\infty(0,\infty)). \tag{3.5}$$

Having used $p = 2$ and the extra condition $\operatorname{Re}\mu \neq 1/2$ to derive (3.5), we now return to the general case $1 < p < \infty$, $\mu \in \Omega_p$. Since $Q \in B(L_\mu^p)$ and $Q\psi = R\psi$ a.e. for $\psi \in C_o^\infty(0,\infty)$, (2.2), (2.9) and (3.5) give, for $\phi \in C_o^\infty(0,\infty)$,

$$\begin{aligned}
\gamma_k^{p,\mu}(R\phi) &= \|x^k\, D^k(x^{-\mu}R\phi)\|_{L^p(0,\infty)} && \text{by (2.9)}\\
&\leq \sum_{r=0}^{k} a_r\, \|x^{-\mu}x^r\, D^r R\phi\|_{L^p(0,\infty)} && \text{by Leibnitz rule}\\
&= \sum_{r=0}^{k} a_r\, \|x^r\, D^r\, R\phi\|_{p,\mu} && \text{by (2.2)}\\
&= \sum_{r=C}^{k} a_r\, \|R\, x^r\, D^r\phi\|_{p,\mu} && \text{by (3.5)}\\
&= \sum_{r=0}^{k} a_r\, \|Q\, x^r\, D^r\phi\|_{p,\mu}\\
&\leq \sum_{r=0}^{k} a_r\, \|Q\|\, \|x^r\, D^r\phi\|_{p,\mu}\\
&= \sum_{r=0}^{k} a_r\, \|Q\|\, \|x^{-\mu}\, x^r\, D^r\phi\|_{L^p(0,\infty)} && \text{by (2.2)}
\end{aligned}$$

$$\leq \sum_{r=0}^{k} a_r \, \|Q\| \left(\sum_{s=0}^{r} b_{rs} \, \|x^s D^s(x^{-\mu}\phi)\|_{L^p(0,\infty)} \right)$$

$$= \|Q\| \sum_{r=0}^{k} a_r \left(\sum_{s=0}^{r} b_{rs} \, \gamma_s^{p,\mu}(\phi) \right) \quad \text{by (2.9)}$$

where a_r and b_{rs} ($s = 0,\ldots, r$; $r = 0,\ldots,k$) are constants independent of ϕ and $\|Q\|$ is the operator norm of $Q \in B(L_\mu^p)$. It follows that, as an operator from $C_o^\infty(0,\infty) \subseteq F_{p,\mu}$ into $F_{p,\mu}$, R is continuous. By Theorem 2.8(ii), we can extend R by continuity to the whole of $F_{p,\mu}$ and the resulting operator, which we also denote by R, is that mentioned at the start of the proof. Indeed, part (i) in Definition 3.1 has been dealt with above. As regards (ii) in Definition 3.1, if $\phi \in C_o^\infty(0,\infty)$,

$(\mathfrak{M}(R\phi))(s) = (\mathfrak{M}(Q\phi))(s) = g(s)\ (\mathfrak{M}\phi)(s)$, by (2.6) applied to Q. Thus, for $1 < p \leq 2$, (3.1) is valid on $F_{p,\mu}$ by the continuity of R on $F_{p,\mu}$ and of the Mellin transform on L_μ^p (Lemma 2.2). Thus, g is an $F_{p,\mu}$ multiplier, as required.

<u>Remark 3.4</u> Theorem 3.3 can be regarded as an extension of ideas in [1, pp. 158-9]. The latter can be used to obtain conditions under which an operator $R \in B(L_\mu^p)$ which has the form

$$(R\phi)(x) = \int_0^\infty k(x/t)\ \phi(t)\ dt/t$$

of a Mellin convolution with kernel k is also a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$. However, there are L_μ^p multipliers g which do not correspond to any such kernel k, a trivial example being $g(s) \equiv 1$.

Theorem 3.3 shows that the class of $F_{p,\mu}$ multipliers is at least as large as the class of L_μ^p multipliers. That it is strictly larger is easily seen.

<u>Example 3.5</u> Let Ω be the entire complex plane and define g on Ω by $g(s) = -s$. Then, by Corollary 2.10, g is an $F_{p,\mu}$ multiplier

corresponding to the continuous linear mapping δ. However g is not an L^p_μ multiplier, in view of the fact that δ is not a member of $B(L^p_\mu)$ for any p and μ. Notice that this is a case where the values $p = 1$ and $p = \infty$ can be included (Remark 3.2) in view of Corollary 2.10.

Example 3.5 is a special case of the next result.

Theorem 3.6 Let g be an L^p_μ multiplier. Then, for any polynomial P, the function Pg is an $F_{p,\mu}$ multiplier. More explicitly, if R is the continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ $(1 < p < \infty$, $\mu \in \Omega_p)$ with symbol g, then Pg is the symbol of the operator $P(-\delta)R = RP(-\delta)$.

Proof:- Let $P(s) = \sum_{k=0}^{n} a_k s^k$ where n is a non-negative integer and $a_0,\ldots,a_n$ are complex numbers. Then $P(-\delta) = \sum_{k=0}^{n} a_k (-\delta)^k = \sum_{k=0}^{n} (-1)^k a_k \delta^k$ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ for $1 \leq p \leq \infty$ and any μ, by Corollary 2.10(i). The existence of the operator R in the statement of the theorem is guaranteed by Theorem 3.3, and the composition of R and $P(-\delta)$, in either order, is also a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ when $1 < p < \infty$ and $\mu \in \Omega_p$. For $1 < p \leq 2$ (also $p = 1$), $\phi \in F_{p,\mu}$ and $\operatorname{Re} s = 1/p - \operatorname{Re} \mu$,

$$(\mathfrak{M}((-\delta)^k\phi))(s) = s^k(\mathfrak{M}\phi)(s)$$

by induction based on (2.14) and this leads in turn to

$$(\mathfrak{M}(P(-\delta)\phi))(s) = P(s)(\mathfrak{M}\phi)(s). \tag{3.6}$$

(3.1) and (3.6) now give

$$(\mathfrak{M}(P(-\delta)R\phi))(s) = P(s)(\mathfrak{M}(R\phi))(s) = P(s)\, g(s)\, (\mathfrak{M}\phi)(s)$$

and, similarly, $(\mathfrak{M}(RP(-\delta)\phi))(s) = P(s)\, g(s)\, (\mathfrak{M}\phi)(s)$. The result follows.

Remark 3.7 Theorem 3.6 shows that, as well as the symbols studied in [2], some of which correspond to integral operators, we now have at our disposal symbols corresponding to certain differential operators and composition of the two types will give rise to certain integro-differential operators. It would be interesting to have a characterisation of all symbols corresponding to continuous linear mappings from $F_{p,\mu}$ into $F_{p,\mu}$ but we shall not pursue this here. Instead, we shall turn to symbols of the form $h(s-\gamma)/h(s)$, as in [2], for which the operator $T : F_{p,\mu} \to F_{p,\mu+\gamma}$ defined by

$$(T\phi)(x) = x^{\gamma}(R\phi)(x),$$

with R as above, satisfies (1.2) and so is suitable for our method of defining general powers T^{α}. The larger class of $F_{p,\mu}$ multipliers means that, within the $F_{p,\mu}$ framework, the class of admissible values of α can usually be extended, by comparison with the L^p_μ theory.

§4 We now present the analogues for $F_{p,\mu}$ of the results for L^p_μ developed in [2, Sections 3 and 4]. For our later convenience, we shall give the appropriate definitions for $F_{p,\mu}$ and indicate in brackets their respective counterparts in L^p_μ. Detailed proofs of the theorems will be omitted as they resemble closely those in [2].

Definition 4.1 H_F (respectively H) will denote the set of all ordered triples (h,Ω,γ) such that

(i) Ω is a domain of the type described in Notation 2.4(i)

(ii) γ is a complex number

(iii) h is a complex-valued function such that the function g defined by

$$g(s) = \frac{h(s-\gamma)}{h(s)} \tag{4.1}$$

is analytic on Ω and is an $F_{p,\mu}$ (respectively L^p_μ)

multiplier in the sense of Definition 3.1 (respectively Definition 2.6).

<u>Theorem 4.2</u> A triple $(h,\Omega,\gamma) \in H_F$ (respectively H) generates a (unique) linear transformation T such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+\gamma}$ (respectively from L^p_μ into $L^p_{\mu+\gamma}$)

(ii) for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in F_{p,\mu}$ (respectively $\phi \in L^p_\mu$),

$$(\mathfrak{M}(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)} (\mathfrak{M}\phi)(s). \tag{4.2}$$

Proof:- The result follows immediately on taking T to be the operator defined by

$$(T\phi)(x) = x^\gamma (R\phi)(x) \tag{4.3}$$

where R is as in Definition 3.1 (respectively Definition 2.6).

<u>Definition 4.3</u> $\mathfrak{J}$ (respectively $\mathfrak{L}$) will denote the class of operators T generated by triples (h,Ω,γ) in H_F (respectively H). More precisely, $T \in \mathfrak{J}$ (respectively $T \in \mathfrak{L}$) if there exists a triple (h,Ω,γ) in H_F (respectively in H) such that (i) and (ii) in Theorem 4.2 are satisfied.

<u>Remark 4.4</u> As in [2, Lemma 3.2 and Remark 3.3], if $T \in \mathfrak{J}$ is generated by the triple $(h,\Omega,\gamma) \in H_F$, where $\gamma \neq 0$, then T is also generated by the triple $(h_k,\Omega,\gamma) \in H_F$ where, for any integer k,

$$h_k(s) = \exp(2\pi ksi/\gamma)\ h(s) \qquad (s \in \Omega). \tag{4.4}$$

From the point of view of defining powers of T, it is essential to know which generating triple is being used and, for that reason, we shall often write $T \equiv T(h,\Omega,\gamma)$ to emphasise that we regard T as being generated by (h,Ω,γ).

Next, we define the admissible set for $T \equiv T(h,\Omega,\gamma)$ which gives the values of α for which we shall be able to define T^α.

<u>Definition 4.5</u> For any triple $(h,\Omega,\gamma) \in H_F$, we define the set A_F of complex numbers by

$$A_F = \{\alpha : (h,\Omega,\alpha\gamma) \in H_F\} \tag{4.5}$$

with an analogous definition of a set A for triples in H [2, Definition 3.4]. Again we could write $A_F \equiv A_F(h,\Omega,\gamma)$ for emphasis, but usually the triple under consideration will be displayed beside T.

Definition 4.6 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{J}$ (respectively $\in \mathfrak{I}$) and let $\alpha \in A_F$ (respectively $\in A$). We define T^α to be the operator in $\mathfrak{J}$ (respectively in $\mathfrak{I}$) generated by the triple $(h,\Omega,\alpha\gamma)$. Thus, for $T \in \mathfrak{J}$ and $\alpha \in A_F$, T^α is the linear operator such that

(i) for $1 < p < \infty$ and $\mu \in \Omega_p$, T^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+\alpha\gamma}$

(ii) for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in F_{p,\mu}$,

$$(\mathfrak{M}(T^\alpha\phi))(s-\alpha\gamma) = \frac{h(s-\alpha\gamma)}{h(s)}(\mathfrak{M}\phi)(s). \tag{4.6}$$

Remark 4.7 Definition 4.6 is meaningful in view of Definitions 4.1 and 4.3. Calculation of the admissible set A_F for a given operator $T(h,\Omega,\gamma) \in \mathfrak{J}$ may, of course, present some difficulties. We shall not attempt to characterise possible admissible sets but merely assume that, in principle, A_F can be calculated. We shall give examples in [3].

The basic properties of the powers of T now follow. In each case, results in [2, §3] carry over on replacing H, $\mathfrak{I}$ and A by H_F, $\mathfrak{J}$ and A_F.

Theorem 4.8 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{J}$ and let $1 < p < \infty$.

(i) $0 \in A_F$ and for $\mu \in \Omega_p$, T^0 is the identity operator on $F_{p,\mu}$.

(ii) If $\{\alpha,\beta,\alpha+\beta\} \subseteq A_F$ and $\{\mu,\mu+\beta\gamma\} \subseteq \Omega_p$, then, as operators on $F_{p,\mu}$,

$$T^\alpha T^\beta = T^{\alpha+\beta}. \tag{4.7}$$

(iii) If $\{\alpha,-\alpha\} \subseteq A_F$ and $\{\mu,\mu+\alpha\gamma\} \subseteq \Omega_p$, then T^α is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu+\alpha\gamma}$ with

$$(T^{\alpha})^{-1} = T^{-\alpha} \tag{4.8}$$

as operators on $F_{p,\mu+\alpha\gamma}$.

Theorem 4.9 If $T \equiv T(h,\Omega,\gamma) \in \mathfrak{T}$, $\{\alpha,\alpha\beta\} \subseteq A_F$, $1 < p < \infty$ and $\mu \in \Omega_p$, then $(T^{\alpha})^{\beta}$ exists as a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+\alpha\beta\gamma}$ and

$$(T^{\alpha})^{\beta} = T^{\alpha\beta} . \tag{4.9}$$

Remark 4.10 As mentioned in [2, Remark 3.12], it is possible to relax the restrictions on the parameters further by a sort of analytic continuation. For instance, in (4.7), the right-hand side is meaningful without the restriction $\mu+\beta\gamma \in \Omega_p$ so that we could use the right-hand side to extend the definition of the left-hand side to cases where $\mu+\beta\gamma \notin \Omega_p$. Similar comments apply to (4.8) and (4.9).

Theorem 4.11 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{T}$ where $\gamma \neq 0$. As usual, for $\alpha \in A_F$, let T^{α} be the α^{th} power of T constructed from the triple (h,Ω,γ). Also, for any integer k, let $[T^{\alpha}]_k$ denote the α^{th} power of T constructed from the triple (h_k,Ω,γ) where h_k is defined by (4.4). Then, for $1 < p < \infty$ and $\mu \in \Omega_p$,

$$[T^{\alpha}]_k = \exp(-2\pi k\alpha i)T^{\alpha} \tag{4.10}$$

as operators from $F_{p,\mu}$ into $F_{p,\mu+\alpha\gamma}$.

Proof:- As mentioned in Remark 4.4, (h_k,Ω,γ) generates T and by (4.6), with h replaced by h_k, we see that, for $1 < p \leq 2$, $\mu \in \Omega_p$ and $\phi \in F_{p,\mu}$,

$$(\mathfrak{M}([T^{\alpha}]_k\phi))(s-\alpha\gamma) = \frac{h_k(s-\alpha\gamma)}{h_k(s)} (\mathfrak{M}\phi)(s)$$

$$= \exp(-2\pi k\alpha i) \frac{h(s-\alpha\gamma)}{h(s)} (\mathfrak{M}\phi)(s)$$

$= \exp(-2\pi k\alpha i) \quad (\mathfrak{M}(T^{\alpha}\phi))(s-\alpha\gamma)$

$= (\mathfrak{M}(\exp(-2\pi k\alpha i)T^{\alpha})\phi)(s-\alpha\gamma)$

from which the result follows easily.

Remark 4.12 As a particular case of Theorems 4.8(ii) and 4.11, we see that if n is a positive integer such that $1/n \in A_F(h,\Omega,\gamma)$, then $T \equiv T(h,\Omega,\gamma)$ has n distinct "n^{th} roots", namely the operators $[T^{1/n}]_k$ for $k = 0,1,\dots,n-1$.

Remark 4.13 We briefly relate the results for $F_{p,\mu}$ just stated to their analogues for L^p_μ in [2]. Formally they look the same but the crux lies in the conditions on the parameters involved. Firstly, Theorem 3.3 and Example 3.5 show that H_F is strictly larger than H (Definition 4.1), so that $\mathfrak{J}$ is strictly larger than $\mathfrak{I}$ (Definition 4.3). Another consequence is that $A_F \supseteq A$ (Definition 4.5) and we might hope that in particular cases A_F would be strictly larger than A. This is so, as we shall show by example in [3]. For such operators we are able to define T^α for a larger class of values of α within the $F_{p,\mu}$ framework than within the L^p_μ framework. Also the conditions on α and β in Theorems 4.8 and 4.9 will be satisfied by a larger range of values and, in particular, there will be a better chance of T^α being a homeomorphism (Theorem 4.8(iii)) relative to the $F_{p,\mu}$ spaces than relative to the L^p_μ spaces. Since obtaining homeomorphisms on the $F_{p,\mu}$ spaces is one of the main objects of the exercise, the reformulation of the results in [2] within the $F_{p,\mu}$ spaces is essential to what follows in [3].

To complete this section we state analogues of results in [2,§4] for adjoint (or conjugate) operators. In [2], we defined the formal adjoint, T', of an operator $T \in \mathfrak{I}$ and showed, among other things, that T' also belongs to $\mathfrak{I}$ and that $(T')^\alpha = (T^\alpha)'$ under appropriate conditions. We shall see there are corresponding results for operators in the class $\mathfrak{J}$. As usual, for $1 < p < \infty$, we write

$$q = p/(p-1) \qquad \text{so that } 1/p + 1/q = 1. \tag{4.11}$$

Definition 4.14 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{F}$. We define T', the formal adjoint of T, to be the (unique) linear mapping such that, for $1 < p < \infty$ and $\mu \in \Omega_p$,

$$\int_0^\infty (T\phi)(x)\ \psi(x)dx = \int_0^\infty \phi(x)\ (T'\psi)(x)dx \tag{4.12}$$

for all $\phi \in F_{p,\mu}$ and all $\psi \in F_{q,-\mu-\gamma}$.

Remark 4.15 In the L^p_μ structure with $T \in \mathfrak{A}$, it was possible to obtain the existence of T' very easily by standard duality arguments. However, the structure of the dual space of $F_{p,\mu}$ is rather more complicated than that of the dual of L^p_μ. Accordingly, we shall obtain the existence of T' by displaying an explicit expression which arose in the proof of [2, Theorem 4.5].

Lemma 4.16 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{F}$. Then, for $1 < p < \infty$ and $\mu \in \Omega_p$, T' exists as a continuous linear mapping from $F_{q,-\mu-\gamma}$ into $F_{q,-\mu}$ and satisfies the relation

$$(T'\psi)(x) = x^\gamma UTUx^\gamma\psi(x) \qquad (\psi \in F_{q,-\mu-\gamma}) \tag{4.13}$$

where, for suitable functions ψ,

$$(U\psi)(x) = x^{-1}\psi(x^{-1}) \qquad (0 < x < \infty). \tag{4.14}$$

Proof:- It is easy to check that, for $1 < p < \infty$ and any complex number μ, U is a homeomorphism from $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ with inverse $U^{-1} = U$. Consequently, if $\psi \in F_{q,-\mu-\gamma}$, $Ux^\gamma\psi \in F_{q,2/q-1+\mu}$. Next, $\mu \in \Omega_p$ if and only if $2/q-1+\mu \in \Omega_q$ (by (2.4)). By Theorem 4.2(i) T is a continuous linear mapping from $F_{q,2/q-1+\mu}$ into $F_{q,2/q-1+\mu+\gamma}$ whence $TUx^\gamma\psi \in F_{q,2/q-1+\mu+\gamma} \Rightarrow UTUx^\gamma\psi \in F_{q,-\mu-\gamma} \Rightarrow x^\gamma UTUx^\gamma\psi \in F_{q,-\mu}$. Thus the right-hand side of (4.13), being the composition of five continuous linear mappings, defines a continuous linear mapping from $F_{q,-\mu-\gamma}$ into $F_{q,-\mu}$. To show that this mapping is T', we must verify (4.12). This is done by using a version of Parseval's theorem for the Mellin transform but the details, being almost identical to those in [2, Theorem 4.5], are omitted. This completes the proof.

With the existence of T' guaranteed, we can proceed to the main results for adjoints in $\mathfrak{J}$.

<u>Theorem 4.17</u> If $T \equiv T(h,\Omega,\gamma) \in \mathfrak{J}$, then $T' \in \mathfrak{J}$ and is generated by the triple (h',Ω',γ') where

$$\Omega' = \{s':1+\gamma-s' \in \Omega\},\ \gamma' = \gamma,\ h'(s') = [h(1-s')]^{-1}(s' \in \Omega'). \tag{4.15}$$

In particular, if $1 < p < \infty$ and $\mu \in \Omega_p$, then

(i) T' is a continuous linear mapping from $F_{q,-\mu-\gamma}$ into $F_{q,-\mu}$

(ii) for $1 < q \leq 2$ and $\psi \in F_{q,-\mu-\gamma}$,

$$(\mathfrak{M}(T'\psi))(s'-\gamma) = \frac{h'(s'-\gamma)}{h'(s')} (\mathfrak{M}\psi)(s') \quad (s' \in \Omega'). \tag{4.16}$$

Proof:- The details are similar to those in [2, Theorem 4.5] and are therefore omitted.

Since $T' \in \mathfrak{J}$, we can define $(T')^{\alpha}$ for values of α in the admissible set for $T' \equiv T'(h',\Omega',\gamma')$. It turns out that this admissible set is the same as the admissible set $A_F \equiv A_F(h,\Omega,\gamma)$ corresponding to $T(h,\Omega,\gamma)$ and that $(T')^{\alpha}$ is related to $(T^{\alpha})'$ in the expected way. As a consequence of this, we can derive the conditions for validity of index laws for powers of T' from the corresponding conditions for T. We state these results without proofs.

<u>Theorem 4.18</u> Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{J}$ and $\alpha \in A_F$ (given by (4.5)). Then $(T')^{\alpha}$ exists as an element of $\mathfrak{J}$ and is generated by the triple $(h',\Omega',\alpha\gamma')$ where h',Ω' and γ' are defined by (4.15). In particular, for $1 < q \leq 2$, $\mu \in \Omega_p$ and $\psi \in F_{q,-\mu-\alpha\gamma}$,

$$(\mathfrak{M}((T')^{\alpha}\psi))(s'-\alpha\gamma) = \frac{h'(s'-\alpha\gamma)}{h'(s')}(\mathfrak{M}\psi)(s') \quad (s'\in\Omega'). \tag{4.17}$$

Theorem 4.19 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{J}$ and let $1 < p < \infty$.

(i) $0 \in A_F$ and for $\mu \in \Omega_p$, $(T')^0$ is the identity operator on $F_{q,-\mu}$.

(ii) If $\alpha \in A_F$ and $\mu \in \Omega_p$, then, as operators on $F_{q,-\mu-\alpha\gamma}$,

$$(T')^{\alpha} = (T^{\alpha})'. \tag{4.18}$$

(iii) If $\{\alpha,\beta,\alpha+\beta\} \subseteq A_F$ and $\{\mu,\mu+\beta\gamma\} \subseteq \Omega_p$ then, as operators on $F_{q,-\mu-\alpha\gamma-\beta\gamma}$,

$$(T')^{\beta}(T')^{\alpha} = (T')^{\alpha+\beta}. \tag{4.19}$$

(iv) If $\{\alpha,-\alpha\} \subseteq A_F$ and $\{\mu,\mu+\alpha\gamma\} \subseteq \Omega_p$, then $(T')^{\alpha}$ is a homeomorphism from $F_{q,-\mu-\alpha\gamma}$ onto $F_{q,-\mu}$ and, as operators on $F_{q,-\mu}$,

$$[(T')^{\alpha}]^{-1} = (T')^{-\alpha}. \tag{4.20}$$

(v) If $\{\alpha,\alpha\beta\} \subseteq A_F$ and $\mu \in \Omega_p$, then $((T')^{\alpha})^{\beta}$ exists and, as operators on $F_{q,-\mu-\alpha\beta\gamma}$,

$$((T')^{\alpha})^{\beta} = (T')^{\alpha\beta}. \tag{4.21}$$

As mentioned earlier, we shall illustrate the results of this paper in [3].

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Applicable Analysis, 1986, Vol. 21, pp. 151–173
0003-6811/86/2102-0151 $18.50/0

Fractional Powers of a Class of Mellin Multiplier Transforms Part III

Communicated by G. F. Roach

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Abstract The theory developed in a previous paper [6] is illustrated by means of examples and extended to generalised functions, producing theorems on the existence of classical and generalised solutions of some equations in weighted L^p spaces.
(Received for Publication August 13, 1985)

INTRODUCTION

This paper is a sequel to [6] to which the reader should refer throughout for notation, terminology, etc. For simplicity, the sections in this paper are numbered 5, 6 and 7 to follow on from 1-4 in [6]. Likewise, reference to (2.10) or Theorem 4.9 will mean the appropriate formula or theorem in [6], unless the contrary is indicated.

In §5, we illustrate the theory in [6] with a number of examples, starting with simple operators and building up to more elaborate differential operators and an integral operator involving Meijer's G-function. These examples illustrate many points in [6], particularly the fact that A_F can be much larger than A; typically A is a half-plane while A_F is the whole complex plane. In §6, we extend the theory to the spaces $F'_{p,\mu}$ of generalised functions, by means of standard theorems on adjoints. Finally, in §7, we apply our theory to the original problem of finding classical solutions in L^p_μ of (1.4) and (1.5) and give an example.

§5. We shall first review the examples in [5, §5] and discover that much greater progress can be made within the $F_{p,\mu}$ spaces than in L^p_μ.

Example 5.1 We consider the triple (h,Ω,γ) where

$$h(s) = \left[\Gamma(1-s)\right]^{-1}, \ \Omega = \{s : \text{Re } s \neq k \text{ for } k=1,2,\ldots\}, \ \gamma=1. \tag{5.1}$$

The expression $h(s-\alpha\gamma)/h(s)$ in (4.6) becomes $\Gamma(1-s)/\Gamma(1-s+\alpha)$ which shows that Ω is the largest s domain of the type we are considering which avoids the poles of the multiplier. For fixed p, the corresponding maximal μ domain is

$$\Omega_p = \{\mu: \text{Re } \mu \neq 1/p - k \text{ for } k = 1,2,\ldots\}. \tag{5.2}$$

In [5, Examples 5.2 and 5.6] we saw that $(h,\Omega,\gamma) \in H$ (as in Definition 4.1) and that the corresponding operator $T \equiv T(h,\Omega,\gamma)$ could be written down explicitly. In connection with the corresponding admissible set A (see Definition 4.5), apart from $\alpha = 0$ which is trivial, we saw that $\{\alpha: \text{Re } \alpha > 0\} \subseteq A$ and obtained explicit expressions for T^α. In the simplest case corresponding to Re $s < 1$ or Re $\mu > 1/p - 1$, T^α was the Riemann-Liouville fractional integral I_1^α of order α defined by

$$(I_1^\alpha\phi)(x) = \left[\Gamma(\alpha)\right]^{-1} \int_0^x (x-t)^{\alpha-1}\phi(t)\,dt \qquad (\phi \in L_\mu^p). \tag{5.3}$$

We shall write I_1^α, rather than T^α, in the case of other ranges of μ (or s) also, although the explicit form of $I_1^\alpha\phi$ $(\phi \in L_\mu^p)$ will then be different [5, formula (5.15)]. We were unable to handle Re $\alpha < 0$ in the L_μ^p case. For $\alpha = -1$, the expression $\Gamma(1-s)/\Gamma(1-s+\alpha)$ collapses to $-s$, which is related to D by (2.12), as we would expect. Since D is not defined on all of L_μ^p let alone bounded, the theory collapses. However, working in $\mathcal{F}$ rather than $\mathcal{L}$ saves the day, as we shall now show.

Let α be any complex number and n a positive integer such that Re $\alpha + n > 0$. Then for $s \in \Omega$ (defined in (5.1)), we may apply the formula $\Gamma(z+1) = z\Gamma(z)$ repeatedly to obtain

$$\frac{\Gamma(1-s)}{\Gamma(1-s+\alpha)} = P(s)\ g(s)$$

where

$$P(s) = (n-s+\alpha)(n-1-s+\alpha)\dots(1-s+\alpha) \tag{5.4}$$

and

$$g(s) = \frac{\Gamma(1-s)}{\Gamma(1-s+\alpha+n)} .$$

Since $\mathrm{Re}(\alpha+n) > 0$, our work in [5] shows that g is an L^p_μ multiplier. Also P is evidently a polynomial. Hence, by Theorem 3.6, $P(s)\,g(s)$ is an $F_{p,\mu}$ multiplier. Thus we have arrived at conditions of great generality, namely, that for $1 < p < \infty$, $\mu \in \Omega_p$ and <u>any</u> complex number α, there is an operator, still denoted by I_1^α, which belongs to the class $\mathfrak{J}$ and satisfies

$$(\mathfrak{M}(I_1^\alpha\phi))(s-\alpha) = \frac{\Gamma(1-s)}{\Gamma(1-s+\alpha)}(\mathfrak{M}\phi)(s) \quad (s \in \Omega) \tag{5.5}$$

for $1 < p \leq 2$ and $\phi \in F_{p,\mu}$. Further,

$$I_1^\alpha\phi = D^n\, I_1^{\alpha+n}\phi \tag{5.6}$$

for any non-negative integer n. To see this, let $\phi \in C_0^\infty(0,\infty)$ be regarded as an element of $F_{2,\mu}$ and let $\mu \in \Omega_2$. By Theorem 2.9, $D^n I_1^{\alpha+n}\phi \in F_{2,\mu+\alpha}$ and, for $\mathrm{Re}\, s = 1/p - \mathrm{Re}\,\mu$, n applications of (2.12) give

$$\begin{aligned}(\mathfrak{M}(D^n\, I_1^{\alpha+n}\phi))(s-\alpha) &= (1-s+\alpha)(\mathfrak{M}(D^{n-1}\, I_1^{\alpha+n}\phi))(s-\alpha-1)\\ &= (1-s+\alpha)(2-s+\alpha)(\mathfrak{M}(D^{n-2}I_1^{\alpha+n}\phi))(s-\alpha-2)\\ &= \dots\\ &= (1-s+\alpha)(2-s+\alpha)\dots(n-s+\alpha)(\mathfrak{M}(I_1^{\alpha+n}\phi))(s-\alpha-n)\\ &= P(s).\ \frac{\Gamma(1-s)}{\Gamma(1-s+\alpha+n)}(\mathfrak{M}\phi)(s) \text{ by (5.5)}\\ &= P(s)\ g(s)\ (\mathfrak{M}\phi)(s)\end{aligned}$$

where $P(s)$ and $g(s)$ are as in (5.4). From (5.5), we have

$$(\mathfrak{M}(I_1^\alpha\phi))(s-\alpha) = (\mathfrak{M}(D^nI_1^{\alpha+n}\phi))(s-\alpha)$$

so that (5.6) holds in this case. Since $C_o^\infty(0,\infty)$ is dense in $F_{p,\mu}$ and the operators involved are all continuous under the appropriate conditions (D^n in view of Theorem 2.9 and I_1^α, $I_1^{\alpha+n}$ in view of their belonging to $\mathfrak{J}$), (5.6) is established for all $\phi \in F_{p,\mu}$ by continuity and density under the stated conditions. In particular, if $\mathrm{Re}\ \mu > 1/p - 1$ and the non-negative integer n is such that $\mathrm{Re}\ \alpha + n > 0$, then

$$(I_1^\alpha\phi)(x) = D^n([\Gamma(\alpha+n)]^{-1}\int_o^x (x-t)^{\alpha+n-1}\ \phi(t)dt) \tag{5.7}$$

for $\phi \in F_{p,\mu}$, with similar expressions in other cases.

Remark 5.2 The ideas presented in Example 5.1 are discussed without the use of multipliers in [3]. The latter approach shows that the results remain true in the cases $p = 1$ and $p = \infty$ also. (See Remark 3.2.) This applies in all the other examples in this section too but this can only be checked because explicit representations of the operators can be obtained, essentially by taking the inverse Mellin transform of their symbols. Such representations as (5.3), (5.6) and (5.7) then enable us to study the cases $p = 1$ and $p = \infty$ along with $1 < p < \infty$ by other means, as in [3]. Having made this point, we shall not pursue it further here.

We shall require a very simple generalisation of Example 5.1.

Example 5.3 Let m be a fixed positive real number. We consider the triple (h,Ω,γ) where

$$h(s) = [\Gamma(1-s/m)]^{-1},\ \Omega = \{s: \mathrm{Re}\ s \neq mk \text{ for } k = 1,2,\ldots\},\ \gamma=m. \tag{5.8}$$

Example 5.1 deals with the case $m = 1$ but the general case is no harder. We find that $(h,\Omega,\gamma) \in H$ and that A contains $\alpha = 0$ and $\{\alpha: \mathrm{Re}\ \alpha > 0\}$ but not $\alpha = -1$, say. However, $(h,\Omega,\gamma) \in H_F$ and A_F is the set of all complex numbers. The corresponding operator $T(h,\Omega,\gamma) \in \mathfrak{J}$ will be denoted by I_m. The multiplier of I_m^α is

$\Gamma(1-s/m)/\Gamma(1-s/m+\alpha)$ by analogy with (5.5), while (5.6) becomes

$$I_m^\alpha \phi = (D_m)^n \, I_m^{\alpha+n} \, \phi \tag{5.9}$$

where α is any complex number, n is a non-negative integer, $1 < p < \infty$, $\mu \in \Omega_p$, $\phi \in F_{p,\mu}$ and D_m is defined by

$$(D_m\phi)(x) = d\phi/dx^m = m^{-1}x^{1-m}(D\phi)(x). \tag{5.10}$$

The explicit form of Ω_p is

$$\Omega_p = \{\mu: \operatorname{Re} \mu \neq 1/p - mk \text{ for } k = 1,2,\ldots\} \tag{5.11}$$

and when $\operatorname{Re} \mu > 1/p - m$, (5.9) gives

$$(I_m^\alpha\phi)(x) = (D_m)^n([\Gamma(\alpha+n)]^{-1} \int_0^x (x^m-t^m)^{\alpha+n-1} mt^{m-1}\phi(t)\,dt) \tag{5.12}$$

with the non-negative integer n such that $\operatorname{Re} \alpha + n > 0$. Let us look at Theorem 4.8 for $T = I_m$. Since A_F is the entire complex plane, A_F puts no restrictions on the parameters. Thus (4.7) and (4.8) are valid subject only to the restrictions involving Ω_p. These are the same restrictions as in [3, Theorem 3.36 and Theorem 3.43(i)] since $\mu + m\eta \in \Omega_p$ if and only if $\eta \in A_{p,\mu,m}$ where the latter set is defined, as in [3, p.60], by

$$A_{p,\mu,m} = \{\eta: \operatorname{Re}(m\eta+\mu)+m \neq 1/p - m\ell \text{ for } \ell = 0,1,2,\ldots\}. \tag{5.13}$$

From the point of view of the semigroup or group property for powers of I_m, our theory reproduces a previous result. However, when we consider Theorem 4.9 we have something new. The problem of defining the β^{th} power of I_m^α for general β was not mentioned in [3]. Here it emerges naturally and (4.9) is established under conditions of great generality, namely,

$$(I_m^\alpha)^\beta = I_m^{\alpha\beta} \tag{5.14}$$

as operators on $F_{p,\mu}$ provided only that ($1 < p < \infty$ and) $\{\mu,\mu+\alpha m\} \subseteq \Omega_p$ or, equivalently, $\{0,\alpha\} \subseteq A_{p,\mu,m}$. This generality is in marked contrast to the conditions required for the validity of (4.9) when the spectral approach is used.

Example 5.4 In connection with adjoints, we shall require the following analogue of Example 5.3. Consider the triple (h,Ω,γ) where

$$h(s) = \Gamma(s/m),\ \Omega = \{s\colon \mathrm{Re}\ s \neq -mk \text{ for } k = 0,1,2,\ldots\},\ \gamma = m. \qquad (5.15)$$

$(h,\Omega,\gamma) \in H$ and we shall denote the operator generated by this triple by K_m. As before, A contains $\alpha = 0$ and $\{\alpha\colon \mathrm{Re}\ \alpha > 0\}$ but not $\alpha = -1$, say. However, if we regard K_m as an element of $\mathfrak{J}$, we can accommodate all complex numbers α. The multiplier of K_m^α is $\frac{\Gamma(s/m-\alpha)}{\Gamma(s/m)}$ and K_m^α will define a continuous linear mapping from $F_{q,-\mu-m\alpha}$ into $F_{q,-\mu}$ provided only that $1/q + \mathrm{Re}\ \mu \in \Omega$. The analogue of (5.9) is

$$K_m^\alpha \phi = K_m^{\alpha+n} (-D_m)^n \phi \qquad (5.16)$$

and, if $\mathrm{Re}\ \mu > -1/q$ and the non-negative integer n is such that $\mathrm{Re}\ \alpha + n > 0$, then, for $\phi \in F_{q,-\mu-m\alpha}$,

$$(K_m^\alpha \phi)(x) = (-1)^n \left[\Gamma(\alpha+n)\right]^{-1} \int_x^\infty (t^m - x^m)^{\alpha+n-1} ((D_m)^n \phi)(t) m t^{m-1} dt \qquad (5.17)$$

which gives a concrete expression in terms of Weyl fractional integrals. (The use of q and $-\mu$ rather than p and μ is to help us in the next example.)

Example 5.5 We now illustrate our results for adjoints in the case of the operator I_m in Example 5.3. According to (4.15) and (5.8) we are concerned with the triple (h',Ω',γ') where

$$h'(s') = \Gamma(s'/m+1-1/m),\ \Omega' = \{s'\colon \mathrm{Re}\ s' \neq 1-mk \text{ for } k = 0,1,\ldots\},\ \gamma' = m. \qquad (5.18)$$

Denoting the adjoint of I_m by I_m', we have formally

$$I_m' = x^{m-1} K_m x^{-m+1}, \qquad (5.19)$$

where K_m is as in Example 5.4, and the result is valid as an operator equation on $F_{q,-\mu-m}$ for $\mu \in \Omega_p$, where Ω is given by (5.8). Indeed, let $\phi \in C_0^\infty(0,\infty)$ be regarded as an element of $F_{2,-\mu-m}$ and let $\mu \in \Omega_2$. Then $x^{-m+1}\phi(x) \in F_{2,-\mu-2m+1}$. By Example 5.4, K_m is a continuous linear mapping from $F_{2,-\mu-2m+1}$ into $F_{2,-\mu-m+1}$ provided that

$$1/2 + \operatorname{Re}(\mu+2m-1) - m \neq -mk \qquad (k = 0,1,2,\ldots)$$

i.e.

$$1/2 - \operatorname{Re}\mu \neq m(k+1) \qquad (k = 0,1,2,\ldots)$$

which is precisely the condition $\mu \in \Omega_2$. Thus, the right-hand side of (5.19) defines a continuous linear mapping from $F_{2,-\mu-m}$ into $F_{2,-\mu}$. Further, if $s' \in \Omega'$ denotes the Mellin transform variable in $F_{2,-\mu-m}$, then $s = 1 + m - s'$ (as given in (5.15)) and for $\psi \in F_{2,-\mu-m}$,

$$\begin{aligned}
&(\mathfrak{M}(x^{m-1} K_m x^{-m+1}\psi))(s'-m)\\
&=(\mathfrak{M}(K_m x^{-m+1}\psi))(s'+m-1-m)\\
&=\frac{\Gamma((s'-1)/m)}{\Gamma((s'+m-1)/m)}(\mathfrak{M}(x^{-m+1}\psi))(s'+m-1) \qquad \text{by } (5.15)\\
&=\frac{\Gamma((s'-1)/m)}{\Gamma((s'+m-1)/m)}(\mathfrak{M}\psi)(s')\\
&=\frac{h'(s'-m)}{h'(s')}(\mathfrak{M}\psi)(s') \qquad \text{by } (5.18).
\end{aligned}$$

Since $\mathfrak{M}$ is one-to-one on $F_{2,-\mu}$, (5.19) is valid when the operators are applied to functions in $C_0^\infty(0,\infty)$. Density and continuity complete the proof as usual. More generally, we can prove in a similar fashion that, for any complex number α,

$$(I_m^\alpha)' = x^{m-1} K_m^\alpha x^{-m+1} \tag{5.20}$$

as operators on $F_{q,-\mu-m\alpha}$ when $\mu \in \Omega_p$, K_m^α being as in Example 5.4. In other words, for $\phi \in F_{p,\mu}$ and $\psi \in F_{q,-\mu-m\alpha}$,

$$\int_0^\infty (I_m^\alpha\phi)(x)\ \psi(x)\ dx = \int_0^\infty \phi(x)(x^{m-1}\ K_m^\alpha x^{-m+1}\psi)(x)dx \tag{5.21}$$

by (4.12). (5.21) may be regarded as a generalisation of "fractional integration by parts". By applying Definition 4.6 to (5.19), we can verify that $((I_m)')^\alpha$ is also given by (5.20), as required by Theorem 4.19(ii). Explicit forms of the operators on the right-hand sides of (5.19) and (5.20) can be obtained from those in Example 5.4. Alternatively it is possible to use the expressions for I_m and I_m^α found in Example 5.3 and to substitute these for T in formula (4.13), this being fairly straightforward if Re $\mu > -1/q$ and Re $\alpha > 0$ (when I_m^α is the simple Riemann-Liouville fractional integral) but rather more complicated otherwise.

In our examples so far we have been looking again at triples generating integral operators in $\mathcal{L}$ and comparing their previous behaviour with that as members of $\mathfrak{J}$. However, we have many more triples available in H_F than in H, particularly, the triples that generate differential operators. In our next example, we look at our previous work from the other end, so to speak.

<u>Example 5.6</u>(i) Consider the triple $(h,\Omega,\gamma) \in H_F$ given by

$$h(s) = [\Gamma(1-s/m)]^{-1},\ \Omega = \{s:\ \text{Re } s \neq mk \text{ for } k = 1,2,\ldots\},\ \gamma=-m. \tag{5.22}$$

The multiplier $h(s+m)/h(s)$ collapses to $-s/m$ and the triple generates D_m, as defined by (5.10). Indeed, by (2.12),

$$(\mathfrak{M}(D_m\phi))(s+m) = (\mathfrak{M}(m^{-1}x^{1-m}D\phi))(s+m) = m^{-1}(\mathfrak{M}(D\phi))(s+m+1-m)$$

$$= m^{-1}(\mathfrak{M}(D\phi))(s+1) = -s/m\ (\mathfrak{M}\phi)(s).$$

(Obviously $(h,\Omega,\gamma)\notin H$.) As before, A_F is the entire complex plane and D_m^α, the α^{th} power of D_m, satisfies

$$(\mathcal{M}(D_m^\alpha\phi))(s+m\alpha) = \frac{\Gamma(1-s/m)}{\Gamma(1-\alpha-s/m)} (\mathcal{M}\phi)(s) \tag{5.23}$$

for $1 < p \leq 2$, $\mu \in \Omega_p$, $\phi \in F_{p,\mu}$ and any complex number α. Comparison with Example 5.3 shows that, under the stated conditions,

$$D_m^\alpha = I_m^{-\alpha} = (I_m^\alpha)^{-1} \tag{5.24}$$

as operators on $F_{p,\mu}$ (as might be expected).

(ii) In a similar vein, consider the triple

$$h(s) = \Gamma(s/m),\ \Omega = \{s\colon \text{Re } s \neq -mk \text{ for } k = 0,1,2,\ldots\},\ \gamma = -m. \tag{5.25}$$

The triple $(h,\Omega,\gamma) \in H_F$ and since $h(s+m)/h(s) = s/m$, the operator in $\mathfrak{J}$ generated by (h,Ω,γ) is $-D_m$, by our calculations above. The admissible set is again the entire complex plane and the α^{th} power of $-D_m$ satisfies

$$(\mathcal{M}((-D_m)^\alpha\phi))(s+m\alpha) = \frac{\Gamma(s/m+\alpha)}{\Gamma(s/m)} (\mathcal{M}\phi)(s) \tag{5.26}$$

for $1 < p \leq 2$, $\mu \in \Omega_p$, $\phi \in F_{p,\mu+m\alpha}$ and any complex number α. Comparison with Example 5.4 shows that, under the stated conditions,

$$(-D_m)^\alpha = K_m^{-\alpha} = (K_m^\alpha)^{-1} \tag{5.27}$$

as operators on $F_{p,\mu+m\alpha}$.

(iii) Use of (5.24) and (5.27) leads to a rather curious state of affairs. Formally it would appear that, in some sense or another,

$$K_m^{-\alpha} = ((\exp i\pi)D_m)^\alpha = \exp(i\pi\alpha)D_m^\alpha = \exp(i\pi\alpha)I_m^{-\alpha}$$

or, on replacing α by $-\alpha$,

$$K_m^\alpha = \exp(-i\pi\alpha) I_m^\alpha . \tag{5.28}$$

This is so provided that some care is taken over the interpretation of the α^{th} powers. Consider the triple $(\hat{h}, \hat{\Omega}, \hat{\gamma})$ where

$$\hat{h}(s) = [\Gamma(1-s/m)]^{-1}\exp(i\pi s/m), \; \hat{\Omega} = \{s: \text{Re } s \neq mk \text{ for any integer } k\},$$
$$\hat{\gamma} = m.$$

For $s \in \hat{\Omega}$,

$$\frac{\hat{h}(s-m)}{\hat{h}(s)} = \frac{\Gamma(1-s/m)}{\Gamma(2-s/m)} \exp(-i\pi) = \frac{1}{s/m-1} = \frac{\Gamma(s/m-1)}{\Gamma(s/m)} = \frac{h(s-m)}{h(s)}$$

where $h(s) = \Gamma(s/m)$ for $s \in \hat{\Omega}$. Hence both triples $(\hat{h}, \hat{\Omega}, \hat{\gamma})$ and $(h, \hat{\Omega}, \hat{\gamma})$ generate a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+m}$ provided that $\mu \in \hat{\Omega}_p$, that is, $1/p - \text{Re } \mu \neq mk$ for any integer k. From (5.15), we see that this mapping is K_m. The α^{th} power of K_m calculated using the triple $(h, \hat{\Omega}, \alpha\hat{\gamma})$ is the operator K_m^α in Example 5.4 (since $\hat{\Omega}$ is a subset of the set of values of s in (5.15)). As regards the triple $(\hat{h}, \hat{\Omega}, \alpha\hat{\gamma})$, we obtain

$$\frac{\hat{h}(s-m\alpha)}{\hat{h}(s)} = \frac{\Gamma(1-s/m)}{\Gamma(1-s/m+\alpha)}\exp(-i\pi\alpha)$$

whence the α^{th} power of K_m, calculated via $(\hat{h}, \hat{\Omega}, \alpha\hat{\gamma})$ is $\exp(-i\pi\alpha) \times I_m^\alpha$ where I_m^α is as in Example 5.3. It is in this sense that (5.28) holds on suitable spaces $F_{p,\mu}$. This is a variant of the theme discussed in Remark 4.4.

Remark 5.7 The results in Example 5.6 can be used to illustrate a general point. Under the conditions stated in Example 5.6(i) $(I_m)^{-1} = D_m$ or $(D_m)^{-1} = I_m$. (2.11) is the particular case $m = 1$. If we compare the triples in (5.8) and (5.22), we see that they are identical apart from the change of sign in γ. Similar comments apply to the triples (5.15) and (5.25). These are instances of

Definition 4.6 applied to the case $\alpha = -1$ (which belongs to A_F in these cases).

In our next example we turn to a class of more interesting differential operators.

Example 5.8 Define the operator T by

$$(T\phi)(x) = x^{a_1} D\, x^{a_2} D \ldots x^{a_n} D x^{a_{n+1}} \phi(x) \tag{5.29}$$

where n is a positive integer, $a_1,\ldots,a_{n+1}$ are complex numbers and

$$m = \left| \sum_{i=1}^{n+1} a_i - n \right| > 0. \tag{5.30}$$

We studied such operators T, sometimes referred to as operators of Bessel type, in [4] without making explicit mention of $\mathfrak{F}$. Nevertheless, our theory above gives an elegant alternative approach. Indeed, there are so many interesting aspects that we shall defer detailed consideration until a later paper and merely outline here a few salient points. T is the composition of n + 1 multiplications by powers of x and n differentiations D. Each of these 2n+1 operators belongs to the class $\mathfrak{F}$ since D has been dealt with in Example 5.6 while the operator of multiplication by x^λ is generated by the triple $(h,\Omega,\gamma) \in H_F$ where $h(s) \equiv 1$, $\Omega = \mathbb{C}$, the complex plane, and $\gamma = \lambda$. However, it is unfortunately the case that

$$T_1 \in \mathfrak{F}\,,\ T_2 \in \mathfrak{F} \not\Rightarrow T_1 T_2 \in \mathfrak{F}$$

as is easily seen by example. Thus, in the first instance, there is no reason why T in (5.29) should belong to $\mathfrak{F}$. That T does belong to $\mathfrak{F}$ only becomes apparent on rewriting T, each D being replaced by $mx^{m-1} D_m$ (see (5.10)), where m is given by (5.30), and the numbers $a_1,\ldots,a_{n+1}$ exchanged for the numbers $b_1,\ldots,b_n$ where

$$b_k = \Big(\sum_{i=k+1}^{n+1} a_i + k - n \Big)/m \qquad (k = 1,\ldots,n). \tag{5.31}$$

The alternative expression for T is different in the two cases $a < n$ and $a > n$ where

$$a = \sum_{i=1}^{n+1} a_i \text{ is assumed to be real} \tag{5.32}$$

($a = n$ being excluded by (5.30)). For simplicity we consider only the case $a < n$. Then, as will be shown in detail in the later paper, $T \in \mathfrak{J}$ and is generated by (h,Ω,γ) where

$$\begin{aligned} h(s) &= m^{ns/m} \Big[\prod_{k=1}^{n} \Gamma(b_k+1-s/m)\Big]^{-1} \\ \Omega &= \{s: \operatorname{Re}(b_k+1-s/m) \neq 0,-1,-2,\ldots \text{ for } k=1,\ldots,n\} \\ \gamma &= -m. \end{aligned} \tag{5.33}$$

Again $A_F = \mathbb{C}$ and, from Definition 4.6(ii), for $1 < p \leq 2$, $\mu \in \Omega_p$, $\phi \in F_{p,\mu}$ and any complex number α,

$$(\mathfrak{M}(T^\alpha\phi))(s+m\alpha) = m^{n\alpha} \prod_{k=1}^{n} \frac{\Gamma(b_k+1-s/m)}{\Gamma(b_k+1-\alpha-s/m)} (\mathfrak{M}\phi)(s) \quad (s \in \Omega). \tag{5.34}$$

Formula (5.34) appears as formula (5.6) in [4] and the condition $\mu \in \Omega_p$ is equivalent to $b_k \in A_{p,\mu,m}$ $(k = 1,\ldots,n)$, which is the condition in [4], as is easily shown via (5.13). We can use (5.23) to deal with each individual quotient of gamma functions, thereby obtaining a representation for T^α in the form

$$(T^\alpha\phi)(x) = m^{n\alpha} x^{-m\alpha}\Big(\prod_{k=1}^{n} x^{-mb_k+m\alpha} D_m^\alpha x^{mb_k}\Big)\phi(x) \tag{5.35}$$

or, by (5.24),

$$(T^\alpha\phi)(x) = m^{n\alpha} x^{-m\alpha}\Big(\prod_{k=1}^{n} x^{-mb_k+m\alpha} I_m^{-\alpha} x^{mb_k}\Big)\phi(x) . \tag{5.36}$$

In the case $\operatorname{Re}\alpha < 0$, (5.36) is essentially an integral operator.

Instead of the product of n operators, we can use the definition of Meijer's G-function as an inverse Mellin transform [1, p.337, formula (14)] together with the multiplier appearing in (5.34) to get, in the simplest case,

$$(T^{\alpha}\phi)(x) = x^{-m\alpha} m^{n\alpha+1} \int_0^x G^{0,n}_{n,n}\left(\frac{x^m}{t^m}\middle| \begin{matrix} -b_1,\ldots,-b_n \\ -b_1+\alpha, \ldots,-b_n+\alpha \end{matrix}\right) \phi(t)dt/t \tag{5.37}$$

the appropriate conditions on the parameters being $\mathrm{Re}\,\alpha < 0$, $\mathrm{Re}(mb_k+\mu) + m > 1/p$ for $k = 1,\ldots,n$. This appears in slightly disguised form as [4, Theorem 6.6].

Example 5.9 As a further illustration of Remark 5.7 and in preparation for §7, consider the triple $(h,\Omega,\gamma) \in H_F$ defined by

$$h(s) = m^{ns/m} \left[\prod_{k=1}^{n} \Gamma(b_k+1-s/m)\right]^{-1}$$

$$\Omega = \{s: \mathrm{Re}(b_k+1-s/m) \neq 0,-1,-2,\ldots \text{ for } k = 1,\ldots,n\} \tag{5.38}$$

$$\gamma = m$$

where m is positive. By Remark 5.7, the operator $T \in \mathfrak{J}$ generated by the triple (5.38) will be the inverse of the differential operator in Example 5.8. The set A_F will be $\mathbb{C}$ and, by (5.37), for $\phi \in F_{p,\mu}$ with $\mathrm{Re}(mb_k+\mu) + m > 1/p$ $(k = 1,\ldots,n)$,

$$(T\phi)(x) = x^m m^{-n+1} \int_0^x G^{0,n}_{n,n}\left(\frac{x^m}{t^m}\middle| \begin{matrix} -b_1,\ldots,-b_n \\ -b_1-1, \ldots,-b_n-1 \end{matrix}\right) \phi(t)dt/t. \tag{5.39}$$

This integral operator belongs to $\mathfrak{I}$ as well as to $\mathfrak{J}$ and we shall return to it in §7 in connection with existence and uniqueness of generalised and classical solutions.

§6 We are now ready to work within the framework of the classes $F'_{p,\mu}$ of generalised functions. Our intention is to extend operators from classical functions to generalised functions by making systematic use of adjoint operators. Obviously, operators in our classes $\mathcal{I}$ and $\mathcal{J}$ are candidates for this treatment. However, if an operator T belongs to the class $\mathcal{J}$, T maps one space of smooth functions into another and because its domain and codomain are so restricted, the mapping properties of T may already be good, as we have seen in the previous section. From one point of view, therefore, there is little to be gained by working distributionally in this case. Much more is to be gained when we start with a operator T in the class $\mathcal{I}$, which will map the whole of one weighted L^p space into another. For the reasons indicated in [6], we do not want to restrict the domain and codomain. In other words, although T could be studied as a member of $\mathcal{J}$ (by Theorem 3.3), it is more convenient to remain within the class $\mathcal{I}$. Accordingly, we make the following remark.

Remark 6.1 In what follows, we shall restrict attention to operators T in the class $\mathcal{I}$ mapping one weighted L^p space into another. Analogous results can be stated for operators T in the class $\mathcal{J}$ mapping one $F_{p,\mu}$ space into another.

Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{I}$, $1 < p < \infty$ and $\mu \in \Omega_p$. Then, by Theorem 4.2, $T \in B(L^p_\mu, L^p_{\mu+\gamma})$. Any element $f \in L^p_\mu$ can be used to generate a functional $\tilde{f}$ on $F_{q,-\mu}$ (where $1/p + 1/q = 1$) which assigns to each function $\phi \in F_{q,-\mu}$ the complex number $(\tilde{f},\phi)$ defined by

$$(\tilde{f},\phi) = \int_0^\infty f(x)\phi(x)\,dx \qquad (\phi \in F_{q,-\mu}). \tag{6.1}$$

Hölder's inequality shows that $\tilde{f} \in F'_{q,-\mu}$ and the mapping $f \to \tilde{f}$ imbeds L^p_μ into $F'_{q,-\mu}$. We propose to extend T to a mapping, $\tilde{T}$ say, defined on all of $F'_{q,-\mu}$. To obtain the appropriate definition, we observe that if $f \in L^p_\mu$, then $Tf \in L^p_{\mu+\gamma}$ so that Tf will generate an element $\widetilde{Tf} \in F'_{q,-\mu-\gamma}$ in the manner of (6.1). For $\tilde{T}$ to be an extension of T, we require

$$\tilde{\tilde{T}}f = \tilde{T}f \quad (f \in L^p_\mu) \tag{6.2}$$

the equality being in the space $F'_{q,-\mu-\gamma}$. To see what (6.2) entails, let $\phi \in F_{q,-\mu-\gamma}$. Then

$$(\tilde{\tilde{T}}f,\phi) = (\tilde{T}f,\phi) = \int_0^\infty (Tf)(x)\phi(x)dx.$$

Since $F_{q,-\mu-\gamma} \subseteq L^q_{-\mu-\gamma}$, we apply [5, Definition 4.2] to obtain

$$(\tilde{\tilde{T}}f,\phi) = \int_0^\infty f(x)(T'\phi)(x)dx = (\tilde{f},T'\phi) \tag{6.3}$$

where T' is the formal adjoint of T. By Lemma 4.16, $T'\phi \in F_{q,-\mu}$ under the stated conditions. Accordingly, the right-hand side of (6.3) remains meaningful if $\tilde{f}$ is replaced by any functional in $F'_{q,-\mu}$, whether the functional be regular (generated from a function in L^p_μ via (6.1)) or not. This suggests that we should define $\tilde{T}$ on $F'_{q,-\mu}$ to be the adjoint of T' on $F_{q,-\mu-\gamma}$. This will produce a mapping from $F'_{q,-\mu}$ into $F'_{q,-\mu-\gamma}$, under the stated conditions.

Remark 6.2 Until further notice we simplify notation by dropping the tildes. Thus we shall write T rather than $\tilde{T}$ for the extended or generalised version of T. Also, f will denote a typical functional, regular or not, and the value assigned by f to a testing-function ϕ will be denoted by (f,ϕ), and similarly for other letters.

Definition 6.3 Let $T \equiv T(h,\Omega,\gamma) \in \mathfrak{L}$, $1 < p < \infty$ and $\mu \in \Omega_p$. We define $T : F'_{q,-\mu} \to F'_{q,-\mu-\gamma}$ by

$$(Tf,\phi) = (f,T'\phi) \quad (f \in F'_{q,-\mu},\ \phi \in F_{q,-\mu-\gamma}). \tag{6.4}$$

Definition 6.3 is meaningful in view of the preamble. The properties of the extended operator T are easily obtained from those of T' by using standard theorems on adjoint operators such as those in [8, §1.10].

Theorem 6.4 Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{L}$, $1 < p < \infty$ and $\mu \in \Omega_p$. Then

(i) the extended operator T is a continuous linear mapping from $F'_{q,-\mu}$ into $F'_{q,-\mu-\gamma}$

(ii) if T' is a homeomorphism from $F_{q,-\mu-\gamma}$ onto $F_{q,-\mu}$, then T is a homeomorphism from $F'_{q,-\mu}$ onto $F'_{q,-\mu-\gamma}$.

Proof:- By Lemma 4.16, T' is a continuous linear mapping from $F_{q,-\mu-\gamma}$ into $F_{q,-\mu}$ under the stated conditions. (i) and (ii) follow immediately from Theorems 1.10-1 and 1.10-2 in [8].

Remark 6.5 Although we are studying T classically within the spaces L^p_μ, we shall exploit to the full the mapping properties of T' relative to the $F_{p,\mu}$ spaces. The assumption in (ii) that T' is a homeomorphism is not blind optimism. As we have seen in §5 for the case of powers of I_m and K_m, it is perfectly feasible to have an operator $T \in B(L^p_\mu, L^p_{\mu+\gamma})$ which is not a homeomorphism from L^p_μ onto $L^p_{\mu+\gamma}$ but is such that T' is a homeomorphism from $F_{q,-\mu-\gamma}$ onto $F_{q,-\mu}$. It is this which provides the motivation for the extension process in the first place.

If $T \in \mathcal{L}$, then $T^\alpha \in \mathcal{L}$ for all $\alpha \in A$ in view of Definitions 4.5 and 4.6. Consequently, we can carry out the extension procedure for T^α. First, we must obtain the formal adjoint of T^α in the L^p_μ structure. This is provided by [5, Theorem 4.10] which states that, under appropriate conditions, $(T^\alpha)' = (T')^\alpha$. (We have the analogue for the $F_{p,\mu}$ structure in Theorem 4.19(ii).) In particular, if $1 < p < \infty$, $\mu \in \Omega_p$ and $\alpha \in A$, then, by analogy with (6.4), $(T^\alpha f,\phi) = (f,(T')^\alpha\phi)$ whenever $f \in F'_{p,\mu}$ and $\phi \in F_{q,-\mu-\alpha\gamma}$. But we can go further and, again, this is part of the justification for the whole process. By Theorem 4.18, the right-hand side remains meaningful if $\alpha \in A_F$ and, while $A_F \supseteq A$ always, A_F may well be much larger than A, as in the examples in §5. Accordingly we can define an extended version of T^α not merely for $\alpha \in A$ (as in the classical L^p_μ treatment) but for $\alpha \in A_F$. Further, since $(T')^\alpha$ is well-behaved under these conditions, so also is T^α in its extended form. We now give the rigorous definition suggested by this preamble.

Definition 6.6 Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{L}$, $1 < p < \infty$, $\mu \in \Omega_p$ and $\alpha \in A_F$. We define $T^\alpha : F'_{q,-\mu} \to F'_{q,-\mu-\alpha\gamma}$ by

$$(T^\alpha f, \phi) = (f, (T')^\alpha \phi) \qquad (f \in F'_{q,-\mu},\ \phi \in F_{q,-\mu-\alpha\gamma}). \tag{6.5}$$

The stated conditions ensure that the definition is meaningful by Theorems 4.17 and 4.18.

We now list some properties of the extended operator T^α.

Theorem 6.7 Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{L}$, $1 < p < \infty$ and $\mu \in \Omega_p$. Then

(i) if $\alpha \in A_F$, T^α is a continuous linear mapping from $F'_{q,-\mu}$ into $F'_{q,-\mu-\alpha\gamma}$

(ii) if $\{\alpha,\beta,\alpha+\beta\} \subseteq A_F$ and $\{\mu,\mu+\beta\gamma\} \subseteq \Omega_p$, then, as operators on $F'_{q,-\mu}$,

$$T^\alpha T^\beta = T^{\alpha+\beta}$$

(iii) if $\{\alpha,-\alpha\} \subseteq A_F$ and $\{\mu,\mu+\alpha\gamma\} \subseteq \Omega_p$, then T^α is a homeomorphism from $F'_{q,-\mu}$ onto $F'_{q,-\mu-\alpha\gamma}$ with inverse $T^{-\alpha}$.

Proof:- All parts follow easily from results in §4. Consider (ii), for instance. Let $f \in F'_{q,-\mu}$ and $\phi \in F_{q,-\mu-\alpha\gamma-\beta\gamma}$. Then

$$(T^\alpha T^\beta f, \phi) = (T^\alpha(T^\beta f), \phi) = (T^\beta f, (T^\alpha)'\phi) = (T^\beta f, (T')^\alpha \phi)$$

the last equality being valid by Theorem 4.19(ii) since $\mu+\beta\gamma \in \Omega_p$ and $\alpha \in A_F$. The same hypotheses ensure that $(T')^\alpha \phi \in F_{q,-\mu-\beta\gamma}$ by Theorems 4.17 and 4.18. Then, since $\beta \in A_F$ and $\mu \in \Omega_p$,

$$(T^\beta f, (T')^\alpha \phi) = (f, (T^\beta)'(T')^\alpha \phi) = (f, (T')^\beta (T')^\alpha \phi)$$

by similar arguments. Next, by Theorem 4.19(iii), the last expression becomes

$$(f, (T')^{\beta+\alpha}\phi) = (f, (T')^{\alpha+\beta}\phi).$$

Finally, by Definition 6.6, we obtain

$$(T^\alpha T^\beta f,\phi) = (f,(T')^{\alpha+\beta}\phi) = (T^{\alpha+\beta}f,\phi)$$

from which the result follows. (iii) is now immediate while (i) follows from Theorem 4.18 and [8, Theorem 1.10-1].

Remark 6.8(i) As regards (iii), we emphasise again that the extended operator T^α may be invertible when the original classical operator is not, as shown in §5. Looked at from another angle, it is possible to solve in a generalised sense certain equations which cannot be solved classically in the setting of the L^p_μ spaces.

(ii) As regards (ii), the restriction $\mu+\beta\gamma \in \Omega_p$ can be dropped if we use analytic continuation. More precisely, we can regard $T^\alpha T^\beta$ as the adjoint of the analytic continuation of $(T')^\beta (T')^\alpha$ in the case $\mu+\beta\gamma \notin \Omega_p$; see Remark 4.10.

Another matter related to analytic continuation arises when we deal with the second index law $(T^\alpha)^\beta = T^{\alpha\beta}$ in the extended sense. There is no problem in interpreting the right-hand side via Definition 6.6. However, on the left-hand side we must first calculate T^α classically which will restrict α to the set A (rather than A_F) in the first instance. We can then calculate $(T^\alpha)^\beta$ via Definition 6.6 as the adjoint of $((T^\alpha)')^\beta$ provided that $\alpha\beta \in A_F$. Since $(T^\alpha)' = (T')^\alpha$ by [5, Theorem 4.10], $(T^\alpha)^\beta$ emerges as the adjoint of $((T')^\alpha)^\beta$ as we would expect. Thus in the first instance we can prove the following result.

Theorem 6.9 Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{I}$, $1 < p < \infty$, $\alpha \in A$, $\alpha\beta \in A_F$ and $\mu \in \Omega_p$. Then $T^\alpha : L^p_\mu \to L^p_{\mu+\alpha\gamma}$ exists as a member of $\mathcal{I}$ and, as operators from $F'_{q,-\mu}$ into $F'_{q,-\mu-\alpha\beta\gamma}$,

$$(T^\alpha)^\beta = T^{\alpha\beta}\,. \tag{6.6}$$

Proof:- The result follows from Theorem 4.19(v) and the preamble.

On closer inspection we see that the result (4.21) used in the proof is valid for $\alpha \in A_F$, $\alpha\beta \in A_F$ by Theorem 4.19(v), the only difficulty being that T^α will not exist classically for any values of α which lie in A_F but not in A. One way to resolve this would be to interpret (6.6) differently. For $\alpha \in A_F$ we can work out the extended operator T^α by means of Definition 6.6. We could then use (6.6) as the definition of the β^{th} power of the extended operator T^α subject only to the extra condition $\alpha\beta \in A_F$. Such an interpretation would not lead to any inconsistencies but we shall not explore it further here.

Finally in this section, if $T \equiv T(h,\Omega,\gamma) \in \mathcal{L}$, then the adjoint T', regarded as an operator from $L^q_{-\mu-\gamma}$ into $L^q_{-\mu}$, can be subjected to the same extension procedure. The rôles of T and T' are interchanged but the basic results are similar. We state the analogue of Theorem 6.7 to show the small changes which have to be made.

<u>Theorem 6.10</u> Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{L}$, $1 < p < \infty$ and $\mu \in \Omega_p$. Then

(i) if $\alpha \in A_F$, $(T')^\alpha$ is a continuous linear mapping from $F'_{p,\mu+\alpha\gamma}$ into $F'_{p,\mu}$

(ii) if $\{\alpha,\beta,\alpha+\beta\} \subseteq A_F$ and $\{\mu,\mu+\beta\gamma\} \subseteq \Omega_p$, then, as operators on $F'_{p,\mu+\alpha\gamma+\beta\gamma}$,

$$(T')^\beta (T')^\alpha = (T')^{\alpha+\beta}$$

(iii) if $\{\alpha,-\alpha\} \subseteq A_F$ and $\{\mu,\mu+\alpha\gamma\} \subseteq \Omega_p$, then $(T')^\alpha$ is a homeomorphism from $F'_{p,\mu+\alpha\gamma}$ onto $F'_{p,\mu}$ with inverse $(T')^{-\alpha}$.

<u>Proof</u>:- This is omitted.

§7 In this final section we return to equation (1.5) and see what our theory has to say about existence and uniqueness of both classical and generalised solutions. Here it is convenient to reinstate tildes to indicate extended operators. Also, we shall write $\tilde{f}$ (as before) to denote the functional generated from the classical function f by means of (6.1). The general position can be summarised as follows.

Theorem 7.1 Let $T \equiv T(h,\Omega,\gamma) \in \mathcal{I}$, $1 < p < \infty$, $\alpha \in A$, $\{\mu,\mu+\alpha\gamma\} \subseteq \Omega_p$ and $g \in L^p_{\mu+\alpha\gamma}$.

(i) If $-\alpha \in A_F$, (1.5) has a unique generalised solution; that is, there exists a unique functional $h \in F'_{q,-\mu}$ such that $\tilde{T}^\alpha h = \tilde{g}$, namely, $h = (\tilde{T}^\alpha)^{-1}\tilde{g}$.

(ii) If $-\alpha \in A_F$, (1.5) has at most one classical solution $f \in L^p_\mu$.

(iii) If $-\alpha \in A$, (1.5) has exactly one classical solution $f \in L^p_\mu$, namely, $f = T^{-\alpha}g$.

Proof:- (i) By definition, a generalised solution of (1.5) satisfies $\tilde{T}^\alpha h = \tilde{g}$. However, under the stated conditions, $\tilde{T}^\alpha$ is a homeomorphism from $F'_{q,-\mu}$ onto $F'_{q,-\mu-\alpha\gamma}$ by Theorem 6.7(iii). The result follows.

(ii) Let $f_1, f_2 \in L^p_\mu$ be two classical solutions of (1.5) and let $h = \widetilde{f_1 - f_2} = \tilde{f}_1 - \tilde{f}_2$. Then $T^\alpha f_1 = g$ and $T^\alpha f_2 = g \Rightarrow \tilde{T}^\alpha \tilde{f}_1 = \tilde{g} = \tilde{T}^\alpha \tilde{f}_2 \Rightarrow \tilde{T}^\alpha(\tilde{f}_1 - \tilde{f}_2) = 0$ or $\tilde{T}^\alpha h = 0$ where 0 denotes the zero functional in $F'_{q,-\mu-\alpha\gamma}$. By (i), $h = 0$, the zero functional in $F'_{q,-\mu} \Rightarrow f_1 - f_2 = 0$ as elements of $L^p_\mu \Rightarrow f_1 = f_2$ as elements of L^p_μ.

(iii) follows from [5, Corollary 3.9] and is repeated here for convenience. This completes the proof of the theorem.

As regards finding a classical solution of (1.5) by the distributional approach, subject to the conditions stated, the position is this. If $-\alpha \in A$, $T^{-\alpha} \in \mathcal{I}$ so that we can construct the extended operator $\widetilde{T^{-\alpha}}$ which is then the inverse of $\tilde{T}^\alpha$ by Theorem 6.7(iii). Thus we may write the generalised solution h in (i) as $(\tilde{T}^\alpha)^{-1}\tilde{g} = (\widetilde{T^{-\alpha}})\tilde{g} = \widetilde{T^{-\alpha}g}$ so that h is regular and generated by $T^{-\alpha}g$, which is therefore the unique classical solution, as promised in (iii). On the other hand, if $-\alpha \in A_F$ but $-\alpha \notin A$, then we are not entitled to write $(\widetilde{T^{-\alpha}})\tilde{g} = (\widetilde{T^{-\alpha}g})$ since $T^{-\alpha}$ will only be defined on $F_{p,\mu+\alpha\gamma}$ and not on the whole of $L^p_{\mu+\alpha\gamma}$. In this case, we are stranded in mid-air and the generalised solution, no longer regular (in the sense of being generated by an element of L^p_μ) is the best we can do, unless we are given extra information about g, such as an appropriate degree of differentiability of g. To illustrate these points we present an example of considerable generality involving the G-function operator of §5.

Example 7.2 For $f \in L^p_\mu$, define Tf on $(0,\infty)$ by

$$(Tf)(x) = x^m m^{-n+1} \int_0^x G^{0,n}_{n,n}\left(\frac{x^m}{t^m}\middle| \begin{matrix} -b_1,\dots,-b_n \\ -b_1 \;\; -1,\dots,-b_n-1 \end{matrix}\right) f(t)\, dt/t \qquad (7.1)$$

as in (5.39). Here m is real and positive, n is a positive integer and $b_1,\dots,b_n$ are given complex numbers. Let (h,Ω,γ) be the triple defined by (5.38). Then $(h,\Omega,\gamma) \in H$ as can be checked using Rooney's class $\mathcal{A}$ [7] and the set A (Definition 4.5) contains $\{\alpha: \operatorname{Re}\alpha > 0\}$ as well as 0, but not -1 since T^{-1} would correspond to the differential operator in Example 5.8. (7.1) represents the form of $T \equiv T(h,\Omega,\gamma)$ when $b_1,\dots,b_n$ satisfy $\operatorname{Re}(mb_k+\mu) + m > 1/p$ $(k = 1,\dots,n)$ and in this case, for $\operatorname{Re}\alpha > 0$, T^α has the form

$$(T^\alpha f)(x) = x^{m\alpha} m^{-n\alpha+1} \int_0^x G^{0,n}_{n,n}\left(\frac{x^m}{t^m}\middle| \begin{matrix} -b_1,\dots,-b_n \\ -b_1 \;\; -\alpha,\dots,-b_n-\alpha \end{matrix}\right) f(t)dt/t. \qquad (7.2)$$

Then, by definition of $\mathcal{L}$, $T \equiv T(h,\Omega,\gamma)$ is a continuous linear mapping from L^p_μ into $L^p_{\mu+m}$ and, for $\operatorname{Re}\alpha > 0$, $T^\alpha \in B(L^p_\mu, L^p_{\mu+m\alpha})$. (Recall that $\gamma = m$ here.) Accordingly, it is meaningful to consider the Volterra integral equation

$$T^\alpha f = g \qquad (7.3)$$

under the above conditions, where g is a given element of $L^p_{\mu+m\alpha}$ and a solution $f \in L^p_\mu$ is sought. The range of T^α is not the whole of $L^p_{\mu+m\alpha}$ and some degree of differentiability is required of g for a solution $f \in L^p_\mu$ to exist. This is fairly clear from the nature of

(7.2) but can also be seen by rewriting the formal inverse of T^{α}, via (5.35), as

$$m^{n\alpha}x^{-m\alpha}\left(\prod_{k=1}^{n} x^{-mb_k+m\alpha} D_m^{\alpha} x^{mb_k}\right)g(x) \tag{7.4}$$

which reveals the need for n derivatives of order α to exist. We therefore have a suitable candidate for our extension process and can apply Theorem 7.1. As regards the conditions of the theorem, Re $\alpha > 0$, assumed above, ensures that $\alpha \in A$ while $\mathrm{Re}(mb_k+\mu)+m>1/p$ $(k = 1,\ldots,n)$ ensures that $\mu \in \Omega_p$. By Example 5.9, $A_F = \mathbb{C}$ so that the restriction $-\alpha \in A_F$ is automatically satisfied. The condition $g \in L^p_{\mu+\alpha m}$ $(=L^p_{\mu+\alpha\gamma})$ was included above, so that the only extra assumption is $\mu + \alpha m \in \Omega_p$. Then, Theorem 7.1 guarantees a unique generalised solution $(\widetilde{T^{\alpha}})^{-1}\tilde{g} = \widetilde{T^{-\alpha}}\tilde{g}$ and at most one classical solution. Since $-\alpha \in A$ is not satisfied when Re $\alpha > 0$, part (iii) of the theorem does not apply. Instead, existence of a (unique) classical solution depends on whether we can write $\widetilde{T^{-\alpha}}\tilde{g} = \widetilde{T^{-\alpha} g}$ or not. This in turn depends on whether or not (7.4) can be evaluated classically. If so, (7.4) provides the unique classical solution; if not, we obtain only a generalised solution. The discussion above gives an alternative treatment of ideas from [4,§8], wherein will also be found an explicit function g for which no classical solution $f \in L^p_{\mu}$ of (7.3) exists. Many special cases can be handled by particular choices of $m,n,b_1,\ldots,b_n$. The case $n = 2$ in particular produces operators involving the ${}_2F_1$ hypergeometric function [4, §9] and results obtained can be related to previous work on classical and generalised solutions of hypergeometric integral equations [2,§6].

Other examples could be given but we shall leave these to the reader. In [5], [6] and the present paper, we have studied, both classically and distributionally, a large class of operators in L^p_{μ} spaces, the operators typically being Volterra or convolution integral operators, for which general powers can be defined in a systematic manner which avoids the complications of spectral theory. We have shown by examples that a distributional treatment allows differential operators to be handled, a larger range of powers to be defined and existence and uniqueness theorems to be stated for generalised solutions of classical equations. The generality of the conditions under which results such as the index law $(T^{\alpha})^{\beta} = T^{\alpha\beta}$ hold would be hard to beat.

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Paper 13

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Vol. 132, No. 2, June 1988
Printed in Belgium

On Relating Two Approaches to Fractional Calculus

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Received October 30, 1986

In this paper, two methods are discussed for defining fractional powers of the nth order ordinary differential expression $L = x^{a_1}Dx^{a_2}D \cdots x^{a_n}Dx^{a_{n+1}}$ $(D = d/dx)$, where each a_i $(i = 1, 2, ..., n+1)$ is a complex number and x is real and positive. An important role is played by the number $m = |\sum_{i=1}^{n+1} a_i - n|$. When $m = 0$, spectral theory is used to obtain a suitable formula for L^α $(\alpha \in \mathbb{C})$ while the case $m > 0$, considered in a previous paper by A. C. McBride (*Proc. London Math. Soc. (3)* **45** (1982), 519–546), is dealt with by expressing L^α in terms of Erdélyi–Kober operators. We give accounts of each approach and go on to consider how the results for $m = 0$ can be obtained as limiting cases of the results for $m > 0$. © 1988 Academic Press, Inc.

1. INTRODUCTION

In any theory of fractional powers of operators, the objective is to obtain a formula which, for suitable operators T and a range of parameters α, will generate a family of operators $\{T^\alpha\}$ possessing properties associated with powers. In particular, T^α should coincide with the iterated power $T^n = T \cdot T \cdots T$ (n terms) when α is a positive integer n and the index law

$$T^\alpha T^\beta = T^{\alpha+\beta}$$

should hold whenever T^α, T^β, and $T^{\alpha+\beta}$ are all defined.

Several methods now exist for constructing such families $\{T^\alpha\}$, each applicable to a particular class of operators. For example, various authors, including Balakrishnan [1], Hövel and Westphal [5], Komatsu [6], and Yosida [14], have examined the problem of deriving an expression for $(-A)^\alpha$ when A is a closed operator on a Banach space X. In each of [1, 5, 6, 14], the starting point is the integral

$$-\pi^{-1}\sin(\pi\alpha)\int_0^\infty \lambda^{\alpha-1}R(\lambda; A)\,Ax\,d\lambda, \qquad x \in X, \tag{1.1}$$

0022-247X/88 $3.00

which essentially defines the fractional power $(-A)^\alpha$ $(0<\operatorname{Re}\alpha<1)$ as an operator on X provided that the resolvent operator $R(\lambda; A)$ exists for each $\lambda>0$ and behaves suitably as $\lambda \to 0+$ and as $\lambda \to \infty$. Integral (1.1) has also been used in [7] to produce a corresponding theory for operators defined on a Fréchet space. A summary of this is given in Section 4.

An alternative to this spectral approach in the case of certain differential operators has recently been discussed in [10], where representations of arbitrary powers of the differential operator

$$L = x^{a_1} D x^{a_2} D \cdots x^{a_n} D x^{a_{n+1}},$$

and other associated operators, have been obtained in terms of the Erdélyi–Kober operators of fractional calculus; see Sections 2, 3 below. A requirement for the formulation of L^α in this theory is the condition that the sum $a=\sum_{k=1}^{n+1} a_k$ of the indices appearing in L is a real number which differs from n, the order of the operator.

The role played by the parameter a can be illustrated simply by considering the action of L on the function x^λ $(x>0, \lambda \in \mathbb{C})$. Since $Lx^\lambda = Cx^{\lambda+a-n}$, where C is a constant, we see that in the exceptional case $a=n$ the operator L, defined on some suitable function space X, will map X into X. This suggests the possibility of using the spectral approach to define L^α when $a=n$. Simple cases have already been considered in the literature. For example, in [8], the first-order operator $\delta = x\,d/dx$ (which corresponds to L with $n=1$, $a_1=1$, $a_2=0$) is examined and an expression for δ^α involving logarithmic-type fractional integrals obtained via (1.1). The exceptional cases $a=n=2, 3, \ldots$, however, present problems mainly due to the difficulty in verifying that the spectral conditions required for the convergence of (1.1) are satisfied. In this paper, we bypass these difficulties by making certain modifications to the spectral approach which allow us to define powers of products of commuting operators and, in particular, enable us to define L^α when $a=n$, for any $n \in \mathbb{N}$. We also demonstrate that the expressions derived for L^α $(a=n)$ are not totally unexpected since they can be obtained as limiting cases of those given in [10] for $n>a$.

It is worth mentioning that the differential operators discussed in [10] have since been shown to belong to a class of Mellin multiplier operators for which a theory of fractional powers exists; see [12, pp. 99–140]. Moreover, the results presented in [10] are simply a special case of those given in [12]. At the end of the paper, we indicate briefly how certain aspects of this Mellin multiplier theory can be used to deduce, in a non-rigorous manner, some of the results given earlier.

2. Notation and Preliminary Results

Various conventions are adhered to throughout the paper. Unless the contrary is explicitly stated, p is a real number satisfying $1 \leqslant p \leqslant \infty$, while μ, η, and α represent appropriate complex numbers. If $x > 0$ and λ is complex then $x^\lambda = \exp(\lambda \log x)$, where $\log x$ is real. Moreover, the expression $x^\lambda \varphi$ will denote the function $(x^\lambda \varphi)(x) = x^\lambda \varphi(x)$, $x > 0$. No confusion should arise from the use of x^λ as both a function and the operator coresponding to multiplication by x^λ. Throughout $\| \ \|_p$ denotes the usual $L^p(0, \infty)$ norm defined by

$$\|\varphi\|_p = \left(\int_0^\infty |\varphi(x)|^p \, dx \right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|\varphi\|_\infty = \operatorname{ess\cdot sup} \{ |\varphi(x)| : x \in (0, \infty) \}.$$

The function spaces which form the domains for the differential and integral operators considered in the paper are the spaces $F_{p,\mu}$ defined as follows.

Definition 2.1. (i) For $1 \leqslant p < \infty$,

$$F_{p,\mu} = \{ \varphi \in C^\infty(0, \infty) : \| x^k D^k (x^{-\mu} \varphi) \|_p < \infty \quad \text{for} \quad k = 0, 1, 2, ... \}, \tag{2.1}$$

where $D \equiv d/dx$.

(ii) $$F_{\infty,\mu} = \{ \varphi \in C^\infty(0, \infty) : x^k D^k (x^{-\mu} \varphi(x)) \to 0 \text{ as } x \to 0+ \text{ and as } x \to \infty \quad \text{for} \quad k = 0, 1, 2, ... \}. \tag{2.2}$$

(iii) For $1 \leqslant p \leqslant \infty$ and $k = 0, 1, 2, ...$, we define $\gamma_k^{p,\mu}$ on $F_{p,\mu}$ by

$$\gamma_k^{p,\mu}(\varphi) = \| x^k D^k (x^{-\mu} \varphi) \|_p. \tag{2.3}$$

For each p and μ, $F_{p,\mu}$ becomes a Fréchet space when equipped with the topology generated by the separating family of seminorms $\{\gamma_k^{p,\mu}\}_{k=0}^\infty$; see [11, Chap. 2].

Remark 2.2. An equivalent topology can be generated on $F_{p,\mu}$ by the separating family of norms $\{\rho_k^{p,\mu}\}_{k=0}^\infty$ defined by

$$\rho_k^{p,\mu}(\varphi) = \max \{ \gamma_i^{p,\mu}(\varphi);\ i = 0, 1, ..., k \} \qquad (k = 0, 1, 2, ...;\ \varphi \in F_{p,\mu}).$$

In this way, $F_{p,\mu}$ becomes a complete countably normed space; see [3, Chap. 1, Section 3]. The significance of this remark will be seen in the proof of Lemma 5.1.

LEMMA 2.3. *Let* $\varphi \in F_{p,\mu}$. *Then*

$$|\varphi(x)| \leqslant M x^{\operatorname{Re}\mu - 1/p}, \qquad \forall x > 0, \tag{2.4}$$

where M *is a constant which depends upon* φ.

Proof. See [11, p. 14, Theorem 2.2].

THEOREM 2.4. *For any complex number* λ, *the mapping* x^{λ} *defined by*

$$(x^{\lambda}\varphi)(x) = x^{\lambda}\varphi(x), \qquad 0 < x < \infty,$$

is a homeomorphism from $F_{p,\mu}$ *onto* $F_{p,\mu+\lambda}$ *with inverse* $x^{-\lambda}$.

Proof. See [11, p. 21, Theorem 2.11].

DEFINITION 2.5. For $\varphi \in F_{p,\mu}$, we define $\delta\varphi$ by $\delta\varphi = xD\varphi$, where $D \equiv d/dx$.

THEOREM 2.6. δ *is a continuous linear mapping from* $F_{p,\mu}$ *into* $F_{p,\mu}$ *and is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu}$ *if and only if* $\operatorname{Re}\mu \neq 1/p$.

Proof. See [11, pp. 25–26, Corollary 2.14].

DEFINITION 2.7. (i) Let $\operatorname{Re}(\eta + \mu) + 1 > 1/p$, and $\varphi \in F_{p,\mu}$. For $\operatorname{Re}\alpha > 0$, we define $H^{\eta,\alpha}\varphi$ by

$$(H^{\eta,\alpha}\varphi)(x) = [\Gamma(\alpha)]^{-1} x^{-\eta-1} \int_0^x [\log(x/t)]^{\alpha-1} t^{\eta}\varphi(t)\, dt, \qquad x > 0. \tag{2.5}$$

This definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by means of the formula

$$H^{\eta,\alpha}\varphi = (\eta + 1) H^{\eta,\alpha+1}\varphi + H^{\eta,\alpha+1}\delta\varphi. \tag{2.6}$$

(ii) Let $\operatorname{Re}(\eta - \mu) > -1/p$, and $\varphi \in F_{p,\mu}$. For $\operatorname{Re}\alpha > 0$, we define $G^{\eta,\alpha}\varphi$ by

$$(G^{\eta,\alpha}\varphi)(x) = [\Gamma(\alpha)]^{-1} x^{\eta} \int_x^{\infty} [\log(t/x)]^{\alpha-1} t^{-\eta-1}\varphi(t)\, dt, \qquad x > 0. \tag{2.7}$$

This definition is extended to $\operatorname{Re}\alpha \leqslant 0$ by

$$G^{\eta,\alpha}\varphi = \eta G^{\eta,\alpha+1}\varphi - G^{\eta,\alpha+1}\delta\varphi. \tag{2.8}$$

THEOREM 2.8. (i) *For* $\alpha \in \mathbb{C}$ *and* $\operatorname{Re}(\eta + \mu) + 1 > 1/p$, $H^{\eta,\alpha}$ *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu}$ *and* $(H^{\eta,\alpha})^{-1} = H^{\eta,-\alpha}$.

(ii) *For* $\alpha \in \mathbb{C}$ *and* $\operatorname{Re}(\eta - \mu) > -1/p$, $G^{\eta,\alpha}$ *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu}$ *and* $(G^{\eta,\alpha})^{-1} = G^{\eta,-\alpha}$.

Proof. This theorem can be proved directly by applying a generalisation of an inequality due to Hardy [4, Theorem 319]; see [9]. It can also be obtained as a particular case of the fractional power theory discussed in Section 4.

To conclude this section, we introduce the Erdélyi–Kober operators of fractional calculus and describe their mapping properties on $F_{p,\mu}$. For brevity, only the salient points are included; further details and proofs can be found in [11, Chap. 3]. The various results we require can be expressed concisely in terms of the following sets of complex numbers.

DEFINITION 2.9. For each $m>0$, μ and p, the sets $A_{p,\mu,m}$ and $A'_{p,\mu,m}$ are given by

$$A_{p,\mu,m}=\left\{\eta\in\mathbb{C}:\operatorname{Re}(m\eta+\mu)+m\neq\frac{1}{p}-mk\quad\text{for}\quad k=0,1,2,\ldots\right\}$$

$$A'_{p,\mu,m}=\left\{\eta\in\mathbb{C}:\operatorname{Re}(m\eta-\mu)\neq-\frac{1}{p}-mk\quad\text{for}\quad k=0,1,2,\ldots\right\}.$$

DEFINITION 2.10. (i) Let $\operatorname{Re}(m\eta+\mu)+m>1/p$, and $\varphi\in F_{p,\mu}$. For $\operatorname{Re}\alpha>0$, we define $I_m^{\eta,\alpha}\varphi$ by

$$(I_m^{\eta,\alpha}\varphi)(x)=[\Gamma(\alpha)]^{-1}\,mx^{-m\eta-m\alpha}$$
$$\times\int_0^x (x^m-t^m)^{\alpha-1}\,t^{m\eta+m-1}\varphi(t)\,dt,\qquad x>0. \tag{2.9}$$

This definition is extended to $\operatorname{Re}\alpha\leqslant 0$ by means of the formula

$$I_m^{\eta,\alpha}\varphi=(\eta+\alpha+1)\,I_m^{\eta,\alpha+1}\varphi+m^{-1}I_m^{\eta,\alpha+1}\delta\varphi. \tag{2.10}$$

(ii) Let $\operatorname{Re}(m\eta-\mu)>-1/p$ and $\varphi\in F_{p,\mu}$. For $\operatorname{Re}\alpha>0$, we define $K_m^{\eta,\alpha}\varphi$ by

$$(K_m^{\eta,\alpha}\varphi)(x)=[\Gamma(\alpha)]^{-1}\,mx^{m\eta}$$
$$\times\int_x^\infty (t^m-x^m)^{\alpha-1}\,t^{-m\eta-m\alpha+m-1}\varphi(t)\,dt,\qquad x>0. \tag{2.11}$$

This definition is extended to $\operatorname{Re}\alpha\leqslant 0$ by means of the formula

$$K_m^{\eta,\alpha}\varphi=(\eta+\alpha)\,K_m^{\eta,\alpha+1}\varphi-m^{-1}K_m^{\eta,\alpha+1}\delta\varphi. \tag{2.12}$$

(iii) Let $\eta\in A_{p,\mu,m}$, $\varphi\in F_{p,\mu}$, and α be any complex number. For $\operatorname{Re}(m\eta+\mu)+m>1/p$, we define $I_m^{\eta,\alpha}$ as in (i). Otherwise, if k is the unique positive integer such that

$$\frac{1}{p}-mk<\operatorname{Re}(m\eta+\mu)+m<\frac{1}{p}-m(k-1),$$

then

$$I_m^{\eta,\alpha}\varphi = (-1)^k I_m^{\eta+k,\alpha-k} K_m^{-\eta-k,k}\varphi, \tag{2.13}$$

where $I_m^{\eta+k,\alpha-k}$ and $K_m^{-\eta-k,k}$ are defined as in (i) and (ii), respectively.

(iv) Let $\eta \in A'_{p,\mu,m}$, $\varphi \in F_{p,\mu}$, and α be any complex number. For $\operatorname{Re}(m\eta-\mu) > -1/p$, we define $K_m^{\eta,\alpha}$ as in (ii). Otherwise, if k is the unique positive integer such that

$$-\frac{1}{p} - mk < \operatorname{Re}(m\eta - \mu) < -\frac{1}{p} - m(k-1),$$

then

$$K_m^{\eta,\alpha}\varphi = (-1)^k K_m^{\eta+k,\alpha-k} I_m^{-\eta-k,k}\varphi, \tag{2.14}$$

where $K_m^{\eta+k,\alpha-k}$ and $I_m^{-\eta-k,k}$ are defined as in (ii) and (i), respectively.

THEOREM 2.11. (i) *If $\eta \in A_{p,\mu,m}$, then $I_m^{\eta,\alpha}$ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$. If also $\eta+\alpha \in A_{p,\mu,m}$, then $I_m^{\eta,\alpha}$ is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ and*

$$(I_m^{\eta,\alpha})^{-1} = I_m^{\eta+\alpha,-\alpha}.$$

(ii) *If $\eta \in A'_{p,\mu,m}$, then $K_m^{\eta,\alpha}$ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$. If also $\eta+\alpha \in A'_{p,\mu,m}$, then $K_m^{\eta,\alpha}$ is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ and*

$$(K_m^{\eta,\alpha})^{-1} = K_m^{\eta+\alpha,-\alpha}.$$

3. FRACTIONAL POWERS OF DIFFERENTIAL OPERATORS— THE DIRECT APPROACH

In [10], formulae are given under appropriate conditions for arbitrary powers of the nth order differential operators

$$L = x^{a_1}Dx^{a_2}Dx^{a_3}\cdots x^{a_n}Dx^{a_{n+1}} \tag{3.1}$$

and

$$M = (-1)^n L. \tag{3.2}$$

In each case, the initial step in obtaining the appropriate formula for the αth power T^α of the operator T $(=L$ or $M)$ under consideration is the derivation of an expression for T^r ($r \in \mathbb{N}$) which exhibits clearly the dependence on r and continues to make sense when r assumes suitably restricted non-integral values. T^α is then defined by this expression with r replaced throughout by α. This approach, which we shall refer to as the direct approach for defining a fractional power, is standard in the theory of fractional calculus. For example, the r-fold repeated integral operator I^r, given by

$$(I^r\varphi)(x) = [\Gamma(r)]^{-1} \int_0^x (x-t)^{r-1}\, \varphi(t)\, dt \qquad (r \in \mathbb{N}),$$

leads immediately to the familiar formula

$$(I^\alpha\varphi)(x) = [\Gamma(\alpha)]^{-1} \int_0^x (x-t)^{\alpha-1}\, \varphi(t)\, dt \qquad (\operatorname{Re} \alpha > 0)$$

for the Riemann–Liouville fractional integral of order α.

If we examine the operator L, given by (3.1), and assume that

$$a = \sum_{k=1}^{n+1} a_k \in \mathbb{R} \tag{3.3}$$

and

$$m = n - a > 0, \tag{3.4}$$

then we can show that

$$L^r\varphi = m^{nr} x^{-mr} \prod_{k=1}^{n} I_m^{b_k, -r}\varphi \qquad (\varphi \in F_{p,\mu},\, r \in \mathbb{N}), \tag{3.5}$$

where, for $k = 1, 2, \ldots, n$,

$$b_k = \left(\sum_{i=k+1}^{n+1} a_i + k - n \right) \Big/ m; \tag{3.6}$$

see [10, pp. 525–526]. It follows from Theorem 2.11 that the right-hand side of (3.5) is well defined as a continuous mapping from $F_{p,\mu}$ into $F_{p,\mu-mr}$ provided that $b_k \in A_{p,\mu,m}$ for $k = 1, 2, \ldots, n$. Formula (3.5) leads immediately to the following definition.

DEFINITION 3.1. Let $m = n - a > 0$, let α be any complex number and let $b_k \in A_{p,\mu,m}$ for $k = 1, 2, \ldots, n$. Then we define L^α on $F_{p,\mu}$ by

$$L^\alpha\varphi = m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k, -\alpha}\varphi \qquad (\varphi \in F_{p,\mu}). \tag{3.7}$$

Note 3.2. (i) Formula (3.7) is obtained formally on replacing r by α in (3.5).

(ii) Since the operators $I_m^{b_k, -\alpha}$ commute [10, Theorem 2.8], the product on the right-hand side of (3.7) is unambiguous.

(iii) The case when $m = a - n > 0$ can also be dealt with. We shall not consider this here but details can be found in [10].

The various properties possessed by the Erdélyi–Kober operators can now be used to establish the following results.

THEOREM 3.3. *Let $m = n - a > 0$, let $b_k \in A_{p,\mu,m}$ for $k = 1, 2, ..., n$ and let $L^\alpha\varphi$ be defined, for $\varphi \in F_{p,\mu}$, by* (3.7).

(i) *L^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m\alpha}$ for all $\alpha \in \mathbb{C}$. If, in addition, $b_k - \alpha \in A_{p,\mu,m}$ for $k = 1, 2, ..., n$, then L^α is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-m\alpha}$ and $(L^\alpha)^{-1} = L^{-\alpha}$.*

(ii) *L^0 is the identity operator on $F_{p,\mu}$.*

(iii) *If $\{b_k - \alpha, b_k - \beta\} \subseteq A_{p,\mu,m}$ for $k = 1, 2, ..., n$, then*

$$L^\alpha L^\beta\varphi = L^{\alpha+\beta}\varphi = L^\beta L^\alpha\varphi.$$

Proof. See [10, pp. 528–529].

Similar results can also be obtained for M^α defined on $F_{p,\mu}$ by

$$M^\alpha\varphi = m^{n\alpha}x^{m\alpha} \prod_{k=1}^{n} K_m^{1-b_k, -\alpha}\varphi \qquad (\varphi \in F_{p,\mu}) \tag{3.8}$$

where, as before, $a = \sum_{k=1}^{n+1} a_k$ is real but m is now given by $a - n$. In this case, it can be shown that M^α is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$ whenever $1 - b_k \in A'_{p,\mu,m}$ $(k = 1, 2, ..., n)$. Other properties are given in [10, Theorem 4.12].

Having considered the case $m > 0$, we now prepare to consider the case $m = 0$.

4. FRACTIONAL POWERS OF OPERATORS—THE SPECTRAL APPROACH

In the sequel, X denotes a Fréchet space with topology generated by a separating family of seminorms $S = \{\nu_k\}_{k=0}^{\infty}$. We say that $A: X \to X$ is in the class $P(X)$ if the following conditions are satisfied:

(I) A is linear and continuous;

(II) $(0, \infty) \subseteq \rho(A)$ (the resolvent set of A);

(III) for each $v_k \in S$, there exists $v_l \in S$ such that

$$v_k([C\lambda R(\lambda; A)]^n x) \leqslant v_l(x) \qquad (\forall x \in X, \lambda > 0 \quad \text{and} \quad n = 1, 2, ...),$$

where C is a positive constant independent of x, λ, and n.

If $A \in P(X)$ has a continuous inverse A^{-1} on X, then it is possible to define $(-A)^\alpha$ on X, for $\alpha \in \mathbb{C}$, by means of the formulae

$$(-A)^\alpha x = -\pi^{-1} \sin(\pi\alpha) \int_0^\infty \lambda^{\alpha-1}[R(\lambda; A) - \lambda/(1+\lambda^2)] Ax\, d\lambda$$
$$- \sin(\pi\alpha/2) Ax, \qquad 0 < \operatorname{Re} \alpha < 2, \tag{4.1}$$
$$(-A)^\alpha x = (-A)^{\alpha-n} (-A)^n x, \qquad n < \operatorname{Re} \alpha < n+2, n = \pm 1, \pm 2, ..., \tag{4.2}$$

where, in (4.2), $(-A)^n = (-A^{-1})^{-n}$ when n is a negative integer. The conditions (I)–(III) listed above ensure that the integral in (4.1), interpreted as an improper Riemann integral of the form

$$\lim_{\varepsilon \to 0+,\, K \to \infty} \int_\varepsilon^K \lambda^{\alpha-1}[R(\lambda; A) - \lambda/(1+\lambda^2)] Ax\, d\lambda,$$

converges in X for each fixed $x \in X$. The main properties of the operators $(-A)^\alpha$ are given below.

THEOREM 4.1. *Let $A \in P(X)$ have a continuous inverse A^{-1} on X and let $(-A)^\alpha$ be defined by* (4.1) *and* (4.2). *Then*

(i) *$(-A)^\alpha$ is a homeomorphism from X onto X for each $\alpha \in \mathbb{C}$ and $[(-A)^\alpha]^{-1} = (-A)^{-\alpha}$;*

(ii) *$(-A)^\alpha x$ is a strongly holomorphic X-valued function of α in $\mathbb{C}$ for each $x \in X$;*

(iii) $(-A)^\alpha x = \pi^{-1} \sin \pi(\alpha + 1 - n) \int_0^\infty \lambda^{\alpha-n} R(\lambda; A)(-A)^n x\, d\lambda$, $n-1 < \operatorname{Re} \alpha < n$;

(iv) $A^{-1} \in P(X)$ *and* $(-A^{-1})^\alpha x = (-A)^{-\alpha} x$, $\forall \alpha \in \mathbb{C}$, $x \in X$;

(v) $(-A)^\alpha (-A)^\beta x = (-A)^{\alpha+\beta} x$, $\forall \alpha, \beta \in \mathbb{C}$, $x \in X$;

(vi) $[(-A)^\alpha]^\beta x = (-A)^{\alpha\beta} x$, $\forall \beta \in \mathbb{C}$, $-1 \leqslant \alpha \leqslant 1$ *and* $x \in X$;

(vii) *$(-A)^\alpha$ is equal to the nth iterate of $-A$ (respectively $-A^{-1}$) when $\alpha = n$ (respectively $\alpha = -n$), where n is a positive integer.*

Proof. See [7].

As an illustration of this approach for defining fractional powers of operators, let $X = F_{p,\mu}$ and consider the Erdélyi–Kober operator $I_1^{\eta, 1}$, When $\mathrm{Re}(\eta+\mu)+1>1/p$, it follows from Theorem 2.11(i) and (2.10) that $I_1^{\eta,1}$ is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ and has inverse given by

$$(I_1^{\eta,1})^{-1}=I_1^{\eta+1,-1}=x^{-\eta-1}\delta x^{\eta+1}=\eta+1+\delta,$$

where $\delta=xd/dx$. Furthermore, since

$$\lambda I+I_1^{\eta,1}=x^{-\eta-1}(\lambda I+\delta^{-1})\,x^{\eta+1},$$

the spectrum of $-I_1^{\eta,1}$ on $F_{p,\mu}$ is identical to the spectrum of $-\delta^{-1}$ on $F_{p,\mu+\eta+1}$. Bearing this in mind, we consider the equation

$$(\lambda I+\delta^{-1})\,\chi=\psi \qquad (\chi,\psi\in F_{p,\mu+\eta+1}). \tag{4.3}$$

Applying δ to each side of (4.3) produces

$$(\delta+\lambda^{-1})\,\chi=\lambda^{-1}\delta\psi.$$

Since this can be written as

$$x^{-1/\lambda}\delta x^{1/\lambda}\chi=\lambda^{-1}\delta\psi,$$

we deduce, from Theorems 2.4 and 2.6, that

$$\begin{aligned}\chi&=\lambda^{-1}x^{-1/\lambda}\delta^{-1}x^{1/\lambda}\delta\psi\\&=\lambda^{-1}\psi-\lambda^{-2}x^{-1/\lambda}\delta^{-1}x^{1/\lambda}\psi\end{aligned}$$

provided that $\mathrm{Re}(\mu+\eta+1+1/\lambda)\neq 1/p$. In particular, when $\mathrm{Re}(\mu+\eta+1+1/\lambda)>1/p$ we obtain

$$\chi=\lambda^{-1}\psi-\lambda^{-2}I_1^{1/\lambda-1,1}\psi$$

and consequently, for $\lambda>0$, $\mathrm{Re}(\eta+\mu)+1>1/p$ and $\varphi\in F_{p,\mu}$,

$$R(\lambda;-I_1^{\eta,1})\,\varphi=\lambda^{-1}\varphi-\lambda^{-2}I_1^{\eta+1/\lambda,1}\varphi \tag{4.4}$$

and

$$\begin{aligned}\gamma_k^{p,\mu}(\lambda R(\lambda;-I_1^{\eta,1})\,\varphi)&\leqslant\|I-\lambda^{-1}I_1^{\eta+1/\lambda,1}\|\,\gamma_k^{p,\mu}(\varphi)\\&\leqslant[1+(1+\lambda[\mathrm{Re}(\eta+\mu)+1-1/p])^{-1}]\,\gamma_k^{p,\mu}(\varphi)\\&<2\gamma_k^{p,\mu}(\varphi).\end{aligned}$$

The norm which appears in the above inequlity is the operator norm of $I-\lambda^{-1}I_1^{\eta+1/\lambda,1}$, regarded as an operator on the Banach space L_μ^p, where the latter is defined as in [10, p. 533]. The upper bound on this norm is easily obtained from the inequality of Hardy [4, Theorem 319] referred to earlier.

It follows that $-I_1^{\eta,1}$ is an invertible operator in $P(F_{p,\mu})$ for $\operatorname{Re}(\eta+\mu)+1>1/p$ and therefore, from Theorem 4.1, $(I_1^{\eta,1})^\alpha$ exists for any $\alpha\in\mathbb{C}$. In particular, for $\operatorname{Re}\alpha\in(0,1)$, we can use Theorem 4.1(iii) and the continuity of the translated delta distribution δ_x on $F_{p,\mu}$ (where $(\delta_x,\varphi)=\varphi(x)$, $\varphi\in F_{p,\mu}$, $x>0$) to obtain

$$\begin{aligned}
&(\delta_x, (I_1^{\eta,1})^\alpha \varphi)\\
&\quad=\pi^{-1}\sin(\pi\alpha)\int_0^\infty \lambda^{\alpha-1}R(\lambda;\,-I_1^{\eta,1})\,I_1^{\eta,1}\varphi(x)\,d\lambda\\
&\quad=\pi^{-1}\sin(\pi\alpha)\int_0^\infty \lambda^{\alpha-1}(I-\lambda R(\lambda;\,-I_1^{\eta,1}))\,\varphi(x)\,d\lambda\\
&\quad=\pi^{-1}\sin(\pi\alpha)\int_0^\infty \lambda^{\alpha-2}(I_1^{\eta+1/\lambda,1}\varphi)(x)\,d\lambda && \text{(from (4.4))}\\
&\quad=\pi^{-1}\sin(\pi\alpha)\int_0^\infty \lambda^{\alpha-2}x^{-\eta-1-1/\lambda}\int_0^x t^{\eta+1/\lambda}\varphi(t)\,dt\,d\lambda && \text{(from (2.9)).}
\end{aligned}\tag{4.5}$$

On applying Fubini's theorem to interchange the order of integration, (4.5) becomes

$$\begin{aligned}
&\pi^{-1}\sin(\pi\alpha)\,x^{-\eta-1}\int_0^x t^\eta\varphi(t)\int_0^\infty \lambda^{\alpha-2}\exp(-\lambda^{-1}\log(x/t))\,d\lambda\,dt\\
&\quad=\pi^{-1}\sin(\pi\alpha)\,x^{-\eta-1}\int_0^x [\log(x/t)]^{\alpha-1}\,t^\eta\varphi(t)\int_0^\infty u^{-\alpha}e^{-u}\,du\,dt\\
&\qquad\qquad\qquad\text{(where } u=\lambda^{-1}\log(x/t))\\
&\quad=[\Gamma(\alpha)]^{-1}\,x^{-\eta-1}\int_0^x [\log(x/t)]^{\alpha-1}\,t^\eta\varphi(t)\,dt\\
&\quad=(H^{\eta,\alpha}\varphi)(x),
\end{aligned}$$

where $H^{\eta,\alpha}\varphi$ is defined by (2.5) and (2.6). Thus

$$(I_1^{\eta,1})^\alpha\,\varphi=H^{\eta,\alpha}\varphi \tag{4.6}$$

for $\varphi\in F_{p,\mu}$, $0<\operatorname{Re}\alpha<1$ and $\operatorname{Re}(\eta+\mu)+1>1/p$, and, since each side of (4.6) is a strongly holomorphic function of α in $\mathbb{C}$, it follows that this identity is valid for all complex values of α.

In a similar manner, it is possible to show that

$$(K_1^{\eta,1})^\alpha\,\varphi=G^{\eta,\alpha}\varphi \tag{4.7}$$

for $\varphi\in F_{p,\mu}$, $\alpha\in\mathbb{C}$, and $\operatorname{Re}(\eta-\mu)>-1/p$.

Formulae (4.6) and (4.7) enable us to define fractional powers of the differential operators L and M, given by (3.1) and (3.2), respectively, when $a=n=1$. To see this, consider the operator L with $n=1$, $a_1=1-\eta$, and $a_2=\eta$. If we denote this operator by T_η, then, for $\varphi \in F_{p,\mu}$,

$$T_\eta \varphi = x^{-\eta} \delta x^\eta \varphi \tag{4.8}$$

and, from Theorem 2.11(i) and Definition 2.10(i),

$$T_\eta \varphi = (I_1^{\eta-1,1})^{-1} \varphi$$

provided that $\mathrm{Re}(\eta+\mu) > 1/p$. Consequently, from the above analysis and Theorem 4.1(i), (iv), we can state that, for $\mathrm{Re}(\eta+\mu)>1/p$, $-T_\eta \in P(F_{p,\mu})$ and $(T_\eta)^\alpha$ exists as a homeomorphism on $F_{p,\mu}$ for all $\alpha \in \mathbb{C}$, with

$$(T_\eta)^\alpha \varphi = [(I_1^{\eta-1,1})^{-1}]^\alpha \varphi = (I_1^{\eta-1,1})^{-\alpha} \varphi = H^{\eta-1,-\alpha} \varphi \;(\varphi \in F_{p,\mu}). \tag{4.9}$$

Similarly, for $\mathrm{Re}(\eta+\mu)<1/p$, $\alpha \in \mathbb{C}$, and $\varphi \in F_{p,\mu}$,

$$(-T_\eta)^\alpha \varphi = [(K_1^{-\eta,1})^{-1}]^\alpha \varphi = G^{-\eta,-\alpha} \varphi. \tag{4.10}$$

It should be noted that the theory outlined in Section 3 cannot be applied to define powers of T_η and $-T_\eta$ since, in each case, $m=0$. Thus the spectral approach enables us to deal with the first-order versions of L and M which fall outside the range of applicability of the direct approach. To cater for the higher order exceptional cases (i.e., $a=n\geqslant 2$) of L and M, the spectral approach must first be modified in such a way that arbitrary powers of products of operators of the form T_η can be defined.

5. Fractional Powers of Products of Operators

We now consider operators T which can be expressed as

$$T=(-1)^N A_1 A_2 \cdots A_N = \prod_{k=1}^{N} (-A_k), \tag{5.1}$$

where N is a positive integer which depends upon T and

(1) A_k is a homeomorphism from X onto X for $k=1, 2, ..., N$;

(2) $A_k \in P(X)$ for $k=1, 2, ..., N$;

(3) $A_k A_l x = A_l A_k x$ for $k, l = 1, 2, ..., N$ and $x \in X$.

The collection of all operators T of this type will be denoted by $P_M(X)$. From Theorem 4.1, fractional powers $(-A_k)^\alpha$ can be defined as homeomorphisms on X for all $\alpha \in \mathbb{C}$ and $k = 1, 2, ..., N$, and therefore the obvious definition for T^α, when $T \in P_M(X)$, is

$$T^\alpha x = (-A_1)^\alpha (-A_2)^\alpha \cdots (-A_N)^\alpha x \qquad (x \in X, \alpha \in \mathbb{C}).$$

To enable us to verify that this formula leads to the usual properties associated with powers, we require the following preliminary results.

LEMMA 5.1. *Let $x \in X$ and let $A_1, A_2 \in P(X)$ be homeomorphisms from X onto X. Then*

(i) *for fixed α, $(-A_1)^\alpha (-A_2)^\beta x$ is strongly holomorphic with respect to β in $\mathbb{C}$;*

(ii) *for fixed β, $(-A_1)^\alpha (-A_2)^\beta x$ is strongly holomorphic with respect to α in $\mathbb{C}$;*

(iii) *$(-A_1)^\alpha (-A_2)^\alpha x$ is strongly holomorphic with respect to α in $\mathbb{C}$.*

Proof. Since (ii) follows from Theorem 4.1(ii), and (i) can be deduced from (iii), it is sufficient to prove (iii). To simplify the proof, we assume that X can be equipped with an equivalent topology generated by a countable collection of norms. Although this restriction is not strictly necessary (see Remark 5.2 below), it is satisfied by the spaces $F_{p,\mu}$ (see Remark 2.2) and, moreover, it allows us to apply standard results on countably normed spaces. In particular, from [3, p. 75], we can deduce that $(-A_1)^\alpha (-A_2)^\alpha x$ is weak* holomorphic for $\alpha \in \mathbb{C}$ and therefore, since weak* holomorphy implies strong holomorphy in a Fréchet space X [13, p. 79], the result follows.

Remark 5.2. Lemma 5.1(iii) is still valid when X is not topologically equivalent to a countably normed space. The proof, however, is more complicated and relies on the fact that for each $v_k \in S$, there exist seminorms $v_{k(i)}$ $(i = 1, 2, ..., m)$ in S and a positive constant $M(k)$ for which

$$v_k((-A)^\alpha x) \leqslant \left\{ \frac{1}{2} \left| \frac{\sin[\pi(\alpha - n)]}{\sin[\pi \operatorname{Re}(\alpha - n)/2]} \right| + |\sin[\pi(\alpha - n)/2]| \right\}$$
$$\times M(k) \max\{v_{k(i)}(x), i = 1, 2, ..., m\}$$

where $x \in X$ and $n < \operatorname{Re} \alpha < n + 2$, $n \in \mathbb{N}$. Further details are omitted.

LEMMA 5.3. *Let $A_1, A_2 \in P(X)$ be homeomorphisms from X onto X and let $A_1 A_2 = A_2 A_1$. Then*

$$(-A_1)^\alpha (-A_2)^\beta x = (-A_2)^\beta (-A_1)^\alpha x, \qquad \forall \alpha, \beta \in \mathbb{C}, x \in X. \tag{5.2}$$

Proof. Suppose initially that $0 < \operatorname{Re} \alpha$, $\operatorname{Re} \beta < 1$. Then, from Theorem 4.1(iii),

$$(-A_1)^{\alpha}(-A_2)^{\beta}x$$
$$= \pi^{-2}\sin(\pi\alpha)\sin(\pi\beta)\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}R(\lambda; A_1)\,A_1R(\mu; A_2)\,A_2x\,d\mu\,d\lambda$$
$$= \pi^{-2}\sin(\pi\alpha)\sin(\pi\beta)\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}R(\mu; A_2)\,A_2R(\lambda; A_1)\,A_1x\,d\mu\,d\lambda$$
$$(x \in X), \qquad (5.3)$$

where the last step follows from the commutativity of the operators and hence of their resolvents. Interchanging the order of integration in (5.3) produces the required result. This can be justified in the following manner. Let f be any continuous linear functional on X and consider

$$\left(f, \int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}R(\mu; A_2)\,A_2R(\lambda; A_1)\,A_1x\,d\mu\,d\lambda\right)$$
$$= \int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}(f, R(\mu; A_2)\,A_2R(\lambda; A_1)\,A_1x)\,d\mu\,d\lambda. \qquad (5.4)$$

Since f, A_1, and A_2 are continuous on X and $A_1, A_2 \in P(X)$, there exist seminorms $v_{k(i)}$, $i = 1, 2, ..., m$, and a positive constant M such that the integral in (5.4) is bounded above by $MB\max\{v_{k(i)}(x),\ i = 1, 2, ..., m\}$, where

$$B = \int_0^1\int_0^1 \lambda^{\alpha-1}\mu^{\beta-1}\,d\mu\,d\lambda + \int_1^{\infty}\int_0^1 \lambda^{\alpha-2}\mu^{\beta-1}\,d\mu\,d\lambda$$
$$+ \int_0^1\int_1^{\infty}\lambda^{\alpha-1}\mu^{\beta-2}\,d\mu\,d\lambda + \int_1^{\infty}\int_1^{\infty}\lambda^{\alpha-2}\mu^{\beta-2}\,d\mu\,d\lambda < \infty.$$

Consequently, Fubini's theorem can be applied to rewrite (5.4) as

$$\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}(f, R(\mu; A_2)\,A_2R(\lambda; A_1)\,A_1x)\,d\lambda\,d\mu$$
$$= \left(f, \int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}R(\mu; A_2)\,A_2R(\lambda; A_1)\,A_1x\,d\lambda\,d\mu\right),$$

and since the dual space X' separates points in X, we conclude that (5.3) is equal to

$$\pi^{-2}\sin(\pi\alpha)\sin(\pi\beta)\int_0^{\infty}\int_0^{\infty}\lambda^{\alpha-1}\mu^{\beta-1}R(\mu; A_2)\,A_2R(\lambda; A_1)\,A_1x\,d\lambda\,d\mu.$$

Finally, the result for all complex values of α and β follows from Lemma 5.1(i), (ii).

DEFINITION 5.4. Let $T \in P_M(X)$ with

$$T = (-1)^N A_1 A_2 \cdots A_N = \prod_{k=1}^{N} (-A_k). \tag{5.5}$$

Then, for any $\alpha \in \mathbb{C}$, we define T^α on X by

$$T^\alpha x = (-A_1)^\alpha (-A_2)^\alpha \cdots (-A_N)^\alpha x \qquad (x \in X), \tag{5.6}$$

where each $(-A_k)^\alpha$ is defined via (4.1) and (4.2).

Note 5.5. It follows from Lemma 5.3, with $\alpha = \beta$, that the definition of T^α given above is independent of the order of the factors in T.

THEOREM 5.6. *Let $T \in P_M(X)$ and let T^α be given by* (5.6). *Then*

(i) *T^α is a homeomorphism from X onto X for each $\alpha \in \mathbb{C}$ and $(T^\alpha)^{-1} = T^{-\alpha}$;*

(ii) *$T^\alpha x$ is a strongly holomorphic X-valued function of α in $\mathbb{C}$ for each $x \in X$;*

(iii) *$T^{-1} \in P_M(X)$ and $(T^{-1})^\alpha x = T^{-\alpha} x$, $\forall \alpha \in \mathbb{C}$, $x \in X$;*

(iv) *$T^\alpha T^\beta x = T^{\alpha+\beta} x$, $\forall \alpha, \beta \in \mathbb{C}$, $x \in X$;*

(v) *$(T^\alpha)^\beta x = T^{\alpha\beta} x$, $\forall \beta \in \mathbb{C}$, $-1 \leqslant \alpha \leqslant 1$ and $x \in X$;*

(vi) *T^α is equal to the nth iterate of T (respectively T^{-1}) when $\alpha = n$ (respectively $\alpha = -n$), $n \in \mathbb{N}$.*

Proof. (i), (iv), and (vi) follow immediately from Theorem 4.1 and Lemma 5.3.

(ii) The case when $N = 2$ has already been proved in Lemma 5.1(iii). The general case follows by induction.

(iii) Since $T^{-1} = (-A_N^{-1})(-A_{N-1}^{-1}) \cdots (-A_1^{-1})$, where each $A_k^{-1} \in P(X)$ (by Theorem 4.1(iv)), is invertible, and $A_k^{-1} A_l^{-1} = A_l^{-1} A_k^{-1}$ $(k, l = 1, 2, ..., N)$, it follows that $T^{-1} \in P_M(X)$ and

$$\begin{aligned}
(T^{-1})^\alpha x &= (-A_N^{-1})^\alpha (-A_{N-1}^{-1})^\alpha \cdots (-A_1^{-1})^\alpha x \qquad (x \in X, \alpha \in \mathbb{C}) \\
&= (-A_N)^{-\alpha} (-A_{N-1})^{-\alpha} \cdots (-A_1)^{-\alpha} x \qquad (= T^{-\alpha} x) \\
&\qquad\qquad \text{(by Theorem 4.1(iv) and Lemma 5.3)} \\
&= [(-A_1)^\alpha (-A_2)^\alpha \cdots (-A_N)^\alpha]^{-1} x \qquad \text{(by Theorem 4.1(i))} \\
&= (T^\alpha)^{-1} x.
\end{aligned}$$

(v) By definition, $T^{\alpha}=(-A_1)^{\alpha}(-A_2)^{\alpha}\cdots(-A_N)^{\alpha}=(-1)^N B_1 B_2\cdots B_N$, where $B_k=-(-A_k)^{\alpha}$ $(k=1, 2, ..., N)$. Now, for $-1\leqslant\alpha\leqslant 1$, each operator B_k is a homeomorphism in $P(X)$ and $B_k B_l x=B_l B_k x$ for $k, l=1, 2, ..., N$, and $x\in X$ (by Lemma 5.3). Consequently, $T^{\alpha}\in P_M(X)$ for $-1\leqslant\alpha\leqslant 1$ and

$$\begin{aligned}(T^{\alpha})^{\beta}x&=(-B_1)^{\beta}(-B_2)^{\beta}\cdots(-B_N)^{\beta}x && (x\in X, \beta\in\mathbb{C})\\ &=[(-A_1)^{\alpha}]^{\beta}[(-A_2)^{\alpha}]^{\beta}\cdots[(-A_N)^{\alpha}]^{\beta}x\\ &=(-A_1)^{\alpha\beta}(-A_2)^{\alpha\beta}\cdots(-A_N)^{\alpha\beta}x && \text{(by Theorem 4.1(vi))}\\ &=T^{\alpha\beta}x.\end{aligned}$$

This completes the proof.

If we now re-examine the operator L, given by (3.1), and assume that $a=n$, then

$$L\varphi=\prod_{k=1}^{n}(-A_k)\varphi \qquad (\varphi\in F_{p,\mu}),$$

where $-A_k=x^{-\eta_k}\delta x^{\eta_k}=T_{\eta_k}$, with $\eta_k=k-\sum_{i=1}^{k}a_i$ $(k=1, 2, ..., n)$. Each operator A_k is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ provided that $\operatorname{Re}(\eta_k+\mu)\neq 1/p$ and, for $\operatorname{Re}(\eta_k+\mu)>1/p$, is in the class $P(F_{p,\mu})$. Also, it can easily be verified that $A_k A_l=A_l A_k$ on $F_{p,\mu}$ for $k, l=1, 2, ..., n$. Consequently, L^{α} exists as a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ if $\operatorname{Re}(\eta_k+\mu)>1/p$ for $k=1, 2, ..., n$, and is given by

$$\begin{aligned}L^{\alpha}\varphi&=(T_{\eta_1})^{\alpha}(T_{\eta_2})^{\alpha}\cdots(T_{\eta_n})^{\alpha} && (\varphi\in F_{p,\mu})\\ &=H^{\eta_1-1,-\alpha}H^{\eta_2-1,-\alpha}\cdots H^{\eta_n-1,-\alpha}\varphi && \text{(by (4.9))}.\end{aligned}\tag{5.7}$$

Applying the results of Theorem 5.6, we can also state that, when $\operatorname{Re}(\eta_k+\mu)>1/p$ $(k=1, 2, ..., n)$ and $\varphi\in F_{p,\mu}$,

$$\begin{aligned}(L^{\alpha})^{-1}\varphi&=L^{-\alpha}\varphi\\ &=(I_1^{\eta_1-1,1})^{\alpha}(I_1^{\eta_2-1,1})^{\alpha}\cdots(I_1^{\eta_n-1,1})^{\alpha}\varphi\\ &=H^{\eta_1-1,\alpha}H^{\eta_2-1,\alpha}\cdots H^{\eta_n-1,\alpha}\varphi && (\forall\alpha\in\mathbb{C});\end{aligned}\tag{5.8}$$

$$L^{\alpha}L^{\beta}\varphi=L^{\alpha+\beta}\varphi \qquad (\forall\alpha, \beta\in\mathbb{C});\tag{5.9}$$

$$(L^{\alpha})^{\beta}\varphi=L^{\alpha\beta}\varphi \qquad (\forall\beta\in\mathbb{C}, -1\leqslant\alpha\leqslant 1).\tag{5.10}$$

The operator M can be handled in a similar manner when $a=n$. In this case an appropriate formula is

$$M^{\alpha}\varphi=G^{-\eta_1,-\alpha}G^{-\eta_2,-\alpha}\cdots G^{-\eta_n,-\alpha}\varphi$$
$$(\varphi\in F_{p,\mu}, \operatorname{Re}(\eta_k+\mu)<1/p, k=1, 2, ..., n).$$

6. Connection between the Spectral and Direct Approaches

The methods described in the preceding sections enable us to define L^α on $F_{p,\mu}$ by the formulae

$$L^\alpha\varphi = m^{n\alpha}x^{-m\alpha}\prod_{k=1}^{n} I_m^{b_k,-\alpha}\varphi \qquad \left(m=n-a>0,\, a=\sum_{k=1}^{n+1} a_k,\, \varphi\in F_{p,\mu}\right) \tag{6.1}$$

$$L^\alpha\varphi = \prod_{k=1}^{n} H^{\eta_k-1,-\alpha}\varphi \qquad (a=n,\, \varphi\in F_{p,\mu}), \tag{6.2}$$

where $b_k=(\sum_{i=k+1}^{n+1} a_i+k-n)/m$ and $\eta_k=k-\sum_{i=1}^{k} a_i=mb_k$ $(k=1,2,...,n)$. Our aim now is to demonstrate that formula (6.2) emerges when m is allowed to converge to 0 in (6.1).

We examine initially the simplest case of a first-order differential operator given by

$$L_\varepsilon\varphi = x^{-\eta-\varepsilon}\delta x^\eta\varphi \qquad (\varphi\in F_{p,\mu},\, \varepsilon\geqslant 0). \tag{6.3}$$

When $\varepsilon=0$, $L_\varepsilon\varphi=T_\eta\varphi$, where $T_\eta\varphi$ is defined by (4.8), and therefore, on applying the spectral approach, $(L_0)^\alpha$ can be defined as a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ by

$$(L_0)^\alpha\,\varphi = H^{\eta-1,-\alpha}\varphi \qquad (\varphi\in F_{p,\mu},\, \mathrm{Re}(\eta+\mu)>1/p,\, \alpha\in\mathbb{C}). \tag{6.4}$$

On the other hand, if $\varepsilon>0$ then $m=n-a=1-(1-\varepsilon)=\varepsilon$ and the direct approach produces

$$(L_\varepsilon)^\alpha\,\varphi = \varepsilon^\alpha x^{-\varepsilon\alpha} I_\varepsilon^{b_1,-\alpha}\varphi \qquad (\varphi\in F_{p,\mu},\, b_1\in A_{p,\mu,\varepsilon},\, \alpha\in\mathbb{C},\, \varepsilon>0), \tag{6.5}$$

where $b_1=\eta/\varepsilon$. If $\mathrm{Re}(\eta+\mu)>1/p$ then $\eta/\varepsilon\in A_{p,\mu,\varepsilon}$ for all $\varepsilon>0$, and, for Re $\alpha<0$, we can write

$$[(L_0)^\alpha\,\varphi](x) = [\Gamma(-\alpha)]^{-1}\int_0^1 [\log(1/u)]^{-\alpha-1}\,u^{\eta-1}\varphi(xu)\,du \qquad (x>0) \tag{6.6}$$

$$[(L_\varepsilon)^\alpha\,\varphi](x) = [\Gamma(-\alpha)]^{-1}\,x^{-\varepsilon\alpha}\int_0^1 [(1-u^\varepsilon)/\varepsilon]^{-\alpha-1}\,u^{\eta+\varepsilon-1}\varphi(xu)\,du \qquad (x>0). \tag{6.7}$$

Now

$$\begin{aligned}&|u^{\eta+\varepsilon-1}[(1-u^\varepsilon)/\varepsilon]^{-\alpha-1}\,\varphi(xu)|\\ &\quad\leqslant Mx^{\mathrm{Re}\,\mu-1/p}u^{\mathrm{Re}(\eta+\mu)+\varepsilon-1-1/p}\,|[(1-u^\varepsilon)/\varepsilon]^{-\alpha-1}|\\ &\qquad(\forall u,\, \varepsilon\in(0,1),\, x>0)\ \text{(by Lemma 2.3)},\end{aligned} \tag{6.8}$$

and, since $(1-u^{\varepsilon})/\varepsilon$, regarded as a function of ε, increases as $\varepsilon \to 0+$, it follows that the quantity in (6.8) is bounded above by

$$Mx^{\operatorname{Re}\mu - 1/p} g(u),$$

where

$$g(u)=\begin{cases} u^{\operatorname{Re}(\eta+\mu)-1-1/p}[\log(1/u)]^{-\operatorname{Re}\alpha-1}, & \operatorname{Re}\alpha \leqslant -1,\ 0<u<1 \\ u^{\operatorname{Re}(\eta+\mu)-1-1/p}(1-u)^{-\operatorname{Re}\alpha-1}, & -1<\operatorname{Re}\alpha<0,\ 0<u<1. \end{cases}$$

Since g is integrable over $(0, 1)$, Lebesgue's dominated convergence theorem applies and, on allowing $\varepsilon \to 0+$, we find that

$$\begin{aligned}[(L_{\varepsilon})^{\alpha} \varphi](x) &\to [\Gamma(-\alpha)]^{-1} \int_0^1 [\log(1/u)]^{-\alpha-1} u^{\eta-1}\varphi(xu)\, du \\ &= [(L_0)^{\alpha} \varphi](x) \qquad (\operatorname{Re}\alpha<0), \end{aligned} \tag{6.9}$$

for fixed $\varphi \in F_{p,\mu}$ and fixed $x>0$.

When $0 \leqslant \operatorname{Re}\alpha < r$, $r \in \mathbb{N}$, and ε is small enough to ensure that $\operatorname{Re}(\eta+\mu) > 1/p + (r-1)\,\varepsilon$, we can use Theorem 3.3(iii) to write

$$(L_{\varepsilon})^{\alpha} \varphi = (L_{\varepsilon})^{\alpha-r} (L_{\varepsilon})^{r} \varphi, \tag{6.10}$$

where

$$\begin{aligned}[(L_{\varepsilon})^{r} \varphi](x) &= [(\delta+\eta+\varepsilon)(\delta+\eta+2\varepsilon)\cdots(\delta+\eta+r\varepsilon)\, x^{-r\varepsilon}\varphi](x) \\ &\to [(\delta+\eta)^{r} \varphi](x) \quad \text{as} \quad \varepsilon \to 0+ \ (\varphi \in F_{p,\mu},\ x>0). \end{aligned} \tag{6.11}$$

Consequently, from (6.4), (6.9)–(6.11), and (2.6), we deduce that

$$\begin{aligned}[(L_{\varepsilon})^{\alpha} \varphi](x) &\to [(L_0)^{\alpha-r} (\delta+\eta)^{r} \varphi](x) \\ &= (H^{\eta-1,\,-\alpha}\varphi)(x) \qquad (\varphi \in F_{p,\mu},\ x>0), \end{aligned}$$

as $\varepsilon \to 0+$. The passage of the limit under the integral sign can be justified by making slight adjustments to the argument used above. Since $r \in \mathbb{N}$ is arbitrary, it follows that, when $\operatorname{Re}(\eta+\mu) > 1/p$,

$$[(L_{\varepsilon})^{\alpha} \varphi](x) \to [(L_0)^{\alpha} \varphi](x) \qquad (\varphi \in F_{p,\mu},\ x>0) \tag{6.12}$$

as $\varepsilon \to 0+$, for all complex values of α.

The nth order case can be dealt with in a similar fashion by considering the operator

$$L_{\varepsilon} = \prod_{k=1}^{n} x^{-\eta_k - \varepsilon/n} \delta x^{\eta_k} \qquad (\varepsilon>0)$$

for which $a=n-\varepsilon$, $m=\varepsilon$, and $b_k=(\eta_k-(n-k)\,\varepsilon/n)/\varepsilon$ $(k=1, 2, ..., n)$. If we assume that $\operatorname{Re}(\eta_k+\mu)>1/p$ for $k=1, 2, ..., n$, then $b_k \in A_{p,\mu,\varepsilon}$ for each k and the direct and spectral approaches lead to

$$(L_\varepsilon)^\alpha\,\varphi=\varepsilon^{n\alpha}x^{-\varepsilon\alpha}\prod_{k=1}^{n} I_\varepsilon^{b_k,\,-\alpha}\varphi \qquad (\varphi\in F_{p,\mu},\ \varepsilon>0) \tag{6.13}$$

$$(L_0)^\alpha\,\varphi=\prod_{k=1}^{n} H^{\eta_k-1,\,-\alpha}\varphi \qquad (\varphi\in F_{p,\mu}). \tag{6.14}$$

When $\operatorname{Re}\alpha<0$, (6.13) can be written as

$$[(L_\varepsilon)^\alpha\,\varphi](x)=[\Gamma(-\alpha)]^{-n}\,x^{-\varepsilon\alpha}\int_0^1\cdots\int_0^1\left(\prod_{k=1}^{n}[(1-u_k^\varepsilon)/\varepsilon]^{-\alpha-1}\,u_k^{\varepsilon b_k+\varepsilon-1}\right)$$
$$\times\,\varphi(xu_1u_2\cdots u_n)\,du_n\cdots du_1,$$

where $\varepsilon b_k+\varepsilon-1\to\eta_k-1$ as $\varepsilon\to 0+$ for each k. Lebesgue's dominated convergence theorem can once again be applied and, on letting $\varepsilon\to 0+$, we find that $[(L_\varepsilon^\alpha)\,\varphi](x)$ converges to

$$[\Gamma(-\alpha)]^{-n}\int_0^1\cdots\int_0^1\left(\prod_{k=1}^{n}[\log(1/u_k)]^{-\alpha-1}\,u_k^{\eta_k-1}\right)\varphi(xu_1\cdots u_n)\,du_n\cdots du_1$$
$$=(H^{\eta_1-1,\,-\alpha}H^{\eta_2-1,\,-\alpha}\cdots H^{\eta_n-1,\,-\alpha}\varphi)(x)$$
$$=[(L_0)^\alpha\,\varphi](x),$$

for each fixed $\varphi\in F_{p,\mu}$ and fixed $x>0$. The extension of this result to $\operatorname{Re}\alpha<r$ follows as above by first writing $(L_\varepsilon)^\alpha$ as $(L_\varepsilon)^{\alpha-r}\,(L_\varepsilon)^r$ and then establishing that $[(L_\varepsilon)^{\alpha-r}\,(L_\varepsilon)^r\,\varphi](x)\to[(L_0)^{\alpha-r}\,(L_0)^r\,\varphi](x)$.

It should be emphasised that the various limits considered above are all pointwise limits, each being valid for fixed $\varphi\in F_{p,\mu}$ and $x>0$. If we look for stronger versions of these limits with, for example, pointwise convergence replaced by convergence in $F_{p,\mu}$, then problems arise because the range of $(L_\varepsilon)^\alpha$ varies with ε.

It is also of interest to note that the results of this section emerge naturally from the fractional power theory for Mellin multiplier operators, mentioned in Section 1. If we define the Mellin transform $\mathcal{M}\varphi$ of a function $\varphi\in F_{p,\mu}$ $(1\leqslant p\leqslant 2)$ by

$$\mathcal{M}\varphi(s)=\int_0^\infty x^{s-1}\varphi(x)\,dx \qquad (\operatorname{Re}s=1/p-\operatorname{Re}\mu),$$

and, for simplicity, examine the first-order operator L_ε given by (6.3), then we can show that

$$\mathcal{M}[(L_\varepsilon)^\alpha\,\varphi](s+\varepsilon\alpha)=\varepsilon^\alpha\,\frac{\Gamma([\eta-s]/\varepsilon+1)}{\Gamma([\eta-s]/\varepsilon+1-\alpha)}\,\mathcal{M}\varphi(s) \tag{6.15}$$

and

$$\mathcal{M}[(L_0)^\alpha \varphi](s) = (\eta - s)^\alpha \mathcal{M}\varphi(s). \tag{6.16}$$

Since $\Gamma(z+\alpha)/\Gamma(z+\beta) = z^{\alpha-\beta}[1 + O(z^{-1})]$ ($|\arg z| < \pi - \tau, 0 < \tau < \pi$) as $z \to \infty$ (see [2, p. 47, (4)]), it follows that

$$\varepsilon^\alpha \frac{\Gamma([\eta - s]/\varepsilon + 1)}{\Gamma([\eta - s]/\varepsilon + 1 - \alpha)} = \varepsilon^\alpha([\eta - s]/\varepsilon)^\alpha [1 + O(\varepsilon/[\eta - s])] \qquad \text{(for small } \varepsilon)$$

$$\to (\eta - s)^\alpha \quad \text{as} \quad \varepsilon \to 0+. \tag{6.17}$$

From (6.15)–(6.17), we are led formally to the conclusion that $(L_\varepsilon)^\alpha$ converges, in some sense, to $(L_0)^\alpha$ as $\varepsilon \to 0+$.

Finally, we remark that analogous results and comments hold for the operator M given by (3.2). For example, if

$$M_\varepsilon \varphi = -x^{-\eta+\varepsilon}\delta x^\eta \varphi \qquad (\varphi \in F_{p,\mu}, \varepsilon \geqslant 0, \operatorname{Re}(\eta + \mu) < 1/p),$$

then

$$[(M_\varepsilon)^\alpha \varphi](x) \to [(M_0)^\alpha \varphi](x)$$
$$= [(-T_\eta)^\alpha \varphi](x) = (G^{-\eta, -\alpha}\varphi)(x) \qquad \text{(from (4.10))},$$

as $\varepsilon \to 0+$, for all complex values of α.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium

Fractional Integrals and Semigroups

Paper 14

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Introduction

In this paper we shall survey some of the connections between operators of fractional integration and semigroups of operators. For simplicity we shall restrict attention to operators related to the Riemann-Liouville fractional integral. On the one hand, such operators provide illustrations of the general theory of semigroups, particularly fractional power semigroups. On the other hand, it could be said that the operators have provided the stimulus for extensions of the general theory.

The paper is divided into four sections as follows.

1. Boundary values of holomorphic semigroups.
2. Fractional powers of certain operators mapping one space into a different space.
3. Fractional powers of certain operators mapping a space into itself.
4. α-times integrated semigroups.

1. For $\operatorname{Re}\alpha > 0$ and a suitable function ϕ we define $I^\alpha\phi$, *the Riemann-Liouville fractional integral of order α of ϕ*, by

$$(I^\alpha\phi)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\phi(t)dt. \tag{1}$$

G. R. Goldstein and J. A. Goldstein (eds.),
Semigroups of Linear and Nonlinear Operations and Applications, 205–224.

When $\alpha = n$, we obtain an operator corresponding to repeated (n-fold) integration.

First consider the properties of I^α relative to the spaces $L^p(0,1)$ so that we take $0 < x < 1$ in (1). From [9, pp. 664 et seq.] we know that $\{I^\alpha : \operatorname{Re}\alpha > 0\}$ gives rise to a holomorphic semigroup of bounded linear operators on $L^p(0,1)$ (for $1 \leq p \leq \infty$) and *the first index law*

$$I^\alpha I^\beta = I^{\alpha+\beta} = I^\beta I^\alpha \quad (\operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0) \tag{2}$$

holds in the sense of operators on $L^p(0,1)$.

On writing $\alpha = \mu + i\nu(\mu, \nu$ real with $\mu > 0)$, we may consider the behaviour of I^α as $\mu = \operatorname{Re}\alpha \to 0+$. For $1 < p < \infty$, it was proved by Kalisch [13] and Fisher [6] that, for each fixed $\phi \in L^p(0,1)$, the limit

$$I^{i\nu}\phi = \lim_{\mu \to 0+} I^{\mu+i\nu}\phi \tag{3}$$

was well-defined, the limit existing in the $L^p(0,1)$ norm. Furthermore, the family of operators $\{I^{i\nu} : \nu \in \mathbb{R}\}$ so defined forms a strongly continuous group of bounded operators on $L^p(0,1)$. We can think of this group as giving the *boundary values* of the original holomorphic semigroup and, as one might expect from (2),

$$I^\alpha I^{i\nu} = I^{\alpha+i\nu} = I^{i\nu} I^\alpha \quad (\operatorname{Re}\alpha > 0,\ \nu \in \mathbb{R}) \tag{4}$$

in the sense of operators on $L^p(0,1)$ for $1 < p < \infty$.

Now let us replace $L^p(0,1)$ by $L^p(0,\infty)$ where $1 < p < \infty$. The operators I^α ($\operatorname{Re}\alpha > 0$) are now unbounded operators whose domains are proper subspaces of $L^p(0,\infty)$. Nevertheless, Fisher [6] proved that it was still possible to obtain a *boundary group*

$\{I^{i\nu} : \nu \in \mathbf{R}\}$ of bounded linear operators on the whole space $L^p(0,\infty)$, although the limit process was more elaborate than that in (3). Equation (4) remains valid in the sense that

$$I^\alpha I^{i\nu}\phi = I^{\alpha+i\nu}\phi = I^{i\nu}I^\alpha\phi \quad (\mathrm{Re}\,\alpha > 0,\ \nu \in \mathbf{R})$$

whenever

$$\phi \in D(I^\alpha) = \{\phi \in L^p(0,\infty) : I^\alpha\phi \in L^p(0,\infty)\}.$$

These results have led to more general investigations into boundary values of holomorphic semigroups of (possibly) unbounded linear operators in a Banach space X. We mention in particular the work of Hughes and Kantorovitz [12] who introduced the concept of a *regular semigroup of operators* and showed that such semigroups $\{T(\alpha) : \mathrm{Re}\,\alpha > 0\}$ gave rise to boundary groups $\{T(i\nu) : \nu \in \mathbf{R}\}$ of bounded linear operators, the boundary groups being strongly continuous. As indicated in [12], this is one instance where results for fractional integrals have suggested an extension of the general theory of semigroups.

2. We have already mentioned that, in the setting of $L^p(0,\infty)$, the Riemann-Liouville fractional integral is an unbounded operator. One way to remedy the situation is to introduce weighted spaces with simple powers as weights. In this section and the next we shall work within the framework of the Banach spaces $L_{p,\mu}$ $(1 < p < \infty, \mu \in \mathbf{C})$ where

$$L_{p,\mu} = \{\phi : \| \phi \|_{p,\mu} < \infty\}$$
$$\| \phi \|_{p,\mu} = \{\int_0^\infty | x^{-\mu}\phi(x) |^p \, dx/x\}^{\frac{1}{p}}. \tag{5}$$

It is clear that $L_{p,\mu}$ is homeomorphic to the usual space $L^p(0,\infty)$ under the mapping $\phi(x) \to x^{-\mu-1/p}\phi(x)$. (We shall exclude the cases $p = 1$ and $p = \infty$ although many of our results apply in such spaces also.)

If we take $\phi(x) = x^\lambda$ with $\operatorname{Re}\lambda > -1$, then

$$I^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\lambda dt = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} x^{\lambda+\alpha}$$

i.e.

$$I^\alpha x^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\alpha+1)} x^{\lambda+\alpha}. \tag{6}$$

The change in power from λ to $\lambda+\alpha$ can easily be accommodated within the structure of our weighted spaces. It can be shown that

I^α is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu+\alpha}$ provided that $\operatorname{Re}\mu > -1, \operatorname{Re}\alpha > 0$. (7)

Thus each I^α (and I^1 in particular) maps from one weighted space into a *different* one.

We shall now consider a class of operators, each of which maps from one $L_{p,\mu}$ space into a different one. For each such operator T we shall define a general power T^α ($\operatorname{Re}\alpha > 0$) and thereby obtain a fractional power semigroup. The Riemann-Liouville fractional integral can be recovered as a special case. However, by restricting attention to smooth functions, we can also treat fractional derivatives and powers of "Bessel type" differential operators. We shall merely outline the theory here. Full details can be found in [21], [22] and [23], with an edited version in [24, pp. 99-139].

We shall make extensive use of the Mellin transform $\mathcal{M}$ defined formally by

$$(\mathcal{M}\phi)(s) = \int_0^\infty x^{s-1}\phi(x)dx. \tag{8}$$

(For $\phi \in L_{p,\mu}$, the integral exists via mean convergence provided that $1 < p \leq 2$ and $\operatorname{Re} s = -\operatorname{Re}\mu$.) The Mellin convolution $k * \phi$ of two functions k and ϕ is defined by

$$(k * \phi)(x) = \int_0^\infty k(\frac{x}{t})\phi(t)\frac{dt}{t} \quad (x > 0). \tag{9}$$

Under appropriate conditions, we obtain

$$(\mathcal{M}(k * \phi))(s) = (\mathcal{M}k)(s)(\mathcal{M}\phi)(s). \tag{10}$$

For fixed k, we can think of the transform S defined by

$$S\phi = k * \phi. \tag{11}$$

It follows from (10) and (11) that

$$(\mathcal{M}(S\phi))(s) = (\mathcal{M}k)(s)(\mathcal{M}\phi)(s). \tag{12}$$

It is then possible to obtain the mapping properties of the transform S by studying its multiplier $\mathcal{M}k$ and extensive investigations have been carried out by Rooney [26], [27]. (A change of variable relates this to symbols of pseudo-differential operators defined via the Fourier transform.) Under appropriate conditions on $\mathcal{M}k$, S will map $L_{p,\mu}$ into itself and we then call $\mathcal{M}k$ *an* $L_{p,\mu}$ *multiplier.*

Now let γ be a non-zero complex number. Let

$$T = x^{-\gamma}S \tag{13}$$

where S is as in (11). (This means $(T\phi)(x) = x^{-\gamma}(S\phi)(x)$.) Then from (12) and (13) we get

$$(\mathcal{M}(T\phi))(s-\gamma) = (\mathcal{M}k)(s)(\mathcal{M}\phi)(s). \tag{14}$$

We now assume that the multiplier $\mathcal{M}k$ has a particular 'factorised" form, namely $h(s-\gamma)/h(s)$ for some function h. Thus we finally arrive at operators T which satisfy an equation of the form

$$(\mathcal{M}(T\phi))(s-\gamma) = \frac{h(s-\gamma)}{h(s)}(\mathcal{M}\phi)(s). \tag{15}$$

A simple induction based on (15) gives formally

$$(\mathcal{M}(T^n\phi))(s-n\gamma) = \frac{h(s-n\gamma)}{h(s)}(\mathcal{M}\phi)(s) \quad (n=1,2,\cdots)$$

and this immediately suggests that an operator T^α can be defined by requiring that

$$(\mathcal{M}(T^\alpha\phi))(s-\alpha\gamma) = \frac{h(s-\alpha\gamma)}{h(s)}(\mathcal{M}\phi)(s). \tag{16}$$

This will work provided that $h(s-\alpha\gamma)/h(s)$ is an acceptable $L_{p,\mu}$ multiplier, in which case T^α will be a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu+\alpha\gamma}$ under appropriate conditions. We may feel justified in referring to T^α as *an αth power of T*.

It is worth remarking that a general αth power defined in this way is not unique. To see this we note that if (15) holds for a particular function h then it will also hold for each of the functions $h_r (r=1,2,\cdots)$ where

$$h_r(s) = \exp(2r\pi i s/\gamma)h(s). \tag{17}$$

However, as regards (16),

$$\frac{h_r(s-\alpha\gamma)}{h_r(s)} = \exp(-2r\pi i\alpha)\frac{h(s-\alpha\gamma)}{h(s)}$$

and, since $\exp(-2r\pi i\alpha) \neq 1$ in general, we may obtain infinitely many possibilities for T^α.

Suppose now that T satisfies (15) and that we use the same choice of h throughout (i.e. for the calculation of all powers). Then it is routine to check the validity of *the first index law*

$$T^{\alpha}T^{\beta} = T^{\alpha+\beta} = T^{\beta}T^{\alpha} \tag{18}$$

under appropriate conditions. Perhaps more surprisingly it is also easy to deal with *the second index law*

$$(T^{\alpha})^{\beta} = T^{\alpha\beta}. \tag{19}$$

Indeed, with the same h throughout we calculate $(T^{\alpha})^{\beta}$ by replacing T, α and γ by T^{α}, β and $\alpha\gamma$ respectively in (16). This produces (16) with α replaced by $\alpha\beta$ and (19) is proved formally. The simplicity contrasts with the difficulties encountered in the spectral approach which is introduced in the next section.

It is time to illustrate our theory and, as promised, we shall show first how to recover I^{α}. All we need to do here (and in subsequent cases) is to identify γ and a suitable h.

We start with $T = I^1$ where

$$(I^1\phi)(x) = \int_0^x \phi(t)dt$$

and find that (under appropriate conditions, e.g. for $\phi \in C_0^{\infty}(0,\infty)$)

$$(\mathcal{M}(I^1\phi))(s-1) = \frac{1}{1-s}(\mathcal{M}\phi)(s).$$

Comparison with (15) gives $\gamma = 1$ and a suitable choice for h is

$$h(s) = \frac{1}{\Gamma(1-s)}. \tag{20}$$

Formula (16) then says that $I^\alpha \equiv (I^1)^\alpha$ has to be such that

$$(\mathcal{M}(I^\alpha \phi))(s-\alpha) = \frac{\Gamma(1-s)}{\Gamma(1-s+\alpha)}(\mathcal{M}\phi)(s). \tag{21}$$

We can check directly that this gives the operator in (1) so that the choice of h in (20) is in a sense canonical.

To discuss derivatives we need to use smooth functions. With $D \equiv \dfrac{d}{dx}$ and $\phi \in C_0^\infty(0,\infty)$, we may integrate by parts to obtain

$$(\mathcal{M}(D\phi))(s+1) = -s(\mathcal{M}\phi)(s).$$

Comparison with (15) gives $\gamma = -1$ and a suitable choice for h is given once again by (20). Then D^α has to be such that

$$(\mathcal{M}(D^\alpha \phi))(s+\alpha) = \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)}(\mathcal{M}\phi)(s). \tag{22}$$

Examination of (21) and (22) leads us to conclude formally that

$$I^\alpha = D^{-\alpha}; \quad D^\alpha = I^{-\alpha} \tag{23}$$

as one might expect.

It is worth commenting briefly on the validity of our formal calculations. The multiplier $\Gamma(1-s)/\Gamma(1-s+\alpha)$ in (21) can serve as an $L_{p,\mu}$ multiplier if $\operatorname{Re}\alpha > 0$ (or if $\alpha = 0$, in which case we obtain the identity operator) but not if $\operatorname{Re}\alpha < 0$. When we restrict attention to smooth functions in a subspace $F_{p,\mu}$ of $L_{p,\mu}$ (see Definition 6.1 in [24], for instance), we discover that (21) can serve as an $F_{p,\mu}$ multiplier for any α and can then proceed to establish (22) and (23). Again see [24] for further details.

Sometimes we may wish to differentiate or integrate with respect to a positive power x^m of the variable rather than x itself.

It is possible to define operators I_m^α and D_m^α which extend the previous results for $m = 1$. Thus, in $L_{p,\mu}$ with $\operatorname{Re}\alpha > 0$, (21) becomes

$$(\mathcal{M}(I_m^\alpha \phi))(s - m\alpha) = \frac{\Gamma(1 - \frac{s}{m})}{\Gamma(1 - \frac{s}{m} + \alpha)}(\mathcal{M}\phi)(s). \tag{24}$$

Such generalisations are important in connection with special functions.

One way of justifying the last remark is to consider an n-th order differential operator of the general form

$$T = x^{a_1} D x^{a_2} D \cdots x^{a_n} D x^{a_{n+1}}. \tag{25}$$

As an example, consider

$$\begin{aligned} B_\nu &= x^{-\nu-1} D x^{2\nu+1} D x^{-\nu} \\ &= x^{-2}(x^2 D^2 + x D - \nu^2). \end{aligned} \tag{26}$$

The equation $B_\nu y = -y$ is *Bessel's equation of order ν*. Consequently the operator (26) is sometimes called *a hyper-Bessel operator.* To define T^α via (16), we assume that $\sum_{j=1}^{n+1} a_j$ is real and that

$$m = \left| \sum_{j=1}^{n+1} a_j - n \right| > 0. \tag{27}$$

(If the modulus is zero we are in the situation of the next section.) On replacing each D in (25) by $m x^{m-1} \frac{d}{dx^m}$, with m as in (27), a concrete expression can be obtained. See [20] and [24], these papers having been motivated by Sprinkhuizen-Kuyper [29]. Notice that for B_ν in (26) we get $m = 2$. This explains the appearance of fractional integrals with respect to x^2 in formulae for the Bessel functions such as Sonine's integral [5, §7.12].

To summarise our theory, we may say that the formula (15) leads to a very simple method of defining T^α and hence of obtaining a fractional power semigroup (or even a group). The fact that T maps between different spaces means that some features are missing by comparison with what will follow in the next section. For example, since $\lambda I - T$ is meaningless, we do not have resolvent operators available. Analyticity with respect to α has to be handled at a lower level, with Fréchet derivatives unavailable. Thus we are forced to fix not only $\phi \in L_{p,\mu}$ but also $x \in (0,\infty)$ and then to investigate $(T^\alpha\phi)(x)$ as a function of α. Our final comment here is that the above results are thirled to the Mellin transform and the weighted spaces $L_{p,\mu}$. In contrast we shall now turn to the spectral approach for operators mapping a general Banach space into itself.

3. We now consider operators mapping a space (or a subspace of it) into the same space. Precisely, let X be a Banach space and let A be a linear operator whose domain $D(A)$ and range $R(A)$ are linear subspaces of X. We shall review the basic method of defining powers of $-A$. See [3], [11] and [17].

First recall that, if A is a positive real number and α is a complex number satisfying $0 < \operatorname{Re}\alpha < 1$, then

$$A^{\alpha-1} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\lambda^{\alpha-1}}{\lambda + A}\,d\lambda. \tag{28}$$

The formal analogue of (28) for operators is

$$(-A)^{\alpha-1} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{\alpha-1}(\lambda I - A)^{-1}\,d\lambda$$

or

$$(-A)^\alpha x = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{\alpha-1}R(\lambda, A)(-A)x\,d\lambda \tag{29}$$

where $R(\lambda, A) = (\lambda I - A)^{-1}$ and $0 < \mathrm{Re}\,\alpha < 1$ as before. For (29) to exist we certainly need $x \in D(A)$ as well as requiring $R(\lambda, A)$ to exist for all $\lambda > 0$. To guarantee convergence of (29) as a Bochner integral we assume that

$$\begin{aligned} &(0,\infty) \subset \rho(A), \text{ the resolvent set of } A \\ &\| \lambda R(\lambda, A) \| \leq M \text{ for all } \lambda > 0 \end{aligned} \tag{30}$$

where M is a positive constant.

We observe that condition (30) is satisfied when A is the infinitesimal generator of a uniformly bounded C_0-semigroup, as a consequence of the Hille-Yosida Theorem.

In order to define $(-A)^\alpha$ for $\mathrm{Re}\,\alpha > 0$ rather than for the restricted range $0 < \mathrm{Re}\,\alpha < 1$, we first observe that (29) can be rewritten in the form

$$\begin{aligned} (-A)^\alpha x = {} & \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1}[R(\lambda, A) - \frac{\lambda}{1+\lambda^2}](-A)x\,d\lambda \\ & + \sin\frac{\pi\alpha}{2}(-A)x. \end{aligned} \tag{31}$$

However the expression (31) is meaningful for the larger range of values $0 < \mathrm{Re}\,\alpha < 2$. (Basically $R(\lambda, A) - \dfrac{\lambda}{1+\lambda^2} I$ behaves like λ^{-2} as $\lambda \to \infty$.) We can therefore use (31) to extend the definition of $(-A)^\alpha x$ to this larger range for $x \in D(A)$. Finally, if α satisfies $n - 1 < \mathrm{Re}\,\alpha < n + 1$ for a positive integer n, we define $(-A)^\alpha$ via

$$(-A)^\alpha x = (-A)^{\alpha-n+1}(-A)^{n-1}x \tag{32}$$

for suitable x. If for simplicity we assume that $D(A) = X$ and that A is bounded, then the family $\{(-A)^\alpha : \mathrm{Re}\,\alpha > 0\}$ is a holomorphic semigroup.

To illustrate this theory we return to our theme of the Riemann-Liouville fractional integral. In considering (6) in the previous section, we introduced a family of different spaces. Alternatively we can modify I^α by considering the operator $I^{\eta,\alpha}$ defined by

$$(I^{\eta,\alpha}\phi)(x) = x^{-\eta-\alpha} I^\alpha x^\eta \phi(x)$$
$$= \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta \phi(t)dt. \qquad (33)$$

In contrast to (6) we find that, for $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}(\eta+\lambda) > -1$

$$I^{\eta,\alpha} x^\lambda = \frac{\Gamma(\eta+\lambda+1)}{\Gamma(\eta+\alpha+\lambda+1)} x^\lambda. \qquad (34)$$

In view of (34) it is no surprise that

$I^{\eta,\alpha}$ is a continuous linear mapping from $L_{p,\mu}$ into itself if $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}(\eta+\mu) > -1$. (35)

The operator (33) is an example of an Erdélyi-Kober operator. Such operators were studied by Erdélyi and Kober in a series of papers [4], [15] and [16]. Modifications involving fractional integrals with respect to x^2 rather than x subsequently led to an elegant method for solving dual integral equations arising in potential theory. (This is related to the emergence of $m = 2$ in connection with the Bessel operator (26).) For further details see the article by Sneddon in [28] as well as Chapter 7 in [19].

Consider the operator $I^{\eta,1}$ on $L^p(0,\infty)$ for simplicity. (The results go through for any $L_{p,\mu}$ space with minor changes.) Take $A = -I^{\eta,1}$ in order to define powers of $I^{\eta,1}$. The results which follow are due to Lamb [18]. As regards (30) we find that, for $\lambda > 0$,

$$R(\lambda, -I^{\eta,1}) = \frac{1}{\lambda} I - \frac{1}{\lambda^2} I^{\eta+\frac{1}{\lambda},1} \qquad (36)$$

provided that $\mathrm{Re}\,\eta > -1$ (where the first I on the right-hand side denotes the identity operator!). We remark that (36) can be checked by showing that both sides have the same Mellin multiplier in the sense of (12), namely $(\eta + 1 - s)/(\lambda\eta + \lambda + 1 - \lambda s)$. We then obtain

$$\begin{aligned} &\| \lambda R(\lambda, -I^{\eta,1}) \| = \| I - \frac{1}{\lambda} I^{\eta+\frac{1}{\lambda},1} \| \\ &\leq 1 + \frac{1}{1+\lambda(\eta+1)} \leq 1 + \frac{1}{1} = 2 \text{ for } \mathrm{Re}\,\lambda > 0,\ \mathrm{Re}\,\eta > -1. \end{aligned}$$

This completes the verification of (30). On substituting (36) into (31) and then using (32) we find that, for $\mathrm{Re}\,\alpha > 0$ and $\mathrm{Re}\,\eta > -1$,

$$(I^{\eta,1})^\alpha = x^{-(\eta+1)} H^\alpha x^{\eta+1}$$

where the operator H^α is defined by

$$(H^\alpha \phi)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x [\log(\frac{x}{t})]^{\alpha-1} \phi(t) \frac{dt}{t}. \tag{37}$$

The operator H^α in (37) corresponds to integrating α times with respect to $\log x$. It is often linked with the name of Hadamard.

In the above example, everything could be calculated explicitly. However the case of $I^{\eta,2}$ presents more difficulty and the case of $(I^{\eta,\alpha})^\beta$ for general α and β seems hopeless.

4. In this final section we shall consider families of bounded operators obtained as fractional integrals of semigroups. Consider a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X and let $A : D(A) \to X$ be its infinitesimal generator. Then the abstract Cauchy problem (ACP)

$$\frac{du}{dt} = Au \quad (t > 0);\ u(0) = u_0 \tag{38}$$

has a unique "classical" solution $u : [0,\infty) \to X$ for any given $u_0 \in D(A)$ and u is given by

$$u(t) = T(t)u_0 \quad (t \geq 0).$$

It is of interest to ask what happens if A does not generate a C_0-semigroup.

Let α be real and positive and let $\{T(t)\}_{t\geq 0}$ be as above. We define a new family $\{S(s)\}_{s\geq 0}$ of bounded linear operators on X by

$$S(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} T(t)dt \tag{39}$$

the integral converging with respect to the operator norm on $B(X)$. We shall refer to the family $\{S(s)\}_{s\geq 0}$ as an α-times integrated semigroup.

This concept was introduced in the case $\alpha = 1$ by Arendt [1] and thereafter the theory has been extended to positive integers α and finally to all positive values of α. Details can be found in the papers of Kellermann and Hieber [14], Neubrander [25], Thieme [30] and Hieber [7], [8]. Basically (39) is an operator version of (1) and we could express this briefly in the form

$$S = I^\alpha T. \tag{40}$$

If A denotes the infinitesimal generator of $\{T(t)\}$, we may apply the convolution theorem for the (operator-valued) Laplace transform to deduce from (39) that

$$\int_0^\infty e^{-\lambda s} S(s)ds = \frac{1}{\lambda^\alpha} R(\lambda, A)$$

i.e.

$$R(\lambda, A) = \lambda^\alpha \int_0^\infty e^{-\lambda s} S(s)ds \tag{41}$$

for all sufficiently large (real) λ. It is evident that A not only generates $\{T(t)\}_{t\geq 0}$ but also generates $\{S(s)\}_{s\geq 0}$ in a sense embodied in (41). However, it is possible to find operators A which will "generate" a family $\{S(s)\}_{s\geq 0}$ satisfying (41) without generating a C_0-semigroup. We shall use the following definition.

Definition A linear operator $A : D(A) \to X$ is said to be the generator of an α-times integrated semigroup (for $\alpha \geq 0$) if

(i) the resolvent set, $\rho(A)$, of A contains (ω, ∞) for some $\omega \in \mathbf{R}$

(ii) there exists a mapping $S : [0, \infty) \to X$ which is strongly continuous and satisfies

$$\| S(s) \| \leq M e^{\omega s} \quad (s \geq 0)$$

(where M is a positive constant) and

$$R(\lambda, A) = \lambda^{\alpha} \int_0^{\infty} e^{-\lambda s} S(s) ds$$

for $\lambda > \max(\omega, 0)$.

For $\alpha = 0$, the above definition coincides with the usual infinitesimal generator in view of the Hille-Yosida Theorem. A "Hille-Yosida Theorem" can be proved for $\alpha > 0$ too and the theory in this case bears some similarities with that for $\alpha = 0$. For example $\{S(s)\}$ is uniquely determined by A. There are also some differences, notably that

$$\textit{for } \alpha > 0, D(A) \textit{ need not be dense in } X. \tag{42}$$

A useful fact is that

if A generates an α-times integrated semigroup, then A generates a β-times integrated semigroup for all $\beta > \alpha$.

This means that we can accommodate more operators as generators by increasing α.

With this in mind let us return to the ACP (38) and suppose that A generates an r-times integrated semigroup for some non-negative integer r. It is well known (see, for instance [2]) that (38) will have a unique "classical" solution for each $u_0 \in D(A^{r+1})$. This has been extended to positive non-integral values of α by Hieber [7].

Theorem Let $\alpha \geq 0$, $\epsilon > 0$ and assume that A generates an α-times integrated semigroup $\{S(s)\}_{s\geq 0}$ satisfying

$$\| S(s) \| \leq M s^{\alpha} e^{\omega s} \quad (s \geq 0)$$

for non-negative constants M, ω. Then there exists a unique classical solution of (38) for all $u_0 \in D((-A)^{\alpha+\epsilon+1})$.

This result indicates a connection with the theory of fractional powers of operators discussed in the previous section. At this juncture, it is legitimate to ask if there are any important applications which require the use of a non-integral value of α, thereby justifying the use of "fractional" integration. To provide this justification, we mention another result of Hieber.

Consider the ACP (38) for the Schrödinger equation

$$\frac{du}{dt} = i\Delta u \quad (t > 0); \quad u(0) = u_0 \tag{43}$$

in the Banach space $L^p(\mathbb{R}^n)$ where we take $1 < p < \infty$ for simplicity. Hörmander [10] proved that (when defined on its natural domain) $i\Delta$ generates a C_0-semigroup on $L^p(\mathbb{R}^n)$ if and only if $p = 2$. In contrast Hieber [8] has shown that $i\Delta$ generates an α-times integrated semigroup on $L^p(\mathbf{R}^n)$ if and only if

$$\alpha \geq n \left| \frac{1}{2} - \frac{1}{p} \right| . \tag{44}$$

(This confirms that $\alpha = 0$ is only possible when $p = 2$.) It is of interest to investigate the space of initial conditions u_0 for which (43) has a unique classical solution u. The "optimal" space turns out to be the Sobolev space $W^{n+2,p}$. To obtain this when n is odd it is necessary to use fractional values of α, as use of r-times integrated semigroups with r a non-negative integer will give a weaker result which only guarantees existence and uniqueness of a classical solution for $u_0 \in W^{n+3,p}$.

Conclusion This survey paper has touched on a few of the interconnections between the Riemann-Liouville fractional integrals, fractional powers of operators and semigroups of operators. It may be expected that all three areas will continue to play a role in the future study of evolution equations and abstract Cauchy problems.

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Paper 15

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On Solving Hyper-Bessel Differential Equations By Means of Meijer's G-Functions II: The Nonhomogeneous Case

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Strathclyde Mathematics Research Report No. 1992/20

ON SOLVING HYPER-BESSEL DIFFERENTIAL EQUATIONS BY MEANS OF MEIJER'S G-FUNCTIONS, II: THE NONHOMOGENEOUS CASE

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ABSTRACT

Ordinary differential equations of m-th order with variable coefficients:

(1) $B\, y(x) = \lambda\, y(x) + f(x)$, $\lambda = \text{const}$, $f(x)$ a given function,

are considered, where B is the so-called "hyper-Bessel"("Bessel-type") operator:

(2) $B = x^{\alpha_0} D x^{\alpha_1} D \ldots x^{\alpha_{m-1}} D x^{\alpha_m}$ $(D = d/dx)$, $0 < x < \infty$,

arising in the problems of Mathematical Physics mainly in the equivalent forms:

(2') $B = x^{-\beta} Q_m(xD) = x^{-\beta}(x^m D^m + a_1 x^{m-1} D^{m-1} + \ldots + a_{m-1} xD + a_m)$, $\beta > 0$.

It is a generalization of the classical 2-nd order operator of Bessel, related to the Bessel functions $J_\nu(x)$. In [24] we have obtained explicit solutions of the equations $B\, y(x) = \lambda\, y(x)$ and $B\, y(x) = f(x)$, satisfying arbitrary initial conditions at $x=0$. All of them are special cases of the generalized hypergeometric G-functions of Meijer. Now, in order to solve the problem entirely, we propose a particular solution to the nonhomogeneous equation (1). It is represented by means of a series of integrals also involving G-functions.The transmutation method, based on a Poisson-type integral transformation is used, combined with the Techniques of the Laplace and Mellin transforms and Generalized Fractional Calculus.Some known special functions such as Lommel and Struve functions are shown to be particular cases of the solution found here. The results of both papers confirm the close relation between the hyper-Bessel operators of arbitrary order and G-functions.

Prepared during the visit of one of us (V.S.K.) at the University of Strathclyde and supported by the British Council.

1991 Mathematics Subject Classification: 34A05, 34A25, 33A35, 26A33.

1. Introduction

Let $m > 1$ be an integer, $\beta > 0$ and $\gamma_1, \gamma_2, \ldots, \gamma_m$ be arbitrary real numbers. Throughout the paper, $D = d/dx$, $D^m = (d/dx)^m$, $m=1, 2, \ldots$ and $xD = x(d/dx)$ is the Euler differential operator.

Definition 1.1. Let $Q_m(\mu) = \prod_{k=1}^{m} (\mu + \beta\gamma_k)$ be a m-th degree polynomial whose zeros $\mu_k = -\beta\gamma_k$, $k=1,\ldots,m$ are counted with their multiplicities. The linear differential operator

$$(1.1) \qquad B = x^{-\beta} Q_m(xD) = x^{-\beta} \prod_{k=1}^{m} (xD + \beta\gamma_k), \quad 0 < x < \infty,$$

is said to be a hyper-Bessel differential operator.

The term "Bessel-type differential operator of order m" is also used for operator (1.1), introduced and studied by Dimovski [5], [6], [7], Dimovski and Kiryakova [10], [11] and McBride [28], [29] in the form

$$(1.2) \qquad B = x^{\alpha_0} D x^{\alpha_1} D \ldots x^{\alpha_{m-1}} D x^{\alpha_m}, \quad 0 < x < \infty,$$

where: $\alpha_0 = -\beta - \beta\gamma_1 + 1$; $\alpha_k = \beta\gamma_k - \beta\gamma_{k+1} + 1$, $k=1,\ldots,m-1$; $\alpha_m = \beta\gamma_m$;

or, $\beta = m - (\alpha_0 + \alpha_1 + \ldots + \alpha_m) > 0$, $\gamma_k = (\alpha_k + \alpha_{k+1} + \ldots + \alpha_m - m + k)/\beta$, $k=1,\ldots,m$.

Depending on the arrangement of the zeros $\mu_k = -\beta\gamma_k$, $k=1,\ldots,m$ of the polynomial $Q_m(\mu)$, operator (1.1) has different representations in form (1.2). To avoid this, we shall assume that

$$\gamma_1 \leqq \gamma_2 \leqq \ldots \leqq \gamma_m, \quad \text{or} \quad \gamma_1 \geqq \gamma_2 \geqq \ldots \geqq \gamma_m,$$

specified according to our convenience.

In general, operators (1.1), (1.2) are singular linear differential operators with variable coefficients, arising in Analysis and Differential Equations usually in the equivalent forms (often with $\beta = m$, or $\beta = 1$):

$$(1.3) \qquad B = x^{-\beta} (x^m D^m + a_1 x^{m-1} D^{m-1} + \ldots + a_{m-1} xD + a_m),$$

where

$$a_{m-j} = (1/j!) \sum_{l=0}^{j} \left[(-1)^l \binom{j}{l} \prod_{i=1}^{m} (\beta\gamma_i + j + 1) \right], \quad j=0,1,\ldots,m-1.$$

Particular cases. The best known example, giving rise to the name of operators (1.1),(1.2),(1.3), is the second order singular differential operator of Bessel:

$$(1.4) \qquad B_\nu = x^{-2} (xD + \nu)(xD - \nu) = x^{\nu-1} D x^{-2\nu+1} D x^{\nu} = x^{-\nu-1} D x^{2\nu+1} D x^{-\nu}$$

$$= D^2 + x^{-1} D - \nu^2 x^{-2}, \quad \text{that is, } \beta = m = 2, \ \gamma_1 = \nu/2, \ \gamma_2 = -\nu/2,$$

related to the Bessel function $y(x) = J_\nu(x)$ which is a solution of the equation

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0 \text{ , that is, } B_\nu y(x) = -y(x).$$

Among the other Bessel-type operators of 2-nd order, we shall mention:

$B = D^2 + (k/x) D$ ($k \geq 1$), related to the "generalized" heat equation;

$B = x^{-n} D^2$ ($n > 0$ integer), arising in P.D.E. of mixed type;

$B = D x D$, $B = x^{-\alpha} D x^{\alpha+1} D$; etc. (see [24] for more details).

The simplest higher order hyper-Bessel differential operator is the operator of the m-fold differentiation ($m > 1$, integer):

$$(1.5) \quad \begin{cases} B = D^m = (d/dx)^m & \text{with "hyper-Bessel" parameters:} \\ \beta = m \ , \ \gamma_k = (k/m)-1 & , \ k=1,\dots,m. \end{cases}$$

Operational Calculi and Integral Transforms for special m-th order Bessel-type differential operators such as

$$B = x^{-1} (xD)^m = D x D x D \dots x D \quad ; \quad B_{m,\nu} = D\, x^{(1/m)-\nu} (x^{1-1/m} D)^{m-1} x^{\nu+1-2/m},$$

have been developed by different authors (see [24]) and in the general case by Dimovski [5], [6], [7], Dimovski and Kiryakova [9], [10], [11]. In the papers of McBride [28], [29] and Dimovski and Kiryakova [11] the negative powers of differential operators (1.2), found earlier in [5] in terms of convolutional products, have been represented explicitly by means of integral operators involving Meijer's G-functions.

It is worth pointing out that the history of the Bessel functions and corresponding differential operators can be traced back to Bernoulli, Euler and Poisson, who associated them to the P.D.E. of the potential, wave motion, diffusion, etc. Actually, in the P.D.E. of Mathematical Physics, hyper-Bessel operators arise often, at least in one of the variables: x, y, z (Cartesian coordinates), or r (Polar, Cylindrical or Spherical coordinates), for example in the forms:

$$(1/r)(\partial/\partial r)r \ , \ (1/r)(\partial/\partial r)r^2 \ , \ (1/r^n)(\partial/\partial r)^2, \ (\partial/\partial r)(1/r)(\partial/\partial r)r(\partial/\partial r) \ .$$

Usually, by separating the variables or applying a suitable integral transform we can reduce these problems to initial value problems for O.D.E. involving hyper-Bessel operators.

Definition 1.2. Let B be an arbitrary Bessel-type differential operator
1.1), (1.2), (1.3) of order $m > 1$. An ordinary differential equation of the form:

$$\text{1.6)} \qquad B\, y(x) = \lambda\, y(x) + f(x)\ , \lambda = \underline{\text{const}}\ ,\ f(x) \text{ a given function,}$$

s said to be a <u>hyper-Bessel differential equation</u>.

<u>Cauchy initial value problems</u> for equations (1.6) can be stated either by means
f the classical initial conditions

$$\text{1.7)} \qquad \lim_{x \to +0} y^{(k-1)}(x) = \beta_k \quad , k=1,\dots,m\ ,$$

r by means of the equivalent set of <u>"Bessel-type" initial conditions</u>

$$\text{1.8)} \qquad \lim_{x \to +0} B_{\varkappa}\, y(x) = b_k \quad , k=1,\dots,m\ ,$$

here B_k are the "truncated" hyper-Bessel operators

$$B_k = x^{\beta\gamma_k} \prod_{j=k+1}^{m} (xD + \beta\gamma_j) = x^{\alpha_k} D\, x^{\alpha_{k+1}} \dots D\, x^{\alpha_m} \quad , k=1,\dots,m-1,$$

$$B_m = x^{\beta\gamma_m} = x^{\alpha_m} .$$

An important problem is to find a fundamental system of solutions of the hyper-
essel differential equation $B\, y(x) = \lambda\, y(x)$ (that is, the eigen-functions of ope-
ator B) as well as to solve an arbitrary initial value problem for the nonhomoge-
eous equation $B\, y(x) = f(x)$. Till now, many efforts by different authors have
een made towards finding solutions of these problems in various special cases: mainly
or $m=2$, or proposing algorithms only (see [1], [2], [4], [15], [17], [25] and
he other references in [24]). Most of these solutions are modifications of the Bessel,
-Bessel functions ([1],[15]), hyper-Bessel functions ([4]), etc. and all of them
an be expressed in terms of the generalized hypergeometric functions ${}_pF_q$ or, in
he most general cases, as Meijer's G-functions. The close relation between the latter
pecial functions and the hyper-Bessel operators of arbitrary order has been observed
lready in [10], [11], [19] . There, a $G^{m,0}_{m,m}$-function has been shown to be a solu-
ion of the equation $B\, y(x) = \lambda\, y(x)$ as well as a kernel-function of the Obrechkoff
ransform ([31]) corresponding to arbitrary hyper-Bessel operator B. Later on, all
he theory of the hyper-Bessel operators was revised from the point of view of the
leijer's G-functions (see [19], [21], [22], [23], [30]). The results in papers [28],[29]

by McBride confirmed the role of the G-functions used as kernel-functions of related integral operators and gave rise to the generalized fractional integrals introduced in [20], [21]. In [24] we gave the explicit solutions of the initial value problems mentioned above, all of them also being Meijer's G-functions. Before summarizing these results, we give the following

Definition 1.3. By a Meijer's G-function we mean a Mellin-Barnes type contour integral of the form

$$(1.9)\qquad G^{m,n}_{p,q}(x) = G^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\right] = G^{m,n}_{p,q}\left[x \,\middle|\, \begin{matrix} (a_j)_1^p \\ (b_k)_1^q \end{matrix}\right] = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathcal{G}^{m,n}_{p,q}(s)\, x^s\, ds,$$

where $\mathcal{L}$ is a suitably chosen contour in $\mathbf{C}$, $x \neq 0$ and the integrand is:

$$\mathcal{G}^{m,n}_{p,q}(s) = \frac{\prod_{k=1}^{m} \Gamma(b_k - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{k=m+1}^{q} \Gamma(1 - b_k + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}\,.$$

The integers $(m,n;p,q)$ $(0 \leqq m \leqq q,\ 0 \leqq n \leqq p)$ are said to be "orders" of the G-function and the parameters a_j, b_k are such that $a_j - b_k \neq 0, \pm 1, \pm 2, \ldots,$ $j=1,\ldots,n$, $k=1,\ldots,m$.

This function is analytic in: $|x| < 1$, $|x| > 1$, or in the whole complex plane $\mathbf{C}$, this depending on the orders and parameters. It includes as special cases most of the Special Functions in Mathematical Physics and thus, provides an uniform approach to them, combined with a succinctness of notations and easy-to-use properties. For all the details we recommend the books [14, v.1], [26, v.1], [27], [32].

Let us define also the basic functional spaces in which we seek solutions of the hyper-Bessel equations (1.6) from a practical point of view.

Definition 1.4. Let $C[0,\infty)$ and $C^{(\ell)}[0,\infty)$ be the spaces of the continuous, respectively ℓ-times smooth functions in $[0,\infty)$. We consider the following linear sets of functions ($\ell \geq 0$, $\alpha \in \mathbf{R}$, $p \in \mathbf{R}$):

$$(1.10)\qquad C_\alpha^{(\ell)} = \left\{ y(x) = x^p\, \tilde{y}(x)\ ;\ p > \alpha,\ \tilde{y} \in C^{(\ell)}[0,\infty) \right\},\quad C_\alpha^{(0)} := C_\alpha,$$

in particular, for the hyper-Bessel operators B (1.1) with

$$(1.11)\qquad \alpha = \max_{1 \leqq k \leqq m} \left[-\beta(\gamma_k + 1) \right].$$

Also, for a sequence $\lambda_1, \lambda_2, \ldots, \lambda_m$ we denote

(1.12) $\quad \underline{\text{span}} \left\{ x^{\lambda_k} \right\}_1^m := \left\{ y(x) = c_1 x^{\lambda_1} + c_2 x^{\lambda_2} + \ldots + c_m x^{\lambda_m} \right\}.$

Then, the following results hold (see Lemma 2.1, Theorem 2.2 , Theorem 2.3 of [24]):

Lemma 1.5. Let condition

(1.13) $\quad \gamma_i - \gamma_j \neq$ integer for any $i, j = 1, \ldots, m$,

for the parameters of B be satisfied. Then, the functions

(1.14) $\quad y_k(x) = x^{-\beta\gamma_k}, \quad k=1,\ldots,m$

form a fundamental system of solutions of the equation

(1.15) $\quad B\, y(x) = 0$

in a neighbourhood of $x = 0$ and

(1.16) $\quad y_0(x) = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_m y_m(x) = \sum_{k=1}^{m} c_k x^{-\beta\gamma_k}$

with coefficients

(1.17) $\quad c_k = \left[\prod_{j=k+1}^{m} \beta(\gamma_j - \gamma_k) \right]^{-1} . \, b_k, \quad k=1,\ldots,m$

is the particular solution satisfying arbitrary initial conditions (1.8).

This means that the kernel-space of the operator B is:

(1.18) $\quad \underline{\ker}\, B = \underline{\text{span}} \left\{ x^{-\beta\gamma_k} \right\}_1^m .$

Theorem 1.6. Let conditions (1.13) be satisfied and $f \in C_\alpha$. The solution of the initial value problem for the nonhomogeneous equation

(1.19) $\quad B\, y(x) = f(x)$,

defined by arbitrary initial conditions (1.8) has the representation

(1.20) $\quad y(x) = y_0(x) + Y(x)$

with $y_0(x)$ defined as in (1.16) and

(1.21) $$Y(x) = L\, f(x) = \beta^{-m} x^{\beta} \int_0^1 G_{0,m}^{m,0}\left[\sigma \left| \begin{matrix} (\gamma_i+1)_1^m \\ (\gamma_i)_1^m \end{matrix} \right.\right] f(x\,\sigma^{1/\beta})\, d\sigma ,$$

L being the linear right-inverse operator of B ($BL = I$, I - the identity operator in C_α), defined by zero initial conditions: $\lim_{x \to +0} B_k L f(x) = 0$, $k=1,\ldots,m$.

Theorem 1.7. Let (1.13) be satisfied again. Then, in a neighbourhood of the point $x = 0$ a fundamental system of solutions of the hyper-Bessel O.D.E.

(1.22) $\quad B\, y(x) = \lambda\, y(x) \quad , \lambda = \text{const} \quad ,$

(that is, eigen-functions of the operator B) consists of the Meijer's G-functions:

(1.23) $\quad y_k(x) = G^{1,0}_{0,m}\left[-(\lambda x^{\beta}/\beta^m) \;\middle|\; -\gamma_k, -\gamma_1, \ldots, -\gamma_{k-1}, -\gamma_{k+1}, \ldots, -\gamma_m \right], \; k=1,\ldots,m,$

the latter representable also in terms of the generalized hypergeometric functions ${}_0F_{m-1}(\lambda x^{\beta}/\beta^m)$ and the hyper-Bessel functions of Delerue $J^{(m-1)}_{\gamma_1,\ldots,*,\ldots,\gamma_m}(x)$ from [4] (for definition see also [24]).

It is seen now that the only problem that remains open for solving initial value problem (1.6),(1.8) in its generality, is to find a particular solution to the general nonhomogeneous equation $B\, y(x) = \lambda\, y(x) + f(x)$, $\lambda \neq 0$ and $f \neq 0$. In the sequel, we use the transmutation method (see [8], [16], [24]): first solving the problem in a simpler case (with a simpler hyper-Bessel operator like D^m) and then transforming it into the general case with arbitrary operator B.

2. Solution to the simpler problem

The simplest hyper-Bessel differential operator of order $m \quad 1$ is the m-fold differentiation $B = D^m$, (1.5). Let us consider the initial value problem

$$\begin{cases} \tilde{y}^{(m)}(x) = \lambda\, \tilde{y}(x) + \tilde{f}(x) \;, & (2.1)\\ \tilde{y}(0) = \tilde{y}'(0) = \ldots = \tilde{y}^{(m-1)}(0) = 0 \;. & (2.2)\end{cases}$$

To solve it, one can use the formal techniques of the Laplace transform ([13]):

(2.3) $\quad Y(s) := L\left\{\tilde{y}(x);s\right\} = \int_0^{\infty} \exp(-xs)\, \tilde{y}(x)\, dx \quad , \quad \mathrm{Re}(s) > \rho \quad ,$

considered in the linear spaces (compare with (1.10)):

(2.4) $\quad C^{(1)}_{\exp} := \left\{ y(x) \in C^{(1)}_{-1};\; y(x) = \mathcal{O}(\exp \rho x) \text{ as } x \to \infty \right\} \subset C^{(1)}_{-1} \;.$

The initial conditions (2.2) imply

(2.5) $\quad L\left\{\tilde{y}^{(m)}(x);s\right\} = s^m\, Y(s) \;,$

and thus, differential equation (2.1) is transformed into the algebraic one:

$$s^m\, Y(s) = \lambda\, Y(s) + F(s) \quad , \quad \text{where} \quad F(s) = L\left\{\tilde{f}(x);s\right\} .$$

Then, the Laplace image of the solution $\tilde{y}(x)$ is:

(2.6) $\quad Y(s) = F(s).H(s) \quad \text{with} \quad H(s) = 1/(s^m - \lambda) = 1/[(s/\lambda^{1/m})^m - 1] \;.$

For $Re(s) > |\lambda|^{1/m}$, that is, for $\rho := |\lambda|^{1/m}$ in (2.3),(2.4), H(s) is the Laplace transform of a function h(x) of following alternative forms (see [14, v.3], p.214, (15), respectively p.216,(32)):

i) if $\lambda > 0$, then

$$h(x) = h_+(x) = \lambda^{-1+1/m} h_{m,m}(\lambda^{1/m} x) ,$$

where in general , we denote by

$$(2.7) \qquad h_{m,i}(x) = \sum_{r=0}^{\infty} x^{mr+i-1} / (mr+i-1)! \quad , \quad i=1,\dots,m \quad ,$$

the <u>hyperbolic functions of order</u> <u>m</u> , the solutions of O.D.E. $\tilde{y}^{(m)} = \tilde{y}$;

ii) if $\lambda < 0$, then

$$h(x) = h_-(x) = (-\lambda)^{-1+1/m} k_{m,m}[(-\lambda)^{1/m} x] ,$$

where

$$(2.8) \qquad k_{m,i}(x) = \underset{r=0}{} (-1)^r \, x^{mr+i-1} / (mr+i-1)! \quad , \quad i=1,\dots,m \, ,$$

are the <u>trigonometric functions of order</u> <u>m</u> , solutions of the equation $\tilde{y}^{(m)} = -\tilde{y}$.

Let us note that in [24] we have already used other functions of this kind, namely: $\cos_m(x) = k_{m,1}(x)$ and $h_m(x) = h_{m,1}(x)$.

Both functions $h_+(x)$, $h_-(x)$ have however the common power series form:

$$(2.9) \qquad h(x) = h_\pm(x) = \sum_{r=0}^{\infty} \lambda^r x^{mr+m-1} / (mr+m-1)! \quad , \quad \text{convergent for } |x| < \infty ,$$

or in terms of the generalized hypergeometric functions,

$$(2.10) \qquad h(x) = (x^{m-1}/(m-1)!) \; {}_0F_{m-1}[(1+i/m)_1^{m-1} \, ; \lambda(x/m)^m]$$

$$= \sqrt{(2\pi)^{m-1}/m} \; . \; G^{1,0}_{0,m}[-\lambda(x/m)^m \,|\, (i/m)_0^{m-1}] \; .$$

Then, (2.6) and the <u>convolution theorem for the Laplace transform</u> ([13]) give the original solution of problem (2.1),(2.2) in terms of the <u>Duhamel convolution</u>:

$$(2.11) \qquad \tilde{y}(x) = \tilde{f}(x) * h(x) = \int_0^x \tilde{f}(x-t)\, h(t)\, dt = \int_0^x \tilde{f}(t)\, h(x-t)\, dt$$

$$= \int_0^x \tilde{f}(t)\Big[\sum_{r=0}^{\infty} \lambda^r (x-t)^{mr+m-1} / (mr+m-1)!\Big]\, dt \; .$$

Thus, we can formulate the following

Lemma 2.1. <u>The solution of initial value problem</u> (2.1),(2.2) <u>with</u> $\tilde{f} \in C_{-1}$ <u>belongs to</u> $C^{(m)}_{m-1} \subset C_{-1}$ <u>and has the integral representation</u>

$$(2.12) \qquad \tilde{y}(x) = \sqrt{(2\pi)^{m-1}/m} \, . \, \lambda^{-1+1/m} \int_0^x \tilde{f}(x-t)\; G^{1,0}_{0,m}[-\lambda(x/m)^m \,|\, (i/m)_0^{m-1}]\, dt \; ,$$

or by means of a series,

$$(2.13) \qquad \tilde{y}(x) = \sum_{r=0}^{\infty} \lambda^r \left[\int_0^x \frac{(x-t)^{mr+m-1}}{(mr+m-1)!} \tilde{f}(t)\, dt \right] = \sum_{r=0}^{\infty} \lambda^r R^{m(r+1)} \tilde{f}(x) ,$$

where R^{δ} denotes the Riemann-Liouville integral of order $\delta > 0$ (if δ is integer, $\Gamma(\delta) \longmapsto (\delta-1)!$):

$$(2.14) \qquad R^{\delta} \tilde{f}(x) = \int_0^x \frac{(x-t)^{\delta-1}}{\Gamma(\delta)} \tilde{f}(t)\, dt .$$

Proof. Representation (2.12) follows immediately from (2.10),(2.11). The Heaviside-Mikusinski Operational Calculus (see [13]) based on the same Duhamel convolution (2.11), however, avoids the formal restriction to the subspace $C_{exp} \subset C_{-1}$, imposed by the use of the Laplace transformation. Since this convolution is an operation in C_{-1}, $\tilde{f} \in C_{-1}$ and $h \in C_{-1}$ would imply $\tilde{f} * h = \tilde{y} \in C_{-1}$ too. In our case, $h(x) \in C^{(m)}_{m-1} \subset C_{-1}$ (see (2.9), (2.10) and asymptotic behaviour of the G-functions as $x \to 0$, [14, v.1]). Then, a more precise general theorem of Bozhinov [3] states that $C_{-1} \times C^{(m)}_{m-1} \xrightarrow{(*)} C^{(m)}_{m-1}$, that is, the solution $\tilde{y} \in C^{(m)}_{m-1}$ and has the integral representation (2.12). Further, we can change the order of the integral and series (both of them absolutely convergent) in the second line of (2.11), namely:

$$\tilde{y}(x) = \sum_{r=0}^{\infty} \lambda^r \left[\int_0^x \frac{(x-t)^{mr+m-1}}{(mr+m-1)!} \tilde{f}(t)\, dt \right] ,$$

or in terms of the Riemann-Liouville operators of fractional integration:

$$\tilde{y}(x) = \sum_{r=0}^{\infty} \lambda^r R^{m(r+1)} \tilde{f}(x) ,$$

the form of solution $\tilde{y}(x)$ we shall use in the sequel (Section 4).

3. The Poisson-Dimovski transformation as a transmutation operator

The transmutation method is based on the idea of transforming a complicated problem to a simpler one whose solution is known or easier to find. This method, dealing with operators of transformation (transmutation operators, similarity operators, etc) is often used in solving initial and boundary value problems for differential and integral equations ([8], [11], [12], [16], see [24] for other references and Definition 4.1). In this paper, we are interested in a transformation, relating the m-th order hyper-Bessel differential operators $D^m = (d/dx)^m$ (with parameters as in (1.5)) and B of form (1.1). Definition 4.2 in [24] provides the general form of the so-called Poisson-Dimovski transformations **P**, transmuting D^m into B. Let us assume now that

$$(3.1) \qquad \gamma_1 \geqq \gamma_2 \geqq \dots \geqq \gamma_m$$

(we can freely rearrange the zeros $\mu_k = -\beta\gamma_k$, k=1,...,m of $Q_m(\mu)$ in (1.1) in order to satisfy (3.1)). Then, we obtain the simplest of these transformations, represented by means of (m-1)-tuple integral (compare with [24]: (4.4) with $\mu = -\gamma_m$, and (4.12)).

Definition 3.1. Let **B** be an arbitrary hyper-Bessel operator of form (1.1) with $\gamma_1 \geqq \gamma_2 \geqq \dots \geqq \gamma_m$. The integral transformation $P: C_{-1} \to C_\alpha$, $\alpha = -\beta(\gamma_m+1)$, defined by

$$(3.2) \qquad P\,f(x) = c\,(x^\beta/\beta^m)^{-\gamma_m} \int_0^1 \dots \int_0^1 \prod_{k=1}^{m} \left[\frac{(1-t_k)^{\gamma_k-\gamma_m-k/m}\, t_k^{-1+k/m}}{\Gamma(1+\gamma_k-\gamma_m-k/m)} \right]$$

$$\times\ f[(m/\beta)\, x^{\beta/m} (t_1 \dots t_m)^{1/m}]\, dt_1 \dots dt_m$$

is said to be a Poisson-Dimovski (P.-D.) transformation, corresponding to the hyper-Bessel operator B (1.1), where for convenience the constant c stands for

$$c = \sqrt{m/(2\pi)^{m-1}} \prod_{k=1}^{m} \Gamma(\gamma_k+1) .$$

It is more convenient to use transformation **P** written in a single-integral form, as a special case of the "generalized fractional integrals" (see [20], [21], [22], [24]) involving Meijer's G-functions (1.9) in kernels.

Definition 3.2. Let $m>1$ be an integer, $\beta>0$, $\delta_k \geqq 0$ and γ_k, k=1,...m be real numbers. The set $\gamma = (\gamma_1, \dots \gamma_m)$ is said to be a multiweight and the set $\delta = (\delta_1, \dots, \delta_m)$ - a multiorder of integration. The integral operator defined by

$$(3.3) \qquad I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = \begin{cases} \displaystyle\int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\delta_k)_1^m \\ (\gamma_k)_1^m \end{matrix}\right] f(x\sigma^{1/\beta})\, d\sigma, & \text{if } \sum_{k=1}^{m} \delta_k > 0, \\ f(x), \quad \text{if } \delta_1 = \delta_2 = \dots = \delta_m = 0 \end{cases}$$

is said to be a generalized fractional integral (of Riemann-Liouville type), or a multiple (m-tuple) Erdélyi-Kober fractional integral.

Operators (3.3) generalize the Erdélyi-Kober operators of fractional integration (see [33],[34]; [28],[21]) with $\gamma \in R$, $\delta > 0$:

$$(3.4) \qquad I_{\beta}^{\gamma,\delta} f(x) = [\Gamma(\delta)]^{-1} \int_0^1 (1-\sigma)^{\delta-1} \sigma^{\gamma} f(x\sigma^{1/\beta})\, d\sigma = I_{\beta,1}^{(\gamma),(\delta)} f(x) ,$$

as well as the Riemann-Liouville integrals (2.14): $R^{\delta} = x^{\delta} I_1^{0,\delta} = x^{\delta} \cdot I_{1,1}^{0,\delta}$ and various integral operators involving special functions as kernels.

By using the Mellin Multipliers technique (see McBride [28],[29]), namely the relation

$$(3.5) \qquad M\left\{ I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x);s \right\} = \left[\prod_{k=1}^{m} \frac{\Gamma(\gamma_k+1-s/\beta)}{\Gamma(\gamma_k+\delta_k+1-s/\beta)} \right] . M\left\{ f(x);s \right\} ,$$

where
$$M\left\{ f(x;s \right\} := \int_0^{\infty} x^{s-1} f(x)\, dx$$

is the Mellin transform, one can easily derive the following "Composition/ Decomposition theorem" ([20]-[23]) in C_{α}, if $\gamma_k > -1-\alpha/\beta$, $\delta_k \geqq 0$, $k=1,\dots,m$:

$$(3.6) \qquad I_{\beta,m}^{(\gamma_k),(\delta_k)} f(x) = I_{\beta}^{\gamma_m,\delta_m} [I_{\beta}^{\gamma_{m-1},\delta_{m-1}} \dots (I_{\beta}^{\gamma_1,\delta_1} f(x))] = [\prod_{k=1}^{m} I_{\beta}^{\gamma_k,\delta_k}] f(x)]$$
$$= \int_0^1 \dots \int_0^1 \prod_{k=1}^{m} [[\Gamma(\delta_k)]^{-1} (1-\sigma_k)^{\delta_k-1} \sigma_k^{\gamma_k}] f[x(\sigma_1 \dots \sigma_m)^{1/\beta}] d\sigma_1 \dots d\sigma_m ,$$

representing the generalized fractional integrals (3.3) as products of commuting classical Erdélyi-Kober operators (3.4).

It is seen now that the latter relation gives rise to the name "multiple" Erdélyi-Kober fractional integrals for (3.3). It happens that all the basic operators, related to the hyper-Bessel operators B and O.D.E. (1.6), can be stated in terms of the generalized fractional integrals (3.3). Thus, the linear right inverse operator L (1.21) of B is:

$$L = \beta^{-m} x^{\beta} I_{\beta,m}^{(\gamma_1,\dots,\gamma_m),(1,\dots,1)} ;$$

its fractional (in particular, integer) powers L^{η}, $\eta > 0$ (see McBride [28],[29]) are multiple fractional integrals too:

$$(3.7) \qquad L^{\eta} f(x) = (x^{\beta}/\beta^m)^{\eta} \int_0^1 G_{m,m}^{m,0} \left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+\eta)_1^m \\ (\gamma_k)_1^m \end{matrix} \right] . f(x\sigma^{1/\beta})\, d\sigma$$

$$= (x^{\beta}/\beta^m)^{\eta}\, I_{\beta,m}^{(\gamma_1,\dots,\gamma_m),(\eta,\dots,\eta)} f(x) ;$$

so are the transmutation operators between two different hyper-Bessel operators $B^{(1)}$ and $B^{(2)}$ of the same order $m>1$ (see Dimovski [7], Dimovski and Kiryakova [11], Kiryakova [21]). In particular, the Poisson-Dimovski transformation $\mathbf{P}$ (3.2) is the fractional $(m-1)$-tuple integral:

(3.8) $$\mathbf{P}\, f(x) = c\ (x^{\beta}/\beta^m)^{-\gamma_m}\ I_{\beta,m-1}^{(-1+k/m),(1+\gamma_k-\gamma_m-k/m)}\ f[(m/\beta)x^{\beta/m}]$$
$$= \sqrt{m/(2\pi)^{m-1}}\ \prod_{k=1}^{m}\Gamma(\gamma_k+1)\ (x^{\beta}/\beta^m)^{-\gamma_m}\int_0^1 G_{m-1,m-1}^{m-1,0}\left[\sigma\left|\begin{matrix}(\gamma_k-\gamma_m)_1^{m-1}\\(-1+k/m)_1^{m-1}\end{matrix}\right.\right] f[\tfrac{m}{\beta}\,x^{\beta/m}\sigma^{1/m}]d\sigma .$$

There is a full list of operational properties of operators (3.3) ([20]-[24], [28]-[29]), derived by using Mellin's transform technique and the properties of the G-functions and they imply, in particular, the necessary corresponding <u>properties of the P.-D. transformation</u> (3.2)=(3.8). We refer to a few of them only with explicit use in the sequel, for example:

(3.9) $$\mathbf{P}\left\{x^j\right\} = d_j\cdot x^{-\beta\gamma_m+j}, \quad j>\tilde{\alpha}=-1 ;$$

(3.10) $$\mathbf{P}\left\{x^{\beta q} f(x)\right\} = c\ x^{\beta(-\gamma_m+q)}\cdot I_{\beta,m-1}^{(-1+q+k/m),(\gamma_k-\gamma_m+1+j/m)}\ f(x) ;$$

(3.11) $$\left\{\mathbf{P}\,\tilde{y}\right\}^{(j)}(0) = d_j\left\{x^{-\beta\gamma_m}\cdot\tilde{y}\right\}^{(j)}(0) = d_j\sum_{i=0}^{j}\frac{x^{-\beta\gamma_m-i}\ \tilde{y}^{(j-i)}(0)}{i!\,(j-i)!\,\Gamma(-\beta\gamma_m-i+1)} ;$$
$j=0,1,2,\ldots$

where the constant $\underline{c}$ is as in Def. 3.1 and

$$d_j = c\prod_{k=1}^{m-1}\left[\ \Gamma((k+j)/m)\ /\ \Gamma(\gamma_k-\gamma_m+1+j/m)\ \right].$$

It is easy to conclude now that:

(3.12) $\mathbf{P}: C_{-1}\overset{\text{into}}{\longmapsto} C_{\alpha}$, $\alpha=-\beta(\gamma_m+1)$ and $\mathbf{P}: C_{m-1}^{(m)}\overset{\text{into}}{\longmapsto} C_{\alpha+\beta}^{(m)}$, $\alpha+\beta=-\beta\gamma_m$;

(3.13) If (2.2) is satisfied, then: $y^{(j)}(0)=\left\{\mathbf{P}\,\tilde{y}\right\}^{(j)}(0)=0$, $j=0,1,2,\ldots$.

According to Dimovski [6], [7], [8] transformation $\mathbf{P}$ is a similarity between the linear right inverse operators of D^m and B, namely: between the integral operators R^m (the m-fold integration defined by (2.14) and L (defined by (1.21)):

(3.14) $$\mathbf{P}\,R^m\,\tilde{f}(x) = L\,\mathbf{P}\,\tilde{f}(x) \qquad \text{for each } \tilde{f}\in C_{-1} .$$

Theorem 4.3 , [24], implies that $\mathbf{P}$ transmutes also D^m into B:

(3.15) $\mathbf{P}\,D^m\,\tilde{y}(x) = B\,\mathbf{P}\,\tilde{y}(x)$ for $\tilde{y}\in C_{m-1}^{(m)}$ (i.e. $\tilde{y}(0)=\tilde{y}'(0)=\ldots=\tilde{y}^{(m-1)}(0)=0$).

Therefore, the Poisson-Dimovski transformation **P** transforms the simpler equation (2.1) into the general hyper-Bessel O.D.E. (1.6) for the image $y(x) = \mathbf{P}\tilde{y}(x)$ with $f(x) = \mathbf{P}\,\tilde{f}(x)$, namely:

(3.16) $\mathbf{P}[\tilde{y}^{(m)}] = B\,\mathbf{P}\,\tilde{y} = B\,y\ ;\quad \mathbf{P}\{\lambda\tilde{y} + \tilde{f}\} = \lambda y + f \implies B\,y = \lambda\,y + f\ ,$

preserving the zero initial conditions (see (3.13)).

4. Solution to the nonhomogeneous hyper-Bessel differential equation

Consider now the initial value problem

$$\begin{cases} B\,y(x) = \lambda\,y(x) + f(x)\ , \quad 0 < x < \infty, \quad f \in C_\alpha\ ,\quad \alpha = -\beta(\gamma_m+1)\ , & (4.1)\\ y(0) = y'(0) = \dots = y^{(m-1)}(0) = 0 & (4.2)\end{cases}$$

with arbitrary hyper-Bessel differential operator B of order $m > 1$ and form (1.1), assuming for convenience arrangement (3.1) for its parameters, that is:

$$\beta > 0\ ;\ \gamma_1 \geqq \gamma_2 \geqq \dots \geqq \gamma_m\ ;\ \alpha = \max_k[-\beta(\gamma_k+1)] = -\beta(\gamma_m+1)\ .$$

Theorem 4.1. Under the conditions stated, initial value problem (4.1),(4.2) has a solution $y \in C^{(m)}_{\alpha+\beta}$, given by the series

(4.3) $$y(x) = (x^{\beta}/\beta^m)\sum_{r=0}^{\infty}(\lambda x^{\beta}/\beta^m)^r\ .\ G_r(x)\ ,$$ convergent for $0 \leqq x\ \infty$,

where $G_r(x)$, $r = 0, 1, 2, \dots$ stand for the following generalized fractional integrals of $f(x)$ (with all the components of multiorder of integration equal to $r+1$)):

(4.4) $$\begin{aligned} G_r(x) &= I^{(\gamma_1,\dots,\gamma_m),(r+1,\dots,r+1)}_{\beta,m}\ f(x)\\ &= \int_0^1 G^{m,0}_{m,m}\left[\sigma\ \middle|\ \begin{matrix}(\gamma_k+r+1)_1^m\\ (\gamma_k)_1^m\end{matrix}\right]\ f(x\,\sigma^{1/\beta})\,d\sigma\ .\end{aligned}$$

Proof. According to the latter conclusions in Section 3, the solution $y(x)$ of (4.1),(4.2) is nothing but the **P**- image of the solution $\tilde{y}(x)$, (2.13) of simpler problem (2.1),(2.2) with $\tilde{f}(x) = \mathbf{P}^{-1} f(x)$, namely:

$$y(x) = \mathbf{P}\,\tilde{y}(x) = \mathbf{P}\ \left[\sum_{r=0}^{\infty}\lambda^r\ R^{m(r+1)}\ \mathbf{P}^{-1}\ f(x)\ \right]\ .$$

Due to the absolute convergence of integral **P** and of series (2.13), we can change their order, and so:

$$y(x) = \sum_{r=0}^{\infty}\lambda^r\ [\ \mathbf{P}\,R^{m(r+1)}\ \mathbf{P}^{-1}\ ]\ f(x).$$

Using repeatedly similarity relation (3.14), we obtain:

$$\mathbf{P}\, R^{m(r+1)}\, \mathbf{P}^{-1} = \mathbf{P}\, \underbrace{R^m R^m \ldots R^m}_{(r+1)\text{-times}} \mathbf{P}^{-1} = L\, \mathbf{P}\, \underbrace{R^m \ldots R^m}_{r\text{-times}} \mathbf{P}^{-1} = \ldots = L^{r+1}\, \mathbf{P}\, \mathbf{P}^{-1} = L^{r+1},$$

and then, we can represent the solution $y(x)$ by series in the integer powers of the hyper-Bessel integral operator L, (1.21), the latter being generalized fractional integrals with integral representations (3.7) involving Meijer's G-functions (McBride [28], [29], also Dimovski and Kiryakova [11]). Hence,

$$(4.5) \qquad y(x) = \sum_{r=0}^{\infty} \lambda^r\, L^{r+1} f(x) = \sum_{r=0}^{\infty} \lambda^r\, (x^{\beta}/\beta^m)^{r+1}\, I_{\beta,m}^{(\gamma_1,\ldots,\gamma_m),(r+1,\ldots,r+1)} f(x)$$

$$= (x^{\beta}/\beta^m) \sum_{r=0}^{\infty} (\lambda x^{\beta}/\beta^m)^r \int_0^1 G_{m,m}^{m,0}\left[\sigma \,\middle|\, \begin{matrix} (\gamma_k+r+1)_1^m \\ (\gamma_k)_1^m \end{matrix}\right] f(x\sigma^{1/\beta})\, d\sigma,$$

and denoting the integrals from 0 to 1 by $G_r(x)$ as in (4.4), we obtain (4.3).

The absolute convergence of this series for all the finite $x \geqq 0$, can be derived from the conditions (3.1), $f \in C_\alpha$ and the asymptotic behaviour of the G-functions involved (see [14, v.1], [26], [32], etc ; this has been done repeatedly for the generalized fractional integrals $I_{\beta,m}^{(\gamma_k),(\delta_k)}$ in C_α in [20]-[24]).

Now we would like to provide another approach from the point of view of the Convolutional Calculus ([8]). To this end, let us come back to integral representation (2.12) of $\tilde{y}(x)$ based on the Duhamel convolution $(*)$: $\tilde{y}(x) = \tilde{f}(x) * h(x)$. Denote by $g(x)$ the $\mathbf{P}$-image of $h(x)$: $g(x) = \mathbf{P}\, h(x)$, that is $h(x) = \mathbf{P}^{-1} g(x)$. It is not a problem to evaluate $g(x)$ using a more general result on the $I_{\beta,m}^{(\gamma_k),(\delta_k)}$– images of an arbitary G-function $G_{\sigma,\tau}^{\mu,\nu} \in C_\alpha$ (in [21]), based on the following useful formula for integral of a product of two G-functions (see [26, v.1], p.159, (1), p.160-164 for the various conditions for its validity):

$$(4.6) \qquad \int_0^\infty z^{\gamma-1}\, G_{p,q}^{m,n}\left[\eta z \,\middle|\, \begin{matrix} (a_i) \\ (b_j) \end{matrix}\right] . \, G_{\sigma,\tau}^{\mu,\nu}\left[\omega z^{k/\ell} \,\middle|\, \begin{matrix} (c_\alpha) \\ (d_\beta) \end{matrix}\right] dz = \eta^{-\gamma} (2\pi)^{\delta(1-k)+\rho(1-l)}$$

$$\times\; k^{U+\gamma(q-p-1)} . \, \ell^V . \, G_{\ell\sigma+kq,\; \ell\tau+kp}^{\ell\mu+kn,\; \ell\nu+km}\left[W \,\middle|\, \begin{matrix} \Delta(\ell,c_1),\ldots,\Delta(\ell,c_\nu);\, \Delta(k,1-b_j-\gamma)\big|_1^q ; \\ \Delta(\ell.d_1),\ldots,\Delta(\ell,d_\mu);\, \Delta(k,1-a_i-\gamma)\big|_1^p ; \end{matrix}\right.$$

$$\left.\ldots \begin{matrix} ;\Delta(\ell,c_{\nu+1}),\ldots,\Delta(\ell,c_\sigma) \\ ;\Delta(\ell,d_{\mu+1}),\ldots,\Delta(\ell,d_\tau) \end{matrix}\right],$$

where: $k,l > 0$ are integers, $\eta \neq 0$, $\omega \neq 0$, $\delta = m+n-(p+q)/2$, $\rho = \mu+\nu-(\sigma+\tau)/2$;
$U = \sum_1^q b_j - \sum_1^p a_i + 1 + (p-q)/2$, $V = \sum_1^\tau d_\beta - \sum_1^\sigma c_\alpha + 1 + (\sigma-\tau)/2$;
$W = [\omega^l \, l^{l(\sigma-\tau)} / \eta^k \, k^{k(p-q)}]$; the symbol $\Delta(k,c)$ stands for the sequence of parameters: $c/k, (c+1)/k, \dots, (c+k-1)/k$; the integral from 0 to ∞ ranges often from 0 to 1 when at least one of the G-functions vanishes outside the unit disk. This is the key-formula for evaluating all the integrals G_r in (4.4).

In particular, for evaluating $g(x)$ it gives:

$$(4.7)\qquad g(x) = (-\lambda)^{\gamma_m} |\lambda|^{-1+1/m} \prod_{k=1}^{m} \Gamma(\gamma_k+1) \, . \, G^{1,0}_{0,m}\left[-\lambda x^\beta/\beta^m \,\middle|\, -\gamma_m; (-\gamma_k)_1^{m-1} \right].$$

The asymptotic behaviour of this G-function is (see [14, v.1], p. 212; [26, v.1]):

$$g(x) = O(x^q) \quad \text{as} \quad x \to 0 \quad \text{with} \quad q = \max_k[-\beta\gamma_k] = -\beta\gamma_m = \alpha+\beta,$$

therefore, $g(x) = \mathbf{P}\, h(x) \in C^{(m)}_{\alpha+\beta}$.

According to this denotation, from $\tilde{y}(x) = \tilde{f}(x) * h(x)$ we obtain:

$$(4.8)\qquad y(x) = \mathbf{P}\,\tilde{y}(x) = \mathbf{P}\,[\,(\mathbf{P}^{-1}f) * (\mathbf{P}^{-1}g)\,] \;:=\; f(x) \,\tilde{*}\, g(x).$$

Since the Duhamel convolution is a convolution of the integration operator R and its powers R^m (2.14) and $\mathbf{P}$ is a similarity between R^m and L (see (3.14)); Theorem 1.3.6 (Dimovski [8], p. 26) states that the operation $(\tilde{*})$ in (4.8) is a convolution (in the sense of [8]) of the hyper-Bessel integral operator L (1.21) in C_α. That is, $(\tilde{*})$ is a commutative, associative and bilinear operation in C_α;
$C_\alpha \times C_\alpha \xrightarrow{(\tilde{*})} C_\alpha$, such that $L[\,f \,\tilde{*}\, g\,] = [\,Lf\,] \,\tilde{*}\, g = f \,\tilde{*}\, [\,Lg\,]$, $f,g \in C_\alpha$.
In particular, we have however: $f \in C_\alpha$, $g \in C^{(m)}_{\alpha+\beta}$ and then, again by the results of Dimovski [8] and Bozhinov [3] it follows that

$$y(x) = \mathbf{P}\,\tilde{y}(x) = f(x) \,\tilde{*}\, g(x) \in C^{(m)}_{\alpha+\beta},$$

and the same for the equivalent series representation (4.3) of solution $y(x)$.
This completes the proof.

Note 4.2. One can guess that the same series (4.3) can be obtained formally by using the Functional Analysis Techniques of Neumann's series (see [35],p.69, Th.2) along the following lines: Equation $B\,y = \lambda\,y + f$ can be rewritten as:

$$y = (B-\lambda)^{-1} f = B^{-1}(1-\lambda B^{-1})^{-1} f = L(1-\lambda L)^{-1} f = \sum_{r=0}^{\infty} \lambda^r L^{r+1} f.$$

However, proving the convergence of this series is related to the problem of boundedness of the operator L in spaces C_α and we find both Convolutional and Transmutation Methods providing a considerable insight.

Note 4.3. Instead of using the Laplace transform technique for solving first the simpler problem (2.1),(2.2), one can approach problem (4.1),(4.2) directly by means of the corresponding Obrechkoff integral transform ([6], [9]-[11]). Then, the function $G(s) = [\, 1 / (\beta^m s^\beta - \lambda)\,]$ is to be interpretted as an Obrechkoff transform of the G-function $g(x)$ in (4.7) by the use of a complex inversion formula in [10]. In this way, the solution $y(x)$ is to be obtained in the form (4.2): $y(x) = g(x) \tilde{*} f(x)$, where $(\tilde{*})$ denotes the convolution of the Obrechkoff integral transform, the same as convolution (4.8) of L in C_α.

Note 4.4. Since most of the right-hand side functions $f(x)$ that could practically arise in equation (4.1) are special cases of the Meijer's G-function, Example 5.1 in next section gives the solution of (4.1),(4.2) in a closed explicit form.

5. Examples

Example 5.1. Most of the elementary and special functions of Mathematical Physics are only special cases of the Meijer's G-function (1.9). Thus, let us consider the case when f(x) is an arbitrary G-function in C_α, that is:

$$(5.1) \qquad f(x) = G^{\mu,\nu}_{\sigma,\tau}\left[x \;\middle|\; \begin{matrix} (c_\alpha)_1^\sigma \\ (d_\beta)_1^\tau \end{matrix} \right], \quad 0 \leqq \sigma \leqq \tau,$$

$$\min_{1\leqq\beta\leqq\mu} d_\beta + \min_{1\leqq k\leqq m} \gamma_k > -1 \qquad ,$$

where γ_k, $k=1,\dots,m$ are the parameters of the operator B (1.1) and for the sake of briefness we assume that $\beta = m$. Using general formula (4.6) one can evaluate the integrals $G_r(x)$, $r = 0, 1, 2, \dots$ in (4.4), namely:

$$(5.2)\quad G_r(x) = A \,.\, G^{m\mu,m\nu+m}_{m\sigma+m,m\tau+m}\left[(x/m^{\sigma-\tau})^m \left| \begin{matrix} \Delta(m,c_\alpha)_1^\nu;(-\gamma_k)_1^m;\Delta(m,c_\alpha)_{\nu+1}^\sigma \\ \Delta(m,d_\beta)_1^\mu;(-\gamma_k-r-1)_1^m;\Delta(m,d_\beta)_{\mu+1}^\tau \end{matrix} \right.\right],$$

where the constant A stands for

$$A = (2\pi)^{(1-m)[\mu+\nu-(\sigma+\tau)/2]} \cdot m^{\sum d_\beta - \sum c_\alpha + 1 + (\sigma-\tau)/2} .$$

Each of these G-functions is in C_α , and the solution $y(x)$ takes the form of a series , convergent for all $x \geqq 0$:

$$(5.3)\quad y(x) = A \,.(\frac{x}{m})^m \sum_{r=0}^{\infty} [\lambda(x/m)^m]^r \; G^{m\mu,m\nu+m}_{m\sigma+m,m\tau+m}\left[\lambda(\frac{x}{m^{\sigma-\tau}})^m \left|\; \ldots \right.\right].$$

More details on series of this kind can be seen in Luke [26, v.1] and some computational methods in Luke [26, v.2].

Example 5.2. The simplest but very common example in the case of equation (4.1) of arbitrary order $m>1$ is with a right-hand side function

$$(5.4)\quad f(x) = x^p \;,\; p > \alpha \quad \Longrightarrow \quad f \in C_\alpha \;.$$

Then integrals (4.4) turn into: $G_r(x) = b_{p,r} \,.\, x^p$, $r = 0, 1, 2, \ldots$ with constants $b_{p,r}$ (see [21]);

$$b_{p,r} = \int_0^1 G^{m,0}_{m,m}\left[\sigma \left| \begin{matrix}(\gamma_k+r+1+p/\beta)_1^m \\ (\gamma_k+p/\beta)_1^m\end{matrix}\right.\right] d\sigma = \left[\prod_{k=1}^m (\gamma_k+1+p/\beta).(\gamma_k+2+p/\beta)_r\right]^{-1}$$

and series (4.3) takes the form

$$(5.5)\quad y(x) = \left[\prod_{k=1}^m(\beta\gamma_k+p+\beta)\right]^{-1} x^{p+\beta} \sum_{r=0}^{\infty} \frac{(1)_r}{\prod_{k=1}^m (\gamma_k+2+p/\beta)_r} \cdot \frac{(\lambda x^{\beta}/\beta^m)^r}{r!}$$

$$= \left[\prod_{k=1}^m(\beta\gamma_k+p+\beta)\right]^{-1} . \; x^{p+\beta} \; {}_1F_m[\,1;(\gamma_k+2+p/\beta)_1^m;\; \lambda x^\beta/\beta^m\,] \;,\quad 0 \leq x < \infty,$$

that is, the solution of the particular problem

$$B\, y = \lambda\, y + x^p \;,\quad y(0)=y'(0)=\ldots=y^{(m-1)}(0)=0$$

is the generalized hypergeometric function $x^{p+\beta}.\; {}_1F_m(\lambda x^\beta/\beta^m)$ in (5.5), and it is also a $G^{1,1}_{1,m+1}$ -function of Meijer.

In the sequel we confine ourselves to some well-known examples related to O.D.E. involving the classical second order operator of Bessel B_ν (1.4) with

$\beta = m = 2$, $\gamma_1 = \nu/2$, $\gamma_2 = -\nu/2$, namely equations of the form

(5.6) $\quad x^2 y'' + x y' - (\nu^2 \mp x^2) y = F(x)$, or: $B_\nu y = \lambda y + f$; $\lambda = \pm 1$, $f(x) = x^{-2} F(x)$.

It is worth pointing out that the corresponding Poisson-Dimovski transformation (3.2),(3.8), transmuting D^2 into B_ν , is the well-known Poisson transformation:

(5.7) $$P_\nu f(x) = [\, 2\,(x/2)^\nu / \sqrt{\pi}\ \Gamma(\nu+1/2)\,] \int_0^1 (1-\sigma^2)^{\nu-1/2} f(x\sigma)\, d\sigma, \nu > -1/2,$$

generated by the Poisson integral for the Bessel functions $J_\nu(x)$ (see [14, v.2], p. 14, 81 ; after a substitution $\sigma := \sin\varphi$):

(5.8) $$J_\nu(x) = P_\nu \{\cos x\}$$

$$= [\, 2\,(x/2)^\nu / \sqrt{\pi}\ \Gamma(\nu+1/2)] \int_0^{\pi/2} (\cos\varphi)^{2\nu} \cos[\, x \cdot \sin\varphi]\, d\varphi,\ \nu > -1/2 .$$

In this case Theorem 4.1 provides the following solutions:

Example 5.3. Consider equation (5.6) with $\lambda = +1$, $f(x) = x^{\mu-1}$, that is $F(x) = x^{\mu+1}$, assuming $\nu > \mu+1 > 0$. Series (4.3) turns into

(5.9) $$y(x) = (x/2)^2 (x^{\mu-1}/4) \sum_{r=0}^{\infty} \frac{(1)_r}{((\mu+\nu+3)/2)_r\ ((\mu-\nu+3)/2)_r} \cdot \frac{(-x^2/4)^r}{r!}$$

$$= [(\mu+\nu+1)(\mu-\nu+1)]^{-1}\ x^{\mu+1}\ {}_1F_2\left(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{x^2}{4}\right)$$

$= s_{\mu,\nu}(x)$, the Lommel function ,

known to be the solution $y(x)$ with $y(0) = y'(0) = 0$ of the equation

(5.10) $\quad x^2 y'' + x y' + (x^2 - \nu^2) y = x^{\mu+1}$

(compare with [26, v.1], p. 217-218, (1),(16)).

Example 5.4. Analogously (even as a special case of (5.9)), the solution $y(x)$ of equation (5.6) with $\lambda = +1$, $f(x) = [\Gamma(1/2)\Gamma(\nu+1/2)]^{-1} (x/2)^{\nu-1}$, that is,

(5.11) $\quad x^2 y'' + x y' + (x^2 - \nu^2) y = 4\,(x/2)^{\nu+1} / \Gamma(1/2)\ \Gamma(\nu+1/2)$, $\nu > 0$,

is obtained as a Struve function $H_\nu(x)$ (see [26, v.1], p. 217-218, (3),(21)):

(5.12) $$y(x) = [\Gamma(3/2)\ \Gamma(\nu+3/2)]^{-1} (x/2)^{\nu+1}\ {}_1F_2(1; 3/2; \nu+3/2; -x^2/4)$$

$$= [\pi . 2^{\nu-1} (1/2)_\nu]^{-1}\, s_{\nu,\nu}(x) = H_\nu(x) ,$$

Example 5.5. An example of Bessel equation (5.6) with $\lambda = -1$ and another kind of right-hand side $f(x) = \exp(-x).x^{\mu-1}$ is the following:

(5.13) $$x^2 y'' + x y' - (x^2 + \nu^2) y = \exp(-x).x^{\mu+1} , \quad \nu > \mu+1 > 0 ,$$

having the solution (as a special case of series (4.3) in Theorem 4.1):

(5.14) $$y(x) = [(\mu-\nu+1)(\mu+\nu+1)]^{-1} \exp(-x).x^{\mu+1}$$

$$\times \; {}_2F_2 (1, \mu+3/2 ; \mu-\nu+2, \mu+\nu+2 ; 2x) = h_{\mu,\nu}(x) ,$$

the so-called "associated Bessel function" (see [26, v.1], p. 219, (25),(27)).

It is easily seen that solutions (5.9), (5.12) and (5.14), related to the Bessel Bessel differential operator B_ν, belong to supspaces $C_\nu^{(\infty)} \subset C_\nu^{(2)}$ of the space $C_{\alpha+\beta}^{(m)} = C_{-\beta\gamma_m}^{(m)} = C_\nu^{(2)}$ (as suggested by Th.4.1).

6. Concluding remarks

The solution of the initial value problem

$$\begin{cases} B\, Y(x) = \lambda Y(x) + f(x) , & 0 < x < \infty, \quad (6.1) \\ \lim\limits_{x\to+0} B_k Y(x) = b_k , \; k=1,\dots,m ; & B_k \text{ as in (1.8)} , \quad (6.2) \end{cases}$$

with arbitrary hyper-Bessel type initial conditions (equivalent to the classical initial conditions (1.7)), belonging to the space $\mathcal{X} = \text{span}\left\{x^{-\beta\gamma_k}\right\}_1^m \oplus C_{\alpha+\beta}^{(m)}$, is:

(6.3) $$Y(x) = y(x) + \sum_{k=1}^{m} a_k\, y_k(x) ,$$

where:

$y(x) \in C_{\alpha+\beta}^{(m)}$ is the solution (4.3) of (4.1),(4.2) ;

$\sum_{k=1}^{m} a_k\, y_k(x) \in \text{span}\left\{x^{-\beta\gamma_k}\right\}_1^m \oplus C_{\alpha+\beta}^{(m)}$ is the solution of the problem:

$$B\, y(x) = \lambda\, y(x) , \quad \lim_{x\to+0} B_k y(x) = b_k , \; k=1,\dots,m ,$$

found in [24], Theorem 2.3 (see here Theorem 1.7) with:

$y_k(x)$ - the G-functions, defined by (2.15),

$$a_k = b_k . \lambda^{\gamma_k} . \beta^{k-m(\gamma_m+1)} . \Big[\prod_{j=k+1}^{m} (\gamma_j - \gamma_k)\Big]^{-1} , \; k=1,\dots,m-1 ; \quad a_m = b_m \lambda^{\gamma_m} . \beta^{-\gamma_m} .$$

The results of [24] and this paper show that in the case of an arbitrary hyper-Bessel differential operator B (1.1)-(1.3) of order $m > 1$, the solutions of the O.D.E. (1.6)=(6.1),(1.19),(1.22) are representable by means of Meijer's G-functions which only in special cases are reduced to other known Special Functions.

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Paper 16

Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences
Tome 46, N° 5, 1993

MATHEMATIQUE
Fonctions spécialles

EXPLICIT SOLUTION OF THE NONHOMOGENEOUS HYPER-BESSEL DIFFERENTIAL EQUATION

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(Submitted by Academician L. Iliev on February 2, 1993)

By a hyper-Bessel differential equation we mean an arbitrary differential equation of m-th order with variable coefficients of the form:

(1) $$By(x)=\lambda y(x)+f(x),\ \lambda=\text{const},\ f(x) \text{ a given function};$$

where

(2) $$B=x^{-\beta}Q_m(xD)=x^{-\beta}\prod_{k=1}^{m}(xD+\beta\gamma_k),\quad 0<x<\infty$$

with $\beta>0$, $\mu_k=-\beta\gamma_k$, $k=1,\dots,m$ denoting the zeros of the m-th degree polynomial $Q_m(\mu)$ and $D=d/dx$.

The so-called hyper-Bessel (Bessel-type) operators B (2) were introduced in the form.

(3) $$B=x^{\alpha_0}Dx^{\alpha_1}\dots x^{\alpha_{m-1}}Dx^{\alpha_m},\quad \beta:=m-(\alpha_0+\alpha_1+\dots+\alpha_m)>0,$$

(Dimovski [3, 4], McBride [11, 12]) and arise in many problems of Analysis and Mathematical Physics in the equivalent form

(4) $$B=x^{-\beta}(x^mD^m+a_1x^{m-1}D^{m-1}+\dots+a_{m-1}xD+a_m),\quad \beta>0.$$

Representations (2), (3), (4) follow each from the other by choosing for example

$$\alpha_0=-\beta-\beta\gamma_1+1;\ \alpha_k=\beta\gamma_k-\beta\gamma_{k+1}+1,\ k=1,\dots m-1;\ \alpha_m=\beta\gamma_m$$

and

$$a_{m-j}=(1/j)!\sum_{l=0}^{i}\left[(-1)^l\binom{j}{l}\prod_{i=1}^{m}(\beta\gamma_i+j+1)\right];\ j=0,1,\dots,m-1.$$

The best known example, giving rise to the name of operators (2), (3), (4), is the second order differential operator of Bessel $\left(\beta=m=2,\ \gamma_{1,2}=\pm\frac{\nu}{2}\right)$:

(5) $$B_\nu=x^{-2}(xD+\nu)(xD-\nu)=x^{\nu-1}Dx^{2\nu+1}Dx^{\nu}=D^2+x^{-1}D-x^{-2}\nu^2,$$

related to the Bessel function $y(x)=J_\nu(x)$ satisfying the equation $B_\nu y(x)=-y(x)$.

Another simple example of higher order is the operator of m-fold differentiation:

(6) $$\widetilde{B}=D^m=(d/dx)^m \text{ with } \beta=m>1,\ \gamma_k=(k/m)-1,\ k=1,\dots,m.$$

This work was partially supported by the Bulgarian Ministry of Science and Education, under grant MM 65/91.

Operational Calculi, Integral Transforms and other tools of Analysis have been recently developed by many authors (see the reference in [4, 5, 9]) for various particular cases of hyper-Bessel operators. This interest is motivated by their frequent appearance in initial or boundary value problems for P. D. E. of Potential Theory, Wave Motion, Diffusion, axially symmetric equations of Elasticity, Hydro-Aerodynamics, etc. Usually, by separating variables or applying a suitable integral transform, one can reduce these problems to initial value problems for the equation $By=\lambda y+f$. For the special case $By=f$ and the homogeneous equation $By=\lambda y$ with arbitrary initial conditions, the problem has been solved in [7, 6, 9] by finding the explicit solutions in terms of the Meijer's G-function (see [1], v. 1).

$$G^{m,n}_{p,q}(x)=G^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right]=G^{m,n}_{p,q}\left[x\,\middle|\,\begin{matrix}(a_j)_1^p\\ (b_k)_1^q\end{matrix}\right]\tag{7}$$

$$=\frac{1}{2\pi i}\int_L \frac{\Pi_{k=1}^m\Gamma(b_k-s)\,\Pi_{j=1}^n\Gamma(1-a_j+s)}{\Pi_{k=m+1}^q\Gamma(1-b_k+s)\,\Pi_{j=n+1}^p\Gamma(a_j-s)}\,x^s\,ds\,.$$

Namely, the following result has been proved:

Theorem 1. Suppose $\gamma_i-\gamma_j\neq 0, \pm1, \pm2, \dots;\ i, j=1, \dots, m$, then in neighbourhood of the point $x=0$ a fundamental system of solutions of the hyper-Bessel O. D. E. $By(x)=\lambda y(x)$ consists of the Meijer's G-functions:

$$y_k(x)=G^{1,0}_{0,m}[-(\lambda x^\beta/\beta^m)\,|-\gamma_k,\ -\gamma_1, \dots, -\gamma_{k-1}, \gamma_{k+1}, \dots, -\gamma_m],\quad k=1, \dots, m,\tag{8}$$

the latter being representable also in terms of the generalized hypergeometric functions ${}_0F_{m-1}(\lambda x^\beta/\beta^m)$ [1] and the hyper-Bessel functions of Delerue $J^{(m-1)}_{\gamma_1,\dots,*,\dots,\gamma_m}(x)$ (see [2])

It is seen now that only a problem that remains open is to solve explicitly the non-homogeneous equation (1) under zero initial conditions. The denotation $C^{(k)}_\alpha$ is further used for the basic functional spaces we consider from practical point of view:

$$C^{(k)}_\alpha:=\{f(x)=x^p\tilde{f}(x);\ p>\alpha,\ \tilde{f}\in C^{(k)}[0, \infty)\},\quad C_\alpha:=C^{(0)}_\alpha\,.\tag{9}$$

We are going to state the following new result to complete the circle of investigations and to show that the G-functions play an important role again.

Theorem 2. Suppose the parameters of hyper-Bessel operator (2) are arranged in a decreasing (increasing) order, e. g. they satisfy

$$\beta>0;\ \gamma_1\geq\gamma_2\geq\dots\geq\gamma_m,\ \text{ i. e. } \alpha:=\max_k[-\beta(\gamma_k+1)]=-\beta(\gamma_m+1).\tag{10}$$

Then the initial value problem

$$\begin{cases}By(x)=\lambda y(x)+f(x), & f\in C_\alpha,\\ y(0)=y'(0)=\dots=y^{(m-1)}(0)=0\end{cases}\tag{11}$$

has a solution $y\in C^{(m)}_{\alpha+\beta}$, given by the series

$$y(x)=(x^\beta/\beta^m)\sum_{r=0}^{\infty}(\lambda x^\beta/\beta^m)^r\,.\,G_r(x),\ \text{ convergent for } 0\leq x<\infty,\tag{12}$$

with $G_r(x)$, $r=0, 1, 2, \dots$ standing for the integrals of G-functions:

$$G_r(x)=\int_0^1 G^{m,0}_{m,m}\left[\sigma\,\middle|\,\begin{matrix}(\gamma_k+r+1)_1^m\\ (\gamma_k)_1^m\end{matrix}\right]f(x\sigma^{1/\beta})\,d\sigma.\tag{13}$$

Idea of the proof. To solve problem (11) we use the transmutation method. It is based on the idea of transforming a new, complicated problem to a simpler one whose solution is known or easier to find. In this special case the techniques of the Generalized Fractional Calculus are useful. The role of transmutation operators (operators of transformation, similarity operators, etc.) is played by the generalized operators of fractional integration [5, 8, 9] of the form

$$(14)\qquad I_{\beta,n}^{(\eta_k),(\delta_k))}f(x)=\begin{cases}\int\limits_0^1 G_{n,n}^{n,0}\left[\sigma\left|\begin{matrix}(\eta_k+\delta_k)_1^n\\(\eta_k)_1^n\end{matrix}\right.\right]f(x\sigma^{1/\beta})\,d\sigma, & \text{if } \Sigma_{k=1}^n\delta_k>0;\\ f(x) & \text{if } \delta_1=\delta_2=\cdots=\delta_n=0.\end{cases}$$

Here $n\geq 1$ is an integer, $\beta>0$, the η_k's are real and the set $\delta=(\delta_1,\ldots,\delta_n)$ is said to be a fractional multiorder of integration. By a suitable choice of these parameters, the classical fractional integrals of Riemann-Liouville and Erdélyi-Kober, the hypergeometric fractional integrals and many other generalized integrals follow from (14). Here we are interested in a transmutation operator, transforming the simplest m-th order hyper-Bessel differential operator (6): $\widetilde{B}=D^m=(d/dx)^m$ into the general one B of form (2). Dimovski [4] has found a Poisson-type integral transformation P: $C_{-1}\rightarrow C_\alpha$ which is a similarity between the integral operators R^m and L, right inverse of $\widetilde{B}$ and B resp.:

$$(15)\qquad R^m\widetilde{f}(x)=\int\limits_0^x\frac{(x-t)^{(m-1)}}{\Gamma(m)}\widetilde{f}(t)\,dt,\quad Lf(x)=(x^\beta/\beta^m)\int\limits_0^1 G_{m,m}^{m,0}\left[\sigma\left|\begin{matrix}(\gamma_i+1)_1^m\\(\gamma_i)_1^m\end{matrix}\right.\right]f(x\sigma^{1/\beta})\,d\sigma,$$

namely:

$$(16)\qquad PR^m\widetilde{f}(x)=L\,P\widetilde{f}(x)\quad\text{for each } \widetilde{f}\in C_{-1}.$$

As it is shown in [6], this Poisson-Dimovski transform can be represented simpler as a generalized fractional integral of form (14), i. e., as an integral transform with a G-function:

$$(17)\qquad Pf(x)=\sqrt{m/(2\pi)^{m-1}}\;\left[\prod_{k=1}^m\Gamma(\gamma_k+1)\right]I_{\beta,\,m-1}^{(-1+k/m),(1+\gamma_k-\gamma_m-k/m)}f((m/\beta)\,x^{\beta/m}).$$

Since P transforms (see [6]) the simpler initial value problem for $\widetilde{y}\in C_{-1}$:

$$(18)\qquad\begin{cases}D^m\widetilde{y}(x)=\widetilde{y}^{(m)}(x)=\lambda\,\widetilde{y}(x)+\widetilde{f}(x), & \widetilde{f}\in C_{-1},\\ \widetilde{y}(0)=\widetilde{y}'(0)=\cdots=\widetilde{y}^{(m-1)}(0)=0\end{cases}$$

into initial value problem (11) for $y\in C_\alpha$, then $P\widetilde{y}(x)=y(x)$ is the sought solution.

By the techniques of Operational Calculus one can find the solution of (18) represented either by an integral involving $G_{0,m}^{1,0}$-function, or by the series

$$(19)\qquad \widetilde{y}(x)=\sum_{r=0}^\infty\lambda^r\left[\int\limits_0^x\frac{(x-t)^{mr+m-1}}{(mr+m-1)!}\widetilde{f}(t)\,dt\right]=\sum_{r=0}^\infty\lambda^r R^{m(r+1)}\widetilde{f}(x)\text{ is:}$$

Then, the solution $y(x)$ as a P-image of (19) with $\widetilde{f}(x)=P^{-1}f(x)$ is:

$$y(x)=P\widetilde{y}(x)=\sum_{r=0}^\infty\lambda^r[P\,R^{m(r+1)}\,P^{-1}]f(x)=\sum_{r=0}^\infty\lambda^r L^{r+1}f(x),$$

since $PR^{m(r+1)}P^{-1}=L^{r+1}$, due to (16). To find $y(x)$ in form (12) it remains only to use the integral representation for the powers L^{r+1}, $r=0, 1, 2, \ldots$ of the hyper-Bessel integral operator L (see (15)), found by McBride [11, 12] and later on in [5] too:

$$L^{\delta} f(x)=(x^{\beta}/\beta^{m})^{\delta} \int_0^1 G_{m,m}^{m,0}\left[\sigma \left| \begin{matrix} (\gamma_k+\delta)_1^m \\ (\gamma_k)_1^m \end{matrix} \right.\right] f(x\sigma^{1/\beta})\, dt, \quad \delta>0.$$

Examples. Solutions to various special cases of hyper-Bessel O. D. E. can be obtained from the above general results. Theorem 1 leads, in particular, to the Bessel functions, hyper-Bessel functions of Delerue, etc. As an illustration, for the second order Bessel operator (5) Theorem 2 gives the Lommel, Struve and associated Bessel functions ([1], v. 2). Let us have in mind that the Meijer's G-functions include as special cases the basic elementary functions and almost all the known Special Functions of Mathematical Physics. That is why, by taking $f(x)$ as an arbitrary Meijer's G-function, we cover all the right-hand sides of nonhomogeneous equation (1) that could arise in practice. In this case integrals G_r (13) are evaluated explicitly and solution (12) of (11) takes the form of a series in G-functions. Methods for their numerical calculations are discussed in [10].

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3

RANGE AND INVERTIBILITY

OF

MELLIN MULTIPLIER TRANSFORMS

CONTENTS

Paper 17

Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS
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Vol. 156, No. 2, April 1991
Printed in Belgium

On the Range and Invertibility of a Class of Mellin Multiplier Transforms, I

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Received October 10, 1989

We study the range and invertibility, in weighted versions of $L^p(0, \infty)$, of the operators N_m^η and M_l^ξ whose Mellin multipliers are $\Gamma(\eta + s/m)$ and $\Gamma(\xi - s/l)$, respectively. We deduce corresponding results for the Laplace transform which generalise those of Widder. The paper is the first of a series concerning operators whose multipliers involve products and quotients of gamma functions and which include as special cases operators with G-function kernels studied by Rooney.

1. This paper is the first of a series in which we shall study operators T satisfying a relation of the form

$$(\mathcal{M}(Tf))(s) = h(s)(\mathcal{M}f)(s). \tag{1.1}$$

Here f is a suitable function defined on $(0, \infty)$, $\mathcal{M}$ denotes the Mellin transform and s is a suitably restricted complex variable. The function h, which is the multiplier of the operator T, will have the general form

$$h(s) = \frac{\prod_{i=k+1}^{K} \Gamma(\eta_i + r_i s) \prod_{j=l+1}^{L} \Gamma(\xi_j - t_j s)}{\prod_{i=1}^{k} \Gamma(\eta_i + r_i s) \prod_{j=1}^{l} \Gamma(\xi_j - t_j s)}, \tag{1.2}$$

where k, l, K, L are non-negative integers satisfying $0 \leqslant k \leqslant K$, $0 \leqslant l \leqslant L$ (empty products being unity by convention), the numbers $r_1, ..., r_K$, $t_1, ..., t_L$ are real and positive, and $\eta_1, ..., \eta_K, \xi_1, ..., \xi_L$ are complex numbers.

Multipliers of the form (1.2) are generalisations of those studied by Rooney in [10]. His multipliers essentially correspond to the case when all the numbers r_i and t_j are unity, and, under appropriate conditions, the

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0022-247X/91 $3.00

corresponding operators T are integral operators with kernels involving Meijer's G-function. Rooney discusses the range of his operators in the setting of certain weighted L^p spaces. The analysis is quite long and there are various different cases to be considered. Naturally, more general values of r_i and t_j will introduce further complications. However, it turns out that a great simplification can be achieved by means of a distributional treatment relative to certain spaces of generalised functions in which the weighted L^p spaces are imbedded. The development of this distributional theory is our ultimate aim.

As a first step, this paper deals with two operators on the weighted spaces $L_{p,\mu}$ with multipliers $\Gamma(\eta + s/m)$ and $\Gamma(\xi - s/l)$, where l and m are real and positive. The operators, denoted by N_m^η and M_l^ξ, respectively, are both related to the classical Laplace transform. We shall concentrate on N_m^η with results for M_l^ξ then following easily. We shall obtain a precise characterisation of the range of N_m^η on $L_{p,\mu}$ under conditions of great generality, namely, $1 < p < \infty$ and $\mathrm{Re}(\eta - \mu/m) \neq 0, -1, -2, \ldots$. Also we shall obtain a family of inversion formulae involving operators of fractional differentiation. From these properties of N_m^η (and hence M_l^ξ) we are able to derive corresponding results for the Laplace transform on $L_{p,\mu}$ which generalise familiar results of Widder [12]. In particular we shall show how the Widder–Post inversion formula is a special case of our family of formulae.

In a future paper we shall use these results to study the product $M_l^\xi N_m^\eta = N_m^\eta M_l^\xi$ on $L_{p,\mu}$. In principle it is possible to study more general finite products on $L_{p,\mu}$ but the situation becomes complicated. Fortunately the results of the present paper enable us to construct certain spaces of testing functions relative to which many of the complications disappear. By the use of adjoint operators we are then able to use the two special cases studied here to develop a comprehensive study of the general operator T in (1.1) within the setting of generalised functions. Thus, although our results may be of interest in their own right, their main role is to provide the starting point for the later distributional theory.

2. For $1 \leqslant p < \infty$ and $\mu \in \mathbb{C}$, we shall denote by $L_{p,\mu}$ the space of (equivalence classes of) complex-valued measurable functions f defined (a.e.) on $(0, \infty)$ such that

$$\| f \|_{p,\mu} \equiv \left\{ \int_0^\infty |x^{-\mu} f(x)|^p \, dx/x \right\}^{1/p} < \infty. \tag{2.1}$$

The expression (2.1) defines a norm on $L_{p,\mu}$ and $(L_{p,\mu}, \| \; \|_{p,\mu})$ is a Banach space. With $D \equiv d/dx$, let

$$F_{p,\mu} = \{ f \in C^\infty(0, \infty) : x^n D^n f \in L_{p,\mu} \quad \text{for} \quad n = 0, 1, 2, \ldots \}. \tag{2.2}$$

The space $F_{p,\mu}$ is a Fréchet space with respect to the topology generated by the seminorms $\{\gamma_n^{p,\mu}\}_{n=0}^{\infty}$, where

$$\gamma_n^{p,\mu}(f) = \| x^n D^n f \|_{p,\mu} \qquad (f \in F_{p,\mu};\, n = 0, 1, 2, \ldots). \tag{2.3}$$

Remark 2.1. For properties of the spaces $F_{p,\mu}$ see [4, Chapter 2] but note that we have replaced μ in [4] by $\mu + 1/p$ here, as a result of introducing the factor $1/x$ in (2.1). The effect is to remove p from the conditions on the parameters.

The Mellin transform $\mathcal{M}$ is defined formally by

$$(\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x)\, dx. \tag{2.4}$$

When interpreted suitably (see [7] or [8], for instance), $(\mathcal{M}f)(s)$ exists (a.e.) on the line

$$\operatorname{Re} s = -\operatorname{Re} \mu \tag{2.5}$$

whenever $f \in L_{p,\mu}$ with $1 \leqslant p \leqslant 2$, $\mu \in \mathbb{C}$. Condition (2.5) will be assumed throughout when working in $L_{p,\mu}$.

We state without proof the following standard result.

THEOREM 2.2. *For fixed $\mu \in \mathbb{C}$ and $k \in L_{1,\mu}$, define the operator K by*

$$(Kf)(x) \equiv (k * f)(x) = \int_0^\infty k(x/t)\, f(t)\, dt/t \qquad (f \in L_{p,\mu}). \tag{2.6}$$

Then K is a bounded linear operator from $L_{p,\mu}$ into itself with

$$\| Kf \|_{p,\mu} \leqslant \| k \|_{1,\mu} \, \| f \|_{p,\mu} \tag{2.7}$$

and for $1 \leqslant p \leqslant 2$, $\mu \in \mathbb{C}$ and $f \in L_{p,\mu}$

$$(\mathcal{M}(Kf))(s) = (\mathcal{M}k)(s)(\mathcal{M}f)(s). \tag{2.8}$$

Formula (2.8) shows that $(\mathcal{M}k)(s)$ is the (Mellin) multiplier corresponding to the operator K. Conversely, we may use suitable multipliers to define corresponding operators.

Notation 2.3. Define the subsets $\Omega_j\, (j = 0, 1, 2, \ldots)$ and Ω of the complex plane by

$$\Omega_0 = \{ z \in \mathbb{C} : \operatorname{Re} z > 0 \}$$

$$\Omega_j = \{ z \in \mathbb{C} : -j < \operatorname{Re} z < -(j-1) \} \qquad (j = 1, 2, \ldots)$$

$$\Omega = \bigcup_{j=0}^{\infty} \Omega_j.$$

DEFINITION 2.4. Let η, μ, and α be complex numbers and let m be real and positive.

(i) For $\eta+1+\mu/m\in\Omega$ and $\alpha\in\mathbb{C}$, $I_m^{\eta,\alpha}$ is the unique continuous linear operator from $F_{p,\mu}$ into itself such that, when $1\leqslant p\leqslant 2$,

$$(\mathcal{M}(I_m^{\eta,\alpha}f))(s)=\frac{\Gamma(\eta+1-s/m)}{\Gamma(\eta+\alpha+1-s/m)}(\mathcal{M}f)(s)\qquad(f\in F_{p,\mu}).\tag{2.9}$$

(ii) For $\eta-\mu/m\in\Omega$ and $\alpha\in\mathbb{C}$, $K_m^{\eta,\alpha}$ is the unique continuous linear operator from $F_{p,\mu}$ into itself such that, when $1\leqslant p\leqslant 2$,

$$(\mathcal{M}(K_m^{\eta,\alpha}f))(s)=\frac{\Gamma(\eta+s/m)}{\Gamma(\eta+\alpha+s/m)}(\mathcal{M}f)(s)\qquad(f\in F_{p,\mu}).\tag{2.10}$$

Remark 2.5. These are the familiar Erdélyi–Kober operators and explicit representations under various conditions on the parameters can be found in [4, Chapter 3]. When $\eta+1+\mu/m\in\Omega_0$ (respectively, $\eta-\mu/m\in\Omega_0$) and $\alpha\in\Omega_0$, the operator $I_m^{\eta,\alpha}$ (respectively, $K_m^{\eta,\alpha}$) is an integral operator belonging to $B(L_{p,\mu})$ for $1\leqslant p<\infty$, $\mu\in\mathbb{C}$. We shall make extensive use of these operators in the sequel.

DEFINITION 2.6. Let η and μ be complex numbers and $m>0$. For $\eta-\mu/m\in\Omega$, N_m^η is the unique operator in $B(L_{p,\mu})$ such that, when $1\leqslant p\leqslant 2$,

$$(\mathcal{M}(N_m^\eta f))(s)=\Gamma(\eta+s/m)(\mathcal{M}f)(s)\qquad(f\in L_{p,\mu}).\tag{2.11}$$

We shall show that such an operator exists by obtaining an explicit representation via (2.6). For convenience, we shall assume until Section 7 that $\eta-\mu/m\in\Omega_0$, so that $\operatorname{Re}(\eta+s/m)>0$ in view of (2.5).

THEOREM 2.7. *For* $1\leqslant p<\infty$ *and* $\operatorname{Re}(\eta-\mu/m)>0$, N_m^η *has the integral representation*

$$(N_m^\eta f)(x)=m\int_0^\infty t^{m\eta}\exp(-t^m)f(x/t)\,dt/t\qquad(f\in L_{p,\mu}).\tag{2.12}$$

Proof. Apply Theorem 2.2 to the kernel $k(x)=mx^{m\eta}\exp(-x^m)$, which belongs to $L_{1,\mu}$ under the stated conditions.

Remark 2.8. (i) It might appear that we could take $m=1$ without loss of generality, since

$$N_m^\eta=P_mN_1^\eta P_m^{-1},\tag{2.13}$$

where

$$(P_m f)(x) = f(x^m). \tag{2.14}$$

However, the parameter m plays a crucial role in subsequent papers.

(ii) N_1^η (and hence N_m^η) is simply related to the Laplace transform $\mathscr{L}$. Indeed,

$$N_1^\eta = x^\eta \mathscr{L} R\, x^{1-\eta} \tag{2.15}$$

as operators on $L_{p,\mu}$, where

$$(Rf)(x) = f(1/x). \tag{2.16}$$

In particular, it follows from Lerch's Theorem [6, Theorem 9.13] that N_m^η is one-to-one on $L_{p,\mu}$.

3. We now set about characterising the range of N_m^η on $L_{p,\mu}$. It is fairly easy to prove that $N_m^\eta(L_{p,\mu})$ is dense in $L_{p,\mu}$, but we can be much more precise. In view of Remark 2.8(ii), our task is equivalent to characterising the range of the Laplace transform on $L_{p,\mu}$. In [12, Chap. VII], Widder discusses the range of $\mathscr{L}$ on $L^p(0, \infty) \equiv L_{p,-1/p}$ but does not cater for a general μ. Widder's results also suggest that the case $p = 1$ has added complications. (Compare Sections 15 and 17 in [12, Chap. VII].) Accordingly we shall restrict attention to $1 < p < \infty$.

For convenience, we shall use the notation

$$\langle f, g \rangle \equiv \int_0^\infty f(t)\, g(1/t)\, dt/t \qquad (f \in L_{p,\mu},\ g \in L_{q,\mu}) \tag{3.1}$$

which is meaningful in view of Hölder's inequality. We can then state a result involving weak compactness.

THEOREM 3.1. *Let $1 < p < \infty$, $\mu \in \mathbb{C}$ and let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in $L_{p,\mu}$. Then there is a subsequence $\{f_{n_i}\}_{i=1}^\infty$ and a function $f \in L_{p,\mu}$ such that*

$$\langle f_{n_i}, g \rangle \to \langle f, g \rangle \quad \text{as} \quad i \to \infty, \qquad \text{for all} \quad g \in L_{q,\mu}.$$

Proof. This follows from [1, p. 130] by simple changes of variables.

In the proof of the main theorem in this section, we shall also use the following result.

LEMMA 3.2. *Let $\mu, \eta \in \mathbb{C}$ with* $\operatorname{Re}(\eta-\mu)>0$ *and let $a \in \mathbb{R}$. For $n=1, 2, \ldots$, define h_n by*

$$h_n(x)=x^{\eta}(1-x/n)^{n+a}\,\chi_{(0,n)}(x) \qquad (x>0),$$

where $\chi_{(0,n)}$ denotes the characteristic function of the interval $(0, n)$. *Then*

$$h_n(x) \to x^{\eta} e^{-x} \qquad \text{in } L_{1,\mu} \quad \text{as} \quad n \to \infty.$$

Proof. Certainly $h_n \in L_{1,\mu}$ for all sufficiently large n, in particular for $n>-a$. We shall assume that $n>-a$ in the rest of the proof. Since $1+y \leqslant e^{y}$ for all $y \in \mathbb{R}$, $(n-x)/(n+a)=1-(x+a)/(n+a) \leqslant \exp(-(x+a)/(n+a))$ for all $x \in \mathbb{R}$. For $0<x<n$, we obtain $[(n-x)/(n+a)]^{n+a} \leqslant \exp(-(x+a))$ whence $(1-x/n)^{n+a} \leqslant (1+a/n)^{n+a} e^{-a} e^{-x}$ or

$$\left(1-\frac{x}{n}\right)^{n+a} \leqslant C_{n,a} e^{-x} \qquad \text{for} \quad 0<x<n, n+a>0, \tag{3.2}$$

where

$$C_{n,a}=\left(1+\frac{a}{n}\right)^{n+a} e^{-a} \to 1 \qquad \text{as} \quad n \to \infty. \tag{3.3}$$

By the triangle inequality

$$\|x^{\eta} e^{-x}-h_n(x)\|_{1,\mu} \leqslant |1-C_{n,a}|\;\|x^{\eta} e^{-x}\|_{1,\mu}+\|C_{n,a} x^{\eta} e^{-x}-h_n(x)\|_{1,\mu}. \tag{3.4}$$

The first term on the right-hand side of (3.4) tends to 0 as $n \to \infty$ by (3.3). So, it only remains to show that the second term on the right-hand side of (3.4) also tends to 0. We split the integral concerned into two parts corresponding to $0<x<n$ and $x \geqslant n$. The latter gives $\int_n^{\infty} |C_{n,a} x^{\eta-\mu} e^{-x}-0|\,(dx/x) \to 0$ as $n \to \infty$. The former gives $\int_0^n |x^{\eta-\mu}[C_{n,a} e^{-x}-(1-x/n)^{n+a}]|\,(dx/x) \leqslant \int_0^n x^{\operatorname{Re}(\eta-\mu)-1}[C_{n,a} e^{-x}-(1-x/n)^{n+a}]\,dx$ by (3.2) $\leqslant \int_0^{\infty} C_{n,a} x^{\operatorname{Re}(\eta-\mu)-1}\,dx-\int_0^1 (ny)^{\operatorname{Re}(\eta-\mu)-1}(1-y)^{n+a}\,n\,dy=C_{n,a}\,\Gamma(\operatorname{Re}(\eta-\mu))-n^{\operatorname{Re}(\eta-\mu)}\,\Gamma(\operatorname{Re}(\eta-\mu))\,\Gamma(n+a+1)/\Gamma(\operatorname{Re}(\eta-\mu)+n+a+1)$. As $n \to \infty$, the first term tends to $\Gamma(\operatorname{Re}(\eta-\mu))$ by (3.3). The second term can be shown to tend to $\Gamma(\operatorname{Re}(\eta-\mu))$ also by using the fact that

$$\frac{\Gamma(n+c)}{n^{c}\Gamma(n)} \to 1 \quad \text{as} \quad n \to \infty \qquad \text{for any real } c \tag{3.5}$$

[3, 1.18(5)]. Hence the integral over $(0, n)$ tends to zero as $n \to \infty$ and in view of our previous comments this completes the proof.

We can now state our characterisation of $N_m^{\eta}(L_{p,\mu})$ in the simplest case.

THEOREM 3.3. *For* $1 < p < \infty$ *and* $\operatorname{Re}(\eta - \mu/m) > 0$, $g \in N_m^\eta(L_{p,\mu})$ *if and only if* $g \in F_{p,\mu}$ *and there exists a constant* A_g *(depending on g but independent of n) such that*

$$\| K_m^{\eta+n, -n} g \|_{p,\mu} \leqslant A_g \, \Gamma(\operatorname{Re}(\eta + n - \mu/m)) \qquad (n = 0, 1, 2, \ldots), \tag{3.6}$$

where $K_m^{\eta+n, -n}$ *is the differential operator* $x^{m\eta + mn}(-D_m)^n x^{-m\eta}$. *(We are dealing with a special case of* (2.10) *and* $D_m \equiv d/dx^m$.)

Proof. We shall only deal with the case $m = 1$. The results for general m then follow easily from (2.13).

(i) $m = 1$. *Necessity.* Assume that $1 < p < \infty$ and $\operatorname{Re}(\eta - \mu) > 0$. Suppose $g = N_1^\eta f$, where $f \in L_{p,\mu}$, so that

$$g(x) = \int_0^\infty (x/u)^\eta \exp(-(x/u)) f(u) \, du/u. \tag{3.7}$$

With $k(x) = x^\eta e^{-x}$, $g(x) = \int_0^\infty k(x/u) f(u) \, du/u$ and since k is smooth we shall differentiate under the integral sign to obtain formally

$$(\delta^j g)(x) = \int_0^\infty (\delta^j k)(x/u) f(u) \, du/u \qquad (\delta \equiv x d/dx, j = 0, 1, 2, \ldots).$$

Since $(\delta^j k)(x) = \sum_{i=0}^j c_i x^{\eta + i} e^{-x}$ for certain constants c_i, $\delta^j g \in L_{p,\mu}$ $(j = 0, 1, 2, \ldots)$, i.e., $g \in F_{p,\mu}$. Similarly, (3.7) leads to

$$\begin{aligned} x^{\eta+n}(-D)^n x^{-\eta} g(x) &= x^{\eta+n}(-D)^n \int_0^\infty u^{-\eta} \exp(-(x/u)) f(u) \, du/u \\ &= x^{\eta+n} \int_0^\infty u^{-\eta}(u^{-1})^n \exp(-(x/u)) f(u) \, du/u \\ &= \int_0^\infty (x/u)^{\eta+n} \exp(-(x/u)) f(u) \, du/u, \end{aligned}$$

i.e.,

$$K_1^{\eta+n, -n} g = K_1^{\eta+n, -n} N_1^\eta f = N_1^{\eta+n} f. \tag{3.8}$$

By (2.7), $\| K_1^{\eta+n, -n} g \|_{p,\mu} = \| N_1^{\eta+n} f \|_{p,\mu} \leqslant \| x^{\eta+n} e^{-x} \|_{1,\mu} \| f \|_{p,\mu} = \Gamma(\operatorname{Re}(\eta + n - \mu)) \| f \|_{p,\mu}$ so that we obtain (3.6) with $A_g = \| f \|_{p,\mu}$ which shows the dependence on f and hence on g.

(ii) $m = 1$. *Sufficiency.* Assume that $1 < p < \infty$ and $\operatorname{Re}(\eta - \mu) > 0$. Let $g \in F_{p,\mu}$ and suppose that g satisfies (3.6). Define f_n $(n = 1, 2, \ldots)$ by

$$f_n(x) = [\Gamma(n)]^{-1} n^{-\eta} (K_1^{\eta+n, -n} g)(nx). \tag{3.9}$$

By [4, Theorems 2.11 and 2.13] $f_n \in F_{p,\mu}$ for all n and (3.6) shows that

$$\| f_n \|_{p,\mu} \leqslant [\Gamma(n)]^{-1} n^{-\operatorname{Re}\eta + \operatorname{Re}\mu} A_g \Gamma(\operatorname{Re}(\eta + n - \mu)) \qquad (n = 1, 2, ...).$$

By (3.5) the right-hand side tends to A_g as $n \to \infty$ so that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $L_{p,\mu}$. By Theorem 3.1, we can find a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ and a function f in $L_{p,\mu}$ such that $\langle f_{n_i}, \psi \rangle \to \langle f, \psi \rangle$ as $i \to \infty$, for all $\psi \in L_{q,\mu}$. In particular, for fixed $x > 0$, let $\psi(t) = (xt)^\eta \exp(-xt)$. This function of t belongs to $L_{q,\mu}$ under the stated conditions. Also $\langle f_{n_i}, \psi \rangle = \int_0^\infty f_{n_i}(t)\, \psi(1/t)\, dt/t = \int_0^\infty (x/t)^\eta \exp(-(x/t)) f_{n_i}(t)\, dt/t = (N_1^\eta f_{n_i})(x)$ and similarly $\langle f, \psi \rangle = (N_1^\eta f)(x)$. Hence $(N_1^\eta f_{n_i})(x) \to (N_1^\eta f)(x)$ for each fixed $x > 0$. Our aim is to show that $N_1^\eta f = g$, thereby establishing sufficiency. On inverting (3.9) we obtain

$$g(x) = [\Gamma(n)]\, n^\eta x^\eta [\Gamma(n)]^{-1} \int_x^\infty (t - x)^{n-1}\, t^{-\eta - n} f_n(t/n)\, dt.$$

Routine manipulations show that $g(x) = \int_0^\infty h_n(x/t) f_n(t)\, dt/t$, where

$$h_n(x) = x^\eta (1 - x/n)^{n-1}\, \chi_{(0,n)}(x) \qquad (n = 1, 2, ...).$$

Hence $N_1^\eta f_n - g = k_n * f_n$, where

$$k_n(x) = x^\eta e^{-x} - h_n(x) = x^\eta e^{-x} - x^\eta (1 - x/n)^{n-1}\, \chi_{(0,n)}(x).$$

By Lemma 3.2 with $a = -1$, $\| k_n \|_{1,\mu} \to 0$ as $n \to \infty$ and hence by Theorem 2.2

$$\| N_1^\eta f_n - g \|_{p,\mu} \leqslant \| k_n \|_{1,\mu}\, \| f_n \|_{p,\mu} \to 0 \qquad \text{as} \quad n \to \infty$$

since $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence in $L_{p,\mu}$. Hence $N_1^\eta f_n$ converges to g in the $L_{p,\mu}$ norm as $n \to \infty$, so that $N_1^\eta f_{n_i}$ converges to g in the $L_{p,\mu}$ norm as $i \to \infty$. By standard results in integration theory, we conclude that $N_1^\eta f = g$ (both functions being continuous). This completes the proof.

Remark 3.4. Under the conditions of Theorem 3.3 we can prove equivalently that $g \in N_m^\eta(L_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there exists a constant A_g such that

$$\| K_m^{\eta + \gamma, -\gamma} g \|_{p,\mu} \leqslant A_g \Gamma(\operatorname{Re}(\eta + \gamma - \mu/m))$$

for all $\gamma \in \mathbb{C}$ such that $\operatorname{Re}(\eta + \gamma - \mu/m) > 0$.

Formula (3.8) is a special case of the second part of our next result, the first part of which sheds some light on ranges. (In both parts, $p = 1$ could be included.)

THEOREM 3.5. *Let* $1<p<\infty$, $\mathrm{Re}(\eta-\mu/m)>0$. *If* $\alpha\in\mathbb{C}$, *then as identities on* $L_{p,\mu}$,

(i) $N_m^{\eta+\alpha}K_m^{\eta,\alpha}=N_m^\eta$ *provided that* $\mathrm{Re}\,\alpha>0$

(ii) $K_m^{\eta,\alpha}N_m^{\eta+\alpha}=N_m^\eta$ *provided that* $\mathrm{Re}(\eta+\alpha-\mu/m)>0$.

Proof. For $1<p\leqslant 2$, both parts are immediate via (2.10) and (2.11). For $p>2$ we can use continuity and density arguments. Note that the weaker condition on α in (ii) arises because if $f\in L_{p,\mu}$ then $N_m^{\eta+\alpha}f\in F_{p,\mu}$ (not merely $L_{p,\mu}$).

COROLLARY 3.6. *Let* $1<p<\infty$, $\mathrm{Re}\,\eta_1<\mathrm{Re}\,\eta_2$ *and* $\mathrm{Re}(\eta_i-\mu/m)>0$ *for* $i=1,2$. *Then* $N_m^{\eta_1}(L_{p,\mu})$ *is a proper subset of* $N_m^{\eta_2}(L_{p,\mu})$.

Proof. Let $g=N_m^{\eta_1}f$, where $f\in L_{p,\mu}$. By Theorem 3.5(i) with $\eta=\eta_1$ and $\alpha=\eta_2-\eta_1$ we obtain $g=N_m^{\eta_2}\phi$, where $\phi=K_m^{\eta_1,\eta_2-\eta_1}f\in L_{p,\mu}$. This establishes inclusion. Now suppose that $g=N_m^{\eta_2}F$, where $F\in L_{p,\mu}$ does not belong to the range of $K_m^{\eta_1,\eta_2-\eta_1}$ on $L_{p,\mu}$. (Such an F exists for $\mathrm{Re}(\eta_2-\eta_1)>0$.) If g were to belong to $N_m^{\eta_1}(L_{p,\mu})$, there would exist $G\in L_{p,\mu}$ such that $g=N_m^{\eta_1}G$ whence $g=N_m^{\eta_2}K_m^{\eta_1,\eta_2-\eta_1}G$ by Theorem 3.5(i). Since $N_m^{\eta_2}$ is one-to-one on $L_{p,\mu}$, we obtain $F=K_m^{\eta_1,\eta_2-\eta_1}G$, a contradiction. This gives strict inclusion as required.

Remark 3.7. Corollary 3.6 shows that $N_m^\eta(L_{p,\mu})$ depends on η (as well as p, μ and m). This may not be surprising but does lead to complications. In contrast we shall discover when we develop the $F_{p,\mu}$ theory in a later paper that $N_m^\eta(F_{p,\mu})$ is independent of η under very general conditions. This simplification provides one of the justifications for the $F_{p,\mu}$ theory.

4. We have noted that, for $1<p<\infty$ and $\mathrm{Re}(\eta-\mu/m)>0$, $N_m^\eta(L_{p,\mu})$ is a subspace of $F_{p,\mu}$ which is dense in $L_{p,\mu}$. Thus $(N_m^\eta(L_{p,\mu}),\|\ \|_{p,\mu})$ is not a Banach space. However, $N_m^\eta(L_{p,\mu})$ can be turned into a Banach space by using a different norm.

DEFINITION 4.1. Let $1<p<\infty$ and $\mathrm{Re}(\eta-\mu/m)>0$, For $g\in N_m^\eta(L_{p,\mu})$ define $\|g\|_{p,\mu,m}^{(;\eta)}$ by

$$\|g\|_{p,\mu,m}^{(;\eta)}=\lim_{n\to\infty}[\mathrm{Re}(\eta+n-\mu/m)]^{-1}\|K_m^{\eta+n,-n}g\|_{p,\mu}. \tag{4.1}$$

Remark 4.2. The slightly curious notation $\|\ \|_{p,\mu,m}^{(;\eta)}$ is to take account of the fact that $\|\ \|_{p,\mu,l}^{(\xi;)}$ and $\|\ \|_{p,\mu,l,m}^{(\xi;\eta)}$ will turn up in connection with M_l^ξ (Section 8) and $M_l^\xi N_m^\eta$ [5], respectively.

THEOREM 4.3. *For* $1<p<\infty$ *and* $Re(\eta-\mu/m)>0$, $\| \ \|^{(;\eta)}_{p,\mu,m}$ *defines a norm on* $N^{\eta}_{m}(L_{p,\mu})$ *and* $(N^{\eta}_{m}(L_{p,\mu}), \| \ \|^{(;\eta)}_{p,\mu,m})$ *is a Banach space.*

Proof. First we must show that the limit in (4.1) is well defined. For $g\in N^{\eta}_{m}(L_{p,\mu})\subset F_{p,\mu}$,

$$K^{\eta+n,-n}_{m} g = K^{\eta+n,1}_{m} K^{\eta+n+1,-(n+1)}_{m} g \qquad [4, \text{Chap. } 3]$$

and by applying Theorem 2.2 to the integral operator $K^{\eta+n,1}_{m}$, we obtain

$$\| K^{\eta+n,-n}_{m} g\|_{p,\mu} \leqslant \frac{\Gamma(\mathrm{Re}(\eta+n-\mu/m))}{\Gamma(\mathrm{Re}(\eta+n+1-\mu/m))} \| K^{\eta+n+1,-(n+1)}_{m} g\|_{p,\mu}.$$

Hence the sequence $\{[\mathrm{Re}(\eta+n-\mu/m)]^{-1} \| K^{\eta+n,-n}_{m} g\|_{p,\mu}\}_{n=0}^{\infty}$ is non-decreasing. By Theorem 3.3, the sequence is bounded above by the constant A_g appearing in (3.6). Hence the sequence converges and $\| g\|^{(;\eta)}_{p,\mu,m}$ is well defined, with $\| g\|^{(;\eta)}_{p,\mu,m}\leqslant A_g$. Indeed we could alternatively write

$$\| g\|^{(;\eta)}_{p,\mu,m} = \inf\{A_g : \text{Eq. (3.6) holds}\} \qquad (g\in N^{\eta}_{m}(L_{p,\mu})) \tag{4.2}$$

and obtain, for $n=0, 1, 2, ...$,

$$\| K^{\eta+n,-n}_{m} g\|_{p,\mu} \leqslant \| g\|^{(;\eta)}_{p,\mu,m}\, \Gamma(\mathrm{Re}(\eta+n-\mu/m)) \qquad (g\in N^{\eta}_{m}(L_{p,\mu})). \tag{4.3}$$

Next we must check that $\| \ \|^{(;\eta)}_{p,\mu,m}$ defines a norm on the range.

(i) $\| g\|^{(;\eta)}_{p,\mu,m}=0 \Rightarrow \| g\|_{p,\mu}=0$ by (4.3) with $n=0 \Rightarrow g=0$ (since $g\in F_{p,\mu}$ and is in particular continuous).

(ii) For $\lambda\in\mathbb{C}$, $\| \lambda g\|^{(;\eta)}_{p,\mu,m}=|\lambda| \, \| g\|^{(;\eta)}_{p,\mu,m}$ by (4.1) and the corresponding property of $\| \ \|_{p,\mu}$.

(iii) Let $g_1, g_2\in N^{\eta}_{m}(L_{p,\mu})$. Then by (4.3),

$$\begin{aligned}\| g_1+g_2\|^{(;\eta)}_{p,\mu,m} &= \lim_{n\to\infty} [\mathrm{Re}(\eta+n-\mu/m)]^{-1} \| K^{\eta+n,-n}_{m}(g_1+g_2)\|_{p,\mu} \\ &\leqslant \lim_{n\to\infty} [\mathrm{Re}(\eta+n-\mu/m)]^{-1} \| K^{\eta+n,-n}_{m} g_1\|_{p,\mu} \\ &\quad + \lim_{n\to\infty} [\mathrm{Re}(\eta+n-\mu/m)]^{-1} \| K^{\eta+n,-n}_{m} g_2\|_{p,\mu} \\ &= \| g_1\|^{(;\eta)}_{p,\mu,m} + \| g_2\|^{(;\eta)}_{p,\mu,m},\end{aligned}$$

as required.

Hence $\| \ \|^{(;\eta)}_{p,\mu,m}$ is a norm on $N^{\eta}_{m}(L_{p,\mu})$.

Finally, to prove completeness let $\{g_j\}_{j=0}^{\infty}$ be a Cauchy sequence in $N^{\eta}_{m}(L_{p,\mu})$ with respect to $\| \ \|^{(;\eta)}_{p,\mu,m}$. For $\varepsilon>0$, $\exists N$ (depending on ε) such that

$$\| g_i-g_j\|^{(;\eta)}_{p,\mu,m}<\varepsilon \qquad \text{for all} \quad i, j\geqslant N.$$

Bearing in mind the monotonicity of the sequence on the right of (4.1) established above, we deduce that

$$\| K_m^{\eta+n,\,-n}(g_i - g_j)\|_{p,\mu} < \varepsilon\Gamma(\mathrm{Re}(\eta + n - \mu/m)) \quad \text{for all} \quad i, j \geqslant N \text{ and all } n \geqslant 0. \tag{4.4}$$

In particular, $\exists C$ (independent of j and n) such that

$$\| K_m^{\eta+n,\,-n} g_j \|_{p,\mu} \leqslant C\Gamma(\mathrm{Re}(\eta + n - \mu/m)) \quad \text{for all} \quad j \geqslant 0 \text{ and all } n \geqslant 0. \tag{4.5}$$

Next we observe that the operator $K_m^{\eta+n,\,-n}$ is a polynomial of degree n in $\delta \equiv x d/dx$. A routine but tedious calculation shows that the seminorms $\{v_n^{p,\mu}\}_{n=0}^{\infty}$ defined on $F_{p,\mu}$ by

$$v_n^{p,\mu}(\phi) = \| K_m^{\eta+n,\,-n} \phi \|_{p,\mu} \qquad (n = 0, 1, 2, ...; \phi \in F_{p,\mu}) \tag{4.6}$$

are equivalent to the seminorms $\{\gamma_n^{p,\mu}\}_{n=0}^{\infty}$ in (2.3). Thus, by (4.4), $\{g_j\}_{j=0}^{\infty}$ is a Cauchy sequence in $F_{p,\mu}$ with its usual topology. As the latter is a complete space, $\exists g \in F_{p,\mu}$ such that $g_j \to g$ in the topology of $F_{p,\mu}$ as $j \to \infty$. In particular, if we use the equivalent seminorms (4.6), we may let $i \to \infty$ in (4.4) to obtain

$$\| K_m^{\eta+n,\,-n}(g - g_j)\|_{p,\mu} \leqslant \varepsilon\Gamma(\mathrm{Re}(\eta + n - \mu/m)) \quad \text{for all} \quad j \geqslant N \text{ and all } n \geqslant 0. \tag{4.7}$$

Hence by (4.5) and the triangle inequality

$$\| K_m^{\eta+n,\,-n} g \|_{p,\mu} \leqslant (C + \varepsilon)\, \Gamma(\mathrm{Re}(\eta + n - \mu/m)) \qquad \text{for } n = 0, 1, 2, ...$$

so that $g \in N_m^{\eta}(L_{p,\mu})$ by Theorem 3.3. Finally by rearranging (4.7) and letting $n \to \infty$, we obtain $\| g - g_j \|_{p,\mu,m}^{(;\eta)} \leqslant \varepsilon$ for all $j \geqslant N \equiv N(\varepsilon)$. Hence $\{g_j\}_{j=0}^{\infty}$ converges to g in $(N_m^{\eta}(L_{p,\mu}), \| \; \|_{p,\mu,m}^{(;\eta)})$ as $j \to \infty$.

This completes the proof of the theorem.

THEOREM 4.4. *Let* $1 < p < \infty$ *and* $\mathrm{Re}(\eta - \mu/m) > 0$. *Then* N_m^{η} *is a homeomorphism from* $(L_{p,\mu}, \| \; \|_{p,\mu})$ *onto* $(N_m^{\eta}(L_{p,\mu}), \| \; \|_{p,\mu,m}^{(;\eta)})$.

Proof. It is obvious that N_m^{η} is one-to-one and onto. Further, if $f \in L_{p,\mu}$,

$$\begin{aligned} \| N_m^{\eta} f \|_{p,\mu,m}^{(;\eta)} &= \lim_{n \to \infty} [\mathrm{Re}(\eta + n - \mu/m)]^{-1} \| K_m^{\eta+n,\,-n} N_m^{\eta} f \|_{p,\mu} \\ &= \lim_{n \to \infty} [\mathrm{Re}(\eta + n - \mu/m)]^{-1} \| N_m^{\eta+n} f \|_{p,\mu} \\ &\leqslant \lim_{n \to \infty} [\mathrm{Re}(\eta + n - \mu/m)]^{-1} \| x^{\eta+n} e^{-x} \|_{1,\mu} \| f \|_{p,\mu}, \end{aligned}$$

where we have used Theorems 3.5(ii) and 2.2. Hence N_m^η is continuous with respect to the norms stated. Finally, since we are now dealing with two Banach spaces, continuity of $(N_m^\eta)^{-1}$ is automatic from the Open Mapping Theorem.

Remark 4.5. We can actually go further and say that, in the situation of Theorem 4.4, N_m^η is an isometry. As this is easily proved using our inversion formulae, we postpone the result until Theorem 5.6.

5. Next we shall consider a family of inversion formulae for N_m^η which are similar in spirit to the familiar Widder–Post inversion formula for the Laplace transform. In this section we shall consider all values of p in the range $1 \leqslant p < \infty$.

DEFINITION 5.1. For each fixed $\mu \in \mathbb{C}$, DK_μ denotes the set of all sequences $\{k_n\}_{n=1}^\infty$ such that

(i) for all sufficiently large n, $k_n \in L_{1,\mu}$

(ii) for all sufficiently large n, $k_n(x) \geqslant 0$ for almost all $x > 0$

(iii) $C_n \equiv \|k_n\|_{1,\mu} \to 1$ as $n \to \infty$

(iv) for each θ satisfying $0 < \theta < 1$, $\int_{1-\theta}^{1+\theta} |x^{-\mu} k_n(x)|\, dx/x \to 1$ as $n \to \infty$.

We can think of DK_μ as a set of sequences of kernels converging in some sense to a delta-type distribution, delta kernels in brief. The following results are proved in [11].

THEOREM 5.2. *Let $\{k_n\}_{n=1}^\infty \in DK_\mu$ for some $\mu \in \mathbb{C}$ and let α be real. Then, for each $f \in L_{p,\mu}$ $(1 \leqslant p < \infty)$, $(x^{i\alpha} k_n) * f$ converges to f with respect to $\| \ \|_{p,\mu}$ as $n \to \infty$, where $*$ is as in* (2.6).

Proof. See [11, Lemma 4.27].

LEMMA 5.3. *For fixed $\eta \in \mathbb{R}$, define k_n $(n = 1, 2, ...)$ by*

$$k_n(x) = \frac{n^n}{\Gamma(n)} x^{n+\eta} e^{-nx} \qquad (x > 0). \tag{5.1}$$

Then $\{k_n\}_{n=1}^\infty \in DK_\mu$ for all $\mu \in \mathbb{C}$.

Proof. See [11, Lemma 4.29].

Our inversion formulae for N_m^η will involve operators of the form λ_a $(a > 0)$, where

$$(\lambda_a \phi)(x) = \phi(ax) \qquad (x > 0). \tag{5.2}$$

THEOREM 5.4. *Let* η, α, $\mu \in \mathbb{C}$ *and* $m > 0$ *with* $\operatorname{Re}(\eta - \mu/m) > 0$ *and let* $g = N_m^\eta f$, *where* $f \in L_{p,\mu}$ $(1 \leqslant p < \infty)$. *Then.*

$$f = (N_m^\eta)^{-1} g = \lim_{n \to \infty} L_m^{n,\eta,\alpha} g,$$

where (for n sufficiently large)

$$L_m^{n,\eta,\alpha} g = \frac{n^{-(\eta+\alpha)}}{\Gamma(n)} \lambda_{n^{1/m}} K_m^{\eta+\alpha+n,\,-(\alpha+n)} g \qquad (g \in F_{p,\mu}) \tag{5.3}$$

and convergence is with respect to $\| \ \|_{p,\mu}$.

Proof. We give details for $m = 1$, the general case then following via (2.13). For $n > -\operatorname{Re}(\eta + \alpha - \mu)$ we obtain, from (5.3) and Theorem 3.5(ii),

$$\begin{aligned} L_1^{n,\eta,\alpha} g &= \frac{n^{-(\eta+\alpha)}}{\Gamma(n)} \lambda_n K_1^{\eta+\alpha+n,\,-(\alpha+n)} N_1^\eta f \\ &= \frac{n^{-(\eta+\alpha)}}{\Gamma(n)} \lambda_n N_1^{\eta+\alpha+n} f. \end{aligned} \tag{5.4}$$

By (2.12) and (5.2), it follows that

$$L_1^{n,\eta,\alpha} g = (x^{i \operatorname{Im}(\eta+\alpha)} k_n) * f,$$

where

$$k_n(x) = \frac{n^n}{\Gamma(n)} x^{\operatorname{Re}(\eta+\alpha)+n} e^{-nx} \qquad (x > 0). \tag{5.5}$$

Since (5.5) is obtained from (5.1) by replacing η by $\operatorname{Re}(\eta + \alpha)$, we see from Lemma 5.3 that $\{k_n\} \in DK_\mu$. Finally, by Theorem 5.2 with α replaced by $\operatorname{Im}(\eta + \alpha)$, we deduce that $L_1^{n,\eta,\alpha} g \to f$ with respect to $\| \ \|_{p,\mu}$ as $n \to \infty$. This completes the proof.

Remark 5.5. Note that Theorem 5.4 holds for $p = 1$ even although we have not obtained a characterisation of $N_1^\eta(L_{p,\mu})$ for $p = 1$. More importantly, for $1 < p < \infty$, we can now add one further piece of information to Theorem 4.4.

THEOREM 5.6. *For* $1 < p < \infty$ *and* $\operatorname{Re}(\eta - \mu/m) > 0$, N_m^η *is an isometry from* $(L_{p,\mu}, \| \ \|_{p,\mu})$ *onto* $(N_m^\eta(L_{p,\mu}), \| \ \|_{p,\mu,m}^{(\,;\,\eta)})$.

Proof. Again it is enough to consider the case $m = 1$.

We use (5.3) with $\alpha=0$, and $m=1$. Then $\operatorname{Re}(\eta+n-\mu)>0$ for all $n \geqslant 1$ and by continuity of $\| \ \|_{p,\mu}$ we obtain

$$\begin{aligned}
\|f\|_{p,\mu} &= \lim_{n\to\infty} \|L_1^{\eta,\eta,0} N_1^\eta f\|_{p,\mu} \\
&= \lim_{n\to\infty} \frac{n^{-\operatorname{Re}\eta}}{\Gamma(\eta)} \|\lambda_n K_1^{\eta+n,-n} N_1^\eta f\|_{p,\mu}
\end{aligned}$$

as in (5.4)

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{n^{-\operatorname{Re}\eta} n^{\operatorname{Re}\mu}}{\Gamma(n)} \|K_1^{\eta+n,-n} N_1^\eta f\|_{p,\mu} \\
&= \lim_{n\to\infty} \frac{\Gamma(\operatorname{Re}(\eta+n-\mu))}{\Gamma(n)\, n^{\operatorname{Re}(\eta-\mu)}} \{\|K_1^{\eta+n,-n} N_1^\eta f\|_{p,\mu}/\Gamma(\operatorname{Re}(\eta+n-\mu))\} \\
&= \|N_1^\eta f\|_{p,\mu,1}^{(\,;\,\eta)}
\end{aligned}$$

by (3.5) and (4.1). This gives the required result.

6. At this stage, we can use our results to obtain information about the Laplace transform on $L_{p,\mu}$. We shall show in particular how our results on the range of $\mathscr{L}$ and our family of inversion formulae contain results of Widder [12] as special cases. A brief summary will suffice.

THEOREM 6.1. *For $1<p<\infty$ and $\operatorname{Re}\mu>-1$, $g\in\mathscr{L}(L_{p,\mu})$ if and only if $g\in F_{p,-\mu-1}$ and there exists a constant A_g (independent of n) such that*

$$\|x^n D^n g\|_{p,-\mu-1} \leqslant A_g \Gamma(\operatorname{Re}(n+1+\mu)) \qquad (n=0,1,2,\ldots). \tag{6.1}$$

Proof. Routine on using (2.15) and Theorem 3.3 with $m=1$.

EXAMPLE 6.2. We shall apply Theorem 6.1 to $L_{p,-1/p} \equiv L^p(0,\infty)$ with $1<p<\infty$. Then we have $\operatorname{Re}\mu=-1/p>-1$ and (6.1) says that

$$[\Gamma(\operatorname{Re}(n+1-1/p))]^{-1} \|x^n D^n g\|_{p,1/p-1} \leqslant A_g \qquad (n=0,1,2,\ldots)$$

for some constant A_g. Writing out the norm explicitly we obtain

$$[\Gamma(n+1-1/p)]^{-p}\left\{\int_0^\infty |x^n D^n g(x)|^p x^{p-2}\,dx\right\} \leqslant A_g^p \qquad (n=0,1,2,\ldots). \tag{6.2}$$

The corresponding formula in [12, p. 312] has the numerical factor $n/(n!)^p$ on the left instead of $[\Gamma(n+1-1/p)]^{-p}$. However, by (3.5)

$$\frac{1}{[\Gamma(n+1-1/p)]^p}\frac{(n!)^p}{n}$$

$$=\left[\frac{\Gamma(n+1)\{(n+1)\}^{-1/p}}{\Gamma(n+1-1/p)}\right]^p\left(1+\frac{1}{n}\right)\to 1 \quad \text{as} \quad n\to\infty.$$

Hence (6.2) is equivalent to Widder's formula.

THEOREM 6.3. *Let* $1\leqslant p<\infty$, $\operatorname{Re}\mu>-1$ *and* $g=\mathscr{L}f$, *where* $f\in L_{p,\mu}$. *Then for any* $\alpha\in\mathbb{C}$

$$f=\lim_{n\to\infty}\frac{n^{-\alpha}}{\Gamma(n)}Rx\lambda_n K_1^{\alpha+n,\,-(\alpha+n)}g, \tag{6.3}$$

where convergence is with respect to the $L_{p,\mu}$ *norm, and* λ_n *is defined via* (5.2).

Proof. The result is immediate on using Theorem 5.4 and (2.15).

EXAMPLE 6.4. We shall apply Theorem 6.3 to $L_{p,-1/p}\equiv L^p(0,\infty)$ for $1<p<\infty$, in which case $\operatorname{Re}\mu=-1/p>-1$. Also we shall choose $\alpha=0$ in (6.3). Then

$$Rx\,\lambda_n K_1^{n,\,-n}g(x)=Rx\,\lambda_n x^n(-1)^n g^{(n)}(x)=(-1)^n\,Rx(nx)^n g^{(n)}(nx)$$

$$\Rightarrow\frac{1}{\Gamma(n)}Rx\,\lambda_n K_1^{n,\,-n}g(x)=\frac{(-1)^n}{(n-1)!}n^n(1/x)^{n+1}g^{(n)}(n/x)$$

$$=\frac{(-1)^n}{n!}\left(\frac{n}{x}\right)^{n+1}g^{(n)}\left(\frac{n}{x}\right).$$

Thus (6.3) becomes

$$f(x)=\lim_{n\to\infty}\frac{(-1)^n}{n!}\left(\frac{n}{x}\right)^{n+1}g^{(n)}\left(\frac{n}{x}\right)$$

with convergence in the $L^p(0,\infty)$ norm. Thus we have recovered the familiar Widder–Post inversion formula [12, Chap. VII, Definition 6 (corrected) and Theorem 15b] as a special case of our family of inversion formulae. As usual the case $p=1$ requires special treatment.

7. We now turn our attention to the strips Ω_j $(j=1,2,\ldots)$ in Notation 2.3 and we shall see that the results obtained above for Ω_0 remain true with minor modifications.

For a fixed positive integer j, we shall be considering s and μ values satisfying

$$-j<\mathrm{Re}(\eta+s/m)<-(j-1), \qquad -j<\mathrm{Re}(\eta-\mu/m)<-(j-1) \quad (7.1)$$

in connection with (2.11).

THEOREM 7.1. *Let* $1\leqslant p<\infty$ *and* $-j<\mathrm{Re}(\eta-\mu/m)<-(j-1)$. *Then the operator* N_m^η *which satisfies* (2.11) *(for* $1\leqslant p\leqslant 2$ *and values of s satisfying* (7.1)*) is given by*

$$N_m^\eta f=(-1)^j N_m^{\eta+j} I_m^{-\eta-j,j} f=(-1)^j I_m^{-\eta-j,j} N_m^{\eta+j} f \qquad (f\in L_{p,\mu}) \quad (7.2)$$

when $I_m^{-\eta-j,j}$ *and* $N_m^{\eta+j}$ *are defined via* (2.10) *and* (2.12), *respectively. Further* $N_m^\eta\in B(L_{p,\mu})$ *with*

$$\| N_m^\eta f\|_{p,\mu}\leqslant(-1)^j\,\Gamma(\mathrm{Re}(\eta-\mu/m))\| f\|_{p,\mu} \qquad (f\in L_{p,\mu}). \quad (7.3)$$

Proof. First observe that, since $\mathrm{Re}(-\eta-j+\mu/m+1)>0$ and $\mathrm{Re}((\eta+j)-\mu/m)>0$, $I_m^{-\eta-j,j}$ and $N_m^{\eta+j}$ can be defined via (2.10) and (2.12) and have multipliers $\Gamma(-\eta-j+1-s/m)/\Gamma(-\eta+1-s/m)$ and $\Gamma(\eta+j+s/m)$, respectively. The operators both belong to $B(L_{p,\mu})$ and they commute since their multipliers do. By using $\Gamma(z+1)=z\Gamma(z)$ we obtain

$$\begin{aligned}
&\frac{(-1)^j\,\Gamma(-\eta-j+1-s/m)}{\Gamma(-\eta+1-s/m)}\,\Gamma(\eta+j+s/m)\\
&\quad=(-1)^j\left\{\prod_{l=1}^{j}(-\eta-l+1-s/m)^{-1}\right\}\Gamma(\eta+j+s/m)\\
&\quad=\left\{\prod_{l=1}^{j}(\eta+l-1+s/m)^{-1}\right\}\Gamma(\eta+j+s/m)\\
&\quad=\frac{\Gamma(\eta+s/m)}{\Gamma(\eta+j+s/m)}\,\Gamma(\eta+j+s/m)=\Gamma(\eta+s/m),
\end{aligned}$$

all the calculations being valid since $\mathrm{Re}(\eta+s/m)$ is not an integer. Equation (7.2) therefore follows, in the first instance for $f\in L_{2,\mu}\cap L_{p,\mu}$ (say) and then generally by continuity and density arguments in the usual way. Finally we can obtain (7.3) by using (2.7) and performing a similar calculation to the above (with s replaced by $-\mu$).

Remark 7.2. The form of N_m^η appearing in (7.2) is reminiscent of the "cut" fractional integrals and Hankel transforms which first appeared in the work of Erdélyi and Kober and have subsequently been discussed by, among others, Braaksma and Schuitman [2], McBride [4, Chaps. 3 and 5], and Rooney [9].

Although the form of N_m^η is now different, our characterisation of its range on $L_{p,\mu}$ remains intact.

THEOREM 7.3. *If* $1<p<\infty$ and $-j<\mathrm{Re}(\eta-\mu/m)<-(j-1)$ *for some* $j=1,2,\ldots$, *then* $g\in N_m^\eta(L_{p,\mu})$ *if and only if* $g\in F_{p,\mu}$ *and there exists a constant* A_g *such that* (3.6) *holds for all* $n=0,1,2,\ldots$, *with* $K_m^{\eta+n,-n}$ *denoting the same differential operator as in Theorem* 3.3.

Proof. We shall make use of the properties of the Erdélyi–Kober operators on the spaces $F_{p,\mu}$ as described in [4, Chap. 3] (with a slight change of notation).

Let $g\in N_m^\eta(L_{p,\mu})$ with $g=N_m^\eta f$ ($f\in L_{p,\mu}$). Since both sides are in $F_{p,\mu}$ by (7.2) and Theorem 3.3, we can apply $I_m^{-\eta,-j}$ to obtain

$$I_m^{-\eta,-j}g=(-1)^j I_m^{-\eta,-j}I_m^{-\eta-j,j}N_m^{\eta+j}f=N_m^{\eta+j}((-1)^jf)$$

so that $I_m^{-\eta,-j}g\in N_m^{\eta+j}(L_{p,\mu})$. Conversely if $I_m^{-\eta,-j}g=N_m^{\eta+j}G$ ($G\in L_{p,\mu}$) then $g=(-1)^jI_m^{-\eta-j,j}N_m^{\eta+j}(-1)^jG=N_m^\eta((-1)^jG)$ by (7.2). Hence

$$g\in N_m^\eta(L_{p,\mu}) \quad \text{iff} \quad I_m^{-\eta,-j}g\in N_m^{\eta+j}(L_{p,\mu}). \tag{7.4}$$

Since $\mathrm{Re}(\eta+j-\mu/m)>0$, we can apply (3.6) with g and η replaced by $I_m^{-\eta,-j}g$ and $\eta+j$ respectively to obtain a necessary and sufficient condition involving A_g in the form

$$\|K_m^{\eta+j+n,-n}I_m^{-\eta,-j}g\|_{p,\mu}\leqslant A_g\Gamma(\mathrm{Re}(\eta+j+n-\mu/m)) \qquad (n=0,1,2,\ldots). \tag{7.5}$$

The composition $K_m^{\eta+j+n,-n}I_m^{-\eta,-j}$ has multiplier

$$\begin{aligned}
&\frac{\Gamma(\eta+j+n+s/m)}{\Gamma(\eta+j+s/m)}\,\frac{\Gamma(-\eta+1-s/m)}{\Gamma(-\eta-j+1-s/m)}\\
&\quad=\frac{\Gamma(\eta+j+n+s/m)}{\Gamma(\eta+j+s/m)}\prod_{l=1}^{j}(-\eta-l+1-s/m)\\
&\quad=\frac{\Gamma(\eta+j+n+s/m)}{\Gamma(\eta+j+s/m)}(-1)^j\prod_{l=1}^{j}(\eta+l-1+s/m)\\
&\quad=\frac{\Gamma(\eta+j+n+s/m)}{\Gamma(\eta+j+s/m)}(-1)^j\frac{\Gamma(\eta+j+s/m)}{\Gamma(\eta+s/m)}\\
&\quad=(-1)^j\frac{\Gamma(\eta+j+n+s/m)}{\Gamma(\eta+s/m)}
\end{aligned}$$

and the latter is the multiplier for $(-1)^j K_m^{\eta+j+n,\,-(j+n)}$. Hence we arrive at (3.6) with n replaced by $n+j$. Equivalently (3.6) holds for all $n \geqslant j$. However, the finite set of numbers $\{[\Gamma(\mathrm{Re}(\eta+n-\mu/m))]^{-1} \|K_m^{\eta+n,\,-n} g\|_{p,\mu} : n=0, 1, ..., j-1\}$ is bounded, so that we can adjust A_g to make (3.6) hold for all $n \geqslant 0$. This completes the proof.

DEFINITION 7.4. Let $1<p<\infty$ and $-j<\mathrm{Re}(\eta-\mu/m)<-(j-1)$ for some $j=1, 2, ...$. For $g \in N_m^\eta(L_{p,\mu})$, define $\|g\|_{p,\mu,m}^{(\,;\,\eta)}$ as in (4.1).

THEOREM 7.5. *For* $1<p<\infty$ *and* $-j<\mathrm{Re}(\eta-\mu/m)<-(j-1)$ $(j=1, 2, ...)$, $(N_m^\eta(L_{p,\mu}), \|\ \|_{p,\mu,m}^{(\,;\,\eta)})$ *is a Banach space.*

Proof. By (7.4), $g \in N_m^\eta(L_{p,\mu})$ iff $I_m^{-\eta,\,-j} g \in N_m^{\eta+j}(L_{p,\mu})$. We then imitate the proof of Theorem 4.3, noting that, since we are working in $F_{p,\mu}$, $I_m^{-\eta,\,-j}$ is a homeomorphism under the given conditions. We omit further details.

In view of the foregoing, the following should come as no surprise.

THEOREM 7.6. *The statements of Theorems* 5.4 *and* 5.6 *remain valid if the condition* $\mathrm{Re}(\eta-\mu/m)>0$ *is replaced by the condition* $-j<\mathrm{Re}(\eta-\mu/m)<-(j-1)$, *where* $j=1, 2, ...$.

Proof. Both results are proved easily. For instance, we may apply Theorem 5.4 for Ω_0, with η and α replaced by $\eta+j$ and $\alpha-j$, respectively, to obtain the corresponding result in Ω_j. We omit further details.

Remark 7.7. We may summarise by saying that the outcome in each strip Ω_j is the same with $j=0$ being a special case. For instance, (7.2) collapses to N_m^η on the right-hand side when $j=0$, since $I_m^{-\eta,0}$ is the identity operator.

8. In this final section we shall briefly discuss the operator M_l^ξ corresponding to the multiplier $\Gamma(\xi-s/l)$. Thus $M_l^\xi \in B(L_{p,\mu})$ satisfies

$$(\mathcal{M}(M_l^\xi f))(s) = \Gamma(\xi-s/l)(\mathcal{M}f)(s) \qquad (f \in L_{p,\mu}) \tag{8.1}$$

under appropriate restrictions on p, μ, ξ, l, and s. An easy calculation shows that

$$M_l^\xi = RN_l^\xi R, \tag{8.2}$$

where R is given by (2.16). The properties of M_l^ξ therefore follow readily from those of N_l^ξ. We shall list the main properties without proof since we shall require them in a subsequent paper concerning the composition of N_m^η

and M_l^ξ (which also explains the change from η to ξ and from m to l here). The proofs make use of the relations

$$K_l^{\eta,\alpha} = RI_l^{\eta-1,\alpha}R; \qquad I_l^{\eta,\alpha} = RK_l^{\eta+1,\alpha}R \tag{8.3}$$

as operator equations on $L_{p,\mu}$ or $F_{p,\mu}$ under appropriate conditions.

THEOREM 8.1. *Let* $1 \leqslant p < \infty$ *and* $\mathrm{Re}(\xi + \mu/l) \neq 0, -1, -2, \ldots$. *Then*

(i) $M_l^\xi \in B(L_{p,\mu})$

(ii) *for* $\mathrm{Re}(\xi + \mu/l) > 0$, M_l^ξ *has the integral representation*

$$(M_l^\xi f)(x) = l \int_0^\infty t^{-l\xi} \exp(-t^{-l}) f(x/t)\, dt/t \tag{8.4}$$

on $L_{p,\mu}$ *and*

$$\| M_l^\xi f \|_{p,\mu} \leqslant \Gamma(\mathrm{Re}(\xi + \mu/l)) \| f \|_{p,\mu} \tag{8.5}$$

(iii) *for* $-j < \mathrm{Re}(\xi + \mu/l) < -(j-1)$, *where* $j = 1, 2, \ldots$, M_l^ξ *has the representation*

$$M_l^\xi f = (-1)^j K_l^{-\xi+1-j,j} M_l^{\xi+j} f = (-1)^j M_l^{\xi+j} K_l^{-\xi+1-j,j} f \tag{8.6}$$

on $L_{p,\mu}$ *and*

$$\| M_l^\xi f \|_{p,\mu} \leqslant (-1)^j \Gamma(\mathrm{Re}(\xi + \mu/l)) \| f \|_{p,\mu} \tag{8.7}$$

(iv) M_l^ξ *is one-to-one on* $L_{p,\mu}$ *and* $M_l^\xi(L_{p,\mu})$ *is dense in* $L_{p,\mu}$.

THEOREM 8.2. *For* $1 < p < \infty$ *and* $\mathrm{Re}(\xi + \mu/l) \neq 0, -1, -2, \ldots$, $g \in M_l^\xi(L_{p,\mu})$ *if and only if* $g \in F_{p,\mu}$ *and there exists a constant* B_g *(depending on* g *but independent of* n*) such that*

$$\| I_l^{\xi-1+n,-n} g \|_{p,\mu} \leqslant B_g \Gamma(\mathrm{Re}(\xi + n + \mu/l)) \qquad (n = 0, 1, 2, \ldots), \tag{8.8}$$

where $I_l^{\xi-1+n,-n}$ *is the differential operator* $x^{-l\xi+l}(D_l)^n x^{l\xi-l+ln}$ *with* $D_l \equiv d/dx^l$.

DEFINITION 8.3. For $1 < p < \infty$ and $\mathrm{Re}(\xi + \mu/l) \neq 0, -1, -2, \ldots$, define $\| \;\|_{p,\mu,l}^{(\xi;)}$ on $M_l^\xi(L_{p,\mu})$ by

$$\| g \|_{p,\mu,l}^{(\xi;)} = \lim_{n\to\infty} [\mathrm{Re}(\xi + n + \mu/l)]^{-1} \| I_l^{\xi-1+n,-n} g \|_{p,\mu}. \tag{8.9}$$

THEOREM 8.4. *For* $1 < p < \infty$ *and* $\mathrm{Re}(\xi + \mu/l) \neq 0, -1, -2, \ldots$,

(i) $(M_l^\xi(L_{p,\mu}), \| \|_{p,\mu,l}^{(\xi;)})$ *is a Banach space*

(ii) M_l^ξ *is an isometric isomorphism from* $(L_{p,\mu}, \| \|_{p,\mu})$ *onto* $(M_l^\xi(L_{p,\mu}), \| \|_{p,\mu,l}^{(\xi;)})$.

THEOREM 8.5. *Let* $\xi, \alpha, \mu \in \mathbb{C}$ *with* $\mathrm{Re}(\xi + \mu/l) \neq 0, -1, -2, \ldots$ *and let* $g = M_l^\xi f$ *for some* $f \in L_{p,\mu}$ $(1 \leqslant p < \infty)$. *Then*

$$f = \lim_{n \to \infty} \frac{n^{-(\xi+\alpha)}}{\Gamma(n)} \lambda_{n^{-1/l}} I_l^{\xi - 1 + \alpha + n, -(\alpha + n)} g, \tag{8.10}$$

where convergence is with respect to $\| \|_{p,\mu}$.

Armed with these results we shall embark on a study of the operator with multiplier $\Gamma(\xi - s/l)\,\Gamma(\eta + s/m)$ relative to the $L_{p,\mu}$ spaces in a subsequent paper [5]. However, we shall find that matters become complicated because of the dependence of $N_m^\eta(L_{p,\mu})$ on η and of $M_l^\xi(L_{p,\mu})$ on ξ, an inheritance from Corollary 3.6 already mentioned in Remark 3.7.

ACKNOWLEDGMENT

One of us (W. J. S.) gratefully acknowledges the award of a Carnegie Scholarship by the Carnegie Trust for the Universities of Scotland to enable this research to be carried out.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium

Paper 18

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ON THE RANGE AND INVERTIBILITY OF A CLASS OF MELLIN MULTIPLIER TRANSFORMS II

by

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Strathclyde Mathematics Research Report No. 1988/7

On the range and invertibility of a class of Mellin multiplier transforms II

A.C. McBride and W.J. Spratt

Abstract

We study the range and invertibility, in weighted versions of $L^p(0,\infty)$, of finite products of operators each of which has the form M_l^ξ or N_m^η studied in a previous paper. As a particular case we examine operators related to the Stieltjes transform and obtain generalisations of results of Widder on the range of the Stieltjes transform on $L^p(0,\infty)$, as well as a family of inversion formulae which includes as special cases several formulae used previously by various authors.

§1 This paper is a sequel to [4] to which the reader should refer for notation and terminology where necessary.

In [4] we discussed, in the setting of the $L_{p,\mu}$ spaces, operators N_m^η and M_l^ξ with corresponding multipliers $\Gamma(\eta+s/m)$ and $\Gamma(\xi-s/l)$ respectively. These operators can be regarded as modifications of the Laplace transform. For each operator we obtained a characterisation of its range on $L_{p,\mu}$ $(1 < p < \infty)$ as well as a family of inversion formulae similar to the Widder-Post formula for the Laplace transform but involving the use of fractional integrals or derivatives. The aim of this paper is to carry through a similar programme for the composition of a finite number of operators each of which is of type M_l^ξ or N_m^η.

The discussion of the most general case leads to a notational nightmare. Fortunately, it transpires that the general case is in principle no different from the particular case of the composition of two of the operators. Accordingly, we shall concentrate on this particular case and choose as a representative the composition of an operator of type M_l^ξ with one of type N_m^η, order being immaterial as the operators commute. The composition of two operators of the same type is similar since

$$N_m^\eta = RM_m^\eta R \quad ; \quad M_l^\xi = RN_l^\xi R \tag{1.1}$$

where

$$(Rf)(x) = f(1/x). \tag{1.2}$$

We indicate briefly what happens in general in §6.

A combination of the form $M_1^\xi N_1^\eta = N_1^\eta M_1^\xi$ produces an operator related to the Stieltjes transform and is of interest in its own right. In particular we shall see that, in a sense to be made clear in §5, our results generalise those of Widder [8] who studied the range of the

Stieltjes transform on $L^p(0,\infty) \equiv L_{p,-1/p}$. Likewise our family of inversion formulae essentially contains a number of particular formulae which have been used by other authors such as Erdélyi [1], Love and Byrne [2], Pollard [6] and Widder [8].

As might be expected, our results look like a combination of those in [4] for M_l^ξ and N_m^η separately. However, there is more to this than meets the eye. For instance, we have studied M_l^ξ on $L_{p,\mu}$ but not on the subset $N_m^\eta(L_{p,\mu})$. So, sometimes we can use results from [4] whereas on other occasions we use techniques developed in [4] rather than actual results.

§2. As in [4] we shall make use of the sets Ω_j $(j = 0,1,2,\ldots)$ defined by

$$\begin{aligned}\Omega_0 &= \{z \in \mathbb{C} \ : \ \operatorname{Re} z > 0\} \\ \Omega_j &= \{z \in \mathbb{C} \ : \ -j < \operatorname{Re} z < -(j-1)\} \quad (j = 1,2,\ldots)\ .\end{aligned} \tag{2.1}$$

Also we shall write

$$\Omega = \bigcup_{j=0}^{\infty} \Omega_j . \tag{2.2}$$

In [4] we defined N_m^η on $L_{p,\mu}$ for $\eta - \mu/m \in \Omega$ [4, Theorems 2.7 and 7.1] and M_l^ξ on $L_{p,\mu}$ for $\xi+\mu/l \in \Omega$ [4, Theorem 8.1]. Accordingly we can make the following definition of their composition.

<u>Definition 2.1</u> Let $\xi,\eta,\mu \in \mathbb{C}$, $l > 0$, $m > 0$ and let $\xi + \mu/l \in \Omega$, $\eta-\mu/m \in \Omega$. For $1 \leq p < \infty$ we define $S_{l,m}^{\xi,\eta}$ on $L_{p,\mu}$ by

$$S_{l,m}^{\xi,\eta} = M_l^\xi N_m^\eta = N_m^\eta M_l^\xi \tag{2.3}$$

where M_l^ξ, N_m^η are as in [4].

<u>Remark 2.2</u> The actual form of $S^{\xi,\eta}_{l,m}$ will vary depending on the sets Ω_j to which $\xi+\mu/l$ and $\eta-\mu/m$ belong. For future use, we describe the simplest situation in the following example.

<u>Example 2.3</u> Let $l = m = 1$, $\mathrm{Re}(\xi+\mu) > 0$, $\mathrm{Re}(\eta-\mu) > 0$. Then M^{ξ}_{1} and N^{η}_{1} have the simple integral representations given by formulae (8.4) and (2.12) in [4]. By inverting the order of integration, we obtain

$$(S^{\xi,\eta}_{1,1}f)(x) = \Gamma(\xi+\eta) \int_0^\infty (x/v)^\eta \, (1+x/v)^{-(\xi+\eta)} f(v)dv/v \tag{2.4}$$

for $f \in L_{p,\mu}$ $(1 \leq p < \infty)$. We may rewrite (2.4) in the form

$$(S^{\xi,\eta}_{1,1}f)(x) = \Gamma(\xi+\eta)x^\eta \, S_{\xi+\eta} \, x^{\xi-1} \, f(x) \tag{2.5}$$

where S_ρ is the generalisation of the Stieltjes transform defined by

$$(S_\rho g)(x) = \int_0^\infty \frac{g(t)}{(x+t)^\rho} \, dt \; . \tag{2.6}$$

In particular $S^{1,0}_{1,1}$ is the usual Stieltjes transform.

We can now consider the mapping properties of $S^{\xi,\eta}_{l,m}$ on $L_{p,\mu}$.

<u>Theorem 2.4</u> Let $\xi,\eta,\mu \in \mathbb{C}$, $\xi+\mu/l \in \Omega$, $\eta-\mu/m \in \Omega$ and $1 \leq p < \infty$. Then

(i) $S^{\xi,\eta}_{l,m} \in B(L_{p,\mu})$

(ii) for $\xi+\mu/l \in \Omega_{j_1}$, $\eta-\mu/m \in \Omega_{j_2}$ and $f \in L_{p,\mu}$,

$$\|S^{\xi,\eta}_{l,m}f\|_{p,\mu} \leq (-1)^{j_1+j_2} \, \Gamma(\mathrm{Re}(\xi+\mu/l))\Gamma(\mathrm{Re}(\eta-\mu/m)) \, \|f\|_{p,\mu} \tag{2.7}$$

(iii) $S^{\xi,\eta}_{l,m}$ is one-to-one on $L_{p,\mu}$.

<u>Proof</u>:- All parts are immediate on using Definition 2.1 above along with Theorems 7.1, 8.1 in [4].

As regards the range of our operator we first observe that, under the hypotheses of Theorem 2.4, $S^{\xi,\eta}_{1,m}(L_{p,\mu})$ is dense in $L_{p,\mu}$, as is easily proved from the corresponding results for N^{η}_{m} and M^{ξ}_{1}.

The dependence of the range on ξ and η is illustrated by the following result.

Lemma 2.5 For $i = 1,2$, let $\xi_i + \mu/1 \in \Omega$, $\eta_i - \mu/m \in \Omega$ where $\operatorname{Re}\xi_1 < \operatorname{Re}\xi_2$, $\operatorname{Re}\eta_1 < \operatorname{Re}\eta_2$. Then for $1 \leq p < \infty$, $S^{\xi_1,\eta_1}_{1,m}(L_{p,\mu})$ is a proper subset of $S^{\xi_2,\eta_2}_{1,m}(L_{p,\mu})$.

Proof:- This is similar to that of [4, Corollary 3.6]. We omit the details.

Remark 2.6 The case $1 = m = 1$ is a special case of Rooney's operators in [7]. With $1 < p < \infty$ in [7, Theorem 7.2], the range of $S^{\xi,\eta}_{1,1}$ is given in terms of the range of the composition of two modified Laplace transforms. On sorting out notation, we find that this composition can be written as $x^{-(\eta-\operatorname{Re}\eta)}N^{\eta}_{1}M^{\xi}_{1}\ x^{\eta-\operatorname{Re}\eta}$ and since multiplication of $f(x)$ by $x^{i\vartheta}$ (ϑ real) is an isomorphism on $L_{p,\mu}$ we simply end up with the range of $M^{\xi}_{1}N^{\eta}_{1}$ on $L_{p,\mu}$ which is reassuring but doesn't constitute progress. In the next section we do obtain something more substantial.

§3 Throughout this section we shall assume that $1 < p < \infty$.

Given that we have obtained characterisations of $N_m^\eta(L_{p,\mu})$ [4, Theorems 3.3 and 7.3] and $M_l^\xi(L_{p,\mu})$ [4, Theorem 8.2], it is no surprise that we can obtain a characterisation of $M_l^\xi N_m^\eta(L_{p,\mu})$, essentially by putting together formulae (3.6) and (8.8) in [4]. This is not an immediate consequence of our earlier results since we have not studied the behaviour of M_l^ξ on $N_m^\eta(L_{p,\mu})$. However the proof is relatively straightforward, as we shall now see.

<u>Theorem 3.1</u> Let $\xi,\eta,\mu \in \mathbb{C}$, $\xi+\mu/l \in \Omega$, $\eta-\mu/m \in \Omega$ and $1 < p < \infty$. Then $g \in S_{l,m}^{\xi,\eta}(L_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there exists a constant C_g (depending on g but independent of the non-negative integers n_1 and n_2) such that

$$\|I_l^{\xi-1+n_1,-n_1} K_m^{\eta+n_2,-n_2} g\|_{p,\mu} \leq C_g \Gamma(\mathrm{Re}(\xi+n_1+\mu/l))\Gamma(\mathrm{Re}(\eta+n_2-\mu/m)) \tag{3.1}$$

for all $n_1,n_2 = 0,1,2,\ldots$.

(Recall that

$$I_l^{\xi-1+n_1,-n_1} = x^{-l\xi+l}(D_l)^{n_1} x^{l\xi-l+ln_1}; \quad K_m^{\eta+n_2,-n_2} = x^{m\eta+mn_2}(-D_m)^{n_2} x^{-m\eta} .)$$

<u>Proof:-</u> <u>Necessity</u> Let $g = S_{l,m}^{\xi,\eta} f$ where $f \in L_{p,\mu}$. Since $g = M_l^\xi N_m^\eta f$, it follows from [4, Theorems 7.3 and 8.2] that $g \in F_{p,\mu}$. Further

$$I_l^{\xi-1+n_1,-n_1} K_m^{\eta+n_2,-n_2} g = I_l^{\xi-1+n_1,-n_1} M_l^\xi K_m^{\eta+n_2,-n_2} N_m^\eta f$$

since all the operators commute. By using multipliers we see that the last expression is $M_l^{\xi+n_1} N_m^{\eta+n_2} f$. (Alternatively, we can use results such as [4, Theorem 3.5 (ii)].) If we now apply [4, (7.3) and (8.7)] with ξ and η replaced by $\xi+n_1$ and $\eta+n_2$ respectively, we obtain (3.1) with $C_g = (-1)^{j_1+j_2} \|f\|_{p,\mu}$ in case $\xi+\mu/l \in \Omega_{j_1}$, $\eta-\mu/m \in \Omega_{j_2}$. (See (2.1).)

<u>Sufficiency</u> Assume that $g \in F_{p,\mu}$ and that g satisfies (3.1) for all

n_1, n_2. Then, for each fixed n_2, $K_m^{\eta+n_2,-n_2}g$ satisfies [4, (8.8)] with $B_g = C_g\,\Gamma(\mathrm{Re}(\eta+n_2-\mu/m))$ and η replaced by n_1. Hence, by [4, Theorem 8.2], for each $n_2 = 0,1,2,\ldots \exists h_{n_2} \in L_{p,\mu}$ such that

$$M_1^{\xi} h_{n_2} = K_m^{\eta+n_2,-n_2} g. \tag{3.2}$$

Further, by [4, Definition 8.3 and Theorem 8.4(ii)],

$$\|h_{n_2}\|_{p,\mu} = \|M_1^{\xi} h_{n_2}\|_{p,\mu,1}^{(\xi;)} \le C_g\,\Gamma(\mathrm{Re}(\eta+n_2-\mu/m)) \quad (n_2=0,1,2,\ldots). \tag{3.3}$$

By standard properties of the Erdélyi-Kober operators on $F_{p,\mu}$ [3, Chapter 3], (3.2) gives

$$g = K_m^{\eta,n_2} M_1^{\xi} h_{n_2} = M_1^{\xi} K_m^{\eta,n_2} h_{n_2} \qquad \text{for all } n_2 = 0,1,2,\ldots$$

so that $M_1^{\xi}[K_m^{\eta,n_2} h_{n_2} - h_o] = 0$ for all n_2. Since M_1^{ξ} is one-to-one on $L_{p,\mu}$ [4, Theorem 8.1(iv)], we obtain

$$h_o = K_m^{\eta,n_2} h_{n_2} \qquad (n_2 = 0,1,2,\ldots).$$

A standard argument now shows that there exists a smooth function $h \in F_{p,\mu}$ such that

$$h_{n_2} = K_m^{\eta+n_2,-n_2} h \quad \text{(a.e.)} \quad \text{for } n_2 = 0,1,2,\ldots\,. \tag{3.4}$$

On substituting in (3.3) and using [4, Theorem 7.3] we deduce that $\exists\, f \in L_{p,\mu}$ such that $N_m^{\eta} f = h$. Finally, from (3.4) and (3.2) with $n_2 = 0$

$$S_{1,m}^{\xi,\eta} f = M_1^{\xi} N_m^{\eta} f = M_1^{\xi} h = M_1^{\xi} h_o = g$$

so that $g \in S_{1,m}^{\xi,\eta}(L_{p,\mu})$ and the proof is complete.

Our aim is to turn the range into a Banach space by renorming. In studying M_1^{ξ} and N_m^{η} separately we immediately defined new norms by means of appropriate limits with respect to an integer variable n. However, this was equivalent (see [4, (4.2)] for instance) to taking the infimum of the set of constants for which certain formulae [4,(3.6) and (8.8)] were true for all n. Since we now have two integer variables n_1 and n_2 to contend with, it is perhaps slightly easier to start with the infimum.

Definition 3.2 Let $\xi,\eta,\mu \in \mathbb{C}$, $\xi+\mu/1 \in \Omega$, $\eta-\mu/m \in \Omega$, $1 < p < \infty$. For $g \in S_{1,m}^{\xi,\eta}(L_{p,\mu})$, define

$$\|g\|_{p,\mu,1,m}^{(\xi;\eta)} = \inf \{C_g : (3.1) \text{ is true for all } n_1,n_2 = 0,1,2,\ldots\}. \quad (3.5)$$

Certainly (3.5) is meaningful in view of Theorem 3.1 and we shall shortly prove that we have a norm on the range. To help us, we show that we can trade in two integer variables n_1 and n_2 for a single integer variable n. In particular, letting $n_1 = n_2 = n$ in (3.1) gives

$$\|I_1^{\xi-1+n,-n}\, K_m^{\eta+n,-n} g\|_{p,\mu} \leq C_g\, \Gamma(\mathrm{Re}(\xi+n+\mu/1))\Gamma(\mathrm{Re}(\eta+n-\mu/m)). \quad (3.6)$$

Lemma 3.3 Under the hypotheses of Definition 3.2,

$$\|g\|_{p,\mu,1,m}^{(\xi;\eta)} = \inf\{C_g : (3.6) \text{ is true for all } n = 0,1,2,\ldots\}. \quad (3.7)$$

Proof:- In view of the preamble it is enough to show that, if C_g is such that (3.6) holds for all $n = 0,1,2,\ldots$, then (3.1) holds for all $n_1,n_2 = 0,1,2,\ldots$. Here we again make use of properties of the Erdélyi-Kober operators on $F_{p,\mu}$. In particular for $n_1 < n_2$

$$\|I_1^{\xi-1+n_1,-n_1}\, K_m^{\eta+n_2,-n_2} f\|_{p,\mu} = \|I_1^{\xi-1+n_1,n_2-n_1}\, I_1^{\xi-1+n_2,-n_2}\, K_m^{\eta+n_2,-n_2} f\|_{p,\mu}$$

$$\leq \frac{\Gamma(\mathrm{Re}(\xi+n_1+\mu/1))}{\Gamma(\mathrm{Re}(\xi+n_2+\mu/1))}\, \|I_1^{\xi-1+n_2,-n_2}\, K_m^{\eta+n_2,-n_2} f\|_{p,\mu}$$

$$= \frac{\Gamma(\mathrm{Re}(\xi+n_1+\mu/1))}{\Gamma(\mathrm{Re}(\xi+n_2+\mu/1))}\, C_g\, \Gamma(\mathrm{Re}(\xi+n_2+\mu/1))\, \Gamma(\mathrm{Re}(\eta+n_2-\mu/m)) \quad \text{by assumption}$$

from which (3.1) holds in this case. The case $n_1 > n_2$ can be handled by writing

$$I_1^{\xi-1+n_1,-n_1} K_m^{\eta+n_2,-n_2} f = K_m^{\eta+n_2,n_1-n_2} I_1^{\xi-1+n_1,-n_1} K_m^{\eta+n_1,-n_1} f$$ (the last

three operators commuting with each other). Since the case $n_1 = n_2$ is trivial, the proof is complete.

Corollary 3.4 Under the hypotheses of Definition 3.2,

$$\|g\|_{p,\mu,1,m}^{(\xi;\eta)} = \lim_{n\to\infty} [\Gamma(\mathrm{Re}(\xi+n+\mu/1))\Gamma(\mathrm{Re}(\eta+n-\mu/m))]^{-1}\|I_1^{\xi-1+n,-n} K_m^{\eta+n,-n} g\|_{p,\mu}. \quad (3.8)$$

Proof:- Since we are dealing with a bounded sequence whose supremum is $\|g\|_{p,\mu,1,m}^{(\xi;\eta)}$ by Lemma 3.3, the result will follow if we can show that the sequence is monotonic non-decreasing. To this end we again use properties of the Erdélyi-Kober operators as in the previous proof to obtain

$$\|I_1^{\xi-1+n,-n} K_m^{\eta+n,-n} g\|_{p,\mu} = \|I_1^{\xi-1+n,1} K_m^{\eta+n,1} I_1^{\xi-1+(n+1),-(n+1)} K_m^{\eta+(n+1),-(n+1)} g\|_{p,\mu}$$

$$\leq \frac{\Gamma(\mathrm{Re}(\xi+n+\mu/1))}{\Gamma(\mathrm{Re}(\xi+n+1+\mu/1))} \frac{\Gamma(\mathrm{Re}(\eta+n-\mu/m))}{\Gamma(\mathrm{Re}(\eta+n+1-\mu/m))} \|I_1^{\xi-1+(n+1),-(n+1)} K_m^{\eta+(n+1),-(n+1)} g\|_{p,\mu}$$

The required monotonicity now follows and the proof is complete.

We have thus obtained an analogue of the definitions of the norms on $M_1^{\xi}(L_{p,\mu})$ and $N_m^{\eta}(L_{p,\mu})$. With this version we can conveniently handle the next result.

Theorem 3.5 Let $\xi,\eta,\mu \in \mathbb{C}$, $\xi+\mu/1 \in \Omega$, $\eta-\mu/m \in \Omega$ and $1 < p < \infty$. Then $\|\ \|_{p,\mu,1,m}^{(\xi;\eta)}$ is a norm on $S_{1,m}^{\xi,\eta}(L_{p,\mu})$ and $(S_{1,m}^{\xi,\eta}(L_{p,\mu}), \|\ \|_{p,\mu,1,m}^{(\xi;\eta)})$ is a Banach space.

Proof:- That we have a norm follows fairly easily. For instance,

$\|g\|^{(\xi;\eta)}_{p,\mu,1,m} = 0$ if and only if $\|I_1^{\xi-1+n,-n} K_m^{\eta+n,-n} g\|_{p,\mu} = 0$ for all $n = 0,1,\ldots$ and taking $n = 0$ gives $\|g\|_{p,\mu} = 0$ so that $g \equiv 0$ (since g is smooth). We omit proofs of the other norm properties.

To prove completeness, we consider a Cauchy sequence $\{g_i\}_{i=0}^{\infty}$ in $S_{1,m}^{\xi,\eta}(L_{p,\mu})$ and let $\varepsilon > 0$. There exists N (depending on ε) such that

$$\|g_i - g_j\|^{(\xi;\eta)}_{p,\mu,1,m} < \varepsilon \qquad \text{for all } i,j \geq N.$$

By Lemma 3.3 and (3.6), for each $n = 0,1,2,\ldots$

$$\|I_1^{\xi-1+n,-n} K_m^{\eta+n,-n}(g_i - g_j)\|_{p,\mu} < \varepsilon\, \Gamma(\mathrm{Re}(\xi+n-\mu/1))\Gamma(\mathrm{Re}(\eta+n-\mu/m)). \quad (3.9)$$

For each fixed $n = 0,1,2,\ldots,$ let

$$g_i^{(n)} = I_1^{\xi-1+n,-n} K_m^{\eta+n,-n} g_i \qquad (i = 0,1,2,\ldots). \quad (3.10)$$

By (3.9), $\{g_i^{(n)}\}_{i=0}$ is a Cauchy sequence in $L_{p,\mu}$ for each fixed n. By completeness of $(L_{p,\mu}, \|\ \|_{p,\mu})$, for each $n = 0,1,2,\ldots,$ $\exists\, g^{(n)} \in L_{p,\mu}$ such that

$$g^{(n)} = \lim_{i\to\infty} g_i^{(n)} \quad . \quad (3.11)$$

We may apply the operator $I_1^{\xi-1,n}\ K_m^{\eta,n} \in B(L_{p,\mu})$ to (3.11) and use (3.10) to obtain

$$I_1^{\xi-1,n} K_m^{\eta,n}\, g^{(n)} = \lim_{i\to\infty} I_1^{\xi-1,n} K_m^{\eta,n}\, g_i^{(n)} = \lim_{i\to\infty} g_i = \lim_{i\to\infty} g_i^{(0)} = g^{(0)}.$$

Once again (as in the derivation of (3.4)) we can deduce that $\exists\, g \in F_{p,\mu}$ such that

$$g^{(n)} = I_1^{\xi-1+n,-n} K_m^{\eta+n,-n} g \quad \text{(a.e.)} \qquad \text{for } n = 0,1,2,\ldots\ . \quad (3.12)$$

Since Cauchy sequences are bounded in norm, we can deduce from (3.9) and (3.10) that there is a constant C (independent of i and n) such that

$$\|g_i^{(n)}\|_{p,\mu} \leq C\Gamma(\mathrm{Re}(\xi+n+\mu/l))\,\Gamma(\mathrm{Re}(\eta+n-\mu/m)) \quad (i,n = 0,1,2,\ldots).$$

On letting $i\to\infty$ and using (3.11), (3.12) and (3.7) in turn, we deduce that $g \in S^{\xi,\eta}_{p,\mu,l,m}(L_{p,\mu})$ with $\|g\|^{(\xi;\eta)}_{p,\mu,l,m} \leq C$. Also from (3.9) and (3.10), for $n = 0,1,2,\ldots,$

$$\|g_i^{(n)}-g_j^{(n)}\|_{p,\mu} < \varepsilon\ \Gamma(\mathrm{Re}(\xi+n+\mu/l))\,\Gamma(\mathrm{Re}(\eta+n-\mu/m)) \qquad (i,j \geq N).$$

On letting $j\to\infty$ and using (3.11) we obtain, for $n = 0,1,2,\ldots,$

$$\|g_i^{(n)}-g^{(n)}\|_{p,\mu} \leq \varepsilon\ \Gamma(\mathrm{Re}(\xi+n+\mu/l))\,\Gamma(\mathrm{Re}(\eta+n-\mu/m)) \qquad (i \geq N).$$

Finally use of (3.10), (3.12) and (3.7) in turn gives

$$\|g_i-g\|^{(\xi;\eta)}_{p,\mu,l,m} \leq \varepsilon \qquad \text{for all } i \geq N.$$

Thus the Cauchy sequence $\{g_i\}_{i=0}^{\infty}$ converges to g with respect to

$\|\ \|^{(\xi;\eta)}_{p,\mu,l,m}$ and this establishes completeness.

<u>Corollary 3.6</u> Let $\xi,\eta,\mu \in \mathbb{C}$, $\xi+\mu/l \in \Omega$, $\eta-\mu/m \in \Omega$ and $1 < p < \infty$. Then $S^{\xi,\eta}_{l,m}$ is a homeomorphism from $(L_{p,\mu},\|\ \|_{p,\mu})$ onto $(S^{\xi,\eta}_{l,m}(L_{p,\mu}),\|\ \|^{(\xi;\eta)}_{p,\mu,l,m})$.

<u>Proof:-</u> Consider only positive integers n such that

$\xi+n+\mu/l \in \Omega_o$, $\eta+n-\mu/m \in \Omega_o$. (See (2.1).)

For such n we obtain

$$\|I_l^{\xi-1+n,-n}\,K_m^{\eta+n,-n}\,M_l^{\xi}N_m^{\eta}\,f\|_{p,\mu} = \|M_l^{\xi+n}\,N_m^{\eta-n}\,f\|_{p,\mu} \text{ (by multipliers)}$$

$$\leq \Gamma(\mathrm{Re}(\xi+n+\mu/l))\,\Gamma(\mathrm{Re}(\eta+n-\mu/m))\ \|f\|_{p,\mu}.$$

On rearranging, letting $n\to\infty$ and using (3.8) we obtain

$$\|S^{\xi,\eta}_{l,m}f\|^{(\xi;\eta)}_{p,\mu,l,m} \leq \|f\|_{p,\mu} \qquad (f \in L_{p,\mu})$$

which shows that $S^{\xi,\eta}_{l,m}$ is continuous. It is one-to-one by Theorem 2.4(iii).

and since we have two Banach spaces $(S^{\xi,\eta}_{l,m})^{-1}$ is continuous by the Open Mapping Theorem.

Remark 3.7 We can go further and show that, in the situation of Corollary 3.6, $S^{\xi,\eta}_{1,m}$ is an isometry. This can be extracted from the proof of Theorem 3.1 but it emerges naturally from our inversion formulae in the next section and so we postpone this result until then.

§4. In this section we shall consider a family of inversion formulae for $S^{\xi,\eta}_{1,m}$. Formally these are found by combining those obtained previously for $(M^{\xi}_{1})^{-1}$ and $(N^{\eta}_{m})^{-1}$ in [4, Theorems 5.4, 7.6 and 8.5]. However, for reasons similar to those outlined at the start of §3, we shall start from scratch and use the results for delta kernels in [4, §5] to derive norm convergence of the appropriate sequences. We observe that p = 1 can be allowed here.

Definition 4.1 For fixed $\xi,\eta,\mu,\alpha,\beta \in \mathbb{C}$, $1 \leq p < \infty$ and all sufficiently large positive integers n_1 and n_2 we define the operator $L^{n_1,n_2,\xi,\eta,\alpha,\beta}_{1,m}$ on $F_{p,\mu}$ by

$$L^{n_1,n_2,\xi,\eta,\alpha,\beta}_{1,m}\, g = \frac{n_1^{-(\xi+\alpha)} n_2^{-(\eta-\beta)}}{\Gamma(n_1)\Gamma(n_2)} \lambda_{n_1}{}^{-1/1} \lambda_{n_2}{}^{1/m} I_1^{\xi-1+\alpha+n_1,-(\alpha+n_1)} K_m^{\eta-\beta+n_2,-(\beta+n_2)} \tag{4.1}$$

where the λ operators are defined as in [4, (5.2)].

Remark 4.2 Note that (4.1) is the composition of the operators in [4, (5.3) and (8.10)] and will certainly be well-defined when $\xi+\alpha+n_1+\mu/1 \in \Omega_0$ and $\eta+\beta+n_2-\mu/m \in \Omega_0$. (See (2.1).) Further the I and K operators then commute with each other.

Theorem 4.3 Under the hypotheses of Definitions 2.1 and 4.1, if $g = S^{\xi,\eta}_{1,m} f$ ($f \in L_{p,\mu}$) then

$$f = (S_{l,m}^{\xi,\eta})^{-1} g = \lim_{n_1\to\infty,\ n_2\to\infty} L_{l,m}^{n_1,n_2,\xi,\eta,\alpha,\beta}\, g \tag{4.2}$$

where n_1 and n_2 tend to infinity independently and convergence is with respect to $\| \ \|_{p,\mu}$.

Proof:- We shall break the proof up into a number of steps.

(i) For $f \in L_{p,\mu}$ we use (4.1) and (2.3) to obtain

$$L_{l,m}^{n_1,n_2,\xi,\eta,\alpha,\beta}\, S_{l,m}^{\xi,\eta} f$$

$$= \frac{n_1^{-(\xi+\alpha)} n_2^{-(\eta+\beta)}}{\Gamma(n_1)\,\Gamma(n_2)}\ \lambda_{n_1^{-1/l} n_2^{1/m}}\, I_l^{\xi-1+\alpha+n_1,-(\alpha+n_1)} M_l^{\xi} K_m^{\eta+\beta+n_2,-(\beta+n_2)} N_m^{\eta} f$$

$$= \frac{n_1^{-(\xi+\alpha)} n_2^{-(\eta+\beta)}}{\Gamma(n_1)\,\Gamma(n_2)}\ \lambda_{n_1^{-1/l}\, n_2^{1/m}}\, M_l^{\xi+\alpha+n_1} N_m^{\eta+\beta+n_2} f \tag{4.3}$$

by using multipliers. (See, for instance, [4, (3.8) and Theorem 3.5].) These manipulations are valid for all n_1,n_2 such that $\xi+\alpha+n_1+\mu/l \in \Omega_o$ and $\eta+\beta+n_2-\mu/m \in \Omega_o$ i.e. for $n_1 > -\mathrm{Re}(\xi+\alpha+\mu/l)$, $n_2 > -\mathrm{Re}(\eta+\beta-\mu/m)$. We shall assume henceforth that n_1 and n_2 satisfy these inequalities. This means in particular that $M_l^{\xi+\alpha+n_1}$ and $N_m^{\eta+\beta+n_2}$ both have integral representations of the forms in [4, (8.4) and (2.12) respectively] which in turn can be written as Mellin convolutions with appropriate kernels. Incorporating the other items, we can thus write (4.3) in the form

$$k_1^{(n_1)} * k_2^{(n_2)} * f \tag{4.4}$$

where the kernels $k_1^{(n_1)}$ and $k_2^{(n_2)}$ are given by

$$k_1^{(n_1)}(x) = \frac{l n_1^{n_1}}{\Gamma(n_1)}\, x^{-l(\xi+\alpha+n_1)} \exp(-n_1 x^{-l}) \tag{4.5}$$

$$k_2^{(n_2)}(x) = \frac{m n_2^{n_2}}{\Gamma(n_2)}\, x^{m(\eta+\beta+n_2)} \exp(-n_2 x^{m}) \tag{4.6}$$

and $*$ denotes the usual Mellin convolution, i.e.

$$(k*f)(x) = \int_0^\infty k(x/t)f(t)dt/t = \int_0^\infty k(t)f(x/t)dt/t \quad . \tag{4.7}$$

For convenience we have suppressed the dependence of the kernels on $\xi,\eta,\alpha,\beta,\underset{\sim}{1}$ and m.

(ii) Next we shall use our results for delta kernels to show that, for any $f \in L_{p,\mu}$, $k_i^{(n)}*f$ converges to f with respect to $\| \ \|_{p,\mu}$ as $n\to\infty$ $(i = 1,2)$. Starting with $k_2^{(n)}$, we write

$$k_2^{(n)}(x) = m\, P_m\, \{x^{i\ \mathrm{Im}(\eta+\beta)}\ k_n(x)\} \tag{4.8}$$

where

$$k_n(x) = \frac{n^n}{\Gamma(n)}\, x^{\mathrm{Re}(\eta+\beta)+n}\, \exp(-nx) \qquad (n = 1,2,\ldots) \tag{4.9}$$

and P_m is given by [$\underset{\sim}{4}$, (2.14)], i.e. $(P_m f)(x) = f(x^m)$. We observe that (4.9) is the same function as in [$\underset{\sim}{4}$, Lemma 5.3] except that $\mathrm{Re}(\eta+\beta)$ now replaces the real number η. Hence k_n satisfies the conditions of [$\underset{\sim}{4}$, Definition 5.1] and so by [$\underset{\sim}{4}$, Theorem 5.2] applied with μ replaced by μ/m we deduce that

$$\|(x^{i\ \mathrm{Im}(\eta+\beta)}k_n*F - F\|_{p,\mu/m} \to 0 \quad \text{as } n\to\infty \quad \text{for all } F \in L_{p,\mu/m}\ .$$

But P_m is a homeomorphism from $L_{p,\mu/m}$ onto $L_{p,\mu}$ and $\|mP_m g\|_{p,\mu} = \|g\|_{p,\mu/m}$ for all $g \in L_{p,\mu/m}$. Hence as $n\to\infty$

$$\|mP_m\{x^{i\ \mathrm{Im}(\eta+\beta+n)}k_n*F\} - mP_m F\|_{p,\mu} \to 0 \quad \text{for all } F \in L_{p,\mu/m}\ . \tag{4.10}$$

A simple calculation based on (4.7) shows that $mP_m\,(k*h) = (mP_m k)*(mP_m h)$ under appropriate circumstances. Applying this to (4.10) and using (4.8), we obtain

$$\|k_2^{(n)} * (mP_m F) - mP_m F\|_{p,\mu} \to 0 \text{ as } n\to\infty \text{ for all } F \in L_{p,\mu/m}.$$

On writing $f = mP_m F$ and using the homeomorphic property of P_m mentioned above, we see that $k_2^{(n)}*f$ converges to f with respect to $\|\ \|_{p,\mu}$ as $n\to\infty$. Recall that this calculation assumed that $\eta-\mu/m \in \Omega$ and $\eta+\beta+n-\mu/m \in \Omega_0$.

To handle $k_1^{(n)}$, we shall use the operator R, defined by (1.2), which is easily seen to be an isometric isomorphism from $L_{p,\mu}$ onto $L_{p,-\mu}$. Hence if $\xi+\mu/1 \in \Omega$ and $\xi+\alpha+n+\mu/1 \in \Omega_0$ we may apply the previous case with $\xi,\alpha,-\mu$ and 1 replacing η,β,μ and m respectively and noting that then $k_2^{(n)}$ becomes $Rk_1^{(n)}$. Thus $\|(Rk_1^{(n)}*F)-F\|_{p,-\mu} \to 0$ as $n\to\infty$ for all $F \in L_{p,-\mu}$ and hence

$$\|R(Rk_1^{(n)}*F)-RF\|_{p,\mu} \to 0 \text{ as } n\to\infty \text{ for all } F \in L_{p,-\mu}. \tag{4.11}$$

Another simple calculation based on (4.7) shows that $R(k*h) = (Rk)*(Rh)$ under appropriate circumstances. Applying this to (4.11), writing $f = RF$ and using the homeomorphic property of R mentioned above gives

$$\|k_1^{(n)}*f-f\|_{p,\mu} \to 0 \text{ as } n\to\infty \text{ for all } f \in L_{p,\mu} .$$

This completes the second step.

(iii) Finally we shall use (ii) along with (4.4) to prove the result stated. Indeed with

$g = S_{1,m}^{\xi,\eta} f$, (4.3) and (4.4) give

$$\|L_{1,m}^{n_1,n_2,\xi,\eta,\alpha,\beta}\, g-f\|_{p,\mu} = \|k_1^{(n_1)}*k_2^{(n_2)}*f-f\|_{p,\mu}$$

$$\leq \|k_1^{(n_1)}*(k_2^{(n_2)}*f-f)\|_{p,\mu} + \|k_1^{(n_1)}*f-f\|_{p,\mu}$$

$$\leq \|k_1^{(n_1)}\|_{1,\mu}\, \|k_2^{(n_2)}*f-f\|_{p,\mu} + \|k_1^{(n_1)}*f-f\|_{p,\mu} . \tag{4.12}$$

$$\text{But } \|k_1^{(n_1)}\|_{1,\mu} = \frac{l n_1^{n_1}}{\Gamma(n_1)} \int_0^\infty x^{-\mathrm{Re}(\mu+l(\xi+\alpha+n_1))} \exp(-n_1 x^{-l}) dx/x$$

$$= \frac{n_1^{n_1}}{\Gamma(n_1)} \int_0^\infty y^{\mathrm{Re}(\xi+\alpha+n_1+\mu/l)} \exp(-n_1 y) dy/y$$

and by our calculations in [4, Lemma 5.3] with η and μ replaced by $\mathrm{Re}(\xi+\alpha)$ and $-\mu/l$ respectively, we deduce that $\|k_1^{(n_1)}\|_{1,\mu} \to 1$ as $n_1 \to \infty$. Hence we may let n_1 and n_2 tend to infinity independently in (4.12) and use part (ii) to obtain the desired result. This finally completes the proof.

Remark 4.4 If we let $n_1 = n_2 = n$ in (4.1) and (4.2) we obtain a particular case which on the one hand relates more closely to (3.6) and (3.8) and on the other hand ties in with the work of several authors mentioned earlier. As an illustration, we deal with the isometric property of $S_{l,m}^{\xi,\eta}$ mentioned in Remark 3.7, although $p = 1$ has to be excluded.

Theorem 4.5 Under the hypotheses of Corollary 3.6, $S_{l,m}^{\xi,\eta}$ is an isometric isomorphism from $(L_{p,\mu}, \|\ \|_{p,\mu})$ onto $(S_{l,m}^{\xi,\eta}(L_{p,\mu}), \|\ \|_{p,\mu,l,m}^{(\xi;\eta)})$.

Proof:- We have only to prove that $S_{l,m}^{\xi,\eta}$ is an isometry. Let $g = S_{l,m}^{\xi,\eta} f$ where $f \in L_{p,\mu}$. By taking $\alpha = \beta = 0$ and $n_1 = n_2 = n$ in (4.2) and using continuity of $\|\ \|_{p,\mu}$, we obtain

$$\|f\|_{p,\mu} = \lim_{n\to\infty} \frac{n^{-\mathrm{Re}(\xi+\eta)}}{[\Gamma(n)]^2} \|\lambda_{n^{-1/l+1/m}} I_l^{\xi-1+n,-n} K_m^{\eta+n,-n} g\|_{p,\mu}$$

$$= \lim_{n\to\infty} \frac{n^{-\mathrm{Re}(\xi+\mu/l+\eta-\mu/m)}}{[\Gamma(n)]^2} \Gamma(\mathrm{Re}(\xi+n+\mu/l))\Gamma(\mathrm{Re}(\eta+n-\mu/m)) \frac{\|I_l^{\xi-1+n,-n} K_m^{\eta+n,-n} g\|_{p,\mu}}{\Gamma(\mathrm{Re}(\xi+n+\mu/l))\Gamma(\mathrm{Re}(\eta+\beta-\mu/m))}$$

We observed in [4, (3.5)] that, for any real c, $\Gamma(n+c)/n^c\Gamma(n)\to 1$ as $n\to\infty$. Applying this to $\mathrm{Re}(\xi+\mu/l)$ and $\mathrm{Re}(\eta-\mu/m)$ and using (3.8) leads immediately to $\|f\|_{p,\mu} = \|g\|^{(\xi;\eta)}_{p,\mu,l,m}$. This completes the proof.

§5. As another illustration we show how our results encompass those of Widder [8], Erdélyi [1], Love and Byrne [2] and Pollard [6].

Example 5.1 The classical Stieltjes transform corresponds to the case

$$\xi = l = m = 1 \ , \ \eta = 0 \tag{5.1}$$

of the operator $S^{\xi,\eta}_{l,m}$ by Example 2.3 and we shall write

$$S_1 \equiv S^{1,0}_{1,1} \tag{5.2}$$

in the sequel. We shall examine Widder's results on the range and inversion of S_1 on the usual $L^p(0,\infty)$ space. Note that this is the space $L_{p,-1/p}$ in our notation so that we take

$$\mu = -1/p. \tag{5.3}$$

(i) Consider first Theorem 3.1. For $1 < p < \infty$, the substitutions (5.1) and (5.3) lead to $\xi + \mu/l = 1-1/p > 0$, $\eta - \mu/m = 1/p > 0$ so that the hypotheses of the theorem are all satisfied. Bearing in mind considerations in Lemma 3.3, let $n_1 = n$, $n_2 = n-1$ in (3.1). We deduce that $g \in S_1(L^p(0,\infty))$ if and only if $g \in F_{p,-1/p}$ and there exists C_g such that

$$\|I_1^{n,-n}\,K_1^{n-1,-(n-1)}g\|_{p,-1/p} \le C_g\ \Gamma(n+1-1/p)\ \Gamma(n-1+1/p) \tag{5.4}$$

for all $n \ge 1$. As noted in the statement of Theorem 3.1, with $D \equiv d/dx$,

$$I_1^{n,-n}\,K_1^{n-1,-(n-1)}g = K_1^{n-1,-(n-1)}I_1^{n,-n}g = x^{n-1}(-D)^{n-1}(D)^n x^n g = (-1)^{n-1}x^{n-1}D^{2n-1}x^n g.$$

Hence (5.4) demands that for all $n = 1,2,\ldots$

$$\left\{\int_0^\infty |x^{1/p}(-1)^{n-1}x^{n-1}D^{2n-1}x^n g(x)|^p dx/x\right\}^{1/p} \le C_g\Gamma(n+1-1/p)\Gamma(n-1+1/p)$$

i.e. $[\Gamma(n+1-1/p)\Gamma(n-1+1/p)]^{-p}\int_0^\infty |x^{np-p}D^{2n-1}(x^n g(x))|^p dx \le C_g^p \quad (n=1,2,\ldots).$ (5.5)

In [8, p.369] we find (with slightly changed notation) the statement

$$c_n^p \int_0^\infty |x^{np-p}\, D^{2n-1}(x^n g(x))|^p dx < M \qquad (n = 1,2,\ldots) \tag{5.6}$$

where $c_1 = 1$, $c_n = [n!(n-2)!]^{-1}$ $(n = 2,3,\ldots)$.

However $\dfrac{n!(n-2)!}{\Gamma(n+1-1/p)\Gamma(n-1+1/p)} = \dfrac{\Gamma(n+1)n^{-1/p}}{\Gamma(n+1-1/p)} \cdot \dfrac{\Gamma(n-1)n^{1/p}}{\Gamma(n-1+1/p)} \to 1$ as $n\to\infty$.

Hence (5.5) and (5.6) are equivalent.

(ii) Now consider the inversion formula (4.2) for $1 < p < \infty$. Let $n_1 = n_2 = n$ and also choose $\alpha = 0$, $\beta = -1$. Together with (5.1) these substitutions lead to

$$f = \lim_{n\to\infty} \frac{1}{\Gamma(n)\Gamma(n)}\, I_1^{n,-n}\, K_1^{n-1,-(n-1)}g = \lim_{n\to\infty} \frac{1}{\{\Gamma(n)\}^2}\, K_1^{n-1,-(n-1)} I_1^{n,-n}g.$$

Arguing as in (i) we see that this gives

$$f = \lim_{n\to\infty} \frac{1}{\{\Gamma(n)\}^2}\,(-1)^{n-1}\, x^{n-1}\, D^{2n-1}\, x^n g \tag{5.7}$$

where convergence is with respect to $\| \;\|_{p,-1/p}$. The inversion formula suggested by the operators in [8, p.345] is

$$f = \lim_{n\to\infty} c_n\, (-1)^{n-1}\, x^{n-1}D^{2n-1}x^n g \tag{5.8}$$

where c_n is given by (5.6). But for $n \ge 2$, $\dfrac{1}{\{\Gamma(n)\}^2} - c_n = \dfrac{1}{\Gamma(n+1)\Gamma(n)}$

and by (5.4)

$$\left\|\left(\frac{1}{\{\Gamma(n)\}^2} - c_n\right) (-1)^{n-1} x^{n-1} D^{2n-1} x^n g\right\|_{p,-1/p}$$

$$\leq \frac{1}{\Gamma(n+1)\Gamma(n)} C_g \Gamma(n+1-1/p) \Gamma(n-1+1/p)$$

$$= \frac{1}{n-1} \frac{\Gamma(n+1-1/p)}{\Gamma(n+1)n^{-1/p}} \frac{\Gamma(n-1+1/p)}{\Gamma(n-1)n^{1/p}} \to 0 \quad \text{as} \quad n\to\infty .$$

Hence the inversion formulae (5.7) and (5.8) are equivalent, both giving convergence with respect to the norm on $L_{p,-1/p}$ which is the usual $L^p(0,\infty)$ norm. Compare this with [8,p.372].

(iii) As regards $p = 1$ we have not obtained a characterisation of the range of S_1 on $L_{1,-1} = L^1(0,\infty)$. The form of [8, Definition 21, pp. 374-5] suggests how the theory for $p > 1$ would need to be modified to obtain this range but we shall not pursue the matter here. With similar modifications, the inversion formulae (5.7) and (5.8) are valid (and equivalent) for $p = 1$ also. Since (5.4) has only been proved for $1 < p < \infty$, we have to use instead the fact that the sequence of functions appearing on the right-hand side of (5.8) is a Cauchy sequence in $L^1(0,\infty)$ and hence bounded in norm [8, p.375].

(iv) We may summarise the position by saying that for $1 < p < \infty$ (and to a certain extent $p = 1$ too) we have greatly extended Widder's results by treating $L_{p,\mu}$ for arbitrary values of μ (not merely $\mu = -1/p$) and by providing a whole family of inversion formulae, one special case of which is equivalent to Widder's inversion formula.

Example 5.2 Next we consider a connection with formulae mentioned by Erdélyi [1] for inverting the generalised Stieltjes transform S_ρ defined in the first instance by (2.6) for suitable values of ρ. However we may extend the definition of S_ρ by rewriting (2.5) in the form

$$(S_\rho f)(x) = [\Gamma(\rho)]^{-1}\, S^{\rho,0}_{1,1}\, x^{1-\rho}\, f(x)\,. \tag{5.9}$$

Defined in this way S_ρ will be a continuous linear mapping from $L_{p,\mu}$ into $F_{p,\mu-\rho+1}$ provided that $\mu+1 \in \Omega$, $\rho-\mu-1 \in \Omega$ and $1 \le p < \infty$. By (4.1) and (4.2), we obtain the inversion formulae for $S_\rho f = g$ in the form

$$f = \lim_{n\to\infty} \Gamma(\rho)\, x^{\rho-1} \frac{n^{-(\rho+\alpha+\beta)}}{\{\Gamma(n)\}^2}\, I_1^{\rho-1+\alpha+n,-(\alpha+n)}\, K_1^{\beta+n,-(\beta+n)} g$$

for any $\alpha,\beta \in \mathbb{C}$. In particular, let α and β be integers (positive, negative or zero). Then the I_1 and K_1 operators are differential operators (for large enough n) and, as they commute,

$$f = \lim_{n\to\infty} \frac{\Gamma(\rho)n^{-(\rho+\alpha+\beta)}}{\{\Gamma(n)\}^2}\, D^{\alpha+n}\, x^{\rho-1+(\alpha+n)+(\beta+n)}(-1)^{\beta+n} D^{\beta+n} g. \tag{5.10}$$

As in Example 5.1, we can replace $n^{-(\rho+\alpha+\beta)}/\{\Gamma(n)\}^2$ by $[\Gamma(\alpha+n+1)\Gamma(\beta+n+\rho-1)]^{-1}$ in (5.10) to obtain

$$f = \lim_{n\to\infty} \frac{\Gamma(\rho)(-1)^{\beta+n}}{\Gamma(\alpha+n+1)\Gamma(\beta+n+\rho-1)}\, D^{\alpha+n}\, x^{\rho-1+(\alpha+n)+(\beta+n)} D^{\beta+n} g \tag{5.11}$$

which agrees with one of the results in [1]; see in particular [1, (6.1) and (6.6)] and replace p and q therein by α and β. Following Erdélyi, let us write (5.11) in abbreviated form as

$$f = \lim_{n\to\infty} L_{n,x} g. \tag{5.12}$$

The formal adjoint $L^*_{n,x}$ of $L_{n,x}$ is obtained by interchanging α and β and multiplying by the factor $\{\Gamma(\alpha+n+\rho-1)\Gamma(\beta+n+1)\}/\{\Gamma(\alpha+n+1)\Gamma(\beta+n+\rho-1)\}$: see [1,(6.2) and (6.6)]. Since the numerical factor tends to 1 as $n\to\infty$ and α and β are arbitrary integers, it follows from (5.12) that

$$f = \lim_{n\to\infty} L^*_{n,x}\, g \tag{5.13}$$

also, [1,p.240]. For us, convergence in formulae (5.10)-(5.13) is in the $L_{\rho,\mu}$ norm whereas results in [1] are with respect to the topology on a space of testing functions. Similar inversion formulae also appear in the work of Love and Byrne [2] and there convergence is merely pointwise. Both [1] and [2] contain other related results but we shall not pursue these.

Example 5.3 As a final indication of the flexibility of our results, we consider another inversion formula for S_ρ due to Pollard [6] which is discussed distributionally by Pathak [5]. The formula is

$$f = \lim_{n\to\infty} \frac{(-1)^{n-1}\, 2^{\rho-1}(2n-1)!\,\Gamma(\rho)}{n!(n-2)!\,\Gamma(2n+\rho-1)}\, D^n\, x^{2n+\rho-2}\, D^{n-1}\, g. \tag{5.14}$$

On taking $\alpha = 0$, $\beta = -1$ in (5.11) we obtain

$$f = \lim_{n\to\infty} \frac{(-1)^{n-1}\Gamma(\rho)}{n!\,\Gamma(n+\rho-2)}\, D^n x^{2n+\rho-2}\, D^{n-1} g. \tag{5.15}$$

Comparing the numerical factors in (5.14) and (5.15) we obtain (for large enough n)

$$\frac{2^{\rho-1}(2n-1)!\,\Gamma(n+\rho-2)}{(n-2)!\,\Gamma(2n+\rho-1)} = \frac{(2n)^{\rho-1}\Gamma(2n)}{\Gamma(2n+\rho-1)} \cdot \frac{\Gamma(n+\rho-2)}{\Gamma(n)n^{\rho-2}} \cdot \frac{\Gamma(n)n^{-1}}{\Gamma(n-1)} \to 1 \text{ as } n\to\infty .$$

By the same argument as in Example 5.1, we deduce that Pollard's formula (5.14) is valid in the sense of convergence with respect to $\|\ \|_{\rho,-1/\rho}$.

§6 As mentioned at the start it is possible to obtain the corresponding results for the composition of two operators of type M_l^ξ or two operators of type N_m^η by using the relations (1.1). More importantly, the techniques used here can be used to study the range and invertibility of the composition of any finite number of operators each of which is either of type M_l^ξ or of type N_m^η. The theory is inherently no more difficult as we shall now indicate briefly.

Let ν_1 and ν_2 be two non-negative integers and introduce multi-indices

$$\underline{\xi} = (\xi_1,\ldots,\xi_{\nu_1}),\quad \underline{\eta} = (\eta_1,\ldots,\eta_{\nu_2}),\quad \underline{l} = (l_1,\ldots,l_{\nu_1}),\quad \underline{m} = (m_1,\ldots,m_{\nu_2})$$

where the components of $\underline{\xi}$ and $\underline{\eta}$ are complex numbers and those of $\underline{l}$ and $\underline{m}$ are positive real numbers. Define the operator $S_{\underline{l},\underline{m}}^{\underline{\xi},\underline{\eta}}$ by

$$S_{\underline{l},\underline{m}}^{\underline{\xi},\underline{\eta}} = \Big(\prod_{i=1}^{\nu_1} M_{l_i}^{\xi_i}\Big)\Big(\prod_{j=1}^{\nu_2} N_{m_j}^{\eta_j}\Big) \equiv M_{\underline{l}}^{\underline{\xi}} N_{\underline{m}}^{\underline{\eta}} \quad . \tag{6.1}$$

We make the convention that if $\nu_1 = 0$ the multi-index $\underline{\xi}$ is absent (corresponding to a product of operators of type N_m^η only) and similarly if $\nu_2 = 0$, while empty products are regarded as unity or the identity operator as appropriate. The products in (6.1) are unambiguous since all operators commute and the operator $S_{\underline{l},\underline{m}}^{\underline{\xi},\underline{\eta}}$ has multiplier

$$h(s) = \prod_{i=1}^{\nu_1} \Gamma(\xi_i - s/l_i) \prod_{j=1}^{\nu_2} \Gamma(\eta_j + s/m_j). \tag{6.2}$$

By examining the proofs of the results in §3 and §4 or proceeding inductively, we can see how to obtain analogues in the more general situation. To take one instance, (4.4) would be replaced by a multiple convolution with $\nu_1+\nu_2$ kernels all related to the standard delta kernel in [4, Lemma 5.3]. For completeness we shall state without proofs the salient results for $S_{\underline{l},\underline{m}}^{\underline{\xi},\underline{\eta}}$ (with the above notation).

<u>Theorem 6.1</u> If $\xi_i+\mu/l_i \in \Omega$ $(i = 1,\ldots,\nu_1)$, $\eta_j-\mu/m_j \in \Omega$ $(j = 1,\ldots,\nu_2)$ and $1 < p < \infty$ then $g \in S_{\underline{l},\underline{m}}^{\underline{\xi},\underline{\eta}}(L_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there exists a constant C_g independent of the components of $\underline{n}^{(1)} = (n_1^{(1)},\ldots,n_{\nu_1}^{(1)})$ and $\underline{n}^{(2)} = (n_1^{(2)},\ldots,n_{\nu_2}^{(2)})$ such that

$$\left\| \prod_{i=1}^{\nu_1} \prod_{j=1}^{\nu_2} I_{l_i}^{\xi_i-1+n_i^{(1)},-n_i^{(1)}} K_{m_j}^{\eta_j+n_j^{(2)},-n_j^{(2)}} g \right\|_{p,\mu}$$

$$\leq C_g \prod_{i=1}^{\nu_1} \prod_{j=1}^{\nu_2} \Gamma(\mathrm{Re}(\xi_i+n_i^{(1)}+\mu/l_i))\Gamma(\mathrm{Re}(\eta_j+n_j^{(2)}-\mu/m_j)) \tag{6.3}$$

for all non-negative integers $n_i^{(1)}$ $(i=1,\ldots,\nu_1)$ and $n_j^{(2)}$ $(j = 1,\ldots,\nu_2)$.

<u>Theorem 6.2</u> Under the hypotheses of Theorem 6.1. $S_{\underline{l},\underline{m}}^{\underline{\xi},\underline{\eta}}(L_{p,\mu})$ is a Banach space with respect to the norm $\| \ \|_{p,\mu,\underline{l},\underline{m}}^{(\underline{\xi};\underline{\eta})}$ defined by

$$\|g\|_{p,\mu,\underline{l},\underline{m}}^{(\underline{\xi};\underline{\eta})} = \inf\{C_g : (6.3) \text{ holds for all } n_i^{(1)}, n_j^{(2)} = 0,1,2,\ldots\}. \tag{6.4}$$

Equivalently (taking all $n_i^{(1)}$ and $n_j^{(2)}$ to be the same non-negative integer n)

$$\|g\|^{(\underline{\xi};\underline{\eta})}_{p,\mu,\underline{1},\underline{m}} = \lim_{n\to\infty}[\prod_{i=1}^{\nu_1}\prod_{j=1}^{\nu_2}\Gamma(Re(\xi_i+n+\mu/l_i))\Gamma(Re(\eta_j+n-\mu/m_j))\ ]^{-1}\ \times \tag{6.5}$$

$$\|\prod_{i=1}^{\nu_1}\prod_{j=1}^{\nu_2} I_{l_i}^{\xi_i-1+n,-n}\ K_{m_j}^{\eta_j+n,-n}\ g\|_{p,\mu}\ .$$

An inversion formula analogous to (4.1) can be obtained involving a limit as all the components of $\underline{n}^{(1)}$ and $\underline{n}^{(2)}$ (as above) tend to infinity independently. We shall content ourselves with the version where, as in (6.5), all the $n_i^{(1)}$ and $n_j^{(2)}$ are the same integer n. We require further multi-indices $\underline{\alpha} = (\alpha_1,\ldots,\alpha_{\nu_1})$ and $\underline{\beta} = (\beta_1,\ldots,\beta_{\nu_2})$ of complex numbers and, as is customary, we write

$$|\underline{\alpha}| = \sum_{i=1}^{\nu_1} |\alpha_i|\quad ,\qquad |\underline{\beta}| = \sum_{j=1}^{\nu_2} |\beta_j|$$

and similarly for $|\underline{\xi}|,|\underline{\eta}|$. Also let

$$\gamma = \sum_{j=1}^{\nu_2}(1/m_j) - \sum_{i=1}^{\nu_1}(1/l_i).$$

Theorem 6.3 With the above notation and under the hypotheses of Theorem 6.1, if $g = S^{\underline{\xi},\underline{\eta}}_{\underline{1},\underline{m}}f$ where $f \in L_{p,\mu}$ then

$$f = \lim_{n\to\infty}\frac{n^{-(|\underline{\xi}|+|\underline{\alpha}|+|\underline{\eta}|+|\underline{\beta}|)}}{\{\Gamma(n)\}^{\nu_1+\nu_2}}\ \lambda_{n^{\gamma}}(\prod_{i=1}^{\nu_1}\prod_{j=1}^{\nu_2} I_{l_i}^{\xi_i-1+\alpha_i+n,-(\alpha_i+n)}\ K_{m_j}^{\eta_j+\beta_j+n,-(\beta_j+n)}\ g) \tag{6.6}$$

Remark 6.4 Formulae such as (6.3), (6.5) and (6.6) can be made to look less frightening by more extensive use of multi-index notation. For instance, with some obvious conventions, (6.3) can be turned into

$$\| I_{\underline{1}}^{\underline{\xi}-\underline{1}+\underline{\eta}^{(1)},-\underline{\eta}^{(1)}} K_{\underline{m}}^{\underline{\eta}+\underline{\eta}^{(2)},-\underline{\eta}^{(2)}} \|_{p,\mu} \le C_g\, \Gamma(\mathrm{Re}(\underline{\xi}+\underline{\eta}^{(1)}+\underline{\mu}/\underline{1}))\Gamma(\mathrm{Re}(\underline{\eta}+\underline{\eta}^{(2)}-\underline{\mu}/\underline{m}))$$

which is reminiscent of (3.1). Nevertheless, the situation remains very complicated with the range $S_{\underline{1},\underline{m}}^{\underline{\xi},\underline{\eta}}(L_{p,\mu})$ depending on many parameters, in particular on all the components of $\underline{\xi}$ and $\underline{\eta}$. Things become even worse if we want to study operators with multipliers of the form $h_1(s)/h_2(s)$ where each of h_1 and h_2 has the form (6.2). Many such operators arise involving special functions and indeed can be regarded as extensions of the G-function operators studied by Rooney [7]. It turns out, however, that such operators can be handled more simply in $F_{p,\mu}$ than in $L_{p,\mu}$. In future papers, we hope to present an $F_{p,\mu}$ investigation and to deal with particular examples.

Acknowledgment One of us (W.J.S.) gratefully acknowledges the award of a Carnegie Scholarship by the Carnegie Trust for the Universities of Scotland to enable this research to be carried out.

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REPRINTED FROM / RÉIMPRIMÉ DE

CANADIAN JOURNAL OF MATHEMATICS

JOURNAL CANADIEN DE MATHÉMATIQUES

The Canadian Mathematical Society

La Société mathématique du Canada

Can. J. Math.Vol. 43 (6), 1991 pp. 1323–1338

ON THE RANGE AND INVERTIBILITY OF A CLASS OF MELLIN MULTIPLIER TRANSFORMS III

To Professor Tim Rooney with best wishes

A. C. MCBRIDE AND W. J. SPRATT

ABSTRACT. We continue to develop the theory of previous papers concerning transforms corresponding to Mellin multipliers which involve products and/or quotients of Γ-functions. We show that, by working with certain subspaces of $L_{p,\mu}$ consisting of smooth functions, we can simplify considerably the restrictions on the parameters which were necessary in the $L_{p,\mu}$ setting. As a result, operators in our class become homeomorphisms on these subspaces under conditions of great generality.

1. In this paper we continue our investigations into Mellin multiplier transforms T satisfying a relation of the form

$$\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s) \tag{1.1}$$

under suitable conditions, where the multiplier h has the form

$$h(s) = \frac{\prod_{i=k+1}^{K} \Gamma(\eta_i + r_i s) \prod_{j=\ell+1}^{L} \Gamma(\xi_j - t_j s)}{\prod_{i=1}^{k} \Gamma(\eta_i + r_i s) \prod_{j=1}^{\ell} \Gamma(\xi_j - t_j s)}. \tag{1.2}$$

Here k, ℓ, K, L are non-negative integers satisfying $0 \leq k \leq K$, $0 \leq \ell \leq L$ (empty products being unity by convention), the numbers $r_1, \ldots, r_K$ and $t_1, \ldots, t_L$ are real and positive, and $\eta_1, \ldots, \eta_K$, $\xi_1, \ldots, \xi_L$ are complex numbers.

In [5] and [6] we characterised the range of the operators corresponding to the multipliers $\Gamma(\eta + s/m)$, $\Gamma(\xi - s/m)$ and $\Gamma(\eta + s/m)\Gamma(\xi - s/m)$, with $m > 0$, on the weighted spaces $L_{p,\mu}$. In the case of the third multiplier, the range was already becoming rather complicated and depended on η and ξ separately. In principle, it might be feasible to analyse the multiplier (1.2) relative to $L_{p,\mu}$ but, in practice, it would be a very tiresome business. A thorough investigation was carried out by Rooney in [9] but was restricted to the special case when all the numbers r_i and t_j are unity.

The purpose of the present paper is to develop a corresponding theory within the framework of certain subspaces $F_{p,\mu}$ of $L_{p,\mu}$ consisting of smooth functions. It turns out that the complexity associated with $L_{p,\mu}$ disappears and is replaced by a set of simple conditions of great generality. In particular the range on $F_{p,\mu}$ of the operator T corresponding to (1.2) is a certain subspace $F_{p,\mu,r}$ which depends essentially on only a particular combination of the numbers r_i and t_j, as given by (5.3).

Received by the editors July 16, 1991.
AMS subject classification: 42A45, 44A15.

In § 2, we introduce the relevant spaces by examining the range on $F_{p,\mu}$ of the operator N_m^η corresponding to the simple multiplier $\Gamma(\eta + s/m)$ with $m > 0$. We obtain a characterisation of $N_m^\eta(F_{p,\mu})$ as a set and then equip it with an appropriate topology which turns the set into a Fréchet space. We then discover that the range is essentially independent of η and relabel it as $F_{p,\mu,r}$ with the number $r = 1/m$ often being more convenient to use than m itself. An equally simple multiplier which we might have tried at the start is $\Gamma(\xi - s/m)$. This gives rise to an operator M_m^ξ which was studied along with N_m^η in [5] but produced a different range for every ξ. In contrast we discover in § 3 that $M_m^\xi(F_{p,\mu})$ is independent of ξ and is just $F_{p,\mu,r}$ back again. This is the first hint of the intrinsic importance of $F_{p,\mu,r}$ for all operators with the same value of r given by (5.3). In § 4, we examine a few simple operators relative to $F_{p,\mu,r}$ and review some known results. Finally, in § 5, we reveal the full details of how the spaces $F_{p,\mu,r}$ emerge as the ranges of operators on $F_{p,\mu}$ with multipliers of the form (1.2). However, as we point out at the end, the theory can be extended in a number of ways and this we hope to do in a future paper.

Throughout the paper we shall make use of notation, terminology and results from [5] and [6] to which the reader should refer as necessary. In particular, we use the notation

$$\Omega = \{z \in \mathbf{C} : \operatorname{Re} z \neq 0, -1, -2, \ldots\}. \tag{1.3}$$

2. We begin by recalling the following result from [5].

THEOREM 2.1. *If $1 < p < \infty$, $m > 0$ and $\eta - \mu/m \in \Omega$, then $g \in N_m^\eta(L_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there exists a constant A_g such that*

$$\left\|\left[\Gamma\left(\operatorname{Re}(\eta + n - \mu/m)\right)\right]^{-1} K_m^{\eta+n,-n} g\right\|_{p,\mu} \le A_g \text{ for } n = 0, 1, 2, \ldots \tag{2.1}$$

where $K_m^{\eta+n,-n}$ denotes the differential operator

$$K_m^{\eta+n,-n} = x^{m\eta+mn}(-D_m)^n x^{-m\eta}; \quad D_m = d/dx^m. \tag{2.2}$$

PROOF. See [5, Theorems 3.3 and 7.3].

Since the topology of $F_{p,\mu}$ is defined by a family of seminorms rather than a single norm as in the case of $L_{p,\mu}$, the next result represents the obvious modification to Theorem 2.1 which makes use of these seminorms.

THEOREM 2.2. *If $1 < p < \infty$, $m > 0$ and $\eta - \mu/m \in \Omega$, then $g \in N_m^\eta(F_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and, for each $i = 0, 1, 2, \ldots$, there exists a constant $A_g^{(i)}$, depending on g but independent of n, such that*

$$\gamma_i^{p,\mu}\left(\left[\Gamma\left(\operatorname{Re}(\eta + n - \mu/m)\right)\right]^{-1} K_m^{\eta+n,-n} g\right) \le A_g^{(i)} \text{ for } n = 0, 1, 2, \ldots. \tag{2.3}$$

PROOF. Let $g = N_m^\eta f$ where $f \in F_{p,\mu}$. Then, for each $i = 0, 1, 2, \ldots$, $\delta^i g = N_m^\eta \delta^i f$ and since $\delta^i f \in L_{p,\mu}$, Theorem 2.1 shows that, for some constant $B_g^{(i)}$,

$$\left\|\left[\Gamma\left(\operatorname{Re}(\eta + n - \mu/m)\right)\right]^{-1} K_m^{\eta+n,-n} \delta^i g\right\|_{p,\mu} \le B_g^{(i)} \text{ for } n = 0, 1, 2, \ldots. \tag{2.4}$$

However the operators $K_m^{\eta+n,-n}$ and δ^i commute and the operator x^iD^i appearing in the seminorm $\gamma_i^{p,\mu}$ is a polynomial of degree i in δ. Hence (2.3) follows from (2.4) with the constant $A_g^{(i)}$ being a linear combination of $B_g^{(k)}$ $(k = 0, 1, \ldots, i)$.

Conversely, let $g \in F_{p,\mu}$ and let g satisfy (2.3) for certain constants $A_g^{(i)}$. Then g satisfies (2.4) for certain constants $B_g^{(i)}$. By Theorem 2.1, for each $i = 0, 1, 2, \ldots$ there exists $f_i \in L_{p,\mu}$ such that

$$\delta^i g = N_m^\eta f_i. \tag{2.5}$$

If $\operatorname{Re}\mu \neq 0$, δ is invertible on $F_{p,\mu}$ and, since δ^{-1} commutes with N_m^η on $L_{p,\mu}$,

$$g = (\delta^{-1})^i N_m^\eta f_i = N_m^\eta (\delta^{-1})^i f_i \quad (i = 0, 1, 2, \ldots). \tag{2.6}$$

Explicitly, δ^{-1} is given by

$$(\delta^{-1}h)(x) = \begin{cases} \int_0^x h(t)\,dt/t & (\operatorname{Re}\mu > 0) \\ -\int_x^\infty h(t)\,dt/t & (\operatorname{Re}\mu < 0) \end{cases} \tag{2.7}$$

and in either case defines a bounded integral operator on $L_{p,\mu}$. Hence $(\delta^{-1})^i f_i \in L_{p,\mu}$ for each $i = 0, 1, 2, \ldots$. Further, N_m^η is one-to-one on $L_{p,\mu}$ and (2.5), (2.6) therefore lead to

$$f_0 = (\delta^{-1})^i f_i \quad (i = 0, 1, 2, \ldots). \tag{2.8}$$

A standard argument based on (2.7) and (2.8) now shows that f_0 is infinitely differentiable and is a function in $F_{p,\mu}$. Hence $g = N_m^\eta f_0 \in N_m^\eta(F_{p,\mu})$ in this case. Finally, to deal with the case $\operatorname{Re}\mu = 0$, notice that (2.3) can be rewritten as

$$\gamma_i^{p,\mu-m}\Big(\big[\Gamma\big(\operatorname{Re}(\eta + n - \mu/m)\big)\big]^{-1} K_m^{\eta-1+n,-n} x^{-m} g\Big) \le A_g^{(i)} \text{ for } n = 0, 1, 2, \ldots.$$

By the previous case with η, μ and g replaced by $\eta - 1$, $\mu - m$ and $x^{-m}g$ respectively, there exists $h \in F_{p,\mu-m}$ such that $x^{-m}g = N_m^{\eta-1}h$. A simple calculation shows that, as operators on $F_{p,\mu}$,

$$N_m^\eta = x^m N_m^{\eta-1} x^{-m}$$

under the stated conditions. Hence $g = N_m^\eta f$ where $f = x^m h \in F_{p,\mu}$. This completes the proof.

REMARK 2.3. The necessary and sufficient condition obtained in Theorem 2.2 is equivalent to another condition in which the non-negative integer n is replaced by a more general complex number λ. More precisely, under the hypotheses of Theorem 2.2, $g \in N_m^\eta(F_{p,\mu})$ if and only if $g \in F_{p,\mu}$ and there are constants $A_g^{(i)}$ independent of λ such that

$$\gamma_i^{p,\mu}\Big(\big[\Gamma\big(\operatorname{Re}(\eta + \lambda - \mu/m)\big)\big]^{-1} K_m^{\eta+\lambda,-\lambda} g\Big) \le A_g^{(i)} \tag{2.9}$$

for all complex numbers λ such that $\operatorname{Re}(\eta + \lambda - \mu/m) \neq 0, -1, -2, \ldots$. The operator $K_m^{\eta+\lambda,-\lambda}$ is a general Erdélyi-Kober operator whose Mellin multiplier is

$\Gamma(\eta + \lambda + s/m)/\Gamma(\eta + s/m)$. The proof of (2.9) is omitted but we shall require this characterisation of the range in the sequel.

In [5], we noted that $N_m^\eta(L_{p,\mu})$ depends on η in the sense that, if $\operatorname{Re}\eta_1 < \operatorname{Re}\eta_2$ then $N_m^{\eta_1}(L_{p,\mu})$ is a subset of $N_m^{\eta_2}(L_{p,\mu})$ under appropriate conditions. See in particular [5, Corollary 3.6]. This situation arose because a certain Erdélyi-Kober operator was not invertible in the $L_{p,\mu}$ setting. However, when we work in $F_{p,\mu}$, this difficulty disappears.

THEOREM 2.4. *If $1 < p < \infty$, $m > 0$ and $\eta_j - \mu/m \in \Omega$ for $j = 1,2$ then, as sets,*

$$N_m^{\eta_1}(F_{p,\mu}) = N_m^{\eta_2}(F_{p,\mu}). \tag{2.10}$$

PROOF. Under the stated conditions, the operator equation

$$N_m^{\eta_1} = N_m^{\eta_2} K_m^{\eta_1,\eta_2-\eta_1}$$

holds on $F_{p,\mu}$, while $K_m^{\eta_1,\eta_2-\eta_1}$ is a homeomorphism from $F_{p,\mu}$ onto itself, with inverse $K_m^{\eta_2,\eta_1-\eta_2}$. The result follows immediately.

Our experience with $N_m^\eta(L_{p,\mu})$ suggests that, to turn N_m^η into a homeomorphism in the $F_{p,\mu}$ setting, we should make use of (2.3) and imitate the construction in [5]. This time, however, we shall obtain a whole family of new seminorms on the range rather than just a single new norm. In what follows, we shall often write

$$r = 1/m \quad (\text{where } m > 0). \tag{2.11}$$

DEFINITION 2.5. Let $m > 0$, $1 < p < \infty$ and $\eta - \mu/m \in \Omega$. For $i = 0, 1, 2, \ldots$ and $g \in N_m^\eta(F_{p,\mu})$ let

$$\gamma_i^{p,\mu,r,\eta}(g) = \inf\{A_g^{(i)} : (2.3) \text{ holds for this fixed } g \text{ and } i\}. \tag{2.12}$$

REMARK 2.6. It is easy to check that, under the stated conditions, $\{\gamma_i^{p,\mu,r,\eta}\}_{i=0}^\infty$ is a countable multinorm in the sense of Zemanian [13]. Having shown in Theorem 2.4 that the set $N_m^\eta(F_{p,\mu})$ is independent of η, under the appropriate conditions, our aim is to show that $N_m^\eta(F_{p,\mu})$ equipped with the multinorm $\{\gamma_i^{p,\mu,r,\eta}\}_{i=0}^\infty$ is independent of such η as a topological vector space.

LEMMA 2.7. *If $1 < p < \infty$, $m > 0$ and $\eta_j - \mu/m \in \Omega$ for $j = 1,2$ then*

$N_m^{\eta_1}(F_{p,\mu})$ is continuously imbedded in $N_m^{\eta_2}(F_{p,\mu})$

with respect to the topologies generated by the multinorms $\{\gamma_i^{p,\mu,r,\eta_1}\}_{i=0}^\infty$ and $\{\gamma_i^{p,\mu,r,\eta_2}\}_{i=0}^\infty$.

PROOF. Let $g \in N_m^{\eta_1}(F_{p,\mu}) = N_m^{\eta_2}(F_{p,\mu})$. We shall make use of basic properties of the Erdélyi-Kober operators. Firstly, for $n = 0, 1, 2, \ldots$

$$K_m^{\eta_2+n,-n} g = K_m^{\eta_1,\eta_2-\eta_1} K_m^{\eta_2+n,\eta_1-\eta_2-n} g.$$

Under the given conditions, $h_n = K_m^{\eta_2+n,\eta_1-\eta_2-n}g \in F_{p,\mu}$ and $K_m^{\eta_1,\eta_2-\eta_1}$ is a continuous linear mapping from $F_{p,\mu}$ into itself. By [13, Lemma 1.10-1], with η_1 and η_2 fixed, for each $i = 0, 1, 2, \ldots$, there exist a non-negative integer N_i and non-negative constants C_j ($j = 0, 1, \ldots, N_i$) which are independent of n such that

$$\gamma_i^{p,\mu}(K_m^{\eta_2+n,-n}g) \leq \sum_{j=0}^{N_i} C_j\gamma_j^{p,\mu}(K_m^{\eta_2+n,\eta_1-\eta_2-n}g)$$

for all $n = 0, 1, 2, \ldots$. Now divide both sides by the quantity

$$\left|\Gamma\big(\mathrm{Re}\,(\eta_2 + n - \mu/m)\big)\right| = \left|\Gamma\Big(\mathrm{Re}\,\big(\eta_1 + (\eta_2 - \eta_1 + n) - \mu/m\big)\Big)\right|,$$

invoke Remark 2.3 with η and λ replaced by η_1 and $\eta_2 - \eta_1 + n$ to handle the righthand side and take infima to get

$$\gamma_i^{p,\mu,r,\eta_2}(g) \leq \sum_{j=0}^{N_i} C_j\gamma_j^{p,\mu,r,\eta_1}(g). \tag{2.13}$$

The result now follows.

THEOREM 2.8. *For fixed m, p and μ such that $m > 0$, $1 < p < \infty$ and $\mu \in \mathbf{C}$, the topological vector space consisting of the set $N_m^\eta(F_{p,\mu})$ and the multinorm $\{\gamma_i^{p,\mu,r,\eta}\}_{i=0}^\infty$ is independent of $\eta \in \mathbf{C}$ satisfying $\eta - \mu/m \in \Omega$.*

PROOF. The two multinorms give equivalent topologies in view of (2.13) and a similar inequality with η_1 and η_2 interchanged. This, together with (2.10), completes the proof.

NOTATION 2.9. Under the conditions of Theorem 2.8, we shall write

$$N_m^\eta(F_{p,\mu}) \equiv F_{p,\mu,r} \text{ and } \gamma_i^{p,\mu,r,\eta} \equiv \gamma_i^{p,\mu,r} \quad (i = 0, 1, 2, \ldots) \tag{2.14}$$

to indicate independence of η, subject to the condition

$$\eta - r\mu \in \Omega \tag{2.15}$$

which will be assumed throughout.

REMARK 2.10.

(i) Since we have lost dependence on η, we have the first indication that the operator N_m^η is not the only candidate which can be used to generate $F_{p,\mu,r}$. The space depends intrinsically on r (equivalently on m) and we have one such space for every $r > 0$. We shall continue to use N_m^η a little longer to develop properties of $F_{p,\mu,r}$ and return later to dependence on r only (for fixed p and μ).

(ii) A fairly routine calculation shows that, with the relevant topologies

$$F_{p,\mu,r} \text{ is homeomorphic to } F_{p,\mu r,1} \tag{2.16}$$

under the mapping P_r where $(P_r\phi)(x) = \phi(x^r)$. Hence any two spaces $F_{p,\mu,r}(r > 0)$ are homeomorphic to each other.

(iii) It will be convenient to write

$$F_{p,\mu} \equiv F_{p,\mu,0} \tag{2.17}$$

i.e. to regard our original $F_{p,\mu}$ space as corresponding in some sense to $r = 0$. The reason for this will become clearer later.

THEOREM 2.11. *The space $F_{p,\mu,r}(r > 0)$ is a Fréchet space with respect to $\{\gamma_i^{p,\mu,r}\}_{i=0}^{\infty}$.*

PROOF. Only completeness has to be established and in view of Remark 2.10(ii) it is sufficient to prove the result for $r = 1$. Choose any $\eta \in \mathbb{C} : \eta - \mu \in \Omega$.

Let $\{g_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $F_{p,\mu,1}$. By (2.3) and (2.12),

$$\gamma_i^{p,\mu}\big(K_1^{\eta+n,-n}(g_j - g_k)\big) \leq \gamma_i^{p,\mu,1}(g_j - g_k)\big|\Gamma\big(\mathrm{Re}\,(\eta + n - \mu)\big)\big| \tag{2.18}$$

for each fixed i, $n = 0, 1, 2, \ldots$ and all $j, k = 1, 2, \ldots$. Hence for each fixed n, $\{K_1^{\eta+n,-n}g_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $F_{p,\mu}$. Since the latter space is complete, for each $n = 0, 1, 2, \ldots$ we may define

$$h_n = \lim_{k\to\infty} K_1^{\eta+n,-n} g_k \tag{2.19}$$

where the limit is with respect to the $F_{p,\mu}$ topology. In particular $g_k \longrightarrow h_0$ as $k \longrightarrow \infty$, and, since $K_1^{\eta+n,-n}$ is a homeomorphism on $F_{p,\mu}$ under the given conditions, (2.19) shows that

$$h_n = K_1^{\eta+n,-n} h_0 \text{ for all } n = 0, 1, 2, \ldots . \tag{2.20}$$

Next observe that, for fixed $i = 0, 1, 2, \ldots$, there is a constant C_i such that, for $k \geq 1$,

$$\gamma_i^{p,\mu}(K_1^{\eta+n,-n}g_k) \leq C_i\big|\Gamma\big(\mathrm{Re}\,(\eta + n - \mu)\big)\big| \text{ for all } n = 0, 1, 2, \ldots .$$

By letting $k \longrightarrow \infty$ and using (2.19) and (2.20), we obtain

$$\gamma_i^{p,\mu}(K_1^{\eta+n,-n}h_0) \leq C_i\big|\Gamma\big(\mathrm{Re}\,(\eta + n - \mu)\big)\big| \text{ for all } n = 0, 1, 2, \ldots$$

so that $h_0 \in F_{p,\mu,1}$, with $\gamma_i^{p,\mu,1}(h_0) \leq C_i$. Also, from (2.18), for any $\varepsilon > 0$, there exists a positive integer N, independent of n, such that

$$\gamma_i^{p,\mu}\big(K_1^{\eta+n,-n}(g_j - g_k)\big) \leq \varepsilon\big|\Gamma\big(\mathrm{Re}\,(\eta + n - \mu)\big)\big| \text{ for all } j, k \geq N.$$

If we let $j \longrightarrow \infty$ and use (2.19) and (2.20) again, we obtain for $n = 0, 1, 2, \ldots$

$$\begin{aligned}&\gamma_i^{p,\mu}\big(K_1^{\eta+n,-n}(h_0 - g_k)\big) \leq \varepsilon\big|\Gamma\big(\mathrm{Re}\,(\eta + n - \mu)\big)\big| \text{ for all } k \geq N\\ &\Rightarrow \gamma_i^{p,\mu,1}(h_0 - g_k) \leq \varepsilon \text{ for all } k \geq N.\end{aligned}$$

Hence $\{g_k\}_{k=1}^{\infty}$ converges to h_0 with respect to $\{\gamma_i^{p,\mu,1}\}_{i=0}^{\infty}$. This completes the proof.

THEOREM 2.12. *Under condition (2.15), N_m^{η} is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$ with respect to the multinorms $\{\gamma_i^{p,\mu}\}_{i=0}^{\infty}$ and $\{\gamma_i^{p,\mu,r}\}_{i=0}^{\infty}$ respectively.*

PROOF. N_m^{η} is one-to-one (by the corresponding result in $L_{p,\mu}$) and onto (by construction). As regards continuity of N_m^{η}, note that, as operators on $F_{p,\mu}$,

$$K_m^{\eta+n,-n}N_m^{\eta} = N_m^{\eta+n}; \quad x^iD^iN_m^{\eta+n} = N_m^{\eta+n}x^iD^i \quad (i = 0, 1, 2, \ldots)$$

so that, for $f \in F_{p,\mu}$,

$$\gamma_i^{p,\mu}(K_m^{\eta+n,-n} N_m^{\eta} f) = \gamma_i^{p,\mu}(N_m^{\eta+n} f) = \|x^i D^i N_m^{\eta+n} f\|_{p,\mu}$$
$$= \|N_m^{\eta+n} x^i D^i f\|_{p,\mu} \leq \left|\Gamma\big(\mathrm{Re}\,(\eta + n - \mu/m)\big)\right| \|x^i D^i f\|_{p,\mu}$$

where we have used [5, Theorem 7.1]. It then follows that

$$\gamma_i^{p,\mu,r}(N_m^{\eta} f) \leq \gamma_i^{p,\mu}(f) \text{ for all } i = 0, 1, 2, \ldots$$

from which continuity of N_m^{η} follows. Finally, continuity of $(N_m^{\eta})^{-1}$ is now automatic by Theorem 2.11 and the Open Mapping Theorem for Fréchet spaces [12, Theorem 17.1].

3. In [5], we obtained characterisations of $N_m^{\eta}(L_{p,\mu})$ and $M_m^{\xi}(L_{p,\mu})$ and discovered that these spaces were not the same. In contrast, we shall show that the space $F_{p,\mu,r}$, which represents the range of N_m^{η} on $F_{p,\mu}$ for all η satisfying (2.15), is also the range of M_m^{ξ} on $F_{p,\mu}$ for all ξ satisfying an analogue of (2.15). We achieve this by making use of the theory of multipliers developed by Rooney [8].

THEOREM 3.1. *Let $1 < p < \infty$, $r > 0$ and $\mu \in \mathbf{C}$. If ξ is any complex number such that $\xi + r\mu \in \Omega$ then M_m^{ξ} is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$ (where, as usual, $m = 1/r$).*

PROOF. Choose $\eta_1 \in \mathbf{C}$ such that $\mathrm{Re}\,(\eta_1 - r\mu) > \max\big(0, \mathrm{Re}\,(\xi + r\mu)\big)$ and let

$$h_1(s) = \Gamma(\xi - rs)/\Gamma(\eta_1 + rs). \tag{3.1}$$

Then we can find a strip $S = \{s \in \mathbf{C} : \alpha < \mathrm{Re}\, s < \beta\}$ such that

(i) S contains the line $\mathrm{Re}\, s = -\mathrm{Re}\,\mu$
(ii) $\mathrm{Re}\,(\eta_1 + rs) > \mathrm{Re}\,(\xi - rs)$ for all $s \in S$
(iii) h is analytic on S.

By using the strip S we can check that h is in the class $\mathcal{A}$ introduced in [8, Definition 3.1]. Indeed, by [1, 1.18(6)]

$$|\Gamma(x + iy)| \sim \sqrt{2\pi}\,|y|^{x-1/2} e^{-\pi|y|/2} \tag{3.2}$$

as $|y| \to \infty$, uniformly with respect to x in a bounded interval, and since $\mathrm{Re}\,(\xi - rs) < \mathrm{Re}\,(\eta_1 + rs)$, we can deduce that h is bounded on any substrip of the form $\alpha' \leq \mathrm{Re}\, s \leq \beta'$ with $\alpha < \alpha' < \beta' < \beta$. The condition that

$$|h'(s)| = O(|\mathrm{Im}\, s|^{-1}) \quad (s \in S)$$

can be checked similarly by using the formula [1, 1.18(7)] for the asymptotic behaviour of the function $\psi = \Gamma'/\Gamma$. Thus, by [8, Theorem 1], h is an $L_{p,\mu}$ multiplier and hence, by [4, Theorem 3.3], an $F_{p,\mu}$ multiplier. Hence there is a continuous linear mapping T_1 from $F_{p,\mu}$ into $F_{p,\mu}$ such that

$$\big(\mathcal{M}(T_1 f)\big)(s) = h(s)(\mathcal{M}f)(s) \tag{3.3}$$

whenever $f \in F_{p,\mu} \cap F_{2,\mu}$ and $\operatorname{Re} s = -\operatorname{Re} \mu$.

Recall that the operators $N_m^{\eta_1}$ and M_m^{ξ} have the respective multipliers

$$\Gamma(\eta_1 + s/m) \equiv \Gamma(\eta_1 + rs) \text{ and } \Gamma(\xi - s/m) \equiv \Gamma(\xi - rs).$$

It follows from (3.1) and (3.3) that

$$\big(\mathcal{M}(N_m^{\eta_1} T_1 f)\big)(s) = \big(\mathcal{M}(M_m^{\xi} f)\big)(s)$$

whenever $f \in F_{p,\mu} \cap F_{2,\mu}$ and $\operatorname{Re} s = -\operatorname{Re} \mu$. By a standard continuity and density argument we deduce that, under the given hypotheses,

$$N_m^{\eta_1} T_1 f = M_m^{\xi} f \text{ for all } f \in F_{p,\mu} \tag{3.4}$$

so that M_m^{ξ} is a continuous linear mapping from $F_{p,\mu}$ into $N_m^{\eta_1}(F_{p,\mu}) \equiv F_{p,\mu,r}$ this being valid since $\eta_1 - r\mu \in \Omega$ by choice of η_1.

To prove that M_m^{ξ} is a homemorphism we choose $\eta_2 \in \mathbf{C}$ such that $\operatorname{Re}(\eta_2 - r\mu) < \operatorname{Re}(\xi + r\mu)$ and $\eta_2 - r\mu \in \Omega$. Consider the multiplier

$$h_2(s) = \Gamma(\eta_2 + rs)/\Gamma(\xi - rs)$$

in a suitable strip containing the line $\operatorname{Re} s = -\operatorname{Re} \mu$, throughout which h_2 is analytic and $\operatorname{Re}(\eta_2 + rs) < \operatorname{Re}(\xi - rs)$. By proceeding as above, we obtain a continuous linear operator T_2 from $F_{p,\mu}$ into $F_{p,\mu}$ such that

$$M_m^{\xi} T_2 f = N_m^{\eta_2} f \text{ for all } f \in F_{p,\mu}$$

by analogy with (3.4). This shows that $F_{p,\mu,r} \equiv N_m^{\eta_2}(F_{p,\mu}) \subseteq M_m^{\xi}(F_{p,\mu})$ and hence from above we obtain $M_m^{\xi}(F_{p,\mu}) = F_{p,\mu,r}$. Furthermore, M_m^{ξ} is a homeomorphism by the Open Mapping Theorem for Fréchet spaces [12, Theorem 17.1] and the proof is complete.

REMARK 3.2. Our proof above required us to choose two separate values of η in order to use the asymptotics of Γ and ψ and obtain a multiplier in the class $\mathcal{A}$ for both h_1 and h_2. An alternative approach can be found in [10, Theorem 6.22]. We shall make further use of multipliers in subsequent work which will reveal that Theorem 3.1 is only the tip of the iceberg. There are many continuous linear mappings and homeomorphisms, all mapping $F_{p,\mu}$ onto $F_{p,\mu,r}$ and all having their behaviour dictated by a common value of a parameter r. Any one of these operators could be used to study the space $F_{p,\mu,r}$ but N_m^{η} and M_m^{ξ} (with $m = 1/r$) are the two which have the simplest multipliers.

4. Having obtained the topological structure of the spaces $F_{p,\mu,r}$, we shall now look at a few simple operators relative to these spaces. Some of our results point the way ahead to more substantial results in § 5.

THEOREM 4.1. *For $1 < p < \infty$, $r > 0$ and any complex numbers λ and μ, the mapping x^λ is a homeomorphism from $F_{p,\mu,r}$ onto $F_{p,\mu+\lambda,r}$ with inverse $x^{-\lambda}$. [Here, as usual, we are talking of the mapping which sends $g(x)$ to $x^\lambda g(x)$.]*

PROOF. Choose any $\eta \in \mathbf{C}$ such that $\eta - r\mu \in \Omega$. Then, with $m = 1/r$ and $g \in F_{p,\mu,r} = N_m^\eta(F_{p,\mu})$, we obtain

$$K_m^{\eta+n,-n} g = x^{-\lambda} K_m^{\eta+r\lambda+n,-n}(x^\lambda g) \text{ for } n = 0, 1, 2, \dots .$$

Hence, for $i = 0, 1, 2, \dots$,

$$\begin{aligned}
\gamma_i^{p,\mu+\lambda}&\Big(\Big[\Gamma\Big(\operatorname{Re}\big((\eta + r\lambda) + n - r(\mu + \lambda)\big)\Big)\Big]^{-1} K_m^{\eta+r\lambda+n,-n}(x^\lambda g)\Big) \\
&= \|x^i D^i\big(\big[\Gamma\big(\operatorname{Re}(\eta + n - r\mu)\big)\big]^{-1} x^{-\lambda} K_m^{\eta+r\lambda+n,-n}(x^\lambda g)\big)\|_{p,\mu} \\
&= \|x^i D^i\big(\big[\Gamma\big(\operatorname{Re}(\eta + n - r\mu)\big)\big]^{-1} K_m^{\eta+n,-n} g\big)\|_{p,\mu} \\
&= \gamma_i^{p,\mu}\big(\big[\Gamma\big(\operatorname{Re}(\eta + n - r\mu)\big)\big]^{-1} K_m^{\eta+n,-n} g\big).
\end{aligned}$$

It now follows that

$$\gamma_i^{p,\mu+\lambda,r,\eta+r\lambda}(x^\lambda g) = \gamma_i^{p,\mu,r,\lambda}(g) \tag{4.1}$$

the lefthand side being well defined, as $\operatorname{Re}\big((\eta + r\lambda) - r(\mu + \lambda)\big) = \operatorname{Re}(\eta - r\mu) \neq 0, -1, -2, \dots$. (4.1) proves that x^λ is a continuous mapping from $F_{p,\mu,r}$ into $F_{p,\mu+\lambda,r}$. A similar argument shows that $x^{-\lambda}$ is a continuous mapping from $F_{p,\mu+\lambda,r}$ into $F_{p,(\mu+\lambda)-\lambda,r}$ $= F_{p,\mu,r}$ and the required result follows at once.

THEOREM 4.2. *For $1 < p < \infty$, $r > 0$ and $\mu \in \mathbf{C}$, the operator U defined by*

$$(Ug)(x) = g(1/x) \quad (x > 0) \tag{4.2}$$

is a homeomorphism from $F_{p,\mu,r}$ onto $F_{p,-\mu,r}$ and $U^{-1} = U$.

PROOF. Choose any $\eta \in \mathbf{C}$ such that $\operatorname{Re}(\eta - r\mu) \neq 0, -1, -2, \dots$. A simple calculation involving multipliers [5, (8.2)] gives

$$M_m^\eta Uf = UN_m^\eta f \quad (f \in F_{p,\mu}). \tag{4.3}$$

(We shall use U rather than R, as used in [5], to avoid any confusion with r.) The operator U is a homeomorphism from $F_{p,\mu}$ onto $F_{p,-\mu}$ so that both sides of (4.3) are well-defined under the given condition on η. We can rewrite (4.3) in the form

$$Ug = M_m^\eta U(N_m^\eta)^{-1} g \quad (g \in F_{p,\mu,r})$$

and by Theorem 2.12, Theorem 3.1 and our previous remark concerning U, the required result now follows.

REMARK 4.3. From earlier work, we know that Theorems 4.1 and 4.2 remain true when $r = 0$ if we make the convention in (2.17). Indeed such results justify the use of (2.17) to some extent and further justification will follow.

Suppose now that we look at things the other way. Under what circumstances will an operator which is well-behaved in the original $F_{p,\mu}$ setting ($r = 0$) continue to be well-behaved relative to the $F_{p,\mu,r}$ spaces for $r > 0$? The simplest situation is when an operator T is a continuous linear mapping (or homeomorphism) from $F_{p,\mu}$ into $F_{p,\mu}$ (i.e. no change in μ). For $r > 0$, $F_{p,\mu,r}$ is a subset of $F_{p,\mu}$ and we may ask when T restricted to $F_{p,\mu,r}$ gives a continuous linear mapping (or homeomorphism) from $F_{p,\mu,r}$ into $F_{p,\mu,r}$. There is a large class of operators T for which this is true, as we shall now see.

THEOREM 4.4. *For $1 < p < \infty$ and appropriate complex numbers μ let T be a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ corresponding to an $F_{p,\mu}$ Mellin multiplier h. Then (the restriction of) T is a continuous linear mapping from $F_{p,\mu,r}$ into $F_{p,\mu,r}$ for all $r > 0$ (under the same conditions on p and μ).*

PROOF. The differential operator $K_m^{\eta+n,-n}$ appearing in (2.3) is a Mellin multiplier transform whose multiplier is $\prod_{j=1}^{n}(\eta + j - 1 + s/m)$ where, as usual, $m = 1/r$ and η is such that $\operatorname{Re}(\eta - r\mu) \neq 0, -1, -2, \ldots$. Since any two multiplier transforms commute, it follows that, in the notation of (2.3),

$$\begin{aligned}
&\gamma_i^{p,\mu}\left(\left[\Gamma\left(\operatorname{Re}(\eta + n - \mu r)\right)\right]^{-1} K_m^{\eta+n,-n} Tg\right)\\
&= \gamma_i^{p,\mu}\left(T\left(\left[\Gamma\left(\operatorname{Re}(\eta + n - \mu/m)\right)\right]^{-1} K_m^{\eta+n,-n} g\right)\right) \qquad (4.4)\\
&\leq \sum_{j=0}^{N(i)} \gamma_j^{p,\mu}\left(\left[\Gamma\left(\operatorname{Re}(\eta + n - \mu/m)\right)\right]^{-1} K_m^{\eta+n,-n} g\right)
\end{aligned}$$

for some non-negative integer $N(i)$ and constants C_j $\big(j = 0, 1, \ldots, N(i)\big)$ by [13, Lemma 1.10-1]. The inequality (4.4) now leads to

$$\gamma_i^{p,\mu,r,\eta}(Tg) \leq \sum_{j=0}^{N(i)} C_j \gamma_j^{p,\mu,r,\eta}(g) \quad \forall g \in N_m^{\eta}(F_{p,\mu}) \equiv F_{p,\mu,r}$$

and the result follows.

COROLLARY 4.5. *Let $1 < p < \infty$ and $\mu \in \mathbf{C}$. If T is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ and T, T^{-1} are Mellin multiplier transforms (corresponding to multipliers h and $1/h$, say) then (the restriction of) T is a homeomorphism from $F_{p,\mu,r}$ onto $F_{p,\mu,r}$ for all $r > 0$.*

PROOF. This is immediate on applying Theorem 4.4 to T and T^{-1}.

REMARK 4.6. Theorem 4.4 and Corollary 4.5 are further instances of results which hold for $r > 0$ as well as for $r = 0$, with the results for $r > 0$ being inherited from those for $r = 0$.

EXAMPLE 4.7. The spaces $F_{p,\mu} \equiv F_{p,\mu,0}$ studied in [2] were developed for the study of the Erdélyi-Kober operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ (the η here not being the same η as in N_m^{η}

necessarily!) In view of results obtained for $r = 0$, we can say that for $1 < p < \infty$, $r > 0$ and appropriate $\mu \in \mathbf{C}$ (and with $m = 1/r$),

(i) $I_m^{\eta,\alpha}$ is a continuous linear mapping from $F_{p,\mu,r}$ into $F_{p,\mu,r}$ if $\eta + 1 + r\mu \in \Omega$ and is a homeomorphism if, in addition, $\eta + \alpha + 1 + r\mu \in \Omega$.

(ii) $K_m^{\eta,\alpha}$ is a continuous linear mapping from $F_{p,\mu,r}$ into $F_{p,\mu,r}$ if $\eta - r\mu \in \Omega$ and is a homeomorphism if, in addition, $\eta + \alpha - r\mu \in \Omega$.

We therefore have a whole family of subspaces of the original $F_{p,\mu}$ spaces which are invariant under the Erdélyi-Kober operators, at least for $1 < p < \infty$.

Use of Theorem 4.1 enables us to handle operators where μ changes but p and r remain the same. An example of such operators is given by certain operators involving the ${}_2F_1$ hypergeometric function which were studied relative to the $F_{p,\mu}$ spaces in [2].

EXAMPLE 4.8. Consider the operator $H_1(a, b; c; m) = x^{mc}T_1(a, b; c; m)$ where $T_1(a, b; c; m)$ is a Mellin multiplier transform with multiplier

$$\Gamma(a+1-s/m)\Gamma(b+1-s/m)/\{\Gamma(a+b+1-s/m)\Gamma(c+1-s/m)\}. \tag{4.5}$$

Here a, b and c are suitably restricted complex numbers and $m > 0$. The multiplier in (4.5) is of a form which we shall be handling later in more generality. For the moment, we can proceed by factorising $H_1(a, b; c; m)$ in the form

$$H_1(a, b; c; m) = x^{mc-mb} I_m^{0,c-b} I_m^{a-b,b} x^{mb}$$

which can easily be checked via multipliers. Manipulations will be valid on $F_{p,\mu}$ provided that $a + 1 + \mu/m \in \Omega$ and $b + 1 + \mu/m \in \Omega$. Theorem 4.1 and Example 4.7 then show that $H_1(a, b; c; m)$ is a continuous linear mapping from $F_{p,\mu,1/m}$ into $F_{p,\mu+mc,1/m}$ for $1 < p < \infty$, $m > 0$ and μ as above. Furthermore, a homeomorphism will be obtained if, in addition,

$$c + 1 + \mu/m \in \Omega \text{ and } a + b + 1 + \mu/m \in \Omega.$$

When $\operatorname{Re}(a + 1 + \mu/m) > 0$ and $\operatorname{Re}(b + 1 + \mu/m) > 0$, the operator $H_1(a, b; c; m)$ is the integral operator given by

$$\big(H_1(a, b; c; m)f\big)(x) = \int_0^x (x^m - t^m)F^*(a, b; c; 1 - x^m/t^m)f(t)\, d(t^m) \tag{4.6}$$

where $F^*(a, b; c; z)$ is an analytic continuation of ${}_2F_1(a, b; c; z)/\Gamma(c)$ [2, p. 88, 93]. These results for $r = 1/m > 0$ accord with those in [2] for $r = 0$. The other operators in [2, Chapter 4] can be treated similarly.

Hypergeometric functions also arise in our next example. In [11], differential operators of the form $x^{a_1}Dx^{a_2}Dx^{a_3}$ (of so-called Bessel type) were studied and in [3] these considerations were extended to n^{th} order expressions. We shall review just one of the results.

EXAMPLE 4.9. Consider the formal differential expression

$$T = x^{a_1}Dx^{a_2}D\cdots x^{a_n}Dx^{a_{n+1}}$$

of order n where $a_1, \ldots, a_{n+1}$ are complex numbers such that

$$m = n - a > 0, \text{ where } a = \sum_{i=1}^{n+1} a_i.$$

We showed that relative to $F_{p,\mu}$ spaces, it was possible to define an α^{th} power T^α of T to be such that

$$\big(\mathcal{M}(T^\alpha f)\big)(s + m\alpha) = m^{n\alpha} \prod_{k=1}^{n} \frac{\Gamma(b_k + 1 - s/m)}{\Gamma(b_k + 1 - \alpha - s/m)} (\mathcal{M}f)(s)$$

under appropriate conditions, where

$$b_k = \left(\sum_{i=k+1}^{n+1} a_i + k - n \right) / m \quad (k = 1, \ldots, n).$$

Again, the product involving gamma functions is a special case of the type of general multiplier we shall discuss later but here we can use the factorisation

$$T^\alpha = m^{n\alpha} x^{-m\alpha} \prod_{k=1}^{n} I_m^{b_k, -\alpha} \tag{4.7}$$

in terms of Erdélyi-Kober operators. Theorem 4.1 and Example 4.7 show that T^α defines a continuous linear mapping from $F_{p,\mu,1/m}$ into $F_{p,\mu-m\alpha,1/m}$ for every $m > 0$ provided that $1 < p < \infty$ and $b_k + 1 + \mu/m \in \Omega$ for $k = 1, \ldots, n$.

REMARK 4.10. Examples 4.8 and 4.9 both lead to multipliers which are special cases of the class we are interested in and, indeed, for $\operatorname{Re}\alpha < 0$ the operator in (4.7) is an integral operator involving Meijer's G-function $G_{n,n}^{n,0}$. However, in neither case do we effect a change in the value of r. This is because the product of quotients of gamma functions is "balanced" in a sense to be made precise. A change in the value of r occurs when the multiplier is not "balanced," as we shall discover when we develop the theory of our general class of Mellin multiplier transforms relative to the $F_{p,\mu,r}$ spaces ($r \geq 0$). We are now ready to embark upon this development.

5. Consider again the multiplier h in (1.2) and let

$$c = \sum_{i=k+1}^{K} r_i + \sum_{j=1}^{\ell} t_j - \sum_{i=1}^{k} r_i - \sum_{j=\ell+1}^{L} t_j \tag{5.1}$$

$$d = \operatorname{Re}\left\{ \sum_{i=k+1}^{K} \eta_i + \sum_{j=\ell+1}^{L} \xi_j - \sum_{i=1}^{k} \eta_i - \sum_{j=1}^{\ell} \xi_j \right\} + k + \ell - \frac{1}{2}(K + L) \tag{5.2}$$

$$r = \sum_{i=k+1}^{K} r_i + \sum_{j=\ell+1}^{L} t_j - \sum_{i=1}^{k} r_i - \sum_{j=1}^{\ell} t_j. \tag{5.3}$$

The use of r in (5.3) is deliberate and we shall reconcile this version of r with the previous version of $r = 1/m$ shortly. Of course, c, d and r all depend on the multiplier h. The relevance of these quantities is shown in the following lemma.

LEMMA 5.1. *For the multiplier h in (1.2), and with $s = \sigma + i\tau$ (σ, τ real)*

(5.4) $$|h(s)| = O(|\tau|^{c\sigma+d}\exp\{-\pi r|\tau|/2\}) \text{ as } |\tau| \to \infty,$$

the estimate holding uniformly for σ in any compact subset of $\mathbf{R}$.

PROOF. The result follows by applying (3.2) to each Γ-function appearing in h.

Recall next the class C of multipliers introduced in [7]. If we use the equivalent version obtained in [7, Theorem 4.4], we can deduce

COROLLARY 5.2. *If $r \equiv r(h) \geq 0$, then the multiplier h in (1.2) belongs to the class C.*

PROOF. Chose any real numbers α and β such that the strip $\alpha < \operatorname{Re} s < \beta$ contains none of the poles of h (which are finite in number). Then h is analytic on this strip and if we choose a positive integer N such that

$$N > \sup\{c\sigma + d : \alpha < \sigma = \operatorname{Re} s < \beta\},$$

then, by (5.4), $|s^{-N}h(s)|$ will be bounded as $|s| \to \infty$, uniformly with respect to σ in any closed substrip $\alpha' \leq \sigma \leq \beta'$ where $\alpha < \alpha' < \beta' < \beta$, i.e. $h(s)$ is uniformly of order $|s|^N$ as $|s| \to \infty$ within such a strip. The result follows.

In view of [7, Theorem 4.3], we can conclude that h is the multiplier of a mapping T which maps $F_{p,\mu}$ into $F_{p,\mu}$ for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$ where α and β are any real numbers such that h is analytic on $\alpha < \operatorname{Re} s < \beta$. Also we may allow $\alpha = -\infty$ or $\beta = \infty$, as appropriate. It is convenient to introduce the following notation.

DEFINITION 5.3. For h as in (1.2), define the set $\Delta \equiv \Delta(h)$ by

(5.5) $$\Delta(h) = \{x \in \mathbf{R} : \text{ no pole of } h(s) \text{ lies on } \operatorname{Re} s = x\}.$$

Our previous statement then becomes the statement that h is an $F_{p,\mu}$ multiplier for $1 < p < \infty$ and $-\operatorname{Re}\mu \in \Delta(h)$, provided that $r(h) \geq 0$.

We can think of multipliers (1.2) having $r(h) = 0$ as being "balanced" while those having $r(h) > 0$ are "top heavy." As might be expected, the properties of the multiplier transform T corresponding to h are simplest in the balanced case.

THEOREM 5.4. *Let T be the multiplier transform corresponding to the multiplier (1.2) with $r(h) = 0$.*

(i) If $1 < p < \infty$ and $-\operatorname{Re}\mu \in \Delta(h)$, then T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$.

(ii) If, in addition, $-\operatorname{Re}\mu \in \Delta(1/h)$, then T is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$ whose inverse is the multiplier transform corresponding to $1/h$.

PROOF. This is almost immediate from the preamble and the observation that, in general,

(5.6) $$r(1/h) = -r(h)$$

as is obvious from (5.3). In our case $r(1/h) = r(h) = 0$.

EXAMPLE 5.5. Individual Erdélyi-Kober operators are balanced as are products of such operators, a particular example being the hypergeometric operator $T_1(a, b; c; m)$ in Example 4.8 corresponding to the balanced multiplier (4.5).

However, the full significance of the $F_{p,\mu,r}$ spaces begins to become apparent when $r(h) > 0$. The crux of the proof of the following theorem is to make the original top-heavy multiplier balanced by introducing another Γ-function in the denominator, the extra Γ-function being the multiplier associated with N_m^η $(m = 1/r)$.

THEOREM 5.6. *Let T be the multiplier transform corresponding to the multiplier (1.2) with $r \equiv r(h) > 0$.*

(i) If $1 < p < \infty$ and $-\operatorname{Re}\mu \in \Delta(h)$, then T is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu,r}$.

(ii) If, in addition, $-\operatorname{Re}\mu \in \Delta(1/h)$, then T is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$.

PROOF. For fixed μ such that $-\operatorname{Re}\mu \in \Delta(h)$, there exist numbers α and β such that $h(s)$ is analytic on a strip $\alpha < \operatorname{Re}s < \beta$ which contains the line $\operatorname{Re}s = -\operatorname{Re}\mu$. Choose any η such that $\operatorname{Re}(\eta + r\alpha) > 0$. Then $\Gamma(\eta + rs)$ is also analytic on the strip $\alpha < \operatorname{Re}s < \beta$. Let

$$g(s) = h(s)/\Gamma(\eta + rs). \tag{5.7}$$

Then $\Delta(h) \subseteq \Delta(g)$, since no new poles have been created and the multiplier g is balanced, since $r(g) = r(h) - r = 0$. By Theorem 5.4(i), g gives rise to a Mellin multiplier transform T_0 which is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ for the given μ and for $1 < p < \infty$. However, (5.7) leads to

$$h(s) = \Gamma(\eta + rs)g(s)$$

and, under the given conditions, $\Gamma(\eta + rs)$ is the multiplier of N_m^η with $m = 1/r$, so that, as operators on $F_{p,\mu}$,

$$T = N_m^\eta T_0 \tag{5.8}$$

and since, by construction, N_m^η maps $F_{p,\mu}$ continuously onto $F_{p,\mu,r}$, (i) of the theorem is proved.

To prove (ii) assume also that $-\mu \in \Delta(1/h)$. Then

$$1/g(s) = \Gamma(\eta + rs)/h(s)$$

so that by choice of η, with α and β as above, $-\mu \in \Delta(1/g)$. By Theorem 5.4(ii), T_0 is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu}$. Also N_m^η is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu,r}$. Part (ii) of the theorem therefore follows from (5.8). This completes the proof of the theorem.

REMARK 5.7. We can now see clearly how the theory of the multiplier (1.2) becomes so much simpler if we work relative to $F_{p,\mu}$ rather than to $L_{p,\mu}$. In the case when the corresponding operator T is as well-behaved as possible, i.e. is a homeomorphism, the range depends only on the combination (5.3) of the parameters r_i, t_j $(1 \leq i \leq K,$ $1 \leq j \leq L)$ and not on the parameters individually. Furthermore, for fixed $\mu \in \mathbf{C}$ and $1 < p < \infty$, the range is independent of the parameters $\eta_i, \xi_j (1 \leq i \leq K, 1 \leq j \leq L)$ provided only that these are chosen so that we avoid poles and ensure that $-\mu \in \Delta(h) \cap \Delta(1/h)$.

Theorem 5.6 can be extended further. Bearing in mind the convention adopted in (2.17), we have obtained a continuous linear mapping from $F_{p,\mu,0}$ into $F_{p,\mu,r}$ and, under additional conditions, a homeomorphism. Since any $F_{p,\mu,r}$ space is a subset of $F_{p,\mu,0}$ (with a different topology), we might enquire as to how the restriction of T to $F_{p,\mu,r'}$ behaves, for any $r' > 0$. This question can be answered completely. We can state the answer rather imprecisely in the form of the final theorem.

THEOREM 5.8. *Let T be the multiplier transform corresponding to the multiplier (1.2) with $r \equiv r(h) \geq 0$. Then under conditions of great generality, T (restricted where appropriate) is a continuous linear mapping from $F_{p,\mu,r'}$ into $F_{p,\mu,r'+r}$ for any $r' \geq 0$ and will be a homeomorphism under additional mild restrictions.*

REMARK 5.9.

(i) We shall not offer a proof of Theorem 5.8 here as a certain amount of extra machinery is needed. One approach is via duality and it seems appropriate to defer further details until a future paper where we hope to present a distributional analogue of the classical $L_{p,\mu}$ theory. Since the $F_{p,\mu}$ spaces are the underlying spaces of test-functions, the simple conditions on parameters exemplified in Theorem 5.6 will be retained in the distributional theory, again in contrast to the classical theory.

(ii) Theorem 4.4 and Example 4.7 provide an illustration of Theorem 5.8, with r' and r being replaced by r and 0 respectively. In general, each $F_{p,\mu,r}$ space will be invariant under any multiplier transform corresponding to a balanced multiplier.

(iii) The question arises as to what can be done when r is negative. The multiplier (1.2) is then bottom heavy, a typical example being $1/\Gamma(\eta + rs)$ which should correspond to $(N_m^\eta)^{-1}$ with $m = 1/r$. Since this operator maps the subspace $F_{p,\mu,r}$ onto $F_{p,\mu,0}$, it seems reasonable that $(N_m^\eta)^{-1}(F_{p,\mu})$ will be a larger set than $F_{p,\mu,0}$. This leads to an attempt to define a "negative space" $F_{p,\mu,r}$ for $r < 0$ as opposed to the "positive spaces" $F_{p,\mu,r}$ for $r > 0$. Such ideas are again related to duality and it is possible to mimic a construction often used for Hilbert spaces. We hope to pursue this topic also in a future paper.

ACKNOWLEDGEMENT. The second author gratefully acknowledges the award of a Carnegie Scholarship by the Carnegie Trust for the Universities of Scotland to enable this research to be carried out.

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CANADIAN JOURNAL OF MATHEMATICS

JOURNAL CANADIEN DE MATHÉMATIQUES

The Canadian Mathematical Society

La Société mathématique du Canada

Canad. Math. Bull.Vol. 35 (2), 1992 pp. 252–260

A CLASS OF MELLIN MULTIPLIERS

A. C. McBRIDE AND W. J. SPRATT

ABSTRACT. We examine a class of functions which can serve as Mellin multipliers in the setting of the spaces $F_{p,\mu}$ which we have used extensively in other papers. The conditions to be satisfied by such a multiplier h do not involve h' explicitly. This means that multipliers involving Γ-functions can be handled by means of the asymptotics of $\Gamma(z)$ alone, without the need to study $\psi = \Gamma'/\Gamma$, thereby saving effort in the case of complicated multipliers.

1. In [7], Rooney introduced a class $\mathcal{A}$ of Mellin multipliers such that each multiplier $h \in \mathcal{A}$ gives rise to a corresponding bounded linear mapping T from $L_{p,\mu}$ into $L_{p,\mu}$ for $1 < p < \infty$ and suitable complex numbers μ. In particular, the relation

$$\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s), \quad \operatorname{Re} s = -\operatorname{Re}\mu$$

holds for all $f \in L_{p,\mu} \cap L_{2,\mu}$, where $\mathcal{M}$ denotes the Mellin transform.

Recently we have been concerned with multipliers for continuous linear mappings from $F_{p,\mu}$ into $F_{p,\mu}$, where $F_{p,\mu}$ is a certain subspace of smooth functions in $L_{p,\mu}$. It was proved in [3, Theorem 3.3] that every multiplier which gives rise to a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$ does likewise for $F_{p,\mu}$, i.e. every $L_{p,\mu}$ multiplier is an $F_{p,\mu}$ multiplier. However, the class of $F_{p,\mu}$ multipliers is strictly larger.

The definition of Rooney's class $\mathcal{A}$ involves a condition on h', the derivative of h, and this condition can be tedious to verify if h is complicated. We have been particularly interested in multipliers involving products and/or quotients of gamma functions where the appropriate condition on h' can be checked via the asymptotics of Γ and $\psi = \Gamma'/\Gamma$. However, the calculations involving ψ are unnecessary. We shall obtain another criterion involving h, but not h', which will guarantee that h is an $F_{p,\mu}$ multiplier and which will be applicable in particular to our Γ-function multipliers. See [4], [5] and [6] for details of this application, along with the necessary background.

2. First let us establish the notation to be used. Throughout we shall assume that $1 < p < \infty$ and that μ is a suitable complex number.

DEFINITION 2.1.

(i) We denote by $L_{p,\mu}$ the set

$$(2.1) \qquad L_{p,\mu} = \{f : \|f\|_{p,\mu} < \infty\}$$

Received by the editors July 24, 1990.
AMS subject classification: 42A45.

where

(2.2) $$\|f\|_{p,\mu} = \left\{ \int_0^\infty |x^{-\mu} f(x)|^p dx/x \right\}^{1/p}.$$

(ii) We denote by $F_{p,\mu}$ the set

(2.3) $$F_{p,\mu} = \{ f \in C^\infty(0,\infty) : \delta^i f \in L_{p,\mu} \text{ for } i = 0,1,2,\ldots \}$$

where

(2.4) $$(\delta f)(x) = xf'(x).$$

For $i = 0,1,2,\ldots$ and $f \in F_{p,\mu}$, define $\nu_i^{p,\mu}(f)$ by

(2.5) $$\nu_i^{p,\mu}(f) = \|\delta^i f\|_{p,\mu}.$$

REMARK 2.2.

(i) The expression $\| \ \|_{p,\mu}$ in (2.2) defines a norm on $L_{p,\mu}$ and $(L_{p,\mu}, \| \ \|_{p,\mu})$ is a Banach space.

(ii) For each $i = 0,1,2,\ldots$, the expression $\nu_i^{p,\mu}(f)$ defines a seminorm on $F_{p,\mu}$ and $\nu_0^{p,\mu}(f)$ defines a norm. The topology generated by the multinorm $\{\nu_i^{p,\mu}\}_{i=0}^\infty$ turns $F_{p,\mu}$ into a Fréchet space.

(iii) The seminorms $\{\nu_i^{p,\mu}\}_{i=0}^\infty$ are more convenient here than the equivalent family of seminorms $\{\gamma_i^{p,\mu}\}_{i=0}^\infty$ defined by

$$\gamma_i^{p,\mu}(f) = \|x^i D^i f(x)\|_{p,\mu}$$

which are used in [4] and elsewhere.

(iv) Although μ is assumed to be real in [7], we can allow μ to be complex without any difficulty. The spaces $(L_{p,\mu}, \| \ \|_{p,\mu})$ and $(L_{p,\mathrm{Re}\,\mu}, \| \ \|_{p,\mathrm{Re}\,\mu})$ are identical, so that there is no loss of generality in taking μ real when it is convenient.

(v) Our set $L_{p,\mu}$ corresponds to $L^p_{-\mu}$ in [7].

DEFINITION 2.3. For suitable functions f, we define $\mathcal{M}f$, the Mellin transform of f, by

(2.6) $$(\mathcal{M}f)(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

for suitable complex numbers s.

THEOREM 2.4. *For $1 < p \leq 2$ and $f \in L_{p,\mu}$, $\mathcal{M}f$ exists almost everywhere on the line*

(2.7) $$\mathrm{Re}\, s = -\mathrm{Re}\,\mu$$

the integral (2.6) being interpreted in terms of mean convergence.

PROOF. See [7] but note Remark 2.2(v). ∎

DEFINITION 2.5. The set $\mathcal{A}$ consists of all functions h for which there exist extended real numbers α and β (depending on h) with $\alpha < \beta$ such that

(i) $h(s)$ is analytic on the strip $\alpha < \mathrm{Re}\, s < \beta$

(ii) $h(s)$ is bounded on every closed substrip $\alpha' \leq \mathrm{Re}\, s \leq \beta'$ where $\alpha < \alpha' \leq \beta' < \beta$

(iii) for $\alpha < \mathrm{Re}\, s < \beta$, $|h'(s)| = 0(|\,\mathrm{Im}\, s|^{-1})$ as $|\,\mathrm{Im}\, s| \to \infty$.

THEOREM 2.6. *Every function $h \in \mathcal{A}$ is an $L_{p,\mu}$ multiplier. More precisely, for α, β as in Definition 2.5, there exists a linear operator T such that*

(i) T is a bounded linear operator from $L_{p,\mu}$ into $L_{p,\mu}$ for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$

(ii)

$$(\mathcal{M}(Tf))(s) = h(s)(\mathcal{M}f)(s) \text{ on the line } \operatorname{Re} s = -\operatorname{Re}\mu \tag{2.8}$$

whenever $f \in L_{p,\mu} \cap L_{2,\mu}$, $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$.

PROOF. See [7, Theorem 1]. ∎

THEOREM 2.7. *Every function $h \in \mathcal{A}$ is an $F_{p,\mu}$ multiplier. More precisely, for α, β as in Definition 2.5, there exists a linear operator T such that*

(i) T is a continuous linear operator from $F_{p,\mu}$ into $F_{p,\mu}$ for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$

(ii) (2.8) holds for $f \in F_{p,\mu} \cap F_{2,\mu}$ where $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$.

PROOF. See [3, Theorem 3.3]. ∎

REMARK 2.8.

(i) In the situation of Theorems 2.6 and 2.7 we shall call T a (Mellin) multiplier transform having h as its multiplier.

(ii) Functions other than those in $\mathcal{A}$ can act as multipliers of continuous operators, simple examples being 1 and $-s$ which correspond to the identity operator and δ, as in (2.4), the latter only being meaningful in $F_{p,\mu}$ rather than in $L_{p,\mu}$.

3. As indicated in § 1 we now introduce a class of functions which can serve as multipliers but which are characterised by conditions which do not involve a"growth" estimate of the derivative. It turns out that the growth estimate in Definition 2.5(iii) is a consequence of the alternative conditions, these being easier to check in certain cases.

DEFINITION 3.1. The set $\mathcal{B}$ consists of all functions h for which there exist real numbers α and β (depending on h) with $\alpha < \beta$ such that

(i) $h(s)$ is analytic on the strip $\alpha < \operatorname{Re} s < \beta$

(ii) $sh(s)$ is bounded on every closed substrip $\alpha' \leq \operatorname{Re} s \leq \beta'$ where $\alpha < \alpha' \leq \beta' < \beta$.

THEOREM 3.2. *$\mathcal{B}$ is a subset of $\mathcal{A}$.*

PROOF. We check the conditions of Definition 2.5. Condition (*i*) for $\mathcal{A}$ follows from condition (*i*) for $\mathcal{B}$. Also boundedness of $sh(s)$ on the strip $\alpha' \leq \operatorname{Re} s \leq \beta'$ guarantees boundedness of $h(s)$ on the same strip with $|h(s)| = 0(\operatorname{Im} s|^{-1})$ as $|\operatorname{Im} s| \to \infty$ within the strip. It remains to get a similar estimate for the derivative. For given α' and β', let

$$\epsilon = \frac{1}{2}\min(\beta - \beta', \alpha' - \alpha)$$

$$M = \sup\{\,|sh(s)| : \alpha' - \epsilon \le \operatorname{Re} s \le \beta' + \epsilon\}.$$

Note that $[\alpha' - \epsilon, \beta' + \epsilon] \subset (\alpha, \beta)$ so that M exists by Definition 3.1(ii). Let $\rho = \epsilon/2$. Then for $\alpha' \le \operatorname{Re} s \le \beta'$, we may write

$$sh'(s) = \frac{d}{ds}\big(sh(s)\big) - h(s) = \frac{1}{2\pi i}\int_{C_\rho} \frac{zh(z)}{(z-s)^2}\,dz - h(s)$$

where C_ρ denotes the circle with centre s and radius ρ, C_ρ lying entirely within the strip $\alpha' - \epsilon \le \operatorname{Re} s \le \beta' + \epsilon$ by choice of ρ. By a standard estimate, we obtain

$$|sh'(s)| \le \frac{1}{2\pi}\frac{M}{\rho^2}\cdot 2\pi\rho + |h(s)| = \frac{M}{\rho} + |h(s)|.$$

As noted above, $|h(s)|$ is bounded on $\alpha' \le \operatorname{Re} s \le \beta'$ and hence so is $|sh'(s)|$. It now follows that $|h'(s)| = O(|\operatorname{Im} s|^{-1})$ as $|\operatorname{Im} s| \to \infty$ within the strip $\alpha' \le \operatorname{Re} s \le \beta'$. This verifies the third condition in Definition 2.5 and therefore completes the proof. ∎

The multiplier transforms corresponding to multipliers in $\mathcal{B}$ form a subset of those corresponding to those in $\mathcal{A}$. For instance we lose the identity transformation whose multiplier h, given by $h(s) \equiv 1$, belongs to $\mathcal{A}$ but not to $\mathcal{B}$. The transforms corresponding to multipliers in $\mathcal{B}$ can be characterised as convolution integral operators by virtue of the following result.

THEOREM 3.3. *Let $h \in \mathcal{B}$ and let α and β be as in Definition 3.1. Then there exists a function k such that*

(i) $k \in L_{1,\mu}$ for all μ satisfying $\alpha < -\operatorname{Re}\mu < \beta$

(ii) $(\mathcal{M}k)(s) = h(s)$ on the strip $\alpha < \operatorname{Re} s < \beta$.

The corresponding multiplier transform T is given by

$$(Tf)(x) = (k * f)(x) \equiv \int_0^\infty k(x/t)f(t)\,dt/t \quad (f \in L_{p,\mu}) \tag{3.1}$$

and is a bounded linear mapping from $L_{p,\mu}$ into itself for $1 < p < \infty$, $\alpha < -\operatorname{Re}\mu < \beta$.

PROOF. See [8, Theorem 2.35]. ∎

EXAMPLE 3.4. Let us review a familiar operator in the context of the class $\mathcal{B}$. Consider the multiplier

$$h(s) = \Gamma(\eta + s)/\Gamma(\eta + \gamma + s)$$

where η and γ are complex numbers with $\operatorname{Re}\gamma > 0$. h is analytic in the half-plane $\operatorname{Re} s > -\operatorname{Re}\eta$. Take $\alpha = -\operatorname{Re}\eta$, β to be any real number such that $\beta > \alpha$. For condition (ii) in Definition 3.1 we may make use of the formula [1, 1.18(6)]

$$|\Gamma(x+iy)| \sim (2\pi)^{1/2}|y|^{x-1/2}e^{-\pi|y|/2} \text{ as } |y| \to \infty. \tag{3.2}$$

Then if we write $\gamma = \gamma_1 + i\gamma_2$, $\eta = \eta_1 + i\eta_2$, $s = \sigma + i\tau$, take $-\eta_1 < \alpha' \le \sigma \le \beta' < \beta$ and note that (3.2) is uniform in x for x in a compact subset of $\mathbb{R}$, we get

$$\left|s\frac{\Gamma(\eta+s)}{\Gamma(\eta+\gamma+s)}\right| \le (C + |\tau|)\frac{|\eta_2+\tau|^{\eta_1+\sigma-1/2}}{|\eta_2+\gamma_2+\tau|^{\eta_1+\gamma_1+\sigma-1/2}} \tag{3.3}$$
$$\times \exp[-\pi\{|\eta_2+\tau| - |\eta_2+\gamma_2+\tau|\}/2]$$

(C a constant) and for boundedness as $|\tau| \longrightarrow \infty$ we require $1 - \gamma_1 \leq 0$ i.e. $\mathrm{Re}\,\gamma \geq 1$. In this case the corresponding multiplier transform is the Erdélyi-Kober operator $K_1^{\eta,\gamma}$ given by

$$(K_1^{\eta,\gamma} f)(x) = [\Gamma(\gamma)]^{-1} x^{\eta} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\eta-\gamma} f(t)\, dt \tag{3.4}$$

which has the form (3.1) with the kernel

$$k(t) = \begin{cases} [\Gamma(\gamma)]^{-1}(1-t)^{\gamma-1} t^{\eta} & 0 < t < 1 \\ 0 & t \geq 1 \end{cases}. \tag{3.5}$$

We deduce that, for $\mathrm{Re}\,\gamma \geq 1$, $K_1^{\eta,\gamma}$ is a bounded linear mapping from $L_{p,\mu}$ into itself whenever $\mathrm{Re}\,\eta > \mathrm{Re}\,\mu$ (as $\beta > \alpha$ was arbitrary). However, it is well-known that the resulting operator remains bounded under the weaker condition $\mathrm{Re}\,\gamma > 0$ (and $\mathrm{Re}\,\eta > \mathrm{Re}\,\mu$ as before). Indeed we can check that the kernel k in (3.5) belongs to $L_{1,\mu}$ under these conditions. Thus the set $\mathcal{B}$ does not tell the whole story in the $L_{p,\mu}$ setting, i.e. $h \in \mathcal{B}$ is sufficient to guarantee a convolution integral operator but not necessary.

REMARK 3.5. At this stage the reader may wonder why we have introduced $\mathcal{B}$ at all. It is true that the multiplier h in Example 3.4 belongs to $\mathcal{A}$ under the weaker condition $\mathrm{Re}\,\gamma > 0$, as can be checked via the asymptotics of the function $\psi = \Gamma'/\Gamma$ given by [1, 1.18(7)]. However, although our class $\mathcal{B}$ may seem to be deficient in the $L_{p,\mu}$ setting, it comes into its own (suitably modified) in the $F_{p,\mu}$ setting. It is in that setting that we can obtain the most elegant theory for multipliers involving products and/or quotients of gamma functions. Accordingly, we shall proceed to $F_{p,\mu}$ for our subsequent discussions.

4. When we are working in $F_{p,\mu}$, polynomials are available to us as multipliers, with the polynomial $P(s) = \sum_{i=0}^{n} a_i s^i$ corresponding to the continuous linear operator $P(-\delta) = \sum_{i=0}^{n} a_i(-\delta)^i$. We shall exploit this to the full in defining our next class of multipliers.

DEFINITION 4.1. The set C consists of all functions h for which there exist extended real numbers α and β (depending on h) such that

(i) $h(s)$ is analytic on the strip $\alpha < \mathrm{Re}\, s < \beta$

(ii) for each α_0 and β_0 satisfying $\alpha < \alpha_0 < \beta_0 < \beta$, there exists a non-negative integer $N \equiv N(\alpha_0, \beta_0, h)$ such that

$$\begin{cases} (\alpha_0 - s)^{-N} s h(s) \text{ is bounded on every closed substrip} \\ \alpha' \leq \mathrm{Re}\, s \leq \beta' \text{ where } \alpha_0 < \alpha' < \beta' < \beta_0. \end{cases} \tag{4.1}$$

REMARK 4.2.

(i) Condition (ii) in Definition 4.1 says that if we restrict attention to $\alpha_0 < \mathrm{Re}\, s < \beta_0$ we can find a non-negative integer N such that $(\alpha_0 - s)^{-N} h(s)$ defines a multiplier in $\mathcal{B}$. However if we change α_0 and β_0, we are allowed to change N in order to control the growth of h.

(ii) Instead of introducing the factor $(\alpha_0 - s)^{-N}$, we could equally well have introduced $(\beta_0 - s)^{-N}$, with the same effect.

Immediately we can prove

THEOREM 4.3. *Let $h \in C$ and let α, β be as in Definition 4.1. Then there exists an operator T such that, for $1 < p < \infty$ and $\alpha < -\operatorname{Re}\mu < \beta$,*

(i) T is a continuous linear operator from $F_{p,\mu}$ into $F_{p,\mu}$

(ii) $\big(\mathcal{M}(Tf)\big)(s) = h(s)(\mathcal{M}f)(s)$ on the line $\operatorname{Re} s = -\operatorname{Re}\mu$ for all $f \in F_{p,\mu} \cap F_{2,\mu}$.

PROOF. Choose α_0 and β_0 such that $\alpha < \alpha_0 < \beta_0 < \beta$. With N as in Definition 4.1(ii), let

$$h_0(s) = (\alpha_0 - s)^{-N} h(s) \quad (\alpha_0 < \operatorname{Re} s < \beta_0). \tag{4.2}$$

By Theorem 3.3 and Remark 4.2(i), there exists a function k_0 such that, for all μ satisfying $\alpha_0 < -\operatorname{Re}\mu < \beta_0$,

$$k_0 \in L_{1,\mu} \text{ and } (\mathcal{M}k)(s) = h_0(s) \text{ for } \operatorname{Re} s = -\operatorname{Re}\mu. \tag{4.3}$$

Let T_0 be the convolution integral operator generated by k_0 via (3.1) and let

$$T = (\alpha_0 + \delta)^N T_0 \tag{4.4}$$

where $\alpha_0 + \delta$ stands for $\alpha_0 I + \delta$, I being the identity operator on $F_{p,\mu}$. Under the stated conditions, T_0 is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$. Also, if $f \in F_{p,\mu}$, a standard result involving the Mellin convolution $*$ allows us to say that

$$\delta^i(T_0 f) = \delta^i(k_0 * f) = k_0 * \delta^i f = T_0(\delta^i f) \text{ for } i = 0, 1, 2, \ldots.$$

Hence T_0 defines a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ under the stated conditions and the same is therefore true of T. For appropriate p and μ and for $f \in F_{p,\mu} \cap F_{2,\mu}$,

$$\begin{aligned}\big(\mathcal{M}(Tf)\big)(s) &= (\alpha_0 - s)^N\big(\mathcal{M}(T_0 f)\big)(s) = (\alpha_0 - s)^N(\mathcal{M}k_0)(s)(\mathcal{M}f)(s)\\ &= (\alpha_0 - s)^N h_0(s)(\mathcal{M}f)(s) = h(s)(\mathcal{M}f)(s)\end{aligned}$$

where we have used successively (4.4), (4.3) and (4.2). Since the above argument applies to any strip $\alpha_0 < \operatorname{Re} s < \beta_0$ where $\alpha < \alpha_0 < \beta_0 < \beta$, we have constructed an operator T satisfying the requirements of the theorem. (That the versions of T coming from different substrips agree on the intersection of the substrips is proved by an argument similar to that in [7, Lemma 3.2].) This completes the proof. ∎

The conditions in Definition 4.1 led to the appearance of δ and choice of α_0 ensured that $\alpha_0 + \delta$ was an invertible operator. The operator δ itself is invertible on $F_{p,\mu}$ iff $\operatorname{Re}\mu \neq 0$, a condition which may or may not be satisfied throughout the range $\alpha < -\operatorname{Re}\mu < \beta$, depending on the values of α and β. Nevertheless, we can now obtain an equivalent characterisation of C which is easier to use in that there is no explicit mention of α_0 and β_0.

THEOREM 4.4. *A function h belongs to the class C if and only if there exist extended real numbers α and β (depending on h) such that*

(i) $h(s)$ is analytic on the strip $\alpha < \operatorname{Re} s < \beta$

(ii) for each closed substrip $\alpha' \leq \operatorname{Re} s \leq \beta'$ with $\alpha < \alpha' \leq \beta' < \beta$, there exists a non-negative integer N such that h(s) is uniformly of order $|s|^N$ as $|s| \longrightarrow \infty$, in the sense that there exist constants M and K such that

$$|s^{-N}h(s)| \leq M \quad \forall s : \alpha' \leq \operatorname{Re} s \leq \beta' \textit{ and } |s| > K. \tag{4.5}$$

PROOF. Let $h \in C$. Then condition (i) of the theorem is satisfied and it remains to check (ii). With α', β' as stated in (ii), choose α_0 and $\beta_0 : \alpha < \alpha_0 < \alpha' \leq \beta' < \beta_0 < \beta$. By Definition 4.1 (ii), there exists a positive integer N' such that $(\alpha_0 - s)^{-N'} sh(s)$ is bounded on the strip $\alpha' \leq \operatorname{Re} s \leq \beta'$ and from this (4.5) follows easily with $N = N' - 1$.

Conversely, let h satisfy the conditions of Theorem 4.4. We need only check that h satisfies Definition 4.1(ii). Given α and β, choose α_0 and $\beta_0 : \alpha < \alpha_0 < \beta_0 < \beta$ and consider the substrip $\alpha' \leq \operatorname{Re} s \leq \beta'$ where $\alpha_0 < \alpha' < \beta' < \beta_0$. The quantity $sh(s)/(\alpha_0 - s)^{N+1}$ is bounded in modulus for $\alpha' \leq \operatorname{Re} s \leq \beta'$ with N as in (4.5), since for such s satisfying $|s| > K$ we can use (4.5) and when $|s| \leq K$, we use boundedness of a continuous function on a compact set. This leads to (4.1) with N replaced by $N + 1$ and the proof is complete. ∎

We shall use Theorem 4.4 to rehabilitate the Erdélyi-Kober operator we discussed in §3.

EXAMPLE 4.5. Consider again the function $h(s)$ in Example 3.4. Let

$$\Omega = \{z \in \mathbb{C} : \operatorname{Re} z \neq 0, -1, -2, \ldots\}. \tag{4.6}$$

Then $h(s)$ is analytic in the region corresponding to $\eta + s \in \Omega$. Suppose that $\eta - \mu \in \Omega$. We can find a strip containing the line $\operatorname{Re} s = -\operatorname{Re} \mu$ where $h(s)$ is analytic. Calling this strip $\alpha < \operatorname{Re} s < \beta$, we see that on any closed substrip $\alpha' \leq \operatorname{Re} s \leq \beta'$ containing $\operatorname{Re} s = -\operatorname{Re} \mu$ in its interior, there is an estimate of the form (3.3) for $\Gamma(\eta + s)/\Gamma(\eta + \gamma + s)$ which shows that its modulus behaves like $|s|^{-\operatorname{Re}\gamma}$ as $|s| \longrightarrow \infty$ in this substrip. Accordingly we may simply choose any integer N such that $N > -\operatorname{Re}\gamma$ to see that h satisfies the conditions of Theorem 4.4 for any $\gamma \in \mathbb{C}$. By Theorems 4.3 and 4.4, h is the multiplier of an operator, called $K_1^{\eta,\gamma}$ as before, which is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ provided only that $1 < p < \infty$ and $\eta - \mu \in \Omega$.

For this operator we can say more. The function $1/h$ has the same form as h with η and γ replaced by $\eta + \gamma$ and $-\gamma$ respectively. Thus $1/h$ will be the multiplier of $K_1^{\eta+\gamma,-\gamma}$ which is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu}$ provided only that $1 < p < \infty$ and $\eta + \gamma - \mu \in \Omega$. Combining our results,

$$\begin{cases} \text{if } 1 < p < \infty, \quad \eta - \mu \in \Omega \text{ and } \eta + \gamma - \mu \in \Omega \text{ then} \\ K_1^{\eta,\gamma} \text{ is a homeomorphism from } F_{p,\mu} \text{ onto } F_{p,\mu} \text{ with inverse } K_1^{\eta+\gamma,-\gamma}. \end{cases} \tag{4.7}$$

This is in accord with known results [2, Chapter 3], which also hold for $p = 1$ and $p = \infty$ (although our theory here has to be modified to handle these values of p.)

REMARK 4.6.

(i) Other Erdélyi-Kober operators can be handled similarly. It is worth repeating the point that in Example 4.5 we have made use of formula (3.2) for the Γ-function but we did not require to use a corresponding result for $\psi = \Gamma'/\Gamma$. When the multiplier h consists of products and quotients of many Γ-functions, the saving in effort becomes well worthwhile, as illustrated in [6].

(ii) Statement (4.7) illustrates another point. In general, a multiplier $h \in \mathcal{C}$ will give rise to a homeomorphism on $F_{p,\mu}$ provided that $1/h$ also belongs to $\mathcal{C}$ and that corresponding strips overlap. We shall summarise the situation briefly in the following theorem.

THEOREM 4.7. *Let h be such that*

(i) $h \in \mathcal{C}$, *with numbers* α *and* β *as in Theorem 4.4*

(ii) $1/h \in \mathcal{C}$, *with corresponding numbers* α_1 *and* β_1

(iii) $S \equiv \{s : \alpha < \operatorname{Re} s < \beta\} \cap \{s : \alpha_1 < \operatorname{Re} s < \beta_1\}$ *is non-empty.*

Then for $1 < p < \infty$ *and* $-\mu \in S$, *h is the multiplier of a homeomorphism T from* $F_{p,\mu}$ *onto* $F_{p,\mu}$.

PROOF. This is almost immediate. ∎

5. In Theorem 3.3 we saw that multipliers in $\mathcal{B}$ gave rise to convolution integral operators, although there were convolution integral operators on $L_{p,\mu}$ which did not arise in this way, such as $K_1^{\eta,\gamma}$ for $0 < \operatorname{Re}\gamma < 1$ in Example 3.4. It turns out that we can give a precise characterisation of the continuous linear operators on $F_{p,\mu}$ which correspond to multipliers in the class $\mathcal{C}$. For this, we need one further simple piece of notation.

DEFINITION 5.1. For any $a > 0$, define the dilation operator λ_a on $F_{p,\mu}$ by

$$(\lambda_a f)(x) = f(ax) \quad (x > 0). \tag{5.1}$$

THEOREM 5.2. *A function h is in the class* $\mathcal{C}$, *with* α, β *as in Definition 4.1, if and only if it is the Mellin multiplier of a mapping T such that*

(i) T is a continuous linear mapping from $F_{p,\mu}$ *into* $F_{p,\mu}$ *for* $1 < p < \infty$ *and* $\alpha < -\operatorname{Re}\mu < \beta$

(ii) T commutes with λ_a *for all* $a > 0$.

PROOF. Certainly if $h \in \mathcal{C}$, (i) will follow from Theorem 4.3 and (ii) is easily checked since, under the appropriate conditions,

$$\big(\mathcal{M}(\lambda_a f)\big)(s) = a^{-s}(\mathcal{M}f)(s)$$

and a^{-s} will commute with $h(s)$. The reverse implication is more complicated and we omit details which can be found in [8, Theorem 3.24]. ∎

REMARK 5.3. Condition (ii) in Theorem 5.2 is the analogue for the Mellin transform of translation invariance for the Fourier transform. The theorem is an analogue of results for Fourier multipliers to be found in [9], and the proof uses techniques such as interpolation which are also found in [9].

This concludes our brief look at a class of multipliers which includes many of the Mellin multipliers which arise in common applications.

ACKNOWLEDGEMENT. One of us (W. J. S.) gratefully acknowledges the award of a Carnegie Scholarship by the Carnegie Trust for the Universities of Scotland to enable this research to be carried out.

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Paper 21

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BOUNDEDNESS OF MELLIN MULTIPLIER TRANSFORMS ON L^p-SPACES WITH POWER WEIGHTS

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Strathclyde Mathematics Research Report No. 1993/20

BOUNDEDNESS OF MELLIN MULTIPLIER TRANSFORMS ON L^p-SPACES WITH POWER WEIGHTS

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ABSTRACT

Necessary and sufficient conditions are known for the boundedness of the Laplace transform from one L^p-space with power weight on $(0, \infty)$ into another such space. We review these conditions from a new angle and show how they extend to a wide class of Mellin multiplier transforms.

AMS Subject Classification: 42 A 45, 44 A 15, 26 D 15, 26 A 33.

§1. In [1, Theorem 3.1], Bloom gives necessary and sufficient conditions under which the Laplace transform is a bounded linear mapping from one L^p space with power weight on $(0,\infty)$ into another such space. These conditions are obtained from properties of the so-called Hardy operator H defined by $(Hf)(x) = \int_0^x f(t)dt$.

In the present paper we shall first discuss these conditions from a different angle and then describe how similar results can be obtained for a large class of Mellin multiplier transforms.

We shall make extensive use of results in [3] and [4] to which the reader should refer as necessary. However, to make our discussions reasonably self-contained, we shall introduce standard notation and quote some basic results in the next section.

§2. For simplicity we assume throughout that $1 < p < \infty$ and that μ is real.

Notation 2.1 (i) $L_{p,\mu}$ will denote the space of (equivalence classes of) Lebesgue measurable functions f for which

$$\|f\|_{p,\mu} = \{\int_0^\infty |x^{-\mu}f(x)|^p \, dx/x\}^{1/p} < \infty . \tag{2.1}$$

(ii) $F_{p,\mu}$ will denote the subspace of $L_{p,\mu}$ consisting of all smooth functions f on $(0,\infty)$ for which

$$\gamma_n^{p,\mu}(f) = \|\delta^n f\|_{p,\mu} < \infty \qquad \text{for } n = 0,1,2,... \tag{2.2}$$

where $(\delta f)(x) \equiv xf'(x)$.

Remark 2.2 Expressions (2.1) and (2.2) provide a convenient way of handling the parameters p and μ. To relate them to other work, note that

(i) our space $L_{p,\mu}$ corresponds to Bloom's space $L^p(x^\beta)$ with

$$\beta = -\mu p - 1 \qquad ; \tag{2.3}$$

(ii) the space $F_{p,\mu}$ here corresponds to $F_{p,\mu+1/p}$ in [2].

The space $L_{p,\mu}$ is a Banach space with respect to the norm $\| \ \|_{p,\mu}$ while the space $F_{p,\mu}$ becomes a Fréchet space when equipped with the topology generated by the countable multinorm $\{\gamma_n^{p,\mu}\}_{n=0}^\infty$. (See [2, Chapter 2].) We can consider the continuity relative to these spaces of the operators R, S and T defined formally by

$$(Rf)(x) = f(\frac{1}{x}) \tag{2.4}$$

$$(Sf)(x) = \int_0^\infty k(xt)f(t)dt \tag{2.5}$$

$$(Tf)(x) = \int_0^\infty k(x/t)f(t)dt/t \tag{2.6}$$

where k is a suitable kernel.

Theorem 2.3

(i) R is a homeomorphism from $L_{p,\mu}$ onto $L_{p,-\mu}$ and from $F_{p,\mu}$ onto $F_{p,-\mu}$.

(ii) When $k \in L_{1,-1-\mu}$, S is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,-1-\mu}$ and from $F_{p,\mu}$ into $F_{p,-1-\mu}$.

(iii) When $k \in L_{1,\mu}$, T is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$ and from $F_{p,\mu}$ into $F_{p,\mu}$.

Remark 2.4

For proofs, see [3] and [4]. The operator T can be handled via the Hausdorff-Young inequality. Results for S then follow easily from (i) and (iii) since

$$(Sf)(x) = TR[xf(x)] \ . \tag{2.7}$$

In what follows we shall concentrate on T rather than S, as our theory in [3] and [4] applies immediately in that case.

§3. To obtain the Laplace transform $\mathcal{L}$ we take $k(u) = e^{-u}$ as the kernel in (2.5). The corresponding operator T in (2.6) is then the special case N_1^0 of the general operator N_m^η studied in [3]. By Theorem 2.3(ii), we see that $\mathcal{L}$ is a continuous linear mapping from $L_{p,\mu}$ into $L_{p,-1-\mu}$ provided that

$$\int_0^\infty |x^{1+\mu}e^{-x}|dx/x \ < \ \infty$$

which happens if and only if $\mu > -1$. (Recall that μ is real here.) By examining the statement and proof of [**3**, Theorem 3.3], we see that $\mathcal{L}$ is a continuous linear mapping from $L_{p,\mu}$ into $F_{p,-1-\mu}$ under the same condition.

At this stage we recall the following result.

Theorem 3.1

Let $1 < p \leq q < \infty$ and let μ, ν be real. Then

$$F_{p,\mu} \subseteq F_{q,\nu} \text{ if and only if } \mu = \nu \ ,$$

in which case $F_{p,\mu}$ is continuously imbedded in $F_{q,\nu}$.

<u>Proof</u>:- This is [**2**, Theorem 2.9] with the modification mentioned in Remark 2.2(ii).

We can combine this with our previous result to show that

$$\mathcal{L} \text{ maps } L_{p,\mu} \text{ continuously into } L_{q,-1-\mu} \tag{3.1}$$

provided that $\mu > -1$ and $1 < p \leq q < \infty$.

To return to the notation in [**1**], recall (2.3) and let

$$\alpha = (1+\mu)q - 1, \quad \beta = -\mu p - 1 . \tag{3.2}$$

Then $\mu > -1$ iff $\alpha > -1$. Further, by solving the equations in (3.2) we get

$$\mu = (\alpha+1)/q - 1 = -(1+\beta)/p$$

i.e.

$$\beta = p - 1 - p(\alpha+1)/q .$$

We have thus recovered one half of [**1**, Theorem 3.1], namely,

Theorem 3.2 For $1 < p \leq q < \infty$, $\alpha > -1$ and $\beta = p-1-p(\alpha+1)/q$, $\mathcal{L}$ is a bounded linear mapping from $L^p(x^\beta)$ into $L^q(x^\alpha)$.

Further investigation would be needed to establish the necessity of the conditions on the parameters in Theorem 3.2. Such an investigation would use the full force of Theorem 3.1 as well as the precise characterisation of the range of N_m^η on $L_{p,\mu}$ which formed the basis of the theory in [**3**] and [**4**]. Rather than discuss $\mathcal{L}$ further here, we shall instead consider how to extend [**1**, Theorem 3.1] from $\mathcal{L}$ to a much larger class of operators.

§4. As in [4] we consider a function h of the form

$$h(s) = \frac{\prod_{i=k+1}^{K} \Gamma(\eta_i + r_i s) \prod_{j=\ell+1}^{L} \Gamma(\xi_j - t_j s)}{\prod_{i=1}^{k} \Gamma(\eta_i + r_i s) \prod_{j=1}^{\ell} \Gamma(\xi_j - t_j s)} . \tag{4.1}$$

Here k, ℓ, K, L are non-negative integers satisfying $0 \leq k \leq K$, $0 \leq \ell \leq L$ (with empty products equal to unity by convention), the numbers $r_1, ..., r_K$ and $t_1, ..., t_L$ are real while the numbers $\eta_1, ..., \eta_K$ and $\xi_1, ..., \xi_L$ are complex.

Under appropriate conditions on the parameters, h can serve as the multiplier of a Mellin multiplier transform T. This means that T is a bounded linear mapping from $L_{p,\mu}$ into itself for $1 < p < \infty$ and suitable values of μ. Further for $f \in L_{p,\mu}$ $(1 < p \leq 2)$,

$$(\mathcal{M}(Tf))(s) = h(s)(\mathcal{M}f)(s) \tag{4.2}$$

where $\mathcal{M}$ denotes the Mellin transform and $\operatorname{Re} s = -\mu$.

In determining the behaviour of T in detail, a crucial role is played by the number $r \equiv r(h)$ defined by

$$r(h) = \sum_{i=k+1}^{K} r_i + \sum_{j=l+1}^{L} t_j - \sum_{i=1}^{k} r_i - \sum_{j=1}^{l} t_j \ . \tag{4.3}$$

We are particularly concerned with the case $r \equiv r(h) > 0$. In this case we may use (4.3) to write

$$h(s) = h_1(s)\Gamma(rs) \quad \text{where} \quad r(h_1) = 0 \ . \tag{4.4}$$

The quantity $\Gamma(rs)$ is the multiplier of an operator E_r of the form (2.6) with

$$k(u) = (1/r)\exp(-u^{1/r}) \ .$$

Note that E_r is the operator denoted by $N^0_{1/r}$ in [**3**] and [**4**]. The case $r = 1$ was mentioned at the start of §3. As before we can prove the following result.

Theorem 4.1

The operator E_r with multiplier $\Gamma(rs)$ is a continuous linear mapping from $L_{p,\mu}$ into $F_{p,\mu}$ for $1 < p < \infty$ and $\mu < 0$.

Proof:- See the statement and proof of [**3**, Theorem 3.3].

Now let T_1 be the operator corresponding to the "balanced" multiplier h_1. Corresponding to the factorisation (4.4) there is the operator equation

$$T = T_1 E_r \ . \tag{4.5}$$

Since T_1 is acting on a subspace of $F_{p,\mu}$ (rather than just $L_{p,\mu}$) it has simple mapping properties therein. In particular, by [4, Theorem 5.4] T_1 acts as a homeomorphism from $F_{p,\mu}$ onto itself under conditions of great generality. In those circumstances T is a continuous linear mapping from $L_{p,\mu}$ into $F_{p,\mu}$ and on applying T_1^{-1} to both sides of (4.5) we obtain

$$E_r = T_1^{-1} T \ . \tag{4.6}$$

Taken together, (4.5) and (4.6) show that T and E_r behave in identical fashion as mappings from $L_{p,\mu}$ into $F_{p,\mu}$. By invoking Theorem 3.1 and making explicit the restrictions needed for T_1 to be a homeomorphism on $F_{p,\mu}$ we arrive at the following.

Theorem 4.2

Let T be the operator corresponding to the Mellin multiplier h given by (4.1). Let $1 < p \leq q < \infty$ and let μ, ν be real with

$$\begin{aligned} \mathrm{Re}(\eta_i - r_i\mu) &\neq 0 \quad (i = 1, ..., K) \\ \mathrm{Re}(\xi_j + t_j\mu) &\neq 0 \quad (j = 1, ..., L) \\ \mathrm{Re}\,\mu &< 0. \end{aligned} \tag{4.7}$$

Then T is a bounded linear mapping from $L_{p,\mu}$ into $L_{q,\nu}$ if and only if $\mu = \nu$.

Proof:- By [**1**, Theorem 3.1] and a version of (2.7), $\mu = \nu$ is a necessary and sufficient condition for E_r to map $L_{p,\mu}$ into $L_{q,\nu}$ when $1 < p \leq q < \infty$. The proof can then be completed as indicated in the preamble by applying [**4**, Theorem 5.4]. Further details are omitted.

Finally, we can apply (2.7) to get information about the Laplace-type transformation S corresponding to the multiplier (4.1). This means that, for $1 < p \leq 2$ and $f \in L_{p,\mu}$ with suitable μ,

$$(\mathcal{M}(Sf))(1-s) = h(1-s)(\mathcal{M}f)(s)\ . \tag{4.8}$$

Conditions (4.7) have to be applied with μ replaced by $-1-\mu$.

Theorem 4.3

Let S be the operator satisfying (4.8) where h is given by (4.1). Let $1 < p \leq q < \infty$ and let μ, ν be real with

$$\begin{aligned} \mathrm{Re}(\eta_i + r_i + r_i\mu) &\neq 0 \quad (i = 1, ..., K) \\ \mathrm{Re}(\xi_j - t_j - t_j\mu) &\neq 0 \quad (j = 1, ..., L) \\ \mathrm{Re}\,\mu &> -1. \end{aligned} \tag{4.9}$$

Then S is a bounded linear mapping from $L_{p,\mu}$ into $L_{q,-1-\mu}$ if and only if $\mu = \nu$.

Proof:- This follows easily from Theorem 4.2 and (2.7) as indicated in the preamble.

Remark 4.4

(i) Theorem 4.3 can be recast in the notation of [**1**] via the substitution (3.2) and the resulting theorem extends [**1**, Theorem 3.1] to a wide class of operators. For relationships with Erdélyi-Kober operators, Stieltjes transforms and Meijer's G-function, see [**3**] and [**4**].

(ii) As a final comment, we would emphasise the importance of the spaces $F_{p,\mu}$ in the above analysis. The condition $r(h) > 0$ means that the operator corresponding to the multiplier T maps $L_{p,\mu}$ into $F_{p,\mu}$ (under appropriate conditions). Theorem 3.1 can then be invoked to great effect. In contrast, no analogue of Theorem 3.1 is available in the $L_{p,\mu}$ spaces, except in the trivial case $p = q, \mu = \nu$.

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Paper 22

REPRINTS from
Univalent Functions, Fractional Calculus,
and their Applications
Edited by H. M. Srivastava and Shigeyoshi Owa

Ellis Horwood Limited, Chichester

11

CONNECTIONS BETWEEN FRACTIONAL CALCULUS AND SOME MELLIN MULTIPLIER TRANSFORMS

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ABSTRACT

We consider a class of transforms in weighted versions of $L^p(0,\infty)$ whose Mellin multipliers involve products and quotients of Γ–functions. The Erdélyi–Kober operators are particular examples of these transforms, as are certain modifications of the Laplace transform. We discuss the range and invertibility of one such modification, indicating the connection between it and the Erdélyi–Kober operators and showing how the situation becomes much simpler when we restrict attention to certain subspaces of smooth functions.

1. INTRODUCTION

Many integral transforms of functions defined on $(0,\infty)$ have the form:

$$(T\varphi)(x) = \int_0^\infty f\left[\frac{x}{t}\right] \varphi(t) \frac{dt}{t}, \tag{1.1}$$

where f is a fixed kernel. Formally, this leads to

$$(\mathcal{M}(T\varphi))(s) = (\mathcal{M}f)(s)(\mathcal{M}\varphi)(s), \tag{1.2}$$

where $\mathcal{M}$ is the Mellin transform, defined (as usual) by

$$(\mathcal{M}\psi)(s) = \int_0^\infty x^{s-1} \psi(x)\, dx.$$

In many cases, the multiplier $h \equiv \mathcal{M}f$ has the form:

$$h(s) = \frac{\prod_{i=k+1}^{K} \Gamma(\eta_i + r_i s) \prod_{j=\ell+1}^{L} \Gamma(\xi_j - t_j s)}{\prod_{i=1}^{k} \Gamma(\eta_i + r_i s) \prod_{j=1}^{\ell} \Gamma(\xi_j - t_j s)}. \tag{1.3}$$

Here k, ℓ, K, L are non-negative integers satisfying $0 \leqq k \leqq K$, $0 \leqq \ell \leqq L$ (empty products being 1, by convention), the numbers $r_1, \cdots, r_K$, $t_1, \cdots, t_L$ are real and positive, while $\eta_1, \cdots, \eta_K$, $\xi_1, \cdots, \xi_L$ are complex numbers. When all the r_i's and t_j's are 1, the corresponding kernel $f = \mathcal{M}^{-1}h$ in (1.1) is related to Meijer's G-function, while Fox's H-function can also be included. However, some light can be shed on these very general kernels and operators by looking at some simple cases.

Example 1.1 (i). Consider

$$h(s) = \frac{\Gamma(\eta + 1 - s/m)}{\Gamma(\eta + \alpha + 1 - s/m)}. \tag{1.4}$$

Here $m > 0$, while η and α are complex. Equation (1.4) is of the form (1.3) with $k = K = 0$, $\ell = 1$, $L = 2$, $t_1 = t_2 = 1/m$, $\xi_1 = \eta+\alpha+1$, and $\xi_2 = \eta+1$. When $\mathrm{Re}(\alpha) > 0$, the corresponding operator T in (1.1) is the familiar Erdélyi–Kober operator $I_m^{\eta,\alpha}$ given by

$$(I_m^{\eta,\alpha}\varphi)(x) = \frac{x^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m - t^m)^{\alpha-1} \, t^{m\eta} \, \varphi(t) \, d(t^m), \tag{1.5}$$

where $d(t^m) = mt^{m-1} \, dt$.

Example 1.1 (ii). Similarly, if

$$h(s) = \frac{\Gamma(\eta + s/m)}{\Gamma(\eta+\alpha+s/m)}, \tag{1.6}$$

we have $k = 1$, $K = 2$, $\ell = L = 0$, $r_1 = r_2 = 1/m$, $\eta_1 = \eta+\alpha$, and $\eta_2 = \eta$. For $\mathrm{Re}(\alpha) > 0$, we obtain formally the Erdélyi–Kober operator $K_m^{\eta,\alpha}$ given by

$$(K_m^{\eta,\alpha}\varphi)(x) = \frac{x^{m\eta}}{\Gamma(\alpha)} \int_x^\infty (t^m - x^m)^{\alpha-1} \, t^{-m\eta-m\alpha} \, \varphi(t) \, d(t^m). \tag{1.7}$$

Example 1.2 (i). Consider

$$h(s) = \Gamma(\eta + s/m), \tag{1.8}$$

which is almost the simplest special case of (1.3). The corresponding operator T in (1.1) will be denoted by N_m^η and is given formally by

$$(N_m^\eta\varphi)(x) = m \int_0^\infty t^{m\eta} \exp(-t^m) \, \varphi(x/t) \, dt/t. \tag{1.9}$$

Example 1.2 (ii). If we take

$$h(s) = [\Gamma(\eta+s/m)]^{-1}\ , \tag{1.10}$$

we get a multiplier of the form (1.3), but the corresponding operator, which we would expect to be $(N_m^\eta)^{-1}$ in some sense or another, will not be of the form (1.1). Indeed, since N_m^η is related to the Laplace transform $\mathscr{L}$, $(N_m^\eta)^{-1}$ will be related to the Widder–Post inversion formula for $\mathscr{L}$.

2. FUNCTION SPACES AND THE ERDÉLYI–KOBER OPERATORS

To provide rigour, we shall study the various multiplier operators in the setting of weighted L^p spaces. The case of the G–function mentioned earlier has been treated in great detail by Rooney [3]. The more general multiplier (1.3) has been studied by Spratt [4] with further extensions by McBride. In the present paper we shall give a flavour of the theory, emphasising in particular the role played by the Erdélyi–Kober operators of fractional calculus. Indeed, the original results of Erdélyi and Kober around 1940 suggest that the following spaces might be useful.

Definition 2.1. For $1 \leqq p \leqq \infty$ and $\mu \in \mathbb{C}$, let $L_{p,\mu}$ be the space of (equivalence classes of) Lebesgue measurable functions $\varphi : (0,\infty) \to \mathbb{C}$ such that $\|\varphi\|_{p,\mu} < \infty$ where

$$\begin{aligned} \|\varphi\|_{p,\mu} &= \left\{ \int_0^\infty |x^{-\mu}\, \varphi(x)|^p \frac{dx}{x} \right\}^{1/p} \qquad (1 \leqq p < \infty) \\ \|\varphi\|_{\infty,\mu} &= \text{ess.sup}\{ |x^{-\mu}\, \varphi(x)| \ :\ x > 0 \}. \end{aligned} \tag{2.1}$$

Note 2.2 (i). As in [4], we insert $1/x$ when $p < \infty$ to harmonise with the Mellin convolution, but this is a slight change of notation by comparison with [1]. The present space $L_{p,\mu}$ is the space $L^p_{\mu+1/p}$ in [1].

Note 2.2 (ii). For $1 \leqq p \leqq \infty$ and $\mu \in \mathbb{C}$, $(L_{p,\mu}, \|\ \|_{p,\mu})$ is a Banach space.

With our notation, the familiar results of Erdélyi and Kober turn into

Theorem 2.3. *For* $1 \leqq p \leqq \infty$, $\mu \in \mathbb{C}$ *and* $\mathrm{Re}(\alpha) > 0$, *the operators* $I_m^{\eta,\alpha}$ *and* $K_m^{\eta,\alpha}$, *given by* (1.5) *and* (1.7), *respectively, are continuous linear mappings from* $L_{p,\mu}$ *into* $L_{p,\mu}$ *provided that* $\mathrm{Re}(\eta+1+\mu/m) > 0$ and $\mathrm{Re}(\eta-\mu/m) > 0$, *respectively.*

Note 2.4. It is well known that the integral defining the Mellin transform of an $L_{p,\mu}$ function exists (almost everywhere) on the line

$$\mathrm{Re}(s) = -\mathrm{Re}(\mu) \tag{2.2}$$

when $1 \leqq p \leqq 2$. (The integral has to be interpreted via mean convergence for $1 < p \leqq 2$.) In the sequel, we shall assume that s and μ are always related via (2.2). The conditions on η in Theorem 2.3 then ensure that we avoid the poles of the gamma functions in the numerators of (1.4) and (1.6).

Under the conditions of Theorem 2.3, $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ do not map $L_{p,\mu}$ onto itself, but merely into. For instance, if $\psi = I_m^{\eta,1}\varphi$, then ψ must be differentiable almost everywhere. In general, the ranges of $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ on $L_{p,\mu}$ are characterized by the existence of certain fractional derivatives. More can be said if we work instead in the spaces $F_{p,\mu}$ introduced by McBride [1, Chapter 2]. In what follows we shall write

$$\delta \equiv x\frac{d}{dx}. \tag{2.3}$$

Definition 2.5. For $1 \leqq p \leqq \infty$ and $\mu \in \mathbb{C}$, let

$$\begin{aligned} F_{p,\mu} &= \{\varphi \in C^\infty(0,\infty) : \delta^k\varphi \in L_{p,\mu} \text{ for } k = 0,1,2,\cdots\} \\ \gamma_k^{p,\mu}(\varphi) &= \|\delta^k\varphi\|_{p,\mu}\, (\varphi \in F_{p,\mu},\ k = 0,1,2,\cdots). \end{aligned} \tag{2.4}$$

Note 2.6 (i). Here we again use the change of notation mentioned in Note 2.2(i).

Note 2.6 (ii). With respect to the topology generated by the seminorms $\{\gamma_k^{p,\mu}\}_{k=0}^\infty$, $F_{p,\mu}$ is a Fréchet space.

As noted in [2, p. 118], $L_{p,\mu}$ multipliers are $F_{p,\mu}$ multipliers. This applies, in particular, to the multipliers (1.4) and (1.6) under the conditions of Theorem 2.3. However, the latter conditions can be relaxed in $F_{p,\mu}$. To avoid the poles, all we need is to ensure that $\eta \in A_{\mu,m}$ (for (1.4)) or $\eta \in A'_{\mu,m}$ (for (1.6)) where

$$A_{\mu,m} = \{\eta : \mathrm{Re}(\eta+\mu/m) \neq -1,-2,-3,\cdots\}, \tag{2.5}$$

$$A'_{\mu,m} = \{\eta : \mathrm{Re}(\eta-\mu/m) \neq 0,-1,-2,\cdots\}. \tag{2.6}$$

It then turns out that we end up with $F_{p,\mu}$ multipliers for any $\alpha \in \mathbb{C}$, not merely for $\mathrm{Re}(\alpha) > 0$. Let us spell out some details for $K_m^{\eta,\alpha}$.

Theorem 2.7. *Let* $1 \leqq p \leqq \infty$, $\mu \in \mathbb{C}$, $m > 0$, $\alpha \in \mathbb{C}$, *and* $\eta \in A'_{\mu,m}$. *Then there exists a unique linear operator* $K_m^{\eta,\alpha}$ *with the properties that* (*i*) $$(\mathcal{M}(K_m^{\eta,\alpha}\varphi))(s) = \frac{\Gamma(\eta+s/m)}{\Gamma(\eta+\alpha+s/m)}(\mathcal{M}\varphi)(s) \tag{2.7}$$

whenever $1 \leqq p \leqq 2$, $\varphi \in F_{p,\mu}$, *and* $\mathrm{Re}(s) = -\mathrm{Re}(\mu)$;

(*ii*) $K_m^{\eta,\alpha}$ *is a continuous linear mapping from* $F_{p,\mu}$ *into* $F_{p,\mu}$;

(*iii*) *when, in addition,* $\eta+\alpha \in A'_{\mu,m}$, $K_m^{\eta,\alpha}$ *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu}$ *with* $(K_m^{\eta,\alpha})^{-1} = K_m^{\eta+\alpha,-\alpha}$.

Note 2.8 (i). The operators $K_m^{\eta,\alpha}$ thus obtained are precisely those obtained in a more elementary but less elegant fashion in [1, Chapter 3], where concrete representations in terms of integrals and/or derivatives can be found. As a simple case, we observe that if $\eta \in A'_{\mu,m}$ and n is a positive integer, then $\eta+n \in A'_{\mu,m}$ so that $K_m^{\eta,n}$ is a homeomorphism with inverse $K_m^{\eta+n,-n}$. The multiplier for the latter operator collapses to $\prod_{j=1}^{n} (\eta+j-1+s/m)$ and the operator itself is simply the differential operator

$$(K_m^{\eta+n,-n}\varphi)(x) = x^{m\eta+mn}(-D_m)^n x^{-m\eta}\varphi(x) \quad (D_m = \frac{d}{dx^m}). \tag{2.8}$$

Note 2.8 (ii). A similar theorem can be stated for $I_m^{\eta,\alpha}$.

3. THE RANGE OF N_m^η ON $L_{p,\mu}$

Now that we have recovered familiar properties for the Erdélyi–Kober operators, let us try out the same process on the operator N_m^η in Example 1.2. For $\varphi \in L_{p,\mu}$, a routine application of the

Hausdorff–Young inequality shows that $N_m^\eta \varphi \in L_{p,\mu}$ with

$$\|N_m^\eta \varphi\|_{p,\mu} \leqq \Gamma(\mathrm{Re}(\eta-\mu/m)) \, \|\varphi\|_{p,\mu}, \tag{3.1}$$

provided that $\mathrm{Re}(\eta-\mu/m) > 0$, so that (1.9) certainly gives a continuous linear mapping from $L_{p,\mu}$ into $L_{p,\mu}$ in this case. However, much more can be said. Since $N_m^\eta \varphi$, given by (1.9), can be written as the Mellin convolution $f\star \varphi$, where f is the smooth kernel

$$f(x) = mx^{m\eta} \exp(-x^m), \tag{3.2}$$

we conclude that, for any $\varphi \in L_{p,\mu}$, $N_m^\eta \varphi$ is smooth with

$$\delta^n N_m^\eta \varphi = (\delta^n f) \star \varphi \qquad (n = 0,1,2,\cdots) \tag{3.3}$$

so that $\delta^n N_m^\eta \varphi \in L_{p,\mu}$, again by the Hausdorff–Young inequality. Bearing in mind Definition 2.5, we see that N_m^η maps us from $L_{p,\mu}$ into $F_{p,\mu}$. That being so, it is no surprise that we can relax the condition $\mathrm{Re}(\eta-\mu/m) > 0$ to the condition $\eta \in A'_{\mu,m}$, as for $K_m^{\eta,\alpha}$. In general, the extended operator N_m^η will no longer have the simple form (1.9). More precisely, for $-j < \mathrm{Re}(\eta-\mu/m) < -(j-1)$ with j a fixed positive integer, we obtain

$$N_m^\eta \varphi = (-1)^j I_m^{-\eta-j,j} N_m^{\eta+j} \varphi = (-1)^j N_m^{\eta+j} I_m^{-\eta-j,j} \varphi, \tag{3.4}$$

where $I_m^{-\eta-j,j}$ and $N_m^{\eta+j}$ are given via (1.5) and (1.9), respectively, in view of the conditions on the parameters.

The next step is to characterize $N_m^\eta(L_{p,\mu})$ as a subset of $F_{p,\mu}$. To avoid technicalities, we shall assume that $1 < p < \infty$ (although some progress can be made for $p = 1$ and $p = \infty$, too). Spratt [4] used techniques such as weak compactness and delta convergent sequences of kernels to deal with the case $\text{Re}(\eta-\mu/m) > 0$. The general case $\eta \in A'_{\mu,m}$ then follows fairly easily. However, we stick to the simplest case.

THEOREM 3.1. *Let $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, and $\text{Re}(\eta-\mu/m) > 0$. Then $\psi \in N_m^\eta(L_{p,\mu})$ if and only if there is a constant $C(\psi)$, depending on ψ but independent of the non-negative integer n, such that*

$$\|K_m^{\eta+n,-n}\psi\|_{p,\mu} \leqq C(\psi)\Gamma(\text{Re}(\eta+n-\mu/m)) \quad (n = 0,1,2,\cdots), \tag{3.5}$$

where $K_m^{\eta+n,-n}$ is given by (2.8).

When (3.5) holds, we can seek the infimum of all possible constants $C(\psi)$ and it turns out that this provides a norm on the range $N_m^\eta(L_{p,\mu})$. An equivalent expression for this norm can be obtained which does not involve $C(\psi)$ explicitly.

Definition 3.2. For $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, and $\text{Re}(\eta-\mu/m) > 0$, define $\|\ \|_{p,\mu,m,\eta}$ on $N_m^\eta(L_{p,\mu})$ by

$$\|\psi\|_{p,\mu,m,\eta} = \lim_{n\to\infty} [\Gamma(\text{Re}(\eta+n-\mu/m))]^{-1} \|K_m^{\eta+n,-n}\psi\|_{p,\mu}. \tag{3.6}$$

THEOREM 3.3. *For $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$,* and *$\text{Re}(\eta-\mu/m) > 0$, the expression* (3.6) *defines a norm on $N_m^\eta(L_{p,\mu})$. Further, $(N_m^\eta(L_{p,\mu}), \|\ \|_{p,\mu,m,\eta})$ is a Banach space.*

Note 3.4. When viewed as a subset of $L_{p,\mu}$, $N_m^\eta\,(L_{p,\mu})$ is dense in $L_{p,\mu}$, with respect to the norm (2.1), and hence does not become a Banach space with respect to the topology induced by (2.1). Thus the new norm (3.6) on the range has great advantages. In particular, as a consequence of the Open Mapping Theorem, we obtain

THEOREM 3.5. *Under the hypotheses of Theorem* 3.3, N_m^η *is a homeomorphism when regarded as a mapping from* $(L_{p,\mu}, \|\ \|_{p,\mu})$ *onto* $(N_m^\eta\,(L_{p,\mu}), \|\ \|_{p,\mu,m,\eta})$.

This leads us naturally to consideration of $(N_m^\eta)^{-1}$, which is continuous under the hypotheses of Theorem 3.5. As previously noted, we can think of this operator as corresponding to the multiplier (1.10) in some sense. However, we would like to have some sort of expression for $(N_m^\eta)^{-1}$. Problems arise with $\mathscr{M}^{-1}h$, with h as in (1.10), so that another approach is needed. We offer a whole family of inversion formulae which involve an operator of the form λ_a, where

$$(\lambda_a\,\varphi)(x) = \varphi(ax) \qquad (a > 0,\ x > 0). \tag{3.7}$$

THEOREM 3.6. *Let* $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, *and* $\mathrm{Re}(\eta - \mu/m) > 0$. *Then for* $\psi \in N_m^\eta\,(L_{p,\mu})$,

$$(N_m^\eta)^{-1}\,\psi = \lim_{n\to\infty} \frac{n^{-(\eta+\alpha)}}{\Gamma(n)}\,\lambda_{n^{1/m}}\,K_m^{\eta+\alpha+n,-(\alpha+n)}\,\psi \tag{3.8}$$

for any $\alpha \in \mathbb{C}$, *the limit existing with respect to* $\|\ \|_{p,\mu}$.

Note 3.7. We emphasize that we may use any $\alpha \in \mathbb{C}$ in (3.8). For all large enough n, $\eta+\alpha+n \in A'_{\mu,m}$ and $K_m^{\eta+\alpha+n,-(\alpha+n)}$ can be interpreted as in Theorem 2.7. This leads to families of inversion formulae for a whole class of transforms which we can think of, roughly speaking, as of Laplace type.

Example 3.8. Consider the usual Laplace transform $\mathscr{L}$ defined by

$$(\mathscr{L}\varphi)(x) = \int_0^\infty e^{-xt}\, \varphi(t)\, dt \qquad (x > 0). \tag{3.9}$$

By comparing (3.9) and (1.9), we see that, under appropriate conditions,

$$(N_1^\eta\, \varphi)(x) = x^\eta \mathscr{L} R\, x^{1-\eta}\, \varphi(x), \tag{3.10}$$

where

$$(R\varphi)(x) = \varphi\left[\frac{1}{x}\right] \quad (x > 0). \tag{3.11}$$

Formally, therefore

$$(\mathscr{L}^{-1}\, \psi)(x) = Rx^{1-\eta}(N_1^\eta)^{-1}\, x^\eta\, \psi(x).$$

The dependence on η on the right–hand side is bogus. So we may take $\eta = 0$, say. Substituting $\eta = 0$, $\alpha = 0$, and $m = 1$, and using (2.8) leads, after a short calculation, to the Widder–Post formula

$$(\mathscr{L}^{-1}\, \psi)(x) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left[\frac{n}{x}\right]^{n+1} \psi^{(n)}\left[\frac{n}{x}\right]. \tag{3.12}$$

This will be valid in $L^p(0,\infty) \equiv L_{p,-1/p}$ for $1 < p < \infty$; since the various conditions required are satisfied when $\mu = -1/p$.

Formula (3.8) gives one instance where the Erdélyi–Kober operators are relevant. We mention two others. Firstly, we may generalize (3.5) to fractional derivatives and integrals.

Theorem 3.9. *For* $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, *and* $\mathrm{Re}(\eta - \mu/m) > 0$, $\psi \in N_m^\eta(L_{p,\mu})$ *if and only if there exists a constant* $C(\psi)$, *independent of* γ, *such that*

$$\|K_m^{\eta+\gamma,-\gamma}\psi\|_{p,\mu} \leqq C(\psi)\,\Gamma(\mathrm{Re}(\eta+\gamma-\mu/m)) \tag{3.13}$$

for all $\gamma \in \mathbb{C}$ *such that* $\mathrm{Re}(\eta+\gamma-\mu/m) > 0$.

Of more significance is the following simple fact.

Theorem 3.10. *Let* $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, *and* $\mathrm{Re}(\eta-\mu/m) > 0$. *Then, as operators on* $L_{p,\mu}$,

$$N_m^{\eta+\alpha} K_m^{\eta,\alpha} = N_m^\eta \tag{3.14}$$

provided that $\mathrm{Re}(\alpha) > 0$.

Proof. For $\varphi \in L_{p,\mu} \cap L_{2,\mu}$ and $\mathrm{Re}(s) = -\mathrm{Re}(\mu)$,

$$\begin{aligned}(\mathcal{M}(N_m^{\eta+\alpha} K_m^{\eta,\alpha}\varphi))(s) &= \Gamma(\eta+\alpha+s/m)\,(\mathcal{M}(K_m^{\eta,\alpha}\varphi))(s)\\ &= \Gamma(\eta+\alpha+s/m)\,\frac{\Gamma(\eta+s/m)}{\Gamma(\eta+\alpha+s/m)}\,(\mathcal{M}\varphi)(s)\\ &= \Gamma(\eta+s/m)(\mathcal{M}\varphi)(s) = (\mathcal{M}(N_m^\eta\varphi))(s).\end{aligned}$$

Since $\mathcal{M}$ is one-to-one on $L_{2,\mu}$, $N_m^{\eta+\alpha} K_m^{\eta,\alpha}\varphi = N_m^\eta\varphi$ for $\varphi \in L_{p,\mu} \cap L_{2,\mu}$. The result then extends to all of $L_{p,\mu}$ by continuity and density.

Corollary 3.11. *Let* $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, *and* $\mathrm{Re}(\eta_i - \mu/m) > 0$ $(i = 1,2)$, *where* $\mathrm{Re}(\eta_1) < \mathrm{Re}(\eta_2)$. *Then* $N_m^{\eta_1}(L_{p,\mu})$ *is a proper subset of* $N_m^{\eta_2}(L_{p,\mu})$.

Proof. We apply (3.14) with $\alpha = \eta_2 - \eta_1$ to deduce that, if $\psi = N_m^{\eta_1}\varphi_1$ then $\psi = N_m^{\eta_2}\varphi_2$ where $\varphi_2 = K_m^{\eta_1,\eta_2-\eta_1}\varphi_1$. Further, if $\varphi_1 \in L_{p,\mu}$ then $\varphi_2 \in L_{p,\mu}$ by Theorem 2.3. Conversely, suppose $\psi = N_m^{\eta_2}\varphi_2$ where $\varphi_2 \in L_{p,\mu}$. If $\psi = N_m^{\eta_1}\varphi_1$, then φ_2 must be $K_m^{\eta_1,\eta_2-\eta_1}\varphi_1$. However, φ_2 need not lie in the range of $K_m^{\eta_1,\eta_2-\eta_1}$, as noted in Section 2, in which case φ_1 will not exist in $L_{p,\mu}$. The result follows.

Note 3.12. Corollary 3.11 shows that even in the simplest case when $\mathrm{Re}(\eta - \mu/m) > 0$, as opposed to $\eta \in A'_{\mu,m}$, $N_m^{\eta}(L_{p,\mu})$ depends on η, as well as on m. If we were to study the operator on $L_{p,\mu}$ with multiplier (1.3), the range would depend on a plethora of parameters including all the η_i's and ξ_j's. Fortunately, we can make life simpler for ourselves by restricting attention from $L_{p,\mu}$ to $F_{p,\mu}$. We have a clue that this will help from the proof of Corollary 3.11. The breakdown which occurred there will no longer trouble us, because $K_m^{\eta_1,\eta_2-\eta_1}$ is invertible on $F_{p,\mu}$ if $\eta_i \in A'_{\mu,m}$ $(i = 1,2)$. Let us see what happens.

4. THE RANGE OF N_m^η ON $F_{p,\mu}$

In view of Theorem 3.1 and the relation between the topologies on $L_{p,\mu}$ and $F_{p,\mu}$, the following result should not come as a great surprise.

THEOREM 4.1. *For* $1 < p < \infty$, $\mu \in \mathbb{C}$, $m > 0$, *and* $\mathrm{Re}(\eta-\mu/m) > 0$, $\psi \in N_m^\eta(F_{p,\mu})$ *if and only if, for each* $k = 0,1,2,\cdots$, *there exists a constant* $C_k(\psi)$, *depending on* ψ *and* k, *but independent of the non-negative integer* n, *such that*

$$\gamma_k^{p,\mu}(K_m^{\eta+n,-n}\,\psi) \leqq C_k(\psi)\,\Gamma(\mathrm{Re}(\eta+n-\mu/m)) \tag{4.1}$$

for all $n = 0,1,2,\cdots$.

It is convenient, in order to handle the multiplier (1.3), to write

$$r = \frac{1}{m} \quad (m > 0) \tag{4.2}$$

and to make the following definition.

Definition 4.2. Let $1 < p < \infty$, $\mu \in \mathbb{C}$, $r > 0$, and $\mathrm{Re}(\eta-r\mu) > 0$.

(i) $F_{p,\mu,r}$ is the space of all functions $\psi \in F_{p,\mu}$ such that for each $k = 0,1,2,\cdots$

$$\gamma_k^{p,\mu}(K_{1/r}^{\eta+n,-n}\,\psi) \leqq C_k(\psi)\,\Gamma(\mathrm{Re}(\eta+n-r\mu)) \tag{4.3}$$

for all $n = 0,1,2,\cdots$, where $C_k(\psi)$ is independent of n.

(ii) For $\psi \in F_{p,\mu,r}$ and $k = 0,1,2,\cdots$, let

$$\gamma_k^{p,\mu,r}(\psi)$$

$$= \inf\{C_k(\psi) : (4.3) \text{ holds for all } n = 0,1,2,\cdots\}. \tag{4.4}$$

Theorem 4.3. *Under the conditions of Definition* 4.2, $F_{p,\mu,r}$ *is a Fréchet space with respect to the topology generated by the multinorm* $\{\gamma_k^{p,\mu,r}\}_{k=0}^{\infty}$.

Note 4.4. An important point now arises, a point to which we have already alluded. Although a number η satisfying $\mathrm{Re}(\eta - r\mu) > 0$ is introduced, the space $F_{p,\mu,r}$ obtained is independent of the choice of such an η. It depends only on p, μ, and r.

As a consequence of Theorem 4.1 and Definition 4.2, we immediately obtain

Theorem 4.5. *For* $1 < p < \infty$, $\mu \in \mathbb{C}$, $r > 0$, *and* $\mathrm{Re}(\eta - r\mu) > 0$, $N_{1/r}^{\eta}$ *is a homeomorphism from* $F_{p,\mu}$ *onto* $F_{p,\mu,r}$, *where these spaces have the topologies generated by* $\{\gamma_k^{p,\mu}\}_{k=0}^{\infty}$ *and* $\{\gamma_k^{p,\mu,r}\}_{k=0}^{\infty}$, *respectively.*

A similar investigation can be carried out to relate $N_{1/r}^{\eta}(L_{p,\mu})$ to the $F_{p,\mu,r}$ spaces when $\eta \in A'_{\mu,1/r}$. However, we shall not dwell on this and instead focus attention on the spaces themselves.

5. GENERAL RESULTS FOR A CLASS OF MELLIN MULTIPLIERS

We have seen how the space $F_{p,\mu,r}$ arises as $N^{\eta}_{1/r}(F_{p,\mu})$. If that was its only occurrence, its usefulness would be limited. However, it turns out that we can now go back and study the operator corresponding to the multiplier (1.3) in this setting. So far, we have taken r to be positive. For convenience, we now add in $r = 0$ by adopting the convention that

$$F_{p,\mu,0} \equiv F_{p,\mu} \text{ (as in Definition 2.5).} \tag{5.1}$$

With the multiplier (1.3) we can associate the number

$$R \equiv R(h) = \left[\sum_{i=k+1}^{K} r_i + \sum_{j=\ell+1}^{L} t_j\right] - \left[\sum_{i=1}^{k} r_i + \sum_{j=1}^{\ell} t_j\right], \tag{5.2}$$

where empty sums are taken to be zero.

Example 5.1 (i). The multiplier (1.4) has $R = (0+1/m) - (0+1/m) = 0$, and the multiplier (1.6) also has $R = 0$. The multipliers in (1.8) and (1.10) have $R = +1/m$ and $R = -1/m$, respectively.

Example 5.1 (ii). We can think of multipliers with $R = 0$ as "balanced", those with $R > 0$ as "top heavy", and those with $R < 0$ as "bottom heavy". In view of (i), we might expect balanced multipliers to produce operators with mapping properties akin to those of $I^{\eta,\alpha}_m$ and $K^{\eta,\alpha}_m$, and this is indeed so. Likewise, N^{η}_m is a prototype for $R > 0$, while $R < 0$ is more complicated but typified by $(N^{\eta}_m)^{-1}$.

Spratt [4] discussed such matters in detail. We shall state a typical result to justify our comments above.

THEOREM 5.2. *Let* h *be given by* (1.3) *and let* $R \geqq 0$ (R *given by* (5.2)). *Assume that, for each* $i = 1,\cdots,K$ *and each* $j = 1,\cdots,L$,

$$\eta_i - r_i\mu\ ,\quad \xi_j + t_j\mu \notin \{0,-1,-2,\cdots\}. \tag{5.3}$$

Then there exists an operator T *such that, for* $1 \leqq p \leqq \infty$,

(i) $(\mathcal{M}(T\varphi))(s) = h(s)(\mathcal{M}\varphi)(s)$ $(\varphi \in F_{p,\mu,0} \cap F_{2,\mu,0},$
$\mathrm{Re}(s) = -\mathrm{Re}(\mu))$

(ii) *for each* $r \geqq 0$, T *is a continuous linear mapping from* $F_{p,\mu,r}$ *into* $F_{p,\mu,r+R}$ *and a homeomorphism in the case* $R = 0$.

Note 5.3. Assumption (5.3) ensures that we avoid the poles of all the gamma functions in $h(s)$. If we hit a pole of the denominator, we retain continuity of T, but lose the homeomorphic property when $R = 0$.

It is possible to take matters further. For instance, spaces $F_{p,\mu,r}$ can be defined for $r < 0$, although the process is rather technical. However, the upshot is that we obtain a family of spaces relative to which a wide class of multiplier operators can be studied. In particular, we have a family of spaces upon which the Erdélyi–Kober operators act as homeomorphisms, thereby extending Theorem 2.7 which corresponds to $r = 0$. A basis then exists for developing an extended theory for generalized functions, but that is another story.

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RADIAL SPHERICAL FOURIER TRANSFORMS

CONTENTS

Paper 23

UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS

MULTIDIMENSIONAL FRACTIONAL INTEGRALS AND DISTRIBUTIONS I

by

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Strathclyde Mathematics Research Report No. 1994/2

MULTIDIMENSIONAL FRACTIONAL INTEGRALS AND DISTRIBUTIONS I

Adam C. McBride and Boris Rubin

SYNOPSIS

New spaces of generalised functions on $\mathbf{R}^n$ are introduced. These spaces generalise corresponding spaces on $\mathbf{R}_+$ studied previously by the first author. Basic properties are obtained in preparation for a study of fractional integrals, including Riesz potentials, in a companion paper.

1. INTRODUCTION

In [1] McBride introduced spaces $F_{p,\mu} \equiv F_{p,\mu}(\mathbf{R}_+)$ of test functions and corresponding spaces $F'_{p,\mu}$ of generalised functions. These spaces provided a natural setting for the study of operators of fractional integration and differentiation [1, Chapter 3]. The spaces are also well suited to the study of other operators such as the Hankel and Laplace transforms. Our aim in the present paper is to generalise the theory from $\mathbf{R}_+$ to n dimensions.

Our work was stimulated by the fact that a number of multidimensional fractional integrals arise in applications for which it is important to have invariant spaces of smooth functions. Examples of such spaces include the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ which is invariant under Bessel potentials and the Lizorkin-Semyanistyi space $\Phi(\mathbf{R}^n)$ which is invariant under Riesz potentials. (See, for example, [5], p.132.) These spaces are suitable for translation invariant operators which may be investigated in the framework of classical Fourier analysis. However, there are examples of fractional integrals which are invariant under quite different groups of transformations. Each such group G generates its own Fourier analysis and so we should seek new function spaces with good properties relative to G-invariant operators.

In the present paper we introduce a multidimensional analogue of the spaces

$F_{p,\mu}(\mathbf{R}_+)$ based on the action of the group $G = \mathbf{R}_+ \times SO(n)$ of dilations and rotations in $\dot{\mathbf{R}}^n \equiv \mathbf{R}^n \setminus \{0\}$. Essentially we use the theory of the one-dimensional spaces $F_{p,\mu}(\mathbf{R}_+)$ from [1] and the harmonic analysis generated by $\mathbf{R}_+ \times SO(n)$, as developed in [4]. In a companion paper [2] we shall show the suitability of these spaces (and the corresponding spaces of generalised functions) for the study of various multidimensional integrals, which are important in applications.

After listing some standard notation below, we shall assemble in Section 2 all relevant properties of the spaces $F_{p,\mu}(\mathbf{R}_+)$ and the basic harmonic analysis from [4] which relies on standard results in [3]. We also list there some auxiliary formulae for future use. In Section 3, we define the multidimensional analogues of $F_{p,\mu}(\mathbf{R}_+)$ and develop their theory. The main result is Theorem 3.9 which characterises functions in these spaces by means of their expansions in terms of spherical harmonics. The stage is then set for the study of multidimensional fractional integrals in [2].

Notation

$\mathbf{Z}$ (respectively $\mathbf{N}, \mathbf{Z}_+$) is the set of all integers (respectively all non-negative integers, all positive integers).

$\mathbf{R}$ (respectively $\mathbf{R}_+$) is the set of all real numbers (respectively all positive real numbers). $\mathbf{C}$ is the set of all complex numbers.

$\mathbf{N}^n = \{\gamma = (\gamma_1, \cdots, \gamma_n) : \gamma_j \in \mathbf{N} \text{ for } j = 1, \cdots, n\}$ is the set of all multi-indices.

$\mathbf{R}^n = \{x = (x_1, \cdots, x_n) : x_j \in \mathbf{R} \text{ for } j = 1, \cdots, n\}$; $\dot{\mathbf{R}}^n = \mathbf{R}^n \setminus \{0\}$.

For any multi-index $\gamma \in \mathbf{N}^n$, write

$$|\gamma| = \gamma_1 + \cdots + \gamma_n$$

$$\partial^\gamma = \partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n} \text{ where } \partial_j = \frac{\partial}{\partial x_j} .$$

$$\sum = \{x \in \mathbf{R}^n : |x| = 1\} \; ; \; |\sum| = 2\pi^{n/2}\Gamma(n/2) .$$

Let Ω be an open subset of $\mathbf{R}^n$.

The notation $C(\Omega)$, $C^\infty(\Omega)$ and $L^p(\Omega)$ (for $1 \le p \le \infty$) is standard.

$C_c^\infty(\Omega)$ is the set of all $C^\infty(\Omega)$ functions with compact support in Ω.

$\mathcal{D}'(\Omega)$ is the dual of $C_c^\infty(\Omega)$ with respect to the usual topology.

$$C_0(\mathbf{R}_+) = \{f \in C(\mathbf{R}_+) : \lim_{r \to 0,\infty} f(r) = 0\}.$$

$$C_0(\dot{\mathbf{R}}^n) = \{\phi \in C(\dot{\mathbf{R}}^n) : \lim_{|x| \to 0,\infty} \phi(x) = 0\} .$$

The spaces $C_0(\mathbf{R}_+)$ and $C_0(\dot{\mathbf{R}}^n)$ are Banach spaces with respect to the appropriate supremum norms.

For $1 \le p < \infty$, write $p' = p/(p-1)$ (with $p' = \infty$ when $p = 1$).

$$\mathcal{L}_p(\mathbf{R}_+) = \{f : \|f\|_p \equiv (\int_0^\infty |f(r)|^p dr/r)^{\frac{1}{p}} < \infty\}$$

$$\mathcal{L}_p(\mathbf{R}^n) = \{\phi : \|\phi\|_p \equiv (\int_{\mathbf{R}^n} |\phi(x)|^p dx/|x|^n)^{\frac{1}{p}} < \infty .$$

The notation $\mathcal{L}_\infty(\mathbf{R}_+)$, $\mathcal{L}_\infty(\mathbf{R}^n)$ will be used for the spaces $C_0(\mathbf{R}_+)$ and $C_0(\dot{\mathbf{R}}^n)$ respectively.

For $1 \le p \le \infty$ and $\mu \in \mathbf{C}$, define the weighted Banach spaces

$$\mathcal{L}_{p,\mu}(\mathbf{R}_+) = \{f : \|f\|_{p,\mu} \equiv \|r^\mu f(r)\|_p < \infty\}$$
$$\mathcal{L}_{p,\mu}(\mathbf{R}^n) = \{\phi : \|\phi\|_{p,\mu} \equiv \||x|^\mu \phi(x)\|_p < \infty\}.$$

(We hope the reader will not be confused by norms $\|.\|_p$, $\|\cdot\|_{p,\mu}$ of functions in $\mathbf{R}_+$ and in $\mathbf{R}^n$.)

I denotes the identity operator (on an appropriate space).

$\square$ denotes the end of a proof.

§2. PRELIMINARIES

Here we present some auxiliary material which will be needed in subsequent sections.

2.1 The spaces $F_{p,\mu}(\mathbf{R}_+)$.

Below we quote basic properties of the spaces $F_{p,\mu}$ in $\mathbf{R}_+$. For more detailed information see [1], where slightly different notation is used.

Definition 2.1 *Let $\mu \in \mathbf{C}, \delta = r\frac{d}{dr}$. We define the spaces $F_{p,\mu}(\mathbf{R}_+)$ for $1 \le p \le \infty$ by*

$$F_{p,\mu}(\mathbf{R}_+) = \{f \in C^\infty(\mathbf{R}_+) : \delta^k f \in \mathcal{L}_{p,\mu}(\mathbf{R}_+)\ \forall k \in \mathbf{N}\} \tag{2.1}$$

when $1 \le p < \infty$, and

$$F_{\infty,\mu}(\mathbf{R}_+) = \{f \in C^\infty(\mathbf{R}_+) : r^\mu \delta^k f \in C_0(\mathbf{R}_+)\ \forall k \in \mathbf{N}\}. \tag{2.2}$$

Recall (see Notation) that $C_0(\mathbf{R}_+)$ is the Banach space of all continuous functions f of $r \in \mathbf{R}_+$ such that $f(r) \to 0$ as $r \to 0$ and as $r \to \infty$.

For $1 \le p \le \infty, \mu \in \mathbf{C}$, the topology on $F_{p,\mu}(\mathbf{R}_+)$ is generated by the family of norms $\{\lambda_k^{p,\mu}\}_{k=0}^\infty$ where

$$\lambda_k^{p,\mu}(f) = \sup_{m \le k} \|\delta^m f\|_{p,\mu} = \sup_{m \le k} \|r^\mu \delta^m f\|_p \ . \tag{2.3}$$

In the case $p = \infty$ we interpret $\|.\|_p$ as the supremum norm on a space of continuous functions.

An equivalent topology on $F_{p,\mu}(\mathbf{R}_+)$ is given by the norms $\{\tilde{\lambda}_k^{p,\mu}\}_{k=0}^{\infty}$ where

$$\tilde{\lambda}_k^{p,\mu}(f) = \sup_{m \le k} \|\delta^k r^\mu f\|_p \ . \tag{2.4}$$

Lemma 2.2 *For $1 \le p \le \infty$ and $\mu \in \mathbf{C}$, the following statements hold.*

(i) *$F_{p,\mu}(\mathbf{R}_+)$ is a Fréchet space.*

(ii) *$C_c^\infty(\mathbf{R}_+)$ is dense in $F_{p,\mu}(\mathbf{R}_+)$.*

(iii) *The mapping $f(r) \to r^\nu f(r)$ is a isomorphism from $F_{p,\mu}(\mathbf{R}_+)$ onto $F_{p,\mu-\nu}(\mathbf{R}_+)$ for any $\nu \in \mathbf{C}$.*

(iv) *δ is a continuous linear operator from $F_{p,\mu}(\mathbf{R}_+)$ into itself and is an automorphism when $Re\ \mu \ne 0$.*

(v) *If $f \in F_{p,\mu}(\mathbf{R}_+)$, then $f(r) = o(r^{-Re\mu})$ as $r \to 0$ and as $r \to \infty$.*

Proof. See [1, Chapter 2]. Note that (v) was formulated in [1, Theorem 2.2] with $O(r^{-Re\mu})$ but analysis of the proof shows that $o(r^{-Re\mu})$ holds. □

The following pointwise estimates for functions in $F_{p,\mu}(\mathbf{R}_+)$ will be used.

Lemma 2.3 *Let $f \in F_{p,\mu}(\mathbf{R}_+)$, $1 \le p \le \infty$, $\mu \in \mathbf{C}$ and let $r > 0$.*

(i) *For $Re\ \mu \ne 0$,*

$$|f(r)| \le (p'|Re\mu|)^{-1/p'} r^{-Re\mu} \|\delta f\|_{p,\mu} \ . \tag{2.5}$$

(ii) *For any $\mu \in \mathbf{C}$ and any $\epsilon > 0$,*

$$|f(r)| < (p'\epsilon)^{-1/p'} r^{-Re\mu} (|\mu - \epsilon| \|f\|_{p,\mu} + \|\delta f\|_{p,\mu}). \tag{2.6}$$

Proof. (i) Write $\mu' = Re\ \mu$. First consider the case $\mu' > 0$. By Lemma 2.2 (v) we have $|f(r)| \le cr^{-\mu'}$ for all $r > 0$, where c is a constant. Thus, $f(r) \to 0$ as $r \to \infty$. It follows that

$$f(r) = -\int_r^\infty f'(t)dt$$

and hence

$$|f(r)| = \left| \int_r^\infty [t^{\mu'-1/p}(\delta f)(t)] t^{-\mu'-1/p'} dt \right|$$

$$\leq \|\delta f\|_{p,\mu} \left\{ \int_r^\infty [t^{-\mu'-1/p'}]^{p'} dt \right\}^{1/p'} = \frac{r^{-\mu'}}{(p'\mu')^{1/p'}} \|\delta f\|_{p,\mu}$$

which gives (2.5) in this case. Similarly, for $\mu' < 0$, we have

$$|f(r)| = \left| \int_0^r f'(t)dt \right| \leq \|\delta f\|_{p,\mu} \left\{ \int_0^r [t^{-\mu'-1/p'}]^{p'} dt \right\}^{1/p'} = \frac{r^{-\mu'}}{(p'(-\mu'))^{1/p'}} \|\delta f\|_{p,\mu} \, .$$

(ii) For $\mu \in \mathbf{C}$ and $\epsilon > 0, r^{\mu-\epsilon} f(r) \in F_{p,\epsilon}(\mathbf{R}_+)$ by Lemma 2.2 (iii). By (2.5) with μ replaced by ϵ, we obtain

$$|r^{\mu-\epsilon} f(r)| \leq \frac{r^{-\epsilon}}{(\epsilon p')^{1/p'}} \|\delta r^{\mu-\epsilon} f\|_{p,\epsilon} \leq \frac{r^{-\epsilon}}{(\epsilon p')^{1/p'}} (|\mu - \epsilon| \; \|f\|_{p,\mu} + \|\delta f\|_{p,\mu}).$$

2.2 Radial-spherical Fourier transforms.

In this section we present some definitions related to the type of harmonic analysis which is natural for operators commuting with rotations and dilations in $\mathbf{R}^n$. Instead of $\mathbf{R}^n$ we consider $\dot{\mathbf{R}}^n \equiv \mathbf{R}^n \backslash \{0\}$ and use the group $G \equiv \mathbf{R}_+ \times SO(n)$ as our basic group of linear transformations in $\mathbf{R}^n$.

Such Fourier analysis was developed in [4] although various fragments of it were used before in many papers. The main object in this analysis is a radial-spherical convolution operator defined by

$$(K\phi)(x) = \int_{R^n} \phi(y) k\left(\frac{|x|}{|y|}, x'.y'\right) \frac{dy}{|y|^n} \tag{2.7}$$

where $x' \equiv x/|x|$ is a point of the unit sphere $\sum$ and $x'.y' \equiv x_1'y_1' + \cdots + x_n'y_n'$ is the usual inner product. Operator (2.7) is G-invariant, i.e., it commutes with rotations and dilations in $\mathbf{R}^n$. Moreover, any G-invariant linear operator which is bounded from

$\mathcal{L}_{p_1,\mu_1}(\mathbf{R}^n)$ to $\mathcal{L}_{p_2,\mu_2}(\mathbf{R}^n)$ may be represented as a radial-spherical convolution in some distributional sense. (See [4] for the details.)

A Fourier analysis in $\dot{\mathbf{R}}^n$ generated by the action of the group G is constructed as follows. Denote by $\{Y_{j,\nu}(x')\}$ an orthonormal basis (in $L^2(\sum)$) of spherical harmonics on $\sum$. Here $j = 0, 1, \cdots$, and $\nu = 1, 2, \cdots, d_n(j)$, where $d_n(j)$ is the dimension of the subspace of spherical harmonics of the order j. Every spherical harmonic is the restriction to $\sum$ of a homogeneous polynomial of degree j (see, e.g., [3] for the details).

Given a function $\phi(x), x \in \dot{\mathbf{R}}^n$, we define

$$\phi_{j,\nu}(r) = \int_{\Sigma} \phi(rx')Y_{j,\nu}(x')dx', \ j \in \mathbf{N}; \ \nu = 1, \cdots, d_n(j); \tag{2.8}$$

$$\tilde{\phi}_{j,\nu}(z) = \int_0^\infty \phi_{j,\nu}(r)r^{z-1}\,dr \ . \tag{2.9}$$

The sequence $\{\tilde{\phi}_{j,\nu}(z)\}$ is called the radial-spherical Fourier transform (RSF-transform) of ϕ. (Note that $\tilde{\phi}_{j,\nu}$ is the Mellin transform of $\phi_{j,\nu}$.)

We complete this sub-section by quoting some auxiliary facts related to spherical harmonics.

Lemma 2.4

(i) With the previous notation

$$|d_n(j)| \leq c(1+j)^{n-2} \qquad \text{where } \ c = c(n) \ .$$

(ii) If $\beta \in \mathbf{N}^n$ then

$$|\partial^\beta Y_{j,\nu}(x/|x|)| \leq c(1+j)^{(n-2)/2+|\beta|} \tag{2.10}$$

where c does not depend on j, ν.

(iii) If Δ_Σ is a Laplace-Beltrami operator on $\sum$, then

$$(\Delta_\Sigma Y_{j,\nu})(x') = -j(j+n-2)Y_{j,\nu}(x'). \tag{2.11}$$

Proof:- See [3], the relevant pages being 218 for (i), 225 for (ii) and 229 for (iii). □

2.3 Auxiliary relations for derivatives.

Partial derivatives

$$\omega^{(\gamma)}(x) = (\partial^\gamma \omega)(x) = \frac{\partial^{|\gamma|}\omega(x)}{\partial x_1^{\gamma_1}\cdots\partial x_n^{\gamma_n}},\ \gamma = (\gamma_1,\cdots,\gamma_n) \in \mathbf{N}^n, \tag{2.12}$$

$|\gamma| = \gamma_1 + \cdots + \gamma_n$, which commute with translations are not so convenient in our investigation where invariance of another sort (under dilations) is dominating. For that reason we shall introduce so-called "homogeneous derivatives"

$$(\mathcal{D}_\gamma\omega)(x) = r^{|\gamma|}(\partial^\gamma\omega)(x), \quad r = |x|, \tag{2.13}$$

and

$$(\delta\omega)(x) = r\frac{\partial}{\partial r}\omega(x), \quad x = rx', x' = x/r \in \Sigma. \tag{2.14}$$

The following relations between (2.12)-(2.14) will be needed in the sequel.

Lemma 2.5 *For $m \in \mathbf{N}$*

$$(\delta^m\omega)(x) = \sum_{k=0}^{m}\sum_{|\gamma|=k} P_\gamma(x)\omega^{(\gamma)}(x), \tag{2.15}$$

where, for each $\gamma, P_\gamma(x)$ is a polynomial of degree $|\gamma|$.

Lemma 2.6 *If $\mu \in \mathbf{C}$, $m \in \mathbf{N}$, $f \equiv f(r)$, then*

$$r^\mu\delta^m r^{-\mu} f = (\delta-\mu)^m f. \tag{2.16}$$

Lemma 2.7 *Let $\gamma \in \mathbf{N}^n, f(x) \equiv f_0(r),\ r \equiv |x|, x' \equiv x/r$. Then*

$$f^{(\gamma)}(x) = r^{-|\gamma|}\sum_{\ell=0}^{|\gamma|}(\delta^\ell f_0)(r)P_{\ell,\gamma}(x'), \tag{2.17}$$

where for each ℓ and $\gamma, P_{\ell,\gamma}(x)$ is a polynomial of degree $|\gamma|$.

We leave the proofs of these simple statements to the reader.

Let us obtain an analogue of (2.17) for functions which may not be radial. Let

$$\omega(x) \sim \sum_{j,\nu} \omega_{j,\nu}(r) Y_{j,\nu}(x'), \ r \equiv |x|, \ x' \equiv x/r. \tag{2.18}$$

Let

$$\psi_{j,\nu}(x) \equiv \omega_{j,\nu}(r) Y_{j,\nu}(x') \equiv r^{-j} \omega_{j,\nu}(r) Y_{j,\nu}(x).$$

By making use of (2.17) we have

$$\begin{aligned}
\psi_{j,\nu}^{(\gamma)}(x) &= \sum_{\beta \le \gamma} \binom{\gamma}{\beta} Y_{j,\nu}^{(\beta)}(x) (r^{-j} \omega_{j,\nu}(r))^{(\gamma-\beta)} \\
&= \sum_{\beta \le \gamma} r^{|\beta-\gamma|} Y_{j,\nu}^{(\beta)}(x) \sum_{\ell=0}^{|\gamma-\beta|} \delta^\ell (r^{-j} \omega_{j,\nu}(r)) P_{\ell,\gamma,\beta}(x') \\
&= r^{-|\gamma|} \sum_{\substack{\beta \le \gamma \\ |\beta| \le j}} Y_{j,\nu}^{(\beta)}(x') \sum_{\ell=0}^{|\gamma-\beta|} (\delta - j)^\ell \omega_{j,\nu}(r) P_{\ell,\gamma,\beta}(x'),
\end{aligned}$$

where $P_{\ell,\gamma,\beta}(x)$ are polynomials of degree $|\gamma - \beta|$. Write

$$Q_{j,\nu}^{\gamma,\beta,\ell}(x') = Y_{j,\nu}^{(\beta)}(x') P_{\ell,\gamma,\beta}(x').$$

Use of estimate (2.10) leads to

Lemma 2.8 *If $\gamma \in \mathbf{N}^n$, then*

$$\mathcal{D}_\gamma [\omega_{j,\nu}(r) Y_{j,\nu}(x')] = \sum_{\substack{\beta \le \gamma \\ |\beta| \le j}} \sum_{\ell=0}^{|\gamma-\beta|} Q_{j,\nu}^{\gamma,\beta,\ell}(x') (\delta - j)^\ell \omega_{j,\nu}(r), \tag{2.19}$$

where

$$|Q_{j,\nu}^{\gamma,\beta,\ell}(x')| \le c j^{(n-2)/2 + |\beta|} \tag{2.20}$$

with the constant c being independent of j, ν.

§3. SPACES $F_{p,\mu}(\mathbf{R}^n)$

In this section we generalise the theory of the one-dimensional spaces $F_{p,\mu}(\mathbf{R}_+)$ to the multidimensional case. Our motivation for this investigation is the following.

(i) There should be a "good" harmonic analysis serving a new function space.

(ii) It is desirable to have non-trivial families of operators under which a new space is invariant.

In the present investigation we give a multidimensional generalisation of $F_{p,\mu}(\mathbf{R}^n)$ which is based on the action of the group $G = \mathbf{R}_+ \times SO(n)$ in $\dot{\mathbf{R}}^n = \mathbf{R}^n \backslash \{0\}$.

For convenience we shall write $\mathbf{R}^n$ without "dot" with the exception of some special cases when the origin should be essentially dropped out.

Definition 3.1 *Given* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$ *we define a space* $F_{p,\mu}(\mathbf{R}^n)$ *by*

$$F_{p,\mu}(\mathbf{R}^n) = \{\phi \in C^\infty(\dot{\mathbf{R}}^n) : \mathcal{D}_\gamma \phi \in \mathcal{L}_{p,\mu}(\mathbf{R}^n)\ \forall \gamma \in \mathbf{N}^n\}, \tag{3.1}$$

where $\mathcal{D}_\gamma$ *stands for the "homogeneous" derivative* (2.13). As in the case of the half-line, if $p = \infty$, *we replace* $\mathcal{L}_\infty(\dot{\mathbf{R}}^n)$ *by the space* $C_0(\dot{\mathbf{R}}^n)$ *of continuous functions vanishing at the origin and at infinity.*

The space (3.1) may be endowed with the topology generated by the sequence of norms

$$\|\phi\|_k^{p,\mu} = \sup_{|\gamma| \leq k} \|\mathcal{D}_\gamma \phi\|_{p,\mu},\ k \in \mathbf{N} \tag{3.2}$$

or

$$|||\phi|||_k^{p,\mu} = \sup_{|\gamma| \leq k} \|\mathcal{D}_\gamma r^\mu \phi\|_p,\ k \in \mathbf{N}. \tag{3.3}$$

It can be checked that the topologies generated by (3.2) and (3.3) are equivalent.

Lemma 3.2 *For any* $\lambda \in \mathbf{C}$, $r^\lambda I$ *is an isomorphism from* $F_{p,\mu}(\mathbf{R}^n)$ *onto* $F_{p,\mu-\lambda}(\mathbf{R}^n)$ *and*

$$|||r^\lambda \phi|||_k^{p,\mu-\lambda} = |||\phi|||_k^{p,\mu},\ k \in \mathbf{N}\ .$$

This statement is an immediate consequence of (3.1), (3.3).

Lemma 3.3 *$F_{p,\mu}(\mathbf{R}^n)$ is a Fréchet space.*

Proof. Let us prove that $F_{p,\mu}(\mathbf{R}^n)$ is complete. By Lemma 3.1, it is sufficient to consider the case $\mu = n/p$, when $F_{p,\mu}(\mathbf{R}^n)$ is a subspace of $L^p(\mathbf{R}^n)$. Let $\{\phi_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $F_{p,n/p}(\mathbf{R}^n)$. Then for each $\gamma \in \mathbf{N}^n$ there exists a function $\phi^\gamma \in L^p(\mathbf{R}^n)$ such that $\lim_{j\to\infty} \mathcal{D}_\gamma \phi_j = \phi^\gamma$ in L^p-norm.

Let $\theta = (0, \cdots, 0)$ and note that $\phi^\gamma = \mathcal{D}_\gamma \phi^\theta$ in the sense of distributions in $\mathcal{D}'(\dot{\mathbf{R}}^n)$. Indeed, for any test function $\omega \in C_c^\infty(\dot{\mathbf{R}}^n)$ we have

$$(\phi^\gamma, \omega) = \lim_{j\to\infty} (\mathcal{D}_\gamma \phi_j, \omega) = \lim_{j\to\infty} (\phi_j, \mathcal{D}_\gamma^* \omega) = (\phi^\theta, \mathcal{D}_\gamma^* \omega) = (\mathcal{D}_\gamma \phi^\theta, \omega) \tag{3.4}$$

where $\mathcal{D}_\gamma^*$ is an adjoint to $\mathcal{D}_\gamma$. Thus, $\phi^\theta \in L^p(\mathbf{R}^n)$ and has generalized derivatives $\mathcal{D}_\gamma \phi^\theta \in L^p(\mathbf{R}^n)$.

Our aim now is to show that there exists a function $\phi \in C^\infty(\dot{\mathbf{R}}^n)$ such that $\phi^\gamma = \mathcal{D}_\gamma \phi$ almost everywhere in $\dot{\mathbf{R}}^n$ for any $\gamma \in \mathbf{N}^n$. This will imply that $\lim_{j\to\infty} [\mathcal{D}_\gamma(\phi_j - \phi)] = 0 \; \forall \gamma \in \mathbf{N}^n$ in L^p-norm and the lemma will be proved.

We proceed to find ϕ. For each $k \in \mathbf{Z}_+$ let

$$\Omega_k = \{x \in \mathbf{R}^n : k^{-1} < |x| < k\}.$$

By (3.4) we have

$$(\partial^\gamma \phi^\theta, \omega) = (r^{-|\gamma|}\phi^\gamma, \omega) \quad \forall \omega \in C_c^\infty(\Omega_k). \tag{3.5}$$

Since ϕ^θ and $r^{-|\gamma|}\phi^\gamma$ both belong to $L^p(\Omega_k)$, then by the Sobolev embedding theorem there exists a function $\psi_k \in C^\infty(\Omega_k)$ such that $\psi_k = \phi^\theta$ almost everywhere on Ω_k. For any $k \in \mathbf{N}$, ψ_{k+1} is an extension of ψ_k, and since $\bigcup_k \Omega_k = \dot{\mathbf{R}}^n$, there exists $\phi \in C^\infty(\dot{\mathbf{R}}^n)$ such that $\phi \equiv \psi_k$ on Ω_k. Hence $\phi = \phi^\theta$ a.e. on $\dot{\mathbf{R}}^n$ so that $\lim_{j\to\infty} \phi_j = \phi$ in L^p-norm. Let us check that $\mathcal{D}_\gamma \phi = \phi^\gamma$ a.e. for all $\gamma \in \mathbf{N}^n$. Indeed, for any $\omega \in C_c^\infty(\dot{\mathbf{R}}^n)$,

$$(\phi^\gamma, \omega) = (\mathcal{D}_\gamma \phi^\theta, \omega) = (\phi^\theta, \mathcal{D}_\gamma^* \omega) = (\phi, \mathcal{D}_\gamma^* \omega) = (\mathcal{D}_\gamma \phi, \omega).$$

Since ϕ^γ and $\mathcal{D}_\gamma \phi$ are locally integrable over $\dot{\mathbf{R}}^n$, they coincide almost everywhere. In view of our earlier comments, $F_{p,\mu}(\mathbf{R}^n)$ has been proved to be complete and hence it is a Fréchet space. □

Lemma 3.4 *$C_c^\infty(\dot{\mathbf{R}}^n)$ is dense in $F_{p,\mu}(\mathbf{R}^n)$.*

Proof. Let $\eta \in C_c^\infty(\mathbf{R})$ be such that

$$\eta(\xi) \equiv 1 \text{ for } |\xi| \leq 1; \ \ \eta(\xi) \equiv 0 \text{ for } |\xi| \geq 2.$$

For $k = 1, 2, \ldots$, define η_k on $\dot{\mathbf{R}}^n$ by

$$\eta_k(x) = \eta(k^{-1}\log r), \quad r = |x| > 0.$$

Then

$$\eta_k(x) = \begin{cases} 1 \text{ for } e^{-k} \leq |x| \leq e^k, \\ 0 \text{ for } |x| \leq e^{-2k} \text{ and } |x| \geq e^{2k}. \end{cases}$$

Also $\eta_k \in C_c^\infty(\dot{\mathbf{R}}^n)$ for each $k \geq 1$. By Lemma 2.7,

$$(\mathcal{D}_\beta \eta_k)(x) = \sum_{j=0}^{|\beta|} (\delta^j \eta_k)(r) P_{j,\beta}(x'), \ x' = x/r\ , \tag{3.6}$$

where $P_{j,\beta}(x)$ are polynomials of degree $|\beta|$.

One can readily see that

$$|(\mathcal{D}_\beta \eta_k)(x)| \leq K_\beta \text{ for some constant } K_\beta \text{ independent of } k. \tag{3.7}$$

This follows from the properties of $\eta(\xi)$ in view of the relation

$$(\delta^j \eta_k)(r) = k^{-j}\eta^{(j)}(k^{-1}\log r).$$

Now we take $\phi \in F_{p,\mu}(\dot{\mathbf{R}}^n)$ and put $\phi_k = \phi\eta_k$, $k \in \mathbf{Z}_+$. Obviously $\phi_k \in C_c^\infty(\dot{\mathbf{R}}^n)$. Let us prove that $\phi_k \to \phi$ in $F_{p,\mu}(\dot{\mathbf{R}}^n)$ as $k \to \infty$. Let

$$V_k = \mathbf{R}^n \backslash \{x \in \dot{\mathbf{R}}^n : e^{-k} \leq |x| \leq e^k\}, \quad \mu' = \text{Re } \mu\ .$$

Then, for any $\gamma \in \mathbf{N}^n$, by making use of (3.7) we have

$$\|\mathcal{D}_\gamma(\phi-\phi_k)\|_{p,\mu} = \left(\int\limits_{V_k} r^{(|\gamma|+\mu')p-n}|\partial^\gamma[\phi(1-\eta_k)]|^p dx\right)^{1/p}$$

$$\leq \sum_{\beta\leq\gamma}\binom{\gamma}{\beta}\left(\int\limits_{V_k} r^{\mu' p-n}|\mathcal{D}_{\gamma-\beta}\phi|^p|\mathcal{D}_\beta(1-\eta_k)|^p dx\right)^{1/p}$$

$$\leq c\sum_{\beta\leq\gamma}\left(\int_{V_k} r^{\mu' p-n}|\mathcal{D}_{\gamma-\beta}\phi|^p dx\right)^{1/p} \to 0 \text{ as } k\to\infty. \qquad \square$$

Lemma 3.5. *Let $1\leq p\leq\infty$, $\mu\in\mathbf{C}$. The Fourier-Laplace coefficients $\phi_{j,\nu}(r)$ of a function $\phi\in F_{p,\mu}(\mathbf{R}^n)$ belong to $F_{p,\mu}(\mathbf{R}_+)$ with the estimate*

$$\|\delta^k\phi_{j,\nu}\|_{p,\mu}\leq c(k,n,p,N)(1+j)^{-2N+(n-2)/2}\|\phi\|^{p,\mu}_{k+2N} \tag{3.8}$$

which holds for any $k,N\in\mathbf{N}$.

Proof. For any positive integers k and N, according to (2.11) we have

$$\begin{aligned}&[(-j)(j+n-2)]^N(\delta^k\phi_{j,\nu})(r)\\ =&[(-j)(j+n-2)]^N\int\limits_\Sigma Y_{j,\nu}(x')(\delta^k\phi)(rx')dx' = \int\limits_\Sigma Y_{j,\nu}(x')(\Delta^N_\Sigma\delta^k\phi)(rx')dx'.\end{aligned}$$

Thus, by (2.10)

$$|(\delta^k\phi_{j,\nu})(r)|\leq c_{n,p}(1+j)^{-2N+(n-2)/2}\left(\int\limits_\Sigma|(\Delta^N_\Sigma\delta^k\phi)(rx')|^p dx'\right)^{1/p}$$

where the constant $c_{n,p}$ depends only on n and p. With $\mu'=Re\ \mu$, we therefore obtain

$$\|\delta^k\phi_{j,\nu}\|_{p,\mu}=\left(\int\limits_0^\infty|r^\mu(\delta^k\phi_{j,\nu})(r)|^p dr/r\right)^{1/p}$$

$$\leq c_{n,p}(1+j)^{-2N+(n-2)/2}\left(\int\limits_0^\infty r^{\mu' p-1}dr\int\limits_\Sigma|(\Delta^N_\Sigma\delta^k\phi)(rx')|^p dx'\right)^{1/p}$$

$$\leq c_{n,p}(1+j)^{-2N+(n-2)/2}\sum_{|\gamma|=k+2N}\alpha_\gamma\left(\int\limits_{\mathbf{R}^n} r^{\mu' p-n+kp+2Np}|\phi^{(\gamma)}(x)|^p dx\right)^{1/p}$$

from which (3.8) follows. □

The following lemma involves pointwise estimates of $\delta^k\omega_{j,\nu}$. These estimates will play an important role in the sequel.

Lemma 3.6 *For $\phi \in F_{p,\mu}(\mathbf{R}^n)$, $1 \le p \le \infty$, and any $k \in \mathbf{N}$,*

$$|\delta^k\phi_{j,\nu}(r)| \le r^{-Re\mu} \begin{cases} c_{\mu,p}\|\delta^{k+1}\phi_{j,\nu}\|_{p,\mu} & \text{for } Re\,\mu \neq 0, \\ c_1\|\delta^k\phi_{j,\nu}\|_{p,\mu} + c_2\|\delta^{k+1}\phi_{j,\nu}\|_{p,\mu} & \text{for any } \mu \in \mathbf{C}, \end{cases} \tag{3.9}$$

where $c_{\mu,p} = (p'|Re\mu|)^{-1/p'}$, $c_1 = c_{1,p}|\mu - 1|$, $c_2 = c_{1,p}$.

Proof. This is immediate from Lemma 2.3 on taking $f = \delta^k\phi_{j,\nu}$ (which belongs to $F_{p,\mu}(\mathbf{R}_+)$ by Lemma 3.5) and using $\epsilon = 1$ in (2.6). □

Corollary 3.7 *If $\phi \in F_{p,\mu}(\mathbf{R}^n)$, $1 \le p \le \infty$, $\mu \in \mathbf{C}$, then for any $k \in \mathbf{N}$ and any $\gamma \in \mathbf{N}^n$, the series*

$$\sum_{j,\nu}(\delta^k\phi_{j,\nu})(r)Y_{j,\nu}(x'), \quad \sum_{j,\nu}\mathcal{D}_\gamma[\phi_{j,\nu}(r)Y_{j,\nu}(x')]$$

converge uniformly in $x \equiv rx'$ in the annulus $a \le |x| \le b$ $\forall[a,b] \subset (0,\infty)$ and represent corresponding derivatives $(\delta^k\phi)(x)$ and $(\mathcal{D}_\gamma\phi)(x)$ therein.

Proof. For the first series the statement follows from (3.9), (3.8). The same estimates imply the required convergence of the second series according to Lemma 2.8. □

Corollary 3.7 justifies term-by-term differentiation of the expansion of $\phi \in F_{p,\mu}(\mathbf{R}^n)$ in terms of spherical harmonics. Now we show how things work in the opposite direction.

Lemma 3.8 *For each $j \in \mathbf{N}$ and $\nu = 1, \cdots, d_n(j)$, let the function $\phi_{j,\nu} \in F_{p,\mu}(\mathbf{R}_+)$ be given. Define ϕ on $\dot{\mathbf{R}}^n$ by*

$$\phi(x) = \sum_{j,\nu}\phi_{j,\nu}(r)Y_{j,\nu}(x') = \sum_{j,\nu}\psi_{j,\nu}(x). \tag{3.10}$$

Assume that, for all $j, k, N \in \mathbf{N}$ and for $\nu = 1, \cdots, d_n(j)$,

$$\|\delta^k\phi_{j,\nu}\|_{p,\mu} \le c_{k,N}(1+j)^{-2N+(n-2)/2} \tag{3.11}$$

where $c_{k,N}$ does not depend on j. Then $\phi \in F_{p,\mu}(\mathbf{R}^n)$.

Proof. To prove that $\phi \in C^\infty(\dot{\mathbf{R}}^n)$, we must show that one can apply $\mathcal{D}_\gamma$ term-by-term in (3.10). Again we need uniform convergence in annuli of the form $0 < a \le r \le b < \infty$. By Lemma 2.8

$$|\mathcal{D}_\gamma \psi_{j,\nu}(x)| \le \sum_{\substack{\beta\le\gamma\\ |\beta|\le j}} \sum_{\ell=0}^{|\gamma|-|\beta|} \sum_{k=0}^{\ell} c_{k,\ell,\beta,\gamma} j^{(n-2)/2+|\beta|+\ell-k} |(\delta^k \omega_{j,\nu})(r)|$$

for certain coefficients $c_{k,\ell,\beta,\gamma}$, i.e.,

$$|\mathcal{D}_\gamma \psi_{j,\nu}(x)| \le c_\gamma j^{(n-2)/2+|\gamma|} \sum_{k=0}^{|\gamma|} (\delta^k \phi_{j,\nu})(r). \tag{3.12}$$

According to (2.6), for $r \in [a,b] \subset (0,\infty)$ we get

$$|(\delta^k \phi_{j,\nu})(r)| \le c_{a,b}(\|\delta^k \phi_{j,\nu}\|_{p,\mu} + \|\delta^{k+1}\phi_{j,\nu}\|_{p,\mu}).$$

By (3.11) we deduce that $|\mathcal{D}_\gamma \phi_{j,\nu}(x)|$ is bounded by a constant multiple of $(1+j)^{-2N+n-2+|\gamma|}$. This estimate gives the required uniform convergence with respect to $x' \in \Sigma$ and $0 < a \le r \le b < \infty$. Hence $\phi \in C^\infty(\dot{\mathbf{R}}^n)$.

We now show that $\|\mathcal{D}_\gamma \phi\|_{p,\mu}$ is finite for each multi-index γ. We have, with $\mu' = Re\ \mu$,

$$\begin{aligned}
\|\mathcal{D}_\gamma \phi\|_{p,\mu} &= \left(\int_{\mathbf{R}^n} r^{\mu' p-n} |(\mathcal{D}_\gamma \phi)(x)|^p dx\right)^{1/p} = \left(\int_{\mathbf{R}^n} r^{\mu' p-n} \Big|\sum_{j,\nu} \mathcal{D}_\gamma(\psi_{j,\nu}(x))\Big|^p\, dx\right)^{1/p} \\
&\le c_\gamma \sum_{k=0}^{|\gamma|} \sum_{j,\nu} j^{(n-2)/2+|\gamma|} \left(\int_{\mathbf{R}^n} r^{\mu' p-n} |(\delta^k \phi_{j,\nu})(r)|^p\, dx\right)^{1/p} \\
&\le c_{n,\gamma} \sum_{k=0}^{|\gamma|} \sum_{j,\nu} j^{(n-2)/2+|\gamma|} \left(\int_0^\infty r^{\mu' p-1} |(\delta^k \phi_{j,\nu})(r)|^p\, dr\right)^{1/p} \\
&\le c_1 \sum_{k=0}^{|\gamma|} \sum_{j,\nu} j^{(n-2)/2+|\gamma|} \|\delta^k \phi_{j,\nu}\|_{p,\mu} \\
&\le c_2 \sum_{k=0}^{|\gamma|} \{\sup_{j,\nu}(1+j)^\ell \|\delta^k \phi_{j,\nu}\|_{p,\mu}\} \left\{\sum_{j=0}^{\infty} (1+j)^{(n-2)/2-\ell+|\gamma|} d_n(j)\right\}.
\end{aligned}$$

(Here c_1 and c_2 are constants independent of j and (3.12) has been used.) By Lemma 2.4(i), $d_n(j) \le c_3(1+j)^{n-2}$ so that the infinite series converges for $3(n-2)/2-\ell+|\gamma| \le -2$ which is true for ℓ sufficiently large. In that case we shall have

$$\|\mathcal{D}_\gamma \phi\|_{p,\mu} \le c_4 \sup_{k\le|\gamma|} \sup_{j,\nu} (1+j)^\ell \|\delta^k \phi_{j,\nu}\|_{p,\mu} \tag{3.13}$$

where c_4 depends on γ, and ℓ depends on n and $|\gamma|$. With ℓ suitably chosen, (3.11) shows that, for any $N \in \mathbf{N}$

$$(1+j)^{\ell}\|\delta^k \phi_{j,\nu}\|_{p,\mu} \leq K(k,n,p,N)(1+j)^{\ell-2N+(n-2)/2}.$$

Choosing N such that $\ell - 2N + (n-2)/2 < 0$ guarantees that the right-hand side of (3.13) is finite. Thus, $\phi \in F_{p,\mu}(\mathbf{R}^n)$, and the theorem is proved. $\square$

The statements above show that there exists "$F_{p,\mu}$-correspondence" between functions on $\dot{\mathbf{R}}^n$ and their Fourier-Laplace coefficients. In this correspondence the behaviour of coefficients with respect to j is very important. In order to give a precise form to these facts we consider a set of triples

$$\Lambda = \{(j,\nu,r) : j \in \mathbf{N}; \nu = 1,\cdots,d_n(j); r > 0\}$$

and introduce a linear topological space $F_{p,\mu}(\Lambda)$ of functions $\psi : \Lambda \to \mathbf{C}$ with the topology generated by the sequence of norms

$$n_k^{p,\mu} = \sup_{\ell,m\leq k} \sup_{\nu} (1+j)^{\ell}\|\delta^k \psi(j,\nu,.)\|_{\mathcal{L}_{p,\mu}(\mathbf{R}_+)}, \quad k = 0,1,2,\cdots.$$

Then Lemmas 3.4, 3.6 with the estimates (3.8), (3.13) lead to the following statement.

Theorem 3.9 *Let $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$. Then the mapping $\phi(x) \to \phi_{j,\nu}(r)$ is an isomorphism from $F_{p,\mu}(\mathbf{R}^n)$ onto $F_{p,\mu}(\Lambda)$.*

This statement enables us to characterise functions in $F_{p,\mu}(\mathbf{R}^n)$ in terms of their Fourier-Laplace coefficients belonging to $F_{p,\mu}(\mathbf{R}_+)$. This is extremely useful for studying operators which are invariant under rotations and dilations in $\mathbf{R}^n$. Examples of such operators, namely fractional integrals, will be presented in [2].

Remark 3.10. By using the estimates above it is not difficult to obtain a multidimensional analogue of statement (v) of Lemma 2.1 and an analogue of Lemma 2.3. Namely, we have the following

Lemma 3.11. *Let $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$, $\phi \in F_{p,\mu}(\mathbf{R}^n)$. Then for any multi-index γ,*

$$\sup_{x' \in \Sigma} |(\mathcal{D}_\gamma \phi)(rx')| \leq cr^{-Re\mu}\|\phi\|^{p,\mu}_{2|\gamma|+2n}; \quad c = c(n,p,\mu,\gamma), \tag{3.14}$$

and hence

$$(\mathcal{D}_\gamma \phi)(rx') = o(r^{-Re\mu}) \text{ as } r \to 0,\infty, \tag{3.15}$$

uniformly in $x' \in \Sigma$.

Proof. Put $\mu' = Re\ \mu$ and use "c" to denote different constants independent of j. According to (3.12), (2.6), (3.8) we have

$$|(\mathcal{D}_\gamma\phi)(rx')| \leq cr^{-\mu'} \sum_{j,\nu}(1+j)^{(n-2)/2+|\gamma|} \sum_{k=0}^{|\gamma|}(\|\delta^k\phi_{j,\nu}\|_{p,\mu} + \|\delta^{k+1}\phi_{j,\nu}\|_{p,\mu})$$

$$\leq cr^{-\mu'} \sum_{j,\nu}(1+j)^{-2N+n-2+|\gamma|} \sum_{k=0}^{|\gamma|} \|\phi\|^{p,\mu}_{k+2N+1}$$

$$\leq cr^{-\mu'} \|\phi\|^{p,\mu}_{|\gamma|+2N+1} \sum_{j=0}^{\infty}(1+j)^{-2N+|\gamma|+2(n-2)}, \quad \forall N \in \mathbf{Z}_+ .$$

By putting $N = n + [(|\gamma| - 1)/2]$ we obtain (3.14). Relation (3.15) is implied by (3.14). Indeed, since $C_c^\infty(\dot{\mathbf{R}}^n)$ is dense in $F_{p,\mu}(\mathbf{R}^n)$, then we can take a sequence $\{\phi_m(x)\}_{m=1}^\infty$ in $C_c^\infty(\dot{\mathbf{R}}^n)$ which converges to ϕ in $F_{p,\mu}(\mathbf{R}^n)$- topology. Then

$$|(\mathcal{D}_\gamma\phi)(rx')| \leq |(\mathcal{D}_\gamma(\phi - \phi_m))(rx')| + |(\mathcal{D}_\gamma\phi_m)(rx')|$$

$$\leq cr^{-\mu'} \|\phi - \phi_m\|^{p,\mu}_{2|\gamma|+2n} + |(\mathcal{D}_\gamma\phi_m)(rx')|,$$

and the result follows since the norm in the first term may be arbitrary small for sufficiently large m. □

This completes the preparations for the study of fractional integrals in the companion paper [2].

Acknowledgements

The paper was written while the second author was visiting the University of Strathclyde in Glasgow. The second author is deeply grateful to the Royal Society for the support during his visit and to his colleagues in the University of Strathclyde for their hospitality.

The research of the second author was also partly funded by the National Council for Research and Development of Israel (grant no. 032-7251) and in part by the Edmund Landau Centre for Research in Mathematical Analysis sponsored by the Minerva Foundation (Germany).

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Paper 24

UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MATHEMATICS

MULTIDIMENSIONAL FRACTIONAL INTEGRALS AND DISTRIBUTIONS II

by

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Strathclyde Mathematics Research Report No. 1994/3

MULTIDIMENSIONAL FRACTIONAL INTEGRALS AND DISTRIBUTIONS II

Adam C. McBride and Boris Rubin

This paper continues the investigation started in [4]. Multidimensional fractional integrals and Riesz potentials are shown to have excellent mapping properties relative to the spaces of test functions and generalised functions introduced in [4].

§1. INTRODUCTION

This paper is a sequel to our previous paper [4] to which the reader should constantly refer. In that paper we provided motivation for the development of a distributional theory of multidimensional fractional integrals and Riesz potentials. We then introduced spaces $F_{p,\mu}(\mathbf{R}^n)$ and $F'_{p,\mu}(\mathbf{R}^n)$ which were suitable analogues in n dimensions of corresponding spaces on the half-line, the latter having being introduced and extensively studied by McBride [3]. Basic properties of our new spaces were established in [4]. In the present paper we shall show how these spaces are very suitable for the study of certain multidimensional fractional integrals, including Riesz potentials.

Our main tactic is to relate the multidimensional fractional integrals to corresponding one-dimensional operators, the so-called Erdélyi-Kober operators, whose properties are developed fully in [3]. By making use of expansions in terms of spherical harmonics, we see that this relation is very simple. We have an example of a radial spherical Fourier transform (RSF-transform) in the sense of Rubin [5], the definition of which involves the use of the Mellin transform.

In §2, we gather together for convenience all the required properties of the Erdélyi-Kober operators on the spaces $F_{p,\mu}(\mathbf{R}_+)$. These properties are then used in §3 to obtain the mapping properties of certain multidimensional fractional integrals relative to our spaces $F_{p,\mu}(\mathbf{R}^n)$. The corresponding properties of Riesz potentials in these spaces are then easily deduced. Finally in §4 we derive the corresponding distributional theory by making use of adjoint operators. Applications similar to those for the corresponding theory in Sobolev spaces may be pursued later.

Notation Throughout we shall use the notation of [4] to which the reader should refer. For convenience, we briefly recall that for $1 \le p \le \infty$ and $\mu \in \mathbf{C}$

$$F_{p,\mu}(\mathbf{R}_+) = \{\phi \in C^\infty(\mathbf{R}_+) : \lambda_k^{p,\mu}(\phi) < \infty \text{ for } k \in \mathbf{N}\} \tag{1.1}$$

where the seminorms $\lambda_k^{p,\mu}$ are defined for $1 \le p < \infty$ by

$$\lambda_k^{p,\mu}(\phi) = \{\int_0^\infty |r^\mu(\delta^k\phi)(r)|^p dr/r\}^{1/p} \tag{1.2}$$

while

$$\lambda_k^{\infty,\mu}(\phi) = \text{ ess } \sup\{r^\mu(\delta^k\phi)(r) : r > 0\} . \tag{1.3}$$

Here $(\delta\phi)(r) \equiv r\ d\phi/dr$.

The spaces $F_{p,\mu}(\mathbf{R}^n)$ are defined in [4, Definition 3.1].

We shall require one additional piece of notation.

For $\phi : \mathbf{R}_+ \to \mathbf{C}$, we define the Mellin transform $\tilde{\phi}$ of ϕ by

$$\tilde{\phi}(z) = \int_0^\infty r^{z-1}\phi(r)dr . \tag{1.4}$$

(An alternative notation for $\tilde{\phi}$ is $\mathcal{M}\phi$.)

§2. ERDÉLYI - KOBER OPERATORS

For $Re\ \alpha > 0$ and suitable functions f defined on $\mathbf{R}_+$ let

$$(I_2^\alpha f)(r) = \frac{2}{\Gamma(\alpha)} \int_0^r (r^2 - t^2)^{\alpha-1} f(t)\ tdt, \tag{2.1}$$

$$(K_2^\alpha f)(r) = \frac{2}{\Gamma(\alpha)} \int_r^\infty (t^2 - r^2)^{\alpha-1} f(t)\ tdt . \tag{2.2}$$

The operators I_2^α, K_2^α map $\mathcal{L}_{p,\mu}(\mathbf{R}_+)$ into $\mathcal{L}_{p,\mu-2\alpha}(\mathbf{R}_+)$ under appropriate conditions. For our purposes it is convenient to use homogeneous modifications of I_2^α, K_2^α. These are the Erdélyi-Kober operators $I_2^{\eta,\alpha}$ and $K_2^{\eta,\alpha}$ ($\eta \in \mathbf{C}$) given by

$$(I_2^{\eta,\alpha} f)(r) = r^{-2\eta-2\alpha} I_2^\alpha r^{2\eta} f(r) , \tag{2.3}$$

$$(K_2^{\eta,\alpha} f)(r) = r^{2\eta} K_2^{\alpha} r^{-2\eta-2\alpha} f(r) \ . \tag{2.4}$$

The operators $I_2^{\eta,\alpha}, K_2^{\eta,\alpha}$ are particular cases of operators studied in [3, Chapter 3] which involve integration with respect to $r^m (m > 0)$. Here $m = 2$. From (2.3) and (2.4) we obtain the representations

$$(I_2^{\eta,\alpha} f)(r) = \frac{2}{\Gamma(\alpha)} \int_0^1 (1-u^2)^{\alpha-1} u^{2\eta+1} f(ru) du, \tag{2.5}$$

$$(K_2^{\eta,\alpha} f)(r) = \frac{2}{\Gamma(\alpha)} \int_1^\infty (u^2-1)^{\alpha-1} u^{-2\eta-2\alpha+1} f(ru) du \tag{2.6}$$

valid under appropriate conditions, as indicated in the following result.

Lemma 2.1 *Let* $1 \le p \le \infty, \mu \in \mathbf{C}$, *Re* $\alpha > 0$. *Then*

(i) $I_2^{\eta,\alpha}$ *is a bounded linear operator from* $\mathcal{L}_{p,\mu}(\mathbf{R}_+)$ *into itself provided that* $Re(\eta + 1 - \mu/2) > 0$;

(ii) $K_2^{\eta,\alpha}$ *is a bounded linear operator from* $\mathcal{L}_{p,\mu}(\mathbf{R}_+)$ *into itself provided that* $Re(\eta + u/2) > 0$.

Proof. These statements are well known and may be obtained by applying the Minkowski inequality to (2.5) and (2.6). □

The Mellin transforms of $I_2^{\eta,\alpha} f$ and $K_2^{\eta,\alpha} f$ may be readily calculated in terms of the Mellin transform of f.

Lemma 2.2 *Let* $f \in \mathcal{L}_{2,\mu}(\mathbf{R}_+)$, *Re* $\alpha > 0$, *Re* $z =$ *Re* μ. *Then*

(i) *for* $Re(\eta + 1 - \mu/2) > 0$

$$(\mathcal{M}(I_2^{\eta,\alpha} f))(z) = \frac{\Gamma(\eta + 1 - z/2)}{\Gamma(\eta + \alpha + 1 - z/2)} (\mathcal{M} f)(z), \tag{2.7}$$

(ii) *for* $Re(\eta + \mu/2) > 0$

$$(\mathcal{M}(K_2^{\eta,\alpha} f))(z) = \frac{\Gamma(\eta + z/2)}{\Gamma(\eta + \alpha + z/2)} (\mathcal{M} f)(z) \ . \tag{2.8}$$

Proof. See [6], for example. □

When the operators $I_2^{\eta,\alpha}$ and $K_2^{\eta,\alpha}$ are applied to smooth functions the range of admissible parameters becomes greatly extended.

Definition 2.3 *Let* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$, $Re(\mu/2 - \eta - 1) \notin \mathbf{N}$. *We define* $I_2^{\eta,\alpha} f$ *for* $f \in F_{p,\mu}(\mathbf{R}_+)$ *as follows.*

(i) *For* $Re(\eta + 1 - \mu/2) > 0$ *and* $\alpha \in \mathbf{C}$, *let* $\ell \in \mathbf{N}$ *be such that* $Re(\alpha + \ell) > 0$. *Then*

$$I_2^{\eta,\alpha} f = I_2^{\eta,\alpha+\ell} P_{\eta,\alpha,\ell}(\delta) f = P_{\eta,\alpha,\ell}(\delta) I_2^{\eta,\alpha+\ell} f \tag{2.9}$$

where $I_2^{\eta,\alpha+\ell}$ *is defined via (2.5) (with* $\alpha + \ell$ *replacing* α*) and*

$$P_{\eta,\alpha,\ell}(\delta) = \prod_{i=1}^{\ell} (\eta + \alpha + i + \delta/2). \tag{2.10}$$

(An empty product is regarded as unity.)

(ii) *Let* $-k < Re(\eta + 1 - \mu/2) < -k + 1$ *for some* $k \in \mathbf{Z}_+$. *Then*

$$I_2^{\eta,\alpha} f = (-1)^k I_2^{\eta+k,\alpha-k} K_2^{-\eta-k,k} f = (-1)^k K_2^{-\eta-k,k} I_2^{\eta+k,\alpha-k} f \tag{2.11}$$

where $I_2^{\eta+k,\alpha-k} f$ *is defined via (2.9) and* $K_2^{-\eta-k,k}$ *via (2.6).*

Lemma 2.4 *Let* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}$, $\alpha \in \mathbf{C}$.

(i) *If* $Re(\mu/2 - \eta - 1) \notin \mathbf{N}$, *then* $I_2^{\eta,\alpha}$ *is a continuous linear mapping from* $F_{p,\mu}(\mathbf{R}_+)$ *into itself.*

(ii) *If, in addition,* $Re(\mu/2 - (\eta + \alpha) - 1) \notin \mathbf{N}$, *then* $I_2^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}(\mathbf{R}_+)$ *and*

$$(I_2^{\eta,\alpha})^{-1} = I_2^{\eta+\alpha,-\alpha}. \tag{2.12}$$

Proof. See [3, Theorem 3.31]. □

We shall require the following estimates with respect to the norms $\lambda_k^{p,\mu}$ given by (1.2) and (1.3).

Lemma 2.5 *Let* $1 \leq p \leq \infty$, $\mu \in \mathbf{C}, \alpha \in \mathbf{C}$, $m > 0$, $f \in F_{p,\mu}(\mathbf{R}_+)$.

(i) *For* $Re(\eta + 1 - \mu/2) > 0$ *and* $Re\ \alpha > 0$

$$\lambda_m^{p,\mu}(I_2^{\eta,\alpha} f) \leq \frac{\Gamma(Re\alpha)\Gamma(Re(\eta + 1 - \mu/2))}{|\Gamma(\alpha)|\ \Gamma(Re(\eta + \alpha + 1 - \mu/2))} \lambda_m^{p,\mu}(f). \tag{2.13}$$

(ii) *For* $Re(\eta+1-\mu/2)>0$ *and* $\alpha\in\mathbf{C}$ *with* $Re(\alpha+\ell)>0$ $(\ell\in\mathbf{N})$,

$$\lambda_m^{p,\mu}(I_2^{\eta,\alpha}f)\leq\frac{\Gamma(Re\alpha+\ell)\ \Gamma(Re(\eta+1-\mu/2))}{|\Gamma(\alpha+\ell)|\Gamma(Re(\eta+\alpha+\ell+1-\mu/2))}\sum_{s=0}^{\ell}c(\eta,\alpha,s)\lambda_{m+s}^{p,\mu}(f). \tag{2.14}$$

(iii) *For* $-k<Re(\eta+1-\mu/2)<-k+1$ *and* $Re\ \alpha-k+\ell>0$ $(k,\ell\in\mathbf{N})$

$$\begin{aligned}\lambda_m^{p,\mu}(I_2^{\eta,\alpha}f)&\leq\frac{\Gamma(Re(-\eta-k+\mu/2))\ \Gamma(Re\alpha-k+\ell)\ \Gamma(Re(\eta+k+1-\mu/2))}{\Gamma(Re(-\eta+\mu/2))|\Gamma(\alpha+\ell)|\Gamma(Re(\eta+\alpha+\ell+1-\mu/2))}\\&\times\sum_{s=0}^{\ell}c(\eta+k,\alpha-k,s)\lambda_{m+s}^{p,\mu}(f)\ .\end{aligned} \tag{2.15}$$

In (ii) *and* (iii) *the notation* $c(\eta,\alpha,s)$ *and* $c(\eta+k,\alpha-k,s)$ *is used for constants which depend on arguments in brackets.*

Proof.

(i) follows from a standard result for Mellin convolutions.

(ii) then follows easily from (2.9).

(iii) follows from (ii) and the same result for Mellin convolutions as in (i) but applied this time to $K_2^{-\eta-k,k}$. □

Remark 2.6 It is important to note that at each stage of the extension process the symbol for $I_2^{\eta,\alpha}$ remains unchanged, i.e. (2.7) continues to hold in all cases in Lemma 2.4. A similar programme can now be carried through for $K_2^{\eta,\alpha}$, at each stage of which (2.8) continues to hold. We shall indicate briefly the main points of the theory.

Definition 2.7 *Let* $1\leq p\leq\infty$, $\mu\in\mathbf{C}$, $\alpha\in\mathbf{C},-Re(\eta+\mu/2)\notin\mathbf{N}$. *We define* $K_2^{\eta,\alpha}f$ *for* $f\in F_{p,\mu}(\mathbf{R}_+)$ *as follows.*

(i) *For* $Re(\eta+\mu/2)>0$ *and* $\alpha\in\mathbf{C}$, *let* $\ell\in\mathbf{N}$ *be such that* $Re(\alpha+\ell)>0$. Then

$$K_2^{\eta,\alpha}f=K_2^{\eta,\alpha+\ell}Q_{\eta,\alpha,\ell}(\delta)f=Q_{\eta,\alpha,\ell}(\delta)K_2^{\eta,\alpha+\ell}f \tag{2.16}$$

where $K_2^{\eta,\alpha+\ell}$ *is defined via (2.6) and*

$$Q_{\eta,\alpha,\ell}(\delta)=\prod_{k=1}^{\ell}(\eta+\alpha+k-1-\delta/2).$$

(ii) *For* $-k < Re(\eta + \mu/2) < -k + 1, k \in \mathbf{Z}_+$,

$$K_2^{\eta,\alpha} f = (-1)^k K_2^{\eta+k,\alpha-k} I_2^{-\eta-k,k} = (-1)^k I_2^{-\eta-k,k} K_2^{\eta+k,\alpha-k} \tag{2.17}$$

where $K_2^{\eta+k,\alpha-k}$ *is defined via (2.16) and* $I_2^{-\eta-k,k}$ *via (2.5).*

Lemma 2.8 *Let* $1 \leq p \leq \infty, \mu \in \mathbf{C}, \alpha \in \mathbf{C}$.

(i) *For* $-Re(\eta + \mu/2) \notin \mathbf{N}$, $K_2^{\eta,\alpha}$ *is a continuous linear mapping from* $F_{p,\mu}(\mathbf{R}_+)$ *into itself.*

(ii) *If, in addition,* $-Re(\eta + \alpha + \mu/2) \notin \mathbf{N}$, *then* $K_2^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}(\mathbf{R}_+)$ *and*

$$(K_2^{\eta,\alpha})^{-1} = K_2^{\eta+\alpha,-\alpha} . \tag{2.18}$$

Estimates of the form obtained in Lemma 2.4 hold for $K_2^{\eta,\alpha}$. For example, in the situation of Definition 2.7(i) we obtain as an analogue of (2.14)

$$\lambda_m^{p,\mu}(K_2^{\eta,\alpha} f) \leq \frac{\Gamma(Re\alpha + \ell)\ \Gamma(Re(\eta + \mu/2))}{|\Gamma(\alpha + \ell)|\Gamma(Re(\eta + \alpha + \ell + \mu/2))} \sum_{s=0}^{\ell} d(\eta, \alpha, s) \lambda_{m+s}^{p,\mu}(f) \tag{2.19}$$

for constants d depending on η, α and s.

§3. MULTIDIMENSIONAL FRACTIONAL INTEGRALS

In this section we exhibit the good mapping properties, relative to $F_{p,\mu}(\mathbf{R}^n)$, of some multidimensional modifications of the usual Riemann-Liouville fractional integrals.

Let $Re\ \alpha > 0$, $\gamma_{n,\alpha} = 2/\{\Gamma(\alpha)|\sum|\}$. For suitable functions $\phi(x)$, $x \in \mathbf{R}^n$, consider the fractional integrals

$$(B_+^\alpha \phi)(x) = \gamma_{n,\alpha} \int\limits_{|y|<|x|} \frac{(|x|^2 - |y|^2)^\alpha}{|x-y|^n} \phi(y) dy, \tag{3.1}$$

$$(B_-^\alpha \phi)(x) = \gamma_{n,\alpha} \int\limits_{|y|>|x|} \frac{(|y|^2 - |x|^2)^\alpha}{|x-y|^n} \phi(y) dy, \tag{3.2}$$

$$(I^\alpha\phi)(x) = c_{n,\alpha} \int\limits_{\mathbf{R}^n} \frac{\phi(y)}{|x-y|^{n-\alpha}} dy \tag{3.3}$$

where

$$c_{n,\alpha} = 2^{-\alpha}\pi^{-n/2}\Gamma((n-\alpha)/2)/\Gamma(\alpha/2) , \quad 0 < Re\ \alpha < n .$$

Integrals (3.1) and (3.2) were introduced in [5] in connection with the inversion problem for Riesz potentials in a ball. Integral (3.3) is the usual Riesz potential; see, for example, [1],[6],[7]. For sufficiently good ϕ, integrals (3.1)-(3.3) possess a semigroup property with respect to α, e.g., $I^\alpha I^\beta \phi = I^{\alpha+\beta}\phi$.

In the sequel it will be convenient to consider the modifications of (3.1)-(3.3) defined by

$$\mathcal{B}^\alpha_\pm \phi = r^{-2\alpha} B^\alpha_\pm \phi, \quad \mathcal{J}^\alpha \phi = r^{-\alpha} I^\alpha \phi . \tag{3.4}$$

These operators commute with rotations and dilations and map $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ into itself under appropriate conditions, which are stated in the following result.

Lemma 3.1 *Let* $1 \le p \le \infty$, $Re\ \alpha > 0$, $\mu \in \mathbf{C}$.

(i) $\mathcal{B}^\alpha_+$ *is bounded on* $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ *provided that* $Re\ \mu < n$.

(ii) $\mathcal{B}^\alpha_-$ *is bounded on* $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ *provided that* $Re(\mu - 2\alpha) > 0$.

(iii) *If* $Re\ \alpha < Re\ \mu < n$ *then* $\mathcal{J}^\alpha$ *is bounded on* $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ *and the following factorisation holds:*

$$\mathcal{J}^\alpha \phi = 2^{-\alpha} \mathcal{B}^{\alpha/2}_+ \mathcal{B}^{\alpha/2}_- \phi = 2^{-\alpha} \mathcal{B}^{\alpha/2}_- \mathcal{B}^{\alpha/2}_+ \phi, \quad \phi \in \mathcal{L}_{p,\mu}(\mathbf{R}^n). \tag{3.5}$$

The statements in this lemma were proved in [5]. Here they are presented in slightly different notations. Note that the restrictions on α and μ above are necessary for the existence of the integrals $\mathcal{B}^\alpha_\pm \phi$, $\mathcal{J}^\alpha \phi$ for $\phi \in \mathcal{L}_{p,\mu}(\mathbf{R}^n)$.

Our purpose is to extend definitions (3.1)-(3.4) to all $\alpha \in \mathbf{C}$ and to investigate mapping properties of the extended operators in the spaces $F_{p,\mu}(\mathbf{R}^n)$. Since the integrals

(3.4) are radial-spherical convolutions it is natural to study them by using corresponding Fourier analysis outlined in [4, Section 2.2].

Lemma 3.2 *Let $\phi \in \mathcal{L}_{2,\mu}(\mathbf{R}^n)$, $Re\ \alpha > 0$. Then the RSF-transforms of $\mathcal{B}^{\alpha}_{\pm}\phi$ are well-defined for $Re\ z = Re\ \mu$ and have the forms*

$$(\widetilde{\mathcal{B}^{\alpha}_{+}\phi})_{j,\nu}(z) = \frac{\Gamma((n+j-z)/2)}{\Gamma((n+j-z)/2+\alpha)}\tilde{\phi}_{j,\nu}(z), \quad Re\ \mu < n; \tag{3.6}$$

$$(\widetilde{\mathcal{B}^{\alpha}_{-}\phi})_{j,\nu}(z) = \frac{\Gamma((j+z)/2-\alpha)}{\Gamma((j+z)/2)}\tilde{\phi}_{j,\nu}(z), \quad Re(\mu - 2\alpha) > 0. \tag{3.7}$$

Proof. This statement follows from Lemma 2.2 since the Fourier-Laplace coefficients $\phi_{j,\nu}(r), (\mathcal{B}^{\alpha}_{\pm}\phi)_{j,\nu}(r)$ belong to $\mathcal{L}_{2,\mu}(\mathbf{R}_+)$ and may be represented via Erdélyi-Kober operators by the formulae

$$(\mathcal{B}^{\alpha}_{+}\phi)_{j,\nu} = I_2^{(n+j)/2-1,\alpha}\phi_{j,\nu}\ , \tag{3.8}$$

$$(\mathcal{B}^{\alpha}_{-}\phi)_{j,\nu} = K_2^{j/2-\alpha,\alpha}\phi_{j,\nu}\ . \tag{3.9}$$

See [5], where (3.8), (3.9) are proved and the right-hand sides are written in the form of Mellin convolutions. □

In order to define $\mathcal{B}^{\alpha}_{\pm}\phi$, $\phi \in F_{p,\mu}(\mathbf{R}^n)$, for all $\alpha \in \mathbf{C}$ we shall use their RSF-transforms (3.6), (3.7) or (what is the same) formulae (3.8), (3.9), the right-hand sides of which are interpreted in accordance with Definitions 2.3 and 2.7. Since $\phi_{j,\nu} \in F_{p,\mu}(\mathbf{R}_+)$, the Erdélyi-Kober integrals in (3.8), (3,9) also belong to $F_{p,\mu}(\mathbf{R}_+)$ and we can define $\mathcal{B}^{\alpha}_{\pm}\phi$ via spherical harmonic expansions with the coefficients (3.8), (3.9) respectively.

Let us proceed with this programme.

We shall study $\mathcal{B}^{\alpha}_{+}$ in detail, outline corresponding results for $\mathcal{B}^{\alpha}_{-}$ and finally get similar statements for the fractional integrals $B^{\alpha}_{\pm}\phi$ and $I^{\alpha}\phi$, where $\phi \in F_{p,\mu}(\mathbf{R}^n)$, $\alpha \in \mathbf{C}$.

Lemma 3.3 *Let $\alpha \in \mathbf{C}$, $1 \le p \le \infty$, $f \in F_{p,\mu}(\mathbf{R}_+)$, $Re\ \mu - n \notin \mathbf{N}$. If $\ell \in \mathbf{N}$ is such that $Re(\alpha + \ell) > 0$, then*

$$\lambda^{p,\mu}_{m}(I_2^{(n+j)/2-1,\alpha} f) \le c(1+j)^{-Re\,\alpha}\lambda^{p,\mu}_{m+\ell}(f), \quad \forall m, j \in \mathbf{N}, \tag{3.10}$$

where $\lambda_m^{p,\mu}(.)$ is given by (1.2) and (1.3) and constant c is independent of j.

Proof. We consider three cases.

(i) $Re\ \alpha > 0$, $Re(n-\mu) > 0, \ell = 0$. Since δ commutes with any Erdélyi-Kober operator, it follows from (2.13) that

$$\lambda_m^{p,\mu}(I_2^{(n+j)/2-1,\alpha} f) \leq c_j \lambda_m^{p,\mu}(f)$$

where

$$c_j = \frac{\Gamma(Re\alpha)\ \Gamma((n+j)/2 - Re\mu/2)}{|\Gamma(\alpha)|\Gamma((n+j)/2 + Re(\alpha-\mu/2))} = O(j^{-Re\alpha}), j \to \infty.$$

Here we used the well-known relation $\dfrac{\Gamma(a+z)}{\Gamma(b+z)} \sim z^{a-b}$, $|z| \to \infty$. (See, for example, [2].)

(ii) $Re\ \alpha > 0$, $\ell = 0$, $Re(n-\mu) < 0$ ($Re(n-\mu) \neq -1, -2, \cdots$). For j sufficiently large, $Re(n+j-\mu) > 0$ (compare Lemma 2.5) and the estimate in (i) applies. For finitely many j we obtain c_j from (2.15) with $\eta = (n+j)/2 - 1$, $\ell = 0$ and k such that $-k < Re((n+j-\mu)/2) < -k+1$. Again the set $\{c_j : j \in \mathbf{N}\}$ is bounded and (3.10) holds.

(iii) $Re\ \alpha \leq 0$, $Re\ \mu - n \notin \mathbf{N}$. With ℓ as stated in the lemma we have

$$I_2^{(n+j)/2-1,\alpha} = I_2^{(n+j)/2-1,\alpha+\ell} P_{n,j,\ell,\alpha}(\delta)$$

where

$$P_{n,j,\ell,\alpha}(\delta) = \prod_{i=0}^{\ell}((n+j)/2 - 1 + \alpha + i + \delta/2)$$

is a polynomial of degree ℓ in δ whose coefficients involve j^i for $i = 0, 1, \cdots, \ell$. (See (2.10).) This polynomial in δ commutes with δ so that

$$\lambda_m^{p,\mu}(I_2^{(n+j)/2-1,\alpha} f) = \lambda_m^{p,\mu}(I_2^{(n+j)/2-1,\alpha+\ell}(P_{n,j,\ell,\alpha}(\delta))f)$$

$$\leq b_j \lambda_m^{p,\mu}(P_{n,j,\ell,\alpha}(\delta) f), \quad b_j = O(j^{-\ell - Re\alpha}), \quad j \to \infty ,$$

by the previous cases. The last norm involves terms of the form $a_i j^i \delta^{k+s}$ for $0 \leq i,\ s \leq \ell,\ 0 \leq k \leq m$. Hence we can obtain powers of δ up to $\ell + m$ and powers of j up to ℓ. From the definition of the norm $\lambda_{m+\ell}^{p,\mu}(f)$, (3.10) now follows. □

Theorem 3.4 *If $1 \leq p \leq \infty$, $\alpha \in \mathbf{C}$, $Re\mu - n \notin \mathbf{N}$, then $\mathcal{B}_+^\alpha$, as defined by (3.8), is a continuous linear mapping from $F_{p,\mu}(\mathbf{R}^n)$ into itself.*

Proof. *Let $\phi \in F_{p,\mu}(\mathbf{R}^n)$, $\psi_{j,\nu} = I_2^{(n+j)/2-1,\alpha} \phi_{j,\nu}$,*

$$(\mathcal{B}_+^\alpha \phi)(x) = \sum_{j,\nu} \psi_{j,\nu}(r) Y_{j,\nu}(x'), \quad r = |x|, \quad x' = x/r. \tag{3.11}$$

By making use of Lemma 3.3 and [4, Lemma 3.5] we have $\psi_{j,\nu} \in F_{p,\mu}(\mathbf{R}_+)$ with the estimate

$$\lambda_m^{p,\mu}(\psi_{j,\nu}) \leq c(1+j)^{-Re\alpha - 2N + (n-2)/2} \|\phi\|_{m+\ell+2N}^{p,\mu}, \quad \forall N \in Z_+ \tag{3.12}$$

where the constant c is independent of j. By [4, Theorem 3.9] it follows that the operator $\mathcal{B}_+^\alpha$ defined by (3.11) is a continuous linear operator from $F_{p,\mu}(\mathbf{R}^n)$ into itself. □

Motivated by (3.4) we make the following definition.

Definition 3.5 For $1 \leq p \leq \infty, \alpha \in \mathbf{C}$, *$Re\ \mu - n \notin \mathbf{N}$, define B_+^α on $F_{p,\mu}(\mathbf{R}^n)$ by*

$$B_+^\alpha = r^{2\alpha} \mathcal{B}_+^\alpha \tag{3.13}$$

where the operator $\mathcal{B}_+^\alpha$ is interpreted via (3.8) and Definition 2.3, as above.

Lemma 3.6 *For $\alpha \in \mathbf{C}$, $Re\ \mu - n \notin \mathbf{N}$ and $\phi \in F_{p,\mu}(\mathbf{R}^n)$,*

$$(\mathcal{M}(B_+^\alpha \omega)_{j,\nu})(z - 2\alpha) = \frac{\Gamma((n+j-z)/2)}{\Gamma((n+j-z)/2+\alpha)} (\mathcal{M}\omega_{j,\nu})(z) \tag{3.14}$$

for $Re\ z = Re\ \mu$.

Proof. Note that $(B_+^\alpha \phi)_{j,\nu} = r^{2\alpha}(\mathcal{B}_+^\alpha \phi)_{j,\nu}$. The result then follows easily from (3.8) and (2.17) by taking into account Remark 2.6. □

Theorem 3.7 *Let* $1 \le p \le \infty$, $\alpha \in \mathbf{C}$.

(i) B_+^α *is a continuous linear mapping from* $F_{p,\mu}(\mathbf{R}^n)$ into $F_{p,\mu-2\alpha}(\mathbf{R}^n)$ *provided that Re* $\mu - n \notin \mathbf{N}$.

(ii) *If, in addition, Re*$(\mu - 2\alpha) - n \notin \mathbf{N}$, *then* B_+^α *is an isomorphism from* $F_{p,\mu}(\mathbf{R}^n)$ onto $F_{p,\mu-2\alpha}(\mathbf{R}^n)$ *with inverse* $B_+^{-\alpha}$.

Proof.

(i) is immediate from Theorem 3.4 and [4, Lemma 3.2].

(ii) Let $\phi \in F_{2,\mu}(\mathbf{R}^n)$. We prove that $(B_+^{-\alpha} B_+^\alpha \phi)_{j,\nu} = \phi_{j,\nu}$ by using symbols. The conditions ensure that B_+^α and $B_+^{-\alpha}$ are continuous linear mappings from $F_{p,\mu}(\mathbf{R}^n)$ into $F_{p,\mu-2\alpha}(\mathbf{R}^n)$ and from $F_{p,\mu-2\alpha}(\mathbf{R}^n)$ into $F_{p,\mu}(\mathbf{R}^n)$ respectively. By repeated applications of (3.14), we obtain

$$
\begin{aligned}
&(\mathcal{M}(B_+^{-\alpha} B_+^\alpha \phi)_{j,\nu})(z) \\
&= \frac{\Gamma((n+j-(z-2\alpha))/2)}{\Gamma((n+j-(z-2\alpha))/2-\alpha)} (\mathcal{M}(B_+^\alpha \phi)_{j,\nu})(z-2\alpha) \\
&= \frac{\Gamma((n+j-z)/2+\alpha)\ \Gamma((n+j-z)/2)}{\Gamma((n+j-z)/2)\ \Gamma((n+j-z)/2+\alpha)} (\mathcal{M}\phi_{j,\nu})(z) = (\mathcal{M}\phi_{j,\nu})(z).
\end{aligned}
$$

Hence for all $j, \nu, (B_+^{-\alpha} B_+^\alpha \phi)_{j,\nu} = \phi_{j,\nu}$ so that $B_+^{-\alpha} B_+^\alpha \phi = \phi$ for all $\phi \in F_{2,\mu}(\mathbf{R}^n)$. By continuity and density, the result holds for $\phi \in F_{p,\mu}(\mathbf{R}^n)$ $\forall p \ge 1$. $\square$

Remark 3.8 This completes the discussion for B_+^α. As indicated above, we shall outline briefly what happens for B_-^α, many of the details being very similar to those for B_+^α. Again $\mathcal{B}_-^\alpha$ is more convenient initially.

The programme for $\mathcal{B}_-^\alpha$ starts with (3.7), which leads to (3.9). The operator $K_2^{j/2-\alpha,\alpha}$ is extended as in Definition 2.7 to a continuous linear operator on $F_{p,\mu}(\mathbf{R}_+)$ provided that $Re(j/2-\alpha+\mu/2) \neq 0, -1, -2, \cdots$. This will be satisfied if $Re(2\alpha-\mu) \notin \mathbf{N}$. Under these conditions we use (3.9) to extend the definition of $\mathcal{B}_-^\alpha$. An estimate for $K_2^{j/2-\alpha,\alpha}$ analogous to (3.10) can be obtained. This leads to the following

Theorem 3.9 *If* $1 \le p \le \infty$, $\alpha \in \mathbf{C}$, $Re(2\alpha - \mu) \notin \mathbf{N}$, *then* $\mathcal{B}_-^\alpha$ *is a continuous linear mapping from* $F_{p,\mu}(\mathbf{R}^n)$ *into itself.*

We then extend the definition of B_-^α by the relation $B_-^\alpha = r^{2\alpha}\mathcal{B}_-^\alpha$, analogous to (3.13). An analogue of Lemma 3.6 leads to

Theorem 3.10 *Let* $1 \leq p \leq \infty$, $\alpha \in \mathbf{C}$.

(i) B_-^α *is a continuous linear mapping from* $F_{p,\mu}(\mathbf{R}^n)$ *into* $F_{p,\mu-2\alpha}(\mathbf{R}^n)$ *provided that* $Re(2\alpha - \mu) \notin \mathbf{N}$.

(ii) *If, in addition,* $-Re\ \mu \notin \mathbf{N}$ *then* B_-^α *is an isomorphism from* $F_{p,\mu}(\mathbf{R}^n)$ *onto* $F_{p,\mu-2\alpha}(\mathbf{R}^n)$ *with inverse* $B^{-\alpha}$.

Now we can consider the Riesz potential $I^\alpha\phi$ defined by (3.3). An extended definition can be given via (3.5) where the operators on the right-hand sides are interpreted in their extended forms.

Theorem 3.11 *Let* $1 \leq p \leq \infty$, $\alpha \in \mathbf{C}$.

(i) I^α *is a continuous linear mapping from* $F_{p,\mu}(\mathbf{R}^n)$ *into* $F_{p,\mu-\alpha}(\mathbf{R}^n)$, *provided that* $Re(\mu - n) \notin \mathbf{N}$, $Re(\alpha - \mu) \notin \mathbf{N}$.

(ii) *If, in addition,* $Re(\mu - \alpha - n) \notin \mathbf{N}, -Re\ \mu \notin \mathbf{N}$ *then* I^α *is an isomorphism from* $F_{p,\mu}(\mathbf{R}^n)$ *onto* $F_{p,\mu-\alpha}(\mathbf{R}^n)$ *with inverse* $I^{-\alpha}$.

Proof. All parts follow easily from Theorems 3.7 and 3.10. To obtain the formula for $(I^\alpha)^{-1}$ we use (3.5) to get formally

$$(I^\alpha)^{-1} = (2^{-\alpha}B_+^{\alpha/2}r^{-\alpha}B_-^{\alpha/2})^{-1} = 2^\alpha B_-^{-\alpha/2}r^\alpha B_+^{-\alpha/2} = I^{-\alpha}.$$

This calculation is valid under the stated conditions. □

Remark 3.12 We have shown that all of B_+^α, B_-^α and I^α are well-behaved in the context of the $F_{p,\mu}$ spaces. It should be emphasised that the extended operators coincide with the original expressions in (3.1)-(3.3) when the latter exist. When the integrals diverge, however, our results can be used to interpret the operators distributionally, a topic which we shall discuss in our final section.

§4. DISTRIBUTIONAL THEORY

Our aim now is to apply $B_\pm^\alpha$ and I^α to distributions in the spaces $F'_{p,\mu}(\mathbf{R}^n)$. As in the one-dimensional case, we can imbed the spaces $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ into these dual spaces by generating regular functionals.

Let $\phi \in F_{p,\mu}(\mathbf{R}^n)$ and $f \in \mathcal{L}_{p',-\mu+n}(\mathbf{R}^n)$ where, as usual, $p' = p/(p-1)$. Then, by Hölder's inequality,

$$\left|\int_{\mathbf{R}^n} f(x)\phi(x)dx\right| = \left|\int_{\mathbf{R}^n} \{|x|^{-\mu+n-n/p'}f(x)\}\{|x|^{\mu-n/p}\phi(x)\}dx\right|$$

$$\leq \||x|^{-\mu+n}f(x)\|_{p'}\||x|^{\mu}\phi(x)\|_p = \|f\|_{p',-\mu+n}\|\phi\|_{p,\mu}. \tag{4.1}$$

Thus f generates a regular functional $\bar{f}$ on $F_{p,\mu}(\mathbf{R}^n)$ via

$$(\bar{f},\phi) = \int_{\mathbf{R}^n} f(x)\phi(x)dx \quad (\phi \in F_{p,\mu}(\mathbf{R}^n)). \tag{4.2}$$

Lemma 4.1 $\mathcal{L}_{p',-\mu+n}(\mathbf{R}^n)$ is imbedded in $F'_{p,\mu}(\mathbf{R}^n)$ under the mapping $f \to \bar{f}$ defined by (4.2).

Proof: This follows at once from the estimate (4.1). $\square$

We now use regular functionals to motivate the definition of $B^{\alpha}_{\pm}$ and I^{α} on $F'_{p,\mu}(\mathbf{R}^n)$. We shall consider B^{α}_{+} in detail and then state the analogous results for B^{α}_{-} and I^{α}. (We could also handle $\mathcal{B}^{\alpha}_{\pm}$ and $\mathcal{J}^{\alpha}$.)

Let us temporarily denote the extension of B^{α}_{+} to $F'_{p,\mu}(\mathbf{R}^n)$ by $\bar{B}^{\alpha}_{+}$. We require that, for a regular functional $\bar{f}$ generated by $f \in \mathcal{L}_{p',-\mu+n}(\mathbf{R}^n)$,

$$\bar{B}^{\alpha}_{+}\bar{f} = \widetilde{B^{\alpha}_{+}f} \tag{4.3}$$

where the right-hand side is the regular functional generated by $B^{\alpha}_{+}f$ on the appropriate space. Under suitable conditions, including $Re\ \alpha > 0$, $B^{\alpha}_{+}f$ is an element of $\mathcal{L}_{p',-\mu+n-2\alpha}(\mathbf{R}^n)$. See, for instance, [5] where slightly different notation is used. Then $\widetilde{B^{\alpha}_{+}f} \in F'_{p,\mu+2\alpha}(\mathbf{R}^n)$ by Lemma 4.1, with μ replaced by $\mu + 2\alpha$. Accordingly let $\phi \in F_{p,\mu+2\alpha}(\mathbf{R}^n)$. Then in the usual notation,

$$(\bar{B}^{\alpha}_{+}\bar{f},\phi) = (\widetilde{B^{\alpha}_{+}f},\phi) = \int_{\mathbf{R}^n} (B^{\alpha}_{+}f)(x)\phi(x)dx\ .$$

Under appropriate conditions on f, we may invert the order of integration and, on noting (3.1) and (3.2), obtain

$$(\widetilde{B_+^\alpha f}, \phi) = \int_{\mathbf{R}^n} f(x)(B_-^\alpha \phi)(x)dx = (\tilde{f}, B_-^\alpha \phi). \tag{4.4}$$

Note that, under appropriate conditions, $B_-^\alpha \phi \in F_{p,\mu}(\mathbf{R}^n)$ by Theorem 3.10 so that the right-hand side of (4.4) is well defined. Thus the extended operator $\tilde{B}_+^\alpha$ on regular functionals is the adjoint of B_-^α on $F_{p,\mu+2\alpha}(\mathbf{R}^n)$. The operator $\tilde{B}_+^\alpha$ is an extension of the classical operator B_+^α in the sense of (4.3). We use (4.4) to define $\tilde{B}_+^\alpha$ on any functional, regular or not. Using f now for any functional and writing the extended operator as B_+^α rather than $\tilde{B}_+^\alpha$, we turn (4.4) into

$$(B_+^\alpha f, \phi) = (f, B_-^\alpha \phi)$$

for $f \in F'_{p,\mu}(\mathbf{R}^n)$, $\phi \in F_{p,\mu+2\alpha}(\mathbf{R}^n)$.

By considering B_-^α similarly, we are led to the following definitions.

Definition 4.2 Let $f \in F'_{p,\mu}(\mathbf{R}^n)$. Under appropriate conditions on α and μ we define $B_+^\alpha f$, $B_-^\alpha f$ as elements of $F'_{p,\mu+2\alpha}(\mathbf{R}^n)$ by

$$(B_+^\alpha f, \phi) = (f, B_-^\alpha \phi) \quad (\phi \in F_{p,\mu+2\alpha}(\mathbf{R}^n)) \tag{4.5}$$

$$(B_-^\alpha f, \phi) = (f, B_+^\alpha \phi) \quad (\phi \in F_{p,\mu+2\alpha}(\mathbf{R}^n)). \tag{4.6}$$

Theorem 4.3 Let $1 \le p \le \infty$, $\alpha \in \mathbf{C}$.

(i) If $-Re\ \mu \notin \mathbf{N}$, then B_+^α is a continuous linear mapping from $F'_{p,\mu}(\mathbf{R}^n)$ into $F'_{p,\mu+2\alpha}(\mathbf{R}^n)$.

(ii) If, in addition, $-Re(\mu + 2\alpha) \notin \mathbf{N}$ then B_+^α is a homeomorphism from $F'_{p,\mu}(\mathbf{R}^n)$ onto $F'_{p,\mu+2\alpha}(\mathbf{R}^n)$ with inverse $B_+^{-\alpha}$.

Proof:

(i) By Theorem 3.10 (i), B_-^α is a continuous linear mapping from $F_{p,\mu+2\alpha}(\mathbf{R}^n)$ into $F_{p,\mu}(\mathbf{R}^n)$ since $Re(2\alpha-(\mu+2\alpha)) \notin \mathbf{N}$. The result follows by standard properties of adjoint operators.

(ii) follows in a similar fashion via Theorem 3.10 (ii). □

Remark 4.4 The conditions in Theorem 4.3 are precisely those in Theorem 3.7, for the classical transform on $F_{p,\mu}(\mathbf{R}^n)$, with μ replaced by $-\mu+n$. This is a legacy of Lemma 4.1 and recurs in the results below.

Theorem 4.5 Let $1 \le p \le \infty$, $\alpha \in \mathbf{C}$.

(i) If $Re(\mu+2\alpha-n) \notin \mathbf{N}$, B_-^α is a continuous linear mapping from $F'_{p,\mu}(\mathbf{R}^n)$ into $F'_{p,\mu+2\alpha}(\mathbf{R}^n)$.

(ii) If, in addition, $Re\ \mu - n \notin \mathbf{N}$, then B_-^α is a homeomorphism from $F'_{p,\mu}(\mathbf{R}^n)$ onto $F'_{p,\mu+2\alpha}(\mathbf{R}^n)$ with inverse $B_-^{-\alpha}$.

Proof: This is similar to that for Theorem 4.3. □

For the Riesz potential, we start with (3.3) and proceed as before to produce the following definition.

Definition 4.6 For $f \in F'_{p,\mu}(\mathbf{R}^n)$, under appropriate conditions on α and μ, define $I^\alpha f$ as a member of $F'_{p,\mu+\alpha}(\mathbf{R}^n)$ by

$$(I^\alpha f, \phi) = (f, I^\alpha \phi) \qquad (\phi \in F_{p,\mu+\alpha}(\mathbf{R}^n)). \tag{4.7}$$

Theorem 4.7 Let $1 \le p \le \infty, \mu \in \mathbf{C}$.

(i) I^α is a continuous linear mapping from $F'_{p,\mu}(\mathbf{R}^n)$ into $F'_{p,\mu+\alpha}(\mathbf{R}^n)$ provided that $Re(\mu+\alpha-n) \notin \mathbf{N}$, $-Re\ \mu \notin \mathbf{N}$.

(ii) If, in addition, $Re(\mu-n) \notin \mathbf{N}$, $-Re(\mu+\alpha) \notin \mathbf{N}$ then I^α is a homeomorphism from $F'_{p,\mu}(\mathbf{R}^n)$ onto $F'_{p,\mu+\alpha}(\mathbf{R}^n)$ with inverse $I^{-\alpha}$.

Proof: This follows from (4.7), Theorem 3.11 and properties of adjoints. □

Theorems 4.3, 4.5 and 4.7 immediately give uniqueness results for classical solutions of certain equations involving $B_+^\alpha, B_-^\alpha, I^\alpha$.

Example 4.8 Let $1 \le p \le \infty$, $Re\ \alpha > 0, \mu \in \mathbf{C}$.

We shall consider the Riesz potential I^α on $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$. From results in [5] we can show that I^α is a bounded linear mapping from $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ into $\mathcal{L}_{p,\mu-\alpha}(\mathbf{R}^n)$ provided that

$$Re\ \alpha < Re\ \mu < n\ . \tag{4.8}$$

Let $g \in \mathcal{L}_{p,\mu-\alpha}(\mathbf{R}^n)$ and consider the integral equation

$$I^\alpha f = g \ . \tag{4.9}$$

As before $\tilde{g} \in F'_{p',-\mu+\alpha+n}(\mathbf{R}^n)$ (by Lemma 4.1). The condition $Re\ \alpha > 0$ and (4.8) enable us to apply Theorem 4.7 with μ replaced by $-\mu+n$ to deduce that there is a unique solution $h \in F'_{p',-\mu+n}(\mathbf{R}^n)$ of the equation

$$I^\alpha h = \tilde{g} \ . \tag{4.10}$$

From (4.7), we see that (4.9) has at most one solution $f \in \mathcal{L}_{p,\mu}(\mathbf{R}^n)$ and the solution exists if and only if the solution h of (4.10) is a regular distribution with $h = \tilde{f}$.

Example 4.9 Let $1 < p < \infty$, $Re\ \alpha \geq n/p$. In this case, when the integral $I^\alpha f$ is generally divergent for $f \in L_p(\mathbf{R}^n)$, it is important to give a correct interpretation to it. Of course, one can consider $I^\alpha f$ as a Φ'-distribution (of Lizorkin-Semyanistyi type, see [6]), but such an interpretation is too general and is not sensitive to concrete values of p and α. The theory presented above enables us to give an essentially different interpretation for $I^\alpha f$ which is free from the last shortcoming. Namely, according to Lemma 4.1 and Theorem 4.7, $I^\alpha f$ may be treated as a distribution in the space $F'_{p',n/p'+\alpha}(\mathbf{R}^n)$ provided that

$$Re\ \alpha - n/p \neq 0, 1, 2, \cdots \tag{4.11}$$

In the framework of Φ'-theory we have no such restriction, so that (4.11) is the price we have to pay. In a similar way one can interpret fractional integrals $B^\alpha_\pm f$ when they are divergent. A Φ'-theory fails for these operators because they are translation invariant. One can also include weighted spaces $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ in our scheme. A Φ'-theory does not cater for them if $Re\ \mu \geq n$ when the functions from $\mathcal{L}_{p,\mu}(\mathbf{R}^n)$ are generally not summable at the origin.

Further applications of Theorems 4.3, 4.5 and 4.7 may be pursued elsewhere.

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APPENDIX

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Fractional Calculus
Research Notes in Mathematics No. 138, Pitman, London, 1985.

Book 25

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Fractional calculus and integral transforms of generalized functions

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Fractional calculus and integral transforms of generalized functions

FEARON PITMAN PUBLISHERS INC.
6 Davis Drive, Belmont, California 94002

PITMAN PUBLISHING LIMITED
39 Parker Street, London WC2B 5PB
North American Editorial Office
1020 Plain Street, Marshfield, Massachusetts 02050

Associated Companies
Copp Clark Pitman, Toronto
Pitman Publishing New Zealand Ltd, Wellington
Pitman Publishing Pty Ltd, Melbourne

AMS Subject Classifications: (main) 36A33, 45A05, 46F10
(subsidiary) 44A15, 46F05

British Library Cataloguing in Publication Data

McBride, Adam C
Fractional calculus and integral transforms of generalised functions. – (Research notes in mathematics; 31).
1. Calculus
I. Title
515 QA303

ISBN 0-273-08415-1

Manufactured in Great Britain

Contents

Preface

In recent years there has been a lot of interest in extending the standard classical integral transforms to classes of generalised functions or distributions. The theory of the Fourier transform has been documented in standard works such as [24] and [75] but it was not until the appearance of [87] that transforms on the half-line $(0,\infty)$ such as the Laplace, Mellin and Hankel transforms received much attention. In the ten years since [87] was published, interest has continued. However, most authors have used classes of generalised functions which are ideal for the particular transform under consideration but for no others. The purpose of this book is to describe a class of spaces of generalised functions which are amenable to the study of a number of important operators and to use the theory to solve in some detail a number of standard problems. In particular, we show how various classical results are incorporated in our distributional theory.

Since an indefinite integral is probably the simplest integral transform of all, no apology is needed for using this as the starting point for a theory of fractional calculus, another topic which has sprung to life in recent years with the publication of [56] and [74]. This distributional fractional calculus is used as a unifying theme in the later chapters of the book. We have concentrated on problems which are of general interest and where the theory is complete. Thus we consider hypergeometric integral equations, Hankel transforms and dual integral equations of Titchmarsh type in detail. In the last chapter we mention how an incomplete theory can be developed in the case of most of the other standard transforms on $(0,\infty)$ and also indicate a number of directions in which the theory may develop in the future.

On a personal note, my interest in this field began during the period 1968-71 when it was my great pleasure and privilege to be a research student at the University of Edinburgh under the supervision of the late Professor Arthur Erdélyi. During that period and right up to his untimely death, his willing help and friendly advice were a great inspiration. Without him, this book would never have existed. I hope that it might serve as my modest tribute to a very great mathematician and friend.

It is perhaps appropriate that the impetus for me to put pen to paper came from another of Professor Erdélyi's former students, David Colton, and I am pleased to record my appreciation of Professor Colton's advice and encouragement. My thanks also go to my colleague Dr. Gary Roach for his interest and for looking through the manuscript. Last but by no means least, I would like to record my sincere thanks to Mrs. Mary Sergeant and Miss Elaine Livsey for preparing the typescript so excellently and coping with my whims and fancies.

0 Notation

Here we introduce a few standard notations and conventions which will be used throughout the book.

1. All functions will be complex-valued.
2. An expression such as x^λ, where x is a positive real number and λ is complex, will be interpreted as $\exp(\lambda \log x)$ with $\log x$ real.
3. All integrals will be Lebesgue integrals.
4. Let I denote either the open interval $(0,\infty)$ or the whole real line R^1. We consider (complex-valued) measurable functions defined almost everywhere (a.e.) on I.

   (i) f is locally integrable on I if it is (Lebesgue) integrable over every compact sub-interval of I.

   (ii) Any equation involving locally integrable functions is to be interpreted as holding a.e. on the appropriate set. Alternatively, we may work with equivalence classes of functions, two functions being in the same equivalence class if they are equal a.e.

   (iii) For $1 \leq p < \infty$, $L^p(I)$ is the set of (equivalence classes of) measurable functions f such that

$$||f||_p = \{\int_I |f(x)|^p dx\}^{1/p} < \infty. \tag{0.1}$$

   $L^\infty(I)$ is the set of (equivalence classes of) measurable functions f such that

$$||f||_\infty = \operatorname*{ess\ sup}_I f < \infty. \tag{0.2}$$

For $1 \leqslant p \leqslant \infty$, $L^p(I)$ is a Banach space with respect to the norm $||\ ||_p$.

(iv) Let $1 \leqslant p \leqslant \infty$ and let μ be a complex number. Then L^p_μ is the space of (equivalence classes of) functions f such that $x^{-\mu}f(x) \in L^p(0,\infty)$, i.e.,

$$L^p_\mu = \{f: x^{-\mu}f(x) \in L^p(0,\infty)\}. \tag{0.3}$$

(We shall not require this definition on $(-\infty,\infty)$.) Occasionally we will write

$$L^p = L^p_0 = L^p(0,\infty). \tag{0.4}$$

L^p_μ is a Banach space with respect to the norm $||\ ||_{p,\mu}$ defined by

$$||f||_{p,\mu} = ||x^{-\mu}f(x)||_p \tag{0.5}$$

where $||\ ||_p$ is given by (0.1) or (0.2).

(v) If $1 \leqslant p \leqslant \infty$, the number q will always be related to p via the relation

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{or} \quad q = \frac{p}{p-1} \tag{0.6}$$

with the convention that if $p = 1$, $q = \infty$ while if $p = \infty$, $q = 1$.

5. Again, let I denote either $(0,\infty)$ or R^1. Here we consider complex-valued functions defined everywhere on I.

(i) $C^\infty(I)$ denotes the set of all (complex-valued) functions on I which are infinitely differentiable on I, i.e. which have derivatives of all orders at all points of I.

(ii) $C_0^\infty(I)$ is the subset of $C^\infty(I)$ consisting of those functions ϕ which have compact support, i.e. which are such that $\phi(x) = 0$ outside some compact subset of I (the compact subset varying with ϕ).

The set $C_0^\infty(I)$ will sometimes be denoted by $\mathfrak{D}(I)$ and, in the case $I = (0,\infty)$, simply by $\mathfrak{D}$.

6. If X is a topological vector space, we denote by X' the dual space of X, i.e. the set of all continuous linear functionals on X. The value assigned to $\phi \in X$ by $f \in X'$ will be denoted by (f,ϕ).

1 Introduction

§1.1 Motivation and Background

Let X, Y be two (non-empty) sets and T a mapping of X into Y. If $g \in Y$ does not lie in the range of T, then the equation

$$Tf = g \tag{1.1}$$

has no solution $f \in X$. Nevertheless, it is sometimes possible to recover something from the wreck.

Suppose, for instance, that it is possible to imbed the sets X and Y in sets $\tilde{X}$ and $\tilde{Y}$ respectively with $f \to \tilde{f}$, $g \to \tilde{g}$ etc. Suppose also that T can be extended to a mapping $\tilde{T}$ of $\tilde{X}$ <u>onto</u> $\tilde{Y}$ in such a way that, for all $f \in X$,

$$\tilde{T}\tilde{f} = \widetilde{Tf}. \tag{1.2}$$

Then, if g is as in (1.1), the equation

$$\tilde{T}h = \tilde{g} \tag{1.3}$$

will have at least one solution $h \in \tilde{X}$. Such an h might be called a <u>generalised</u> (or <u>weak</u>) solution of (1.1) since, if (1.1) has a solution $f \in X$, $h = \tilde{f}$ will satisfy (1.3) in view of (1.2). Although the word "imbed" was used above, there may not be any topologies involved initially. However if X, Y, $\tilde{X}$, $\tilde{Y}$ are topological spaces and the imbeddings are continuous, it would be ideal if $\tilde{T}$ turned out to be a homeomorphism of $\tilde{X}$ onto $\tilde{Y}$. Then we would have a unique generalised solution of (1.1).

We shall be concerned with the case when X and Y are topological vector spaces of functions (justifying the choice of f, g above) and T is an integral operator from X into Y but not onto. Indeed X and Y will be spaces of the form L^p_μ as defined in (0.3). We give two instances for the case $\mu = 0$ involving operators which will attract much of our attention in the sequel.

Example 1.1

Let η and α be complex numbers with Re $\alpha > 0$ and define $I_1^{\eta,\alpha}$ by

$$(I_1^{\eta,\alpha}f)(x) = \frac{x^{-\eta-\alpha}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta f(t)dt \qquad (0 < x < \infty).$$

$I_1^{\eta,\alpha}$ is one of the Erdélyi-Kober operators of fractional integration introduced in [32] and [34]. Indeed in [32, Theorem 2], Kober showed that $I_1^{\eta,\alpha}$ is a continuous linear mapping of $L^p (= L^p(0,\infty))$ into itself provided that Re $\eta > -1/q$. However $I_1^{\eta,\alpha}$ does not map L^p onto L^p. For instance, if

$$I_1^{\eta,1} f = g \qquad (f, g \in L^p(0,\infty))$$

(equality holding almost everywhere on $(0,\infty)$), then $x^{\eta+1}g(x)$ must be differentiable almost everywhere on $(0,\infty)$.

Example 1.2

For suitable complex numbers ν, $1 < p \leqslant 2$ and $f \in L^p$, we may define $H_\nu f$, the Hankel transform of f of order ν, by

$$(H_\nu f)(x) = \underset{n \to \infty}{\text{l.i.m.}(q)} \int_0^n \sqrt{xt}\, J_\nu(xt) f(t)dt \qquad (0 < x < \infty)$$

where l.i.m.(q) denotes the limit in the $L^q(0,\infty)$ norm. Then by standard results in [1] and [78], H_ν is a continuous linear mapping of L^p into L^q when Re $\nu > -3/2 + 1/p$. However, except in the very special case of

$p = 2$, H_ν does not map L^p onto L^q and a useful characterisation of the range $H_\nu(L^p)$ does not seem to be known.

An even more extreme situation arises with the Laplace transform .

Example 1.3

For $f \in L^p$ $(1 \leqslant p \leqslant \infty)$, define Lf by

$$(Lf)(x) = \int_0^\infty e^{-xt} f(t)dt \qquad (0 < x < \infty).$$

Then from [85, pp.312-3], if $1 < p < \infty$, $g = Lf$ $(f \in L^p)$ if and only if g is infinitely differentiable and, for some constant M,

$$\frac{k}{(k!)^p} \int_0^\infty \left| \frac{d^k f}{dx^k} \right|^p x^{kp+p-2} dx < M \qquad (k = 0,1,2, \ldots).$$

Thus L maps L^p into $L^p_{2/p-1}$ but clearly not onto. The cases $p = 1$ and $p = \infty$ produce analogous results.

Similar comments can be made about other standard integral transforms on the half-line $(0,\infty)$ but we have enough to be getting on with.

Returning to the general case, we have to consider how to imbed our space X of functions in a suitable larger set. One method is to take $\tilde{X} = Z'$, the dual space of some space Z of infinitely differentiable testing-functions. If Z is chosen appropriately, we might hope that each element f of X would generate a functional $\tilde{f} \in Z'$ according to the prescription

$$(\tilde{f},\phi) = \int_E f(x)\phi(x)dx \qquad (1.4)$$

where, for our present discussion, E will be either $(0,\infty)$ or $(-\infty,\infty)$. Examples of such spaces Z' are the spaces $\mathfrak{D}'(-\infty,\infty)$ and $\mathfrak{D}'(0,\infty)$ of Schwartz distributions and the space $\mathcal{S}'$ of tempered distributions discussed in, for instance, [24], [75], [79] and [86]. We cannot hope for a single space Z'

which will be ideal for every operator T we care to consider. Hence, in the literature, many different spaces are introduced which are tailor-made for the problem in hand. However, in the case of functions defined on $(0,\infty)$, it is usual to demand that $\mathfrak{D}(0,\infty)$ be dense in Z so that the restrictions of the elements of Z' to $\mathfrak{D}(0,\infty)$ form a subspace of the Schwartz space $\mathfrak{D}'(0,\infty)$ by [87, Corollary 1.8 - 2a]. Similar comments apply to $(-\infty,\infty)$. Thus Z' is a space of generalised functions in the sense of Zemanian [87, p.39].

The extension of integral transformations from classical functions to generalised functions has attracted a lot of attention in recent years and mention must be made of the work of Zemanian which appears in his book [87]. Since [87] was published, further developments have taken place and we shall mention briefly a few of these below, although no attempt has been made to make the list comprehensive. We do this in the course of outlining three methods which have been successfully used to carry out the extension process.

The first method might be called "the adjoint operator method". Suppose we are dealing with two spaces X and Y of functions on $(0,\infty)$ which are imbedded in spaces Z_1', Z_2' of generalised functions respectively and let T map X into Y. Then if $f \in X$, (1.2) decrees that, for any testing-function $\phi \in Z_2$,

$$(\widetilde{Tf},\phi) = \int_0^\infty Tf(x)\phi(x)dx = \int_0^\infty f(x)T^*\phi(x)dx = (\tilde{f},T^*\phi) \tag{1.5}$$

where $T^*: Z_2 \to Z_1$ is the formal adjoint of T ([87], §1.10). (1.5) suggests that we define $\tilde{T}: Z_1' \to Z_2'$ by

$$(\tilde{T}h,\phi) = (h,T^*\phi) \quad (h \in Z_1',\ \phi \in Z_2). \tag{1.6}$$

The properties of $\tilde{T}$ then follow from those of T^* by standard theorems ([87], p. 29). Thus in this approach, we try to choose spaces Z_1 and Z_2 such that $T^*: Z_2 \to Z_1$ is a homeomorphism. Then, by [87, Theorem 1.10 - 2], $\tilde{T}: Z_1' \to Z_2'$ is also a homeomorphism. This approach is used in [87] for a study of the Hankel transform. An earlier example of the same method (on $(-\infty,\infty)$) is the now standard theory of the Fourier transform on the space $\mathcal{S}'$ of tempered distributions; see, for instance, [24], [75] and [79].

A second, more specialised method might be called "the convolution method". Again, we shall work on $(0,\infty)$ only. The method treats an integral transform T of the form

$$(Tf)(x) = \int_0^\infty k(x-t)f(t)dt \quad (0 < x < \infty) \tag{1.7}$$

so that Tf is the convolution of the kernel k and the unknown function f. Convolution is an operation which is meaningful for distributions in $\mathcal{D}'(-\infty,\infty)$ whose support is bounded on the left ([24, Ch. 1, §5], [86, Ch. 5]) and in particular for elements of $\mathcal{D}'(0,\infty)$. Hence if k generates a distribution $\tilde{k} \in \mathcal{D}'(0,\infty)$, we are led to define $\tilde{T}$ on $\mathcal{D}'(0,\infty)$ by

$$\tilde{T}h = \tilde{k} * h \tag{1.8}$$

where * denotes distributional convolution. An example of this approach is afforded by the extension of the Riemann-Liouville fractional integral I_1^α defined for Re $\alpha > 0$ by

$$I_1^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt$$

so that I_1^α is one of the constituents of the operator in Example 1.1. In this case, the kernel generates the distribution $x_+^{\alpha-1}/\Gamma(\alpha)$ described in [24, p. 47] so that the extended operator, $\tilde{I}_1^\alpha$ say, is given by

$$\tilde{I}_1^{\alpha} h = x_+^{\alpha-1}/\Gamma(\alpha) * h \quad (h \in \mathcal{D}'(0,\infty)).$$

Using basic properties of convolution, a modest theory of fractional integration can be developed ([24], pp. 115 - 122).

The third method might be called the "kernel method". This is somewhat different in that it maps a generalised function into a classical function rather than another generalised function. Again, to fix ideas, consider the operator T defined by

$$(Tf)(x) = \int_0^\infty k(x,t)f(t)dt \quad (0 < x < \infty) \tag{1.9}$$

where k is a known kernel and $f \in X$. To imbed f in Z', we choose Z in such a way that, as a function of t, $k(x,t) \in Z$ for each fixed $x \in (0,\infty)$. Then, under appropriate conditions, $f \in X$ will generate a functional $\tilde{f} \in Z'$ and the right-hand side of (1.9) can be regarded as $(\tilde{f},k_x)$ where

$$k_x(t) = k(x,t) \quad (0 < t < \infty). \tag{1.10}$$

This suggests that if $h \in Z'$, we take $\tilde{T}h$ to be the classical function defined by

$$\tilde{T}h(x) = (h,k_x) \quad (0 < x < \infty). \tag{1.11}$$

This method is extensively used by Zemanian in [87] where we find applications to the Laplace, Mellin, K, I and Weierstrass transforms (the K and I transforms being analogues of the Hankel transform with J_ν replaced by the modified Bessel functions K_ν and I_ν). Perhaps paradoxically, a general convolution transform is also treated by this method in [87, Chapter 8].

In recent years, the adjoint operator method or kernel method has been applied to all the standard integral transforms on $(0,\infty)$ as well as many more off-beat generalisations. There have been studies using the adjoint operator method of

fractional calculus by Erdélyi and McBride [17], [46], [47], [50], [74],

Hankel transforms by Dube and Pandey [6], Koh [35], [36] and Lee [38], [39],

Mellin, Hankel and Watson transforms and fractional integrals by Braaksma and Schuitman [2].

On the other hand, there have been studies using the kernel method of

Stieltjes transforms by Erdélyi [15], Pandey [65] and Pathak [66],

Hardy transforms by Pathak and Pandey [67].

However, as hinted above, there is a snag. In many cases the spaces of generalised functions introduced in the references quoted are expressly geared to one particular transform and seem to be of little or no use for any other transform. This is hardly surprising since the kernels of the various transforms behave so differently. Nevertheless, in all but the simplest problems, it will be necessary to apply a succession of operators in order to obtain a solution and we therefore need spaces of generalised functions relative to which all the relevant operators are well-behaved.

The object of this book is to introduce and study certain spaces of generalised functions which, we believe, are of interest as regards the L^p theory of a number of important operators on the positive half-line $(0,\infty)$. We have chosen to study a few operators in considerable detail rather than to deal sketchily with a lot of transforms. To provide some continuity, we have chosen fractional calculus as a unifying theme and we deal with a number of problems which are connected to this theme. Except in the last chapter, we use the "adjoint operator" method as described above.

Perhaps the nearest relative of our approach in the literature is that in [2]. The spaces used therein are closely related to our spaces in the case $p = \infty$. (Indeed, most of the spaces of testing-functions in the literature have topologies defined by "L^∞-type" seminorms.) However, we will handle all values of p in the range $1 \leq p \leq \infty$ simultaneously. Our spaces are homeomorphic images of the spaces D_{L^p} of Schwartz [75,pp.199-201] and the latter spaces can in turn be related to the Sobolev spaces $W^{m,p}(-\infty,\infty)$ (see [75], p. 199). Thus we may think loosely of our spaces as modified Sobolev spaces on the half-line $(0,\infty)$.

Much of the material of Chapters 2 - 6 has appeared in a series of papers [47], [48], [49], [50] and [51]. However, the opportunity has been taken to carry out revisions which we hope will make the text more readable. At various stages, there was a great temptation to wander off at a tangent to discuss something of perhaps marginal interest. We have resisted this as much as possible. We have tried to develop the theory as concisely as possible stating only those results which are relevant. We have tried to show its applicability to various types of problems and, in particular, we show how classical L^p-type results can be deduced from it. We have tried to show how the general theory incorporates the classical theory and, at the same time, provides a framework wherein the formal analysis found in many books and papers can be justified rigorously. It is up to the reader to judge whether we have been successful or not.

To keep the bibliography as short as possible, we have only mentioned those works which are of direct relevance and no attempt has been made to give a comprehensive list of references. For instance, we shall frequently refer to Zemanian's book [87] which is unique in the field and contains an extensive bibliography which it would be pointless to repeat. Likewise, full lists of references covering the development and use of fractional calculus will be found in [56] and [74]. Again, in Chapter 4, we do not list all the special cases of Love's work as these can be found in [40] and [41].

§1.2 Plan of Campaign

In Chapter 2, we introduce the spaces $F_{p,\mu}$ of testing-functions and $F'_{p,\mu}$ of generalised functions and obtain their basic algebraic and topological properties. We also study a number of simple operators in these spaces.

As indicated above, we will make extensive use of fractional calculus. Chapter 3, the longest in the book, develops the theory which we require later as well as giving a first example of our extension process at work. We start with a simple generalisation of the operator $I_1^{\eta,\alpha}$ of Example 1.1 and go on to study Erdélyi-Kober operators as well as the Riemann-Liouville and Weyl fractional integrals. As we go along, the conditions on the parameters are relaxed more and more and the operators change their character from straight integrals to integro-differential operators. For instance, starting from Example 1.1, we end up with an operator $I_1^{\eta,\alpha}$ defined on $F_{p,\mu}$ for all complex α and values of η such that Re $(\eta+\mu) \neq - 1/q - \ell$ $(\ell = 0,1,2, \ldots)$. The properties of the operators on $F'_{p,\mu}$ are obtained using the adjoint operator approach.

Immediately, in Chapter 4, we apply our theory of fractional calculus to the solution of certain integral equations involving the ${}_2F_1$ hypergeometric function. As we shall see, use of generalised functions again enables the restrictions necessary for a classical treatment to be relaxed greatly and interesting results emerge quite naturally. At the end of the chapter, we show how a typical classical result can be deduced.

Chapter 5 is devoted to a treatment of the Hankel transform. We have chosen to deal with this transform for a number of reasons. Firstly, it is an important and useful transform. Secondly, the methods are similar to those which would be tried for other transforms. Thirdly, the results obtained are particularly complete in the case of the Hankel transform. Finally, the connections with fractional calculus are very strong. Again we travel a long way from the simple situation of Example 1.2. We also discuss a simple modification of the Hankel transform which we use extensively later.

Chapter 6 is devoted to stating conditions under which various connections between fractional calculus and the Hankel transform are valid in $F_{p,\mu}$ and $F'_{p,\mu}$.

Chapter 7 sees the application of the material in Chapters 3, 5 and 6 to dual integral equations of Titchmarsh type. Questions of existence and uniqueness are posed and solved for equations in $F'_{p,\mu}$ and we then show how this information can be transferred back to classical problems, dealing in detail with the simplest case and indicating briefly what happens otherwise.

We might regard Chapter 7 as the climax of the theory described in the earlier chapters. Nevertheless, in the final chapter, we mention briefly a few other operators which can be studied in our spaces or in simple modifications of these spaces.

2 The spaces $F'_{p,\mu}$

§2.1 Definition and Simple Properties of the Spaces $F_{p,\mu}$

We will be concerned with infinitely-differentiable complex-valued functions ϕ defined on the positive real line $(0,\infty)$ and will use the notation described in Chapter 0.

Definition 2.1

(i) For $1 \leqslant p < \infty$, we define $F_{p,0}$ by

$$F_{p,0} = \{\phi \in C^{\infty}(0,\infty):\ x^k d^k\phi/dx^k \in L^p(0,\infty) \text{ for } k = 0,1,2,\ \ldots\}.$$

(ii) We define $F_{\infty,0}$ by

$$F_{\infty,0} = \{\phi \in C^{\infty}(0,\infty): x^k d^k\phi/dx^k \to 0 \text{ as } x \to 0+ \text{ and as } x \to \infty \text{ for } k = 0,1,2,\ \ldots\}.$$

(iii) For $1 \leqslant p \leqslant \infty$ and any complex number μ, we define $F_{p,\mu}$ by

$$F_{p,\mu} = \{\phi \in C^{\infty}(0,\infty):\ x^{-\mu}\phi(x) \in F_{p,0}\}.$$

The reason for the slightly different definition in the case $p = \infty$ will perhaps be a little clearer after the following theorem which gives a very simple, but nevertheless very useful, growth estimate for functions in $F_{p,\mu}$.

Theorem 2.2

Let $1 \leqslant p \leqslant \infty$ and let μ be any complex number. Then, if $\phi \in F_{p,\mu}$, $x^{1/p-\mathrm{Re}\mu}\phi(x)$ is bounded on $(0,\infty)$.

<u>Proof</u>:- It is sufficient to establish the result for $\mu = 0$ in view of Definition 2.1(iii) and the fact that $|x^{\mathrm{Im}\mu}| = 1$ for all $x \in (0,\infty)$. Since the case $p = \infty$ is trivial we take $1 \leqslant p < \infty$ and, without loss of generality, take ϕ to be real-valued.

Let $0 < a < b < \infty$ and let $\phi \in F_{p,0}$. Integration by parts gives

$$\int_a^b px\phi'(x)\{\phi(x)\}^{p-1}dx = \left[x\{\phi(x)\}^p\right]_a^b - \int_a^b \{\phi(x)\}^p dx. \tag{2.1}$$

Since $\phi \in L^p(0,\infty)$, the integral on the right-hand side remains bounded as $a \to 0+$ or $b \to \infty$ (or both). As regards the integral on the left, we note that $x\phi'(x) \in L^p(0,\infty)$ while $\{\phi(x)\}^{p-1} \in L^q(0,\infty)$ $(q = \frac{p}{p-1})$. Hence, by Hölder's inequality, the left-hand side of (2.1) remains bounded as $a \to 0+$ or as $b \to \infty$. Thus the same is true of $x\{\phi(x)\}^p$ as $x \to 0+$ or $x \to \infty$ and the proof is complete in view of our opening remarks.

To return to Definition 2.1, we now see that the functions in $F_{p,0}(1 \leqslant p < \infty)$ satisfy the conditions required in (ii). On the other hand, the conditions $x^k d^k\phi/dx^k \in L^\infty(0,\infty)$ for $k = 0,1,2, \ldots$ (which would be the analogue of (i)) do not imply that ϕ satisfies the conditions in (ii). For technical reasons, the conditions in (ii) are essential for our theory.

It is immediately desirable to introduce a topology on $F_{p,\mu}$.

<u>Definition 2.3</u>

For $1 \leqslant p \leqslant \infty$ and any complex μ, $F_{p,\mu}$ is equipped with the topology generated by the semi-norms $\gamma_k^{p,\mu}$ $(k = 0,1,2, \ldots)$ defined by

$$\gamma_k^{p,\mu}(\phi) = ||x^k d^k/dx^k\, (x^{-\mu}\phi)||_p \qquad (\phi \in F_{p,\mu}).$$

<u>Notes</u>

1. Here $||\ ||_p$ denotes the $L^p(0,\infty)$ norm $(1 \leqslant p \leqslant \infty)$.

2. That each $\gamma_k^{p,\mu}$ is a semi-norm on $F_{p,\mu}$ is easily checked.

3. For an explanation of how the topology is generated, see [87, p.9]. It would be possible to develop the properties of $F_{p,\mu}$ from scratch but it is much quicker to make use of results in [75]. For convenience we recall the following definitions from [75, pp. 199-201].

<u>Definition 2.4</u>

(i) For $1 \leqslant p < \infty, D_{L^p}$ is the set defined by

$$D_{L^p} = \{\phi \in C^\infty(-\infty,\infty): d^k\phi/dx^k \in L^p(-\infty,\infty) \text{ for } k = 0,1,2, \ldots\}.$$

D_{L^p} is equipped with the topology generated by the semi-norms ν_k^p $(k = 0,1,2, \ldots)$ defined by

$$\nu_k^p(\phi) = ||d^k\phi/dx^k||_{L^p(-\infty,\infty)} \qquad (\phi \in D_{L^p}).$$

(ii) $\dot{B}$ is the set

$$\dot{B} = \{\phi \in C^\infty(-\infty,\infty): d^k\phi/dx^k \to 0 \text{ as } |x| \to \infty \text{ for } k = 0,1,2, \ldots\}.$$

$\dot{B}$ is equipped with the topology generated by the semi-norms $\nu_k^\infty(k = 0,1,2, \ldots)$ defined by

$$\nu_k^\infty(\phi) = ||d^k\phi/dx^k||_{L^\infty(-\infty,\infty)} \qquad (\phi \in \dot{B}).$$

To connect these Schwartz spaces with the $F_{p,\mu}$ spaces, we introduce operators $T_{p,\mu}$ as follows.

<u>Definition 2.5</u>

Let $1 \leqslant p \leqslant \infty$ and let μ be any complex number. If ϕ is a complex-valued function defined (a.e.) on $(0,\infty)$, we define $T_{p,\mu}\phi$ on $(-\infty,\infty)$ by

$$(T_{p,\mu}\phi)(x) = e^{(1/p-\mu)x}\phi(e^x) \qquad (-\infty < x < \infty).$$

We then have the following theorem.

Theorem 2.6

(i) For $1 \leqslant p < \infty$ and any complex number μ, $T_{p,\mu}$ is a homeomorphism of $F_{p,\mu}$ onto D_{L^p}.

(ii) For any complex number μ, $T_{\infty,\mu}$ is a homeomorphism of $F_{\infty,\mu}$ onto $\dot{B}$.

Proof:- (i) For $1 \leqslant p < \infty$ and $\phi \in F_{p,\mu}$

$$\int_{-\infty}^{\infty} |T_{p,\mu}\phi(x)|^p dx = \int_{-\infty}^{\infty} |e^{-\mu x}\phi(e^x)|^p e^x dx = \int_0^{\infty} |t^{-\mu}\phi(t)|^p dt$$

so that

$$\nu_0^p(T_{p,\mu}\phi) = \gamma_0^{p,\mu}(\phi). \tag{2.2}$$

By induction, it follows easily that for $k = 0,1,2, \ldots ,$

$$\begin{aligned} D^k(T_{p,\mu}\phi) &= T_{p,\mu}\left[(1/p-\mu)I + x\, d/dx\right]^k\phi \\ &= \sum_{j=0}^{k} c_j T_{p,\mu}\,(x^j d^j\phi/dx^j) \end{aligned} \tag{2.3}$$

for certain constants c_j $(j = 0,1, \ldots, k)$. Hence, using (2.2) and (2.3), we get

$$\begin{aligned} \nu_k^p(T_{p,\mu}\phi) &= \nu_0^p(D^k T_{p,\mu}\phi) \\ &\leqslant \sum_{j=0}^{k} |c_j|\,\nu_0^p(T_{p,\mu}(x^j d^j\phi/dx^j)) \\ &= \sum_{j=0}^{k} |c_j|\,\gamma_0^{p,\mu}(x^j d^j\phi/dx^j) \end{aligned}$$

$$=> \nu_k^p(T_{p,\mu}\phi) \leq \sum_{j=0}^{k} d_j \gamma_j^{p,\mu}(\phi) \tag{2.4}$$

for some constants d_j $(j = 0,1, \ldots, k)$, the last step following by routine algebra using the definition of $\gamma_j^{p,\mu}$. (2.4) shows that $T_{p,\mu}$ maps $F_{p,\mu}$ continuously into D_{L^p}. Indeed, $T_{p,\mu}$ maps $F_{p,\mu}$ onto D_{L^p} with inverse

$$(T_{p,\mu}^{-1}\psi)(x) = x^{\mu-1/p}\psi(\log x) \quad (0 < x < \infty) \tag{2.5}$$

where $\psi \in D_{L^p}$. A calculation similar to the above shows that $T_{p,\mu}^{-1} : D_{L^p} \to F_{p,\mu}$ is continuous. This completes the proof of (i).

(ii) is proved similarly but requires the extra observation that

$$x^k d^k\phi/dx^k \to 0 \text{ as } x \to 0+ \text{ and } x \to \infty \qquad (\text{for } k = 0,1,2, \ldots)$$

$$\text{iff} \quad d^k(T_{p,\mu}\phi)/dx^k \to 0 \text{ as } |x| \to \infty \qquad (\text{for } k = 0,1,2, \ldots).$$

The details are omitted.

This completes the proof of Theorem 2.6.

Theorem 2.6 states that from the topological point of view, $F_{p,\mu}$ and D_{L^p} have the same structure. Hence properties of $F_{p,\mu}$ are easily obtained from known properties of D_{L^p}. We mention some which are of importance for us.

Corollary 2.7

$C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$ for $1 \leq p \leq \infty$ and any complex number μ.

Proof:- From [75, p.199], $C_0^\infty(-\infty,\infty)$ is dense in D_{L^p} $(1 \leq p < \infty)$ and in $\dot{B}$. Since $T_{p,\mu}^{-1}(C_0^\infty(-\infty,\infty)) = C_0^\infty(0,\infty)$ for $1 \leq p \leq \infty$ and any μ, the result follows at once from Theorem 2.6.

Corollary 2.8

For $1 \leqslant p \leqslant \infty$ and any complex number μ, $F_{p,\mu}$ is a Fréchet space.

Proof:- Since D_{L^p} $(1 \leqslant p < \infty)$ and $\dot{B}$ are complete [75, p.199], so is $F_{p,\mu}$ by Theorem 2.6. That the space $F_{p,\mu}$ is locally convex follows at once from the definition of its topology [87, p.9]. Finally, $F_{p,\mu}$ is metrizable, a suitable metric being given by

$$d(\phi,\psi) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\gamma_k^{p,\mu}(\phi-\psi)}{1+\gamma_k^{p,\mu}(\phi-\psi)} \qquad (\phi,\psi \in F_{p,\mu})$$

[79, p.71].

Theorem 2.6 is also used in proving our final result in this section.

Theorem 2.9

Let μ_1 and μ_2 be complex numbers and let $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$. Then

(i) $F_{p_1,\mu_1} \subseteq F_{p_2,\mu_2}$ if and only if $\mathrm{Re}(\mu_1-\mu_2) = 1/p_1 - 1/p_2$ with strict inclusion if $p_1 < p_2$.

(ii) If $\mathrm{Re}(\mu_1-\mu_2) = 1/p_1-1/p_2$, the imbedding of F_{p_1,μ_1} into F_{p_2,μ_2} is continuous.

Proof:- First we mention that if μ is any complex number and $1 \leqslant p \leqslant \infty$,

$$F_{p,\mu} = F_{p,\mathrm{Re}\,\mu} \,. \tag{2.6}$$

This follows from the fact that $x^{\mathrm{Im}\,\mu} \in C^{\infty}(0,\infty)$ and $|x^{\mathrm{Im}\,\mu}| = 1$. Hence, without loss of generality, we may take μ_1,μ_2 to be real.

Suppose then that $\mu_1 - \mu_2 = 1/p_1 - 1/p_2$. The case $p_1 = p_2$ is now trivial. We are left with the case $1 \leqslant p_1 < p_2 \leqslant \infty$. Assume also that $p_2 < \infty$. Since $1/p_1 - 1/p_2 = \mu_1 - \mu_2$, T_{p_1,μ_1} and T_{p_2,μ_2} are formally the same. Hence

$$D_{L^{p_1}} \subset D_{L^{p_2}} \qquad \text{by [75, p.200]}$$

$$\Rightarrow T^{-1}_{p_1,\mu_1}(D_{L^{p_1}}) \subset T^{-1}_{p_2,\mu_2}(D_{L^{p_2}})$$

$$\Rightarrow F_{p_1,\mu_1} \subset F_{p_2,\mu_2} \qquad \text{by Theorem 2.6.}$$

The case $p_2 = \infty$ proceeds similarly with $\dot{B}$ replacing $D_{L^{p_2}}$. The strict inclusion and continuity of the imbedding follow from the corresponding properties of $D_{L^{p_1}}$ and $D_{L^{p_2}}$ along with the homeomorphic properties of T_{p_1,μ_1} and T_{p_2,μ_2}.

Now suppose that $\mu_1 - \mu_2 \neq 1/p_1 - 1/p_2$ and let

$$b = (1/p_2 - \mu_2) - (1/p_1 - \mu_1)$$

so that $b \neq 0$. Also, let $\theta \in C^\infty(0,\infty)$ be such that

$$\theta(x) = \begin{cases} 0 & x < 1 \\ 1 & x > 2 \end{cases} .$$

If $b > 0$, we consider the function

$$\phi(x) = \theta(x)x^{\mu_1 - 1/p_1 - b/2} \qquad (0 < x < \infty).$$

It is easy to check that $\phi \in F_{p_1,\mu_1}$. However, since

$$x^{-\mu_2}\phi(x) = \theta(x)x^{b/2 - 1/p_2}$$

$\phi \notin F_{p_2,\mu_2}$ so that $F_{p_1,\mu_1} \not\subset F_{p_2,\mu_2}$ in this case. The case $b < 0$ is handled similarly using

$$\psi(x) = \theta(1/x)x^{\mu_1 - 1/p_1 - b/2} .$$

This completes the proof of Theorem 2.9.

Remarks 2.10

(i) The importance of Theorem 2.9 will become clear later when we attempt to find the range of fractional integrals and the Hankel transform on the $F_{p,\mu}$ spaces.

(ii) In view of (2.6) we could restrict our attention to $F_{p,\mu}$ spaces with μ real. However, since other parameters later on will be complex (so that, for instance, we can use analytic continuation), it seems sensible to take μ complex as well.

§2.2 Simple Operators on the $F_{p,\mu}$ Spaces

In this section we discuss some simple operators which are used extensively later. We have already met the simplest of all namely the identity on $F_{p,\mu}$ and, more generally, the imbedding of F_{p_1,μ_1} into F_{p_2,μ_2} in Theorem 2.9. Another simple operator is x^λ defined by

$$(x^\lambda\phi)(x) = x^\lambda\phi(x) \qquad (0 < x < \infty) \quad . \tag{2.7}$$

(We will use the notation x^λ for the operation taking $\phi(x)$ to $x^\lambda\phi(x)$; there should be no confusion between the operator x^λ and the function x^λ.)

Theorem 2.11

For any complex numbers λ, μ and $1 \leqslant p \leqslant \infty$, x^λ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+\lambda}$ with inverse $x^{-\lambda}$.

Proof:- The result is trivial since $\gamma_k^{p,\mu+\lambda}(x^\lambda\phi) = \gamma_k^{p,\mu}(\phi)$ for $\phi \in F_{p,\mu}$ and $k = 0,1,2, \ldots$.

More interesting is the behaviour of the differentiation operator which we shall often denote by D, i.e.

$$(D\phi)(x) = d\phi/dx \tag{2.8}$$

for all suitable functions ϕ. We shall also need the operators δ, δ' defined by

$$(\delta\phi)(x) = x d\phi/dx \tag{2.9}$$

$$(\delta'\phi)(x) = d/dx(x\phi) = (\delta\phi)(x) + \phi(x). \tag{2.10}$$

Remark 2.12

It is easy to show that $\phi \in C^{\infty}(0,\infty)$ belongs to $F_{p,\mu}$ iff for $k = 0,1,2, \ldots$

$$\delta^k(x^{-\mu}\phi) \in L^p(0,\infty) \qquad (1 \leqslant p < \infty)$$

or $\quad \delta^k(x^{-\mu}\phi)(x) \to 0$ as $x \to 0+$ and as $x \to \infty \qquad (p = \infty)$

with similar comments for δ'.

Theorem 2.13

(i) For any complex number μ and $1 \leqslant p \leqslant \infty$, D is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-1}$.

(ii) D is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-1}$ iff $\mathrm{Re}\,\mu \neq 1/p$ and, if $\psi \in F_{p,\mu-1}$, then

$$(D^{-1}\psi)(x) = \begin{cases} \int_0^x \psi(t)dt & (\mathrm{Re}\,\mu > 1/p) \\ -\int_x^{\infty} \psi(t)dt & (\mathrm{Re}\,\mu < 1/p)\ . \end{cases} \tag{2.11}$$

Proof:- (i) For $k = 0,1,2, \ldots,$ and $\phi \in F_{p,\mu}$,

$$\gamma_k^{p,\mu-1}(D\phi) = ||x^k D^k(x^{1-\mu}D\phi)||_p$$

$$= ||x^k D^k(x\{D(x^{-\mu}\phi) + \mu x^{-\mu-1}\phi\})||_p$$

$$= ||x^k\{xD^{k+1}(x^{-\mu}\phi) + kD^k(x^{-\mu}\phi) + \mu D^k(x^{-\mu}\phi)\}||_p$$

$$\leqslant \gamma_{k+1}^{p,\mu}(\phi) + (k+|\mu|)\gamma_k^{p,\mu}(\phi)$$

and in the case $p = \infty$ we obtain similarly that $x^kD^k(x^{1-\mu}D\phi)(x) \to 0$ as $x \to 0+$ and as $x \to \infty$. The continuity of D is thus established.

(ii) We will consider in detail the cases $\mathrm{Re}\ \mu > 1/p$ (the case $\mathrm{Re}\ \mu < 1/p$ being similar) and $\mathrm{Re}\ \mu = 1/p$.

(a) <u>$\mathrm{Re}\ \mu > 1/p$</u>

If $\phi \in F_{p,\mu}$ and $D\phi = 0$, then $\phi(x) = c$ for all $x \in (0,\infty)$ where c is a constant. Since $\mathrm{Re}\ \mu > 1/p$, $x^{-\mu} \notin L^p(0,\infty)$ $(1 \leqslant p < \infty)$ and in the case $p = \infty$, $x^{-\mu} \not\to 0$ as $x \to 0+$. Hence $cx^{-\mu} \in F_{p,\mu}$ iff $c = 0$ so that D is one-one.

To show that D maps $F_{p,\mu}$ onto $F_{p,\mu-1}$, let $\psi \in F_{p,\mu-1}$. Then $\int_0^x \psi(t)\,dt$ belongs to $F_{p,\mu}$. Indeed we may write

$$\int_0^x \psi(t)dt = xI_1^{0,1}\psi(x)$$

where $I_1^{0,1}$ is as in Example 1.1 and our result is then a very simple special case of Lemma 3.3 below. Thus although we could proceed from scratch, we merely refer the reader to that result. Now, in view of Theorem 2.2, for some constant C,

$$\left|\int_0^x \psi(t)dt\right| \leqslant \int_0^x |\psi(t)|dt \leqslant C\int_0^x t^{-1/p+\mathrm{Re}\ \mu-1}dt \ .$$

Thus since $\mathrm{Re}\ \mu > 1/p$, the integral $\int_0^x \psi(t)dt$ converges absolutely and uniformly on compact subsets of $(0,\infty)$. In particular, we conclude that

$$D\left(\int_0^x \psi(t)dt\right) = \psi(x).$$

Hence D is onto.

Finally, the continuity of D^{-1} follows again from Lemma 3.3 below but is also an automatic consequence of the Open Mapping Theorem for Fréchet spaces [79 , Theorem 17.1]. This completes the proof in case $\operatorname{Re} \mu > 1/p$.

(b) <u>$\operatorname{Re} \mu = 1/p$</u>

Here we shall prove that D does not map $F_{p,\mu}$ <u>onto</u> $F_{p,\mu-1}$. Firstly, we assume that $1 \leqslant p < \infty$. Choose a such that $-1/p < a < 0$ and let $\theta \in C^{\infty}(0,\infty)$ be such that

$$\theta(x) = \begin{cases} 0 & 0 < x \leqslant 2 \\ 1 & x \geqslant e . \end{cases}$$

Next define ϕ by

$$\phi(x) = \theta(x) \ (\log x)^{a} \quad (0 < x < \infty). \tag{2.12}$$

Finally we shall write

$$\psi(x) = (\delta\phi)(x)$$

where δ is given by (2.9). For $k = 0,1,2, \ldots$

$$(\delta^{k}\psi)(x) = (\delta^{k+1}\phi)(x) = \begin{cases} 0 & 0 < x \leqslant 2 \\ a(a-1) \ \ldots \ (a-k)(\log x)^{a-k-1} & x \geqslant e \end{cases}$$

and since for $\operatorname{Re} \mu = 1/p$,

$$\int_{e}^{\infty} |x^{-\mu}(\log x)^{a-k-1}|^{p} dx = \int_{1}^{\infty} t^{(a-k-1)p} \, dt < \infty,$$

$\psi \in F_{p,\mu}$ so that $\chi \in F_{p,\mu-1}$ where

$$\chi(x) = x^{-1}\psi(x) \quad (0 < x < \infty).$$

Notice that $(D\phi)(x) = \chi(x)$. However,

$$\int_e^\infty |x^{-\mu}\phi(x)|^p dx = \int_e^\infty x^{-1}(\log x)^{ap} dx = \int_1^\infty u^{ap} du$$

is divergent since $ap > -1$, so that $\phi \notin F_{p,\mu}$. Furthermore, if there were a function $\phi_1 \in F_{p,\mu}$ such that $D\phi_1(x) = \chi(x)$, we would obtain

$$\phi_1(x) = \phi(x) + c \tag{2.13}$$

where c is a constant. By letting $b \to \infty$ in (2.1) we deduce that $x^{1/p-\mathrm{Re}\,\mu}\phi_1(x) = \phi_1(x)$ tends to a limit, L say, as $x \to \infty$. If $L \neq 0$, $\int_X^\infty |x^{-\mu}\phi_1(x)|^p dx \geq |L/2| \int_X^\infty (x^{-1/p})^p dx$ for X sufficiently large and we have divergence. Hence $L = 0$, i.e. $\phi_1(x) \to 0$ as $x \to \infty$. Also, as $a < 0$, $\phi(x) \to 0$ as $x \to \infty$. Letting $x \to \infty$ in (2.13) gives $c = 0 \Rightarrow \phi_1 = \phi$ $\Rightarrow \phi \in F_{p,\mu}$, a contradiction. Hence there is no function $\phi \in F_{p,\mu}$ such that $D\phi = \chi$ and hence D does not map $F_{p,\mu}$ onto $F_{p,\mu-1}$ in this case.

When $p = \infty$, $\mathrm{Re}\,\mu = 0$ and we consider ϕ as defined by (2.12) but with $0 < a < 1$ now. The proof is similar and is omitted; details are to be found in [50].

Corollary 2.14

Let μ be any complex number, $1 \leq p \leq \infty$ and δ, δ' as in (2.9), (2.10).

(i) δ, δ' are continuous linear mappings from $F_{p,\mu}$ into itself.

(ii) δ is a homeomorphism of $F_{p,\mu}$ onto itself iff $\mathrm{Re}\mu \neq 1/p$.

(iii) δ' is a homeomorphism of $F_{p,\mu}$ onto itself iff $\mathrm{Re}\,\mu \neq -1/q$.

Proof:- The results follow easily from Theorems 2.11 and 2.13.

Later we shall require differentiation with respect to a positive power of the variable. For $m > 0$, we define the operator D_m by

$$(D_m\phi)(x) = d/dx^m(\phi(x)). \tag{2.14}$$

For reference, we state the following result.

<u>Corollary 2.15</u>

Let μ be any complex number, $1 \leqslant p \leqslant \infty$, $m > 0$.

(i) D_m is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\mu-m}$.

(ii) D_m is a homeomorphism from $F_{p,\mu}$ onto $F_{p,\mu-m}$ iff $\operatorname{Re}\mu \neq 1/p$.

<u>Proof</u>:- These also follow from Theorems 2.11 and 2.13 on writing

$$(D_m\phi)(x) = m^{-1}x^{1-m}D\phi(x) .$$

<u>Remark 2.16</u>

It is perhaps worthwhile to examine the exceptional cases occurring in the last few results. For simplicity, we shall look in detail at D_1 on $F_{p,\mu}$.

Let $\phi \in F_{p,\mu}$ have compact support. Then so does $\psi = D_1\phi \in F_{p,\mu-1}$. Further

$$\phi(x) = \int_0^x \psi(t)dt + c(\psi) \qquad (0 < x < \infty)$$

where $c(\psi)$ is a constant depending on ψ. There are three cases again.

(i) <u>$\operatorname{Re}\mu > 1/p$</u>. On taking $c(\psi) \equiv 0$, the right-hand side becomes $\int_0^x \psi(t)dt$ as in (2.11). This is the only choice which leads to an operator which can be extended from $C_0^\infty(0,\infty)$ to a continuous linear mapping from all of $F_{p,\mu-1}$ into $F_{p,\mu}$; the uniqueness follows from the fact that no non-zero constant function belongs to $F_{p,\mu}$.

(ii) <u>$\operatorname{Re}\mu < 1/p$</u>. This time the only $c(\psi)$ giving an operator which can be extended as above is

$$c(\psi) = -\int_0^\infty \psi(t)dt$$

which again gives (2.11).

(iii) Re $\mu = 1/p$. If the support of ϕ lies in $[a,b]$ $(0 < a < b < \infty)$, then

$$\int_0^x \psi(t)dt + c(\psi) = \begin{cases} c_1 & (0 < x \leqslant a) \\ c_2 & (x \geqslant b) \end{cases}$$

where c_1 and c_2 are constants and since $|c_i x^{-\mu}|^p = |c_i|^p x^{-1}$ $(i = 1,2)$ in this case we must have $c_1 = c_2 = 0$ in order to obtain an element of $F_{p,\mu}$. Letting $x \to 0$ gives $c(\psi) = 0$ and letting $x \to \infty$ gives

$$\int_0^\infty \psi(t)dt = 0.$$

Only functions $\psi \in C_0^\infty(0,\infty)$ satisfying this last condition can belong to the range of D_1 on $F_{p,\mu}$ (Re $\mu = 1/p$) and for such ϕ, the values of $c(\psi)$ in (i) and (ii) agree. This shows how case (iii) is a half-way house between (i) and (ii). Also the proof of Theorem 2.13 shows that when Re $\mu = 1/p$ neither of the expressions

$$\int_0^x \psi(t)dt, \quad -\int_x^\infty \psi(t)dt \quad (\psi \in C_0^\infty(0,\infty))$$

can be extended to continuous linear mappings from $F_{p,\mu-1}$ into $F_{p,\mu}$.

Similar comments apply to D_m, δ and δ'. Formulae for the inverse operators can be written down in the non-exceptional cases. For instance,

$$\delta^{-1}\phi = I_1^{-1,1}\phi \quad (\phi \in F_{p,\mu})$$

when Re $\mu > 1/p$, $I_1^{-1,1}$ being as in Example 1.1. In the exceptional cases, there are complications. It would be possible, for instance, to study

$(D_m)^{-1}$ as an operator from $D_m(F_{p,\mu})$ onto $F_{p,\mu}$ when Re $\mu = 1/p$. However, we will not conduct this investigation any further here. We merely observe that the non-invertibility of D_m, δ and δ' on the $F_{p,\mu}$ spaces in the exceptional cases has considerable repercussions in the sequel; see §3.3 and §5.3.

We now have sufficient knowledge of the spaces $F_{p,\mu}$ to proceed to a study of their dual spaces.

§2.3 The Spaces $F'_{p,\mu}$ of Generalised Functions

We shall use the following standard notation.

Notation 2.17

Let $1 \leqslant p \leqslant \infty$ and let μ be any complex number.

(i) $F'_{p,\mu}$ will denote the set of all continuous linear functionals on $F_{p,\mu}$ equipped with the topology of pointwise (or weak) convergence.

(ii) Typical elements of $F'_{p,\mu}$ will be denoted by f, g etc. The value assigned by $f \in F'_{p,\mu}$ to a function $\phi \in F_{p,\mu}$ will be denoted by (f,ϕ).

For an explanation of the terminology used in (i), see [87, p.21].

Example 2.18

For $1 \leqslant p \leqslant \infty$ and any complex μ, let $f \in L^q_{-\mu}$ (see (0.3)). Then f generates an element, $\tilde{f}$ say, of $F'_{p,\mu}$ via the formula

$$(\tilde{f},\phi) = \int_0^\infty f(x)\phi(x)dx \quad (\phi \in F_{p,\mu}). \tag{2.15}$$

Indeed this is a simple consequence of Hölder's inequality. Any functional $\tilde{f}$ in $F'_{p,\mu}$ generated from a classical function f via (2.15) will be called a regular functional on $F_{p,\mu}$. Often we will write τf rather than $\tilde{f}$ so that

$$(\tau f, \phi) = \int_0^\infty f(x)\phi(x)dx \quad (\phi \in F_{p,\mu}). \tag{2.16}$$

The mapping τ defined by (2.16) gives an imbedding of $L^q_{-\mu}$ into $F'_{p,\mu}$ and, although strictly τ depends on p and μ, we will omit specific mention of the latter.

Results for $F'_{p,\mu}$ are easily obtained from those for $F_{p,\mu}$ using standard theorems on duals of Fréchet spaces. Typical is the following result.

Theorem 2.19

For $1 \leqslant p \leqslant \infty$ and any complex μ, $F'_{p,\mu}$ is complete.

Proof:- The result is immediate from Corollary 2.8 above and [87, Theorem 1.8-3].

We should also note the following fact.

Theorem 2.20

For $1 \leqslant p \leqslant \infty$ and any complex μ, $F'_{p,\mu} \subseteq \mathfrak{D}'$ in the sense that the restriction of any $f \in F'_{p,\mu}$ to $\mathfrak{D}$ defines an element of $\mathfrak{D}'$. (The notation is as in Chapter 0.)

Proof:- The result follows at once from Corollary 2.7 above and [87, Corollary 1.8-2a]. Roughly speaking, the theorem states that our generalised functions are distributions in the sense of Schwartz.

We will obtain much more information shortly but, first, we consider some simple operators on $F'_{p,\mu}$.

§2.4 Simple Operators in $F'_{p,\mu}$

At this stage we return to the question on extendability of operators which we discussed informally in §1.1 and show what happens in the case of the simple operators x^λ, δ, δ' and D_m of §2.2. Since the situation in each case is similar we shall consider only one in detail, δ being a suitable choice.

As a temporary notation, we shall write

$$M_\mu^p = \{f \in L_\mu^p : \delta f \in L_\mu^p\} \quad (1 \leqslant p \leqslant \infty,\ \mu \text{ complex}).$$

Thus $f \in M_\mu^p$ if it is differentiable (almost everywhere) on $(0,\infty)$ and $x\, df/dx \in L_\mu^p$. The operator δ maps M_μ^p into L_μ^p. On the other hand both $M_{-\mu}^q$, $L_{-\mu}^q$ can be imbedded in $F'_{p,\mu}$ via (2.15). We would like to extend δ to an operator, $\tilde{\delta}$ say, mapping $F'_{p,\mu}$ into itself. For $\tilde{\delta}$ to be an extension of δ we would require that

$$\tilde{\delta}\tilde{g} = \widetilde{\delta g} \quad (g \in M_{-\mu}^q) \ . \tag{2.17}$$

To see what this entails, let $\phi \in C_0^\infty(0,\infty)$. Then,

$$(\tilde{\delta}\tilde{g},\phi) = (\widetilde{\delta g},\phi) = \int_0^\infty (\delta g)(x)\phi(x)dx = \int_0^\infty xg'(x)\phi(x)dx$$

and since ϕ has compact support, we may integrate by parts to obtain

$$(\tilde{\delta}\tilde{g},\phi) = -\int_0^\infty g(x)d/dx(x\phi)dx = (\tilde{g},-\delta'\phi)$$

where δ' is defined by (2.10). Since $C_0^\infty(0,\infty)$ is dense in $F_{p,\mu}$ by Corollary 2.7, we may extend $\tilde{\delta}\tilde{g}$ by continuity to the whole of $F_{p,\mu}$ and we then have

$$(\tilde{\delta}\tilde{g},\phi) = (\tilde{g},-\delta'\phi) \quad (\phi \in F_{p,\mu}).$$

The last equation suggests that for any functional $f \in F'_{p,\mu}$, regular or not we should require that

$$(\tilde{\delta}f,\phi) = (f,-\delta'\phi) \quad (\phi \in F_{p,\mu}).$$

From now on we shall denote the extended operator by δ rather than δ. This should not cause any confusion, the context indicating whether δ is the classical or extended operator.

A similar argument can be applied to the other operators mentioned and we are led to the following definitions.

Definition 2.21

Let $1 \leqslant p \leqslant \infty$, let μ be any complex number and let $f \in F'_{p,\mu}$.

(i) For any complex λ, we define the functional $x^\lambda f$ on $F_{p,\mu-\lambda}$ by

$$(x^\lambda f,\phi) = (f,x^\lambda\phi) \quad (\phi \in F_{p,\mu-\lambda}). \tag{2.18}$$

(ii) The functionals δf and $\delta' f$ are defined on $F_{p,\mu}$ by

$$(\delta f,\phi) = (f,-\delta'\phi) \quad (\phi \in F_{p,\mu}) \tag{2.19}$$

$$(\delta' f,\phi) = (f,-\delta\phi) \quad (\phi \in F_{p,\mu}). \tag{2.20}$$

(iii) For $m > 0$, we define the functional $D_m f$ on $F_{p,\mu+m}$ by

$$(D_m f,\phi) = (f,-m^{-1}D(x^{1-m})\phi) \quad (\phi \in F_{p,\mu+m}). \tag{2.21}$$

Notes

1. The expressions on the right-hand sides of (2.18) - (2.21) are meaningful in view of Theorem 2.11, Corollary 2.14 (i) and Corollary 2.15 (i).
2. When $m = 1$, (2.21) takes the simple form

$$(Df,\phi) = (f,-D\phi) \quad (\phi \in F_{p,\mu+1}).$$

Using the fact that our extended operators on $F'_{p,\mu}$ are formal adjoints of known operators on $F_{p,\mu}$, we can make use of standard results on adjoint operators in complete countably multinormed spaces (of which the spaces $F_{p,\mu}$ are examples).

Theorem 2.22

Let $1 \leq p \leq \infty$ and let μ be any complex number.

(i) For any complex number λ, x^{λ} (as defined by (2.18)) is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$ with inverse $x^{-\lambda}$.

(ii) δ is a continuous linear mapping of $F'_{p,\mu}$ into itself and is a homeomorphism in case Re $\mu \neq -1/q$.

(iii) δ' is a continuous linear mapping of $F'_{p,\mu}$ into itself and is a homeomorphism in case Re $\mu \neq 1/p$.

(iv) D_m is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+m}$ and is a homeomorphism in case Re $\mu \neq -1/q$.

Proof:- The results follow almost at once on using Theorem 2.11, Corollary 2.14 and Corollary 2.15 above in conjunction with [87, Theorems 1.10-1 and 1.10-2].

Remark 2.23

We notice that the restrictions on the parameters in the results for $F'_{p,\mu}$ are obtained from those for $F_{p,\mu}$ by interchanging the pairs p and q, μ and $-\mu$; for example "Re $\mu \neq 1/p$" becomes "Re $\mu \neq -1/q$". This is a simple consequence of the duality in (2.15) and the trend continues throughout the sequel.

As an example of the use of our results to date, we end this chapter with a structure theorem which gives a description of a typical element of $F'_{p,\mu}$. To prove this theorem, we return to the spaces D_{L^p} and $\dot{B}$ (in Definition 2.4) whose dual spaces will be denoted by $(D_{L^p})'$ and $(\dot{B})'$

respectively.

Lemma 2.24

(i) Any element $f \in (D_{L^p})'$ $(1 \leqslant p < \infty)$ is of the form

$$f = \sum_{k=0}^{n} D^k \tilde{f}_k \tag{2.22}$$

where n is a non-negative integer, $f_k \in L^q(-\infty,\infty)$ for $k = 0,1, \ldots, n$, $\tilde{f}_k$ $(\in (D_{L^p})')$ is defined by

$$(\tilde{f}_k,\phi) = \int_{-\infty}^{\infty} f_k(x)\phi(x)dx \qquad (\phi \in D_{L^p})$$

and D denotes generalised differentiation.

(ii) Any element $f \in (\dot{B})'$ is of the form (2.22) where $f_k \in L^1(-\infty,\infty)$ $(k = 0,1, \ldots, n)$ and $\tilde{f}_k$ is defined by

$$(\tilde{f}_k,\phi) = \int_{-\infty}^{\infty} f_k(x)\phi(x)dx \qquad (\phi \in \dot{B}).$$

Proof:- See [75, p. 201].

We can now obtain our structure theorem easily.

Theorem 2.25

Let $1 \leqslant p < \infty$ and let μ be any complex number. Then any element $f \in F'_{p,\mu}$ is of the form

$$f = \sum_{k=0}^{n} \delta^k \tilde{f}_k \tag{2.23}$$

where n is a non-negative integer, $f_k \in L^q_{-\mu}$ $(k = 0,1, \ldots, n)$, $\tilde{f}_k$ is defined via (2.15) and δ is defined via (2.19).

Proof:- We shall consider the case $1 \leqslant p < \infty$, the case $p = \infty$ being similar.

We define $T'_{p,\mu}: (D_{L^p})' \to F'_{p,\mu}$ by

$$(T'_{p,\mu}g,\phi) = (g,T_{p,\mu}\phi) \qquad (g \in (D_{L^p})',\ \phi \in F_{p,\mu})$$

where $T_{p,\mu}$ is as in Definition 2.5. From Theorem 2.6 (i) above and [87, Theorem 1.10-2], $T'_{p,\mu}$ is a homeomorphism of $(D_{L^p})'$ onto $F'_{p,\mu}$.

Let $f \in F'_{p,\mu}$. Then $\exists\, g \in (D_{L^p})'$: $T'_{p,\mu}\, g = f$, and using Lemma 2.24 (i),

$$f = T'_{p,\mu}\left(\sum_{k=0}^{n} D^k\tilde{g}_k\right)$$

for some non-negative integer n and $g_k \in L^q(-\infty,\infty)$ $(k = 0,1, \ldots, n)$.

Thus, if $\phi \in F_{p,\mu}$,

$$(f,\phi) = \sum_{k=0}^{n} (\tilde{g}_k, (-1)^k D^k T_{p,\mu}\phi)$$

$$= \sum_{k=0}^{n} (\tilde{g}_k, T_{p,\mu}(-\delta'+(\mu+1/q))^k\phi)$$

from (2.3) and (2.10). Let us write

$$\psi_k = (-\delta'+(\mu+1/q))^k\phi \qquad (k = 0,1, \ldots, n) \tag{2.24}$$

so that $\psi_k \in F_{p,\mu}$ by Corollary 2.14 (iii). We consider the single term

$$(\tilde{g}_k,T_{p,\mu}\psi_k) = \int_{-\infty}^{\infty} g_k(x)T_{p,\mu}\psi_k(x)dx$$

$$= \int_{-\infty}^{\infty} g_k(x)e^{(1/p-\mu)x}\psi_k(e^x)dx$$

$$= \int_0^{\infty} g_k(\log t)t^{1/p-\mu-1}\psi_k(t)dt$$

$$= \int_0^{\infty} t^{-\mu}h_k(t)\psi_k(t)dt \tag{2.25}$$

where $h_k(t) = t^{-1/q}g_k(\log t)$ $(0 < t < \infty)$. Now for $1 < q < \infty$,

$$\int_0^\infty |h_k(t)|^q dt = \int_0^\infty |g_k(\log t)|^q \frac{dt}{t} = \int_{-\infty}^\infty |g_k(u)|^q du < \infty$$

and if $q = \infty$, $\underset{t\varepsilon(0,\infty)}{\text{ess sup}} |h_k(t)| = \underset{x\varepsilon(-\infty,\infty)}{\text{ess sup}} |g_k(x)| < \infty$. Thus, in either case, $h_k \varepsilon L^q(0,\infty)$. From (2.24) and (2.25) we obtain

$$\begin{aligned}(f,\phi) &= \sum_{k=0}^{n} (\widetilde{x^{-\mu}h_k}, (\sum_{\ell=0}^{k} a_{k\ell}(-\delta')^\ell\phi)) \\ &= \sum_{\ell=0}^{n} (\sum_{k=\ell}^{n} \widetilde{a_{k\ell}x^{-\mu}h_k}, (-\delta')^\ell\phi) \\ &= \sum_{\ell=0}^{n} (\delta^\ell \tilde{f}_\ell, \phi)\end{aligned}$$

where the $a_{k\ell}$'s are constants and $f_\ell = \sum_{k=\ell}^{n} a_{k\ell}x^{-\mu}h_k \varepsilon L^q_{-\mu}$. Thus f is of the form (2.23).

Conversely any expression of the form (2.23) defines an element of $F'_{p,\mu}$ in view of Example 2.18. This completes the proof of Theorem 2.25, which we will require occasionally in the sequel, notably in §7.2.

We now have all the basic results for $F_{p,\mu}$ and $F'_{p,\mu}$ which we shall need. As we have seen, some proofs required us to separate the cases $1 \leqslant p < \infty$ and $p = \infty$ although the end results were the same. To save repetition we make the following convention.

Convention 2.26

From now on, unless the contrary is stated explicitly, p will lie in the range $1 \leqslant p \leqslant \infty$ and μ will be any complex number.

3 Fractional calculus

§3.1 Introduction

The development of fractional calculus within the framework of classical functions is now well-known and no purpose would be served here by a detailed exposition. Most of the basic formal analysis can be found in [56] while some of the more theoretical issues as well as historical information can be found in [74]. We merely make a few comments relevant to our discussions.

There are many different starting points for a discussion of classical fractional calculus. One development begins with a generalisation of repeated integration. If f is, for instance, locally integrable on $(0,\infty)$, we find that (a.e.)

$$\int_0^x dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt \qquad (3.1)$$

for $n = 1,2, \ldots$ and $0 < x < \infty$. On writing $(n-1)! = \Gamma(n)$ we obtain an immediate generalisation in the form of the operator I_1^α defined for $\mathrm{Re}\,\alpha > 0$ and suitable functions f by

$$(I_1^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt \qquad (0 < x < \infty). \qquad (3.2)$$

$I_1^\alpha f$ is the Riemann-Liouville fractional integral of order α of the function f. Similarly, there is the Weyl fractional integral of order α, denoted by K_1^α, and defined for $\mathrm{Re}\,\alpha > 0$ and suitable functions f by

$$(K_1^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1} f(t)dt \qquad (0 < x < \infty). \qquad (3.3)$$

The conditions on f which ensure the existence of $I_1^{\alpha}f$, $K_1^{\alpha}f$ ($\text{Re}\ \alpha > 0$) are well-known. For instance if f is locally integrable over $(0,\infty)$, then $I_1^{\alpha}f$ exists almost everywhere on $(0,\infty)$ and is again locally integrable; see for instance [43]. We can then define a fractional derivative of order α of a locally integrable function f to be a (locally integrable) solution g of the equation

$$I_1^{\alpha}g = f$$

if such a solution exists and in that case we write formally $g = I_1^{-\alpha}f$. Questions of existence and uniqueness of fractional derivatives defined in this way are discussed in, for instance, [40], [43]. Similar comments apply to K_1^{α}.

An entirely different approach begins with fractional derivatives, taking as starting point the Cauchy Integral Formula of complex analysis, namely

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(z-w)^{n+1}}\,dw$$

(valid under the usual conditions) and defining a fractional derivative of order α by means of the expression

$$f^{(\alpha)}(z) = \frac{\Gamma(\alpha+1)}{2\pi i}\int_C \frac{f(w)}{(z-w)^{\alpha+1}}\,dw$$

interpreted in an appropriate way. This approach has been used in a series of papers by Osler [60], [61], [62], [63], [64]. Yet another method defines fractional derivatives by differentiating power series formally "α times" so that

$$\frac{d^\alpha}{dz^\alpha}\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} a_n z^{n-\alpha} .$$

These approaches and others are discussed and compared in the articles by Gaer and Rubel, by Lavoie, Tremblay and Osler and by Ross in [74].

In the classical framework, the approach to be used depends on the class of functions under consideration, since, clearly, functions of a real variable locally integrable over $(0,\infty)$ and functions of a complex variable analytic in a domain are very different objects. In particular the question arises as to which is most suitable for the spaces L^p_μ. A clue is given by the work of Kober who showed in [32] that, under suitable conditions, I^α_1 and K^α_1, as defined by (3.2) and (3.3), are continuous linear mappings from L^p_μ into $L^p_{\mu+\alpha}$ (although his results were stated slightly differently). Therefore our approach will be to start with integrals and obtain derivatives later. Further, we shall wish to extend I^α_1, K^α_1 to $F'_{p,\mu}$ in the sense described in §1.1, using the imbedding defined by (2.16).

However, we are not quite ready to begin at once, since I^α_1 and K^α_1 can be generalised in at least two useful ways. Firstly, we may wish to integrate with respect to a continuously differentiable, increasing function ρ of a positive real variable producing expressions such as

$$\frac{1}{\Gamma(\alpha)} \int_0^x (\rho(x)-\rho(t))^{\alpha-1} f(t)\rho'(t)dt.$$

We shall only be concerned with the case where $\rho(x) = x^m$ (m real, $m > 0$). We therefore introduce the operators I^α_m, K^α_m defined for Re $\alpha > 0$ and suitable functions f by

$$(I^\alpha_m f)(x) = \frac{m}{\Gamma(\alpha)} \int_0^x (x^m-t^m)^{\alpha-1} t^{m-1} f(t)dt \qquad (0 < x < \infty) \tag{3.4}$$

$$(K_m^{\alpha} f)(x) = \frac{m}{\Gamma(\alpha)} \int_x^{\infty} (t^m - x^m)^{\alpha-1} t^{m-1} f(t)\,dt \qquad (0 < x < \infty). \tag{3.5}$$

The case m = 1, of course, takes us back to (3.2) and (3.3) again. Secondly, the results of Kober in [32] indicate that it is useful to consider operators $I_1^{\eta,\alpha}$, $K_1^{\eta,\alpha}$ defined for Re $\alpha > 0$, suitable complex numbers η and suitable functions f by

$$(I_1^{\eta,\alpha} f)(x) = x^{-\eta-\alpha} I_1^{\alpha} x^{\eta} f(x) \qquad (0 < x < \infty) \tag{3.6}$$

$$(K_1^{\eta,\alpha} f)(x) = x^{\eta} K_1^{\alpha} x^{-\eta-\alpha} f(x) \qquad (0 < x < \infty). \tag{3.7}$$

Combining these two generalisations we arrive at the operators $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$ defined by

$$(I_m^{\eta,\alpha} f)(x) = x^{-m\eta-m\alpha} I_m^{\alpha} x^{m\eta} f(x) \qquad (0 < x < \infty) \tag{3.8}$$

$$(K_m^{\eta,\alpha} f)(x) = x^{m\eta} K_m^{\alpha} x^{-m\eta-m\alpha} f(x) \qquad (0 < x < \infty). \tag{3.9}$$

Using (3.4), (3.5) and putting t = xu produces the expressions

$$(I_m^{\eta,\alpha} f)(x) = \frac{m}{\Gamma(\alpha)} \int_0^1 (1-u^m)^{\alpha-1} u^{m\eta+m-1} f(xu)\,du \tag{3.10}$$

$$(K_m^{\eta,\alpha} f)(x) = \frac{m}{\Gamma(\alpha)} \int_1^{\infty} (u^m-1)^{\alpha-1} u^{-m\eta-m\alpha+m-1} f(xu)\,du \tag{3.11}$$

which show that the operators $I_m^{\eta,\alpha}, K_m^{\eta,\alpha}$ are, in a certain sense, "homogeneous". They are usually referred to as the Erdélyi-Kober operators after the two mathematicians who pioneered their systematic use.

Two things should perhaps be said about these extensions. Firstly, although the results for a general value of m > 0 can be deduced from those for m = 1, it is no harder to deal with the general value from the start and the case m = 2 turns out to be very important in connection with Hankel transforms; see Chapter 6. Secondly, the use of complex values of η and α is no more complicated than using real values and has the important advantage that powerful results on analytic continuation can then be applied.

Although, as indicated in §1.1, some progress can be made using the idea of convolution for distributions, it is more convenient in view of Kober's results to start with the homogeneous operators given by (3.10) and (3.11) and then to derive results for the "inhomogeneous" operators I_m^{α}, K_m^{α} from them. The extension process is quite long but our aim is to define the operators for as large a set of values of η and α as possible. We start with integrals and end up with more complicated integro-differential operators. We will split the process up into stages, the first of which we are ready to tackle now.

§3.2 Fractional Calculus in $F_{p,\mu}$: Stage I

Initially in this section we assume that Re $\alpha > 0$ and consider the operators $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$ as given by (3.10) and (3.11). These operators have been studied relative to the L_μ^p spaces by a number of authors, for instance, Erdélyi [16], Flett [23], Okikiolu [52], [53] and Rooney [72], [73]. However, all we require at this stage is a result of Kober [32]. We shall deal in detail with $I_m^{\eta,\alpha}$ and merely state the corresponding facts for $K_m^{\eta,\alpha}$. Also we recall that Convention 2.26 is operational.

Lemma 3.1

If Re $\alpha > 0$, Re $\eta > -1/q$, then $I_1^{\eta,\alpha}$ is a continuous linear mapping of L^p $(= L_0^p)$ into $L_{1/r-1/p}^r$ under the following additional alternative hypotheses:

(i) $1 \leqslant p \leqslant \infty$, $p = r$

(ii) $1 < p < r < \infty$, $1/p - 1/r < \text{Re}\,\alpha \leqslant 1/p$

(iii) $1 \leq p \leq r < \infty$, $\mathrm{Re}\,\alpha > 1/p$.

Proof:- Given in [32, Theorem 2].

In the case of the spaces L^p_μ, the results obtained in (i), (ii), (iii) are independent. But in the case of the $F_{p,\mu}$ spaces, the situation is very different. We might expect that under the same conditions, $I_1^{\eta,\alpha}$ would map $F_{p,0}$ into $F_{r,1/r-1/p}$ and this is indeed the case. However, Theorem 2.9 (i) shows that when $1 \leq p \leq r \leq \infty$, $F_{p,0} \subseteq F_{r,1/r-1/p}$ with strict inclusion if $p < r$. Thus, as regards the range of $I_1^{\eta,\alpha}$ on $F_{p,0}$, we see that the spaces $F_{r,1/r-1/p}$ $(p < r)$ are ruled out, leaving only one candidate $F_{p,0}$ itself. We shall see that $F_{p,0}$ is indeed the range, a consequence of our more general theorem below. We will use the conditions in (i) above; our results will then incorporate those which can be obtained using (ii) and (iii).

Because of the homogeneity of (3.10), it seems reasonable to expect that $I_m^{\eta,\alpha}$ will map L^p_μ into L^p_μ under suitable conditions and this is so.

Lemma 3.2

If $\mathrm{Re}\,\alpha > 0$, then $I_m^{\eta,\alpha}$ is a continuous linear mapping of L^p_μ into itself provided that $\mathrm{Re}(m\eta+\mu) + m > 1/p$.

Proof:- We merely remark that the result is established by Rooney in [72] using slightly different notation. Alternatively, we may deduce the result from Lemma 3.1 (i) by using some simple changes of variable.

It is fairly clear that $I_m^{\eta,\alpha}$ does not map L^p_μ onto L^p_μ. For instance, if $I_1^{\eta,1}f = g$ $(f \in L^p_\mu)$, then $x^{-\eta-1}I_1^1x^\eta f = g$ (a.e.) and this would require, for instance, that $x^{\eta+1}g(x)$ be differentiable almost everywhere on $(0,\infty)$. In general, a characterisation of the range of $I_m^{\eta,\alpha}$ on L^p_μ, other than some statement using fractional derivatives, is not obvious. This makes $I_m^{\eta,\alpha}$ a candidate for the treatment outlined in §1.1. It would be possible to develop the properties of all our operators relative to the L_μ^p spaces obtaining, for instance, the so-called index laws; again we should mention Rooney's papers [72], [73]. However, we will only discuss these properties in $F_{p,\mu}$ and $F'_{p,\mu}$.

With the aid of Lemma 3.2, we can obtain our first results for $I_m^{\eta,\alpha}$ in $F_{p,\mu}$.

<u>Lemma 3.3</u>

If Re $\alpha > 0$, then $I_m^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided that $\mathrm{Re}(m\eta+\mu) + m > 1/p$.

<u>Proof</u>:- First we assume that $1 \leqslant p < \infty$ and that $\mu = 0$. We may use (3.10) and differentiate under the integral sign to obtain, for $\phi \in F_{p,0}$,

$$(\delta I_m^{\eta,\alpha}\phi)(x) = \frac{m}{\Gamma(\alpha)} \int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+m-1} (\delta\phi)(xt)dt.$$

Since $\delta\phi \in F_{p,0}$ by Corollary 2.14 (i), there exists a constant C such that

$$|(\delta\phi)(xt)| \leqslant C(xt)^{-1/p} \quad (0 < x,\ t < \infty)$$

in view of Theorem 2.2. It follows easily that the integral above is uniformly convergent when x lies in a compact subset of $(0,\infty)$ so that the differentiation under the integral sign is justified. Indeed, we may repeat the process to deduce that, for $k = 0,1,2, \ldots,$

$$(\delta^k I_m^{\eta,\alpha}\phi)(x) = \frac{m}{\Gamma(\alpha)} \int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+m-1} (\delta^k\phi)(xt)dt$$

or

$$\delta^k I_m^{\eta,\alpha}\phi = I_m^{\eta,\alpha}\delta^k\phi \quad (\phi \in F_{p,0}). \tag{3.12}$$

Since $x^k d^k/dx^k$ is a polynomial of degree k in δ, we can deduce from (3.12) that, for $k = 0,1,2, \ldots,$ and $\phi \in F_{p,\mu}$,

$$x^k d^k/dx^k (I_m^{\eta,\alpha}\phi) = I_m^{\eta,\alpha}(x^k d^k \phi/dx^k). \tag{3.13}$$

$x^k d^k \phi/dx^k \in L_0^p \Rightarrow x^k d^k/dx^k (I_m^{\eta,\alpha}\phi) \in L_0^p$ $(k = 0,1,2, \ldots)$ by Lemma 3.2. Hence $I_m^{\eta,\alpha} \in F_{p,0}$. Also by (3.13) and Lemma 3.2

$$\gamma_k^{p,0}(I_m^{\eta,\alpha}\phi) \leq C_1 \gamma_k^{p,0}(\phi)$$

for some constant C_1. Thus $I_m^{\eta,\alpha}$ maps $F_{p,0}$ continuously into $F_{p,0}$.

Next we consider the case $p = \infty$, $\mu = 0$. The proof proceeds as above but in addition we must show that, if $\phi \in F_{\infty,0}$,

$$x^k d^k/dx^k (I_m^{\eta,\alpha}\phi)(x) \to 0 \text{ as } x \to 0+ \text{ and as } x \to \infty \quad (k = 0,1,2, \ldots).$$

It is sufficient to deal with $k = 0$ as the general result will then follow via (3.13). Now, for $\phi \in F_{\infty,0}$ and $\mathrm{Re}\,\alpha > 0$,

$$|I_m^{\eta,\alpha}\phi(x)| = \left|\frac{m}{\Gamma(\alpha)}\int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+m-1}\phi(xt)dt\right|$$

$$\leq \sup_{0<u<x} |\phi(u)| \frac{m}{|\Gamma(\alpha)|}\int_0^1 (1-t^m)^{\mathrm{Re}\,\alpha-1} t^{m\,\mathrm{Re}\,\eta+m-1} dt\ .$$

The integral is finite under the given conditions and since $\phi(v) \to 0$ as $v \to 0+$ we deduce that $I_m^{\eta,\alpha}\phi(x) \to 0$ as $x \to 0+$. On the other hand, let $\{x_n\}_{n=1}^{\infty}$ be any sequence of positive numbers tending to ∞ and let

$$f_n(t) = (1-t^m)^{\alpha-1} t^{m\eta+m-1}\phi(x_n t) \quad (0 < t < 1).$$

Since $\phi \in F_{\infty,0}$, $f_n(t) \to 0$ as $n \to \infty$ for each fixed t so that f_n converges pointwise to zero on (0,1). Also

$$|f_n(t)| \leqslant (1-t^m)^{\mathrm{Re}\ \alpha-1} t^{m\ \mathrm{Re}\ \eta+m-1} \gamma_0^{\infty,0}(\phi)$$

and the right-hand side is integrable over (0,1). Hence, by the Lebesgue Dominated Convergence Theorem, $\int_0^1 f_n(t)dt \to 0$ as $n \to \infty$ so that $I_m^{\eta,\alpha}\phi(x_n) \to 0$ as $n \to \infty$. Since $\{x_n\}_{n=1}^{\infty}$ was an arbitrary sequence tending to ∞, we conclude that $I_m^{\eta,\alpha}\phi(x) \to 0$ as $x \to \infty$. In view of our previous remarks, this completes the proof for $p = \infty$, $\mu = 0$.

We have now established the result for $1 \leqslant p \leqslant \infty$, $\mu = 0$. The case of a general value of μ follows from the previous cases on writing

$$I_m^{\eta,\alpha}\phi = x^{\mu} I_m^{\eta+\mu/m,\alpha} x^{-\mu}\phi \qquad (\phi \in F_{p,\mu})$$

and using Theorem 2.11. This finally completes the proof of Lemma 3.3.

We shall want to remove the restriction Re $\alpha > 0$ very soon and, for this, we need a result on analyticity of $I_m^{\eta,\alpha}$ with respect to α. It is convenient to recall the definition of the Fréchet derivative of an operator.

Definition 3.4

Let V_1, V_2 be Hausdorff topological vector spaces and suppose that, for each α in some domain D of the complex plane, T_α is a continuous linear mapping from V_1 into V_2. The Fréchet derivative $\partial T_\alpha/\partial\alpha$ of T_α is the mapping from V_1 into V_2 defined by

$$(\partial T_\alpha/\partial\alpha)\phi = \lim_{h\to 0} h^{-1}\left[T_{\alpha+h}\phi - T_\alpha\phi\right] \qquad (\phi \in V_1)$$

where the limit is taken with respect to the topology on V_2 as the (complex) increment $h \to 0$ in any manner.

Occasionally we will require some simple facts about Fréchet derivatives in topological vector spaces, all of which can be found in the article by

Nashed in [70]. Here we simply state the following result.

Lemma 3.5

(i) For fixed $m > 0$ and η such that $\mathrm{Re}(m\eta+\mu) + m > 1/p$, $I_m^{\eta,\alpha}$ as a mapping from $F_{p,\mu}$ into itself has a Fréchet derivative for all α in the half-plane $\mathrm{Re}\,\alpha > 0$.

(ii) Under the same conditions and for fixed $\phi \in F_{p,\mu}$ and fixed $x \in (0,\infty)$, $I_m^{\eta,\alpha}\phi(x)$ is an analytic function of α in the half-plane $\mathrm{Re}\,\alpha > 0$.

Proof:- We shall omit this proof, details of which can be found in [46].

Remark 3.6

As regards Fréchet derivatives in general, there will be a few occasions in the sequel when we will require this strong form of analyticity but in most cases, the weaker "pointwise" version, typified by (ii) above, will suffice.

We are now ready to relax the restriction $\mathrm{Re}\,\alpha > 0$. Obviously we will have to leave the straightforward integral version (3.10) behind. The appropriate generalisation is suggested by the following simple result.

Lemma 3.7

Let $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}(m\eta+\mu)+ m > 1/p$, $\phi \in F_{p,\mu}$. Then

$$I_m^{\eta,\alpha}\phi = (\eta+\alpha+1)I_m^{\eta,\alpha+1}\phi + m^{-1}I_m^{\eta,\alpha+1}\delta\phi. \tag{3.14}$$

Proof:- Integrating by parts we obtain

$$I_m^{\eta,\alpha+1}\delta\phi(x) = \frac{m}{\Gamma(\alpha+1)}\int_0^1 (1-t^m)^\alpha t^{m\eta+m-1} xt\phi'(xt)dt$$

$$=\left[\frac{m}{\Gamma(\alpha+1)}(1-t^m)^\alpha t^{m\eta+m}\phi(xt)\right]_0^1 - \frac{m}{\Gamma(\alpha+1)}\int_0^1 \frac{d}{dt}\left[(1-t^m)^\alpha t^{m\eta+m}\right]\phi(xt)dt.$$

The integrated terms vanish at 0 (using Theorem 2.2) and at 1 (since $\mathrm{Re}\ \alpha > 0$) and after differentiation (and a little algebra) the integral becomes

$$m\, I_m^{\eta,\alpha}\phi - m(\eta+\alpha+1)\, I_m^{\eta,\alpha+1}\phi$$

from which the result follows.

The right-hand side of (3.14) is meaningful provided only that $\mathrm{Re}\ \alpha > -1$ (and $\mathrm{Re}\ (m\eta+\mu) + m > 1/p$). We can therefore use (3.14) to extend the definition of $I_m^{\eta,\alpha}$ to all values of α in the half-plane $\mathrm{Re}\ \alpha > -1$ and, by repeated application, to all complex values of α. We can immediately state the following facts.

<u>Lemma 3.8</u>

Let $\mathrm{Re}\ (m\eta+\mu) + m > 1/p$. Then

(i) $I_m^{\eta,\alpha}$, defined by (3.10) for $\mathrm{Re}\ \alpha > 0$ and extended via (3.14), is a continuous linear mapping of $F_{p,\mu}$ into itself.

(ii) $I_m^{\eta,\alpha}$ has a Fréchet derivative with respect to α for all complex α. In particular, for each fixed $\phi \in F_{p,\mu}$ and each fixed $x \in (0,\infty)$, $I_m^{\eta,\alpha}\phi(x)$ is an entire function of α. Similarly $I_m^{\eta,\alpha}$ has a Fréchet derivative with respect to η in the half-plane $\mathrm{Re}(m\eta+\mu) + m > 1/p$.

(iii) $I_m^{\eta,0}$ is the identity operator on $F_{p,\mu}$.

<u>Proof</u>:- (i) and (ii) follow easily on using (3.14) in conjunction with Lemmas 3.3 and 3.5.

As regards (iii), if $\phi \in F_{p,\mu}$, (3.14) gives

$$I_m^{\eta,0}\phi(x) = \int_0^1 \{(m\eta+m)t^{m\eta+m-1}\phi(xt) + t^{m\eta+m-1}xt\phi'(xt)\}dt$$

$$= \left[t^{m\eta+m}\phi(xt)\right]_0^1 = \phi(x)$$

the integrated terms vanishing at 0 in view of Theorem 2.2. The result follows.

Lemma 3.8 (i) is a modest start while Lemma 3.8 (iii) is of importance for the next stage in the operation, as is the following result.

Lemma 3.9

Let $\mathrm{Re}(m\eta+\mu) + m > 1/p$, $\mathrm{Re}(m\eta+m\alpha+\mu) + m > 1/p$ and let β be any complex number. Then, for $\phi \in F_{p,\mu}$,

$$I_m^{\eta+\alpha,\beta} I_m^{\eta,\alpha}\phi = I_m^{\eta,\alpha+\beta}\phi. \tag{3.15}$$

Proof:- First we assume also that $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$. In this case (3.15) can be deduced from [32, Theorem 4] by simple changes of variable. Alternatively we may note that, in view of Corollary 2.7 and Lemma 3.8 (i), it is sufficient to establish (3.15) for $\phi \in C_0^\infty(0,\infty)$. In this case we may easily justify inverting the order of integration on the left-hand side and, on evaluating the inner integral in terms of Γ-functions, the result follows.

Next we see that if we evaluate both sides of (3.15) at a fixed point $x \in (0,\infty)$, we obtain two entire functions of β, because of Lemma 3.8 (ii). Thus the restriction $\mathrm{Re}\,\beta > 0$ may be removed by the principle of analytic continuation.

Finally we would like to remove the restriction $\mathrm{Re}\,\alpha > 0$ (while still retaining $\mathrm{Re}(m\eta+m\alpha+\mu) + m > 1/p$). This is slightly more complicated in that α appears twice on the left-hand side. However Lemma 3.8 (ii) and some routine calculations (using either Fréchet derivatives or the weaker "pointwise" version of analyticity) show that the condition $\mathrm{Re}\,\alpha > 0$ may be removed by analytic continuation. This completes the proof.

Remark 3.10

We have described the analytic continuation process at some length above. In future, however, we shall usually just mention that analytic continuation applies and omit details.

At last, we are in a position to give a reasonably complete description of our operators $I_m^{\eta,\alpha}$ as defined so far.

Theorem 3.11

(i) If $\mathrm{Re}(m\eta+\mu) + m > 1/p$, then $I_m^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself.

(ii) If also $\mathrm{Re}(m\eta+m\alpha+\mu) + m > 1/p$, then $I_m^{\eta,\alpha}$ is a homeomorphism of $F_{p,\mu}$ onto itself and

$$(I_m^{\eta,\alpha})^{-1} = I_m^{\eta+\alpha,-\alpha} \quad . \tag{3.16}$$

Proof:- In view of Lemma 3.8 (i) only the last statement requires comment. From (3.15) and Lemma 3.8 (iii), for $\phi \in F_{p,\mu}$,

$$I_m^{\eta+\alpha,-\alpha} I_m^{\eta,\alpha}\phi = I_m^{\eta,\alpha-\alpha}\phi = I_m^{\eta,0}\phi = \phi$$

$$I_m^{\eta,\alpha} I_m^{\eta+\alpha,-\alpha}\phi = I_m^{(\eta+\alpha)-\alpha,\alpha} I_m^{\eta+\alpha,-\alpha}\phi = I_m^{\eta+\alpha,0}\phi = \phi$$

from which the result follows at once.

Perhaps we should give one very simple example at this stage which ties in with earlier work.

Example 3.12

For simplicity, take $m = 1$, $\eta = -1$, $\alpha = 0$ and assume that $\mathrm{Re}\,\mu > 1/p$. Then by (3.14) and Lemma 3.8 (iii)

$$I_1^{-1,1}\delta\phi = I_1^{-1,0}\phi - (-1+0+1)I_1^{-1,1}\phi = I_1^{-1,0}\phi = \phi \tag{3.17}$$

and since, by (3.12), $\delta I_1^{-1,1}\phi = I_1^{-1,1}\delta\phi$, we deduce that δ is a homeomorphism of $F_{p,\mu}$ onto itself if Re $\mu > 1/p$ and that

$$\delta^{-1}\phi(x) = I_1^{-1,1}\phi(x) = \int_0^x t^{-1}\phi(t)dt \quad (\phi \in F_{p,\mu})$$

all of which is in accord with, for example, formula (2.11) and Corollary 2.14 (ii). Alternatively we may use (3.17) to say that $I_1^{-1,1}$ is a homeomorphism of $F_{p,\mu}$ onto itself with

$$(I_1^{-1,1})^{-1} = \delta.$$

When taken in conjunction with (3.16) we find that if Re $\mu + 1 > 1/p$,

$$I_1^{0,-1} = \delta.$$

<u>Remark 3.13</u>

This agrees with the intuitive idea that $I_m^{\eta,-n}$ should have some connection with differentiating n times with respect to x^m. This connection appears more clearly below when we discuss the "inhomogeneous" operators I_m^{α} and K_m^{α}.

The next logical step would seem to be to attempt to relax the restriction Re$(m\eta+\mu) + m > 1/p$ which appears regularly above. However, our method of tackling this requires the properties of $K_m^{\eta,\alpha}$. So, at this stage, we turn our attention to $K_m^{\eta,\alpha}$. As indicated above, we only mention the salient points and omit the details.

We start with the integral representation (3.11) and find, using results in [32] or [72], that $K_m^{\eta,\alpha}$, as so defined, gives a continuous linear mapping of $F_{p,\mu}$ into itself provided that Re$(m\eta-\mu) > -1/p$ and Re $\alpha > 0$. The restriction Re $\alpha > 0$ is removed using

$$K_m^{\eta,\alpha}\phi = (\eta+\alpha)K_m^{\eta,\alpha+1}\phi - m^{-1}K_m^{\eta,\alpha+1}\delta\phi, \tag{3.18}$$

an analogue of (3.14) valid for $\phi \in F_{p,\mu}$ and Re $(m\eta-\mu) > -1/p$; in particular, $K_m^{\eta,0}$ is the identity operator on $F_{p,\mu}$. The extended operators satisfy the relation

$$K_m^{\eta,\alpha}K_m^{\eta+\alpha,\beta}\phi = K_m^{\eta,\alpha+\beta}\phi \quad (\phi \in F_{p,\mu}) \tag{3.19}$$

provided that Re $(m\eta-\mu) > -1/p$ and Re $(m\eta+m\alpha-\mu) > -1/p$. From these facts we can obtain the following information.

Theorem 3.14

(i) If Re $(m\eta-\mu) > -1/p$, then $K_m^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself. If also Re $(m\eta+m\alpha-\mu) > -1/p$, then $K_m^{\eta,\alpha}$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu}$ and

$$(K_m^{\eta,\alpha})^{-1} = K_m^{\eta+\alpha,-\alpha}. \tag{3.20}$$

(ii) If Re $(m\eta-\mu) > -1/p$ and α is complex, then $K_m^{\eta,\alpha}$ has Fréchet derivatives with respect to η (for fixed α) and α (for fixed η).

Proof:- Omitted.

Remark 3.15

We shall see below that $I_m^{\eta,\alpha}$ and $K_m^{\eta+1-m^{-1},\alpha}$ are formal adjoints. Results such as (3.18), (3.19) and (3.20) are essentially adjoint versions of the corresponding results for $I_m^{\eta,\alpha}$. Also the restrictions on the parameters for $K_m^{\eta,\alpha}$ are obtained from those for $I_m^{\eta,\alpha}$ by replacing η by $\eta - 1 + m^{-1}$, μ by $-\mu$ and p by q; we are thus continuing the trend referred to in Remark 2.23.

We could state further properties of $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$ at this stage but in §3.4 we shall discuss these under conditions of greater generality. Here, we merely quote one which we will require shortly.

Lemma 3.16

Let Re $(m\eta+\mu) + m > 1/p$, Re $(m\xi+\mu) + m > 1/p$ and let α,β be any complex numbers. Then for $\phi \in F_{p,\mu}$,

$$I_m^{\eta,\alpha} I_m^{\xi,\beta}\phi = I_m^{\xi,\beta} I_m^{\eta,\alpha}\phi. \tag{3.21}$$

Proof:- When Re $\alpha > 0$, Re $\beta > 0$, the left-hand side, evaluated at $x \in (0,\infty)$, is

$$\frac{m}{\Gamma(\alpha)} \int_0^1 (1-t^m)^{\alpha-1} t^{m\eta+m-1} dt \; \frac{m}{\Gamma(\beta)} \int_0^1 (1-u^m)^{\beta-1} u^{m\xi+m-1} \phi(xtu)du.$$

Using Theorem 2.2 and Fubini's theorem, we may interchange the order of integration and the result follows at once in this case. The restrictions Re $\alpha > 0$ and Re $\beta > 0$ can then be removed using analytic continuation.

We shall have much more to say about the composition of Erdélyi-Kober operators in §3.4. However, it is convenient now to return to the operators I_m^α and K_m^α.

From (3.8) and (3.9), we see that for Re $\alpha > 0$ and suitable functions ϕ,

$$I_m^\alpha\phi(x) = x^{m\alpha} I_m^{0,\alpha}\phi(x); \quad K_m^\alpha\phi(x) = K_m^{0,\alpha} x^{m\alpha}\phi(x). \tag{3.22}$$

However, under appropriate conditions, the right-hand sides are meaningful for any complex number α and $\phi \in F_{p,\mu}$, using the extended definitions of $I_m^{0,\alpha}$ and $K_m^{0,\alpha}$. Thus, using Theorems 3.11 and 3.14, we can define I_m^α on $F_{p,\mu}$ for any α if Re $\mu + m > 1/p$ while K_m^α requires Re $(\mu+m\alpha) < 1/p$.

Remark 3.17

It is worth mentioning a minor technicality at this stage. (3.8) and (3.9) hold for Re $\alpha > 0$; indeed that is how we defined $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$ to start with. However, if we extend $I_m^{\eta,\alpha}$, $K_m^{\eta,\alpha}$ via (3.14) and (3.18) and extend I_m^{α}, K_m^{α} via (3.22), then (3.8) and (3.9) still hold even if Re $\alpha \leqslant 0$. To see this we note that if Re $(m\eta+\mu) + m > 1/p$, $\phi \in F_{p,\mu}$, then for any complex λ,

$$I_m^{\eta,\alpha}\phi = x^{\lambda} I_m^{\eta+\lambda/m,\alpha} x^{-\lambda}\phi \tag{3.23}$$

using (3.8) for Re $\alpha > 0$ and analytic continuation otherwise, which is valid in view of Lemma 3.8 (ii). Thus if $\phi \in F_{p,\mu}$, Re $(m\eta+\mu) + m > 1/p$ and α is arbitrary, then

$$\begin{aligned} I_m^{\eta,\alpha}\phi(x) &= x^{-m\eta} I_m^{0,\alpha} x^{m\eta}\phi(x) \\ &= x^{-m\eta-m\alpha} x^{m\alpha} I_m^{0,\alpha} x^{m\eta}\phi(x) \\ &= x^{-m\eta-m\alpha} I_m^{\alpha} x^{m\eta}\phi(x) \end{aligned}$$

as required. (3.9) is handled using

$$K_m^{\eta,\alpha}\phi = x^{-\lambda} K_m^{\eta+\lambda/m,\alpha} x^{\lambda}\phi \tag{3.24}$$

valid for $\phi \in F_{p,\mu}$, Re $(m\eta-\mu) > -1/p$ and any complex numbers α and λ.

We shall prove the standard results for I_m^{α} and merely state the analogues for K_m^{α}.

Theorem 3.18

(i) If α is any complex number and Re $\mu + m > 1/p$, then I_m^{α} is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$.

(ii) If also $\mathrm{Re}\,(\mu+m\alpha) + m > 1/p$, then I_m^{α} is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+m\alpha}$ with $(I_m^{\alpha})^{-1} = I_m^{-\alpha}$.

Proof:- (i) follows easily from (3.22), Theorem 3.11 and Theorem 2.11. The same results show that I_m^{α} is invertible under the hypotheses in (ii) and that, for $\psi \in F_{p,\mu+m\alpha}$, $(I_m^{\alpha})^{-1}\psi = I_m^{\alpha,-\alpha}x^{-m\alpha}\psi$. But from (3.22) and (3.23), $(I_m^{\alpha})^{-1}\psi = x^{-m\alpha}I_m^{0,-\alpha}\psi = I_m^{-\alpha}\psi$ and the proof is complete.

Example 3.19

Before proceeding, we examine the operators I_m^{α} ($\mathrm{Re}\,\alpha \leqslant 0$) in more detail, extending the ideas of Example 3.12.

(i) If $\mathrm{Re}\,\mu + m > 1/p$, I_m^{0} is the identity operator on $F_{p,\mu}$ in view of (3.22) and Lemma 3.8 (iii).

(ii) Next we can prove that, for any complex α, $\phi \in F_{p,\mu}$ and $\mathrm{Re}\,\mu + m > 1/p$,

$$I_m^{\alpha}\phi = D_m I_m^{\alpha+1}\phi$$

where D_m is given by (2.14). To see this, we use (3.22), (3.14) and (3.12) in that order and obtain

$$I_m^{\alpha}\phi = x^{m\alpha}I_m^{0,\alpha}\phi = (\alpha+1)x^{m\alpha}I_m^{0,\alpha+1}\phi + m^{-1}x^{m\alpha}\delta I_m^{0,\alpha+1}\phi.$$

On the other hand, noting that $D_m = m^{-1}x^{1-m}D_1 = m^{-1}x^{-m}\delta$, we have

$$D_m(I_m^{\alpha+1}\phi) = D_m(x^{m(\alpha+1)}I_m^{0,\alpha+1}\phi)$$

$$= (\alpha+1)x^{m\alpha}I_m^{0,\alpha+1}\phi + x^{m(\alpha+1)}m^{-1}x^{-m}\delta I_m^{0,\alpha+1}\phi$$

from which the result follows at once.

(iii) Still assuming $\mathrm{Re}\,\mu + m > 1/p$, it follows easily from (i) and (ii) that on $F_{p,\mu}$, $I_m^{-1} = D_m$ and hence, by induction, that

$$I_m^{-n} = D_m^n \text{ on } F_{p,\mu} \qquad (n = 0,1,2, \ldots).$$

(iv) (iii) is in agreement with Theorem 3.18 (ii) which states roughly that the operator $I_m^{-\alpha}$ should correspond to differentiating α times with respect to x^m if $\mathrm{Re}\,\alpha \geqslant 0$. (ii) and (iii) enable us to obtain an explicit expression for I_m^α if required; namely if $\phi \in F_{p,\mu}$, $\mathrm{Re}\,\mu + m > 1/p$ and n is a non-negative integer such that $\mathrm{Re}\,\alpha + n > 0$, then

$$I_m^\alpha \phi(x) = \frac{m}{\Gamma(\alpha+n)} (D_m)^n \int_0^x (x^m - u^m)^{\alpha+n-1} u^{m-1} \phi(u) du.$$

Remark 3.20

The results in Example 3.19 are not surprising. Indeed, as mentioned earlier, we could have used these properties to develop the theory of I_m^α first and then used this theory to study $I_m^{\eta,\alpha}$. However, the point is that, although some familiar results, such as those in Example 3.19, appear somewhat later than might otherwise be the case, they emerge simply and naturally from the analytic continuation process. Of course this method relies heavily on the differentiability of the functions in $F_{p,\mu}$. Consequently the approach in the case of, for instance, locally integrable functions is much more complicated. Indeed, defining such an operator as I_m^α with $\mathrm{Re}\,\alpha = 0$ for such functions is a full-scale operation; see for instance [22], [30], [31], [33], [42]. Further, the restrictions on the parameters required also emerge clearly in our development whereas a glance at [40], [41] or [43] shows how complicated things can be in studying I_m^α first.

Similar comments to the above apply to our treatment of the next results which are the so-called index laws for the operators I_m^α. These have been discussed in great detail for certain classes of classical functions by Love [43] and for certain classes of generalised functions by Erdélyi [14]. In the spaces $F_{p,\mu}$ we obtain the following.

Theorem 3.21

Let $\phi \in F_{p,\mu}$ and let α, β be any complex numbers.

(i) (First Index Law) If $1/p - m - \text{Re}\,\mu < \min(0, m\,\text{Re}\,\alpha, m\,\text{Re}\,\beta)$, then

$$I_m^{\alpha} I_m^{\beta} \phi = I_m^{\alpha+\beta} \phi = I_m^{\beta} I_m^{\alpha} \phi. \tag{3.25}$$

(ii) (Second Index Law). If $\alpha + \beta + \gamma = 0$ and $1/p - m - \text{Re}\,\mu < \min(0, m\,\text{Re}\,\gamma)$, then

$$x^{m\alpha} I_m^{\beta} x^{m\gamma} \phi = I_m^{-\gamma} x^{-m\beta} I_m^{-\alpha} \phi. \tag{3.26}$$

Proof:- (i) Using (3.22), (3.23) and (3.15) in that order, we obtain

$$I_m^{\alpha} I_m^{\beta} \phi = x^{m\alpha} I_m^{0,\alpha} x^{m\beta} I_m^{0,\beta} \phi = x^{m\alpha+m\beta} I_m^{\beta,\alpha} I_m^{0,\beta} \phi$$

$$= x^{m\alpha+m\beta} I_m^{0,\alpha+\beta} \phi = I_m^{\alpha+\beta} \phi.$$

This is valid if $1/p - m - \text{Re}\,\mu < \min(0, m\,\text{Re}\,\beta)$. The second equality holds similarly if $1/p - m - \text{Re}\,\mu < \min(0, m\,\text{Re}\,\alpha)$.

(ii) Starting with the right-hand side, we obtain

$$I_m^{-\gamma} x^{-m\beta} I_m^{-\alpha} \phi = I_m^{\alpha+\beta} x^{-m\beta} I_m^{-\alpha} \phi$$

$$= x^{m\alpha+m\beta} I_m^{0,\alpha+\beta} x^{-m\beta} I_m^{-\alpha} x^{m\alpha+m\beta} x^{-m\alpha-m\beta} \phi \qquad \text{by (3.22)}$$

$$= x^{m\alpha+m\beta} I_m^{0,\alpha+\beta} I_m^{\alpha+\beta,-\alpha} x^{-m\alpha-m\beta} \phi \qquad \text{by Remark 3.17}$$

$$= x^{m\alpha+m\beta} I_m^{\alpha+\beta,-\alpha} I_m^{0,\alpha+\beta} x^{-m\alpha-m\beta} \phi \qquad \text{by (3.21)}$$

$$= x^{m\alpha+m\beta} I_m^{0,\beta} x^{-m\alpha-m\beta}\phi \qquad \text{by (3.15)}$$

$$= x^{m\alpha} I_m^{\beta} x^{m\gamma}\phi \qquad \text{by (3.22)}$$

each step being justified under the stated conditions.

Remark 3.22

(i) In the second law, there is, of course, no need to introduce γ. However the use of γ makes the statement slightly more elegant.

(ii) An interesting point arises in connection with the conditions under which the first index law holds. The three expressions occurring are meaningful when

$$1/p - m - \mathrm{Re}\,\mu < \begin{cases} \min(0, m\,\mathrm{Re}\,\beta) & \text{for the first} \\ 0 & \text{for the second} \\ \min(0, m\,\mathrm{Re}\,\alpha) & \text{for the third.} \end{cases}$$

This suggests the possibility that the first and third expressions can be extended to a wider range of validity. This is tied up with what we might call "removable singularities", a topic we will return to later in §3.4.

We now have all the basic facts about I_m^α and can state the corresponding results for K_m^α, defined for $\mathrm{Re}\,\alpha > 0$ by (3.5) and extended via (3.22).

Theorem 3.23

(i) If $\mathrm{Re}\,(\mu+m\alpha) < 1/p$, then K_m^α is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$.

(ii) If also $\mathrm{Re}\,\mu < 1/p$, then K_m^α is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+m\alpha}$ with

$$(K_m^{\alpha})^{-1} = K_m^{-\alpha}.$$

(iii) If Re $\mu < 1/p$, then K_m^0 is the identity operator on $F_{p,\mu}$.

(iv) If n is a non-negative integer and Re $\mu - mn < 1/p$, then

$$K_m^{-n} = (-D_m)^n \quad \text{on} \quad F_{p,\mu}.$$

Proof:- Omitted.

Theorem 3.24

Let $\phi \in F_{p,\mu}$ and let α, β be any complex numbers.

(i) (First Index Law) If $1/p - \text{Re}\,\mu > \max(m\,\text{Re}\,\alpha, m\,\text{Re}\,\beta, m\,\text{Re}(\alpha+\beta))$, then

$$K_m^{\alpha} K_m^{\beta}\phi = K_m^{\alpha+\beta}\phi = K_m^{\beta}K_m^{\alpha}\phi. \tag{3.27}$$

(ii) (Second Index Law) If Re $\mu - 1/p < \min(0, m\,\text{Re}\,\gamma)$, then

$$x^{m\gamma} K_m^{\beta} x^{m\alpha}\phi = K_m^{-\alpha} x^{-m\beta} K_m^{-\gamma}\phi. \tag{3.28}$$

Proof:- Omitted.

This completes the first stage of our theory during which the integral operators have been extended to integro-differential operators by means of analytic continuation. We are now ready to start the second stage during which the operators will again change their natures.

§3.3 Fractional Calculus in $F_{p,\mu}$: Stage II

In describing the second stage of our operation, we will again concentrate on $I_m^{\eta,\alpha}$ and only mention the salient points as regards the other operators.

We have defined $I_m^{\eta,\alpha}$ on $F_{p,\mu}$ for all complex numbers α and values of η such that Re $(m\eta+\mu) + m > 1/p$. Treating p and μ as fixed, we now investigate how to relax this restriction on η. We can get no further using formulae such as (3.14) as they stand and some recasting is necessary. Our approach starts with (3.15) which shows that

$$I_m^{\eta,\alpha}\phi = I_m^{\eta+1,\alpha-1}I_m^{\eta,1}\phi$$

for $\phi \in F_{p,\mu}$ provided that Re $(m\eta+\mu) + m > 1/p$. Next, from (3.8)

$$I_m^{\eta,1}\phi = x^{-m\eta-m}I_m^1 x^{m\eta}\phi.$$

Also, from Example 3.19 (ii), $D_m I_m^1 x^{m\eta}\phi = I_m^0 x^{m\eta}\phi = x^{m\eta}\phi$ under the given conditions and similarly we can show that, for $\psi \in F_{p,m\eta+\mu+m}$, $I_m^1 D_m\psi = \psi$. It follows that, as a mapping from $F_{p,m\eta+\mu}$ into $F_{p,m\eta+\mu+m}$, I_m^1 is a homeomorphism and $I_m^1 = (D_m)^{-1}$. Finally, therefore, we may write

$$I_m^{\eta,\alpha}\phi = I_m^{\eta+1,\alpha-1}x^{-m\eta-m}(D_m)^{-1}x^{m\eta}\phi \tag{3.29}$$

for any complex α, $\phi \in F_{p,\mu}$ and Re $(m\eta+\mu) + m > 1/p$. The point of this manoeuvre is that the right-hand side is meaningful under more general conditions; namely

$$\text{Re } (m\eta+m+\mu) + m > 1/p, \quad \text{Re } (m\eta+\mu) + m \neq 1/p \tag{3.30}$$

by virtue of Theorem 3.11 and Corollary 2.15. We may therefore use (3.29) to define an operator, again denoted by $I_m^{\eta,\alpha}$, on $F_{p,\mu}$ subject only to (3.30). In view of the previous discussion, the new operator agrees with the old on spaces $F_{p,\mu}$ with Re $(m\eta+\mu) + m > 1/p$. However, if $1/p - m <$ Re $(m\eta+\mu) + m < 1/p$, $(D_m)^{-1}$ has to be replaced by $-K_m^1$ by Theorem 3.23 and we obtain

$$I_m^{\eta,\alpha}\phi = - I_m^{\eta+1,\alpha-1}K_m^{-\eta-1,1}\phi \quad (\phi \in F_{p,\mu})$$

in this case.

Remark 3.25

Before continuing this process, we are naturally led to ask what happens when Re $(m\eta+\mu) + m = 1/p$. From Corollary 2.15 (ii), we know that the range $D_m(F_{p,m\eta+\mu+m})$ of D_m on $F_{p,m\eta+\mu+m}$ is a proper subset of $F_{p,m\eta+\mu}$. Certainly (3.29) is meaningful if ϕ is such that $x^{m\eta}\phi \in D_m(F_{p,m\eta+\mu+m})$, but we are unable to define, by this method, an analogue of $I_m^{\eta,\alpha}$ as a continuous linear operator from all of $F_{p,\mu}$ into $F_{p,\mu}$ if Re $(m\eta+\mu+m) = 1/p$. As in Remark 2.16, we will eliminate this exceptional case from our enquiries.

Returning to our main theme, we have defined $I_m^{\eta,\alpha}$ on $F_{p,\mu}$ subject to (3.30). However the right-hand side of (3.30) involves $I_m^{\eta+1,\alpha-1}$ on $F_{p,\mu}$ which now has a meaning provided only that

$$\text{Re } (m(\eta+1) + m + \mu) + m > 1/p \;, \quad \text{Re } (m(\eta+1) + \mu + m) \neq 1/p$$

$$\text{or} \quad \text{Re } (m\eta + \mu + m) > 1/p - 2m, \qquad \text{Re } (m\eta + \mu + m) \neq 1/p - m. \tag{3.31}$$

We can therefore use (3.29) to define $I_m^{\eta,\alpha}$ on $F_{p,\mu}$ subject only to (3.31) and the condition Re $(m\eta+\mu+m) \neq 1/p$. For $\phi \in F_{p,\mu}$, we obtain

$$I_m^{\eta,\alpha}\phi = I_m^{\eta+2,\alpha-2}x^{-m(\eta+1)-m}(D_m)^{-1}x^{m(\eta+1)}x^{-m\eta-m}(D_m)^{-1}x^{m\eta}\phi$$

$$\text{or} \quad I_m^{\eta,\alpha}\phi = I_m^{\eta+2,\alpha-2}x^{-m(\eta+2)}(D_m)^{-2}x^{m\eta}\phi.$$

The process can be repeated indefinitely and we can therefore define $I_m^{\eta,\alpha}$ on $F_{p,\mu}$ except in certain exceptional cases where D_m fails to be invertible at some stage.

Definition 3.26

If m, p and μ are fixed, we define the set $A_{p,\mu,m}$ of complex numbers by

$$A_{p,\mu,m} = \{\eta: \ \mathrm{Re}(m\eta+\mu+m) \neq 1/p - m\ell, \quad \ell = 0,1,2, \ldots\} .$$

For fixed p,μ and m, $A_{p,\mu,m}$ consists of all complex numbers η except those lying on a family of equally spaced lines parallel to the imaginary axis. If η lies on one of these lines, say $\mathrm{Re}\ (m\eta+\mu+m) = 1/p - mk$, then our process will fail at the $(k+1)^{th}$ stage. Remark 3.25 dealt with the case $k = 0$ and in the spirit of our comments there, we shall only consider $I_m^{\eta,\alpha}$ on $F_{p,\mu}$ if $\eta \in A_{p,\mu,m}$. We can now give the formal definition.

<u>Definition 3.27</u>

For $\eta \in A_{p,\mu,m}$ and any complex number α, we define $I_m^{\eta,\alpha}$ on $F_{p,\mu}$ by

$$I_m^{\eta,\alpha}\phi(x) = I_m^{\eta+k,\alpha-k} x^{-m(\eta+k)} (D_m)^{-k} x^{m\eta}\phi \qquad (\phi \in F_{p,\mu}) \tag{3.32}$$

where k is any non-negative integer such that $\mathrm{Re}\ (m\eta+\mu) + m > 1/p - mk$ and $I_m^{\eta+k,\alpha-k}$ is defined via (3.10) and (3.14).

<u>Notes</u>

1. Since $\mathrm{Re}\ (m(\eta+k) + \mu) + m > 1/p$, $I_m^{\eta+k,\alpha-k}$ can, indeed, be defined via (3.10) and (3.14) by virtue of Lemma 3.8 (i).
2. The definition is independent of the choice of the non-negative integer k satisfying $\mathrm{Re}\ (m\eta+\mu) + m > 1/p - mk$. To see this, suppose also that $\mathrm{Re}\ (m\eta+\mu) + m > 1/p - m\ell$ and that, without loss of generality, $\ell > k$. Then for $\phi \in F_{p,\mu}$,

$$I_m^{\eta+k,\alpha-k} x^{-m(\eta+k)} (D_m)^{-k} x^{m\eta}\phi$$

$$= I_m^{\eta+\ell,\alpha-\ell} I_m^{\eta+k,\ell-k} x^{-m(\eta+k)} (D_m)^{-k} x^{m\eta}\phi \qquad \text{by (3.15)}$$

$$= I_m^{\eta+\ell,\alpha-\ell} x^{-m(\eta+\ell)} I_m^{\ell-k} (D_m)^{\ell-k} (D_m)^{-\ell} x^{m\eta}\phi \qquad \text{by Remark 3.17}$$

$$= I_m^{\eta+\ell,\alpha-\ell} x^{-m(\eta+\ell)} (D_m)^{-\ell} x^{m\eta} \phi \qquad \text{from Example 3.19}$$

and the result follows.

3. From Note 2, it follows that $I_m^{\eta,\alpha}$ agrees with the previous definition for spaces $F_{p,\mu}$ with Re $(m\eta+\mu) + m > 1/p$ since in this case we can take $k = 0$.

4. If $\eta \in A_{p,\mu,m}$ satisfies Re $(m\eta+\mu) + m < 1/p$ and if k is the unique positive integer such that

$$1/p - mk < \text{Re } (m\eta+\mu) + m < 1/p - m(k-1),$$

then Theorem 3.23 shows that (3.32) may be written in the form

$$I_m^{\eta,\alpha}\phi(x) = (-1)^k I_m^{\eta+k,\alpha-k} K_m^{-\eta-k,k}\phi \qquad (\phi \in F_{p,\mu}) \tag{3.33}$$

where, since Re $(m(-\eta-k) - \mu) > -1/p$, $K_m^{-\eta-k,k}$ is defined via (3.11) and (3.18). (3.33) is, for many purposes, the easiest form of the definition to handle.

<u>Remark 3.28</u>

(3.33) shows clearly how the nature of $I_m^{\eta,\alpha}$ has now changed. Roughly speaking, we now have to integrate k times from x to ∞ and "integrate" $(\alpha-k)$ times from 0 to x instead of "integrating" α times over $(0,x)$. Such operators have been considered before, for instance, by Erdélyi [8] in his work on cut Hankel transforms. To relate our work to his, we have to put $\mu = 0$ and $m = 1$ and then we see that the exceptional strips in [8], namely Re $\eta = -1/q - \ell$ emerge quite naturally as the complement of $A_{p,0,1}$.

(3.33) would seem to indicate that, under the conditions stated, our I operator begins to look more and more like a K operator. Indeed, we have the following curious particular case.

Example 3.29

If $1/p - mk < \mathrm{Re}\,(m\eta+\mu) + m < 1/p - m(k-1)$, then for $\phi \in F_{p,\mu}$,

$$I_m^{\eta,k}\phi = (-1)^k K_m^{-\eta-k,k}\phi \ . \tag{3.34}$$

On the left $I_m^{\eta,k}$ is defined via (3.32) and on the right $K_m^{-\eta-k,k}$ is defined by (3.11). This is, of course, a special case and in spite of it we will continue to use the notation $I_m^{\eta,\alpha}$ for the operator defined by (3.32), remembering that occasionally it is a K operator ! This is not unrelated to Remark 2.16.

As regards the general properties of our extended operator, continuity presents no problems as we shall state in a moment. However, the validity of (3.16) under more general conditions is a little more awkward. Although $I_m^{\eta,\alpha}$ is analytic on $A_{p,\mu,m}$, as we shall also see in a moment, we cannot use the principle of analytic continuation since $A_{p,\mu,m}$ is not simply connected. Hence an alternative approach is needed. It seems that (3.33) is not much help because of the fact that the value of k such that

$$1/p - mk < \mathrm{Re}\,(m\eta+\mu) + m < 1/p - m(k-1)$$

does not ensure that

$$1/p - mk < \mathrm{Re}\,(m\eta+m\alpha+\mu) + m < 1/p - m(k-1).$$

We therefore fall back on (3.32) and will use the following result.

Lemma 3.30

If $\eta \in A_{p,\mu,m}$ and k is any non-negative integer, then, for $\phi \in F_{p,\mu}$,

$$I_m^{\eta,\alpha}\phi(x) = x^{-m(\eta+\alpha)}(D_m)^k x^{m(\eta+\alpha+k)} I_m^{\eta,\alpha+k}\phi(x) \quad . \tag{3.35}$$

<u>Proof</u>:- First suppose that Re $(m\eta+\mu) + m > 1/p$. If we put $k = 1$ in the expression on the right and use Example 3.19 (ii), we get

$$x^{-m(\eta+\alpha)}(D_m) I_m^{\alpha+1} x^{m\eta}\phi = x^{-m(\eta+\alpha)} I_m^{\alpha} x^{m\eta}\phi$$

which is $I_m^{\eta,\alpha}\phi$ as required. A simple induction argument now establishes (3.35) in this case.

Now let Re $(m\eta+\mu) + m < 1/p$ and let ℓ be any integer such that Re $(m\eta+\mu) + m > 1/p - m\ell$. Let $\phi \varepsilon F_{p,\mu}$ and write

$$\psi = x^{-m(\eta+\ell)}(D_m)^{-\ell} x^{m\eta}\phi.$$

Then $\psi \ \varepsilon \ F_{p,\mu}$ and by (3.32),

$$I_m^{\eta,\alpha}\phi = I_m^{\eta+\ell,\alpha-\ell}\psi. \tag{3.36}$$

By the previous case with η,α and ϕ replaced by $\eta+\ell$, $\alpha-\ell$ and ψ,

$$I_m^{\eta,\alpha}\phi = I_m^{\eta+\ell,\alpha-\ell}\psi$$

$$= x^{-m(\eta+\alpha)}(D_m)^k x^{m(\eta+\alpha+k)} I_m^{\eta+\ell,\alpha+k-\ell}\psi$$

$$= x^{-m(\eta+\alpha)}(D_m)^k x^{m(\eta+\alpha+k)} I_m^{\eta,\alpha+k}\phi$$

where in the last line we have used (3.36) with α replaced by $\alpha+k$. This completes the proof.

Now we can state the final version of our results for $I_m^{\eta,\alpha}$ on $F_{p,\mu}$.

Theorem 3.31

(i) If $\eta \in A_{p,\mu,m}$, then $I_m^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself.

If also $\eta + \alpha \in A_{p,\mu,m}$, then $I_m^{\eta,\alpha}$ is a homeomorphism of $F_{p,\mu}$ onto itself and

$$(I_m^{\eta,\alpha})^{-1} = I_m^{\eta+\alpha,-\alpha}.$$

(ii) If p, μ and m are fixed, then $I_m^{\eta,\alpha}$ has Fréchet derivatives with respect to η on $A_{p,\mu,m}$ (for fixed α) and with respect to α (for fixed $\eta \in A_{p,\mu,m}$).

Proof:- (i) The continuity of $I_m^{\eta,\alpha}$ follows from (3.33), Theorem 3.11 and Theorem 3.14. As regards invertibility, suppose also that $\text{Re}\,(m\eta+m\alpha+\mu) + m > 1/p$. Then Theorem 3.11 and (3.32) show that $(I_m^{\eta,\alpha})^{-1}$ exists and that

$$(I_m^{\eta,\alpha})^{-1}\phi = x^{-m\eta}(D_m)^k x^{m(\eta+k)} I_m^{\eta+\alpha,-\alpha+k}\phi \qquad (\phi \in F_{p,\mu})$$

where $\text{Re}(m\eta+\mu) + m > 1/p - mk$. The latter expression is just $I_m^{\eta+\alpha,-\alpha}\phi$ by (3.35) with η,α replaced by $\eta+\alpha$, $-\alpha$ respectively. This completes the proof in this case. Finally, if $\text{Re}\,(m\eta+m\alpha+\mu) + m < 1/p$, choose ℓ such that $\text{Re}\,(m\eta+m\alpha+\mu) + m > 1/p - m\ell$. Then, using (3.35) with k replaced by ℓ together with the previous case, we find that $(I_m^{\eta,\alpha})^{-1}$ exists and is given by

$$(I_m^{\eta,\alpha})^{-1}\phi = I_m^{\eta+\alpha+\ell,-\alpha-\ell} x^{-m(\eta+\alpha+\ell)}(D_m)^{-\ell} x^{m(\eta+\alpha)}\phi \quad (\phi \in F_{p,\mu})$$

and the latter expression is just $I_m^{\eta+\alpha,-\alpha}\phi$ in view of (3.32).

(ii) follows easily from (3.33), Theorem 3.11 and Theorem 3.14 and this completes the proof.

We now deal briefly with the second extension of $K_m^{\eta,\alpha}$. Starting with the definition valid for Re $(m\eta-\mu) > -1/p$, and obtained from (3.11) and (3.18), we can show that

$$K_m^{\eta,\alpha}\phi = -x^{m\eta}(D_m)^{-1}x^{-m(\eta+1)}K_m^{\eta+1,\alpha-1}\phi$$

which is essentially an adjoint version of (3.29). The right-hand side is meaningful under the weaker conditions

$$\text{Re }(m\eta-\mu) > -1/p - m, \qquad \text{Re }(m\eta-\mu) \neq -1/p.$$

Repeating the process step-by-step leads to the following definitions.

Definition 3.32

If m, p and μ are fixed, we define the set $A'_{p,\mu,m}$ of complex numbers by

$$A'_{p,\mu,m} = \{\eta: \text{ Re }(m\eta-\mu) \neq -1/p - m\ell, \quad \ell = 0,1,2, \ldots\}.$$

We note in passing that

$$\eta \in A_{p,\mu,m} \text{ if and only if } \eta + 1 - 1/m \in A'_{q,-\mu,m} \tag{3.37}$$

a fact which is related to Remark 2.23 as we shall see in §3.5.

Definition 3.33

For $\eta \in A'_{p,\mu,m}$ and any complex number α, we define $K_m^{\eta,\alpha}$ on $F_{p,\mu}$ by

$$K_m^{\eta,\alpha}\phi(x) = (-1)^k x^{m\eta}(D_m)^{-k}x^{-m(\eta+k)}K_m^{\eta+k,\alpha-k}\phi \tag{3.38}$$

where k is a non-negative integer such that Re $(m\eta-\mu) > -1/p - mk$ and $K_m^{\eta+k,\alpha-k}$ is defined via (3.11) and (3.18).

Note

Comments analogous to Notes 1-4 following Definition 3.27 are in order here.

In particular

1. The definition is independent of the choice of non-negative integer k satisfying Re $(m\eta-\mu) > -1/p - mk$.

2. The extended definition agrees with the previous definition when Re $(m\eta-\mu) > -1/p$.

3. If Re $(m\eta-\mu) < -1/p$, and k is the unique positive integer such that

$$-1/p - mk < \text{Re}\,(m\eta-\mu) < -1/p - m(k-1)\,,$$

then we obtain

$$K_m^{\eta,\alpha}\phi = (-1)^k I_m^{-\eta-k,k} K_m^{\eta+k,\alpha-k}\phi \qquad (\phi \in F_{p,\mu}).$$

The final version of our results for $K_m^{\eta,\alpha}$ on $F_{p,\mu}$ is as follows.

Theorem 3.34

(i) If $\eta \in A'_{p,\mu,m}$, then $K_m^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself. If also $\eta + \alpha \in A'_{p,\mu,m}$, then $K_m^{\eta,\alpha}$ is a homeomorphism of $F_{p,\mu}$ onto itself and

$$(K_m^{\eta,\alpha})^{-1} = K_m^{\eta+\alpha,-\alpha}.$$

(ii) If p,μ and m are fixed, then $K_m^{\eta,\alpha}$ has Fréchet derivatives with respect to η on $A'_{p,\mu,m}$ (for fixed α) and with respect to α (for fixed $\eta \in A'_{p,\mu,m}$).

Proof:- This is similar to that of Theorem 3.31 and is omitted.

To complete the extension process for I_m^α and K_m^α, we fall back again on (3.22).

Definition 3.35

(i) If $0 \in A_{p,\mu,m}$, I_m^{α} is defined on $F_{p,\mu}$ by

$$I_m^{\alpha}\phi = x^{m\alpha}I_m^{0,\alpha}\phi \qquad (\phi \in F_{p,\mu})$$

where $I_m^{0,\alpha}$ is as in Definition 3.27.

(ii) If $-\alpha \in A'_{p,\mu,m}$, K_m^{α} is defined on $F_{p,\mu}$ by

$$K_m^{\alpha}\phi = K_m^{0,\alpha}x^{m\alpha}\phi \qquad (\phi \in F_{p,\mu})$$

where $K_m^{0,\alpha}$ is as in Definition 3.33.

The following results are almost immediate.

Theorem 3.36

If $0 \in A_{p,\mu,m}$, then I_m^{α} is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$. If also $\alpha \in A_{p,\mu,m}$, then I_m^{α} is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+m\alpha}$ and

$$(I_m^{\alpha})^{-1} = I_m^{-\alpha}.$$

Proof:- The first statement follows from Theorems 3.31 (i) and 2.11.

For the second statement, we note that $\alpha \in A_{p,\mu,m}$ if and only if $0 \in A_{p,\mu+m\alpha,m}$ and in this case, Theorem 3.31 shows that I_m^{α} is invertible and that

$$(I_m^{\alpha})^{-1}\psi = I_m^{\alpha,-\alpha}x^{-m\alpha}\psi \qquad (\psi \in F_{p,\mu+m\alpha}).$$

Using (3.23), (3.24) and (3.33) we then deduce easily that

$$(I_m^{\alpha})^{-1}\psi = x^{-m\alpha}I_m^{0,-\alpha}\psi = I_m^{-\alpha}\psi$$

as required and this completes the proof.

Theorem 3.37

If $-\alpha \in A'_{p,\mu,m}$, then K_m^α is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$. If also $0 \in A'_{p,\mu,m}$, then K_m^α is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+m\alpha}$ and

$$(K_m^\alpha)^{-1} = K_m^{-\alpha}.$$

Proof:- This is similar to that of Theorem 3.36 and is omitted.

§3.4 Some Consequences

Now that we have completed the extension process on $F_{p,\mu}$, we can pause and look at a few simple consequences.

Results proved in §3.2 remain true when the operators are interpreted in terms of their extensions as in §3.3. We give a few simple examples.

Example 3.38

(i) If $\eta \in A_{p,\mu,m}$, then $I_m^{\eta,0}$ is the identity mapping on $F_{p,\mu}$. Indeed, choosing k as in Definition 3.27 we obtain, for $\phi \in F_{p,\mu}$,

$$I_m^{\eta,0}\phi = I_m^{\eta+k,-k} x^{-m(\eta+k)} (D_m)^{-k} x^{m\eta}\phi$$

$$= x^{-m\eta} I_m^{-k} (D_m)^{-k} x^{m\eta}\phi \qquad \text{by Remark 3.17}$$

$$= x^{-m\eta} (D_m)^{k} (D_m)^{-k} x^{m\eta}\phi \qquad \text{by Example 3.19 (iii)}$$

and the result follows.

(ii) Similarly if $\eta \in A'_{p,\mu,m}$, $K_m^{\eta,0}$ is the identity mapping on $F_{p,\mu}$.

Example 3.39

(i) (3.15) holds if $\eta \in A_{p,\mu,m}$ and $\eta + \alpha \in A_{p,\mu,m}$. To see this, we choose a non-negative integer k such that Re $(m\eta+\mu) + m > 1/p - mk$ and Re $(m\eta+m\alpha+\mu) + m > 1/p - mk$. Then, by (3.32) and (3.35), for

$\phi \in F_{p,\mu}$,

$$I_m^{\eta+\alpha,\beta} I_m^{\eta,\alpha} \phi$$

$$= I_m^{\eta+\alpha+k,\beta-k} x^{-m(\eta+\alpha+k)} (D_m)^{-k} x^{m(\eta+\alpha)} x^{-m(\eta+\alpha)} (D_m)^k x^{m(\eta+\alpha+k)} I_m^{\eta,\alpha+k} \phi$$

$$= I_m^{\eta+\alpha+k,\beta-k} I_m^{\eta+k,\alpha} x^{-m(\eta+k)} (D_m)^{-k} x^{m\eta} \phi$$

$$= I_m^{\eta+k,\alpha+\beta-k} x^{-m(\eta+k)} (D_m)^{-k} x^{m\eta} \phi \qquad \text{by Lemma 3.9}$$

$$= I_m^{\eta,\alpha+\beta} \phi \qquad \text{by (3.32).}$$

Similarly (3.19) holds provided $\eta \in A'_{p,\mu,m}$ and $\eta + \alpha \in A'_{p,\mu,m}$.

(ii) An interesting point arises at this stage and again we will concentrate on (3.15). The left-hand side of (3.15) is well-defined if $\eta \in A_{p,\mu,m}$ and $\eta + \alpha \in A_{p,\mu,m}$, whereas the right-hand side only requires $\eta \in A_{p,\mu,m}$. This seems to indicate that the restriction $\eta + \alpha \in A_{p,\mu,m}$ is, in some sense, removable here. In order to investigate we shall temporarily write

$$T_m^{\eta,\alpha,\beta} \phi = I_m^{\eta+\alpha,\beta} I_m^{\eta,\alpha} \phi \quad (\phi \in F_{p,\mu}) \ . \tag{3.39}$$

Let $\eta \in A_{p,\mu,m}$ and let k be a non-negative integer such that $\mathrm{Re}\ (m\eta+m\alpha+\mu) + m > 1/p - mk$. Then the calculation in (i) shows that

$$T_m^{\eta,\alpha,\beta} \phi = I_m^{\eta+\alpha+k,\beta-k} I_m^{\eta,\alpha+k} \phi = T_m^{\eta,\alpha+k,\beta-k} \phi. \tag{3.40}$$

Since $(\eta+\alpha+k) \in A_{p,\mu,m}$, the last expression defines a continuous linear mapping of $F_{p,\mu}$ into itself. Since an appropriate k can always be found, we may use (3.40) to extend the definition of $T_m^{\eta,\alpha,\beta}$ to values of η, α with $\eta \in A_{p,\mu,m}$, $\eta + \alpha \notin A_{p,\mu,m}$. As usual, this extension is independent of the non-negative integer k satisfying

$\eta + \alpha + k \in A_{p,\mu,m}$. Furthermore, by Theorem 3.31 (ii), $T_m^{\eta,\alpha,\beta}\phi$ is analytic in η on $A_{p,\mu,m}$ for fixed $\phi \in F_{p,\mu}$ so that we can regard (3.40) as providing an analytic continuation of $T_m^{\eta,\alpha,\beta}$ to the whole of $A_{p,\mu,m}$ and the singularities corresponding to $\eta + \alpha \notin A_{p,\mu,m}$ are thereby removed. In exactly the same way we find that (3.19) is valid for $\eta \in A'_{p,\mu,m}$ provided that, if $\eta + \alpha \notin A'_{p,\mu,m}$, the left-hand side is interpreted in terms of the appropriate analytic continuation.

Remark 3.40

The phenomenon of "removable singularities" mentioned above will turn up again later, for instance in §6.2 in connection with the Hankel transform.

Example 3.39 deals with a special case of the composition of two of our operators and in Chapter 4, we shall examine in some detail operators arising from a more general composition. However, it is convenient to state here the following theorem on commutativity.

Theorem 3.41

Let $\phi \in F_{p,\mu}$, let α, β be complex numbers and let m, n be positive real numbers. Then

(i) $I_m^{\eta,\alpha} I_n^{\xi,\beta}\phi = I_n^{\xi,\beta} I_m^{\eta,\alpha}\phi$ if $\eta \in A_{p,\mu,m}$, $\xi \in A_{p,\mu,n}$

(ii) $I_m^{\eta,\alpha} K_n^{\xi,\beta}\phi = K_n^{\xi,\beta} I_m^{\eta,\alpha}\phi$ if $\eta \in A_{p,\mu,m}$, $\xi \in A'_{p,\mu,n}$

(iii) $K_m^{\eta,\alpha} K_n^{\xi,\beta}\phi = K_n^{\xi,\beta} K_m^{\eta,\alpha}\phi$ if $\eta \in A'_{p,\mu,m}$, $\xi \in A'_{p,\mu,n}$.

Proof:- All are similar and, since we have already seen a simple version of (i) in Lemma 3.16, we shall consider (ii) for variety.

First suppose that $\mathrm{Re}\,(m\eta+\mu) + m > 1/p$, $\mathrm{Re}\,(n\xi-\mu) > -1/p$. If also $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$, both sides are given by absolutely convergent repeated integrals via (3.10) and (3.11) and hence the result follows easily from

Fubini's theorem in this case. The restrictions Re $\alpha > 0$, Re $\beta > 0$ can then be removed using analytic continuation.

Next suppose $\eta \in A_{p,\mu,m}$ with Re $(m\eta+\mu) + m < 1/p$ and let k be the unique positive integer such that

$$1/p - mk < \text{Re } (m\eta+\mu) + m < 1/p - m(k-1).$$

Then, still assuming that Re $(n\xi-\mu) > -1/p$,

$$I_m^{\eta,\alpha}K_n^{\xi,\beta}\phi = (-1)^k I_m^{\eta+k,\alpha-k}K_m^{-\eta-k,k}K_n^{\xi,\beta}\phi \qquad \text{by (3.33).}$$

The two K operators commute since, if Re $\beta > 0$, we have an absolutely convergent repeated integral and the case Re $\beta \leq 0$ follows by analytic continuation. Thus

$$I_m^{\eta,\alpha}K_n^{\xi,\beta}\phi = (-1)^k I_m^{\eta+k,\alpha-k}K_n^{\xi,\beta}K_m^{-\eta-k,k}\phi$$

$$= K_n^{\xi,\beta}(-1)^k I_m^{\eta+k,\alpha-k}K_m^{-\eta-k,k}\phi \qquad \text{by the previous case}$$

$$= K_n^{\xi,\beta}I_m^{\eta,\alpha}\phi \qquad \text{by (3.33).}$$

The condition Re $(n\xi-\mu) > -1/p$ can be relaxed to $\xi \in A'_{p,\mu,n}$ similarly and this completes the proof.

<u>Remark 3.42</u>

We have given more details of the last proof than necessary because its anatomy is typical of many which follow. Namely, for results for $I_m^{\eta,\alpha}$,

(i) we deal with the integral version for Re $(m\eta+\mu) + m > 1/p$, Re $\alpha > 0$

(ii) then remove Re $\alpha > 0$ using analytic continuation

(iii) finally relax Re $(m\eta+\mu) + m > 1/p$ to $\eta \in A_{p,\mu,m}$ using, for instance, (3.32) and the previous cases.

Similar comments apply to $K_m^{\eta,\alpha}$ and in future we shall not give such full details.

To complete our present discussions in $F_{p,\mu}$, we use the last few results to obtain the final versions of the index laws for I_m^α and K_m^α.

Theorem 3.43

Let $\phi \in F_{p,\mu}$ and let $\alpha + \beta + \gamma = 0$.

(i) If $0 \in A_{p,\mu,m}$, then

$$I_m^\alpha I_m^\beta \phi = I_m^{\alpha+\beta}\phi = I_m^\beta I_m^\alpha \phi$$

where if $\alpha \notin A_{p,\mu,m}$ or $\beta \notin A_{p,\mu,m}$, the third or first expression respectively is to be interpreted in terms of its analytic continuation via (3.40).

(ii) If $\gamma \in A_{p,\mu,m}$, then

$$x^{m\alpha} I_m^\beta x^{m\gamma}\phi = I_m^{-\gamma} x^{-m\beta} I_m^{-\alpha}\phi$$

where if $0 \notin A_{p,\mu,m}$, the right-hand side is to be interpreted in terms of its analytic continuation.

Proof:- The proof of Theorem 3.21 goes over by virtue of Example 3.39 (i) and Theorem 3.41 (i) and the statements concerning analytic continuation emerge from Example 3.39 (ii). We omit the details.

Theorem 3.44

Let $\phi \in F_{p,\mu}$, $\alpha + \beta + \gamma = 0$, $\gamma \in A'_{p,\mu,m}$. Then

(i) $K_m^\alpha K_m^\beta \phi = K_m^{\alpha+\beta}\phi = K_m^\beta K_m^\alpha \phi$

where, if $-\alpha \notin A'_{p,\mu,m}$ or $-\beta \notin A'_{p,\mu,m}$, the third or first expression respectively is to be interpreted in terms of its analytic continuation.

(ii) $x^{m\gamma}K_m^{\beta}x^{m\alpha}\phi = K_m^{-\alpha}x^{-m\beta}K_m^{-\gamma}\phi$

where if $0 \notin A'_{p,\mu,m}$, the expression on the right is interpreted in terms of its analytic continuation.

Proof:- This is omitted.

§3.5 Definition of the Operators in $F'_{p,\mu}$

At last, we are ready to extend our operators to $F'_{p,\mu}$ in the manner suggested in Chapter 1. Fortunately this is relatively easy after all the work of the previous sections and boils down to properties of adjoint operators in countably multinormed spaces as in §2.4.

Again we will consider first $I_m^{\eta,\alpha}$ on $F'_{p,\mu}$ and go right back to the simplest case where $I_m^{\eta,\alpha}$ is defined via the integral (3.10). To follow the motivation in §1.1, let $g \in L^q_{-\mu}$ and let $\mathrm{Re}\,\alpha > 0$. Then according to Lemma 3.2, $I_m^{\eta,\alpha}g$ exists a.e. on $(0,\infty)$ and is an element of $L^q_{-\mu}$ provided that $\mathrm{Re}\,(m\eta-\mu) + m > 1/q$. In this case g and $I_m^{\eta,\alpha}g$ generate elements of $F'_{p,\mu}$ and, in the previous notation, we require that

$$\tau I_m^{\eta,\alpha}g = I_m^{\eta,\alpha}\tau g \tag{3.41}$$

where the operator $I_m^{\eta,\alpha}$ on the right is the desired extended operator. To see what this entails, let $\phi \in F_{p,\mu}$. Then, from (2.16),

$$(I_m^{\eta,\alpha}\tau g,\phi) = (\tau I_m^{\eta,\alpha}g,\phi) = \int_0^\infty I_m^{\eta,\alpha}g(x)\phi(x)dx.$$

The right-hand side can be written as a repeated integral to which Fubini's theorem can be applied and we find that

$$\int_0^\infty I_m^{\eta,\alpha}g(x)\phi(x)dx = \int_0^\infty g(x)K_m^{\eta+1-1/m,\alpha}\phi(x)dx.$$

Alternatively we can regard this as "fractional integration by parts"; the case $m = 1$, $\mu = 0$ is contained in the work of Love and Young [44] and the general case follows by simple changes of variable. We are therefore led to the equation

$$(I_m^{\eta,\alpha}\tau g,\phi) = (\tau g, K_m^{\eta+1-1/m,\alpha}\phi) \tag{3.42}$$

valid for $\phi \in F_{p,\mu}$, $\mathrm{Re}(m\eta-\mu) + m > 1/q$, $\mathrm{Re}\,\alpha > 0$. This in turn suggests that we should define $I_m^{\eta,\alpha}$ on $F'_{p,\mu}$, under these conditions, by

$$(I_m^{\eta,\alpha}f,\phi) = (f, K_m^{\eta+1-1/m,\alpha}\phi)$$

for all $\phi \in F_{p,\mu}$ and for all $f \in F'_{p,\mu}$, regular or not. In view of Theorem 3.14, the right-hand side is well-defined under the given conditions since $\mathrm{Re}[m(\eta+1-1/m)-\mu] > -1/p$. However, using our extended definition of $K_m^{\eta+1-1/m,\alpha}$ on $F_{p,\mu}$, we can remove the restriction $\mathrm{Re}\,\alpha > 0$ and replace $\mathrm{Re}(m\eta-\mu) + m > 1/q$ by $\eta + 1 - 1/m \in A'_{p,\mu,m}$, the latter being equivalent to $\eta \in A_{q,-\mu,m}$ in view of (3.37). Hence we are led to the following definition.

Definition 3.45

For $\eta \in A_{q,-\mu,m}$ and any complex number α, we define $I_m^{\eta,\alpha}$ on $F'_{p,\mu}$ by

$$(I_m^{\eta,\alpha}f,\phi) = (f, K_m^{\eta+1-1/m,\alpha}\phi) \tag{3.43}$$

where $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu}$ and $K_m^{\eta+1-1/m,\alpha}$ is defined via (3.38).

Remark 3.46

In arriving at our definition we ensured that (3.41) held for appropriate classical functions when $\mathrm{Re}\,(m\eta-\mu) + m > 1/q$, $\mathrm{Re}\,\alpha > 0$. In fact, (3.41) will still hold for functions $g \in L^q_{-\mu}$ such that $I_m^{\eta,\alpha}g \in L^q_{-\mu}$ even if these conditions do not necessarily hold. In this case the operator $I_m^{\eta,\alpha}$ has to be interpreted in an appropriate extended sense incorporating, for instance, any differentiability properties of f and these extensions follow the same trend as those in the previous sections; see, for instance, [8]. Furthermore, our results are in accord with the theory of the extendability of the operators as discussed by Rooney in [73].

The properties of $I_m^{\eta,\alpha}$ on $F'_{p,\mu}$ are easily obtained from those of $K_m^{\eta+1-1/m,\alpha}$ using the theory of adjoints mentioned above.

Theorem 3.47

(i) If $\eta \in A_{q,-\mu,m}$, then $I_m^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself.

If also $\eta + \alpha \in A_{q,-\mu,m}$, then $I_m^{\eta,\alpha}$ is a homeomorphism of $F'_{p,\mu}$ onto itself and

$$(I_m^{\eta,\alpha})^{-1} = I_m^{\eta+\alpha,-\alpha}.$$

(ii) For fixed $f \in F'_{p,\mu}$ and fixed $\phi \in F_{p,\mu}$, $(I_m^{\eta,\alpha}f,\phi)$ is an analytic function of η on $A_{q,-\mu,m}$ (for fixed α) and an entire function of α (for fixed $\eta \in A_{q,-\mu,m}$).

Proof:- (i) follows on using Theorem 3.34 (i) (with η replaced by $\eta + 1 - 1/m$) in conjunction with Theorems 1.10-1 and 1.10-2 in [87]. (ii) follows from Theorem 3.34 (ii) along with the continuity of f.

Remark 3.48

Note once again that the conditions in Theorem 3.47 are derived from those in Theorem 3.31 by interchanging p and q, μ and $-\mu$ (see Remark 2.23).

Similar motivation leads to the following definition.

Definition 3.49

For $\eta \in A'_{q,-\mu,m}$ and any complex number α, we define $K_m^{\eta,\alpha}$ on $F'_{p,\mu}$ by

$$(K_m^{\eta,\alpha}f,\phi) = (f, I_m^{\eta-1+1/m,\alpha}\phi) \qquad (3.44)$$

where $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu}$ and $I_m^{\eta-1+1/m,\alpha}$ is defined via (3.32).

Theorem 3.50

(i) If $\eta \in A'_{q,-\mu,m}$, then $K_m^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself.

If also $\eta + \alpha \in A'_{q,-\mu,m}$, then $K_m^{\eta,\alpha}$ is a homeomorphism of $F'_{p,\mu}$ onto itself and

$$(K_m^{\eta,\alpha})^{-1} = K_m^{\eta+\alpha,-\alpha}.$$

(ii) For fixed $f \in F'_{p,\mu}$ and fixed $\phi \in F_{p,\mu}$, $(K_m^{\eta,\alpha}f,\phi)$ is an analytic function of η on $A'_{q,-\mu,m}$ (for each fixed α) and an entire function of α (for fixed $\eta \in A'_{q,-\mu,m}$).

Proof:- By (3.37) $\eta \in A'_{q,-\mu,m}$ iff $\eta - 1 + 1/m \in A_{p,\mu,m}$. With this comment, the proof is analogous to that of Theorem 3.47.

Finally we consider I_m^α, K_m^α on $F'_{p,\mu}$. Again we can go through a preamble similar to the above starting with fractional integration by parts. In this case we find that, for $g \in L^q_{-\mu}$, $\phi \in F_{p,\mu-m\alpha}$ and Re $\alpha > 0$,

$$\int_0^\infty I_m^\alpha g(x)\phi(x)dx = \int_0^\infty g(x)x^{m-1}K_m^\alpha x^{-m+1}\phi(x)dx$$

provided in the first instance that Re $\mu - m < -1/q$. This provides the motivation for the following definition.

Definition 3.51

If $0 \in A_{q,-\mu,m}$ and if α is any complex number, we define I_m^α on $F'_{p,\mu}$ by

$$(I_m^\alpha f,\phi) = (f,x^{m-1}K_m^\alpha x^{-m+1}\phi) \tag{3.45}$$

where $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu-m\alpha}$ and K_m^α is as in Definition 3.35 (ii).

Theorem 3.52

If $0 \in A_{q,-\mu,m}$, then I_m^α is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$.

If also $\alpha \in A_{q,-\mu,m}$, then I_m^α is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and

$$(I_m^\alpha)^{-1} = I_m^{-\alpha}.$$

Proof:- $0 \in A_{q,-\mu,m} \Rightarrow 1 - 1/m \in A'_{p,\mu,m} \Rightarrow -\alpha \in A'_{p,\mu-m\alpha-m+1}$ by (3.37). Hence by Theorems 2.11 and 3.37, $x^{m-1}K_m^\alpha x^{-m+1}$ is a continuous linear mapping of $F_{p,\mu-m\alpha}$ into $F_{p,\mu}$ so that (3.45) is meaningful. The results now follow from Theorem 3.37 above and Theorems 1.10-1 and 1.10-2 in [87].

Likewise we have the following.

Definition 3.53

If $-\alpha \in A'_{q,-\mu,m}$, we define K_m^α on $F'_{p,\mu}$ by

$$(K_m^\alpha f,\phi) = (f,x^{m-1}I_m^\alpha x^{-m+1}\phi)$$

where $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu-m\alpha}$ and I_m^α is as in Definition 3.35 (i).

Theorem 3.54

If $-\alpha \in A'_{q,-\mu,m}$, then K_m^α is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$.

If also $0 \in A'_{q,-\mu,m}$, then K_m^α is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and

$$(K_m^{\alpha})^{-1} = K_m^{-\alpha}.$$

Proof:- This is similar to that of Theorem 3.52 and is omitted.

Finally we list a few simple consequences which we shall need later. Since all of these are obtained from the corresponding results above for $F_{p,\mu}$ by taking adjoints, we shall omit proofs.

Theorem 3.55

Let $f \varepsilon F'_{p,\mu}$, let α, β be complex numbers and let m, n be positive real numbers. Then

(i) $I_m^{\eta,\alpha} I_n^{\xi,\beta} f = I_n^{\xi,\beta} I_m^{\eta,\alpha} f$ if $\eta \varepsilon A_{q,-\mu,m}$, $\xi \varepsilon A_{q,-\mu,n}$

(ii) $I_m^{\eta,\alpha} K_n^{\xi,\beta} f = K_n^{\xi,\beta} I_m^{\eta,\alpha} f$ if $\eta \varepsilon A_{q,-\mu,m}$, $\xi \varepsilon A'_{q,-\mu,n}$

(iii) $K_m^{\eta,\alpha} K_n^{\xi,\beta} f = K_n^{\xi,\beta} K_m^{\eta,\alpha} f$ if $\eta \varepsilon A'_{q,-\mu,m}$, $\xi \varepsilon A'_{q,-\mu,n}$.

Theorem 3.56

Let $f \varepsilon F'_{p,\mu}$ and let α, β be complex numbers.

(i) If $\eta \varepsilon A_{q,-\mu,m}$, then

$$I_m^{\eta,\alpha} I_m^{\eta+\alpha,\beta} f = I_m^{\eta+\alpha,\beta} I_m^{\eta,\alpha} f = I_m^{\eta,\alpha+\beta} f$$

where, if $\eta + \alpha \notin A_{q,-\mu,m}$, the first two expressions have to be interpreted in terms of their analytic continuations.

(ii) If $\eta \varepsilon A'_{q,-\mu,m}$, then

$$K_m^{\eta,\alpha} K_m^{\eta+\alpha,\beta} f = K_m^{\eta+\alpha,\beta} K_m^{\eta,\alpha} f = K_m^{\eta,\alpha+\beta} f$$

where, if $\eta + \alpha \notin A'_{q,-\mu,m}$, the first two expressions have to be interpreted in terms of their analytic continuations.

To amplify the statements about analytic continuation, let $f \varepsilon F'_{p,\mu}$,

$\phi \in F_{p,\mu}$ and $\eta \in A'_{q,-\mu,m}$. Then if, also, $\eta + \alpha \in A'_{q,-\mu,m}$,

$$(K_m^{\eta,\alpha}K_m^{\eta+\alpha,\beta}f,\phi) = (f, I_m^{\eta+\alpha-1+1/m,\beta}I_m^{\eta-1+1/m,\alpha}\phi) \tag{3.46}$$

from (3.44). But from Example 3.39 with η replaced by $\eta - 1 + 1/m$, as well as (3.37), $I_m^{\eta+\alpha-1+1/m,\beta}I_m^{\eta-1+1/m,\alpha}\phi$ can be continued analytically in η to all of $A'_{q,-\mu,m}$ as $I_m^{\eta-1+1/m,\alpha+\beta}\phi$. From the continuity of f or Theorem 3.50 (ii), the right-hand side of (3.46) can be continued analytically to all of $A'_{q,-\mu,m}$ as $(f, I_m^{\eta-1+1/m,\alpha+\beta}\phi) = (K_m^{\eta,\alpha+\beta}f,\phi)$, which gives Theorem 3.56 (ii). Similar comments apply to Theorem 3.56 (i) and also to the statements on analytic continuation in the following theorems.

<u>Theorem 3.57</u>

Let $f \in F'_{p,\mu}$ and let $\alpha + \beta + \gamma = 0$.

(i) If $0 \in A_{q,-\mu,m}$, then

$$I_m^{\alpha}I_m^{\beta}f = I_m^{\alpha+\beta}f = I_m^{\beta}I_m^{\alpha}f$$

where if $\alpha \notin A_{q,-\mu,m}$ or $\beta \notin A_{q,-\mu,m}$, the third or first expression respectively is to be interpreted in terms of its analytic continuation.

(ii) If $\gamma \in A_{q,-\mu,m}$, then

$$x^{m\alpha}I_m^{\beta}x^{m\gamma}f = I_m^{-\gamma}x^{-m\beta}I_m^{-\alpha}f$$

where if $0 \notin A_{q,-\mu,m}$, the right-hand side is to be interpreted in terms of its analytic continuation.

<u>Theorem 3.58</u>

Let $f \in F'_{p,\mu}$, let $\alpha + \beta + \gamma = 0$ and let $\gamma \in A'_{q,-\mu,m}$. Then

(i) $K_m^\alpha K_m^\beta f = K_m^{\alpha+\beta} f = K_m^\beta K_m^\alpha f$

where if $-\alpha \notin A'_{q,-\mu,m}$ or $-\beta \notin A'_{q,-\mu,m}$, the third or first expression respectively is to be interpreted in terms of its analytic continuation.

(ii) $x^{m\gamma} K_m^\beta x^{m\alpha} f = K_m^{-\alpha} x^{-m\beta} K_m^{-\gamma} f$

where if $0 \notin A'_{q,-\mu,m}$, the right-hand side is to be interpreted in terms of its analytic continuation.

§3.6 A Simple Application

Although the main applications of our fractional calculus will come in the later chapters, we will discuss here one very simple application.

For any complex number η we define the differential operator L_η by

$$(L_\eta \phi)(x) = \frac{d^2\phi}{dx^2} + \frac{2\eta+1}{x}\frac{d\phi}{dx} \tag{3.47}$$

where ϕ is a suitable classical function. L_η is the one-dimensional analogue of the partial differential operator

$$\frac{\partial^2}{\partial r^2} + \frac{2\eta+1}{r}\frac{\partial}{\partial r} \tag{3.48}$$

and the latter turns up in a number of important problems. One well-known instance is in connection with spherically symmetric solutions of Laplace's equation in n dimensions or axially symmetric solutions in n + 1 dimensions, in which (3.48) turns up with $\eta = n/2 - 1$ (and where r denotes the usual radial co-ordinate). Weinstein [83], [84] considered (3.48) for general real values of η in the process of developing his generalized axially symmetric potential theory (GASPT). (3.48) also occurs in the Euler-Poisson-Darboux equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2\eta+1}{r}\frac{\partial\phi}{\partial r} = \frac{\partial^2\phi}{\partial t^2} \quad . \tag{3.49}$$

In [11], [12] and [13], Erdélyi has examined the role played by Erdélyi-Kober operators in these two situations. For instance, he shows that, by the application of certain Erdélyi-Kober operators, the equation (3.49) can be transformed into the one-dimensional wave equation

$$\frac{\partial^2\psi}{\partial r^2} = \frac{\partial^2\psi}{\partial t^2} \tag{3.50}$$

where ψ is simply related to ϕ. We will establish connections between L_ν and Erdélyi-Kober operators within the spaces $F'_{p,\mu}$ and, although we will be working with a single variable, it is not unreasonable to expect the results to carry over to higher dimensions and thus to be of use in the theory of the partial differential equations mentioned above.

A number of trivial observations can be made about L_ν on $F_{p,\mu}$. For instance, since (as is easily verified)

$$L_\eta = x^{-2\eta-2}\delta x^{2\eta}\delta = x^{-2}\delta(\delta+2\eta) \tag{3.51}$$

L_η is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-2}$ for any complex numbers μ and η (by Theorem 2.11 and Corollary 2.14 (i)) and is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-2}$ provided that Re $\mu \neq 1/p$ and Re $(2\eta+\mu) \neq 1/p$ (by Corollary 2.14 (ii)). Using (2.14), (3.51) can be written in the form

$$L_\eta = x^{-2\eta-1}D_1x^{2\eta+1}D_1 = 4x^{-2\eta}D_2x^{2\eta+2}D_2 \tag{3.52}$$

and it is also easy to check that

$$L_\eta = x^{-1}D_1x^{1-2\eta}D_1x^{2\eta} = 4D_2x^{2-2\eta}D_2x^{2\eta}. \tag{3.53}$$

As a simple consequence of (3.52) and (3.53), we see that for any η,

$$x^{2\eta}L_\eta = L_{-\eta}x^{2\eta} \tag{3.54}$$

while $D_2L_\eta = 4D_2x^{-2\eta}D_2x^{2\eta+2}D_2 = L_{\eta+1}D_2$. By induction, it follows that for any complex η and any non-negative integer k,

$$(D_2)^kL_\eta = L_{\eta+k}(D_2)^k \tag{3.55}$$

and hence, if $\phi \in F_{p,\mu}$,

$$L_\eta(D_2)^{-k}\phi = (D_2)^{-k}L_{\eta+k}\phi \tag{3.56}$$

provided that Re $\mu \neq 1/p - 2\ell$ $(\ell = 0,1, \ldots, k)$ by Corollary 2.15 (ii).

With all these preliminaries, we can now establish something of more substance.

<u>Theorem 3.59</u>

Let α be any complex number and let $\phi \in F_{p,\mu}$.

(i) If $\eta \in A_{p,\mu-2,2}$, then

$$I_2^{\eta,\alpha}L_\eta\phi = L_{\eta+\alpha}I_2^{\eta,\alpha}\phi. \tag{3.57}$$

(ii) If $\eta \in A'_{p,\mu,2}$, then

$$L_{-\eta}K_2^{\eta,\alpha}\phi = K_2^{\eta,\alpha}L_{-\eta-\alpha}\phi. \tag{3.58}$$

<u>Proof</u>:- (i) Since $\eta \in A_{p,\mu-2,2} \Rightarrow \eta \in A_{p,\mu,2}$, both sides of (3.57) define continuous linear mappings of $F_{p,\mu}$ into $F_{p,\mu-2}$. To establish equality, assume first that Re $(2\eta+\mu) > 1/p$. Then we may use the theory in §3.2. In particular, from (3.14), we obtain

$$I_2^{\eta,\alpha}(\delta+2\eta+2\alpha)\phi = 2I_2^{\eta,\alpha-1}\phi$$

$$\Rightarrow L_{\eta+\alpha} I_2^{\eta,\alpha}\phi = x^{-2} I_2^{\eta,\alpha}(\delta+2\eta+2\alpha)\delta\phi = 2x^{-2} I_2^{\eta,\alpha-1}\delta\phi$$

by (3.12) and (3.51). Now $2I_2^{\eta,\alpha-1}\delta\phi = I_2^{\eta-1,\alpha}(\delta+2\eta)\delta\phi$, using integration by parts when Re $\alpha > 1$ and analytic continuation otherwise. Hence by (3.23) and (3.51),

$$L_{\eta+\alpha} I_2^{\eta,\alpha}\phi = x^{-2} I_2^{\eta-1,\alpha}(\delta+2\eta)\delta\phi = I_2^{\eta,\alpha} x^{-2}(\delta+2\eta)\delta\phi = I_2^{\eta,\alpha} L_\eta \phi$$

and (3.57) is proved in this case.

Now let $\eta \in A_{p,\mu-2,2}$ and let k be a non-negative integer such that $\mathrm{Re}(2\eta+\mu) > 1/p - 2k$. Then

$$\begin{aligned}
& L_{\eta+\alpha} I_2^{\eta,\alpha}\phi \\
&= L_{\eta+\alpha} I_2^{\eta+k,\alpha-k} x^{-2(\eta+k)} (D_2)^{-k} x^{2\eta}\phi && \text{by (3.32)} \\
&= I_2^{\eta+k,\alpha-k} L_{\eta+k} x^{-2(\eta+k)} (D_2)^{-k} x^{2\eta}\phi && \text{by the previous case} \\
&= I_2^{\eta+k,\alpha-k} x^{-2(\eta+k)} L_{-\eta-k} (D_2)^{-k} x^{2\eta}\phi && \text{by (3.54)} \\
&= I_2^{\eta+k,\alpha-k} x^{-2(\eta+k)} (D_2)^{-k} L_{-\eta} x^{2\eta}\phi && \text{by (3.56)} \\
&= I_2^{\eta+k,\alpha-k} x^{-2(\eta+k)} (D_2)^{-k} x^{2\eta} L_\eta\phi && \text{by (3.54)} \\
&= I_2^{\eta,\alpha} L_\eta \phi && \text{by (3.32).}
\end{aligned}$$

This completes the proof of (i). That of (ii) is similar and is omitted.

<u>Remark 3.60</u>

(3.57) and (3.58) enable us to relate any operator L_η to the simple operator $L_{-\frac{1}{2}} = \dfrac{d^2}{dx^2}$ via fractional calculus. For instance, (3.57) shows that under appropriate conditions

$$I_2^{\eta,-1/2-\eta} L_\eta \phi = L_{-\frac{1}{2}} I_2^{\eta,-1/2-\eta} \phi$$

which is typical of the results exploited by Erdélyi in, for instance, transforming (3.49) into (3.50).

To obtain the analogue of Theorem 3.59 for $F'_{p,\mu}$ we take adjoints as usual. Equivalently, we want to extend L_η to $F'_{p,\mu}$ in such a way that, for functions $g \in L^q_{-\mu}$ with $L_\eta g \in L^q_{-\mu-2}$,

$$L_\eta \tau g = \tau L_\eta g \tag{3.59}$$

as an equality in $F'_{p,\mu+2}$. Then if $\phi \in C_0^\infty(0,\infty)$ is regarded as an element of $F_{p,\mu+2}$, integration by parts gives

$$\begin{aligned}(L_\eta \tau g, \phi) &= \int_0^\infty (L_\eta g)(x)\phi(x)dx = \int_0^\infty x^{-2\eta-1} D_1 x^{2\eta+1} D_1 g(x)\phi(x)dx \\ &= \int_0^\infty g(x) D_1 x^{2\eta+1} D_1 x^{-2\eta-1} \phi(x)dx \\ &= (\tau g, xL_{-\eta} x^{-1} \phi)\end{aligned}$$

using (3.52) and (3.53). As in §2.4, we are thus led to define L_η on $F'_{p,\mu}$ by

$$(L_\eta f, \phi) = (f, xL_{-\eta} x^{-1}\phi) \quad (f \in F'_{p,\mu},\ \phi \in F_{p,\mu+2})\ . \tag{3.60}$$

L_η as so defined is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+2}$ for all complex η and μ and is a homeomorphism under appropriate circumstances. However, we shall content ourselves with the following result, the proof of which shows that (3.57) and (3.58) are essentially adjoint versions of each other.

Theorem 3.61

Let α be any complex number and let $f \in F'_{p,\mu}$.

(i) If $\eta \in A_{q,-\mu-2,2}$, then

$$I_2^{\eta,\alpha} L_\eta f = L_{\eta+\alpha} I_2^{\eta,\alpha} f.$$

(ii) If $\eta \in A'_{q,-\mu,2}$, then

$$L_{-\eta} K_2^{\eta,\alpha} f = K_2^{\eta,\alpha} L_{-\eta-\alpha} f.$$

Proof: (i) Both sides define continuous linear mappings of $F'_{p,\mu}$ into $F'_{p,\mu+2}$ under the given conditions since $A_{q,-\mu-2,2} \subset A_{q,-\mu,2}$. Let $\phi \in F_{p,\mu+2}$. Then,

$$(I_2^{\eta,\alpha} L_\eta f, \phi)$$

$$= (f, xL_{-\eta} x^{-1} K_2^{\eta+1/2,\alpha} \phi) \qquad \text{by (3.43) and (3.60)}$$

$$= (f, xL_{-\eta} K_2^{\eta,\alpha} x^{-1} \phi) \qquad \text{by (3.24)}$$

$$= (f, xK_2^{\eta,\alpha} L_{-\eta-\alpha} x^{-1} \phi) \qquad \text{by (3.58)}$$

$$= (f, K_2^{\eta+1/2,\alpha} xL_{-\eta-\alpha} x^{-1} \phi) \qquad \text{by (3.24)}$$

$$= (L_{\eta+\alpha} I_2^{\eta,\alpha} f, \phi) \qquad \text{by (3.43) and (3.60).}$$

In the above (3.58) is applied with ϕ and μ replaced by $x^{-1}\phi$ and $\mu + 1$ respectively and this is valid since $A'_{p,\mu+1,2} = A_{q,-\mu-2,2}$. This completes the proof of (i) and that of (ii) is similar.

The possible uses of this result have been suggested above and we will not elaborate here. In conclusion, we remark that the function $x^{-\eta} J_\eta(x)$ is a solution of the equation $L_\eta \phi + \phi = 0$, as is easily checked using properties of the Bessel function J_η of the first kind and order η. It is therefore no surprise to find strong connections between fractional calculus and Hankel transforms and, in particular, results such as (6.4) and (6.8) in Chapter 6.

4 Hypergeometric integral equations

§4.1 Introduction

For the first main application of the theory in Chapter 3 we consider in detail some integral equations involving the Gauss or ${}_2F_1$ hypergeometric function. We shall be concerned with four operators $H_i(a,b;c;m)$ $(i = 1,2,3,4)$, typical of which is $H_1(a,b;c;m)$. For suitable functions ϕ, Re $c > 0$, $m > 0$ and suitably restricted complex numbers a and b, we define $H_1(a,b;c;m)\phi$ on $(0,\infty)$ by

$$(H_1(a,b;c;m)\phi)(x) = \int_0^x \frac{(x^m-t^m)^{c-1}}{\Gamma(c)} \; {}_2F_1(a,b;c;1-x^m/t^m)mt^{m-1}\phi(t)dt \; . \quad (4.1)$$

The other three operators are similar.

It is well-known that there are intimate connections between the ${}_2F_1$ hypergeometric function and fractional calculus; see, for instance, [7], [9], [24 p.118]. It is therefore no surprise that equations involving hypergeometric functions can be solved via fractional calculus. In two comprehensive papers [40] and [41], Love studied our four operators for the case m = 1 in the context of locally integrable classical functions. These papers unified the work of several authors who had dealt with various special cases using various different methods; references can be found in [40] and [41]. Our object is to study the operators in the context of generalised functions and to extend them in the manner of Chapter 3. However, at the end of the section, we give an instance of how classical results can be recovered from this extension.

For any complex numbers a, b, c and z, we shall write

$$F^*(a,b;c;z) = \frac{1}{\Gamma(c)} \; {}_2F_1\,(a,b;c;z) \tag{4.2}$$

so that, for $|z| < 1$,

$$F^*(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{\Gamma(c+n)} \frac{z^n}{n!} \tag{4.3}$$

where, for example, $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$ $(n \geqslant 1)$. $F^*(a,b;c;z)$, as defined by (4.3), is an entire function of a,b and c and an analytic function of z for $|z| < 1$. Since (4.1) requires $F^*(a,b;c;z)$ for real $z \in (-\infty,0)$, we extend the definition of $F^*(a,b;c;z)$ to the half-plane Re $z < \frac{1}{2}$ by one of Kummer's relations ([18], p.105)

$$F^*(a,b;c;z) = (1-z)^{-a} F^*(a,c-b;c;z/(z-1)) \tag{4.4}$$

where the principal branch of $(1-z)^{-a}$ is used. By (4.2) and analytic continuation we note that

$$F^*(a,b;c;z) = F^*(b,a;c;z) \tag{4.5}$$

for Re $z < \frac{1}{2}$. Further, the extended function is an entire function of a,b and c and an analytic function of z for Re $z < \frac{1}{2}$.

Most of the properties of $F^*(a,b;c;z)$ we require are in [18]. However, it is convenient to mention the following result.

<u>Lemma 4.1</u>

Let a, b and c be complex numbers and let $\delta > 0$. Then there is a constant M, independent of $v \in (0,1)$, such that the four expressions

$$|F^*(a,b;c;1-1/v^m)|, \qquad \left|\frac{\partial}{\partial a} F^*(a,b;c;1-1/v^m)\right|$$

$$\left|\frac{\partial}{\partial b} F^*(a,b;c;1-1/v^m)\right|, \quad \left|\frac{\partial}{\partial c} F^*(a,b;c;1-1/v^m)\right|$$

are all less than or equal to $Mv^{\min(m \text{ Re } a,\ m \text{ Re } b) - \delta}$.

<u>Proof</u>:- The details for m = 1, which are rather involved, are to be found in [46] and the case for general $m > 0$ follows similarly.

With these preliminaries we can now start and, to save repetition, we assume throughout this chapter that a,b and c are complex numbers and that m is real and positive.

§4.2 The Operators $H_i(a,b;c;m)$ on $F_{p,\mu}$

For convenience we recall the definition of $H_1(a,b;c;m)$.

Definition 4.2

For Re c > 0 and suitable functions ϕ, we define $H_1(a,b;c;m)\phi$ on $(0,\infty)$ by

$$H_1(a,b;c;m)\phi(x) = \int_0^x (x^m-t^m)^{c-1} F^*(a,b;c;1-x^m/t^m)mt^{m-1}\phi(t)dt \quad . \quad (4.6)$$

As a first step we can prove

Lemma 4.3

Let Re c > 0, $-\text{Re}\,\mu - m + 1/p < \min(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$, $\phi \in F_{p,\mu}$. Then

(i) $H_1(a,b;c;m)\phi(x)$ exists for all $x \in (0,\infty)$ and defines a continuous function on $(0,\infty)$

(ii) for each fixed $x \in (0,\infty)$, $H_1(a,b;c;m)\phi(x)$ is analytic in a,b and c in the region $-\text{Re}\,\mu - m + 1/p < \min(m\,\text{Re}\,a,\ m\,\text{Re}\,b)$, Re c > 0.

Proof:- By (4.6),

$$H_1(a,b;c;m)\phi(x) = x^{mc}\int_0^1 (1-v^m)^{c-1} F^*(a,b;c;1-1/v^m)mv^{m-1}\phi(xv)dv. \qquad (4.7)$$

By Theorem 2.2 and Lemma 4.1, for any $\delta > 0$, there exists M, independent of $v \in (0,1)$, such that

$$|(1-v^m)^{c-1} F^*(a,b;c;1-1/v^m)mv^{m-1}\phi(xv)|$$
$$\leq M(1-v^m)^{\text{Re}\,c-1}\, v^{\min(m\,\text{Re}\,a,\ m\,\text{Re}\,b) - \delta + m - 1}(xv)^{\text{Re}\,\mu - 1/p}$$

for $v \in (0,1)$. Thus, under the given conditions, the integral in (4.7) converges uniformly on compact subsets of $(0,\infty)$ and (i) follows. (ii) follows similarly since, by Lemma 4.1, we may differentiate under the integral sign with respect to a,b or c under the given conditions.

To make further progress, we show how $H_1(a,b;c;m)$ is connected to our operators in Chapter 3. In Theorem 3.41 we saw that, under appropriate conditions, $I_m^{\eta,\alpha}$ and $I_n^{\xi,\beta}$ commute. In the case m = n we discover the following.

Lemma 4.4

Let $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$, $-\mathrm{Re}\,\mu - m + 1/p < \min(m\,\mathrm{Re}\,\xi,\ m\,\mathrm{Re}\,\eta)$, $\phi \in F_{p,\mu}$. Then

$$I_m^{\eta,\alpha}\, I_m^{\xi,\beta}\phi(x) = x^{-m\eta-m\alpha}\, H_1(\xi+\beta-\eta,\beta;\alpha+\beta;m)\, x^{m\eta-m\beta}\phi(x) \quad . \tag{4.8}$$

Proof:- From (3.5) and (3.8), $I_m^{\eta,\alpha} I_m^{\xi,\beta}\phi(x)$

$$= \frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m-u^m)^{\alpha-1}\, u^{m\eta+m-1}\, du\, \frac{mu^{-m\xi-m\beta}}{\Gamma(\beta)} \int_0^u (u^m-t^m)^{\beta-1} t^{m\xi+m-1}\phi(t)dt.$$

Using Theorem 2.2, we see that, under the given conditions, this repeated integral is absolutely convergent and hence, by Fubini's theorem, is equal to

$$\frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x t^{m\xi+m-1}\phi(t)dt \int_t^x (x^m-u^m)^{\alpha-1}(u^m-t^m)^{\beta-1} u^{m\eta-m\xi-m\beta} mu^{m-1} du\, .$$

Under the substitution $w = \dfrac{u^m-t^m}{x^m-t^m}$, the inner integral becomes

$$(x^m-t^m)^{\alpha+\beta-1} t^{m\eta-m\xi-m\beta} \int_0^1 (1-w)^{\alpha-1}\, w^{\beta-1}\left[1-w(1-x^m/t^m)\right]^{\eta-\xi-\beta} dw$$

$$= \Gamma(\alpha)\Gamma(\beta)\ (x^m-t^m)^{\alpha+\beta-1}\ t^{m\eta-m\xi-m\beta}\ F^*(\xi+\beta-\eta,\beta;\alpha+\beta;1-x^m/t^m)$$

using Euler's Integral [18, p.59]. The result now follows easily.

Corollary 4.5

Let $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$, $-\mathrm{Re}\,\mu - m + 1/p < \min(m\,\mathrm{Re}\,a,\ m\,\mathrm{Re}\,b)$, $\phi \in F_{p,\mu}$. Then

$$H_1(a,b;c;m)\phi = I_m^{c-b}\ x^{-ma}\ I_m^{b}\ x^{ma}\phi \quad . \tag{4.9}$$

Proof:- In Lemma 4.4, we take $\alpha = c-b$, $\beta = b$, $\xi = \eta+a-b$ and replace μ and $\phi(x)$ by $\mu - m\eta + mb$ and $x^{-m\eta+mb}\phi(x)$. The conditions of Lemma 4.4 are satisfied so that

$$H_1(a,b;c;m)\phi = x^{m\eta+mc-mb}\ I_m^{\eta,c-b}\ I_m^{\eta+a-b,b}\ x^{-m\eta+mb}\phi \ . \tag{4.10}$$

The free parameter η disappears on using (3.8) and the result follows.

(4.9) indicates how we can extend the definition of $H_1(a,b;c;m)$ on $F_{p,\mu}$ to other values of μ. First of all, the restriction Re $c >$ Re $b > 0$ can be removed. However, there is a slight technicality here because $H_1(a,b;c;m)$ has already been defined for Re $c > 0$, not just for Re $c >$ Re $b > 0$. Fortunately, the extension given by (4.9) coincides with (4.6) in this case by virtue of Lemma 4.3(ii) and the principle of analytic continuation. (Indeed, this was the sole purpose of stating Lemmas 4.1 and 4.3.). However, because of Theorem 3.36, the right-hand side of (4.9) is meaningful provided only that $0 \in A_{p,\mu+ma,m}$ and $0 \in A_{p,\mu+mb,m}$ or, equivalently, $\{a,b\} \subseteq A_{p,\mu,m}$. Hence we make the following definition.

Definition 4.6

For $\{a,b\} \subseteq A_{p,\mu,m}$, we define $H_1(a,b;c;m)$ on $F_{p,\mu}$ by

$$H_1(a,b;c;m)\phi = I_m^{c-b}\ x^{-ma}\ I_m^{b}\ x^{ma}\phi \ . \tag{4.11}$$

In view of the above motivation, this gives an extension of Definition 4.2.

We can immediately obtain very full information about this extended operator.

Theorem 4.7

If $\{a,b\} \subseteq A_{p,\mu,m}$, then $H_1(a,b;c;m)$ is a continuous linear mapping of $F_{p,\mu}$

into $F_{p,\mu+mc}$.

If also $\{c,a+b\} \subseteq A_{p,\mu,m}$, then $H_1(a,b;c;m)$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+mc}$ and, for $\psi \in F_{p,\mu+mc}$,

$$[H_1(a,b;c;m)]^{-1}\psi = x^{-ma}\, H_1(-a,b-c;-c;m)x^{ma}\psi\ . \tag{4.12}$$

Proof:- The first statement follows from Theorems 2.11 and 3.36. For the second, we observe that under the given conditions, $H_1(a,b;c;m)$ is invertible and

$$[H_1(a,b;c;m)]^{-1}\psi = x^{-ma}\, I_m^{-b}\, x^{ma}\, I_m^{b-c}\, \psi\ .$$

(4.12) now follows easily from (4.11).

An interesting point arises here. Because of (4.5) we might expect to have

$$H_1(a,b;c;m) = H_1(b,a;c;m) \tag{4.13}$$

for $\{a,b\} \subseteq A_{p,\mu,m}$. This is not obvious from (4.11) which is not symmetrical in a and b. However,

$$I_m^{c-b}\, x^{-ma}\, I_m^{b}\, x^{ma-mb}\, x^{mb}\, \phi$$

$$= I_m^{c-b}\, I_m^{b-a}\, x^{-mb}\, I_m^{a}\, x^{mb}\, \phi \qquad \text{by Theorem 3.43(ii)}$$

$$= I_m^{c-a}\, x^{-mb}\, I_m^{a}\, x^{mb}\, \phi \qquad \text{by Theorem 3.43(i)}$$

so that (4.13) is indeed true if $\{a,b\} \subseteq A_{p,\mu,m}$. We have used the index laws to verify (4.13) but it is also possible to derive the index laws as special cases of results for hypergeometric operators.

The other three operators $H_i(a,b;c;m)$ $(i = 2,3,4)$ can now be dealt with fairly quickly. As regards $H_2(a,b;c;m)$, we start with the integral representation

$$H_2(a,b;c;m)\phi(x) = \int_0^x (x^m-t^m)^{c-1} F^*(a,b;c;1-t^m/x^m)mt^{m-1}\phi(t)dt \quad (4.14)$$

valid for Re $c > 0$ and appropriate functions ϕ. Imitating Love [40, p.195] we soon find that

$$H_2(a,b;c;m)\phi(x) = x^{ma} H_1(a,c-b;c;m)x^{-ma}\phi(x)$$

or $$H_2(a,b;c;m)\phi(x) = x^{ma} I_m^b x^{-ma} I_m^{c-b} \phi(x) .$$

In particular, these are valid for $\phi \in F_{p,\mu}$ provided that $-\mathrm{Re}\,\mu - m + 1/p < \min(0, m\mathrm{Re}(c-b-a))$. Extending as before, we arrive at the following.

Definition 4.8

If $\{0,c-b-a\} \subseteq A_{p,\mu,m}$, we define $H_2(a,b;c;m)$ on $F_{p,\mu}$ by

$$H_2(a,b;c;m)\phi = x^{ma} I_m^b x^{-ma} I_m^{c-b}\phi . \quad (4.15)$$

In view of the above motivation, this extends (4.14) and we can quickly establish the main properties of the extended operator.

Theorem 4.9

If $\{0,c-b-a\} \subseteq A_{p,\mu,m}$, then $H_2(a,b;c;m)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$.

If also $\{c-a, c-b\} \subseteq A_{p,\mu,m}$, then $H_2(a,b;c;m)$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+mc}$ and, for $\psi \in F_{p,\mu+mc}$,

$$[H_2(a,b;c;m)]^{-1}\psi = H_1(-a, -b; -c; m)\psi . \quad (4.16)$$

Proof:- The results follow easily from Theorem 3.36. As regards (4.16), we observe that $[H_2(a,b;c;m)]^{-1}\psi = I_m^{b-c} x^{ma} I_m^{-b} x^{-ma}\psi$ and then use (4.11).

Remark 4.10

(4.16) is a rather interesting result which emerges quite naturally in our setting of generalised functions but which does not appear at all clear in a classical approach such as that in [40].

For $H_3(a,b;c;m)$ we start with the integral representation

$$H_3(a,b;c;m)\phi(x) = mx^{m-1}\int_x^\infty (t^m-x^m)^{c-1} F^*(a,b;c;1-x^m/t^m)\phi(t)dt \tag{4.17}$$

valid for Re $c > 0$ and suitable functions ϕ. We then use an adjoint version of (4.8), namely

$$K_m^{\xi,\beta} K_m^{\eta,\alpha}\phi(x) = x^{m\xi-m+1}H_3(\xi+\beta-\eta,\alpha;\alpha+\beta;m)x^{-m\xi-m\beta-m\alpha+m-1}\phi(x) \tag{4.18}$$

valid for $\phi \in F_{p,\mu}$ provided that Re $\alpha > 0$, Re $\beta > 0$ and Re $\mu - 1/p < \min(m \text{ Re } \xi, m \text{ Re } \eta)$. Disentangling (4.18) produces

$$H_3(a,b;c;m)\phi = x^{m-1} K_m^{c-b} x^{-ma} K_m^b x^{ma-m+1}\phi$$

provided $\phi \in F_{p,\mu}$, Re $c >$ Re $b > 0$ and Re $\mu - m + 1/q < \min(-m \text{ Re } c, -m \text{ Re}(a+b))$. Extending as before, we obtain the following definition.

<u>Definition 4.11</u>

If $\{-c, -a-b\} \in A_{q,-\mu,m}$, we define $H_3(a,b;c;m)$ on $F_{p,\mu}$ by

$$H_3(a,b;c;m)\phi = x^{m-1} K_m^{c-b} x^{-ma} K_m^b x^{ma-m+1}\phi \; . \tag{4.19}$$

For $H_4(a,b;c;m)$, we start with

$$H_4(a,b;c;m)\phi(x) = mx^{m-1}\int_x^\infty (t^m-x^m)^{c-1} F^*(a,b;c;1-t^m/x^m)\phi(t)dt \tag{4.20}$$

valid for Re $c > 0$ and suitable functions ϕ and are led, using Kummer's relations, to

$$H_4(a,b;c;m)\phi = x^{ma+m-1} K_m^b x^{-ma} K_m^{c-b} x^{-m+1}\phi$$

for Re $c >$ Re $b > 0$, $\phi \in F_{p,\mu}$ and Re $\mu - m + 1/q < \min(m \text{ Re }(a-c), m \text{ Re}(b-c))$. This in turn suggests the following definition.

Definition 4.12

If $\{a-c,\ b-c\} \subseteq A_{q,-\mu,m}$, we define $H_4(a,b;c;m)$ on $F_{p,\mu}$ by

$$H_4(a,b;c;m)\phi = x^{ma+m-1} K_m^b x^{-ma} K_m^{c-b} x^{-m+1}\phi \quad . \tag{4.21}$$

The properties of $H_i(a,b;c;m)$ $(i = 3,4)$ are obtained from Theorem 3.44 and we state them without proof.

Theorem 4.13

If $\{-c,\ -a-b\} \subseteq A_{q,-\mu,m}$, then $H_3(a,b;c;m)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$.

If also $\{-a,\ -b\} \subseteq A_{q,-\mu,m}$, then $H_3(a,b;c;m)$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+mc}$ and

$$\left[H_3(a,b;c;m)\right]^{-1} = H_4(-a,-b;-c;m).$$

Theorem 4.14

If $\{a-c,\ b-c\} \subseteq A_{q,-\mu,m}$, then $H_4(a,b;c;m)$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+mc}$.

If also $\{0,\ a+b-c\} \subseteq A_{q,-\mu,m}$, then $H_4(a,b;c;m)$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu+mc}$ and

$$\left[H_4(a,b;c;m)\right]^{-1} = H_3(-a,-b;-c;m) \ .$$

Remark 4.15

This completes our discussion of the four operators on $F_{p,\mu}$. Perhaps it would be in order to mention that, under appropriate conditions, an operator of the form $I_m^{\eta,\alpha} K_m^{\xi,\beta}$ produces another type of hypergeometric operator. However, we shall not deal with this here and merely refer to [72] where there is a discussion in a classical setting.

§4.3 The Operators $H_i(a,b;c;m)$ on $F'_{p,\mu}$

We now consider how to extend the operators $H_i(a,b;c;m)$ to $F'_{p,\mu}$ and once again we will concentrate on $H_1(a,b;c;m)$.

As in previous cases, we begin with regular functionals. Let $g \in L^q_{-\mu}$ and let Re $c > 0$. Then, under appropriate conditions, $H_1(a,b;c;m)g \in L^q_{-\mu+mc}$. Indeed, if we use (4.10) along with Lemma 3.2, we find that $H_1(a,b;c;m)$ is a continuous linear mapping from $L^q_{-\mu}$ into $L^q_{-\mu+mc}$ provided that Re $c >$ Re $b > 0$ and Re $\mu - m + 1/q < \min$ (m Re a, m Re b). On closer examination we find that the condition Re $c >$ Re $b > 0$ can be replaced by Re $c > 0$; see, for instance, [48]. If our new operator $H_1(a,b;c;m)$ is to be an extension of the previous one, we would require that

$$H_1(a,b;c;m)\tau g = \tau\, H_1(a,b;c;m)g \tag{4.22}$$

whenever both sides are meaningful; here we are using the notation of Example 2.18 again and (4.22) is to hold in the sense of equality in $F'_{p,\mu-mc}$. To see what (4.22) entails, let $\phi \in F_{p,\mu-mc}$, Re $c > 0$ and Re $\mu - m + 1/q < \min$ (m Re a, m Re b). Then

$$(H_1(a,b;c;m)\tau g,\phi) = (\tau H_1(a,b;c;m)g,\phi)$$

$$= \int_0^\infty H_1(a,b;c;m)g(x)\phi(x)\,dx$$

$$= \int_0^\infty \left(\int_0^x (x^m-t^m)^{c-1} F^*(a,b;c;1-x^m/t^m)mt^{m-1}g(t)dt\right)\phi(x)dx .$$

By our comments above and Hölder's inequality, the repeated integral is absolutely convergent. Hence, by Fubini's theorem, the last expression becomes

$$\int_0^\infty g(t)\ (mt^{m-1}\int_t^\infty (x^m-t^m)^{c-1}\ F^*(a,b;c;1-x^m/t^m)\phi(x)dx)dt$$

$$= \int_0^\infty g(t)\ H_4(a,b;c;m)\phi(t)dt$$

so that

$$(H_1(a,b;c;m)\tau g,\phi) = (\tau g,\ H_4(a,b;c;m)\phi) \quad (\phi \in F_{p,\mu-mc}) .$$

This suggests that we would wish

$$(H_1(a,b;c;m)f,\phi) = (f,H_4(a,b;c;m)\phi) \qquad (\phi \in F_{p,\mu-mc}) \qquad (4.23)$$

for all $f \in F'_{p,\mu}$, regular or not. Further, by Theorem 4.14, the right-hand side of (4.23) is meaningful provided only that $\{a-c,\ b-c\} \subseteq A_{q,-\mu+mc,m}$ or, equivalently, $\{a,b\} \subseteq A_{q,-\mu,m}$. Hence we are led to define $H_1(a,b;c;m)$ on $F'_{p,\mu}$ as the adjoint of $H_4(a,b;c;m)$ on $F_{p,\mu-mc}$ under these conditions.

Just as $H_1(a,b;c;m)$ and $H_4(a,b;c;m)$ are formal adjoints, so are $H_2(a,b;c;m)$ and $H_3(a,b;c;m)$. Hence, by similar arguments, we are led to the following definition.

Definition 4.16

Let $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu-mc}$. We define $H_i(a,b;c;m)$ $(i = 1,2,3,4)$ on $F'_{p,\mu}$ by the following equations under the conditions stated.

(i) If $\{a,b\} \subseteq A_{q,-\mu,m}$, then

$$(H_1(a,b;c;m)f,\phi) = (f,\ H_4(a,b;c;m)\phi).$$

(ii) If $\{0,c-b-a\} \subseteq A_{q,-\mu,m}$, then

$$(H_2(a,b;c;m)f,\phi) = (f,\ H_3(a,b;c;m)\phi).$$

(iii) If $\{-c,\ -a-b\} \subseteq A_{p,\mu,m}$, then

$$(H_3(a,b;c;m)f,\phi) = (f,\ H_2(a,b;c;m)\phi).$$

(iv) If $\{a-c,\ b-c\} \subseteq A_{p,\mu,m}$, then

$$(H_4(a,b;c;m)f,\phi) = (f,H_1(a,b;c;m)\phi).$$

The properties of our extended operators follow at once from Theorems 4.7, 4.9, 4.13 and 4.14 above together with Theorems 1.10-1 and 1.10-2 in [87]. Rather than list all the results separately, we give the details in tabular form, and omit details of the proof.

Theorem 4.17

The properties of the operators $H_i(a,b;c;m)$ on $F'_{p,\mu}$ are given in the following table. Column 2 gives the conditions under which the operators are continuous linear mappings (c.l.m.) from $F'_{p,\mu}$ into $F'_{p,\mu-mc}$, column 3 the conditions under which the operators are homeomorphisms (homeo.) and column 4 the inverse operators, which exist under the conditions of column 3.

	c.l.m.	homeo.	inverse
$H_1(a,b;c;m)$	$\{a,b\} \subseteq A_{q,-\mu,m}$	$\{a,b,c,a+b\} \subseteq A_{q,-\mu,m}$	$H_2(-a,-b;-c;m)$
$H_2(a,b;c;m)$	$\{0,c-a-b\} \subseteq A_{q,-\mu,m}$	$\{0,c-a,c-b,c-a-b\} \subseteq A_{q,-\mu,m}$	$H_1(-a,-b;-c;m)$
$H_3(a,b;c;m)$	$\{-c,-a-b\} \subseteq A_{p,\mu,m}$	$\{-a,-b,-c,-a-b\} \subseteq A_{p,\mu,m}$	$H_4(-a,-b;-c;m)$
$H_4(a,b;c;m)$	$\{a-c,b-c\} \subseteq A_{p,\mu,m}$	$\{0,a-c,b-c,a+b-c\} \subseteq A_{p,\mu,m}$	$H_3(-a,-b;-c;m)$

Again, as a check, we see that the conditions above are derived from those appropriate to $F_{p,\mu}$ by interchanging p and q, μ and $-\mu$.

§4.4 The Classical Case

We conclude this chapter with an indication of how our theory sheds some light on the classical problems from which the whole discussion started.

As mentioned above, Love in [40] and [41] has discussed the operators relative to certain classes of locally integrable classical functions. We will work in the L^p_μ spaces and to provide a close analogy with Love's work, we take $m = 1$ for the remainder of this chapter. Again, we will concentrate on $H_1(a,b;c;1)$ and return to the case when we can use the simple integral representation (4.6).

Our problem therefore is as follows; for a given classical function g defined (almost everywhere) on $(0,\infty)$, does there exist f defined (almost everywhere) on $(0,\infty)$ such that

$$\int_0^x (x-t)^{c-1} F^*(a,b;c;1-x/t)\, f(t)dt = g(x) \tag{4.24}$$

for almost all $x \in (0,\infty)$? We mentioned in the previous section that if $f \in L^p_\mu$, then the left-hand side of (4.24) defines a function in $L^p_{\mu+c}$ under appropriate conditions (such as Re $c > 0$, $-\text{Re}\,\mu - 1/q < \min$ (Re a, Re b)). Thus, we will assume that $g \in L^p_{\mu+c}$.

As regards uniqueness, we can prove the following.

Theorem 4.18

If $-\text{Re}\,\mu - 1/q < \min$ (Re a, Re b, Re c, Re$(a+b)$), Re $c > 0$ and $g \in L^p_{\mu+c}$, then (4.24) has at most one solution $f \in L^p_\mu$.

Proof :-

Let $f \in L^p_\mu$ satisfy (4.24). Then $H_1(a,b;c;1)f \in L^p_{\mu+c}$ and so

$$\tau\, H_1(a,b;c;1)f = \tau g$$

in the sense of equality in $F'_{q,-\mu-c}$. But under the given conditions, (4.22) holds so that

$$H_1(a,b;c;1)\tau f = \tau g\ . \tag{4.25}$$

Also, by Theorem 4.17, with p and μ replaced by q and $-\mu$ respectively, the equation

$$H_1(a,b;c;1)h = \tau g \tag{4.26}$$

has a unique solution $h \in F'_{q,-\mu}$ and by (3.45) and (4.21)

$$h = x^{-a} I_1^{-b} x^a I_1^{b-c} \tau g \; . \tag{4.27}$$

Hence, from (4.25) and (4.26), our equation (4.24) will have either one solution in L^p_μ or no solution in L^p_μ according as there does or does not exist $f \in L^p_\mu$ such that $\tau f = h$. (We are, of course, identifying any two functions which differ only on a set of measure zero.)

As regards existence, the proof of Theorem 4.18 shows that (4.24) has a solution $f \in L^p_\mu$ if and only if h, as defined by (4.27), is regular (in the sense of Example 2.18). To guarantee this regularity, we have to impose extra conditions on the parameters and on g. This can be done in various ways and we thereby obtain analogues of Love's results. Here is one possibility which is our equivalent of [40, Theorem 11].

Theorem 4.19

Let Re $b < 0$, Re $c > 0$, $-\text{Re}(\mu+b) - 1/q < \min(0, \text{Re } a)$. Furthermore, assume that there is a function $G \in L^p_{\mu+b}$ such that $g = I_1^{c-b} G$. Then (4.24) has a unique solution $f \in L^p_\mu$ given by

$$f(x) = x^{-a} I_1^{-b} x^a G(x) \; . \tag{4.28}$$

Proof :-

Since $\text{Re}(c-b) > 0$, and $\text{Re}(\mu+b) > -1/q$, we see from Lemma 3.2 that if $G \in L^p_{\mu+b}$, then $g = I_1^{c-b} G = x^{c-b} I_1^{0,c-b} G \in L^p_{\mu+c}$ as we would like. Also $-\text{Re } \mu - 1/q < \min(\text{Re } a, \text{Re } b, \text{Re } c, \text{Re}(a+b))$ under the given conditions.

Hence, as in Theorem 4.18, we find that (4.24) has a solution if and only if h, as defined by (4.27), is regular and generated by f. We shall show that h is indeed regular in the given situation. To do this, we recall that if η is any complex number, $\mathrm{Re}\,\alpha > 0$, $F \in L^p_\nu$ (ν complex) and $\mathrm{Re}\,\nu > -1/q$, then

$$\tau(x^\eta f) = x^\eta \tau f \qquad (\text{in } F'_{q,-\nu-\eta}) \tag{4.29}$$

$$\text{and } \tau(I_1^\alpha f) = I_1^\alpha \tau f \quad (\text{in } F'_{q,-\nu-\alpha}). \tag{4.30}$$

Indeed (4.29) supplied the motivation for Definition 2.21(i) while (4.30) is a consequence of (3.22) and (3.41). Thus,

$$\begin{aligned}
& x^{-a}\, I_1^{-b}\, x^a\, I_1^{b-c}\, \tau g \\
&= x^{-a}\, I_1^{-b}\, x^a\, I_1^{b-c}\, \tau I_1^{c-b}\, G \\
&= x^{-a}\, I_1^{-b}\, x^a\, I_1^{b-c}\, I_1^{c-b}\, \tau G && \text{since } \mathrm{Re}(c-b) > 0,\ \mathrm{Re}(\mu+b) > -1/q \\
&= x^{-a}\, I_1^{-b}\, x^a\, \tau G && \text{by Theorem 3.52} \\
&= x^{-a}\, I_1^{-b}\, \tau x^a G && \text{by (4.29)} \\
&= x^{-a}\, \tau I_1^{-b}\, x^a G && \text{since } \mathrm{Re}(-b) > 0,\ \mathrm{Re}(\mu+a+b) > -1/q \\
&= \tau\, x^{-a}\, I_1^{-b}\, x^a G\,.
\end{aligned}$$

Finally, since $\mathrm{Re}(-b) > 0$, $\mathrm{Re}(\mu+a+b) > -1/q$ and $G \in L^p_{\mu+b}$, $x^{-a}\, I_1^{-b}\, x^a\, G \in L^p_\mu$ by Lemma 3.2. Thus $h = \tau f$ where f is given by (4.28) and hence f satisfies (4.24) in view of our previous remarks. That the solution is unique in L^p_μ follows from Theorem 4.18. This completes the proof.

To say, as in Theorem 4.19, that $g = I_1^{c-b}\, G$ where $G \in L^p_{\mu+b}$ effectively says that g has a fractional derivative of order $c-b$ belonging to $L^p_{\mu+b}$. Such a condition is typical of the type required in order to guarantee a classical solution; this theme is also discussed by Higgins in [29]. If such a guarantee is not forthcoming, the process above breaks down. We can obtain h in (4.27) as before but now it may no longer be regular and we have to be content with a "generalised solution" of (4.24), namely the functional h.

Finally, we should say that results for the special cases investigated by various authors (such as Jacobi polynomials and Legendre functions) can be obtained from the above by suitable choices of a, b, c and m.

5 The Hankel transform

§5.1 Introduction

The second main application of the theory in Chapter 3 will be in connection with dual integral equations of Titchmarsh type which can be expressed in terms of operators simply related to the Hankel transform; see Chapter 7. Thus our immediate task is to study the Hankel transform in $F_{p,\mu}$ and $F'_{p,\mu}$. This is a fairly extensive investigation but is justified on two counts; firstly, very full information can be obtained about the Hankel transform and, secondly, the method is typical of that which can be applied to other integral transforms (although such full information is not always obtained, as we shall see in Chapter 8).

The Hankel transform is an ideal candidate for our treatment. If the transform is studied classically, there are problems as regards describing its range in general. To illustrate this point, we consider briefly the Fourier cosine transform (the Hankel transform of order − 1/2) on L^p $(= L^p_0)$. Formally, the Fourier cosine transform $H_{-\frac{1}{2}}$ is defined by

$$(H_{-\frac{1}{2}}\phi)(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(xt)\phi(t)dt \qquad (0 < x < \infty) \ . \tag{5.1}$$

However, if $\phi \in L^p$, the integral does not converge pointwise (almost everywhere) in general and instead we have to use mean convergence. For instance, using Theorem 74 in [78] for the Fourier transform, it can easily be shown that, if $1 < p \leqslant 2$ and $H_{-\frac{1}{2}}\phi$ is defined by

$$(H_{-\frac{1}{2}}\phi)(x) = \underset{n \to \infty}{\text{l.i.m.}(q)} \sqrt{\frac{2}{\pi}} \int_0^n \cos(xt)\phi(t)dt \qquad (5.2)$$

where l.i.m.(q) denotes the limit in the L^q norm, then $H_{-\frac{1}{2}}$ is a continuous linear mapping of L^p into L^q. On the other hand, Theorem 80 in [78] shows that

$$(H_{-\frac{1}{2}}\phi)(x) = x^{2/p-1} \underset{n \to \infty}{\text{l.i.m.}(p)} x^{1-2/p} \sqrt{\frac{2}{\pi}} \int_0^n \cos(xt)\phi(t)dt \qquad (5.3)$$

defines a continuous linear mapping of L^p into $L^p_{2/p-1}$ under the same conditions. If $p = 2$, the situation is very clear since then $H_{-\frac{1}{2}}$ is an involutory homeomorphism of L^2 onto itself (so that $H^{-1}_{-\frac{1}{2}} = H_{-\frac{1}{2}}$). However, if $1 < p < 2$, there does not appear to be a simple characterisation of the range $H_{-\frac{1}{2}}(L^p)$ as is noted in [78] again. What can be said is that the range is not the whole of L^q (for (5.2)) or $L^p_{2/p-1}$ (for (5.3)). The case $p > 2$ as well as the case of $L^p_\mu(\mu \neq 0)$ produces further difficulties which we will not dwell on here.

A number of authors have extended the Hankel transform to various classes of generalised functions. One notable theory was developed by Zemanian and is described in detail in [87, Chapter 5]. This theory has since been extended by Dube and Pandey [6] and Lee [38], [39]. However, there is a drawback to this theory from our point of view. The spaces used are ideally suited to the Hankel transform but seem remarkably awkward from the point of view of other operators which are likely to arise in conjunction with the Hankel transform. The precise behaviour of even simple differentiation operators, such as D itself, is often troublesome to find while the behaviour of an operator such as $I_1^{\eta,\alpha}$ is not at all clear or simple. These points are dealt with in a little more detail in [51]. Another approach has been adopted by Braaksma and Schuitman [2]. They use the Mellin transform to produce a theory for the case $p = \infty$ using spaces very similar to our spaces $F'_{\infty,\mu}$. Not surprisingly, our results for $p = \infty$ are in accord with theirs. Our theory has the advantages over those mentioned above of dealing with all values of p in the range $1 \leq p \leq \infty$ and allowing us to relate the Hankel transform to other operators, such as fractional integrals, very easily.

As before, we start with the simplest case when we have a straightforward integral operator and then extend this operator in a way analogous to that used in Chapter 3. Again the process is quite long so that we defer connections with fractional calculus to the next chapter. However, we will deal with a particularly important modification of the Hankel transform which will be used extensively in Chapters 6 and 7.

§5.2 H_ν on $F_{p,\mu}$: the Simplest Case

For any complex number ν and suitable functions ϕ, $H_\nu\phi$, the Hankel transform of order ν of ϕ is defined formally by

$$H_\nu\phi(x) = \int_0^\infty \sqrt{xt}\, J_\nu(xt)\phi(t)dt \qquad (0 < x < \infty) \tag{5.4}$$

where, as usual, J_ν is the Bessel function of the first kind of order ν. In view of our comments above, there is a problem as to how (5.4) is to be interpreted. Fortunately, if ϕ is a function in $F_{p,\mu}$, and not merely in L^p_μ, the differentiability of ϕ enables us to use pointwise convergence in (5.4) under reasonably generous conditions, as we shall see in a moment. Our comments also indicate a possible difficulty in pinning down the range of H_ν on $F_{p,\mu}$; we would expect H_ν to map $F_{p,0}$ into both $F_{q,0}$ and $F_{p,2/p-1}$ under appropriate conditions, including $1 < p \leq 2$, and this is so. Fortunately, as in §3.2, Theorem 2.9(i) comes to the rescue; if $1 < p \leq 2$, $F_{p,2/p-1} \subseteq F_{q,0}$ so that $F_{p,2/p-1}$ is indicated as a possible candidate for $H_\nu(F_{p,0})$. This, too, will be borne out in practice.

Our first little result sets the ball rolling.

Lemma 5.1

Let $\phi \in F_{p,\mu}$ with $-\operatorname{Re}\nu - 3/2 + 1/p < \operatorname{Re}\mu < 1/p$. Then $H_\nu\phi$, as defined by (5.4), is infinitely differentiable on $(0,\infty)$ and, for $k = 0,1,2,\ldots$,

$$\delta^k H_\nu\phi = (-1)^k H_\nu(\delta+I)^k \phi \tag{5.5}$$

(where, as usual, $\delta = xD$ and I is the identity operator on $F_{p,\mu}$).

Proof: - Integrating by parts, and using the fact [19, p.11] that

$$z^{\nu+1} J_\nu(z) = d/dz(z^{\nu+1} J_{\nu+1}(z)) ,$$

we find that

$$(H_\nu\phi)(x) = -\int_0^\infty (xt)^{-\frac{1}{2}} J_{\nu+1}(xt)\{(\delta\phi)(t) - (\nu + 1/2)\phi(t)\}dt. \tag{5.6}$$

The integrated terms vanish by virtue of Theorem 2.2 and properties of J_ν, while the latter also show that, since $\delta\phi - (\nu + 1/2)\phi \in F_{p,\mu}$, the integral in (5.6) converges absolutely and uniformly on compact subsets of $(0,\infty)$ under the given restrictions on the parameters. Hence $H_\nu\phi$ is continuous on $(0,\infty)$. Further, if, temporarily, we write

$$R_\nu\phi = \delta\phi - (\nu + 1/2)\phi \quad (\phi \in F_{p,\mu}) , \tag{5.7}$$

then

$$(H_\nu\phi)(x) = -\int_0^\infty u^{-1/2} J_{\nu+1}(u)(R_\nu\phi)(u/x)du/x . \tag{5.8}$$

Again we may differentiate under the integral sign to obtain

$$D(H_\nu\phi)(x) = x^{-1}\int_0^\infty u^{-1/2} J_{\nu+1}(u)\left[(u/x)(DR_\nu\phi)(u/x) + R_\nu\phi(u/x)\right]du/x$$

$$\Rightarrow \delta H_\nu\phi\ (x) = \int_0^\infty u^{-1/2} J_{\nu+1}(u)\left[(\delta+I)R_\nu\phi\right](u/x)du/x. \tag{5.9}$$

Finally since $(\delta+I)R_\nu\phi = R_\nu(\delta+I)\phi$, (5.8) and (5.9) give

$$\delta H_\nu\phi = -H_\nu(\delta+I)\phi$$

which is (5.5) for $k = 1$. A simple induction argument completes the proof.

To obtain the continuity of H_ν as a mapping from $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$, we will study the operator G_ν defined for suitable functions ϕ by

$$G_\nu\phi(x) = -\int_0^\infty u^{-1/2} J_{\nu+1}(u)\phi(u/x)\, du/x. \tag{5.10}$$

Lemma 5.2

If $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$, then G_ν is a continuous linear mapping of L^p_μ into $L^p_{2/p-1-\mu}$.

Proof:- First we observe that, for $1 \leq p < \infty$ and $f \in L^p_{2/p-1-\mu}$,

$$\int_0^\infty |x^{\mu+1-2/p} f(x)|^p dx = \int_0^\infty |x^{-\mu-1} f(1/x)|^p dx$$

so that $||f(x)||_{p,2/p-1-\mu} = ||x^{-1}f(1/x)||_{p,\mu}$, using the notation of (0.5). Since the corresponding result for $p = \infty$ is trivial, it is sufficient to prove that $(1/x)G_\nu\phi(1/x) \in L^p_\mu$ for all $\phi \in L^p_\mu$. But, from (5.10),

$$(1/x)(G_\nu\phi)(1/x) = -\int_0^\infty u^{-1/2} J_{\nu+1}(u)\phi(ux)du$$

$$= -\int_0^\infty (v/x)^{-1/2} J_{\nu+1}(v/x)\phi(v)dv/x$$

$$\Rightarrow x^{-\mu-1}(G_\nu\phi)(1/x) = -\int_0^\infty (v/x)^{\mu+1/2} J_{\nu+1}(v/x)\, v^{-\mu}\phi(v)dv/v.$$

Since $x^{-\mu}\phi(x) \in L^p_0$, the result will follow from, for instance [72, Lemma 3.1], provided that

$$\int_0^\infty x^{1/p-1}|x^{-\mu-1/2}J_{\nu+1}(1/x)|dx < \infty$$

$$\text{or} \int_0^\infty t^{\mathrm{Re}\,\mu - 1/p - 1/2}|J_{\nu+1}(t)| < \infty .$$

This is so since $\mathrm{Re}\,\mu - 1/p - 1/2 + \mathrm{Re}\,\nu + 1 > -1$ (for $t = 0$) and $\mathrm{Re}\,\mu - 1/p - 1/2 - 1/2 < -1$ (for $t = \infty$). This completes the proof.

This enables us to prove the following result.

Corollary 5.3

If $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$, then H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$.

Proof:- Let $\phi \in F_{p,\mu}$. We note that $\delta^k\phi$ is a linear combination of the functions $x^\ell\, d^\ell\phi/dx^\ell$ ($\ell = 0,1,2,\dots k$) and $x^k d^k\phi/dx^k$ is a linear combination of the functions $\delta^\ell\phi$ ($\ell = 0,1,2,\dots,k$). Also, from (5.7), (5.8) and (5.10), $H_\nu = G_\nu R_\nu$. Hence

$$\begin{aligned}
&\gamma_k^{p,2/p-1-\mu}(H_\nu\phi)\\
&\leq \sum_{\ell=0}^{k} a_\ell\, \gamma_o^{p,2/p-1-\mu}(\delta^\ell H_\nu\phi)\\
&= \sum_{\ell=0}^{k} a_\ell\, \gamma_o^{p,2/p-1-\mu}\,(H_\nu(\delta+I)^\ell\phi) && \text{by } (5.5)\\
&\leq \sum_{\ell=0}^{k} b_\ell\, \gamma_o^{p,\mu}\,(R_\nu(\delta+I)^\ell\phi) && \text{by Lemma 5.2}\\
&\leq \sum_{\ell=0}^{k+1} c_\ell\, \gamma_o^{p,\mu}\,(\delta^\ell\phi) && \text{by } (5.7)\\
&\leq \sum_{\ell=0}^{k+1} d_\ell\, \gamma_\ell^{p,\mu}(\phi)
\end{aligned}$$

where a_ℓ, b_ℓ ($\ell = 0,1,\dots,k$) and c_ℓ, d_ℓ ($\ell = 0,1,\dots,k+1$) are constants independent of ϕ. This completes the proof.

It is natural to ask whether, in the circumstances of Corollary 5.3, H_ν maps $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$. This is so, under certain additional conditions. To obtain the result we again fall back on analytic continuation.

Theorem 5.4

Let $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$ and let $\phi \in F_{p,\mu}$. Then, H_ν has Fréchet derivatives $\partial^k H_\nu/\partial\nu^k$ ($k = 0,1,2,\dots$) with respect to ν on $F_{p,\mu}$ and

$$[\partial^k H_\nu/\partial\nu^k]\phi(x) = \int_0^\infty \sqrt{xt}\,\partial^k J_\nu/\partial\nu^k(xt)\phi(t)dt\,. \tag{5.11}$$

In particular for fixed $x \in (0,\infty)$ and fixed $\phi \in F_{p,\mu}$ with $\mathrm{Re}\,\mu < 1/p$, $H_\nu\phi(x)$ is an analytic function of ν in the half-plane $\mathrm{Re}\,\nu > -\mathrm{Re}\,\mu - 3/2 + 1/p$.

<u>Proof</u>:- We omit the details. A proof using the Cauchy integral formula is given in [51, Theorem 3.3].

<u>Corollary 5.5</u>

Let $-1/q < \mathrm{Re}\,\mu < 1/p$. Then, for fixed $\phi \in F_{p,\mu}$ and fixed $x \in (0,\infty)$, $H_\nu H_\nu\phi(x)$ is analytic in ν in the half-plane

$$\mathrm{Re}\,\nu > \max(-\mathrm{Re}\,\mu - 3/2 + 1/p,\ \mathrm{Re}\,\mu - 3/2 + 1/q) .$$

<u>Proof</u>:- If $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$, $H_\nu\phi \in F_{p,2/p-1-\mu}$ and if also $-\mathrm{Re}\,\nu - 3/2 + 1/p < 2/p - 1 - \mathrm{Re}\,\mu < 1/p$, $H_\nu H_\nu\phi \in F_{p,\mu}$, by Corollary 5.3. Hence under the given conditions and with ν in the stated half-plane, $H_\nu H_\nu\phi \in F_{p,\mu}$ for all ν. Then, using Theorem 5.4 and the chain rule for Fréchet derivatives [70, p.164], we deduce that $H_\nu H_\nu$ as an operator from $F_{p,\mu}$ into $F_{p,\mu}$ is Fréchet differentiable for the stated values of ν and the result follows.

The point of considering $H_\nu H_\nu$ is that if, in addition to the various other conditions, $\mathrm{Re}\,\nu > -1/2$, the Hankel inversion theorem [82, p.456] states that $H_\nu H_\nu$ is the identity operator. We can use analytic continuation to remove this extra restriction as follows.

<u>Lemma 5.6</u>

If $\max(-\mathrm{Re}\,\nu - 3/2 + 1/p,\ -1/q) < \mathrm{Re}\,\mu < \min(1/p,\ \mathrm{Re}\,\nu + 3/2 - 1/q)$, then

$$H_\nu H_\nu\phi = \phi \quad (\phi \in F_{p,\mu}) \cdot \tag{5.12}$$

<u>Proof</u>:- First let $\phi \in C_o^\infty(0,\infty)$. Then, since $\phi \in L^1(0,\infty)$, (5.12) follows by the Hankel inversion theorem if $\mathrm{Re}\,\nu > -1/2$ and by Corollary 5.5 otherwise. Finally, since $H_\nu H_\nu$ is a continuous linear mapping from $F_{p,\mu}$ into itself under the given conditions, (5.12) holds for all $\phi \in F_{p,\mu}$ by Corollary 2.7.

We can therefore summarise our progress so far.

Theorem 5.7

If $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$, then H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$. If, also, $-1/q < \mathrm{Re}\,\mu < \mathrm{Re}\,\nu + 3/2 - 1/q$, then H_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ and

$$H_\nu^{-1} = H_\nu . \tag{5.13}$$

Proof:- The results follow at once from Corollary 5.3 and Lemma 5.6.

The possible values of μ and ν in Theorem 5.7 are somewhat restricted. For instance, the condition $-1/q < \mathrm{Re}\,\mu < 1/p$ restricts μ to a strip of width 1. However, the next stage in our operation relaxes the restrictions considerably.

§5.3 H_ν on $F_{p,\mu}$: the Extended Operator.

It turns out that there is a fairly simple and natural way to remove the restriction $\mathrm{Re}\,\mu < 1/p$ which figures prominently in the previous section. We return to (5.6) and instead of using R_ν, as given by (5.7), we will find it more convenient to use the operator N_ν, defined for any complex ν and suitable functions ϕ by

$$N_\nu\phi(x) = x^{\nu+1/2}d/dx(x^{-\nu-1/2}\phi) = d\phi/dx - (\nu + 1/2)\, x^{-1}\phi . \tag{5.14}$$

Obviously $N_\nu\phi(x) = x^{-1}R_\nu\phi(x)$ and from Theorem 2.11 and Corollary 2.15, N_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-1}$ for any complex numbers μ and ν. (5.6) can then be written in the form

$$H_\nu\phi(x) = -x^{-1}\, H_{\nu+1}\, N_\nu\phi(x) \tag{5.15}$$

provided that $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$ in the first instance. However, by Corollary 5.3, the right-hand side is meaningful provided only that

$$-\mathrm{Re}(\nu+1) - 3/2 + 1/p < \mathrm{Re}(\mu-1) < 1/p$$

$$\text{or } -\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p + 1.$$

We can therefore use (5.15) to define an extension of the operator H_ν to spaces $F_{p,\mu}$ such that $1/p \leq \text{Re}\,\mu < 1/p + 1$ (and $-\text{Re}\,\nu - 3/2 + 1/p < \text{Re}\,\mu$ still) and for these spaces H_ν is an integro-differential operator. When $\text{Re}\,\mu < 1/p$, H_ν as given by (5.15) coincides with H_ν as given by (5.4) because of the above motivation. Having replaced $\text{Re}\,\mu < 1/p$ by $\text{Re}\,\mu < 1/p + 1$, we can now repeat the process and eventually obtain an operator H_ν on $F_{p,\mu}$ subject only to the restriction $-\text{Re}\,\nu - 3/2 + 1/p < \text{Re}\,\mu$. Here is the appropriate definition.

Definition 5.8

Let $\text{Re}\,\nu > -\text{Re}\,\mu - 3/2 + 1/p$. For $\phi \in F_{p,\mu}$, we define $H_\nu\phi$ by

$$H_\nu\phi(x) = (-1)^k x^{-k} H_{\nu+k} N_{\nu+k-1} \cdots N_{\nu+1} N_\nu\phi(x) \qquad (5.16)$$

where k is any non-negative integer k such that $\text{Re}\,\mu < 1/p + k$.

Notes

1. Since $-\text{Re}(\nu+k) - 3/2 + 1/p < \text{Re}\,\mu - k < 1/p$, the right-hand side of (5.16) exists and defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ by Theorem 2.11, Corollary 2.15 and Corollary 5.3.

2. As usual, we must check that this definition is independent of the choice of the non-negative integer k satisfying $\text{Re}\,\mu < 1/p + k$. To see this, suppose also that $\text{Re}\,\mu < 1/p + \ell$ where $k > \ell$ (without loss of generality) and that $-\text{Re}\,\nu - 3/2 + 1/p < \text{Re}\,\mu$. Then for $\phi \in F_{p,\mu}$,

$$\begin{aligned}
&(-1)^k x^{-k} H_{\nu+k} N_{\nu+k-1} \cdots N_\nu\phi \\
&= (-1)^\ell x^{-\ell} (-1)^{k-\ell} x^{-(k-\ell)} H_{\nu+k} N_{\nu+k-1} \cdots N_{\nu+\ell}(N_{\nu+\ell-1} \cdots N_\nu\phi) \\
&= (-1)^\ell x^{-\ell} H_{\nu+k-(k-\ell)} N_{\nu+\ell-1} \cdots N_\nu\phi .
\end{aligned}$$

The last step follows by $(k-\ell)$ applications of (5.15) with ϕ replaced by $N_{\nu+\ell-1} \cdots N_\nu\phi$, this being justified since $-\text{Re}(\nu+\ell)-3/2+1/p < \text{Re}\,\mu - \ell < 1/p + (k-\ell)$. The required result follows at once.

3. By Note 2, if $\phi \in F_{p,\mu}$ where $-\text{Re}\,\nu - 3/2 + 1/p < \text{Re}\,\mu < 1/p$, $H_\nu\phi$, as given by (5.16), agrees with $H_\nu\phi$, as given by (5.4).

4. If $-\text{Re}\,\nu - 3/2 + 1/p < \text{Re}\,\mu < 1/p + k$, then (5.16) can be written explicitly in the form

$$(H_\nu\phi)(x) = (-1)^k x^{-k} \int_0^\infty \sqrt{xt}\, J_{\nu+k}(xt) t^{\nu+k+1/2} (t^{-1}d/dt)^k\, t^{-\nu-1/2}\phi(t)dt$$

$$= (-2)^k\, x^{-k}\, H_{\nu+k}\, x^{\nu+k+1/2}\, (D_2)^k\, x^{-\nu-1/2}\phi(x) \tag{5.17}$$

where D_2 is defined via (2.14).

5. For $\phi \in C_0^\infty(0,\infty)$ (regarded as a subspace of $F_{p,\mu}$), (5.16) agrees with (5.4).

6. A similar process has been carried through by Zemanian [87, p.163] for his spaces.

To help us to study the invertibility of our extended operator, we need

<u>Lemma 5.9</u>

Let $\text{Re}\,\nu > -\text{Re}\,\mu - 3/2 + 1/p$, $\phi \in F_{p,\mu}$ and let ℓ be any non-negative integer. Then

$$H_{\nu+\ell}(x^\ell\phi) = (-1)^\ell\, N_{\nu+\ell-1} \cdots N_{\nu+1}\, N_\nu H_\nu\phi\,. \tag{5.18}$$

If also $\text{Re}(\mu+\nu) \neq -3/2 + 1/p - 2s$ $(s = 0, 1, \ldots, \ell-1)$, then

$$H_\nu\phi = (-1)^\ell\, N_\nu^{-1}\, N_{\nu+1}^{-1} \cdots N_{\nu+\ell-1}^{-1}\, H_{\nu+\ell}\,(x^\ell\phi). \tag{5.19}$$

<u>Proof</u>:- Let $\phi \in C_0^\infty(0,\infty)$. Then, by Definition 5.8, Note 5, $H_\nu\phi$ is given via (5.4) and since $x^\ell\phi \in C_0^\infty(0,\infty)$, the same applies to $H_{\nu+\ell}(x^\ell\phi)$. Hence in this case, (5.18) can be established by differentiating under the integral sign in (5.4), the details being as in [87, p.139]. Also, since $-\text{Re}(\nu+\ell) - 3/2 + 1/p < \text{Re}\,\mu + \ell$, Definition 5.8, Note 1, shows that both sides of (5.18) define continuous mappings of $F_{p,\mu}$ into $F_{p,2/p-1-\mu-\ell}$. Hence (5.18) holds for all $\phi \in F_{p,\mu}$ by Corollary 2.7.

To establish (5.19), we recall that from (5.14),

$$N_\nu = x^{\nu+1/2}\, D\, x^{-\nu-1/2}\,.$$

Hence by Corollary 2.15, N_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ provided that $\mathrm{Re}(\mu-\nu-1/2) \neq 1/p$. Thus, if $s = 0,1, \ldots ,\ell -1$, $N_{\nu+s}$ is a homeomorphism of $F_{p,2/p-1-\mu-s}$ onto $F_{p,2/p-1-\mu-s-1}$ since $\mathrm{Re}(2/p - 1 - \mu - s - \nu - s - 1/2) \neq 1/p$ by hypothesis. (5.19) now follows at once from (5.18).

We will find two uses for Lemma 5.9 in what follows. The first use is in proving the following result.

Theorem 5.10

(i) If $\mathrm{Re}\,\nu > -\mathrm{Re}\,\mu - 3/2 + 1/p$, then H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p - 1 - \mu}$.

(ii) H_ν is Fréchet differentiable on $F_{p,\mu}$ for $\mathrm{Re}\,\nu > - \mathrm{Re}\,\mu - 3/2 + 1/p$.

(iii) If $\mathrm{Re}\,\nu > \max(-\mathrm{Re}\,\mu - 3/2 + 1/p,\ \mathrm{Re}\,\mu - 3/2 + 1/q)$, then H_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ and $H_\nu^{-1} = H_\nu$.

Proof:- (i) Follows from Definition 5.8, Note 1.

(ii) Since N_ν is Fréchet differentiable on $F_{p,\mu}$ for all complex μ and ν, the result follows from (5.16), Theorem 5.4 and [70, p.164].

(iii) Since $\mathrm{Re}\,\nu > - \mathrm{Re}\,\mu - 3/2 + 1/p$ and $\mathrm{Re}\,\nu > - \mathrm{Re}(2/p - 1 - \mu) - 3/2 + 1/p$, $H_\nu H_\nu$ is a continuous linear mapping from $F_{p,\mu}$ into itself and Fréchet differentiable under the given conditions, by (i) and (ii). We will show that $H_\nu H_\nu$ is the identity operator on $F_{p,\mu}$. Let k be a non-negative integer such that $\mathrm{Re}\,\mu > - 1/q - k$ and let $\phi \in F_{p,\mu}$. By Definition 5.8 and (5.18)

$$H_\nu(H_\nu\phi) = (-1)^k x^{-k} H_{\nu+k} N_{\nu+k-1} \cdots N_\nu H_\nu \phi$$
$$= x^{-k} H_{\nu+k} H_{\nu+k} x^k \phi . \tag{5.20}$$

Now $H_{\nu+k} x^k\phi \in F_{p,2/p-1-\mu-k}$ and since $-\mathrm{Re}(\nu+k) - 3/2 + 1/p < \mathrm{Re}(2/p - 1 - \mu - k) < 1/p$, we obtain, from (5.4),

$$H_{\nu+k}(H_{\nu+k} x^k\phi) = \int_0^\infty \sqrt{xt}\, J_{\nu+k}(xt) \, [H_{\nu+k}(x^k\phi)](t)dt. \tag{5.21}$$

If, in particular, $\phi \in C_o^\infty(0,\infty)$ (as a subset of $F_{p,\mu}$), $x^k\phi \in C_o^\infty(0,\infty)$ (as a subset of $F_{p,\mu+k}$) and since $\mathrm{Re}(\nu+k) > -\mathrm{Re}(\mu+k) - 3/2 + 1/p$,

$$[H_{\nu+k}(x^k\phi)](t) = \int_0^\infty \sqrt{tu}\, J_{\nu+k}(tu)\, u^k\phi(u)du \qquad (5.22)$$

by Definition 5.8, Note 5. Since $x^k\phi(x) \in L^1(0,\infty)$ in this case, the Hankel inversion theorem [82, p.456] shows that, if also $\mathrm{Re}\,\nu > -1/2 - k$, then

$$H_{\nu+k}(H_{\nu+k}\, x^k\phi) = x^k\phi$$

from (5.21) and (5.22). Hence from (5.20), we have $H_\nu H_\nu\phi = \phi$ for $\phi \in C_o^\infty(0,\infty)$ and under the additional restriction $\mathrm{Re}\,\nu > -1/2 - k$. The latter can be removed by analytic continuation and then, by Corollary 2.7 and continuity, we find that $H_\nu H_\nu\phi = \phi$ for all $\phi \in F_{p,\mu}$ under the stated conditions. Similarly $H_\nu H_\nu\psi = \psi$ for $\psi \in F_{p,2/p-1-\mu}$ and this completes the proof of the theorem.

Remark 5.11

This proof is typical of many we shall encounter and, in future, we shall not give such full details.

Having dealt with the restriction $\mathrm{Re}\,\mu < 1/p$, we now turn to the other restriction which figures prominently in §5.2, namely $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu$. (5.16) is no help to us because if $H_{\nu+k}$ is interpreted via Definition 5.8, both sides of (5.16) are meaningful under the same condition, namely that which we are trying to remove! This is where we come to the second use of Lemma 5.9.

Both sides of (5.18) have a meaning when $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p$. However the left-hand side is meaningful under the weaker condition $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2\ell$. To exploit this, we "solve" for $H_\nu\phi$ obtaining (5.19). But now we run up against the same problems as in §3.3 because of the non-invertibility of N_ν on $F_{p,\mu}$ when $\mathrm{Re}(\mu-\nu-1/2) = 1/p$. Nevertheless, we can make some progress here. The conditions under which (5.19) is valid suggest the following definition.

Definition 5.12

For $1 \leqslant p \leqslant \infty$ and any complex number μ, we define the set $\Omega_{p,\mu}$ of complex numbers by

$$\Omega_{p,\mu} = \{\nu: \ \mathrm{Re}(\mu+\nu) \neq -3/2 + 1/p - 2\ell \text{ for } \ell = 0,1,2, \ldots\} . \tag{5.23}$$

Remark 5.13

As we shall see later, there are close relationships between $\Omega_{p,\mu}$ and the sets $A_{p,\mu,2}$ and $A'_{p,\mu,2}$ as given by Definitions 3.26 and 3.32 (with $m = 2$). These emerge when we discuss the connections between fractional calculus and the Hankel transform in Chapter 6.

(5.19) and the conditions for validity also suggest the following definition.

Definition 5.14

Let $\nu \in \Omega_{p,\mu}$ and let k be a non-negative integer such that $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2k$. We define H_ν on $F_{p,\mu}$ by

$$H_\nu\phi(x) = (-1)^k N_\nu^{-1} N_{\nu+1}^{-1} \cdots N_{\nu+k-1}^{-1} H_{\nu+k}(x^k\phi) \quad (\phi \in F_{p,\mu}) \tag{5.24}$$

where $H_{\nu+k}(x^k\phi)$ is defined via (5.16).

Notes

1. Since $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2k$ and $\mathrm{Re}(\mu+\nu) \neq -3/2 + 1/p - 2s$ $(s = 0,1, \ldots, k-1)$ the right-hand side defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ by Theorem 5.10(i) and Lemma 5.9.

2. The definition is independent of the non-negative integer k satisfying $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2k$. The proof is similar to that in Definition 5.8, Note 2 but uses (5.19) rather than (5.15). We omit the details.

3. As a special case of Note 2, for spaces $F_{p,\mu}$ such that $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p$, (5.24) agrees with (5.16).

4. If desired, an explicit expression for H_ν can be obtained using (5.17). In passing we note that an alternative expression for $H_\nu\phi$ is given by

$$H_\nu\phi(x) = (-2)^{-k}\, x^{\nu+1/2}(D_2)^{-k}\, x^{-\nu-k-1/2}\, H_{\nu+k}\, x^k\phi\ . \tag{5.25}$$

In particular, if $\mathrm{Re}(\mu+\nu) < -3/2 + 1/p$ and k is the unique positive integer such that

$$-3/2 + 1/p - 2k < \mathrm{Re}(\mu+\nu) < -3/2 + 1/p - 2(k-1),$$

then (5.25) takes on the form

$$H_\nu\phi(x) = (-2)^{-k}\, x^k\, I_2^{-\nu/2-1/4-k/2,k}\, H_{\nu+k}\, x^k\phi \tag{5.26}$$

where $I_2^{-\nu/2-1/4-k/2,\,k}$ is defined via (3.11). (5.26) foreshadows the results in Chapter 6.

5. If $\phi \in C_o^\infty(0,\infty)$ is regarded as an element of $F_{p,\mu}$ where $\mathrm{Re}(\mu+\nu) < -3/2 + 1/p$, then $H_\nu\phi$ as given by (5.24) does not collapse to (5.4), for reasons similar to those discussed in Example 3.29. The situation is analogous to "cut" Hankel transforms as discussed by, for instance, Erdélyi [8].

To obtain the mapping properties of this version of H_ν we need

<u>Lemma 5.15</u>

Let $\nu \in \Omega_{p,\mu}$, $\phi \in F_{p,\mu}$ and let k be a non-negative integer.

Then (5.16) holds if H_ν and $H_{\nu+k}$ are interpreted in the sense of Definition 5.14.

<u>Proof</u>:- Let ℓ be a non-negative integer such that $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2\ell$. Since also $\mathrm{Re}(\mu - k + \nu + k) > -3/2 + 1/p - 2\ell$, we have, for $\phi \in F_{p,\mu}$,

$$\begin{aligned}
&(-1)^k\, x^{-k}\, H_{\nu+k}\, N_{\nu+k-1} \cdots N_\nu\phi \\
&= (-1)^k\, x^{-k}\, (-1)^\ell\, N_{\nu+k}^{-1} \cdots N_{\nu+k+\ell-1}^{-1}\, H_{\nu+k+\ell}\, x^\ell N_{\nu+k-1} \cdots N_\nu\phi \\
&= (-1)^{k+\ell}\, N_\nu^{-1} \cdots N_{\nu+\ell-1}^{-1}\, x^{-k}\, H_{\nu+k+\ell}\, N_{\nu+k+\ell-1} \cdots\cdots N_{\nu+\ell}\, x^\ell\phi
\end{aligned} \tag{5.27}$$

since, by (5.14), $x^\lambda N_\nu = N_{\nu+\lambda} x^\lambda$, $N_\nu^{-1} x^{-\lambda} = x^{-\lambda} N_{\nu+\lambda}^{-1}$ under the appropriate conditions. By (5.16) and Definition 5.8, Note 3, (5.27) can be written as $(-1)^\ell N_\nu^{-1} \cdots N_{\nu+\ell-1}^{-1} H_{\nu+\ell}\, x^\ell\phi$ which is $H_\nu\phi$ by (5.24). This completes the proof.

Remark 5.16

We can prove similarly that formulae such as (5.18), (5.19) and (5.24) still hold when each Hankel transform H_ν on $F_{p,\mu}$ is interpreted via

(5.4) if $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < 1/p$

(5.16) if $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu$ but $\mathrm{Re}\,\mu \geq 1/p$

and (5.24) if $\nu \in \Omega_{p,\mu}$ but $\mathrm{Re}(\mu+\nu) < -3/2 + 1/p$.

However we shall omit details.

We are now ready for the fullest description of H_ν on $F_{p,\mu}$.

Theorem 5.17

(i) If $\nu \in \Omega_{p,\mu}$, then H_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$. If also $\nu \in \Omega_{q,-\mu}$, then H_ν is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-1-\mu}$ and $H_\nu^{-1} = H_\nu$.

(ii) H_ν is Fréchet differentiable on $F_{p,\mu}$ for $\nu \in \Omega_{p,\mu}$.

Proof:- (i) The first statement is Note 1 following Definition 5.14. For the second, let k be a non-negative integer such that $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2k$. Assume also that $\mathrm{Re}(-\mu+\nu) > -3/2 + 1/q$. Then $H_{\nu+k}$ is invertible on $F_{p,\mu+k}$ with $H_{\nu+k}^{-1} = H_{\nu+k}$ by Theorem 5.10 (iii). Hence from (5.24), H_ν is invertible on $F_{p,\mu}$ and, for $\psi \in F_{p,2/p-1-\mu}$,

$$H_\nu^{-1}\psi = (-1)^k x^{-k} H_{\nu+k} N_{\nu+k-1} \cdots N_\nu \psi = H_\nu \psi$$

by Lemma 5.15. The restriction $\mathrm{Re}(-\mu+\nu) > -3/2 + 1/q$ can be relaxed to $\nu \in \Omega_{q,-\mu}$ by similar arguments. Notice that since $\Omega_{p,\mu}$ is not simply connected, we cannot merely use Theorem 5.10(iii) and analytic continuation.

(ii) can be proved using Theorem 5.10(ii) and [70, p.164] but we shall omit the details. This completes the proof.

It should be noted that there are other equivalent methods of carrying out the above programme. For instance, in [51], we made use of the operator

M_ν defined for any complex number ν and suitable functions ϕ by

$$M_\nu\phi = x^{-\nu-1/2}\, D\, x^{\nu+1/2}\phi = d\phi/dx + (\nu + 1/2)\, x^{-1}\phi\,. \tag{5.28}$$

Thus M_ν is the formal adjoint of $-N_\nu$ and the following analogues of (5.16) and (5.24) come as no surprise.

Theorem 5.18

Let $\nu \in \Omega_{p,\mu}$, let $\phi \in F_{p,\mu}$ and let k be a non-negative integer. Then

$$H_\nu\phi(x) = M_\nu M_{\nu+1} \cdots M_{\nu+k-1}\, H_{\nu+k}(x^{-k}\phi) \tag{5.29}$$

$$H_\nu\phi(x) = x^k\, H_{\nu+k}\, M_{\nu+k-1}^{-1} \cdots M_\nu^{-1}\phi(x). \tag{5.30}$$

Proof:- (i) If $\phi \in C_0^\infty(0,\infty)$ and $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p$, (5.29) can be proved as in [87, p.140] since we can then make use of (5.4). By continuity and Corollary 2.7, (5.29) holds for all $\phi \in F_{p,\mu}$ in this case. Otherwise, we have to use (5.24) and the proof is similar to those of Lemma 5.15 and Theorem 5.17 and uses Theorem 3.41. (ii) is similar. We merely note that M_ν is invertible on $F_{p,\mu}$ if and only if $\mathrm{Re}(\mu + \nu + 1/2) \neq 1/p$. Thus (5.30) is well-defined if $\nu \in \Omega_{p,\mu}$.

Remark 5.19

Formula (5.30) was used as a definition in our approach in [51]. We now see that the two approaches are, indeed, equivalent.

§5.4 H_ν on $F'_{p,\mu}$

Now that we have a complete description of H_ν on $F_{p,\mu}$, we can obtain our results for $F'_{p,\mu}$ very easily by using adjoint operators. The extension process takes on the familiar form.

By standard classical results, the operator H_ν, as defined using mean convergence in a manner analogous to (5.3), is a continuous linear mapping from L^q_0 into $L^q_{2/q-1}$ provided that $\mathrm{Re}\,\nu > -3/2 + 1/q$ and $1 < q \leq 2$. In fact,

H_ν can be extended to a continuous linear mapping from $L^q_{-\mu}$ into $L^r_{1/q + 1/r - 1 + \mu}$ provided that $1 < q \leq r < \infty$ and $\max(1/q, 1 - 1/r) \leq 1/q + \text{Re}\,\mu < \text{Re}\,\nu + 3/2$ as was shown by Rooney in [73]. We merely mention that our extension is consistent with Rooney's but allows the range of values of the parameters to be enlarged by passing to generalised functions. For simplicity, we will deal with the case $\mu = 0$, $\text{Re}\,\nu > -3/2 + 1/q$ and $1 < q \leq 2$. Let $g \in L^q_0$. Then g generates $\tau g \in F'_{p,0}$ according to (2.16). Also $H_\nu g \in L^q_{2/q-1}$ and generates $\tau H_\nu g \in F'_{p,1-2/q} = F'_{p,2/p-1}$. We require

$$H_\nu \tau g = \tau H_\nu g \tag{5.31}$$

as an equation in $F'_{p,2/p-1}$, where the H_ν on the left is the desired extension. To see what (5.31) entails, let $\phi \in C_0^\infty(0,\infty)$ be regarded as an element of $F_{p,\, 2/p - 1}$. Then we have

$$(H_\nu \tau g, \phi) = (\tau H_\nu g, \phi) = \int_0^\infty H_\nu g(x)\phi(x)dx = \int_0^\infty g(x) H_\nu \phi(x) dx$$

where we have used a version of Parseval's formula. Since $-\text{Re}\,\nu - 3/2 + 1/p < 2/p - 1 < 1/p$ under the given conditions, Corollary 5.3 and Hölder's inequality lead to

$$(H_\nu \tau g, \phi) = (\tau g, H_\nu \phi)$$

for all $\phi \in F_{p,2/p-1}$. For $g \in L^q_{-\mu}$ we arrive at the same formal expression (under the appropriate conditions) but with $\phi \in F_{p,2/p-1-\mu}$. This in turn suggests that we require

$$(H_\nu f, \phi) = (f, H_\nu \phi)$$

for all $\phi \in F_{p,2/p-1-\mu}$ and all $f \in F'_{p,\mu}$, regular or not. Further, the expression on the right is well-defined provided only that $\nu \in \Omega_{p,2/p-1-\mu} = \Omega_{q,-\mu}$, by Theorem 5.17. Hence we finally arrive at the following definition.

Definition 5.20

Let $1 \leq p \leq \infty$ and let μ be any complex number. For $\nu \in \Omega_{q,-\mu}$, we define

H_ν on $F'_{p,\mu}$ by

$$(H_\nu f, \phi) = (f, H_\nu \phi) \tag{5.32}$$

for $f \in F'_{p,\mu}$ and $\phi \in F_{p,2/p-1-\mu}$, where $H_\nu\phi$ is as in Definition 5.14.

As we mentioned above, (5.32) incorporates (5.31) for classical functions f for which $H_\nu f$ exists under the conditions of, for instance, Rooney [73].

The properties of H_ν on $F'_{p,\mu}$ are easily obtained.

<u>Theorem 5.21</u>

(i) If $\nu \in \Omega_{q,-\mu}$, then H_ν is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,2/p-1-\mu}$. If also $\nu \in \Omega_{p,\mu}$, H_ν is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,2/p-1-\mu}$ and $H_\nu^{-1} = H_\nu$.

(ii) For fixed $f \in F'_{p,\mu}$ and $\phi \in F_{p,2/p-1-\mu}$, $(H_\nu f, \phi)$ is an analytic function of ν in $\Omega_{q,-\mu}$.

<u>Proof</u>:- The results follow immediately from Theorem 5.17 above and Theorems 1.10-1 and 1.10-2 in [87] together with the fact that $\Omega_{q,-\mu} = \Omega_{p,2/p-1-\mu}$.

We mention in passing that obvious analogues of formulae such as (5.16), (5.18), (5.19) and (5.24) can be established on $F'_{p,\mu}$ by taking adjoints of the results in the previous section and noting that the adjoints of M_ν and N_ν are $-N_\nu$ and $-M_\nu$ respectively. We omit the details.

§5.5 The Modified Hankel Transform $S^{\eta,\alpha}$

It is a routine matter to deal with several operators closely related to H_ν. For instance, there are obvious analogues of Rooney's results in [72] for the composition of two Hankel transforms and of Okikiolu's results in [54], [55] for operators of the form $x^{-\nu-1/2} H_\nu x^{-\nu-1/2}$. However, for our purposes, there is one modification which is of great importance and for which we will display results explicitly. This is the operator $S^{\eta,\alpha}$ defined initially for suitable complex numbers η and α and suitable functions ϕ by

$$S^{\eta,\alpha}\phi(x) = 2^{\alpha}x^{-\alpha}\int_0^{\infty} t^{1-\alpha} J_{2\eta+\alpha}(xt)\phi(t)dt. \tag{5.33}$$

This operator appears in the work of Kober and Erdélyi [32], [34] and is extensively used by Sneddon [76] in connection with dual integral equations. It is this application which will also concern us in Chapter 7. Meanwhile we observe that, formally,

$$S^{\eta,\alpha}\phi(x) = 2^{\alpha}x^{-1/2-\alpha} H_{2\eta+\alpha}x^{1/2-\alpha}\phi(x)$$

where $H_{2\eta+\alpha}$ is defined via (5.4). However, the right-hand side is well-defined for $\phi \in F_{p,\mu}$ provided only that $2\eta+\alpha \in \Omega_{p,\mu+1/2-\alpha}$, that is,

$$\mathrm{Re}(2\eta+\alpha+\mu+1/2-\alpha) \neq -3/2 + 1/p - 2\ell \qquad (\ell = 0,1,2, \ldots)$$

$$\text{or} \quad \mathrm{Re}(2\eta + \mu) + 2 \neq 1/p - 2\ell \qquad (\ell = 0,1,2, \ldots)$$

so that $\eta \in A_{p,\mu,2}$ in the notation of Definition 3.26. Because of what comes later, it is convenient to drop the suffix 2 from $A_{p,\mu,2}$ and $A'_{p,\mu,2}$ (as given by Definition 3.32).

Notation 5.22

For $1 \leq p \leq \infty$ and any complex number μ,

$$A_{p,\mu} = \{\eta\colon \mathrm{Re}(2\eta+\mu) + 2 \neq 1/p - 2\ell \qquad (\ell = 0,1,2, \ldots)\} \tag{5.34}$$

$$A'_{p,\mu} = \{\eta\colon \mathrm{Re}(2\eta-\mu) \neq -1/p - 2\ell \qquad (\ell = 0,1,2, \ldots)\}\,. \tag{5.35}$$

We then make the following definition, suggested by the preamble above.

Definition 5.23

For $\eta \in A_{p,\mu}$ and any complex α, we define $S^{\eta,\alpha}$ on $F_{p,\mu}$ by

$$S^{\eta,\alpha}\phi(x) = 2^{\alpha}x^{-1/2-\alpha} H_{2\eta+\alpha}\, x^{1/2-\alpha}\phi(x) \tag{5.36}$$

where $H_{2\eta+\alpha}$ is defined via (5.24).

The following theorem is almost immediate.

Theorem 5.24

If $\eta \in A_{p,\mu}$ and α is any complex number, then $S^{\eta,\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-2-\mu}$. If also $\eta+\alpha \in A'_{p,\mu}$, then $S^{\eta,\alpha}$ is a homeomorphism of $F_{p,\mu}$ onto $F_{p,2/p-2-\mu}$ and $(S^{\eta,\alpha})^{-1} = S^{\eta+\alpha,-\alpha}$.

Proof:- This follows in a routine manner from (5.36) and Theorem 5.17 together with the observation that $A_{p,2/p-2-\mu} = A'_{p,\mu}$.

To see how to extend $S^{\eta,\alpha}$ to $F'_{p,\mu}$, we proceed formally. From (5.36) and (5.32), we have

$$(S^{\eta,\alpha}f,\phi) = (f,\ 2^{\alpha}x^{1/2-\alpha}H_{2\eta+\alpha}\ x^{-1/2-\alpha}\phi) = (f,x\ S^{\eta,\alpha}x^{-1}\phi)$$

and the right-hand side is meaningful if $f \in F'_{p,\mu}$, $\phi \in F_{p,2/p-\mu}$ and $\eta \in A_{p,2/p-1-\mu} = A_{q,-\mu}$, by Theorem 5.24. Hence, we are led to the following.

Definition 5.25

For $\eta \in A_{q,-\mu}$ and any complex number α, we define $S^{\eta,\alpha}$ on $F'_{p,\mu}$ by

$$(S^{\eta,\alpha}\ f,\phi) = (f,\ xS^{\eta,\alpha}\ x^{-1}\phi) \quad (\phi \in F_{p,2/p-\mu}) \ . \tag{5.37}$$

This definition ensures that

$$S^{\eta,\alpha}\tau g = \tau\ S^{\eta,\alpha}g \tag{5.38}$$

for $g \in L^q_{-\mu}$ under the appropriate conditions.

Theorem 5.26

If $\eta \in A_{q,-\mu}$ and α is any complex number, then $S^{\eta,\alpha}$ is a continuous linear mapping from $F'_{p,\mu}$ into $F'_{p,2/p-\mu}$. If also $\eta+\alpha \in A'_{q,-\mu}$, then $S^{\eta,\alpha}$ is a homeomorphism of $F'_{p,\mu}$ onto $F'_{p,2/p-\mu}$ and $(S^{\eta,\alpha})^{-1} = S^{\eta+\alpha,-\alpha}$.

Proof:- This follows easily from Theorem 5.24 above and Theorems 1.10-1 and 1.10-2 in [87].

We now have all the information we require about H_ν and $S^{\eta,\alpha}$ on $F'_{p,\mu}$ and, whereas it would be possible to mention some consequences of a classical nature at this stage, there will be enough of these in Chapter 7.

6 Fractional calculus and the Hankel transform

§6.1 Introduction

The purpose of this chapter is to use results in Chapters 3 and 5 to establish some connections between fractional calculus and the operators H_ν and $S^{\eta,\alpha}$ within the context of our spaces $F'_{p,\mu}$. There are many such connections but we shall restrict attention to those which illustrate points of the theory already developed and to those which will be needed in the next chapter.

It is not at all surprising that there are such connections, since many properties of the Bessel function J_ν can be written in terms of fractional integrals and derivatives. To give one instance, we see from [78, formula (7.1.11)] that

$$x^{1/2-\nu}\, J_{\nu-1/2}(x) = 2^{-\nu}\, I_2^{-1/2,\nu}\, (\surd(2/\pi)\cos x) \quad (x > 0) \tag{6.1}$$

if $\operatorname{Re}\nu > 0$. Many others are to be found throughout [19], [78] and [82]. Again, as we saw in §3.6, there are relationships between Erdélyi-Kober operators and the operators L_ν, the latter in turn being related to Bessel functions. A further selection of relevant material can be found in, for instance, [8], [32], [34], [54], [55], [72] and [73].

§6.2 Erdélyi-Kober Operators and H_ν

We take as our starting point the work of Kober [32]. He studied the Hankel transform on $L^2(0,\infty)$ and used Tricomi's form $\mathfrak{h}_\nu$ defined by

$$\mathfrak{h}_\nu\phi(x) = \underset{n\to\infty}{\text{l.i.m.}}(2)\int_0^n J_\nu(2\sqrt{xt})\,\phi(t)\,dt \qquad (0 < x < \infty)\,. \tag{6.2}$$

Using Mellin transforms, he showed that, if $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\nu > -1$ and

$\phi \in L^2(0,\infty)$, then

$$I_1^{\nu/2,\alpha}\mathcal{H}_\nu\phi = \mathcal{H}_{\nu+2\alpha} I_1^{\nu/2,\alpha}\phi . \tag{6.3}$$

The operator $\mathcal{H}_\nu$ is not suitable for functions ϕ in a general L^p_μ space. However, by simple changes of variables such as $u = t^2/2$, we find, after some completely routine algebra, that (6.3) is equivalent to

$$I_2^{\nu/2-1/4,\alpha} H_\nu\phi = H_{\nu+2\alpha} I_2^{\nu/2-1/4,\alpha}\phi . \tag{6.4}$$

Of course, both sides of (6.4) can be interpreted under conditions of great generality using our previous theory and this we will explore in a moment. But first, we notice that in changing from $\mathcal{H}_\nu$ to H_ν we have gone from an operator of the form $I_1^{\eta,\alpha}$ to one of the form $I_2^{\xi,\beta}$ (which has already appeared in (6.1)). Indeed, all our relations in the rest of this chapter will involve $I_m^{\eta,\alpha}$ or $K_m^{\eta,\alpha}$ with $m = 2$, rather than $m = 1$.

<u>Remark 6.1</u>

From now on we shall make extensive use of Notation 5.22, wherein the subscript 2 is dropped from $A_{p,\mu,2}$ and $A'_{p,\mu,2}$.

We now consider in detail the conditions under which (6.4) might hold for $\phi \in F_{p,\mu}$. By Theorems 3.31 and 5.17, the left-hand side of (6.4) defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ provided that $\nu \in \Omega_{p,\mu}$ and $\nu/2 - 1/4 \in A_{q,-\mu}$ or, equivalently, that $\nu \in \Omega_{p,\mu}$ and $\nu \in \Omega_{q,-\mu}$. The corresponding conditions for the right-hand side are that $\nu \in \Omega_{p,\mu}$ and $\nu + 2\alpha \in \Omega_{p,\mu}$. Since these pairs of conditions are not the same, some further investigation is called for. It would seem that the potential "singularities" for values of $\nu(\in \Omega_{p,\mu})$ such that $\nu \notin \Omega_{q,-\mu}$ or $\nu + 2\alpha \notin \Omega_{p,\mu}$ are perhaps removable. (See Example 3.39 and Remark 3.40.) We now show that this is so.

Let k be a non-negative integer and let $\nu \in \Omega_{p,\mu}$. Then, if $\nu \in \Omega_{q,-\mu}$ also, we may write

$$I_2^{\eta,\alpha}\psi(x) = I_2^{\eta+k,\alpha-k} x^{-2(\eta+k)} (D_2)^{-k}\, x^{2\eta}\, \psi \quad (\psi \in F_{p,2/p-1-\mu}). \tag{6.5}$$

Formally this is (3.32) which remains valid under the conditions stated; see our comments in §3.4. Hence if $\phi \in F_{p,\mu}$, (6.5), (5.28) and (5.29) give

$$I_2^{\nu/2-1/4,\alpha}\, H_\nu\phi$$

$$= 2^k\, I_2^{\nu/2-1/4+k,\alpha-k}\, x^{-\nu+1/2-2k}(x^{-1}D)^{-k} x^{\nu-1/2}\, x^{-\nu+1/2}\, (x^{-1}D)^k\, x^{\nu-1/2+k}\, H_{\nu+k} x^{-k}\phi$$

$$= 2^k\, x^{-k}\, I_2^{(\nu+k)/2-1/4,\alpha-k}\, H_{\nu+k}\, x^{-k}\phi .$$

However the last expression is meaningful provided only that $\nu + k \in \Omega_{p,\mu-k}$ and $\nu + k \in \Omega_{q,-\mu+k}$ or, equivalently, that $\nu \in \Omega_{p,\mu}$ and $\nu + 2k \in \Omega_{q,-\mu}$. The second condition can be satisfied even if $\nu \notin \Omega_{q,-\mu}$ by taking k sufficiently large. Hence we may define $I_2^{\nu/2-1/4,\alpha} H_\nu$ on $F_{p,\mu}$ subject only to the condition $\nu \in \Omega_{p,\mu}$ as follows.

<u>Definition 6.2</u>

If $\nu \in \Omega_{p,\mu}$, we define $I_2^{\nu/2-1/4,\alpha} H_\nu$ on $F_{p,\mu}$ by

$$I_2^{\nu/2-1/4,\alpha}\, H_\nu\phi = 2^k\, x^{-k}\, I_2^{(\nu+k)/2-1/4,\alpha-k}\, H_{\nu+k}\, x^{-k}\phi \quad (\phi \in F_{p,\mu}) \tag{6.6}$$

where k is any non-negative integer such that $\nu + 2k \in \Omega_{q,-\mu}$ and where the operators on the right are as in Definitions 3.27 and 5.14.

<u>Notes</u>

1. It is easy to check that this definition is independent of the choice of non-negative integer k.
2. If $\nu \in \Omega_{p,\mu}$ and $\nu \in \Omega_{q,-\mu}$, (6.6) coincides with the original definition since, in this case, we may take k = 0.
3. (6.6) provides an analytic continuation of the expression $I_2^{\nu/2-1/4,\alpha}\, H_\nu\phi(x)$ from $\Omega_{p,\mu} - \Omega_{q,-\mu}$ to all of $\Omega_{p,\mu}$.

4. The operator is still a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ by Theorems 3.31 and 5.17.

The operator on the right-hand side of (6.4) requires similar treatment. Without going into the details, we merely say that the following extension is suggested.

Definition 6.3

If $\nu \in \Omega_{p,\mu}$, we define $H_{\nu+2\alpha} I_2^{\nu/2-1/4,\alpha}$ on $F_{p,\mu}$ by

$$H_{\nu+2\alpha}\, I_2^{\nu/2-1/4,\alpha}\phi = 2^{-k}\, x^k\, H_{(\nu-k)+2(\alpha+k)}\, I_2^{(\nu-k)/2-1/4,\alpha+k}\, x^k \phi \qquad (\phi \in F_{p,\mu}) \tag{6.7}$$

where k is a non-negative integer such that $\nu+2\alpha+2k \in \Omega_{p,\mu}$ and where the operators on the right-hand side are as in Definitions 3.27 and 5.14.

Note

Comments analogous to Notes 1-4 above apply. This time the singularities for $\nu+2\alpha \notin \Omega_{p,\mu}$ have been removed.

Having sorted out what we mean by (6.4), we can now find when equality holds.

Lemma 6.4

If $\phi \in F_{p,\mu}$ and $\nu \in \Omega_{p,\mu}$, then (6.4) holds, the operators being interprete via Definitions 6.2 and 6.3.

Proof:- When $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p$, $\mathrm{Re}(\mu+\nu+2\alpha) > -3/2 + 1/p$ and $\mathrm{Re}(-\mu+\nu) > -3/2 + 1/q$, the result can be obtained from Kober's by taking $\phi \in C_o^\infty(0,\infty)$ and then using continuity, denseness and analytic continuatior as in several previous proofs. The three conditions can be relaxed successively to $\nu \in \Omega_{p,\mu}$, $\nu + 2\alpha \in \Omega_{p,\mu}$ and $\nu \in \Omega_{q,-\mu}$ respectively. For instance, let $\mathrm{Re}(\mu+\nu+2\alpha) > -3/2 + 1/p$, $\mathrm{Re}(-\mu+\nu) > -3/2 + 1/q$ and $\nu \in \Omega_{p,\mu}$

and choose a non-negative integer k such that $\mathrm{Re}(\mu+\nu) > -3/2 + 1/p - 2k$. Then

$$I_2^{\nu/2-1/4,\alpha} H_\nu \phi$$

$$= 2^k I_2^{\nu/2-1/4,\alpha} x^k H_{\nu+k} x^{-\nu-k+1/2}(D_2)^{-k} x^{\nu-1/2} \phi \qquad \text{by (5.30)}$$

$$= 2^k x^k I_2^{(\nu+k)/2-1/4,\alpha} H_{\nu+k} x^{-\nu-k+1/2}(D_2)^{-k} x^{\nu-1/2}\phi$$

$$= 2^k x^k H_{\nu+k+2\alpha} I_2^{(\nu+k)/2-1/4,\alpha} x^{-\nu-k+1/2}(D_2)^{-k} x^{\nu-1/2}\phi$$

by the previous case

$$= H_{\nu+2\alpha} x^{-\nu-2\alpha+1/2}(D_2)^k x^{\nu+2\alpha+k-1/2} I_2^{(\nu+k)/2-1/4,\alpha} x^{-\nu-k+1/2}(D_2)^{-k} x^{\nu-1/2}\phi \qquad \text{by (5.30)}$$

$$= H_{\nu+2\alpha} I_2^{\nu/2-1/4,\alpha}\phi$$

using the theory of §3.3. The other two conditions are relaxed similarly and can finally be removed using analytic continuation as described above. We omit the details.

The following adjoint version of Lemma 6.4 comes as no surprise.

<u>Lemma 6.5</u>

If $\phi \in F_{p,\mu}$ and $\nu \in \Omega_{p,\mu}$, then

$$H_\nu K_2^{\nu/2+1/4,\alpha}\phi = K_2^{\nu/2+1/4,\alpha} H_{\nu+2\alpha} \phi \qquad (6.8)$$

where if $\nu \notin \Omega_{q,-\mu}$ or $\nu + 2\alpha \notin \Omega_{p,\mu}$, the left-hand or right-hand expression respectively is to be interpreted in terms of its analytic continuation.

<u>Proof</u>:- We omit the details merely remarking that the appropriate analytic continuations are adjoint versions of (6.6) and (6.7).

The final piece of this little jigsaw is provided by Rooney. Using Mellin transforms also, he shows in [72, Lemma 8.1], with slightly changed notation, that under appropriate conditions,

$$H_{\nu+\gamma} H_\nu\phi = (K_2^{\nu/2+1/4,\ \gamma/2})^{-1} I_2^{\nu/2-1/4,\ \gamma/2}\phi. \qquad (6.9)$$

(6.9) is true, for instance, for $\phi \in L^2(0,\infty)$ if Re $\nu > -1$ and Re $\gamma > 0$.

(6.9) can be recast in the form

$$I_2^{\nu/2-1/4,\alpha} H_\nu \psi = K_2^{\nu/2+1/4,\alpha} H_{\nu+2\alpha} \psi \qquad (\psi \in L^2(0,\infty))$$

in view of the Hankel Inversion Theorem. By applying the usual method we can prove the following result.

Theorem 6.6

If $\phi \in F_{p,\mu}$ and $\nu \in \Omega_{p,\mu}$, then

$$I_2^{\nu/2-1/4,\alpha} H_\nu \phi = H_{\nu+2\alpha} I_2^{\nu/2-1/4,\alpha}\phi = K_2^{\nu/2+1/4,\alpha} H_{\nu+2\alpha}\phi = H_\nu K_2^{\nu/2+1/4,\alpha}\phi \tag{6.10}$$

the operators being interpreted in terms of analytic continuation where appropriate.

Proof:- We use Lemmas 6.4 and 6.5 along with (6.9). The details are omitted.

The corresponding result for $F'_{p,\mu}$ is easily obtained.

Theorem 6.7

If $f \in F'_{p,\mu}$ and $\nu \in \Omega_{q,-\mu}$, then

$$I_2^{\nu/2-1/4,\alpha} H_\nu f = H_{\nu+2\alpha} I_2^{\nu/2-1/4,\alpha} f = K_2^{\nu/2+1/4,\alpha} H_{\nu+2\alpha} f = H_\nu K_2^{\nu/2+1/4,\alpha} f \tag{6.11}$$

the operators being interpreted in terms of analytic continuation where appropriate.

Proof:- (6.11) follows on taking adjoints in (6.10). For instance, if $\phi \in F_{p,2/p-1-\mu}$, $(H_\nu K_2^{\nu/2+1/4,\alpha} f,\phi) = (f, I_2^{\nu/2-1/4,\alpha} H_\nu\phi)$ and we can extend the definition of the left-hand expression to all values of ν in $\Omega_{q,-\mu}$ via (6.6). Again we omit further details.

As an application of our last result, we prove the following result for three Hankel transforms, often attributed to Plancherel [68].

Theorem 6.8

(i) Let $\alpha \in \Omega_{p,\mu}$, $\beta \in \Omega_{q,-\mu}$, $\gamma \in \Omega_{p,\mu}$ and $\phi \in F_{p,\mu}$. Then

$$H_\alpha H_\beta H_\gamma \phi = H_\gamma H_\beta H_\alpha \phi. \tag{6.12}$$

(ii) Let $\alpha \in \Omega_{q,-\mu}$, $\beta \in \Omega_{p,\mu}$, $\gamma \in \Omega_{q,-\mu}$ and $f \in F'_{p,\mu}$. Then

$$H_\alpha H_\beta H_\gamma f = H_\gamma H_\beta H_\alpha f. \tag{6.13}$$

Proof:- (i) $H_\alpha H_\beta H_\gamma \phi$

$$= H_\alpha K_2^{\beta/2+1/4,\gamma/2-\beta/2} I_2^{\gamma/2-1/4,\beta/2-\gamma/2}\phi \qquad \text{by (6.9)}$$

$$= H_\alpha K_2^{\alpha/2+1/4,\gamma/2-\alpha/2} K_2^{\beta/2+1/4,\alpha/2-\beta/2} I_2^{\gamma/2-1/4,\beta/2-\gamma/2}\phi$$

$$= H_\gamma I_2^{\alpha/2-1/4,\gamma/2-\alpha/2} I_2^{\gamma/2-1/4,\beta/2-\gamma/2} K_2^{\beta/2+1/4,\alpha/2-\beta/2} \qquad \text{by (6.10)}$$

$$= H_\gamma I_2^{\alpha/2-1/4,\beta/2-\alpha/2} K_2^{\beta/2+1/4,\alpha/2-\beta/2}$$

$$= H_\gamma H_\beta H_\alpha \phi \qquad \text{by (6.9).}$$

Here we have also made use of Example 3.39 and Theorem 3.41. Each step can be justified under the given conditions. (ii) follows on taking adjoints in (i) and the proof is complete.

We should say that there are other ways of obtaining (6.12) for classical functions ϕ. For instance, de Snoo [5] uses Mellin transforms. Also, (6.12) and (6.13) are particular cases of a more general result for so-called Watson transforms.

§6.3 Erdélyi-Kober Operators and $S^{\eta,\alpha}$

Our second set of results concerns the well-known connections between the Erdélyi-Kober operators and the modified Hankel transform operator $S^{\eta,\alpha}$ discussed in §5.5. We shall make extensive use of these connections in the next chapter and our main concern here is to state the conditions for their validity relative to the spaces $F_{p,\mu}$ and $F'_{p,\mu}$. The results concerned can be found in [76, p.274 formulae 13A-18A] as well as in the survey article in [74, p.41].

<u>Theorem 6.9</u>

Let $\phi \in F_{p,\mu}$. Then

(i) $I_2^{\eta+\alpha,\beta}\, S^{\eta,\alpha}\phi = S^{\eta,\alpha+\beta}\phi$ if $\eta \in A_{p,\mu}$, $\eta+\alpha \in A'_{p,\mu}$

(ii) $K_2^{\eta,\alpha}\, S^{\eta+\alpha,\beta}\phi = S^{\eta,\alpha+\beta}\phi$ if $\eta \in A_{p,\mu}$, $\eta+\alpha \in A_{p,\mu}$

(iii) $S^{\eta+\alpha,\beta}\, S^{\eta,\alpha}\phi = I_2^{\eta,\alpha+\beta}\phi$ if $\eta \in A_{p,\mu}$, $\eta+\alpha \in A'_{p,\mu}$

(iv) $S^{\eta+\alpha,\beta}\, I_2^{\eta,\alpha}\phi = S^{\eta,\alpha+\beta}\phi$ if $\eta \in A_{p,\mu}$, $\eta+\alpha \in A_{p,\mu}$

(v) $S^{\eta,\alpha}\, K_2^{\eta+\alpha,\beta}\phi = S^{\eta,\alpha+\beta}\phi$ if $\eta \in A_{p,\mu}$, $\eta+\alpha \in A'_{p,\mu}$

(vi) $S^{\eta,\alpha}\, S^{\eta+\alpha,\beta}\phi = K_2^{\eta,\alpha+\beta}\phi$ if $\eta \in A'_{p,\mu}$, $\eta+\alpha \in A_{p,\mu}$.

The right-hand sides provide analytic continuations of the left-hand sides to all values of η in $A_{p,\mu}$ (in cases (i)-(v)) or in $A'_{p,\mu}$ (in case (vi)).

<u>Proof</u>:- All parts are similar; we consider only (i).

If $\eta \in A_{p,\mu}$, the right-hand side of (i) is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-2-\mu}$ by Theorem 5.24. The same is true of the left-hand side if $\eta+\alpha \in A_{p,2/p-2-\mu} = A'_{p,\mu}$ initially. However, using (5.36) we can write the left-hand side in the form

$$2^{\alpha}\, x^{-\alpha-1/2}\, I_2^{\eta+\alpha/2-1/4,\beta}\, H_{2\eta+\alpha}\, x^{1/2-\alpha}\phi$$

to which we can apply Definition 6.2 with ν and μ replaced by $2\eta+\alpha$ and $\mu+1/2-\alpha$ respectively. This removes the restriction $\eta+\alpha \in A_{p,2/p-2-\mu}$ and it now remains to establish equality with the right-hand side for $\eta \in A_{p,\mu}$. This follows the familiar pattern. We first establish equality for

$\phi \in C_o^\infty(0,\infty)$ and $\mathrm{Re}(2\eta+\mu) + 2 > 1/p$ by proceeding as in [76], then extend to all $\phi \in F_{p,\mu}$ by continuity and density and finally handle the case $\eta \in A_{p,\mu}$, $\mathrm{Re}(2\eta+\mu) + 2 < 1/p$ via Definitions 3.27 and 5.14. We omit the details.

Finally, by taking adjoints we obtain the following result.

Theorem 6.10

Let $f \in F'_{p,\mu}$. Then

(i) $I_2^{\eta+\alpha,\beta} \; S^{\eta,\alpha} \; f = S^{\eta,\alpha+\beta} f$ if $\eta \in A_{q,-\mu}$, $\eta+\alpha \in A'_{q,-\mu}$

(ii) $K_2^{\eta,\alpha} \; S^{\eta+\alpha,\beta} \; f = S^{\eta,\alpha+\beta} f$ if $\eta \in A_{q,-\mu}$, $\eta+\alpha \in A_{q,-\mu}$

(iii) $S^{\eta+\alpha,\beta} \; S^{\eta,\alpha} \; f = I_2^{\eta,\alpha+\beta} f$ if $\eta \in A_{q,-\mu}$, $\eta+\alpha \in A'_{q,-\mu}$

(iv) $S^{\eta+\alpha,\beta} \; I_2^{\eta,\alpha} \; f = S^{\eta,\alpha+\beta} f$ if $\eta \in A_{q,-\mu}$, $\eta+\alpha \in A_{q,-\mu}$

(v) $S^{\eta,\alpha} \; K_2^{\eta+\alpha,\beta} \; f = S^{\eta,\alpha+\beta} f$ if $\eta \in A_{q,-\mu}$, $\eta+\alpha \in A'_{q,-\mu}$

(vi) $S^{\eta,\alpha} \; S^{\eta+\alpha,\beta} \; f = K_2^{\eta,\alpha+\beta} f$ if $\eta \in A'_{q,-\mu}$, $\eta+\alpha \in A_{q,-\mu}$.

The right-hand sides provide analytic continuations of the left-hand sides to all values of η in $A_{q,-\mu}$ (in cases (i)-(v)) or in $A'_{q,-\mu}$ (in case (vi)).

Proof:- These results follow easily from Theorem 6.9. For instance, if $\phi \in F_{p,2/p-\mu}$,

$$
\begin{aligned}
&(S^{\eta,\alpha} \; K_2^{\eta+\alpha,\beta} f, \phi) \\
&= (f, \; I_2^{\eta+\alpha-1/2,\beta} \; x \; S^{\eta,\alpha} \; x^{-1} \phi) && \text{by (3.44) and (5.37)} \\
&= (f, \; x \; I_2^{\eta+\alpha,\beta} \; S^{\eta,\alpha} \; x^{-1} \phi) \\
&= (f, \; x \; S^{\eta,\alpha+\beta} \; x^{-1} \phi) && \text{by Theorem 6.9(i)} \\
&= (S^{\eta,\alpha+\beta} f, \phi).
\end{aligned}
$$

as required. The other parts are similar.

Various classical results could be deduced from the above. However, we shall proceed to the most important application, namely dual integral equations.

7 Dual integral equations of Titchmarsh type

§7.1 Introduction

In this chapter we will apply our theory to obtain results concerning the existence and uniqueness of classical solutions of dual integral equations of Titchmarsh type. Expressed very briefly and imprecisely, our problem is as follows.

Problem 7.1

Let g_1 and g_2 be functions defined (almost everywhere) on $(0,\infty)$ and let ν_1, ν_2, α_1 and α_2 be complex numbers. Find functions f defined (a.e.) on $(0,\infty)$ such that

$$\begin{aligned} S^{\nu_1/2-\alpha_1,2\alpha_1} f(x) &= g_1(x) && \text{on } (0,1) \\ S^{\nu_2/2-\alpha_2,2\alpha_2} f(x) &= g_2(x) && \text{on } (1,\infty) \ . \end{aligned} \tag{7.1}$$

More precisely, g_1 and g_2 will be functions from some space whose values are known, at least on $(0,1)$ and $(1,\infty)$ respectively. f is our unknown function and initially, since we are working classically, $S^{\nu_i/2-\alpha_i,2\alpha_i} f$ $(i = 1,2)$ is defined, via (5.33), by

$$S^{\nu_i/2-\alpha_i,2\alpha_i} f(x) = 2^{2\alpha_i} x^{-2\alpha_i} \int_0^\infty t^{1-2\alpha_i} J_{\nu_i}(xt)f(t)dt \tag{7.2}$$

interpreted in the appropriate way. In our case f, g_1 and g_2 will be elements of L^p_μ spaces.

Many problems in mathematical physics can be formulated as such a pair of dual integral equations. This matter is treated comprehensively in [76] to which we shall refer frequently. Suffice to say that, for instance, the choice of values

$$\nu_1 = 0 \quad \nu_2 = 0 \quad \alpha_1 = 1 \quad \text{and } \alpha_2 = 1/2 \tag{7.3}$$

gives rise to the classic problem of the electrified disk [76, Chapter 3] while the values

$$\nu_1 = -1/2 \quad \nu_2 = -1/2 \quad \alpha_1 = -1/4 \quad \text{and} \quad \alpha_2 = 1/4 \tag{7.4}$$

give rise to a problem involving the stress distribution across a Griffiths crack; see [81] and also [76, §4.5] where other similar problems are discussed.

Associated with Problem 7.1 are the basic questions of existence and uniqueness and to deal with these we will use our generalised functions $F'_{p,\mu}$. In addition to the motivation given in Chapter 1, there seems another natural reason why generalised functions should be employed here. The formulation of Problem 7.1 demands that something happens on $(0,1)$ and something else on $(1,\infty)$ and it seems conceivable that something fairly drastic may occur at the point 1. Possibly a Heaviside function of the form $H(x-1)$ might suffice but it seems probable (and is indeed the case as we shall see) that a central role is played by the distribution δ_1 defined by

$$(\delta_1,\phi) = \phi(1) \tag{7.5}$$

and distributional derivatives of δ_1. (There should not be any confusion between δ_1 and the operator $\delta = xD$!)

Our first task is to set up the analogue of Problem 7.1 for generalised functions in $F'_{p,\mu}$. For a classical function F such that $S^{\eta,\alpha}F$ exists classically, our definitions in Chapter 5 and in particular (5.32) and (5.37) ensure that under appropriate conditions $\tau S^{\eta,\alpha}F = S^{\eta,\alpha}\tau F$. Hence, if we now regard f, g_1 and g_2 as generalised functions, we will interpret $S^{\nu_i/2-\alpha_i,2\alpha_i}f$ as in Definition 5.25. If our proposed solution $f \in F'_{p,\mu}$, then Theorem 5.26 indicates that $g_i \in F'_{p,2/p-\mu}$ $(i = 1,2)$ under appropriate conditions and we will build this into our assumptions. Also, we need the usual idea of equality of two generalised functions on an open set.

Definition 7.2

Let G be an open subset of $(0,\infty)$ and let $h_1, h_2 \in F'_{p,\mu}$. Then $h_1 = h_2$ on G if $(h_1,\phi) = (h_2,\phi)$ for all $\phi \in F_{p,\mu}$ whose support is a compact subset of G. We will require this for $G = (0,1)$ and $G = (1,\infty)$.

With this preamble, we are led to the following problem.

Problem 7.3

Let $g_i \in F'_{p,2/p-\mu}$ $(i = 1,2)$ and let ν_1,ν_2,α_1 and α_2 be complex numbers. Find $f \in F'_{p,\mu}$ such that

$$\begin{aligned} S^{\nu_1/2-\alpha_1,2\alpha_1} f &= g_1 \quad \text{on } (0,1) \\ S^{\nu_2/2-\alpha_2,2\alpha_2} f &= g_2 \quad \text{on } (1,\infty) \quad . \end{aligned} \tag{7.6}$$

By studying this problem, we might hope to obtain information about Problem 7.1. In particular we will want to study existence and uniqueness for Problem 7.3.

It should be mentioned that a similar investigation has been carried out by Walton [80], [81]. He uses Zemanian's theory and spaces and consequently has to do some preliminary work in order to obtain analogues of our Theorems 6.9 and 6.10. Our theory is consistent with his but the use of extended operators allows us to handle more general conditions on the parameters. As Walton's work suggests, it is the uniqueness investigation which is, in many ways, the more interesting part of the exercise.

§7.2 A Technicality

Before dealing with the existence of solutions of Problem 7.3, we need to resolve a technicality.

Problem 7.4

Let $f_1, f_2 \in F'_{p,\mu}$. Does there exist $f \in F'_{p,\mu}$ such that, in the sense of Definition 7.2,

$$f = f_1 \text{ on } (0,1); \quad f = f_2 \text{ on } (1,\infty) \ ?$$

The slight difficulty here is that standard results such as those in [24] or [75] cannot be applied as they stand since $(0,\infty) \neq (0,1) \cup (1,\infty)$. Nevertheless, we can fall back on our structure theorem (Theorem 2.25) which reduces the problem to gluing together classical functions.

Theorem 7.5

Problem 7.4 has infinitely many solutions $f \in F'_{p,\mu}$, any two of which differ by a distribution whose support is concentrated at the point 1.

Proof:- Let f_1, f_2 be as in Problem 7.4. By Theorem 2.25, there exist a non-negative integer n and functions $f_k^{(i)} \in L^q_{-\mu}$ $(i = 1,2;\ k = 0,1,\ldots,n)$ such that

$$f_1 = \sum_{k=0}^{n} \delta^k \tilde{f}_k^{(1)} \quad \text{and} \quad f_2 = \sum_{k=0}^{n} \delta^k \tilde{f}_k^{(2)}$$

using the notation of (2.15). (The same integer n can obviously be used for f_1, f_2 by inserting zero functions if necessary.) Define f_k (almost everywhere) on $(0,\infty)$ by

$$f_k(x) = \begin{cases} f_k^{(1)}(x) & 0 < x \leqslant 1 \\ f_k^{(2)}(x) & 1 < x < \infty \end{cases} .$$

Then $f_k \in L^q_{-\mu}$ $(k = 0,1,\ldots n)$ so that f defined by

$$f = \sum_{k=0}^{n} \delta^k \tilde{f}_k$$

is an element of $F'_{p,\mu}$ which clearly solves Problem 7.4 and we have existence. The second statement of the theorem is more or less immediate. We can be more explicit and say that any two solutions of Problem 7.4 differ by an expression of the form $\sum_{k=0}^{N} a_k \delta_1^{(k)}$ where $\delta_1^{(k)}$ denotes the k^{th} distributional derivative of δ_1 (defined by (7.5)) and $a_o,\ldots a_N$ are constants.

We shall require Theorem 7.5 almost immediately.

§7.3 Existence of Solutions of Problem 7.3

We are now ready to study Problem 7.3 and it is convenient to fix some notation. For the remainder of this chapter, ν_1, ν_2, α_1 and α_2 are fixed complex numbers and we shall write

$$\lambda = (\nu_1+\nu_2)/2 - (\alpha_1-\alpha_2) \quad . \tag{7.7}$$

We shall also require four non-negative integers k_1, k_2, ℓ_1 and ℓ_2.

Notation 7.6

(i) Let $\nu_i/2 + \alpha_i \in A'_{q,-\mu}$ $(i = 1,2)$. We define k_i $(i = 1,2)$ as follows.

If $\mathrm{Re}(\nu_i + 2\alpha_i + \mu) > -1/q$, then $k_i = 0$.

Otherwise, k_i is the unique positive integer such that

$\mathrm{Re}\,(\nu_i + 2\alpha_i + \mu) + 2(k_i - 1) < -1/q < \mathrm{Re}(\nu_i + 2\alpha_i + \mu) + 2k_i$.

(ii) Let $\nu_i/2 - \alpha_i \in A_{q,-\mu}$ $(i = 1,2)$. We define ℓ_i $(i = 1,2)$ as follows.

If $\mathrm{Re}(\nu_i - 2\alpha_i - \mu) + 2 > 1/q$, then $\ell_i = 0$.

Otherwise, ℓ_i is the unique positive integer such that

$\mathrm{Re}(\nu_i - 2\alpha_i - \mu) + 2\ell_i < 1/q < \mathrm{Re}(\nu_i - 2\alpha_i - \mu) + 2\ell_i + 2$.

The need for these definitions will become clear as we proceed.

The idea now is to modify the methods in [74, p.47] and [76] to take account of the extended operators and to incorporate k_1, k_2, ℓ_1 and ℓ_2 in an appropriate way. As regards existence, we prove the following result.

Theorem 7.7

Let $g_i \in F'_{p,2/p-\mu}$, $\nu_i/2 + \alpha_i \in A'_{q,-\mu}$ and $\nu_i/2-\alpha_i \in A_{q,-\mu}$ for $i = 1,2$. Then Problem 7.3 has a solution $f \in F'_{p,\mu}$.

Proof:- By the hypotheses and the facts that $A'_{q,-\mu} = A_{q,\mu-2/p}$, $A_{q,-\mu} = A'_{q,\mu-2/p}$, we may conclude from Theorems 3.47 and 3.50 that $I_2^{\nu_1/2+\alpha_1,\lambda+k_2-\nu_1} K_2^{\nu_2/2-\alpha_2,\ell_2} g_1$ and $K_2^{\nu_1/2-\alpha_1,\nu_2-\lambda+\ell_2} I_2^{\nu_2/2+\alpha_2,k_2} g_2$ exist as elements of $F'_{p,2/p-\mu}$. Hence, by Theorem 7.5, $\exists h \in F'_{p,2/p-\mu}$ such that

$$h = \begin{cases} I_2^{\nu_1/2+\alpha_1,\ \lambda+k_2-\nu_1} K_2^{\nu_2/2-\alpha_2,\ell_2} g_1 & \text{on } (0,1) \\ K_2^{\nu_1/2-\alpha_1,\nu_2-\lambda+\ell_2} I_2^{\nu_2/2+\alpha_2,\ k_2} g_2 & \text{on } (1,\infty) \ . \end{cases} \tag{7.8}$$

There are infinitely many such functionals h but any one will do here.

Next we define H by

$$H = K_2^{\nu_2/2-\alpha_2+\ell_2,-\ell_2} h \ . \tag{7.9}$$

Since $\mathrm{Re}(\nu_2-2\alpha_2+2\ell_2-1+2/p-\mu) + 2 > 1/p$, $H \in F'_{p,2/p-\mu}$ also .

Finally, we define f by

$$f = S^{\nu_2/2+\alpha_2+k_2,\nu_1-2\alpha_1-\lambda-k_2} H. \tag{7.10}$$

Because of our choice of k_2, $f \in F'_{p,\mu}$ by Theorem 5.26. We will show that f is a solution of Problem 7.3.

By Theorems 3.56 and 6.10, we have

$$\begin{aligned} & S^{\nu_1/2-\alpha_1,2\alpha_1} f \\ &= I_2^{\lambda-\nu_1/2+\alpha_1,\ \nu_1-\lambda} S^{\nu_1/2-\alpha_1,\lambda-\nu_1+2\alpha_1} f \\ &= I_2^{\lambda-\nu_1/2+\alpha_1,\nu_1-\lambda} I_2^{\lambda-\nu_1/2+\alpha_1+k_2,-k_2} H \\ &= I_2^{\lambda-\nu_1/2+\alpha_1+k_2,\nu_1-\lambda-k_2} H \\ &= I_2^{\lambda-\nu_1/2+\alpha_1+k_2,\nu_1-\lambda-k_2} K_2^{\nu_2/2-\alpha_2+\ell_2,-\ell_2} h \ . \end{aligned} \tag{7.11}$$

If $\phi \in F_{p,2/p-\mu}$ has support in (0,1), then so does $I_2^{\nu_2/2-\alpha_2+\ell_2-1/2,-\ell_2} K_2^{\lambda-\nu_1/2+\alpha_1+k_2+1/2,\ \nu_1-\lambda-k_2} \phi$ since $\mathrm{Re}(\nu_2 + 2\alpha_2 + \mu) + 2k_2 > -1/q$ (so that the K operator is given by (3.11)). Hence, for such functions ϕ,

$$(S^{\nu_1/2-\alpha_1,2\alpha_1}f,\phi)$$

$$= (h,\ I_2^{\nu_2/2-\alpha_2+\ell_2-1/2,-\ell_2}K_2^{\lambda-\nu_1/2+\alpha_1+k_2+1/2,\nu_1-\lambda-k_2}\phi) \qquad \text{by (7.11)}$$

$$= (I_2^{\nu_1/2+\alpha_1,\lambda+k_2-\nu_1}K_2^{\nu_2/2-\alpha_2,\ell_2}g_1,\ I_2^{\nu_2/2-\alpha_2+\ell_2-1/2,-\ell_2}K_2^{\lambda-\nu_1/2+\alpha_1+k_2+1/2,\nu_1-\lambda-k_2}\phi)$$

$$= (g_1,\phi)$$

on using Theorems 3.55 and 3.56. Thus $S^{\nu_1/2-\alpha_1,2\alpha_1}f = g_1$ on $(0,1)$

Similarly,

$$S^{\nu_2/2-\alpha_2,\ 2\alpha_2}f$$

$$= K_2^{\nu_2/2-\alpha_2,\lambda-\nu_2}S^{\lambda-\nu_2/2-\alpha_2,\ 2\alpha_2+\nu_2-\lambda}f$$

$$= K_2^{\nu_2/2-\alpha_2,\lambda-\nu_2}I_2^{\nu_2/2+\alpha_2+k_2,-k_2}H$$

$$= K_2^{\nu_2/2-\alpha_2,\lambda-\nu_2}I_2^{\nu_2/2+\alpha_2+k_2,-k_2}K_2^{\nu_2/2-\alpha_2+\ell_2,-\ell_2}h$$

$$= K_2^{\nu_2/2-\alpha_2+\ell_2,\lambda-\nu_2-\ell_2}I_2^{\nu_2/2+\alpha_2+k_2,-k_2}h$$

where we have used Theorems 3.50 and 3.55. As before, if $\phi \in F_{p,2/p-\mu}$ has support in $(1,\infty)$, then, since $\mathrm{Re}(\nu_2-2\alpha_2+2\ell_2-1+2/p-\mu)+2 > 1/p$, so does

$$K_2^{\nu_2/2+\alpha_2+k_2+1/2,-k_2}I_2^{\nu_2/2-\alpha_2+\ell_2-1/2,\lambda-\nu_2-\ell_2}\phi$$

(so that we may use (3.10) and (3.14)). Hence, for such functions ϕ,

$$(S^{\nu_2/2-\alpha_2,2\alpha_2}f,\phi)$$

$$= (h,\ K_2^{\nu_2/2+\alpha_2+k_2+1/2,-k_2}I_2^{\nu_2/2-\alpha_2+\ell_2-1/2,\lambda-\nu_2-\ell_2}\phi)$$

$$= (K_2^{\nu_1/2-\alpha_1,\nu_2-\lambda+\ell_2} I_2^{\nu_2/2+\alpha_2,k_2} g_2,$$
$$K_2^{\nu_2/2+\alpha_2+k_2+1/2,-k_2} I_2^{\nu_2/2-\alpha_2+\ell_2-1/2,\lambda-\nu_2-\ell_2}\phi)$$

$= (g_2,\phi)$ as before so that $S^{\nu_2/2-\alpha_2,2\alpha_2} f = g_2$ on $(1,\infty)$. Hence f, as given by (7.8) (7.9) and (7.10), is a solution of Problem 7.3.

Remark 7.8

Three comments are in order concerning the proof of Theorem 7.7.

1. The hypotheses $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i = 1,2)$ ensure that all the steps in the above analysis are valid.
2. The integers k_1 and ℓ_1 in Notation 7.6 are not required in this existence proof, but they will appear in the uniqueness investigation. However, k_2 and ℓ_2 play a vital role.
3. The fact that we chose any admissible h suggests that non-uniqueness is almost certain. This is borne out by the results in the next section.

§7.4 Uniqueness for Problem 7.3

To handle the question of uniqueness for Problem 7.3, it is clearly necessary and sufficient to decide whether there exists a non-zero solution of the corresponding homogeneous problem or not; in other words we seek a non-zero element f in $F'_{p,\mu}$ satisfying

$$\begin{aligned} &S^{\nu_1/2-\alpha_1,2\alpha_1} f = 0 && \text{on } (0,1) \\ &S^{\nu_2/2-\alpha_2,2\alpha_2} f = 0 && \text{on } (1,\infty) \end{aligned} \tag{7.12}$$

We shall proceed in two stages first identifying possible candidates for f and then finding which of these do indeed satisfy (7.12). We therefore begin with the following lemma.

Lemma 7.9

Let $\nu_i/2+\alpha_i \in A'_{q,-\mu}$ and $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i = 1,2)$. If $f \in F'_{p,\mu}$ satisfies (7.12), then

$$I_2^{\nu_1/2+\alpha_1+k_1,-k_1} S^{\nu_1/2-\alpha_1+\ell_1,\lambda-\nu_1+2\alpha_1-\ell_1} f = \sum_{s=0}^{r} a_s (\delta')^s \delta_1 \tag{7.13}$$

where r is a non-negative integer, $a_o,\ldots,a_r$ are constants, δ_1 is defined by (7.5) and δ' is defined by (2.20).

Proof:- Using Theorems 3.56 and 6.10, we have

$$I_2^{\nu_1/2+\alpha_1+k_1,-k_1} S^{\nu_1/2-\alpha_1+\ell_1,\lambda-\nu_1+2\alpha_1-\ell_1} f$$

$$= I_2^{\nu_1/2+\alpha_1+k_1,-k_1} I_2^{\nu_1/2+\alpha_1,\lambda-\nu_1} S^{\nu_1/2-\alpha_1+\ell_1,2\alpha_1-\ell_1} f$$

$$= I_2^{\nu_1/2+\alpha_1+k_1,\lambda-\nu_1-k_1} K_2^{\nu_1/2-\alpha_1+\ell_1,-\ell_1} S^{\nu_1/2-\alpha_1,2\alpha_1} f .$$

If $\phi \in F_{p,2/p-\mu}$ has support in $(0,1)$, then so does $I_2^{\nu_1/2-\alpha_1+\ell_1-1/2,-\ell_1}$ $K_2^{\nu_1/2+\alpha_1+k_1+1/2,\lambda-\nu_1-k_1}\phi$ since $\mathrm{Re}(\nu_1 + 2\alpha_1 + 2k_1 + 1 - 2/p + \mu) > -1/p$ (so that the theory in §3.2 applies). Hence, for such functions ϕ,

$$(I_2^{\nu_1/2+\alpha_1+k_1,-k_1} S^{\nu_1/2-\alpha_1+\ell_1,\lambda-\nu_1+2\alpha_1-\ell_1} f, \phi)$$

$$= (S^{\nu_1/2-\alpha_1,2\alpha_1} f, I_2^{\nu_1/2-\alpha_1+\ell_1-1/2,-\ell_1} K_2^{\nu_1/2+\alpha_1+k_1+1/2,\lambda-\nu_1-k_1}\phi) = 0$$

by (7.12). Thus the left-hand side of (7.13) is zero on $(0,1)$. In a similar way, we can show that it vanishes on $(1,\infty)$. Hence the support of the left-hand side of (7.13) is $\{1\}$ so that by [86, §3.5],

$$I_2^{\nu_1/2+\alpha_1+k_1,-k_1} S^{\nu_1/2-\alpha_1+\ell_1,\lambda-\nu_1+2\alpha_1-\ell_1} f = \sum_{s=0}^{r} b_s D^s \delta_1 \tag{7.14}$$

where r is a non-negative integer, $b_o, \ldots, b_r$ are constants and $D^s\delta_1$ is the s^{th} distributional derivative of δ_1. Although (7.14) is quite acceptable, there is a snag. To obtain an element of $F'_{p,2/p-\mu}$ on the right we have to regard δ_1 as an element of $F'_{p,2/p-\mu-s}$ in using $D^s\delta_1$. To avoid this, we merely note that formally

$$((\delta')\delta_1,\phi) = (\delta_1, -\delta\phi) = -\phi'(1) = (\delta_1, -D\phi) = (D\delta_1, \phi)$$

so that we can rewrite (7.14) in the form (7.12) and regard δ_1 as an element of $F'_{p,2/p-\mu}$ throughout. This completes the proof.

Using Example 3.29, Theorem 6.10 and Lemma 7.9, we see that possible solutions of (7.12) must be of the form

$$f = S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2+\ell_1} K_2^{-\nu_1/2-\alpha_1-k_1,k_1} \left(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1\right) \qquad (7.15)$$

and we must now identify the possible choices of $a_o,\ldots, a_r$. For simplicity, we will state our results in two parts.

<u>Lemma 7.10</u>

If $\nu_i/2+\alpha_i \in A'_{q,-\mu}$ and $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i = 1,2)$, then f, as given by (7.15), satisfies

$$S^{\nu_1/2-\alpha_1,\ 2\alpha_1} f = 0 \text{ on } (0,1)$$

in the following circumstances.

(i) If $k_1 = k_2 = 0$, r and $a_o,\ldots, a_r$ arbitrary.

(ii) If $k_1 = 0$, $k_2 > 0$, $a_o,\ldots, a_r$ such that

$$\sum_{s=0}^{r} a_s(-1)^s(\nu_2+2\alpha_2+1+2h_2)^s = 0 \qquad \text{for } h_2 = 0,1,\ldots,k_2-1\ . \qquad (7.16)$$

(iii) If $k_1 > 0$, $k_2 = 0$, $a_o,\ldots, a_r$ such that

$$\sum_{s=0}^{r} a_s(-1)^s\,(\nu_1+2\alpha_1+1+2h_1)^s = 0 \qquad \text{for } h_1 = 0,1,\ldots k_1-1\ . \qquad (7.17)$$

(iv) If $k_1 > 0$, $k_2 > 0$, $a_o,\ldots, a_r$ such that both (7.16) and (7.17) hold.

<u>Proof</u>:- With f as in (7.15), we find from Theorem 6.10 that

$$S^{\nu_1/2-\alpha_1,2\alpha_1} f$$

$$= K_2^{\nu_1/2-\alpha_1,\ell_1} S^{\nu_1/2-\alpha_1+\ell_1,2\alpha_1-\ell_1} S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2+\ell_1} K_2^{-\nu_1/2-\alpha_1-k_1,k_1} (\sum_{s=0}^{r} a_s(\delta')^s\delta_1)$$

$$= K_2^{\nu_1/2-\alpha_1,\ell_1} I_2^{\nu_2/2+\alpha_2,\nu_1-\lambda} K_2^{-\nu_1/2-\alpha_1-k_1,k_1} (\sum_{s=0}^{r} a_s(\delta')^s\delta_1)$$

$$= (-1)^{\ell_1} I_2^{-\nu_1/2+\alpha_1-\ell_1,\ell_1} (-1)^{k_2} I_2^{\nu_2/2+\alpha_2+k_2,\nu_1-\lambda-k_2} K_2^{-\nu_2/2-\alpha_2-k_2,k_2} K_2^{-\nu_1/2-\alpha_1-k_1,k_1} (\sum_{s=0}^{r} a_s(\delta')^s\delta_1)$$

where at the last stage we have used Definitions 3.27 and 3.33 (and their adjoints). Hence if $\phi \in F_{p,2/p-\mu}$, we may write

$$(S^{\nu_1/2-\alpha_1,2\alpha_1} f,\phi)$$
$$= (\sum_{s=0}^{r} a_s(\delta')^s\delta_1, I_2^{-\nu_1/2-\alpha_1-k_1-1/2,k_1} I_2^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi)$$

where

$$\psi = (-1)^{k_2+\ell_1} K_2^{\nu_2/2+\alpha_2+k_2+1/2,\nu_1-\lambda-k_2} K_2^{-\nu_1/2+\alpha_1-\ell_1+1/2,\ell_1}\phi .$$

If ϕ has support in $(0,1)$, so does ψ as $\mathrm{Re}(\nu_2 + 2\alpha_2 + 2k_2 + 1 - 2/p + \mu) > - 1/p$ and $\mathrm{Re}(-\nu_1 + 2\alpha_2 - 2\ell_1 + 1 - 2/p + \mu) > - 1/p$. We now split the proof into four parts, corresponding to the cases listed above.

(i) $\underline{k_1 = k_2 = 0}$ In this case $(S^{\nu_1/2-\alpha_1,2\alpha_1} f,\phi) = (\sum_{s=0}^{r} a_s(\delta')^s \delta_1,\psi) = 0$ for all choices of r and $a_o,\ldots,a_r$ since ψ and all its derivatives vanish at 1.

(ii) $\underline{k_1 = 0,\ k_2 > 0.}$ In this case, it is easy to see from (3.8) that, for $x \geq 1$,

$$I_2^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi(x) = x^{\nu_2+2\alpha_2+1} P(x^2)$$

where $P(t)$ is a polynomial of degree at most k_2-1 in t.

Notice that since $\mathrm{Re}(\nu_2+2\alpha_2+1+2k_2-2+\mu-2/p) < -1/p$, all such polynomials can arise from suitable functions ψ (and hence ϕ) using the homeomorphic properties of the I and K operators. By taking $P(t) = t^{h_2}$ $(h_2 = 0,1,\ldots,k_2-1)$ we deduce that $S^{\nu_1/2-\alpha_1,2\alpha_1} f = 0$ on (0,1) if and only if

$$\left(\sum_{s=0}^{r} a_s(\delta')^s \delta_1,\ x^{\nu_2+2\alpha_2+1+2h_2}\right) = 0 \text{ for } h_2 = 0,1,\ldots,k_2-1$$

from which (7.16) readily follows. (Strictly, in the last bracket we mean a function in $F_{p,2/p-\mu}$ which equals $x^{\nu_2+2\alpha_2+1+2h_2}$ for $x \geq 1$.)

(iii) $\underline{k_1 > 0,\ k_2 = 0}$ is similar with suffices 1,2 interchanged.

(iv) $\underline{k_1 > 0,\ k_2 > 0}$. As in (ii) we have $I_2^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi(x) = x^{\nu_2+2\alpha_2+1} P(x^2)$ (where $P(t)$ is a polynomial of degree $\leq k_2-1$ in t) for $x \geq 1$. A fairly routine calculation now shows that if $\mathrm{Re}(\nu_1+2\alpha_1) + 2k_1 \neq \mathrm{Re}(\nu_2+2\alpha_2) + 2k_2$, then for $x \geq 1$,

$$I_2^{-\nu_1/2-\alpha_1-k_1-1/2,k_1}\ I_2^{-\nu_2/2-\alpha_2-k_2-1/2,k_2}\psi(x)$$

$$= x^{\nu_1+2\alpha_1+1} Q_1(x^2) + x^{\nu_2+2\alpha_2+1} Q_2(x^2)$$

where $Q_1(t)$, $Q_2(t)$ are polynomials in t of degrees at most k_1-1, k_2-1 respectively. Hence from (ii) and (iii), we obtain solutions by taking $a_o,\ldots a_r$ as stated. In the case $\mathrm{Re}(\nu_1+2\alpha_1)+2k_1 = \mathrm{Re}(\nu_2+2\alpha_2) + 2k_2$, there is an added complication because logarithms appear in the integration and we have to add on a term of the form $x^{\nu_1+2\alpha_1+2k_1-1} \times Q_3(1/x^2)\log(x^2)$ where $Q_3(t)$ is a polynomial in t of degree at most $\min(k_1-1,k_2-1)$. However, since $\log 1 = 0$, the problem again reduces to evaluating polynomials at $x = 1$ and the proof goes through, but we shall omit the details of these calculations which are a little tedious.

This completes the proof of Lemma 7.10.

The complementary result for $(1,\infty)$ is as follows.

Lemma 7.11

If $\nu_i/2+\alpha_i \in A'_{q,-\mu}$ and $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i = 1,2)$, then f, as given by (7.15), satisfies

$$S^{\nu_2/2-\alpha_2,\, 2\alpha_2} f = 0 \text{ on } (1,\infty)$$

in the following circumstances.

(i) If $\ell_1 = \ell_2 = 0$, r and $a_o,\ldots,a_r$ arbitrary.

(ii) If $\ell_1 = 0$, $\ell_2 > 0$, $a_o,\ldots,a_r$ such that

$$\sum_{s=0}^{r} a_s(\nu_2-2\alpha_2+1+2j_2)^s = 0 \text{ for } j_2 = 0,1,\ldots,\ell_2-1. \tag{7.18}$$

(iii) If $\ell_1 > 0$, $\ell_2 = 0$, $a_o,\ldots,a_r$ such that

$$\sum_{s=0}^{r} a_s(\nu_1-2\alpha_1+1+2j_1)^s = 0 \text{ for } j_1 = 0,1,\ldots,\ell_1-1. \tag{7.19}$$

(iv) If $\ell_1 > 0$, $\ell_2 > 0$, $a_o,\ldots,a_r$ such that both (7.18) and (7.19) hold.

Proof:- This is similar to the previous lemma and is omitted.

Remark 7.12

The conditions in Lemmas 7.10 and 7.11 put some constraints on r. For instance (7.16)-(7.19) demand respectively that $r \geq k_2, r \geq k_1, r \geq \ell_2$ and $r \geq \ell_1$. However it is clear that each condition can be met in infinitely many ways. Furthermore we can combine our results to obtain the following theorem.

Theorem 7.13

Let $\nu_i/2+\alpha_i \in A'_{q,-\mu}$, $\nu_i/2-\alpha_i \in A_{q,-\mu}$ $(i = 1,2)$. Then there are infinitely many solutions $f \in F'_{p,\mu}$ satisfying (7.12).

<u>Proof</u>: Any of the four cases in Lemma 7.10 can occur with any of the four in Lemma 7.11. The most restrictive situation (corresponding to (iv) in each lemma) imposes $k_1+k_2+\ell_1+\ell_2$ constraints which can be met in infinitely many ways if r is such that $r+1>k_1+k_2+\ell_1+\ell_2$. We omit the details.

As a consequence of the last result we can finally state the following non-uniqueness theorem for Problem 7.3.

<u>Theorem 7.14</u>

Let $g_i \in F'_{p,2/p-\mu}$, $\nu_i/2+\alpha_i \in A'_{q,-\mu}$ and $\nu_i/2-\alpha_i \in A_{q,-\mu}$ for $i = 1,2$. Then Problem 7.3 has infinitely many solutions $f \in F'_{p,\mu}$, each of which is of the form

$$S^{\nu_2/2+\alpha_2+k_2,\nu_1-2\alpha_1-\lambda-k_2} H + S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2+\ell_1} K_2^{-\nu_1/2-\alpha_1-k_1,k_1}\left(\sum_{s=0}^{r}{}' a_s(\delta')^s\delta_1\right) \tag{7.20}$$

where H is given by (7.8) and (7.9) and $a_o,\ldots,a_r$ are described by the appropriate cases in Lemmas 7.10 and 7.11.

<u>Proof</u>:- The result is immediate from Theorems 7.7 and 7.13 and Lemmas 7.10 and 7.11.

This completes our study of Problem 7.3. Now we must try to see what light it sheds on Problem 7.1.

<u>§7.5 Uniqueness for Problem 7.1: The Simplest Case</u>

As indicated earlier, the uniqueness investigation for the classical problem is rather interesting. We shall not go into this in all its details but, rather, try to show how some well-known results emerge quite naturally from our work in the previous section.

We shall seek non-zero solutions $f \in L^q_{-\mu}$ of Problem 7.1 in the homogeneous case $g_1 = g_2 = 0$. Our first problem is to ascertain the conditions under which $S^{\eta,\alpha}f$ can be sensibly defined for $f \in L^q_{-\mu}$. We mentioned in §5.4 that H_ν can be extended to a bounded linear mapping from $L^q_{-\mu}$ into $L^q_{2/q-1+\mu}$ provided that $1 < p < \infty$ and $\max(1/p,1/q) \leqslant 1/q + \mathrm{Re}\,\mu < \mathrm{Re}\,\nu + 3/2$; this was proved by Rooney in [73, p.1100]. Although we could go further in the manner of §5.3, we shall not do so here and instead keep things as simple as possible. Using (5.36), it follows easily that $S^{\eta,\alpha}$ can be defined on $L^q_{-\mu}$ provided that

$$\max(1/p,1/q) \leqslant 1/q + \mathrm{Re}(\mu-1/2+\alpha) < \mathrm{Re}(2\eta+\alpha) + 3/2 \quad \text{and } 1 < p < \infty.$$

Theorem 7.15

$S^{\eta,\alpha}$ is a continuous linear mapping of $L^q_{-\mu}$ into $L^q_{2/q-2+\mu}$ provided that

$$\max(1/p,1/q) \leqslant 1/q + \mathrm{Re}(\mu-1/2+\alpha) < \mathrm{Re}(2\eta+\alpha) + 3/2 \quad \text{and } 1 < p < \infty.$$

Proof:- Immediate from [73, p.1100] and the above.

Throughout this section, we shall use this classical interpretation of $S^{\eta,\alpha}f$. Our work in §§5.4 and 5.5 ensures that if $f \in L^q_{-\mu}$ and the conditions in Theorem 7.15 hold, then

$$\tau S^{\eta,\alpha}f = S^{\eta,\alpha}\tau f \tag{7.21}$$

as an equation in $F'_{p,2/p-\mu}$. Hence we can immediately state the following result.

Lemma 7.16

If $\max(1/p,1/q) \leqslant 1/q + \mathrm{Re}(\mu-1/2+2\alpha_i) < \mathrm{Re}\,\nu_i + 3/2\,(i = 1,2)$, if $1 < p \leqslant \infty$ and if $f \in L^q_{-\mu}$ satisfies (7.1) with $g_1 = g_2 = 0$, then τf satisfies (7.12).

Proof:- This follows immediately from (7.21) and Theorem 7.15 together with the observation that, since $\mathrm{Re}(\nu_i-2\alpha_i-\mu) + 2 > 1/q$, $\nu_i/2-\alpha_i \in A_{q,-\mu}\,(i = 1,2)$. This also prompts the following comment.

Remark 7.17

In the sense of Notation 7.6, $\ell_1 = \ell_2 = 0$ until further notice. To simplify matters still further, we will assume until further notice that $k_1 = k_2 = 0$ also; in other words we assume also that $\mathrm{Re}(\nu_i+2\alpha_i+\mu) > -1/q$ for $i = 1,2$.

We can then state the following lemma.

Lemma 7.18

Let $\max(1/p, 1/q) \leqslant 1/q + \mathrm{Re}(\mu-1/2+2\alpha_i) < \mathrm{Re}\ \nu_i+3/2$ and $\mathrm{Re}(\nu_i+2\alpha_i+\mu) > -1/q$ $(i = 1,2)$ and let $1 < p < \infty$. If $f \in L^q_{-\mu}$ satisfies (7.1) with $g_1 = g_2 = 0$, then τf is of the form

$$\tau f = S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}\left(\sum_{s=0}^{r} a_s(\delta')^s\,\delta_1\right). \tag{7.22}$$

Proof:- This follows from Lemma 7.9 and (7.15). Note that the right-hand side of (7.22) satisfies (7.12) for all choices of r and $a_o,\ldots a_r$ by Lemmas 7.10 and 7.11 and Remark 7.17.

We should observe that the conditions on the parameters in Lemma 7.18 put restrictions on λ. Indeed, on adding the inequalities

$$\mathrm{Re}(\nu_1-2\alpha_1-\mu) + 2 > 1/q \text{ and } \mathrm{Re}(\nu_2+2\alpha_2+\mu) > -1/q$$

we see that in this case

$$\mathrm{Re}\ \lambda > -1. \tag{7.23}$$

Fortunately, it is still possible to satisfy all the constraints so that we don't have a vacuous problem! We will give an example shortly.

It is clear that we must now examine (7.22) in some detail and to do this, we first choose $\phi \in C_o^\infty(0,\infty)$, regarding ϕ as an element of $F_{p,\mu}$. Then

$$(S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}(\delta')^s\,\delta_1,\phi)$$

$$= (\,(\delta')^s\delta_1,\ x\,S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}\,x^{-1}\phi) \qquad \text{by (5.37)}$$

$$= (\,(\delta')^s\,\delta_1,\ 2^{\lambda-\nu_2-2\alpha_2}\,x^{1-\lambda+\nu_2+2\alpha_2}\int_0^\infty t^{-\lambda+\nu_2+2\alpha_2}\,J_\lambda(xt)\phi(t)dt)$$

by (5.33)

If we use the formula [19, p.11]

$$d/dz\ (z^\gamma J_\gamma(z)) = z^\gamma J_{\gamma-1}(z)$$

and perform some routine algebra, we obtain

$$(S^{\nu_2/2+\alpha_2,\lambda-\nu_2-2\alpha_2}(\sum_{s=0}^{r} a_s(\delta')^s\delta_1),\phi)$$

$$= \sum_{s=0}^{r} b_s(\delta_1,\ 2^{\lambda-\nu_2-2\alpha_2}\,x^{1-\lambda+\nu_2+2\alpha_2+s}\int_0^\infty t^{-\lambda+\nu_2+2\alpha_2+s}\,J_{\lambda-s}(xt)\phi(t)dt)$$

$$= \sum_{s=0}^{r} c_s\int_0^\infty t^{-\lambda+\nu_2+2\alpha_2+s}\,J_{\lambda-s}(t)\phi(t)dt$$

where $b_o,\ldots,b_r$ and $c_o,\ldots\ldots,c_r$ are constants. Thus, using $\phi \in C_o^\infty(0,\infty)$, we are led to the candidates

$$f(x) = \sum_{s=0}^{r} c_s\,x^{-\lambda+\nu_2+2\alpha_2+s}\,J_{\lambda-s}(x). \tag{7.24}$$

The above analysis will hold for all functions $\phi \in F_{p,\mu}$ if f, as defined by (7.24), is an element of $L^q_{-\mu}$ since, in that case, f will generate a continuous linear functional on all of $F_{p,\mu}$. However, from properties of the Bessel function, it is fairly easy to prove that $x^a\,J_b(x) \in L^q_{-\mu}$ if and only if $-\mathrm{Re}\ b-3/2+1/p < \mathrm{Re}(\mu+a) - 1/2 < -1/q$. Hence (7.24) defines an element of $L^q_{-\mu}$ if and only if $-\mathrm{Re}(\lambda-s)-3/2+1/p < \mathrm{Re}(\mu-\lambda+\nu_2+2\alpha_2+s)-1/2 < -1/q$ for each $s = 0,1,\ldots,$ r or equivalently,

$$-1/q < \mathrm{Re}(\nu_2+2\alpha_2+\mu) < 1/2-1/q+\mathrm{Re}\,\lambda-r \quad . \tag{7.25}$$

We can therefore prove the following theorem.

Theorem 7.19

Let $\max(1/p,1/q) \leqslant 1/q + \mathrm{Re}(\mu-1/2+2\alpha_i) < \mathrm{Re}\,\nu_i + 3/2$ and $\mathrm{Re}(\nu_i+2\alpha_i+\mu) > -1/q$ for $i = 1,2$ and let $1 < p < \infty$. Then

(i) f, as defined by (7.24), satisfies (7.1) with $g_1 = g_2 = 0$ if and only if $\mathrm{Re}(\nu_2+2\alpha_2+\mu) < 1/2-1/q+\mathrm{Re}\,\lambda-r$.

(ii) The homogeneous problem (7.1) has a non-zero solution $f \in L^q_{-\mu}$ if and only if $\mathrm{Re}(\nu_2+2\alpha_2+\mu) < 1/2-1/q+\mathrm{Re}\,\lambda$.

Proof:- Almost immediate from the preamble and Lemma 7.18.

Remark 7.20

Perhaps it is worth mentioning that f in (7.24) generates a regular functional in $F'_{p,\mu}$ if and only if

$$-1/q < \mathrm{Re}(\nu_2+2\alpha_2+\mu) < 1/2+1/p+\mathrm{Re}\,\lambda-r \tag{7.26}$$

as can be proved using, for instance, Corollary 5.3. (7.26) is slightly weaker than (7.25), the extra scope being provided by the differentiability of the functions in $F_{p,\mu}$. However, we need the full force of (7.25) in order to have a function f in $L^q_{-\mu}$.

It seems worthwhile to check from first principles that the proposed solutions are indeed solutions, as Theorem 7.19 suggests. To do this, we first make the following observation.

Lemma 7.21

The conditions on the parameters in Theorem 7.19 have the following consequences:

$\operatorname{Re} \lambda > -1$, $\operatorname{Re} \nu_i > -1$ $(i = 1,2)$, $r < \operatorname{Re}(\nu_1-\lambda)$ and $r < \operatorname{Re}(\lambda-\nu_2)$.

<u>Proof</u>:- $\operatorname{Re} \lambda > -1$ is (7.23). $\operatorname{Re} \nu_i > -1$ follows from the inequalities

$$\operatorname{Re}(2\alpha_i+\mu) > -\operatorname{Re} \nu_i-1/q \text{ and } \operatorname{Re}(2\alpha_i+\mu) < \operatorname{Re} \nu_i+2-1/q .$$

Next, $\max(1/p,1/q) \leq 1/q+\operatorname{Re}(\mu-1/2+2\alpha_i) \Rightarrow \operatorname{Re}(2\alpha_i+\mu) > 1/2-1/q$.

Also $\nu_2 + 2\alpha_2 + \mu - \lambda = \lambda - \nu_1 + 2\alpha_1 + \mu$. Hence

$$\operatorname{Re}(\nu_2+2\alpha_2+\mu) < 1/2 - 1/q + \operatorname{Re} \lambda - r$$

$$\Rightarrow \operatorname{Re}(\lambda-\nu_1+2\alpha_1+\mu) < 1/2 - 1/q - r$$

$$\Rightarrow \operatorname{Re} \nu_1 > \operatorname{Re} \lambda + \operatorname{Re}(2\alpha_1+\mu) - 1/2 + 1/q + r > \operatorname{Re} \lambda + r$$

whence $r < \operatorname{Re}(\nu_1-\lambda)$. $r < \operatorname{Re}(\lambda-\nu_2)$ is proved similarly.

We can now proceed with the verification. For $s = 0,1,\ldots,r$

$$S^{\nu_i/2-\alpha_i,2\alpha_i}(x^{-\lambda+\nu_2+2\alpha_2+s} J_{\lambda-s}(x))$$

$$= 2^{2\alpha_i} x^{-1/2-2\alpha_i} \int_0^\infty \sqrt{xt}\, J_{\nu_i}(xt)\, t^{1/2-2\alpha_i-\lambda+\nu_2+2\alpha_2+s} J_{\lambda-s}(t)dt .$$

<u>i=1</u> We must consider $\int_0^\infty t^{1+\lambda-\nu_1+s} J_{\lambda-s}(t) J_{\nu_1}(xt)dt$. For $0 < x < 1$, this integral is $cx^{\nu_1} \dfrac{\Gamma(\lambda+1)}{\Gamma(\nu_1+1)\Gamma(-s)}\ {}_2F_1(\lambda+1, s+1; \nu_1+1; x^2)$ where c is a constant by [19, p.51, formula (29)]. Note that the conditions for convergence, namely $-1 < \operatorname{Re} \lambda < \operatorname{Re} \nu_1-s$ are satisfied by Lemma 7.21. Finally, using $\{\Gamma(s+1)\Gamma(-s)\}^{-1} = \pi^{-1} \sin \pi(s+1)$, we find that the integral is 0 for $0 < x < 1$. Alternatively, by integrating by parts s times and using properties of the Bessel functions, we can reduce the integral to a sum of integrals of the form $\int_0^\infty t^{1+a-b} J_a(t) J_b(xt)dt$ which are zero by a simpler case of the Weber-Schafheitlin integral [19, p.92, formula (34)]. Thus

$$S^{\nu_1/2-\alpha_1,2\alpha_1}(x^{-\lambda+\nu_2+2\alpha_2+s} J_{\lambda-s}(x)) = 0 \text{ for } 0 < x < 1. \tag{7.27}$$

$\underline{i=2}$ Now we have to consider $\int_0^\infty t^{1-\lambda+\nu_2+s} J_{\lambda-s}(t) J_{\nu_2}(xt)\,dt$. For $x > 1$, we may use [19, p.92, formula (34)] again to obtain 0, since $-1 < \mathrm{Re}\ \nu_2 < \mathrm{Re}\ \lambda-s$ by Lemma 7.21. Thus

$$S^{\nu_2/2-\alpha_2,2\alpha_2}(x^{-\lambda+\nu_2+2\alpha_2+s} J_{\lambda-s}(x)) = 0 \text{ for } x > 1. \tag{7.28}$$

Hence, under the conditions of Theorem 7.19, f, as given by (7.24), does indeed satisfy (7.1) with $g_1 = g_2 = 0$ by virtue of (7.27) and (7.28). Our verification is complete.

Remarks 7.22

1. (7.27) and (7.28) may be obvious to anyone with intimate knowledge of the Weber-Schafheitlin discontinuous integral. However, our object was to show how the non-trivial solutions of the homogeneous problem (7.1) emerge naturally from our theory.
2. The solution (7.24) can be written in various different ways. For instance, if we use the formula $d/dz\,(z^{-\gamma} J_\gamma(z)) = -z^{-\gamma} J_{\gamma+1}(z)$ instead of $d/dz(z^\gamma J_\gamma(z)) = z^\gamma J_{\gamma-1}(z)$, we obtain an expression
$$\sum_{s=0}^{r} d_s\, x^{-\lambda+\nu_2+2\alpha_2+s} J_{\lambda+s}(x) \tag{7.29}$$
valid under appropriate conditions. However (7.29) can be reconciled with (7.24) by using [19, p.99, Formula (10)].

In order to show that our theory is not vacuous, we give the following example .

Example 7.23

Let c be any complex number with Re $c > 1/2$.

(i) Let $\nu_1 = 3c$, $\nu_2 = c$, $\alpha_1 = \alpha_2 = 0$ so that $\lambda = 2c$. The conditions of Lemma 7.18 require values of μ and q such that ($1 < q < \infty$ and)

$\max(1/p,1/q) \leq 1/q + \text{Re}\,\mu - 1/2 < \text{Re}\,c + 3/2$ and $\text{Re}\,c + \mu > -1/q$

or $\max(1/p + 1/2,\ 1/q + 1/2) \leq \text{Re}\,\mu + 1/q < \text{Re}\,c + 2$ (since $\text{Re}\,c > 1/2$).

These conditions can always be satisfied when $\text{Re}\,c > 1/2$ as $\max(1/p + 1/2,\ 1/q + 1/2) \leq 3/2$. To get a non-trivial solution of the form (7.24) we need

$\text{Re}(c + \mu) < 1/2 - 1/q + \text{Re}\,2c - r$ or $\text{Re}(\mu + 1/q) < 1/2 + \text{Re}\,c - r$.

Since $1/2 + \text{Re}\,c - r < \text{Re}\,c + 2$, we seek μ, q and r such that

$\max(1/p + 1/2,\ 1/q + 1/2) \leq \text{Re}\,\mu + 1/q < 1/2 + \text{Re}\,c - r$.

Taking $p = q = 2$ we then seek μ and r such that $1 \leq \text{Re}\,\mu + 1/2 < 1/2 + \text{Re}\,c - r$. We may take any value of r such that $r < \text{Re}\,c - 1/2$ and since $\text{Re}\,c > 1/2$, $r = 0$ is admissible. Finally, choosing μ such that $1/2 \leq \text{Re}\,\mu < \text{Re}\,c$, we see that the homogeneous Problem 7.1 has a non-trivial solution f in $L^q_{-\mu}$ for the stated values of the parameters, namely $f(x) = x^{-c} J_{2c}(x)$ from (7.24).

(ii) Let $\nu_1 = c$, $\nu_2 = 3c$, $\alpha_1 = \alpha_2 = 0$ so that $\lambda = 2c$. Proceeding as above, we seek r such that $\text{Re}(3c + \mu) < 1/2 - 1/q + \text{Re}\,2c - r$ or $\text{Re}(\mu + 1/q) < 1/2 - \text{Re}\,c - r$ in addition to the previous conditions. Hence we require

$1 \leq \max(1/p+1/2, 1/q+1/2) \leq \text{Re}\,\mu + 1/q < 1/2 - \text{Re}\,c - r$

which is impossible if $\text{Re}\,c > 1/2$ and r is a non-negative integer. Hence, under the hypotheses of Lemma 7.18, the homogeneous Problem 7.1 has no non-trivial solution in $L^q_{-\mu}$ for the stated values of the parameters.

Finally, in this section, we look at Problem 7.1 in the non-homogeneous case still assuming that $k_1 = k_2 = \ell_1 = \ell_2 = 0$. We state the following theorem.

Theorem 7.24

Let $\max(1/p,1/q) \leqslant 1/q + \mathrm{Re}(\mu-1/2+2\alpha_i) < \mathrm{Re}\ \nu_i + 3/2$, $\mathrm{Re}(\nu_i+2\alpha_i+\mu) > -1/q$ and $g_i \in L^q_{2/q-2+\mu}$ $(i = 1,2)$ and let $1 < p < \infty$. Then Problem 7.1 has a unique solution $f \in L^q_{-\mu}$ if also $\mathrm{Re}(\lambda-\nu_1) > 0$, $\mathrm{Re}(\nu_2-\lambda) > 0$ and $\mathrm{Re}(\nu_2+2\alpha_2+\mu) \leqslant \mathrm{Re}\ \lambda + 3/2 - 1/q - \max(1/p,1/q)$.

Proof: g_i generates $\tau g_i \in F'_{p,-2/q+2-\mu} = F'_{p,2/p-\mu}$ for $i = 1,2$. Let h be constructed as in (7.8) with g_i replaced by τg_i for $i = 1,2$. Since $k_1 = k_2 = \ell_1 = \ell_2 = 0$, Remark 3.46 shows that h can be chosen to be a regular element of $F'_{p,2/p-\mu}$ under the given conditions; this is where we require $\mathrm{Re}(\lambda-\nu_1) > 0$ and $\mathrm{Re}(\nu_2-\lambda) > 0$. Next, observing that H in (7.9) coincides with h here, we proceed to (7.10) and find, using (5.38) and Theorem 7.15 that $S^{\nu_2/2+\alpha_2,\nu_1-2\alpha_1-\lambda}h$ is a regular element of $F'_{p,\mu}$; this requires in particular that $\mathrm{Re}(\nu_2 + 2\alpha_2 + \mu) \leqslant \mathrm{Re}\ \lambda + 3/2 - 1/q - \max(1/p, 1/q)$. Hence using Theorem 7.7, this functional is generated by a function in $L^q_{-\mu}$ which is a solution of Problem 7.1. This gives existence.

For uniqueness we note that any two solutions in $L^q_{-\mu}$ of Problem 7.1 would differ by an expression of the form (7.24) where, by Lemma 7.21, r would satisfy $0 \leqslant r < \mathrm{Re}(\nu_1-\lambda)$, $0 \leqslant r < \mathrm{Re}(\lambda-\nu_2)$ which is impossible if $\mathrm{Re}(\lambda-\nu_1) > 0$ and $\mathrm{Re}(\nu_2-\lambda) > 0$. This completes the proof.

Remark 7.25

Since $\mathrm{Re}\ \lambda > -1$ by (7.23) and $\mathrm{Re}(\nu_2+2\alpha_2+\mu) > -1/q$, the condition $\mathrm{Re}(\nu_2+2\alpha_2+\mu) < \mathrm{Re}\ \lambda + 3/2 - 1/q - \max(1/p,1/q)$ can be made non-vacuous by choosing $p = q = 2$, for instance.

If any one of the conditions stated in Theorem 7.24 is violated, the proof breaks down and various possibilities occur. For instance, we have the following.

(i) If $\mathrm{Re}(\lambda-\nu_1) < 0$, $\mathrm{Re}(\nu_2-\lambda) > 0$, h as defined by (7.8) (with τg_i instead of g_i) will only be regular if g_1 satisfies an appropriate condition involving fractional derivatives (Compare Theorem 4.19.) Thus, since $0 \leqslant r < \mathrm{Re}(\lambda-\nu_2)$ is impossible, we get at most one classical solution in this case.

(ii) If $\mathrm{Re}(\lambda-\nu_1) < 0$, $\mathrm{Re}(\nu_2-\lambda) < 0$, we now require restrictions on g_1 and g_2 involving fractional derivatives. Hence we may get no solution in $L^q_{-\mu}$. But if we find one, we will have infinitely many under the conditions of Theorem 7.19.

Other results may be obtained as required but we shall not give details here.

§ 7.6 Uniqueness for Problem 7.1: Other Cases

To end this chapter, we mention briefly what can happen if one or more of the integers k_1, k_2, ℓ_1 and ℓ_2 is positive. For simplicity we shall restrict attention to the homogeneous case of Problem 7.1 with $g_1 = g_2 = 0$. If $\ell_1 = \ell_2 = 0$ so that we can interpret $S^{\nu_i/2-\alpha_i,2\alpha_i}$ $(i = 1,2)$ as in Theorem 7.15, the arguments for $k_1 > 0$ or $k_2 > 0$ or both are fairly similar to the above. We may have to apply one or two Erdélyi-Kober operators to the Bessel functions J_ν obtaining either ${}_1F_2$ or ${}_2F_3$ hypergeometric functions [20, pp.193-195]. By using results in [18] we can easily describe the behaviour of these functions at 0 and ∞ and results similar to those in the previous section can be obtained. We omit details.

If $\ell_1 > 0$ or $\ell_2 > 0$, we run into difficulties over the classical definition of $S^{\nu_i/2-\alpha_i,2\alpha_i}$. To illustrate this point, we consider the following.

Example 7.26 (The (un)electrified disk)

By (7.3) this corresponds to the values

$\nu_1 = \nu_2 = 0$, $\alpha_1 = 1$, $\alpha_2 = 1/2$ and $\lambda = -1/2$. If we were to try to use $\ell_1 = \ell_2 = 0$, we would require a space $L^q_{-\mu}$ with $\max(1/p,1/q) \leq 1/q + \mathrm{Re}\,\mu - 1/2 + 2 < 3/2$ and $\max(1/p,1/q) \leq 1/q + \mathrm{Re}\,\mu - 1/2 + 1 < 3/2$ which requires $-1/2 + \max(1/p,1/q) \leq 1/q + \mathrm{Re}\,\mu < 0$. However $\max(1/p,1/q) - 1/2 \geq 0$ and we cannot find a suitable μ and q. Thus we have to find an alternative definition for at least one of $S^{\nu_i/2-\alpha_i,2\alpha_i}$ $(i = 1,2)$.

By analogy with Definition 5.14, Note 4 and (5.36), we define $S^{\nu_i/2-\alpha_i,2\alpha_i}$ on $L^q_{-\mu}$, with $\ell_i > 0$, by

$$S^{\nu_i/2-\alpha_i,2\alpha_i} f(x) =$$

$$(-1)^{\ell_i} 2^{2\alpha_i-\ell_i} x^{-1/2-2\alpha_i+\ell_i} I_2^{-\nu_i/2-1/4-\ell_i/2,\ell_i} H_{\nu+\ell_i} x^{1/2-2\alpha_i+\ell_i} f$$

or equivalently

$$S^{\nu_i/2-\alpha_i,2\alpha_i} f(x) = (-1)^{\ell_i} I_2^{-\nu_i/2+\alpha_i-\ell_i,\ell_i} S^{\nu_i/2-\alpha_i+\ell_i,2\alpha_i-\ell_i} f. \quad (7.30)$$

If $\max(1/p,1/q) \leq 1/q + \mathrm{Re}(\mu-1/2+2\alpha_i-\ell_i) < \mathrm{Re}(\nu_i+\ell_i) + 3/2$ and $1 < p < \infty$, the right-hand side is a continuous linear mapping of $L^q_{-\mu}$ into $L^q_{2/q-2+\mu}$ by Lemma 3.2 and Theorem 7.15. In particular we shall use this interpretation for $i = 1$, with $\ell_1 = 1$.

Now take $k_1 = k_2 = 0$, $\ell_1 = 1$, $\ell_2 = 0$. The conditions to be satisfied by μ and q are ($1 < q < \infty$ and) $\max(1/p,1/q) \leq 1/q + \mathrm{Re}\,\mu + 1/2 < 3/2$, $\mathrm{Re}(\mu+1/q) > 0$ or $\max(1/p-1/2,1/q-1/2,0) \leq \mathrm{Re}(\mu+1/q) < 1$. Such values of μ,q can be found and for these values we try to find a non-trivial solution $f \in L^q_{-\mu}$ of the equations

$$S^{-1,2} f = 0 \text{ on } (0,1); \quad S^{-1/2,1} f = 0 \text{ on } (1,\infty)$$

where $S^{-1,2}$ is interpreted via (7.30) and $S^{-1/2,1}$ as in Theorem 7.15. The argument now proceeds as before and we obtain as a possible solution

$$f = \sum_{s=0}^{r} c_s \, x^{1/2+s} \, J_{1/2-s} \tag{7.31}$$

where the admissible constants $c_o,\dots,c_r$ are restricted by Lemma 7.11. For the right-hand side to belong to $L^q_{-\mu}$ we need $-1/q < \mathrm{Re}(1+\mu) < 1 - 1/q - r$; in particular, we need $\mathrm{Re}(\mu+1/q) < -r$. However, as $\mathrm{Re}(\mu+1/q) > 0$, no non-negative integer is feasible. Hence our problem only has the trivial solution $f = 0$. This is in accord with the physics since when the disk is not charged, the potential at any point must be zero, in the absence of an external field.

A uniqueness investigation for the charged disk can be carried through in a similar way using the above in conjunction with Theorem 7.7. Likewise, the crack problem corresponding to the values in (7.4) can be studied using $k_1 = k_2 = \ell_1 = 0$, $\ell_2 = 1$, for instance. However, perhaps we have gone as far as necessary in outlining the usefulness of generalised functions in the study of dual integral equations of Titchmarsh type. It would seem probable that similar methods can be applied to more general problems such as those producing triple integral equations of the type treated by Cooke [4] .

8 Other integral transforms

§8.1 Introduction

This final chapter is different from its predecessors. In it we will prove little but will state a few facts about the behaviour of certain other standard transforms relative to $F'_{p,\mu}$. We will discover that the theory is by no means as complete as that for the operators discussed above. We will also indicate how some simple modifications of our spaces are suggested by the work of various authors.

§8.2 Other Transforms

Let T be an integral transform defined for suitable functions ϕ by

$$T\phi(x) = \int_0^\infty k(xt)\phi(t)dt \qquad (0 < x < \infty) . \tag{8.1}$$

Then we can state the following result.

Theorem 8.1

If $1 \leqslant p \leqslant \infty$ and μ is any complex number, then T is a continuous linear mapping of L^p_μ into $L^p_{2/p-1-\mu}$ provided that

$$\int_0^\infty x^{\mathrm{Re}\,\mu - 1/p}\,|k(x)|dx < \infty \quad . \tag{8.2}$$

Proof:- For ϕ defined almost everywhere on $(0,\infty)$, define $R\phi$ by

$$R\phi(x) = (1/x)\phi(1/x) \qquad (0 < x < \infty). \tag{8.3}$$

It is easy to check that $\phi \in L^p_\mu \Rightarrow R\phi \in L^p_{2/p-1-\mu}$ and that

$$||\phi||_{p,\mu} = ||R\phi||_{p,2/p-1-\mu} \quad . \tag{8.4}$$

Also

$$T\phi(x) = \int_0^\infty k(x/u)R\phi(u)du/u \tag{8.5}$$

so that the result now follows from (8.4) and Lemma 3.1 in [72] (with a slight change of notation).

The simple ideas behind the proof of Theorem 8.1 have already been used in Lemma 5.1 in connection with H_ν. Other ideas from Chapter 5 easily go over. For instance, let $\phi \in F_{p,\mu}$ and write

$$T\phi(x) = \int_0^\infty k(v)\phi(v/x)dv/x.$$

We may differentiate under the integral sign to obtain

$$\delta^n T\phi = T(-\delta')^n\phi \qquad (n = 0, 1, 2, \ldots)$$

where δ' is defined via (2.10); the differentiation can be justified using uniform convergence and Theorem 2.2 provided that (8.2) holds. Hence we have the following result.

<u>Corollary 8.2</u>

T, as defined by (8.1), is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ provided that (8.2) holds.

<u>Proof</u>:- This is immediate from the above.

So far we have not demanded any continuity or differentiability of the kernel k, but obviously if k is, say, continuously differentiable, we may be able to relax (8.2). Indeed, for the Hankel transform, $k(x) = \sqrt{x}\, J_\nu(x)$ and (8.2) is satisfied provided that $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu < -1/q$. But, using integration by parts, we saw in Lemma 5.1 that $\mathrm{Re}\,\mu < -1/q$ may be relaxed to $\mathrm{Re}\,\mu < 1/p$ in the first instance and then, by repeating the process, we removed the restriction $\mathrm{Re}\,\mu < 1/p$ completely in Definition 5.8. Finally, the restriction $-\mathrm{Re}\,\nu - 3/2 + 1/p < \mathrm{Re}\,\mu$ was relaxed in

Definition 5.14. We can carry out similar programmes for other transforms. For instance, let us consider briefly the Laplace transform .

For $\phi \in F_{p,\mu}$, the Laplace transform, $L\phi$, of ϕ is defined in the first instance by

$$L\phi(x) = \int_0^\infty e^{-xt}\,\phi(t)dt \qquad (0 < x < \infty). \tag{8.6}$$

By Corollary 8.2, L, as so defined, is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ provided that $\operatorname{Re}\mu > -1/q$. However, if we integrate by parts, we get

$$L\phi(x) = \left[e^{-xt}\,D^{-1}\phi(t)\right]_0^\infty + \int_0^\infty xe^{-xt}\,D^{-1}\phi(t)dt \tag{8.7}$$

($D = d/dx$ of course). Since $\operatorname{Re}\mu > -1/q, \phi \in F_{p,\mu} \Rightarrow D^{-1}\phi \in F_{p,\mu+1}$ by Theorem 2.13 and hence, by Theorem 2.2, the integrated terms vanish, leaving

$$L\phi(x) = x\,L\,D^{-1}\phi(x) \; . \tag{8.8}$$

However the right-hand side defines a continuous linear mapping of $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ provided only that $\operatorname{Re}\mu + 1 > -1/q$ (for L) and $\operatorname{Re}\mu \neq -1/q$ (for D^{-1} using Theorem 2.13(ii)). Repeating the process we can extend L to spaces $F_{p,\mu}$ such that $\operatorname{Re}\mu \neq -1/q - \ell$ ($\ell = 0, 1, 2, \ldots$), or equivalently, such that $0 \in A_{p,\mu,1}$, using the notation in Definition 3.26. Namely, if $0 \in A_{p,\mu,1}$, we define L on $F_{p,\mu}$ by

$$L\phi(x) = x^n L(D^{-1})^n\phi(x) \tag{8.9}$$

where n is a non-negative integer such that $\operatorname{Re}\mu + n > -1/q$ and L is given by (8.6). In particular, if $\operatorname{Re}\mu < -1/q$ and n is chosen such that

$$\operatorname{Re}\mu + (n-1) < -1/q < \operatorname{Re}\mu + n \; ,$$

then (8.9) takes on the form

$$L\phi(x) = (-x)^n\,L\,K_1^n\phi(x) \tag{8.10}$$

in the notation of (3.5). (8.10) can be regarded as the analogue of formulae such as (3.33) or (5.26).

This is all very well but unfortunately there is a snag. If we return to (8.6), we find that, for Re $\mu > -1/q$, L maps $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ but not onto $F_{p,2/p-1-\mu}$. For instance, under the given conditions, the function e^{-x} is an element of $F_{p,2/p-1-\mu}$ but there is no $\phi \in F_{p,\mu}$ such that

$$\int_0^\infty e^{-xt}\,\phi(t)dt = e^{-x} \quad (0 < x < \infty). \tag{8.11}$$

In the case $\mu = 0$, this follows from [85, p.313] and the general case is similar. We deduce immediately from (8.9) that L maps $F_{p,\mu}$ into $F_{p,2/p-1-\mu}$ when $0 \in A_{p,\mu,1}$ also. In some ways we are now worse off than if we had started with L^p_μ instead of $F_{p,\mu}$. For instance, when $\mu = 0$, the range of L on $L^p = L^p_0$ is characterised in [85, pp.313, 318]. All we have said so far is that the range of L on $F_{p,0}$ is some vaguely characterised subset of the range on L^p. Nor does it seem too hopeful to imbed L^p_μ in $F'_{q,-\mu}$. Since L is formally self-adjoint, we are led to define L on $F'_{p,\mu}$ by

$$(Lf,\ \phi) = (f,\ L\phi) \qquad (f \in F'_{p,\mu},\ \phi \in F_{p,2/p-1-\mu}). \tag{8.12}$$

Certainly, (8.12) is meaningful if $0 \in A_{q,-\mu,1}$ in view of [87, Theorem 1.10-1] and our previous remarks. However, we can only deduce that L maps $F'_{p,\mu}$ into $F'_{p,2/p-1-\mu}$ and it would appear that at best some rather incomplete results could be obtained.

(8.11) gives a hint as to how things might be made better. If the unknown function ϕ were replaced by a distribution, then we could solve (8.11) obtaining the (unique) solution δ_1, as defined by (7.5). More precisely, we are led to abandon the adjoint operator approach and to try instead the "kernel method" as described in §1.1. First we note that, for fixed $x \in (0,\infty)$, e^{-xt} (as a function of t) is an element of $F_{p,\mu}$ provided that Re $\mu < 1/p$. Hence, for such values of p and μ, we may define the Laplace transform, $\tilde{L}f$ say, of $f \in F'_{p,\mu}$ by

$$\tilde{L}f(x) = (f(t), e^{-xt}) \qquad (0 < x < \infty) \tag{8.13}$$

where, with the customary abuse of notation, f(t) indicates that f acts on functions of the variable t. By the structure theorem (Theorem 2.25),

$$f = \sum_{k=0}^{n} \delta^k \tilde{f}_k$$

for some non-negative integer n and $f_k \in L^q_{-\mu}$ $(k = 0,1,\ldots,n)$. (8.13) then becomes

$$\tilde{L}f(x) = \sum_{k=0}^{n} \int_0^\infty f_k(t)(-\delta')^k e^{-xt}dt = \sum_{k=0}^{n} x^k \int_0^\infty t^k g_k(t)\, e^{-xt}\, dt$$

for certain functions $g_k \in L^q_{-\mu}$. Thus

$$\tilde{L}f(x) = \sum_{k=0}^{n} x^k\, L\, x^k g_k \tag{8.14}$$

where L denotes the classical Laplace transform.(Hence the use of $\tilde{L}$ for the generalised transform here.) By Theorem 8.1, $\tilde{L}f \in L^q_{2/q-1+\mu}$ provided that $\text{Re}(-\mu+k) > -1/p$ for $k = 0,1, \ldots, n$ and this is satisfied if, as above, Re $\mu < 1/p$. Indeed $\tilde{L}f \in F_{q,2/q-1+\mu}$ and, from a characterisation of the range of L on $L^q_{2/q-1+\mu}$ (analogous to those in [85, Chapter 7])we could easily obtain a description of the range of $\tilde{L}$ on $F'_{p,\mu}$. As mentioned in §1.1, Zemanian [87, Chapter 3] adopts this approach but uses different spaces. We might expect to be able to develop an analogous theory for $F'_{p,\mu}$ but we will not embark on this here.

Closely related to the Laplace transform is the K transform of order ν, defined formally for suitable functions ϕ and complex numbers ν by

$$\mathcal{K}_\nu\phi(x) = \int_0^\infty \sqrt{xt}\;\; K_\nu(xt)\phi(t)dt \qquad (0 < x < \infty)$$

where K_ν denotes the modified Bessel function of the third kind and order ν. The connection with L is simply explained using fractional calculus. From [19, p.18, formula (15)] we see that for Re $\nu > -1/2$ and $y > 0$,

$$\sqrt{y}\;K_\nu(y) = \sqrt{\pi/2}\;(y/2)^{\nu+1/2}\;K_2^{-\nu,\nu+1/2}\;e^{-y}\,.$$

Thus formally we have

$$\begin{aligned}\mathcal{K}_\nu\phi(x) &= \int_0^\infty \sqrt{xt}\;K_\nu(xt)\phi(t)dt\\ &= \sqrt{\frac{\pi}{2}}\int_0^\infty e^{-xt}\;I_2^{-\nu-1/2,\nu+1/2}\;(xt/2)^{\nu+1/2}\phi(t)dt\\ &= \sqrt{\pi/2}\;(x/2)^{\nu+1/2}\;L\;I_2^{-\nu-1/2,\nu+1/2}\;x^{\nu+1/2}\;\phi\,. \qquad (8.15)\end{aligned}$$

(8.15) is a special case of results of Okikiolu [54], [55] and the paper [10] by Erdélyi is also relevant here. Conditions under which (8.15) is valid can easily be obtained from Lemma 3.1 and Theorem 8.1 and these are very reminiscent of the conditions encountered in §5.2 for H_ν, which is perhaps not surprising as [19, pp4-6]

$$J_\nu(x) = \frac{i}{\pi}\left[e^{\nu\pi i/2}K_\nu(ix) - e^{-\nu\pi i/2}K_\nu(-ix)\right].$$

Using either the adjoint operator method or the kernel method, we can develop a theory of the K transform, proceeding either as in Chapter 5 or using results for L (or $\tilde{L}$) and (8.15). However, we will omit the details.

Hypergeometric functions of one kind or another are never far away and it is worth mentioning that transforms involving the ${}_1F_1$ confluent hypergeometric function have also attracted the attention of various authors. In [10], Erdélyi discusses a transform of the form

$$\frac{\Gamma(a)}{\Gamma(b)}\int_0^\infty (xt)^\beta\;{}_1F_1\;(a;b;\,-xt)\phi(t)dt \qquad (8.16)$$

where $a = \beta + \gamma + 1$, $b = \alpha + \beta + \gamma + 1$ and α, β, γ are suitably restricted complex numbers. Since, by [20, p.187, formula (14)],

$$I_1^{\eta,\alpha} e^{-x} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\alpha+1)} \; {}_1F_1 \; (\eta+1; \; \eta+\alpha+1; \; -x) \; ,$$

it is no surprise that the transform can be written as the composition of an Erdélyi-Kober operator and the Laplace transform. The transform has been extended to distributions by Rao [71]. In view of our remarks on the Laplace transform, we could develop a theory for this ${}_1F_1$ transform in $F'_{p,\mu}$ but the theory would have the shortcomings described above. It might be thought, since

$${}_1F_1 \; (a; \; c; \; x) = \lim_{b\to\infty} {}_2F_1 \; (a, \; b; \; c; \; x/b) \; ,$$

that we might be able to deduce some properties of certain ${}_1F_1$ operators from our results in Chapter 4 but eventually troublesome factors involving e^{-x} or e^{x} appear. As an instance of this, we mention the integral equation

$$\frac{1}{\Gamma(c)} \int_0^x (x-t)^{c-1} \; {}_1F_1 \; (a; \; c; \; \lambda(x-t))\phi(t)dt = \psi(x) \quad (0 < x < \infty). \quad (8.17)$$

In two almost identical papers, Habibullah [25] and Prabhakar [69] show that, under appropriate conditions, the solution of (8.17) can be written in the form

$$\psi(x) = e^{\lambda x} \; I_1^{-a} \; e^{-\lambda x} \; I_1^{a-c} \; \phi(x) \; . \qquad (8.18)$$

We may say that, although some results can be obtained in $F'_{p,\mu}$ using standard results for convolutions e.g. [26, pp.396-7] these will be of a somewhat incomplete nature as regards the range of the operators on $F_{p,\mu}$ or $F'_{p,\mu}$.

§8.3 Modifications of the Spaces

We have probably said enough about how we might try to study most of the standard integral transforms in $F_{p,\mu}$ and $F'_{p,\mu}$, with varying degrees of success. However, the work of several authors suggests that it might be fruitful to examine certain extensions and/or modifications of the spaces $F_{p,\mu}$ and $F'_{p,\mu}$. We give a brief indication of some possibilities.

To provide a little motivation, let us consider the classical Mellin transform, which we will denote by M. Thus, for a classical function ϕ on $(0,\infty)$, we define $M\phi$ by

$$M\phi(x) = \int_0^\infty t^{x-1}\,\phi(t)dt \qquad (0 < x < \infty) \tag{8.19}$$

where the integral is interpreted appropriately. Since the formal adjoint of M involves integration with respect to a power, the adjoint operator approach does not look very promising. However, if we try the kernel approach in $F'_{p,\mu}$ we are also in trouble. For fixed x, the function t^{x-1} does not belong to $F_{p,\mu}$ for any p and μ; there is trouble either at t=0 or t=∞. To get round this, Zemanian in [87, Chapter 4] introduces spaces of testing-functions whose behaviour at 0 may differ from that at ∞. Using the notation of Erdélyi in [74, p.162], for real numbers a and b,

$$\mathcal{M}_{a,b} = \{\phi \in C^\infty(0,\infty) : \mu_{a,b,k}(\phi) < \infty \text{ for } k = 0,1,2,\ldots\} \tag{8.20}$$

where

$$\mu_{a,b,k}(\phi) = \sup_{0 < x < \infty} x^{1-a}(1+x)^{a-b}\,|x^k d^k\phi/dx^k|. \tag{8.21}$$

$\mathcal{M}_{a,b}$ is given the topology generated by the semi-norms $\{\mu_{a,b,k}\}_{k=0}^\infty$.

When b = a, $F_{\infty,a-1}$ is a proper subspace of $\mathcal{M}_{a,a}$, since

$$\phi \in \mathcal{M}_{a,a} \not\Rightarrow x^{1-a+k}\; d^k\phi/dx^k \to 0 \text{ as } x \to 0+ \text{ or } x \to \infty.$$

The space $\mathcal{M}(a,b)$ is then defined as the countable union space (in the sense of [87, §4.2]) of spaces $\mathcal{M}(a_n,b_n)$ where $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ are real sequences with $a_n \downarrow a$, $b_n \uparrow b$ as $n \to \infty$ (with $a = -\infty$ or $b = \infty$ or both being allowed). It is not hard to see that

$$t^{x-1} \in \mathcal{M}_{a,b} \text{ when } a < x < b.$$

More generally, replacing x by a complex variable s, we have

$$t^{s-1} \in \mathcal{M}_{a,b} \text{ when } a < \operatorname{Re} s < b \tag{8.22}$$

which of course demands that $a < b$. We can then use the kernel method to define $\mathcal{M}f$ for $f \in \mathcal{M}'(a,b)$ by

$$(\mathcal{M}f)(s) = (f(t), t^{s-1}) \qquad (a < \operatorname{Re} s < b)\ . \tag{8.23}$$

This produces a classical function of the complex variable s analytic in the strip stated. In [74], Erdélyi shows also how a theory of fractional calculus can be developed in $\mathcal{M}'(a,b)$ and establishes generalisations of familiar classical results such as

$$\mathcal{M}I_m^{\eta,\alpha} f(s) = \frac{\Gamma(\eta+1-s/m)}{\Gamma(\alpha+\eta+1-s/m)} \mathcal{M}f(s) \qquad (f \in \mathcal{M}'\ (a,b)\ )$$

$(a < \operatorname{Re} s < b < m \operatorname{Re} \eta + m)$ where $I_m^{\eta,\alpha}$ is interpreted in its generalised sense.

Another transform which has been studied in the spaces $\mathcal{M}'_{a,b}$ is the Stieltjes transform. The classical Stieltjes transform is defined for suitable functions ϕ by

$$S\phi(x) = \int_0^\infty \frac{\phi(t)}{x+t}\, dt \qquad (0 < x < \infty) \tag{8.24}$$

the integral being interpreted appropriately. A generalisation of (8.24) is the transform S_ρ defined formally by

$$S_\rho\phi(x) = \int_0^\infty \frac{\phi(t)}{(x+t)^\rho}\, dt \qquad (0 < x < \infty) \tag{8.25}$$

where ρ is a complex parameter. The operator S_ρ has been studied classically by Byrne and Love who note ([3], p.331, formula (2.2)) that

$$S\phi = S_1\phi = K_1^{\rho-1}\ S_\rho\phi \tag{8.26}$$

under appropriate conditions, where $K_1^{\rho-1}$ is as in Chapter 3. Since S is essentially the square of the Laplace transform L, it is clear that we could use results for L along with (8.26) to obtain a theory of S_ρ in $F'_{p,\mu}$. However, several authors have proceeded from scratch. In [65], Pandey extended S to certain spaces which, in the above notation, are of the form $\mathcal{M}'_{1,1-a}$ and Pathak [66] studied S_ρ in the same spaces. Both used the "kernel method".

In [15], Erdélyi carries out a similar investigation for S_ρ in $\mathcal{M}'(a,b)$ and combines it with his fractional calculus mentioned above to deal with a hypergeometric integral equation, which takes the form

$$\int_0^\infty t^{-b}\ {}_2F_1\ (a,\ b;\ c;\ -\ x/t)\phi(t)dt = \psi(x) \quad . \tag{8.27}$$

In [74], Love studies this equation classically and shows how the operator on the left is expressible in terms of fractional integrals and the Stieltjes transform S. The extension to generalised functions is then easy using Erdélyi's theory. The operator in (8.27) may look fairly similar to those we studied at length in Chapter 4 but its mapping properties on $F_{p,\mu}$ are rather different and imprecise because of the corresponding properties of S.

Two general points are in order concerning the results obtained using the kernel method in the various cases quoted above. Firstly, the spaces used all involve L^∞-type semi-norms; (8.21) is typical. It seems reasonable to expect that a corresponding L^p theory could be obtained by making appropriate modifications. For instance, the analogues of (8.20) and (8.21) would be

$$\{\phi \in C^\infty(0,\infty) : \mu^p_{a,b,k}(\phi) < \infty \text{ for } k = 0,1,2,\ldots.\}$$

where

$$\mu^{p}_{a,b,k}(\phi) = ||x^{1-a}(1+x)^{a-b}\ x^k d^k\phi/dx^k||_p \quad .$$

The second remark concerns the inversion formulae for the transforms. It is a slight drawback in this approach that most such formulae hold for $\phi \in C_0^\infty(0,\infty)$ and not for all ϕ. To be more precise, let us consider again the Mellin transform in $\mathcal{M}'(a,b)$. (8.23) associates with each $f \in \mathcal{M}'(a,b)$ the classical function $\mathcal{M}f$, analytic in the strip $a < \text{Re } s < b$. A precise characterisation of the range of $\mathcal{M}$ on $\mathcal{M}'(a,b)$ is given in [87, Theorem 4.3-5]. Conversely, suppose we are given a function, g say, in this range. Then, according to [87, Theorem 4.3-3], $g = \mathcal{M}f$ where $f \in \mathcal{M}'(a,b)$ is such that

$$f(x) = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{\sigma-ir}^{\sigma+ir} g(s)\ x^{-s} ds$$

where σ is a real number satisfying $a < \sigma < b$ and convergence is in the sense of the space $\mathcal{D}'(0,\infty)$ of distributions rather than in the sense of $\mathcal{M}'_{a,b}$. It remains to be seen whether anything can be done to get round this.

To summarise our discussion in this section, we may say that there is scope for studying fractional calculus in spaces more general than $F'_{p,\mu}$ (i.e. the L^p versions of $\mathcal{M}'(a,b)$), for studying various integral transforms in these spaces and, finally, connecting the transforms with the fractional calculus in order to solve practical problems.

§8.4 Subspaces of $F_{p,\mu}$

As well as generalising $F_{p,\mu}$ on the lines indicated in the previous section, there is another idea which arises naturally, namely, the study of certain subspaces of $F_{p,\mu}$. For instance, we might want the subset

$$\{\phi \in F_{p,\mu} : \phi(x) = 0 \text{ for } x \in I\} \tag{8.28}$$

(I a sub-interval of $(0,\infty)$), which, when given the topology induced by that on $F_{p,\mu}$, is clearly a closed subspace of $F_{p,\mu}$ and a Fréchet space in its own right.

To give one instance of how this may arise, we return to the equation (8.17) which, as we mentioned, presents some difficulties when studied in $F_{p,\mu}$. However, let us define an operator R as follows; for a function ϕ defined (almost everywhere) on $(-\infty,\infty)$, define $R\phi$ on $(0,\infty)$ by

$$R\phi(x) = \phi(\log x) \qquad (0 < x < \infty) \tag{8.29}$$

so that R is a particular case of the operator $T_{p,\mu}^{-1}$ defined in (2.5). A number of simple observations can be made.

(i) For any complex λ, $Re^{\lambda x}\phi(x) = x^{\lambda}\ R\phi(x)$.

(ii) If $\phi \in C^{\infty}(-\infty,\infty)$, then for $k = 0,1,2\ldots,$

$$\delta^k\ R\phi(x) = R\ D^k\phi(x) \quad .$$

(iii) If ϕ is continuous on $(-\infty,\infty)$ and $\phi(x) = 0$ for $x \leqslant 0$, then for $\mathrm{Re}\ \alpha > 0$,

$$\begin{aligned} R\ I_1^{\alpha}\phi(x) &= \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\log x} (\log x - t)^{\alpha-1}\ \phi(t)dt \\ &= \frac{1}{\Gamma(\alpha)}\int_0^x (\log x - \log u)^{\alpha-1}\ \phi(\log u)\ du/u \\ &= I_{\log}^{\alpha}\ R\phi(x) \end{aligned}$$

where $I_{\log}^{\alpha}$ is defined as in the preamble to (3.4) with $\rho(x) = \log x$. Using this operational calculus, we can transform (8.17) into a more amenable form. First we extend (8.17) to the whole real line by setting $\phi(x) = 0$ and $\psi(x) = 0$ for $-\infty < x \leqslant 0$. Next (8.17) can be written in the form

$$I_1^{c-a}\ e^{\lambda x}\ I_1^a\ e^{-\lambda x}\phi(x) = \psi(x) \qquad (0 < x < \infty) \tag{8.30}$$

as is shown in [25] or [69]. Interpreting both sides of (8.30) as zero for $x \leqslant 0$, we obtain formally

$$R\, I_1^{c-a}\ e^{\lambda x}\ I_1^a\ e^{-\lambda x}\phi(x) = R\psi(x)$$

$$\Rightarrow I_{\log}^{c-a}\, x^{\lambda}\, I_{\log}^{a}\ x^{-\lambda}\ R\phi(x) = R\psi(x) \qquad (0 < x < \infty).$$

Since $R\phi(x) = R\psi(x) = 0$ for $0 < x \leqslant 1$, if $\phi(x) = \psi(x) = 0$ for $-\infty < x \leqslant 0$, we are led to study the equation

$$I_{\log}^{c-a}\, x^{\lambda}\, I_{\log}^{a}\ x^{-\lambda}\ \Phi(x) = \Psi(x) \tag{8.31}$$

in a set of functions defined on $(0,\infty)$ and vanishing on $(0,1]$. Further, since

$$\delta = x\, d/dx = d/d(\log x)\ ,$$

we might expect (taking courage in both hands!) that

$$I_{\log}^{\alpha} = (\delta^{-1})^{\alpha} \tag{8.32}$$

in some sense or another and that (8.31) might have the unique solution

$$\Phi(x) = x^{\lambda}\ \delta^{a}\ x^{-\lambda}\ \delta^{c-a}\Psi(x) \quad .$$

Since we have studied x^{λ} and δ in $F_{p,\mu}$ and we also require functions which vanish on $(0,1]$ we are led to the set (8.28) where $I = (0,1]$.

The above discussion involves much hand-waving and a lot requires checking. We won't give the details here but merely remark that, from (2.11), δ^{-1} is given by Erdélyi-Kober operators on $F_{p,\mu}$ in the cases Re $\mu > 1/p$ and Re $\mu < 1/p$ so that (8.32) involves the study of fractional powers of Erdélyi-Kober operators. This seems to be fairly straightforward using results in [21] on the spectrum of the operators and the general theory of fractional powers of operators as developed by Komatsu in [37]. At the end we have to get back from Φ, Ψ in (8.31) to ϕ, ψ as in (8.30) or, indeed, (8.17). This requires the homeomorphic image under R^{-1} of our subspace of $F_{p,\mu}$. This programme has been successfully carried through by my research student, Wilson Lamb.

Similar comments apply to a number of other integral equations of convolution type which can be solved via fractional calculus and for which solutions are given formally in the literature.

Various other subspaces arise naturally when we consider fractional integrals from a to x or x to b ($a > 0$, $b < \infty$) but we will not elaborate further here.

§8.5 Concluding Remarks

We seem almost to have come back to where we started, talking about fractional calculus per se so perhaps it is time to stop before we start a second circuit.

We have tried to give an idea of some of the work which has been and is being done in extending integral transforms from classical functions to generalised functions and have used fractional calculus as a unifying theme. We also concentrated on a few topics which led to the explicit solution of certain problems of interest. We made no attempt to go into the subject in great generality, preferring the concrete to the abstract.

We ignored fractional integrals on $(-\infty,\infty)$ and, even on $(0,\infty)$, we have not dealt with some useful generalisations such as operators introduced by Lowndes [45] to solve a diffraction problem and studied analytically by Heywood and Rooney [27], [28]. Likewise we have not ventured into R^n and grappled with results such as those of Stein and Weiss [77] nor into Orlicz spaces for the work of O'Neil [57] and others. Further, we only mentioned the most basic integral transforms on $(0,\infty)$ and did not discuss their more abstruse generalisations. Nor did we look at transforms on $(-\infty,\infty)$ including such as the Hilbert transform which has been studied in a distributional setting by Orton [58], [59] and others. These topics are of considerable interest but were not part of our general theme and so we will leave them to some other author preaching some other sermon.

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