THE GEOMETRY OF CUBIC SURFACES, AND GRACE'S EXTENSION OF THE DOUBLE-SIX, OVER FINITE FIELDS

by

J. W. P. HIRSCHFELD

Thesis presented for the Degree of
Doctor of Philosophy of the University
of Edinburgh in the Faculty of Science
1st October, 1965.

INTRODUCTION

Between March 1962 and March 1963 I did an M.Sc. thesis entitled "The double-six of lines over PG(3, 4)" under the supervision of Professor T.G. Room at Sydney. The results of this are embodied in a paper [30]: the numbers in square brackets refer throughout to the bibliography at the end of this thesis. From October 1963 to October 1965 I have been doing this work under the supervision of Dr. W.L. Edge at Edinburgh.

Π_n is used to denote the projective space of n dimensions, GF(q) the Galois field of q elements and PG(n, q) the projective geometry in Π_n over GF(q).

The main aim is to clarify and extend the earlier thesis, which investigated the double-six of lines over the smallest field for which it could be defined, and to find out how Grace's extension of the double-six can occur in a finite geometry. In fact, all the projectively distinct cubic surfaces with 27 lines over GF(q) have been classified for $q \leq 9$. Each of these surfaces is denoted by F_q^n as it is found, where n is the number of points on no line of the surface (this symbol is omitted if there is no ambiguity) and q is the order of the field. Two surfaces are defined to be projectively distinct in PG(n, q) if there is no non-singular linear homogeneous transformation of the space transforming the one into the other, cf. Segre [57] Chapter 16. Grace's extension of the double-six is also considered for all GF(q) $q \leq 9$.

Two main features distinguish PG(n, q) from the geometry over the complex field. Firstly, the number of Π_r's in Π_n is known, viz. it is

$$\prod_{i=0}^{r} (q^{n+1-i} - 1) \Big/ \prod_{i=0}^{r} (q^{r+1-i} - 1) \qquad \text{Segre [ ] p.257;}$$

for $r = 0$, this is the number of points in Π_n, which is therefore $(q^{n+1} - 1)/(q - 1)$. Secondly, the number of roots of an equation in one variable of degree m that lie in the field is known only to be between 0 and m.

Four references occur most frequently: Baker [4] and Segre [48] for the classical theory of the general cubic surface, Todd [61] for classical projective geometry, and Segre [57] for the theory of Galois fields and finite projective geometries. A model of the double-six of lines is given by Hilbert and Cohn-Vossen [29] p. 165.

Finally, a finite projective geometry should not be confused with the branch of mathematics known as "Finite geometry", in which a comparable problem is the consideration of the 27 lines of a cubic surface over the real field which does not necessarily have an equation, e.g. Marchaud [36].

CONTENTS

CHAPTER I. Historical summary.

§1. The double-six and the cubic surface in classical projective geometry.

Throughout this short account, the emphasis is on the particular discoveries relevant to the remainder of the thesis. There is no intention of providing a general survey of the cubic surface: for this, extensive references are given by Meyer [37] and Henderson [28].

There are 27 lines on a general cubic surface, F, over the real field. This result was first published by Cayley [7] in 1849. He observed that through each line of F, there are five planes meeting it in two other lines. Since these planes, described as treble tangent, named as triple tangent, which we will call tritangent, each contain three of the 27 lines, they number 45. Further, he showed that the equation of F can be written as $LMN + PQR = 0$, where L, M, N, P, Q, R are linear forms in the four coordinates, in 120 ways. The planes $L = 0$ etc. are all tritangent planes. The six planes form a pair of trihedra, the planes of each one containing the same set of nine lines. He also proved that each pair of trihedra, later to be called a Steiner trihedral pair, is associated with two others, dividing the 27 lines into three sets of nine; and he notated the lines accordingly. However he said "There is great difficulty in conceiving the complete figure formed by the twenty-seven lines, indeed this can hardly be accomplished till a more perfect notation is discovered" and, at the end of the paper, "I may mention in conclusion that the whole subject of this memoir was developed in correspondence with Mr. Salmon, and in particular, that I am indebted to him for the determination of the number of lines upon the surface....."

In 1851 Sylvester [60] stated, without a proper proof, that the general cubic surface can be expressed as $\sum_{i=0}^{i=4} z_i^3 = 0$, $\sum \lambda_i z_i \equiv 0$ where the z_i are linear functions of the coordinates x_0, x_1, x_2, x_3 and the 5 planes $z_i = 0$ form a pentahedron. The result was proved in 1861 by Clebsch [10] firstly using the method indicated by Sylvester and more directly [11] later that year.

The projective generation of F as the locus of intersection of corresponding planes of three collinear stars was made by Grassmann [26] in 1855. In 1857, Steiner [59] systematically expounded a long list of enumerative properties of the 27 lines, most of which are implicit in Cayley's paper. In 1858, Schläfli [46] proved the existence of the double-six of lines by considering when three corresponding planes of the stars meet in a line, and showed that there are 36 double-sixes on F. He then suggested the double-six notation for the 27 lines - a_i, b_i, c_{ij} $i,j = 1, \ldots, 6$ $i \neq j$ which illustrates all their properties except their symmetry. This notation has since been standard and still seems the simplest devised. After enunciating the double-six theorem, he asked for a proof of its existence independent of the cubic surface on which it lies. He mentioned a surface with fifteen real lines and six pairs of conjugate lines which form a double-six: thus he anticipated Clebsch's diagonal surface, though not the reason for its name.

In 1864, Cayley [8] discussed a special cubic surface which isolates a particular trihedral pair. The three planes of one trihedron meet in a line: the three planes of the other trihedron, which he named tritom planes, each meet the surface in three concurrent lines. The points of concurrency were named tritom points. The respective properties of the two trihedra imply each other. The existence of the tritom points and their collinearity in certain sets of three are properties intrinsic to the smaller finite geometries.

In 1866, using the projective definition, Clebsch [12] mapped F onto a plane, plane sections of F being mapped by cubic curves through six base points. The correspondence is birational with some exceptional elements. The six base points do not lie on a conic and each represents one of the lines of half a double-six a_i $i = 1, \ldots, 6$ say. In 1871, considering equations of the fifth degree, Clebsch [13] found a curve which is the intersection of a quadric and a cubic surface; written in pentahedral coordinates it is $\Sigma \xi_i = 0$, $\Sigma \xi_i^2 = 0$, $\Sigma \xi_i^3 = 0$. The cubic surface contains the diagonals of the quadrilaterals cut out on each of the five faces of the pentahedron by the other four: hence his name "diagonal surface of the pentahedron". These fifteen lines lie in threes in fifteen planes, in ten of which the three lines are concurrent, being the joins of the ten vertices to opposite edges. The remaining twelve lines form a double-six. The existence of the double-six on the surface depends on the "Golden Section" whose proportions are given by the solution of $\mu^2 - \mu - 1 = 0$. In the plane representation the six base points form a Brianchon hexagon in ten ways. He characterized the quadric by showing that its section by a plane of the pentahedron contains the three pairs of double-points of the involutions determined by the intersection of any diagonal with the other two. Clebsch just missed the polarity of the double-six since the above quadric is the one required; neither did he point out the invariance of the double-six under any permutation of the faces of the pentahedron. But in 1911, Burnside [5] constructed this special double-six dually from a skew pentagon whose vertices are actually the poles of the pentahedral faces with respect to the polarity of the double-six, and showed the double-six to be invariant under the group of permutations of the vertices of the pentagon. In the plane representation, the five vertices of the skew pentagon become five collinear points; these points are five of the fifteen diagonal points of a pentastigm, whose vertices are taken from the six base points. The collinearity of five diagonal points of a pentastigm being subject to $\mu^2 - \mu - 1 = 0$ was observed by B. Segre [56] in 1959.

Eckardt [18], in 1876, proved that if two of the coefficients a_i of the equation of F written in pentahedral coordinates - $\Sigma a_i x_i^3 = 0$ $\Sigma x_i = 0$ - are equal, then one of the vertices of the pentahedron is a tritom point. The tritom points were afterwards called Eckardt points: we shall call them E-points. He showed that these cases comprised surfaces having 1, 2, 3, 4, 6 and 10 E-points. The surface with three is Cayley's: the surface with ten is Clebsch's. Sylvester's form for the cubic surface was $\Sigma x_i^3 = 0$ $\Sigma b_i x_i = 0$, which is equivalent to the form above only when the pentahedral faces are 4 by 4 independent. Eckardt then considered the two cases of four and three faces of the pentahedron being dependent. In the former case when four faces have a common point, the surface, which is described by Segre [48] §§ 85-88 as cyclic and non-equianharmonic, has the canonical equation $x_0^3 + K = 0$, where K is a canonical cubic form in x_1, x_2, x_3. The surface has 9 E-points which are all the points of inflexion of the cubic curve $x_0 = 0$ $K = 0$ and which therefore lie in threes on 12 lines. The tritangent planes at the 9 E-points are concurrent at the vertex of the cone $K = 0$. The second case when three faces of the pentahedron are collinear can be reduced to the canonical equation $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. Eckardt showed that over the complex field it contains 18 E-points lying in threes on the edges of the fundamental tetrahedron. The existence of the E-points depends on solutions of $\mu^2 + \mu + 1 = 0$. This surface was named the equianharmonic surface by Segre [48] p.128 after the general cubic or equianharmonic curve in the plane with which it is connected. The complete determination of surfaces with a degenerate pentahedron was given by Rodenberg [41] in 1879. Further, Segre [48] §§ 89-94 showed that there exist non-singular cubic surfaces with no Sylvester representation.

"Die sechs Geraden g und c der F^3, welche mit den beiden erzeugenden Netzen der (c) und (g) derselben zusammenhängen und die bekannte Schlaefli'sche Dopplesechs bilden, besitzen eine Eigenthümlichkeit, die den Geometern bisher entgangen zu sein scheint," wrote Schur [47] in 1881, and proved, using the projective generation of F, the existence of a quadric with respect to which the double-six is self-polar.

In this paper Schur showed the existence of the double-six by using a porism of the plane cubic curve. This proof was not independent of the cubic surface. However, in 1870, Cayley [9] achieved the result using line coordinates. This proof was of inordinate length and one in a simpler form after the same style and also proving the polarity was given by Kasner [32] in 1903. In 1908, Richmond [40] gave a short and elegant proof by showing that "if in space of five dimensions a quadric passes through all the vertices of a hexahedron and touches five of its faces, it must touch the sixth face also. Finally in 1911, Baker [1] gave a purely geometrical proof in three dimensions of the double-six theorem which depended only on the incidence of properties of the lines and was quite independent of the cubic surface. This proof showed clearly that, given five skew lines with a common transversal, the necessary and sufficient condition for the formation of the double-six is that each set of four out of the five lines shall have a unique second transversal. Further in 1921, Baker [2] gave a proof by projection from four dimensions.

§ 2. Grace's extension of the double-six.

In 1898, Grace [25] showed the inter-dependence of the theorems below and proved an extension of the double-six theorem.

1) "We take six hyperplanes in four dimensions; any four of them meet in a point, consequently omitting one of them we get five points through which there is a hypersphere, then the six hyperspheres so obtained by omitting each hyperplane in turn meet in a point."

2) "Taking six linear complexes having a line in common then any four of them have another line in common, and therefore from a set of five of them we get five lines through which one linear complex may be made to pass; then from the six complexes we get six sets of five, and as from each five we get another complex, we thus derive six new complexes, then the theorem is that these six complexes have one common line."

3) "If we have five lines meeting a given line, as in the figure, a, b, c, d, e meet t, then any four of them as a, b, c, d have another line in common; thus we get five such lines and ..... these five lines are met by another line f'. Now ..... if we take six lines a, b, c, d, e, f, then from each set of five we get a line like f', and the property is that these six lines are all met by one and the same straight line."

Grace studied the correspondence between line geometry in three dimensions and sphere geometry in four dimensions. Then by considering cubic threefolds in four dimensions he proved 1). 2) is equivalent to 1) by the above correspondence. The extension theorem 3) is a special case of 2).

Another proof, which brought out the relations of all the lines implicit in Grace's figure, was given by Wren [65] in 1916. This involved 44 lines - the original line, its six transversals, the fifteen second transversals of sets of four of these six, the six completing lines of the six double-sixes, the transversal of these six, and the fifteen second transversals of sets of four of these six. A diagram showed that these lines form 32 double-sixes and that the Grace figure starting from six lines and a transversal is formed in 16 ways. In 1917, Kubota [33] gave a shorter proof after the same style.

The theorem appeared again in 1922 in a dramatic way. E.K. Wakeford had been corresponding with Baker while a soldier in the First World War. He was killed in action and a manuscript found in his kitbag was sent to Baker, who had it published [63] with a paper of his own [3] expanding some of Wakeford's arguments. Wakeford proved Grace's extension by considering the unique twisted cubic which has six lines having a common transversal as chords and establishing the polarity between the original six lines and the six lines obtained from the construction. Thus the required transversal of the six derived lines is the polar of the transversal of the original six lines.

It is implicit in all the proofs of the extension theorem that the existence, given the six skew lines with a transversal, of a unique second transversal for each set of four out of the six lines is a necessary condition for the theorem. However all fail to point out that the above condition is not always sufficient, as shall be seen subsequently.

§ 3. Finite projective geometry.

Projective geometry over a finite field was given an impetus by Veblen and Bussey [62] in 1906. Previous considerations had been mainly group theoretical. In particular there is a vast deal of geometry hidden in the pages of Burnside [6] and Dickson [17] which were first published in 1897 and 1900 respectively. Coble [14] in 1908 described a configuration in the geometry over the field of three elements isomorphic with the twenty-seven lines of a cubic surface. Frame [24] in 1938 observed the isomorphism between the 27 lines of a real cubic surface and a configuration in the geometry over the field of four elements; however, the accent was on the isomorphism and not on the finite space itself.

Up to 1948, all the work done had been concerned with either the axiomatics of the subject or the properties of particular finite geometries, usually not for their own sake. The first more general and detailed study was made by Segre [49] in his "Lezioni di Geometria Moderna". Then came two papers, by Qvist [39] in 1952 and Segre [51] in 1955, which seem to me to have been chiefly responsible for recent interest and progress; for these papers showed the elegance of the results obtainable and the interest of the subject for its own sake, not dependent on other branches of mathematics such as group theory, classical projective geometry or statistics with which it had previously been connected. Quite independently of these, there followed a series of papers by Edge, which showed that an understanding of the simplicity of certain classical groups was not to be had without studying the finite geometrical structures of which they were the groups. Various papers on the existence of the twenty-seven lines in a finite geometry have been published by Segre [50], Rosati [43] [44], Edge [21] [22] [23] and Coxeter [15], all since 1940.

In what follows it is proposed to blend the techniques of finite geometry, in which the problems lie, and those of classical geometry, whence the problems are derived!

CHAPTER II. The double-six and its cubic surface over GF(4).

§ 4. The double-six over an arbitrary field.

In Π_3 over an algebraically closed field, 4 independent skew lines have 2 transversals. Therefore, given a line b_6 with 5 skew transversals a_1, a_2, a_3, a_4, a_5, there exist lines b_1, b_2, b_3, b_4, b_5 such that b_i is the second transversal besides b_6 of a_j, a_k, a_ℓ, a_m. Then

(i) the lines b_1, b_2, b_3, b_4, b_5 have a transversal a_6;

(ii) there exists a unique polarity with respect to which the double-six is self polar, that is for which a_i is polar to b_i $\quad i = 1, \cdots, 6$.

Although (i) is formulated in terms of an algebraically closed field, the proof of (i) and consequently (ii) given by Kasner [32] is true for all fields except perhaps GF(q) $q \leq 5$, since the proof depends on only four parameters not being equal to the zero or the unit element of the field. Since lines over GF(q) contain $q + 1$ points and since at least 5 points on a line are required for the double-six, GF(4) and GF(5) demand first attention.

The diagonal surface of Clebsch [13] contains 15 lines over any field, diagonals of the quadrangles cut out on each of the 5 planes of a pentahedron by the other 4. If $\mu^2 = \mu + 1$ has two roots in the field, the surface contains another 12 lines forming a double-six which Burnside [5] showed to be invariant under 120 collineations of the space. Baker's description [4] p.168 of this double-six is sufficient to show that such a Burnside double-six does indeed exist over GF(4).

Over a field of characteristic other than two, an involutory reciprocity is either (i) a polarity with respect to a quadric

or (ii) a null polarity.

Over a field of characteristic two, an involutory reciprocity is

either (i) a null polarity

or (ii) a pseudopolarity, Segre [57] pp. 238-245.

It will now be shown that a double-six self-polar with respect to a linear complex exists over GF(4) and in fact only over GF(4) and its extensions; moreover every double-six over GF(4) is such a one. Then it will follow that such a double-six is a special Burnside double-six and cannot exist over $GF(2^n)$ where n is odd since $\mu^2 = \mu + 1$ has then no roots: $\mu^3 = 1$ has 3 roots in GF(q) only if q - 1 is divisible by 3, and $2^n - 1$ is not divisible by 3 for n odd.

§ 5. Existence of the double-six over GF(4).

Over any field, the coordinates of the line p through $X(x_0, x_1, x_2, x_3)$ and $Y(y_0, y_1, y_2, y_3)$ are $(p_{01}, p_{02}, p_{03}, p_{12}, p_{31}, p_{23})$ where $p_{ij} = x_i y_j - x_j y_i$ and $p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0$. Such line coordinates uniquely determine a line and are uniquely determined by the line. If two points are conjugate with respect to a linear complex, the line joining them is self-polar and belongs to the complex. If two lines a, b are polar with respect to a linear complex C, denote this by $a \mathcal{P}_C b$ or more briefly $a \mathcal{P} b$.

The mutual invariant of two lines r, s is

$\tilde{\omega}(r, s) = r_{01} s_{23} + r_{02} s_{31} + r_{03} s_{12} + r_{12} s_{03} + r_{31} s_{02} + r_{23} s_{01}$;

r, s intersect if and only if $\tilde{\omega}(r, s) = 0$.

Let C be the linear complex

$\Sigma a_{ij} p_{ij} \equiv a_{01} p_{01} + a_{02} p_{02} + a_{03} p_{03} + a_{12} p_{12} + a_{31} p_{31} + a_{23} p_{23} = 0$

with $a_{01} a_{23} + a_{02} a_{31} + a_{03} a_{12} \neq 0$ so that C is not special: put $a_{01} a_{23} + a_{02} a_{31} + a_{03} a_{12} = 1$. Let $R = \Sigma a_{ij} r_{ij}$ and $R' = \Sigma a_{ij} r'_{ij}$ where r_{ij}, r'_{ij} are the coordinates of the lines r, r'. If $r \mathcal{P} r'$ then

$$r_{ij} + r'_{ij} = R a_{k\ell}$$

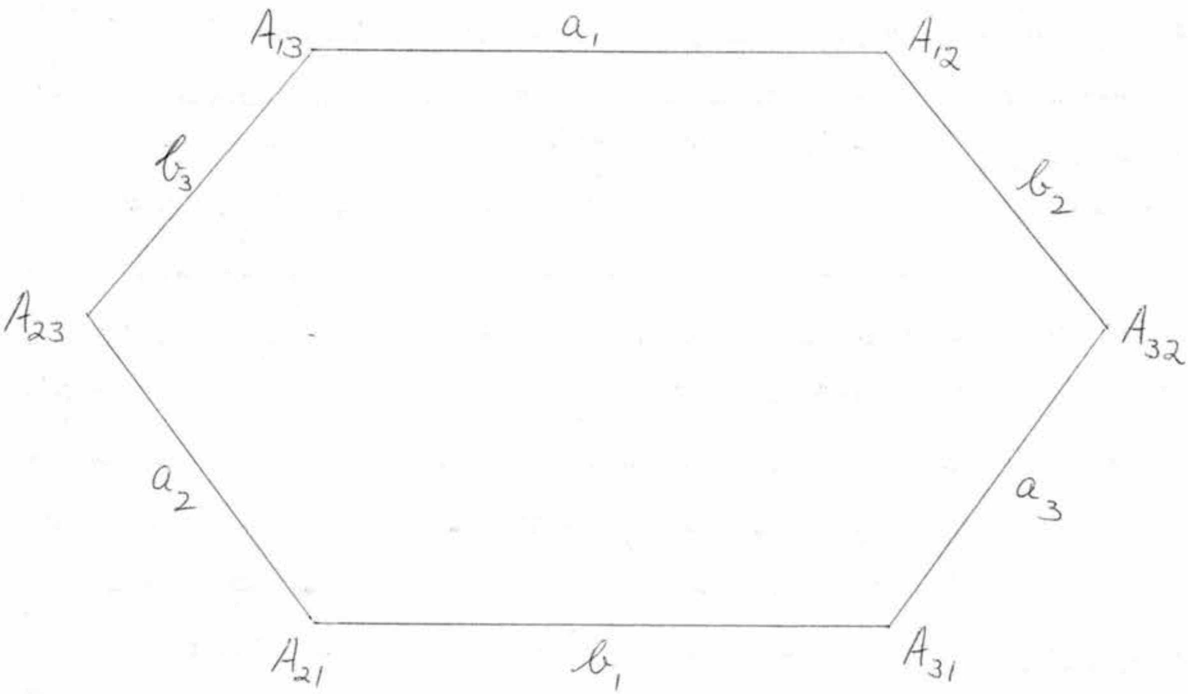
as shown by Baker [4] p.64 in a proof valid over any field. If this result is multiplied by a_{ij} and summed, it follows immediately that $R = R'$. Let s, s' be polar lines with coordinates s_{ij}, s'_{ij}; then $s_{ij} + s'_{ij} = S\, a_{k\ell}$ where $S = \Sigma\, a_{ij}\, s_{ij}$. If r, s are conjugate so that r meets s', then $\tilde{\omega}(r, s') = 0$; but $\tilde{\omega}(r, s') = \Sigma\, r_{ij} s'_{k\ell} = \Sigma\, r_{ij}(S\, a_{ij} - s_{k\ell}) = RS - \tilde{\omega}(r, s)$ therefore

$$\tilde{\omega}(r, s) = RS .$$

The mutual invariant of r', s' is

$$\tilde{\omega}(r', s') = \Sigma\, r'_{ij}\, s'_{k\ell} = \Sigma(R\, a_{k\ell} - r_{ij})(S\, a_{ij} - s_{k\ell})$$

$$= 2\,RS - RS - RS + \Sigma\, r_{ij}\, s_{k\ell} = \tilde{\omega}(r, s)$$



Consider the skew hexagon H : $a_1\, b_3\, a_2\, b_1\, a_3\, b_2$ with vertices $A_{ij} = (a_i, b_j)$. Let H be self-polar with respect to a linear complex, i.e. $a_i \,\mathcal{P}\, b_i$ $i = 1, 2, 3$; then A_{ij} lies in the opposite plane $[a_j, b_i]$, its polar plane. Thus the planes $[a_2, b_1]$, $[a_1, b_3]$, $[a_3, b_2]$ all contain the points A_{12}, A_{31}, A_{23} which are therefore collinear. Similarly the planes $[a_1, b_2]$,

$[a_3, b_1]$, $[a_2, b_3]$ intersect in the line $A_{21}\, A_{13}\, A_{32}$. Thus if a skew hexagon is self-polar with respect to a linear complex, sets of alternate vertices are collinear. Let these two lines be called the axes of H.

The converse if also true. Take A_{13}, A_{12}, A_{31}, A_{21} as the reference points X_0, X_1, X_2, X_3 respectively, and let the unit point be on $A_{32}\, A_{23}$. Then, A_{32} being on $A_{13}\, A_{21}$ and A_{23} being on $A_{31}\, A_{12}$, the equation of the complex is $p_{01} + p_{23} = 0$: the hexad is the one in the next diagram. Thus there is a unique linear complex with respect to which the hexad is self-polar.

Now take H: $a_1\, b_3\, a_2\, b_1\, a_3\, b_2$ with sets of alternate vertices collinear and the unique linear complex C such that $a_i \, \mathcal{P} \, b_i \quad i = 1, 2, 3.$ Take a transversal b_6 of a_1, a_2, a_3, skew to b_1, b_2, b_3. Restricting the field so that $\mu^2 = \mu + 1$ has two roots, take the two transversals a_4, a_5 of b_1, b_2, b_3, b_6. Then let b_4, b_5, a_6 be the polars of a_4, a_5, b_6 respectively: both b_4 and b_5 meet a_1, a_2, a_3, a_6 and a_6 meets b_1, b_2, b_3, b_4, b_5. Let $\tilde{\omega}_{ij} = \tilde{\omega}(a_i, a_j) = \tilde{\omega}(b_i, b_j)$ and C be $\Sigma\, c_{ij}\, p_{ij} = 0$ with $c_{01}\, c_{23} + c_{02}\, c_{31} + c_{03}\, c_{12} = 1$. Put $A_i = \Sigma\, c_{jk}\, a^i_{jk}$ where a^i_{jk} are the line coordinates of a_i; then by the conjugacy conditions described above,

$$\tilde{\omega}_{ij} = A_i A_j \quad \text{except for } ij = 45.$$

If a_4 is proved conjugate to a_5, and therefore b_4 to b_5, the theorem postulated in §4 will be established. This will be proven if $\tilde{\omega}_{45} = A_4\, A_5$.

Consider $W = (\tilde{\omega}_{ij})$ $\quad i, j = 1, \dots, 5$; W is a symmetric matrix with diagonal elements zero. Let the cofactor of $\tilde{\omega}_{ij}$ be π_{ij}. Then the condition that the 5 lines $a_i \quad i = 1, \dots, 5$ have a transversal is

$$w = 3|W| - \sum_{i<j}{}^{i,j=5}_{i,j=1} \tilde{\omega}_{ij}\, \pi_{ij} = 0 \qquad \text{(Appendix I).}$$

This only reduces to $|W| = 0$ over fields of characteristic other than two. Substituting in the formula $w = 0$ for the $\tilde{\omega}_{ij}$, excluding $\tilde{\omega}_{45}$, gives

$\tilde{\omega}_{45} = 3A_4\ A_5$ (Appendix I). Therefore such a double-six exists over $GF(2^{2n})$ and only over $GF(2^{2n})$, i.e. $GF(4)$ and its extensions.

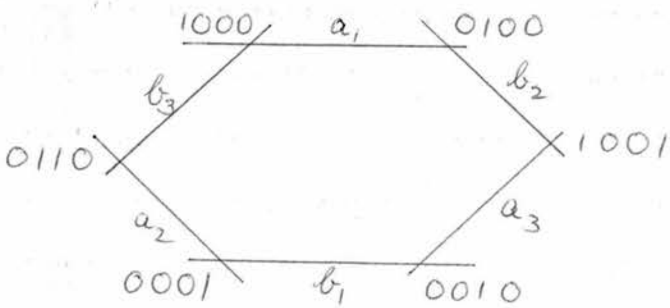
It will now be shown that over $GF(4)$ every skew hexagon which gives rise to a non-degenerate double-six has sets of alternate vertices collinear. Then it follows that every double-six over $GF(4)$ is as above.

Take H: $a_1\ b_3\ a_2\ b_1\ a_3\ b_2$ again. A regulus over $GF(4)$ consists of 5 skew lines. There is a regulus R complementary to the regulus $a_1\ a_2\ a_3$. If the double-six is to be non-degenerate, 3 lines of R, namely the lines b_4, b_5, b_6 of the double-six, meet each of the lines a_1, a_2, a_3 in no vertices of the hexagon. As lines over $GF(4)$ contain only 5 points, the remaining two lines of R meet a_1, a_2, a_3 in the 6 vertices of the hexagon. Therefore these two lines must be $A_{12}\ A_{31}\ A_{23}$ and $A_{21}\ A_{13}\ A_{32}$.

It is worth pointing out that a hexagon with alternate vertices collinear can be self-polar with respect to a quadric. From §4, a polarity can have two forms

(i) $\sum a_{ii}\, x_i\, y_i + \sum_{i<j} a_{ij}(x_i\, y_j + x_j\, y_i) = 0$

(ii) $\sum_{i<j} a_{ij}(x_i\, y_j - x_j\, y_i) = 0$



With respect to this hexagon, for fields of characteristic two, both (i) and (ii) become

$$(x_0\, y_1 + x_1\, y_0) + (x_2\, y_3 + x_3\, y_2) = 0.$$

However for other fields, (i) becomes

$$(x_0 y_1 + x_1 y_0) - (x_2 y_3 + x_3 y_2) = 0$$

and a double-six may well exist containing a_i, b_i $i = 1, 2, 3$ with this as its polarity; whereas (ii) becomes

$$(x_0 y_1 - x_1 y_0) + (x_2 y_3 - x_3 y_2) = 0$$

and it has been shown that a double-six does not exist.

§6. The diagonal surface.

PG(3, 4) contains 85 points, 357 lines, 85 planes. We have established the existence of the double-six of lines a_i, b_i $i = 1, \ldots, 6$ and of its polarity with respect to a linear complex C in the smallest field in which the double-six could be defined.

Let c_{ij} be the intersection of the planes $[a_i, b_j]$, $[a_j, b_i]$ and c'_{ij} the join of the points (a_i, b_j), (a_j, b_i). From any hexagon ijk $c_{ij} = c'_{ij}$, this line being a diagonal of the hexagon. The c_{ij} are self-polar lines and thus lie in C; c_{ij} meets a_i, b_j and as in classical geometry c_{ij} meets $c_{k\ell}$ only for $k, \ell \neq i, j$, Baker [4] p.160.

Any further intersections among the lines would imply a degeneration of the double-six. Each line is thus met by 10 and only 10 lines, 2 through each of its 5 points. There are 6 lines a_i, 6 lines b_i, 15 lines c_{ij}. Each line contains 5 points. Each of these points lies on 3 lines. Therefore the 27 lines comprise $27 \times 5/3 = 45$ points.

The unique cubic surface F_4 on which the double-six lies consists only of the 45 points lying on the 27 lines; for the 5 planes through a_1 cover the space and meet F_4 in the cubic curves $a_1 b_j c_{1j}$ $j = 2, \ldots, 6$. The 3 lines of such a cubic curve, being two adjacent sides and the diagonal meeting them of a hexagon, are concurrent. Thus the 45 points of F_4 are all points of concurrency of the 3 lines in which the surface is met by the 45 tritangent planes and so are all E-points (Eckardt points). The 45 tritangent planes are 30 of the type $[a_i, b_j, c_{ij}]$, 15 of the type $[c_{ij}, c_{k\ell}, c_{mn}]$.

As in classical geometry, the 27 lines form 36 double-sixes of the types

D	a_1	a_2	a_3	a_4	a_5	a_6	1
	b_1	b_2	b_3	b_4	b_5	b_6	
D_{12}	a_1	b_1	c_{23}	c_{24}	c_{25}	c_{26}	${}^6C_2 = 15$
	a_2	b_2	c_{13}	c_{14}	c_{15}	c_{16}	
D_{123}	a_1	a_2	a_3	c_{56}	c_{46}	c_{45}	${}^6C_3 = 20.$
	c_{23}	c_{13}	c_{12}	b_4	b_5	b_6	

Each of the 36 double-sixes D, D_{ij}, D_{ijk} is self-polar with respect to a unique non-special linear complex d, d_{ij}, d_{ijk}.

Now since, as is seen from Baker's description, the double-six over GF(4) appears from Burnside's construction on 5 arbitrary points and since it has been shown that all double-sixes over GF(4) are of the same type, they are all projectively equivalent. F_4 has a projective group A(4, 3) of order 25, 920 as shown by Frame [24]; this group is transitive on the 36 double-sixes. Therefore each double-six has a projective group of order 720. This is S_6, Edge [22], and is isomorphic to the symplectic group in 4 variables over GF(2). This raises the problem of why S_6 appears as the projective group instead of S_5 as with Burnside.

In complex space the Sylvester pentahedron of the diagonal surface

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \qquad x_0 + x_1 + x_2 + x_3 + x_4 \equiv 0$$

has faces $x_i = 0$ $i = 0, \cdots, 4$ and its 10 edges lie on the Hessian of the surface. Over GF(4) the Hessian is not defined. However, apart from the 5 faces of the pentahedron, the other 10 planes of PG(3, 2) $x_i + x_j = 0$ $i, j = 0, \cdots, 4$ are also tritangent planes of the surface and pass one through each of the 10 edges of the pentahedron. Any of the 5 faces, let it be $x_0 = 0$, and the 4 of the 10 planes through the 4 edges of the pentahedron on $x_0 = 0$, viz. $x_0 + x_1 = 0$, $x_0 + x_2 = 0$, $x_0 + x_3 = 0$, $x_0 + x_4 = 0$, also form a Sylvester pentahedron of the surface as

$$x_0^3 + (x_0 + x_1)^3 + (x_0 + x_2)^3 + (x_0 + x_3)^3 + (x_0 + x_4)^3$$
$$\equiv 5x_0^3 + x_0^2(x_1 + x_2 + x_3 + x_4) + x_0(x_1^2 + x_2^2 + x_3^2 + x_4^2) + x_1^3 + x_2^3 + x_3^3 + x_4^3$$
$$\equiv x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3$$

and $x_0 + (x_0 + x_1) + (x_0 + x_2) + (x_0 + x_3) + (x_0 + x_4) \equiv \Sigma x_i$.

There are 4 other pentahedra formed in this way and hence the 15 planes of the PG (3, 2) contained in PG (3, 4) form 6 mutually interwoven pentahedra. Thus there are 6 Sylvester pentahedra belonging to the surface and to a particular double-six on it. The intersections of any 2 planes of different pentahedra viz. $x_i = x_j + x_k = 0$ may be taken as the 15 lines c_{ij}. The other 30 tritangent planes $[a_i, b_j]$ pass 5 through each of the other 12 lines, which form the double-six D. The 6 pentahedra show that the group of the double-six is S_6.

Dually the double-six is generated by 6 Burnside pentagons which form a set of 6 mutually interwoven pentagons any two of which have a common vertex. The 15 vertices are all the points of a PG(3, 2), Edge [22]. The 15 faces of the pentahedra above being the $[c_{ij}, c_{k\ell}, c_{mn}]$, the 15 vertices of the pentagons are their poles $(c_{ij}, c_{k\ell}, c_{mn})$ with respect to the polarity of the double-six D, which is $\Sigma(x_i y_j + x_j y_i) = 0$.

All the double-sixes are projectively equivalent; so, for each of the 36 double-sixes, the 15 tritangent planes of F_4 containing no line of a double-six form 6 Sylvester pentahedra which give 6! projectivities of the double-six. Thus F_4 is a diagonal surface in $6 \times 36 = 216$ ways and, as before, its projective group is of order $6! \times 36 = 25,920$.

§7. The equianharmonic surface.

A Steiner trihedral pair is two sets of 3 planes whose 9 lines of intersection can be arranged as, for example,

$$\begin{array}{ccc} c_{23} & b_3 & a_2 \\ a_3 & c_{13} & b_1 \\ b_2 & a_1 & c_{12} \end{array} .$$

Let S_{123} denote the array of 9 lines as well as the set of 6 planes containing the 3 lines of a row or column of the array. The 3 planes of each trihedron have a line in common in contrast to the classical case where the 3 planes have mostly only a point in common. These two lines will be called the axes of the trihedral pair. The lines of any determinantal product in the array e.g. $c_{23}\ c_{13}\ c_{12}$ are the diagonals of the hexagon formed by the remaining 6 lines. The 6 hexagons so obtainable all have the same 6 vertices and the same two axes, which are also the axes of S_{123}. Thus the 9 lines are the joins of two sets of 3 points lying on two skew lines as in the diagram below.

The 45 tritangent planes form 120 Steiner trihedral pairs, 20 of the type S_{123} as well as

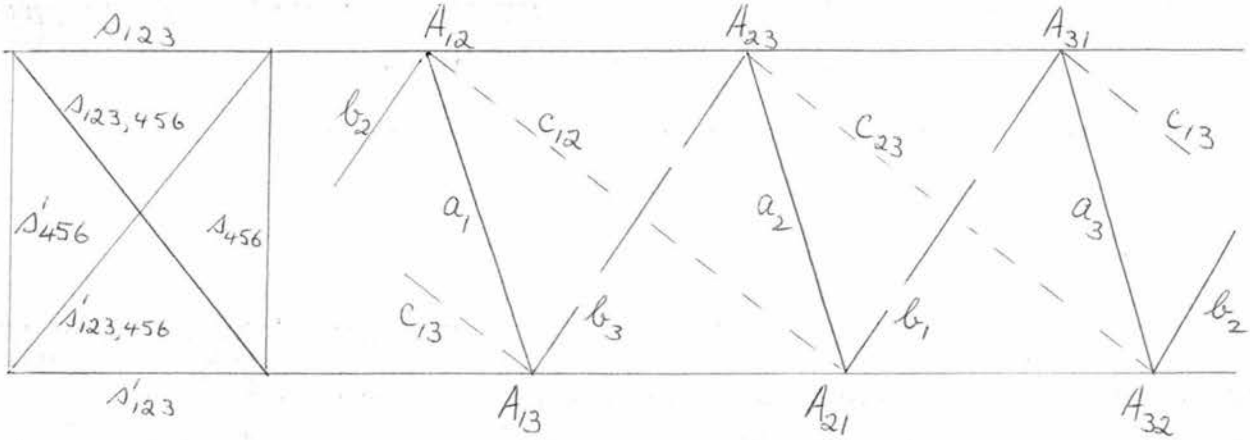
$${}^6C_2 \times {}^4C_2 = 90 \text{ of the type} \qquad S_{12,34} \quad \begin{array}{ccc} a_1 & b_4 & c_{14} \\ b_3 & a_2 & c_{23} \\ c_{13} & c_{24} & c_{56} \end{array}$$

$$\text{and } {}^6C_3/2 = 10 \text{ of the type} \qquad S_{123,456} \quad \begin{array}{ccc} c_{14} & c_{25} & c_{36} \\ c_{26} & c_{34} & c_{15} \\ c_{35} & c_{16} & c_{24} \end{array}$$

Each tritangent plane lies in $240 \times 3/45 = 16$ trihedra, of which the 32 other faces are those tritangent planes containing none of the 3 lines of the original tritangent plane.

Let s_{ijk}, s'_{ijk}; $s_{ij,k\ell}, s'_{ij,k\ell}$; $s_{ijk,\ell mn}, s'_{ijk,\ell mn}$ be the axes of the trihedral pairs S_{ijk}, $S_{ij,k\ell}$, $S_{ijk,\ell mn}$. The 120 trihedral pairs fall into 40 triads, each of which provides a trichotomy of the 27 lines -

10 triads like S_{123}, S_{456}, $S_{123,456}$ and 30 like $S_{12,34}$, $S_{34,56}$, $S_{56,12}$. The axes of such a triad of trihedral pairs are the 3 pairs of opposite edges of a tetrahedron.



The figure contains 6 reguli $a_1\ a_2\ a_3$, $b_1\ b_2\ b_3$, $c_{23}\ c_{13}\ c_{12}$ each with s_{456}, s'_{456} and $a_1\ b_1\ c_{23}$, $a_2\ b_2\ c_{13}$, $a_3\ b_3\ c_{12}$ each with $s_{123,456}$ $s'_{123,456}$, as for example s_{456}, s'_{456} are in the complementary regulus to $b_4\ b_5\ b_6$ which is the complementary regulus to $a_1\ a_2\ a_3$. The 6 hexagons arise from the 6 permutations of one of the sets of 3 vertices whilst the other remains fixed. The 3 pairs of opposite lines of each hexagon are polar lines in a double-six. These are D, D_{123}, D_{456}; D_{23}, D_{13}, D_{12}. The double-sixes formed from the two sets of 6 hexagons whose axes are s_{456}, s'_{456} ; $s_{123,456}$, $s'_{123,456}$ are D, D_{123}, D_{456}, D_{56}, D_{46}, D_{45}; D_{23}, D_{13}, D_{12}, D_{56}, D_{46}, D_{45}.

If the faces of the tetrahedron, whose edges are the 6 axes of the triad of trihedral pairs, are $x_i = 0$ $i = 0, \cdots, 3$ then the equation of the surface is

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0 ;$$

this is the equianharmonic surface of Segre [48] p.149, which is given over GF(4) by Coxeter [15]. In classical geometry the surface has only the 18

E-points on the edges of the tetrahedron, which is the Hessian of the surface; it has a projective group of order $3^3 \times 4! = 648$. In this case, the Hessian is undefined. However, each of the 40 triads of trihedral pairs will give such a tetrahedron whose edges contain 18 E-points. Thus F_4 is equianharmonic in 40 ways and has a projective group of order $40 \times 648 = 25{,}920$ as before. Another tetrahedron giving the same surface is apparent from the identity

$$(x_1 + x_2 + x_3)^3 + (x_0 + x_2 + x_3)^3 + (x_0 + x_1 + x_3)^3 + (x_0 + x_1 + x_2)^3 \equiv x_0^3 + x_1^3 + x_2^3 + x_3^3$$

Consistently, the equianharmonic surface can be transformed into the diagonal surface as, for example,

$$(\omega x_0 + x_1 + x_2 + x_3)^3 + (x_0 + \omega x_1 + x_2 + x_3)^3 + (x_0 + x_1 + \omega x_2 + x_3)^3 + (x_0 + x_1 + x_2 + \omega x_3)^3 \equiv x_0^3 + x_1^3 + x_2^3 + x_3^3 + (x_0 + x_1 + x_2 + x_3)^3$$

where $\omega^2 + \omega + 1 = 0$.

Thus, over GF(4), the cubic surface containing 27 lines is simultaneously diagonal, depending on $\mu^2 = \mu + 1$, in 216 ways and equianharmonic, depending on $\mu^2 + \mu + 1 = 0$, in 40 ways, and has a projective group of order 25,920.

The projective group of PG(3, 4) has order

$$(4^4 - 1)(4^4 - 4)(4^4 - 4^2)(4^4 - 4^3)/(4 - 1) = 2^{12}.\ 3^4.\ 5^2.\ 7.\ 17.$$

Thus the number of double-sixes over GF(4) is

$$2^{12}.\ 3^4.\ 5^2.\ 7.\ 17/720 = 2^8.\ 3^2.\ 5.\ 7.\ 17.$$

Alternatively the number of double-sixes is one sixth the number of pentahedrons

$= 85.\ 84.\ 80.\ 64.\ 27/6.\ 5!$

$= 2^8.\ 3^2.\ 5.\ 7.\ 17$

$= 1,\ 370,\ 880.$

§ 8. A representation in Π_5 .

There are two classical representations of the cubic surface, which have a particular appropriateness here: firstly, either the Plücker-Klein representation of lines in Π_3 by points of a quadric in Π_5 or its dual; secondly, the Clebsch mapping of a cubic surface onto a plane.

Consider the following representation in Π_5 in which lines of Π_3 become tangent primes to a quadric Ω in Π_5 and non-special linear complexes become points of Π_5 not on Ω. The line $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$ becomes the prime $p_{23} x_0 + p_{13} x_1 + p_{12} x_2 + p_{03} x_3 + p_{02} x_4 + p_{01} x_5 = 0$. This is a tangent prime to Ω: $x_0 x_5 + x_1 x_4 + x_2 x_3 = 0$ at the point $(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})$. If p, q are two intersecting lines, then the point of contact of the tangent prime representing q lies in the tangent prime representing p and vice versa. The linear complex C

$$a_{23} p_{01} + a_{13} p_{02} + a_{12} p_{03} + a_{03} p_{12} + a_{02} p_{13} + a_{01} p_{23} = 0$$

becomes the point $(a_{01}, a_{02}, a_{03}, a_{12}, a_{13}, a_{23})$. From §5, if two lines r, s are such that $r \mathscr{P} s$, then $r_{ij} + s_{ij} = \lambda a_{ij}$. Therefore the points of contact of the primes representing r, s are collinear with the point representing C. If three coplanar lines are concurrent, they are linearly dependent; so their representing primes as well as the latters' points of contact are collinear.

Let the primes representing a_i, b_i, c_{ij} be A_i, B_i, Γ_{ij} with points of contact α_i, β_i, γ_{ij} and the points representing d, d_{ij}, d_{ijk} be δ, δ_{ij}, δ_{ijk} whose polar primes with respect to Ω are Δ, Δ_{ij}, Δ_{ijk}. The row $a_1\ a_2\ a_3\ a_4\ a_5\ a_6$ of the double-six D is represented by the simplex with faces A_i and vertices β_i . The row $b_1\ b_2\ b_3\ b_4\ b_5\ b_6$ is represented by the simplex with faces B_i and vertices α_i. Thus the two rows of the double-six become two simplexes inscribed and circumscribed to each other and to the quadric primal and in perspective from the point δ. Also, as all the lines c_{ij} lie in the complex d, all the primes Γ_{ij} pass through δ. All the other double-sixes are similarly represented.

This figure of 63 points α_i, β_i, γ_{ij}, δ, δ_{ij}, δ_{ijk} is equivalent but not projective to the entire space over GF(2) as described by Edge [21]. There are 651 lines in PG(5, 2) and 651 linear relations can be obtained from the polar relations like $\alpha_i + \beta_i = \delta$ and the relations dependent upon the E-points like $\alpha_i + \beta_j = \gamma_{ij}$, as the correspondence with the configuration over GF(2) shows that these linear relations can all be written with unit coefficients. This correspondence is perhaps not so surprising. The set of 27 points in PG(5, 2) corresponding to the 27 lines of the cubic surface is all the points of an elliptic quadric which can be taken as $\sum_{i<j} x_i x_j = 0$. When the field is extended by a root of $\mu^2 = \mu + 1$, this quadric can be transformed into the Klein quadric $\Omega : x_0 x_5 + x_1 x_4 + x_2 x_3 = 0$ which is ruled. The extension of GF(2) to GF(4) to produce properties connected with the double-six has already appeared in § 6, where the diagonal surface was considered.

Let x, y be points in Π_5. Define $x \circ y = \sum_{i=0}^{i=5} x_i y_{5-i}$. Then the intersection of lines in Π_3 is expressible by $\alpha_i \circ \beta_j = 0$ etc. Other properties of the configuration can now be derived. For example, as $\alpha_k + \beta_k = \delta$ and $\gamma_{ik} + \gamma_{jk} = \delta_{ij}$, so $\delta \circ \delta_{ij} = 0$; thus the points δ_{ij} all lie in Δ, which implies that exactly 5 out of the 16 linear complexes d, d_{ij} are independent.

One can also derive linear properties of the δ's. As $\delta = \alpha_1 + \beta_1$, $\delta_{123} = \alpha_1 + \gamma_{23}$, $\delta_{456} = \beta_1 + \gamma_{23}$, so $\delta + \delta_{123} + \delta_{456} = 0$.
The existence of this line in Π_5 is equivalent to the existence in Π_3 of the Steiner trihedral pair $S_{123,456}$, as the 9 primes Γ_{ij} $i = 1, 2, 3$; $j = 4, 5, 6$ representing the 9 lines c_{ij} of the array $S_{123,456}$ all

contain the points δ, δ_{123}, δ_{456}. This line with the lines $\delta_{12} + \delta_{13} + \delta_{23} = 0$, $\delta_{45} + \delta_{46} + \delta_{56} = 0$ span the Π_5; this fact is equivalent to the triad of Steiner trihedral pairs S_{123}, S_{456}, $S_{123,456}$ providing a trichotomy of the 27 lines. The 6 axes of this triad were in Π_8 the edges of a simplex; they have become in Π_5 the vertices of a simplex inscribed in Ω.

These results may be elaborated and others formulated by further exposition of the above linear $(x + y + z = 0)$ and multiplicative $(x \circ y = 0)$ properties. The relations among the linear complexes are similar to those in complex space among the Schur quadrics as shown by Room [42].

§ 9. The plane over GF(4) and the mapping of F_4 onto it.

In the plane over GF(4) there are 21 points, 21 lines, 5 points on a line, 5 lines through a point. The sides of a triangle contain its vertices and 9 other points; any one of the remaining 9 points of the plane with the 3 vertices of the triangle are vertices of a quadrangle Q. Q has 3 diagonal points and each of its 6 sides contains two of its vertices and one of its diagonal points leaving two points on each side. Thus the points in the plane not on any side of Q number $21 - 4 - 3 - 6.2 = 2$. One of these 2 points and the vertices of Q form a pentad P, a set of 5 points no 3 collinear. The points of P are the vertices of 5 quadrangles whose 15 diagonal points are necessarily distinct. Each of the 10 chords of P contains 2 of its points and meets the 3 chords through its other 3 points in the remaining 3 points of the chord. Thus the chords of P contain only its points and its diagonal points which number $5 + 15 = 20$. Hence the 4 vertices of Q and the 2 points not on its sides form a hexad H, a set of 6 points no 3 collinear.

Since the join of 2 points of H meets the 6 sides of the quadrangle Q whose vertices are the other 4 points of H in none of these vertices, none of the 3 other points of this join can lie on 3 sides of Q and therefore must each lie on 2 and only 2 of the 6 sides of Q: i.e. it passes through the 3 diagonal points of Q. Thus the 3 diagonal points of a quadrangle are collinear and lie on the line through the 2 points not on any side of the quadrangle. Q determines H uniquely and H is fixed by any of its 15 tetrads. The 15 lines of diagonal points of quadrangles of H are the 15 chords of H. H may be partitioned into 3 pairs of points in 15 different ways; any 2 chords of such a trichotomy, being sides of a quadrangle whose vertices they contain, meet at a diagonal point which lies on the third chord. Thus the 15 sets of 3 chords of H are concurrent at the 15 diagonal points of the quadrangles which are therefore all Brianchon points of H. This means that every hexagon in the plane is a Brianchon hexagon fifteenfold and the plane is exhausted by the 6 points of the hexad and its 15 Brianchon points.

Each of the 6 vertices of H lies on 5 of its 15 chords, thus the remaining 6 lines of the plane are skew to H. Hence the 15 Brianchon points of H are collinear in 6 sets of 5 on the above 6 lines. These 6 lines form a hexagram dual to H.

A conic, being the set of intersections of two projectively related pencils in the plane, is a pentad. The tangent at each point is uniquely determined as the line through the point not passing through the other four points; it is also the line of diagonal points of the quadrangle whose vertices are these 4 points. Hence the 5 tangents are concurrent at the sixth point of the hexad containing the 5 points of the conic.

In the mapping of the cubic surface onto a plane, plane sections of the surface become cubic curves through 6 base points A_i in the plane; a curve of order λ in the plane which has a "general multiple point" of order λ_i at the base point A_i maps a curve of order $3\lambda - \Sigma \lambda_i$ on the surface, Baker [4] p.191. The lines a_i become the hexad H of base points A_i, the lines b_i become conics B_i through the 5 points of H which exclude A_i, and the lines c_{ij} become chords $C_{ij} = A_iA_j$ of H. Since A_1A_2 is the tangent at A_2 to the conic $A_2A_3A_4A_5A_6$, the point (a_2, b_1) is an Eckardt point of the surface; in this fashion there appear the 30 E-points (a_i, b_j). The 15 Brianchon points of H map the 15 E-points $(c_{ij}, c_{k\ell}, c_{mn})$. The 6 Burnside pentagons become the points of the 6 lines skew to H. This is not surprising since a line in the plane not through any of the base points A_i is the map of a twisted cubic on the surface. The 5 points of a skew pentagon, which incidentally form an elliptic quadric over GF(2), form a twisted cubic over GF(4).

§ 10. The group of order 25, 920 in the plane.

Since the hexad H is fixed by any of its tetrads, there are 6.5.4.3 = 360 projectivities in the plane leaving H fixed. These projectivities impose the 360 even permutations on the points of H. To impose odd permutations, the automorphism that replaces every mark of GF(4) by its square must be used. The projectivities leaving H fixed transform all the cubic curves through H into one another and, therefore, plane sections of F will also be mapped into one another; thus, these projectivities leaving H fixed map projectivities in Π_3 that leave each half of the double-six D invariant. If there is any projectivity in Π_3 transposing the two halves of D , it will be mapped by Cremona transformation in the plane, which transforms cubic curves through H into one another.

The points A_i $(i = 1, \cdots 6)$ of H may be taken as 100, 010, 001, 111, $\omega\omega^2 1$, $\omega^2\omega 1$ respectively; then the conics B_i are

$x^2 = yz$, $\qquad y^2 = zx$, $\qquad z^2 = xy$

$xy + yz + zx = 0$, $\qquad \omega xy + \omega^2 yz + zx = 0$, $\qquad \omega^2 xy + \omega yz + zx = 0$,

and the lines C_{ij} $\quad i, j = 1, 2, 3$ are

C_{23} $\;x = 0$, $\qquad C_{13}$ $\;y = 0$, $\qquad C_{12}$ $\;z = 0$.

The only projectivity in Π_3 such that $a_i \to b_i$ (all i) and therefore, since the polarity of D will remain unchanged, $b_i \to a_i$ (all i), is the identity; for then the c_{ij} remain fixed and hence a pentahedron of the surface. If a projectivity, other than the identity, transposes the two halves of the double-six, take, since 4 points in the plane can be selected arbitrarily, $a_i \to b_i$, $i = 1, \cdot\cdot, 4$, $\quad a_5 \to b_6$, $\quad a_6 \to b_5$ which, again by the constancy of the polarity, implies $b_i \to a_i$, $\quad i = 1, \cdots, 4$, $\quad b_5 \to a_6$, $b_6 \to a_5$.

The plane sections $a_i b_j c_{ij} \longleftrightarrow b_i a_j c_{ij}$ $\quad i, j = 1, 2, 3$; then in the plane $B_j + C_{ij} \longleftrightarrow B_i + C_{ij}$ $\quad i, j = 1, 2, 3$. Thus in the transformation from the (x, y, z) plane to the (u, v, w) plane, writing

$X = x^2 + yz$, $\;\; Y = y^2 + zx$, $\;\; Z = z^2 + xy$, $\;\; U = u^2 + vw$, $\;\; V = v^2 + wu$, $\;\; W = w^2 + uv$

$$\begin{array}{llll} zY = 0 \to wU = 0 & & zX = 0 \to wV = 0 & \\ yZ = 0 \to vU = 0 & & yX = 0 \to vW = 0 & \ldots\ldots(A) \\ xZ = 0 \to uV = 0 & & xY = 0 \to uW = 0 & \end{array}$$

Each of the 6 cubic curves in the (x, y, z) plane is, as the arrows indicate, to become a corresponding cubic curve in the (u, v, w) plane. When the proper functions of u, v, w are substituted for x, y, z, each resulting polynomial must have the indicated cubic as a factor. If the residual factor is the same in all 6 instances

$y : z = vW : wV$, $\qquad z : x = wV : uW$, $\qquad x : y = uV : vU$

all of which relations hold if

$$x : y : z \;=\; uVW : vWU : wUV \qquad \ldots\ldots\ldots\ldots\ldots\ldots(B)$$

These relations (B) do, in fact, achieve the desired transformation of the composite cubic curves as, for example, X is then proportional to

$$u^2V^2W^2 + vwU^2VW \equiv VW(v^3w^3 + w^3u^3 + u^3v^3 + u^2v^2w^2)$$

and therefore yX is proportional to the product of vW and a symmetric function Σ of u, v, w.

$$\begin{aligned}\Sigma &\equiv UVW(v^3w^3 + w^3u^3 + u^3v^3 + u^2v^2w^2)\\ &\equiv UVW(vw + wu + uv)(\omega^2 vw + wu + \omega uv)(\omega vw + wu + \omega^2 uv)\end{aligned}$$

is the product of 6 quadratic factors which, equated to zero, are the 6 conics containing 5 of the 6 fundamental points. Σ is never zero except at these 6 points.

The relations (B) imply that at any point other than the 6 fundamental points

$$u : v : w = xYZ : yZX : zXY .$$

This would be expected from (A) if the arrows were reversible; its independent establishment shows that they are reversible. Direct calculation from (B) proves xYZ to be proportional to $u\Sigma^2$.

Thus the correspondence is an involutory Cremona transformation, which transforms lines into quintics through the A_i, and cubics through the A_i into one another.

There are 360 Cremona transformations and these, with the 360 plane projectivities, form a group of 720 birational transformations mapping isomorphically the group of projectivities of the double-six over GF(4).

Other projectivities of the cubic surface may be revealed by transforming D into other double-sixes of F_4. Plane sections of F_4 are transformed into others and equivalently cubic curves through H become other cubic curves through H. To transform, for example, D to D_{123}, suppose

$$\left.\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right) \rightarrow \left(\begin{matrix} a_1 & a_2 & a_3 \\ c_{23} & c_{13} & c_{12} \end{matrix}\right.$$

and $a_i b_j c_{ij} \rightarrow a_i c_{ik} b_k \qquad i, j, k = 1, 2, 3 .$

Thus in the plane

$$zY = 0 \rightarrow vW = 0 \qquad zX = 0 \rightarrow uW = 0$$
$$yZ = 0 \rightarrow wV = 0 \qquad yX = 0 \rightarrow uV = 0 \qquad \ldots\ldots(C)$$
$$xZ = 0 \rightarrow wU = 0 \qquad xY = 0 \rightarrow vU = 0$$

Following the previous argument, the transformation of the cubic curves is produced by

$$x : y : z = U : V : W \qquad \ldots\ldots(D)$$

as X is then proportional to

$$U^2 + VW \equiv u(u^3 + v^3 + w^3 + uvw) \quad \text{and}$$

therefore yX is proportional to the product of uV and a symmetric function of u,v,w; this function is the product of three factors which, equated to zero are the 3 lines A_4A_5, A_4A_6, A_5A_6. The relations (D) imply that, except at A_4, A_5, A_6,

$$u : v : w = X : Y : Z.$$

As before, this shows that the arrows of (C) are reversible.

The correspondence is again an involutory Cremona transformation which transforms lines into conics through A_4, A_5, A_6 and cubics through the A_i into one another.

To transform D to D_{12}, suppose

$$\left.\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}\right\} \rightarrow \left\{\begin{matrix} a_1 & b_1 & c_{23} \\ a_2 & b_2 & c_{13} \end{matrix}\right.$$

therefore

$$a_1\ b_2\ c_{12} \rightarrow a_1\ b_2\ c_{12}\ ; \qquad a_2\ b_1\ c_{12} \rightarrow b_1\ a_2\ c_{12}$$
$$a_1\ b_3\ c_{13} \rightarrow a_1\ c_{13}\ b_3\ ; \qquad a_3\ b_1\ c_{13} \rightarrow c_{23}\ a_2\ b_3$$
$$a_2\ b_3\ c_{23} \rightarrow b_1\ c_{13}\ a_3\ ; \qquad a_3\ b_2\ c_{23} \rightarrow c_{23}\ b_2\ a_3\ .$$

In the plane

$$zY = 0 \rightarrow wV = 0 \qquad zX = 0 \rightarrow wU = 0$$
$$yZ = 0 \rightarrow vW = 0 \qquad yX = 0 \rightarrow uW = 0 \qquad \ldots\ldots(E)$$
$$xZ = 0 \rightarrow vU = 0 \qquad xY = 0 \rightarrow uV = 0$$

The transformation of the cubic curves is achieved by

$$x : y : z = uU : uW : wU \qquad ....(F)$$

Then X is proportional to

$$u^2U^2 + uwUW = uU\,(u^3 + w^3)$$

so that yX is proportional to the product of uW and $uU(u^3 + w^3)$. The latter expression occurs in each case and is the product of 5 factors which, equated to zero, give the lines A_2A_3, A_2A_4, A_2A_5, A_2A_6 and the conic B_1.

The relations (F) imply that except at A_2, A_3, A_4, A_5, A_6

$$u : v : w = xX : xZ : zX .$$

This shows that the arrows of (E) are reversible.

The correspondence is again an involutory Cremona transformation, which transforms lines into cubics through A_2, A_3, A_4, A_5, A_6 and cubics through the A_i into one another.

Accordingly, by transforming D into itself and the 35 other double-sixes of F_4, we have 25,920 plane Cremona transformations forming a group which is the projective group of F_4. This plane over GF(4) and the group of order 25,920 were recently studied by Edge [23].

CHAPTER III. The double-six over GF(5) and arithmetical properties of cubic surfaces.

§ 11. Existence of the double-six over GF(5).

There is no double-six over GF(5) since every hexad in the plane is a conic.

Lines over GF(5) contain 6 points. Each chord of a pentad P meets 3 chords at two of the points of P, the other 3 chords at distinct points and thus no chord at its remaining point. The chords of P therefore consist of $5 + 10 \times 3/2 + 10 = 30$ points, leaving a single point in the plane which is therefore the remaining point of the conic, as well as of the hexad, containing P. Thus each hexad is a conic. This is also a particular case of Segre's theorem [51] that, in a Desarguesian plane of odd characteristic, every oval is a conic.

An argument solely in Π_3 is worthwhile. Suppose $D(a_i, b_i \quad i = 1, \cdots, 6)$ exists. Then it lies on a cubic surface F_5 containing 27 lines, each one met by 10 others. Each line, comprising exactly 6 points, contains at least 4 E-points. Therefore F_5 contains at least $27 \times 4/3 = 36$ E-points.

It will now be shown that a cubic surface F with 27 lines over any field of characteristic other than two has at most 18 E-points. D is determined by the skew hexagon $a_1 b_3 a_2 b_1 a_3 b_2$ and the line b_6 say. The hexagon determines a polarity d given by the bilinear form

$$\sum a_{ii} x_i y_i + \sum_{i<j} a_{ij}(x_i y_j + x_j y_i) = 0.$$

The polarity is always unique as even the most restricted case, where sets of alternate vertices are collinear as in § 5, gives 9 independent conditions, viz. the 8 a_{ij} excluding a_{01}, a_{23} are zero and $a_{01} + a_{23} = 0$. D is completed by constructing successively a_4, a_5, b_4, b_5, a_6. Thus d is necessarily a polarity of the double-six whatever the field.

Suppose (a_1, b_i) is an E-point. Then (a_1, b_i) lies on c_{1i}, hence in its own polar plane $[a_i, b_1]$ and hence on the polarising quadric $\Sigma a_{ij} x_i x_j = 0$. If a_1 contains 3 E-points, it lies entirely on this quadric and is self-polar. Thus a_1 and similarly all the lines of F have at most 2 E-points, and F contains at most $27 \times 2/3 = 18$ E-points. Thus there is again no double-six over GF(5).

In contrast to the above, let F be a cubic surface with 27 lines over a field of characteristic two. Then if a line on F contains 2 E-points it contains 5. Take again $D(a_i, b_i \quad i = 1, \ldots, 6)$ with its polarity

$$d : \sum a_{ii} x_i y_i + \sum_{i<j} a_{ij}(x_i y_j + x_j y_i) = 0$$

If all the a_{ii} are zero, the polarity is null and, as in §5, over $GF(2^{2n})$ every line of F contains 5 E-points whence F contains 45 E-points. Over $GF(2^{2n+1})$, with all the a_{ii} zero, F does not exist. Suppose then that not all the a_{ii} are zero and that a_1 contains 2 E-points. These two points, being self-conjugate, both lie in the plane $\Sigma \sqrt{a_{ii}}\, x_i = 0$ which therefore contains all the points of a_1. Thus each point (a_1, b_i) of a_1 lies in its polar plane $[a_i, b_1]$ and hence on c_{1i}. Thus a_1 contains 5 E-points.

§ 12. Arithmetical properties of the cubic surface and the plane over GF(q).

The compulsory presence of E-points on the cubic surfaces with 27 lines over small fields largely determines the structure of the surfaces. A cubic surface F with 27 lines over GF(q) comprises $q^2 + 7q + 1$ points. This appears from the plane mapping as each point of F is mapped to a separate point of the plane except for the lines $a_i \quad i = 1, \ldots, 6$ say, which are each mapped to a single point. So the number of points on F is

$$(q^2 + q + 1) - 6 + 6(q + 1) = q^2 + 7q + 1.$$

This number is obtained differently by Rosati [43].

Let the lines of F be ℓ_i $i = 1, \cdots, 27$ and let N be the total number of points on the ℓ_i. Suppose F has e E-points. Let e_i, d_i, p_i be the respective numbers of points on ℓ_i where it meets two, one and no other lines of F. Then

$d_i + 2e_i = 10$, $\quad p_i + d_i + e_i = q + 1$, $\quad \Sigma e_i = 3e$

$$N = \Sigma p_i + \frac{\Sigma d_i}{2} + \frac{\Sigma e_i}{3}.$$

Therefore

$$p_i + d_i/2 = q - 4$$

$$N = 27(q - 4) + e.$$

If a double-six is to exist

$$q^2 + 7q + 1 \geqslant 27(q - 4) + e.$$

So

$$e \leqslant q^2 - 20q + 109 = (q - 10)^2 + 9.$$

(For $q = 5$ this means $e \leqslant 34$, whereas if F_5 exists $e = 36$.)

If n is the number of points on F off the lines

$$e + n = (q - 10)^2 + 9.$$

Since each line meets 10 others, if $q \leqslant 9$ then

$$e \geqslant 27\{10 - (q + 1)\}/3 = 9(9 - q).$$

The difference between the upper and lower bounds for e is

$$q^2 - 11q + 28 = (q - 4)(q - 7).$$

q	$q^2 + 7q + 1$	$27(q - 4)$	$e + n =$ upper bound for e	lower bound for e
4	45	0	45	45
7	99	81	18	18
8	121	108	13	9
9	145	135	10	0
11	199	189	10	0
13	261	243	18	0
16	369	324	45	0

There are also some arithmetical properties connected with F in the plane. Consider a hexad of points in PG(2, q). It has 15 chords, each of which meets 4 others at two of its points and the remaining 6 at separate points if the hexad has no Brianchon points, which shall henceforth be called B- points. If 3 coplanar lines form a triangle they comprise 3q points: if concurrent, 3q + 1 points. Thus each B-point adds one point to the points on the chords of the hexad. Let b be the number of B-points, P the number of points in the plane and M the number of points on the chords. Let the chords be c_i $i = 1, .., 15$ and let c_i contain b_i, r_i, s_i points where it meets exactly 2, 1, 0 other chords respectively. Then

$b_i + r_i + s_i = q - 1$ $\qquad$ $2b_i + r_i = 6$ $\qquad$ $\Sigma\, b_i = 3b$

$$\begin{aligned} M &= \Sigma\, s_i + (\Sigma\, r_i)/2 + (\Sigma\, b_i)/3 + 6 \\ &= 15(q - 4) + b + 6 \\ &= 15q - 54 + b \\ P &= q^2 + q + 1 \end{aligned}$$

Therefore $b \leqslant q^2 - 14q + 55 = (q - 7)^2 + 6$;

and if $q \leqslant 7$, $\quad b \geqslant 15\,\{6 - (q - 1)\}/3 = 5(7 - q)$.

The difference between the upper and lower bounds for b is

$$q^2 - 9q + 20 = (q - 4)(q - 5)$$

q	P	15q - 54	upper bound for b	lower bound for b
4	21	6	15	15
5	31	21	10	10
7	57	51	6	0
8	73	66	7	0
9	91	81	10	0

As has been seen in particular cases, the chords of a triad contain 3q points, the chords of a tetrad contain $6q - 5$ points and the chords of a pentad contain $10q - 20$ points. Thus the number of hexads in $PG(2, q)$ not lying on a conic is

$$\{q^2+q+1\}\{(q^2+q+1) - 1\}\{(q^2+q+1)-(q+1)\}\{q^2+q+1)-3q\}\{(q^2+q+1) - (6q - 5)\}$$
$$\times\{(q^2+q+1) - (10q-20) - (q-4)\}/6!$$
$$= (q^2+q+1)(q^2+q)\ q^2\ (q^2-2q+1)(q^2-5q+6)(q^2-10q+25)/6!$$
$$= q^3(q+1)(q-1)^2\ (q-2)(q-3)(q-5)^2\ (q^2+q+1)/6!$$

CHAPTER IV. The double-six, its cubic surface and its extension over GF(7).

§ 13. The cubic surface over GF(7).

Since a line over GF(7) contains 8 points, every line of a cubic surface with 27 lines contains at least two E-points. So the surface contains at least $2 \times 27/3 = 18$ E-points. Over GF(q), as shown in § 12, a cubic surface with 27 lines comprises $q^2 + 7q + 1$ points which, for $q = 7$, is 99. The points lying on the 27 lines number $27(q - 4) + e$, where e is the number of E-points. For $q = 7$, this is $e + 81$: thus e is at most 18. Consequently a cubic surface over GF(7) containing 27 lines has exactly 18 E-points, two on each line, and contains no points not on the 27 lines. Since $\mu^2 + \mu + 1 = 0$ has two roots over GF(7), the equianharmonic surface

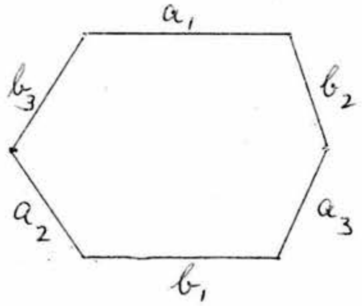
$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0, \qquad \text{Segre [48] p. 149,}$$

contains 27 lines and 18 E-points, two on each line. The E-points lie in threes on the six edges of a tetrahedron which is analogous to one of the 40 tetrahedra that occurred in § 7 over GF(4), whose edges were the axes of a triad of Steiner trihedral pairs. Here in GF(7), of the 120 trihedral pairs, only 3 have the property that the three faces of each trihedron are collinear. Any two vertices of the tetrahedron each complete equianharmonic tetrads with the 3 E-points on their join.

Let F be any cubic surface over GF(7) with 27 lines. To prove that F is always equianharmonic, it will first be shown that the 18 E-points, 2 on each of the 27 lines, lie in threes on the edges of a tetrahedron.

To do this we show that 2 E-points not lying on the same line of F are collinear with a third E-point. This means that in the array of lines determined by a Steiner trihedral pair, e.g. S_{123}, if the lines of two rows are

are concurrent at E-points then the lines of the third row are also concurrent at an E-point and the 3 E-points are collinear. So consider once more the skew hexagon $a_1\ b_3\ a_2\ b_1\ a_3\ b_2$



the lines belong to S_{123}

c_{23}	a_3	b_2
b_3	c_{13}	a_1
a_2	b_1	c_{12}

Let (a_1, b_3) and (a_2, b_1) be E-points: (a_1, b_3) is on c_{13} and therefore lies in $[a_3, b_1]$; similarly (a_2, b_1) lies in $[a_1, b_2]$. Therefore the 3 planes $[a_3, b_1]$, $[a_1, b_2]$, $[a_2, b_3]$ contain the line ℓ, the join of (a_1, b_3) and (a_2, b_1). But $[a_3, b_1]$, $[a_1, b_2]$ both contain (a_3, b_2) which therefore lies on ℓ and in $[a_2, b_3]$. Thus (a_3, b_2) lies in both $[a_3, b_2]$ and $[a_2, b_3]$ and so on their intersection c_{23}. Hence (a_3, b_2) is also an E-point and the 3 E-points (a_3, b_2, c_{23}), (a_1, b_3, c_{13}), (a_2, b_1, c_{12}) are collinear.

Let the E-points on a_1 and b_1 be (a_1, b_i), (a_1, b_j), (a_k, b_1), (a_ℓ, b_1). Then there are three cases to consider, namely when the two pairs i, j and k, ℓ have both, one or no members in common.

(i) Let (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_3, b_1) be E-points; then (a_2, b_3), (a_3, b_2) are also E-points. The 6 points (a_i, b_j) $i, j = 4, 5, 6$ are all E-points since, for example, a_4 can only contain the E-points (a_4, b_5), (a_4, b_6) as b_1, b_2, b_3 each have 2 E-points already. In the same way the remaining E-points are the 6 points (c_{1i}, c_{2j}, c_{3k}) $i, j, k = 4, 5, 6$. Thus the edges of the appropriate tetrahedron are the axes of the triad S_{123}, S_{456}, $S_{123,456}$.

(ii) Let $(a_1, b_3), (a_1, b_4), (a_5, b_1), (a_6, b_1)$ be E-points; then $(a_3, b_5), (a_3, b_6), (a_4, b_5), (a_4, b_6)$ are also E-points. So $a_1, a_3, a_4, b_1, b_5, b_6$ have their full complement of 2 E-points. So the E-points on a_2 and b_2 are $(a_2, b_3), (a_2, b_4), (a_5, b_2), (a_6, b_2)$ giving 2 E-points on each a_i, b_i. From the array $S_{12,34}$, (a_1, b_3, c_{13}) and (a_2, b_4, c_{24}) being E-points implies that (c_{14}, c_{23}, c_{56}) is an E-point. Similarly we have the other E-points (c_{13}, c_{24}, c_{56}), (c_{12}, c_{35}, c_{46}), (c_{12}, c_{36}, c_{45}), (c_{15}, c_{26}, c_{34}). The edges of the appropriate tetrahedron are then the axes of the triad $S_{12,34}$, $S_{34,56}$, $S_{56,12}$.

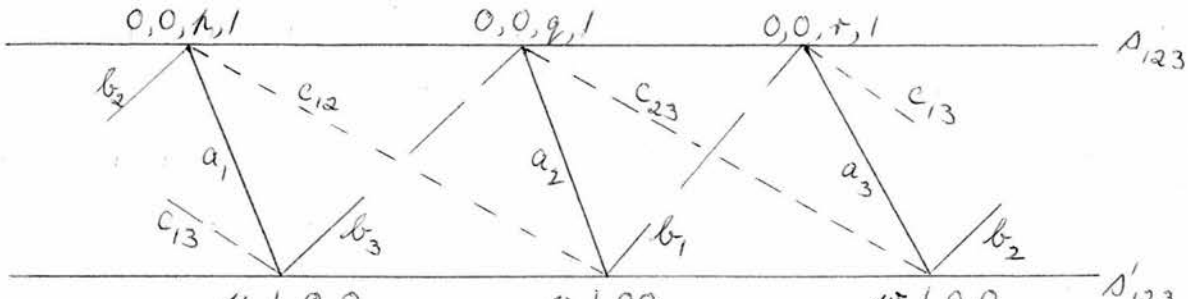
(iii) Let $(a_1, b_2), (a_1, b_3), (a_2, b_1), (a_4, b_1)$ be E-points.

Then (a_2, b_4) is an E-point as $(a_1, b_2), (a_4, b_1)$ are E-points,
(a_3, b_4) is an E-point as $(a_1, b_3), (a_4, b_1)$ are E-points,
(a_3, b_2) is an E-point as $(a_1, b_3), (a_2, b_1)$ are E-points.
So (a_4, b_3) is an E-point as $(a_2, b_4), (a_3, b_2)$ are E-points.

Thus each of the 8 lines a_i, b_i $i = 1, \ldots, 4$ contains 2 of the above E-points; this leaves a_5 with only one E-point (a_5, b_6). So this case does not occur.

Accordingly, the 18 E-points always lie in threes on the edges of a tetrahedron.

Now it will be shown that F is projectively equivalent to the equianharmonic surface. Let the lines of F be transversals of the axes of the triad S_{123}, S_{456}, $S_{123,456}$ and take these axes as the edges of the unit simplex. Then the 9 lines a_i, b_i, c_{ij} $i, j = 1, 2, 3$ may be given Plücker coordinates as below



$a_1 : 0, pu, u, p, -1, 0$ $b_1 : 0, rv, v, r, -1, 0$ $c_{23} : 0, qw, w, q, -1, 0$

$a_2 : 0, qv, v, q, -1, 0$ $b_2 : 0, pw, w, p, -1, 0$ $c_{13} : 0, ru, u, r, -1, 0$

$a_3 : 0, rw, w, r, -1, 0$ $b_3 : 0, qu, u, q, -1, 0$ $c_{12} : 0, pv, v, p, -1, 0$

where p, q, r as well as u, v, w are unequal. As the 18 E-points of F are distinct and as no 4 are collinear none of the E-points is a vertex of the tetrahedron, so $pqruvw \neq 0$. Let the axes s_{456}, s'_{456} which meet a_i, b_i, c_{ij} $i, j = 4, 5, 6$ be $(0, 0, 0, 0, 1, 0)$ and $(0, 1, 0, 0, 0, 0)$; as in § 7, they form reguli with $a_1\ a_2\ a_3$, with $b_1\ b_2\ b_3$, and with $c_{23}\ c_{13}\ c_{12}$ so that any 4 lines like s_{456}, s'_{456}, a_1, a_2 are linearly dependent. Therefore

$$\frac{p}{u} = \frac{q}{v} = \frac{r}{w}, \quad \frac{r}{v} = \frac{p}{w} = \frac{q}{u}, \quad \frac{q}{w} = \frac{r}{u} = \frac{p}{v}$$

so $$p^3 = q^3 = r^3, \qquad u^3 = v^3 = w^3 .$$

If m is a non-zero mark of GF(7), m^3 is either $+1$ or -1.

Suppose then that F has equation

$$\sum_{i,j=0}^{i,j=3} d_{ij}\, x_i^2\, x_j + \sum_{i \neq j \neq k \neq \ell} d_i\, x_j\, x_k\, x_\ell = 0$$

If $p^3 = q^3 = r^3 = -1$ so that p, q, r are $-1, -2, 3$ in some order, then the conditions that the 3 E-points on s_{123} are on F are

$$-d_{22} + d_{23} - d_{32} + d_{33} = 0$$
$$-d_{22} - 3d_{23} - 2d_{32} + d_{33} = 0$$
$$-d_{22} + 2d_{23} + 3d_{32} + d_{33} = 0$$

$\therefore\ d_{23} = d_{32} = 0 \quad d_{22} = d_{33}$

If $p^3 = q^3 = r^3 = 1$ so that p, q, r are $1, 2, -3$ in some order, then the conditions that the 3 E-points on s_{123} are on F are

$$d_{22} + d_{23} + d_{32} + d_{33} = 0$$
$$d_{22} - 3d_{23} + 2d_{32} + d_{33} = 0$$
$$d_{22} + 2d_{23} - 3d_{32} + d_{33} = 0$$

$\therefore\ d_{23} = d_{32} = 0 \quad d_{22} = -d_{33}$.

From s'_{123}, $\qquad d_{01} = d_{10} = 0 \qquad d_{00} = \pm\, d_{11}$.

Similarly from the other 2 pairs of opposite edges,

$d_{02} = d_{20} = d_{13} = d_{31} = 0 \qquad d_{00} = \pm\, d_{22} \qquad d_{11} = \pm\, d_{33}$

$d_{03} = d_{30} = d_{12} = d_{21} = 0 \qquad d_{00} = \pm\, d_{33} \qquad d_{11} = \pm\, d_{22}$

The d_{ii} must, for an existent F, take values that make the above conditions consistent. Thus F has the equation

$$x_0^3 + e\,x_1^3 + f\,x_2^3 + g\,x_3^3 + x_0\,x_1\,x_2\,x_3\,\Sigma\,d_i\,x_i^{-1} = 0$$

where each of e, f, g is either 1 or -1. Such surfaces for fixed e, f, g but varying d_i meet the edges of the tetrahedron of reference in the same 18 points; however, none contains the 27 lines except those having $d_0 = d_1 = d_2 = d_3 = 0$. So F is one of the surfaces

$$x_0^3 \pm x_1^3 \pm x_2^3 \pm x_3^3 = 0$$

all of which are projectively equivalent to

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0 \quad .$$

It has now been proved that every cubic surface over GF(7) containing 27 lines is equianharmonic.

§ 14. The mapping of F_7 onto the plane.

The projective uniqueness of F, hence to be called F_7, and the exhaustive covering by its 27 lines are mapped by corresponding geometry in the plane.

A plane k-arc, a set of k points in a plane with no 3 collinear, is complete if it is contained in no (k + 1) - arc: or equivalently, if the joins of the k points fill the plane. Over GF(q), where q is odd, every q-arc is contained in a (q + 1) - arc, Segre [52]; every (q + 1) - arc is a conic, Segre [51]. Thus in the plane over GF(7), every 7-arc is contained in a conic. Hence all 6-arcs not lying on a conic are complete. From the last table in §12, a complete 6-arc has 6 B-points, namely when the upper bound for b is achieved. Were no chord of the 6-arc to contain more than one B-point, there would be at most 5 B-points. Thus at least one chord contains 2 B-points. Therefore the 6-arc or hexad can be formed from the four vertices of a quadrangle Q and two points on the join of two of its diagonal points.

Let the vertices of Q be $(\pm 1, \pm 1, 1)$ so that its diagonal point triangle is the triangle of reference; the hexad is to be completed by two points on $x = 0$. The pencil of conics through the vertices of Q is

$$ax^2 + by^2 + cz^2 = 0 \quad \text{where} \quad a + b + c = 0.$$

Of the 8 conics in the pencil, 3 are line pairs. Of the remaining 5 conics, 3 are skew to $x = 0$. $(0, 1, 0)$ and $(0, 0, 1)$ are the double points of an involution on $x = 0$ whose pairs $(0, \pm\alpha, 1)$ each lie on one conic of the pencil. The pairs $(0, \pm 1, 1)$, $(0, \pm 2, 1)$, $(0, \pm 3, 1)$ lie on the line pair $y^2 - z^2 = 0$ and the conics $x^2 - 2y^2 + z^2 = 0$, $x^2 + y^2 - 2z^2 = 0$ respectively. Thus the only possible pairs of points for the hexad are

(a)	0 2 1	(b)	0 2 1	(c)	0 -2 1	(d)	0 -2 1
	0 3 1		0 -3 1		0 3 1		0 -3 1

By the harmonic inversion $y \longleftrightarrow -y$, which leaves Q fixed, the hexads obtained from (a), (d) and (b), (c) are equivalent. Thus there are only two projectively distinct complete hexads

α : -1 1 1 , 1 -1 1 , -1 -1 1 , 1 1 1 , 0 2 1 , 0 3 1

β : -1 1 1 , 1 -1 1 , -1 -1 1 , 1 1 1 , 0 2 1 , 0 -3 1 .

This is reflected in Π_3 by the two notationally different triads of trihedral pairs e.g. S_{123}, S_{456}, $S_{123,456}$ and $S_{12,34}$, $S_{34,56}$, $S_{56,12}$. The hexad α is related to the former type, the hexad β to the latter. If the points of α are $A_1, \dots, A_6$ in the order

-1 1 1 , 1 1 1 , 0 2 1 , 1 -1 1 , -1 -1 1 , 0 3 1 ,

they fall into two triads $A_1 A_2 A_3$, $A_4 A_5 A_6$ in sextuple perspective from the 6 B-points. Each of the 9 lines $A_i A_j$ where A_i and A_j are from different triads has two B-points, whereas if A_i and A_j are from the same triad $A_i A_j$ has no B-points. On the other hand, if the points of β are $B_1, \dots, B_6$ in the order

-1 1 1 , 1 1 1 , 1 -1 1 , -1 -1 1 , 0 2 1 , 0 -3 1 , they fall into three pairs B_1B_2, B_3B_4, B_5B_6. Each of the 3 lines B_iB_j, where B_iB_j is one of the pairs, contains two B-points; the other 12 lines B_iB_j contain one B-point apiece.

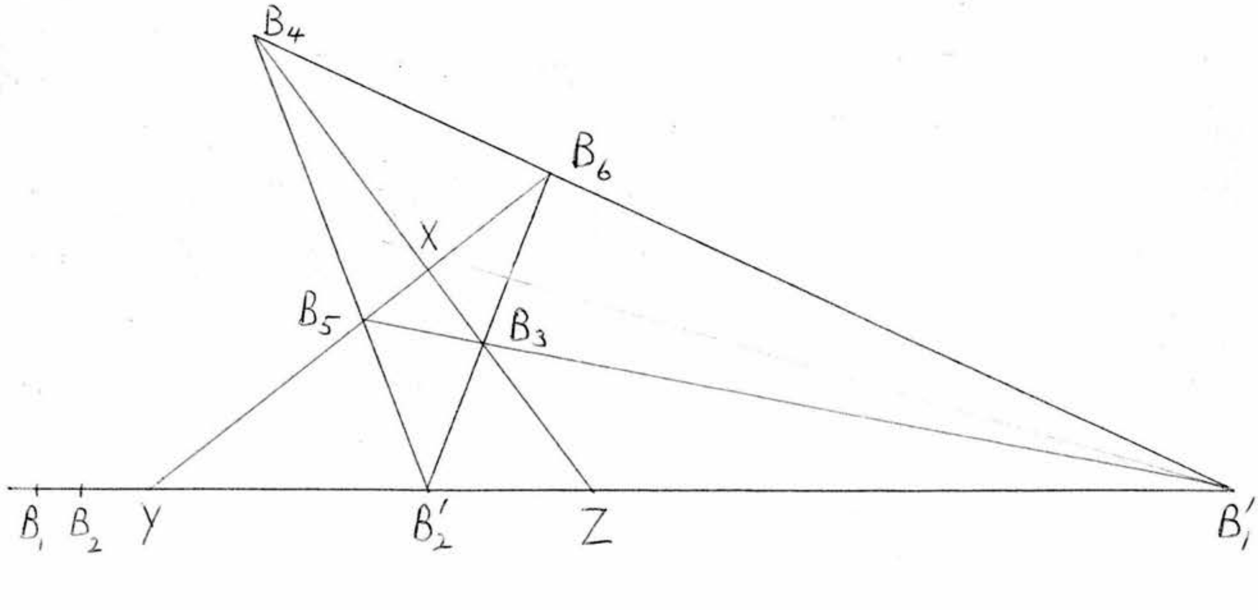
The completeness of the hexads maps the covering of F_7 by the 27 lines, or in fact by the 21 lines a_i, c_{jk}, and the two types of hexad map the two different types of notations for the properties of F_7.

The 6 B-points of each hexad form a hexad of the same type as its originator. For α, the B-points $H_1, \dots, H_6$ are

H_1 : A_1A_4, A_2A_5, A_3A_6 $\quad$ H_4 : A_1A_4, A_2A_6, A_3A_5
H_2 : A_1A_5, A_2A_6, A_3A_4 $\quad$ H_5 : A_1A_5, A_2A_4, A_3A_6
H_3 : A_1A_6, A_2A_4, A_3A_5 $\quad$ H_6 : A_1A_6, A_2A_5, A_3A_4

They form a hexad α' partitioned into two triads $H_1H_2H_3$, $H_4H_5H_6$. The B-points of α' are the points of α as, for example, H_1H_4, H_2H_5, H_3H_6 are the respective lines A_1A_4, A_1A_5, A_1A_6 which meet at A_1. Thus $H_1H_2H_3$, $H_4H_5H_6$ are in sextuple perspective from the 6 A_i. In this way the hexads like α occur in closed pairs.

The hexad β depends on the triangle with sides B_1B_2, B_3B_4, B_5B_6. Let this be XYZ and label the B-points of β as $B'_1, \dots, B'_6$ where $B'_1 = (B_1B_2, B_3B_5, B_4B_6)$ $B'_2 = (B_1B_2, B_3B_6, B_4B_5)$, B'_3, B'_4 lie on B_3B_4 and B'_5, B'_6 lie on B_5B_6.



From the quadrilateral B_3B_5, B_5B_4, B_4B_6, B_6B_3 with diagonal line triangle XYZ, the pair B_3, B_4 is harmonically separated by X, Z as is the pair B_5, B_6 by X,Y and the pair B_1', B_2' by Y,Z. Similarly B_1, B_2 are harmonically conjugate to Y,Z as are B_3', B_4' to X,Z and B_5', B_6' to X,Y. Thus the 6 points of β as well as the 6 B-points of β lie on the sides of XYZ and are also harmonic conjugates of the 3 pairs of vertices of the triangle.

A converse can also be obtained from the above figure. Begin with B_3, B_4, B_5, B_6, X and select Y,Z so that B_3, B_4 are harmonically separated by X,Z and B_5, B_6 by X,Y : thus the line YZ is the polar of X with respect to any conic through B_3, B_4, B_5, B_6 and in particular the line pairs B_3B_5, B_4B_6 and B_3B_6, B_4B_5. Therefore the vertices B_1', B_2' of these line pairs are points of YZ and as before harmonic conjugates of Y,Z.

Suppose the 6 B-points of β do not form a hexad; then some pair B_i', B_j' lying on different sides of XYZ are collinear with a point B_k' on the third side. It will suffice to consider one pair as the argument is the same for any other. So, letting $B_3' = (B_1B_5, B_2B_6)$, $B_5' = (B_1B_3, B_2B_4)$, $B_6' = (B_1B_4, B_2B_3)$, either B_1', B_3', B_5' or B_1', B_3', B_6' are collinear. If B_1', B_3', B_5',

$$(YZ,\ B_1B_1')\ \overset{B_5}{\barwedge}\ (XZ,\ B_3'B_3)\ \overset{B_5'}{\barwedge}\ (YZ,\ B_1'B_1)$$

whence B_1, B_1' are harmonic conjugates of Y,Z; this is impossible as the harmonic conjugate of B_1 with respect to Y, Z is B_2 which is not on B_3B_5 or B_4B_6 and so is distinct from B_1'. If B_1', B_3', B_6' are collinear,

$$(YZ,\ B_1B_1')\ \overset{B_5}{\barwedge}\ (XZ,\ B_3'B_3)\ \overset{B_6'}{\barwedge}\ (YZ,\ B_1'B_2)$$

which implies a solution of $\{0, \infty, 1, \lambda\} = \{0, \infty, \lambda, -1\}$, whence $\lambda^2 + 1 = 0$ which is insoluble over GF(7). Thus the 6 B-points of β form a hexad β'.

From the converse stated **above**, the B-points of β' also lie on the sides of the triangle and are harmonic conjugates of the vertices. As B_5', B_6' differ from B_5, B_6 so the points where $B_1'' B_5'$, $B_1' B_6'$ meet XZ differ from B_3, B_4. So the B-points of β' are not the points of β. Thus the 6 B-points of β' — B_1'', .., B_6'' form a hexad β'' where B_1, B_2; B_1', B_2'; B_1'', B_2'' are the three pairs of harmonic conjugates of Y,Z; similarly for X,Z and X,Y.

Finally the B-points of β'' are the points of the original hexad β. So the hexads like β occur in closed triads. In coordinates, β is

$$-1\ 1\ 1,\quad 1\ 1\ 1,\quad 1\ {-1}\ 1,\quad -1\ {-1}\ 1,\quad 0\ 2\ 1,\quad 0\ {-3}\ 1.$$

Applying the projectivity $y + z \to y$, $y - z \to z$ so that XYZ is the triangle of reference, this becomes

$$3\ 1\ 0,\quad -3\ 1\ 0,\quad 3\ 0\ 1,\quad -3\ 0\ 1,\quad 0\ 3\ 1,\quad 0\ {-3}\ 1,$$

and the successive hexads are

$$-1\ 1\ 0,\quad 1\ 1\ 0,\quad -2\ 0\ 1,\quad 2\ 0\ 1,\quad 0\ {-1}\ 1,\quad 0\ 1\ 1$$

$$-2\ 1\ 0,\quad 2\ 1\ 0,\quad -1\ 0\ 1,\quad 1\ 0\ 1,\quad 0\ {-2}\ 1,\quad 0\ 2\ 1.$$

Given the triangle XYZ, there are $3 \times 3 \times 2 = 18$ hexads like β whose points lie on the sides of the triangle. The 18 hexads fall into 6 triads. Since there are $7^3(7^3 - 1)(7^2 - 1)$ projectivities in the plane and 57.56.49/6 triangles, XYZ has a group of order 6^3. Thus each hexad β has a group of order 12 and each triad a group of order 36. The number of hexads of type β is

$$18 \times 57.56.49/6 = 2^3.3^2.7^3.19.$$

The numbers of the two types of hexads in the plane can be calculated simultaneously. Having selected the vertices of Q, there are $3 \times 2 = 6$ ways of selecting the remaining two points on the sides of the diagonal point triangle of Q for each type of hexad. However for each hexad, the number of ways of selecting the original tetrad is the number of residual pairs whose join contains 2 B-points, viz. 9 for hexads like α, 3 for hexads like β. Thus there are thrice the number of hexads of type β as of type α.

Of type α there are $\frac{57.56.49.36}{4!} \times \frac{6}{9} = 2^3.3.7^3.19.$

Of type β there are $\frac{57.56.49.36}{4!} \times \frac{6}{3} = 2^3.3^2.7^3.19;$

totalling $2^5.3.\ 7^3.19$

which is, by § 12, the number of hexads in the plane not lying on a conic.

That there are 3 hexads like β to one like α corresponds in Π_3 to there being 30 triads of trihedral pairs like $S_{12,34}$, $S_{34,56}$, $S_{56,12}$ and 10 like S_{123}, S_{456}, $S_{123,456}$.

As there are $7^3(7^3 - 1)(7^2 - 1)$ projectivities in the plane, the groups which leave α,β invariant are of respective orders 36 and 12, the latter as above.

The subgroup of A_6 which consists of all the permutations of 123,456 such that the two triads are invariant or interchanged has order $3! \times 3! \times 2 = 72$. The projective group of α of order 36 consists therefore of the subgroup of those of the 72 permutations which are even, as the operation (56), say, cannot be effected by a projectivity.

The group of β has, as a subgroup, the 4-group consisting of the unit and the harmonic inversions with respect to each of the three sides of the triangle, each of which inversions interchanges the members of two of the pairs B_1B_2, B_3B_4, B_5B_6. As the pairs must be left invariant, some other operation in the group of β is a permutation of the three pairs of order 3 i.e. a complete permutation, none of which commute with the elements of the 4-group. Hence the 4-group is a normal subgroup of the group of β, which is therefore A_4, Dickson [17] p.268.

§ 15. Existence of Grace's extension of the double-six.

Grace's extension of the double-six requires the existence of 6 skew lines with a common transversal such that any 4 have a unique second transversal. A double-six includes sets of 5 suitable lines. We will firstly approach the problem by seeking the existence over $GF(7)$ of a sixth line.

Let $a_1, a_2, a_3, a_4, a_5, b$ be the 5 lines and their transversal; denote by a_6 the line sought. In the representation of lines in Π_3 by points of a quadric in Π_5, let the points (w, x, y, z, t, u) lie on $K : wu + xt + yz = 0$. Denote a point of K by the same symbol as the line in Π_3 it represents. Take b as $(1, 0, 0, 0, 0, 0)$: then the tangent prime at b, $u = 0$, contains the a_i $i = 1, \cdots, 6$ and meets K in a quadric cone with vertex b and base $Q : xt + yz = 0$.

If 4 skew lines have a single transversal then the polar line with respect to K of the Π_3 spanned by the 4 points representing the 4 lines touches K at a point which lies in the Π_3 ([31] p.217). Thus the transversal is linearly dependent on the 4 lines.

Take the a_i as $(w_i, x_i, y_i, z_i, t_i, 0)$ $i = 1, \cdots, 6$. In the Π_3 $w = u = 0$, let the points $(0, x_i, y_i, z_i, t_i, 0)$ on Q be called A_i. Let the planes T_i, S_{ijk} be

$$T_i : xt_i + yz_i + zy_i + tx_i = 0 \qquad i = 1, \cdots, 5$$

$$S_{ijk} : \begin{vmatrix} x & y & z & t \\ x_i & y_i & z_i & t_i \\ x_j & y_j & z_j & t_j \\ x_k & y_k & z_k & t_k \end{vmatrix} = 0 \qquad i, j, k = 1, \cdots, 5$$

T_i is the tangent plane at A_i meeting Q in the two generators through A_i and S_{ijk} is a non-tangent plane meeting Q in the 8 points of a conic, no two on the same generator. As a_i does not meet a_j, A_i does not lie in

T_j $j \neq i$. As each 4 of the a_i $i = 1, \ldots, 5$ have two distinct transversals and no more, b is independent of any 4 of the a_i and any 4 of the a_i are themselves independent: thus A_i does not lie in $S_{jk\ell}$ $j, k, \ell \neq i$. A_6 is not to be either in any of the 5 planes T_i or in any of the 10 planes S_{ijk}.

Suppose there is such a point A_6 and project the points of Q stereographically from A_6 onto a plane π following Todd [61] p.110. Let ℓ, g be the generators of Q through A_6 and m the intersection of π with the tangent plane at A_6. The point A of Q corresponds to the point α in which A_6A meets π. The exceptional elements are A_6 which corresponds to the line m and the lines ℓ, g which correspond to the points L, G of m in which they meet π. The $(q + 1)^2 - (2q + 1) = q^2$ points A of Q not on ℓ or g correspond 1 - 1 with the $(q^2 + q + 1) - (q + 1) = q^2$ points α of π not on m. The two generators of Q through A correspond to the line pair αL, αG. A conic section of Q through A_6 corresponds to a line pair one of which is m. A conic section of Q not through A_6 corresponds to a conic through L, G.

As none of the A_i $i = 1, \ldots, 5$ lie in the tangent plane at A_6, they are represented by points α_i not on m. As S_{ijk} does not contain A_6, the section of Q by S_{ijk} is represented by a conic through L, G, α_i, α_j, α_k. So no three α_i can be collinear. As A_j does not lie in the tangent plane at A_i, α_j does not lie on either of the lines $\alpha_i L$, $\alpha_i G$. Thus the points L, G, α_1, α_2, α_3, α_4, α_5 form a 7-arc. These 7 points therefore lie on a conic so the A_i $i = 1, \ldots, 5$ lie in a plane. So the construction is impossible. So there is no suitable A_6. So there is no suitable a_6. So Grace's extension of the double-six does not exist over GF(7).

That a 7-arc lies on a conic was taken in § 14 from Segre's theorem as the discussion of the cubic surface was started 'ab initio' in the plane; but it is deducible directly from the cubic surface F_7 of § 13. In both types of notations for the triads of trihedral pairs which determine the 18 E-points of F_7, there are 6 E-points which are intersections of three c_{ij} lines. Thus in the plane the only 6-arcs not lying on a conic have 6 B-points and so are complete. So any 7-arc lies on a conic.

CHAPTER V. Preliminaries and general data on Grace's extension.

§ 16. The twisted cubic over GF(q).

Before attempting to discover by an inductive method whether Grace's extension of the double-six exists over GF(7), it is here necessary to discuss the notion of a "twisted cubic" over GF(q).

In the subsequent arguments, n "points" refer to the roots of an equation of degree n, which may or may not lie in the field being used.

Segre [52] defines the normal rational curve in an arbitrary dimension for fields of odd characteristic, and shows that any set of q + 1 points in three dimensions with no four coplanar forms such a curve; in [53] the twisted cubic over a field of characteristic two is also defined. However, to provide a uniform definition of the twisted cubic over any field GF(q), we will follow Todd [61] p.117 and consider the corresponding planes of three projective pencils. It is shown, by taking a projectivity with two distinct self-corresponding points on one of the axes, how to set up the correspondence so that three corresponding planes never meet in a line. Thus a set of points {P} is derived, of which any point P_λ is given by

$$x_0 : x_1 : x_2 : x_3 = f_0(\lambda) : f_1(\lambda) : f_2(\lambda) : f_3(\lambda)$$

where the $f_i(\lambda)$ are all homogeneous cubic polynomials in $\lambda_0 : \lambda_1$ with no common factor. Let u be the plane

$$u_0x_0 + u_1x_1 + u_2x_2 + u_3x_3 = 0.$$

P_λ lies in u if and only if

$$u_0f_0(\lambda) + u_1f_1(\lambda) + u_2f_2(\lambda) + u_3f_3(\lambda) = 0.$$

It is desired to prove that the discriminant of the cubic form in the above equation does not vanish identically, so that not every plane meets the curve in a double point. Assume the contrary. For all fields except those of characteristic two and three, Todd's argument may be followed. In the exceptional cases the discriminant of a cubic form takes a different shape. Let $ax^3 + bx^2 + cx + d$ be an arbitrary cubic form with a repeated factor; it can then be written as $a(x + e)^2(x + f)$. Hence

$$b = a(2e + f) \qquad c = a(e^2 + 2ef) \qquad d = ae^2f.$$

Thus by elimination,

$$27a^2d^2 + 4ac^3 + 4b^3d - b^2c^2 - 18\,abcd = 0$$

for any field. The expression on the left of the equation is the discriminant. This becomes

$$(ad + bc)^2 \qquad \text{over } GF(2),$$

$$ac^3 + b^3d - b^2c^2 \qquad \text{over } GF(3).$$

Following Todd, $f_0(\lambda)$ can be taken as one of the forms λ_0^3, $\lambda_0^2\lambda_1$. Let

$$g(\lambda) \equiv u_1f_1(\lambda) + u_2f_2(\lambda) + u_3f_3(\lambda) = A\lambda_0^3 + B\lambda_0^2\lambda_1 + C\lambda_0\lambda_1^2 + D\lambda_1^3,$$

where A, B, C, D are linear forms in u_1, u_2, u_3. Firstly over $GF(2^n)$, let $f_0(\lambda) = \lambda_0^3$; then the discriminant of $u_0f_0(\lambda) + g(\lambda)$ is $\{(u_0 + A)D + BC\}^2$. If this vanishes identically, then $D = 0$ and either $B = 0$ or $C = 0$. If $f_0(\lambda) = \lambda_0^2\lambda_1$ and $\{AD + (u_0 + B)C\}^2 \equiv 0$, then $C = 0$ and either $A = 0$ or $D = 0$. Over $GF(3^n)$, if $f_0(\lambda) = \lambda_0^3$ then we require $(u_0 + A)C^3 + B^3D - B^2C^2 \equiv 0$; thus $C = 0$ and either $B = 0$ or $D = 0$. If $f_0(\lambda) = \lambda_0^2\lambda_1$ and $AC^3 + (u_0 + B)^3D - (u_0 + B)^2C^2 \equiv 0$, then $D = C = 0$. In each of these seven cases, two of A, B, C, D are zero; thus only two of the $f_i(\lambda)$ are linearly independent and there are two linear relations between them. All the points P_λ would then lie on a line, contrary to the initial arrangement of the correspondence. Thus the discriminant does not vanish identically.

If u is any plane of the space such that the discriminant of $\Sigma\, u_if_i(\lambda)$

is not zero, then the plane contains exactly three distinct "points" of the curve given by the set of points $\{P\}$.

Finally it is shown that the polynomials are linearly independent. Thus the set of points can be given by

$$x_0 : x_1 : x_2 : x_3 = \lambda^3 : \lambda^2 : \lambda : 1.$$

This is now our twisted cubic Γ.

Over $GF(q)$, Γ comprises $q + 1$ points joined in pairs by $q(q + 1)/2$ chords of which at most $[(q + 1)/2]$ are skew. Γ also has $(q^2 - q)/2$ other chords, each the join of a pair of conjugate points whose coordinates belong to $GF(q^2)$. These chords are mutually skew and skew to all the other chords. At each of its $q + 1$ points, Γ also has one tangent and q^2 secants. The chord through points $\lambda = r, s$ of Γ has coordinates

$$p_{01} : p_{02} : p_{03} : p_{12} : p_{31} : p_{23}$$
$$= r^2s^2 : rs(r + s) : r^2 + rs + s^2 : rs : -(r + s) : 1 .$$

Thus the tangent at $\lambda = r$ has coordinates

$$r^4 : 2r^3 : 3r^2 : r^2 : -2r : 1 .$$

Γ separates the points of the space into 4 mutually exclusive classes:

$q + 1$	points	on Γ
$(q + 1)q$		off Γ, on the tangents
$\frac{1}{2}q(q + 1)(q - 1)$		off Γ, on the "real" chords
$\frac{1}{2}(q^2 - q)(q + 1)$		on the "conjugate" chords,

giving all $(q^2 + 1)(q + 1)$ points of the space. This shows that through each point not on Γ there is a unique line with two-point contact on Γ. In the dual case the situation is different: the planes of the space fall into 5 different classes:

$q + 1$	osculating planes
$q(q + 1)$	planes containing a tangent of Γ and one other point besides the contact of the tangent
$\frac{1}{6}(q^3 - q)$	planes containing 3 distinct points and 3 distinct "real" chords of Γ
$\frac{1}{2}(q^3 - q)$	planes containing one point and one "conjugate" chord of Γ,

leaving $\frac{1}{3}(q^3 - q)$ which do not meet Γ. This is then the number of irreducible cubic equations over $GF(q)$.

Over fields of characteristics two and three, Γ has peculiarities. Over any field the tangents to Γ lie in the linear complex

$$p_{03} = 3p_{12} .$$

Over $GF(2^n)$ the tangents to Γ form a regulus

$$p_{02} = p_{31} = 0 \qquad p_{03} = p_{12} .$$

Over $GF(3^n)$ the tangents to Γ lie in a special linear complex

$$p_{03} = 0.$$

More strikingly, over $GF(2^n)$ and $GF(3^n)$, Γ is not a properly self-dual construct. The plane which contains the points $\lambda = r, s, t$ of Γ is

$$x_0 - (r + s + t)x_1 + (rs + rt + st)x_2 - rst\, x_3 = 0 .$$

Thus the osculating planes over $GF(2^n)$ form a proper developable given by

$$x_0 + \theta x_1 + \theta x_2 + \theta^3 x_3 = 0 ,$$

while those over $GF(3^n)$ merely form a pencil

$$x_0 - \theta^3 x_3 = 0 ,$$

whose axis is the axis of the special linear complex $p_{03} = 0$, the transversal of all the tangents of Γ. However, Γ is also the residual intersection of two quadric cones with a common generator: for example, the quadric cones $x_1^2 - x_0x_2 = 0$ with vertex X_3 and $x_2^2 - x_1x_3 = 0$ with vertex X_0 meet in X_0X_3 and Γ. Dually, consider the planes which touch two conics with a common tangent, say the conic $x_1^2 - x_0x_2 = 0$ in $x_3 = 0$ and $x_2^2 - x_1x_3 = 0$

in $x_0 = 0$. Over $GF(2^n)$ the planes form a pencil $x_0 + \theta x_3 = 0$, whereas over $GF(3^n)$ they form a proper developable

$$x_0 + \theta x_1 + \theta^2 x_2 + \theta^3 x_3 = 0.$$

§ 17. Grace's extension in general and over GF(7).

Over the complex field, given 6 skew lines a_i $i = 1, \cdots, 6$ with a common transversal b such that any 4 a_k, a_ℓ, a_m, a_n have a unique second transversal $b_{ij} = b_{ji}$, then from b and $a_j, a_k, a_\ell, a_m, a_n$ the double-six D_i can be formed with the completing line α_i meeting $b_{ij}, b_{ik}, b_{i\ell}, b_{im}, b_{in}$ as in Appendix II. Each D_i $i = 1, \cdots, 6$ lies on a cubic surface F_i, which contains another 15 lines c^i_{jk}, where c^i_{ij} meets α_i, a_j and c^i_{jk} meets a_j, a_k. In this field there is a unique twisted cubic t with the 6 lines a_i as chords, e.g. Wakeford [63] p.112 footnote. This cubic t contains 10 points of F_i, therefore lies on it, and is the residual intersection of F_i and F_j besides b, b_{ij}, a_k, a_ℓ, a_m, a_n. Baker [4] p.195 proves that "given six skew lines with a common transversal, the locus of a point, such that the planes joining it to the seven lines touch a quadric cone, is a cubic curve, having the six lines as chords but not meeting the transversal." For fields of odd characteristic, as the twisted cubic and the lines that are chords of it have been properly defined, Baker's theorem and the proof, as it stands, are true. The theorem breaks down over characteristic two because of the phenomenon mentioned in § 16 that the common tangent planes to two conics with a common tangent form a pencil and not a cubic developable.

Of the fields of characteristic two, I will concern myself at the moment with GF(8) and show that six lines having a common transversal with each four having a unique second transversal can be chosen as chords of a twisted cubic.

A point of Γ is given by

$$x_0 : x_1 : x_2 : x_3 = r^3 : r^2 : r : 1 .$$

A chord of Γ has coordinates

$$p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23}$$

$$= r^2s^2 : rs(r + s) : r^2 + rs + s^2 : rs : r + s : 1$$

The line u with coordinates $(0,0,0,1,0,0)$ is skew to Γ and meets the chords of Γ whose parameters satisfy $r^2 + rs + s^2 = 0$ i.e. $(r + s)^2 = rs$. Let $r + s = v$. Then u meets the chords v of Γ with coordinates $(v^4, v^3, 0, v^2, v , 1)$. As $GF(4)$ is not a subfield of $GF(8)$, $r^2 + rs + s^2$ is irreducible over $GF(8)$; therefore these chords all meet Γ in pairs of conjugate points with r, s in $GF(8^2)$. Thus these chords are all mutually skew. The two with parameters $0, \infty$ are tangents.

Any four of these chords a, b, c, d have exactly two transversals if

$$\begin{vmatrix} a^4 & a^3 & a & 1 \\ b^4 & b^3 & b & 1 \\ c^4 & c^3 & c & 1 \\ d^4 & d^3 & d & 1 \end{vmatrix} \neq 0,$$

i.e.
$$(a + b)(a + c)(a + d)(b + c)(b + d)(c + d) \times (ab + ac + ad + bc + bd + cd) \neq 0$$

Thus, to find a line and six transversals such that any 4 have a unique second transversal and so that the six lines are chords of a twisted cubic, necessarily unique, it is sufficient to find six non-zero elements of $GF(8)$ such that no four satisfy $ab + ac + ad + bc + bd + cd = 0$. Conveniently the 10 planes $x_i = 0$, $x_j + x_k = 0$ $i,j,k = 0, \cdots 3$ cover the quadric $\sum_{i<j} x_i x_j = 0$. To show this, consider the quadric over $GF(q)$ where $q = 2^{2n+1}$. Over these fields it is non-ruled and therefore consists of $q^2 + 1$ points, Primrose [38]. The 10 planes are each spanned by three of the five points 1000, 0100, 0010, 0001, 1111; the line of intersection of any two of the 10 planes meets the

quadric in either one or two of these five points, but never in any other point. For, of these 45 intersections of pairs of the 10 planes, 30 are the 10 lines through pairs of the 5 points and the remaining 15 are intersections of pairs of planes with one of the 5 points in common. Thus the number of points on the quadric lying in none of the 10 planes is

$$(q^2 + 1) - 10(q - 2) - 5 = q^2 - 10q + 16 = (q - 2)(q - 8).$$

Therefore the 10 planes cover the quadric over $GF(8)$, and any 6 of the 7 non-zero elements of $GF(8)$ may be taken as parameters of the six chords of Γ. Then these six chords have a common transversal and each four have a unique second transversal.

To show the necessity of its existence, I would like to approach the question of the twisted cubic lying on all six F_i from another point of view. Firstly, it must be shown that all the c^i_{jk} are distinct. It is sufficient to consider F_1 and F_2, and to prove that every line c^1_{jk} is distinct from every line $c^2_{\ell m}$.

$c^1_{1i} = c^2_{2j}$ implies that the planes $[b, a_i]$, $[b, a_j]$ meet in c^1_{1i}, so $\quad c^1_{1i} = b$;

$c^1_{2i} = c^2_{1j}$ implies similarly that $\quad c^1_{2i} = b_{12}$;

$c^1_{1i} = c^2_{jk}$ $\quad j \neq 2 \quad k \neq 2, i$ implies that $\quad c^1_{1i}$ meets a_k;

$c^1_{2i} = c^2_{\ell j}$ $\quad i, j \neq 1, 2$ implies that $\quad c^1_{2i}$ meets a_j;

$c^1_{jk} = c^2_{\ell m}$ $\quad j, k, l, m \neq 1, 2 \quad m \neq j, k$ implies that $\quad c^1_{jk}$ meets a_m.

All these cases contradict the intersections of lines on F_1 imposed by the initial conditions on the a_i.

F_1, F_2 have 8 common tritangent planes - those containing b, b_{12} and each of a_3, a_4, a_5, a_6. Thus, besides these six lines, F_1, F_2 have 8 points in common viz. where c^1_{1i} meets c^2_{2i} $\quad i = 3, \dots, 6$ and where c^1_{2j} meets c^2_{1j} $\quad j = 3, \dots, 6$. It is these points which lie on the residual intersection of F_1, F_2 and must therefore be part of a twisted cubic.

Now consider any two cubic surfaces over $GF(q)$ intersecting in a curve C of degree 9 which may be degenerate. A line lying on only one of the surfaces meets the other and also their intersection in 3 "points". Suppose a line ℓ is part of C. A plane through ℓ, containing no other part of C, will meet the remainder of C, degenerate or not, in 8 "points". The plane meets each of the surfaces residually in a conic. The two conics meet in 4 "points", and ℓ meets the two conics and hence the residual part of C in another 4 "points".

Over $GF(q)$, where q is odd or 2^{2n}, take as before the line b, the 6 lines a_i, the twisted cubic t of which the a_i are chords, the 15 lines b_{ij}, the 6 lines α_i, the 6 double-sixes D_i and the 6 cubic surfaces F_i. F_i, F_j meet in a_k, a_ℓ, a_m, a_n, b, b_{ij} and t. Each F_i contains 15 lines c^i_{jk} j, k = 1, ..., 6. All 90 c^i_{jk} are distinct as previously shown; or otherwise, the coincidence of any two would make them a part of the intersection of two of the F_i.

The lines a_i are chords of t. From the previous argument, the α_i are also chords of t, the c^i_{jk} are secants of t, the b_{ij} are skew to t; b is also skew to t since in the intersection of F_i and F_j, it meets a_k, a_ℓ, a_m, a_n. The lines c^i_{jk}, c^j_{ki}, c^k_{ij}, being the intersections of pairs of the planes $[a_i, b_{jk}]$, $[a_j, b_{ki}]$, $[a_k, b_{ij}]$ are concurrent at a point L_{ijk} of t. The plane $[b, a_i]$ contains the 5 lines c^j_{ij} $j \neq i$, all of which meet t; they are therefore concurrent at a point L_i of t. In this way there are 6 points L_i and 20 points L_{ijk} on t.

As c^k_{ik} does not meet c^k_{jk}, $L_i \neq L_j$ $i \neq j$;

As c^j_{ij} does not meet c^j_{ik}, $L_i \neq L_{ijk}$;

As c^i_{jk} does not meet $c^i_{j\ell}$, $L_{ijk} \neq L_{ij\ell}$ $k \neq \ell$.

Suppose then that, over GF(7), there exist 6 skew lines with a common transversal such that any 4 have a unique second transverdsl; the construction of the figure, described above, may then be carried out. The line b lies on each surface F_i, therefore contains 2 E-points of each; these comprise 12 E-points altogether, distributed among the 6 points (b, a_i) of b. This means that 12 lines c^{j}_{jk} pass through these 6 points. Since no more than 5 of these lines can pass through any one of the 6 points, at least 2 of the points have 2 or more of these lines through them. Therefore, at least 2 points of b are points L_i of t; consequently b is a chord of t. This gives a contradiction as it was previously seen that b is skew to t.

Thus, over GF(7), it is not possible to find a set of 6 skew lines with a common transversal such that any 4 have a unique second transversal. So Grace's extension of the double-six does not exist over GF(7).

It can also be shown that, for fields GF(q) where $q < 11$, a necessary condition for the existence of Grace's extension is that not all the surfaces F_i are completely covered by their 27 lines.

Suppose all the F_i are covered by their lines. The point L_1, the meet of the c^{i}_{1i} $i = 2, .., 6$ is on t and therefore on F_1. It does not lie on a_j $j = 2, ..., 6$, for then a_j would meet c^{i}_{1i} on F_i $i \neq j$. As $L_1 \neq L_i$ $i = 2, .., 6$, L_1 does not lie on c^{1}_{1i}. As $L_1 \neq L_{1ij}$, L_1 does not lie on c^{1}_{ij} $i, j = 2, .., 6$. The b_{1i} $i = 2, .., 6$ are skew to t and therefore also do not contain L_1. Thus L_1, which must lie on some line of F_1, lies on α_1. Thus α_1 meets t in two real points. Similarly all the α_i meet t in two real points. As $q < 11$, t contains less than 12 points. Therefore if the a_i exist then the α_i exist but are not mutually skew. Consequently, Grace's extension will not exist.

CHAPTER VI. The cubic surface, and Grace's extension, over GF(8).

§ 18. The cubic surface over GF(8).

Over GF(8) lines contain 9 points, so each line on a cubic surface F with 27 lines has at least one E-point and it is seen from the table in § 12 that F has between 9 and 13 E-points. All the sets of 9 E-points readily imply the existence of 13 except an arrangement of 3 on each sides of a triangle as, for example,

$$\begin{matrix} (a_1, b_2, c_{12}) & (a_4, b_5, c_{45}) & (c_{15}, c_{24}, c_{36}) \\ (a_2, b_3, c_{23}) & (a_5, b_6, c_{56}) & (c_{14}, c_{26}, c_{35}) \\ (a_3, b_1, c_{13}) & (a_6, b_4, c_{46}) & (c_{16}, c_{25}, c_{34}), \end{matrix}$$

a figure which turns out to be non-existent. These 9 points form Maclaurin's figure lying 3 on each of 12 lines, 4 lines through each point. As there seems no obvious reason why such a set of 9 points cannot lie on a plane cubic over GF(8), it will be easier to consider 6-arcs (hexads) in the plane than the arrangement of the E-points on the surface directly.

In the plane over GF(8), there are 73 points, 73 lines, 9 points on a line, 9 lines through a point. All 7, 8 and 9-arcs belong to 10-arcs, Segre [55] p.45; such a 10-arc, which is an oval, always comprises the points of a conic C and its nucleus N, the meet of all the tangents of C, [55] p.37. Thus there are two possible types of 6-arc not lying on a conic.

1) A 5-arc (pentad) plus the nucleus of the conic C it determines.

2) A complete 6-arc.

Consider a pentad 100, 010, 001, 111, abc.

C is $a(b + c)yz + b(c + a)zx + c(a + b)xy = 0$:

N is $\{a(b + c), \quad b(c + a), \quad c(a + b)\}$, which lies on the line of diagonal points, $x + y + z = 0$, of the quadrangle Q with vertices 100, 010, 001, 111. Thus N lies on the line of diagonal points of every quadrangle inscribed in C: or equivalently, the line of diagonal points of Q passes through the nucleus of every circumscribed conic. (The latter replaces the theorem, over fields of characteristic other than two, that the diagonal point triangle of Q is self-polar with respect to every conic circumscribing Q.)

Let $A_1A_2A_3A_4A_5$ N be a hexad H of type (1). Let d_i be the line of diagonal points of the quadrangle whose vertices are the A_j residual to A_i. The 5 lines d_i all pass through N; the 9 lines through N are all the tangents to the conic C containing the A_i $i = 1, .. 5$. As there are only 4 points on C besides the A_i and as no vertices of a quadrangle lie on its line of diagonal points, at least one of the d_i passes through the corresponding A_i: let it be d_5. So d_5 is NA_5 and contains 3 B-points of H viz. (A_1A_2,A_3A_4), (A_1A_3,A_2A_4), (A_1A_4,A_2A_3). Moreover H has no other B-points. For, from § 12, the chords of a hexad over GF(8) with 3 B-points contain $15.8 - 54 + 3 = 69$ points. This leaves 4 points in the plane which are therefore the residual points on C to the A_i; none of these 4 points can be B-points of H. Thus H consists of the vertices of a quadrangle Q and two points on its line of diagonal points, which contains the 3 B-points of H; conversely H determines Q uniquely.

Consider a hexad K of type (2) - $A_1A_2A_3A_4A_5A_6$; being complete, it has 7 B-points as shown in § 12. If no chord of K were to contain more than one B-point, K would have only 5 B-points. So at least one chord, let it be A_1A_2, contains 2 B-points. These two points are diagonal points of the quadrangle Q with vertices A_3,A_4,A_5,A_6; so A_1A_2 contains all three diagonal points of Q, which are all B-points of K. Thus the hexad, whether of type (1) or (2), consists of the vertices of a quadrangle Q and two points on the latter's line of diagonal points, which contains 3 B-points. In case (1), Q is unique: in case (2), as will be seen, Q can be selected in 3 ways, the 3 lines of diagonal points being concurrent. Most of these remarks on plane 6-arcs over GF(8) have also been made by Segre [54] and Scafati [45] .

The two types of hexads can be given canonical coordinates. GF(8) is the cubic extension of GF(2) by a root of either $x^3 + x^2 + 1 = 0$ or $x^3 + x + 1 = 0$, the roots of either equation being reciprocals of the other. We will take the former equation so that the elements of GF(8) are

$$0,\ \epsilon,\ \epsilon^2,\ \epsilon^3,\ \epsilon^4,\ \epsilon^5,\ \epsilon^6,\ \epsilon^7 = 1$$

where

$$1 + 1 = 0,$$

$$\epsilon^3 + \epsilon^2 + 1, \qquad \epsilon^6 + \epsilon^4 + 1 = 0, \qquad \epsilon^5 + \epsilon + 1 = 0.$$

The only automorphisms of the field are ϕ, ϕ^2, $\phi^3 = 1$, where ϕ replaces each element by its square, Segre [57] p.99. GF(8) has the further property that each element admits a unique n-th root, viz.

$$x^{1/n} = x^m \qquad nm \equiv 1 \mod 7 \qquad \text{Segre [57] p.100}$$

Let the vertices of Q be 100, 010, 001, 111; the hexads have their remaining two points on $x + y + z = 0$. The two points are to be chosen from the 6 points on this line apart from the diagonal points, viz. from

$$\epsilon^3\epsilon^2 1, \quad \epsilon^6\epsilon^4 1, \quad \epsilon^5\epsilon 1, \quad \epsilon^2\epsilon^3 1, \quad \epsilon^4\epsilon^6 1, \quad \epsilon\epsilon^5 1:$$

15 hexads can be thus selected. Those of type (1) include the vertices

$$\left.\begin{matrix}\epsilon^3\epsilon^2 1\\ \epsilon^6\epsilon^4 1\end{matrix}\right) \quad \left.\begin{matrix}\epsilon^6\epsilon^4 1\\ \epsilon^5\epsilon\ 1\end{matrix}\right) \quad \left.\begin{matrix}\epsilon^5\epsilon\ 1\\ \epsilon^3\epsilon^2\ 1\end{matrix}\right); \quad \left.\begin{matrix}\epsilon^2\epsilon^3 1\\ \epsilon^4\epsilon^6 1\end{matrix}\right) \quad \left.\begin{matrix}\epsilon^4\epsilon^6 1\\ \epsilon\ \epsilon^5 1\end{matrix}\right) \quad \left.\begin{matrix}\epsilon\ \epsilon^5 1\\ \epsilon^2\epsilon^3 1\end{matrix}\right).$$

From the formula previously given, the lower point is the nucleus of the conic containing the vertices of Q and the upper point. The 6 pairs of points are arranged in two cycles of 3: in each cycle the nucleus of each of the 3 conics is a point of the next. For, as the nucleus of the conic through the vertices of Q and (a, b, c) is $\{a(b + c), b(c + a), c(a + b)\}$, when $a + b + c = 0$ the nucleus is (a^2, b^2, c^2). So the nucleus is obtained by applying ϕ to (a, b, c); ϕ having period 3 induces the cycles of three.

The pairs completing hexads of type (2) are

$$\begin{matrix} \begin{matrix}\epsilon^3\epsilon^2 1\\ \epsilon^2\epsilon^3 1\end{matrix}\Big\} & \begin{matrix}\epsilon^6\epsilon^4 1\\ \epsilon\ \epsilon^5 1\end{matrix}\Big\} & \begin{matrix}\epsilon^5\epsilon\ 1\\ \epsilon^4\epsilon^6 1\end{matrix}\Big\} \\ \begin{matrix}\epsilon^6\epsilon^4 1\\ \epsilon^4\epsilon^6 1\end{matrix}\Big\} & \begin{matrix}\epsilon^5\epsilon\ 1\\ \epsilon^2\epsilon^3 1\end{matrix}\Big\} & \begin{matrix}\epsilon^3\epsilon^2 1\\ \epsilon\ \epsilon^5 1\end{matrix}\Big\} \\ \begin{matrix}\epsilon^5\epsilon\ 1\\ \epsilon\ \epsilon^5 1\end{matrix}\Big\} & \begin{matrix}\epsilon^3\epsilon^2 1\\ \epsilon^4\epsilon^6 1\end{matrix}\Big\} & \begin{matrix}\epsilon^6\epsilon^4 1\\ \epsilon^2\epsilon^3 1\end{matrix}\Big\} . \end{matrix}$$

That the 6 hexads of type (1) are equivalent up to collineation, as well as the 9 hexads of type (2), is seen by applying the six permutation matrices and the automorphisms ϕ, ϕ^2.

Let the points of the first hexad H of type (1) be named $A_1, \cdots, A_6$ in the order

$$100,\ 010,\ 001,\ 111,\ \epsilon^3\epsilon^2 1,\ \epsilon^6\epsilon^4 1$$

They are the base points of a map of a cubic surface F with 13 coplanar E-points on the lines a_6, b_6, c_{56}. For, as A_6 is the meet of the tangents at the A_i $i = 1, \cdot\cdot, 5$ to the conic containing these points, b_6 has the 5 E-points (b_6, a_i, c_{i6}); as A_5A_6 contains 3 B-points of H, c_{56} contains 3 E-points $(c_{56}, c_{ij}, c_{k\ell})$ of F; as A_5A_6 is the line of diagonal points of Q, it contains the nucleus of any conic circumscribing Q - in particular A_5A_6 is a tangent at A_6 to the conic through A_1, A_2, A_3, A_4, A_6, so (c_{56}, a_6, b_5) is also an E-point. So far there are 9 E-points lying on b_6 and c_{56}; but (c_{12}, c_{34}, c_{56}), say, and each of (b_6, a_i, c_{i6}) $i = 1, \cdot\cdot, 4$ are collinear with a third E-point on a_6. Thus the points (a_6, b_j, c_{j5}) $j = 1, \cdot\cdot, 4$ are also E-points. Hence F has 13 coplanar E-points lying 5 on each of the 3 lines a_6, b_6, c_{56}, including their point of concurrency.

Let the points of the first hexad K of type (2) be named $A_1, \cdot\cdot, A_6$ in the order

$$100,\ 010,\ 001,\ 111,\ \epsilon^3\epsilon^2 1,\ \epsilon^2\epsilon^3 1.$$

The three lines

$$A_5A_6 \ : \ x + y + z = 0$$
$$A_3A_4 \ : \ x + y \quad = 0$$
$$A_1A_2 \ : \qquad z = 0$$

are the lines of diagonal points of the 3 quadrangles whose vertices are the 4 points of K they do not contain. The 7 B-points of K therefore lie 3 on each of A_1A_2, A_3A_4, A_5A_6, including their intersection. Thus K exists in 3 ways as the vertices of a quadrangle and two points on the latter's line of diagonal points.

The points of K are the base points of a map of a cubic surface G with 13 coplanar E-points lying on the lines c_{12}, c_{34}, c_{56}. For, as each of A_1A_2, A_3A_4, A_5A_6 contains 3 B-points of K, each of c_{12}, c_{34}, c_{56} contains 3 E-points of G. Also, A_1A_2 contains the nucleus of any conic through A_3, A_4, A_5, A_6. Therefore A_1A_2 is a tangent at A_1 to the conic $A_1A_3A_4A_5A_6$ and at A_2 to the conic $A_2A_3A_4A_5A_6$. Thus c_{12} has two further E-points (a_1, b_2, c_{12}) and (a_2, b_1, c_{12}): similarly for c_{34}, c_{56}. So each of c_{12}, c_{34}, c_{56} has 5 E-points on G.

If G is now mapped onto a plane so that the lines a_1, a_2, a_3, c_{45}, c_{46}, c_{56}, comprising one half of the double-six D_{123}, are all mapped to a point, these six points form a hexad of type (1). Hence the two types of hexad map the one type of cubic surface.

It has now been shown that, over $GF(8)$, a cubic surface F with 27 lines always contains 13 coplanar E-points; from the table in § 12, the 27 lines comprise the whole of F. It remains to discover whether F is projectively unique. This will be done by developing the properties implied by the E-points and thus seeking a canonical form for F.

Call the plane of the E-points f. Of the 27 lines, 24 contain exactly one E-point while the remaining 3 each contain five. The self-conjugate points with respect to the polarity of any double-six on F lie, by § 11, in a plane. As each line contains at least one E-point, the 12 lines of the double-six contain at least 6 E-points. These 6 all lie in their polar planes. As these E-points also all lie in f, the plane of the self-conjugate points in the polarities of all the double-sixes on F is f.

Let us now examine f and particularly the 3 lines of F in f: let them be c_{12}, c_{34}, c_{56}. They are concurrent and each meets two other lines of F at 4 more points and no other lines at its remaining 4 points. As has been shown in §13, if the lines meeting at 3 E-points form an array associated with a Steiner trihedral pair, the 3 E-points are collinear. Thus the 12 points on c_{12}, c_{34}, c_{56}, excluding (c_{12}, c_{34}, c_{56}), which are E-points, are collinear in sets of 3 in 16 ways; for example, through one of the 12 points, say (c_{12}, a_1, b_2), there are 4 lines which each contain an E-point on c_{34} and c_{56}, viz.

$$
\begin{array}{ll}
(c_{34},\ a_3,\ b_4\ ), & (c_{56},\ c_{14},\ c_{23}); \\
(c_{34},\ a_4,\ b_3\ ), & (c_{56},\ c_{13},\ c_{24}); \\
(c_{34},\ c_{16},\ c_{25}), & (c_{56},\ a_5,\ b_6\ ); \\
(c_{34},\ c_{15},\ c_{26}), & (c_{56},\ a_6,\ b_5\ ).
\end{array}
$$

Let these 16 lines be called m-lines.

The remaining 12 points on the 3 lines, call them G-points, do not have this property. If a line other than c_{12}, c_{34}, c_{56} passes through 2 E-points, it passes through 3 and is an m-line. Through each E-point there are 4 m-lines; so the remaining 4 lines in f, excluding the line of F, through the E-point all contain 2 G-points: call these n-lines. Thus through each E-point there are 4 m-lines and 4 n-lines, totalling 16 m-lines and 48 n-lines. These comprise the 64 lines of f apart from the pencil through (c_{12}, c_{34}, c_{56}). Thus through each G-point there are 8 n-lines.

In any Desarguesian plane, given 3 lines ℓ_i $i = 1, 2, 3$ through a point U each containing 4 points A_i, B_i, C_i, D_i such that A_1,B_1,C_1,D_1 and A_2,B_2,C_2,D_2 are in perspective from a point P_3 of ℓ_3 and such that A_2,B_2,C_2,D_2 and A_3,B_3,C_3,D_3 are in perspective from a point P_1 of ℓ_1, then if P_1P_3 meets ℓ_2 at P_2, A_1,B_1,C_1,D_1 and A_3,B_3,C_3,D_3 are in perspective from a point V where V,P_2 are harmonic conjugates of P_1,P_3. To prove this, consider the quadrangle $UA_1A_2A_3$; two of its diagonal points, the intersections of two pairs of its opposite sides UA_1, A_2A_3 and UA_3, A_1A_2 are P_1 and P_3. So the third pair UA_2, A_1A_3 meets P_1P_3 in points P_2, V which are harmonic conjugates of P_1,P_3, Todd [61] p.45. Thus A_1,B_1,C_1,D_1 and A_3,B_3,C_3,D_3 are in perspective from V, where V is determined as the harmonic conjugate of P_2 with respect to P_1,P_3. However, over fields of characteristic two, P_2 and V coincide, Todd [61] p.40.

The 12 points A_i,B_i,C_i,D_i $i = 1, 2, 3$ can be taken as the G-points on c_{12}, c_{34}, c_{56} so that U is (c_{12}, c_{34}, c_{56}); then P_1,P_2,P_3 are suitable E-points. The line $P_1P_2P_3$ is an m-line and the configuration can be obtained in exactly 16 ways, one for each m-line. It is natural to ask how the 8 points of ℓ_i residual to U are partitioned into E-points and G-points.

On a line — and similarly in what Todd [61] p. 15 calls " a primitive geometric form of dimension one " — over a field of characteristic two, 4 points are pairs in 3 involutions all of which have the same double-point; this is called the associated point to the other four, Segre [53]. If a,b,c,d and x are the parameters of 4 points on a line and their associated point, then

$$x^2 = (bcd + cda + dab + abc) / (a + b + c + d);$$

in particular $\infty, 0, 1, \alpha$ are associated with $\sqrt{\alpha}$. Over GF(8), if S_1, S_2, S_3, S_4, S_5 are 5 points on a line M and T_1, T_2, T_3, T_4 the remaining 4 points, then the T_i are associated with one of the S_j, which is also the associated point of the other 4 S_j, and each of the remaining 4 tetrads of the S_j has a different T_i as its associated point, Segre [54].

The parameters of the 9 points on M are

$$\infty, 0, 1, \epsilon, \epsilon^2, \epsilon^3, \epsilon^4, \epsilon^5, \epsilon^6.$$

A set of 6 points on M can be partitioned into two triads; if α is any one of the ϵ^i $i = 1, \cdots, 6$, then

$$\begin{array}{llll}
\infty, 0, 1, \alpha & : \alpha^4 & : \alpha^2 & \alpha^3\alpha^6\alpha^5 \\
\infty, 0, 1, \alpha^4 & : \alpha^2 & : \alpha & \alpha^3\alpha^6\alpha^5 \\
\infty, 0, 1, \alpha^2 & : \alpha & : \alpha^4 & \alpha^3\alpha^6\alpha^5 \\
\infty, 0, 1, \alpha^3 & : \alpha^5 & : \alpha^6 & \alpha\ \alpha^2\alpha^4 \\
\infty, 0, 1, \alpha^5 & : \alpha^6 & : \alpha^3 & \alpha\ \alpha^2\alpha^4 \\
\infty, 0, 1, \alpha^6 & : \alpha^3 & : \alpha^5 & \alpha\ \alpha^2\alpha^4
\end{array}$$

where the middle parameters are associated to both tetrads in the same row. The hexad residual to $\infty, 0, 1$ is uniquely partitioned into two triads $\alpha, \alpha^2, \alpha^4$; $\alpha^3, \alpha^6, \alpha^5$ such that the tetrads consisting of the one triad and any element of the other is associated with an element of the second triad. Let this be written

$$\alpha, \alpha^2, \alpha^4 \mathcal{A} \alpha^3, \alpha^5, \alpha^6.$$

$\mathcal{A}$ is an equivalence relation:

(i) it is trivially reflexive as from the given formula a,b,c,c are associated with c;

(ii) $\mathcal{A}$ is symmetric from above;

(iii) As a triad is arbitrary on M, each residual hexad has the same property, so from (ii)

$$\infty, 0, 1 \ \mathcal{A}\ \alpha, \alpha^4, \alpha^2; \qquad \text{this and}$$

$$\alpha, \alpha^2, \alpha^4 \ \mathcal{A}\ \alpha^3, \alpha^6, \alpha^5 \qquad \text{give}$$

$$\infty, 0, 1 \ \mathcal{A}\ \alpha^3, \alpha^5, \alpha^6.$$

It is to be noted that two associations determine the partitioning of a hexad, viz. if

x_1, x_2, x_3, x_4 are associated with x_5

and x_1, x_2, x_3, x_5 are associated with x_6,

then x_1, x_2, x_3, x_6 are associated with x_4

and $x_1, x_2, x_3 \mathcal{A} x_4, x_5, x_6$.

The three triads $\infty, 0, 1$; $\alpha, \alpha^2, \alpha^4$; $\alpha^3, \alpha^5, \alpha^6$ are all in the relation $\mathcal{A}$ to one another and M can be partitioned into 3 such triads in ${}^9C_3/3 = 28$ ways. The projective group $LF(2, 2^3)$ of M has recently been studied in another connection by Macbeath [35].

The previous geometry would indicate that U is the associated point of the E-points as well as the G-points on each ℓ_i. For, as a line meets the sides of a quadrangle Q in pairs of an involution, if a line passes through exactly one diagonal point P of Q, it meets the other 4 sides in pairs of an involution for which P is a double-point. So consider the quadrangle $A_2B_2A_3C_3$; one of its diagonal points is U. Now A_2A_3 meets ℓ_1 in P_1 and B_2C_3 cannot meet ℓ_1 in P_1 as B_2B_3 does. So the line of diagonal points passes through U but is not ℓ_1. Therefore the 4 sides A_2A_3, A_2C_3, B_2A_3, B_2C_3 meet ℓ_1 in the 4 E-points which have U as their fifth associated point. Thus the 4 G-points also have U as their fifth associated point.

The plane of the E-points, f, is one of the 45 tritangent planes of F; let the other 12 through the lines of F in f be v-planes and the remaining 32 be w-planes. Each tritangent plane is in $3 \times 240/45 = 16$ trihedra. Thus f is in the 16 trihedra whose other faces are pairs of the 32 w-planes. The faces of these 16 trihedra are collinear in the 16 m-lines. The 16 conjugate trihedra are composed of the 12 v-planes, each being in $3.16/12 = 4$ of them.

The 12 v-planes have the property that the sets of 4 through c_{12}, c_{34}, c_{56}

all have f as their associated plane; for f and the 4 v-planes through c_{12}, say, are the polar planes of U and the other 4 E-points on c_{12} with respect to any double-six in which c_{12} lies.

A similar property holds for the other 24 lines of F. One of the 5 tritangent planes through such a line of F also contains one of c_{12}, c_{34}, c_{56}; this plane is then associated with the other 4. For example, the 5 planes $[b_1, a_i]$ $i = 2, \cdot\cdot, 6$ all pass through (b_1, c_{12}) and meet f in c_{12} and the 4 m-lines through it; c_{34} meets c_{12} in U and the 4 m-lines in its other 4 E-points, which are associated with U. Therefore $[b_1, a_3], \cdots\cdots, [b_1, a_6]$ are associated with $[b_1, a_2]$. Dually, in any double-six of F, the 5 points in which a line is met by the other lines of the double-six form a set of 4 and its associated point according to their polar planes. In D $(a_i, b_i \quad i = 1, \cdots 6)$, the 5 points (a_i, b_i) $i = 2, \cdot\cdot, 6$ have polar planes $[b_1, a_i]$ so that (a_1, b_2) is the associated point of (a_1, b_3), (a_1, b_4), (a_1, b_5), (a_1, b_6). However, it should be noted that, of the 36 double-sixes of F, 24 contain two of c_{12}, c_{34}, c_{56}, and 12 contain none. Thus with respect to one of the 12, D for example, the associated point among the 5 on a_1 is the E-point (a_1, b_2, c_{12}), but with respect to one of the 24, it is not; for example in

$$D_{123}: \begin{matrix} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{matrix}$$ it is (a_1, b_4) that is associated with

(a_1, c_{13}), (a_1, c_{12}), (a_1, b_5), (a_1, b_6).

One of the 16 trihedral pairs containing f is

$$S_{135,246} \qquad \begin{matrix} c_{12} & c_{34} & c_{56} \\ c_{36} & c_{25} & c_{14} \\ c_{45} & c_{16} & c_{23} \end{matrix}$$

From such a trihedral pair a canonical form for F may be derived. Let f be $x_0 = 0$; this is then the face of a trihedron, another of whose faces may be taken as $x_1 = 0$ so that the third is $Ax_0 + Bx_1 = 0$. Two faces of the conjugate trihedron may be taken as $x_2 = 0$, $x_3 = 0$ so that two of the lines of F in f are $x_0 = 0$, $x_2 = 0$ and $x_0 = 0$, $x_3 = 0$; then the third face can be taken as $x_0 + x_2 + x_3 = 0$. The equation of F now is

$$x_0x_1(Ax_0 + Bx_1) = x_2x_3(x_0 + x_2 + x_3).$$

By substituting AB^6x_1 for x_1, the equation becomes

$$Ax_0x_1(x_0 + x_1) = x_2x_3(x_0 + x_2 + x_3).$$

Over characteristic two, consider the reducible ternary quadratic form

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + a_{12}xy + a_{13}xz + a_{23}yz$$
$$= (ax + by + cz)(dx + ey + fz)$$

$a_{11} = ad$ $\qquad$ $a_{22} = be$ $\qquad$ $a_{33} = cf$

$a_{23} = bf + ce$ $\qquad$ $a_{13} = af + cd$ $\qquad$ $a_{12} = ae + bd$

By substitution it can be seen that

$$a_{11}a_{23}^2 + a_{22}a_{13}^2 + a_{33}a_{12}^2 + a_{23}a_{13}a_{12} = 0 .$$

If y, $ax + by + cz$, $dx + ey + fz$ are linearly dependent, then $af + cd = 0$. So

$$a_{13} = 0 \qquad a_{11}a_{23}^2 + a_{33}a_{12}^2 = 0.$$

$x_0 = \lambda x_2$ meets F in 3 concurrent lines for 5 values of λ, two of which are 0 and ∞. Substituting in the equation for F,

$$A\lambda x_2x_1\ (\lambda x_2 + x_1) = x_2x_3(\overline{\lambda + 1}\ x_2 + x_3)$$
$$x_2(A\lambda x_1^2 + x_3^2 + A\lambda^2x_1x_2 + \overline{\lambda + 1}\ x_2x_3) = 0$$

By the above conditions,

$$A\lambda(\lambda + 1)^2 + (A\lambda^2)^2 = 0$$
$$A\lambda(A\lambda^3 + \lambda^2 + 1) = 0.$$

As $A\lambda^3 + \lambda^2 + 1$ has distinct roots, $A\lambda^3 + \lambda^2 + 1$ divides $\lambda^7 + 1$. But over $GF(2^n)$

$$\lambda^7 + 1 \equiv (\lambda + 1)(\lambda^3 + \lambda + 1)(\lambda^3 + \lambda^2 + 1).$$

Therefore $A = 1$. Thus the canonical form for F, hence to be called F_8 is

$$x_0x_1(x_0 + x_1) = x_2x_3(x_0 + x_2 + x_3)$$

$$\text{or} \quad x_0^2 x_1 + x_0 (x_1^2 + x_2x_3) + x_2x_3(x_2 + x_3) = 0,$$

and every cubic surface over $GF(8)$ with 27 lines is projectively equivalent to this form.

This equation of F_8 enables us to determine the order of its projective group. Any projectivity leaving F_8 invariant will leave $x_0 = 0$ invariant, and so will transform the trihedral pair $S_{135,246}$ into itself or one of the 15 other pairs which have $x_0 = 0$ as a face. If $S_{135,246}$ is left fixed then both its trihedra are also left fixed: in the trihedron containing $x_0 = 0$ the remaining two faces can only interchanged, while in the conjugate trihedron the three faces can only be permuted. The 12 possible such operations can all be achieved by projectivities. The following three generate the others.

(i) Leave x_0, x_2, x_3 fixed and substitute $x_0 + x_1$ for x_1, thus interchanging $x_1 = 0$ and $x_0 + x_1 = 0$.

(ii) Leave x_0, x_1 fixed and transpose x_2, x_3, thus interchanging $x_2 = 0$ and $x_3 = 0$.

(iii) Leave x_0, x_1, x_2 fixed and substitute $x_0 + x_2 + x_3$ for x_3, thus interchanging $x_3 = 0$ and $x_0 + x_2 + x_3 = 0$.

These operations generate a group. So the trihedral pair has a projective group, and F_8 a subgroup, of order 12, the direct product of a cyclic group of order 2, generated by (i), and a symmetric group of degree 3, generated by (ii) and (iii).

The trihedral pair $S_{135,246}$ now has to be transformed into the other 15 containing the plane $[c_{12}, c_{34}, c_{56}]$. These fall into two types: the 9 that have another plane in common with $S_{135,246}$, for example $S_{13,24}$, and the 6 that do not, for example $S_{14,23}$.

$$S_{135,246} \quad \begin{array}{ccc} c_{12} & c_{34} & c_{56} \\ c_{36} & c_{25} & c_{14} \\ c_{45} & c_{16} & c_{23} \end{array} \qquad S_{13,24} \quad \begin{array}{ccc} c_{12} & c_{34} & c_{56} \\ a_1 & b_4 & c_{14} \\ b_2 & a_3 & c_{23} \end{array} \qquad S_{14,23} \quad \begin{array}{ccc} c_{12} & c_{34} & c_{56} \\ a_1 & b_3 & c_{13} \\ b_2 & a_4 & c_{24} \end{array}$$

Both types of transformation can be done by projectivities. For example,

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \epsilon^5 & 0 \\ 0 & 0 & 1 & 0 \\ \epsilon^3 & 0 & 0 & 1 \end{bmatrix}$ gives the trihedral pair

$$x_0(x_1 + \epsilon^5 x_2)(x_0 + x_1 + \epsilon^5 x_2) = x_2(\epsilon^3 x_0 + x_3)(\epsilon^2 x_0 + x_2 + x_3)$$

$$x_0^2(x_1 + \epsilon^5 x_2) + x_0(x_1^2 + \epsilon^3 x_2^2) = \epsilon^5 x_0^2 x_2 + x_0 x_2(\epsilon^3 x_2 + x_3) + x_2 x_3(x_2 + x_3)$$

$$x_0 x_1(x_0 + x_1) = x_2 x_3(x_0 + x_2 + x_3)$$

$$\left|\begin{array}{l} 1 + \epsilon^2 + \epsilon^3 = 0 \\ \epsilon + \epsilon^3 + \epsilon^4 = 0 \\ \epsilon^2 + \epsilon^4 + \epsilon^5 = 0 \\ \epsilon^3 + \epsilon^5 + \epsilon^6 = 0 \\ 1 + \epsilon^4 + \epsilon^6 = 0 \\ 1 + \epsilon + \epsilon^5 = 0 \end{array}\right.$$

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \epsilon^4 & 1 & \epsilon^5 & \epsilon^3 \\ \epsilon^6 & 0 & 1 & 0 \\ \epsilon^3 & 0 & 0 & 1 \end{bmatrix}$ gives the trihedral pair

$$x_0(\epsilon^4 x_0 + x_1 + \epsilon^5 x_2 + \epsilon^3 x_3)(\epsilon^6 x_0 + x_1 + \epsilon^5 x_2 + \epsilon^3 x_3) = (\epsilon^6 x_0 + x_2)(\epsilon^3 x_0 + x_3)(\epsilon x_0 + x_2 + x_3)$$

$$\epsilon^3 x_0^3 + x_0^2(x_1 + \epsilon^5 x_2 + \epsilon^3 x_3) + x_0(x_1^2 + \epsilon^3 x_2^2 + \epsilon^6 x_3^2) = \epsilon^3 x_0^3 + x_0^2(\epsilon^5 x_2^2 + \epsilon^3 x_3^2) + x_0(\epsilon^3 x_2^2 + x_2 x_3 + \epsilon^6 x_3^2) + x_2 x_3(x_2 + x_3)$$

$$x_0 x_1(x_0 + x_1) = x_2 x_3(x_0 + x_2 + x_3)$$

Thus $S_{135,246}$ can be projectively transformed into all 16 trihedral pairs containing $[c_{12}, c_{34}, c_{56}]$ and into no others. Therefore the order of the projective group of F_8 is $16 \times 12 = 192$.

Where does F_8 fit into Segre's classification of the cubic surfaces over the real and complex fields? It is a degenerate case of the non-equianharmonic cyclic surface, [48] §§ 85-89, 100, where the fundamental plane is f and the centre, the intersections of the tangent planes at all the E-points, is

(c_{12}, c_{34}, c_{56}); the centre does not normally lie on the cyclic surface, much less on one of its lines. In the complex case the cyclic non-equianharmonic surfaces have groups of order either 54 or 108, whereas F_8 has a group of order 192.

§ 19. Grace's extension over GF(8).

When it is said that Grace's extension exists, it is meant that the six lines α_i as constructed in § 17 are skew and have a unique transversal. Over GF(8), from the example in § 17, six lines a_i with a transversal b can be selected with any 4 having a unique second transversal, so that the α_i exist. However, as all cubic surfaces over GF(8) are covered by their 27 lines, the theorem at the end of § 17 shows that there is no proper extension.

What does happen to the six lines α_i? It was shown in § 17 that, if the surface F_i is covered by its 27 lines, α_i passes through L_i. Thus over GF(8), α_i passes through L_i for $i = 1, \cdot\cdot, 6$. The twisted cubic t comprises 9 points over GF(8), but must contain the 26 points L_i, $L_{jk\ell}$. From § 17, two of the $L_{jk\ell}$ may coincide with L_i, e.g. L_1, L_{234}, L_{256}. Four of the $L_{jk\ell}$ may coincide with one another e.g. L_{123}, L_{145}, L_{246}, L_{356}. Thus, if two $L_{jk\ell}$ coincide with each L_i and the remaining 8 $L_{jk\ell}$ are equivalent in two sets of four, the 26 points are 8 distinct ones: less there cannot be. This leaves one point L on t. Each α_i is a chord of t and contains L_i; L is the only other point of t that may lie on α_i. Therefore the 6 lines α_i are all concurrent at L. An example of this figure is given in Appendix III.

To discover more of the figure it is necessary to find the plane of the E-points for each F_i; let these planes be called f_i $i = 1, \cdots, 6$. α_1 meets the 10 lines b_{1i}, $c_1^1{}_i$ $i = 2, \cdots, 6$ of F_1 and cuts t in L, L_1; b_{1i}

is skew to t and $c_1^1{}_i$ meets t in L_i, which does not coincide with either L or L_1. So these 10 lines of F_1, meet α_1 in at most 7 points. So α_1 contains at least 3 E-points and therefore 5. Thus f_i is one of the five tritangent planes of F_i through α_i.

These 6 planes, one through each α_i, form cycles like $f_i f_j \ldots f_m$, which represents

$$[\alpha_i, b_{ij}]\,[\alpha_j, b_{jk}] \,\ldots\ldots\, [\alpha_m, b_{mi}].$$

If there is a cycle of two, say $[\alpha_1, b_{12}]\,[\alpha_2, b_{21}]$, then (b_{12}, a_3, c_{23}^1) and (b_{12}, a_3, c_{13}^2) are both E-points: so b_{12} contains L_{123}, the meet of c_{23}^1 and c_{13}^2; but b_{12} is skew to t. So there are no cycles of two. Therefore, suppose that f_1 and f_2 are $[\alpha_1, b_{12}, c_{12}^1]$ and $[\alpha_2, b_{23}, c_{23}^2]$. Now let $K_i = (b, a_i)$ $i = 1, \ldots, 6$. Then, from D_1 and D_2 respectively,

K_3, K_4, K_5, K_6 are associated with K_2

and K_1, K_4, K_5, K_6 are associated with K_3:

therefore K_2, K_4, K_5, K_6 are associated with K_1

and $K_1, K_2, K_3 \mathcal{A} K_4, K_5, K_6$. Thus $[\alpha_3, b_{31}, c_{13}^3]$ is f_3 and the 6 f_i form two cycles $f_1f_2f_3$ and $f_4f_5f_6$ or $f_4f_6f_5$. In fact, in the example given in Appendix III, the cycles are $f_1f_6f_4$ and $f_2f_3f_5$, so that on b

$$K_1, K_4, K_6 \mathcal{A} K_2, K_3, K_5.$$

Each line b_{ij} meets 6 of the 12 lines a_k, α_k, but the partitioning into triads of these points on b_{ij} is not always determined by f_i and f_j. There are two cases to be considered: either b_{ij} lies in one of f_i, f_j or in none. Let K_k^{ij} be (b_{ij}, α_k) for $k = i,j$ and (b_{ij}, a_k) for $k \neq i,j$. Consider b_{23}; f_2 and f_3 are $[\alpha_2, b_{23}]$ and $[\alpha_3, b_{35}]$. So, from D_2 and D_3 respectively,

$K_1^{23}, K_4^{23}, K_5^{23}, K_6^{23}$ are associated with K_2^{23}

and $K_1^{23}, K_3^{23}, K_4^{23}, K_6^{23}$ are associated with K_5^{23}

therefore $K_1^{23}, K_4^{23}, K_6^{23} \mathcal{A} K_2^{23}, K_3^{23}, K_5^{23}$.

However, b_{12} is in a different case; f_1 and f_2 are $[\alpha_1, b_{16}]$ and $[\alpha_2, b_{23}]$. From D_1 and D_2 respectively,

$$K_1^{12}, K_3^{12}, K_4^{12}, K_5^{12} \text{ are associated with } K_6^{12}$$

and $$K_2^{12}, K_4^{12}, K_5^{12}, K_6^{12} \text{ are associated with } K_3^{12},$$

so that either $K_1^{12}, K_3^{12}, K_4^{12} \mathcal{A} K_2^{12}, K_5^{12}, K_6^{12}$

or $K_1^{12}, K_3^{12}, K_5^{12} \mathcal{A} K_2^{12}, K_4^{12}, K_6^{12}$.

Each line a_i meets the 11 lines b, b_{jk} $j, k \neq i$ and therefore must accommodate 11 points $(a_i, b), (a_i, b_{jk})$ among the 9 it contains as well as 11 planes $[a_i, b], [a_i, b_{jk}]$ among the 9 that contain it. As L_{246} coincides with L_{345}, the planes $[a_4, b_{26}], [a_4, b_{35}]$ contain both a_4 and this point, which lies on t and therefore not on a_4. Thus

$$[a_4, b_{26}] = [a_4, b_{35}].$$

Further, as b_{26}, b_{35} meet and as a_1 meets both these lines but not a_4, a_1 must pass through their intersection; so

$$(a_1, b_{26}) = (a_1, b_{35}).$$

In this way, the 11 lines which meet a_i occupy 8 points, given on a_1 by

b	b_{23}	b_{24}	b_{25}	b_{26}	b_{34}	b_{36}	b_{45}
	b_{46}			b_{35}	b_{56}		

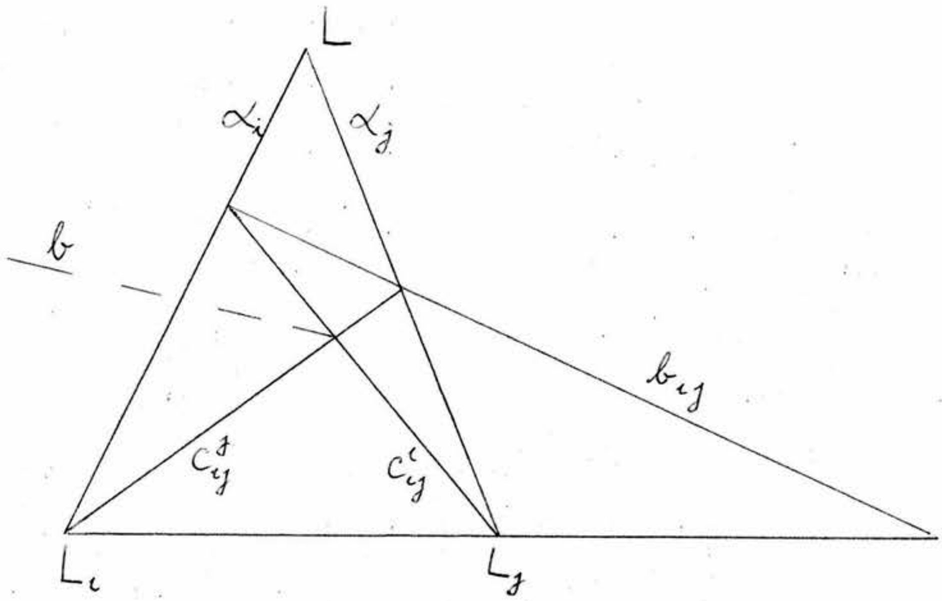
Similarly, these 11 lines are in 8 planes through a_1

b	b_{23}	b_{24}	b_{25}	b_{26}	b_{35}	b_{45}	b_{56}
		b_{36}	b_{34}		b_{46}		

From each D_i $i \neq 1$, one can pick out a set of 4 points on a_1 and their associated point, and similarly for the planes, but this is not reflected in the above arrangements.

As the final piece of geometry of the figure, consider the plane LL_iL_j. The concurrent sides of the diagonal line triangle of the quadrilateral $\alpha_i\alpha_jc_i^jc_j^i$ are L_iL_j, b_{ij} and the transversal of b, L_iL_j through L; b

meets c_{ij}^{i} and c_{ij}^{j}, cannot lie in this plane and therefore contains the point (c_{ij}^{j}, c_{ij}^{i}).



So b contains 15 points (c_{ij}^{j}, c_{ij}^{i}) $i, j = 1, \ldots, 6$. Each of 6 c_{ij}^{j} contains 5 E-points and so passes through $(b, a_i, c_{ij}^{j}) = K_i = (c_{ij}^{j}, c_{ij}^{i})$. The remaining 9 points (c_{ij}^{j}, c_{ij}^{i}) therefore coincide in threes in the 3 remaining points of b.

The search for Grace's extension must now move to PG(3,9).

CHAPTER VII. The cubic surface, and Grace's extension, over GF(9).

§ 20. The Eckardt points on a cubic surface over GF(9).

Over GF(4), GF(7) and GF(8), cubic surfaces with 27 lines have been found to be projectively unique and completely covered by their lines. Over these fields Grace's extension of the double-six does not exist, a necessary condition being the existence of a point on the cubic surface off the lines. Over GF(9), a line has 10 points so that a cubic surface with 27 lines does not 'a priori' have E-points. From § 12, a cubic surface with 27 lines has at most 10 E-points, in which case it is covered by its lines.

As mentioned in § 1, over the complex field Eckardt showed, by considering Sylvester's form for a cubic surface, that a surface F with 27 lines has its E-points at the vertices of its pentahedron; so that, if F has any E-points at all, it has 1, 2, 3, 4, 6 or 10 and in the case of a degenerate pentahedron 9 or 18. Although Eckardt [18] did not identify all the cubic surfaces with 27 lines, a complete classification, e.g. Segre [48] pp 125 - 162, did not reveal any further arrangements of E-points.

Over any field, the arrangement of E-points on F is governed by the lemma of §§ 5 and 13 that two E-points not on the same line of F are collinear with a third E-point. Over fields of characteristic other than two, it was seen in § 11 that each line of F has at most two E-points and F itself has at most 18 E-points. With the help of these two lemmas let us examine, over fields of characteristic other than two, the possible arrangements of E-points on F.

If there is one E-point (a_1, b_2, c_{12}), a second E-point is either on one of the same lines of F, e.g. (a_2, b_1, c_{12}), and implies no other E-points, or is on different lines of F, e.g. (a_2, b_3, c_{23}), and implies a third collinear E-point (a_3, b_1, c_{13}). If another E-point is added to (a_1, b_2, c_{12}) and (a_2, b_1, c_{12}), the existence of at least one more is implied and a closed set of three is formed. Thus to construct any number of E-points beyond three, one can start with a collinear set of three,

$$\text{e.g. } (a_1, b_2, c_{12}), \ (a_2, b_3, c_{23}), \ (a_3, b_1, c_{13})$$

without any loss of generality. A fourth E-point can be chosen in 3 ways: of the three lines of F through it, either none or one or three will have occurred in the above 3 E-points.

(i) Choose (a_4, b_5, c_{45}); this and the other 3 E-points imply the further 3 E-points (c_{15}, c_{24}, c_{36}), (c_{16}, c_{25}, c_{34}), (c_{14}, c_{26}, c_{35}). These then induce the E-points (a_6, b_4, c_{46}) and (a_5, b_6, c_{56}). These 9 E-points are a closed set and form a plane Maclaurin set lying in threes on 12 lines, 4 through each E-point. From § 1, such surfaces exist over the complex field where the 9 E-points are all the inflexions of a plane cubic.

(ii) In this case choose (c_{12}, c_{34}, c_{56}); this implies two further E-points (b_1, a_4, c_{14}) and (a_2, b_4, c_{24}). The 6 E-points are vertices of a quadrangle lying two on each of the sides b_1, a_2, c_{12} of its diagonal line triangle. This configuration occurs when the 6 E-points are all the vertices of the pentahedron of F which lie in one of its faces.

(iii) Finally the choice of (a_2, b_1, c_{12}) gives a closed set of 4

$$(a_1, b_2, c_{12}), \quad (a_2, b_3, c_{23}), \quad (a_3, b_1, c_{13})$$

$$(a_2, b_1, c_{12}).$$

The addition of a further E-point to (iii) is accompanied by the original three choices: the 3 lines of F through the point can have none or one or three lines in common with the 9 that occur above.

(iv) Take (a_4, b_5, c_{45}); this implies the E-points (a_3, b_2, c_{23}),

(a_1, b_3, c_{13}) and the 11 others lying on none of the 9 lines a_i, b_i, c_{ij} $i, j = 1, 2, 3$ as in case (i) of § 15. These are the 18 E-points of an equianharmonic surface lying in threes on the edges of a tetrahedron which are the axes of the trihedral pairs $S_{123}, S_{456}, S_{123,456}$.

(v) Now take (a_1, b_4, c_{14}); this implies the further E-points (a_4, b_2, c_{24}), (a_4, b_3, c_{34}), (c_{13}, c_{24}, c_{56}), (a_3, b_4, c_{34}), (c_{14}, c_{23}, c_{56}). These are the 10 E-points of a diagonal surface, lying two on each of the 15 lines residual to the double-six D_{56}.

(vi) Finally take (a_3, b_2, c_{23}); this implies only one more E-point (a_1, b_3, c_{13}). Thus there are 6 E-points lying in threes on two skew lines as in the diagram of § 7. So the faces of both trihedra of the pair S_{123} are collinear and F has the equation

$$A\, x_0x_1(x_0 + x_1) = x_2x_3(x_2 + x_3).$$

In fields GF(q) where $(q-1, 3) = 1$, each element has a cube root; so the equation can be transformed to

$$x_0x_1(x_0 + x_1) = x_2x_3(x_2 + x_3).$$

Segre [58] p. 224 gives the theorem that, over GF(q) where $(q - 1, 3) = 1$, the surface $f(x_0, x_1) = f(x_2, x_3)$ contains $[q^2 + \{(d - 1)^2 + 1\}q + 1]$ points, where d is the number of zeros in GF(q) of the cubic form $f(x, y)$. Thus, when $(q - 1, 3) = 1$, the above surface contains $q^2 + 5q + 1$ points and therefore cannot contain 27 lines. When $(q - 1, 3) = 3$, $x^3 - 1$ divides $x^q - 1$ and $x^2 + x + 1$ has two roots in GF(q). Then the equation of F can be transformed to

$$A\, x_0^3 + A\, x_1^3 + x_2^3 + x_3^3 = 0.$$

If A has a cube root in GF(q), F is equianharmonic and contains 18 E-points as (iv). If A has no cube root, F consists of $q^2 + 4q + 1$ points, Segre [58] p.237, and cannot contain 27 lines. Thus it is not possible that

F has exactly 6 such E-points.

The addition of further E-points to any of the cases (i), (ii), (v) implies the cases (iv) or (v). Thus we can conclude that F, a cubic surface with 27 lines over a field of odd characteristic, has 0, 1, 2, 3, 4, 6, 9, 10, or 18 E-points and that these E-points are arranged exactly as over the complex field as described by Eckardt [18] .

GF(9) is a quadratic extension of GF(3) by a root of $\mu^2 = \mu + 1$ or of $\mu^2 + \mu = 1$, Segre [57] p.84. Taking the former equation, the elements of GF(9) are

$$0, \pm 1, \pm \sigma, \pm \sigma^2, \pm \sigma^3$$

where $$1 + 1 + 1 = 0$$

$$\sigma^2 - \sigma - 1 = 0 = \sigma^4 + 1$$

and $\sigma^3 - \sigma^2 - \sigma = 0 \qquad \sigma^3 + \sigma^2 + 1 = 0 \qquad \sigma^3 + \sigma - 1 = 0.$

The only automorphisms of GF(9) are ψ and $\psi^2 = 1$, where ψ replaces each element by its cube.

Over GF(9), a cubic surface F with 27 lines has between 0 and 10 E-points, § 12. If F has 10 E-points, they are the 10 vertices of a pentahedron and lie two on each of 15 lines as in (v). A diagonal surface, although it does not have its usual form

$$\sum_0^4 x_i^3 = 0 \qquad \sum_0^4 x_i = 0,$$

can be obtained by evaluating the above as a cubic in x_0, x_1, x_2, x_3 . The terms in x_i^3 disappear and a factor of 3 can be cancelled. Then, over GF(9), the surface is

$$\sum_{\substack{i,j=0 \\ i \neq j}}^{3} x_i^2 x_j - x_0x_1x_2x_3 \sum_0^3 x_i^{-1} = 0$$

or more symmetrically

$$\sum_{\substack{i,j,k=0 \\ i<j<k}}^{4} x_i x_j x_k = 0 \qquad\qquad \sum_{0}^{4} x_i = 0.$$

Thus, as $\mu^2 = \mu + 1$ has two roots in GF(9), the surface has 27 lines and 10 E-points, Baker [4] p.168, and is entirely covered by its lines. The faces of the pentahedron are $x_i = 0 \quad i = 0, \cdots, 4$ as over any other field. Therefore, by transforming the pentahedron of any surface over GF(9) with 27 lines and 10 E-points into this one, all such surfaces are projectively equivalent to the one given, which will henceforth be called F_9^o. The projective group of F_9^o, as over the complex field, is the symmetric group of degree 5 consisting of all permutations of the faces of the pentahedron and has order 120.

To discover other surfaces, hexads in the plane could be considered. From the diagonal surface, hexads with 10, 4, 2 B-points can be obtained by mapping F_9^o via the 3 different types of double-sixes. The hexad with 10 B-points is, by § 12, complete. Segre [55] p. 49 has shown that over GF(9) there exist complete 6, 7, 8-arcs. A 9-arc is contained in a 10-arc, which is always complete. Thus a hexad is either a complete 6-arc or is contained in a 7-arc or an 8-arc. The difficulty in considering the hexads in complete 7 or 8-arcs theoretically (practically, enumeration is straightforward giving only hexads with 2, 3, 4 or 10 B-points) suggests examining further possibilities in three dimensions.

For a cubic surface F with 27 lines over GF(9), let e be the number of E-points and n the number of points not on any line; then

$$e + n = 10 \qquad\qquad \S\, 12$$

The case of F_9^o where $e = 10 \quad n = 0$ has just been considered. It will now be shown that $n \leq 1$.

To do this we require an estimate of the number of points on a plane cubic curve over GF(9). Unfortunately, there is no estimate in the same comparatively elementary vein as this work. Segre [58] p.228 foll. discusses several cases which he gives as particular examples of a general theorem of Hasse [27] and Weil [64] —

"An absolutely irreducible curve of genus g and order d over a field GF(q) has N points, where

$$|N - (q + 1)| \leq 2g\sqrt{q}, \qquad g \leq \tfrac{1}{2}(d - 1)(d - 2)."$$

The theorem is given precisely in this form and generalised by Lang and Weil [34]. (It is also equivalent to the Riemann hypothesis in function-fields over a finite field). In particular, this theorem shows that a plane cubic curve over GF(9) has at most 16 points.

From § 12, F contains 145 points. Suppose F has at least two points P, Q off its lines. Either PQ meets F in no further points or in one further point. In the first case, if a plane section of F through PQ contains a line ℓ of F, then ℓ would meet PQ in a point of F other than P or Q, contrary to the hypothesis; so all plane sections of F through PQ are irreducible cubics. Therefore F contains at most $10(16 - 2) + 2 = 142$ points, which is contrary to there being 145.

Suppose then that PQ meets F in a further point R. So, if a plane section of F through PQ is a line and a conic, the line will pass through R. Thus at most 3 sections of F through PQ can consist of a line and a conic, in which case the 3 lines form an E-point at R. Thus in the cases where there are 0, 1, 2, 3 such plane sections, the number of points on F is respectively at most

$$\begin{array}{llll} 10(16-3) & & +\ 3 & = 133 \\ 9(16-3) & +\ 1(20-3) & +\ 3 & = 137 \\ 8(16-3) & +\ 2(20-3) & +\ 3 & = 141 \\ 7(16-3) & +\ 3(20-3) & +\ 3 & = 145\ . \end{array}$$

Only the last case is admissible where R is an E-point. So, for every pair of points not on any line of F, there is an E-point on their join. As there are n points of F on none of its lines, there are $n(n-1)/2$ lines through pairs of these points and each line contains an E-point, though these are not necessarily distinct. However $n < 6$; for if $n \geq 6$, $n(n-1)/2 \geq 15$ so that at least 5 E-points would be required, whereas there is room for at most 4 since $e + n = 10$. For $n = 2, 3, 5$, e is respectively 8, 7, 5 and in each case we have seen that there is no possible way of F containing exactly this number of E-points.

When $n = 4$, $e = 6$ and the 6 E-points are vertices of a plane quadrilateral as in (ii), e.g.

$$(c_{12}, c_{35}, c_{46}) \qquad (c_{34}, c_{15}, c_{26}) \qquad (c_{56}, c_{13}, c_{24})$$
$$(c_{12}, c_{36}, c_{45}) \qquad (c_{34}, c_{16}, c_{25}) \qquad (c_{12}, c_{34}, c_{56});$$

these are the points where the plane $[c_{12}, c_{34}, c_{56}]$ meets the edges of the tetrahedron whose vertices are the 4 points of F on none of its lines. Let the tetrahedron be the simplex of reference and the plane $[c_{12}, c_{34}, c_{56}]$ be $x_0 + x_1 + x_2 + x_3 = 0$. Then, as c_{12}, c_{34}, c_{56} are joins of opposite vertices of the quadrilateral, they have equations

$$x_0 + x_1 = x_2 + x_3 = 0, \qquad x_0 + x_2 = x_1 + x_3 = 0, \qquad x_0 + x_3 = x_1 + x_2 = 0.$$

The 6 tritangent planes containing 3 concurrent lines of F have equations

$[c_{12}, c_{35}, c_{46}]$	$x_0 + x_1 = a(x_2 + x_3)$
$[c_{12}, c_{36}, c_{45}]$	$b(x_0 + x_1) = x_2 + x_3$
$[c_{34}, c_{15}, c_{26}]$	$x_0 + x_2 = c(x_1 + x_3)$
$[c_{34}, c_{16}, c_{25}]$	$d(x_0 + x_2) = x_1 + x_3$
$[c_{56}, c_{13}, c_{24}]$	$x_0 + x_3 = e(x_1 + x_2)$
$[c_{56}, c_{14}, c_{23}]$	$f(x_0 + x_3) = x_1 + x_2$

As $[c_{12}, c_{34}, c_{56}]$ meets F in c_{12}, c_{34}, c_{56} and as X_0, X_1, X_2, X_3 lie on F, its equation is

$$(x_0 + x_1 + x_2 + x_3)(A_{01}x_0x_1 + A_{02}x_0x_2 + A_{03}x_0x_3 + A_{12}x_1x_2 + A_{13}x_1x_3 + A_{23}x_2x_3) + (x_2 + x_3)(x_1 + x_3)(x_1 + x_2) = 0.$$

c_{12}, c_{35}, c_{46} all lie in planes through X_0X_1. So $x_0 + x_1 = a(x_2 + x_3)$ meets F in $x_2 + x_3 = 0$, $a'x_2 + x_3 = 0$, $x_2 + a''x_3 = 0$. Substituting $x_0 = ax_2 + ax_3 - x_1$ in the equation

$$(a + 1)(x_2 + x_3)\{(ax_2 + ax_3 - x_1)(A_{01}x_1 + A_{02}x_2 + A_{03}x_3) + A_{12}x_1x_2 + A_{13}x_1x_3 + A_{23}x_2x_3\} + (x_2 + x_3)(x_1^2 + x_1x_2 + x_1x_3 + x_2x_3) = 0 .$$

Then, suppressing the factor $(x_2 + x_3)$, the coefficients of x_1^2, x_1x_2, x_1x_3 in the remaining conic must be zero.

$$-(a + 1)A_{01} + 1 = 0$$

$$a(a + 1)A_{01} - (a + 1)A_{02} + (a + 1)A_{12} + 1 = 0$$

$$a(a + 1)A_{01} - (a + 1)A_{03} + (a + 1)A_{13} + 1 = 0$$

Substituting 1 for $(a + 1)A_{01}$ in the last two equations and, as $a \neq -1$, cancelling $a + 1$

$$1 - A_{02} + A_{12} = 0$$
$$1 - A_{03} + A_{13} = 0 .$$

Similarly from the section of F by $x_0 + x_2 = c(x_1 + x_3)$

$$1 - A_{01} + A_{12} = 0$$
$$1 - A_{03} + A_{23} = 0$$

and from $x_0 + x_3 = e(x_1 + x_2)$

$$1 - A_{02} + A_{23} = 0$$
$$1 - A_{01} + A_{13} = 0 .$$

Therefore $A_{01} = A_{02} = A_{03} = (a + 1)^{-1} = (c + 1)^{-1} = (e + 1)^{-1} = A$

$$A_{12} = A_{13} = A_{23} = A - 1 .$$

So the equation of F is

$$(x_0 + x_1 + x_2 + x_3)\{Ax_0(x_1 + x_2 + x_3) + (A - 1)(x_1x_2 + x_1x_3 + x_2x_3)\}$$
$$+ (x_2 + x_3)(x_1 + x_3)(x_1 + x_2) = 0$$

$x_0 + x_1 = a(x_2 + x_3)$ meets F in

$$(a + 1)(x_2 + x_3)\{A(-x_1 + ax_2 + ax_3)(x_1 + x_2 + x_3) + (A - 1)(x_1x_2 + x_1x_3 + x_2x_3)\}$$
$$+ (x_2 + x_3)(x_1 + x_3)(x_1 + x_2) = 0$$

which, since $(a + 1)A = 1$, is

$$(x_2 + x_3)(ax_2^2 + \overline{a + 1}\, x_2x_3 + ax_3^2) = 0$$

The quadratic expression has two different zeros, so its discriminant

$$\Delta = (a + 1)^2 - a^2 = 1 - a = -(A + 1)/A$$

is a non-zero square. So $A \neq 0, -1$ and $\Delta^4 = 1$;

$$\text{so} \quad (A + 1)^4 = A^4$$
$$A^3 + A + 1 = 0$$
$$A = 1 \text{ or } -\sigma \text{ or } -\sigma^3$$

c_{12} is $x_0 + x_1 = x_2 + x_3 = 0$ so $x_0 + x_1 = \lambda(x_2 + x_3)$ meets F in 3 lines for 5 values of λ.

$$(\lambda + 1)(x_2 + x_3)\{A(-x_1 + \lambda x_2 + \lambda x_3)(x_1 + x_2 + x_3) + (A - 1)(x_1x_2 + x_1x_3 + x_2x_3)\}$$
$$+ (x_2 + x_3)(x_1 + x_3)(x_1 + x_2) = 0$$

Suppressing the factor $(x_2 + x_3)$

$$\{1 - A(\lambda + 1)\}x_1^2 + A\lambda(\lambda + 1)(x_2^2 + x_3^2)$$
$$+ \{A(\lambda^2 - 1) + (A - 1)(\lambda + 1) + 1\}(x_1x_2 + x_1x_3)$$
$$+ \{- A\lambda(\lambda + 1) + (A - 1)(\lambda + 1) + 1\}x_2\, x_3 = 0$$

So λ satisfies
$$\begin{vmatrix} 1 - A\,\overline{\lambda + 1} & \lambda(1 - A\,\overline{\lambda + 1}) & \lambda(1 - A\,\overline{\lambda + 1}) \\ \lambda(1 - A\,\overline{\lambda + 1}) & A\lambda(\lambda + 1) & A(\lambda^2 - 1) + \lambda \\ \lambda(1 - A\,\overline{\lambda + 1}) & A(\lambda^2 - 1) + \lambda & A\lambda(\lambda + 1) \end{vmatrix} = 0$$

$$(1 - A\,\overline{\lambda + 1}) \begin{vmatrix} 1 & \lambda & \lambda \\ \lambda(1 - A\,\overline{\lambda + 1}) & A\lambda(\lambda + 1) & A(\lambda^2 - 1) + \lambda \\ \lambda(1 - A\,\overline{\lambda + 1}) & A(\lambda^2 - 1) + \lambda & A\lambda(\lambda + 1) \end{vmatrix} = 0$$

$$(1 - A\,\overline{\lambda + 1}) \begin{vmatrix} A\lambda(\lambda+1)^2 - \lambda^2 & A(\lambda+1)(\lambda^2+\lambda-1) + \lambda(1-\lambda) \\ A(\lambda+1)(\lambda^2+\lambda-1) + \lambda(1-\lambda) & A\lambda(\lambda+1)^2 - \lambda^2 \end{vmatrix} = 0$$

$$(1 - A\,\overline{\lambda + 1})\,[\{A\lambda(\lambda+1)^2 - \lambda^2\}^2 - \{A(\lambda+1)(\lambda^2+\lambda-1) + \lambda(1-\lambda)\}^2] = 0$$

$$(1 - A\,\overline{\lambda + 1})\{-A(\lambda+1)(\lambda-1)^2 + \lambda(\lambda+1)\}\{A(\lambda+1) - \lambda\} = 0$$

$$(1 - A\,\overline{\lambda + 1})(\lambda+1)(\lambda - A\,\overline{\lambda+1})\{A(\lambda-1)^2 - \lambda\} = 0$$

The quadratic $A(\lambda - 1)^2 - \lambda = A\lambda^2 + \overline{A - 1}\,\lambda + A$ has two different factors so its discriminant

$$V = (A - 1)^2 - A^2 = A + 1$$

satisfies $V^4 = 1$; so

$$\begin{aligned} (A + 1)^4 &= 1 \\ A^4 + A^3 + A &= 0 \end{aligned}$$

and as $A \neq 0$ $\quad A^3 + A^2 + 1 = 0$

so $\quad A = 1$ or σ or σ^3.

Thus, by comparison with the previous values, $A = 1$. Therefore $a = c = e = 0$: the planes $x_0 + x_1 = 0$, $x_0 + x_2 = 0$, $x_0 + x_3 = 0$ are tritangent planes and contain the points X_1, X_2, X_3, which now lie on lines of F. F is, in fact,

$$x_0(x_1 + x_2 + x_3)(x_0 + x_1 + x_2 + x_3) + (x_2 + x_3)(x_1 + x_3)(x_1 + x_2) = 0$$

Putting $x_0 + x_1 + x_2 + x_3 + x_4 \equiv 0$, this becomes

$$x_0x_4(x_1 + x_2 + x_3) + (x_0 + x_4 + x_1)(x_0 + x_4 + x_2)(x_0 + x_4 + x_3) = 0$$

$$x_0x_4(x_1 + x_2 + x_3) + (x_0 + x_4)^3 + (x_0 + x_4)^2(x_1 + x_2 + x_3)$$
$$+ (x_0 + x_4)(x_1x_2 + x_1x_3 + x_2x_3) + x_1x_2x_3 = 0$$

$$x_0x_4(x_1 + x_2 + x_3) + (x_0 + x_4)(x_1x_2 + x_1x_3 + x_2x_3) + x_1x_2x_3 = 0$$

$$\sum_{i<j<k} x_ix_jx_k = 0 \qquad \sum_0^4 x_i \equiv 0.$$

This is the diagonal surface F_9° on which X_0, X_1, X_2, X_3 are 4 of the 10 E-points.

It has now been shown that $n \leqslant 1$.

§ 21. The cubic surface over GF(9) with 9 E-points.

From § 20, there remains the possibility that n = 1 and F has 9 E-points as in (i), e.g.

$R_1 : (a_1, b_2, c_{12})$ $R_4 : (a_4, b_5, c_{45})$ $R_7 : (c_{15}, c_{24}, c_{36})$
$R_2 : (a_2, b_3, c_{23})$ $R_5 : (a_5, b_6, c_{56})$ $R_8 : (c_{14}, c_{26}, c_{35})$
$R_3 : (a_3, b_1, c_{13})$ $R_6 : (a_6, b_4, c_{46})$ $R_9 : (c_{16}, c_{25}, c_{34})$

which lie in a plane section of F on the 12 lines

$R_1R_2R_3$, $R_4R_5R_6$, $R_7R_8R_9$, $R_1R_4R_7$, $R_2R_5R_8$, $R_3R_6R_9$

$R_1R_5R_9$, $R_1R_6R_8$, $R_2R_4R_9$, $R_2R_6R_7$, $R_3R_5R_7$, $R_3R_4R_8$.

These lines are given by the rows, columns and determinantal products in the above array. If there is such an F with only one point off its lines, this point lies in the plane section containing the E-points; this plane section therefore contains 10 points and is rational. To show this, coordinates will be given to the 9 E-points. Eight constants can be chosen arbitrarily in a plane; as a point has freedom 2, we may select 3 points and then one more on both the lines joining two of the points to the third, viz.

$R_1\ (1, 0, 0)$; $R_2\ (0, 1, 0)$; $R_3\ (1, 1, 0)$

$R_1\ (1, 0, 0)$; $R_4\ (0, 0, 1)$; $R_7\ (1, 0, 1)$.

The remaining 4 points can be given coordinates by the following collinearities

$R_2\ (0, 1, 0)$; $R_4\ (0, 0, 1)$; $\underline{R_9\ (0, \alpha, 1)}$

$R_3\ (1, 1, 0)$; $R_7\ (1, 0, 1)$; $\underline{R_5\ (\beta + 1, \beta, 1)}$

$R_4\ (0, 0, 1)$; $R_5\ (\beta + 1, \beta, 1)$; $\underline{R_6\ (\beta + 1, \beta, \gamma)}$

$R_7\ (1, 0, 1)$; $R_9\ (0, \alpha, 1)$; $\underline{R_8\ (\delta, \alpha, \delta + 1)}$

The other 6 collinearities are

$R_1\ (1, 0, 0)$; $R_6\ (\beta + 1, \beta, \gamma)$; $R_8\ (\delta, \alpha, \delta + 1)$

$R_1\ (1, 0, 0)$; $R_5\ (\beta + 1, \beta, 1)$; $R_9\ (0, \alpha, 1)$

$R_2\ (0, 1, 0)$; $R_5\ (\beta + 1, \beta, 1)$; $R_8\ (\delta, \alpha, \delta + 1)$

$R_2\ (0, 1, 0)$; $R_6\ (\beta + 1, \beta, \gamma)$; $R_7\ (1, 0, 1)$

$R_3\ (1, 1, 0)$; $R_4\ (0, 0, 1)$; $R_8\ (\delta, \alpha, \delta + 1)$

$R_3\ (1, 1, 0)$; $R_6\ (\beta + 1, \beta, \gamma)$; $R_9\ (0, \alpha, 1)$

The conditions for these 6 collinearities are respectively $\beta(\delta + 1) = \alpha\gamma$, $\beta = \alpha$, $\beta(\delta + 1) = -1$, $\beta + 1 = \gamma$, $\alpha = \delta$, $\alpha\gamma = -1$. These give $\alpha = \beta = -\gamma = \delta = 1$ so that the 9 points are

R_1 (1, 0, 0)	R_4 (0, 0, 1)	R_7 (1, 0, 1)
R_2 (0, 1, 0)	R_5 (−1, 1, 1)	R_8 (−1, −1, 1)
R_3 (1, 1, 0)	R_6 (1, −1, 1)	R_9 (0, 1, 1)

They lie on the cubic curve

$$x^2(y - z) + y^2(z - x) + z^2(x - y) = 0$$

whose only other point is (1, 1, 1). Thus the plane section of F containing the 9 E-points also contains the only point of F on none of its lines and is rational. Before determining whether such a surface actually exists, it will be helpful to find out more about the rational plane cubic.

Over $GF(q)$, $q = 3^n$, consider a rational cubic T in the plane by projecting the twisted cubic Γ given by

$$x_0 : x_1 : x_2 : x_3 = \lambda^3 : \lambda^2 : \lambda : 1$$

from a point P on the line $\ell : x_0 = x_3 = 0$, which is, by § 16, the meet of the osculating planes of Γ, onto a plane u which does not meet Γ in any points belonging to $GF(q)$. The $q + 1$ tangents of Γ meet ℓ in distinct points. Thus, through P, there is one tangent, at Q_0 say, and q secants, at $Q_1, \ldots, Q_q$, of Γ. As the osculating planes of Γ pass through P, the tangents to T, which are the intersections of the osculating planes of Γ with u, all have 3-point contact with T. Thus the projections $P(Q_i)$ of the Q_i are either inflexions or cusps. As PQ_0 is a tangent to Γ, the planes through PQ_0 meet Γ in the Q_i $i = 0, \ldots, q$ so that all the lines of u through $P(Q_0)$ meet ~~Γ~~ T in a $P(Q_i)$ $i = 0, \ldots, q$: thus $P(Q_0)$ is a cusp. As the PQ_i $i = 1, \ldots, q$ are secants, the $P(Q_i)$ $i \neq 0$ are inflexions. So T consists of q inflexions and one cusp.

As the osculating planes of Γ are collinear in ℓ, the tangents of T are concurrent in the meet of ℓ and u. Any plane containing two of the secants of Γ through P contains a third; so the q inflexions of T are collinear in threes in $q(q-1)/6$ ways. The osculating plane at Q_0 contains

the point P			
the other q	points on	ℓ	
the other q	points on	PQ_0	
$\frac{1}{2}(q^2 - q)$	points on	"conjugate"	chords of Γ
$\frac{1}{2}(q^2 - q)$	points on	"real"	chords of Γ,

the last being collinear in threes. In this way we have all $q^2 + q + 1$ points of the plane. If Q_0 is the point $\lambda = v$, then PQ_0, the tangent to Γ at Q_0, has coordinates

$$v^4 : -v^3 : 0 : v^2 : v : 1 \qquad \S\ 16\ ;$$

ℓ is $0 : 0 : 0 : 1 : 0 : 0$

so a line α through P in the osculating plane at Q_0 is

$$v^4 : -v^3 : 0 : v^2 - \phi : v : 1\ .$$

A chord β of Γ is

$$r^2s^2 : rs(r+s) : r^2 + rs + s^2 : rs : -r-s : 1.$$

If β meets α

$$v^4 + v^3(r+s) + (v^2 - \phi)(r^2 + rs + s^2) + vrs(r+s) + r^2s^2 = 0.$$

So
$$\phi(r-s)^2 = \{v^2 - v(r+s) + rs\}^2$$

Thus the chords of Γ meeting α are all "real" or all "conjugate" according as ϕ is a square or a non-square. Therefore, through each point of the $(q-1)/2$ chords α given by square ϕ, there is a real chord of Γ. Thus the $q(q-1)/6$ planes through P and 3 points of Γ are collinear in sets of $q/3$ on $(q-1)/2$ concurrent lines of the osculating plane ℓQ_0. Thus $q/3$ lines through the q inflexions of T are concurrent at a point of the cuspidal tangent in $(q-1)/2$ ways.

Over GF(9) in particular, T consists of 9 inflexions and one cusp with the 10 tangents concurrent; the 9 inflexions are collinear in 12 sets of 3, any set of 3 lines through the 9 inflexions being concurrent at a point of the cuspidal tangent. Herein lies a difference from the complex field where, although a plane cubic has 9 inflexions collinear in 12 sets of 3, a set of 3 lines through the 9 inflexions forms a triangle.

So, the surface F with 9 E-points at the inflexions of a plane cubic contains 145 points of which 144 are on the 27 lines. The remaining point of F is the double point of the plane cubic.

It remains to find out whether such a surface F with 27 lines and 9 E-points actually exists. Keeping in mind the previous discussion of the plane cubic over GF(9) let us aim at a surface like

$$x_0 \sum_{i \leq j} a_{ij} x_i x_j + x_1^3 - x_2^2 x_3 = 0$$

where the 9 E-points are to lie on $x_0 = x_1^3 - x_2^2 x_3 = 0$. The tangent plane at $(0, y_1, y_2, y_3)$ is

$$x_0 \sum_{\substack{i \leq j}}{}_{i,j=1}^{i,j=3} a_{ij} y_i y_j + y_2 y_3 x_2 - y_2^2 x_3 = 0.$$

The points of $x_0 = x_1^3 - x_2^2 x_3 = 0$ are given by $(0, t, 1, t^3)$ and 3 points $t = p, q, r$ are collinear if $p + q + r = 0$. To show that F has 27 lines, it is sufficient to find its equation in terms of each member of a triad of Steiner trihedral pairs, e.g. S_{123}, S_{456}, $S_{123,456}$. Consider S_{123} $\begin{matrix} c_{23} & a_2 & b_3 \\ b_2 & c_{12} & a_1 \\ a_3 & b_1 & c_{13} \end{matrix}$;

let the rows be the tangent planes at the E-points at $t = 0, 1, -1$ so that the columns are tritangent planes, collinear in the line containing these E-points.

Thus F can be given the equation

$$(Ax_0 + x_1 - x_3)(Bx_0 + x_1 - x_3)(Cx_0 + x_1 - x_3) = (x_0 + x_3)(x_2 - x_3)(x_2 + x_3)$$

where $x_0 = 0$ has been chosen as the plane of E-points, $x_0 + x_3 = 0$, $x_2 - x_3 = 0$, $x_2 + x_3 = 0$ as arbitrary planes through $t = 0, 1, -1$ respectively, and $x_0 = x_1 - x_3 = 0$ as the line through these 3 points. Thus 14 parameters have been chosen; the fifteenth appears from the unit element being taken as the ratio of the products on either side of the equation. Expanding the equation gives

$$ABC\,x_0^3 + x_0^2\,\{AB + AC + BC)(x_1 - x_3)\} + x_0\{(A + B + C)(x_1 - x_3)^2 - x_2^2 + x_3^2\}$$
$$+ x_1^3 - x_2^2 x_3 = 0$$

The tangent plane at $(0, t, 1, t^3)$ is therefore

$$x_0\{D(t - t^3)^2 + t^6 - 1\} + t^3 x_2 - x_3 = 0 \qquad \text{where } D = A + B + C\,.$$

Now similarly take the equation of F from the trihedral pair S_{456}, three of whose planes are tritangent planes at the E-points $t = \sigma^3, -\sigma^2, -\sigma$ and whose other 3 planes contain the line of these points, $x_0 = x_1 - \sigma^2 x_2 - x_3 = 0$.

$$(\alpha x_0 + x_1 - \sigma^2 x_2 - x_3)(\beta x_0 + x_1 - \sigma^2 x_2 - x_3)(\gamma x_0 + x_1 - \sigma^2 x_2 - x_3)$$
$$= \{(-D + \sigma)x_0 + \sigma x_2 - x_3\}\{(-D + 1)x_0 + \sigma^2 x_2 - x_3\}\{(-D + \sigma^3)x_0 - \sigma^3 x_2 - x_3\}$$
$$x_0^3\,\{\alpha\beta\gamma + (D - \sigma)(D - 1)(D - \sigma^3)\} + x_0^2\,\{(\alpha\beta + \alpha\gamma + \beta\gamma)(x_1 - \sigma^2 x_2 - x_3) + \sigma^2 x_2 - Dx_3\}$$
$$+ x_0\{(\alpha + \beta + \gamma)(x_1 - \sigma^2 x_2 - x_3)^2 - (D + 1)x_2^2 + x_3^2\} + x_1^3 - x_2^2 x_3 = 0\,.$$

Compare the two equations of F.

From the coefficient of x_0, $\quad \alpha + \beta + \gamma = A + B + C = 0$.

Then from x_0^2, $\quad \alpha\beta + \alpha\gamma + \beta\gamma = AB + AC + BC = 1$.

Therefore from x_0^3, $\quad \alpha\beta\gamma + 1 = ABC$.

Let $ABC = k$. Then A, B, C are the roots of

$$x^3 + x - k = 0$$

and α, β, γ are the roots of $x^3 + x - k + 1 = 0$. However $x^9 - x = (x^3 + x)(x^3 + x - 1)(x^3 + x + 1)$; thus $k = 0, 1, -1$ i.e. $k^3 = k$. So F has equation

$$k\,x_0^3 + x_0^2(x_1 - x_3) - x_0(x_2^2 - x_3^2) + x_1^3 - x_2^2 x_3 = 0\,.$$

Substituting $k\,x_0 + x_1$ for x_1 gives

$$x_0^2(x_1 - x_3) - x_0(x_2^2 - x_3^2) + x_1^3 - x_2^2 x_3 = 0\,.$$

To verify that F has 27 lines, consider its equation via the trihedral pair $S_{123,456}$ derived from the E-points at $t = -\sigma^3, \sigma^2, \sigma$.

$$(a x_0 + x_1 + \sigma^2 x_2 - x_3)(b x_0 + x_1 + \sigma^2 x_2 - x\ )(c x_0 + x_1 + \sigma^2 x_2 - x_3)$$

$$= (\sigma x_0 - \sigma x_2 - x_3)(x_0 - \sigma^2 x_2 - x_3)(\sigma^3 x_0 + \sigma^3 x_2 - x_3)$$

$$(abc + 1)x_0^3 + x_0^2 \{(ab + ac + bc)(x_1 + \sigma^2 x_2 - x_3) - \sigma^2 x_2\}$$

$$+ x_0\{(a + b + c)(x_1 + \sigma^2 x_2 - x_3)^2 - x_2^2 + x_3^2\} + x_1^3 - x_2^2 x_3 = 0$$

So $abc + 1 = 0$, $ab + ac + bc = 1$, $a + b + c = 0$; and a, b, c are the roots of $x^3 + x + 1 = 0$.

Thus all cubic surfaces over GF(9) with 27 lines and 9 E-points are projectively equivalent to F, hence to be called F_9^1, which is

$$x_0^2 (x_1 - x_3) - x_0(x_2^2 - x_3^2) + x_1^3 - x_2^2 x_3 = 0$$

This equation enables us to determine the order of the projective group of F_9^1. Any projectivity leaving F_9^1 fixed also leaves the cubic curve containing the E-points $x_0 = x_1^3 - x_2^2 x_3 = 0$ fixed, as well as the cusp (0, 0, 0, 1) of the curve and the cuspidal tangent $x_0 = x_2 = 0$. The meet of the inflexional tangents (0, 1, 0, 0) is also fixed as well as the set of 4 points (0, a, 0, 1) $a^4 = 1$ in which the 12 lines through sets of three inflexions meet the cuspidal tangent. Each of these 4 points is the meet of the 3 lines in which the faces of halves of the pairs in a triad of trihedral pairs are concurrent; for example, from the derivation of the equation of F_9^1, the faces of one trihedron of S_{123} meet in $x_0 = x_1 - x_3 = 0$, of S_{456} meet in $x_0 = x_1 - \sigma^2 x_2 - x_3 = 0$, of $S_{123,456}$ meet in $x_0 = x_1 + \sigma^2 x_2 - x_3 = 0$. These 3 lines meet in (0, 1, 0, 1) on $x_0 = x_2 = 0$. The other 3 such triads of trihedral pairs are

$S_{14,25}$	$S_{25,36}$	$S_{36,14}$
$S_{15,26}$	$S_{26,34}$	$S_{34,15}$
$S_{16,24}$	$S_{24,35}$	$S_{35,16}$

To determine the order of the group of F_9^1 we will consider in how many ways S_{123} can be transformed into itself and whether it can be transformed into another trihedral pair, both of its own triad and of one of the other triads. If, under a projectivity of F_9^1, the point of $x_0 = x_2 = 0$ which "belongs" to a triad — in the sense that $(0, 1, 0, 1)$ belongs to S_{123}, S_{456}, $S_{123,456}$ — is left fixed, then the triad is also left fixed.

F_9^1 $\qquad x_0^2(x_1 - x_3) - x_0(x_2^2 - x_3^2) + x_1^3 - x_2^2 x_3 = 0$

S_{123} $\qquad (x_1 - x_3)(\sigma^2 x_0 + x_1 - x_3)(-\sigma^2 x_0 + x_1 - x_3) = (x_0 + x_3)(x_2 - x_3)(x_2 + x_3)$

Any projectivity leaving S_{123} fixed also keeps both its trihedra fixed. Call the trihedra indicated by the left and right sides of the equation L and R respectively. Consider the following three transformations.

(i) Leave x_0, x_2, x_3 fixed and substitute $\sigma^2 x_0 + x_1$ for x_1, thus giving a complete permutation of the faces of L and leaving those of R fixed.

(ii) Leave x_0, x_1, x_3 fixed and substitute $-x_2$ for x_2, thus interchanging the faces $x_2 - x_3 = 0$ and $x_2 + x_3 = 0$ of R while leaving those of L fixed.

(iii) Leave x_0 fixed and substitute $-x_0 + x_1 + x_2$ for x_1, $-x_0 - x_2$ for x_2 and $-x_0 + x_2 + x_3$ for x_3, thus interchanging the faces $x_0 + x_3 = 0$ and $x_2 + x_3 = 0$ of R while leaving those of L fixed.

Thus L has a cyclic group of order 3, generated by (i), and R has a symmetric group of degree 3, generated by (ii) and (iii). S_{123} has a non-abelian group of order 18 and these are all the projectivities which leave it invariant.

As the axis of one trihedron of S_{456} is $x_0 = x_1 - \sigma^2 x_2 - x_3 = 0$ and of $S_{123,456}$ is $x_0 = x_1 + \sigma^2 x_2 - x_3 = 0$, the projectivity (ii) transposes S_{456} and $S_{123,456}$. Similarly S_{123} can be transformed into both S_{456} and $S_{123,456}$. Thus the triad of trihedral pairs has a group of order $3 \times 18 = 54$. The projectivities (i), (ii), (iii) all leave each point of $x_0 = x_2 = 0$ fixed and so keep all four triads invariant. However, the triads can be transformed into one another. The projectivity, which leaves x_0 fixed and replaces x_1 by $x_0 - x_1$, x_2 by $\sigma^3 x_2$, x_3 by $\sigma^3 x_0 - \sigma^2 x_3$, makes F_9^1

$$x_0(x_0 - x_1 - \sigma^3 x_0 + \sigma^2 x_3) - x_0(-\sigma^2 x_2^2 + \sigma^2 x_0^2 + \sigma x_0 x_3 + x_3^2)$$
$$+ x_0^3 - x_1^3 + \sigma^2 x_2^2(\sigma^3 x_0 - \sigma^2 x_3) = 0$$
$$- x_0^2(x_1 - x_3) + x_0(x_2^2 - x_3^2) - x_1^3 + x_2^2 x_3 = 0 .$$

This projectivity transforms $(0, 1, 0, 1)$ into $(0, -\sigma^2, 0, 1)$ and so S_{123} into a member of one of the other triads. In this way, S_{123} can be transformed into all 12 trihedral pairs one of whose trihedra has an axis. Thus the order of the projective group of F_9^1 is $4 \times 3 \times 18 = 216$.

This surface differs from the cyclic non-equianharmonic surfaces over the complex field, Segre [48] § 100, in that its centre, the meet of the tangent planes at the E-points, lies on the surface and in its fundamental plane, neither being true in the complex case. Also, the group over the complex field is of order 54 or 108, not 216 as for F_9^1.

It was seen on F_8 that the 5 points in which a line of a double-six meets the other lines are partitioned into 4 and 1, as are the 5 tritangent planes through the line. A similar property holds on F_9^1.

Over any field not of characteristic two, the condition that x_1, x_2 and x_3, x_4 are harmonic conjugates is

$$(x_1 + x_2)(x_3 + x_4) = 2(x_1 x_2 + x_3 x_4).$$

Over fields of characteristic three, this becomes

$$\sum_{i<j} x_i x_j = 0$$

so that any two pairs from such a set of x_i are harmonic conjugates; thus any permutation of the $4 x_i$ can be effected by a projectivity. This also shows that any four elements which are the roots of a quartic whose middle term is zero are harmonic. Over GF(9) in particular, the squares, satisfying $x^4 - 1 = 0$, as well as the non-squares, satisfying $x^4 + 1 = 0$, are harmonic. The 6 points on a line over GF(9) residual to a harmonic tetrad fall into 3 pairs, any two of which form a harmonic tetrad, Edge [20]. Taking the initial tetrad as $\sigma, -\sigma, \sigma^3, -\sigma^3$, the sextuplet consists of the three pairs

$$\infty, 0;\ 1, -1;\ \sigma^2, -\sigma^2 .$$

On the rational plane cubic over GF(9), the 4 points of concurrency of sets of 3 lines through the 9 inflexions lie on the cuspidal tangent and are given by 4 parameters which are the 4 non-zero squares in GF(9): thus these 4 points are harmonic. Let them be S_1, S_2, S_3, S_4 and suppose, as before, the 9 inflexions R_i $i = 1, .., 9$ are such that $R_1R_2R_3$, $R_4R_5R_6$, $R_7R_8R_9$, are concurrent at S_1. Then, as the join of any two inflexions passes through a third and also one of the S_i, the range $(S_1R_1R_2R_3)$ is in sextuple perspective from the points R_n $n = 4, .., 9$ with the range $(S_1\ S_i\ S_j\ S_k)$, where ijk is a permutation of 234. As $(S_1\ S_2\ S_3\ S_4)$ is harmonic, so is $(S_1\ R_1\ R_2\ R_3)$. Similarly any 3 collinear inflexions plus the point where their join meets the cuspidal tangent form a harmonic tetrad.

Now consider F_9^1 whose 9 E-points, one on each of the 27 lines, lie on a plane cubic f. The 9 tritangent planes at the E-points contain the inflexional tangents to f at these points and have a common point R. The 12 lines a_i, b_i $i = 1, .., 6$ of the double-six D meet in pairs at the 6 E-points

$$\begin{array}{llll} R_1 : (a_1, b_2, c_{12}) & \qquad & R_4 : (a_4, b_5, c_{45}) \\ R_2 : (a_2, b_3, c_{23}) & & R_5 : (a_5, b_6, c_{56}) \\ R_3 : (a_3, b_1, c_{13}) & & R_6 : (a_6, b_4, c_{46}) \end{array}$$

This arrangement of E-points on a surface projective to F_9^1 will be described by saying that the surface is E(123,456): the numbers give the cycles of E-points, 123 representing $(a_1, b_2), (a_2, b_3), (a_3, b_1)$. Consider the 5 planes

$[a_1, b_i]$ through a_1; they meet the plane of f in the lines R_1R, R_1R_3, R_1R_6, R_1R_4, R_1R_5. The 4 lines R_1R_3, R_1R_4, R_1R_5, R_1R_6 form a harmonic range $(S_1R_4R_5R_6)$ on $R_4R_5R_6$; thus the planes

$$[a_1, b_3],\ [a_1, b_4],\ [a_1, b_5],\ [a_1, b_6]$$

are harmonic.

In contrast, consider the 5 points (a_1, b_i) on a_1. Their polar planes $[b_1, a_i]$ with respect to the polarity of D meet the plane of f in the lines R_3R_2, R_3R, R_3R_4, R_3R_5, R_3R_6; the 4 lines R_3R_2, R_3R_4, R_3R_5, R_3R_6 form a harmonic range $(S_1R_4R_5R_6)$ on $R_4R_5R_6$. Thus the 4 points

$$(a_1, b_2),\ (a_1, b_4),\ (a_1, b_5),\ (a_1, b_6)$$

form a harmonic set on a_1. Thus the 5 tritangent planes through any line of F_9^1 contain a harmonic set; the residual plane is the only one of the 5 which contains 3 concurrent lines of F_9^1. Dually, the 5 points in which any line of any double-six on F_9^1 is met by the other lines of the double-six form a harmonic set of 4 and a residual point according to their polar planes.

§ 22. Existence of Grace's extension over GF(9).

There are two necessary conditions for a proper Grace figure.

(i) the line b must have 6 skew transversals a_i such that any 4 of the a_i have one further transversal and such that any 5 of the a_i are linearly independent.

(ii) The 6 α_i, as constructed in Appendix II, must be skew.

When these two conditions are fulfilled, the proofs of Wren [65] and Kubota [33] show that the α_i have a transversal β; thus (i) and (ii) are also sufficient conditions.

Suppose (i) is satisfied - b and a_i $i = 1, \cdots, 6$ exist. Then, as in § 15, let them correspond to points b and a_i on the quadric K in Π_5

such that the a_i lie in the tangent prime at $b = (1, 0, 0, 0, 0, 0)$. The points a_i are projected from b to the points A_i on the ruled quadric $Q : xt + yz = 0$ lying in $w = u = 0$. No A_j lies in the tangent plane at A_i to Q and no 4 of the A_j are coplanar. The points on the line bA_i in Π_5 correspond in Π_3 to the lines in the plane $[b, a_i]$ through the point (b, a_i). Let the reguli of Q be ρ, ρ' consisting respectively of the lines $g_i, h_i, i = 1, \cdots, 10$. So the lines g_i of ρ represent the points of the line b in Π_3 and the lines h_i of ρ' represent the planes through b. As indicated in § 21, 6 points on a line or 6 lines of a regulus over $GF(9)$ occur most systematically as those residual to a harmonic tetrad.

Let g_7, g_8, g_9, g_{10} be a harmonic set with parameters $\sigma, \sigma^3, -\sigma, -\sigma^3$ and suppose any two of the pairs g_1, g_6 ; g_2, g_5; g_3, g_4 also form a harmonic set; let them have parameters $\infty, 0, 1, -1, \sigma^2, -\sigma^2$ in the order given. Further, let h_7, h_8, h_9, h_{10} have parameters $\sigma, \sigma^3, -\sigma, -\sigma^3$ in some order and let any two of the pairs h_1, h_4; h_2, h_6; h_3, h_5 form a harmonic set. These $6h_i$ have $2^3 . 3! = 48$ substitutions which preserve the pairs. Any collineation leaving the set of $6h_i$ fixed will also preserve the pairs. The residual harmonic set has 24 projectivities into itself, § 21 ; the automorphism ψ of § 20 also leaves the residual harmonic set invariant. Thus the 48 substitutions of the sextuplet can all be effected by collineations. Thus the parameters of h_i $i = 1, \cdots, 6$ can be selected as $\infty, 0, 1, -1, \sigma^2, -\sigma^2$ in the order given above. So

$$g_1 = h_1 = \infty \qquad g_6 = h_4 = 0$$
$$g_2 = h_2 = 1 \qquad g_5 = h_6 = -1$$
$$g_3 = h_3 = \sigma^2 \qquad g_4 = h_5 = -\sigma^2$$

Let A_i be (g_i, h_i) $i = 1, \cdots, 6$. The condition that A_i, A_j, A_k, A_ℓ are not coplanar is that the cross-ratios $\{g_i, g_j; g_k, g_\ell\}$, $\{h_i, h_j; h_k, h_\ell\}$ are unequal.

$\{g_1, g_2; g_3, g_i\}$	$\neq$	$\{h_1, h_2; h_3, h_i\}$	$i = 4, 5, 6$
$\{g_4, g_5; g_6, g_i\}$	$\neq$	$\{h_4, h_5; h_6, h_i\}$	$i = 1, 2, 3$
$\{g_1, g_6; g_2, g_5\}$	$\neq$	$\{h_1, h_6; h_2, h_5\}$	the cross-ratios
$\{g_2, g_5; g_3, g_4\}$	$\neq$	$\{h_2, h_5; h_3, h_4\}$	on the left
$\{g_3, g_4; g_1, g_6\}$	$\neq$	$\{h_3, h_4; h_1, h_6\}$	are -1 not
$\{g_1, g_4; g_2, g_6\}$	$\neq$	$\{h_1, h_4; h_2, h_6\}$	the cross-ratios
$\{g_2, g_6; g_3, g_5\}$	$\neq$	$\{h_2, h_6; h_3, h_5\}$	on the right
$\{g_3, g_5; g_1, g_4\}$	$\neq$	$\{h_3, h_5; h_1, h_4\}$	are -1

$\{g_1, g_2; g_4, g_5\} = -\sigma$ $\qquad\qquad$ $\{h_1, h_2; h_4, h_5\} = -\sigma^3$

$\{g_1, g_3; g_5, g_6\} = \sigma^3$ $\qquad\qquad$ $\{h_1, h_3; h_5, h_6\} = \sigma$

$\{g_2, g_3; g_4, g_6\} = \sigma^3$ $\qquad\qquad$ $\{h_2, h_3; h_4, h_6\} = \sigma$

So $\{g_i, g_j; g_k, g_\ell\} \neq \{h_i, h_j; h_k, h_\ell\}$ for all 15 sets $ijk\ell$. So no 4 of the $A_i = (g_i, h_i)$ are coplanar.

Thus, choose the points (b, a_i) in a sextuplet as the g_i and the planes $[b, a_i]$ in a sextuplet as the h_i; also let the first coordinates u_i of the a_i be such that any 5 a_j are linearly independent (this is only 6 linear conditions on the u_i). Then b and the 6 a_i satisfy (i). Also, of any 5 planes $[b, a_i]$, one set of 4 is harmonic; and similarly for the points (b, a_i).

From these a_i, the lines $b_{ij} = b_{ji}$, the double-sixes D_i and their cubic surfaces F_i, and the lines α_i are constructed as in Appendix II. The 12 lines a_i, α_i are chords of a twisted cubic t, to which b, b_{ij} are skew and c^i_{jk} is secant, § 17.

The surfaces F_i are projectively equivalent to F^0_9 or F^1_9. Suppose that at least one F_i is equivalent to the diagonal surface F_9^0. The group of F_9^0 is transitive on the 12 lines of the double-six D, as the equations given by Baker [4] p. 168 show; the other 15 lines c_{ij} of F_9^0 all contain 2 E-points. One of the 12 lines of D is

$$R : \sigma y + z + x = 0 \qquad S : \sigma z + y + t = 0$$

and the 5 tritangent planes through it are

$$R, \; S, \; R + S, \quad \sigma^3 R + S, \quad \sigma R - S$$

no 4 of which are harmonic. So b cannot be one of the lines of D containing no E-points, but must be one of the lines c_{ij} containing two E-points. Thus, on any of the surfaces F_i like F_9^0, b will contain 2 E-points; on any like F_9^1, only one E-point. As at least one F_i is like F_9^0, b contains at least 7 E-points on the F_i; so two of the lines c_{jk}^j pass through the same point (b, a_k) of b. This point is then L_k on t, which contradicts that b is skew to t, § 17. Therefore, all the surfaces F_i are like F_9^1: each line of F_i contains exactly one E-point on F_i.

Before considering whether (ii) is satisfied, it must be shown that the α_i are either all concurrent or all skew. From the 6 a_k, $b_{ij} = b_{ji}$ was constructed meeting the 4 a_k besides a_i, a_j; b_{ij} also meets α_i, α_j. Kubota [33] proved that the 4 reguli

$$(b_{12}, b_{13}, b_{14}), \quad (b_{21}, b_{23}, b_{24}), \quad (b_{31}, b_{32}, b_{34}), \quad (b_{41}, b_{42}, b_{43})$$

have a line in common: let this be $\beta_{56} = \beta_{65}$. The lines $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ lie in the 4 respective complementary reguli and thus meet β_{56}. The lines a_5, a_6 lie in all 4 complementary reguli and also meet β_{56}. Similarly there are 15 lines $\beta_{ij} = \beta_{ji}$ each meeting 6 of the 12 lines a_i, α_i.

b_{ij} meets $\alpha_i, \alpha_j, a_k, a_\ell, a_m, a_n$.

β_{ij} meets $a_i, a_j, \alpha_k, \alpha_\ell, \alpha_m, \alpha_n$.

Suppose α_1 meets α_2 at M. Then, as α_1, α_2 are both chords of the twisted cubic t, M lies on t. b_{12}, which is skew to t and meets both α_1 and α_2, cannot pass through M and therefore lies in the plane m of α_1, α_2. The 6 lines β_{ij} $i,j \neq 1, 2$ meet both α_1 and α_2. β_{56}, as it meets α_1 and α_2, lies in m or passes through M. As β_{56} is in the same regulus as b_{12}, it cannot lie in m; therefore β_{56} passes through M. Similarly the other β_{ij} $i,j \neq 1, 2$ pass through M.

However, α_3 meets β_{46}, β_{56} and therefore lies in the plane $[\beta_{46}, \beta_{56}]$ or passes through their intersection, which is M. Similarly a_6 lies in $[\beta_{46}, \beta_{56}]$ or passes through M. As a_6 does not meet α_1, it does not pass through M. Therefore a_6 lies in $[\beta_{46}, \beta_{56}]$. As α_3 cannot meet a_6, it must pass through M. Similarly α_4, α_5, α_6 pass through M and the 15 β_{ij} as well. So the α_i are all concurrent.

When the α_i are all concurrent, β_{jk} is also determined as the unique line through M, a_j, a_k, as was the case in § 19 over GF(8). If, initially, β_{56} meets β_{46}, then α_1, α_2, α_3 meet these two lines and so two of them intersect; the situation is then as above. Thus, if no two of the α_i meet, then β_{jk} does not meet $\beta_{j\ell}$ and vice versa.

To see that (ii) is satisfied, suppose the α_i are concurrent at the point L of t. The 26 points L_i, L_{ijk} of § 17 also lie on t. As α_i does not meet c^i_{jk} $j, k \neq i$, α_i does not contain L_{ijk}; so L_{ijk} cannot coincide with L. Suppose L_1 coincides with L. Let $H_i = (b, a_i)$; then any two pairs of H_1, H_6; H_2, H_5; H_3, H_4 are harmonic. As (H_2, H_3, H_4, H_5) is a harmonic set, by the property of § 21 of any double-six on F^1_9,

$$D_6 : \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & \alpha_6 \\ b_{16} & b_{26} & b_{36} & b_{46} & b_{56} & b \end{matrix}$$ gives that $(\alpha_6, b_{16}, c^6_{16})$ is an E-point. But

c^6_{16} meets t at L_1. So, as L and L_1 are coincident, α_6 and c^6_{16}

both pass through it; thus b_{16} also contains this point. But b_{16} is skew to t, so L_1 cannot coincide with L. Therefore the 26 points L_i, L_{ijk} occupy the 9 points of t residual to L in some arrangement similar to that of the 8 points in Appendix III over GF(8); no more than one of the 26 points could be isolated. As F_i is like F_9^1, α_i has exactly one E-point and exactly one point through which no other line of F_i passes. From § 17, L_1 is either on α_1 or it is the point of F_1 on none of the lines; also, none of the lines b_{1i}, c_{1i}^1 $i \neq 1$ pass through L_1. If L_1 is on α_1, the 10 lines b_{1i}, c_{1i}^1 occupy the 8 points of α_1 residual to L and L_1; so α_1 contains at least 2 E-points of F_1. This is impossible as F_1 is like F_9^1. Thus L_i does not lie on α_i. So L_i is the point of F_i on none of its lines. As α_i cannot contain L_{ijk}, none of the $6\alpha_j$ can pass through three coincident points such as L_1, L_{246}, L_{236} or L_{124}, L_{136}, L_{235}. If the 26 points L_i, L_{ijk} occupy only 8 points of t as in Appendix III, no α_i can pass through any of them. If they occupy the 9 points residual to L, only 3 points of t can have only two of the 26 points coincident and so at most $3\alpha_i$ could be chords. Even if a further α_i is the tangent at L, there are two α_i which have only one-point contact with t. So it is impossible that the α_i are concurrent.

Thus the 6 α_i are skew and their transversal β is a line of the 6 double-sixes

$$\Delta_i \qquad \begin{matrix} a_i & \alpha_j & \alpha_k & \alpha_\ell & \alpha_m & \alpha_n \\ \beta & \beta_{ij} & \beta_{ik} & \beta_{i\ell} & \beta_{im} & \beta_{in} \end{matrix}$$

of Appendix IV. So Grace's extension exists over GF(9).

An example of the configuration is given in Appendix VI.

§ 23. Description of Grace's extension over GF(9).

Let us now take any case of Grace's extension over GF(9). There are 44 lines —

$$1\, b,\ 6\, a_i,\ 15\, b_{jk},\ 15\, \beta_{\ell m},\ 6\, \alpha_n,\ 1\, \beta .$$

There are $2^6 = 64$ sets of 6 lines obtained by selecting one of each pair a_i, α_i $i = 1, \dots, 6$; 32 of these sets have a single transversal viz.

b	meets the	$6\ a_i$	1
β	meets the	$6\ \alpha_i$	1
b_{ij}	meets	$\alpha_i, \alpha_j, \alpha_k, a_\ell, \alpha_m, \alpha_n$	15
β_{ij}	meets	$a_i, a_j, \alpha_k, \alpha_\ell, \alpha_m, \alpha_n$	15 .

Also a_i meets the 16 lines $b, b_{jk}, \beta_{i\ell}$ $j, k, \ell \neq i$
α_i meets the 16 lines $\beta, \beta_{jk}\ b_{i\ell}$ $j, k, \ell \neq i$.

The incidence relations of these 44 lines are displayed in Appendix V in a table taken from Wren [65]. The other 32 sets of 6 lines are rows of the 32 double-sixes of Appendix IV, viz.

D_i	α_i	a_j	a_k	a_ℓ	a_m	a_n	6
	b	b_{ij}	b_{ik}	$b_{i\ell}$	b_{im}	b_{in}	
Δ_i	a_i	α_j	α_k	α_ℓ	α_m	α_n	6
	β	β_{ij}	β_{ik}	$\beta_{i\ell}$	β_{im}	β_{in}	
V_{ijk}	a_i	a_j	a_k	α_ℓ	α_m	α_n	20
	β_{jk}	β_{ki}	β_{ij}	b_{mn}	$b_{n\ell}$	$b_{\ell m}$	

Wakeford [63] proved the existence of a polarity W reciprocating b into β, a_i into α_i, b_{ij} into β_{ij}. Thus the construction can begin from any of the 32 lines $b, \beta, b_{ij}, \beta_{ij}$ and the completing line will be the polar of the initial line.

In the figure, there are 32 cubic surfaces — call them F-surfaces,

F_i	containing the double-six	D_i	$i = 1, \dots, 6$
Φ_i	containing the double-six	Δ_i	$i = 1, \dots, 6$
F_{ijk}	containing the double-six	V_{ijk}	$i, j, k = 1, \dots, 6$ $i < j < k$.

Each surface contains

	6	of the 12 A - lines	a_i, α_i
	6	of the 32 B - lines	b, β, b_{ij}, β_{ij}
and	15	of the 480 C - lines	c^i_{jk}, γ^i_{jk}, $\partial^{ijk}_{\ell m}$

The last are, on each F_i, 15 lines c^i_{jk} $j, k = 1, \cdot\cdot, 6$
on each Φ_i, 15 lines γ^i_{jk} $j, k = 1, \cdot\cdot, 6$
on each F_{ijk}, 15 lines $\partial^{ijk}_{\ell m}$ $\ell, m = 1, \cdot\cdot, 6$

where the lower suffixes refer to the indices of the two A-lines whence the C-line is constructed. The 32 F-surfaces all contain the twisted cubic t: the A-lines are chords of t; the B-lines are all skew to t; the C-lines are all secants of t, i.e. they have one-point contact with t. Each A-line lies on 16 and each B-line lies on 6 of the F-surfaces.

The intersections of the plane $[b, a_1]$ with the 5 planes $[\alpha_1, b_{1i}]$ are c^i_{1i}; these lines all meet t once and $[b, a_1]$ meets t in three "points", two of which are on a_1. So the c^i_{1i} are concurrent at a point L_1 of t. As α_i is a chord of t, L_1 is the other point in which $[\alpha_i, b_{1i}]$ meets t. Hence the intersection of any two of $[\alpha_i, b_{1i}]$ $i = 2, \cdot\cdot, 6$ also passes through L_1. Thus the planes

$[a_1, b]$, $[\alpha_2, b_{12}]$, $[\alpha_3, b_{13}]$, $[\alpha_4, b_{14}]$, $[\alpha_5, b_{15}]$, $[\alpha_6, b_{16}]$ meet in pairs in the following 15 lines through the point L_1

c^2_{12}, c^3_{13}, c^4_{14}, c^5_{15}, c^6_{16}, ∂^{456}_{23}, ∂^{356}_{24}, ∂^{346}_{25}, ∂^{345}_{26}, ..., ∂^{234}_{56} .

Similarly the planes

$[\alpha_1, \beta]$, $[a_2, \beta_{12}]$, $[a_3, \beta_{13}]$, $[a_4, \beta_{14}]$, $[a_5, \beta_{15}]$, $[a_6, \beta_{16}]$

meet in pairs in the following 15 lines through the point A_1

γ^2_{12}, γ^3_{13}, γ^4_{14}, γ^5_{15}, γ^6_{16}, ∂^{123}_{23}, ∂^{124}_{24}, ∂^{125}_{25}, ∂^{126}_{26}, ... ∂^{156}_{56}

and the planes

$[a_1, b_{23}]$, $[a_2, b_{13}]$, $[a_3, b_{12}]$, $[\alpha_4, \beta_{56}]$, $[\alpha_5, \beta_{46}]$, $[\alpha_6, \beta_{45}]$ meet in pairs in the following 15 lines through the point L_{123} $(=\Lambda_{456})$

$$c^1_{23},\ c^2_{13},\ c^3_{12},\ \gamma^4_{56},\ \gamma^5_{46},\ \gamma^6_{45},\ \partial^{156}_{14},\ \partial^{146}_{15},\ \partial^{145}_{16},\ \partial^{256}_{24},\ \ldots\ldots,\ \partial^{345}_{36}\ .$$

As each A-line meets 16 B-lines, there are $12 \times 16 = 192$ planes $[A, B]$, 6 of which contain one of the 32 L-points L_i, Λ_i, L_{ijk}. As each of the 192 planes meets the 5 others through the same L-point in a C-line, the $192 \times 5/2 = 480$ C-lines are all accounted for and pass $480/32 = 15$ through each L-point.

From the double-six V_{123}, a_1 cannot meet ∂^{123}_{23} which is the intersection of $[a_2, \beta_{12}]$ and $[a_3, \beta_{13}]$; ∂^{123}_{23} contains Λ_1 so a_1 does not contain Λ_1. Similarly

a_1	cannot contain	L_{ijk}	$i, j, k \neq 1$
		L_m	$m \neq 1$
		Λ_1	
α_1	cannot contain	L_{1ij}	
		Λ_k	$k \neq 1$
		L_1	

From § 17, L_1 can only lie on α_1 of the lines of F_1. From above, L_1 does not lie on α_1; so F_1 is projectively equivalent to F^1_9 and L_1 is the point on none of its lines. Similarly Λ_1 is the point of Φ_1 on none of its lines and L_{123} that of F_{456}. All the F-surfaces are projectively equivalent to F^1_9. Hence on each surface each line has exactly one E-point. Each B-line meets 6 A-lines, 5 on each of the 6 F-surfaces on which the B-line lies; and on each of these 6 surfaces, the B-line contains a different E-point, e.g. if (b, a_1) were an E-point on both F_2 and F_3, then b would contain the point $(b, a_1) = (c^2_{12}, c^3_{13}) = L_1$ of t.

With regard to a_1, the L-points are of two types: either an L-point lies in one of the 16 planes through a_1 and the B-lines which meet it or it does not. The above shows that a_1 does not contain an L-point of the second type. Suppose it contains an L-point of the first type, e.g. L_1; then the 5 lines c_1^i, $i \neq 1$ are concurrent at a point of a_1. But (b, a_1) is an E-point for one of the surfaces F_i; so b also contains this point L_1, which is impossible as b is skew to t. So a_1 does not contain L_1. So no A-line contains any L-point.

The 6 A-lines through a B-line form a configuration like that of § 22. For example,

on	F_1,	b meets		a_2	a_3	a_4	a_5	a_6
on	F_2,	b meets	a_1		a_3	a_4	a_5	a_6
on	F_3,	b meets	a_1	a_2		a_4	a_5	a_6
on	F_4,	b meets	a_1	a_2	a_3		a_5	a_6
on	F_5,	b meets	a_1	a_2	a_3	a_4		a_6
on	F_6,	b meets	a_1	a_2	a_3	a_4	a_5	

Let $h_i = [b, a_i]$ and $H_i = (b, a_i)$. Suppose that H_4 is an E-point on F_1. Then, by § 21, (h_2, h_3, h_5, h_6) is a harmonic set; thus, on F_4, H_1 is an E-point. On F_1, α_1 must contain an E-point; as c_{14}^1 cannot contain two E-points, it cannot be (α_1, b_{14}): let it be (α_1, b_{16}). Then (H_2, H_3, H_4, H_5) is a harmonic set; so, on F_6, (α_6, b_{16}) is an E-point. Now, let H_6 be an E-point on F_2; then (h_1, h_3, h_4, h_5) is a harmonic set: so H_2 is an E-point on F_6. Therefore, from F_3 and F_5, (h_1, h_2, h_4, h_6) is a harmonic set and H_5 and H_3 are E-points on F_3 and F_5 respectively. On F_2, as H_6 is an E-point, (α_2, b_{26}) is not; let the E-point on α_2 be (α_2, b_{25}). Then (H_1, H_3, H_4, H_6) is harmonic and (α_5, b_{25}) is an E-point on F_5. So, from F_3 and F_4, (H_1, H_2, H_5, H_6) is harmonic and (α_3, b_{34}), (α_4, b_{34}) are E-points on F_3 and F_4 respectively. Thus the h_i form a sextuplet h_1, h_4; h_2, h_6; h_3, h_5 and the H_i form a sextuplet H_1, H_6; H_2, H_5; H_3, H_4. So, for any Grace extension over $GF(9)$, the 6 points, as well as the 6 planes, in which b meets the lines a_i

are the sextuplet residual to a harmonic tetrad. And so it is for all 32 B-lines and the 6 A-lines meeting them.

The partitioning of the sextuplets for all the B-lines is known when all the E-points on the 6 F_i are known. So far, in the notation of § 21,

F_1	is	$E(164, i_1 j_1 k_1)$	where	$i_1 j_1 k_1$	is	253	or 235
F_2	is	$E(256, i_2 j_2 k_2)$		$i_2 j_2 k_2$	is	134	or 143
F_3	is	$E(345, i_3 j_3 k_3)$		$i_3 j_3 k_3$	is	162	or 126
F_4	is	$E(143, i_4 j_4 k_4)$		$i_4 j_4 k_4$	is	265	or 256
F_5	is	$E(235, i_5 j_5 k_5)$		$i_5 j_5 k_4$	is	146	or 164
F_6	is	$E(126, i_6 j_6 k_6)$		$i_6 j_k k_6$	is	354	or 345

Suppose $i_1 j_1 k_1 = 253$ i.e. (a_2, b_{15}), (a_5, b_{13}), (a_3, b_{12}) are E-points; then, on F_2, (a_3, b_{21}) is not an E-point so $i_2 j_2 k_2$ is not 143 but 134. This means that (a_1, b_{23}) is an E-point on F_2, so it is not an E-point on F_3. Therefore $i_3 j_3 k_3$ is not 126 but 162. Similarly, let $i_4 j_4 k_4 = 265$; then $i_5 j_5 k_5 = 146$ and $i_6 j_6 k_6 = 354$. The $i\,j\,k$ are therefore given by the first column above. These figures are the correct ones for the example of Appendix VI.

Consider b_{12}; (a_3, b_{12}) is an E-point on F_1 and (a_4, b_{12}) is an E-point on F_2. So

$$[\alpha_1, b_{12}],\ [a_4, b_{12}],\ [a_5, b_{12}],\ [a_6, b_{12}] \text{ are harmonic}$$

and

$$[\alpha_2, b_{12}],\ [a_3, b_{12}],\ [a_5, b_{12}],\ [a_6, b_{12}] \text{ are harmonic.}$$

Therefore the pairs of the sextuplet are

$$[\alpha_1, b_{12}],\ [a_4, b_{12}];\ [\alpha_2, b_{12}],\ [a_3, b_{12}];\ [a_5, b_{12}],\ [a_6, b_{12}].$$

As (a_2, b_{15}) is an E-point on F_1 and (a_1, b_{23}) on F_2, so

$$(\alpha_1, b_{12}),\ (a_3, b_{12}),\ (a_4, b_{12}),\ (a_6, b_{12}) \text{ are harmonic}$$

and

$$(\alpha_2, b_{12}),\ (a_4, b_{12}),\ (a_5, b_{12}),\ (a_6, b_{12}) \text{ are harmonic.}$$

So the pairs of the sextuplet are

$(\alpha_1, b_{12}), (a_3, b_{12}); (\alpha_2, b_{12}), (a_5, b_{12}); (a_4, b_{12}), (a_6, b_{12})$

Similarly the sextuplets of points and planes can be determined for the other b_{ij}. With respect to W, the polar of (b, a_i) is $[\beta, \alpha_i]$, of (b_{ij}, α_i) is $[\beta_{ij}, a_i]$, of (b_{ij}, a_k) is $[\beta_{ij}, \alpha_k]$. So the sextuplets of points and planes for β, β_{ij} are given by the polars of the sextuplets of planes and points for b, b_{ij}. The partitioning of the sextuplets of points is given in Appendix V.

These sextuplets determine the E-points on all the F-surfaces. Define the lines $\bar{c}^i_{jk}, \bar{\gamma}^i_{jk}, \bar{\partial}^{\ell mn}_{jk}$ as the polars with respect to the double-sixes $D_i, \Delta_i, \Delta_{\ell mn}$ of the lines $c^i_{jk}, \gamma^i_{jk}, \partial^{\ell mn}_{jk}$, i.e. c^1_{23} is the meet of $[a_2, b_{13}], [a_3, b_{12}]$ and so $\bar{c}^1_{23}$ is the join of $(a_2, b_{13}), (a_3, b_{12})$.
For W, the polars of $c^i_{jk}, \gamma^i_{jk}, \partial^{\ell mn}_{jk}$ are $\bar{\gamma}^i_{jk}, \bar{c}^i_{jk}, \partial^{xyz}_{jk}$ $x, y, z \neq \ell, m, n$.
On F_1, (a_6, b_{14}) is an E-point, i.e. a_6, b_{14}, c^1_{46} are concurrent. From W, $\alpha_6, \beta_{14}, \bar{\gamma}^1_{46}$ are coplanar and, from the polarity of Δ_1, $\beta_{16}, \alpha_4, \gamma^1_{46}$ are concurrent. So the E-points on Φ_i, F_{ijk} are in the reverse cycles to those on $F_i, F_{\ell mn}$; e.g. F_1 is E(164, 253) and Φ_1 is E(146, 235).

The 32 L-points L_i, Λ_i, L_{ijk} of t can be at most 10 distinct points. From the G-lines which pass through the L-points

L_1	may coincide	with	Λ_i or L_{jkm}		$j, k, m \neq 1$
	but <u>not</u>	with	L_n or L_{1jk}		$n \neq 1$
Λ_1	may coincide	with	L_i or L_{1jk}		
	but <u>not</u>	with	Λ_n or L_{jkm}		$j, k, m, n \neq 1$

L_{123} may coincide	with	$L_4, L_5, L_6, \Lambda_1, \Lambda_2, \Lambda_3, L_{1ij}, L_{2ij}, L_{3ij}, L_{456}$	$i, j \neq 1,2,3$
but <u>not</u>	with	$L_1, L_2, L_3, \Lambda_4, \Lambda_5, \Lambda_6, L_{12k}, L_{13k}, L_{23k}$,	$k \neq 1,2,3$

At most 4 of the L-points can coincide,

e.g. $L_1, \Lambda_2, L_{245}, L_{236}$ or $L_{124}, L_{136}, L_{235}, L_{456}$.

If L_1, Λ_1 or L_{123}, L_{456} coincide, the pair has no further coincident L-point.

The L-points must occupy at least 8 points of t. It will be shown that in fact they occupy all 10 points of t.

From the double-sixes D_i, a_1 is skew to a_j, α_j $j \neq 1$. Suppose a_1 meets α_1; as they are both chords of t, this point (a_1, α_1) is on t and, from before, is not an L-point. Either a_1 and α_1 are both "real" chords of t or one is a "real" chord and the other a tangent. Let α_1 be a chord; then, as the point M of α_1 on t other than (a_1, α_1) is not an L-point, M cannot lie on a C-line. It also cannot lie on a B-line; but M is on Φ_1 and so must lie on an A-line. However, it is not (a_1, α_1) and α_1 does not meet α_i $i \neq 1$. So M cannot lie on Φ_1. So a_1 cannot meet α_1.

Suppose an A-line meets t in a point whose coordinates lie in GF(9); let the line be α_1. As it has been shown that a_1 does not meet α_1 and that α_1 does not contain any L-point, this point cannot lie on Φ_1. So all A-lines are "conjugate" chords of t, i.e. they meet t in two points whose coordinates belong, not to GF(9), but to GF(9^2).

The 32 L-points must therefore occupy all 10 points of t; for, if there is a spare point, it cannot lie on an A-line, B-line or C-line; and the only remaining point on any F-surface is an L-point.

At this stage, the E-points on the F-surfaces and hence the harmonic sets among the 6 planes and the 6 points of the 6 A-lines through a B-line are known. It remains to show how each A-line is met in its 10 points by 16 B-lines and how the 32 L-points are distributed on t. These results are connected.

For each of the 16 F-surfaces containing a particular A-line, there is a harmonic set of planes and one of points associated with the A-line. However, these 16 harmonic sets of planes and of points do not clarify matters. Consequently, let us first consider which B-lines may intersect. From the double-sixes,

b does not meet b_{ij}, but may meet β_{ij}, β

b_{ij} does not meet b, b_{ik}, β_{mn} $\quad k, m, n \neq i, j$
but may meet β, $b_{k\ell}$, β_{im} $\quad k, \ell \neq i, j$

β_{ij} does not meet β, β_{ik}, b_{mn} $\quad k, m, n \neq i, j$
but may meet b, $\beta_{k\ell}$, b_{im} $\quad k, \ell \neq i, j$

β does not meet β_{ij}, but may meet b_{ij}, b . Suppose $L_1 = \Lambda_2$; then the planes $[a_1, b]$ and $[a_1, \beta_{12}]$ have this point and a_1 in common. As the point cannot lie on a_1, these two planes are the same; hence b meets β_{12}. The line a_2 also meets b and β_{12} but not a_1, so a_2 passes through the intersection of b and β_{12}. A similar result holds for β and b_{12} either by W or argued from the planes $[\alpha_2, b_{12}]$, $[\alpha_2, \beta]$.
In all, if $L_1 = \Lambda_2$ then $[a_1, b] = [a_1, \beta_{12}]$, $[\alpha_2, \beta] = [\alpha_2, b_{12}]$
$(\alpha_1, \beta) = (\alpha_1, b_{12})$, $(a_2, b) = (a_2, \beta_{12})$.

Conversely, if one of the 4 equalities holds, the others do also and $L_1 = \Lambda_2$.
Similarly

$L_{134} = L_{156}$ <—> $[a_1, b_{34}] = [a_1, b_{56}]$, $[\alpha_2, \beta_{34}] = [\alpha_2, \beta_{56}]$
$(\alpha_1, \beta_{34}) = (\alpha_1, \beta_{56})$, $(a_2, b_{34}) = (a_2, b_{56})$

$L_1 = L_{234}$ <—> $[\alpha_5, b_{15}] = [\alpha_5, \beta_{16}]$, $[\alpha_6, b_{16}] = [\alpha_6, \beta_{15}]$
$(a_5, \beta_{15}) = (a_5, b_{16})$, $(a_6, \beta_{16}) = (a_6, b_{15})$

$\Lambda_1 = L_{123}$ <—> $[a_2, \beta_{12}] = [a_2, b_{13}]$, $[a_3, \beta_{13}] = [a_3, b_{12}]$
$(\alpha_2, b_{12}) = (\alpha_2, \beta_{13})$, $(\alpha_3, b_{13}) = (\alpha_3, \beta_{12})$

These are all the possible coincidences of L-points which are different within the notation, except for a pair such as L_1 and Λ_1; these do not imply the coincidence of two planes [A, B] and so do not imply the intersection of two B-lines. Any two B-lines which intersect, if they are not polars in W, are met by two skew A-lines; so the intersection of the B-lines must lie on an A-line. That two non-polar B-lines may intersect is wrongly denied by Wren [65] p. 164.

The distribution of the 32 L-points on t can be determined. L_1 is the meet of the 15 lines c_{1i}^{i}, $\partial_{jk}^{\ell mn}$ $i, j, k, \ell, m, n \neq 1$. Therefore, on the 15 surfaces F_i, $F_{\ell mn}$, L_1 lies on a C-line; on F_1 it is the point on no line of F_1. On any other F-surface, L_1 is either on a C-line or is the point on no line of the surface. So L_1 must coincide with some other L-point. In the case where it coincides with A_1, there are 30 C-lines all from different F-surfaces through this point. For the other two F-surfaces F_1 and Φ_1, it is the point on none of their lines. Thus when L_1 and A_1 coincide, the position of this point on all 32 F-surfaces is known.

If L_1 coincides with some other L-point, say A_2, then the 30 C-lines through L_1 and A_2 do not include one from the F-surface F_{134}, for example. So this point $L_1 = A_2$ of t, which must lie on a C-line or be the point on no line of F_{134}, coincides with a further L-point. So now take 3 coincident L-points — L_{123}, L_{145}, L_{256}. The C-lines through them are

L_{123}: $c_{23}^{1}, c_{13}^{2}, c_{12}^{3}, \gamma_{56}^{4}, \gamma_{46}^{5}, \gamma_{45}^{6}, \partial_{14}^{156}, \partial_{15}^{146}, \partial_{16}^{145}, \partial_{24}^{256}, \partial_{25}^{246}, \partial_{26}^{245}, \partial_{34}^{356}, \partial_{35}^{346}, \partial_{36}^{345}$

L_{145}: $c_{45}^{1}, c_{15}^{4}, c_{14}^{5}, \gamma_{36}^{2}, \gamma_{26}^{3}, \gamma_{23}^{6}, \partial_{12}^{136}, \partial_{13}^{126}, \partial_{16}^{123}, \partial_{24}^{346}, \partial_{34}^{246}, \partial_{46}^{234}, \partial_{25}^{356}, \partial_{35}^{256}, \partial_{56}^{235}$

L_{256}: $c_{56}^{2}, c_{26}^{5}, c_{25}^{6}, \gamma_{34}^{1}, \gamma_{14}^{3}, \gamma_{13}^{4}, \partial_{12}^{234}, \partial_{23}^{124}, \partial_{24}^{123}, \partial_{15}^{345}, \partial_{35}^{145}, \partial_{45}^{135}, \partial_{16}^{346}, \partial_{36}^{146}, \partial_{46}^{136}$

These 45 C-lines include one from all the F-surfaces except F_{125}, F_{134}, F_{236}, F_{456}; the last three have L_{256}, L_{145}, L_{123} as the respective points on none of their lines. The only L-point which can coincide with L_{123}, L_{145}, L_{256} is L_{346}, which does indeed lie on F_{125}, being the point on none of its lines.

The 32 L-points therefore coincide in twos or in fours so as to exclude any further coincidences. The solution of $4a + 2b = 32$ and $a + b = 10$ is $a = 6$ and $b = 4$. So the L-points lie 4 at each of 6 points and 2 at each of 4 points of t. In the example of Appendix VI, their distribution is

T_1 : $L_1, A_2, L_{245}, L_{236}$
T_2 : $L_2, A_4, L_{146}, L_{345}$
T_3 : $L_4, A_5, L_{125}, L_{356}$
T_4 : $L_5, A_1, L_{126}, L_{134}$
T_5 : $L_{123}, L_{145}, L_{256}, L_{346}$
T_6 : $L_{124}, L_{136}, L_{235}, L_{456}$
T_7 : L_3, A_3
T_8 : L_6, A_6
T_9 : L_{135}, L_{246}
T_{10}: L_{156}, L_{234}

Now consider the points in which each A-line is met by 16 B-lines. No 3 B-lines are concurrent, so the 16 B-lines meet the A-line in 8, 9 or 10 distinct points: if 8, there are 8 coincidences among the 16 points of intersection; if 10, 6 coincidences. So there are between 72 and 96 coincidences in all. Now each coincidence of two L-points, apart from a pair like L_3 and A_3, gives two coincidences of points where an A-line is met by two B-lines. So the 6 points of t in which 4 L-points coincide give $2 \times 6 \times {}^4C_2 = 72$ such coincidences on the A-lines. Any concurrency of 2 B-lines and an A-line is given by the coincidence of 2 L-points. Thus there are exactly 6 coincidences on each A-line, i.e. the 16 B-lines which meet an A-line occur as 2 through 6 points and one through 4 points of the A-line. For the example, these coincidences are given by the table and the diagram of Appendix VII. The 16 planes through an A-line occur similarly, the planes through a_i being the polars in W of the points on α_i.

Four points on t have a cross-ratio given by that of the planes through the 4 points and a chord of t. If t is

$$x_0 : x_1 : x_2 : x_3 = \lambda^3 : \lambda^2 : \lambda : 1$$

then the plane through $\lambda = r, s, \theta$ is

$$x_0 - (r + s + \theta)x_1 + (rs + r\theta + s\theta)x_2 - rs\theta x_3 = 0$$

$$x_0 - \overline{r + s}\, x_1 + rsx_2 - \theta(x_1 - \overline{r + s}\, x_2 + rsx_3) = 0$$

So the cross-ratio of the 4 planes through r, s and $\theta_1, \theta_2, \theta_3, \theta_4$ is $\{\theta_1, \theta_2; \theta_3, \theta_4\}$ independently of r and s.

As (a_1, b_{23}) is an E-point on F_2, the planes $[a_1, b]$, $[a_1, b_{24}]$, $[a_1, b_{25}]$, $[a_1, b_{26}]$ are harmonic; so the points L_1, L_{124}, L_{125}, L_{126}, which are T_1, T_6, T_3, T_4, are harmonic. Similarly (a_1, b) is an E-point on F_4, so the planes $[a_1, b_{24}]$, $[a_1, b_{34}]$, $[a_1, b_{45}]$, $[a_1, b_{46}]$ are harmonic, as are L_{124}, L_{134}, L_{145}, L_{146}, which are T_6, T_4, T_5, T_2. Thus the 6 points T_i $i = 1, \ldots, 6$ form a sextuplet residual to the harmonic tetrad (T_7, T_8, T_9, T_{10}) in the pairs T_1, T_3; T_2, T_5; T_4, T_6. Thus on any A-line, the 16 points in which it is met by the B-lines being distributed as 6 sets of 2 and 4 of 1, the 4 are harmonic and the 6 are a sextuplet divisible into 3 pairs any two of which are harmonic; similarly for the 16 planes through the A-line.

To summarise, $GF(9)$ is the smallest field over which Grace's extension exists. Its peculiar properties depend on the partitions

$$10 = 6 + 4, \qquad 16 = 2 \times 6 + 4, \qquad 32 = 4 \times 6 + 2 \times 4,$$

where the final digit 4 in each equation indicates a harmonic set, and on all 32 cubic surfaces involved being projectively equivalent to F_9^1.

CHAPTER VIII. Conclusion.

§ 24. Summary.

The aim on starting the thesis was to investigate the existence of the double-six and Grace's extension over finite fields.

Over GF(4) a new type of double-six was found, self-polar with respect to a linear complex and which only exists over $GF(4^m)$. No double-six exists over GF(5). The projectively distinct cubic surfaces with 27 lines over GF(q) were all investigated for $q \leq 9$ and, on discovery, were denoted by F_q^n where q is the order of the field and n the number of points on the surface on no line of it. When $n = 0$, the symbol was omitted if there was no ambiguity. The surfaces are all in the following table where e is the number of E-points and g the order of the projective group of the surfaces.

F_q^n	e	g		
F_4	45	25,920	=	$2^6.3^4.5$
F_7	18	648	=	$2^3.3^4$
F_8	13	192	=	$2^6.3$
F_9^0	10	120	=	$2^3.3.5$
F_9^1	9	216	=	$2^3.3^3$

As there is no double-six over GF(5), there is no Grace's extension. Over GF(7) no line has 6 transversals such that each set of 4 has only one further transversal. Over GF(8), a line and 6 transversals can be found so that each set of four has a second transversal, but the 6 completing lines of the 6 double-sixes obtained from the original line and sets of 5 of its transversals are concurrent; this figure is not limited to GF(8).

It is over GF(9) that Grace's extension is first found to exist and the points in which the 6 transversals meet the original line are necessarily a set of 6 residual to a harmonic tetrad; the 32 cubic surfaces involved are all projectively equivalent to F_9^1.

§ 25. Epilogue.

Several problems arise from the thesis of which the most immediate follow.

(i) What is the smallest field over which a general cubic surface with 27 lines exists — "general" in the sense that it has no E-points?

(ii) What is the smallest field over which a general Grace's extension exists — "general" in the sense that none of the particular coincidences of Appendix VII occur?

(iii) What are the group of substitutions of Grace's extension and the groups of projectivities of the figures over GF(8) and GF(9)?

(iv) Does Grace's extension exist over $GF(4^m)$ such that all 32 F-surfaces have 45 E-points?

(v) Is there an extension to Grace's extension over any field at all?

I can give a definite answer to only one of these questions, viz. (iv), but I will make some remarks on all.

(i) The smallest possible field for this is GF(11). A classification of 6-arcs in PG(2, 11) would decide the question.

(ii) The smallest possible field is GF(31), since the 32 L-points of the twisted cubic t are required to be distinct.

(iii) I would expect the order of the group of substitutions to be 32 × 51, 840, the order of the group of projectivities of the figure over GF(8) to be 6 × 192 and that of GF(9) to be 32 × 216.

(iv) For $GF(4^m)$ it was by no means established that the 6 transversals of a line are chords of a twisted cubic t. If t exists, then Grace's extension is not obtainable from b and a_i $i = 1, \ldots, 6$ so that all 32 F-surfaces have 45 E-points, as b would contain the 6 L-points L_i $i = 1, \cdots, 6$ of t.

If t does not exist, then the 15 C-lines which were given in § 23 as passing through L_1 are not only the intersections of pairs of the 6 planes $[a_1, b]$, $[\alpha_i, b_{1i}]$ $i = 2, \cdots, 6$ but also the joins of pairs of the 6 points (a_1, b), (α_i, b_{1i}). So the 5 lines c_{1i}^{i} all lie in $[a_1, b]$ and contain these 6 points, which are therefore coplanar. Hence the 6 planes $[a_1, b]$, $[\alpha_i, b_{1i}]$ are all the same: this gives unpermissible intersections of lines. Thus it is not possible that the 32 F-surfaces of Grace's extension all have 45 E-points.

(v) To the best of my knowledge, there is no mention in print of any extension to Grace's figure, which itself has received sparse attention. Any opinion I have heard has been contrary to a further extension, but the start of such a figure will be briefly considered.

Let b meet $a_1, a_2, a_3, a_4, a_5, a_6, a_7$; each set of $4a_i$ has a further transversal $b_{jk\ell}$. Thus from the 21 double-sixes like

$$\begin{matrix} \alpha_{12} & a_3 & a_4 & a_5 & a_6 & a_7 \\ b & b_{123} & b_{124} & b_{125} & b_{126} & b_{127} \end{matrix}$$

there are 21 lines α_{ij}. There are also 7 lines β_i such that β_1 is the transversal of $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}$. Do the β_i $i = 1, \cdots, 7$ have a transversal β?

Baker [4] p. 195 proves the three theorems that given a line and 5, 6, 7 transversals the locus of a point such that the planes joining it to the 6, 7, 8 lines touch a quadric cone is a cubic surface, a twisted cubic, a point. Thus there are 7 twisted cubics t_i with a common point T such that t_1 has chords $a_2, a_3, a_4, a_5, a_6, a_7, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}, \alpha_{17}$. Two cubic curves with a common point have 6 common chords, Cremona [16]; so the 6 chords $\alpha_{12}, a_3, a_4, a_5, a_6, a_7$ of t_1 and t_2 are all their common chords. A quartic surface is determined by 34 constants. The number of conditions for a line and 7 transversals to lie on a quartic surface is $5 + 7 \times 4 = 33$.

Let Q be the surface containing b, a_1, a_2, a_3, a_4, a_5, a_6, a_7 and T. Then, as there are 13 common points required to put a twisted cubic on Q, the 7 cubics t_i also lie on Q.

There is also a quartic surface connected with Grace's extension. The chords of a twisted cubic belonging to a special linear complex lie on a ruled quartic surface of the type IIB discussed by Edge [19] §§ 60, 80. It is possible that the t_i coincide so that the 7 lines a_i are chords of one cubic t and Q is then one of these ruled surfaces. (From §§ 17, 19 such a figure can be constructed over GF(8), in which case the α_{ij} are all real chords of t.)

There are $7 \times {}^6C_2 = 105$ lines $\beta^i_{jk} = \beta^i_{kj}$ involved in the construction: the four reguli

$(b_{345}, b_{346}, b_{347})$, $(b_{354}, b_{356}, b_{357})$, $(b_{364}, b_{365}, b_{367})$, $(b_{374}, b_{375}, b_{376})$ all have the common line β^3_{12} and the 5 lines $\beta^3_{12}, \beta^4_{12}, \beta^5_{12}, \beta^6_{12}, \beta^7_{12}$ all belong to a regulus, Kubota [33]. There are $7 \times {}^6C_3 = 140$ double-sixes like

$$\begin{matrix} a_1 & a_2 & a_3 & \alpha_{47} & \alpha_{57} & \alpha_{67} \\ \beta^7_{23} & \beta^7_{13} & \beta^7_{12} & b_{567} & b_{467} & b_{457} \end{matrix}$$

and 42 like

$$\begin{matrix} a_2 & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} & \alpha_{17} \\ \beta_1 & \beta^1_{23} & \beta^1_{24} & \beta^1_{25} & \beta^1_{26} & \beta^1_{27} \end{matrix}$$

If the transversal β of the β_i does exist, there seems no simple way of obtaining it either as the line belonging to certain double-sixes or from the quartic surface Q. The clear symmetry of Grace's extension of the double-six is no longer present.

There is a further problem which suggests itself.

(vi) As there are various relations between the 36 Schur quadrics of a cubic surface, described by Room [42] and which are almost the same as those of

§ 8, what are the relations between the polarities of the 32 double-sixes D_i, Δ_i, ∇_{ijk} and how are these polarities related to W?

APPENDIX I. The condition that 5 skew lines have a transversal.

Take 5 skew lines a, b, c, d, e met by the line x. The line a has coordinates a_i $i = 0, \cdots, 5$ such that $a_0a_5 + a_1a_4 + a_2a_3 = 0$; similarly for the other lines. Any two lines, e.g. a, b, have a mutual invariant $\tilde{\omega}(a, b) = a_0b_5 + a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1 + a_5b_0$. Let a, b, c, d, e correspond to 1, 2, 3, 4, 5 respectively so that $\tilde{\omega}(a, b)$ can be written $\tilde{\omega}_{12}$ and so on.

Take the symmetric matrix $W = (\tilde{\omega}_{ij})$ $i, j = 1, \cdots, 5$ and let the cofactor of $\tilde{\omega}_{ij}$ in $|W|$ be π_{ij}. In classical geometry the condition that the lines a, b, c, d, e have a common transversal is $|W| = 0$, e. g. Todd [61] p. 145 ex. 41. However, over $GF(2^n)$, since W is symmetric with diagonal elements zero, $|W| = 0$. Yet sets of 5 lines with no common transversal do exist even over GF(2) so there must be a more fundamental condition valid over any field.

Since the diagonal elements of W are zero, $|W|$ has, not 5!, but $5 ¡ = 5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$ terms. These form 22 pairs of equal terms. Thus the condition required ought to be found as the sum of these 22 terms.

Since a, b, c, d, e are all met by x,

$$a_0x_5 + a_1x_4 + a_2x_3 + a_3x_2 + a_4x_1 + a_5x_0 = 0$$

$$b_0x_5 + b_1x_4 + b_2x_3 + b_3x_2 + b_4x_1 + b_5x_0 = 0$$

$$c_0x_5 + c_1x_4 + c_2x_3 + c_3x_2 + c_4x_1 + c_5x_0 = 0$$

$$d_0x_5 + d_1x_4 + d_2x_3 + d_3x_2 + d_4x_1 + d_5x_0 = 0$$

$$e_0x_5 + e_1x_4 + e_2x_3 + e_3x_2 + e_4x_1 + e_5x_0 = 0$$

$$x_0x_5 + x_1x_4 + x_2x_3 = 0 .$$

Solving for the x_i from the linear equation and substituting in the quadratic

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ d_0 & d_1 & d_2 & d_3 & d_4 \\ e_0 & e_1 & e_2 & e_3 & e_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} + \quad + \quad = 0$$

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ d_0 & d_1 & d_2 & d_3 & d_4 & d_5 \\ e_0 & e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} \begin{vmatrix} 0 & a_5 & b_5 & c_5 & d_5 & e_5 \\ 0 & a_4 & b_4 & c_4 & d_4 & e_4 \\ 0 & a_3 & b_3 & c_3 & d_3 & e_3 \\ 0 & a_2 & b_2 & c_2 & d_2 & e_2 \\ 0 & a_1 & b_1 & c_1 & d_1 & e_1 \\ 1 & a_0 & b_0 & c_0 & d_0 & e_0 \end{vmatrix} + \quad + \quad = 0$$

$$\begin{vmatrix} 1 & a_0 & b_0 & c_0 & d_0 & e_0 \\ a_5 & 0 & \tilde{\omega}_{12} & \tilde{\omega}_{13} & \tilde{\omega}_{14} & \tilde{\omega}_{15} \\ b_5 & \tilde{\omega}_{21} & 0 & \tilde{\omega}_{23} & \tilde{\omega}_{24} & \tilde{\omega}_{25} \\ c_5 & \tilde{\omega}_{31} & \tilde{\omega}_{32} & 0 & \tilde{\omega}_{34} & \tilde{\omega}_{35} \\ d_5 & \tilde{\omega}_{41} & \tilde{\omega}_{42} & \tilde{\omega}_{43} & 0 & \tilde{\omega}_{45} \\ e_5 & \tilde{\omega}_{51} & \tilde{\omega}_{52} & \tilde{\omega}_{53} & \tilde{\omega}_{54} & 0 \end{vmatrix} + \quad + \quad = 0$$

Expanding the determinants by the first row and then, except for $|W|$, by the first column

$$w \equiv 3|W| - \sum_{i<j}{}_{i,j=1}^{i,j=5} \tilde{\omega}_{ij}\, \pi_{ij} = 0;$$

for π_{ij} occurs twice in each of the three expansions, the coefficient of π_{12} in the displayed determinant is $-a_0b_5 - a_5b_0$, the sum of two of the six products of $-\tilde{\omega}_{12}$.

Over fields not of characteristic 2, $2\,\Sigma\, \tilde{\omega}_{ij}\, \pi_{ij}$ is $|W|$ expanded 5 times so that $\Sigma\, \tilde{\omega}_{ij}\, \pi_{ij} = 5|W|/2$ and the condition becomes $|W|/2 = 0$ as expected.

Now substitute $\tilde{\omega}_{ij} = A_iA_j$ except for $ij = 45$ and put $\tilde{\omega}_{45} = \lambda A_4A_5$; then π_{ij} is the product of $A_1^2 A_2^2 A_3^2 A_4^2 A_5^2 / A_iA_j$ and the corresponding cofactor in $\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & \lambda \\ 1 & 1 & 1 & \lambda & 0 \end{vmatrix} = 2\lambda(3 - \lambda)$

so that $\tilde{\omega}_{ij}\pi_{ij}$ can be replaced in $w = 0$ by this cofactor save when ij is 45; $\tilde{\omega}_{45}\pi_{45}$ will be replaced by $-\lambda \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & \lambda \end{vmatrix} = 3\lambda - 2\lambda^2$.

The replacements for $\tilde{\omega}_{12}\pi_{12}$, $\tilde{\omega}_{13}\pi_{13}$, $\tilde{\omega}_{23}\pi_{23}$ are all

$$- \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & \lambda \\ 1 & 1 & \lambda & 0 \end{vmatrix} = 2\lambda - \lambda^2$$

and for $\tilde{\omega}_{14}\pi_{14}$ $\tilde{\omega}_{15}\pi_{15}$, $\tilde{\omega}_{24}\pi_{24}$, $\tilde{\omega}_{25}\pi_{25}$, $\tilde{\omega}_{34}\pi_{34}$, $\tilde{\omega}_{35}\pi_{35}$ are all

$$\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & \lambda \end{vmatrix} = \lambda .$$

Thus, using all the above replacements in $w = 0$,

$$6\lambda(3 - \lambda) - 6\lambda - 3(2\lambda - \lambda^2) - (3\lambda - 2\lambda^2) = 0$$

$$\lambda(3 - \lambda) = 0 .$$

The line d does not meet e, thus $\tilde{\omega}_{45} \neq 0$; this means that $\lambda \neq 0$.

Therefore $\lambda = 3$.

APPENDIX II. The double-sixes produced by a line and 6 transversals.

b	a_1	a_2	a_3	a_4	a_5	a_6		
D_1	α_1	a_2	a_3	a_4	a_5	a_6	c^{1}_{jk}	F_1
	b	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}		
D_2	a_1	α_2	a_3	a_4	a_5	a_6	c^{2}_{jk}	F_2
	b_{21}	b	b_{23}	b_{24}	b_{25}	b_{26}		
D_3	a_1	a_2	α_3	a_4	a_5	a_6	c^{3}_{jk}	F_3
	b_{31}	b_{32}	b	b_{34}	b_{35}	b_{36}		
D_4	a_1	a_2	a_3	α_4	a_5	a_6	c^{4}_{jk}	F_4
	b_{41}	b_{42}	b_{43}	b	b_{45}	b_{46}		
D_5	a_1	a_2	a_3	a_4	α_5	a_6	c^{5}_{jk}	F_5
	b_{51}	b_{52}	b_{53}	b_{54}	b	b_{56}		
D_6	a_1	a_2	a_3	a_4	a_5	α_6	c^{6}_{jk}	F_6
	b_{61}	b_{62}	b_{63}	b_{64}	b_{65}	b		

APPENDIX III. Grace's extension of the double-six over GF(8).

b	1	0	0	0	0	0
a_1	0	1	0	0	0	0
a_2	0	0	0	0	1	0
a_3	0	1	1	1	1	0
a_4	0	ϵ^5	ϵ	ϵ^4	1	0
a_5	1	ϵ^3	ϵ^2	ϵ	1	0
a_6	ϵ	ϵ	ϵ^3	ϵ^5	1	0
b_{12}	0	ϵ^4	ϵ^4	ϵ^3	ϵ^3	1
b_{13}	ϵ^5	0	ϵ^3	ϵ^2	ϵ^4	1
b_{14}	ϵ	0	ϵ^5	ϵ^3	ϵ^6	1
b_{15}	ϵ^3	0	ϵ^4	ϵ^6	1	1
b_{16}	ϵ^6	0	ϵ^2	ϵ^4	ϵ^5	1
b_{23}	ϵ	1	ϵ	1	0	1
b_{24}	ϵ^2	ϵ^5	ϵ^6	ϵ^3	0	1
b_{25}	ϵ^2	ϵ^2	ϵ^4	ϵ^6	0	1
b_{26}	ϵ^4	ϵ^3	ϵ^5	ϵ^6	0	1
b_{34}	0	0	0	ϵ^5	0	1
b_{35}	ϵ^6	0	ϵ^5	ϵ	0	1
b_{36}	ϵ^3	0	1	ϵ^3	0	1
b_{45}	ϵ^4	0	ϵ^2	ϵ^2	0	1
b_{46}	ϵ^2	0	ϵ	ϵ	0	1
b_{56}	0	0	0	0	0	1
α_1	ϵ	ϵ^4	ϵ	ϵ^3	ϵ^6	1
α_2	ϵ^3	ϵ^2	ϵ^4	ϵ^2	ϵ^3	1
α_3	ϵ	1	ϵ^3	1	ϵ^4	1
α_4	ϵ^3	ϵ^5	ϵ^5	ϵ^3	ϵ^6	1
α_5	0	ϵ^2	ϵ^4	ϵ^5	1	1
α_6	0	ϵ^3	ϵ^2	ϵ^6	ϵ^5	1

$1 + 1 = 0$
$\epsilon^7 + 1 = 0$
$1 + \epsilon^2 + \epsilon^3 = 0$
$\epsilon + \epsilon^3 + \epsilon^4 = 0$
$\epsilon^2 + \epsilon^4 + \epsilon^5 = 0$
$\epsilon^3 + \epsilon^5 + \epsilon^6 = 0$
$1 + \epsilon^4 + \epsilon^6 = 0$
$1 + \epsilon + \epsilon^5 = 0$
$\epsilon + \epsilon^2 + \epsilon^6 = 0$

$L_1 = L_{246} = L_{345} = \epsilon \;\; \epsilon^6 \;\; 1 \;\; 0$

$L_2 = L_{145} = L_{356} = \epsilon^2 \;\; \epsilon^2 \;\; 0 \;\; 1$

$L_3 = L_{126} = L_{245} = \epsilon^2 \;\; \epsilon^6 \;\; 1 \;\; 1$

$L_4 = L_{156} = L_{236} = \epsilon^4 \;\; 0 \;\; \epsilon^4 \;\; 1$

$L_5 = L_{123} = L_{346} = \epsilon^4 \;\; 1 \;\; \epsilon \;\; 1$

$L_6 = L_{125} = L_{134} = \epsilon^2 \;\; \epsilon^5 \;\; \epsilon^5 \;\; 1$

$L_{124} = L_{136} = L_{235} = L_{456} = \epsilon^4 \;\; \epsilon^3 \;\; \epsilon^6 \;\; 1$

$L_{135} = L_{146} = L_{234} = L_{256} = 0 \;\; \epsilon \;\; \epsilon^3 \;\; 1$

$L = \epsilon \;\; \epsilon^4 \;\; \epsilon^2 \;\; 1$

APPENDIX IV. The 32 double-sixes intrinsic to Grace's extension.

D_1	α_1	a_2	a_3	a_4	a_5	a_6
	b	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}
D_2	a_1	α_2	a_3	a_4	a_5	a_6
	b_{21}	b	b_{23}	b_{24}	b_{25}	b_{26}
D_3	a_1	a_2	α_3	a_4	a_5	a_6
	b_{31}	b_{32}	b	b_{34}	b_{35}	b_{36}
D_4	a_1	a_2	a_3	α_4	a_5	a_6
	b_{41}	b_{42}	b_{43}	b	b_{45}	b_{46}
D_5	a_1	a_2	a_3	a_4	α_5	a_6
	b_{51}	b_{52}	b_{53}	b_{54}	b	b_{56}
D_6	a_1	a_2	a_2	a_4	a_5	α_6
	b_{61}	b_{62}	b_{63}	b_{64}	b_{65}	b
∇_{123}	a_1	a_2	a_3	α_4	α_5	α_6
	β_{23}	β_{31}	β_{12}	b_{56}	b_{64}	b_{45}
∇_{124}	a_1	a_2	a_4	α_3	α_5	α_6
	β_{24}	β_{41}	β_{12}	b_{56}	b_{63}	b_{35}
∇_{125}	a_1	a_2	a_5	α_3	α_4	α_6
	β_{25}	β_{51}	β_{12}	b_{46}	b_{63}	b_{34}
∇_{126}	a_1	a_2	a_6	α_3	α_4	α_5
	β_{26}	β_{61}	β_{12}	b_{45}	b_{53}	b_{34}
∇_{134}	a_1	a_3	a_4	α_2	α_5	α_6
	β_{34}	β_{41}	β_{13}	b_{56}	b_{62}	b_{25}
∇_{135}	a_1	a_3	a_5	α_2	α_4	α_6
	β_{35}	β_{51}	β_{13}	b_{46}	b_{62}	b_{24}
Δ_1	a_1	α_2	α_3	α_4	α_5	α_6
	β	β_{12}	β_{13}	β_{14}	β_{15}	β_{16}
Δ_2	α_1	a_2	α_3	α_4	α_5	α_6
	β_{21}	β	β_{23}	β_{24}	β_{25}	β_{26}
Δ_3	α_1	α_2	a_3	α_4	α_5	α_6
	β_{31}	β_{32}	β	β_{34}	β_{35}	β_{36}
Δ_4	α_1	α_2	α_3	a_4	α_5	α_6
	β_{41}	β_{42}	β_{43}	β	β_{45}	β_{46}
Δ_5	α_1	α_2	α_3	α_4	a_5	α_6
	β_{51}	β_{52}	β_{53}	β_{54}	β	β_{56}
Δ_6	α_1	α_2	α_3	α_4	α_5	a_6
	β_{61}	β_{62}	β_{63}	β_{64}	β_{65}	β
∇_{456}	a_4	a_5	a_6	α_1	α_2	α_3
	β_{56}	β_{64}	β_{45}	b_{23}	b_{31}	b_{12}
∇_{356}	a_3	a_5	a_6	α_1	α_2	α_4
	β_{56}	β_{63}	β_{35}	b_{24}	b_{41}	b_{12}
∇_{346}	a_3	a_4	a_6	α_1	α_2	α_5
	β_{46}	β_{63}	β_{34}	b_{25}	b_{51}	b_{12}
∇_{345}	a_3	a_4	a_5	α_1	α_2	α_6
	β_{45}	β_{53}	β_{34}	b_{26}	b_{61}	b_{12}
∇_{256}	a_2	a_5	a_6	α_1	α_3	α_4
	β_{56}	β_{62}	β_{25}	b_{34}	b_{41}	b_{13}
∇_{246}	a_2	a_4	a_6	α_1	α_3	α_5
	β_{46}	β_{62}	β_{24}	b_{35}	b_{51}	b_{13}

V_{136}	a_1	a_3	a_6	α_2	α_4	α_5
	β_{36}	β_{61}	β_{13}	b_{45}	b_{52}	b_{24}
V_{145}	a_1	a_4	a_5	α_2	α_3	α_6
	β_{45}	β_{51}	β_{14}	b_{36}	b_{62}	b_{23}
V_{146}	a_1	a_4	a_6	α_2	α_3	α_5
	β_{46}	β_{61}	β_{14}	b_{35}	b_{52}	b_{23}
V_{156}	a_1	a_5	a_6	α_2	α_3	α_4
	β_{56}	β_{61}	β_{15}	b_{34}	b_{42}	b_{23}
V_{245}	a_2	a_4	a_5	α_1	α_3	α_6
	β_{45}	β_{52}	β_{24}	b_{36}	b_{61}	b_{13}
V_{236}	a_2	a_3	a_6	α_1	α_4	α_5
	β_{36}	β_{62}	β_{23}	b_{45}	b_{51}	b_{14}
V_{235}	a_2	a_3	a_5	α_1	α_4	α_6
	β_{35}	β_{52}	β_{23}	b_{46}	b_{61}	b_{14}
V_{234}	a_2	a_3	a_4	α_1	α_5	α_6
	β_{34}	β_{42}	β_{23}	b_{56}	b_{61}	b_{15}

APPENDIX V. Wren's diagram .

The pairs of the sextuplet of points in which a B-line is met by the A-lines are x, x; y, y; z, z. Any two pairs are harmonic .

	a_1	a_2	a_3	a_4	a_5	a_6	α_1	α_2	α_3	α_4	α_5	α_6
b	x	y	z	z	y	x	•	•	•	•	•	•
b_{12}	•	•	x	y	z	y	x	z	•	•	•	•
b_{13}	•	x	•	y	y	z	z	•	x	•	•	•
b_{14}	•	x	x	•	y	y	z	•	•	z	•	•
b_{15}	•	x	y	z	•	x	z	•	•	•	y	•
b_{16}	•	x	y	z	y	•	x	•	•	•	•	z
b_{23}	x	•	•	y	z	z	•	x	y	•	•	•
b_{24}	x	•	y	•	y	z	•	z	•	x	•	•
b_{25}	x	•	y	x	•	z	•	y	•	•	z	•
b_{26}	x	•	x	y	y	•	•	z	•	•	•	z
b_{34}	x	y	•	•	z	y	•	•	x	z	•	•
b_{35}	x	x	•	y	•	y	•	•	z	•	z	•
b_{36}	x	y	•	x	z	•	•	•	z	•	•	y
b_{45}	x	y	x	•	•	z	•	•	•	z	y	•
b_{46}	x	x	y	•	z	•	•	•	•	y	•	z
b_{56}	x	y	y	z	•	•	•	•	•	•	z	x
β_{56}	•	•	•	•	x	y	z	z	x	y	•	•
β_{46}	•	•	•	x	•	y	z	y	z	•	x	•
β_{45}	•	•	•	x	y	•	x	z	z	•	•	y
β_{36}	•	•	x	•	•	y	y	z	•	x	z	•
β_{35}	•	•	x	•	y	•	z	x	•	y	•	z
β_{34}	•	•	x	x	•	•	y	y	•	•	z	z
β_{26}	•	x	•	•	•	y	x	•	z	z	y	•
β_{25}	•	x	•	•	x	•	y	•	y	z	•	z
β_{24}	•	x	•	y	•	•	z	•	y	•	x	z
β_{23}	•	x	y	•	•	•	y	•	•	z	z	x
β_{16}	x	•	•	•	•	x	•	y	y	z	z	•
β_{15}	x	•	•	•	y	•	•	y	z	z	•	x
β_{14}	x	•	•	y	•	•	•	z	x	•	z	y
β_{13}	x	•	y	•	•	•	•	x	•	z	y	z
β_{12}	x	y	•	•	•	•	•	•	y	x	z	z
β	•	•	•	•	•	•	x	y	z	x	z	y

APPENDIX VI. Grace's extension of the double-six over GF(9).

b	1	0	0	0	0	0
a_1	0	1	0	0	0	0
a_2	0	0	0	0	1	0
a_3	σ	σ^2	σ	$-\sigma$	1	0
a_4	$-\sigma^2$	$-\sigma^3$	σ^3	1	1	0
a_5	σ^2	σ^3	1	$-\sigma^3$	1	0
a_6	σ^2	-1	-1	-1	1	0
b_{12}	$-\sigma^2$	σ^3	0	$-\sigma$	$-\sigma^2$	1
b_{13}	$-\sigma^3$	0	$-\sigma^2$	$-\sigma$	-1	1
b_{14}	-1	0	σ^2	$-\sigma^3$	σ^2	1
b_{15}	σ^3	0	$-\sigma$	σ^2	σ	1
b_{16}	0	0	0	0	σ^3	1
b_{23}	$-\sigma$	σ^2	-1	$-\sigma$	0	1
b_{24}	$-\sigma^3$	σ	σ	σ^2	0	1
b_{25}	σ^3	-1	1	$-\sigma^3$	0	1
b_{26}	0	$-\sigma^3$	0	σ	0	1
b_{34}	$-\sigma$	0	1	σ	0	1
b_{35}	0	0	σ^2	0	0	1
b_{36}	σ	0	σ^3	σ^2	0	1
b_{45}	-1	0	σ^3	$-\sigma$	0	1
b_{46}	1	0	-1	1	0	1
b_{56}	$-\sigma$	0	$-\sigma^2$	σ^3	0	1

β	$-\sigma^3$	σ	$-\sigma^3$	σ^2	σ	1
α_1	-1	$-\sigma$	$-\sigma$	σ^2	σ	1
α_2	$-\sigma$	σ	σ	σ^2	$-\sigma$	1
α_3	σ^2	1	0	-1	$-\sigma^2$	1
α_4	σ^3	$-\sigma$	0	$-\sigma^2$	σ^2	1
α_5	$-\sigma$	-1	0	$-\sigma^3$	$-\sigma$	1
α_6	$-\sigma^2$	$-\sigma^3$	0	$-\sigma$	σ^3	1
β_{12}	0	0	0	1	0	0
β_{13}	$-\sigma$	$-\sigma^2$	$-\sigma$	-1	0	1
β_{14}	$-\sigma$	σ	σ	1	0	1
β_{15}	σ^3	σ^3	1	0	0	0
β_{16}	σ	$-\sigma$	$-\sigma^3$	$-\sigma^2$	0	1
β_{23}	σ	0	σ	-1	1	1
β_{24}	$-\sigma$	0	σ^3	0	1	0
β_{25}	-1	0	$-\sigma$	σ^3	σ	1
β_{26}	σ^2	0	$-\sigma^3$	$-\sigma^3$	$-\sigma$	1
β_{34}	1	σ^3	σ	$-\sigma^3$	$-\sigma$	1
β_{35}	σ	$-\sigma$	$-\sigma$	$-\sigma^2$	$-\sigma^3$	1
β_{36}	$-\sigma^2$	$-\sigma$	0	σ^2	$-\sigma$	1
β_{45}	$-\sigma^2$	-1	1	1	1	0
β_{46}	$-\sigma^2$	-1	0	σ^3	$-\sigma^2$	1
β_{56}	$-\sigma^2$	1	0	1	σ^2	1

$$1 + 1 + 1 = 0$$
$$\sigma^4 + 1 = 0$$
$$\sigma^2 - \sigma - 1 = 0$$

$$\sigma^3 - \sigma^2 - \sigma = 0$$
$$\sigma^3 + \sigma^2 + 1 = 0$$
$$\sigma^3 + \sigma - 1 = 0$$

APPENDIX VII. The special concurrencies of lines implicit in Grace's extension of the double-six over GF(9).

The 72 coincidences of two B-lines on an A-line are given for the lines of Appendix VI .

a_1	b	b_{23}	b_{24}	b_{25}	b_{26}	b_{36}	b_{35}	b_{56}	β_{13}	β_{16}
	β_{15}	b_{46}	β_{14}	b_{34}	β_{12}	b_{45}				
α_1	β	β_{23}	β_{24}	β_{25}	β_{26}	β_{46}	β_{35}	β_{56}	b_{13}	b_{16}
	b_{12}	β_{45}	β_{36}	b_{15}	β_{34}	b_{14}				
a_2	b	b_{13}	b_{14}	b_{15}	b_{16}	b_{56}	b_{34}	b_{46}	β_{23}	β_{26}
	β_{12}	b_{45}	β_{24}	b_{36}	b_{35}	β_{25}				
α_2	β	β_{13}	β_{14}	β_{15}	β_{16}	β_{36}	β_{34}	β_{46}	b_{23}	b_{26}
	b_{24}	β_{56}	β_{35}	b_{25}	b_{12}	β_{45}				
a_3	b_{12}	b_{14}	b_{16}	b_{25}	b_{45}	b_{46}	b	b_{15}	b_{24}	β_{36}
	b_{56}	b_{26}	β_{13}	β_{23}	β_{36}	β_{34}				
α_3	β_{12}	β_{14}	β_{16}	β_{26}	β_{45}	β_{56}	β	β_{15}	β_{24}	b_{36}
	β_{46}	b_{13}	β_{25}	b_{23}	b_{34}	b_{35}				
a_4	b	b_{12}	b_{13}	b_{15}	b_{16}	b_{35}	b_{23}	b_{26}	β_{34}	β_{46}
	β_{24}	b_{36}	b_{56}	β_{14}	b_{25}	β_{45}				
α_4	β	β_{12}	β_{13}	β_{15}	β_{16}	β_{25}	β_{23}	β_{26}	b_{34}	b_{46}
	b_{45}	β_{56}	b_{14}	β_{36}	β_{35}	b_{24}				
a_5	b	b_{12}	b_{14}	b_{23}	b_{24}	b_{26}	b_{13}	b_{16}	β_{35}	β_{56}
	β_{45}	b_{46}	β_{15}	β_{25}	b_{36}	b_{34}				
α_5	β	β_{12}	β_{14}	β_{23}	β_{24}	β_{34}	β_{13}	β_{16}	b_{35}	b_{56}
	b_{15}	β_{36}	β_{26}	β_{46}	b_{25}	b_{45}				
a_6	b_{12}	b_{13}	b_{14}	b_{23}	b_{25}	b_{34}	b	b_{15}	b_{24}	β_{36}
	β_{26}	β_{16}	b_{35}	b_{45}	β_{56}	β_{46}				
α_6	β_{12}	β_{13}	β_{14}	β_{23}	β_{25}	β_{35}	β	β_{15}	β_{24}	b_{36}
	b_{16}	β_{45}	b_{46}	b_{26}	β_{34}	b_{56}				

	b	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{23}	b_{24}	b_{25}	b_{26}	b_{34}	b_{35}	b_{36}	b_{45}	b_{46}	b_{56}	β_{56}	β_{46}	β_{45}	β_{36}	β_{35}	β_{34}	β_{26}	β_{25}	β_{24}	β_{23}	β_{16}	β_{15}	β_{14}	β_{13}	β_{12}	β
b																			a_5						a_4			a_1			a_2	
b_{12}													a_4		a_5	a_3							a_6				α_2					α_1
b_{13}														a_2		a_4											a_6		α_3			
b_{14}										a_3		a_6						α_1							a_2			a_5		α_4		
b_{15}													a_2											α_1					a_4			α_5
b_{16}									a_4			a_2																		a_3	α_6	
b_{23}														a_6	a_1								α_3	a_5								
b_{24}													a_5											α_4					a_1			α_2
b_{25}						a_4					a_1						a_6								α_5	a_3		α_2				
b_{26}				a_3							a_5															α_6					a_1	
b_{34}									a_1	a_5								a_6	α_3													
b_{35}				a_6		a_2											α_3		a_4													
b_{36}		a_4			a_2			a_6						a_1																		
b_{45}			a_2				a_6						a_1								a_3	α_5										α_4
b_{46}		a_5					a_1															a_3							α_6			
b_{56}		a_3	a_4																		α_6			a_2								
β_{56}									a_6			α_3																		α_2	α_4	
β_{46}				α_1							a_6															α_5					α_3	
β_{45}	a_5										α_3	a_4								α_2						α_1				α_6		
β_{36}																			a_2						α_1			α_4			α_5	
β_{35}														a_3		α_6											α_4		α_2			
β_{34}														α_5	a_3								α_1	α_6								
β_{26}		a_6					α_3															α_1							α_5			
β_{25}					α_1		a_5	α_4								a_2						α_6					α_3					
β_{24}	a_4			a_2					α_5											α_1												
β_{23}									a_3	α_6								α_5	α_1													
β_{16}		α_2	a_6																		α_4			α_3								
β_{15}	a_1			a_5					α_2											α_4												
β_{14}			α_3		a_4			a_1							α_6						α_2		α_5									
β_{13}				α_4		a_3											α_2		α_6													
β_{12}	a_2					α_6				a_1							α_4	α_3		α_5												
β		α_1			α_5			α_2						α_4																		

BIBLIOGRAPHY

The abbreviations for extant periodicals are those of Mathematical Reviews 28(1964).

[1] BAKER H.F. : A Geometrical Proof of the Theorem of a Double Six of Straight Lines. Proc. Roy. Soc. Ser. A 84(1911), 597.

[2] —— : On a proof of the theorem of a double six of lines by projection from four dimensions. Proc. Cambridge Philos. Soc. 20(1921), 133.

[3] —— : Remarks on Mr. Wakeford's Paper. Proc. London Math. Soc. 21(1922), 114.

[4] —— : Principles of Geometry vol. III. Cambridge, 1923.

[5] BURNSIDE W.: On the double-six which admits a group of 120 collineations into itself. Proc. Cambridge Philos. Soc. 16(1911), 418.

[6] —— : Theory of Groups of Finite Order. Cambridge, 1911.

[7] CAYLEY A. : On the triple tangent planes of surfaces of the third order. The Cambridge and Dublin Mathematical Journal 4(1849), 118.

[8] —— : Note on the Theory of Cubic Surfaces. Philos. Mag. 27(1864), 493.

[9] —— : On the Double-Sixers of a Cubic Surface. Quart. J. Maths. 10(1870), 58.

[10] CLEBSCH A. : Ueber eine Transformation der homogen Functionen dritter Ordnung mit vier Veränderlichen. J. Reine Angew. Math. 58(1861), 109.

[11] —— : Ueber die Knotenpunkte der Hesseschen Fläche insbesondere bei Oberflächen dritter Ordnung. J. Reine Angew. Math. 59(1861), 193.

[12] —— : Die Geometrie auf den Flächen dritter Ordnung. J. Reine Angew. Math. 65(1866), 359.

[13] —— : Über die Anwendung der quadratischen Substitution auf die Gleichungen 5ten Grades und die geometrische Theorie des ebenen Fünfseits. Math. Ann. 4(1871), 284.

[14] COBLE A.B. : A configuration in finite geometry isomorphic with that of the twenty-seven lines of a cubic surface. Johns Hopkins University Circulars (1908) no. 7, 80.

[15] COXETER H.S.M. : The polytope 2_{21}, whose twenty-seven vertices correspond to the lines on the general cubic surface. Amer. J. Math. 62(1940), 457.

[16] CREMONA L. : Note sur les cubiques gauches. J. Reine Angew Math. 60(1862), 188.

[17] DICKSON L.E. : Linear Groups with an exposition of the Galois Field Theory. Teubner, Leipzig, 1901.

[18] ECKARDT F.E. : Ueber diejenigen Flächen dritter Grades, auf denen sich drei gerade Linien in einem Punkte schneiden. Math. Ann. 10(1876), 227.

[19] EDGE W. L. : The Theory of Ruled Surfaces. Cambridge, 1931.

[20] —— : The isomorphism between LF(2, 3^2) and $\mathcal{A}_6$. J. London Math. Soc. 30(1955), 172.

[21] —— : Quadrics over GF(2) and their relevance for the cubic surface group. Canad. J. Math. 11(1959), 625.

[22] —— : Fundamental figures, in four and six dimensions, over GF(2). Proc. Cambridge Philos. Soc. 60(1964), 183.

[23] —— : Some implications of the geometry of the 21-point plane. Math. Z. 87(1965), 348.

[24] FRAME J.S. : A symmetric representation of the twenty-seven lines on a cubic surface by lines in a finite geometry. Bull. Amer. Math. Soc. 44(1938), 658.

[25] GRACE J.H. : Circles, Spheres, and Linear Complexes. Transactions of the Cambridge Philosophical Society 16(1898), 153.

[26] GRASSMANN H. : Die stereometrische Gleichung dritten Grades und die dadurch dargestellten Oberflächen. J. Reine Angew. Math. 49(1955),47 .

[27] HASSE H. : Punti razionali sopra curve algebriche a congruenze. Atti dei Convegni (1939). Reale Accademia d'Italia, Roma, 1943.

[28] HENDERSON A. : The Twenty-Seven Lines upon the Cubic Surface. Cambridge Tracts no. 13. Cambridge, 1911.

[29] HILBERT D. and COHN-VOSSEN S. : Geometry and the Imagination. Chelsea, New York, 1952.

[30] HIRSCHFELD J.W.P. : The double-six of lines over PG(3, 4). J. Austral. Math. Soc. 4(1964), 83.

[31] HODGE W.V.D. and PEDOE D. : Methods of Algebraic Geometry vol. II. Cambridge, 1952.

[32] KASNER E. : The Double-six Configuration Connected with the Cubic Surface, and a Related Group of Cremona Transformations. Amer. J. Math. 25(1903), 107.

[33] KUBOTA T. : On Double Sixers. Sci. Rep. Tôhoku Univ. Ser. I 6(1917), 89.

[34] LANG S. and WEIL A.: Number of points of varieties in finite fields. Amer. J. Math. 76(1954), 819.

[35] MACBEATH A.M. : On a curve of genus 7. Proc. London Math. Soc. 25(1965), 527.

[36] MARCHAUD A. : Sur les droites de la surface du troisième ordre en géométrie finie (second mémoire). Ann. Sc. Ecole Norm. Sup. 81(1964), 207.

[37] MEYER W.F. : Flächen dritter Ordnung. Encyklopädie der mathematischen Wissenschaften IIIC 10a.

[38] PRIMROSE E.J.F. : Quadrics in finite geometries. Proc. Cambridge Philos. Soc. 47(1951), 299.

[39] QVIST B. : Some remarks concerning curves of the second degree in a finite plane. Ann. Acad. Sci. Fenn. Ser. AI. no.134(1952).

[40] RICHMOND H.W. : On the property of a double-six of lines, and its meaning in hypergeometry. Proc. Cambridge Philos. Soc. 14(1908), 475.

[41] RODENBERG C. : Zur Classification der Flächen dritter Ordnung. Math. Ann. 14(1879), 46.

[42] ROOM T.G. : The Schur quadrics of a cubic surface I, II. J. London Math. Soc. 7(1932), 147.

[43] ROSATI L.A. : Sul numero dei punti di una superficie cubica in uno spazio lineare finito. Boll. Un. Mat. Ital. 11(1956), 412.

[44] —— : L'equazione delle 27 retta della superficie cubica generale in un corpo finito .
I. Boll. Un. Mat. Ital. 12(1957), 612.
II. Boll. Un. Mat. Ital. 13(1958), 84.

[45] SCAFATI M. : Sui 6-archi completi di un piano lineare $S_{2,8}$. Convegno internazionale, Palermo (1957), 128. Cremonese, Roma, 1958.

[46] SCHLÄFLI L. : An Attempt to Determine the Twenty-Seven Lines upon a Surface of the Third Order and to divide such Surfaces into Species in reference to the Reality of the Lines upon the Surface. Quart. J. Math. 2(1858), 55; 110.

[47] SCHUR F. : Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen. Math. Ann. 18(1881), 1.

[48] SEGRE B. : The non-singular cubic surfaces. Oxford., 1942.

[49] —— : Lezioni di geometria moderna vol. I. Zanichelli, Bologna, 1948.

[50] —— : Le rette delle superficie cubiche nei corpi commutativi. Boll. Un. Mat. Ital. 4(1949), 223.

[51] —— : Ovals in a finite projective plane. Canad. J. Math. 7(1955), 414.

[52] —— : Curve razionali normali e k-archi negli spazi finiti. Ann. Mat. Pura Appl. 39(1955), 357.

[53] —— : Intorno alla geometria sopra un campo di caratterisitica due. Istanbul Üniv. Fen Fak. Mec. Ser. A 21(1956), 97.

[54] —— : Sui k-archi nei piani finiti di caratterisitica due. Rev. Math. Pures Appl. 2(1957), 289.

[55] —— : Le geometrie di Galois. Ann. Mat. Pura Appl. 48(1959), 1.

[56] —— : Le geometrie di Galois. Confer. Sem. Mat. Univ. Bari nos. 43 - 44(1959).

[57] —— : Lectures on modern geometry. Cremonese, Rome, 1961.

[58] —— : Arithmetische Eigenschaften von Galois-Räumen I. Math. Ann. 154(1964), 195.

[59] STEINER J. : Über die Flächen dritten Grades. J. Reine Angew. Math. 53(1857), 133.

[60] SYLVESTER J.J. : Sketch of a memoir on elimination, transformation, and canonical forms. The Cambridge and Dublin Mathematical Journal 6(1851), 186.

[61] TODD J.A. : Projective and Analytical Geometry. Pitman, London, 1958.

[62] VEBLEN O. and BUSSEY W.H. : Finite projective geometries. Trans. Amer. Math. Soc. 7(1906), 241.

[63] WAKEFORD E.K. : Chords of Twisted Cubics. Proc. London Math. Soc. 21(1922), 98.

[64] WEIL A. : Sur les courbes algébriques et les variétés qui s'en déduisent. Hermann, Paris, 1948.

[65] WREN T.L. : Some Applications of the Two-Three Birational Space Transformation. Proc. London Math. Soc. 15(1916), 144.