MIXING CONDITIONS

AND

WEIGHT FUNCTIONS ON THE REAL LINE

by

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Presented for the degree of
Doctor of Philosophy in Mathematics
at the
University of Edinburgh

September, 1977.

CONTENTS

ABSTRACT

The thesis is concerned with two problems from the prediction theory of continuous parameter stationary stochastic processes, and the related questions concerning the measure μ on the real line which is associated with the process via Bochner's theorem.

In Section 1 of Chapter 1, we describe briefly the background required from the theory of Hardy spaces in the upper half-plane, and some facts about entire functions of exponential type are given. Then, in Section 2, we discuss stationary processes and describe the main problems, motivating their study by a brief description of the classical prediction problems of Wiener and Kolmogorov and the work of Helson, Sarason and Szegö.

Chapter 2 is devoted to the proof of two representation theorems for weight functions satisfying the strong mixing condition $\rho_\lambda \to 0$ and the positive angle criterion $\rho_\lambda < 1$. The proof uses a result on analytic continuation and a characterisation of the algebra $H^\infty + BUC$. These results generalise the known results for discrete parameter processes.

Chapter 3 consists of a discussion of the spaces BMO and VMO and their relationship to the strong mixing condition; and the Helson-Szegö condition of Chapter 2. We prove a result characterising those positive functions f on $\mathbb{R}$ for which $\log f \in$ VMO, and derive a connection between BMO, the condition $\rho_\lambda < 1$, and the boundedness of the conjugation operator on a subset of $L^2(\mu)$, depending on λ. This generalises the discrete version which is due to Helson and Szegö.

In Chapter 4, we consider the mixing conditions for a multivariate stationary process. The main result is an example of an hermitian 2×2 matrix G, all of whose entries are real VMO functions, which

is such that exp G does not satisfy the strong mixing condition $\rho_\lambda \to 0$. The proof depends on the construction of a VMO function which goes off to infinity at the origin, and the fact that no VMO function can have a jump discontinuity.

ACKNOWLEDGEMENTS

I should like to express my deepest gratitude to my supervisor, Dr A.M. Davie, for his patience, guidance and constant encouragement. I am also grateful to Professor F.F. Bonsall for his help and interest, and to other members of staff and postgraduates in the Department of Mathematics for some stimulating conversations. I should like to thank the Carnegie Trust for the scholarship which made this research possible; and Mrs Pam Rowe for her careful typing of this thesis.

PREFACE

The material presented in this thesis is claimed as original with the exception of those sections where specific mention is made to the contrary.

CHAPTER 1

Introduction

Our aim in this thesis is to investigate a problem in analysis which has its roots in prediction theory. We shall describe the problem and its probabilistic significance in Section 2 of this chapter, but for now a brief outline seems desirable.

Prediction Theory really comes from time series analysis, and is concerned with the relative independence of sets of random variables $\{X_t: t \in T\}$, where T is to be thought of as a set representing time. Typically, you start with a collection of random variables $\{X_t: t \in \mathbb{R}\}$ which are stationary in the sense that the probabilistic properties of a subset $\Lambda_S = \{X_t: t \in S \subseteq \mathbb{R}\}$ are indistinguishable from those of any other subset of the form $\Lambda_{S+r} = \{X_{t+r}: t \in S\}$ for fixed $r \in \mathbb{R}$, that is, translation in time has no effect on the properties of the random variables X_t, probabilistically speaking. The basic problem is to try to obtain as much information as possible about X_r for $r > 0$, knowing something about the X_t for $t \leq 0$. In other words, we try to assess the dependence of X_r for $r > 0$ on the set $\{X_t: t \leq 0\}$. Of course, as you would expect, there are many criteria for estimating the dependence, and each gives rise to a different prediction problem. In Section 2, we describe four different problems, but our attention will be devoted to only two of these, one being studied first by Rosenblatt [21] in 1956, and the other by Helson and Szegö in 1960 [10]. It will turn out that they can be reformulated as questions about a given finite measure on the line and we shall look at this in Chapter 2.

In Chapter 3 we shall be concerned with the recently studied spaces BMO and VMO (See [17] and [24]) which crop up naturally in the course of the work in Chapter 2. Finally, in Chapter 4, we shall look briefly at multidimensional prediction theory, giving an example with interesting properties.

This programme involves us in the theory of Hardy spaces and entire functions, and Section 1 of this chapter is devoted to the necessary background material. A good reference for Chapter 1 is [4], and for the theory of Hardy spaces we refer the reader to [11] and [3]. For entire function theory, see [19], especially Chapter 5.

Section 1 Hardy Spaces in the Upper Half Plane, and Entire Functions

(A) Hardy Spaces

Let us fix some notation. Π^+ will denote the open upper half-plane, $\overline{\Pi^+}$ the closed upper half plane. For $f: \mathbb{R} \to \mathbb{C}$, let $f^\wedge$, $f^\vee$ denote the Fourier transform, and inverse Fourier transform respectively, i.e.

$$f^\wedge(x) = \int e^{-ixt} f(t)\,dt, \qquad f^\vee(x) = \frac{1}{2\pi} \int e^{ixt} f(t)\,dt.$$

Note that throughout this thesis, $\int$ will denote the integral $\int_{-\infty}^{\infty}$, unless explicitly stated otherwise. For $f: \Pi^+ \to \mathbb{C}$, and $y > 0$ define $f_y: \mathbb{R} \to \mathbb{C}$ by $f_y(x) = f(x+iy)$. Let $e_\lambda: \overline{\Pi^+} \to \mathbb{C}$ denote the function $e_\lambda(z) = e^{i\lambda z}$, $(z \in \overline{\Pi^+})$.

Definition 1.1 (a) A function h, analytic in Π^+, is in the Hardy space H^p, $1 \leq p < \infty$, if $\|h\|_p^+ = \sup_{y>0}\|h_y\|_p < \infty$ where $\|\cdot\|_p$ is the usual L^p norm;

(b) A function h, analytic in Π^+, is in the Hardy space H^∞ if

$$\|h\|_\infty^+ = \sup_{y>0}\|h_y\|_\infty < \infty$$

where $\|\cdot\|_\infty$ is the usual essential supremum norm.

It is easily checked that $(H^p, \|\cdot\|_p^+)$ is a Banach space, for $1 \leq p \leq \infty$. In fact, one can define H^p for $0 < p < 1$ in the same way, but $\|\cdot\|_p^+$ is not a norm if $p < 1$. We shall only once need to refer to $H^{\frac{1}{2}}$ and this lack of norm will cause us no problem. We now summarise briefly the results we need from H^p-theory, and we begin with H^2. We shall briefly glance at H^1 and H^∞.

H^2

Theorem 1.2 A function $h \in H^2$ iff $\exists f \in L^2(\mathbb{R})$ with $\hat{f}(t) = 0$ for $t \leq 0$ such that

$$h(z) = \frac{1}{2\pi}\int_0^\infty e^{itz} f^\wedge(t)\,dt \qquad (*) \quad (z \in \Pi^+)$$

Remark Notice that the integrand in (*) is integrable for $z \in \Pi^+$, since $f^\wedge$ is bounded on $\mathbb{R}$ and $|e^{itz} f^\wedge(t)| \leq e^{-ty}|f^\wedge(t)|$ $(z = x + iy \in \Pi^+)$.

From this theorem it is easy to deduce that $\lim_{y\downarrow 0}\|h_y - f\|_2 = 0$ and $\|h\|_2^{+} = \|f\|_2$. Thus h has a limit on $\underset{\sim}{R}$ in the sense that the map $h \to h_0 = \lim_{y\downarrow 0} h_y$ is an isometry of H^2 onto the class $L^2[0,\infty]^{\vee}$ of inverse Fourier transforms of functions $f^{\wedge} \in L^2(\underset{\sim}{R})$, vanishing on the left half-line. Generally, we shall not distinguish between h and h_0, and often we shall think of H^2 as a subspace of $L^2(\underset{\sim}{R})$. A proof of Theorem 1.2 can be found in Dym and McKean [4, p.30-32]. There is a natural projection of L^2 onto H^2 given by

$$Pf(z) = \frac{1}{2\pi i}\int \frac{f(t)}{t - z}\,dt = \frac{1}{2\pi}\int_0^{\infty} f^{\wedge}(t)e^{itz}dt \qquad (f \in L^2,\quad z \in \Pi^{+})$$

and, on $\underset{\sim}{R}$,

$$Pf(x) = \lim_{y\downarrow 0} Pf(z) = \frac{1}{2\pi}\int_0^{\infty} e^{itx}f^{\wedge}(t)dt$$

the limit being in the L^2-sense. In particular, for $h \in H^2$ and $z \in \Pi^{+}$, we have

$$h(z) = \frac{1}{2\pi i}\int \frac{h_0(t)}{t - z}\,dt \quad \text{and} \quad 0 = \frac{1}{2\pi i}\int \frac{h_0(t)}{t - \bar{z}}\,dt.$$

Combining these gives the Poisson formula:

$$h(z) = \frac{y}{\pi}\int \frac{h_0(t)}{(x-t)^2 + y^2}\,dt = (P_y * h_0)(x),$$

where $P_y(x) = \frac{y}{\pi}\frac{1}{1 + x^2}$ is the so-called Poisson kernel for Π^{+}, and $*$ denotes convolution, as usual. Of course, $h_0(x) = \lim_{y\downarrow 0} h_y(x)$

pointwise a.e. on $\underset{\sim}{R}$.

Lemma 1.3 (a) If $h \in H^2$, then

$$\log|h(z)| \leq P_y * \log|h_0| = \frac{y}{\pi}\int \frac{\log|h_0(t)|}{(x-t)^2 + y^2}\,dt \quad \text{for} \quad z = x + iy \in \Pi^+.$$

(b) If $h \not\equiv 0$, then $\int \frac{\log|h_0(t)|}{1 + t^2}\,dt > -\infty$.

Remark (b) is a consequence of (a): for if $y \geq 1$, then

$$\log|h(iy)| \leq (P_y * \log|h_0|)(0) = \frac{y}{\pi}\int \frac{\log|h_0(t)|}{t^2 + y^2}\,dt \leq \frac{1}{\pi}\int \frac{\log|h_0(t)|}{1 + t^2}\,dt.$$

Thus $\int \frac{\log|h_0(t)|}{1 + t^2}\,dt = -\infty \Rightarrow h(iy) = 0 \ \forall y \geq 1 \Rightarrow h \equiv 0$, by analyticity.
Of course, the integral can only diverge to $-\infty$, since

$$\int \frac{\log^+|h_0(t)|}{1 + t^2}\,dt \leq \tfrac{1}{2}\int \frac{|h_0(t)|^2}{1 + t^2}\,dt < \infty$$

(where, as usual, f^+ denotes the function defined by $f^+(t) = \max(|f(t)|,0)$).

The next concept of outer function, is central to our work.

Definition 1.4 A function $h \in H^2$ is *outer* if $h \not\equiv 0$ and $\log|h(z)| = (P_y * \log|h_0|)(x)$ for $z = x + iy \in \Pi^+$. A function $j \in H^\infty$ is *inner* if $|j(z)| \leq 1$ $(z \in \Pi^+)$ and $|j_0| = 1$ a.e. on $\underset{\sim}{R}$.

A proof of the next theorem can be found in Hoffman [11] or Dym-McKean [4].

Theorem 1.5 The following are equivalent for $h \in H^2$, $h \not\equiv 0$.

(i) h is outer

(ii) $\log|h(i)| = \frac{1}{\pi} \int \frac{\log|h_0(t)|}{1+t^2}\,dt$

(iii) the set $\{e_\lambda h: \lambda \geq 0\}$ is dense in H^2.

Remark It is the characterisation (iii) of outer function which is most useful for our purposes.

The following result is well-known.

Proposition 1.6 Every $g \not\equiv 0$ in H^2 can be written as a product

$$g = jh$$

where h is an outer function in H^2 and j is inner. Moreover, the factorisation is unique up to multiplication of the factors by a constant of unit modulus. In fact, h is given by the formula

$$h(z) = \exp\left[\frac{1}{\pi i} \int \frac{tz+1}{t-z} \frac{\log|g_0(t)|}{1+t^2}\,dt\right].$$

Theorem 1.7 Suppose $f \geq 0$ a.e. on $\mathbb{R}$, $f \in L^1$. Then we may write

$$f = |h_0|^2 \quad \text{for some} \quad h \in H^2$$

iff $\int \frac{\log f(t)}{1+t^2}\,dt > -\infty$.

Proof ($\Rightarrow$) is immediate from 1.3 (b).

($\Leftarrow$) Define $h(z) = \exp\left[\frac{1}{2\pi i}\int \frac{tz+1}{t-z}\frac{\log f(t)}{1+t^2}\,dt\right]$. Then h is clearly analytic in Π^+, and $|h(x+iy)|^2 = \exp\{\mathrm{Re}\,\frac{1}{\pi i}\int \frac{tz+1}{t-z}\frac{\log f(t)}{1+t^2}\,dt\}$.

$$= \exp\{P_y * \log f\}$$

so $\int |h_y(x)|^2 dx = \int \exp(P_y * \log f)dx \leq \int (P_y * f)(x)dx$ and so

$\int |h_y|^2 dx \leq \int f(t)dt < \infty$. Thus $h \in H^2$ and

$\lim\limits_{y\downarrow 0} |h_y(x)|^2 = \lim\limits_{y\downarrow 0} \exp[(P_y * \log f)(x)] = f(x)$ a.e. on $\mathbb{R}$.

Having looked at H^2, we turn our attention to H^1 and H^∞, where most of the above results have obvious analogues.

H^1

The obvious analogues of all the H^2 results hold for H^1, the idea of outer function in H^1 being defined just as the H^2 case. The Cauchy formula

$$h(z) = \frac{1}{2\pi i}\int \frac{h_0(t)}{t-z}\,dt \qquad (z \in \Pi^+)$$

holds for H^1 functions, as does the Poisson integral formula, h_0 being defined as the limit of h_y both in the L^1 sense and pointwise a.e.. Also we have the following result.

Proposition 1.8 Every $g \in H^1$ can be written as a product of two H^2 functions α, β

$$g = \alpha\beta$$

with $\|\alpha\|_2^+ = \|\beta\|_2^+ = (\|g\|_1^+)^{\frac{1}{2}}$.

Proof $\exists$ j inner, h outer in H^1 such that $g = jh$. Let $\alpha = jh^{\frac{1}{2}}$, $\beta = h^{\frac{1}{2}}$. Then $\alpha, \beta \in H^2$ and $\|\alpha\|_2^+ = \|\beta\|_2^+ = (\|g\|_1^+)^{\frac{1}{2}}$.

H^∞

Again we have obvious analogues of the Poisson formula and inner-outer factorisation theorem. In addition we have the following result analogous to 1.2.

Lemma 1.9 H^∞ is the annihilator of H^1 in L^∞; more precisely $H^\infty \cong \{f \in L^\infty: \int fh = 0 \ \forall h \in H^1\}$, ~~and~~ $(H^1)^* \cong \frac{L^\infty}{H^\infty}$, where $*$ denotes the dual space, as usual, and

$$\|f+H^\infty\|_\infty = \inf\{\|f+h\|_\infty: \ h \in H^\infty\} = \sup\{|\int fh|: \ h \in H^1, \ \|h\|_1^+ \le 1\}.$$

This completes our survey of H^p space theory, except for the closely associated definition of conjugate function. We shall define both the conjugate function for real-valued f satisfying $\int \frac{|f(x)|}{1+x^2}\,dx < \infty$, and the Hilbert Transform for real $f \in L^1$. In fact for real $f \in L^1$, their difference is a constant function.

If f is real-valued and in L^1, define h analytic in Π^+ by

$$h(z) = \frac{1}{\pi i}\int \frac{f(t)}{t - z}\,dt \qquad (z \in \Pi^+)$$

Then $h(z) = (P_y*f)(x) + i(Q_y*f)(x)$, where $P_y * f$ is the Poisson integral of f, and $Q_y * f = \frac{1}{\pi}\int \frac{x-t}{(x-t)^2+y^2} f(t)\,dt$.

We have

<u>Theorem 1.10</u> $h_0(x) = \lim_{y\to 0} h_y(x)$ exists almost everywhere on $\underset{\sim}{R}$.

<u>Proof</u> See Dym and McKean [4, p.49-50].

<u>Definition 1.11</u> The function $(Hf)(x) = \lim_{y\downarrow 0} (Q_y*f)(x)$ is called the <u>Hilbert Transform</u> of f.

<u>Notation</u> Let L^1_c denote the set of functions $f: \underset{\sim}{R} \to \underset{\sim}{C}$ such that $\int \frac{|f(t)|}{1+t^2}dt < \infty$.

<u>Proposition 1.12</u> Suppose $f \in L^1_c$ is real-valued. Then the function

$$h(z) = \frac{1}{\pi i}\int \frac{tz+1}{t-z} f(t) \frac{dt}{1+t^2} \qquad (*)$$

is analytic in Π^+, and the limit $h_0(x) = \lim_{y\downarrow 0} h(z) = f(x) + i\,\tilde{f}(x)$ exists, almost everywhere on $\underset{\sim}{R}$.

$\tilde{f}$ is called the <u>conjugate function</u> of f. A proof of 1.12 is also in Dym and McKean [p.50-51].

<u>Remarks</u> (i) If $f \in L^1$ is real, then $\tilde{f}$ and Hf differ by the constant function $\frac{1}{\pi}\int \frac{tf(t)}{1+t^2}dt$.

(ii) The concept of the conjugate function for a half-plane is the exact analogue of the more familiar conjugate function on the circle, and one can investigate all the same classical problems concerning L^p-boundedness of the conjugation operator $f \to \tilde{f}$. We touch briefly on this in Chapter 3, but for now we only need the definition of $\tilde{f}$ and the following result.

<u>Lemma 1.13</u> Suppose $s \in L^\infty$ is real valued. Then $\tilde{s} \in L^1_c$.

<u>Proof</u> This follows immediately from the circle result which says that if $g \in L^\infty(\Gamma)$ is real valued, then $\tilde{g} \in L^1(\Gamma)$. The link is provided by the linear fractional map $z \to \xi = \frac{z-i}{z+i}$ which maps the upper half plane Π^+ onto the open unit disc $\mathbb{D}$. It is easy to check that this map 'lifts' the conjugate function on Γ to the conjugate function on $\underset{\sim}{R}$, in the sense that if $s \in L^\infty$ and we define $g(e^{i\theta}) = s(x)$ where $e^{i\theta} = \frac{x-i}{x+i}$, then $g \in L^\infty(\Gamma)$, so $\tilde{g} \in L^1(\Gamma)$ and this implies $\tilde{s} \in L^1_c$, since under this map $d\theta$ corresponds to $\frac{1}{\pi}\frac{dx}{1+x^2}$. The result that $g \in L^\infty(\Gamma) \Rightarrow \tilde{g} \in L^1(\Gamma)$ is obvious from the M. Riesz Theorem [3, p.54].

<u>Remark</u> The linear fractional map $z \to \xi = \frac{z-i}{z+i}$ will be of use to us again. In fact, we use it in the next result to derive a representation for positive harmonic functions on Π^+.

<u>Theorem 1.14</u> Suppose g is a positive harmonic function in Π^+. Then $\exists$ a representation

$$g(z) = ky + \frac{y}{\pi}\int \frac{d\,F(x)}{|x-z|^2} \qquad (1)$$

with $k \geq 0$ constant, and a non-decreasing function F satisfying $\int \frac{d\,F(x)}{1+x^2} < \infty$.

Proof Map Π^+ onto $D = \{\xi = re^{i\theta}: \ r < 1\}$ by the linear fractional map $z \to \xi = \frac{z-i}{z+i}$. Then $g(z) = u(\xi)$ is positive and harmonic in D, and by the Poisson formula,

$$u(R\xi) = \frac{1}{2\pi} \int_{0+}^{2\pi-} \frac{1-|\xi|^2}{|e^{i\theta}-\xi|^2}\, u(Re^{i\theta})\,d\theta \qquad (R < 1)$$

The mass distribution $dF_R(\theta) = u(Re^{i\theta})d\theta$ is non-negative and of total mass

$$\int_0^{2\pi} u(Re^{i\theta})d\theta = 2\pi u(0) = 2\pi g(i) < \infty,$$

so you can make $R \uparrow 1$ through a sequence of values, so that

$$u(\xi) = \lim_{R\uparrow 1} u(R\xi) = \frac{1}{2\pi} \int \frac{1-|\xi|^2}{|e^{i\theta}-\xi|^2}\, dF_1(\theta)$$

with a non-negative mass distribution dF_1 of the same total mass. Now, isolate the jump of F_1 at 0 if any, so that

$$u(\xi) = \frac{1}{2\pi}\frac{1-|\xi|^2}{|1-\xi|^2}\,[F_1(0+)-F_1(0-)] + \frac{1}{2\pi}\int_{0+}^{2\pi-} \frac{1-|\xi|^2}{|e^{i\theta}-\xi|^2}\, dF_1(\theta).$$

Now map back from the circle to Π^+. The inverse image x of $e^{i\theta}$ runs through $-\infty$ to ∞ as θ runs from $0+$ to $2\pi-$, and $\frac{1}{2}dF_1(\theta)$ is carried onto a non-negative mass distribution $(x^2+1)^{-1}dF(x)$ of finite total mass. Moreover, the Poisson kernel is transformed by

$$\frac{1}{2\pi}\frac{1-|\xi|^2}{|e^{i\theta}-\xi|^2} = \frac{1}{2\pi}\frac{1-\left|\frac{z-i}{z+i}\right|^2}{\left|\frac{x-i}{x+i}-\frac{z-i}{z+i}\right|^2} = \frac{y}{2\pi}\frac{x^2+1}{|z-x|^2}$$

so that

$$g(z) = \frac{y}{2\pi}\,[F_1(0+)-F_1(0-)] + \frac{y}{\pi}\int\frac{d\,F(x)}{|x-z|^2},$$

as required.

Lemma 1.15 Suppose g is a positive harmonic function in Π^+. Then $\exists c > 0$ such that

$$g(iy) \leq cy \qquad (\forall y \geq 1).$$

Proof By the representation theorem above, we have if $y \geq 1$

$$g(iy) = ky + \frac{y}{\pi}\int\frac{d\,F(t)}{|t-iy|^2} = ky + \frac{y}{\pi}\int\frac{d\dot{F}(t)}{t^2+y^2}\,dt \leq ky + \frac{y}{\pi}\int\frac{d\,F(t)}{1+t^2} \leq cy.$$

Remark Theorem 1.14 is well-known and the proof given above appears in Dym and McKean [4, p13]. Our interest is twofold; firstly, the result of Lemma 1.15 is used in Chapter 2 to prove a result on conjugate functions which we need. Secondly, the linear fractional transformation $z \to \xi = \frac{z-i}{z+i}$ given above maps Π^+ onto D in such a way that the real line maps onto $\Gamma \setminus \{1\}$. It is very useful in transferring results on Hardy spaces from the disc to the upper half-plane and vice-versa. Also, as we shall see in the next section, it allows us to compare the properties of discrete and continuous parameter processes. Chapter 8 of Hoffman [11] gives an idea of its usefulness. We require the notion of subharmonic function in Chapter 2.

Definition 1.16 A upper semi-continuous function g is subharmonic in a domain $U \subseteq \mathbb{C}$ if (i) $\forall z_0 \in U \; \exists \delta_0 > 0$ such that $D(z_0, \delta_0) \subset U$ and

$$g(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} g(z_0 + \delta e^{i\theta})\, d\theta, \qquad \forall \delta < \delta_0 .$$

where $D(z_0, \delta_0)$ denotes the disc of centre z_0, of radius δ_0.
and (ii) Each integral above is greater than $-\infty$.

Lemma 1.17 If f is analytic in a domain $U \subseteq \mathbb{C}$, and $f \not\equiv 0$, then $\log|f|$ is subharmonic in U, as are $\log^+|f|$ and $|f|^p$ $(0 < p < \infty)$.

Proof That $\log|f|$ is subharmonic follows from the well-known inequality

$$\log|f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta$$

To get that $\log^+|f|$ and $|f|^p$ $(0 < p < \infty)$ are subharmonic, just apply the result that g real and subharmonic in U and ϕ increasing and convex on $\mathbb{R} \Rightarrow \phi \circ g$ is subharmonic, with ϕ defined respectively by $\phi(t) = \max(0,t)$ and $\phi(t) = e^{pt}$ and g given by $g = \log|f|$.

Remark We are only really interested in the result that $|f|^{\frac{1}{2}}$ is subharmonic if f is analytic.

(B) Entire Functions

Definition 1.18 An entire function F is of (finite) exponential type λ if

$$\lim_{R\uparrow\infty} \frac{1}{R} \sup_{0\le\theta<2\pi} \log|F(Re^{i\theta})| = \lambda < \infty$$

i.e. $|F(Re^{i\theta})| <$ constant $\times\, e^{\mu R}$ $\forall\mu > \lambda$, but for no $\mu < \lambda$.

Functions of exponential type are amongst the most tractable of entire functions and the theory is extensive. We shall encounter functions of exponential type which are also of class A, which we now define. Functions of class A are studied by Levin in [19, Chapter 5].

Definition 1.19 An entire function F is of class A if the zeros $\{z_k\}$ satisfy the condition

$$\sum_{k=1}^{\infty} \left| \operatorname{Im} \frac{1}{z_k} \right| < \infty \qquad \text{(A)}$$

Remark (i) An entire function of exponential type with zeros $\{z_k\}$ satisfies Lindelöf's criterion, namely that $\left| \sum_{k=1}^{\infty} \frac{1}{z_k} \right| < \infty$. Thus, in particular, any entire function of exponential type whose zeros all lie in the upper half plane must automatically be of class A.

(ii) The condition (A) is satisfied if the entire function F is sufficiently well-behaved on the real line; indeed we quote the following result characterising some functions of class A, which can be found in Levin [19, Chapter 5, Theorem 11].

Theorem 1.20 If an entire function F of exponential type satisfies one of the following conditions (a) $\int \frac{\log|F(x)|}{1+x^2}\,dx$ exists

(b) $|F|$ is bounded on $\underset{\sim}{R}$

(c) $F|_{\underset{\sim}{R}} \in L^p(\underset{\sim}{R})$ for some p

then F is of class A and we have the representation

$$F(z) = cz^m e^{i\alpha z} \lim_{R\uparrow\infty} \prod_{|z_k|<R} \left(1 - \frac{z}{z_k}\right) \qquad (1)$$

where $\{z_k\}$ are the zeros of F, and c, α are constants.

It is not our intention to become embroiled in the complexities of entire function theory, so we do not prove Theorem 1.20. However, entire functions of exponential type and of class A crop up naturally in the course of our investigations of the strong mixing condition on $\underset{\sim}{R}$; they take the place of the finite trigonometric polynomials on Γ, the unit circle. We shall need the following result of Achieser; for a proof see Levin [19, p.438].

Theorem 1.21 An entire function F of exponential type λ has a representation

$$F(x) = |G(x)|^2 \qquad (x \in \underset{\sim}{R})$$

for some entire function G of exponential type $\frac{\lambda}{2}$ with zeros only in $\overline{\Pi^+}$, iff F is of class A and $F(x) \geq 0$ $(x \in R)$.

Remark Theorem 1.21 is a direct generalisation of the Fejér–Riesz theorem which states that a non-negative trigonometric polynomial

$T(x) = \sum_{-n}^{n} c_k e^{ikx}$ such that $T(x) \geq 0$ $(\forall x \in \mathbb{R})$ can be written in the form $T(x) = |S(x)|^2$ $(x \in \mathbb{R})$ where $S(x) = \sum_{0}^{n} b_k e^{ikx}$, and S can be chosen so that all its zeros are in $\overline{\Pi^+}$. This was generalised by Krein to entire functions of exponential type which are bounded on $\mathbb{R}$, and then by Achieser to a general function of class A.

Our last result in this section is the well-known Paley-Wiener Theorem.

Theorem 1.22 Suppose A and C are positive constants and f is an entire function such that $|f(z)| \leq Ce^{A|z|}$, for all $z \in \mathbb{C}$, and

$$\int |f(x)|^2 dx < \infty.$$

Then $\exists F \in L^2(-A,A)$ such that $f(z) = \int_{-A}^{A} F(t)e^{itz}dt$.

Proof Dym and McKean [4, p.28].

This completes Section 1 and the preliminaries. In the next section we shall introduce the idea of stationary stochastic process, and prediction theory. We shall give the strong-mixing condition for such a process and set up the machinery for its study in Chapter 2.

Section 2 Stationary Processes and Prediction Problems

(A) Stationary Processes

In this section we give a brief general introduction to the theory of stationary random processes before looking at prediction theory. We shall then be able to formulate the strong mixing condition and the positive angle criterion of Helson and Szegö which is the main topic of this thesis. Then in Chapter 2 we shall reformulate these problems in prediction theory as problems about weight functions on the real line and the techniques we use thenceforth will be those of analysis. First we need to set the scene. Let (Ω,Σ,P) be a probability space.

Definition 1.23 A random variable X on Ω is a complex-valued Σ-measurable function on Ω. X is said to be square-summable if $E(|X|^2) = \int |X(\omega)|^2 dP(\omega) < \infty$.

Notation Let $L^2(\Omega,\Sigma,P)$ denote the set of all square-summable random variables on Ω. If we identify functions which are equal a.e. (dP), then $L^2(P)$ becomes a Hilbert space with inner product $\langle X_1, X_2 \rangle_P = E(X_1 \bar{X}_2)$.

Definition 1.24 (a) A stochastic process (s.p.) is a collection $\{X_t\colon\ t \in T\} \subseteq L^2(P)$, where T is either $\underset{\sim}{R}$ or $\underset{\sim}{Z}$. If $T = \underset{\sim}{Z}$, then the process is called discrete; if $T = \underset{\sim}{R}$ it is a continuous s.p..

(b) A discrete s.p. $\{X_n: n \in \underset{\sim}{Z}\}$ is (weakly) stationary if

$$E(X_{m+n}\bar{X}_m) = R(n)$$

is independent of $m \in \underset{\sim}{Z}$.

A continuous process $\{X_t: t \in \underset{\sim}{R}\}$ is stationary if

$$E(X_{s+t}\bar{X}_s) = R(t)$$

is independent of $s \in \underset{\sim}{R}$, and $R(t)$ is a continuous function of t.

Remarks (i) Square-summability is included in the definition of s.p. purely for convenience sake.

(ii) There is a concept of strong stationarity which says that not only is $R(t)$ independent of t, but all the joint distributions of a finite number of the X_t are unchanged by translations in T.

(iii) Notice that for a continuous parameter process we require $R(t)$ to be continuous. This restriction is essential if any meaning is to be given to prediction problems for a continuous stationary s.p.. Indeed, it is automatically satisfied by any s.p. which is weakly continuous in the sense that $\lim_{t-s\to 0} E\{|X_s - X_t|^2\} = 0$.

Lemma 1.25 If $\{X_t: t \in \underset{\sim}{R}\}$ is a stationary s.p., then $R(t)$ is positive definite.

Proof $R(-t) = E(X_{-t}\bar{X}_0) = \overline{E(X_0\bar{X}_{-t})} = \overline{E(X_t\bar{X}_0)}$, by stationarity

$= \overline{R(t)}$.

Also if $t_1, \ldots, t_n \in \underset{\sim}{R}$ and $c_1, \ldots, c_n \in \underset{\sim}{C}$, then

$$\sum R(t_j - t_k)c_j\bar{c}_k = \sum E(X_{t_j}\bar{X}_{t_k})c_j\bar{c}_k = E(\sum X_{t_j}c_j\bar{X}_{t_k}\bar{c}_k)$$

$$= E(|\sum X_{t_j}c_j|^2) \geq 0$$

Thus $R(t)$ is positive definite on $\underset{\sim}{R}$.

The following well-known theorem can be found in almost any text on stochastic processes, or indeed harmonic analysis; for example Doob [2, p.519].

Theorem 1.26 (Bochner) A function $r(t)$ is positive definite iff $\exists$ a finite positive Borel measure μ on $\underset{\sim}{R}$ of total mass $r(0)$ such that

$$r(t) = \int_{-\infty}^{\infty} e^{ixt}\mu(dx).$$

Remark For a discrete process, $R(n)$ is a positive definite sequence and a forerunnerof the above theorem, due to Herglotz, tells us that $\exists$ a finite Borel measure μ on Γ, the unit circle, such that $R(n) = \int_\Gamma e^{in\theta}\mu(d\theta)$. All this is described in Doob [2].

The above theorem tells us that to any stationary process we may associate a +ve measure μ (on either $\underset{\sim}{R}$ or Γ) which we call the spectral measure of the process. Of course, the converse is true: given a measure μ generating a positive definite function $R(t)$

we can find a stationary s.p. $\{X_t\}$ with $R(t) = E(X_t\bar{X}_0)$. For, we may suppose, without loss of generality, that μ is a probability measure. Then we need only define X_t on $(\underset{\sim}{R},B,\mu)$ by $X_t(x) = e^{itx}$. Then $\{X_t: t \in \underset{\sim}{R}\}$ is a stationary s.p. and $E(X_{t+s}\bar{X}_s) = R(t)$. (Here B denotes the Borel σ-algebra on $\underset{\sim}{R}$). In fact it can be shown that given a positive definite function, we can find a Gaussian stationary s.p. $\{X_t: t \in \underset{\sim}{R}\}$ such that $E(X_{t+s}\bar{X}_s) = R(t)$. For a proof, see for example, Doob [2, p.72].

Let us now turn to the prediction theory of a stationary s.p. $\{X_t: t \in \underset{\sim}{R}\}$.

(B) Prediction Theory

Notation (i) Let M be the closed subspace of $L^2(P)$ generated by $\{X_t: t \in \underset{\sim}{R}\}$.

(ii) Let $L^2(\mu)$ denote the set of all functions $u: \underset{\sim}{R} \to \underset{\sim}{C}$ such that $\|u\|_\mu^2 = \int |u(x)|^2\mu(dx) < \infty$, again identifying functions equal a.e. $(d\mu)$.

Lemma 1.27 M is isometrically isomorphic to $L^2(\mu)$ under the correspondence

$$X_t \leftrightarrow e^{itx}.$$

Proof
$$\langle e^{itx},e^{isx}\rangle_\mu = \int_{\underset{\sim}{R}} e^{ix(t-s)}d\mu(x) = R(t-s) = E(X_t\bar{X}_s)$$
$$= \langle X_t,X_s\rangle_P$$

The lemma is completed by observing that $\{e^{itx}: t \in \underset{\sim}{R}\}$ is dense in $L^2(\mu)$.

A similar result is true for discrete processes.

Prediction problems are concerned with the following situation: Let P^t = closed span $\{X_s: s \leq t\}$, F^t = closed span $\{X_s: s \geq t\}$ and let $P^{-\infty} = \bigcap_{t \in \underset{\sim}{R}} P^t$. P^{-1} is called the 'past' of the process, $P^{-\infty}$ its 'remote past'. The basic problem is to obtain information about the process X_t for $t \geq t_0$ when we are given information only about X_t for $t < t_0$. One such problem might be: given X_t for $t \leq 0$ compute the distance in $L^2(P)$ of X_1 to P^0, or more generally from X_{t_0} to P^0 where $t_0 > 0$. This is historically one of the earliest prediction problems and we shall have more to say about it in a moment.

Firstly, however, notice that the isometry of Lemma 1.27 tells us that $\exists$ subspaces of $L^2(\mu)$ corresponding to P^t and F^t etc., and that any statement about P^t or F^t involving only the structure of $L^2(P)$ as a Hilbert space is mirrored exactly in a similar statement about the corresponding subspaces of $L^2(\mu)$, which we denote by P_t, F_t, $P_{-\infty}$ etc.. It is natural to ask why we restrict attention to the relationship between P_t and F_t in the $L^2(P)$-sense, and the short answer is that for Gaussian processes, the most important class of processes, approximation in the norm of $L^2(P)$ is especially significant. The reason is that the best least-squares approximation is achieved by the best linear least squares approximation for a set of Gaussian random variables. This, and the fact that, in a practical sense, linear approximation is easiest to deal with, explains why work in prediction theory is often restricted to Gaussian processes.

Though we shall not make this assumption, the reader may assume that all processes are Gaussian.

Now the same is true for the discrete case that we can define P^n, F^n, $P^{-\infty}$ and the corresponding spaces P_n, F_n etc. in $L^2(\mu)$. Not surprisingly the space F^1 (correspondingly F_1) is called the 'future' of the process. The first problem in prediction theory was investigated by Kolmogorov and Wiener, and its solution is due to Szegö, Kolmogorov, and Krein who each proved part of it, and to Wiener independently. In the discrete case it is simply stated as

"What is the distance from the 'past' to the 'future'?"

More precisely, compute $\sigma = \inf_{f \in F_1} \|1-f\|_{L^2(\mu)}$. From now on we look only at the concrete space $L^2(\mu)$. The solution of this problem is contained in the celebrated Szegö's Theorem.

Theorem 1.28 Let $\{X_n : n \in \underset{\sim}{Z}\}$ be a discrete process with spectral measure μ on Γ. Let μ have Lebesgue decomposition $d\mu = w d\theta + d\mu_s$, where $w \geq 0$ is in $L^1(\Gamma)$, and $d\mu_s$ is the singular part of μ, $d\theta$ being normalised Lebesgue measure on Γ. Then

$$\sigma^2 = \exp\left[\int_\Gamma \log w(\theta) d\theta \right].$$

For a proof see Hoffman [11, p.48-50].

Remarks (i) Notice that the singular part of μ has no effect on σ.

(ii) If $\log w \notin L^1(\Gamma)$, the result is interpreted as saying that the distance σ is zero, i.e. $1 \in F_1$ and all the P_n are equal to P_0 $(n \in \underset{\sim}{Z})$.

(iii) Szegö (1920) proved this for an absolutely continuous measure μ, and the general case is due to Kolmogorov (1941). The proof in Hoffman is due to Helson and Lowdenslager [8].

Definition If $\sigma = 0$, we say that the process is deterministic. If $P^{-\infty} = \{0\}$, it is called purely non-deterministic.

The definition is motivated by the following well-known result.

Theorem 1.29 For a discrete stationary s.p. $\{X_n : n \in \underset{\sim}{Z}\}$ with spectral measure μ the following alternative holds:-

Either (a) $\log w \in L^1(\Gamma)$ and $L^2(\mu) \neq P_0 \neq P_{-\infty}$

or (b) $\log w \notin L^1(\Gamma)$ and $L^2(\mu) = P_0 = P_{-\infty}$.

Proof See Doob [2, p.579].

We omit the proof of Theorem 1.29, because it is well-known and because we shall prove the corresponding result for a continuous s.p.. Notice that part of the theorem is that if σ_k is the distance between P_0 and F_k, then either $\sigma_k = 0$ $\forall k \in \underset{\sim}{Z}$, $k \geq 1$ or $\sigma_k > 0$ $\forall k \geq 1$. If $\sigma = \sigma_1 = 0$, then the past determines the future, i.e. perfect prediction is possible. For the analogue of Theorem 1.29 for the continuous case we need a preliminary lemma.

Lemma 1.30 Either $e^{ixt} \in P_s$ for every $t \geq s$ or for no $t > s$. In the former situation $P_s = L^2(\mu)$ $\forall s \in \underset{\sim}{R}$.

Proof By definition, $P_t = e^{ix(t-s)}P_s$ $(\forall s, t \in \underset{\sim}{R})$. Suppose $e^{ixt_0} \in P_s$ for some $t_0 > s$. Then, for $t \in [s, t_0]$, $e^{ixt} = e^{ix(t-t_0)}e^{ixt_0} \in e^{ix(t-t_0)}P_s = P_{s+t-t_0} \subseteq P_s$. Thus $P_{t_0} = P_s$. Suppose $t > t_0 > s$. Then

$$P_t = e^{ix(t-t_0)}P_{t_0} = e^{ix(t-t_0)}P_s = P_{t+(s-t_0)}.$$

This holds for every $t > t_0$. If $t + (s-t_0) \leq t_0$, then $P_t = P_{t+(s-t_0)} \subseteq P_{t_0} = P_s$. If $t + (s-t_0) > t_0$, then repeat the above argument with t replaced by $t + (s-t_0)$ to get $P_t = P_{t+(s-t_0)} = P_{t+2(s-t_0)}$. Continuing in this way, since there exists n s.t. $t + n(s-t_0) \leq t_0$ we get eventually that $P_t = P_s$ $\forall t \geq s$. The second part is obvious.

Theorem 1.31 Suppose $\{X_t: t \in \underset{\sim}{R}\}$ is a stationary s.p. with spectral measure μ defined as in Theorem 1.26, and suppose $d\mu = w dx + d\mu_s$ is the Lebesgue decomposition. Then the following alternative holds:-

Either (a) $\int \frac{\log w(x)}{1 + x^2} dx > -\infty$ and $P_{-\infty} \neq P_0 \neq L^2(\mu)$

or (b) $\int \frac{\log w(x)}{1 + x^2} dx = -\infty$ and $P_{-\infty} = P_0 = L^2(\mu)$.

Proof (Dym and McKean [4, p.84]) Let P_T denote the projection onto P_T $(T \in \underset{\sim}{R})$. The proof is split into two parts.

(1) $P_0 \neq L^2(\mu) \Rightarrow \int \frac{\log w(x)}{1 + x^2} dx > -\infty$, and $P_{-\infty} \neq P_0$.

<u>Proof of (1)</u> If $P_0 \neq L^2(\mu)$, then $\exists s < t$ so that $\alpha_{st} = (1-P_s)e^{ixt} \neq 0$. Now α_{st} is perpendicular to e^{ixr} for $r \leq s$, by definition, so $\int e^{-ixs}\alpha_{st}(x)e^{-ixr}d\mu(x) = 0$ for $r \leq 0$. By the F. and M. Riesz theorem (just the same as the circle version) $e^{-ixs}\alpha_{st}$ vanishes on the singular set of μ, so that

$$e^{-ixs}\alpha_{st}(x)d\mu(x) = e^{-ixs}\alpha_{st}(x)w(x)dx$$

and $e^{-ixs}\alpha_{st}(x)w(x) \in H^1$.

In particular, $\int \frac{\log|\alpha_{st}w(x)|}{1+x^2}dx > -\infty$, and so we have

$$\int \frac{\log w(x)}{1+x^2}dx = \int \frac{\log[\,|\alpha_{st}(x)|w(x)]^2}{1+x^2}dx - \int \frac{\log[\,|\alpha_{st}(x)|^2 w(x)]}{1+x^2}dx$$

$$\geq 2\int \frac{\log[\,|\alpha_{st}(x)|w(x)]}{1+x^2}dx - \|\alpha_{st}\|^2 > -\infty$$

(2) $P_0 = L^2(\mu) \Rightarrow P_0 = P_{-\infty}$ and $\int \frac{\log w(x)}{1+x^2}dx = -\infty$.

<u>Proof of (2)</u> If $P_0 = L^2(\mu)$, then $L^2(\mu) = \bigcap_{T<0} e^{ixT}P_0 = \bigcap_{T<0} P_T = P_{-\infty}$. It remains to show that $\int \frac{\log w(x)}{1+x^2}dx = -\infty$. Suppose not. Then $\exists h \in H^2$ with $w = |h|^2$ a.e. on $\underset{\sim}{R}$, and if we define $\frac{h}{\bar{h}} = 1$ on the singular set of μ, it follows from $P_{-\infty} = L^2(\mu)$ that if $f = c_1e^{ixt_1} + \ldots + c_ne^{ixt_n}$ with $t_1, \ldots, t_n \leq 0$, then

$$\inf_{\text{all such } f} \int |f(x)\bar{h}(x)-h(x)|^2dx \leq \inf_{\text{all such } f} \int |f(x)-\frac{h}{\bar{h}}(x)|^2d\mu(x) = 0$$

since $\frac{h}{\bar{h}} \in L^2(\mu) = P_0$.

Thus $h \in H^2 \cap \bar{H}^2 = \{0\}$ contradicting $\int \frac{\log w(x)}{1+x^2}dx > -\infty$.

This completes the proof.

Remarks (i) The F. and M. Riesz theorem quoted in the above theorem is given in Dym and McKean [4, p.45]. Its statement is as follows:-

A function F of total variation $\int |dF(x)| < \infty$ with $\int e^{-ixt} dF(x) = 0 \quad \forall t \leq 0$ must be of the form $F(x) = \text{constant} + \int_{-\infty}^{x} h_0(y)dy$ for some $h \in H^1$.

(ii) The proof given above is modelled on that for the discrete case which is given in Doob [2]. Notice that $d\mu_s$, the singular part, plays no part in the prediction, and its significance is shown in the following result.

Lemma 1.32 If $\int \frac{\log w(x)}{1 + x^2} dx > -\infty$, then $P_{-\infty} = L^2(d\mu_s)$, $d\mu_s$ being the singular part of μ which has Lebesgue decomposition $d\mu = wdx + d\mu_s$.

Proof (Dym and McKean [4]) Suppose $\int \frac{\log w(x)}{1 + x^2} dx > -\infty$. Then $P_{-\infty} \neq P_0 \neq L^2(\mu)$. Suppose $f \in P_{-\infty}$. Then $e^{ixr} f \in P_{-\infty}$, $\forall r \in \mathbb{R}$. Now, if $s < t$, the function $\alpha_{st} = (1-P_s)e^{ixt}$ is perpendicular to $P_s \supseteq P_{-\infty}$, and since $\alpha_{st} d\mu(x) = \alpha_{st} w(x)dx$ (as in the proof of Theorem 1.31, part (1)), we have that

$$\int e^{ixr} f(x)\overline{\alpha_{st}(x)} d\mu(x) = \int e^{ixr} f(x)\overline{\alpha_{st}(x)} w(x)dx = 0 \quad (\forall r \in \mathbb{R}).$$

Thus $f\overline{\alpha_{st}}w$ vanishes a.e. (dx). Therefore f vanishes a.e. (dx), since $\int \frac{\log|\alpha_{st}(x)|w(x)}{1 + x^2} dx > -\infty$ stops $\overline{\alpha_{st}}w$ from vanishing a.e. (dx). This proves $P_{-\infty} \subset L^2(d\mu_s)$.

Conversely, suppose $f \in L^2(d\mu_s)$. Then, as $f = 0$ a.e. (dx) and $\alpha_{st} d\mu(x) = \alpha_{st}(x)w(x)dx$, we must have

$$\int [(1-P_s)f(x)]e^{-ixt}d\mu(x) = \int f(x)\overline{[(1-P_s)e^{ixt}]}\,d\mu(x)$$
$$= \int f(x)\overline{\alpha_{st}}(x)w(x)\,dx = 0, \qquad \forall t \geq s.$$

Thus $(1-P_s)f$ is perpendicular to both P_s and F_s and so must vanish. Thus $f = P_s f \in P_s$ $\forall s \in \underset{\sim}{R}$, i.e. $f \in P_{-\infty}$.

Corollary 1.33 $\{X_t: t \in \underset{\sim}{R}\}$ is purely non-deterministic iff μ is absolutely continuous and $\int \frac{\log w(x)}{1+x^2}\,dx > -\infty$.

Proof $P_{-\infty} = \{0\} \Rightarrow P_{-\infty} \neq L^2(\mu) \Rightarrow \int \frac{\log w(x)}{1+x^2}\,dx > -\infty$ and $P_{-\infty} = L^2(d\mu_s)$ by Lemma 1.32 so $d\mu_s = 0$. Conversely,
$\int \frac{\log w(x)}{1+x^2}\,dx > -\infty \Rightarrow P_{-\infty} = L^2(d\mu_s) = \{0\}$, so X_t is purely non-deterministic.

The proofs of Theorem 1.31 and Lemma 1.32 can be found in Dym and McKean as we have indicated above. However, these results can be derived more directly from the corresponding results for discrete processes on Γ, the unit circle, by means of the linear functional map of Theorem 1.14, namely $z \to \xi = \frac{z-i}{z+i}$ $(z \in \Pi^+, \xi \in D)$. As we remarked, then, this transformation is very useful for deriving results about the H^p spaces in Π^+ and indeed about conjugate functions from the corresponding results on D, as in Hoffman [11, Chapter 8]. Let us look at it briefly.

If we start with a continuous stationary s.p. $\{X_t: t \in \underset{\sim}{R}\}$ with spectral measure μ, then there exists an associated discrete process with spectral measure ν on Γ and, more importantly, the subspaces $P_0(\mu)$ and $P_0(\nu)$ correspond, as do $P_{-\infty}(\mu)$ and $P_{-\infty}(\nu)$.

To see this consider the linear fractional map, Λ, from Π^+ to D, the unit disc, defined, as above, by

$$z \to \Lambda(z) = \xi = \frac{z - i}{z + i} \qquad (z \in \Pi^+)$$

with inverse

$$\xi \to \Lambda^{-1}(\xi) = z = \frac{\xi + 1}{i(\xi - 1)} \qquad (\xi \in D)$$

Define ν on Γ by $\nu(\theta) = \mu(\tan \frac{\theta}{2})$. Then $\frac{d\theta}{dx} = \frac{2}{1 + x^2}$, so ν is the spectral measure of some discrete process, and the integrals $\int_{-\infty}^{\infty} \frac{\log w_\mu(x)}{1 + x^2}\, dx$ and $\int_\Gamma \log w_\nu(\theta) d\theta$ are finite or infinite together, where of course $d\mu = w_\mu dx + d\mu_s$ and $d\nu = w_\nu d\theta + d\nu_s$ are the respective Lebesgue decompositions of μ and ν. To see that $P_0(\mu)$ and $P_0(\nu)$ correspond under this transformation, we have

$$e^{-i\theta} = \frac{x + i}{x - i} = 1 - 2 \int_{-\infty}^{0} e^{-itx} e^{t} dt$$

so that $e^{-i\theta}$, and hence $e^{-in\theta}$ $\forall n \geq 1$, corresponds to an element of $P_0(\mu)$, since the integral $\int_{-\infty}^{0} e^{-itx} e^{t} dt$ may be approximated in $L^2(\mu)$ by a linear combination of the functions $e^{-itx} \in P_0(\mu)$ $(t \geq 0)$, and so belongs to $P_0(\mu)$. Thus the image of $P_0(\nu)$ is in $P_0(\mu)$.

Conversely, the function $e^{izt} = \exp(t \frac{\xi+1}{\xi-1})$ is analytic in $|\xi| > 1$ and has modulus < 1 if $t < 0$, since

$$\left|\exp(t \frac{\xi+1}{\xi-1})\right| = \exp(t\mathrm{Re} \frac{\xi+1}{\xi-1}) = \exp(t \frac{|\xi|^2-1}{|\xi-1|^2})$$

and this is less than 1 if $t < 0$ and $|\xi| > 1$. Thus e^{izt} may be expanded in a series of non-positive powers of ξ. On Γ, it is $\exp(ixt)$, $t < 0$, the bounded pointwise limit of $\exp(ixt)$ for $t < 0$. Thus e^{ixt}, $t < 0$ can be approximated boundedly and in $L^2(\mu)$ by the image of an element of $P_0(\nu)$ and so the image of $P_0(\nu)$ is $P_0(\mu)$.

This is enough to prove Theorem 1.31. The assertion that $P_{-\infty}(\nu)$ and $P_{-\infty}(\mu)$ correspond is proved similarly.

A second prediction problem which is quite important is that of interpolation. Suppose $\{X_n : n \in \underset{\sim}{Z}\}$ is a discrete stationary s.p. with spectral measure μ, and $d\mu = w d\theta + d\mu_s$. The problem is to find the best approximation to the function 1 from the combined past and future $P_{-1} \cup F_1$; more precisely, to compute

$$\tau = \inf\{\|1-g\|_{L^2(\mu)} : \ g \in P_{-1} \cup F_1\}.$$

For this discrete case, the solution was given by Kolmogorov and says that

$$\tau = \left(\int_\Gamma \frac{d\theta}{w(\theta)} \right)^{-1} .$$

Notice once again that the singular part of μ is unimportant for computing τ. The interpolation problem for a continuous parameter process, i.e. if the behaviour of X_t for $|t| \geq \lambda > 0$ is known, what can we say about X_t for $|t| < \lambda$, in particular about X_0 ?, is much harder to solve and a complete solution has only recently been provided in the book of Dym and McKean.

Both problems mentioned so far, the Kolmogorov-Wiener problem, and the interpolation problem say roughly that if w is not too 'little' i.e. if $\sigma > 0$ or $\tau > 0$, then the functions $\{e^{itx}: t \geq 0\}$ have a measure of independence in $L^2(\mu)$. The condition which we are interested in is stronger than either of the conditions $\sigma > 0$ and $\tau > 0$, and says that the functions $\{e^{itx}: t \in \underset{\sim}{R}\}$ are "asymptotically" independent in $L^2(\mu)$. We need some definitions.

Definition 1.34 (a) A subspace of a Hilbert space is a closed linear manifold.

(b) Let M, N be subspaces of a Hilbert space H. Then the quantity $\rho(M,N)$ between M and N is given by $\rho(M,N) = \sup\{|\langle \xi,\eta \rangle|: \xi \in M, \eta \in N, \|\xi\| \leq 1, \|\eta\| \leq 1\}$.

It is obvious that $\rho \leq 1$. $\cos^{-1}\rho$ is sometimes called the angle between M and N.

(c) We say that M, N are at positive angle if $\rho < 1$.

Clearly $M \cap N \neq \{0\} \Rightarrow \rho(M,N) = 1$, and if $\dim H < \infty$, then $\rho(M,N) = 1$ iff $M \cap N \neq \{0\}$. This is not true if $\dim H = \infty$. Obviously $\rho = 0$ iff $M \perp N$.

In the context of the prediction theory of a discrete process $\{X_n: n \in \underset{\sim}{Z}\}$, Helson and Szegö [10] in 1960 posed and answered the question: When are P_0 and F_1 at positive angle, i.e. when is $\rho_1 = \rho(P_0,F_1) < 1$? In fact, they proved the following result.

Theorem 1.35 P_0 and F_1 are at positive angle in $L^2(\mu)$ iff μ is absolutely continuous and $\exists r, s$ real L^∞-functions such that

$\|s\|_\infty < \frac{\pi}{2}$ and a.e. on Γ,

$$w = \exp(r+\tilde{s})$$

where $\tilde{s}$ denotes the conjugate function of s.

It is of course natural to ask about $\rho_n = \rho(P_0, F_n)$ for $n > 1$ (ρ_n depends only on n by stationarity, of course). This was solved by Helson and Sarason [9] in 1967 as a corollary of their method of solving the strong-mixing problem which we shall describe in a moment. They proved

Theorem 1.35 P_0 and F_n are at positive angle in $L^2(\mu)$ iff μ is absolutely continuous and $\exists$ a representation

$$w = |P|^2 \exp(r+\tilde{s})$$

where r, s are real L^∞-functions with $\|s\|_\infty < \frac{\pi}{2}$, and P is a polynomial in $e^{i\theta}$ of degree $\leq n-1$ with all its roots on the unit circle.

We shall obtain a complete analogue of Theorem 1.35 in the continuous case, i.e. when is $\rho_\lambda = \rho(P_0, F_\lambda) < 1$ for $\lambda > 0$? However our main task is to study the strong-mixing condition, first suggested by Rosenblatt, for a continuous stationary s.p.. The discrete case was solved by Helson and Sarason [9], and finally by Sarason [23] in 1972. Let us state the problem.

"For what finite positive Borel measures μ on $\mathbb{R}$ is it true that $\rho_\lambda = \rho(P_0, F_\lambda) \to 0$ as $\lambda \to \infty$?"

Of course, ρ_λ is non-increasing, and $\rho_\lambda \to 0 \Rightarrow \rho_\lambda < 1 \ \forall \lambda > \lambda_0$ for some $\lambda_0 > 0$. It turns out that this condition is stronger than the two previous ones we have considered. In the discrete case the following result was proved by Helson and Sarason.

Theorem 1.37. $\rho_n = \rho(P_0, F_n) \to 0$ as $n \to \infty$ iff μ is absolutely continuous and $\log w \in L^1(\Gamma)$, and $\exists$ a representation

$$w = |P|^2 \exp(u + \tilde{v})$$

where u, v are real continuous functions on Γ, and P is a polynomial in $e^{i\theta}$ with all its zeros on Γ.

We shall give a result like Theorem 1.36 in Chapter 2 for a continuous stationary s.p., but it is not so complete. In fact, it corresponds to the result given in the 1967 paper of Helson and Sarason for a discrete process, before it was improved in the 1972 paper.

At first sight it is not obvious that the strong-mixing condition $\rho_n \to 0$ has much probabilistic significance, and again the justification comes when one assumes that the process $\{X_n : n \in \mathbb{Z}\}$ is Gaussian (for convenience we consider the discrete situation). For if $\{X_n\}$ is Gaussian, it was proved by Kolmogorov and Rozanov [18] in 1960 that $4\alpha_n \le \rho_n \le \sin(2\pi\alpha_n)$ where α_n is given by

$$\alpha_n = \alpha(M_{-\infty}^{0}, M_n^{\infty}) = \sup_{E \in M_{-\infty}^{0},\ F \in M_n^{\infty}} |P(EF) - P(E)P(F)|$$

where M_s^t denotes the σ-algebra of events generated by the set

$\{X_r: s \leq r \leq t\}$, and EF denotes the intersection $E \cap F$ (recall that the X_r are defined on (Ω, Σ, P) a probability space). The condition $\alpha_n \to 0$ was first suggested by Rosenblatt [21] in 1956. In the same paper of Kolmogorov and Rozanov [18], ρ_n is related to the maximal correlation coefficient between P_0 and F_n. We shall not pursue this further.

Another important reason for looking at the condition $\rho_\lambda \to 0$ (or $\rho_n \to 0$ in the discrete case) is that processes which satisfy this condition, satisfy a central limit theorem; we refer the interested reader to Ibragimov [16], or Rozanov [22, Chapter IV].

Our main task in Chapter 2 is to derive analogues of Theorems 1.36 and 1.37 for the continuous case. The ideas in the proof of Theorem 1.37 recur there, so we shall not prove it, but refer the reader instead to the papers of Helson and Sarason [9], and Sarason [23]. However, to finish Chapter 1, we give a proof of part of Theorem 1.37 because it is elegant and involves techniques from the theory of uniform algebras.

<u>Lemma 1.38</u> If $\{X_n: n \in \underset{\sim}{Z}\}$ is a stationary s.p. with spectral measure μ on Γ, and if $\rho_n \to 0$ as $n \to \infty$, then μ is absolutely continuous.

<u>Proof</u> Without loss of generality, we may assume that $\mu(\Gamma) = 1$. Suppose μ is not absolutely continuous. Then there exists a set E of Lebesgue measure zero such that $\mu(E) > 0$. By regularity of μ, we may assume that E is closed. Define $g = z^{-1}\chi_E$ on Γ, where χ_E denotes the characteristic function of E. Then $g \in C(E)$. By the Rudin-Carleson theorem (see Gamelin [6, p.58]), $\exists$ a function f

in the disc algebra $A(D)$ such that $f = g$ on E and $\|f\|_\infty = 1$. Also since E is a peak set for $A(D)$ we may assume that $|f| < 1$ off E. By choice, therefore $f^k \to 0$ off E, and we have $\int_\Gamma |f^k| d\mu \to \int_E d\mu = \mu(E) \leq 1$. But also, $|\int f^k z^k d\mu| \to \mu(E)$ as $k \to \infty$. Thus, by definition, $\rho_k \geq |\int f^k z^k d\mu| \not\to 0$ as $k \to \infty$, and this contradiction shows that μ must be absolutely continuous.

Remarks (i) The regularity of μ is obvious, since it comes from a positive definite sequence via Herglotz's theorem.

(ii) The Rudin-Carleson Theorem [6, p.58] characterises closed sets of Lebesgue measure zero in terms of the peak sets of the algebra $A(D)$.

(iii) For our purposes, the neatest proof of Lemma 1.38 is obtained by showing that $\rho_n \to 0 \Rightarrow P_{-\infty} = \{0\}$ and using the result of Theorem 1.31. This is done for a continuous process in Chapter 2.

As a concluding remark, we should point out that for a given measure μ on $\underset{\sim}{R}$ it is usually difficult to calculate ρ_λ for $\lambda > 0$. One which is known is $d\mu = w(x)dx$ where $w(x) = \frac{1}{1 + x^2}$. In this case $\rho_\lambda = e^{-\lambda}$ $(\lambda \geq 0)$, so that w is a Helson-Sarason function. Naturally the question arises of just how $\rho_\lambda \to 0$ in a particular example, and work has been done on this by I.A. Ibragimov [13], [14], [15]. We shall not pursue this question.

CHAPTER 2

Mixing Conditions on the Line

In this chapter we shall study the strong mixing condition of Helson and Sarason, and the condition of positive angle first introduced by Helson and Szegö, for a continuous parameter stationary process. The machinery which we develop for the mixing condition, $\rho_\lambda \to 0$, will be useful in considering the Helson-Szegö problem. Before tackling these individual problems we present a couple of results useful in both contexts. Recall the setting: we have a stationary continuous parameter process $\{X_t: t \in \underset{\sim}{R}\}$ and an associated spectral measure $d\mu = w dx + d\mu_s$, on $\underset{\sim}{R}$. Here w is non-negative and integrable, and dx denotes Lebesgue measure on $\underset{\sim}{R}$. From now on we concern ourselves only with the analytic properties of μ, and the underlying process is not specified. By Theorem 1.31, if $\log w \notin L^1_c$, then $P_{-\infty} = P_0 = L^2(d\mu)$ and so $\rho_\lambda = 1$, $\forall\lambda > 0$. Henceforth we assume that $\log w \in L^1_c$. In this situation recall that by Lemma 1.32, $P_{-\infty} = L^2(d\mu_s)$. We now prove that a process satisfying either the strong mixing condition, $\rho_\lambda \to 0$, or the Helson-Szegö condition, $\rho_\lambda < 1$, must be purely non-deterministic.

<u>Lemma 2.1</u> $P_{-\infty} \neq \{0\} \Rightarrow \rho_\lambda = 1$, $\forall\lambda > 0$.

<u>Proof</u> Suppose $g_0 \in P_{-\infty}$ with $\|g_0\|_\mu = 1$. Then, by definition,

$$\begin{aligned}\rho_\lambda &= \sup\{|\langle f,g\rangle|: \ f \in F_\lambda, \ g \in P_0, \ \|f\|_\mu \le 1, \ \|g\|_\mu \le 1\} \\ &= \sup\{|\langle f,g\rangle|: \ f \in F_{\lambda+\nu}, \ g \in P_\nu, \ \|f\|_\mu \le 1, \ \|g\|_\mu \le 1\}\end{aligned}$$

so $\rho_\lambda \geq \sup\{|\langle f, g_0 \rangle| : \ f \in F_r, \ \|f\|_\mu \leq 1\}$ $(\forall r \in \underset{\sim}{R})$.
But $\overline{\bigcup_{r \in R} F_r} = L^2(\mu)$, and $g_0 \in L^2(\mu)$, so $\rho_\lambda \geq \|g_0\|_\mu^{\ 2} = 1$, i.e. $\rho_\lambda = 1$.

The following is an immediate consequence of Lemma 2.1 and the fact that $P_{-\infty} = L^2(d\mu_s)$.

<u>Corollary 2.2</u> If $\log w \in L^1_c$ and $\rho_\lambda < 1$ for some $\lambda > 0$, then μ is absolutely continuous.

From now on we shall assume that μ is absolutely continuous, $\mu = wdx$ where w is non-negative and integrable, and $\log w \in L^1_c$.

<u>Notation</u> Let $\mathcal{W} = \{w \in L^1, \ w \geq 0: \ \log w \in L^1_c$ and $\rho_\lambda \to 0$ as $\lambda \to \infty\}$. For each $\lambda > 0$, let $\mathcal{W}_\lambda = \{w \in L^1, w \geq 0: \ \log w \in L^1_c$ and $\rho_\lambda < 1\}$.

Obviously $\mathcal{W} \subseteq \bigcup_{\lambda > 0} \mathcal{W}_\lambda$, and $\mathcal{W}_\lambda \subseteq \mathcal{W}_\mu$ if $\mu > \lambda$. Before we can determine the nature of $\mathcal{W}$ and $\mathcal{W}_\lambda$ we need a more concrete description of ρ_λ. Suppose $w \in L^1$, $w \geq 0$ and $\log w \in L^1_c$. Then, by Theorem 1.7, there exists an outer function $h \in H^2$ such that $w = |h|^2$ a.e. on $\underset{\sim}{R}$, and we may write $h = |h|e^{i\phi}$ with ϕ real, chosen so that $\phi(i) = 0$. Of course, h is unique up to multiplication by a constant of modulus 1. Recall that for $\lambda \in \underset{\sim}{R}$, e_λ denotes the function defined on $\overline{\Pi^+}$ by $e_\lambda(z) = e^{i\lambda z}$, so that $e_\lambda \in H^\infty$ for $\lambda \geq 0$,

<u>Lemma 2.3</u> For $\lambda > 0$, $\rho_\lambda = \inf_{A \in H^\infty} \|e^{-2i\phi} - e_{-\lambda} A\|_\infty$ i.e. ρ_λ is the distance in L^∞ from the function $e^{-2i\phi}$ to the subset $e_{-\lambda} H^\infty$ of L^∞.

<u>Proof</u> By definition, $\rho_\lambda = \sup\{|\int fge_\lambda|h|^2dx| : f, g \in B(F_0)\}$

$$= \sup\{|\int (fh)(gh)e_\lambda e^{-2i\phi}dx| : f, g \in B(F_0)\} \quad (1).$$

Since h is outer in H^2, $\{fh: f \in B(F_0)\}$ is dense in the unit ball of H^2, and so by Theorem 1.5, $\{(fh)(gh): f, g \in B(F_0)\}$ is dense in the unit ball of H^1. Thus (1) expresses ρ_λ as the norm of the linear functional R_λ which is defined on H by $R_\lambda(\alpha) = \int \alpha e_\lambda e^{-2i\phi}dx \quad (\alpha \in H^1)$. Now, since $(H^1)' \cong \frac{L^\infty}{H^\infty}$ by Lemma 1.9, as easy application of the Hahn-Banach theorem gives that $\rho_\lambda = \inf_{A \in H^\infty} \|e_\lambda e^{-2i\phi} - A\|_\infty = \inf_{A \in H^\infty} \|e^{-2i\phi} - e_{-\lambda}A\|_\infty$.

The following theorem is based on a result of Sarason [24], and characterises W. It is the continuous case analogue of Theorem 2 in [9].

<u>Theorem 2.4</u> $w \in W$ iff $e^{-2i\phi} \in H^\infty + BUC$, where BUC denotes the set of bounded uniformly continuous functions on $\mathbb{R}$.

<u>Proof</u> By Lemma 2.3, $\rho_\lambda \to 0$ iff $\lim_{\lambda\to\infty} \inf_{A \in H^\infty} \|e^{-2i\phi} - e_{-\lambda}A\|_\infty = 0$. Let R denote the uniform closure of the set $\bigcup_{\lambda>0} \{e_{-\lambda}H^\infty\}$. We prove that $R = H^\infty + BUC$. This is done in two parts.

<u>Step I:- $H^\infty + BUC \subseteq R$.</u> Obviously $H^\infty \subseteq R$. Let $f \in BUC$. For $\lambda > 0$, define g_λ on $\overline{\Pi^+}$ by $g_\lambda(z) = \lambda \int K(\lambda(t-z))f(t)dt$ where $K(t) = \frac{2}{\pi}\left(\frac{\sin t/2}{t}\right)^2$. Then it is easy to see that g_λ is analytic in Π^+. We show that $e_\lambda g_\lambda \in H^\infty$, $\forall\lambda > 0$ and that $\lim_{\lambda\to\infty} \|f - g_\lambda\|_\infty = 0$ which will prove (I).

(a) $\underline{e_\lambda g_\lambda \in H^\infty}$ It suffices to prove that $|g_\lambda(z)| < Ke^{\lambda y}$ for $z = x + iy \in \overline{\Pi^+}$ and some constant K. By translation we need only look at $z = iy$. Now

$$|g_\lambda(iy)| = \frac{2\lambda}{\pi}\left|\int \left(\frac{\sin\left[\frac{\lambda t - i\lambda y}{2}\right]}{\lambda t - i\lambda y}\right)^2 f(t)\,dt\right| \le \frac{2\lambda}{\pi}\|f\|_\infty \int \left|\frac{\sin\left[\frac{\lambda t - i\lambda y}{2}\right]}{\lambda t - i\lambda y}\right|^2 dt$$

Now

$$\left|\frac{\sin\frac{\lambda t - i\lambda y}{2}}{\lambda t - i\lambda y}\right|^2 = \begin{cases} \dfrac{\sin^2\frac{\lambda t}{2}\cosh^2\left(-\frac{\lambda y}{2}\right) + \cos^2\frac{\lambda t}{2}\sinh^2\left(-\frac{\lambda y}{2}\right)}{\lambda^2(t^2+y^2)} & \text{if } t^2 + y^2 \neq 0 \\ \frac{1}{4}, \text{ if } t = y = 0 \end{cases}$$

So, if we fix $\delta > 0$, then

$$t \in (-\delta,\delta) \Rightarrow \left|\frac{\sin\left[\frac{\lambda t - i\lambda y}{2}\right]}{\lambda t - i\lambda y}\right|^2 \le \begin{cases} \frac{1}{4}\cos h^2\left(-\frac{\lambda y}{2}\right) + \dfrac{\sin h^2\left(-\frac{\lambda y}{2}\right)}{\lambda^2 y^2}, & \text{if } y \neq 0 \\ \frac{1}{4}\cos h^2(0), & \text{if } y = 0 \end{cases}$$

Hence $$\left|\frac{\sin\left[\frac{\lambda t - i\lambda y}{2}\right]}{\lambda t - i\lambda y}\right|^2 \le \frac{1}{2}e^{\lambda y}.$$

If $t \notin (-\delta,\delta)$, then
$$\left|\frac{\sin\left[\frac{\lambda t - i\lambda y}{2}\right]}{\lambda t - i\lambda y}\right|^2 \le \frac{\cos h^2\left(-\frac{\lambda y}{2}\right) + \sin h^2\left(-\frac{\lambda y}{2}\right)}{\lambda^2 t^2}$$
$$\le \frac{e^{\lambda y}}{\lambda^2 t^2}$$

Thus
$$\int_{-\infty}^{\infty}\left|\frac{\sin\left[\frac{\lambda t - i\lambda y}{2}\right]}{\lambda t - i\lambda y}\right|^2 dt \le \delta e^{\lambda y} + \frac{2e^{\lambda y}}{\lambda^2}\int_\delta^\infty \frac{dt}{t^2}$$
$$= \left(\delta + \frac{2}{\lambda^2\delta}\right)e^{\lambda y}, \text{ proving (a).}$$

(b) $g_\lambda \to f$ in L^∞ Notice that

$$g_\lambda(x) = \lambda \int K(\lambda(t-x))f(t)dt = \int K(t)f(x+t/\lambda)dt \qquad (x \in \mathbb{R}) .$$

Let $\varepsilon > 0$ and $x \in \mathbb{R}$. Choose $N > \frac{16\|f\|_\infty}{\pi\varepsilon}$. Since f is uniformly continuous, $\exists\delta > 0$ such that $|s-t| < \delta \Rightarrow |f(s)-f(t)| < \frac{\varepsilon}{2}$. With this δ, choose $\lambda > \frac{N}{\delta}$. Then
$f(x) - g_\lambda(x) = \int K(t)\{f(x)-f(x+t/\lambda)\}dt$ so that

$$|f(x)-g_\lambda(x)| \le \int_{-N}^{N} K(t)|f(x)-f(x+t/\lambda)|dt + 4\|f\|_\infty \int_N^\infty K(t)dt$$

Since $\int K(t)dt = 1$ and $\frac{N}{\lambda} < \delta$, $\int_{-N}^{N} K(t)|f(x)-f(x+t/\lambda)|dt < \frac{\varepsilon}{2}$. Also $\int_N^\infty K(t)dt \le \frac{8\|f\|_\infty}{\pi}\int_N^\infty \frac{dt}{t^2} < \frac{\varepsilon}{2}$. Thus $\|f-g_\lambda\|_\infty < \varepsilon$, proving (b).

Step II:- $R \subseteq H^\infty + BUC$ (Sarason [24, p.404])

(a) $e_{-\lambda}h \in H^\infty + BUC$ for $\lambda > 0$ and $h \in H^\infty$ Let $h \in H^\infty$ and $\lambda > 0$. Choose a C^∞ function v of compact support with $v = 1$ on $[-\lambda, 0]$. Let u be the inverse Fourier transform of v, so that $\hat{u} = v$. We have, defining $f = e_{-\lambda}h$, that $f = (u*f) + (f-u*f)$. By definition of u, it is easy to see that $u * f \subset BUC$. It remains to prove that $f - (u*f) \in H^\infty$, and it suffices to show that it annihilates H^1. Thus suppose $g \in H^1$, and let $u_1(x) = u(-x)$ $(x \in \mathbb{R})$. Then, by Fubini, $\int (u*f)(x)g(x)dx = \int (u_1*g)(x)f(x)dx$ so

$$\int [f(x)-(u*f)(x)]g(x)dx = \int [g(x)-(u_1*g)(x)]e^{-i\lambda x}h(x)dx.$$

Now $\hat{u}_1(x) = \hat{v}(-x) = 1$ on $[0,\lambda]$, so the Fourier transform of

$g - u_1 * g$ vanishes on $[0,\lambda]$. But $g \in H^1 \Rightarrow u_1 * g \in H^1$, so the Fourier transform of $g - u_1 * g$ vanishes on $(-\infty,\lambda]$. Thus the function $e_{-\lambda}[g-u_1 * g]$ lies in H^1 and so annihilates H^∞. Hence the last integral above vanishes and (a) is proved.

(b) $H^\infty + BUC$ is closed in L^∞ The proof of this is just like that of the well-known result for the circle that $H^\infty + C$ is closed in L^∞, and is only included for the sake of completeness. Consider the natural map $\eta: \frac{BUC}{H^\infty \cap BUC} \to \frac{L^\infty}{H^\infty}$ given by $f + (H^\infty \cap BUC) \to f + H^\infty$ $(f \in BUC)$. The distance estimate,

$$\operatorname{dist}(f,H^\infty) = \operatorname{dist}(f,H^\infty \cap BUC) \qquad (f \in BUC) \qquad (2)$$

shows that η is an isometry and so has closed range. But $H^\infty + BUC$ is simply the inverse image in L^∞ of this range under the quotient map $L^\infty \to \frac{L^\infty}{H^\infty}$, and so is closed. To check the distance estimate (2) it is enough to prove the inequality $\operatorname{dist}(f,H^\infty \cap BUC) \leq \operatorname{dist}(f,H^\infty)$ $(f \in BUC)$. Let $h \in H^\infty$ and, for $y > 0$, let f_y, h_y be the functions $f_y(x) = f(x+iy)$, $h_y(x) = h(x+iy)$ obtained, as usual via the Poisson extensions of f and h to Π^+. Then, for $y > 0$,

$$\|f-h_y\|_\infty \leq \|f-f_y\|_\infty + \|f_y-h_y\|_\infty$$

But $f \in BUC \Rightarrow \|f-f_y\|_\infty \to 0$ as $y \to 0$, and also $\|f_y-h_y\|_\infty \leq \|f-h\|_\infty$. Thus, since $h_y \in H^\infty \cap BUC$, we have $\operatorname{dist}(f,H^\infty \cap BUC) \leq \|f-h\|_\infty$. Taking the infimum over all $h \in H^\infty$ yields the result (b) and completes the proof of the theorem.

<u>Corollary 2.5</u> $H^\infty + BUC$ is a closed subalgebra of L^∞.

<u>Remark</u> The characterisation given in Theorem 2.4 is sometimes useful in identifying specific elements of W, as we shall see shortly. However, its usefulness is limited and we are seeking a more elaborate description of W along the lines of the Helson-Sarason result for the discrete case given in Theorem 1.37. Accordingly our first step is to identify some elements of W to be the "building blocks" out of which we construct arbitrary elements of W. To this end we identify the analogues of the factors $|P|^2$ and $\exp(u+\tilde{v})$ which appear in Theorem 1.36. This we now do.

<u>Lemma 2.6</u> Suppose $w = \exp(u+\tilde{v})$ is integrable, where u, v are real functions in BUC. Then $w \in W$.

<u>Remark</u> $\tilde{v}$ denotes the conjugate function of v, as defined in Chapter 1.

<u>Proof</u> By hypothesis $w \in L^1$ and $\log w = u + \tilde{v} \in L^1_{\mathbb{C}}$ by Lemma 1.13. Thus $w = |h|^2$ for some outer function $h \in H^2$, and, in fact, h is given by

$$h^2(z) = \exp \frac{1}{\pi i} \int \frac{zx + 1}{x - z} (u+\tilde{v}) \frac{dx}{1+x^2}$$

But, by definition of conjugate function,

$$\lim_{\operatorname{Im} z \to 0} \frac{1}{\pi i} \int \frac{xz + 1}{x - z} (u+\tilde{v}) \frac{dx}{1 + x^2} = u + i\tilde{u} + \tilde{v} - iv$$

Thus,

$$e^{-2i\phi} = \frac{w}{h^2} = \frac{e^{u+\tilde{v}}}{e^{u+i\tilde{u}+\tilde{v}-iv}} = e^{i(v-\tilde{u})}.$$

But $e^{i(v-\tilde{u})} = e^{u+iv}e^{-u-i\tilde{u}}$, and $e^{u+iv} \in BUC$, $e^{-u-i\tilde{u}} \in H^\infty$ so that $e^{-2i\phi} \in H^\infty + BUC$.

In identifying the class of functions on $\underset{\sim}{R}$ which corresponds to the factors $|P|^2$ in the discrete case, it is useful to notice that if $w = |P|^2$ on Γ where the degree of P is n_0, then $\rho_n(w) = 0$ $\forall n > n_0$, and, conversely, if $w \in L^1(\Gamma)$, $w \geq 0$ and $\rho_n = 0$ $\forall n > n_0$, then $\exists$ a polynomial P of degree $\leq n_0$ such that $w = |P|^2$. This is an easy consequence of Lemma 2.3 and the definition of $H^1(\Gamma)$ in terms of the vanishing of the negative Fourier coefficients. The argument is given below for the continuous case and involves only elementary properties of Fourier transforms.

<u>Lemma 2.7</u> (a) Let $w \in L^1(\underset{\sim}{R})$, $w \geq 0$ and suppose that

$$w(x) = \int e^{ixt}\phi(t)dt \tag{3}$$

for some $\phi \in L^1$ of compact support in $[-\lambda_0, \lambda_0]$. Then $\rho_\lambda(w) = 0$ $\forall\lambda > \lambda_0$.

(b) Conversely, if $w \in L^1$, $w \geq 0$ and $\rho_\lambda(w) = 0$ $\forall\lambda > \lambda_0$, then w has a representation in the form (3).

<u>Proof</u> (a) By definition of ρ_λ, it is enough to prove that $\int \alpha e_\lambda w dx = 0$ for every function of the form $\alpha = \sum_{r=1}^{n} a_r e_{\lambda_r}$, where

$\lambda_r \geq 0$, $r = 1, \ldots, n$. Thus it suffices to prove that $\int e_\nu w dx = 0 \ \ \forall \nu \geq \lambda$. But

$$\begin{aligned}\int e_\nu w dx &= \int e^{i\nu x}\hat{\phi}(-x)dx, \quad \text{since} \quad w(x) = \hat{\phi}(-x) \quad \text{by (3)} \\ &= \int e^{-i\nu x}\hat{\phi}(x)dx \\ &= \phi(-\nu).\end{aligned}$$

The last step follows by the inversion theorem for Fourier transforms which is applicable since ϕ and w are both integrable. Since ϕ vanishes off $[-\lambda_0, \lambda_0]$, $\rho_\lambda = 0$.

(b) If $w \geq 0$, $w \in L^1$ satisfies $\rho_\lambda = 0 \ \ \forall \lambda > \lambda_0$, then the function $g = e_\lambda w \in H^1$ by Lemma 2.3. Since $w = \bar{w}$, we have $w = e_\lambda \bar{g}$ and so $g = e_{2\lambda}\bar{g}$ a.e. on $\underset{\sim}{R}$. But $g \in H^1 \Rightarrow \hat{g}$ vanishes on $(-\infty, 0]$, and $\widehat{e_{2\lambda}\bar{g}}(y) = \hat{g}(2\lambda - y)$, so $\widehat{e_{2\lambda}\bar{g}}$ vanishes on $[2\lambda, \infty)$. Thus $\hat{g}$ has support in $(0, 2\lambda)$. Hence $\hat{w} = \widehat{e_{-\lambda}g}$ has support in $(-\lambda, \lambda)$. This is true for every $\lambda > \lambda_0$, and so $\hat{w}$ vanishes off $[-\lambda_0, \lambda_0]$. Since $w \in L^1$, $\hat{w}$ is continuous, and hence in L^1. Thus, by the inversion theorem, $w(x) = \int e^{ixt}\hat{w}(t)dt$ a.e. and (3) holds.

By the Paley-Wiener theorem (Theorem 1.22), the equation

$$\text{(A)} \qquad \Phi(z) = \int_{-\lambda_0}^{\lambda_0} e^{itz}\phi(t)dt \qquad (\phi \in L^1[-\lambda_0, \lambda_0])$$

defines an entire function, and obviously

$$\text{(B)} \qquad |\Phi(z)| \leq Ke^{\lambda_0 |y|}, \qquad z = x + iy$$

for some constant K independent of z (in fact $K = \|\phi\|_1$ in this case). Conversely, any entire function $\Phi(z)$ satisfying (B) and such that $\Phi|_{\tilde{R}} \in L^1$ has a representation of the form (A). So, from Lemma 2.7, we obtain

<u>Theorem 2.8</u> If $w \in L^1$, $w \geq 0$, then $\rho_\lambda = 0 \ \forall\lambda > \lambda_0$ iff there exists an entire function Φ satisfying (B) such that $w = \Phi|_{\tilde{R}}$.

<u>Corollary 2.9</u> (a) $w \in L^1$, $w \geq 0$, $\rho_\lambda = 0 \ \forall\lambda > \lambda_0 \Rightarrow w$ is bounded and continuous on $\tilde{R}$.

(b) If $w \geq 0$, $w \in L^1$ is the restriction to $\tilde{R}$ of an entire function satisfying (B), then $\log w \in L^1_{\mathbb{C}}$.

<u>Proof</u> Immediate from Theorem 2.8.

We have thus obtained the analogues of the factors $|P|^2$ in Theorem 1.37, for, by Theorem 1.21, entire functions satisfying (B) with $\Phi|_{\tilde{R}} \geq 0$, $\Phi|_{\tilde{R}} \in L^1$ have the property that $\exists\Psi$ satisfying (B) such that $\Phi|_{\tilde{R}} = |\Psi|_{\tilde{R}}|^2$. If we combine the results of Theorem 2.8 and Lemma 2.6, we obtain a subset of W, the properties of which are fairly representative of W as a whole.

<u>Theorem 2.10</u> Suppose $w = f \exp(u+\tilde{v})$, where f is a real L^1-function which is the restriction to $\tilde{R}$ of an entire function Φ satisfying $|\Phi(z)| \leq Ke^{\lambda|\text{Im}z|}$ for some constants $\lambda, K > 0$, and u and v are real functions in BUC such that $e^{\tilde{v}} \in L^1$. Then $w \in W$.

<u>Proof</u> By Corollary 2.9 (b), $\log f \in L^1_{\mathbb{C}}$ so that $\log w \in L^1_{\mathbb{C}}$. Also

f is bounded on $\mathbb{R}$ by hypothesis, and so $w \in L^1$. Thus $f = |h_1|^2$, h_1 outer in H^∞ and $\frac{f}{h_1{}^2} \in H^\infty + \mathrm{BUC}$. Similarly, $e^{u+\tilde{v}} = |h_2|^2$, h_2 outer in H^2 and $\frac{e^{u+\tilde{v}}}{h_2{}^2} \in H^\infty + \mathrm{BUC}$. Clearly $w = |h_1 h_2|^2$, and it is easy to see that $h_1 h_2$ is outer, so that

$$e^{-2i\phi} = \frac{w}{h_1{}^2 h_2{}^2} = \frac{f}{h_1{}^2}\,\frac{e^{u+\tilde{v}}}{h_2{}^2} \in H^\infty + \mathrm{BUC}, \quad \text{as required.}$$

<u>Remark</u> Of course, the method of proof of Theorem 2.10 actually shows that if $w_1 \in W$, $w_2 \in W$ and $w_1 w_2 \in L^1$, then $w_1 w_2 \in W$.

For the moment, this is as far as we want to go in identifying particular elements of W. The approach now is to break up an arbitrary element of W into its constituent parts of the above sort. We need to put Lemma 2.3 into a more useful form, which we do in Lemma 2.12. First we notice that multiplication by $\frac{1}{1+x^2}$ preserves W. We have

<u>Lemma 2.11</u> (a) The function $g(x) = \frac{(i+x)^2}{1+x^2}$ is in BUC.

(b) $w \in W \Rightarrow \frac{w}{1+x^2} \in W$.

(c) $w \in W$, $(1+x^2)w \in L^1 \Rightarrow (1+x^2)w \in W$.

<u>Proof</u> (a) $\left|\frac{(i+x)^2}{1+x^2}\right| = 1 \quad (x \in \mathbb{R})$, and

$$\left|\frac{(i+x)^2}{1+x^2} - \frac{(i+y)^2}{1+y^2}\right|^2 = 4(x-y)^2 \left\{\frac{(x+y)^2+(xy-1)^2}{(1+x^2)^2(1+y^2)^2}\right\}$$
$$\leq 4(x-y)^2.$$

so g is in BUC.

(b) if $w \in W$, then $\frac{w}{1+x^2} \in L^1$ and since $w = |h|^2$, h outer in H^2, we have that $\frac{w}{1+x^2} = \left|\frac{h^2}{(i+x)^2}\right|$ and $\frac{h}{i+z}$ is outer in H^2. By Theorem 2.4, it suffices to show that

$\frac{w}{1+x^2} \cdot \frac{(i+x)^2}{h^2} \in H^\infty + BUC$. But this follows, since $\frac{w}{h^2} \in H^\infty + BUC$ and, by (a), $\frac{(i+x)^2}{1+x^2} \in BUC$.

(c) is proved in a similar way using the fact that $\frac{1+x^2}{(i+x)^2}$ is also in BUC.

Lemma 2.12 Suppose $w \in L^1$, $w \geq 0$. Then $w \in W$ iff $\forall \varepsilon > 0$, $\exists \lambda = \lambda(\varepsilon) > 0$, $A = A(\varepsilon) \in H^\infty$ and a real L^∞-function $s = s(\varepsilon)$ such that a.e. on $\mathbb{R}$

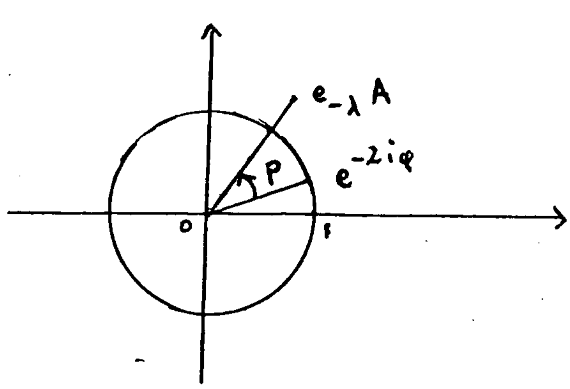
$$|s| < \varepsilon, \quad |\log|A|| < \varepsilon \quad \text{and} \quad s + \arg(Ah^2 e_{-\lambda}) \equiv 0 (\text{mod} 2\pi).$$

Remark The statement $\alpha \equiv 0(\text{mod}2\pi)$ means that for almost all $x \in \mathbb{R}$, $\exists k(x) \in \mathbb{Z}$ such that $\alpha(x) = 2\pi k(x)$.

Proof ($\Rightarrow$) Suppose that $w \in W$. By Lemma 2.3, given $\varepsilon > 0$ $\exists A \in H^\infty$ and $\lambda > 0$ such that $\|e^{-2i\phi} - e_{-\lambda}A\|_\infty < \varepsilon$. Then we have a.e.

$$|A| = |e_{-\lambda}A| = |(e_{-\lambda}A - e^{-2i\phi}) + e^{-2i\phi}| \leq 1 + \varepsilon.$$

Similarly $|A| \geq 1 - \varepsilon$, so we may assume, without loss of generality, that $|\log|A|| < \varepsilon$. Consider the diagram below where P is such that $0 \leq P < 2\pi$.



We have, $\arg(Ah^2e_{-\lambda}) = \arg(e_{-\lambda}A) + \arg(h^2) = \arg(e_{-\lambda}A) + 2\phi$ $= \arg(e_{-\lambda}A) - \arg(e^{-2i\phi})$. So that $\arg(Ah^2e_{-\lambda}) \equiv P(\text{mod}2\pi)$. Now $|e_{-\lambda}A-e^{-2i\phi}| < \varepsilon \Rightarrow \cos P > 0$, for, otherwise, $|e^{-2i\phi}-e_{-\lambda}A|$ is too large. By the cosine rule,

$$\cos P = \frac{|e_{-\lambda}A|^2 + 1 - |e_{-\lambda}A-e^{-2i\phi}|^2}{2|e_{-\lambda}A|} \geq \frac{1+(1-\varepsilon)^2 - \varepsilon^2}{2(1+\varepsilon)} = \frac{1-\varepsilon}{1+\varepsilon},$$

which is close to 1. Thus, modifying λ and A if necessary, we have $|P| < \varepsilon$. Let $s = -P$. Then $|s| < \varepsilon$, and $s + \arg(Ah^2e_{-\lambda}) = \arg(Ah^2e_{-\lambda}) - P \equiv 0(\text{mod}2\pi)$.

($\Leftarrow$) This is straightforward, since

$$|e^{-2i\phi}-e_{-\lambda}A|^2 = 1 + |A|^2 - 2|A|\cos s \leq 1 + (1+\varepsilon)^2 - 2(1-\varepsilon)^2 \leq 6\varepsilon.$$

The next step is to formulate a result on analytic continuation across $\underset{\sim}{R}$ of a function defined on $\overline{\Pi^+}$, and for this we need a definition.

<u>Definition 2.13</u> Suppose α, β and R are real numbers, and U is the interior of the rectangle $S = \{z \in \underset{\sim}{C}: 0 \leq \text{Im}z \leq R, \alpha \leq \text{Re}z \leq \beta\}$. We say that $f: U \to \underset{\sim}{C}$ is in $H^1(U)$ if f is analytic in U, $f_y \in L^1(\alpha,\beta)$ $\forall y \in (0,R)$, and $\int_\alpha^\beta |f_0(t)-f_y(t)|dt \to 0$ as $y \to 0$ for some $f_0 \in L^1(\alpha,\beta)$. This entails that $f_y \to f_0$ pointwise a.e. on (α,β).

<u>Proposition 2.14</u> Suppose $f \in H^1(U)$, and f_0 is real a.e. on (α,β). Then f can be extended analytically across (α,β).

Proof The idea of the proof is to mimic the proof of the Schwarz reflection principle where normally one assumes that f_0 is continuous. The condition $f \in H^1(U)$ satisfactorily replaces this assumption.

Choose and fix $\xi \in U$. Consider a rectangle J symmetric about $\underset{\sim}{R}$ and containing ξ so that $J \subset \text{closure}(U \cup \bar{U})$ as in Figure 1. (Here $\bar{U} = \{\bar{z}: \ z \in U\}$.)

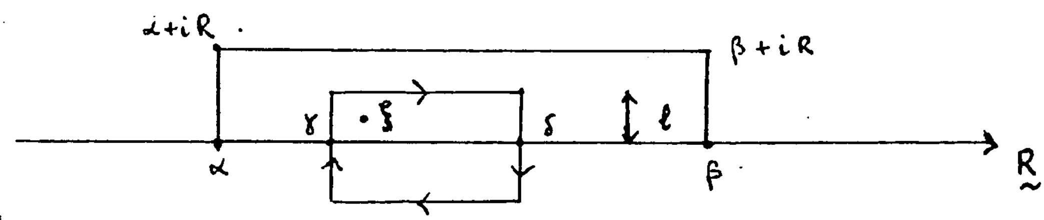


Fig 1

Since $f \in H^1(U)$, extending f to int J by $f(z) = \overline{f(\bar{z})}$ we may assume makes f continuous on the boundary of J. Define g on int J by

$$g(\zeta) = \frac{1}{2\pi i} \int_J \frac{f(z)}{z - \zeta}\, dz \qquad (\zeta \in \text{int } J)$$

Then it is obvious that g is analytic inside J. For each $n > \frac{1}{\text{Im}\xi}$, construct the contour J_n of Figure 2. Then it is clear that

$$f(\xi) = \frac{1}{2\pi i} \int_{J_n} \frac{f(z)dz}{z - \xi} \qquad (1)$$

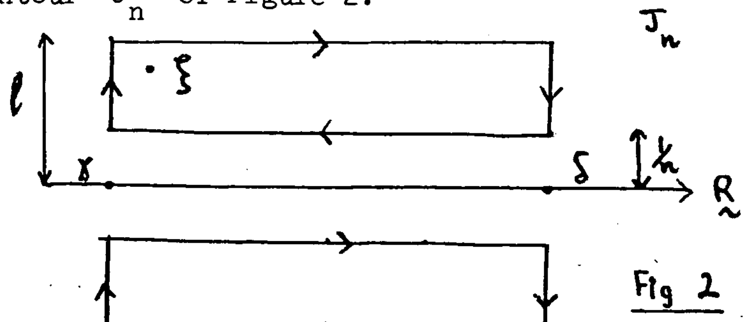
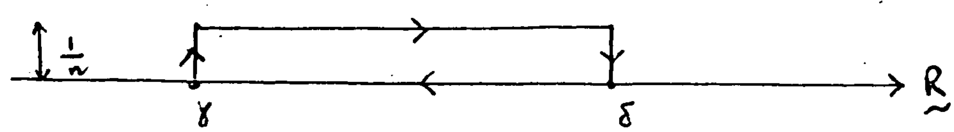


Fig 2

since the integral round the upper contour is $f(\xi)$ and that round the lower is 0. (1) holds for every $n > \frac{1}{\mathrm{Im}\xi}$, so to see that $f(\xi) = g(\xi)$, it will suffice to prove that $\int_{J \setminus J_n} \frac{f(z)}{z-\xi}\,dz \to 0$ as $n \to \infty$. By symmetry it suffices to prove that $\int_{C_n} \frac{f(z)}{z-\xi}\,dz \to 0$ as $n \to \infty$, where C_n is shown in Figure 3.

Fig 3



Now

$$\int_{C_n} \frac{f(z)}{z-\xi}\,dz = \int_{\gamma}^{\delta} \left[\frac{f(x+i/n)}{x+i/n-\xi} - \frac{f_0(x)}{x-\xi}\right]dx + i\int_0^{1/n} \left[\frac{f(\gamma+iy)}{\gamma+iy-\xi} - \frac{f(\delta+iy)}{\delta+iy-\xi}\right]dy.$$

But

$$\int_0^{1/n} \left|\frac{f(\gamma+iy)}{\gamma+iy-\xi} - \frac{f(\delta+iy)}{\delta+iy-\xi}\right| dy \le \int_0^{1/n} \frac{|f(\gamma+iy)|+|f(\delta+iy)|}{\mathrm{Im}\xi - \frac{1}{n}}\,dy \qquad (2)$$

$$\le \frac{1}{n(\mathrm{Im}\xi - 1/n)} \sup\{|f(x+iy)| : 0 \le y \le \tfrac{1}{n},\ x = \gamma \text{ or } x = \delta\}$$

But f is continuous on the boundary of J, hence bounded there and so the supremum above is bounded for all large n. Thus the integral in (2) tends to 0 as $n \to \infty$.

For the integral along the horizontal sides of C_n, we need to use the fact that $f_y \to f_0$ in $L^1(\alpha,\beta)$ and pointwise a.e. on (α,β).

$$\int_{\gamma}^{\delta} \frac{f(x+i/n)}{x+(i/n)-\xi} - \frac{f_0(x)}{x-\xi}\,dx = \int_{\gamma}^{\delta} \frac{f(x+i/n) - f_0(x)}{x+(i/n)-\xi} - \frac{i}{n}\int_{\gamma}^{\delta} \frac{f_0(x)}{(x-\xi)(x-\xi+i/n)}\,dx$$

so

$$\left|\int_{\gamma}^{\delta}\left[\frac{f(x+i/n)}{x+(i/n)-\xi} - \frac{f_0(x)}{x-\xi}\right]dx\right| \le \frac{1}{\mathrm{Im}\xi - 1/n}\left\{\int_{\gamma}^{\delta} |f(x+i/n)-f_0(x)|\,dx + \frac{1}{n\,\mathrm{Im}\xi}\int_{\gamma}^{\delta} |f_0(x)|\,dx\right\}$$

$$= \frac{1}{\mathrm{Im}\xi - 1/n} \{ \| f_{1/n} - f_0 \|_1^{(\alpha,\beta)} + \frac{\| f_0 \|_1^{(\alpha,\beta)}}{n \, \mathrm{Im}\xi} \} ,$$

(where $\|\cdot\|_1^{(\alpha,\beta)}$ denotes $\int_\alpha^\beta |\cdot| dx$). Since $f \in H^1(U)$, $\| f_{1/n} - f_0 \|_1^{(\alpha,\beta)} \to 0$ as $n \to \infty$, and so the last expression above also tends to zero.

<u>Proposition 2.15</u> Suppose f is analytic in Π^+, $f \geq 0$ on $\underset{\sim}{R}$ and $f = \alpha\beta$ where $e_\lambda \alpha \in H^1$ and $\frac{\beta}{(i+z)^2} \in H^1$. Then f continues analytically across $\underset{\sim}{R}$.

<u>Proof</u> $e_\lambda \alpha \in H^1 \Rightarrow \exists u, v \in H^2$ with $|u| = |v|$ and $e_\lambda \alpha = uv$. Similarly, $\exists u', v' \in H^2$ with $|u'| = |v'|$ and $\frac{\beta}{(i+z)^2} = u'v'$. Define γ, δ in $\overline{\Pi^+}$ by

$$e_\lambda \gamma = u(i+z)u' \quad \text{and} \quad \delta = v(i+z)v'.$$

Then $|\gamma| = |\delta|$ on $\underset{\sim}{R}$ and $\gamma\delta = \alpha\beta = f$ a.e. on $\underset{\sim}{R}$. Since $f \geq 0$ on $\underset{\sim}{R}$, we must have $\delta = \bar{\gamma}$ a.e. on $\underset{\sim}{R}$. Let $f_1 = \gamma + \delta$, $f_2 = i(\gamma - \delta)$. It is straightforward to check (*) that $f_1, f_2 \in H^1(U)$ for any rectangle U of the above form, and so f_1, f_2 both continue analytically across $\underset{\sim}{R}$. Hence γ and δ do also, and f does.

(*)<u>Remark</u> To see that $g = \frac{e_\lambda f}{(i+z)^2} \in H^1 \Rightarrow f \in H^1(U)$ for any rectangle of the given form proceed as follows:- with the notation of Proposition 2.14, since $|f_y(x)| = e^{\lambda y}[x^2 + (1+y)^2] |g_y(x)|$, we have

$$\int_\alpha^\beta |f_y(x)| dx \leq \{[\max(|\alpha|^2, |\beta|^2)] + (1+y)^2\} e^{\lambda y} \int_\alpha^\beta |g_y(x)| dx < \infty$$

so $f_y \in L^1(\alpha,\beta)$. Moreover,

$$\begin{aligned}|f_0(x)-f_y(x)| &= |e^{-i\lambda x}e^{\lambda y}(x^2+(1+y)^2)g_y(x)-e^{-i\lambda x}(x^2+1)g_0(x)| \\ &\leq (x^2+1)|e^{\lambda y}g_y(x)-g_0(x)| + e^{\lambda y}|y^2+2y||g_y(x)| \\ &\leq (x^2+1)[e^{\lambda y}|g_y(x)-g_0(x)|+(e^{\lambda y}-1)|g_0(x)|]+e^{\lambda y}|2y+y^2||g_y(x)|\end{aligned}$$

so

$$\begin{aligned}\|f_0-f_y\|_1^{(\alpha,\beta)} &= \int_\alpha^\beta |f_0-f_y|\,dx \\ &\leq [\max(|\alpha|^2,|\beta|^2)+1]\{e^{\lambda y}\|g_y-g_0\|_1^{(\alpha,\beta)}+(e^{\lambda y}-1)\|g_0\|_1^{(\alpha,\beta)}\} \\ &\qquad + e^{\lambda y}|y^2+2y|\,\|g_y\|_1^{(\alpha,\beta)}\end{aligned}$$

and this last expression tends to zero as $y \to 0$ since $g \in H^1$. Thus $f \in H^1(U)$.

The full strength of Proposition 2.15 is not needed immediately, but it is useful when we look at the Helson-Szegö problem. Before we apply it to our investigation of W, we need the following result which is basically the half-plane version of a well-known result of Zygmund [26, p.254].

<u>Lemma 2.16</u> Suppose $s \in L^\infty$ is real and $\|s\|_\infty < \frac{\pi}{2}$. Define g in Π^+ by

$$g(z) = \frac{1}{\pi i}\int \frac{xz+1}{x-z}\, s(x)\,\frac{dx}{1+x^2} \qquad (z \in \Pi^+)$$

Then $\dfrac{e^{ig}}{(i+z)^2} \in H^1$.

Proof Recall from Chapter 1, that $\text{Re}\,g$ is simply the Poisson extension of s to Π^+ and $\lim_{b\downarrow 0} g(a+ib) = s(a) + i\tilde{s}(a)$, by definition. Thus $\|s\|_\infty < \frac{\pi}{2} \Rightarrow |\text{Re}\,g(z)| \leq \|s\|_\infty < \frac{\pi}{2}$. Hence if we define G by

$$G(z) = \text{Re}\,e^{ig(z)} = |e^{ig(z)}| \times \cos \text{Re}\,g(z)$$

then G is a non-negative harmonic function on Π^+, and $|e^{ig(z)}| \leq \frac{G(z)}{\cos\|s\|_\infty}$. By Lemma 1.15, we know that $G((2y+1)i) \leq C(2y+1)$ $(\forall y > 0)$ for some constant C. But

$$\cos\|s\|_\infty \left\| \left[\frac{e^{ig(x)}}{(x+i)^2}\right]_y \right\|_1 \leq \int \frac{G(x+iy)}{x^2+(1+y)^2}\,dx \leq \frac{\pi}{y+1}\, G((2y+1)i)$$

so

$$\|[(x+i)^{-2}e^{ig}]_y\|_1 \leq \pi C\,\frac{2y+1}{y+1} \leq 2\pi C \qquad (\forall y > 0).$$

So $\frac{e^{ig}}{(i+z)^2} \in H^1$, as required.

We are now at last in a position to apply the result described above to the problem of characterising W.

Suppose $w \in W$, $w = |h|^2$, h outer in H^2. By Lemma 2.11, $\frac{w}{1+x^2} \in W$ and $\frac{w}{1+x^2} = |k|^2$ where $k = \frac{h}{i+z}$ is outer in H^2. By Lemma 2.12, given $\varepsilon > 0$ $\exists A \in H^\infty$ and $\lambda > 0$ and a real-valued L^∞-function s with $\|s\|_\infty < \varepsilon$ such that $|\log|A|| < \varepsilon$ and $s + \arg(Ak^2e_{-\lambda}) \equiv 0 (\text{mod}\, 2\pi)$. Define g as in Lemma 2.16, i.e.

$$g(z) = \frac{1}{\pi i}\int \frac{xz+1}{x-z}\, s(x)\,\frac{dx}{1+x^2}$$

Then define f in Π^+ by $f = Ak^2 e_{-\lambda} e^{ig}$. Now $e_\lambda f = (Ah^2)\dfrac{e^{ig}}{(i+z)^2}$, and the functions Ah^2 and $\dfrac{e^{ig}}{(i+z)^2}$ are both in H^1. But we notice that $f \geq 0$ on $\underset{\sim}{R}$ a.e. because

$\arg f = \arg(Ak^2 e_{-\lambda}) + \arg(e^{ig}) = \arg(Ak^2 e_{-\lambda}) + s \equiv 0 (\mathrm{mod}\ 2\pi)$. Now, by definition, $\dfrac{w}{1 + x^2} = |k|^2 = |A|^{-1} f e^{\tilde{s}} = f e^{r+\tilde{s}}$, where $r = -\log|A|$. By Proposition 2.15, f extends analytically across $\underset{\sim}{R}$, and the reflection principle then says that the extension, which we denote by F, is entire. In order to say a little more about F we need the following simple lemma.

Lemma 2.17 Suppose $g \geq 0$ is subharmonic in $\underset{\sim}{C}$, and satisfies

$$\int g(x+iy)\,dx \leq Ke^{\mu|y|} \qquad (y \in \underset{\sim}{R}) \dagger$$

Then g is bounded on $\underset{\sim}{R}$, and $g(z) \leq Ce^{\mu|\mathrm{Im} z|}$ $(z \in \underset{\sim}{C})$.

Proof Suppose $z_0 = x_0 + iy_0$. Then, if $D(z_0)$ is the unit disc centre z_0, we have, by subharmonicity,

$$\pi g(z_0) \leq \iint_{D(z_0)} g(z)\,dx\,dy \leq \int_{y_0-1}^{y_0+1} dy \int_{x_0-\sqrt{1-(y-y_0)^2}}^{x_0+\sqrt{1-(y-y_0)^2}} g(x+iy)\,dx$$

$$\leq K \int_{y_0-1}^{y_0+1} e^{\mu|y|}\,dy \leq Ce^{\mu|y_0|}, \text{ as required.}$$

We can now state one of our main results.

<u>Theorem 2.18</u> Suppose $w \in W$. Then $\forall \varepsilon > 0$, $\exists$ a representation

$$\frac{w}{1+x^2} = f \exp(r+\tilde{s})$$

where (i) $f = F|_{\underset{\sim}{R}}$ is the restriction to $\underset{\sim}{R}$ of an entire function which satisfies

$$|F(z)| \leq Ce^{\lambda|\mathrm{Im} z|} \quad \text{for some} \quad \lambda > 0,$$

f is bounded and non-negative on $\underset{\sim}{R}$, and $f \in L^p(\forall p \geq \frac{1}{2})$. In general, F and λ depend on ε.

and (ii) r and s are real L^∞-functions, depending on $\varepsilon > 0$, with $\|r\|_\infty < \varepsilon$, $\|s\|_\infty < \varepsilon$.

<u>Proof</u> Since $e_\lambda f$ is the product of two H^1-functions, we have that

$$\int |f(x+iy)|^{\frac{1}{2}} dx \leq Ke^{\lambda y} \qquad (\forall y > 0)$$

The reflection principle tells us that $\int |F(x+iy)|^{\frac{1}{2}} dx \leq Ke^{\lambda|y|}$ $(\forall y \in \underset{\sim}{R})$. Now apply Lemma 2.17 to the subharmonic function $g = |F|^{\frac{1}{2}}$, to get that $F(z) \leq Ce^{\lambda|\mathrm{Im} z|}$, in particular, $f = F|_{\underset{\sim}{R}}$ is bounded on $\underset{\sim}{R}$. Since $f^{\frac{1}{2}} \in L^1$ it is immediate that $f \in L^p$ $\forall p \geq \frac{1}{2}$. (ii) is obvious.

<u>**Remark**</u> The construction of F which is described above entails that in Π^+, $F = Ah^2 e_{-\lambda} \dfrac{e^{ig}}{(i+z)^2}$, so it is immediate that $e_\lambda F e^{-ig} \in H^1$. This suggests how we might obtain a converse to Theorem 2.18. In

fact we have the following theorem

Theorem 2.19 Suppose $w \in L^1$, $w \geq 0$ and $\log w \in L^1_c$. Suppose that for each $\varepsilon > 0$ we have a representation $\frac{w}{1+x^2} = f_\varepsilon \exp(r+\tilde{s})$, where

(I) $f_\varepsilon = F_\varepsilon|\mathbb{R}$ for some entire function F_ε satisfying

$|F_\varepsilon(z)| \leq Ce^{\lambda|\text{Im}z|}$ for some $C, \lambda > 0$, depending on ε

and $f_\varepsilon \geq 0$ on $\mathbb{R}$ with $f_\varepsilon \in L^p$ $\forall p \geq \frac{1}{2}$.

(II) r, s are real L^∞-functions, depending on ε, such that $\|r\|_\infty < \varepsilon$, $\|s\|_\infty < \varepsilon$. Then we can conclude that $w \in W$.

Proof The argument falls into two parts; firstly we show that the function $G_\varepsilon(z) = e_\lambda F_\varepsilon e^{-ig}$ is in H^1, $\forall \varepsilon > 0$, and then we deduce that $\frac{w}{1+x^2} \in W$, the result then following by Lemma 2.11 (c).

$G_\varepsilon \in H^1$:- Clearly G_ε is analytic. Also $e_\lambda F_\varepsilon \in H^\infty$ and $\frac{e^{-ig}}{(i+z)^2} \in H^1 \Rightarrow \frac{G_\varepsilon}{(i+z)^2} \in H^1$. So $\exists$ inner j and outer $k \in H^1$ such that $G_\varepsilon = (i+z)^2 jk$. But a.e. on $\mathbb{R}$, $|G_\varepsilon| = |F_\varepsilon e^{\tilde{s}}| = e^{-r}\frac{w}{1+x^2}$.
Now let $w' = e^{-r}\frac{w}{1+x^2}$. Then $w' \in L^1$ and $\log w' = -r + \log w - \log(1+x^2) \in L^1_c$, so $\exists$ h' outer in H^1 such that $w' = |h'|$. But then we have a.e. on $\mathbb{R}$

$$|k| = \frac{|G_\varepsilon|}{1+x^2} = \frac{e^{-r}w}{(1+x^2)^2} = \left|\frac{h'}{(i+x)^2}\right|$$

Now k and $\frac{h'}{(i+x)^2}$ are both outer in H^1, so by uniqueness modulo a constant, $\exists \gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that $k = \frac{\gamma h'}{(i+z)^2}$, and so $G_\varepsilon = (i+z)^2 jk = \gamma j h' \in H^1$.

$\frac{w}{1+x^2} \in W$. We use Lemma 2.12. Again let $w' = \frac{e^{-r}w}{1+x^2}$, and G_ε be as above, so that $G_\varepsilon = jh'$ for some outer $h' \in H^1$ by the first part, and $w' = |h'|$. Now $|j| = 1$ a.e. on $\underset{\sim}{R}$, and

$$|\arg jh'e_{-\lambda}| = |\arg G_\varepsilon e_{-\lambda}| = |\arg f_\varepsilon e^{-i(s+i\tilde{s})}| = |-s| < \varepsilon.$$

Since $e^r > 0 \in \text{Inv}(L^\infty)$ (the invertible elements of L^∞), $\exists B \in \text{Inv}(H^\infty)$, B outer such that $e^r = |B|$ a.e.. Of course, $\frac{w}{1+x^2} = e^r w' = |Bh'|$, and Bh' is outer. Now put $\Gamma = \frac{j}{B} \in H^\infty$. Then, a.e. on $\underset{\sim}{R}$, we have

$$|\log|\Gamma|| = |-\log|B|| = |-r| < \varepsilon, \quad \text{and} \quad |\arg\Gamma(Bh')e_{-\lambda}| = |\arg jh'e_{-\lambda}| < \varepsilon.$$

Thus, for each $\varepsilon > 0$, $\exists\Gamma \in H^\infty$ and $\lambda > 0$ satisfying the condition of Lemma 2.12 with respect to the function $\frac{w}{1+x^2}$, and so $\frac{w}{1+x^2}$, hence w, is in W.

Theorems 2.18 and 2.19 together give necessary and sufficient conditions for w to belong to W. They are the main result of this chapter together with a similar representation theorem for W_λ which we present shortly (Theorem 2.24). The situation is similar to that which obtained prior to the 1972 paper of Sarason for weight functions on the unit circle. Unfortunately for our purposes, the method of his paper, which removes the dependence on ε and ties in W with the set of so-called functions of vanishing mean oscillation, VMO, is not immediately applicable. The difficulty is in deciding how much the entire function F_ε can vary as $\varepsilon \to 0$. On the circle, the corresponding problem concerns only polynomials $|P_\varepsilon|^2$ and Sarason, together with Helson, was able to deduce that $P_\varepsilon = PQ_\varepsilon$ where P has zeros only on Γ, and does not depend on ε, and Q_ε has zeros only off Γ. Then we only need look at functions of the form $e^{r+\tilde{s}+t}$ where $t = 2\log|Q_\varepsilon|$ is a continuous function on Γ.

We have not been able to carry this through completely for the line, the difficulty being in the properties of entire functions. However, we do have the following which says that the entire functions F_ε and F_η occurring in two different representations must have the same real zeros.

Lemma 2.20 Suppose $w \in W$ and f_1, f_2 are two functions with the property (I) of Theorem 2.19 which occur in representations of $\frac{w}{1+x^2}$, i.e.

$$\frac{w}{1+x^2} = f_1 e^{r_1+\tilde{s}_1} = f_2 e^{r_2+\tilde{s}_2}$$

and suppose that $\|s_1\|_\infty < \frac{\pi}{4}$, $\|s_2\|_\infty < \frac{\pi}{4}$.

Then f_1 and f_2 must have the same real zeros.

Proof Notice that the assumption $\|s_1\|_\infty < \frac{\pi}{4}$, $\|s_2\|_\infty < \frac{\pi}{4}$ is no real restriction.

Let $W = \frac{w}{1+x^2}$. Then $\left(\frac{W}{f_1}\right)^2 = e^{2r_1+2\tilde{s}_1} \in L^1{}_C$, by Lemma 2.16, and similarly $\left(\frac{f_2}{W}\right)^2 \in L^1{}_C$. Thus

$$\int \frac{f_2}{f_1}\frac{dx}{1+x^2} \le \left[\int \left(\frac{f_2}{W}\right)^2 \frac{dx}{1+x^2}\right]^{\frac{1}{2}} \left[\int \left(\frac{W}{f_1}\right)^2 \frac{dx}{1+x^2}\right]^{\frac{1}{2}} \quad (\dagger).$$

From $(\dagger)$ we deduce that every zero of f_1 is a zero of f_2; for suppose $f_1(x) = 0$ and $f_2(x) \neq 0$. Since f_2 is continuous on $\underset{\sim}{R}$, $\exists$ a neighbourhood $(x-\delta,x+\delta)$ on which $f_2(y) \geq \varepsilon > 0$. Since f_1 is differentiable at x, the mean-value theorem gives that $f_1(y) = f_1'(\xi)(y-x)$ for some ξ between x and y, and the continuity of f_1' says that $|f_1'(\xi)| \leq K$ for every ξ $(x-\delta,x+\delta)$. Thus

$$\int \frac{f_2}{f_1}\frac{dx}{1+x^2} \geq \frac{1}{1+(|x|+\delta)^2}\int_{x-\delta}^{x+\delta}\frac{f_2}{f_1}\,dy \geq \frac{\varepsilon}{K(1+(|x|+\delta)^2)}\int_{x-\delta}^{x+\delta}\frac{dy}{|y-x|} = \infty,$$

since $\int_0^\delta \frac{dx}{x} = \infty \quad \forall\delta > 0$. Thus (†) is contradicted and we have the result.

The class of entire functions F which can occur in the representations given above is obviously quite special. In fact they are of exponential type and class A as is clear from Theorem 1.20 of Chapter 1, and so they have a representation in the form $F(z) = cz^m e^{iaz} \lim_{R\downarrow\infty} \prod_{|z_k|<R} \left(1 - \frac{z}{z_k}\right)$. The fact that $F(x) \geq 0$, $\forall x \in \mathbb{R}$ implies that m is even, all the real zeros of F are of even multiplicity, and $a = 0$. Let us now summarise our representation theorem in one convenient form.

<u>Theorem 2.21</u> Suppose $w \in L^1$, $w \geq 0$ and $\log w \in L^1_{\mathbb{C}}$. Then $w \in W$ iff $\forall\varepsilon > 0$ $\exists$ a representation

$$\frac{w}{1+x^2} = |g_\varepsilon|^2 \exp(r+\tilde{s})$$

where (i) r, s are real L^∞-functions with $\|r\|_\infty < \varepsilon$, $\|s\|_\infty < \varepsilon$ and (ii) g_ε is the restriction to $\mathbb{R}$ of an entire function of exponential type of class A [illegible] such that

$g_\varepsilon \in L^p \ \forall p \geq 1$. Also, for $x \in \mathbb{R}$,

$$g_\varepsilon(x) = cx^k \prod_{n=1}^{\infty} \left(1 - \frac{x}{\gamma_n}\right) \prod_{m=1}^{\infty} \left(1 - \frac{x}{a_m}\right)$$

where $\{\gamma_n\}$ are the real roots of g_ε, and $\{a_m\}$ are the non-real roots, both counted according to multiplicity. Of course we may assume $\operatorname{Im} a_m > 0 \ \forall m$.

Of course, the set $\{\gamma_n\}$ is not dependent on ε, but we cannot necessarily say the same for the set $\{a_m\}$.

Before we look further at W, we need to know a little about functions of bounded mean oscillation, BMO and this we do in Chapter 3. Firstly however we derive a characterisation of W_λ for fixed $\lambda > 0$ in a similar way. We shall obtain a complete analogue of the Helson-Szegö result of 1960 as extended by Helson and Sarason in 1967. In this case also there is a connection with BMO. The first step is a lemma like 2.12.

<u>Lemma 2.22</u> Suppose $w \in L^1$, $w \geq 0$ and $\log w \in L^1_c$. Then $w \in W_\lambda$ iff $\exists \varepsilon > 0$ and $A \in H^\infty$ such that a.e. on $\mathbb{R}$

$$|A| \geq \varepsilon \quad \text{and} \quad |\arg Ah^2 e_{-\lambda}| < \frac{\pi}{2} - \varepsilon \qquad (*)$$

<u>Proof</u> $(\Rightarrow)$ This is similar to the first part of Lemma 2.12 and is omitted.

$(\Leftarrow)$ Suppose $\varepsilon > 0$ and $A \in H^\infty$ satisfies $(*)$. Let $c > 0$ be a constant. We have, a.e. on $\mathbb{R}$,

$|e^{-2i\phi} - e_{-\lambda} cA|^2 = 1 + c|A|(c|A| - 2\cos \arg Ah^2 e_{-\lambda}) < 1 + c|A|(c|A| - 2\sin\varepsilon)$

Put $c = \frac{\sin \varepsilon}{\|A\|_\infty}$. Then $|e^{-2i\phi} - e_{-\lambda} cA|^2 < 1 - \frac{\varepsilon \sin^2 \varepsilon}{\|A\|_\infty} < 1$. Thus $\rho_\lambda = \inf_{B \in H^\infty} \|e^{-2i\phi} - e_{-\lambda} B\|_\infty < 1$ and $w \in W_\lambda$.

<u>Corollary 2.23</u> Suppose $w \in W_\lambda$ and $k > 0 \in \mathrm{Inv}(L^\infty)$. Then $kw \in W_\lambda$.

<u>Proof</u> Suppose $\rho_\lambda(w) < 1$. Then by Lemma 2.22, $\exists \varepsilon > 0$ and $A \in H^\infty$ satisfying (*). Since $k > 0$ and is invertible in L^∞, $\exists$ an outer function $B \in \mathrm{Inv}(H^\infty)$ such that $k = |B|$ a.e. on $\underset{\sim}{R}$. Let $\Gamma = \frac{A}{B}$. Then $\Gamma \in H^\infty$ and if $h_0{}^2 = Bh$ so that $kw = |h_0|^2$, h_0 outer in H^2, we have $|\Gamma| = \frac{|A|}{|B|} \geq \frac{\varepsilon}{\|k\|_\infty}$, and

$$|\arg \Gamma h_0{}^2 e_{-\lambda}| = |\arg A h^2 e_{-\lambda}| < \frac{\pi}{2} - \varepsilon$$

Thus $kw \in W_\lambda$.

<u>Remark</u> A careful look at the proof of Lemma 2.22 suggests that one can obtain an inequality for $\rho_\lambda(kw)$ in terms of $\rho_\lambda(w)$; in fact, some crude estimation will give the result that

$$\rho_\lambda{}^2(kw) \leq 1 - \frac{4\gamma\delta\mu^2}{(\gamma+\delta)^2}$$

where $\gamma = \frac{1 - \rho_\lambda(w)}{\|k\|_\infty}$, $\delta = \|k^{-1}\|_\infty (1 + \rho_\lambda(w))$, and $\mu = \frac{1 - \rho_\lambda(w)}{1 + \rho_\lambda(w)}$.

We are now in a position to characterise W_λ as we did W. Notice that everything in W is eventually in W_λ for some λ.

<u>Theorem 2.24</u> Suppose $w \in L^1$, $w \geq 0$ and $\log w \in L^1_{\mathfrak{C}}$. Then $w \in W_\lambda$ iff $\exists$ a representation

$$w = f \exp(r+\tilde{s})$$

where, (i) $f = F|_{\mathbb{R}}$ is the restriction of an entire function $F(z)$ which satisfies

$$|F(z)| \leq C|i+z|^2 e^{\lambda \operatorname{Im} z} \qquad (z \in \Pi^+) \quad (1)$$

f being non-negative a.e. on $\mathbb{R}$,

and (ii) r, s are real L^∞-functions satisfying $\|s\|_\infty < \frac{\pi}{2}$.

<u>Proof</u> ($\Rightarrow$) Suppose $w \in \mathcal{W}_\lambda$. By Lemma 2.22, $\exists \varepsilon > 0$ and $A \in H^\infty$ such that a.e. on $\mathbb{R}$

$$|A| \geq \varepsilon \quad \text{and} \quad |\arg Ah^2 e_{-\lambda}| < \frac{\pi}{2} - \varepsilon.$$

As in the case of $\mathcal{W}$, choose s real such that $s + \arg Ah^2 e_{-\lambda} \equiv 0 (\operatorname{mod} 2\pi)$, so that $\|s\|_\infty \leq \frac{\pi}{2} - \varepsilon$. Define F in Π^+ by $F = Ah^2 e_{-\lambda} e^{ig}$ where $g(z) = \frac{-1}{\pi i} \int \frac{xz+1}{x-z} s(x) \frac{dx}{1+x^2}$. Then F is analytic in Π^+ and, on $\mathbb{R}$, $\arg F = s + \arg Ah^2 e_{-\lambda} \equiv 0 (\operatorname{mod} 2\pi)$, so that $f = F|_{\mathbb{R}}$ is non-negative a.e.. Also $F = Ah^2 e_{-\lambda} e^{ig}$ and the functions $\tilde{A}h^2$ and $\frac{e^{ig}}{(i+z)^2}$ are in H^1, so, by Proposition 2.15, F extends analytically across $\mathbb{R}$, and the reflection principle says that F is entire satisfying $F(\bar{z}) = \overline{F(z)}$. We have

$$w = |h|^2 = f . |A|^{-1} |e^{-ig}| = f e^{r+\tilde{s}}$$

where $r = -\log|A| \in L^\infty$. It remains to prove that

$|F(z)| \leq C|i+z|^2 e^{\lambda \text{Im} z}$ $(z \in \Pi^+)$. This is not quite as straightforward as the corresponding result for W, and we state it as a separate result:-

Proposition 2.25 Suppose F is entire, real on $\underset{\sim}{R}$, and satisfies

$$\frac{e_\lambda F}{(i+z)^2} = \alpha\beta \quad \text{for} \quad \alpha, \beta \in H^1 \tag{2}$$

Then F satisfies (1) for some constant $C > 0$.

Proof By (2),

$$\int \frac{|F(x+iy)|^{\frac{1}{2}}}{|i+z|}\, dx \leq K e^{(\lambda/2)y} \qquad (y = \text{Im} z \geq 0) \tag{3}$$

Now $\frac{F}{(i+z)^2}$ is analytic in Π^+, so $g(z) = \frac{|F(z)|^{\frac{1}{2}}}{|i+z|}$ is subharmonic in Π^+. Thus for $y_0 \geq 1$, we may apply Lemma 2.17 to get $\frac{\pi|F(z_0)|^{\frac{1}{2}}}{|i+z_0|} \leq C_0 e^{(\lambda/2)y_0}$ and so $|F(z_0)| \leq C_1 e^{\lambda y_0}|i+z_0|^2$ $(z_0 = x_0 + iy_0,\ y_0 \geq 1)$. Notice that the conclusion of Lemma 2.17 is valid since when considering a unit disc round z_0 we remain in Π^+ and (3) is valid there.

To get (1) for $0 \leq \text{Im} z_0 \leq 1$ we must be more careful. Suppose $y_0 \in [0,1)$. Then the unit disc centre z_0 lies inside the strip $|\text{Im} z| \leq 2$. Since $F(\bar{z}) = \overline{F(z)}$ we must have, corresponding to (3), $\int \frac{|F(x+iy)|^{\frac{1}{2}}}{|i-z|} \leq Ke^{-(\lambda/2)y}$ $(y \leq 0)$ so that, in general, $\int \frac{|F(x+iy)|^{\frac{1}{2}}}{[x^2+(1+|y|)^2]^{\frac{1}{2}}}\, dx \leq Ke^{(\lambda/2)|y|}$ (4) $(z \in \underset{\sim}{C})$. Now, for $|\text{Im} z| \leq 2$, z in the unit ball round z_0, we have

$$x^2 + (1+|y|/2)^2 \leq 3 + |x_0|^2$$

and so, since $|F|^{\frac{1}{2}}$ is subharmonic in $\underset{\sim}{C}$, we have

$$\pi|F(z_0)|^{\frac{1}{2}} \leq \int_{y_0-1}^{y_0+1} dy \int_{x_0-\sqrt{1-(y-y_0)^2}}^{x_0+\sqrt{1-(y-y_0)^2}} |F(x+iy)|^{\frac{1}{2}}dx \leq [3+|x_0|^2]^{\frac{1}{2}} K \int_{y_0-1}^{y_0+1} e^{(\lambda/2)|y|}dy$$

$$\leq K'(3+|x_0|^2)^{\frac{1}{2}}e^{(\lambda/2)|y_0|}$$

whence we get

$$|F(z_0)| \leq C'(3+|x_0|^2)e^{\lambda y_0} \qquad (0 \leq y_0 \leq 1)$$

Thus $\forall z \in \Pi^+$, $|F(z_0)| \leq C|i+z|^2e^{\lambda \text{Im} z}$ proving the proposition.

This proves the implication ($\Rightarrow$) of Theorem 2.24.

($\Leftarrow$) Suppose there is a representation $w = f \exp(r+\tilde{s})$ of the stated form. We proceed as in the proof of Theorem 2.19. We show that $G = e_\lambda F e^{-ig} \in H^1$ and deduce that $w' = e^{-r}w \in W_\lambda$.

Now G is analytic, $\frac{e_\lambda F}{(i+z)^2} \in H^\infty$ by hypothesis, and $\frac{e^{-ig}}{(i+z)^2} \in H^1$ (by Lemma 2.16 applied to $-s$), so that we have $\frac{G}{(i+z)^4} \in H^1$. Thus $\exists$ inner j, outer $k \in H^1$ such that $G = (i+z)^4jk$. Again $w' = |h'|$ for some h' outer in H^1 and, a.e. on $\underset{\sim}{R}$,

$$|k| = \frac{|G|}{(1+x^2)^2} = \frac{w'}{(1+x^2)^2} = \left|\frac{h'}{(i+z)^4}\right|$$

Now h' outer $\Rightarrow \frac{h'}{(i+z)^4}$ is outer, and so the uniqueness of outer functions up to constants of modulus 1 allows us to conclude that $k = \frac{\gamma h'}{(1+z)^4}$ for $\gamma \in \underset{\sim}{C}$ with $|\gamma| = 1$. Thus $G = (i+z)^4jk = \gamma jh' \in H^1$.

To see that $w' \in W_\lambda$ just put $A = \gamma j$ so that $A \in H^\infty$, $|A| = 1$ a.e. on $\underset{\sim}{R}$ and $\arg Ah'e_{-\lambda} = \arg Ge_{-\lambda} = -s$, and the result follows by Lemma 2.22.

CHAPTER 3

The Spaces BMO and VMO

In this chapter we study the spaces of functions of bounded mean oscillation, BMO and vanishing mean oscillation, VMO. These can be defined on both Γ and $\mathbb{R}$. On the circle they arise naturally when considering the Helson-Sarason and Helson-Szegö conditions for a discrete stationary process, and our original motivation for studying them was to try and improve our representation theorem of W (Theorem 2.21) on $\mathbb{R}$ by removing the dependence on ε. While we have not been completely successful, we shall present in this chapter some results on BMO and VMO, both on Γ and $\mathbb{R}$, and we shall establish a connection between W and these spaces. We also prove a result connecting the Helson-Szegö condition $\rho_\lambda < 1$ and the boundedness of the conjugation operator on a subspace of $L^2(\mu)$ depending on $\lambda > 0$. The main results are Theorems 3.9 and 3.16.

Notation For a measurable subset E of $\mathbb{R}$, we let $|E|$ denote its Lebesgue measure. $L^1_{loc}(\mathbb{R})$ will denote the set of locally integrable functions $f: \mathbb{R} \to \mathbb{C}$, i.e. functions for which $\int_I |f|\,dx$ exists for every finite interval $I \subseteq \mathbb{R}$, dx, as usual denoting Lebesgue measure on $\mathbb{R}$.

Definition 3.1 Suppose $f \in L^1_{loc}(\mathbb{R})$ and I is a finite interval. Let f_I denote the average of f over I, i.e. $f_I = \frac{1}{|I|}\int_I f(x)dx$. Define, for $a > 0$, $M_a(f)$ by

$$M_a(f) = \sup_{|I|\leq a} \frac{1}{|I|} \int |f-f_I| dx.$$

Define $\|\cdot\|_* = \sup_{a>0} M_a(f)$, $M_0(f) = \lim_{a\to 0} M_a(f)$. We say that f is of bounded mean oscillation, $f \in BMO$, if $\|f\|_* < \infty$. If $f \in BMO$, and $M_0(f) = 0$, we say that f is of vanishing mean oscillation, VMO.

Now if we identify functions which differ by a constant, then BMO becomes a Banach space under the norm $\|\cdot\|_*$, and it is straightforward to prove that VMO is a closed subspace. We shall be interested in both VMO and BMO, and indeed in certain other subsets of BMO. It is sometimes useful to think of VMO sitting inside BMO in the same way as the continuous functions sit inside L^1_{loc}. BMO was first introduced by John and Nirenberg [17], and later work has been done by Fefferman and Stein [5], and Sarason [24], although the latter focuses attention more on VMO. The paper of Fefferman and Stein [5] is concerned with deriving a real-variable theory of H^p spaces, and the space BMO crops up as the dual of their version of H^1. We shall need a result from this paper (Theorem 3.3 below).

We now state some results from these various papers which we shall need. No proofs will be given.

<u>Theorem 3.2</u> Suppose $I \subseteq \mathbb{R}$ is a finite interval, and $f \in BMO$. Let E_σ be defined for $\sigma > 0$ by $E_\sigma = \{x \in I: |f-f_I| > \sigma\}$. Then $\exists$ positive constants c_1, c_2, not depending on I, such that

$$|E_\sigma| \leq c_1 e^{-c_2\sigma/\|f\|_*}|I| \qquad \forall\sigma > 0$$

<u>Proof</u> John and Nirenberg [17, p.415].

Theorem 3.3 (a) If H is the map taking an L^∞-function g to its conjugate function then $\exists$ a constant A such that $\|Hg\|_* \leq A\|g\|_\infty$ i.e. H is a bounded map from L^∞ into BMO.

(b) For each $f \in$ BMO, $\exists$ functions $r, s \in L^\infty$ with $f = r + \tilde{s}$, and moreover, $\exists$ a constant B such that $\|r\|_\infty \leq B\|f\|_*$ and $\|s\|_\infty \leq B\|f\|_*$. B does not depend on f, r or s.

Proof Fefferman and Stein [5].

Remarks (i) The set E_σ of Theorem 3.2 will be met again when we give necessary and sufficient conditions for a positive function f to satisfy $\log f \in$ VMO.

(ii) Notice that since functions differing by a constant are identified in BMO, it is irrelevant whether we use the conjugate function or Hilbert transform in Theorem 3.3 (a), except of course that the latter is only defined for $g \in L^1$.

Notation (i) For $y > 0$, we let T_y denote the translation operator $(T_y f)(x) = f(x-y)$.

(ii) Let UC denote the space of uniformly continuous functions on $\underset{\sim}{R}$, and BUC, as before, the space of bounded functions in UC.

The following result is proved in Sarason [24, Theorem 1].

Theorem 3.4 Suppose $f \in$ BMO. The following are equivalent:-

(i) $f \in$ VMO

(ii) f is in the BMO closure of UC $\cap$ BMO

(iii) $\|T_y f - f\|_* \to 0$ as $y \to 0$

(iv) $f = u + \tilde{v}$, where u, v are in BUC, and, as usual, $\tilde{v}$ denotes the conjugate function of v.

Remarks (i) The proof of Theorem 3.4 is based on Fourier transform techniques and the result of Theorem 3.3 (b), and the fact that BMO is a homogeneous Banach space.

(ii) It follows from Lemma 2.6 that $w \geq 0$, $w \in L^1$, $\log w \in \text{VMO} \Rightarrow w \in W$, so we immediately have a connection between W and VMO. We shall give necessary and sufficient conditions for $f \geq 0$ to satisfy $\log f \in \text{VMO}$ in Theorem 3.9.

(iii) Obviously we can define BMO, VMO etc. on the circle, and Theorems 3.2 and 3.3 are unchanged, the place of $L^1_{loc}(\mathbb{R})$ being taken by $L^1(\Gamma)$, of course. Theorem 3.4 is also unchanged except that UC and BUC are both replaced by C, the space of continuous functions on Γ. Also, the hypothesis that $f \in \text{BMO}$ is redundant as the following lemma shows, via a covering argument.

Lemma 3.5 Let I_0 be a finite interval. Suppose $f \in L^1(I_0)$ and $M_0(f) = 0$. Then $\|f\|_* = \sup_{I \subseteq I_0} \frac{1}{|I|} \int_I |f - f_I| dx < \infty$.

Proof Since $M_0(f) = 0$, $\exists \delta > 0$ such that $|I| < \delta \Rightarrow \frac{1}{|I|} \int_I |f - f_I| < 1$. Partition I_0 into n disjoint equal intervals $I_1, \ldots, I_n$ with $n > \frac{1}{\delta}$. We may assume $|I_0| = 1$

Let $M = \max_{1 \leq i, j \leq n} |f_{I_i} - f_{I_j}|$, and $K = 3(3+4M)$. Suppose $I \subseteq I_0$. If $|I| \leq \frac{1}{n}$, then $\frac{1}{|I|} \int_I |f - f_I| dx < 1$. If $|I| > \frac{1}{n}$, we can choose $I_1, \ldots, I_k$ (relabelling if necessary) such that $I \subseteq \bigcup_{j=1}^{k} I_j$ and $\sum_{j=1}^{k} |I_j| \leq 3|I|$. Now we have

$$\frac{1}{|I|}\int_I |f-f_I|dx \le \frac{1}{|I|}\sum_{j=1}^{k}\int_{I_j} |f-f_I|dx$$

$$\le \frac{1}{|I|}\sum_{j=1}^{k}\int_{I_j} |f-f_{I_j}|dx + \frac{1}{|I|}\sum_{j=1}^{k}|I_j||f_{I_j}-f_I|$$

$$\le 3 + \frac{1}{|I|}\sum_{j=1}^{k}|I_j||f_{I_j}-f_I|$$

But $|f_{I_j}-f_I| \le \frac{1}{|I|}\int_I |f_{I_j}-f|dx \le \frac{1}{|I|}\sum_{i=1}^{k}\int_{I_i} |f_{I_j}-f|dx$

$$\le \frac{1}{|I|}\sum_{i=1}^{k}\left(\int_{I_i} |f_{I_j}-f_{I_i}|dx + \int_{I_i} |f_{I_i}-f|dx\right)$$

$$< \frac{1}{|I|}\sum_{i=1}^{k}(|I_i|\times(M+1)) \le 3(M+1).$$

Thus $\frac{1}{|I|}\int_I |f-f_I|dx < 3 + 9(M+1) = K < \infty$.

In particular, on the circle for an L^1-function f we have $f \in VMO \Leftrightarrow M_0(f) = 0$ which shows that $C \subseteq VMO(\Gamma)$. In the case of the line, since it is obvious from the definition that $L^\infty \subseteq BMO$, we see that VMO contains all uniformly continuous functions in BMO, hence also BUC as Theorem 3.4 said. Of course, for $f \in L^1_{loc}(\underset{\sim}{R})$ we can have $M_0(f) = 0$ and $\|f\|_* = \infty$, e.g. $f(x) = x$ is in UC but not in BMO. BMO does, however, contain unbounded functions as Theorem 3.3 indicates; for example the function $f(x) = \log|x|$ is in BMO. Notice that if $f \in L^1(\underset{\sim}{R})$ then $M_0(f) = 0 \Rightarrow f \in VMO$.

Lemma 3.6 For $f \in L^1(\underset{\sim}{R})$, $M_0(f) = 0 \Leftrightarrow f \in VMO$.

Proof ($\Rightarrow$) Choose δ so that $|I| < \delta \Rightarrow \frac{1}{|I|}\int_I |f-f_I|dx < 1$. If $|I| \ge \delta$, then since $|f_I| \le \frac{\|f\|_1}{|I|}$, we have

$$\frac{1}{|I|}\int_I |f-f_I|\,dx \leq \frac{2\|f\|_1}{|I|} \leq \frac{2\|f\|_1}{\delta}$$

Thus $\sup_{I\subseteq R} \frac{1}{|I|}\int_I |f-f_I|\,dx \leq \max(1, \frac{2\|f\|_1}{\delta})$, so $\|f\|_* < \infty$. The other implication is trivial.

As we indicated in the remark preceding Lemma 3.5, it will be useful to have a criterion for judging when $\log f \in VMO$ for a given positive function $f \in L^1_{loc}(R)$. First we need a preliminary measure theoretic result.

Proposition 3.7 Suppose $f > 0$ is integrable on the finite interval I. Let $s > 0$ and define $E = \{x \in I: |\log f| > s\}$, $F = \{x \in I: |\log f| > 1 + s\}$. Suppose $J \subseteq I$ is an interval such that

$$|J\setminus E| \geq \frac{|J|}{e^2}, \qquad |J\cap F| \geq \frac{|J|}{e^3}.$$

Then $\inf_{a>0} \frac{1}{|J|}\int_J |a^{-1}f-1|\,dx \geq \frac{e-1}{e^4}$.

Proof Let $\lambda(a) = \frac{1}{|J|}\int_J |a^{-1}f-1|\,dx$. We consider five separate cases

(I) $a \in (0,e^{-(1+s)})$

On $J \setminus E$, $e^{-s} \leq f \leq e^s$, so $|a^{-1}f-1| \geq \frac{e^{-s}-a}{a} \geq \frac{e^{-s}-e^{-(1+s)}}{e^{-(1+s)}}$.
Thus $\lambda(a) \geq (e-1)\frac{|J\setminus E|}{|J|} \geq \frac{e-1}{e^2}$.

(II) $a \in [e^{-(1+s)},e^{-s})$

Again, on $J \setminus E$, $|a^{-1}f-1| \geq \frac{e^{-s}-a}{a}$.

$$\left.\begin{array}{l}\text{If } f < e^{-(1+s)}, \text{ then } |a^{-1}f-1| \geq \dfrac{a-e^{-(1+s)}}{a} \\ \text{If } f > e^{1+s}, \text{ then } |a^{-1}f-1| \geq \dfrac{e^{1+s}-a}{a}\end{array}\right\}$$

Now notice that

$$\frac{e^{1+s}-a}{a} - \frac{a-e^{-(1+s)}}{a} = \frac{e^{1+s}+e^{-(1+s)}}{a} - 2 \geq \frac{e^{1+s}+e^{-(1+s)}}{e^{-s}} - 2 > 0$$

for a in the range $[e^{-(1+s)}, e^{-s})$. Thus, on $J \cap F$,

$|a^{-1}f-1| \geq \frac{a-e^{-(1+s)}}{a}$. So we have

$$\lambda(a) \geq \frac{e^{-s}-a}{a}\frac{|J\setminus E|}{|J|} + \frac{a-e^{-(1+s)}}{a}\frac{|J\cap F|}{|J|} \geq \frac{e^{1-s} - ae + a - e^{-(1+s)}}{e^3 a}$$

$\geq \frac{e-1}{e^4}$, since the last function above has a minimum when $a = e^{-s}$.

(III) $a \in [e^{-s}, e^{s}]$

Again, $f < e^{-(1+s)} \Rightarrow |a^{-1}f-1| \geq \frac{a-e^{-(1+s)}}{a}$ and $f > e^{1+s} \Rightarrow |a^{-1}f-1| \geq \frac{e^{1+s}-a}{a}$. As in case (II),

$\frac{e^{1+s}-a}{a} - \frac{a-e^{-(1+s)}}{a} \geq \frac{e^{1+s}+e^{-(1+s)}}{e^{s}} - 2 > 0$. So, on $J \cap F$,

$|a^{-1}f-1| \geq \frac{a-e^{-(1+s)}}{a}$, and $\lambda(a) \geq \frac{a-e^{-(1+s)}}{a}\frac{|J\cap F|}{|J|}$

$$\geq \frac{e-1}{e^4}.$$

(IV) $a \in (e^{s}, e^{s+1}]$

On $J \setminus E$, $|a^{-1}f-1| \geq \frac{a-e^{s}}{a}$. Let $F(a) = \frac{a-e^{s}}{a}$, $a \in (e^{s}, e^{s+1}]$.

$$\left.\begin{aligned} f < e^{-(1+s)} &\Rightarrow |a^{-1}f-1| \geq \frac{a-e^{-(1+s)}}{a} \\ f > e^{1+s} &\Rightarrow |a^{-1}f-1| \geq \frac{e^{1+s}-a}{a} \end{aligned}\right\} \text{so, on } J \cap F, \text{ we have}$$

$|a^{-1}f-1| \geq G(a) = \min\left(\frac{a-e^{-(1+s)}}{a}, \frac{e^{1+s}-a}{a}\right)$. Thus

$$\lambda(a) \geq \min_{a\in(e^{s},e^{s+1}]}\left\{\frac{F(a)}{e^2} + \frac{G(a)}{e^3}\right\} = \frac{1}{e^3}\min_{a\in(e^{s},e^{s+1}]}\{eF(a)+G(a)\}.$$

If we consider the graph of $eF(a) + G(a)$ in the given range $a \in (e^{s}, e^{s+1}]$ as in Figure 1 below, it is easy to check that the minimum occurrs when $a = e^{s}$ so we have

$$\lambda(a) \geq \frac{e^{s} - e^{-(1+s)}}{e^{3+s}} = \frac{e - e^{-2s}}{e^{4}} \geq \frac{e - 1}{e^{4}} .$$

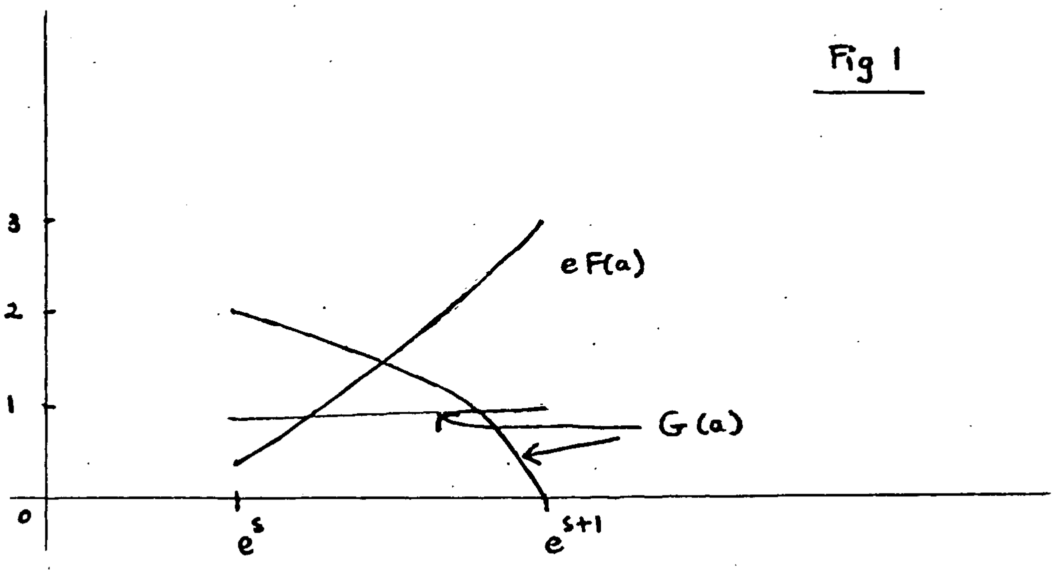


Fig 1

(V) $a \in (e^{s+1}, \infty)$

On $J \setminus E$, $|a^{-1}f-1| \geq \frac{a - e^{s}}{a} \geq \frac{e - 1}{e}$, so $\lambda(a) \geq \frac{e - 1}{e^{3}}$.

This completes the proof that $\inf_{a>0} \lambda(a) \geq \frac{e - 1}{e^{4}}$

Before we use this result to get necessary and sufficient conditions we need to state the Calderon-Zygmund Lemma. This lemma provides a decomposition of a finite interval I in terms of a given function f integrable on I in just the right form for our purposes. It was used by John and Nirenberg in their original paper on BMO for a similar purpose.

Lemma 3.8 Let u be integrable over some finite interval I, and let s be given such that

$$\frac{1}{|I|} \int_{I} |u| \, dx \leq s.$$

Then, $\exists$ a countable number of disjoint open intervals $J_k \subset I$ such that

a) $|u| \leq s$ a.e. in $I \setminus \bigcup_k J_k$

b) $s \leq \frac{1}{|J_k|} \int_{J_k} |u|\,dx \leq 2s.$

For a proof see John and Nirenberg [17, p.418-419].

Theorem 3.9 Suppose $f \in L^1_{loc}(\underset{\sim}{R})$, $f > 0$ a.e.. Then $M_0(\log f) = 0 \Leftrightarrow \forall\varepsilon > 0 \;\; \exists\delta > 0$ such that

$|I| < \delta \Rightarrow \exists a_I > 0$, depending on I, such that $\frac{1}{|I|} \int_I |a_I^{-1} f - 1|\,dx < \varepsilon$

Proof ($\Rightarrow$) Suppose $M_0(\log f) = 0$. Let $g = \log f$. Since $M_0(g) = 0$, given $\eta > 0 \;\; \exists\delta > 0$ such that $|I| < \delta \Rightarrow \frac{1}{|I|} \int_I |g - g_I|\,dx < \eta$. It follows from Theorem 3.2 that, if $E_\sigma = \{x \in I: \; |g - g_I| > \sigma\}$, then $\exists$ constants c_1, c_2, depending only on g and not on I, such that $|E_\sigma| \leq c_1 e^{-c_2 \sigma/\eta} |I|$. Let $a = \exp g_I$. Then $|a^{-1} f - 1| \leq \exp(|g - g_I|) - 1$, as is easily checked.

So $\int_I |a^{-1} f - 1|\,dx \leq \int_I [\exp(|g - g_I|) - 1]\,dx$

$$= \int_0^\infty |E_s|\,dF(s)$$

where $F(s) = e^s - 1$ is continuously differentiable and vanishes at 0. (see John and Nirenberg [17, p.415].)

$$\text{But} \int_0^\infty |E_s|\,dF(s) \leq c_1 |I| \int_0^\infty e^{-c_2 s/\eta} e^s\,ds$$

$$= c_1 |I| \int_0^\infty e^{(1 - (c_2/\eta))s}\,ds$$

$$= \frac{c_1 |I|}{c_2/\eta - 1} = \frac{c_1 \eta |I|}{c_2 - \eta}.$$

Now, if we choose η so small that $\frac{c_1 \eta}{c_2 - \eta} < \varepsilon$, then we get that

$$|I| < \delta \Rightarrow \frac{1}{|I|} \int_I |a^{-1}f-1|\,dx < \varepsilon, \quad \text{as required.}$$

($\Leftarrow$) Suppose $\eta > 0$. Let $\eta_1 = \min(\eta, \frac{3}{e^3})$. Fix $s \in (0, \frac{\eta}{2})$ and suppose $\varepsilon < \min(\frac{\eta_1}{3}(1-e^{-s}), \frac{e-1}{e^4})$. Then, by hypothesis, $\exists \delta > 0$ such that $|I| < \delta \Rightarrow \exists a_I > 0$ such that $\frac{1}{|I|} \int_I |a_I^{-1}f-1|\,dx < \varepsilon$. Choose I with $|I| < \delta$. Without loss of generality we may assume that $a_I = 1$ (otherwise replace f by $a_I^{-1}f$). Then we have

$$\frac{1}{|I|} \int_I |f-1|\,dx < \varepsilon \tag{1}$$

From (1), $\varepsilon > \frac{1}{|I|} \int_I |f-1|\,dx \geq \frac{1}{|I|} \int_E |f-1|\,dx \geq \frac{1-e^{-s}}{|I|} |E|$ so $|E| < \frac{\varepsilon}{1-e^{-s}} |I| < \frac{\eta_1}{3} |I|$, where $E = \{x \in I: |\log f| > s\}$. Let $E_n = \{x \in I: |\log f| > n + s\}$, so that $E = E_0$. Clearly $|E_n| \geq |E_{n+1}|$. We want to show that $|E_n| \geq (e^2-1)|E_{n+1}|$ $(n \geq 0)$ (2)
Fix $n \geq 0$. Let $u = \chi_{E_{n+1}}$, the characteristic function of E_{n+1}. Then

$$\int_I u\,dx = |E_{n+1}| \leq |E_n| < \frac{\eta_1}{3} |I| < \frac{|I|}{e^3}, \quad \text{by choice of } \eta_1.$$

We apply the Calderon-Zygmund Lemma (Lemma 3.8) to u. Thus, $\exists$ a countable number of disjoint open subintervals $J_k \subseteq I$ satisfying

(i) $|u| \leq \frac{1}{e^3}$ in $I \setminus \bigcup_k J_k$

(ii) $\frac{1}{e^3} |J_k| \leq \int_{J_k} u\,dx \leq \frac{1}{e^2} |J_k|$

Since u is a characteristic function, (i) $\Rightarrow u = 0$ on $I \setminus \bigcup_k J_k$ i.e. $|E_{n+1} \cap (I \setminus \bigcup_k J_k)| = 0$. Let $S = \bigcup_k J_k$. Then $|E_{n+1} \cap S'| = 0$. So $|E_{n+1}| = |E_{n+1} \cap S| \leq \frac{1}{e^2} |S|$ from (ii). To prove (2) we need only show that

$$|E_n \cap S| \geq \frac{e^2 - 1}{e^2} |S| \qquad (3)$$

for then $|E_n| \geq |E_n \cap S| \geq \frac{e^2 - 1}{e^2} |S| \geq (e^2-1)|E_{n+1}|$.

Suppose (3) is false. Then $\exists$ at least one J_k, J say, such that

$$|J \setminus E_n| > \frac{|J|}{e^2} \quad \text{and} \quad |J \cap E_{n+1}| \geq \frac{|J|}{e^3} \qquad (4)$$

This follows from (ii). Now just apply Proposition 3.7 with s replaced by $n + s$ and E, F replaced by E_n and E_{n+1} respectively. We deduce that

$$\inf_{a>0} \frac{1}{|J|} \int_J |a^{-1}f-1|\,dx \geq \frac{e - 1}{e^4} > \varepsilon.$$

But this contradicts our hypothesis, and so (3) must hold. Now to finish the proof we use an argument similar to that used in the implication ($\Rightarrow$) above.

Let $F(t) = |\{x \in I: \ |\log f| > t\}|$. Then

$$\begin{aligned}\int_I |\log f|\,dx &= \int_0^\infty F(t)\,dt = \int_0^s F(t)\,dt + \int_s^\infty F(t)\,dt \\ &\leq sF(0) + \sum_{k=0}^\infty F(k+s) \leq s|I| + \sum_{k=0}^\infty |E_k| \\ &\leq s|I| + \sum_{k=0}^\infty (e^2-1)^{-k}|E_0| \\ &= (s + \frac{\eta_1}{3}\frac{e^2-1}{e^2-2})|I| < \eta|I|.\end{aligned}$$

This proves that $M_0(\log f) = 0$, and completes the proof.

Remarks (i) This criterion for $\log f$ to be in VMO was suggested to me by A.M. Davie.

(ii) The theorem has two simple corollaries which we now state.

Corollary 3.10 (a) Suppose $\log f \in L^1(\underset{\sim}{R})$, $f > 0$ a.e. and the condition of Theorem 3.9 is satisfied. Then $\log f \in$ VMO.

(b) If $f > 0$ and $\log f \in$ BMO, then $\log f \in$ VMO iff the condition of 3.9 holds.

Proof (a) follows from Lemma 3.6 and (b) is immediate.

Remark Notice that in the circle case, Lemma 3.5 tells us immediately that the condition of Theorem 3.9 is equivalent to saying that $\log f \in$ VMO, so we have proved the following result for the discrete case.

Lemma 3.11 For a discrete stationary s.p. with spectral measure $d\mu = w(\theta)d\theta$ we have $\rho_n \to 0$ as $n \to \infty$ iff $w = |P|^2 f$ where P is a polynomial in $e^{i\theta}$ with all its roots on Γ, and $f > 0$ a.e. satisfies the condition of Theorem 3.9.

Recall from Chapter 2 that for $w \in W$ we have a representation for each $\varepsilon > 0$, $\frac{w}{1 + x^2} = F \exp(r+\tilde{s})$, where F has a representation on $\underset{\sim}{R}$.

$$F(x) = c^2 x^{2k} \prod_{n=1}^{\infty} |1 - \frac{x}{\gamma n}|^2 \prod_{m=1}^{\infty} |1 - \frac{x}{a_m}|^2 \qquad (x \in \underset{\sim}{R})$$

where the γ_n are real and $\text{Im } a_m > 0 \ \forall m$. Of course, F depends on ε. Recall, however, that from Lemma 2.20, the set $\{\gamma_n\}$ is independent of $\varepsilon > 0$. Thus we may write

$$\frac{w}{1 + x^2} = |G_0|^2 \exp(r_\varepsilon + \tilde{s}_\varepsilon + T_\varepsilon) \qquad (**)$$

where $G_0 = cx^k \prod_{n=1}^{\infty} (1 - \frac{x}{\gamma_n})$ is not dependent on ε and $T_\varepsilon = \log \prod_{m=1}^{\infty} |1 - \frac{x}{a_m}|^2$ where $\{a_m\}$ depends on ε.

Suppose we knew that $T_\varepsilon \in UC \cap BMO$ for a sequence of $\varepsilon \to 0$. Then we would have that the BMO distance from the function $\log \frac{w_1}{|G_0|^2}$ to the set $UC \cap BMO$ was zero, since

$$\underset{BMO}{\text{dist}}(\log \frac{w_1}{|G_0|^2}, UC \cap BMO) = \inf_{T \in UC \cap BMO} \|\log \frac{w_1}{|G_0|^2} - T\|_* \leq \|r_\varepsilon + \tilde{s}_\varepsilon\|_* < M\varepsilon \to 0$$

Here $\underset{\sim}{w_1}$ denotes the function $\frac{w}{1 + x^2}$.

Unfortunately, we do not know if $T_\varepsilon \in UC \cap BMO$ for a sequence of $\varepsilon \to 0$. Ideally, we require necessary and sufficient conditions on the roots $\{a_m\}$ of T_ε which ensure that $T_\varepsilon \in VMO$. We do know that if T_ε has only a <u>finite</u> number $\{a_m\}_{m=1}^{k}$ of roots of F_ε associated with it, then $T_\varepsilon \in UC \cap BMO$, as follows from the following lemma.

<u>Lemma 3.12</u> Suppose $\alpha \in \Pi^+$. Then the function u defined on $\underset{\sim}{R}$ by

$$u(x) = \log|x - \alpha| \qquad (x \in \underset{\sim}{R})$$

is in the space $UC \cap BMO$, and so is a VMO function.

<u>Proof</u> It is easy to see that u is uniformly continuous, since it is differentiable on $\underset{\sim}{R}$ and its derivative is bounded by $\max(1, \frac{1}{(\mathrm{Im}\alpha)^2})$. To see that $u \in \mathrm{BMO}$ we proceed as follows. Suppose I is an interval in $\underset{\sim}{R}$, and denote its length by $|I|$. Choose $\lambda, \mu \in I$ so that $|\lambda-\alpha| = \max_{x\in I}|x-\alpha|, |\mu-\alpha| = \min_{x\in I}|x-\alpha|$ and define $a_I = \log|\lambda-\alpha|$. Then $|u-a_I| = |\log|x-\alpha|-\log|\lambda-\alpha|| = \log \frac{|\lambda-\alpha|}{|x-\alpha|}$. Let $E_\sigma = \{x \in I: \ |u-a_I| > \sigma\} = \{x \in I: \ |x-\alpha| < |\lambda-\alpha|e^{-\sigma}\}$ $(\forall\sigma > 0)$. If $E_\sigma \neq \emptyset$, then it must contain μ so that

$$|\lambda-\alpha|e^{-\sigma} > |\mu-\alpha| \geq |\lambda-\alpha| - |\lambda-\mu| \geq |\lambda-\alpha| - |I|$$

whence $|\lambda-\alpha| \leq \frac{|I|}{1 - e^{-\sigma}}$. It follows that E_σ is contained in the interval $|x-\alpha| \leq \frac{|I|e^{-\sigma}}{1 - e^{-\sigma}} = \frac{|I|}{e^{\sigma} - 1}$. Thus $|E_\sigma| \leq \frac{2|I|}{e^{\sigma} - 1}$. But, of course, $|E_\sigma| \leq |I| \ \forall\sigma > 0$ and so we obtain the inequality

$$|E_\sigma| \leq |I|\min\left(\frac{2}{e^{\sigma}-1}, 1\right) \leq 4e^{-\sigma}|I| \qquad (\forall\sigma > 0)$$

Now proceed exactly as in the last proof of Theorem 3.9 to get that

$$\int_I |u-a_I|dx = \int_0^\infty |E_s|ds \leq 4|I|$$

and so $\int_I |u-u_I|dx \leq \int_I |u-a_I|dx + \int_I |u_I-a_I|dx \leq 8|I|$, so $u \in \mathrm{BMO}$.

<u>Corollary 3.13</u> (a) The function $\log(1+x^2) \in \mathrm{VMO}$.

(b) In the representation (**) preceding Lemma 3.12, namely

$$W = |G_0|^2 \exp(r+\tilde{s}+\log(1+x^2)+T_\varepsilon)$$

if we know that $\exists$ a sequence of $\varepsilon \to 0$ for which each T_ε has only finitely many $\{a_m\}$ occurring in its definition, then $W = |G_0|^2 \exp f$ for some real VMO function f.

Proof (a) follows from Lemma 3.12 with $\alpha = i$.

(b) is immediate from (a) and 3.12.

We now leave consideration of the connection between W and VMO for continuous stationary s.p.'s by stating the main problem still to be solved, and making a conjecture.

Problem Given a function G of the form

$$G(x) = \log \prod_{m=1}^{\infty} \left|1 - \frac{x}{a_m}\right|$$

where $\operatorname{Im} a_m > 0 \ \forall m$, find conditions on the set $\{a_m\}$ such that $G \in \text{VMO}$.

Conjecture For a weight function $w \geq 0$, $w \in L^1$ such that $\log w \in L^1_c$, a necessary and sufficient condition for $w \in W$ is that w have a representation in the form

$$w = |G|^2 \exp f$$

where f is a real VMO function and G is an entire function which satisfies

$$|G(z)| \leq Ke^{\lambda|\text{Im} z|} \qquad (z \in \underset{\sim}{C})$$

for some positive constants λ and K.

Now we turn our attention to the relationship between BMO and W_λ for fixed $\lambda > 0$. By Theorem 2.24, $w \in W_\lambda$ iff $\exists$ an entire function F satisfying $|F(z)| \leq C|i+z|^2 e^{\lambda \text{Im} z}$, for $z \in \Pi^+$ and which is positive a.e. on $\underset{\sim}{R}$, and real L^∞-functions r and s with $\|s\|_\infty < \frac{\pi}{2}$ such that on $\underset{\sim}{R}$

$$w = F \exp(r+\tilde{s})$$

Let $S = \{r + \tilde{s}: \ r \in L^\infty, \ \|s\|_\infty < \frac{\pi}{2}\}$. Then, by Theorem 3.3, $S \subseteq \text{BMO}$. To see just exactly how S sits inside BMO seems fairly difficult. For instance S contains some ball round the origin in BMO, for, by Theorem 3.3 again, $\exists$ constant B such that $\forall f \in \text{BMO} \ \exists r, s \in L^\infty$ with $f = r + \tilde{s}$ and $\|r\|_\infty \leq B\|f\|_*$, $\|s\|_\infty \leq B\|f\|_*$. Hence if $\|f\|_* < \frac{\pi}{2B}$, then $\|s\|_\infty < \frac{\pi}{2}$ and $f \in S$. Notice from the last remark that f real, $\|f\|_* < \frac{\pi}{2B} \Rightarrow e^f \in L^1{}_c$; for we may write $f = r + \tilde{s}$ with $\|s\|_\infty \leq B\|f\|_* < \frac{\pi}{2}$ so, by Lemma 2.16, $e^{\tilde{s}} \in L^1{}_c$ and so $e^f = e^r . e^{\tilde{s}} \in L^1{}_c$. This can be proved more directly, namely that if $\|f\|_*$ is sufficiently small we have $e^f \in L^1{}_c$, using a method similar to that used by Fefferman and Stein [5, p.141-142] to prove that

$BMO \subseteq L^1_c$. Notice, in passing, that this latter result $BMO \subseteq L^1_c$ includes the result of Lemma 1.13 that $s \in L^1 \Rightarrow \tilde{s} \in L^1_c$.

In their 1960 paper, Helson and Szegö established a connection between the condition $\rho_1(w) < 1$ and boundedness of the conjugation operator on $L^2(w)$, where w is some weight function on Γ. Is there some analogue of this result for weight functions on $\underset{\sim}{R}$? In 1974, Coifman and Fefferman [1] proved the following result

Theorem 3.14 Suppose g is a locally integrable function which is positive a.e. on $\underset{\sim}{R}$. Then the following are equivalent:

(a) $\log g \in S$

(b) $\sup_{I \subseteq R} \frac{1}{|I|^2} (\int_I g dx)(\int_I g^{-1} dx) < \infty$ where the supremum is over all the intervals $I \subseteq \underset{\sim}{R}$.

(c) The conjugation operator is bounded on $L^2(g)$.

Remark (i) The equivalence of (a) and (c) for the circle case was known to Helson and Szegö (where $g \in L^1_{loc}(\underset{\sim}{R})$ is replaced by $g \in L^1(\Gamma)$, of course).

(ii) (b) $\Leftrightarrow$ (c) is due to Hunt, Muckenhoupt and Wheeden [12].

From our representation theorem 2.24 we know that a given weight function $w \in L^1(\underset{\sim}{R})$ is in W_λ iff $\frac{w}{F} \in \exp(S)$ for some entire function F satisfying certain growth conditions. The role played by F is essentially to "force" w into L^1, because elements of $\exp(S)$ are never integrable as we see from

Lemma 3.15 $f \in S \Rightarrow \exp(f) \notin L^1(\underset{\sim}{R})$.

Proof $f \in S \Rightarrow \exp(f) \in L^1{}_c$, and $\exp(-f) \in L^1{}_c$. If $\exp(f) \in L^1$, then

$$\frac{1}{1+x^2} = \exp(f)\,\frac{\exp(-f)}{1+x^2} \in L^{\frac{1}{2}}, \quad \text{i.e.} \quad (1+x^2)^{-\frac{1}{2}} \in L^1.$$

But this is not true, and so $\exp f \notin L^1$.

Remarks (i) This behaviour is in contrast to the circle case where $\exp(S) \subseteq L^1(\Gamma)$.

(ii) It is a consequence of Lemma 3.15 that the conjugation operator is never bounded on $L^2(w)$ if $w \in L^1(\mathbb{R})$ and $\log w \in L^1{}_c$. Again this is different from the circle case where $f \in S \Leftrightarrow$ conjugation is a bounded operator on $L^2(e^f)$.

$$\Leftrightarrow \rho_1(e^f) < 1$$

However, there is a connection between boundedness of the conjugation operator and the positivity of the angle ρ_λ for fixed $\lambda > 0$. We have the following result

Theorem 3.16 Suppose $w \in L^1$, $w \geq 0$, $\log w \in L^1{}_c$. The following are equivalent for $\lambda > 0$

(a) The conjugation operator H is a bounded operator, in the norm of $L^2(w)$, on the set of real functions in $\overline{P_{-\lambda} \cup F_\lambda}$

(b) $\rho_{2\lambda} < 1$

(c) $\exists$ a representation $w = F \exp f$ where $f \in S$ is real, and F is an entire function which is positive on $\mathbb{R}$ and satisfies

$$|F(z)| \le C|i+z|^2 e^{2\lambda|\text{Im}z|} \qquad (z \in \Pi^+)$$

Proof (b) ⇔ (c) is the statement of Theorem 2.24. To prove (a) ⇔ (b) we show that (a) ⇔ (A) ⇔ (b), where (A) is the statement that the operator T defined by

$$T(\sum a_n e^{i\mu_n x}) = \sum_{\mu_n \ge 0} a_n e^{i\mu_n x} \qquad (a_n \in \mathbb{C}, \quad \mu_n \in \mathbb{R})$$

is bounded on $\overline{P_{-\lambda} \cup F_\lambda}$.

(a) ⇔ (A) Let us suppose that T is bounded. Then, for each polynomial f_λ in F_λ we have $\|f_\lambda\| \le K\|\text{Re}f_\lambda\|$. But $\|\text{Im}f_\lambda\| \le \|f_\lambda\|$, so we have $\|\text{Im}f_\lambda\| \le K\|\text{Re}f_\lambda\|$. If $g = \text{Re}f_\lambda$, this just says that $\|Hg\| \le K\|g\|$ proving (a). Conversely, if (a) holds, we have $\|\text{Im}f_\lambda\| \le K\|\text{Re}f_\lambda\|$ $\forall$ polynomials $f_\lambda \in F_\lambda$: thus $\|f_\lambda\|^2 = \|\text{Re}f_\lambda\|^2 + \|\text{Im}f_\lambda\|^2 \le (K^2+1)\|\text{Re}f_\lambda\|^2$ and T is bounded on the real parts of polynomials on F_λ. But if $f = g_1 + ig_2$ where g_1 and g_2 are real polynomials in $P_{-\lambda} \cup F_\lambda$, then

$$\|Tf\| = \|T(g_1+ig_2)\| \le \|Tg_1\| + \|Tg_2\| \le B(\|g_1\|+\|g_2\|) \le B\sqrt{2}\|g_1+ig_2\|$$

i.e. $\|Tf\| \le B\sqrt{2}\|f\|$ and (A) holds.

(A) ⇔ (b) Suppose (A) holds. Then $\exists\alpha > 0$ such that

$$\|f_\lambda+p_{-\lambda}\|^2 \ge \alpha\|f_\lambda\|^2 \qquad (\forall f_\lambda \in F_\lambda, \quad p_{-\lambda} \in P_{-\lambda})$$

Suppose $\|f_\lambda\| = \|p_{-\lambda}\| = 1$. Then $\|f_\lambda+p_{-\lambda}\|^2 \ge \alpha > 0$. But

$2 - 2\rho_{2\lambda} = \inf\{\|f_\lambda + p_{-\lambda}\|^2 : \|f_\lambda\| = \|p_{-\lambda}\| = 1\} \geq \alpha > 0$, so $\rho_{2\lambda} < 1$.

Conversely, suppose $\rho_{2\lambda} < 1$. Then $\rho(P_{-\lambda}, F_\lambda) < 1$. Now we have

$$\|f_\lambda + p_{-\lambda}\|^2 \geq \|f_\lambda\|^2 + \|p_{-\lambda}\|^2 - 2\rho_{2\lambda}\|f_\lambda\|\|p_{-\lambda}\|$$

so we must show that $\|f_\lambda\|^2 + \|p_{-\lambda}\|^2 - 2\rho_{2\lambda}\|f_\lambda\|\|p_{-\lambda}\| \geq \alpha\|f_\lambda\|^2$ for some $\alpha > 0$, if $\|f_\lambda\| \neq 0$. But

$$\|f_\lambda\| > 0 \Rightarrow \frac{\|f_\lambda\|^2 + \|p_{-\lambda}\|^2 - 2\rho_{2\lambda}\|f_\lambda\|\|p_{-\lambda}\|}{\|f_\lambda\|^2} = 1 + \frac{\|p_{-\lambda}\|}{\|f_\lambda\|}\left(\frac{\|p_{-\lambda}\|}{\|f_\lambda\|} - 2\rho_{2\lambda}\right)$$

$$\geq 1 - \rho_{2\lambda}^2 > 0$$

and so T is bounded. This completes the proof.

The following is an immediate corollary of Theorem 3.14 and 3.16.

<u>Corollary 3.17</u> Suppose $w \in L^1$, $w \geq 0$, $\log w \in L^1_c$. Then the conjugation operator is bounded on $\overline{P_{-\lambda} \cup F_\lambda}$ iff $\exists$ an entire function F, positive on $\mathbb{R}$, which satisfies $|F(z)| \leq C|i+z|^2 e^{2\lambda \operatorname{Im} z}$ $(z \in \Pi^+)$, and is such that $\frac{w}{F}$ satisfies the (A_2) condition, i.e.,

$$\sup_{I \subset \mathbb{R}} \frac{1}{|I|}\left(\int_I \frac{w}{F}\,dx\right)\left(\int_I \frac{F}{w}\,dx\right) < \infty.$$

This completes Chapter 3.

CHAPTER 4

Multidimensional Prediction: An Example

Since the fundamental papers of Masami and Wiener [25] and Helson and Lowdenslager [8] on multivariate prediction theory were published, a lot of attention has been devoted to the subject. Most recently, attempts have been made to study vector-valued processes $\{X_n: n \in \underset{\sim}{Z}\}$ where the random variables take values in separable Hilbert space, H. It turns out that, even in this generality, one can formulate the Helson-Sarason conditions, and some progress has been made. Moore and Page [20] in 1970 gave an analogue of the result Lemma 2.3 for a discrete process $\{X_n: n \in \underset{\sim}{Z}\}$, taking values in a separable Hilbert space H. Their result expresses ρ_n in terms of an L^1-norm of operator valued functions on Γ and uses ideas from the theory of Hankel operators as well as the Sz-Nagy-Foias Lifting Theorem. In the finite dimensional case, $H = \underset{\sim}{C}^p$, $p \geq 2$, the condition $\rho_n \to 0$ has been investigated by I.A. Ibragimov [14], who, as in the scalar case, has given several sufficient conditions for strong-mixing.

Throughout this chapter we shall deal with discrete processes $\{X_n: n \in \underset{\sim}{Z}\}$, whose spectral measures are defined on Γ, the unit circle, and which take values in $\underset{\sim}{C}^p$ for some fixed $p \geq 2$. We shall not consider the case of infinite dimensional H because we believe that the most important problems already exist in the finite dimensional case.

Recall that for a scalar discrete stationary process with spectral measure μ defined on Γ, Helson and Sarason ([9], [23]) proved that $\rho_n \to 0$ iff μ is absolutely continuous and we may represent w in the form

$$w = |P|^2 \exp g,$$

where $d\mu = w d\theta$, P is a polynomial in $e^{i\theta}$ with all its roots on Γ, and g is a real VMO function ($g \in$ VMO by Theorem 3.4 for the circle case).

The major problem for the multidimensional case seems to be to find the right analogue of the set exp(VMO). Our contribution is to give an example of a hermitian 2×2 matrix G, whose entries are all real VMO functions, such that $w = \exp G$ is of full rank and purely non-deterministic, but for which $\rho_n \not\to 0$. Throughout this section, we shall deal only with discrete processes, whose spectral measures are therefore defined on Γ. A readable account of multi-dimensional prediction theory can be found in either of Rozanov [22] or Hannan [7].

We start with some definitions and notation.

<u>Definition 4.1</u> (a) Let (Ω,Σ,P) be a measure space. Denote by L_2 the set of complex p-vector valued functions X with components $X^{(j)} \in L^2(P)$ $(j = 1, \ldots, p)$. Then L_2 is a Hilbert space under the inner product $\langle X,Y \rangle = \int_\Omega \sum_{j=1}^{p} X^{(j)}(w)Y^{(j)}(w)dP(w)$. For $X, Y \in L_2$ we define the <u>Gramian matrix</u> of X and Y by

$$((X,Y))_{i,j} = \int_\Omega X^{(i)}(w)Y^{(j)}(w)dP(w) \quad (i,j = 1, \ldots, p)$$

(b) A <u>subspace</u> of L_2 is a non-empty subset M such

$$X, Y \in M \Rightarrow AX + BY \in M \qquad \forall\ p \times p\text{-matrices}\ \ A, B$$

and such that M is closed in the norm of L_2.

Compare the following with Definition 1.1.

<u>Definition 4.2</u> <u>A p-variate stationary (discrete) process</u> is a sequence $\{X_n\}_{n\in \underset{\sim}{Z}}$ of random variables $X_n \in L_2$ satisfying

$$((X_r, X_s)) = \Gamma_{r-s} = [\gamma_{ij}^{(r-s)}]$$

is dependent only on the difference $r-s$ and not on r and s separately.

<u>Notation</u> Let $P^n = \text{span}\{X_k\colon\ k \leq n\}$, $F^n = \text{span}\{X_k\colon\ k \geq n\}$ and let $P^{-\infty} = \bigcap_{-\infty}^{\infty} P^n$. By span we mean in the sense of Definition 4.1 (b) above. Let $\rho_n = \rho(P^{0}, F^{n})$, as before, and the stationarity hypothesis says that ρ_n is dependent only on n.

<u>Definition 4.3</u> The process $\{X_n\}$ is <u>purely non-deterministic</u> if $P^{-\infty} = \{0\}$. If $P_{P^{-1}}(X_0)$ denotes the projection of X_0 onto the subspace P^{-1}, then the Gramian

$$G = ((X_0 - P_{P^{-1}}(X_0), X_0 - P_{P^{-1}}(X_0)))$$

is called the <u>prediction error matrix</u>. The process $\{X_n\}$ is <u>deterministic</u> if $G = 0$, i.e. if $X_0 \in P^{-1}$. The <u>rank</u> of the process is the rank of G, and $\{X_n\}$ is said to be of <u>full rank</u> if rank G = p.

We now can quote two theorems which are at the heart of the multivariate theory. They are the analogues of Theorems 1.26 and 1.28 in Chapter 1. A proof may be found in Masani and Wiener [25] or Hannan [7].

<u>Theorem 4.4</u> To every stationary discrete p-variate process $\{X_n: n \in \underset{\sim}{Z}\}$, there corresponds a bounded p×p-matrix valued function M on Γ such that

$$\Gamma_n = ((X_n, X_0)) = \int_0^{2\pi} e^{in\theta} dM(\theta)$$

and M has the following properties.

(a) M is right continuous and non-decreasing in the sense that for almost all $\theta \in \Gamma$, $M(\theta)$ is hermitian non-negative definite, and $M(\theta_1) - M(\theta_2)$ is non-negative definite a.e. (dθ).

(b) M has a derivative $\underline{w}$ a.e. (dθ), and $w(\theta)$ is non-negative definite a.e. with $w_{ij} \in L^1(\Gamma)$, i, j = 1, ..., p.

M is called the spectral measure of the process and, of course, we have a Lebesgue decomposition $dM(\theta) = w(\theta)d\theta + dM_s(\theta)$ a.e..

<u>Theorem 4.5</u> Suppose $\{X_n: n \in \underset{\sim}{Z}\}$ is a stationary discrete p-variate process with spectral measure $dM = wd\theta + dM_s$. Then

(a) $\{X_n\}$ is of full rank iff $\log \det w \in L^1(\Gamma)$ and, in this case, we have the equality $\det G = \exp[\int_0^{2\pi} \log \det wd\theta]$

(b) $\{X_n\}$ is of full rank and purely non-deterministic iff M is absolutely continuous (i.e. $dM_s = 0$), and $\log \det w \in L^1(\Gamma)$.

Remark This last result is the main result of the fundamental papers of Wiener and Masani [25] and Helson and Lowdenslager [8]. We give no proof.

Henceforth we shall consider only discrete p-variate processes $\{X_n : n \in \underset{\sim}{Z}\}$ whose spectral measure $dM = w d\theta$ is absolutely continuous and satisfies $\log \det w \in L^1(\Gamma)$.

Definition 4.6 An analytic trigonometric polynomial is a function of the form $\sum_{k=0}^{n} \alpha_k e^{ik\theta}$, where $(n \geq 0)$ and α_k is a complex p-vector, $k = 0, \ldots, n$.

Following Moore and Page [20, p.1012], as in the scalar case, we may express ρ_n in the form ($\langle \cdot \rangle_p$ denotes the usual inner product on $\underset{\sim}{C}^p$)

$$\rho_n = \sup \left| \int_0^{2\pi} \langle w(e^{i\theta}) f(e^{i\theta}), g(e^{-i\theta}) \rangle_p e^{in\theta} d\theta \right| \qquad (1)$$

where the supremum ranges over all analytic trigonometric polynomials f and g which satisfy

$$\left.\begin{aligned} &\int_0^{2\pi} \langle w(e^{i\theta}) f(e^{i\theta}), f(e^{i\theta}) \rangle_p d\theta \leq 1 \\ &\int_0^{2\pi} \langle w(e^{-i\theta}) g(e^{i\theta}), g(e^{i\theta}) \rangle_p d\theta \leq 1 \end{aligned}\right\} \qquad (2)$$

Of course, it is clear that (1) is unchanged if in (2) we assume equalities instead of $\leq$.

Now suppose $w(\theta)$ is diagonal, say with diagonal elements $w_1(\theta), \ldots, w_p(\theta)$ a.e. $(d\theta)$. Then the problem of deciding when $\rho_n(w) \to 0$ is solved simply using the scalar case, as the following result shows

Lemma 4.7 Suppose $w(\theta) = \operatorname{diag}_{1\leq j\leq p} (w_j(\theta))$ a.e.. Then $\rho_n(w) = \max_{1\leq j\leq p} \rho_n(w_j)$.

Proof For any w, it is easy to see that $\rho_n(w) \geq \max_{1\leq j\leq p} \rho_n(w_j)$ straight from the definition of ρ_n. To obtain the converse, we use the diagonality of w to link $\rho_n(w)$ with the $\rho_n(w_j)$. Suppose f, g are analytic trigonometric polynomials with components $f^{(j)}$, $g^{(j)}$, $j = 1, \ldots, p$, respectively. Then we have

$$\int_0^{2\pi} \langle w(e^{i\theta})f(e^{i\theta}),g(e^{-i\theta})\rangle_p e^{in\theta}d\theta = \sum_{j=1}^{p} \int_0^{2\pi} \langle w_j(e^{i\theta})f^{(j)}(e^{i\theta}),g^{(j)}(e^{-i\theta})\rangle e^{in\theta}d\theta$$

But $f^{(j)}$, $g^{(j)}$ are scalar analytic trigonometric polynomials, so we have, by definition,

$$\left| \int_0^{2\pi} \langle w_j(e^{i\theta})f^{(j)}(e^{i\theta}),g^{(j)}(e^{-i\theta}) \rangle e^{in\theta}d\theta \right| \leq \rho_n(w_j)\|f^{(j)}\|_{w_j} \|g^{(j)}\|_{w_j}$$

where we use the notation, for a scalar function $k(e^{i\theta})$,

$$\|k\|_{w_j}^{\,2} = \int_0^{2\pi} w_j|k|^2 d\theta.$$

Thus we have

$$\left| \int_0^{2\pi} \langle w(e^{i\theta})f(e^{i\theta}), g(e^{-i\theta}) \rangle e^{in\theta} d\theta \right| \leq \sum_{j=1}^{p} \rho_n(w_j) \|f^{(j)}\|_{w_j} \|g^{(j)}\|_{w_j} .$$

Thus

$$\rho_n(w) \leq \sup\{ \sum_{j=1}^{p} \rho_n(w_j) \|f^{(j)}\|_{w_j} \|g^{(j)}\|_{w_j} : \sum_{j=1}^{p} \|f^{(j)}\|_{w_j}{}^2 = \sum_{j=1}^{p} \|g^{(j)}\|_{w_j}{}^2 = 1\}$$

We may suppose, without loss of generality, that $\rho_n(w_j) \leq \rho_n(w_k)$ for $j = 1, \ldots, p$. Then

$$\rho_n(w) \leq \rho_n(w_k) \sup\{ \sum_{j=1}^{p} \|f^{(j)}\|_{w_j} \|g^{(j)}\|_{w_j} : \sum_{j=1}^{p} \|f^{(j)}\|_{w_j}{}^2 = \sum \|g^{(j)}\|_{w_j}{}^2 = 1\}$$

$\leq \rho_n(w_k)$ by the Cauchy-Schwarz inequality.

This proves the lemma.

Corollary 4.8 Suppose $w(\theta)$ is diagonal a.e. $(d\theta)$. Then

$$\rho_n \to 0 \quad \text{iff} \quad W = \begin{bmatrix} |P_1|^2 & & 0 \\ & \ddots & \\ 0 & & |P_p|^2 \end{bmatrix} \exp \begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_p \end{bmatrix}$$

where P_j is a polynomial in $e^{i\theta}$ with all its roots on Γ, and each g_j is a real VMO function.

Proof Immediate from Lemma 4.7 and the scalar discrete Helson-Sarason theorem.

When you move away from the diagonal case, it is not obvious what to try and replace the factor $\exp\begin{bmatrix} g_1 & & 0 \\ & \ddots & \\ 0 & & g_p \end{bmatrix}$ by, to achieve

$\rho_n \to 0$. We shall now give an example which shows that the most obvious candidate, namely $\exp G$ where G is hermitian and all its entries are in VMO, is <u>not</u> the right one! More precisely, we shall construct a weight function w on Γ with the following properties. w is a 2×2 matrix and

(a) $w(\theta) = \exp G(\theta)$, where G is hermitian with all its entries real and in VMO.

(b) $\log \det w \in L^1(\Gamma)$, and all the entries of w are in $L^1(\Gamma)$.

(c) w_{11} has a jump discontinuity at $\theta = 0$.

Such a weight function cannot have $\rho_n \to 0$, for the following reason. If $\rho_n(w) \to 0$, then $\rho_n(w_{11}) \to 0$, so we may write $w_{11} = |P|^2 \exp k$, P is a polynomial and k is real and in VMO. If w_{11} has a jump at $\theta = 0$, then k must have a jump. But it is straightforward from the definition of a VMO function to deduce that no VMO function can have a jump discontinuity. For if k had a jump at $\theta = 0$, then its "oscillation" over small intervals round $\theta = 0$ would not be small. Thus we deduce that $\rho_n(w_{11}) \not\to 0$ and so $\rho_n(w) \not\to 0$.

Before we actually construct w we shall need a couple of preliminary results of a technical nature. First we need a positive VMO function on Γ which is symmetric about $\theta=0$ in the sense that $f(e^{-i\theta}) = f(e^{i\theta})$ $(\theta \in [-\pi,\pi])$, which is monotone decreasing on $(0,\pi]$ and which tends to $+\infty$ as $\theta \to 0$. The following result provides such a function. It will be more convenient to work on the interval $[-\pi,\pi]$ instead of Γ.

<u>Proposition 4.9</u> Define f on $[-\pi,\pi]$ by $f(\theta) = (\log \frac{2\pi}{|\theta|})^{\frac{1}{2}}$. Then $f \in \text{VMO}$.

Proof To prove this we must consider the oscillation of f over intervals $I \subseteq [-\pi,\pi]$. We consider three separate cases depending on the type of interval I.

Case 1 $I = [0,\alpha]$, where $0 < \alpha \leq \pi$.

Let $a_I = (\log \frac{2\pi}{\alpha})^{\frac{1}{2}}$. Then we have

$$\frac{1}{|I|}\int_I |f-a_I|\,d\theta = \frac{1}{\alpha}\int_0^\alpha [(\log \frac{2\pi}{\theta})^{\frac{1}{2}}-(\log \frac{2\pi}{\alpha})^{\frac{1}{2}}]\,d\theta = \frac{1}{\alpha}\int_0^\alpha \frac{\log \frac{2\pi}{\theta} - \log\frac{2\pi}{\alpha}}{(\log\frac{2\pi}{\theta})^{\frac{1}{2}}+(\log\frac{2\pi}{\alpha})^{\frac{1}{2}}}\,d\theta$$

$$\leq \frac{1}{2(\log \frac{2\pi}{\alpha})^{\frac{1}{2}}} \cdot \frac{1}{\alpha}\int_0^\alpha (\log \frac{2\pi}{\theta} - \log\frac{2\pi}{\alpha})\,d\theta = \frac{1}{2(\log \frac{2\pi}{\alpha})^{\frac{1}{2}}} .$$

Thus if $I = [0,\alpha]$, and $a_I = (\log \frac{2\pi}{\alpha})^{\frac{1}{2}}$, then

$\frac{1}{|I|}\int_I |f-a_I|\,d\theta \leq \frac{1}{2(\log \frac{2\pi}{\delta})^{\frac{1}{2}}}$, if $|I| < \delta$.

Case 2 $I = [-\beta,\alpha]$, where $0 < \beta \leq \alpha \leq \pi$.

Let $a_I = (\log \frac{2\pi}{\alpha})^{\frac{1}{2}}$. Then we have

$$\frac{1}{|I|}\int_I |f-a_I|\,d\theta = \frac{1}{\alpha+\beta}\{\int_0^\alpha [f-a_I]\,d\theta + \int_0^\beta [f-a_I]\,d\theta\}$$

$$\leq \frac{2}{\alpha+\beta}\int_0^\alpha [(\log \frac{2\pi}{\theta})^{\frac{1}{2}}-(\frac{\log 2\pi}{\alpha})^{\frac{1}{2}}]\,d\theta$$

$$\leq \frac{\alpha}{\alpha+\beta}\frac{1}{(\log \frac{2\pi}{\alpha})^{\frac{1}{2}}}, \text{ as in Case 1.}$$

Thus, if $|I| = \alpha + \beta < \delta$, then $\frac{1}{|I|}\int_I |f-a_I|\,d\theta \leq \frac{1}{(\log \frac{2\pi}{\delta})^{\frac{1}{2}}} \to 0$ as $\delta \to 0$.

Case 3 $I = [\beta,\alpha]$, where $0 < \beta \leq \alpha \leq \Pi$.

Let $a_I = (\log \frac{2\pi}{\alpha})^{\frac{1}{2}}$. Then

$$\frac{1}{|I|}\int_I |f-a_I|\,d\theta = \frac{1}{\alpha-\beta}\int_\beta^\alpha \frac{\log\frac{2\pi}{\theta} - \log\frac{2\pi}{\alpha}}{(\log\frac{2\pi}{\theta})^{\frac{1}{2}}+(\log\frac{2\pi}{\alpha})^{\frac{1}{2}}}\,d\theta, \quad \text{as in Case 1}$$

$$\leq \frac{1}{(\alpha-\beta)(\log\frac{2\pi}{\alpha})^{\frac{1}{2}}}\int_\beta^\alpha [\log\frac{2\pi}{\theta} - \log\frac{2\pi}{\alpha}]\,d\theta$$

But $\int_\beta^\alpha (\log\frac{2\pi}{\theta} - \log\frac{2\pi}{\alpha})d\theta = \alpha - \beta + \beta\log\frac{\beta}{\alpha}$, so that

$$\frac{1}{|I|}\int_I |f-a_I|\,d\theta \leq \frac{1}{(\log\frac{2\pi}{\alpha})^{\frac{1}{2}}}\{1 + \beta\,\frac{\log\beta-\log\alpha}{\alpha-\beta}\}$$

now $1 + \beta\,\frac{\log\beta-\log\alpha}{\alpha-\beta} \leq 1 - \frac{\beta}{\alpha} = \frac{\alpha-\beta}{\alpha}$, so $\frac{1}{|I|}\int_I |f-a_I|\,d\theta \leq \frac{\alpha-\beta}{\alpha(\log\frac{2\pi}{\alpha})^{\frac{1}{2}}}$ (†)
$\alpha \in (0,\delta^{\frac{1}{2}}) \Rightarrow$ (†) is bounded by $\frac{1}{(\log\frac{2\pi}{\delta^{\frac{1}{2}}})^{\frac{1}{2}}}$, while $\alpha > \delta^{\frac{1}{2}}$ and

$\alpha - \beta < \delta \Rightarrow$(†) is bounded by $\frac{\delta^{\frac{1}{2}}}{(\log 2)^{\frac{1}{2}}}$; so

$\alpha - \beta < \delta \Rightarrow \frac{1}{|I|}\int_I |f-a_I| < \max\left(\frac{1}{(\log\frac{2\pi}{\delta^{\frac{1}{2}}})^{\frac{1}{2}}}, \left(\frac{\delta}{\log 2}\right)^{\frac{1}{2}}\right)$ and this tends to zero as $\delta \to 0$.

Since every subinterval I of $[-\pi,\pi]$ is of type 1, 2 or 3 or is the reflection of such in the origin, we have proved that $M_0(f) = 0$, so $f \in VMO$.

We need one last technical result.

Lemma 4.10 Suppose $\psi(x,y)$ is a power series in two variables with non-negative coefficients which converges for all $x, y \in \mathbb{R}$. Suppose $\psi(0,y) = 1$, $\forall y > 0$ and $\psi(x,y) \geq \frac{x^2y}{6}$, $\forall x.y > 0$. Define τ, for $y > 0$, by $\psi(\tau(y),y) = 2$. Then τ is continuous and $\tau(y) \to 0$ as $y \to \infty$.

<u>Proof</u> Since $\psi(x,y) \geq \frac{x^2y}{6}$, we have $(\tau(y))^2 \leq \frac{12}{y} \to 0$ as $y \to \infty$. Suppose τ is not continuous at $y_0 > 0$. Then $\exists\ \delta > 0$ and a sequence $y_n \to y_0$ with $|\tau(y_n)-\tau(y_0)| > \delta$. By taking a subsequence, if necessary, we may assume that either (1) y_n is monotone increasing or (2) y_n is monotone decreasing to y_0.

Since ψ is differentiable, by the mean-value theorem

$$|\psi(\tau(y_0),y_n)-\psi(\tau(y_n),y_n)| = \left|\frac{\partial}{\partial x}\psi(x,y_n)\right||\tau(y_0)-\tau(y_n)|$$

for some x between $\tau(y_0)$ and $\tau(y_n)$. Thus we have

$$|\psi(\tau(y_0),y_n)-2| \geq \frac{\delta y_n}{3}\min(\tau(y_0),\tau(y_n)), \quad \text{since} \quad \psi(x,y) \geq \frac{x^2y}{6} \qquad (*)$$

Since $y_n \to y_0$, we may assume that $y_n \geq \frac{y_0}{2}$, $\forall n$. In case (1), $\tau(y_n) \geq \tau(y_0)$ for all n, so (*) gives us that

$$|\psi(\tau(y_0),y_n)-2| \geq \frac{\delta y_0}{6}\tau(y_0), \quad \forall n \qquad (3)$$

In case (2), $\tau(y_n) \leq \tau(y_0)$ and, by monotonicity, $\tau(y_n) \geq \tau(y_1)$, so (*) gives

$$|\psi(\tau(y_0),y_n)-2| \geq \frac{\delta y_0}{6}\tau(y_1), \quad \forall n \qquad (4)$$

Since $\tau(y_0) > 0$ and $\tau(y_1) > 0$ (otherwise $\psi(\tau(y_0),y_0) = \psi(\tau(y_1),y_1) = 1$), (3) and (4) say that the set $\{\psi(\tau(y_0),y_n)\}$ is bounded away from 2, which since $y_n \to y_0$ would mean that ψ is not continuous, a contradiction. Thus τ is continuous in y.

We now proceed to construct w. Define the 2×2 matrix G by $G = \begin{bmatrix} 0 & g \\ g & f \end{bmatrix}$, where f is given in Proposition 4.9, and g is a continuous function on $[-\pi,\pi]$ still to be determined. Let $g(\theta) = 0$ $(\theta \in [0,\pi])$, and define $w = \exp G$. Then it is easy to check that $w_{11}(\theta) = 1$ $(\theta \in [0,\pi])$. Define ψ by $\psi(x,y) = (\exp \begin{bmatrix} 0 & x \\ x & y \end{bmatrix})_{11}$. Then ψ satisfies the hypotheses of Lemma 4.10 and so τ is defined and continuous. Define g on $[-\frac{\pi}{2},0)$ by $g(\theta) = \tau(f(\theta))$. Then g is continuous on $[-\frac{\pi}{2},\pi]$ since $\lim_{\substack{\theta\to 0\\ \theta<0}} g(\theta) = \lim_{y\to\infty} \tau(y) = 0 = g(0)$. Now just extend g to be continuous on $[-\pi,\pi]$ with $g(-\pi) = g(\pi) = 0$. This completes the definition of w. Notice that w_{11} has a jump at $\theta = 0$ by construction, since for $\theta \in (-\frac{\pi}{2},0)$ we have $w_{11}(\theta) = \psi(\tau(f(\theta)),f(\theta)) = 2$ and $w_{11}(0) = 1$. It remains only to check that the entries of w are integrable and that $\log \det w \in L(\Gamma)$. But this is straightforward since $\log \det w = f \in VMO \subseteq L'$, and, with the notation of Lemma 4.10, it is easily checked that each entry of $\exp\begin{bmatrix} 0 & x \\ x & y \end{bmatrix}$ is bounded by e^{x+y} (since each entry of $\begin{bmatrix} 0 & x \\ x & y \end{bmatrix}^n$ is bounded by $(x+y)^n$ by induction) and this suffices to prove the integrability of each entry of w.

It is by no means obvious how to decide upon the 'right' class to take the place of exp VMO. We shall suggest two possibilities. In his paper on VMO, Sarason characterised real VMO functions among all real BMO functions as those satisfying $N_0(\exp f) = 1$, where for a positive function $w \geq 0$, $N_0(w) = \lim_{a\to 0} N_a(w)$ and

$$N_a(w) = \sup_{|I|\leq a} \frac{1}{|I|^2} \left(\int_I w(\theta)d\theta\right)\left(\int_I w^{-1}(\theta)d\theta\right)$$

We suggest that, for the multivariate case, the following class might bear investigation. For hermitian G, consider those which satisfy $N_0(W) = 1$ where $W = \exp G$, and $N_0(W) = \lim_{a\to 0} N_a(W)$, with

$$N_a(W) = \sup_{|I|\leq a} \left\| I_p - \frac{1}{|I|^2} \left(\int_I W(\theta)d\theta\right)\left(\int_I W^{-1}(\theta)d\theta\right)\right\| \qquad (1)$$

where I_p is the $p \times p$ identity matrix and $\|\cdot\|$ is the usual matrix norm.

Another possibility is to generalise our criterion for $\log f \in VMO$ from the scalar to the vector case, i.e. consider hermitian G for which $\exp G = W$ satisfies the following condition:

$$\left.\begin{array}{l} \forall\varepsilon > 0 \quad \exists\delta > 0 \quad \text{s.t.} \quad |I| < \delta \Rightarrow \exists \text{ a positive definite matrix } A \text{ such that} \\ \qquad \dfrac{1}{|I|} \displaystyle\int_I \|A^{-\frac{1}{2}}WA^{-\frac{1}{2}}-I\|d\theta < \varepsilon \end{array}\right\} \quad (2)$$

<u>Remark</u> It is not obvious that (1) and (2) are the right generalisations of their scalar counterparts; for example, is $N_a(W)$ the same as

$$\sup_{|I|\leq a} \left\| I_p - \frac{1}{|I|^2} \left(\int_I W^{-1}(\theta)d\theta\right)\left(\int_I W(\theta)d\theta\right)\right\| ?$$

or should (2) be replaced by

$$\frac{1}{|I|} \int_I \|A^{-1}W-I\|d\theta < \varepsilon \quad \text{or} \quad \frac{1}{|I|} \int_I \|WA^{-1}-I\|d\theta < \varepsilon ?$$

This completes Chapter 4.

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