# A NEW CLASS OF RESOLVABLE BLOCK DESIGNS

by

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# DECLARATION

The following record of research work is submitted as a thesis for the degree of Doctor of Philosophy in the University of Edinburgh, having been submitted for no other degree. Except where acknowledgement is made, the work is original.

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## SUMMARY

The main purpose of this thesis is to introduce a new class of resolvable block designs (called $\underset{\sim}{\alpha}$-designs). These designs are intended particularly for variety trials where v, the number of varieties, is often large (say $v = 60$) but r, the number of replications of each variety, is small ($r \leq 4$).

Given a suitable $k \times r$ array $\underset{\sim}{\alpha}$, a simple algorithm is available for constructing an $\underset{\sim}{\alpha}$-design for any v that is a multiple of k, the number of units in each block. Much of the thesis is concerned with the problem of choosing $\underset{\sim}{\alpha}$ to give designs with high efficiency factors.

Two separate subclasses of $\underset{\sim}{\alpha}$-designs are considered in detail. In one subclass no two varieties concur more than once; in the other some pairs of varieties concur twice but no pair concurs more often. Existence conditions are established for the two subclasses and criteria developed for the choice of $\underset{\sim}{\alpha}$.

When the number of varieties does not factorize conveniently in the form $v = ks$, resolvable designs with block sizes differing by not more than one unit can be derived from $\underset{\sim}{\alpha}$-designs.

The class of $\underset{\sim}{\alpha}$-designs and the derived designs with unequal block sizes together meet virtually any requirement for variety trials with equal replication of each variety.

Tables are provided giving generating arrays for parameter combinations in the ranges $v \leq 100$, $2 \leq r \leq 4$ and $4 \leq k \leq 20$, together with average, minimum and pairwise efficiency factors. Guidance is also given on the allocation of control varieties.

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## NOTATION

$\underset{\sim}{1}$ : unit vector of appropriate dimension.

$\underset{\sim}{I}_v$ : $v \times v$ identity matrix.

$\underset{\sim}{J}_v$ : $v \times v$ matrix of ones.

$\underset{\sim}{K}_v$ : $v \times v$ matrix whose $(l,m)^{th}$ element is $(-1)^{l+m}$,

$(l,m = 0,1,\ldots,v-1)$.

$\underset{\sim}{r}'$ : transpose of $\underset{\sim}{r}$.

tr $\underset{\sim}{A}$ : trace of $\underset{\sim}{A}$.

$[\frac{k}{s}]$ : integer part of $\frac{k}{s}$.

$\underset{\sim}{A} \otimes \underset{\sim}{B}$: direct product of $\underset{\sim}{A}$ and $\underset{\sim}{B}$.

$\bar{\underset{\sim}{Y}}$ : complex conjugate of $\underset{\sim}{Y}$.

$\dotplus$, $\dot{-}$ : addition and subtraction, modulo s.

Hence $s \dotplus h = h$,

and $\dot{-} h = s - h$,

$(h = 0,1,\ldots,s-1)$.

## INTRODUCTION

The main aim of this thesis is to produce incomplete block designs for statutory variety trials on agricultural crops. In this situation, the experimenter usually has a relatively large number of varieties (say $v = 60$) to be compared, but is limited by the amount of material or land available to say 2, 3 or 4 replications of each variety.

In practice, resolvable designs have to be used, i.e. designs with blocks grouped into complete replicates. When there is a large number of varieties, it may sometimes be necessary to sow or harvest replicates on different occasions. Again in some four replicate variety trials, one may want to confound two levels of nitrogen with replicates. Further, some measurements are expensive and have to be restricted to just one or two replicates.

There are at present four main sources of incomplete block designs for unstructured treatments:

(i) Balanced incomplete block (BIB) designs (Fisher and Yates, 1963).

(ii) Square and rectangular lattice designs (Cochran and Cox, 1957, Chapter 10).

(iii) Two-associate class partially balanced incomplete block (PBIB(2)) designs (Bose, Clatworthy and Shrikhande, 1954 and Clatworthy, 1973).

(iv) Cyclic incomplete block (CIB) designs (John, Wolock and David, 1972).

Many BIB and PBIB(2) designs have too many replications to be of much use for variety trials; CIB designs are very useful in other applications but relatively few are resolvable. Only the lattice designs are automatically resolvable. They were originally intended for variety trial work and still have an important part to play. But they are available only for limited combinations of parameters. Thus $v$ must be of the form $v = s^2$ for square lattices and $v = (s-1)s$ for rectangular lattices.

Possibly because suitable incomplete block designs are not readily available, experimenters in the United Kingdom have tended to use simple randomized block designs with one replication per block, where in fact incomplete block designs (with block size say around 6 - 12 units) could be used to advantage.

The construction of resolvable designs with two replicates has been considered by Bose and Nair (1962) but their method does not generalize to larger replication numbers. David (1967) investigated the conditions under which CIB designs are resolvable.

The thesis sets out to extend basic principles of standard lattice designs to produce tables of useful resolvable designs for parameters in the ranges $v \leq 100$ and $2 \leq r \leq 4$.

Chapter 1 deals with some general properties of block designs. We define a modified matrix of reduced normal equations which provides the basis for a classification of block designs, described in Chapter 2. This classification is more general than that given by Pearce (1963), and includes a new class of resolvable designs which are defined in Chapter 3.

The new designs (called $\underset{\sim}{\alpha}$-designs), are for ks varieties, r replications and block size k units; they are constructed from a $k \times r$ array $\underset{\sim}{\alpha}$. In Chapter 3 we also discuss some of the properties of $\underset{\sim}{\alpha}$-designs.

Many $\underset{\sim}{\alpha}$-designs exist for each combination of parameters. In choosing between them, we would intuitively prefer designs with a narrow range of numbers of concurrences. All $\underset{\sim}{\alpha}$-designs must include some zero concurrences. Therefore the most acceptable designs are those with only zeros and ones off-diagonal in the concurrence matrix. We call these $\underset{\sim}{\alpha}(0,1)$-designs. In Chapter 4 we show that a necessary condition for the existence of $\underset{\sim}{\alpha}(0,1)$-designs is $k \leq s$.

In practice we also require designs with $k > s$. For this we introduce $\underset{\sim}{\alpha}(0,1,2)$-designs in Chapter 5. We show that $\underset{\sim}{\alpha}(0,1,2)$-designs with $r > 2$ can only exist for $k \leq s^2$.

When the number of varieties does not factorize conveniently in the form $v = ks$, we must relax either the condition of resolvability or that of equal block size. We feel that resolvability is essential for variety trials; unequal block sizes can be tolerated so long as they do not differ by more than one unit. Chapter 6 deals with the construction of almost equiblock-sized resolvable designs from $\underset{\sim}{\alpha}$-designs.

# CHAPTER 1

## BLOCK DESIGNS IN GENERAL

### 1.1 Summary

This chapter gives a general introduction to block designs and some of their properties. In section 1.2 several types of block designs are defined including BIB and PBIB(m) designs.

The intrablock analysis is given in section 1.3 via the modified matrix of reduced normal equations, $\underset{\sim}{A}$ whose non-zero latent roots are known to be the canonical efficiency factors for the design.

In section 1.4, a comparison is made of $\underset{\sim}{A}$ with other matrices that have previously been suggested. The harmonic mean canonical efficiency factor is defined and compared with other definitions of average efficiency.

The dual of a block design is defined in section 1.5. It is shown that the non-unit canonical efficiency factors of a design and its dual are the same. A relationship is given between $\underset{\sim}{A}^{+}$, the Moore-Penrose generalized inverse of $\underset{\sim}{A}$, and $\underset{\sim}{B}^{+}$, the Moore-Penrose generalized inverse of the modified matrix of reduced normal equations, $\underset{\sim}{B}$ for the dual design.

Finally, in section 1.6, a model and full analysis combining all intrablock and interblock information is given. It is shown how this section contains the theory of section 1.3 as a special case.

## 1.2 Block Designs

### 1.2.1. Basic Definitions

Definition 1.2.1. A block design is an arrangement of $v$ varieties in $b$ blocks, where the $i^{th}$ variety is replicated $r_i$ times and the $j^{th}$ block contains $k_j$ units./

Hence if $\underset{\sim}{r}$ is the $v \times 1$ vector of numbers of replications and $\underset{\sim}{k}$ is the $b \times 1$ vector of block sizes then

$$\underset{\sim}{r}'\underset{\sim}{1} = \underset{\sim}{k}'\underset{\sim}{1} . \qquad (1.2.1)$$

A block design is said to be

(i) equiblock-sized if

$$\underset{\sim}{k} = k\,\underset{\sim}{1} , \qquad (1.2.2)$$

(ii) equireplicated if

$$\underset{\sim}{r} = r\,\underset{\sim}{1} . \qquad (1.2.3)$$

Definition 1.2.2. The incidence matrix, $\underset{\sim}{N}$ is a $v \times b$ matrix whose $(i,j)^{th}$ element is the number of times the $i^{th}$ variety appears in the $j^{th}$ block of the design./

Definition 1.2.3. The concurrence matrix, $\underset{\sim}{N}\underset{\sim}{N}'$ is a $v \times v$ matrix whose $(i,j)^{th}$ element is the number of times the $i^{th}$ variety appears in the same block as the $j^{th}$ variety of the design (Pearce, 1963)./

A block design is said to be binary if all the elements of $\underset{\sim}{N}$ are either zero or one.

If the $b$ blocks of a design can be subdivided into $r$ groups such that each variety appears once in each group, then the design is said to be resolvable. Hence a resolvable design must be equireplicated, and each group is called a replicate.

Example 1.2.1. The design given in Table 1.2.1 for $v = 9$, $b = 12$ is an equiblock-sized $(k = 3)$, equireplicated $(r = 4)$, binary resolvable design. /

Table 1.2.1

A binary resolvable block design

| Replicate | 1 | | | 2 | | | 3 | | | 4 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Block | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 0 | 3 | 6 | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| | 1 | 4 | 7 | 3 | 4 | 5 | 4 | 5 | 3 | 5 | 3 | 4 |
| | 2 | 5 | 8 | 6 | 7 | 8 | 8 | 6 | 7 | 7 | 8 | 6 |

This thesis will be concerned only with binary block designs so that

$$k_j \leq v, \qquad (j = 1,2,\ldots,b). \tag{1.2.4}$$

A binary block design for which

$$k_j = v, \qquad (j = 1,2,\ldots,b), \tag{1.2.5}$$

is simply a randomized block design; otherwise the design is said to be an incomplete block design. In other words a randomized block design is a resolvable complete block design.

### 1.2.2. Balanced Incomplete Block (BIB) Designs

Yates (1936b) first proposed the use of balanced arrangements as experimental designs.

Definition 1.2.4. A BIB design is an equiblock-sized, equireplicated binary design such that each pair of varieties appears together within blocks the same number, $\lambda$ of times./

Example 1.2.2. The design given in Table 1.2.1 for $v = 9$, $b = 12$ is a BIB design with $\lambda = 1$./

1.2.3. Partially Balanced Incomplete Block (PBIB) Designs

These designs were first used by Yates (1936a) as lattice designs, and were formally defined by Bose and Nair (1939). The following definitions are given by Raghavarao (1971).

Definition 1.2.5. Given $v$ symbols 0, 1, ..., $v-1$, a relation satisfying the following conditions is said to be an association scheme with $m$ classes:

(i) Any two symbols are either $1^{st}$, $2^{nd}$, ..., or $m^{th}$ associates, the relation being symmetrical.

(ii) Each symbol $\alpha$ has $n_i$ $i^{th}$ associates, the number $n_i$ being independent of $\alpha$.

(iii) If any two symbols $\alpha$ and $\beta$ are $i^{th}$ associates, then the number of symbols that are $j^{th}$ associates of $\alpha$, and $k^{th}$ associates of $\beta$ is $p^i_{jk}$ and is independent of the pair of $i^{th}$ associates $\alpha$ and $\beta$.

The numbers $v$, $n_i$ $(i = 1,2,\ldots,m)$ and $p^i_{jk}$ $(i,j,k = 1,2,\ldots,m)$ are called the parameters of the association scheme./

Definition 1.2.6. If we have an association scheme with $m$ classes, then we get a PBIB design with $m$ associate classes if the $v$ symbols can be arranged as an equiblock-sized, equireplicated binary design such that the number of concurrences of any pair of $i^{th}$ associates is $\lambda_i$ $(i=1,2,\ldots,m)$./

Example 1.2.3. By taking the last 3 replicates of the design given in Table 1.2.1, we obtain a PBIB(2) design for $v = b = 9$, $r = k = 3$, $\lambda_1 = 0$, $\lambda_2 = 1$. The association scheme is given in Table 1.2.2./

Table 1.2.2

Association scheme of Example 1.2.3

| Symbol | First Associates |
|---|---|
| 0 | 1, 2 |
| 1 | 0, 2 |
| 2 | 0, 1 |
| 3 | 4, 5 |
| 4 | 3, 5 |
| 5 | 3, 4 |
| 6 | 7, 8 |
| 7 | 6, 8 |
| 8 | 6, 7 |

## 1.3 Intrablock Analysis

For the intrablock analysis, we assume a model of the form

$$\underset{\sim}{y} = \mu \underset{\sim}{1} + \underset{\sim}{X}\underset{\sim}{\alpha} + \underset{\sim}{Z}\underset{\sim}{\beta} + \underset{\sim}{\varepsilon}\,, \qquad (1.3.1)$$

where $\underset{\sim}{y}$ is an $n \times 1$ vector of observations, $\mu$ is the general mean, $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ are vectors of variety and block effects respectively, $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ are corresponding design matrices, i.e. the $(i,j)^{th}$ element of $\underset{\sim}{X}$ is 1 if the $j^{th}$ variety is applied to the $i^{th}$ unit of the design, and $\underset{\sim}{\varepsilon}$ is a vector of random effects with $E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0}$ and $E(\underset{\sim}{\varepsilon}\underset{\sim}{\varepsilon}') = \sigma^2 \underset{\sim}{I}_n$. The reduced normal equations for estimation of variety effects are well known (Raghavarao, 1971, page 48) to be of the form

$$\underset{\sim}{C}\hat{\underset{\sim}{\alpha}} = \underset{\sim}{Q} , \qquad (1.3.2)$$

where

$$\underset{\sim}{C} = (\underset{\sim}{r}^{\delta} - \underset{\sim}{N}\,\underset{\sim}{k}^{-\delta}\,\underset{\sim}{N}') , \qquad (1.3.3)$$

$$\underset{\sim}{Q} = (\underset{\sim}{X}' - \underset{\sim}{N}\,\underset{\sim}{k}^{-\delta}\,\underset{\sim}{Z}')\underset{\sim}{y} , \qquad (1.3.4)$$

and $\underset{\sim}{N} = \underset{\sim}{X}'\underset{\sim}{Z}$ is the incidence matrix of the design (Definition 1.2.2), $\underset{\sim}{r}^{\delta}$ is $\underset{\sim}{r}$ written as a diagonal matrix and $\underset{\sim}{k}^{-\delta}$ is the inverse of $\underset{\sim}{k}^{\delta}$. The $v \times v$ matrix $\underset{\sim}{C}$ is called the matrix of the reduced normal equations.

It follows that $\underset{\sim}{N}'\underset{\sim}{1} = \underset{\sim}{k}$ and $\underset{\sim}{N}\,\underset{\sim}{1} = \underset{\sim}{r}$, hence

$$\underset{\sim}{C}\,\underset{\sim}{1} = \underset{\sim}{0} . \qquad (1.3.5)$$

Throughout this thesis we will assume all block designs are connected, hence $\underset{\sim}{C}$ will be positive semidefinite with rank $v - 1$ (Raghavarao, 1971, page 49).

Let $\underset{\sim}{\rho}$ be a vector whose $i^{th}$ element is $\sqrt{r_i}$, and define

$$\underset{\sim}{A} = \underset{\sim}{\rho}^{-\delta}\underset{\sim}{C}\,\underset{\sim}{\rho}^{-\delta} . \qquad (1.3.6)$$

The matrix $\underset{\sim}{A}$ is very important in investigating the properties of a design, and in sections 1.4 and 2.2, $\underset{\sim}{A}$ is compared with other matrices that have previously been suggested.

Premultiplying (1.3.3) by $\underset{\sim}{\rho}^{-\delta}$ gives

$$\underset{\sim}{A}(\underset{\sim}{\rho}^{\delta}\hat{\underset{\sim}{\alpha}}) = (\underset{\sim}{\rho}^{-\delta}\underset{\sim}{Q}) , \qquad (1.3.7)$$

where

$$\underset{\sim}{A}\,\underset{\sim}{\rho} = \underset{\sim}{0} .$$

The matrix $\underset{\sim}{A}$ may be written in the spectral form

$$\underset{\sim}{A} = \sum_{u=0}^{v-1} e_u\, \underset{\sim}{\tau}_u\, \underset{\sim}{\tau}_u{}' , \qquad (1.3.8)$$

where $e_0 = 0$ and $\underset{\sim}{\tau}_0 = \frac{1}{\sqrt{n}}\underset{\sim}{\rho}$. The $e_i$ $(i = 1,2,\ldots,v-1)$ are called the canonical efficiency factors of the design and play a fundamental role in the structure of the design (James and Wilkinson, 1971).

The Moore-Penrose generalized inverse of $\underset{\sim}{A}$ is given by

$$\underset{\sim}{A}^{+} = \sum_{u=1}^{v-1} e_u^{-1} \underset{\sim}{\tau}_u \underset{\sim}{\tau}_u' \, . \qquad (1.3.9)$$

Thus the solution of the reduced normal equations becomes

$$\hat{\underset{\sim}{\alpha}} = (\underset{\sim}{\rho}^{-\delta} \underset{\sim}{A}^{+} \underset{\sim}{\rho}^{-\delta})\underset{\sim}{Q} , \qquad (1.3.10)$$

$$= \underset{\sim}{V} \underset{\sim}{Q} ,$$

where

$$\underset{\sim}{V} = \underset{\sim}{\rho}^{-\delta} \underset{\sim}{A}^{+} \underset{\sim}{\rho}^{-\delta} \, . \qquad (1.3.11)$$

Then

$$\underset{\sim}{V} \underset{\sim}{r} = \underset{\sim}{0} , \qquad (1.3.12)$$

and

$$\mathrm{Var}(\hat{\underset{\sim}{\alpha}}) = \underset{\sim}{V} \sigma^2 \, . \qquad (1.3.13)$$

## 1.4 Efficiency Factors

The importance of the canonical efficiency factors has been realized by Jones (1959) and Calinski (1971). Jones defines the matrix

$$\underset{\sim}{M} = \underset{\sim}{r}^{-\delta} (\underset{\sim}{N} \underset{\sim}{k}^{-\delta} \underset{\sim}{N}') , \qquad (1.4.1)$$

which has latent roots $1 - e_i$ $(i = 0, 1, \ldots, v-1)$.
Calinski considers the related matrix

$$\underset{\sim}{M}_0 = \underset{\sim}{M} - \frac{1}{n} \underset{\sim}{1} \underset{\sim}{r}' , \qquad (1.4.2)$$

with latent roots 0, and $1 - e_i$ $(i = 1,2,\ldots,v-1)$, as pointed out by Pearce, Calinski and Marshall (1974). These matrices are less convenient than $\underset{\sim}{A}$. Unlike $\underset{\sim}{A}$, they are in general not symmetric when the design is unequally replicated.

We now define an average canonical efficiency factor.

<u>Definition 1.4.1.</u> The harmonic mean canonical efficiency factor (h.m.c.e.f.), $\bar{\bar{E}}$ of a design is defined as the harmonic mean of the canonical efficiency factors, i.e.

$$\bar{\bar{E}} = \frac{v-1}{\sum_{u=1}^{v-1} e_u^{-1}} \quad . \tag{1.4.3}$$

For equireplicated designs, $\bar{\bar{E}}$ is identical to (i) the average efficiency factor, E given by Bose, Clatworthy and Shrikhande (1954), John, Wolock and David (1972) and (ii) the harmonic mean of the pairwise efficiency factors, $\mathcal{E}'$ given by Pearce (1970). In general, however, in unequally replicated designs $\bar{\bar{E}} \neq \mathcal{E}'$ but is a function of $\mathcal{E}$ defined by Pearce (1970) as

$$\mathcal{E} = \frac{v}{tr(\underset{\sim}{\rho}^{\delta} \underset{\sim}{\Omega} \underset{\sim}{\rho}^{\delta})} , \tag{1.4.4}$$

where

$$\underset{\sim}{\Omega} = \underset{\sim}{V} + \frac{1}{n} \underset{\sim}{J}_v . \tag{1.4.5}$$

Hence from (1.3.11),

$$\frac{v}{\mathcal{E}} = tr(\underset{\sim}{A}^{+}) + \frac{1}{n} tr(\underset{\sim}{\rho} \underset{\sim}{\rho}') . \tag{1.4.6}$$

But $tr(\underset{\sim}{A}^{+})$ is equal to the sum of the latent roots of $\underset{\sim}{A}^{+}$, which by (1.3.9) and (1.4.3) is equal to $\frac{v-1}{\bar{\bar{E}}}$. Hence

$$\frac{v}{\mathcal{E}} = \frac{v-1}{\bar{\bar{E}}} + 1 . \tag{1.4.7}$$

Theorem 1.4.1. The smallest canonical efficiency factor ($e_{min}$) is the minimum efficiency that any varietal contrast $\underset{\sim}{l}$ $(\underset{\sim}{l}'\underset{\sim}{1} = 0)$ can attain.

Proof. We compare the variance of a contrast $\underset{\sim}{l}$ with the randomized block situation, i.e.

$$\frac{\underset{\sim}{l}'\underset{\sim}{\rho}^{-\delta}\ \underset{\sim}{\rho}^{-\delta}\ \underset{\sim}{l}\ \sigma^2}{\underset{\sim}{l}'\ \underset{\sim}{V}\ \underset{\sim}{l}\ \sigma^2}\ , \qquad (1.4.8)$$

is the efficiency factor for the contrast $\underset{\sim}{l}$.

Put

$$\underset{\sim}{q} = \underset{\sim}{\rho}^{-\delta}\ \underset{\sim}{l}\ , \qquad (1.4.9)$$

$$\text{i.e.}\ \ \underset{\sim}{q}'\underset{\sim}{\rho} = 0\ . \qquad (1.4.10)$$

Then from (1.4.8), we want to minimize

$$\frac{\underset{\sim}{q}'\underset{\sim}{q}}{\underset{\sim}{q}'\ \underset{\sim}{A}^{+}\ \underset{\sim}{q}}\ , \qquad (1.4.11)$$

i.e. we want the inverse of the maximum latent root of $\underset{\sim}{A}^{+}$ which is $e_{min}$ (Rao, 1973, page 62)./

1.5 Dual Designs

Definition 1.5.1. The dual design of a block design with incidence matrix $\underset{\sim}{N}$, is the block design with incidence matrix $\underset{\sim}{N}'$./

Thus, the dual is obtained by interchanging the block and variety symbols in the original design.

Let $\underset{\sim}{\kappa}$ be a vector whose $j^{th}$ element is $\sqrt{k_j}$, and analagous to (1.3.6), let the $b \times b$ matrix $\underset{\sim}{B}$ for the dual design be given by

$$\underset{\sim}{B} = \underset{\sim}{I}_b - \underset{\sim}{D}'\underset{\sim}{D}, \tag{1.5.1}$$

where

$$\underset{\sim}{D} = \underset{\sim}{\rho}^{-\delta}\,\underset{\sim}{N}\,\underset{\sim}{\kappa}^{-\delta} . \tag{1.5.2}$$

Note that (1.3.6) could be written as

$$\underset{\sim}{A} = \underset{\sim}{I}_v - \underset{\sim}{D}\underset{\sim}{D}' . \tag{1.5.3}$$

Theorem 1.5.1. A block design and its dual design have the same non-unit canonical efficiency factors.

Proof. Since $\underset{\sim}{D}'\underset{\sim}{D}$ and $\underset{\sim}{D}\underset{\sim}{D}'$ have the same non-zero latent roots then from (1.5.1) and (1.5.3), $\underset{\sim}{A}$ and $\underset{\sim}{B}$ have the same non-unit latent roots. But the latent roots of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are the canonical efficiency factors of the design and its dual respectively./

In later chapters we will be particularly concerned with designs for variety trials; in this situation $v$ is usually larger than $b$. Hence Theorem 1.5.1 is very important since to find the canonical efficiency factors and $\bar{E}$ for the design, it is only necessary to find the latent roots of the smaller matrix $\underset{\sim}{B}$. The following result is also particularly useful when $v > b$ as it enables $\underset{\sim}{V}$ to be obtained from $\underset{\sim}{B}^+$.

Theorem 1.5.2. The Moore-Penrose generalized inverse of $\underset{\sim}{A}$ may be written in the form

$$\underset{\sim}{A}^+ = \underset{\sim}{I}_v + \underset{\sim}{D}\,\underset{\sim}{B}^+\underset{\sim}{D}' - \frac{1}{n}\underset{\sim}{\rho}\underset{\sim}{\rho}' . \tag{1.5.4}$$

Proof. Provided the design is connected, we may form the non-singular matrix

$$\underset{\sim}{A} + \frac{1}{n}\underset{\sim}{\rho}\underset{\sim}{\rho}' = \underset{\sim}{I}_v - (\underset{\sim}{D}\underset{\sim}{D}' - \frac{1}{n}\underset{\sim}{\rho}\underset{\sim}{\rho}') \tag{1.5.5}$$

$$= \underset{\sim}{I}_v - \underset{\sim}{G}, \text{ say.} \tag{1.5.6}$$

Hence

$$(\underset{\sim}{A} + \frac{1}{n}\underset{\sim\sim}{\rho\rho'})^{-1} = (\underset{\sim}{I}_v - \underset{\sim}{G})^{-1}. \qquad (1.5.7)$$

From (1.3.8), $\underset{\sim}{G}$ may be written in the spectral form

$$\underset{\sim}{G} = \sum_{u=1}^{v-1} (1 - e_u)\, \underset{\sim}{\tau}_u \underset{\sim}{\tau}_u'. \qquad (1.5.8)$$

Thus provided the design is connected, all the latent roots of $\underset{\sim}{G}$ are strictly less than 1, so we may expand the right hand side of (1.5.7) in a power series, i.e.

$$\begin{aligned}(\underset{\sim}{I}_v - \underset{\sim}{G})^{-1} &= \underset{\sim}{I}_v + \underset{\sim}{G} + \underset{\sim}{G}^2 + \ldots \\ &= \underset{\sim}{I}_v + \underset{\sim\sim}{DD'} + (\underset{\sim\sim}{DD'})^2 + \ldots, \ (\text{mod } \underset{\sim\sim}{\rho\rho'}) \\ &= \underset{\sim}{I}_v + \underset{\sim\sim}{DB}^+\underset{\sim}{D}', \ (\text{mod } \underset{\sim\sim}{\rho\rho'}). \qquad (1.5.9)\end{aligned}$$

Since the left hand side of (1.5.7) is equal to $\underset{\sim}{A}^+$ (mod $\underset{\sim\sim}{\rho\rho'}$), we have

$$\underset{\sim}{A}^+ = \underset{\sim}{I}_v + \underset{\sim\sim}{DB}^+\underset{\sim}{D}, \ (\text{mod } \underset{\sim\sim}{\rho\rho'}). \qquad (1.5.10)$$

Then (1.5.4) may be obtained since from (1.3.9), $\underset{\sim}{A}^+\underset{\sim}{\rho} = 0$./ Hence it may be more convenient to substitute (1.5.4) into (1.3.11) if $\underset{\sim}{B}^+$ is easier to obtain than $\underset{\sim}{A}^+$.

## 1.6 Recovery of Interblock Information

Instead of (1.3.1), we assume the model

$$\underset{\sim}{y} = u\underset{\sim}{1} + \underset{\sim}{X}\underset{\sim}{\alpha} + \underset{\sim}{\varepsilon}, \qquad (1.6.1)$$

where now $\underset{\sim}{\varepsilon}$ is an $n \times 1$ vector of random effects with $E(\underset{\sim}{\varepsilon}) = \underset{\sim}{0}$ and

$$E(\underset{\sim\sim}{\varepsilon\varepsilon'}) = \sigma^2(\underset{\sim}{I}_n + \phi\, \underset{\sim}{Z}\, \underset{\sim}{k}^{-\delta}\underset{\sim}{Z}'). \qquad (1.6.2)$$

Since we will be concerned with designs for variety trials, we assume a within-block correlation inversely proportional to block size as suggested by Patterson and Thompson (1971). The constants to be estimated in this model are $\mu$, $\underset{\sim}{\alpha}$, $\sigma^2$ and $\phi$.

If $\phi$ is known, then the vector $\hat{\alpha}$ which minimizes

$$(\underset{\sim}{y} - \underset{\sim}{X\alpha})'(\underset{\sim}{I}_n + \phi\, \underset{\sim}{Z}\, \underset{\sim}{k}^{-\delta} \underset{\sim}{Z}')^{-1}(\underset{\sim}{y} - \underset{\sim}{X\alpha}), \qquad (1.6.3)$$

estimates $\underset{\sim}{\alpha}$ efficiently. By differentiation, the equations for $\hat{\underset{\sim}{\alpha}}$ are of the form

$$\underset{\sim}{C}^* \hat{\underset{\sim}{\alpha}} = \underset{\sim}{Q}^* , \qquad (1.6.4)$$

where

$$\underset{\sim}{C}^* = (\underset{\sim}{r}^{\delta} - \frac{1}{\phi^{-1}+1} \underset{\sim}{N}\, \underset{\sim}{k}^{-\delta} \underset{\sim}{N}') , \qquad (1.6.5)$$

$$\underset{\sim}{Q}^* = (\underset{\sim}{X}' - \frac{1}{\phi^{-1}+1} \underset{\sim}{N}\, \underset{\sim}{k}^{-\delta} \underset{\sim}{Z}')\underset{\sim}{y} . \qquad (1.6.6)$$

Define

$$\underset{\sim}{A}^* = \underset{\sim}{\rho}^{-\delta} \underset{\sim}{C}^* \underset{\sim}{\rho}^{-\delta} . \qquad (1.6.7)$$

Premultiplying (1.6.4) by $\underset{\sim}{\rho}^{-\delta}$ gives

$$\underset{\sim}{A}^*(\underset{\sim}{\rho}^{\delta} \hat{\underset{\sim}{\alpha}}) = (\underset{\sim}{\rho}^{-\delta} \underset{\sim}{Q}^*) . \qquad (1.6.8)$$

From (1.3.6)

$$\underset{\sim}{A}^* = \frac{1}{\phi + 1}\, \underset{\sim}{I}_v + \frac{1}{\phi^{-1}+1} \underset{\sim}{A} , \qquad (1.6.9)$$

thus from (1.3.8), $\underset{\sim}{A}^*$ may be written in the spectral form

$$A^* = \sum_{u=0}^{v-1} e_u^* \, \underset{\sim}{\tau}_u \underset{\sim}{\tau}_u' , \qquad (1.6.10)$$

where

$$e_u^* = \frac{1}{\phi + 1} + \frac{1}{\phi^{-1}+1} e_u .$$

Hence $\underset{\sim}{A}^*$ is non-singular unless $\phi^{-1} = 0$, when (1.6.8) reduces to (1.3.7). However for the inversion of $\underset{\sim}{A}^*$, since $e_0^*$ may be very small, it is better to use an effective inverse, $\underset{\sim}{A}^e$, of $\underset{\sim}{A}^*$ given by

$$\underset{\sim}{A}^e = \sum_{u=1}^{v-1} (e_u^*)^{-1}\, \underset{\sim}{\tau}_u \underset{\sim}{\tau}_u' . \qquad (1.6.11)$$

An effective inverse does not necessarily satisfy the condition $\underset{\sim}{A}^*\underset{\sim}{A}^e\underset{\sim}{A}^* = \underset{\sim}{A}^*$ for a generalized inverse of $\underset{\sim}{A}^*$, but it allows the solution of a set of equations with redundant specification. Note that when $\phi^{-1} = 0$, $\underset{\sim}{A}^e = \underset{\sim}{A}^+$. Thus the solution of (1.6.8) becomes

$$\hat{\underset{\sim}{\alpha}} = (\underset{\sim}{\rho}^{-\delta}\underset{\sim}{A}^e\underset{\sim}{\rho}^{-\delta})\underset{\sim}{Q}^* , \qquad (1.6.12)$$
$$= \underset{\sim}{V}^*\underset{\sim}{Q}^* ,$$

where

$$\underset{\sim}{V}^* = \underset{\sim}{\rho}^{-\delta}\underset{\sim}{A}^e\underset{\sim}{\rho}^{-\delta} . \qquad (1.6.13)$$

Then

$$\underset{\sim}{V}^*\underset{\sim}{r} = \underset{\sim}{0} , \qquad (1.6.14)$$

and

$$\mathrm{Var}(\hat{\underset{\sim}{\alpha}}) = \underset{\sim}{V}^*\sigma^2 . \qquad (1.6.15)$$

<u>Example 1.6.1.</u> For a BIB design

$$\underset{\sim}{A} = \bar{\bar{E}}\,(\underset{\sim}{I}_v - \frac{1}{v}\underset{\sim}{J}_v) , \qquad (1.6.16)$$

i.e. all the canonical efficiency factors are equal to $\bar{\bar{E}}$. Hence from (1.6.9),

$$\underset{\sim}{A}^* = \frac{\phi^{-1} + \bar{\bar{E}}}{\phi^{-1} + 1}\underset{\sim}{I}_v - \frac{\bar{\bar{E}}}{v(\phi^{-1} + 1)}\underset{\sim}{J}_v . \qquad (1.6.17)$$

Then

$$\underset{\sim}{A}^e = \frac{\phi^{-1} + 1}{\phi^{-1} + \bar{\bar{E}}}\underset{\sim}{I}_v , \quad (\mathrm{mod}\ \underset{\sim}{J}_v) , \qquad (1.6.18)$$

and from (1.6.15), the variance for the comparison of two estimated variety means is

$$\frac{2\sigma^2}{r}\left(\frac{\phi^{-1} + 1}{\phi^{-1} + \bar{\bar{E}}}\right) . \qquad (1.6.19)$$

This is equivalent to the expression obtained by Cochran and Cox (1957, page 445)./

Finally, the following points should be noted:

(i) In this section we have assumed that $\phi$ is known. In practice $\phi$ must be estimated either by equating sums of squares in the analysis of variance table, to expectation (Cochran and Cox, 1957), or by some other method, (e.g. Patterson and Thompson, 1971). Errors in the estimation of $\phi$ will have an effect on $\underset{\sim}{V}^*$, usually small (Cochran and Cox, 1957, page 399).

(ii) The quantities $e_u^*$ given by (1.6.10) may be thought of as modified canonical efficiency factors. When $\phi^{-1} = 0$, $e_u^* = e_u$ $(u = 1,2,\ldots,v-1)$, and the combined analysis reduces to the intrablock analysis. When $\phi = 0$, $e_u^* = 1$ $(u = 1,2,\ldots,v-1)$ and the combined analysis reduces to a randomized blocks analysis (Cochran and Cox, 1957, page 384).

# CHAPTER 2

## CLASSIFICATION OF BLOCK DESIGNS

### 2.1 Summary

This chapter is concerned with defining a system of classification of block designs, much more general than that given by Pearce (1963), and including several structures which will be used extensively in later chapters.

A brief history of block designs is given in section 2.2, and the effect of increased computing power on the development of block designs is discussed.

In section 2.3, the theory in Appendix A on matrix structures is applied to the $\underset{\sim}{A}$ matrices of equiblock-sized equireplicated designs. A system of classification of block designs is given according to the structure in $\underset{\sim}{A}$ and it is shown that many of the common designs can be fitted into the classification.

Some designs, for example the square and rectangular lattices, may be defined according to the structure they exhibit in the dual matrix $\underset{\sim}{B}$ and these designs are investigated in section 2.4.

### 2.2 Development of Block Designs

After Yates (1936b) introduced BIB designs, it soon became apparent that they existed for only a limited number of parameter combinations. Hence PBIB(m) designs were formally defined by Bose and Nair (1939), which greatly increased the number of available designs.

At that time, desk machines were used for statistical analysis and hence the emphasis was on constructing designs allowing simple analysis, and in particular, with readily obtainable variance matrices. The conditions for an association scheme with m classes (Definition 1.2.5) guarantee that the variance matrix will have the same element down the lead diagonal and m distinct elements off-diagonal (Shah, 1959). Hence after BIB designs (which may be thought of as PBIB(1) designs), the main designs of interest were PBIB(2) designs.

Since then PBIB(2) designs have been studied in great detail and many interesting combinatorial problems have developed (Raghavarao, 1971). Extensive tables of PBIB(2) designs have been produced (Bose, Clatworthy and Shrikhande, 1954 and Clatworthy, 1973). However, due to the complex nature of higher-order association schemes, not much detailed work has been done on PBIB designs for $m > 2$.

Even with the addition of PBIB designs, there were still many parameter combinations for which there was no design tabulated. However, with the advent of powerful computers, there was not the same need for the variance matrix to have a simple structure, as the actual analysis of experiments became much easier. Thus attention was directed more to the $\underset{\sim}{\Omega}^{-1}$ matrix defined by Tocher (1952) as

$$\underset{\sim}{\Omega}^{-1} = \underset{\sim}{C} + \frac{1}{n}\underset{\sim}{r}\,\underset{\sim}{r}' \, . \qquad (2.2.1)$$

Pearce (1963) defined a classification of designs based on the structure of $\underset{\sim}{\Omega}^{-1}$. John (1966) and John, Wolock and David (1972) produced extremely useful tables of cyclic incomplete block (CIB) designs by concentrating on the off-diagonal elements of the concurrence matrix.

In the next sections the ideas of Pearce (1963) are extended by classifying a large range of designs according to the structure possessed by the matrix $\underset{\sim}{A}$.

## 2.3 Matrix Structures

Instead of basing the classification of block designs on the pattern of $\underset{\sim}{\Omega}^{-1}$, as done by Pearce (1963), we will use the matrix $\underset{\sim}{A}$ which was shown in section 1.4 to have an important interpretation in terms of the canonical efficiency factors. From (1.3.6) and (2.2.1)

$$\underset{\sim}{A} = \underset{\sim}{\rho}^{-\delta}\underset{\sim}{\Omega}^{-1}\underset{\sim}{\rho}^{-\delta} - \frac{1}{n}\underset{\sim}{\rho}\,\underset{\sim}{\rho}', \qquad (2.3.1)$$

so that for equireplicated designs, the pattern of $\underset{\sim}{\Omega}^{-1}$ and $\underset{\sim}{A}$ will be the same. If in addition the design is equiblock-sized then

$$\underset{\sim}{A} = \underset{\sim}{I}_v - \frac{1}{rk}\underset{\sim}{N}\,\underset{\sim}{N}' . \qquad (2.3.2)$$

Hence the pattern of $\underset{\sim}{A}$ depends on the concurrence matrix, $\underset{\sim}{N}\,\underset{\sim}{N}'$ (Definition 1.2.3). For the rest of this section we will be concerned only with equiblock-sized, equireplicated designs. Thus (1.3.11) becomes

$$\underset{\sim}{V} = \frac{1}{r}\underset{\sim}{A}^{+} . \qquad (2.3.3)$$

Appendix A contains the definitions and properties of several matrix structures based mainly on the circulant matrix (Definition A.2.2 ). We shall now describe how these structures can be applied to the $\underset{\sim}{A}$ matrices of various block designs, and show that Appendix A includes many of the structures given by

Pearce (1963). If the $\underset{\sim}{A}$ matrix of a design has a certain structure then we will say that the design also has that structure. However in adopting this convention it is important to note that a permutation of the variety symbols may alter the structure possessed by the $\underset{\sim}{A}$ matrix and hence by the design.

2.3.1. C Structure (Definition A.2.2).

This structure is exhibited by CIB designs (John, 1966). These designs are obtained by cyclically developing some number of initial blocks, modulo $v$.

Example 2.3.1. For $v = b = 6$, $r = k = 3$ consider the CIB design obtained from the initial block $(0,1,2)$ and given in Table 2.3.1(i). Here

$$\underset{\sim}{A} = \underset{\sim}{I}_6 - \frac{1}{9}\underset{\sim}{N}\,\underset{\sim}{N}' ,$$

where

$$\underset{\sim}{N}\,\underset{\sim}{N}' = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \end{bmatrix} .$$

Hence $\underset{\sim}{A}$ is of the form (A.2.1) with $\theta_0 = \frac{6}{9}$, $\theta_1 = \theta_5 = -\frac{2}{9}$, $\theta_2 = \theta_4 = -\frac{1}{9}$, $\theta_3 = 0$./

From (2.3.3) and Theorem A.2.3, it follows that if $\underset{\sim}{A}$ has C structure then $\underset{\sim}{V}$ will also have C structure. Pearce (1963) has called designs with C structure, Type C designs (see Table 2.3.2).

2.3.2. BC Structure (Definition A.3.1)

This structure will be used extensively in resolvable designs for variety trials (Chapters 3 - 5).

Example 2.3.2. For $v = 12$, $r = 3$, $b = 9$, $k = 4$ consider the resolvable design given in Table 2.3.1(ii). Here $\underset{\sim}{A}$ is of the form (A.3.1) with $t = 4$, $s = 3$ and

$$\underset{\sim}{\theta}_0 = \frac{1}{12}\begin{bmatrix} 9 & -2 & -2 & -1 \\ -2 & 9 & -1 & -1 \\ -2 & -1 & 9 & -1 \\ -1 & -1 & -1 & 9 \end{bmatrix}, \quad \underset{\sim}{\theta}_1 = \frac{1}{12}\begin{bmatrix} 0 & 0 & 0 & -2 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix},$$

$$\underset{\sim}{\theta}_2 = \frac{1}{12}\begin{bmatrix} 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ -2 & -1 & -1 & 0 \end{bmatrix}. /$$

From (2.3.3) and Theorem A.3.4 it follows that if $\underset{\sim}{A}$ has BC structure then $\underset{\sim}{V}$ will also have BC structure. The Type E designs defined by Pearce (1963) are special cases of BC structure with $\underset{\sim}{\theta}_1 = \underset{\sim}{\theta}_2 = \dots = \underset{\sim}{\theta}_{s-1}$ (see Table 2.3.2).

2.3.3. GC(n) Structure (Definition A.4.1) and BGC(n) Structure (Definition A.5.1)

These structures are generalizations of C structure and BC structure respectively. They are very useful for confounded factorial experiments. For example, John and Smith (1972) consider BGC(2) structure, John (1973) considers GC(n) structure and Cotter, John and Smith (1973) consider BGC(n) structure.

Example 2.3.3. For v = 9, r = 2, b = 6, k = 3 consider the resolvable design given in Table 2.3.1(iii). Here

$$\underset{\sim}{A} = \underset{\sim}{I}_9 - \frac{1}{6}\underset{\sim}{N}\,\underset{\sim}{N}',$$

where

$$\underset{\sim}{N}\,\underset{\sim}{N}' = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

Hence $\underset{\sim}{A}$ is of the form (A.4.1) with n = 2, i.e. GC(2) structure, and $s_1 = s_2 = 3$, $\theta_{00} = \frac{4}{6}$, $\theta_{10} = \theta_{20} = \theta_{21} = \theta_{12} = -\frac{1}{6}$, $\theta_{01} = \theta_{11} = \theta_{02} = \theta_{22} = 0$. /

From (2.3.3) and Theorems A.4.3 and A.5.4, it follows that if $\underset{\sim}{A}$ has GC(n) structure or BGC(n) structure, then $\underset{\sim}{V}$ will also have respectively GC(n) structure or BGC(n) structure.

2.3.4. DC Structure (Definition A.6.2)

Designs with this structure provide a useful extension to CIB designs. Given an initial block, suppose we cyclically add 1 to each even element of the initial block (as is done with CIB designs) but cyclically subtract 1 from each odd element of the initial block, then we obtain a design with DC structure.

Example 2.3.4. For $v = b = 6$, $r = k = 3$ and with the initial block (0, 1, 5) we get the design given in Table 2.3.1(iv). Here

$$\underset{\sim}{A} = \underset{\sim}{I}_6 - \frac{1}{9}\underset{\sim}{N}\,\underset{\sim}{N}' ,$$

where

$$\underset{\sim}{N}\,\underset{\sim}{N}' = \begin{bmatrix} 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 2 & 1 & 2 & 3 & 0 \\ 2 & 1 & 2 & 1 & 0 & 3 \end{bmatrix} .$$

Hence $\underset{\sim}{A}$ is of the form (A.6.2) with $\theta_{00} = \frac{6}{9}$, $\theta_{02} = \theta_{04} = -\frac{1}{9}$, $\theta_{01} = \theta_{03} = \theta_{05} = 0$, $\theta_{11} = \theta_{15} = -\frac{2}{9}$, $\theta_{10} = \theta_{12} = \theta_{13} = \theta_{14} = 0$./
Note that designs of this type are binary only if $v$ is even. When (A.6.3) is satisfied, then from (2.3.3) and Theorem A.6.3 it follows that if $\underset{\sim}{A}$ has DC structure then $\underset{\sim}{V}$ will also have DC structure.

2.3.5. BDC Structure (Definition A.7.1)

This structure will be used in Chapters 3 - 5 as a supplement for BC structure, in cases where the latter is not suitable.

Example 2.3.5. For $v = b = 12$, $r = k = 3$ consider the resolvable design given in Table 2.3.1(v). Here $\underset{\sim}{A}$ is of the form (A.7.1) with $t = 3$, $s = 4$ and

$$\underset{\sim}{\theta}_{00} = \frac{1}{9}\begin{bmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{bmatrix}, \quad \underset{\sim}{\theta}_{02} = \frac{1}{9}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}, \quad \underset{\sim}{\theta}_{01} = \underset{\sim}{\theta}_{03} = \underset{\sim}{0} ,$$

$$\underset{\sim}{\theta}_{11} = \frac{1}{9}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \underset{\sim}{\theta}_{13} = \frac{1}{9}\begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \underset{\sim}{\theta}_{10} = \underset{\sim}{\theta}_{12} = \underset{\sim}{0} \;./$$

When (A.7.2) is satisfied, then from (2.3.3) and Theorem A.7.3 it follows that if $\underset{\sim}{A}$ has BDC structure then $\underset{\sim}{V}$ will also have BDC structure.

2.3.6. <u>BF(n) Structure</u> (Definition A.8.1)

<u>Example 2.3.6.</u> For $v = 9$, $r = 2$, $b = 6$, $k = 3$ consider the resolvable design given in Table 2.3.1(vi). Here

$$\underset{\sim}{A} = \underset{\sim}{I}_9 - \frac{1}{6}\underset{\sim}{N}\,\underset{\sim}{N}' ,$$

where

$$\underset{\sim}{N}\,\underset{\sim}{N}' = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \end{bmatrix} .$$

By Definition A.8.1, $\underset{\sim}{A}$ has BF(2) structure with $s_1 = s_2 = 3$ and $\theta_{00} = \frac{4}{6}$, $\theta_{01} = \theta_{10} = 0$, $\theta_{11} = -\frac{1}{6}$ ./

BF(n) structure is exhibited by n factor balanced factorial experiments (Raghavarao, 1971, page 257) as well as by many of the structures defined by Pearce (1963) and given in Table 2.3.2.

Note that the examples in this section have been chosen to illustrate the various structures as best as possible for small $v$, and in this situation particularly, it is likely that a permutation of the variety symbols in one structure will produce another structure. For example, by applying the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 5 & 3 & 4 & 7 & 8 & 6 \end{pmatrix}$$

we may convert the design with GC(2) structure given in Table 2.3.1(iii), into the design with BF(2) structure given in Table 2.3.1(vi). However this will not be possible in general.

2.3.7. <u>OF(n) Structure</u> (Definition A.9.1)

This structure occurs when a complete set of latent vectors for $\underset{\sim}{A}$ can be identified as a complete set of contrasts for a factorial design with $n$ factors.

<u>Example 2.3.7.</u> Consider the design with BF(2) structure given in Table 2.3.1(vi). A complete set of latent vectors for $\underset{\sim}{A}$ is given by the normalized columns of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 0 & -2 & 1 & -1 & 0 & 0 & 2 & -2 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 2 & 0 & 2 & 0 & -2 \\ 1 & 0 & -2 & 0 & 2 & 0 & 0 & 0 & 4 \\ 1 & -1 & 1 & 0 & 2 & 0 & -2 & 0 & -2 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 0 & -2 & -1 & -1 & 0 & 0 & -2 & -2 \\ 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

But with the mapping

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 00 & 01 & 02 & 10 & 11 & 12 & 20 & 21 & 22 \end{pmatrix},$$

the columns of the above matrix can be recognized as a complete set of contrasts for a $3 \times 3$ factorial design, hence the design has OF(2) structure. Alternatively the result is given directly by Theorem A.9.2./

2.3.8. P Structure (Definition A.10.1)

This structure corresponds to that given by Pearce (1963) for Type P designs (see Table 2.3.2), the only difference being that we define the structure on $\underset{\sim}{A}$ rather than on $\underset{\sim}{\Omega}^{-1}$. All the examples considered in this section have P structure except the design with BC structure given in Table 2.3.1(ii).

It does not necessarily follow that if $\underset{\sim}{A}$ has P structure then $\underset{\sim}{V}$ will also have P structure. For example, for most of the designs which will be constructed in Chapters 3 - 4, $\underset{\sim}{V}$ will not have P structure. However it follows from Definition 1.2.6 that for all PBIB designs, $\underset{\sim}{A}$ and $\underset{\sim}{V}$ both have P structure.

Table 2.3.1

(i) A design with C structure

| Block | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| | 1 | 2 | 3 | 4 | 5 | 0 |
| | 2 | 3 | 4 | 5 | 0 | 1 |

(ii) A resolvable design with BC structure

| Replicate | | 1 | | | 2 | | | 3 | |
|---|---|---|---|---|---|---|---|---|---|
| Block | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | 0 | 1 | 2 | 0 | 1 | 2 | 0 | 1 | 2 |
| | 3 | 4 | 5 | 3 | 4 | 5 | 5 | 3 | 4 |
| | 6 | 7 | 8 | 8 | 6 | 7 | 6 | 7 | 8 |
| | 9 | 10 | 11 | 10 | 11 | 9 | 10 | 11 | 9 |

(iii) A resolvable design with GC(2) structure

| Replicate | | 1 | | | 2 | |
|---|---|---|---|---|---|---|
| Block | 1 | 2 | 3 | 4 | 5 | 6 |
| | 0 | 1 | 2 | 0 | 1 | 2 |
| | 3 | 4 | 5 | 5 | 3 | 4 |
| | 6 | 7 | 8 | 7 | 8 | 6 |

(iv) A design with DC structure

| Block | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| | 1 | 0 | 5 | 4 | 3 | 2 |
| | 5 | 4 | 3 | 2 | 1 | 0 |

Table 2.3.1 (continued)

(v) A resolvable design with BDC structure

| Replicate | 1 | | | | 2 | | | | 3 | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Block | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| | 0 | 1 | 2 | 3 | 1 | 0 | 3 | 2 | 2 | 3 | 0 | 1 |
| | 4 | 5 | 6 | 7 | 6 | 7 | 4 | 5 | 7 | 6 | 5 | 4 |
| | 8 | 9 | 10 | 11 | 11 | 10 | 9 | 8 | 9 | 8 | 11 | 10 |

(vi) A resolvable design with BF(2) structure

| Replicate | 1 | | | 2 | | |
|---|---|---|---|---|---|---|
| Block | 1 | 2 | 3 | 4 | 5 | 6 |
| | 0 | 1 | 2 | 0 | 1 | 2 |
| | 4 | 5 | 3 | 5 | 3 | 4 |
| | 8 | 6 | 7 | 7 | 8 | 6 |

Table 2.3.2

Comparison of present classification with that of Pearce (1963)

| Pearce classification | Present classification |
| --- | --- |
| Type C | C structure |
| Type E | BC structure with $\underset{\sim}{\theta}_1 = \underset{\sim}{\theta}_2 = \ldots = \underset{\sim}{\theta}_{s-1}$ |
| Type O | BF(1) structure with $\theta_0 = 1 - \frac{1}{v}, \theta_1 = -\frac{1}{v}$ |
| Type T | BF(1) structure |
| Type G | BF(2) structure with $\theta_{10} = \theta_{11}$ |
| Type F | BF(2) structure |
| Type P | P structure |

## 2.4 Dual Matrix Structures

Sometimes a block design will possess a simpler structure in its dual matrix $\underset{\sim}{B}$ (given by (1.5.1)) than in $\underset{\sim}{A}$.

Example 2.4.1. For the design given in Table 2.3.1(ii) we can show that $\underset{\sim}{B}$ has BF(2) structure with $s_1 = s_2 = 3$ and $\theta_{00} = \frac{8}{12}$, $\theta_{01} = 0$, $\theta_{10} = -\frac{2}{12}$, $\theta_{11} = -\frac{1}{12}$.
Thus although $\underset{\sim}{A}$ does not even have P structure, $\underset{\sim}{B}$ is highly structured./

Some well-known designs may be defined according to the structure they exhibit in $\underset{\sim}{B}$. For example, we consider the square and rectangular lattice designs (Cochran and Cox, 1957, chapter 10).

### 2.4.1. Square Lattice Designs

These are equireplicated equiblock-sized resolvable designs with parameters

$$v = s^2, \quad b = rs, \quad k = s \; . \qquad (2.4.1)$$

More common cases are the simple square lattice ($r = 2$), triple square lattice ($r = 3$) and balanced square lattice ($r = s + 1$).

Definition 2.4.1. If for a resolvable design with parameters (2.4.1), $\underset{\sim}{B}$ can be written in a form with BF(2) structure, where $s_1 = r$, $s_2 = s$ and $\theta_{00} = 1 - \frac{1}{r}$, $\theta_{01} = 0$, $\theta_{10} = \theta_{11} = 1 - \frac{1}{rk}$, then the design is a square lattice design./
For example, the design given in Table 2.3.1(iii) is the $3^2$ simple square lattice design.

2.4.2. Rectangular Lattice Designs

These are equireplicated, equiblock-sized resolvable designs with parameters

$$v = (s-1)s, \quad b = rs, \quad k = s-1 . \qquad (2.4.2)$$

These designs were first introduced by Harshbarger (1947) to provide useful designs for variety numbers between the allowable values for square lattices. Common cases are simple $(r = 2)$ and triple $(r = 3)$ rectangular lattices.

Definition 2.4.2. If for a resolvable design with parameters (2.4.2), $\underset{\sim}{B}$ can be written in a form with BF(2) structure where $s_1 = r$, $s_2 = s$ and $\theta_{00} = 1 - \frac{1}{r}$, $\theta_{01} = \theta_{10} = 0$, $\theta_{11} = 1 - \frac{1}{rk}$, then the design is a rectangular lattice design./

For example, we can show that the design given in Table 2.3.1(v) is the $3 \times 4$ triple rectangular lattice design.

Note that we could have written (2.4.1) and (2.4.2) in the form

$$v = ks, \quad b = rs , \qquad (2.4.3)$$

for $k = s,\ s - 1$ respectively.

The study of resolvable designs with parameters (2.4.3) but for other values of $k \geq r$ (i.e. $v \geq b$) will be the subject of Chapters 3 - 5 of this thesis.

# CHAPTER 3

## A NEW CLASS OF RESOLVABLE BLOCK DESIGNS

### 3.1 Introduction and Summary

For the remainder of this thesis, we will be concerned mainly with the construction of resolvable block designs for use as variety trial designs. The number of varieties is usually large, but the number of replications of each variety is small. We will mainly be concerned with the ranges $b \leq v \leq 100$ and $2 \leq r \leq 4$.

The square and rectangular lattice designs are well known resolvable designs which may conveniently be used as variety trial designs, and by writing their parameters in the form (2.4.3), we may obtain a possible extension for $v \geq b$, i.e.

$$\begin{aligned} v &= ks, \\ b &= rs, \qquad\qquad (3.1.1) \\ k &\geq r. \end{aligned}$$

In this chapter, a new class of resolvable designs with parameters satisfying (3.1.1) is defined. These designs are obtained by cyclic development (mod s) of the columns of a $k \times r$ array; in section 3.2 $\underset{\sim}{\alpha}$-designs are defined and it is shown that these designs have BC structure. The construction of $\underset{\sim}{\alpha}$-designs is modified in section 3.3 to produce resolvable designs (called $\underset{\sim}{\beta}$-designs) with BDC structure.

A method for the speedy calculation of the h.m.c.e.f. of $\underset{\sim}{\alpha}$-designs from the elements of $\underset{\sim}{\alpha}$ is given in section 3.4, and in section 3.5, the reduced form for an array $\underset{\sim}{\alpha}$ is defined.

In section 3.6, a useful computational form is given for the variance matrix, and a procedure for conveniently representing the pairwise variances of $\underset{\sim}{\alpha}$-designs is outlined.

Finally, in section 3.7, the problem of allocation of control varieties to the symbols of $\underset{\sim}{\alpha}$-designs is considered.

## 3.2 Generation of $\underset{\sim}{\alpha}$-Designs

Let $\underset{\sim}{\alpha}$ be a $k \times r$ array whose $(l,m)^{th}$ element is $a_{lm}$, where $0 \leq a_{lm} < s$ for some positive integer $s$, $(l = 0, 1, \ldots, k-1;\ m = 0, 1, \ldots, r-1)$. We construct a resolvable block design from $\underset{\sim}{\alpha}$ in the following way.

__Construction 3.2.1.__ (i) From each column of $\underset{\sim}{\alpha}$ we produce $s - 1$ additional columns by cyclically adding 1 to each element of the column and reducing modulo $s$. This will produce a $k \times rs$ array, say $\underset{\sim}{\alpha}^*$.

(ii) To the elements in the $l^{th}$ row of $\underset{\sim}{\alpha}^*$ we add the integer $ls$ $(l = 0, 1, \ldots, k-1)$./ By interpreting the columns of the resulting array as blocks, we obtain a block design for $ks$ varieties, $rs$ blocks and block size $k$. Furthermore the design is resolvable since each of the $r$ columns of $\underset{\sim}{\alpha}$ has generated a complete replicate of the design.

__Example 3.2.1.__ For $r = 3$, $k = 4$, $s = 3$ let $\underset{\sim}{\alpha}$ be given by

$$\underset{\sim}{\alpha} = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{matrix}\ , \qquad (3.2.1)$$

then

$$\underset{\sim}{\alpha}^* = \begin{matrix} 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 1 & 2 & 0 \end{matrix} .$$

The resulting block design for $v = 12$, $r = 3$, $b = 9$, $k = 4$ is given in Table 2.3.1 (ii)./

<u>Definition 3.2.1.</u> A resolvable equiblock-sized design obtained from an array $\underset{\sim}{\alpha}$ using Construction 3.2.1 is called an $\underset{\sim}{\alpha}$-design./

We now obtain the concurrence matrix of an $\underset{\sim}{\alpha}$-design. For particular $l, l'$ ($l \neq l'$; $l, l' = 0, 1, \ldots, k-1$), let $\underset{\sim}{\mu}_{ll'}$ be the vector whose $h^{th}$ element ($h = 0, 1, \ldots, s-1$) is the number of times $h$ appears among the $r$ differences of the form

$$a_{l'm} \dot{-} a_{lm} , \qquad (3.2.2)$$

$$(m = 0, 1, \ldots, r-1).$$

Suppose we let $(\underset{\sim}{\mu}'_{ll'})$ represent the $s \times s$ circulant matrix with first row $\underset{\sim}{\mu}'_{ll'}$, then we may show the following result.

<u>Theorem 3.2.1.</u> The concurrence matrix of an $\underset{\sim}{\alpha}$-design is given by

$$\underset{\sim}{N}\underset{\sim}{N}' = \begin{bmatrix} r\underset{\sim}{I}_s & (\underset{\sim}{\mu}'_{01}) & \cdots & (\underset{\sim}{\mu}'_{0k-1}) \\ (\underset{\sim}{\mu}'_{10}) & r\underset{\sim}{I}_s & & \\ \vdots & & \ddots & \\ (\underset{\sim}{\mu}'_{k-10}) & & & r\underset{\sim}{I}_s \end{bmatrix} , \qquad (3.2.4)$$

and hence an $\underset{\sim}{\alpha}$-design has BC structure./

<u>Example 3.2.2.</u> For $r = 3$, $k = 4$, $s = 3$ let $\underset{\sim}{\alpha}$ be given by (3.2.1). Then $\underset{\sim}{\mu}_{01} = \underset{\sim}{\mu}_{02} = (2,0,1)'$, $\underset{\sim}{\mu}_{03} = (1,2,0)'$, and $\underset{\sim}{\mu}_{12} = \underset{\sim}{\mu}_{13} = \underset{\sim}{\mu}_{23} = (1,1,1)'$. Hence for the $\underset{\sim}{\alpha}$-design obtained from (3.2.1),

$$\underset{\sim}{N}\underset{\sim}{N}' = \begin{bmatrix} 3\underset{\sim}{I}_3 & (2\ 0\ 1) & (2\ 0\ 1) & (1\ 2\ 0) \\ (2\ 1\ 0) & 3\underset{\sim}{I}_3 & (1\ 1\ 1) & (1\ 1\ 1) \\ (2\ 1\ 0) & (1\ 1\ 1) & 3\underset{\sim}{I}_3 & (1\ 1\ 1) \\ (1\ 0\ 2) & (1\ 1\ 1) & (1\ 1\ 1) & 3\underset{\sim}{I}_3 \end{bmatrix} . /$$

We define the dual array of $\underset{\sim}{\alpha}$ as follows:

<u>Definition 3.2.2.</u> Let $\underset{\sim}{\alpha}'$ be an $r \times k$ array whose $(m,l)^{th}$ element, $a'_{ml} = \dot{-}a_{lm}$ $(l = 0, 1, \ldots, k-1;\ m = 0, 1, \ldots, r-1)$, then $\underset{\sim}{\alpha}'$ is called the dual array of $\underset{\sim}{\alpha}$. /

For example for the array $\underset{\sim}{\alpha}$ given by (3.2.1),

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{matrix} . \qquad (3.2.5)$$

<u>Theorem 3.2.2.</u> The dual design of an $\underset{\sim}{\alpha}$-design is the $\underset{\sim}{\alpha}'$-design obtained from the dual array $\underset{\sim}{\alpha}'$.

<u>Proof.</u> The element $ls + h$ in the $\underset{\sim}{\alpha}$-design yields the $ls + h^{th}$ block of the dual design $(l = 0, 1, \ldots, k-1;$ $h = 0, 1, \ldots, s-1)$. Hence the elements of the $ls + h^{th}$ block of the dual design are $ms + (h \dot{-} a_{lm})$, $(m = 0,1,\ldots,r-1)$, which by Construction 3.2.1 are the elements of the design obtained from $\underset{\sim}{\alpha}'$. / Thus if $\underset{\sim}{\eta}_{mm'}$ is the vector whose $h^{th}$ element $(h = 0, 1, \ldots, s-1)$ is the number of times $h$ appears among the $k$ differences of the form

$$a'_{m'l} \doteq a'_{ml} \; (= a_{lm} \doteq a_{lm'}), \qquad (3.2.6)$$

$$(l = 0, 1, \ldots, k-1),$$

we have the following corollary.

Corollary 3.2.3. For an $\underset{\sim}{\alpha}$-design,

$$\underset{\sim}{N}'\underset{\sim}{N} = \begin{bmatrix} k\underset{\sim}{I}_s & (\underset{\sim}{\eta}'_{01}) & \cdots & (\underset{\sim}{\eta}'_{0r-1}) \\ (\underset{\sim}{\eta}'_{10}) & k\underset{\sim}{I}_s & & \\ \vdots & & \ddots & \\ (\underset{\sim}{\eta}'_{r-10}) & & & k\underset{\sim}{I}_s \end{bmatrix}, \qquad (3.2.7)$$

and hence the dual design of an $\underset{\sim}{\alpha}$-design has BC structure. /

## 3.3 Generation of $\underset{\sim}{\beta}$-Designs

In this section we give a modified form of Construction 3.2.1 which for certain parameter values, generates more desirable resolvable designs than $\underset{\sim}{\alpha}$-designs.

Let $\underset{\sim}{\beta}$ be a $k \times r$ array whose $(l,m)^{th}$ element is $b_{lm}$, where $0 \leq b_{lm} < s$ for some positive integer $s$,

$$(l = 0, 1, \ldots, k-1; \; m = 0, 1, \ldots, r-1).$$

We construct a resolvable block design from $\underset{\sim}{\beta}$ in the following way.

Construction 3.3.1. (i) From each column of $\underset{\sim}{\beta}$ we produce $s-1$ additional columns by cyclically adding 1 to each even element of the column and cyclically subtracting 1 from each odd element of the column, all operations being carried out modulo $s$. This will produce a $k \times rs$ array, say $\underset{\sim}{\beta}^*$.

(ii) To all the elements in the $l^{th}$ row of $\underset{\sim}{\beta}^*$ we add the integer $ls$ $(l = 0,1,\dots,k-1)$./

By interpreting the columns of the resulting array as blocks we obtain a resolvable block design for $ks$ varieties, $rs$ blocks and block size $k$.

<u>Example 3.3.1.</u> For $r = 3$, $k = 3$, $s = 4$, let $\underset{\sim}{\beta}$ be given by

$$\underset{\sim}{\beta} = \begin{matrix} 0 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 3 & 1 \end{matrix} \quad , \qquad (3.3.1)$$

then

$$\underset{\sim}{\beta}^* = \begin{matrix} 0 & 1 & 2 & 3 & \quad 1 & 0 & 3 & 2 & \quad 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 & \quad 2 & 3 & 0 & 1 & \quad 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & \quad 3 & 2 & 1 & 0 & \quad 1 & 0 & 3 & 2 \end{matrix} \ ,$$

i.e. we cyclically subtract 1 from the odd elements 1 and 3 in (3.3.1). The resulting block design for $v = 12$, $r = 3$, $b = 12$, $k = 3$ is given in Table 2.3.1(v)./

<u>Definition 3.3.1.</u> A resolvable equiblock-sized design obtained from an array $\underset{\sim}{\beta}$ using Construction 3.3.1 is called a $\underset{\sim}{\beta}$-design./

Analagous to Theorem 3.2.1, we may prove the following result.

<u>Theorem 3.3.1.</u> A $\underset{\sim}{\beta}$-design has BDC structure./

We define the dual array of $\underset{\sim}{\beta}$ as follows:

<u>Definition 3.3.2.</u> Let $\underset{\sim}{\beta}'$ be an $r \times k$ array whose $(m,l)^{th}$ element, $b'_{ml} = b_{lm}$, if $b_{lm}$ is odd, and

$$= \dot{-}b_{lm}, \text{ if } b_{lm} \text{ is even},$$

$$(l = 0,1,\dots,k-1\ ;\ m = 0,1,\dots,r-1)\ ,$$

then $\underset{\sim}{\beta}'$ is called the dual array of $\underset{\sim}{\beta}$./

Example 3.3.2. For $r = 3$, $k = 7$, $s = 8$ let $\underset{\sim}{\beta}$ be given by

$$\underset{\sim}{\beta} = \begin{matrix} 0 & 1 & 4 \\ 0 & 6 & 5 \\ 0 & 3 & 2 \\ 0 & 4 & 7 \\ 0 & 5 & 3 \\ 0 & 2 & 6 \\ 0 & 7 & 1 \end{matrix}$$

then

$$\underset{\sim}{\beta}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 3 & 2 & 1 \end{matrix} \ ./$$

Analagous to Theorem 3.2.2, we may prove the following result.

Theorem 3.3.2. The dual design of a $\underset{\sim}{\beta}$-design is the $\underset{\sim}{\beta}'$-design obtained from the dual array $\underset{\sim}{\beta}'$ ./

Corollary 3.3.3. The dual design of a $\underset{\sim}{\beta}$-design has BDC structure./

We will not consider $\underset{\sim}{\beta}$-designs in detail in this thesis, but they will be used in certain cases to supplement the $\underset{\sim}{\alpha}$-designs.

## 3.4 Efficiency Factors of $\underset{\sim}{\alpha}$-Designs

In constructing their tables of CIB designs, John, Wolock and David (1972) chose, for each set of parameters $v$, $k$ and $r$, the CIB design with the highest $\bar{E}$. We will also use this criterion for choosing $\underset{\sim}{\alpha}$-designs. This will involve a comparison of $\bar{E}$ for a large number of $\underset{\sim}{\alpha}$-designs, so that an efficient way of calculating $\bar{E}$ is needed. In this section we show that $\bar{E}$ can be obtained directly from the elements of the array $\underset{\sim}{\alpha}$.

We assume $v \geq b$, and hence by virtue of Theorem 1.5.1, it is easier to obtain the canonical efficiency factors of the

$\alpha$'-design, which by Corollary 3.2.3, has BC structure. Hence, let $\underset{\sim}{B}$ be given by (A.3.1) with $t = r$. By Theorem A.3.1, the latent roots of $\underset{\sim}{B}$ are given by the solution of the determinantal equations

$$\det(\underset{\sim}{\theta}^*_u - \xi \underset{\sim}{I}_r) = 0, \qquad (3.4.1)$$

$$(u = 0, 1, \ldots, s-1),$$

where from (3.2.7),

$$(\underset{\sim}{\theta}^*_u)^{lm} = 1 - \frac{1}{r}, \text{ if } l = m,$$
$$= -\frac{1}{rk} \sum_{h=0}^{s-1} \omega_u^h (\underset{\sim}{\eta}_{lm})^h, \text{ otherwise}. \qquad (3.4.2)$$

Hence when $u = 0$, (3.4.1) may be solved directly since

$$\underset{\sim}{\theta}_0^* = \underset{\sim}{I}_r - \frac{1}{r}\underset{\sim}{J}_r, \qquad (3.4.3)$$

which has latent roots

$$\xi_{00} = 0,$$

and $\qquad (3.4.4)$

$$\xi_{0f} = 1,$$

$$(f = 1, \ldots, r-1).$$

Now from (1.4.3), (3.4.4) and Theorem 1.5.1 we have

$$\bar{\bar{E}} = \frac{ks-1}{(k-r)s + r-1 + \Sigma}, \qquad (3.4.5)$$

where

$$\Sigma = \sum_{f=0}^{r-1} \sum_{u=1}^{s-1} \frac{1}{\xi_{uf}}, \qquad (3.4.6)$$

and if we let $\bar{\bar{E}}'$ be the h.m.c.e.f. of the $\alpha$'-design, then

$$\bar{\bar{E}}' = \frac{rs-1}{r-1 + \Sigma}. \qquad (3.4.7)$$

Define

$$\xi_{u.} = \frac{1}{\sum_{f=0}^{r-1} \frac{1}{\xi_{uf}}} , \qquad (3.4.8)$$

$$(u = 1, 2, \ldots, s-1),$$

then

$$\Sigma = \sum_{u=1}^{s-1} \frac{1}{\xi_{u.}} . \qquad (3.4.9)$$

But

$$\frac{1}{\xi_{u.}} = \frac{-\text{coefficient of } \xi \text{ in the } u^{th} \text{ characteristic equation}}{\text{constant term in the } u^{th} \text{ characteristic equation}} , \qquad (3.4.10)$$

$$(u = 1, 2, \ldots, s-1) .$$

Thus $\bar{E}$ can be calculated directly from the characteristic equations (3.4.1) without having to calculate the latent roots of $\underset{\sim}{B}$. In Appendix B, the equations (3.4.1) are simplified for the cases $r = 2, 3, 4$ and explicit expressions are obtained for (3.4.10) in terms of functions of the elements of $\underset{\sim}{\alpha}$. They provide an effective procedure for computing $\bar{E}$ for $\underset{\sim}{\alpha}$-designs when $r = 2, 3, 4$.

## 3.5 A Reduced Form for $\underset{\sim}{\alpha}$

Definition 3.5.1. Two arrays, say $\underset{\sim}{\alpha}_1$ and $\underset{\sim}{\alpha}_2$, with the same parameters k, r and s are said to be equivalent ($\underset{\sim}{\alpha}_1 \equiv \underset{\sim}{\alpha}_2$) if the $\underset{\sim}{\alpha}$-designs produced by each array have the same set of canonical efficiency factors./ Hence equivalent arrays produce $\underset{\sim}{\alpha}$-designs with the same $\bar{E}$. The following theorem simplifies the problem of trying to find the $\underset{\sim}{\alpha}$-design with the highest $\bar{E}$.

<u>Theorem 3.5.1.</u> Every array $\underset{\sim}{\alpha}$ is equivalent to a special array, called the reduced array for $\underset{\sim}{\alpha}$, with all elements in (i) the first row and (ii) the first column equal to zero.

<u>Proof.</u> (i) It follows from Construction 3.2.1 that we may substitute each column of $\underset{\sim}{\alpha}$ with a column starting with 0, the result simply being a rearrangement of the blocks of the $\underset{\sim}{\alpha}$-design.

(ii) Follows by applying argument (i) to the dual array $\underset{\sim}{\alpha}'$. /

Note that for equivalence operation (i), the latent vectors of $\underset{\sim}{A}$ remain the same, but this is not true in general for equivalence operation (ii). This is important if OF(2) structure is considered for $\underset{\sim}{\alpha}$-designs.

Clearly Theorem 3.5.1 also applies for any dual array $\underset{\sim}{\alpha}'$.

<u>Example 3.5.1.</u> For r = 3, k = 4, s = 3 let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 \end{matrix} \qquad (3.5.1)$$

$$\equiv \begin{matrix} 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{matrix} \qquad (3.5.2)$$

(add (2,1,1,2), mod 3 to each row of (3.5.1)),

$$\equiv \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \end{matrix} \qquad (3.5.3)$$

(add (0,1,2)', mod 3 to each column of (3.5.2)),

and now (3.5.3) is in reduced form. /

## 3.6 Variance Matrix of $\alpha$-Designs

In general for variety trial designs, b will be much smaller than v and hence to obtain $\underset{\sim}{V}$, it is better to use (1.5.4), i.e. from (2.3.3),

$$\underset{\sim}{V} = \frac{1}{r}(\underset{\sim}{I}_v + \frac{1}{rk}\underset{\sim}{N}\,\underset{\sim}{B}^+\,\underset{\sim}{N}'), \quad (\text{mod } \underset{\sim}{J}_v) . \qquad (3.6.1)$$

From Theorem A.3.4, it follows that for $\alpha$-designs, $\underset{\sim}{V}$ has BC structure; however in general $\underset{\sim}{V}$ does not have P structure. Hence $\underset{\sim}{V}$ may contain many distinct elements on and off the lead diagonal. However from the proof of Theorem 1.5.2 we see that $\underset{\sim}{V}$ can be expressed as the power series

$$\underset{\sim}{V} = \frac{1}{r}(\underset{\sim}{I}_v + (\frac{1}{rk}\underset{\sim}{N}\,\underset{\sim}{N}') + (\frac{1}{rk}\underset{\sim}{N}\,\underset{\sim}{N}')^2 + \ldots), \ (\text{mod } \underset{\sim}{J}_v), \quad (3.6.2)$$

which could be approximated by the first two terms, i.e.

$$\underset{\sim}{V} \approx \frac{1}{r}(\underset{\sim}{I}_v + \frac{1}{rk}\underset{\sim}{N}\,\underset{\sim}{N}') \quad (\text{mod } \underset{\sim}{J}_v). \qquad (3.6.3)$$

It follows that pairwise variances can be grouped according to the number of distinct off-diagonal elements in $\underset{\sim}{N}\,\underset{\sim}{N}'$; a convenient summary of the main features of $\underset{\sim}{V}$ is provided by the arithmetic mean and range of the pairwise variances for each group. This idea has been used by Cochran and Cox (1957, page 422) for the rectangular lattice designs.

## 3.7 The Incorporation of Control Varieties into $\alpha$-Designs

Commonly in variety trial experiments, several (say, t) control or standard varieties are included. The main contrasts of interest are then the comparisons of the mean of the control varieties with each new variety. It is clearly desirable that the control varieties be spread as evenly as possible over the b blocks of the design. For some methods of construction a satisfactory allocation of control varieties to the symbols of the design for $t > 1$, may be hard to find. Construction 3.2.1 for $\underset{\sim}{\alpha}$-designs allows a suitable set of controls to be readily identified. Suppose for example, the symbols 0, 1, ..., t-1 are taken to represent the t control varieties. Then it follows that (i) when $t < s$, each block will contain either zero or one control variety,

(ii) when $t = s$, each block will contain exactly one control variety,

(iii) when $s < t < 2s$, each block will contain either one or two control varieties, etc.

Generally, the number of control varieties will be less than s, hence any choice of t symbols from ms, ms + 1, ..., (m+1)s-1 for a particular m (m = 0, 1, ..., k-1), could be used for the control varieties. We try to choose the set of t symbols which gives the highest harmonic mean of the efficiencies for the comparison of the mean of the control varieties with, in turn, all other varieties. This problem can be simplified for some types of $\underset{\sim}{\alpha}$-designs (see section 5.6).

# CHAPTER 4

# CONSTRUCTION OF $\underset{\sim}{\alpha}(0,1)$-DESIGNS

## 4.1 Introduction and Summary

It has been shown by John (1966) that a meaningful procedure for improving the h.m.c.e.f. of an equireplicated, equiblock-sized design is to minimize the range of the off-diagonal elements of the concurrence matrix. This procedure is also intuitively appealing, as it represents an approach towards balance, in which case all the off-diagonal elements of the concurrence matrix are equal.

In this chapter, $\underset{\sim}{\alpha}$-designs for which the concurrence matrix has only ones and zeros off-diagonal, are considered. Such designs are called $\underset{\sim}{\alpha}(0,1)$-designs, and in section 4.2 it is shown that for $\underset{\sim}{\alpha}(0,1)$-designs, $k \leq s$. The conditions under which square lattice and rectangular lattice designs can be written as $\underset{\sim}{\alpha}(0,1)$-designs, are discussed, and for some parameter values, alternatives to the rectangular lattice designs are given.

In section 4.3, the construction and use of tables of $\underset{\sim}{\alpha}(0,1)$-designs given in Table A, is discussed. The use of CIB designs in constructing two-replicate $\underset{\sim}{\alpha}(0,1)$-designs is noted in section 4.3.1.

The dual design of an $\underset{\sim}{\alpha}(0,1)$-design for $r = 2$, is a resolvable paired comparison design. In section 4.4, a comparison is made between the h.m.c.e.f. of paired comparison designs constructed in this thesis, and corresponding designs given by other authors.

## 4.2 Restrictions on $\underset{\sim}{\alpha}$

Definition 4.2.1. An $\underset{\sim}{\alpha}$-design for which the $\underset{\sim}{N}\underset{\sim}{N}'$ matrix has only zeros and ones off-diagonal is called an $\underset{\sim}{\alpha}(0,1)$-design./

Theorem 4.2.1. An array $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1)$-design iff

$$a_{lm} \dot{-} a_{l'm} \neq a_{lm'} \dot{-} a_{l'm'} , \qquad (4.2.1)$$

$$(l \neq l' ; \ m \neq m' ; \ l,l' = 0,1,\dots,k-1 ; m,m' = 0,1,\dots,r-1).$$

Proof. Suppose for some $l \neq l'$, $m \neq m'$,

$$a_{lm} \dot{-} a_{l'm} = a_{lm'} \dot{-} a_{l'm'} = g \text{, say.} \quad (4.2.2)$$

Let $f = \dot{-} a_{l'm}$ and $f' = \dot{-} a_{l'm'}$. Then (4.2.2) implies that in the $\underset{\sim}{\alpha}$-design, symbols 0 and g appear together in blocks $ms + f$ and $m's + f'$, which is a contradiction. The reverse argument completes the proof./

Using Definition 3.2.2, we have the following corollary.

Corollary 4.2.2. A dual array $\underset{\sim}{\alpha}'$, produces an $\underset{\sim}{\alpha}(0,1)$-design iff

$$a'_{ml'} \dot{-} a'_{ml} \neq a'_{m'l'} \dot{-} a'_{m'l} , \qquad (4.2.3)$$

$$(m \neq m' ; \ l \neq l' ; \ m,m' = 0,1,\dots,r-1 ; l,l' = 0,1,\dots,k-1)./$$

For example, the design in Table 2.3.1(iii) is an $\underset{\sim}{\alpha}(0,1)$-design for $r = 2$, $k = 3$, $s = 3$ with

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{matrix} .$$

By writing $\underset{\sim}{\alpha}'$ in reduced form and using (4.2.3), we can deduce the following theorem.

Theorem 4.2.3. For an $\underset{\sim}{\alpha}(0,1)$-design ,

$$k \leq s \ ./ \qquad (4.2.4)$$

In fact (4.2.4) is necessary for the existence of any resolvable design with $v = ks$, block size $k$ and having only zeros and ones off-diagonal in the concurrence matrix.

4.2.1. $\underset{\sim}{\alpha}(0,1)$-Designs for $k = s$

It was pointed out in section 2.4 that when $k = s$, we have the parameters of the square lattice designs. We now study the relation between these designs and $\underset{\sim}{\alpha}(0,1)$-designs.

We first define an orthogonal array (Raghavarao, 1971, page 10).

Definition 4.2.2. An $r \times q$ array $\underset{\sim}{Q}$ with entries from a set of $s$ $(\geq 2)$ elements is called an orthogonal array of size $q$, $r$ constraints, $s$ levels, strength $t$ and index $\lambda$ if any $t \times q$ sub-array of $\underset{\sim}{Q}$ contains all possible $t \times 1$ column vectors with the same frequency $\lambda$. Such an array is denoted by $(q, r, s, t)$./

Theorem 4.2.4. If $k = s$, and $\underset{\sim}{\alpha}'$ satisfies (4.2.3), then

(i) the array $\underset{\sim}{\alpha}'^*$ obtained from $\underset{\sim}{\alpha}'$ using Construction 3.2.1(i), is the orthogonal array $(s^2, r, s, 2)$.

(ii) the $\underset{\sim}{\alpha}(0,1)$-design obtained from $\underset{\sim}{\alpha}'$ is the $r$ replicate square lattice design for $v = s^2$.

Proof. (i) This is a special case $(\lambda = 1)$ of Theorem 3 given by Bose and Bush (1952).

(ii) It follows from (i) that for the $\underset{\sim}{\alpha}(0,1)$-design, $\underset{\sim}{N}'\underset{\sim}{N}$ has BF(2) structure with $\theta_{00} = s$, $\theta_{01} = 0$, $\theta_{10} = \theta_{11} = 1$.

Thus from (1.5.1), $\underset{\sim}{B}$ has the form required by Definition 2.4.1 for a square lattice design./

Example 4.2.1. For $r = 3$, $k = s = 5$, let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{matrix} , \qquad (4.2.5)$$

which satisfies (4.2.3). Hence

$$\underset{\sim}{\alpha}'^{*} = \begin{matrix} 01234 & 01234 & 01234 & 01234 & 01234 \\ 01234 & 12340 & 23401 & 34012 & 40123, \\ 01234 & 40123 & 34012 & 23401 & 12340 \end{matrix}$$

is the orthogonal array (25, 3, 5, 2), and the $\underset{\sim}{\alpha}(0,1)$-design obtained from (4.2.5) is the triple square lattice for $v = 25$./

We now consider for $k = s$, the existence of an array $\underset{\sim}{\alpha}'$ satisfying (4.2.3) for $r = 2$, 3 or 4.

Theorem 4.2.5. For $r = 2$, $k = s$ an array $\underset{\sim}{\alpha}'$ satisfying (4.2.3) exists for all $s \geq 2$, e.g.

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & s-1 \end{matrix} \quad ./ \qquad (4.2.6)$$

Theorem 4.2.6. For $r = 3$, $k = s$ an array $\underset{\sim}{\alpha}'$ satisfying (4.2.3) exists iff $s$ is odd, and then an example would be

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \cdots & s-1 \\ 0 & s-1 & s-2 & \cdots & 1 \end{matrix} . \qquad (4.2.7)$$

Proof. For $r = 3$, $k = s$ conditions (4.2.3) on an array $\underset{\sim}{\alpha}'$ in reduced form become,

$$a'_{2l} \dot{-} l \neq a'_{2l'} \dot{-} l' \neq 0, \qquad (4.2.8)$$

$$(l \neq l'\,;\; l,l' = 1,2,\ldots,s-1),$$

where $a'_{21}, a'_{22}, \ldots, a'_{2(s-1)}$ is a permutation of

1, 2, ..., s-1. Hence from (4.2.8),

$$\sum_{l=1}^{s-1}(a_l \mathbin{\dot{-}} l) = \sum_{l=1}^{s-1} l + zs, \qquad (4.2.9)$$

(z an integer),

i.e.

$$\sum_{l=1}^{s-1} l = \frac{s(s-1)}{2} = z's, \qquad (4.2.10)$$

(z' an integer).

Thus $z' = \frac{s-1}{2}$ is an integer iff s is odd./

Theorem 4.2.6 can be thought of as a special case of Theorem 3.3 given by Hedayat and Federer (1969).

<u>Theorem 4.2.7.</u> For r = 4, k = s an array $\underset{\sim}{\alpha}'$ satisfying (4.2.3) for $s \not\equiv 0 \pmod 3$ and odd, is given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & \cdots & s-3 & s-2 & s-1 \\ 0 & s-1 & s-2 & s-3 & s-4 & \cdots & 3 & 2 & 1 \\ 0 & \frac{s+1}{2} & 1 & \frac{s+3}{2} & 2 & \cdots & \frac{s-3}{2} & s-1 & \frac{s-1}{2} \end{matrix} \qquad (4.2.11)$$ ./

It has not been possible to find arrays $\underset{\sim}{\alpha}'$, satisfying (4.2.3), other than those given by Theorems 4.2.5 - 7; hence only for these cases can the square lattice designs be constructed as $\underset{\sim}{\alpha}(0,1)$-designs.

However, we can obtain some additional constructions by using $\underset{\sim}{\beta}$-designs, i.e. there is an analagous theory to the results of this section for arrays $\underset{\sim}{\beta}$. For example, we may define $\underset{\sim}{\beta}(0,1)$-designs and then it is possible to show that for k = s, the following examples of dual arrays $\underset{\sim}{\beta}'$ produce $\underset{\sim}{\beta}(0,1)$-designs which are square lattice designs:

(i) For $r = 3$, $k = s$ and $s \equiv 0 \pmod 4$, let $\underset{\sim}{\beta}'$ be given by

$$\underset{\sim}{\beta}' = \begin{matrix} 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & \dots & \frac{s}{2} & \frac{s}{2}+1 & \dots & s-2 & s-1 \\ 0 & \frac{s}{2} & \frac{s}{2}+1 & \dots & s-1 & \frac{s}{2}-1 & \dots & 2 & 1 \end{matrix} \quad (4.2.12)$$

(ii) For $r = 4$, $k = s = 4$ let $\underset{\sim}{\beta}'$ be given by

$$\underset{\sim}{\beta}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \\ 0 & 3 & 1 & 2 \end{matrix} \quad (4.2.13)$$

4.2.2. $\underset{\sim}{\alpha}(0,1)$-Designs for $k = s - 1$

When $k = s - 1$, we have the parameters of the rectangular lattice designs. We now study the relation between these designs and $\underset{\sim}{\alpha}(0,1)$-designs.

4.2.2.1. $\underset{\sim}{\alpha}(0,1)$-Designs which are Rectangular Lattices

Let $\underset{\sim}{\alpha}'_1$ be an $r \times s$ array in reduced form satisfying (4.2.3), and write

$$\underset{\sim}{\alpha}'_1 = \underset{\sim}{0} \;\vdots\; \underset{\sim}{\alpha}' , \quad (4.2.14)$$

i.e. $\underset{\sim}{\alpha}'$ is an $r \times (s-1)$ array obtained by deleting the first column of $\underset{\sim}{\alpha}'_1$.

We then have the following result.

Theorem 4.2.8. The $\underset{\sim}{\alpha}(0,1)$-design obtained from the array $\underset{\sim}{\alpha}'$ given by (4.2.14) is the $r$ replicate rectangular lattice design for $v = (s-1)s$.

Proof. It follows from Theorem 4.2.4(ii) that for the $\underset{\sim}{\alpha}(0,1)$-design, $\underset{\sim}{B}$ has the form required by Definition 2.4.2 for a rectangular lattice design./

Hence from Theorems 4.2.5 - 7, rectangular lattice designs may be constructed as $\underset{\sim}{\alpha}(0,1)$-designs for

(i) $r = 2$, $k = s-1$.

(ii) $r = 3$, $k = s-1$, $s$ odd. (4.2.15)

(iii) $r = 4$, $k = s-1$, $s \neq 0 \pmod 3$ and odd.

For example, by deleting the first column of (4.2.5), the resulting $\underset{\sim}{\alpha}(0,1)$-design is the triple rectangular lattice design for $v = 20$.

Similarly, it can be shown that by deleting the first column of (4.2.12) and (4.2.13), the resulting $\underset{\sim}{\beta}(0,1)$-design is a rectangular lattice design.

4.2.2.2. $\underset{\sim}{\alpha}(0,1)$-Designs which are not Rectangular Lattices

Rectangular lattice designs are desirable because their BF(2) structure in the dual matrix $\underset{\sim}{B}$, leads to a high value for $\bar{\bar{E}}$. On the other hand $\underset{\sim}{\alpha}(0,1)$-designs and $\underset{\sim}{\beta}(0,1)$-designs are desirable because they are easy to construct.

In section 4.2.2.1, we obtained the parameters for which rectangular lattice designs could be constructed as $\underset{\sim}{\alpha}(0,1)$-designs or $\underset{\sim}{\beta}(0,1)$-designs. For the following parameters, $\underset{\sim}{\alpha}(0,1)$-designs can be constructed which are not rectangular lattice designs.

Theorem 4.2.9. For $r = 3$, $k = s-1$ an array $\underset{\sim}{\alpha}'$ satisfying (4.2.3) for $s$ even, is given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 \ldots 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \ldots s-3 & s-2 \\ 0 & \frac{s}{2} & 1 & \frac{s}{2}+1 & 2 \ldots s-2 & \frac{s}{2}-1 \end{matrix} \quad ./ \qquad (4.2.16)$$

The minimum canonical efficiency factor for the $\underset{\sim}{\alpha}(0,1)$-design constructed from (4.2.16) is $\frac{2}{3}\left(\frac{s-2}{s-1}\right)$, compared with $\frac{2s-3}{3(s-1)}$ for the rectangular lattice design with the same parameters. In expression (3.4.5) for $\bar{\bar{E}}$,

$$\Sigma = 3(s-1)\left\{\frac{s-2}{4s} + \frac{s}{2s-1} + \frac{s-2}{2s-3} + \frac{s}{4(s-2)}\right\} = \Sigma_1\text{, say,}$$

for the $\underset{\sim}{\alpha}(0,1)$-design whereas for the rectangular lattice design,

$$\Sigma = 3(s-1)\left\{\frac{2(s-1)}{2s-3} + \frac{s-1}{2s}\right\} = \Sigma_2\text{, say.}$$

The difference,

$$\Sigma_2 - \Sigma_1 = \frac{-9(s-1)}{2(s-2)(2s-1)(2s-3)} \quad ,$$

will be small compared with the denominator of (3.4.5). Thus, $\bar{\bar{E}}$ for the $\underset{\sim}{\alpha}(0,1)$-design will be very close to that for the rectangular lattice design.

For $r = 4$, $k = s-1$ the following examples of arrays $\underset{\sim}{\alpha}'$ satisfy (4.2.3) and produce $\underset{\sim}{\alpha}(0,1)$-designs which are not rectangular lattice designs:

(i) For $s = 6$,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 2 & 5 & 1 & 3 \end{matrix} \quad . \qquad (4.2.17)$$

(ii) For $s = 8$,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 4 & 1 & 5 & 2 & 6 & 3 \\ 0 & 2 & 4 & 6 & 1 & 3 & 5 \end{matrix} \; . \qquad (4.2.18)$$

(iii) For $s = 10$,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 7 & 5 & 1 & 6 & 4 & 3 & 8 & 2 \\ 0 & 6 & 3 & 2 & 1 & 7 & 9 & 5 & 4 \end{matrix} \; . \qquad (4.2.19)$$

Construction of the analagous rectangular lattice designs depends on the existence of a pair of $s \times s$ mutually orthogonal latin squares. Thus example (i) above for $s = 6$ is of special interest since the rectangular lattice design with the same parameters cannot be constructed.

## 4.3 Construction of Tables of $\underset{\sim}{\alpha}(0,1)$-Designs

In sections 4.2.1 - 2, we have shown that analytical results are available to help in the construction of $\underset{\sim}{\alpha}(0,1)$-designs for $k = s, s - 1$. However, for $r \leq k < s-1$, there are many inequivalent arrays $\underset{\sim}{\alpha}'$ satisfying (4.2.3), from which we must choose the one which produces the $\underset{\sim}{\alpha}(0,1)$-design with the highest $\bar{E}$ for a particular set of parameters.

A computer program has been written for systematic construction and testing of arrays $\underset{\sim}{\alpha}'$. Given a particular array, the following operations are carried out:

(i) Check that (4.2.3) is satisfied.

(ii) Calculate $\bar{E}$ using the method described in section 3.4.

The parameters are restricted to the ranges,

$$2 \le r \le 4, \qquad r \le s \le 15. \tag{4.3.1}$$

By Theorem 3.5.1, it is only necessary to consider arrays $\underset{\sim}{\alpha}'$ in reduced form, however for certain values of r, s and k there are too many different arrays $\underset{\sim}{\alpha}'$ for an exhaustive search to be carried out.

In Table A, a list of recommended $\underset{\sim}{\alpha}(0,1)$-designs is given for parameters in the above ranges. For each combination of r, s and k, Table A contains an array $\underset{\sim}{\alpha}'$ in reduced form. Hence it is only necessary to tabulate the last $r-1$ rows of $\underset{\sim}{\alpha}'$. For example, for $r=3$, $s=10$, $k=6$ we obtain from Table A.2,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 5 & 6 \\ 0 & 7 & 9 & 5 & 6 & 4 \end{matrix} . \tag{4.3.2}$$

The $\underset{\sim}{\alpha}(0,1)$-design may then be obtained from $\underset{\sim}{\alpha}'$.

Since we have tabulated $\underset{\sim}{\alpha}'$ in Table A, we also list $\bar{E}'$, the h.m.c.e.f. for the $\underset{\sim}{\alpha}'$-design, itself a useful resolvable design for situations where $v \le b$. For example, for the array (4.3.2), the $\underset{\sim}{\alpha}'$-design is a resolvable design for ~~18~~ 30 varieties, 6 replicates and block size 3 with $\bar{E}' = .6605$.

The properties of the $\underset{\sim}{\alpha}(0,1)$-designs are tabulated in detail in Table B. At the moment we are interested only in the cases where $v = ks$. For example, for the array (4.3.2), the $\underset{\sim}{\alpha}(0,1)$-design is a resolvable design for $r = 3$, $v = ks = 60$, $s = 10$, $k = 6$. Thus from Table B.2 we find that,

(i) the smallest canonical efficiency factor, $e_{min} = .5212$,

(ii) the h.m.c.e.f., $\bar{\bar{E}} = .7983$,

(iii) the harmonic mean of the pairwise efficiencies for varieties that do not appear together within blocks, $\bar{\bar{E}}(0) = .7829$ and the range of these pairwise efficiencies is 2%,

(iv) the harmonic mean of the pairwise efficiencies for varieties that appear together once within blocks, $\bar{\bar{E}}(1) = .8475$ and the range of these pairwise efficiencies is 1%.

The final column of the tables will be explained in Chapter 6.

In Table A we use the $\underset{\sim}{\beta}(0,1)$-designs described in sections 4.3.1 - 2 in the few cases when we have shown them to be more appropriate. The arrays, $\underset{\sim}{\beta}'$, are identified in the tables by means of an asterisk, for example, for $r = 3$, $s = 8$, $k = 7$ we obtain from Table A.2,

$$\underset{\sim}{\beta}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 5 & 4 & 3 & 2 & 6 \\ 0 & 1 & 3 & 7 & 6 & 5 & 2 \end{matrix} . \qquad (4.3.3)$$

4.3.1. A Relation with CIB Designs

For $r = 2$, let $\underset{\sim}{\alpha}'$ be written in the reduced form

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & a'_{11} & \cdots & a'_{1(k-1)} \end{matrix} . \qquad (4.3.4)$$

Suppose we interpret the second row of (4.3.4) as the initial block of a CIB design for $s$ varieties, $s$ blocks, block size $k$ and let $\bar{\bar{E}}_{CIB}$ be the h.m.c.e.f. for this design. We then have the following theorem.

Theorem 4.3.1. For the $\underset{\sim}{\alpha}(0,1)$-design obtained from (4.3.4),

$$\Sigma = \frac{4(s-1)}{\bar{\bar{E}}_{CIB}} , \qquad (4.3.5)$$

and hence from (3.4.5), $\bar{\bar{E}}$ can be expressed as a function of $\bar{\bar{E}}_{CIB}$.

Proof. From (B.2.5), we see that $\underset{\sim}{\nu}'_2$ is in fact the first row of the concurrence matrix for the CIB design, which has C structure. Hence it follows from (B.2.8) and (A.2.2) that the canonical efficiency factors, $e_u$, for the CIB design are given by

$$e_u = 1 - 4\,\nu_{2u}^{*} , \qquad (4.3.6)$$

$$(u = 1,2,\ldots,s-1) .$$

Then, from (B.2.12),

$$\frac{1}{\xi_{u.}} = \frac{4}{e_u} , \qquad (4.3.7)$$

$$(u = 1,2,\ldots,s-1) .$$

Hence by (3.4.9),

$$\Sigma = 4 \sum_{u=1}^{s-1} \frac{1}{e_u}$$

$$= \frac{4(s-1)}{\bar{\bar{E}}_{CIB}} \quad ./$$

It follows from Theorem 4.3.1 that the CIB design with the highest $\bar{E}_{CIB}$ may be substituted into (4.3.4) to produce the $\underset{\sim}{\alpha}(0,1)$-design with the highest $\bar{E}$. Thus the tables of CIB designs produced by John, Wolock and David (1972) have proved useful in constructing some of the designs in Table A.1.

Example 4.3.1. For 8 varieties, 8 blocks and block size 4, John, Wolock and David (1972) list the initial block (0,1,2,4) with $\bar{E}_{CIB}$ = .8498. So in Table A.1 for $s = 8$, $k = 4$ we give the array,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \end{matrix} .$$

For the resulting $\underset{\sim}{\alpha}(0,1)$-design,

$$\Sigma = \frac{4 \times 7}{.8498} = 32.95 .$$

Hence from (3.4.14),

$$\bar{E} = \frac{31}{17 + 32.95}$$

$$= .6206 ,$$

which agrees with the value given in Table B.1 for $v = 32$, $s = 8$ and $k = 4$ ./

## 4.4 Paired Comparison Designs

Cyclic paired comparison designs were studied by David (1963), and extensive tables have been produced by John, Wolock and David (1972). Resolvable paired comparison designs are particularly useful for round robin tournaments, and tables of resolvable cyclic paired comparison designs have been given by David (1967). For $r = 2$, the $\underset{\sim}{\alpha}'$-designs obtained from the arrays $\underset{\sim}{\alpha}'$ in Table A.1 are resolvable paired comparison designs for 2s varieties, k replicates and with h.m.c.e.f., $\bar{E}'$.

In Table 4.4.1, we compare the h.m.c.e.f. for,

(i) $\underset{\sim}{\alpha}'$-designs from Table A.1,

(ii) resolvable cyclic paired comparison designs given by David (1967),

(iii) cyclic paired comparison designs given by John, Wolock and David (1972).

We see that the $\underset{\sim}{\alpha}'$-designs generally give higher values for the h.m.c.e.f. than the resolvable cyclic designs given by David (1967). In fact, the $\underset{\sim}{\alpha}'$-designs compare favourably with the cyclic designs given by John, Wolock and David (1972), which in general do not have the restriction of resolvability.

Table 4.4.1 has been restricted to $s \leq 7$. However, many other direct comparisons for $s > 7$, can be made between Table A.1 and the tables of cyclic paired comparison designs given by John, Wolock and David (1972).

Table 4.4.1

Comparison of the h.m.c.e.f. for paired comparison designs

| No. of varieties | No. of replicates | $\underset{\sim}{\alpha}'$-design | David design | John et al. design |
|---|---|---|---|---|
| 6 | 3 | .5556 | .556 | .5556 |
| 8 | 3 | .4828 | .487 | .4876 |
| 8 | 4 | .5385 | .507 | .5385 |
| 10 | 3 | .4361 | .436 | .4361 |
| 10 | 4 | .4982 | .472 | .5000 |
| 10 | 5 | .5294 | .529 | .5294 |
| 12 | 3 | .4151 | .395 | .3945 |
| 12 | 4 | .4705 | .479 | .4793 |
| 12 | 5 | .5038 | .491 | .5018 |
| 12 | 6 | .5238 | .491 | .5238 |
| 14 | 3 | .4081 | .360 | .3599 |
| 14 | 4 | .4573 | .446 | .4588 |
| 14 | 5 | .4853 | .485 | .4874 |
| 14 | 6 | .5061 | .506 | .5061 |
| 14 | 7 | .5200 | .520 | .5200 |

# CHAPTER 5

# CONSTRUCTION OF $\underset{\sim}{\alpha}(0,1,2)$-DESIGNS

## 5.1 Introduction and Summary

It was shown in Chapter 4 that for $\underset{\sim}{\alpha}(0,1)$-designs, $k \leq s$. Thus, to construct $\underset{\sim}{\alpha}$-designs for $k > s$, we must relax the conditions on the range of the off-diagonal elements of the concurrence matrix. In Chapter 5, $\underset{\sim}{\alpha}$-designs for which the concurrence matrix has only zeros, ones and twos off-diagonal, are considered. Such designs are called $\underset{\sim}{\alpha}(0,1,2)$-designs.

Whereas $\underset{\sim}{\alpha}(0,1)$-designs always have P structure, an extra restriction is needed for $\underset{\sim}{\alpha}(0,1,2)$-designs to have P structure. The conditions that $\underset{\sim}{\alpha}$ must satisfy to produce

(i) $\underset{\sim}{\alpha}(0,1,2)$-designs,

(ii) $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure,

are considered in section 5.2.

In section 5.3, minimal $\underset{\sim}{\alpha}(0,1,2)$-designs are defined; these designs are preferable to others in terms of maximizing $\bar{E}$.

Section 5.4 deals with the construction of tables of $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure and section 5.5 with minimal $\underset{\sim}{\alpha}(0,1,2)$-designs.

Finally, in section 5.6, we consider the incorporation of control varieties into $\underset{\sim}{\alpha}(0,1,2)$-designs.

## 5.2 Restrictions on $\underset{\sim}{\alpha}$

### 5.2.1. $\underset{\sim}{\alpha}(0,1,2)$-Designs

Definition 5.2.1. An $\underset{\sim}{\alpha}$-design for which the $\underset{\sim}{N}\underset{\sim}{N}'$ matrix has only zeros, ones and twos off-diagonal or only zeros and twos, is called an $\underset{\sim}{\alpha}(0,1,2)$-design./

Hence from Theorem 3.2.1, we have the following result.

Theorem 5.2.1. An array $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design iff the vectors $\underset{\sim}{\mu}_{ll'}$ defined in section 3.2 consist of the elements 0, 1 and 2 ($l \neq l'$ ; $l,l' = 0,1,\ldots,k-1$)./

The vectors $\underset{\sim}{\mu}_{ll'}$ can never contain elements greater than $r$; hence it follows that for $r = 2$, any array $\underset{\sim}{\alpha}$ will produce either an $\underset{\sim}{\alpha}(0,1)$-design or an $\underset{\sim}{\alpha}(0,1,2)$-design. For $r > 2$, we have the following result.

Theorem 5.2.2. For an $\underset{\sim}{\alpha}(0,1,2)$-design with $r > 2$,

$$k \leq s^2 \, . \qquad (5.2.1)$$

Proof. We prove this theorem by showing that no $\underset{\sim}{\alpha}(0,1,2)$-design exists when $k > s^2$. Let $\underset{\sim}{\alpha}$ be in reduced form and suppose that in the second column of $\underset{\sim}{\alpha}$, the element $h$ appears more than $s$ times. Hence, in the third column of $\underset{\sim}{\alpha}$, there must be at least one element, say $h'$, which appears in the same row as $h$ at least twice, i.e. there are at least two rows in $\underset{\sim}{\alpha}$, say rows $l$ and $l'$; with first three elements 0, h, h'. But this means that the first element of $\underset{\sim}{\mu}_{ll'}$ must be at least 3, i.e. some pairs of varieties concur more than twice./

If $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design, we define a count vector for $\underset{\sim}{\alpha}$ in the following way.

Definition 5.2.2. Let $\underset{\sim}{\gamma}$ be the vector whose $l^{th}$ element is the total number of twos which appear as elements of the $k-1$ vectors $\underset{\sim}{\mu}_{ll'}$ for $l' \neq l$ $(l = 0,1,\ldots,k-1)$./

Example 5.2.1. For $r = 3$, $k = 4$, $s = 3$ let $\underset{\sim}{\alpha}'$ be given by

$$\begin{array}{cccccc} & & 0 & 0 & 0 & 0 \\ \underset{\sim}{\alpha}' & = & 0 & 2 & 1 & 0 \\ & & 0 & 1 & 2 & 1 \end{array} . \qquad (5.2.2)$$

Then $\underset{\sim}{\mu}_{01} = \underset{\sim}{\mu}_{02} = \underset{\sim}{\mu}_{12} = (1,1,1)'$,

$$\underset{\sim}{\mu}_{03} = \underset{\sim}{\mu}_{13} = (2,0,1)', \ \underset{\sim}{\mu}_{23} = (1,2,0)'.$$

Hence the $\underset{\sim}{\alpha}$-design obtained from (5.2.2) is an $\underset{\sim}{\alpha}(0,1,2)$-design and

$$\underset{\sim}{\gamma} = (1,1,1,3)' ./ \qquad (5.2.3)$$

Define

$$c = \underset{\sim}{\gamma}'\underset{\sim}{1}. \qquad (5.2.4)$$

Since $\underset{\sim}{N}\underset{\sim}{N}'$ is given by (3.2.4), and is symmetric, we have the following result.

Theorem 5.2.3. (i) The total number of twos which appear as off-diagonal elements in the concurrence matrix of an $\underset{\sim}{\alpha}(0,1,2)$-design is $cs$.

(ii) $c$ is an even number./

Theorem 5.2.4. Suppose $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design. Then

$$c = \sum_{m \neq m'} \sum_{h=0}^{s-1} \binom{(\underset{\sim}{\eta}_{mm'})^h}{2}, \qquad (5.2.5)$$

where $(\underset{\sim}{\eta}_{mm'})^h$ is the $h^{th}$ element of the vector $\underset{\sim}{\eta}_{mm'}$ $(m \neq m' ;\ m,m' = 0,1,\ldots,r-1)$ defined in section 3.2.

Proof. From (3.2.6), $(\underset{\sim}{\eta}_{mm'})^h$ is the number of times the difference $a_{lm} \dot{-} a_{lm'}$ is equal to $h$. Hence it follows from Corollary 4.2.2 that when $(\underset{\sim}{\eta}_{mm'})^h$ is equal to zero or one, there is a corresponding element in one of the vectors $\underset{\sim}{\mu}_{ll'}$, taking the same value $(l \neq l' ;\ l,l' = 0,1,\ldots,k-1)$. Thus suppose $(\underset{\sim}{\eta}_{mm'})^h \geq 2$. Since the elements of $\underset{\sim}{\mu}_{ll'}$ cannot be greater than two, each element $(\underset{\sim}{\eta}_{mm'})^h$ must produce $\binom{(\underset{\sim}{\eta}_{mm'})^h}{2}$ twos among the elements of $\underset{\sim}{\mu}_{ll'}$ $(l \neq l' ;\ l,l' = 0,1,\ldots,k-1)$. Hence the result (5.2.5) is obtained./

5.2.2. $\underset{\sim}{\alpha}(0,1,2)$-Designs with P Structure

Since $\underset{\sim}{N}\underset{\sim}{N}'\underset{\sim}{1} = rk\underset{\sim}{1}$, it follows that for an $\underset{\sim}{\alpha}(0,1)$-design, each row of $\underset{\sim}{N}\underset{\sim}{N}'$ has $r(k-1)$ ones off-diagonal and hence all $\underset{\sim}{\alpha}(0,1)$-designs must have P structure (Definition A.10.1). However, some $\underset{\sim}{\alpha}(0,1,2)$-designs, for example, the $\underset{\sim}{\alpha}(0,1,2)$-design obtained from (5.2.2), do not have P structure. We now give a condition for an array $\underset{\sim}{\alpha}$ to produce an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure.

Theorem 5.2.5. An $\underset{\sim}{\alpha}(0,1,2)$-design has P structure iff

$$\underset{\sim}{\gamma} = z\underset{\sim}{1} , \qquad (5.2.6)$$

where $z$ is the number of twos per row in $\underset{\sim}{N}\underset{\sim}{N}'$.

Proof. Suppose $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure, i.e. each row of $\underset{\sim}{N}\underset{\sim}{N}'$ has the same number of zeros, ones and twos. Hence from Definition 5.2.2, all the elements of $\underset{\sim}{\gamma}$ must be equal to $z$. The reverse argument completes the proof./

Example 5.2.2. For $r = 3$, $k = 4$, $s = 3$ let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{matrix} \quad . \qquad (5.2.7)$$

Then $\underset{\sim}{\mu}_{01} = \underset{\sim}{\mu}_{02} = \underset{\sim}{\mu}_{23} = (2,0,1)'$,

$\underset{\sim}{\mu}_{03} = \underset{\sim}{\mu}_{12} = (1,1,1)'$, $\underset{\sim}{\mu}_{13} = (1,0,2)'$.

Hence the $\underset{\sim}{\alpha}$-design obtained from (5.2.7) is an $\underset{\sim}{\alpha}(0,1,2)$-design, and

$$\underset{\sim}{\gamma} = (2,2,2,2)' , \qquad (5.2.8)$$

i.e. the $\underset{\sim}{\alpha}(0,1,2)$-design has P structure with 2 twos in each row of the concurrence matrix./

For an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure, (5.2.4) becomes

$$c = k\,z . \qquad (5.2.9)$$

Hence by Theorem 5.2.3(ii), we have the following result.

Theorem 5.2.6. For an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure, $k$ and $z$ cannot both be odd numbers./

## 5.3 Minimal $\underset{\sim}{\alpha}(0,1,2)$-Designs

By virtue of Theorem 1.5.1, it is convenient when $v \geq b$, to calculate the canonical efficiency factors, and hence $\bar{E}$, from $\underset{\sim}{B}$. Thus from (1.5.1), for equireplicated, equiblock-sized designs, the matrix $\underset{\sim}{N}'\underset{\sim}{N}$ is of interest for the calculation of $\bar{E}$.

In section 4.1, we discussed the desirability of minimizing the range of off-diagonal elements of $\underset{\sim}{N}\underset{\sim}{N}'$. By considering the $\underset{\sim}{\alpha}'$-design, we may argue that to improve $\bar{E}$, it is also meaningful to minimize the range of the off-diagonal elements of $\underset{\sim}{N}'\underset{\sim}{N}$.

For $\underset{\sim}{\alpha}(0,1)$-designs, it follows from (4.2.3) that not only $\underset{\sim}{NN}'$ but also $\underset{\sim}{N}'\underset{\sim}{N}$ has only zeros and ones off-diagonal. By contrast, no analagous result exists for $\underset{\sim}{\alpha}(0,1,2)$-designs. Hence it is necessary to consider the form of $\underset{\sim}{N}'\underset{\sim}{N}$ for $\underset{\sim}{\alpha}(0,1,2)$-designs.

From (3.2.7), minimizing the range of the off-diagonal elements of $\underset{\sim}{N}'\underset{\sim}{N}$ can be done by minimizing the range of the elements in the vectors $\underset{\sim}{\eta}_{mm'}$, $(m \neq m' ; m,m' = 0,1,\ldots,r-1)$. Since $\underset{\sim}{1}'\underset{\sim}{\eta}_{mm'} = k$, it follows that if the elements of each vector $\underset{\sim}{\eta}_{mm'}$ are

$$\begin{aligned} x &= [\tfrac{k}{s}], \text{ with multiplicity } (x+1)s-k \\ \text{and} \quad & x+1, \text{ with multiplicity } k-xs, \end{aligned} \tag{5.3.1}$$

then the range of the off-diagonal elements of $\underset{\sim}{N}'\underset{\sim}{N}$ is minimized.

<u>Definition 5.3.1.</u> Suppose $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design. If the vectors $\underset{\sim}{\eta}_{mm'}$, $(m \neq m' ;\ m,m' = 0,1,\ldots,r-1)$, all have elements of the form (5.3.1), then $\underset{\sim}{\alpha}$ is said to produce a minimal $\underset{\sim}{\alpha}(0,1,2)$-design./

For example, for the array given by (5.2.2), $\underset{\sim}{\eta}_{01} = (2,1,1)'$, $\underset{\sim}{\eta}_{02} = \underset{\sim}{\eta}_{12} = (1,2,1)'$. Hence the $\underset{\sim}{\alpha}$-design obtained from (5.2.2) is a minimal $\underset{\sim}{\alpha}(0,1,2)$-design. However for the array given by (5.2.7), $\underset{\sim}{\eta}_{01} = (2,2,0)'$, $\underset{\sim}{\eta}_{02} = (2,1,1)'$, $\underset{\sim}{\eta}_{12} = (1,2,1)'$. Hence the $\underset{\sim}{\alpha}$-design obtained from (5.2.7) is not a minimal $\underset{\sim}{\alpha}(0,1,2)$-design.

By substituting the values (5.3.1) into (5.2.5), we can show the following result.

<u>Theorem 5.3.1.</u> If $\underset{\sim}{\alpha}$ produces a minimal $\underset{\sim}{\alpha}(0,1,2)$-design, then

$$c = \binom{r}{2} x[2k - (x+1)s]\ ./ \qquad (5.3.2)$$

In fact (5.2.5) is minimized when $\underset{\sim}{\alpha}$ produces a minimal $\underset{\sim}{\alpha}(0,1,2)$-design. Thus from Theorem 5.2.3(i), we have the following result.

<u>Theorem 5.3.2.</u> Over the class of $\underset{\sim}{\alpha}(0,1,2)$-designs for a particular set of parameters, a minimal $\underset{\sim}{\alpha}(0,1,2)$-design has the smallest number of twos in the concurrence matrix./

Thus, minimal $\underset{\sim}{\alpha}(0,1,2)$-designs are intuitively desirable since, because $\underset{\sim}{N}\underset{\sim}{N}'\underset{\sim}{1} = rk\underset{\sim}{1}$, they minimize the number of zero concurrences of pairs of varieties within blocks.

## 5.4 Construction of $\underset{\sim}{\alpha}(0,1,2)$-Designs with P Structure

It is desirable that an $\underset{\sim}{\alpha}$-design has P structure when the assignment of varieties to the symbols of the design is arbitrary, since the permutability property of the rows and columns of the concurrence matrix reflects this arbitrariness. From Theorem 5.3.1 and (5.2.9), it follows that for an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure,

$$z \geq \frac{\binom{r}{2} x[2k - (x+1)s]}{k} = y, \text{ say.} \qquad (5.4.1)$$

Hence the number of twos per row in the concurrence matrix of an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure must be an integer greater than or equal to $y$. Further, by Theorem 5.2.6, if $k$ is odd then $z$ must be an even integer greater than or equal to $y$.

Since $\bar{E}$ decreases as $z$ (and hence $c$) increases, these lower bounds on $z$ are useful, as we would prefer to construct $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure for $z$ as small as possible.

Example 5.4.1. For $r = 4$, $s = 3$, $k = 4$ from (5.4.1), $y = 3$. Consider the following two arrays $\underset{\sim}{\alpha}'$ which both produce $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure:

(i)

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{matrix} \quad . \qquad (5.4.2)$$

(ii)

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{matrix} \quad . \qquad (5.4.3)$$

For (i) $z = 3$ and $\bar{E} = .7891$, whereas for (ii) $z = 4$ and $\bar{E} = .7820$./

5.4.1. Construction of Minimal $\underset{\sim}{\alpha}(0,1,2)$-Designs with P Structure

A necessary condition for the existence of a minimal $\underset{\sim}{\alpha}(0,1,2)$-design with P structure is that $y$ be an integer. We only consider the special case when $k = xs$, i.e. from (5.4.1),

$$y = \binom{r}{2}(x-1) . \qquad (5.4.4)$$

If $\underset{\sim}{\alpha}$ produces a minimal $\underset{\sim}{\alpha}(0,1,2)$-design for $k = xs$, then from (5.3.1),

$$\underset{\sim}{\eta}_{mm'} = x\,\underset{\sim}{1} , \qquad (5.4.5)$$

$$(m \neq m' ; \; m,m' = 0,1,\ldots,r-1).$$

Thus we can make use of the results of sections 4.2.1 - 2. Suppose $\underset{\sim}{\alpha}_0'$ is an $r \times s$ array satisfying (4.2.3). We may use this array as a building block to produce an $r \times xs$ array, $\underset{\sim}{\alpha}'$ in the following way.

Construction 5.4.1. (i) Obtain $\underset{\sim}{\alpha}_h'$ from $\underset{\sim}{\alpha}_0'$ by adding $h$ to the elements of each of the last $r-2$ rows of $\underset{\sim}{\alpha}_0'$, and reducing modulo $s$, $(h = 1,2,\ldots,x-1)$.

(ii) Form the $r \times xs$ array

$$\underset{\sim}{\alpha}' = (\underset{\sim}{\alpha}_0' \,\vdots\, \underset{\sim}{\alpha}_1' \,\vdots\, \cdots \,\vdots\, \underset{\sim}{\alpha}_{x-1}'). / \qquad (5.4.6)$$

Then we may show the following result.

Theorem 5.4.1. The $\underset{\sim}{\alpha}$-design obtained from (5.4.6) is a minimal $\underset{\sim}{\alpha}(0,1,2)$-design with P structure and $z = y$./

The constructions given by Theorems 4.2.5 - 7 may be used to produce (5.4.6) for $2 \leq x \leq s-1$.

Example 5.4.2. For $r = 3$, $s = 5$, $x = 3$ from (4.2.7),

$$\underset{\sim}{\alpha}_0' = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 \end{matrix} , \qquad (5.4.7)$$

and hence using Construction 5.4.1,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ 0 & 4 & 3 & 2 & 1 & 1 & 0 & 4 & 3 & 2 & 2 & 1 & 0 & 4 & 3. \end{matrix}$$

(5.4.8)

The $\underset{\sim}{\alpha}$-design obtained from (5.4.8) is a minimal $\underset{\sim}{\alpha}(0,1,2)$-design with P structure for $v = 75$, $r = 3$ and $k = 15$./

Similarly, we may use (4.2.12) and (4.2.13) as building blocks to give an $r \times xs$ array $\underset{\sim}{\beta}'$, producing a minimal $\underset{\sim}{\beta}(0,1,2)$-design with P structure.

From Theorem 4.2.9, we have the following result.

<u>Theorem 5.4.2.</u> For $r = 3$, $k = 2s$ and $s$ even, let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & (s-2) & (s-1) & 0 & 1 & 2 & \cdots & (s-2) & (s-1) \\ 0 & \frac{s}{2} & 1 & \cdots & (\frac{s}{2}-1) & (s-1) & \frac{s}{2} & 0 & (\frac{s}{2}+1) & \cdots & (s-1) & (\frac{s}{2}-1). \end{matrix} \quad (5.4.9)$$

The $\underset{\sim}{\alpha}$-design obtained from this array is a minimal $\underset{\sim}{\alpha}(0,1,2)$-design with P structure and $z = y$./

Examples of arrays $\underset{\sim}{\alpha}'$ producing minimal $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure for $k = xs$ are included in Table 5.4.1. Arrays $\underset{\sim}{\beta}'$ have been used in the cases when no array $\underset{\sim}{\alpha}'$ could be constructed, e.g. for $r = 4$, $s = 4$, $k = 12$.

### 5.4.2. <u>Construction of $\underset{\sim}{\alpha}(0,1,2)$-Designs with P Structure when $k \neq xs$</u>

#### 5.4.2.1. <u>The case $r = 2$</u>

<u>Theorem 5.4.3.</u> For $r = 2$, if $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure then,

(i) $\frac{k}{z}$ is an integer (= g, say),

(ii) there exists a $g \times 2$ array $\underset{\sim}{\alpha}_0$ satisfying (4.2.1) such that

$$\underset{\sim}{\alpha}' \equiv (\underset{\sim}{\alpha}_0' \,\vdots\, \underset{\sim}{\alpha}_0' \,\vdots\, \cdots \,\vdots\, \underset{\sim}{\alpha}_0') \,. \quad (5.4.10)$$

Proof. (i) If for $\underset{\sim}{\alpha}$ in reduced form, $\underset{\sim}{\gamma}$ satisfies (5.2.6), then each distinct row of $\underset{\sim}{\alpha}$ must have multiplicity z, i.e. $\frac{k}{z}$ is an integer.

(ii) Let $\underset{\sim}{\alpha}_0$ be constructed from the g distinct rows of $\underset{\sim}{\alpha}$, i.e. $\underset{\sim}{\alpha}_0$ satisfies (4.2.1), and from (i), (5.4.10) follows after possible permutation of the columns of $\underset{\sim}{\alpha}'$./ Hence Table A.1 can be used to provide arrays $\underset{\sim}{\alpha}_0'$ for values of g such that $k = gz$, since if $\underset{\sim}{\alpha}_0'$ produces the $\underset{\sim}{\alpha}(0,1)$-design with the highest h.m.c.e.f., then for a particular value of g, the $\underset{\sim}{\alpha}(0,1,2)$-design with P structure obtained from (5.4.10) will have the highest $\bar{\bar{E}}$.

For $r = 2$, arrays $\underset{\sim}{\alpha}'$ obtained using Theorem 5.4.3 are included in Table 5.4.1, together with $\bar{\bar{E}}$ for the resulting $\underset{\sim}{\alpha}(0,1,2)$-design with P structure. When there is a choice of $\underset{\sim}{\alpha}_0'$ available from Table A.1, the $\underset{\sim}{\alpha}_0'$ with the largest value of g (i.e. smallest value of z) is used for the construction of $\underset{\sim}{\alpha}'$ in Table 5.4.1.

5.4.2.2. <u>The cases $r = 3, 4$</u>

When $k \neq xs$ and $r > 2$, it is more difficult to obtain $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure. One method is to permute the symbols of a resolvable CIB design. David (1967) has shown that if an initial block generates a resolvable CIB design for $v = ks$ varieties and $r = k$ replicates, it must be of the form

$$(d_0k,\ d_1k+1,\ \ldots,\ d_{k-1}k + (k-1)), \qquad (5.4.11)$$
$$(0 \leq d_j < s;\ j = 0,1,\ldots,k-1).$$

Let $\underset{\sim}{\alpha}$ be given by

$$\underset{\sim}{\alpha} = \begin{array}{cccc} d_0 & d_{k-1} \dotplus 1 & \cdots & d_1 \dotplus 1 \\ d_1 & d_0 & \cdots & d_2 \dotplus 1 \\ d_2 & d_1 & \cdots & d_3 \dotplus 1 \\ \vdots & \vdots & & \vdots \\ d_{k-1} & d_{k-2} & \cdots & d_0 \end{array} . \qquad (5.4.12)$$

The $\underset{\sim}{\alpha}$-design obtained from this array is equivalent to the resolvable CIB design generated by (5.4.11), and hence must have P structure. Further, if the CIB design has only zeros, ones and twos off-diagonal in the concurrence matrix, then an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure is obtained.

<u>Example 5.4.3.</u> For $v = 12$, $r = k = 4$, David (1967) lists the initial block (0,1,3,6). Hence in (5.4.11), $d_0 = d_1 = d_3 = 0$, $d_2 = 1$ and thus $\underset{\sim}{\alpha}$ is given by

$$\underset{\sim}{\alpha} = \begin{array}{cccc} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array} \equiv \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} . /$$

No other general constructions have been obtained for $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure when $k \neq xs$.

A computer program has been written for systematic construction and testing of arrays $\underset{\sim}{\alpha}'$. Given a particular array, the following operations are carried out:

(i) Use Theorem 5.2.1 to check that $\underset{\sim}{\alpha}'$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design.

(ii) Use Theorem 5.2.5 to determine whether the design has P structure.

(iii) Calculate $\bar{\bar{E}}$ using the method described in section 3.4.

Arrays $\underset{\sim}{\alpha}'$ producing $\underset{\sim}{\alpha}$-designs with the lowest value of z, and highest $\bar{\bar{E}}$ are included in Table 5.4.1. When $k \neq xs$, the tabulated arrays in general do not produce minimal $\underset{\sim}{\alpha}(0,1,2)$-designs, and hence when $z > y$, we pay a price in terms of a smaller value of $\bar{\bar{E}}$ when we require the design to have P structure. More importantly, for many parameter combinations, it was not possible to obtain any solutions, and hence Table 5.4.1 does not provide the range of designs that would be desirable from a practical point of view.

The study of systematic methods for the construction of $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure is a topic for further research. However, in this thesis, we now relax the condition of P structure and consider the construction of minimal $\underset{\sim}{\alpha}(0,1,2)$-designs.

Table 5.4.1

$\underset{\sim}{\alpha}(0,1,2)$-Designs with P Structure, $2 \leq r \leq 4$, $2 \leq s \leq 9$, $3 \leq k \leq 20$

| | r | s | k | z | $\bar{\bar{E}}$ | Array $\underset{\sim}{\alpha}'$ |
|---|---|---|---|---|---|---|
| - | 2 | 2 | 4 | 2 | .7778 | 0 1 0 1 |
| | 2 | 3 | 4 | 2 | .6226 | 0 1 0 1 |
| - | 2 | 3 | 6 | 2 | .8095 | 0 1 2 0 1 2 |
| | 2 | 3 | 8 | 4 | .7753 | 0 1 0 1 0 1 0 1 |
| - | 2 | 3 | 9 | 3 | .8667 | 0 1 2 0 1 2 0 1 2 |
| | 2 | 4 | 6 | 2 | .7541 | 0 1 2 0 1 2 |
| - | 2 | 4 | 8 | 2 | .8378 | 0 1 2 3 0 1 2 3 |
| | 2 | 4 | 9 | 3 | .8235 | 0 1 2 0 1 2 0 1 2 |
| - | 2 | 4 | 12 | 3 | .8868 | 0 1 2 3 0 1 2 3 0 1 2 3 |
| | 2 | 4 | 15 | 5 | .8872 | 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 |
| - | 2 | 4 | 16 | 4 | .9130 | 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| | 2 | 5 | 6 | 2 | .7136 | 0 1 2 0 1 2 |
| | 2 | 5 | 8 | 2 | .8114 | 0 1 2 3 0 1 2 3 |
| | 2 | 5 | 9 | 3 | .7908 | 0 1 2 0 1 2 0 1 2 |
| - | 2 | 5 | 10 | 2 | .8596 | 0 1 2 3 4 0 1 2 3 4 |
| | 2 | 5 | 12 | 3 | .8668 | 0 1 2 3 0 1 2 3 0 1 2 3 |
| - | 2 | 5 | 15 | 3 | .9024 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 |
| | 2 | 5 | 16 | 4 | .8970 | 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| | 2 | 5 | 18 | 6 | .8844 | 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 0 1 2 |
| - | 2 | 5 | 20 | 4 | .9252 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 |
| | 2 | 6 | 8 | 2 | .7915 | 0 1 2 3 0 1 2 3 |
| | 2 | 6 | 9 | 3 | .7737 | 0 1 3 0 1 3 0 1 3 |
| | 2 | 6 | 10 | 2 | .8449 | 0 1 2 3 4 0 1 2 3 4 |
| - | 2 | 6 | 12 | 2 | .8765 | 0 1 2 3 4 5 0 1 2 3 4 5 |
| | 2 | 6 | 15 | 3 | .8915 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 |
| | 2 | 6 | 16 | 4 | .8847 | 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| | 2 | 7 | 8 | 2 | .7809 | 0 1 2 4 0 1 2 4 |
| | 2 | 7 | 9 | 3 | .7668 | 0 1 3 0 1 3 0 1 3 |
| | 2 | 7 | 10 | 2 | .8334 | 0 1 2 3 4 0 1 2 3 4 |
| | 2 | 7 | 12 | 2 | .8674 | 0 1 2 3 4 5 0 1 2 3 4 5 |
| - | 2 | 7 | 14 | 2 | .8899 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6 |
| | 2 | 8 | 9 | 3 | .7513 | 0 1 3 0 1 3 0 1 3 |
| | 2 | 8 | 10 | 2 | .8257 | 0 1 2 3 5 0 1 2 3 5 |
| | 2 | 8 | 12 | 2 | .8602 | 0 1 2 3 4 5 0 1 2 3 4 5 |
| | 2 | 9 | 10 | 2 | .8187 | 0 1 2 3 6 0 1 2 3 6 |
| - | 3 | 2 | 3 | 2 | .7435 | 0 1 0<br>0 0 1 |
| - | 3 | 2 | 4 | 3 | .8235 | 0 1 0 1<br>0 1 1 0 |
| | 3 | 3 | 4 | 2 | .7566 | 0 0 1 1<br>0 1 0 2 |
| - | 3 | 3 | 6 | 3 | .8500 | 0 1 2 0 1 2<br>0 2 1 1 0 2 |
| - | 3 | 3 | 9 | 6 | .8966 | 0 1 2 0 1 2 0 1 2<br>0 2 1 1 0 2 2 1 0 |
| | 3 | 4 | 5 | 2 | .7840 | 0 0 1 1 2<br>0 1 2 3 0 |

Table 5.4.1 (continued)

| | r | s | k | z | $\bar{\bar{E}}$ | Array $\underset{\sim}{\alpha}'$ |
|---|---|---|---|---|---|---|
| | 3 | 4 | 6 | 3 | .8163 | 0 0 1 1 2 2<br>0 1 0 2 0 3 |
| - | 3 | 4 | 8 | 3 | .8732 | 0 1 2 3 0 1 2 3<br>0 2 1 3 2 0 3 1 |
| - | 3 | 4 | 9 | 4 | .8841 | 0 0 0 1 1 2 2 3 3<br>0 2 3 1 2 1 3 0 1 |
| | 3 | 4 | 10 | 6 | .8878 | 0 0 0 0 1 1 1 2 2 3<br>0 1 2 3 0 2 3 0 3 0 |
| *- | 3 | 4 | 12 | 6 | .9126 | 0 1 2 3 0 1 2 3 0 1 2 3<br>0 2 3 1 1 3 2 0 3 1 0 2 |
| - | 3 | 4 | 16 | 9 | .9333 | 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3<br>0 2 1 3 2 0 3 1 1 3 2 0 3 1 0 2 |
| - | 3 | 5 | 6 | 1 | .8221 | 0 1 1 2 3 4<br>0 4 3 1 2 0 |
| | 3 | 5 | 8 | 3 | .8569 | 0 0 1 1 2 2 3 3<br>0 1 2 4 2 4 0 1 |
| | 3 | 5 | 9 | 4 | .8693 | 0 0 0 1 1 1 2 3 4<br>0 1 4 2 3 4 4 0 0 |
| - | 3 | 5 | 10 | 3 | .8909 | 0 1 2 3 4 0 1 2 3 4<br>0 4 3 2 1 1 0 4 3 2 |
| - | 3 | 5 | 15 | 6 | .9250 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4<br>0 4 3 2 1 1 0 4 3 2 2 1 0 4 3 |
| - | 3 | 5 | 20 | 9 | .9429 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4<br>0 4 3 2 1 1 0 4 3 2 2 1 0 4 3 3 2 1 0 4 |
| | 3 | 6 | 7 | 2 | .8300 | 0 0 1 1 2 2 3<br>0 5 5 1 4 3 4 |
| | 3 | 6 | 8 | 2 | .8540 | 0 0 1 1 3 3 4 5<br>0 3 0 2 1 5 5 3 |
| - | 3 | 6 | 9 | 2 | .8727 | 0 0 1 1 2 2 3 4 5<br>0 2 4 5 1 3 5 1 0 |
| | 3 | 6 | 10 | 3 | .8817 | 0 0 1 1 2 2 3 3 4 4<br>0 2 2 5 3 5 1 3 0 1 |
| | 3 | 6 | 11 | 4 | .8891 | 0 0 0 1 1 1 2 2 3 3 4<br>0 1 5 2 3 4 1 4 0 2 0 |
| - | 3 | 6 | 12 | 3 | .9045 | 0 1 2 3 4 5 0 1 2 3 4 5<br>0 3 1 4 2 5 3 0 4 1 5 2 |
| | 3 | 6 | 14 | 5 | .9139 | 0 0 0 1 1 1 2 2 2 3 3 3 4 5<br>0 4 5 2 3 5 1 3 4 0 1 2 0 0 |
| | 3 | 6 | 15 | 6 | .9178 | 0 0 0 1 1 1 2 2 2 3 3 3 4 4 4<br>0 1 4 2 3 4 2 3 4 0 1 3 0 1 2 |
| | 3 | 7 | 8 | 2 | .8454 | 0 0 1 1 2 3 4 6<br>0 5 1 3 5 2 3 2 |
| | 3 | 7 | 9 | 2 | .8654 | 0 0 1 1 2 2 3 4 5<br>0 3 5 6 3 4 1 1 0 |
| | 3 | 7 | 10 | 2 | .8805 | 0 0 1 1 2 2 3 3 4 5<br>0 6 2 5 3 4 1 6 1 0 |
| | 3 | 7 | 11 | 4 | .8823 | 0 0 0 1 1 1 2 2 3 3 4<br>0 1 6 2 4 5 1 5 0 2 0 |

Table 5.4.1 (continued)

| r | s | k | z | $\bar{\bar{E}}$ | Array $\underset{\sim}{\alpha}'$ |
|---|---|---|---|---|---|
| - 3 | 7 | 14 | 3 | .9151 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6 |
| | | | | | 0 6 5 4 3 2 1 1 0 6 5 4 3 2 |
| 3 | 8 | 9 | 2 | .8593 | 0 0 1 1 2 2 3 4 5 |
| | | | | | 0 6 3 4 3 7 1 1 0 |
| - 4 | 2 | 3 | 4 | .7692 | 0 1 0 |
| | | | | | 0 0 1 |
| | | | | | 0 1 1 |
| - 4 | 2 | 4 | 6 | .8400 | 0 1 0 1 |
| | | | | | 0 0 1 1 |
| | | | | | 0 1 1 0 |
| - 4 | 3 | 4 | 3 | .7891 | 0 0 1 2 |
| | | | | | 0 2 1 0 |
| | | | | | 0 1 2 1 |
| - 4 | 3 | 6 | 6 | .8644 | 0 1 2 0 1 2 |
| | | | | | 0 2 1 2 1 0 |
| | | | | | 0 0 1 1 2 2 |
| - 4 | 3 | 9 | 12 | .9070 | 0 1 2 0 1 2 0 1 2 |
| | | | | | 0 2 1 1 0 2 2 1 0 |
| | | | | | 0 0 0 1 1 1 2 2 2 |
| 4 | 4 | 5 | 4 | .8051 | 0 0 0 1 3 |
| | | | | | 0 2 3 2 0 |
| | | | | | 0 3 1 1 2 |
| 4 | 4 | 6 | 5 | .8400 | 0 0 0 1 1 3 |
| | | | | | 0 2 3 1 2 0 |
| | | | | | 0 3 2 2 1 2 |
| 4 | 4 | 7 | 6 | .8637 | 0 0 0 1 1 3 3 |
| | | | | | 0 2 3 0 1 0 1 |
| | | | | | 0 3 1 3 2 2 1 |
| - 4 | 4 | 8 | 6 | .8857 | 0 1 2 3 0 1 2 3 |
| | | | | | 0 1 3 0 2 3 1 2 |
| | | | | | 0 0 1 1 2 2 3 3 |
| 4 | 4 | 9 | 10 | .8908 | 0 0 0 0 1 1 2 2 3 |
| | | | | | 0 1 2 3 0 2 0 1 0 |
| | | | | | 0 3 2 1 3 1 2 1 1 |
| 4 | 4 | 10 | 10 | .9046 | 0 0 0 0 1 1 2 2 3 3 |
| | | | | | 0 1 2 3 0 1 0 3 1 2 |
| | | | | | 0 3 2 1 1 0 3 2 2 1 |
| 4 | 4 | 11 | 12 | .9117 | 0 0 0 0 1 1 1 2 2 2 3 |
| | | | | | 0 1 2 3 0 2 3 0 1 3 2 |
| | | | | | 0 3 2 1 3 1 0 2 1 0 3 |
| *- 4 | 4 | 12 | 12 | .9216 | 0 1 2 3 0 1 2 3 0 1 2 3 |
| | | | | | 0 2 3 1 1 3 2 0 3 1 0 2 |
| | | | | | 0 3 1 2 1 2 0 3 3 0 2 1 |
| - 4 | 4 | 16 | 18 | .9403 | 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| | | | | | 0 1 2 3 1 2 3 0 2 3 0 1 3 0 1 2 |
| | | | | | 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 4 | 5 | 6 | 4 | .8275 | 0 0 0 1 2 3 |
| | | | | | 0 3 1 3 1 4 |
| | | | | | 0 2 3 1 1 2 |

Table 5.4.1 (continued)

| | r | s | k | z | $\bar{E}$ | Array $\alpha'$ |
|---|---|---|---|---|---|---|
| | 4 | 5 | 7 | 6 | .8459 | 0 0 0 1 1 2 2<br>0 1 2 0 1 0 1<br>0 4 3 3 2 1 0 |
| | 4 | 5 | 8 | 7 | .8669 | 0 0 0 1 1 1 2 2<br>0 2 3 0 1 4 1 2<br>0 3 2 4 3 0 2 1 |
| | 4 | 5 | 9 | 8 | .8818 | 0 0 0 1 1 1 2 2 2<br>0 2 3 0 1 4 0 1 3<br>0 3 2 4 3 0 3 2 0 |
| - | 4 | 5 | 10 | 6 | .9018 | 0 1 2 3 4 0 1 2 3 4<br>0 4 3 2 1 1 0 4 3 2<br>0 3 1 4 2 1 4 2 0 3 |
| | 4 | 5 | 11 | 10 | .9040 | 0 0 0 0 1 1 1 2 2 3 3<br>0 1 2 4 0 1 4 1 2 0 1<br>0 4 3 1 4 3 0 2 1 2 1 |
| | 4 | 5 | 13 | 12 | .9194 | 0 0 0 0 1 1 1 2 2 2 3 3 3<br>0 1 2 3 0 1 2 0 1 2 0 1 2<br>0 4 3 2 4 3 2 3 2 1 2 1 0 |
| - | 4 | 5 | 15 | 12 | .9328 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4<br>0 4 3 2 1 1 0 4 3 2 2 1 0 4 3<br>0 3 1 4 2 1 4 2 0 3 2 0 3 1 4 |
| - | 4 | 5 | 20 | 18 | .9489 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4<br>0 4 3 2 1 1 0 4 3 2 2 1 0 4 3 3 2 1 0 4<br>0 3 1 4 2 1 4 2 0 3 2 0 3 1 4 3 1 4 2 0 |
| | 4 | 6 | 7 | 6 | .8303 | 0 0 0 1 1 2 2<br>0 1 4 2 5 0 1<br>0 3 2 2 1 3 2 |
| - | 4 | 6 | 12 | 6 | .9142 | 0 1 2 3 4 5 0 1 2 3 4 5<br>0 3 1 4 2 5 3 0 4 1 5 2<br>0 4 0 2 5 1 3 1 3 5 2 4 |
| | 4 | 7 | 8 | 3 | .8653 | 0 0 1 1 2 3 4 5<br>0 4 1 4 5 1 3 6<br>0 3 5 6 3 2 4 4 |
| - | 4 | 7 | 14 | 6 | .9238 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6<br>0 6 5 4 3 2 1 1 0 6 5 4 3 2<br>0 4 1 5 2 6 3 1 5 2 6 3 0 4 |
| | 4 | 8 | 9 | 4 | .8736 | 0 0 1 1 2 2 3 4 5<br>0 7 7 4 4 1 4 1 1<br>0 1 2 5 4 7 3 2 0 |

- means a minimal design is produced,

* means an array $\beta'$ is tabulated.

## 5.5 Construction of Tables of Minimal $\underset{\sim}{\alpha}(0,1,2)$-Designs

### 5.5.1. The case r = 2

Let $x = [\frac{k}{s}]$, $f = k - xs$ and suppose $\underset{\sim}{\alpha}_0'$ is a $2 \times f$ array satisfying (4.2.3).

Define

$$\underset{\sim}{\alpha}_1' = \begin{matrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & s-1 \end{matrix} , \qquad (5.5.1)$$

then from (5.3.1), we have the following result.

Theorem 5.5.1. The $2 \times k$ array $\underset{\sim}{\alpha}'$ given by

$$\underset{\sim}{\alpha}' = (\underbrace{\underset{\sim}{\alpha}_1' \;\vdots\; \cdots \;\vdots\; \underset{\sim}{\alpha}_1'}_{x \text{ terms}} \;\vdots\; \underset{\sim}{\alpha}_0') , \qquad (5.5.2)$$

produces a minimal $\underset{\sim}{\alpha}(0,1,2)$-design./

Table A.1 has been used to provide arrays $\underset{\sim}{\alpha}_0'$ which, when substituted into (5.5.2) produce minimal $\underset{\sim}{\alpha}(0,1,2)$-designs with the highest $\bar{E}$.

For $r = 2$, arrays $\underset{\sim}{\alpha}'$ obtained using Theorem 5.5.1 are given in Table C.1, and the properties of the resulting minimal $\underset{\sim}{\alpha}(0,1,2)$-designs are given in Table D.1. The format of Tables C and D is the same as that of Tables A and B respectively (see section 4.3). The only difference is that for $\underset{\sim}{\alpha}(0,1,2)$-designs, there is an extra column for the harmonic mean of the pairwise efficiencies for varieties that appear together twice within blocks. When $r = 2$, the pairwise comparison of varieties that appear together twice within blocks will be made with full efficiency, hence in Table D.1, the harmonic mean of the pairwise efficiencies is one, with range zero.

5.5.2. The cases $r = 3,4$

When possible, the arrays $\underset{\sim}{\alpha}'$ constructed in section 5.4 are substituted into Table C. These arrays produce minimal $\underset{\sim}{\alpha}(0,1,2)$-designs with P structure.

A computer program has been written for systematic construction and testing of arrays $\underset{\sim}{\alpha}'$. Given a particular array, the following operations are carried out:

(i) Use Theorem 5.2.1 to check that $\underset{\sim}{\alpha}'$ produces an $\underset{\sim}{\alpha}(0,1,2)$-design.

(ii) Check that $\underset{\sim}{\alpha}'$ produces a minimal $\underset{\sim}{\alpha}(0,1,2)$-design.

(iii) Calculate $\bar{E}$ using the method described in section 3.4.

For parameters in the ranges $ks \leq 100$, $s < k < s^2$ arrays $\underset{\sim}{\alpha}'$ producing minimal $\underset{\sim}{\alpha}(0,1,2)$-designs with the highest $\bar{E}$ are given in Table C.

For $r = 4$, $s = 3$, $k = 7$ it can be shown by enumeration of all possible arrays $\underset{\sim}{\alpha}'$, that no minimal $\underset{\sim}{\alpha}(0,1,2)$-design can be constructed, but for completeness, an array producing a non-minimal $\underset{\sim}{\alpha}(0,1,2)$-design is given in Table C.3.

The properties of the minimal $\underset{\sim}{\alpha}(0,1,2)$-designs obtained from Table C are given in Table D for $p = 0$.

## 5.6 The Incorporation of Control Varieties into $\underset{\sim}{\alpha}(0,1,2)$-Designs

In section 3.7, we discussed the desirability of choosing the $t$ control varieties so that they were spread evenly over the $b$ blocks. Further, it is desirable that each control variety should appear in blocks with as many other varieties as possible, as this will have the effect of increasing the harmonic mean of the efficiencies for the comparison of the mean of the control varieties with all other varieties.

From Definition 5.2.2 and (3.2.4), we have that rows $ms$, $ms+1$, ..., $(m+1)s-1$ of $\underset{\sim}{NN}'$ each contain $(\underset{\sim}{\gamma})^m$ twos $(m = 0,1,\ldots,k-1)$. If $(\underset{\sim}{\gamma})^{m'}$ is the smallest element of $\underset{\sim}{\gamma}$, then since $\underset{\sim}{NN}'\underset{\sim}{1} = rk\underset{\sim}{1}$, it follows that varieties $m's$, $m's+1$, ..., $(m'+1)s-1$ each appear in blocks with more different varieties than for any larger $(\underset{\sim}{\gamma})^m$; hence the control varieties should be chosen from this set. By interchanging say, rows 0 and $m'$ of $\underset{\sim}{\alpha}$, we may always arrange for $(\underset{\sim}{\gamma})^0$ to be the smallest element of $\underset{\sim}{\gamma}$, and hence when $t < s$, the control varieties can be allocated to the symbols 0, 1, ..., $t-1$ of the design. This has been done for all the arrays in Table C.

<u>Example 5.6.1.</u> For $r = 3$, $k = 5$, $s = 4$ let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 2 & 3 & 0 & 1 \end{matrix}, \qquad (5.6.1)$$

hence

$$\underset{\sim}{\gamma} = (2,2,1,1,0)' .$$

In other words, the last 4 rows of $\underset{\sim}{NN}'$ do not contain any twos, and thus each of the symbols 16, 17, 18, 19 appear in blocks with the maximum number of $r(k-1) = 12$, other symbols. By interchanging the first and last column of (5.6.1), we obtain,

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 3 & 0 \\ 1 & 2 & 3 & 0 & 0 \end{matrix}$$

$$= \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 2 \\ 0 & 1 & 2 & 3 & 3 \end{matrix} \quad , \qquad (5.6.2)$$

and now $\underset{\sim}{Y} = (0,2,1,1,2)'$, hence the control varieties should be allocated to one or more of the symbols 0,1,2,3 for the $\underset{\sim}{\alpha}(0,1,2)$-design obtained from (5.6.2)./

## 5.6.1. $\underset{\sim}{\alpha}(0,1,2)$-Designs with P Structure

When the design has P structure then from (5.2.6), we may simply allocate control varieties to symbols 0, 1, ..., t-1 of the design.

### 5.6.1.1. $\underset{\sim}{\alpha}(0,1,2)$-Designs with Supplemented P Structure

The partition of the varieties into control and new varieties, allows us to introduce the idea of supplemented P structure.

**Definition 5.6.1.** If for an $\underset{\sim}{\alpha}(0,1,2)$-design,

$$\underset{\sim}{Y} = s(0,1,1,\ldots,1)' , \qquad (5.6.3)$$

then the design is said to have supplemented P structure./ Hence, if we have $s$ control varieties, we may allocate them to the symbols 0, 1, ..., s-1 of an $\underset{\sim}{\alpha}(0,1,2)$-design with supplemented P structure. Note that by removing the control varieties from the design, we obtain an $\underset{\sim}{\alpha}(0,1,2)$-design with P structure for $v = (k-1)s$.

<u>Example 5.6.2.</u> For $r = 3$, $k = 7$, $s = 5$ let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 & 1 & 2 \end{matrix} \quad (5.6.4)$$

then $\underset{\sim}{\gamma} = 2(0,1,1,1,1,1,1)'$, and hence the $\underset{\sim}{\alpha}(0,1,2)$-design obtained from (5.6.4) has supplemented P structure with $z = 2$./

Designs with supplemented P structure are only likely to be useful when exactly $s$ control varieties are required.

# CHAPTER 6

# ALMOST EQUIBLOCK-SIZED $\alpha$-DESIGNS

## 6.1 Introduction and Summary

In the previous chapters we have considered the construction of resolvable block designs for $ks$ varieties with block size $k$. However in many practical cases, the number of varieties cannot be conveniently factorized in the form $v = ks$ and hence to construct suitable block designs, we must either relax the condition of resolvability or that of equal block sizes. We feel that from the practical point of view, resolvability is more important than equal block sizes, so long as the block sizes differ by not more than one unit.

In this chapter, almost equiblock-sized (AE) $\alpha$-designs are defined for $ks - p$ varieties ($p = 1, 2, \ldots, s-1$) with block sizes $k$, $k-1$.

The construction of AE, $\alpha$-designs from $\alpha$-designs is considered in section 6.2 and the compilation of tables in section 6.3.

Finally, in section 6.4, a method is given for the effective allocation of control varieties to the symbols of AE $\alpha$-designs.

## 6.2 Generation of AE $\alpha$-Designs

We first define an AE design.

**Definition 6.2.1.** An unequal block-sized design whose block sizes differ by at most one unit is called an almost equiblock-sized (AE) design./

In previous chapters we have constructed $\underset{\sim}{\alpha}$-designs from a $k \times r$ array $\underset{\sim}{\alpha}$, and a knowledge of the value of $s$. We now introduce a further parameter, $p$ $(0 < p < s)$ and construct an AE design in the following way.

<u>Construction 6.2.1.</u> (i) Obtain the $\underset{\sim}{\alpha}$-design from $\underset{\sim}{\alpha}$ using Construction 3.2.1.

(ii) Delete the $p$ symbols $v-p-1$, $v-p, \ldots, v-1$ from this $\underset{\sim}{\alpha}$-design./

We thus obtain a resolvable block design for $ks-p$ varieties, with $r(s-p)$ blocks of size $k$ and $rp$ blocks of size $k-1$. It is for convenience that we choose to delete symbols from the highest downwards. In section 6.3.2 we will see that this may necessitate a rearrangement of the rows of $\underset{\sim}{\alpha}$.

For some experimental designs it is difficult to delete more than one or two symbols and still maintain an AE design. However, for $\underset{\sim}{\alpha}$-designs Construction 6.2.1 automatically copes with this problem. Note that if $p = 0$, we obtain an $\underset{\sim}{\alpha}$-design with block size $k$, whereas setting $p = s$ is equivalent to removing the last row of the array $\underset{\sim}{\alpha}$ to give an $\underset{\sim}{\alpha}$-design with block size $k-1$.

<u>Definition 6.2.2.</u> The resolvable AE design obtained from $\underset{\sim}{\alpha}$ using Construction 6.2.1 is called an AE $\underset{\sim}{\alpha}$-design./

AE $\underset{\sim}{\alpha}$-designs provide resolvable designs for situations when $v$ does not conveniently factorize in the form $v = ks$. Thus the construction of AE $\underset{\sim}{\alpha}$-designs considerably increases the number of available designs.

## 6.3 Construction of Tables of AE $\underset{\sim}{\alpha}$-Designs

The theory developed in the previous chapters cannot be directly used for AE $\underset{\sim}{\alpha}$-designs, however the following intuitive ideas seem to work well in general:

(i) If $\underset{\sim}{\alpha}$ produces an $\underset{\sim}{\alpha}$-design with a high h.m.c.e.f., then the AE $\underset{\sim}{\alpha}$-designs obtained from $\underset{\sim}{\alpha}$ for $p = 1,2,\ldots,s-1$ will each have a high h.m.c.e.f. . Hence the arrays given in Tables A and C may be used to produce a wide range of AE $\underset{\sim}{\alpha}$-designs. The technique of deleting symbols from standard designs has been used by Hunter (1974).

(ii) From (1.5.3), for an AE $\underset{\sim}{\alpha}$-design,

$$\underset{\sim}{A} = \underset{\sim}{I}_v - \frac{1}{r}\underset{\sim}{N}\,\underset{\sim}{k}^{-\delta}\underset{\sim}{N}', \qquad (6.3.1)$$

and hence the concurrence matrix, $\underset{\sim}{N}\underset{\sim}{N}'$ does not enter explicitly into the analysis, however since the elements of $\underset{\sim}{k}$ differ by not more than one, the elements of $\underset{\sim}{N}\,\underset{\sim}{k}^{-\delta}\underset{\sim}{N}'$ may be approximately related to the elements of $\underset{\sim}{N}\underset{\sim}{N}'$. Then, analagous to (3.6.3), we may write

$$\underset{\sim}{V} \approx \frac{1}{r}\left(\underset{\sim}{I}_v + \frac{1}{r}\underset{\sim}{N}\,\underset{\sim}{k}^{-\delta}\underset{\sim}{N}'\right), \quad (\text{mod } \underset{\sim}{J}_v)\,; \qquad (6.3.2)$$

it follows that pairwise variances may still be grouped according to the number of distinct off-diagonal elements of $\underset{\sim}{N}\underset{\sim}{N}'$, as has been done for $\underset{\sim}{\alpha}$-designs.

### 6.3.1. AE $\underset{\sim}{\alpha}$-Designs obtained from $\underset{\sim}{\alpha}(0,1)$-Designs

The $\underset{\sim}{\alpha}(0,1)$-designs produced by the arrays $\underset{\sim}{\alpha}'$ given in Table A have been used in Construction 6.2.1 to provide AE

$\underset{\sim}{\alpha}$-designs for $ks - p$ varieties $(p = 1,2,\dots,k-1)$. For example, for $r = 3$, $s = 7$, $k = 4$ the array $\underset{\sim}{\alpha}'$ given in Table A.2 is used to produce 3 replicate resolvable designs for

(i) 28 varieties, block size 4 $(p = 0)$,

(ii) 27 varieties, block sizes 3,4 $(p = 1)$,

(iii) 26 varieties, block sizes 3,4 $(p = 2)$,

(iv) 25 varieties, block sizes 3,4 $(p = 3)$.

The properties of all AE $\underset{\sim}{\alpha}$-designs derived in this way are given in Table B.

### 6.3.2. AE $\underset{\sim}{\alpha}$-Designs obtained from $\underset{\sim}{\alpha}(0,1,2)$-Designs

The minimal $\underset{\sim}{\alpha}(0,1,2)$-designs produced by the arrays $\underset{\sim}{\alpha}'$ given in Table C have also been used in Construction 6.2.1 to provide AE $\underset{\sim}{\alpha}$-designs for $ks - p$ varieties $(p = 1,2,\dots,s-1)$. Since these minimal $\underset{\sim}{\alpha}(0,1,2)$-designs in general do not have P structure, we would like to remove the symbols which occur twice in blocks with other symbols, the greatest number of times, i.e. we try to keep the number of twos in the concurrence matrix of an AE $\underset{\sim}{\alpha}$-design at a minimum, as this has the effect of increasing the h.m.c.e.f.. For this reason we have arranged the columns of $\underset{\sim}{\alpha}'$ in Table C so that the largest element of $\underset{\sim}{\gamma}$ is the last element, $(\underset{\sim}{\gamma})^{k-1}$.

<u>Example 6.3.1.</u> For $r = 3$, $k = 4$, $s = 3$ let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{matrix}\,, \qquad (6.3.3)$$

so that

$$\underset{\sim}{\gamma} = (1,1,1,3)' .$$

Let $p = 1$ ; then for the AE $\underset{\sim}{\alpha}$-design obtained from (6.3.3), $\bar{E} = .7585$ .

Now interchange, say the first and last columns of (6.3.3), i.e. let $\underset{\sim}{\alpha}'$ be given by

$$\underset{\sim}{\alpha}' = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 \end{matrix} ,$$

$$\equiv \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{matrix} , \qquad (6.3.4)$$

then

$$\underset{\sim}{\gamma} = (3,1,1,1)' .$$

Let $p = 1$ ; then for the AE $\underset{\sim}{\alpha}$-design obtained from (6.3.4), $\bar{E} = .7378$.

Thus $\bar{E}$ depends on the number of twos in the concurrence matrix of the AE $\underset{\sim}{\alpha}$-design./

## 6.4 The Incorporation of Control Varieties into AE $\underset{\sim}{\alpha}$-Designs

For $\underset{\sim}{\alpha}$-designs, we have adopted the convention of allocating the control varieties to $t$ of the first $s$ symbols of the design, where because $\underset{\sim}{N}\underset{\sim}{N}'$ has BC structure, it does not matter much which combination of $t$ symbols are chosen. However for AE $\underset{\sim}{\alpha}$-designs, some particular combinations of $t$ symbols from the first $s$ may be preferred for allocation of control varieties.

To illustrate this, let us consider an AE $\underset{\sim}{\alpha}$-design with $p = 1$, obtained from the array $\underset{\sim}{\alpha}$. The concurrence matrix of the AE $\underset{\sim}{\alpha}$-design may be obtained by removing the last row and column from the concurrence matrix of the $\underset{\sim}{\alpha}$-design. If the $h^{th}$ element of this last column is 0, and the $h'^{th}$ element is, say 1 $(h,h' = 0,1,\ldots,s-1)$, then in the AE $\underset{\sim}{\alpha}$-design, symbol $h$ would occur in blocks with more of the other symbols than symbol $h'$, and so would be preferred for allocation of a control variety.

Hence from (3.2.4), it follows that the row sums of the last $p$ columns of the circulant matrix $(\underset{\sim}{\mu}'_{0\,k-1})$ determine the order in which the first $s$ symbols of the AE $\underset{\sim}{\alpha}$-design are to be allocated the $t$ control varieties, i.e. symbols corresponding to rows with the smallest row sums are most desirable for allocation of control varieties.

<u>Example 6.4.1.</u> For $r = 3$, $k = 4$, $s = 3$ let $\underset{\sim}{\alpha}'$ be given by (6.3.3), then $\underset{\sim}{\mu}_{03} = (2,0,1)'$ and hence

$$(\underset{\sim}{\mu}'_{03}) = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} . \qquad (6.4.1)$$

(i) When $p = 1$, the row sums of the last column of (6.4.1) are $(1,0,2)'$ and hence the control varieties should be allocated to symbols in the order 1,0,2, i.e. when $t = 1$, the control variety should be allocated to symbol 1 ; when $t = 2$, the control varieties should be allocated to symbols 1 and 0, etc.

(ii) When $p = 2$, the row sums of the last two columns of (6.4.1) are $(1,2,3)'$ and hence the order for control varieties should be 0,1,2 ./

In Tables B and D, the order for control varieties for $t \leq \min(s,5)$ is given, which should be sufficient to cover most experimental situations.

## REFERENCES

BELLMAN, R. (1970). Introduction to matrix analysis. McGraw-Hill, New York.

BOSE, R.C. and BUSH, K.A. (1952). Orthogonal arrays of strength two and three. Ann. Math. Statist., 23, 508-24.

BOSE, R.C., CLATWORTHY, W.H. and SHRIKHANDE, S.S. (1954). Tables of partially balanced block designs with two associate classes. North Carolina Agricultural Experimental Bulletin, No. 107.

BOSE, R.C. and NAIR, K.R. (1939). Partially balanced incomplete block designs. Sankhyā, 4, 337-72.

BOSE, R.C. and NAIR, K.R. (1962). Resolvable incomplete block designs with two replications. Sankhyā, A24, 9-24.

CALINSKI, T. (1971). On some desirable patterns in block designs. Biometrics, 27, 275-92.

CLATWORTHY, W.H. (1973). Tables of two-associate-class partially balanced designs. Nat. Bur. Standards Appl. Math. Ser., 63.

COCHRAN, W.G. and COX, G.M. (1957). Experimental Designs. Wiley, New York.

COTTER, S.C., JOHN, J.A. and SMITH, T.M.F. (1973). Multifactor experiments in non-orthogonal designs. J.R. Statist. Soc., B35, 361-7.

DAVID, H.A. (1963). The structure of cyclic paired comparison designs. J. Aust. Math. Soc., 3, 117-27.

DAVID, H.A. (1967). Resolvable cyclic designs. Sankhyā, A29, 191-8.

DAVIS, A.W. and HALL, W.B. (1969). Cyclic change-over designs. Biometrika, 56, 283-93.

FISHER, R.A. and YATES, F. (1963). Statistical tables for biological, agricultural and medical research. Oliver and Boyd, London.

HARSHBARGER, B. (1947). Rectangular lattices. Virginia Agricultural Experimental Station Memoir 1.

HEDAYAT, A. and FEDERER, W.T. (1969). An application of group theory to the existence and nonexistence of orthogonal latin squares. Biometrika, 56, 547-51.

HINKELMANN, K. (1964). Extended group divisible partially balanced incomplete block designs. Ann. Math. Statist., 35, 681-95.

HUNTER, E.A. (1974). Block designs for variety trials. M.Phil. Thesis, Edinburgh University.

JAMES, A.T. and WILKINSON, G.N. (1971). Factorization of the residual operator and canonical decomposition of nonorthogonal factors in the analysis of variance. Biometrika, 58, 279-94.

JOHN, J.A. (1966). Cyclic incomplete block designs. J. R. Statist. Soc., B28, 345-60.

JOHN, J.A. (1973). Factorial experiments in cyclic designs. Ann. Statist., 1, 188-94.

JOHN, J.A. and SMITH, T.M.F. (1972). Two factor experiments in non-orthogonal designs. J.R. Statist. Soc., B34 401-9.

JOHN, J.A., WOLOCK, F.W. and DAVID, H.A. (1972). Cyclic designs. Nat. Bur. Standards Appl. Math. Ser., 62.

JONES, R.M. (1959). On a property of incomplete blocks. J.R. Statist. Soc., B21, 172-9.

KSHIRSAGAR, A.M. (1966). Balanced factorial designs. J.R. Statist. Soc., B28, 559-67.

PATTERSON, H.D. and THOMPSON, R. (1971). Recovery of inter-block information when block sizes are unequal. Biometrika, 58, 545-54.

PEARCE, S.C. (1963). The use and classification of non-orthogonal designs. J.R. Statist. Soc., A126, 353-69.

PEARCE, S.C. (1970). The efficiency of block designs in general. Biometrika, 57, 339-46.

PEARCE, S.C., CALINSKI, T. and MARSHALL, T.F.de C. (1974). The basic contrasts of an experimental design with special reference to the analysis of data. Biometrika, 61, 449-60.

RAGHAVARAO, D. (1971). Constructions and combinatorial problems in design of experiments. Wiley, New York.

RAO, C.R. (1973). Linear statistical inference and its applications. Wiley, New York.

SHAH, B.V. (1959). A generalization of partially balanced incomplete block designs. Ann. Math. Statist., 30, 1041-50.

TOCHER, K.D. (1952). The design and analysis of block experiments. J.R. Statist. Soc., B14, 45-91.

YATES, F. (1936a). A new method of arranging variety trials involving a large number of varieties. J. Agr. Sci., 26, 424-55.

YATES, F. (1936b). Incomplete randomized blocks. Ann. Eugen., 7, 121-40.

# APPENDIX A

## MATRIX STRUCTURES

### A.1 Summary

In this appendix several structures are defined for a square matrix $\underset{\sim}{A}$ with real elements. In sections A.2 and A.3, circulant matrices and block circulant matrices are defined and some properties are derived.

It is shown in sections A.4 and A.5 how the theory of sections A.2 and A.3 may be generalized to n factors, and generalized circulant matrices and block generalized circulant matrices are defined. Several theorems are stated without proof, they being straightforward generalizations of earlier results.

A generalization of sections A.2 and A.3 in a different direction is given in sections A.6 and A.7 by defining double circulant matrices and block double circulant matrices. Under certain conditions, some properties of double circulant matrices are proved in section A.6, and in section A.7 similar results for block double circulant matrices are simply stated, their proofs not involving any new ideas.

Balanced factorial structure is defined in section A.8 and it is shown that this structure is a special case of generalized circulant structure. Some properties of this structure are compared with previous results for the same structure.

In section A.9, orthogonal factorial structure is defined and several theorems are stated. A comparison is made with recent work by other authors.

Finally in section A.10, a definition of permutation structure is given.

## A.2 Circulant Matrices

Definition A.2.1. An $s \times s$ matrix $\underset{\sim}{\Gamma}_h$ $(h = 0,1,\ldots,s-1)$, whose $(l,m)^{th}$ element,

$$(\underset{\sim}{\Gamma}_h)^{lm} = 1 \text{, if } m \dot{-} l = h, \quad = 0 \text{, otherwise}, \quad (l,m = 0,1,\ldots,s-1),$$

is called a basic circulant./

Definition A.2.2. An $s \times s$ matrix $\underset{\sim}{A}$ is said to be a circulant matrix if it can be written as a linear combination of basic circulants, i.e.

$$\underset{\sim}{A} = \sum_{h=0}^{s-1} \theta_h \underset{\sim}{\Gamma}_h . \qquad (A.2.1)$$

$\underset{\sim}{A}$ is said to have C structure./

The latent roots and latent vectors of a circulant matrix involve the $s^{th}$ roots of unity (Bellman, 1970, page 242).

Theorem A.2.1. Let $\underset{\sim}{A}$ have C structure and be given by (A.2.1). The latent roots of $\underset{\sim}{A}$ are given by

$$\theta_u^* = \sum_{h=0}^{s-1} \omega_u^{\,h} \theta_h , \qquad (A.2.2)$$

with corresponding latent vectors

$$\underset{\sim}{\gamma}_u' = \frac{1}{\sqrt{s}} (\omega_u^{\,0}, \omega_u^{\,1}, \ldots, \omega_u^{\,s-1}), \qquad (A.2.3)$$

where $\omega_u$ is the $u^{th}$, $s^{th}$ root of unity, i.e.

$$\omega_u^{\,h} = \exp\left(\frac{2\pi i u h}{s}\right), \qquad (A.2.4)$$

$$= \cos\left(\frac{2\pi u h}{s}\right) + i \sin\left(\frac{2\pi u h}{s}\right), \qquad (A.2.5)$$

$$(u = 0,1,\ldots,s-1).$$

Proof. Following Davis and Hall (1969), define mutually orthogonal idempotent matrices

$$\underset{\sim}{\Gamma}_u^* = \frac{1}{s} \sum_{h=0}^{s-1} \omega_u^{-h} \underset{\sim}{\Gamma}_h , \qquad (A.2.6)$$

$$(u = 0,1,\ldots,s-1).$$

Then (A.2.1) may be written as

$$\underset{\sim}{A} = \sum_{u=0}^{s-1} \theta_u^* \underset{\sim}{\Gamma}_u^* . \qquad (A.2.7)$$

The $\underset{\sim}{\Gamma}_u^*$ have rank 1, with the $(l,m)^{th}$ element equal to $\frac{1}{s} \omega_u^{(m-l)}$ $(l,m = 0,1,\ldots,s-1)$. Hence

$$\underset{\sim}{\Gamma}_u^* = \bar{\underset{\sim}{\gamma}}_u \underset{\sim}{\gamma}_u' , \qquad (A.2.8)$$

$$(u = 0,1,\ldots,s-1),$$

and thus $\underset{\sim}{A}$ may be written in the spectral form

$$\underset{\sim}{A} = \sum_{u=0}^{s-1} \theta_u^* \bar{\underset{\sim}{\gamma}}_u \underset{\sim}{\gamma}_u' . / \qquad (A.2.9)$$

Theorem A.2.2. Let $\underset{\sim}{A}$ have C structure and be given by (A.2.1). If $\underset{\sim}{A}$ is symmetric then:

(i) $$\theta_h = \theta_{-h} , \qquad (A.2.10)$$

$$(h = 0,1,\ldots,s-1).$$

(ii) $$\theta_u^* = \theta_{-u}^* = \sum_{h=0}^{s-1} \theta_h \cos\left(\frac{2\pi uh}{s}\right), \qquad (A.2.11)$$

$$(u = 0,1,\ldots,s-1).$$

(iii) We can find a complete set of real latent vectors for the real latent roots $\theta_u^*$.

Proof. (i) If $\underset{\sim}{A}$ is symmetric then the $(l,m)^{th}$ element of $\underset{\sim}{A}$, $\theta_{(m \dot{-} l)}$, will be equal to the $(m,l)^{th}$ element of $\underset{\sim}{A}$, $\theta_{(l \dot{-} m)}$, i.e. $\theta_h = \theta_{\dot{-}h}$.

(ii) From (A.2.5) and (A.2.10),

$$\theta_h \, \omega_u^{\;h} + \theta_{\dot{-}h} \, \omega_u^{\;-h} = 2\,\theta_h \cos\left(\frac{2\pi uh}{s}\right), \qquad \text{(A.2.12)}$$

$$(u,h = 0,1,\ldots,s-1).$$

Substitution in (A.2.2) gives (A.2.10).

(iii) This is a fundamental property of real symmetric matrices (Bellman, 1970, page 54). For example we may obtain a complete set of real latent vectors from the $\underset{\sim}{\gamma}_u$ given by (A.2.3) as follows:

Define

$$\begin{bmatrix} \underset{\sim}{\delta}_u{}' \\ \underset{\sim}{\delta}_{\dot{-}u} \end{bmatrix} = \underset{\sim}{U} \begin{bmatrix} \underset{\sim}{\gamma}_u{}' \\ \underset{\sim}{\gamma}_{\dot{-}u} \end{bmatrix}, \qquad \text{(A.2.13)}$$

$$(u = 1,2,\ldots,[\tfrac{s-1}{2}]),$$

where $\underset{\sim}{U}$ is the unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}. \qquad \text{(A.2.14)}$$

Then $\underset{\sim}{\delta}_u$ and $\underset{\sim}{\delta}_{\dot{-}u}$ are two real orthogonal vectors for the latent root $\theta_u^*$ $(u = 1,2,\ldots,[\frac{s-1}{2}])$, i.e.

$$\underset{\sim}{\delta}_u{}' = \sqrt{\tfrac{2}{s}}\,\{1, \cos(\tfrac{2\pi u}{s}), \ldots, \cos(\tfrac{2\pi(s-1)u}{s})\}$$

$$\underset{\sim}{\delta}_{\dot{-}u}{}' = \sqrt{\tfrac{2}{s}}\,\{0, \sin(\tfrac{2\pi u}{s}), \ldots, \sin(\tfrac{2\pi(s-1)u}{s})\}. \qquad \text{(A.2.15)}$$

Further, let

$$\underset{\sim}{\delta}_0{}' = \underset{\sim}{\gamma}_0{}' = \frac{1}{\sqrt{s}}(1,1,\ldots,1), \qquad \text{(A.2.16)}$$

and when $s$ is even, let

$$\underset{\sim}{\delta}_{\frac{s}{2}}' = \underset{\sim}{\gamma}_{\frac{s}{2}}' = \frac{1}{\sqrt{s}}(1,-1,1,\ldots,-1). \qquad (A.2.17)$$

Hence the $\underset{\sim}{\delta}_u$ are a complete set of real latent vectors for $\underset{\sim}{A}$, i.e.

$$\begin{aligned}\underset{\sim}{\delta}_u' \underset{\sim}{\delta}_{u'} &= 1, \text{ if } u = u', \\ &= 0, \text{ otherwise}, \\ &(u,u' = 0,1,\ldots,s-1)./\end{aligned} \qquad (A.2.18)$$

<u>Theorem A.2.3.</u> If $\underset{\sim}{A}$ has C structure, then the Moore-Penrose generalized inverse, $\underset{\sim}{A}^+$ also has C structure.

<u>Proof.</u> Let $\underset{\sim}{A}$ be given by (A.2.1) and define

$$\begin{aligned}\phi_u^* &= 0, \text{ if } \theta_u^* = 0, \\ &= (\theta_u^*)^{-1}, \text{ otherwise}, \\ &(u = 0,1,\ldots,s-1).\end{aligned} \qquad (A.2.19)$$

Then

$$\underset{\sim}{A}^+ = \sum_{u=0}^{s-1} \phi_u^* \underset{\sim}{\Gamma}_u^* . \qquad (A.2.20)$$

Now define

$$\phi_h = \frac{1}{s}\sum_{u=0}^{s-1} \omega_u^{-h} \phi_u^* . \qquad (A.2.21)$$

But from (A.2.6),

$$\underset{\sim}{\Gamma}_h = \sum_{u=0}^{s-1} \omega_u^{h} \underset{\sim}{\Gamma}_u^* , \qquad (A.2.22)$$

$$(h = 0,1,\ldots,s-1).$$

Hence

$$\underset{\sim}{A}^+ = \sum_{h=0}^{s-1} \phi_h \underset{\sim}{\Gamma}_h , \qquad (A.2.23)$$

which by Definition A.2.2, has C structure./

A.3 Block Circulant Matrices

Definition A.3.1. A $ts \times ts$ matrix $\underset{\sim}{A}$ is said to be a block circulant matrix if it can be written in the form

$$\underset{\sim}{A} = \sum_{h=0}^{s-1} \underset{\sim}{\theta}_h \otimes \underset{\sim}{\Gamma}_h , \qquad (A.3.1)$$

where the $\underset{\sim}{\theta}_h$ are now $t \times t$ matrices.

A is said to have BC structure./

Analagous to (A.2.2), define

$$\underset{\sim}{\theta}_u^* = \sum_{h=0}^{s-1} \omega_u^h \underset{\sim}{\theta}_h , \qquad (A.3.2)$$

$$(u = 0,1,\ldots,s-1),$$

and let the $\underset{\sim}{\theta}_u^*$ be written in the spectral form

$$\underset{\sim}{\theta}_u^* = \sum_{f=0}^{t-1} \xi_{uf} \bar{\underset{\sim}{\tau}}_{uf} \underset{\sim}{\tau}_{uf}' . \qquad (A.3.3)$$

Theorem A.3.1. Let $\underset{\sim}{A}$ have BC structure and be given by (A.3.1). The latent roots of $\underset{\sim}{A}$ are $\xi_{uf}$ with corresponding latent vectors $\underset{\sim}{\tau}_{uf} \otimes \underset{\sim}{\gamma}_u$ $(f = 0,1,\ldots,t-1 ; u = 0,1,\ldots,s-1)$.

Proof. Using (A.2.6), (A.3.1) may be written as

$$\underset{\sim}{A} = \sum_{u=0}^{s-1} \underset{\sim}{\theta}_u^* \otimes \underset{\sim}{\Gamma}_u^* , \qquad (A.3.4)$$

$$= \sum_{u=0}^{s-1} \left(\sum_{f=0}^{t-1} \xi_{uf} \bar{\underset{\sim}{\tau}}_{uf} \underset{\sim}{\tau}_{uf}'\right) \otimes \bar{\underset{\sim}{\gamma}}_u \underset{\sim}{\gamma}_u' , \qquad (A.3.5)$$

(using (A.3.3) and (A.2.8)).

Hence $\underset{\sim}{A}$ may be written in the spectral form

$$\underset{\sim}{A} = \sum_{u=0}^{s-1} \sum_{f=0}^{t-1} \xi_{uf} \overline{(\underset{\sim}{\tau}_{uf} \otimes \underset{\sim}{\gamma}_u)} (\underset{\sim}{\tau}_{uf} \otimes \underset{\sim}{\gamma}_u)' ./ \qquad (A.3.6)$$

<u>Theorem A.3.2.</u> Let $\underset{\sim}{A}$ have BC structure and be given by (A.3.1). If $\underset{\sim}{A}$ is symmetric then:

(i) $$\underset{\sim}{\theta}_h = \underset{\sim}{\theta}_{-h}' , \qquad \text{(A.3.7)}$$

$$(h = 0,1,\ldots,s-1) .$$

(ii) The $\underset{\sim}{\theta}_u^*$ are hermitian matrices and

$$\underset{\sim}{\theta}_u^* = \underset{\sim}{\theta}_{-u}^{*\prime} , \qquad \text{(A.3.8)}$$

$$(u = 0,1,\ldots,s-1) .$$

<u>Proof.</u> (i) Writing $l = l's + l''$ ; $m = m's + m''$,

$$(l',m' = 0,1,\ldots,t-1 \ ; \ l'',m'' = 0,1,\ldots,s-1) ,$$

then the $(l,m)^{th}$ element of $\underset{\sim}{A}$ is $(\underset{\sim}{\theta}_{(m''-l'')})^{l'm'}$ .
Thus if $\underset{\sim}{A}$ is symmetric, then

$$(\underset{\sim}{\theta}_{(m''-l'')})^{l'm'} = (\underset{\sim}{\theta}_{(l''-m'')})^{m'l'} , \qquad \text{(A.3.9)}$$

i.e. $$\underset{\sim}{\theta}_h = \underset{\sim}{\theta}_{-h}' .$$

(ii) From (A.3.2), the $(l,m)^{th}$ element of $\underset{\sim}{\theta}_u^*$ is

$$\sum_{h=0}^{s-1} \omega_u^{\,h} (\underset{\sim}{\theta}_h)^{lm} ,$$

$$= \sum_{h=0}^{s-1} \omega_u^{\,h} (\underset{\sim}{\theta}_{-h})^{ml} , \qquad (\text{by (A.3.7)}) ,$$

$$= \sum_{h=0}^{s-1} \omega_u^{-h} (\underset{\sim}{\theta}_u)^{ml} , \qquad (\text{by (A.2.4)}) ,$$

$$= \sum_{h=0}^{s-1} \omega_u^{-h} (\underset{\sim}{\theta}_{-u})^{lm} .$$

Hence

$$\underset{\sim}{\theta}_u^* = \underset{\sim}{\theta}_{-u}^{*\prime} = \underset{\sim}{\bar{\theta}}_u^{*\prime} = \underset{\sim}{\bar{\theta}}_{-u}^* , \qquad \text{(A.3.10)}$$

$$(u = 0,1,\ldots,s-1) ,$$

i.e. the $\theta_u^*$ are hermitian and $\theta_u^* = \theta_{-u}^{*\prime}$, thus $\theta_u^*$ and $\theta_{-u}^*$ will have the same real latent roots. If $u = 0$, (or when $s$ is even, $u = \frac{s}{2}$), the $\theta_u^*$ will be real symmetric matrices since $\omega_0^h = 1$ and $\omega_{\frac{s}{2}}^h = (-1)^h$ $(h = 0,1,\ldots,s-1)$./

<u>Theorem A.3.3.</u> Let $A$ have BC structure and be given by (A.3.1). If $A$ is symmetric and the rows and columns of $\theta_h$ add to the same number, $Z_h$ $(h = 0,1,\ldots,s-1)$, then the $\theta_u^*$ will each have a latent vector, say $\tau_{u0}$, equal to $\frac{1}{\sqrt{t}}\mathbf{1}$, with corresponding latent root

$$\sum_{h=0}^{s-1} Z_h \cos\left(\frac{2\pi uh}{s}\right) . \tag{A.3.11}$$

<u>Proof.</u> Since $\theta_h \mathbf{1} = Z_h$, then by (A.3.2)

$$\theta_u^* \mathbf{1} = \left(\sum_{h=0}^{s-1} \omega_u^h Z_h\right) \mathbf{1} . \tag{A.3.12}$$

Hence $\frac{1}{\sqrt{t}}\mathbf{1}$ is a latent vector of $\theta_u^*$. Now

$$\begin{aligned} Z_h &= \theta_h \mathbf{1} = \theta_h' \mathbf{1}, && \text{(given)} \\ &= \theta_{-h} \mathbf{1}, && \text{(by (A.3.7))} \\ &= Z_{-h}, \\ & (h = 0,1,\ldots,s-1) . \end{aligned}$$

Hence (A.3.11) follows in a similar manner to (A.2.11)./

<u>Theorem A.3.4.</u> If $A$ has BC structure, then the Moore-Penrose generalized inverse, $A^+$ also has BC structure.

<u>Proof.</u> Let $A$ be given by (A.3.1) and from (A.3.3), define

$$\begin{aligned} \beta_{uf} &= 0, \text{ if } \xi_{uf} = 0, \\ &= \xi_{uf}^{-1}, \text{ otherwise,} \end{aligned} \tag{A.3.13}$$

$$(f = 0,1,\ldots,t-1 \ ; \ u = 0,1,\ldots,s-1).$$

Then define

$$\underset{\sim}{\phi}_u{}^* = \sum_{f=0}^{t-1} \beta_{uf}\, \bar{\underset{\sim}{\tau}}_{uf}\, \underset{\sim}{\tau}_{uf}{}' \,. \tag{A.3.14}$$

Then

$$\underset{\sim}{A}^+ = \sum_{u=0}^{s-1} \underset{\sim}{\phi}_u{}^* \otimes \underset{\sim}{\Gamma}_u{}^* \,. \tag{A.3.15}$$

Analagous to (A.2.21), define

$$\underset{\sim}{\phi}_h = \frac{1}{s} \sum_{u=0}^{s-1} \omega_u{}^{-h}\, \underset{\sim}{\phi}_u{}^* \,. \tag{A.3.16}$$

Hence using (A.2.22),

$$\underset{\sim}{A}^+ = \sum_{h=0}^{s-1} \underset{\sim}{\phi}_h \otimes \underset{\sim}{\Gamma}_h \,, \tag{A.3.17}$$

which by Definition A.3.1, has BC structure./

A.4 Generalized Circulant Matrices

Definition A.4.1. Let $s$ be factorized into some number, $n$ of factors, i.e. $s = s_1\, s_2 \ldots s_n$. An $s \times s$ matrix $\underset{\sim}{A}$ is said to be a generalized circulant matrix with $n$ factors if it can be written in the form

$$\underset{\sim}{A} = \sum_{h_1=0}^{s_1-1} \sum_{h_2=0}^{s_2-1} \cdots \sum_{h_n=0}^{s_n-1} \theta_{h_1 h_2 \ldots h_n}\, \underset{\sim}{\Gamma}_{h_1} \otimes \underset{\sim}{\Gamma}_{h_2} \otimes \cdots \otimes \underset{\sim}{\Gamma}_{h_n} \,, \tag{A.4.1}$$

where the $\theta_{h_1 h_2 \ldots h_n}$ are real numbers and the $\underset{\sim}{\Gamma}_{h_j}$ are $s_j \times s_j$ basic circulants.

$\underset{\sim}{A}$ is said to have GC(n) structure./

Hence C structure can be thought of as GC(1) structure.

Theorem A.4.1. Let $\underset{\sim}{A}$ have GC(n) structure and be given by (A.4.1). The latent roots of $\underset{\sim}{A}$ are given by

$$\theta^*_{u_1u_2\ldots u_n} = \sum_{h_1=0}^{s_1-1} \sum_{h_2=0}^{s_2-1} \cdots \sum_{h_n=0}^{s_n-1} \omega_{h_1}^{u_1} \omega_{h_2}^{u_2} \cdots \omega_{h_n}^{u_n} \theta_{h_1h_2\ldots h_n}, \quad (A.4.2)$$

with corresponding latent vectors

$$\underset{\sim}{Y}_{u_1} \otimes \underset{\sim}{Y}_{u_2} \otimes \ldots \otimes \underset{\sim}{Y}_{u_n}, \quad (A.4.3)$$

where $\omega_{u_j}$ is the $u_j^{th}$, $s_j^{th}$ root of unity, and

$$\underset{\sim}{Y}'_{u_j} = \frac{1}{\sqrt{s_j}} (\omega_{u_j}^{0}, \omega_{u_j}^{1}, \ldots, \omega_{u_j}^{s_j-1}), \quad (A.4.4)$$

$$(u_j = 0,1,\ldots,s_j-1; \quad j = 1,2,\ldots,n)./$$

Theorem A.4.2. Let $\underset{\sim}{A}$ have GC(n) structure and be given by (A.4.1). If $\underset{\sim}{A}$ is symmetric then:

(i) $$\theta_{h_1h_2\ldots h_n} = \theta_{\pm h_1 \pm h_2 \ldots \pm h_n}, \quad (A.4.5)$$

$$(h_j = 0,1,\ldots,s_j-1 \,; \quad j = 1,2,\ldots,n).$$

(ii) $$\theta^*_{u_1u_2\ldots u_n} = \theta^*_{\pm u_1 \pm u_2 \ldots \pm u_n}$$

$$= \sum_{h_1=0}^{s_1-1} \sum_{h_2=0}^{s_2-1} \cdots \sum_{h_n=0}^{s_n-1} \theta_{h_1h_2\ldots h_n} \cos\left(\frac{2\pi u_1h_1}{s_1}\right)\cos\left(\frac{2\pi u_2h_2}{s_2}\right)\ldots \cos\left(\frac{2\pi u_nh_n}{s_n}\right), \quad (A.4.6)$$

$$(u_j = 0,1,\ldots,s_j-1 \,; \quad j = 1,2,\ldots,n)./$$

Theorem A.4.3. If $\underset{\sim}{A}$ has GC(n) structure, then the Moore-Penrose generalized inverse, $\underset{\sim}{A}^{+}$ also has GC(n) structure./

## A.5 Block Generalized Circulant Matrices

Definition A.5.1. Let $s$ be factorized into some number, $n$ of factors, i.e. $s = s_1 s_2 \dots s_n$. A $ts \times ts$ matrix $\underset{\sim}{A}$ is said to be a block generalized circulant matrix with $n+1$ factors if it can be written in the form

$$\underset{\sim}{A} = \sum_{h_1=0}^{s_1-1} \sum_{h_2=0}^{s_2-1} \dots \sum_{h_n=0}^{s_n-1} \underset{\sim}{\theta}_{h_1 h_2 \dots h_n} \otimes \underset{\sim}{\Gamma}_{h_1} \otimes \underset{\sim}{\Gamma}_{h_2} \otimes \dots \otimes \underset{\sim}{\Gamma}_{h_n}, \quad (A.5.1)$$

where the $\underset{\sim}{\theta}_{h_1 h_2 \dots h_n}$ are now $t \times t$ matrices.

$\underset{\sim}{A}$ is said to have BGC(n+1) structure./

Hence BC structure can be thought of as BGC(2) structure. Also if all the $\underset{\sim}{\theta}_{h_1 h_2 \dots h_n}$ are circulant matrices, then BGC(n+1) structure becomes GC(n+1) structure.

Analagous to (A.4.2), define

$$\underset{\sim}{\theta}^{*}_{u_1 u_2 \dots u_n} = \sum_{h_1=0}^{s_1-1} \sum_{h_2=0}^{s_2-1} \dots \sum_{h_n=0}^{s_n-1} \omega_{h_1}^{u_1} \omega_{h_2}^{u_2} \dots \omega_{h_n}^{u_n} \underset{\sim}{\theta}_{h_1 h_2 \dots h_n}, \quad (A.5.2)$$

$$(u_j = 0, 1, \dots, s_j - 1 \,;\, j = 1, 2, \dots, n),$$

and let the $\underset{\sim}{\theta}^{*}_{u_1 u_2 \dots u_n}$ be written in the spectral form

$$\underset{\sim}{\theta}^{*}_{u_1 u_2 \dots u_n} = \sum_{f=0}^{t-1} \xi_{u_1 u_2 \dots u_n f} \, \bar{\underset{\sim}{\tau}}_{u_1 u_2 \dots u_n f} \, \underset{\sim}{\tau}'_{u_1 u_2 \dots u_n f} . \quad (A.5.3)$$

Theorem A.5.1. Let $\underset{\sim}{A}$ have BGC(n+1) structure and be given by (A.5.1). The latent roots of $\underset{\sim}{A}$ are $\xi_{u_1 u_2 \ldots u_n f}$ with corresponding latent vectors

$$\underset{\sim}{\tau}_{u_1 u_2 \ldots u_n f} \otimes \underset{\sim}{\gamma}_{u_1} \otimes \underset{\sim}{\gamma}_{u_2} \otimes \ldots \otimes \underset{\sim}{\gamma}_{u_n}$$

$$(f = 0,1,\ldots,t-1 \; ; \; u_j = 0,1,\ldots,s_j-1 \; ; \; j = 1,2,\ldots,n)./$$

Theorem A.5.2. Let $\underset{\sim}{A}$ have BGC(n+1) structure and be given by (A.5.1). If $\underset{\sim}{A}$ is symmetric then:

(i) $$\underset{\sim}{\theta}_{h_1 h_2 \ldots h_n} = \underset{\sim}{\theta}'_{\dot{-}h_1 \dot{-}h_2 \ldots \dot{-}h_n} , \qquad (A.5.4)$$

$$(h_j = 0,1,\ldots,s_j-1 \; ; \; j = 1,2,\ldots,n).$$

(ii) The $\underset{\sim}{\theta}^*_{u_1 u_2 \ldots u_n}$ are hermitian matrices and

$$\underset{\sim}{\theta}^*_{u_1 u_2 \ldots u_n} = \underset{\sim}{\theta}^{*\prime}_{\dot{-}u_1 \dot{-}u_2 \ldots \dot{-}u_n} , \qquad (A.5.5)$$

$$(u_j = 0,1,\ldots,s_j-1 \; ; \; j = 1,2,\ldots,n)./$$

Theorem A.5.3. Let $\underset{\sim}{A}$ have BGC(n+1) structure and be given by (A.5.1). If $\underset{\sim}{A}$ is symmetric and the rows and columns of $\underset{\sim}{\theta}_{h_1 h_2 \ldots h_n}$ add to the same number $z_{h_1 h_2 \ldots h_n}$ $(h_j = 0,1,\ldots,s_j-1 \; ;$ $j = 1,2,\ldots,n)$ then the $\underset{\sim}{\theta}^*_{u_1 u_2 \ldots u_n}$ will each have a latent vector, say $\underset{\sim}{\tau}_{u_1 u_2 \ldots u_n 0}$ equal to $\frac{1}{\sqrt{t}} \underset{\sim}{1}$, with corresponding latent root

$$\sum_{h_1=0}^{s_1-1} \sum_{h_2=0}^{s_2-1} \cdots \sum_{h_n=0}^{s_n-1} z_{h_1 h_2 \ldots h_n} \cos\left(\frac{2\pi u_1 h_1}{s_1}\right) \cos\left(\frac{2\pi u_2 h_2}{s_2}\right) \ldots \cos\left(\frac{2\pi u_n h_n}{s_n}\right) ./ \qquad (A.5.6)$$

Theorem A.5.4. If $\underset{\sim}{A}$ has BGC(n+1) structure, then the Moore-Penrose generalized inverse, $\underset{\sim}{A}^{+}$ also has BGC(n+1) structure./

A.6 Double Circulant Matrices

We modify the notation of section A.2 by representing the basic circulants of Definition A.2.1 by

$$\underset{\sim}{\Gamma}_{0h} = \underset{\sim}{\Gamma}_{h}, \qquad (h = 0,1,\ldots,s-1), \tag{A.6.1}$$

and calling them forward basic circulants.

Definition A.6.1. An $s \times s$ matrix $\underset{\sim}{\Gamma}_{1h}$ $(h = 0,1,\ldots,s-1)$, whose $(l,m)^{th}$ element, $(\underset{\sim}{\Gamma}_{1h})^{lm} = 1$, if $m \dot{+} l = h$,

$= 0$, otherwise,

$(l,m = 0,1,\ldots,s-1)$,

is called a backward basic circulant./

Definition A.6.2. An $s \times s$ matrix $\underset{\sim}{A}$ is said to be a double circulant matrix if it can be written as a linear combination of forward and backward basic circulants, i.e.

$$\underset{\sim}{A} = \sum_{g=0}^{1} \sum_{h=0}^{s-1} \theta_{gh} \underset{\sim}{\Gamma}_{gh} . \tag{A.6.2}$$

$\underset{\sim}{A}$ is said to have DC structure./

We only consider DC structure for which

$$\theta_{gh} = \theta_{g \dot{-} h}, \qquad (g = 0,1 ; \; h = 0,1,\ldots,s-1). \tag{A.6.3}$$

Analagous to (A.2.2) and (A.2.6) define

$$\underset{\sim}{\Gamma}^{*}_{gu} = \frac{1}{s} \sum_{h=0}^{s-1} \omega_u^{-h} \underset{\sim}{\Gamma}_{gh}, \tag{A.6.4}$$

and

$$\theta^*_{gu} = \sum_{h=0}^{s-1} \omega_u^{\,h}\, \theta_{gh}\,, \qquad (A.6.5)$$

$$(g = 0,1;\; u = 0,1,\ldots,s-1).$$

Hence (A.6.2) may be written as

$$\underset{\sim}{A} = \sum_{g=0}^{1} \sum_{u=0}^{s-1} \theta^*_{gu}\, \underset{\sim}{\Gamma}^*_{gu}\,. \qquad (A.6.6)$$

We would like to convert the 2s matrices $\underset{\sim}{\Gamma}^*_{gu}$ into a basis for $\underset{\sim}{A}$.

Define

$$\underset{\sim}{\Delta}_u = \underset{\sim}{\delta}_u\, \underset{\sim}{\delta}_u'\,, \qquad (A.6.7)$$

where the $\underset{\sim}{\delta}_u$ are given by (A.2.15 - 17),

$$(u = 0,1,\ldots,s-1).$$

Thus by (A.2.18), the $\underset{\sim}{\Delta}_u$ are mutually orthogonal idempotent matrices.

Theorem A.6.1.

(i) $\underset{\sim}{\Delta}_0 = \underset{\sim}{\Gamma}_{00}{}^* = \underset{\sim}{\Gamma}_{10}{}^* = \frac{1}{s}\, \underset{\sim}{J}_s\,.$ (A.6.8)

(ii) When s is even,

$$\underset{\sim}{\Delta}_{\frac{s}{2}} = \underset{\sim}{\Gamma}_{0\frac{s}{2}}{}^* = \underset{\sim}{\Gamma}_{1\frac{s}{2}}{}^* = \frac{1}{s}\, \underset{\sim}{K}_s\,. \qquad (A.6.9)$$

(iii)

$$\underset{\sim}{\Delta}_u = \frac{1}{2}\left(\underset{\sim}{\Gamma}_{0u}{}^* + \underset{\sim}{\Gamma}_{0\dot{-}u}{}^* + \underset{\sim}{\Gamma}_{1u}{}^* + \underset{\sim}{\Gamma}_{1\dot{-}u}{}^*\right),$$

$$\underset{\sim}{\Delta}_{\dot{-}u} = \frac{1}{2}\left(\underset{\sim}{\Gamma}_{0u}{}^* + \underset{\sim}{\Gamma}_{0\dot{-}u}{}^* - \underset{\sim}{\Gamma}_{1u}{}^* - \underset{\sim}{\Gamma}_{1\dot{-}u}{}^*\right), \qquad (A.6.10)$$

$$(u = 1,2,\ldots,[\tfrac{s-1}{2}]).$$

Proof. (i) and (ii) follow from (A.2.16 - 17) and (A.6.4).

(iii) From (A.2.13), the $(l,m)^{th}$ element of $\underset{\sim}{\Delta}_u$ is

$$(\underset{\sim}{\Delta}_u)^{lm} = (\underset{\sim}{\delta}_u \underset{\sim}{\delta}_u{}') = \frac{1}{2}(\omega_u{}^l + \omega_{\dot{-}u}{}^l)(\omega_u{}^m + \omega_{\dot{-}u}{}^m),$$

$$= \frac{1}{2}(\omega_u{}^{m-l} + \omega_u{}^{l-m} + \omega_u{}^{l+m} + \omega_u{}^{-(l+m)}),$$

$$= \frac{1}{2}(\underset{\sim}{\Gamma}_{0u}{}^* + \underset{\sim}{\Gamma}_{0\dot{-}u}{}^* + \underset{\sim}{\Gamma}_{1u}{}^* + \underset{\sim}{\Gamma}_{1\dot{-}u}{}^*)^{lm},$$

$$(u = 1,2,\ldots,[\tfrac{s-1}{2}]\ ;\ l,m = 0,1,\ldots,s-1),$$

and the second part of (A.6.10) may be proved in a similar manner./

Define

(i) $\pi_0 = \theta_{00}{}^* + \theta_{10}{}^*$, (A.6.11)

(ii) when $s$ is even,

$\pi_{\frac{s}{2}} = \theta_{0\frac{s}{2}}^* + \theta_{1\frac{s}{2}}^*$, (A.6.12)

(iii) $\pi_u = \theta_{0u}{}^* + \theta_{1u}{}^*$,

$\pi_{\dot{-}u} = \theta_{0u}{}^* - \theta_{1u}{}^*$, (A.6.13)

$$(u = 1,2,\ldots,[\tfrac{s-1}{2}]).$$

<u>Theorem A.6.2.</u> Let $\underset{\sim}{A}$ have DC structure and be given by (A.6.2). If $\underset{\sim}{A}$ satisfies (A.6.3), then the latent roots of $\underset{\sim}{A}$ are $\pi_u$ with corresponding latent vectors $\underset{\sim}{\delta}_u$ $(u = 0,1,\ldots,s-1)$.

<u>Proof.</u> Condition (A.6.3) implies that $\underset{\sim}{A}$ is symmetric and that

$$\theta_{gu}{}^* = \theta_{g\dot{-}u}{}^*, \qquad (A.6.14)$$

$$(g = 0,1\ ;\ u = 0,1,\ldots,s-1).$$

Hence, using (A.6.8 - 10) and (A.6.11 - 13), we see that (A.6.6) may be written in the spectral form

$$\underset{\sim}{A} = \sum_{u=0}^{s-1} \pi_u \underset{\sim}{\delta}_u \underset{\sim}{\delta}'_u \text{ ./} \qquad (A.6.15)$$

<u>Theorem A.6.3.</u> If $\underset{\sim}{A}$ has DC structure and satisfies (A.6.3), then the Moore-Penrose generalized inverse, $\underset{\sim}{A}^+$ also has DC structure.

<u>Proof</u>. Define $\epsilon_u = 0$, if $\pi_u = 0$,

$$= \pi_u^{-1}, \text{ otherwise },$$

$$(u = 0, 1, \ldots, s-1).$$

Then

$$\underset{\sim}{A}^+ = \sum_{u=0}^{s-1} \epsilon_u \underset{\sim}{\Delta}_u \text{ .} \qquad (A.6.16)$$

Define

(i) $\quad \phi^*_{00} = \phi^*_{10} = 0$ , $\qquad$ (A.6.17)

(ii) $\quad$ when $s$ is even,

$$\phi^*_{0\frac{s}{2}} = \phi^*_{1\frac{s}{2}} = 0 , \qquad (A.6.18)$$

(iii) $\quad \phi^*_{gu} = \frac{1}{2}(\epsilon_u + \epsilon_{\dot{-}u})$ ,

$$\phi^*_{g\dot{-}u} = \frac{1}{2}(\epsilon_u - \epsilon_{\dot{-}u}) , \qquad (A.6.19)$$

$$(g = 0,1 \, ; \; u = 1, 2, \ldots, [\tfrac{s-1}{2}]).$$

(iv) $\quad \underset{\sim}{A}^{+*} = \underset{\sim}{A}^+ - \epsilon_0 \underset{\sim}{\Delta}_0 - \epsilon_{\frac{s}{2}} \underset{\sim}{\Delta}_{\frac{s}{2}}$ , if $s$ is even,

$$= \underset{\sim}{A}^+ - \epsilon_0 \underset{\sim}{\Delta}_0 , \text{ if } s \text{ is odd .} \qquad (A.6.20)$$

Then from (A.6.8-10), it follows that

$$\underset{\sim}{A}^{+*} = \sum_{g=0}^{1} \sum_{u=0}^{s-1} \phi^*_{gu} \underset{\sim}{\Gamma}^*_{gu} \text{ .} \qquad (A.6.21)$$

Hence using (A.6.4), $\underset{\sim}{A}^{+*}$ may be written in the form

$$\underset{\sim}{A}^{+*} = \sum_{g=0}^{1} \sum_{h=0}^{s-1} \phi_{gh} \underset{\sim}{\Gamma}_{gh} , \qquad (A.6.22)$$

where

$$\phi_{gh} = \frac{1}{s} \sum_{u=0}^{s-1} \omega_u^{-h} \phi^*_{gh} ,$$

$$(g = 0,1 \; ; \; h = 0,1,\ldots,s-1).$$

Thus by Definition A.6.2, (A.6.22) has DC structure, and hence from (A.6.8), (A.6.9) and (A.6.20), it follows that $\underset{\sim}{A}^+$ also has DC structure./

## A.7 Block Double Circulant Matrices

**Definition A.7.1.** A $ts \times ts$ matrix $\underset{\sim}{A}$ is said to be a block double circulant matrix if it can be written in the form

$$\underset{\sim}{A} = \sum_{g=0}^{1} \sum_{h=0}^{s-1} \underset{\sim}{\theta}_{gh} \otimes \underset{\sim}{\Gamma}_{gh} , \qquad (A.7.1)$$

where the $\underset{\sim}{\theta}_{gh}$ are now $t \times t$ matrices.

$\underset{\sim}{A}$ is said to have BDC structure./

We only consider BDC structure for which

$$\underset{\sim}{\theta}_{gh} = \underset{\sim}{\theta}_{g\dot{-}h} , \qquad (A.7.2)$$

$$(g = 0,1 \; ; \; h = 0,1,\ldots,s-1).$$

Analagous to (A.6.5), define

$$\underset{\sim}{\theta}^*_{gu} = \sum_{h=0}^{s-1} \omega_u^{h} \underset{\sim}{\theta}_{gh} , \qquad (A.7.3)$$

$$(g = 0,1 \; ; \; u = 0,1,\ldots,s-1).$$

**Theorem A.7.1.** Let $\underset{\sim}{A}$ have BDC structure and be given by (A.7.1). If $\underset{\sim}{A}$ satisfies (A.7.2) then the $\underset{\sim}{\theta}^*_{gu}$ are real symmetric matrices and

$$\underset{\sim}{\theta}^{*}_{gu} = \underset{\sim}{\theta}^{*}_{g\cdot u} , \qquad (A.7.4)$$

$$(g = 0,1 ; \ u = 0,1,\ldots,s-1) ./$$

Define

(i) $$\underset{\sim}{\pi}_0 = \underset{\sim}{\theta}^{*}_{00} + \underset{\sim}{\theta}^{*}_{10} , \qquad (A.7.5)$$

(ii) when $s$ is even,

$$\underset{\sim}{\pi}_{\frac{s}{2}} = \underset{\sim}{\theta}^{*}_{0\frac{s}{2}} + \underset{\sim}{\theta}^{*}_{1\frac{s}{2}} , \qquad (A.7.6)$$

(iii) $$\underset{\sim}{\pi}_u = \underset{\sim}{\theta}^{*}_{0u} + \underset{\sim}{\theta}^{*}_{1u} ,$$

$$\underset{\sim}{\pi}_{\cdot u} = \underset{\sim}{\theta}^{*}_{0u} - \underset{\sim}{\theta}^{*}_{1u} , \qquad (A.7.7)$$

$$(u = 1,2,\ldots,[\tfrac{s-1}{2}]) .$$

By Theorem A.7.1, the $\underset{\sim}{\pi}_u$ are real symmetric matrices. Let the $\underset{\sim}{\pi}_u$ be written in the spectral form

$$\underset{\sim}{\pi}_u = \sum_{f=0}^{t-1} \xi_{uf} \underset{\sim}{\tau}_{uf} \underset{\sim}{\tau}'_{uf} , \qquad (A.7.8)$$

$$(u = 0,1,\ldots,s-1).$$

<u>Theorem A.7.2.</u> Let $\underset{\sim}{A}$ have BDC structure and be given by (A.7.1). If $\underset{\sim}{A}$ satisfies (A.7.2), then the latent roots of $\underset{\sim}{A}$ are $\xi_{uf}$ with corresponding latent vectors $\underset{\sim}{\tau}_{uf} \otimes \underset{\sim}{\delta}_u$ $(f = 0,1,\ldots t-1 ; \ u = 0,1,\ldots,s-1)$./

<u>Theorem A.7.3.</u> If $\underset{\sim}{A}$ has BDC structure and satisfies (A.7.2), then the Moore-Penrose generalized inverse, $\underset{\sim}{A}^{+}$ also has BDC structure./

## A.8 <u>Balanced Factorial Structure</u>

<u>Definition A.8.1.</u> Let $\underset{\sim}{A}$ have GC(n) structure and be given

by (A.4.1). If all $\theta_{h_1h_2\cdots h_n}$ (where if $h_j \neq 0$, then $h_j = 1,2,\ldots,s_j-1$), are equal, say to $\theta_{z_1z_2\cdots z_n}$, i.e.

$$\theta_{h_1h_2\cdots h_n} = \theta_{z_1z_2\cdots z_n}, \tag{A.8.1}$$

where

$$\begin{aligned} z_j &= 0, \text{ if } h_j = 0, \\ &= 1, \text{ otherwise}, \\ &\qquad (j = 1,2,\ldots,n), \end{aligned}$$

then $\underset{\sim}{A}$ is said to have balanced factorial structure for n factors, i.e. BF(n) structure./

<u>Theorem A.8.1.</u> Let $\underset{\sim}{A}$ have BF(n) structure. The latent roots of $\underset{\sim}{A}$ are given by

$$\theta^*_{x_1x_2\cdots x_n} = \sum_{z_1=0}^{1}\sum_{z_2=0}^{1}\cdots\sum_{z_n=0}^{1} c_{z_1}^{x_1} c_{z_2}^{x_2}\cdots c_{z_n}^{x_n}\,\theta_{z_1z_2\cdots z_n}, \tag{A.8.2}$$

with multiplicity

$$\prod_{j=1}^{n}(s_j-1)^{x_j}, \tag{A.8.3}$$

where the $c_{z_j}^{x_j}$ are given by the table

| $z_j$ \ $x_j$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | $s_j-1$ | $-1$ |

(A.8.4)

$$(x_j, z_j = 0,1\,;\ j = 1,2,\ldots,n).$$

<u>Proof.</u> From (A.2.4),

$$\sum_{h_j=0}^{s_j-1} \omega_{u_j}^{h_j} = \sum_{h_j=0}^{s_j-1} \{\exp(\frac{2\pi i u_j}{s_j})\}^{h_j} , \qquad (A.8.5)$$

$$= \begin{cases} s_j , & \text{if } u_j = 0 , \\ 0 , & \text{otherwise} , \end{cases} \qquad (A.8.6)$$

$$(j = 1,2,\ldots,n).$$

Since $\omega_{u_j}^{0} = 1 \quad (u_j = 0,1,\ldots,s_j-1)$,

$$\sum_{h_j=1}^{s_j-1} \omega_{u_j}^{h_j} = s_j - 1 , \quad \text{if } u_j = 0 , \qquad (A.8.7)$$

$$= -1 , \quad \text{otherwise},$$

$$(j = 1,2,\ldots,n).$$

Hence, using (A.8.1), it follows that (A.4.2) reduces to (A.8.2)./

Expression (A.8.2) is identical to expression 4.28 given by Kshirsagar (1966) and expression 4.10 given by Hinkelmann (1964).

## A.9 Orthogonal Factorial Structure

Definition A.9.1. Let $s$ be factorized into some number, $n$ of factors, i.e. $s = s_1 s_2 \ldots s_n$. If $\underset{\sim}{A}$ is an $s \times s$ symmetric matrix with real latent vectors of the form

$$\underset{\sim}{\delta}_1 \otimes \underset{\sim}{\delta}_2 \otimes \ldots \otimes \underset{\sim}{\delta}_n , \qquad (A.9.1)$$

where $\underset{\sim}{\delta}_j$ is an $s_j \times 1$ vector, and either $\underset{\sim}{\delta}_j = \frac{1}{\sqrt{s_j}} \underset{\sim}{1}$ or $\underset{\sim}{\delta}_j' \underset{\sim}{1} = 0$, then $\underset{\sim}{A}$ is said to have orthogonal factorial structure for $n$ factors, i.e. OF(n) structure./

With this definition, the following results are immediate:

<u>Theorem A.9.1.</u> The matrix $\underset{\sim}{A}$ of Theorem A.3.3 has OF(2) structure./

This has been shown by John and Smith (1972).

<u>Theorem A.9.2.</u> If $\underset{\sim}{A}$ has GC(n) structure, then $\underset{\sim}{A}$ also has OF(n) structure./

This has been shown by John (1973).

<u>Theorem A.9.3.</u> The matrix $\underset{\sim}{A}$ of Theorem A.5.3 has OF(n+1) structure./

This has been shown by Cotter, John and Smith (1973).

## A.10 <u>Permutation Structure</u>

<u>Definition A.10.1.</u> Let $\underset{\sim}{A}$ be an $s \times s$ matrix. If (i) all the diagonal elements of $\underset{\sim}{A}$ have the same value, and (ii) the elements of any row or column can be obtained by a permutation of the elements of any other row or column, then $\underset{\sim}{A}$ is said to have permutation structure, i.e. P structure./

## APPENDIX B

## COMPUTATION OF THE H.M.C.E.F. OF $\underset{\sim}{\alpha}$-DESIGNS

B.1 Summary. In section 3.4, it was shown that $\bar{E}$ for an $\underset{\sim}{\alpha}$-design may be obtained from the characteristic equations (3.4.1). In this appendix, these equations are simplified for the cases $r = 2, 3, 4$.

B.2 Simplification of the Characteristic Equations

We first define a difference set on $\underset{\sim}{\alpha}$.

Definition B.2.1. Let

$$D = (m_{11}, m_{12}, \ldots, m_{1d_1}; m_{21}, m_{22}, \ldots, m_{2d_2}; \ldots; m_{g1}, m_{g2}, \ldots, m_{gd_g}), \tag{B.2.1}$$

where $\sum_{j=1}^{g} d_j = d$, denote the set of $k^d$ differences of the form

$$(a_{l_{12}m_{11}} \dot{-} a_{l_{11}m_{11}}) \dot{+} (a_{l_{13}m_{12}} \dot{-} a_{l_{12}m_{12}}) \dot{+} \ldots \dot{+} (a_{l_{11}m_{1d_1}} \dot{-} a_{l_{1d_1}m_{1d_1}})$$
$$\dot{+} (a_{l_{22}m_{21}} \dot{-} a_{l_{21}m_{21}}) \dot{+} (a_{l_{23}m_{22}} \dot{-} a_{l_{22}m_{22}}) \dot{+} \ldots \dot{+} (a_{l_{21}m_{2d_2}} \dot{-} a_{l_{2d_2}m_{2d_2}})$$
$$\vdots$$
$$\dot{+} (a_{l_{g2}m_{g1}} \dot{-} a_{l_{g1}m_{g1}}) \dot{+} (a_{l_{g3}m_{g2}} \dot{-} a_{l_{g2}m_{g2}}) \dot{+} \ldots \dot{+} (a_{l_{g1}m_{gd_g}} \dot{-} a_{l_{gd_g}m_{gd_g}}),$$

obtained from the array $\underset{\sim}{\alpha}$,

$(m_{ij} = 0, 1, \ldots, r-1\ ;\ l_{ij} = 0, 1, \ldots, k-1\ ;$

$$j = 1, 2, \ldots, d_i\ ;\ i = 1, 2, \ldots, g)\ ./$$

For example, the difference set $D = (m, m')$ will involve differences of the form

$$(a_{l'm} \dot{-} a_{lm}) \dot{+} (a_{lm'} \dot{-} a_{l'm'}) , \qquad (B.2.2)$$

$$(l, l' = 0, 1, \ldots, k-1) .$$

Suppose the element $h$ appears $\lambda_h$ times in the difference set $D$ $(h = 0, 1, \ldots, s-1)$, and let

$$\underset{\sim}{\lambda}(D) = (\lambda_0, \lambda_1, \ldots, \lambda_{s-1})' . \qquad (B.2.3)$$

Then

$$\underset{\sim}{1}'\underset{\sim}{\lambda}(D) = k^d . \qquad (B.2.4)$$

Definition B.2.2. (i) For $r = 2$, define

$$\underset{\sim}{\nu}_2 = \underset{\sim}{\lambda}(0,1) . \qquad (B.2.5)$$

(ii) For $r = 3$, define

$$\begin{aligned} \underset{\sim}{\nu}_2 &= \underset{\sim}{\lambda}(0,1) + \underset{\sim}{\lambda}(0,2) + \underset{\sim}{\lambda}(1,2) , \\ \underset{\sim}{\nu}_3 &= 2\underset{\sim}{\lambda}(0,1,2) . \end{aligned} \qquad (B.2.6)$$

(iii) For $r = 4$, define

$$\begin{aligned} \underset{\sim}{\nu}_2 &= \underset{\sim}{\lambda}(0,1) + \underset{\sim}{\lambda}(0,2) + \underset{\sim}{\lambda}(0,3) + \underset{\sim}{\lambda}(1,2) + \underset{\sim}{\lambda}(1,3) + \underset{\sim}{\lambda}(2,3), \\ \underset{\sim}{\nu}_3 &= 2(\underset{\sim}{\lambda}(0,1,2) + \underset{\sim}{\lambda}(0,1,3) + \underset{\sim}{\lambda}(0,2,3) + \underset{\sim}{\lambda}(1,2,3)), \\ \underset{\sim}{\nu}_4 &= 2(\underset{\sim}{\lambda}(0,1,2,3) + \underset{\sim}{\lambda}(0,1,3,2) + \underset{\sim}{\lambda}(0,2,1,3)) \\ &\quad - (\underset{\sim}{\lambda}(0,1;2,3) + \underset{\sim}{\lambda}(0,2;1,3) + \underset{\sim}{\lambda}(0,3;1,2)) ./ \end{aligned} \qquad (B.2.7)$$

Let the elements of $\underset{\sim}{\nu}_d$ be $\nu_{dh}$ $(h = 0, 1, \ldots, s-1;$ $d = 2, \ldots, r)$, and define

$$\nu_{du}^* = \frac{1}{(rk)^d} \sum_{h=0}^{s-1} \cos\left(\frac{2\pi uh}{s}\right) \nu_{dh} , \qquad (B.2.8)$$

$$(u = 0, 1, \ldots, s-1 ; \; d = 2, \ldots, r ; \; r = 2, 3, 4) .$$

Then the characteristic equations (3.4.1) may be written in terms of the $\nu_{du}^*$ , in particular,

(i) for $r = 2$,

$$x^2 - \nu_{2u}^* = 0, \tag{B.2.9}$$

(ii) for $r = 3$,

$$x^3 - \nu_{2u}^* x - \nu_{3u}^* = 0, \tag{B.2.10}$$

(iii) for $r = 4$,

$$x^4 - \nu_{2u}^* x^2 - \nu_{3u}^* x - \nu_{4u}^* = 0, \tag{B.2.11}$$

$$(u = 0,1,\ldots,s-1),$$

where

$$x = 1 - \frac{1}{r} - \xi .$$

Hence for $r = 2,3,4$ we may express (3.4.10) in terms of the $\nu_{du}^*$, i.e.

(i) for $r = 2$,

$$\frac{1}{\xi_u} = \frac{1}{\frac{1}{4} - \nu_{2u}^*}, \tag{B.2.12}$$

(ii) for $r = 3$,

$$\frac{1}{\xi_u} = \frac{\frac{4}{3} - \nu_{2u}^*}{\frac{8}{27} - \frac{2}{3}\nu_{2u}^* - \nu_{3u}^*}, \tag{B.2.13}$$

(iii) for $r = 4$,

$$\frac{1}{\xi_u} = \frac{\frac{27}{16} - \frac{3}{2}\nu_{2u}^* - \nu_{3u}^*}{\frac{81}{256} - \frac{9}{16}\nu_{2u}^* - \frac{3}{4}\nu_{3u}^* - \nu_{4u}^*}, \tag{B.2.14}$$

$$(u = 1,2,\ldots,s-1).$$

## GUIDE TO THE TABLES

### Tables A and C

For each combination of r, s, k, Tables A and C contain the last $r - 1$ rows of an array $\underset{\sim}{\alpha}'$ in reduced form, from which $\underset{\sim}{\alpha}(0,1)$-designs and $\underset{\sim}{\alpha}(0,1,2)$-designs respectively may be obtained. In a few cases, arrays $\underset{\sim}{\beta}'$ have been used and these are denoted by an asterisk. Also included is the h.m.c.e.f. ($\bar{E}'$) for the dual design (or $\underset{\sim}{\alpha}'$-design).

### Tables B and D

These tables contain properties of most of the designs obtained from Tables A and C respectively, including almost equiblock-sized $\underset{\sim}{\alpha}$-designs. For each design we give the minimum canonical efficiency factor (E(MIN)); the h.m.c.e.f. ($\bar{E}$); the harmonic mean of the pairwise efficiency factors ($\bar{E}(i)$) for varieties that appear together i times within blocks, together with the range (as a percentage) of these pairwise efficiencies; and the recommended order in which the symbols of the design are to be allocated control varieties (up to min(s,5) symbols).

# TABLE A.1

## $\underset{\sim}{\alpha}$(0,1)-DESIGNS , R=2 , 3≤S≤15 , 3≤K≤10

| S | K | Ē' | ARRAY $\underset{\sim}{\alpha}$' | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | .5556 | 0 | 1 | 2 | | | | | | | |
| 4 | 3 | .4828 | 0 | 1 | 2 | | | | | | | |
| 4 | 4 | .5385 | 0 | 1 | 2 | 3 | | | | | | |
| 5 | 3 | .4361 | 0 | 1 | 2 | | | | | | | |
| 5 | 4 | .4982 | 0 | 1 | 2 | 3 | | | | | | |
| 5 | 5 | .5294 | 0 | 1 | 2 | 3 | 4 | | | | | |
| 6 | 3 | .4151 | 0 | 1 | 3 | | | | | | | |
| 6 | 4 | .4705 | 0 | 1 | 2 | 3 | | | | | | |
| 6 | 5 | .5038 | 0 | 1 | 2 | 3 | 4 | | | | | |
| 6 | 6 | .5238 | 0 | 1 | 2 | 3 | 4 | 5 | | | | |
| 7 | 3 | .4081 | 0 | 1 | 3 | | | | | | | |
| 7 | 4 | .4573 | 0 | 1 | 2 | 4 | | | | | | |
| 7 | 5 | .4853 | 0 | 1 | 2 | 3 | 4 | | | | | |
| 7 | 6 | .5061 | 0 | 1 | 2 | 3 | 4 | 5 | | | | |
| 7 | 7 | .5200 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | | |
| 8 | 3 | .3896 | 0 | 1 | 3 | | | | | | | |
| 8 | 4 | .4418 | 0 | 1 | 2 | 4 | | | | | | |
| 8 | 5 | .4735 | 0 | 1 | 2 | 3 | 5 | | | | | |
| 8 | 6 | .4928 | 0 | 1 | 2 | 3 | 4 | 5 | | | | |
| 8 | 7 | .5070 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | | |
| 8 | 8 | .5172 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| 9 | 3 | .3753 | 0 | 1 | 3 | | | | | | | |
| 9 | 4 | .4318 | 0 | 1 | 3 | 4 | | | | | | |
| 9 | 5 | .4631 | 0 | 1 | 2 | 3 | 6 | | | | | |
| 9 | 6 | .4832 | 0 | 1 | 2 | 3 | 4 | 6 | | | | |
| 9 | 7 | .4969 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | | |
| 9 | 8 | .5073 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| 9 | 9 | .5152 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
| 10 | 3 | .3623 | 0 | 1 | 3 | | | | | | | |
| 10 | 4 | .4247 | 0 | 1 | 2 | 5 | | | | | | |
| 10 | 5 | .4562 | 0 | 1 | 2 | 3 | 6 | | | | | |
| 10 | 6 | .4755 | 0 | 1 | 2 | 3 | 5 | 6 | | | | |
| 10 | 7 | .4892 | 0 | 1 | 2 | 3 | 4 | 5 | 7 | | | |
| 10 | 8 | .4994 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | | |
| 10 | 9 | .5073 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
| 10 | 10 | .5135 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 11 | 3 | .3491 | 0 | 1 | 3 | | | | | | | |
| 11 | 4 | .4201 | 0 | 1 | 2 | 5 | | | | | | |
| 11 | 5 | .4521 | 0 | 1 | 2 | 4 | 7 | | | | | |
| 11 | 6 | .4705 | 0 | 1 | 2 | 4 | 5 | 7 | | | | |
| 11 | 7 | .4832 | 0 | 1 | 2 | 3 | 4 | 6 | 7 | | | |
| 11 | 8 | .4931 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | | |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 9 | .5009 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | |
| 11 | 10 | .5072 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 12 | 3 | .3463 | 0 | 1 | 4 | | | | | | | |
| 12 | 4 | .4175 | 0 | 1 | 3 | 7 | | | | | | |
| 12 | 5 | .4458 | 0 | 1 | 2 | 4 | 7 | | | | | |
| 12 | 6 | .4646 | 0 | 1 | 2 | 3 | 5 | 8 | | | | |
| 12 | 7 | .4782 | 0 | 1 | 2 | 3 | 4 | 7 | 9 | | | |
| 12 | 8 | .4882 | 0 | 1 | 2 | 3 | 4 | 6 | 8 | 9 | | |
| 12 | 9 | .4957 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 9 | |
| 12 | 10 | .5019 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 13 | 3 | .3425 | 0 | 1 | 4 | | | | | | | |
| 13 | 4 | .4161 | 0 | 1 | 3 | 9 | | | | | | |
| 13 | 5 | .4408 | 0 | 1 | 2 | 4 | 7 | | | | | |
| 13 | 6 | .4601 | 0 | 1 | 2 | 3 | 6 | 10 | | | | |
| 13 | 7 | .4737 | 0 | 1 | 2 | 3 | 5 | 6 | 9 | | | |
| 13 | 8 | .4838 | 0 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | | |
| 13 | 9 | .4917 | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 9 | 10 | |
| 13 | 10 | .4975 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 10 |
| 14 | 3 | .3347 | 0 | 1 | 4 | | | | | | | |
| 14 | 4 | .4106 | 0 | 1 | 4 | 6 | | | | | | |
| 14 | 5 | .4368 | 0 | 1 | 2 | 4 | 9 | | | | | |
| 14 | 6 | .4568 | 0 | 1 | 2 | 3 | 6 | 10 | | | | |
| 14 | 7 | .4699 | 0 | 1 | 2 | 3 | 5 | 6 | 9 | | | |
| 14 | 8 | .4802 | 0 | 1 | 2 | 3 | 4 | 6 | 10 | 11 | | |
| 14 | 9 | .4878 | 0 | 1 | 2 | 3 | 4 | 5 | 7 | 9 | 10 | |
| 14 | 10 | .4940 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 8 | 10 | 11 |
| 15 | 3 | .3281 | 0 | 1 | 4 | | | | | | | |
| 15 | 4 | .4062 | 0 | 1 | 3 | 7 | | | | | | |
| 15 | 5 | .4339 | 0 | 1 | 2 | 4 | 10 | | | | | |
| 15 | 6 | .4543 | 0 | 1 | 2 | 3 | 6 | 10 | | | | |
| 15 | 7 | .4679 | 0 | 1 | 2 | 4 | 5 | 8 | 10 | | | |
| 15 | 8 | .4775 | 0 | 1 | 2 | 3 | 7 | 10 | 11 | 13 | | |
| 15 | 9 | .4848 | 0 | 1 | 2 | 3 | 8 | 9 | 11 | 12 | 13 | |
| 15 | 10 | .4907 | 0 | 1 | 2 | 3 | 4 | 6 | 7 | 8 | 9 | 13 |

## TABLE A.2

### $\underset{\sim}{\alpha}(0,1)$-DESIGNS , R=3 , $3 \leq S \leq 15$ , $3 \leq K \leq 9$

| S | K | $\bar{E}'$ | ARRAY $\underset{\sim}{\alpha}'$ |
|---|---|---|---|
| 3 | 3 | .7273 | 0 2 1 |
| | | | 0 1 2 |
| * 4 | 3 | .6801 | 0 3 2 |
| | | | 0 1 3 |
| * 4 | 4 | .7097 | 0 1 2 3 |
| | | | 0 2 3 1 |
| 5 | 3 | .6477 | 0 3 4 |
| | | | 0 2 1 |
| 5 | 4 | .6824 | 0 2 3 1 |
| | | | 0 3 2 4 |
| 5 | 5 | .7000 | 0 1 2 3 4 |
| | | | 0 4 3 2 1 |
| 6 | 3 | .6293 | 0 5 4 |
| | | | 0 3 1 |
| 6 | 4 | .6625 | 0 1 5 4 |
| | | | 0 4 3 5 |
| 6 | 5 | .6807 | 0 1 2 5 4 |
| | | | 0 3 1 2 5 |
| 7 | 3 | .6199 | 0 1 2 |
| | | | 0 3 6 |
| 7 | 4 | .6546 | 0 1 2 4 |
| | | | 0 3 6 5 |
| 7 | 5 | .6691 | 0 1 2 3 4 |
| | | | 0 3 6 1 5 |
| 7 | 6 | .6814 | 0 5 6 2 3 1 |
| | | | 0 2 1 5 4 6 |
| 7 | 7 | .6897 | 0 1 2 3 4 5 6 |
| | | | 0 6 5 4 3 2 1 |
| 8 | 3 | .6060 | 0 7 6 |
| | | | 0 4 1 |
| 8 | 4 | .6444 | 0 7 1 3 |
| | | | 0 2 5 1 |
| 8 | 5 | .6604 | 0 1 2 3 5 |
| | | | 0 3 7 4 1 |
| 8 | 6 | .6730 | 0 2 4 5 6 1 |
| | | | 0 3 7 4 2 6 |
| * 8 | 7 | .6804 | 0 7 5 4 3 2 6 |
| | | | 0 1 3 7 6 5 2 |
| * 8 | 8 | .6866 | 0 1 2 3 4 5 7 6 |
| | | | 0 4 5 6 7 3 1 2 |
| 9 | 3 | .5958 | 0 8 7 |
| | | | 0 4 1 |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 4 | .6399 | 0 | 8 | 2 | 6 | | | | | |
| | | | 0 | 1 | 7 | 3 | | | | | |
| 9 | 5 | .6536 | 0 | 8 | 1 | 2 | 4 | | | | |
| | | | 0 | 2 | 7 | 6 | 3 | | | | |
| 9 | 6 | .6662 | 0 | 2 | 3 | 4 | 5 | 1 | | | |
| | | | 0 | 1 | 5 | 7 | 6 | 8 | | | |
| 9 | 7 | .6731 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | |
| | | | 0 | 8 | 1 | 5 | 7 | 6 | 2 | | |
| 9 | 8 | .6794 | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 1 | |
| | | | 0 | 7 | 6 | 5 | 4 | 3 | 2 | 8 | |
| 9 | 9 | .6842 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | | | 0 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
| 10 | 3 | .5869 | 0 | 9 | 8 | | | | | | |
| | | | 0 | 5 | 1 | | | | | | |
| 10 | 4 | .6324 | 0 | 1 | 3 | 4 | | | | | |
| | | | 0 | 2 | 8 | 7 | | | | | |
| 10 | 5 | .6485 | 0 | 1 | 3 | 4 | 8 | | | | |
| | | | 0 | 4 | 9 | 2 | 5 | | | | |
| 10 | 6 | .6605 | 0 | 1 | 2 | 3 | 5 | 6 | | | |
| | | | 0 | 7 | 9 | 5 | 6 | 4 | | | |
| 10 | 7 | .6676 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | | |
| | | | 0 | 3 | 1 | 8 | 2 | 9 | 7 | | |
| 10 | 8 | .6728 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
| | | | 0 | 5 | 1 | 9 | 6 | 8 | 7 | 2 | |
| 10 | 9 | .6783 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | | | 0 | 5 | 1 | 6 | 2 | 7 | 3 | 8 | 4 |
| 11 | 3 | .5824 | 0 | 6 | 7 | | | | | | |
| | | | 0 | 9 | 2 | | | | | | |
| 11 | 4 | .6261 | 0 | 1 | 3 | 5 | | | | | |
| | | | 0 | 2 | 8 | 9 | | | | | |
| 11 | 5 | .6431 | 0 | 1 | 2 | 4 | 5 | | | | |
| | | | 0 | 8 | 10 | 7 | 6 | | | | |
| 11 | 6 | .6558 | 0 | 1 | 2 | 4 | 8 | 9 | | | |
| | | | 0 | 6 | 10 | 3 | 4 | 2 | | | |
| 11 | 7 | .6629 | 0 | 1 | 2 | 3 | 5 | 8 | 9 | | |
| | | | 0 | 2 | 5 | 10 | 9 | 6 | 8 | | |
| 11 | 8 | .6698 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | |
| | | | 0 | 9 | 1 | 10 | 6 | 8 | 7 | 5 | |
| 11 | 9 | .6738 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | | | 0 | 9 | 1 | 10 | 6 | 8 | 7 | 5 | 2 |
| 12 | 3 | .5762 | 0 | 5 | 7 | | | | | | |
| | | | 0 | 6 | 4 | | | | | | |
| 12 | 4 | .6207 | 0 | 1 | 7 | 9 | | | | | |
| | | | 0 | 2 | 5 | 6 | | | | | |
| 12 | 5 | .6393 | 0 | 1 | 3 | 7 | 8 | | | | |
| | | | 0 | 4 | 10 | 5 | 9 | | | | |
| 12 | 6 | .6513 | 0 | 1 | 2 | 6 | 7 | 10 | | | |
| | | | 0 | 8 | 4 | 5 | 10 | 11 | | | |
| 12 | 7 | .6582 | 0 | 1 | 2 | 4 | 5 | 8 | 10 | | |
| | | | 0 | 2 | 6 | 3 | 10 | 11 | 8 | | |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 12 | 8 | .6650 | 0 1 2 3 6 7 8 9 |
| | | | 0 10 5 11 7 6 1 4 |
| 12 | 9 | .6695 | 0 1 2 3 4 6 7 8 9 |
| | | | 0 10 5 11 2 7 6 1 4 |
| 13 | 3 | .5716 | 0 1 8 |
| | | | 0 3 4 |
| 13 | 4 | .6160 | 0 1 4 6 |
| | | | 0 3 5 12 |
| 13 | 5 | .6369 | 0 1 2 4 9 |
| | | | 0 2 7 12 11 |
| 13 | 6 | .6476 | 0 1 2 3 8 11 |
| | | | 0 5 8 12 11 10 |
| 13 | 7 | .6551 | 0 1 3 5 6 9 10 |
| | | | 0 3 12 10 7 8 4 |
| 13 | 8 | .6617 | 0 1 2 4 6 7 10 11 |
| | | | 0 8 11 7 5 2 3 12 |
| 14 | 3 | .5682 | 0 3 8 |
| | | | 0 4 7 |
| 14 | 4 | .6117 | 0 1 9 11 |
| | | | 0 8 10 13 |
| 14 | 5 | .6330 | 0 1 4 6 7 |
| | | | 0 2 3 11 10 |
| 14 | 6 | .6431 | 0 1 3 4 9 10 |
| | | | 0 9 6 8 1 5 |
| 14 | 7 | .6525 | 0 1 2 4 5 7 9 |
| | | | 0 3 1 10 12 11 5 |
| 15 | 3 | .5649 | 0 1 11 |
| | | | 0 3 4 |
| 15 | 4 | .6093 | 0 2 6 12 |
| | | | 0 13 5 7 |
| 15 | 5 | .6295 | 0 1 2 8 11 |
| | | | 0 6 14 9 13 |
| 15 | 6 | .6414 | 0 1 2 7 8 11 |
| | | | 0 5 14 6 9 13 |
| 15 | 7 | .6488 | 0 13 4 5 6 11 12 |
| | | | 0 3 2 7 1 8 11 |

# TABLE A.3

## $\underset{\sim}{\alpha}(0,1)$-DESIGNS , R=4 , $4 \leq S \leq 15$ , $4 \leq K \leq 9$

| S | K | $\bar{E}'$ | ARRAY $\underset{\sim}{\alpha}'$ |
|---|---|---|---|
| * 4 | 4 | .7895 | 0 1 2 3 |
| | | | 0 2 3 1 |
| | | | 0 3 1 2 |
| 5 | 4 | .7686 | 0 2 3 1 |
| | | | 0 4 1 2 |
| | | | 0 1 4 3 |
| 5 | 5 | .7808 | 0 1 3 4 2 |
| | | | 0 3 4 2 1 |
| | | | 0 4 2 1 3 |
| 6 | 4 | .7533 | 0 3 4 2 |
| | | | 0 4 2 5 |
| | | | 0 1 3 4 |
| 6 | 5 | .7655 | 0 1 2 3 5 |
| | | | 0 4 1 5 3 |
| | | | 0 3 5 1 4 |
| 7 | 4 | .7432 | 0 1 2 6 |
| | | | 0 6 4 3 |
| | | | 0 4 3 5 |
| 7 | 5 | .7567 | 0 1 2 3 6 |
| | | | 0 2 6 5 4 |
| | | | 0 4 3 1 5 |
| 7 | 6 | .7655 | 0 1 2 3 4 6 |
| | | | 0 2 4 6 1 5 |
| | | | 0 3 6 2 5 4 |
| 7 | 7 | .7714 | 0 1 2 3 4 5 6 |
| | | | 0 2 4 6 1 3 5 |
| | | | 0 3 6 2 5 1 4 |
| 8 | 4 | .7340 | 0 5 6 7 |
| | | | 0 7 1 5 |
| | | | 0 1 4 6 |
| 8 | 5 | .7488 | 0 1 2 7 4 |
| | | | 0 2 7 3 6 |
| | | | 0 5 3 1 7 |
| 8 | 6 | .7576 | 0 1 2 5 7 4 |
| | | | 0 3 6 2 5 7 |
| | | | 0 5 1 3 2 6 |
| 8 | 7 | .7636 | 0 1 2 3 4 5 7 |
| | | | 0 5 1 6 2 7 4 |
| | | | 0 2 4 7 1 3 6 |
| 9 | 4 | .7269 | 0 6 7 8 |
| | | | 0 8 1 6 |
| | | | 0 1 4 7 |

| S | K | $\bar{E}'$ | ARRAY $\underset{\sim}{\alpha}'$ |
|---|---|---|---|
| 9 | 5 | .7419 | 0 5 6 8 7 |
| | | | 0 2 4 3 8 |
| | | | 0 7 1 4 6 |
| 9 | 6 | .7520 | 0 1 2 5 6 8 |
| | | | 0 7 4 8 5 6 |
| | | | 0 3 8 6 2 7 |
| 9 | 7 | .7581 | 0 2 3 5 6 7 8 |
| | | | 0 8 5 6 4 1 7 |
| | | | 0 3 8 7 1 6 5 |
| 10 | 4 | .7240 | 0 6 7 9 |
| | | | 0 2 4 8 |
| | | | 0 8 5 3 |
| 10 | 5 | .7382 | 0 1 2 4 9 |
| | | | 0 4 9 5 8 |
| | | | 0 6 3 1 7 |
| 10 | 6 | .7467 | 0 2 4 5 8 9 |
| | | | 0 5 6 3 4 8 |
| | | | 0 6 3 1 9 7 |
| 10 | 7 | .7532 | 0 3 4 6 7 8 9 |
| | | | 0 9 6 7 1 5 8 |
| | | | 0 2 1 9 8 3 7 |
| 10 | 8 | .7580 | 0 1 3 4 5 6 7 8 |
| | | | 0 7 8 3 2 9 1 6 |
| | | | 0 8 6 2 7 5 3 9 |
| 10 | 9 | .7617 | 0 1 2 3 4 5 8 9 7 |
| | | | 0 5 3 2 7 1 6 4 9 |
| | | | 0 9 5 7 3 2 4 1 8 |
| 11 | 4 | .7181 | 0 1 7 8 |
| | | | 0 7 1 10 |
| | | | 0 10 8 1 |
| 11 | 5 | .7325 | 0 1 2 5 6 |
| | | | 0 5 1 2 8 |
| | | | 0 2 6 8 1 |
| 11 | 6 | .7432 | 0 1 2 3 5 10 |
| | | | 0 3 6 2 1 7 |
| | | | 0 6 8 1 2 3 |
| 11 | 7 | .7485 | 0 1 2 3 4 5 9 |
| | | | 0 2 4 6 3 1 7 |
| | | | 0 7 1 4 6 3 2 |
| 11 | 8 | .7544 | 0 1 2 3 4 5 9 10 |
| | | | 0 7 4 6 3 1 2 8 |
| | | | 0 8 7 2 1 6 4 3 |
| 11 | 9 | .7580 | 0 1 2 3 4 5 6 8 9 |
| | | | 0 5 9 1 6 4 3 2 10 |
| | | | 0 8 5 2 1 3 7 10 4 |
| 12 | 4 | .7122 | 0 1 4 9 |
| | | | 0 4 1 11 |
| | | | 0 7 5 1 |
| 12 | 5 | .7287 | 0 1 2 5 10 |
| | | | 0 2 6 1 7 |
| | | | 0 6 8 2 1 |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 12 | 6 | .7395 | 0 1 2 4 5 10 |
| | | | 0 4 1 2 6 7 |
| | | | 0 2 7 6 1 4 |
| 12 | 7 | .7452 | 0 1 2 3 4 6 7 |
| | | | 0 4 6 1 3 2 9 |
| | | | 0 9 3 2 6 4 1 |
| 12 | 8 | .7511 | 0 1 2 3 4 5 6 10 |
| | | | 0 3 5 7 1 4 2 8 |
| | | | 0 5 8 4 7 2 1 3 |
| 12 | 9 | .7547 | 0 4 5 6 7 8 9 10 2 |
| | | | 0 3 6 8 10 4 7 5 11 |
| | | | 0 2 7 10 6 9 4 3 5 |
| 13 | 4 | .7100 | 0 1 5 10 |
| | | | 0 5 1 12 |
| | | | 0 8 6 1 |
| 13 | 5 | .7250 | 0 1 2 5 6 |
| | | | 0 5 1 2 8 |
| | | | 0 2 5 9 1 |
| 13 | 6 | .7363 | 0 1 2 3 6 7 |
| | | | 0 6 3 1 2 9 |
| | | | 0 2 6 9 3 1 |
| 13 | 7 | .7427 | 0 1 2 3 4 6 7 |
| | | | 0 4 6 1 3 2 9 |
| | | | 0 6 4 9 2 3 1 |
| 13 | 8 | .7481 | 0 5 6 7 8 9 11 12 |
| | | | 0 8 12 1 9 11 10 4 |
| | | | 0 2 8 6 11 4 5 3 |
| 14 | 4 | .7043 | 0 1 4 5 |
| | | | 0 4 1 7 |
| | | | 0 5 6 1 |
| 14 | 5 | .7236 | 0 1 2 6 7 |
| | | | 0 6 1 2 9 |
| | | | 0 2 6 9 1 |
| 14 | 6 | .7330 | 0 1 2 3 6 7 |
| | | | 0 6 3 1 2 9 |
| | | | 0 9 1 6 3 2 |
| 14 | 7 | .7394 | 0 1 2 3 4 5 9 |
| | | | 0 4 3 5 1 11 2 |
| | | | 0 3 1 11 5 2 4 |
| 15 | 4 | .7045 | 0 1 4 13 |
| | | | 0 5 1 6 |
| | | | 0 7 5 1 |
| 15 | 5 | .7206 | 0 1 2 4 9 |
| | | | 0 4 1 2 11 |
| | | | 0 11 4 1 2 |
| 15 | 6 | .7310 | 0 1 2 3 7 12 |
| | | | 0 7 3 1 2 14 |
| | | | 0 3 1 7 14 2 |
| 15 | 7 | .7386 | 0 2 3 4 5 9 14 |
| | | | 0 6 13 9 7 8 5 |
| | | | 0 7 10 8 14 6 9 |

# TABLE B.1

## PROPERTIES OF $\alpha(0,1)$-DESIGNS , R=2 , 13≤V≤100

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 13 | 4 | 4 | 3 | .3333 | .6306 | .5738(12) | .7496( 4) | 0312 |
| 14 | 4 | 4 | 2 | .3333 | .6628 | .6081(17) | .7684( 7) | 0132 |
| 15 | 4 | 4 | 1 | .3750 | .6908 | .6396( 9) | .7851( 3) | 0123 |
| 16 | 4 | 4 | 0 | .5000 | .7143 | .6667( 0) | .8000( 0) | 0123 |
| 17 | 5 | 4 | 3 | .2572 | .6090 | .5620(17) | .7491( 8) | 01342 |
| 18 | 5 | 4 | 2 | .2996 | .6353 | .5899(13) | .7638( 6) | 01234 |
| 19 | 5 | 4 | 1 | .2977 | .6566 | .6127(11) | .7774( 4) | 01324 |
| 20 | 5 | 4 | 0 | .3750 | .6770 | .6352( 5) | .7895( 0) | 01234 |
| 21 | 6 | 4 | 3 | .2205 | .5898 | .5493(17) | .7495( 8) | 01234 |
| 21 | 5 | 5 | 4 | .3750 | .6942 | .6529( 9) | .7999( 2) | 04123 |
| 22 | 6 | 4 | 2 | .2202 | .6109 | .5713(15) | .7615( 6) | 03124 |
| 22 | 5 | 5 | 3 | .3750 | .7101 | .6698(11) | .8093( 4) | 01423 |
| 23 | 6 | 4 | 1 | .2310 | .6306 | .5923(15) | .7724( 6) | 01342 |
| 23 | 5 | 5 | 2 | .3750 | .7247 | .6858(11) | .8179( 4) | 01243 |
| 24 | 6 | 4 | 0 | .2835 | .6501 | .6134( 8) | .7826( 1) | 01234 |
| 24 | 5 | 5 | 1 | .4000 | .7380 | .7007( 6) | .8259( 2) | 01234 |
| 25 | 7 | 4 | 3 | .2209 | .5863 | .5526(14) | .7486( 7) | 01234 |
| 25 | 5 | 5 | 0 | .5000 | .7500 | .7143( 0) | .8333( 0) | 01234 |
| 26 | 7 | 4 | 2 | .2240 | .6038 | .5706(13) | .7591( 5) | 01423 |
| 26 | 6 | 5 | 4 | .2943 | .6803 | .6447(11) | .7997( 5) | 01452 |
| 27 | 7 | 4 | 1 | .2500 | .6207 | .5883(12) | .7690( 4) | 01245 |
| 27 | 6 | 5 | 3 | .3161 | .6940 | .6592(12) | .8072( 5) | 01245 |
| 28 | 7 | 4 | 0 | .3232 | .6364 | .6050( 5) | .7778( 0) | 01234 |
| 28 | 6 | 5 | 2 | .3346 | .7063 | .6724( 9) | .8144( 4) | 01234 |
| 29 | 8 | 4 | 3 | .1785 | .5719 | .5414(17) | .7490( 7) | 04123 |
| 29 | 6 | 5 | 1 | .3382 | .7174 | .6844( 8) | .8213( 2) | 01243 |
| 30 | 8 | 4 | 2 | .1842 | .5887 | .5589(16) | .7579( 5) | 01452 |
| 30 | 6 | 5 | 0 | .4000 | .7280 | .6962( 4) | .8276( 0) | 01234 |
| 31 | 8 | 4 | 1 | .2128 | .6054 | .5764(15) | .7665( 5) | 01245 |
| 31 | 7 | 5 | 4 | .2474 | .6683 | .6367(14) | .7996( 6) | 01245 |
| 31 | 6 | 6 | 5 | .4000 | .7376 | .7062( 6) | .8333( 1) | 05123 |
| 32 | 8 | 4 | 0 | .2500 | .6206 | .5924( 8) | .7742( 1) | 01234 |
| 32 | 7 | 5 | 3 | .2686 | .6806 | .6497(11) | .8061( 5) | 01234 |
| 32 | 6 | 6 | 4 | .4000 | .7467 | .7158( 8) | .8386( 3) | 01523 |
| 33 | 9 | 4 | 3 | .2116 | .5715 | .5449(16) | .7499( 7) | 04512 |
| 33 | 7 | 5 | 2 | .2670 | .6912 | .6610(10) | .8123( 4) | 01423 |
| 33 | 6 | 6 | 3 | .4000 | .7552 | .7251( 8) | .8436( 3) | 01253 |
| 34 | 9 | 4 | 2 | .2127 | .5846 | .5583(16) | .7576( 7) | 01456 |
| 34 | 7 | 5 | 1 | .2796 | .7015 | .6721(10) | .8181( 3) | 01245 |
| 34 | 6 | 6 | 2 | .4000 | .7632 | .7339( 8) | .8484( 3) | 01235 |
| 35 | 9 | 4 | 1 | .2087 | .5973 | .5715(15) | .7646( 5) | 01245 |
| 35 | 7 | 5 | 0 | .3198 | .7115 | .6829( 5) | .8235( 1) | 01234 |
| 35 | 6 | 6 | 1 | .4167 | .7707 | .7422( 4) | .8528( 1) | 01234 |
| 36 | 9 | 4 | 0 | .2500 | .6101 | .5848( 7) | .7714( 1) | 01234 |
| 36 | 8 | 5 | 4 | .2388 | .6612 | .6333(14) | .7992( 5) | 01235 |
| 36 | 6 | 6 | 0 | .5000 | .7778 | .7500( 0) | .8571( 0) | 01234 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 37 | 10 | 4 | 3 | .1560 | .5618 | .5375(18) | .7491( 7) | 01562 |
| 37 | 8 | 5 | 3 | .2471 | .6722 | .6450(11) | .8049( 4) | 01234 |
| 37 | 7 | 6 | 5 | .3249 | .7285 | .7007( 8) | .8332( 3) | 01562 |
| 38 | 10 | 4 | 2 | .1686 | .5760 | .5522(18) | .7563( 6) | 01256 |
| 38 | 8 | 5 | 2 | .2536 | .6820 | .6553(10) | .8104( 3) | 01253 |
| 38 | 7 | 6 | 4 | .3349 | .7364 | .7091( 9) | .8376( 3) | 01256 |
| 39 | 10 | 4 | 1 | .1962 | .5899 | .5666(16) | .7630( 5) | 01235 |
| 39 | 8 | 5 | 1 | .2763 | .6915 | .6654( 9) | .8157( 3) | 01235 |
| 39 | 7 | 6 | 3 | .3519 | .7438 | .7171( 9) | .8419( 3) | 01235 |
| 40 | 10 | 4 | 0 | .2314 | .6024 | .5796( 8) | .7692( 1) | 01234 |
| 40 | 8 | 5 | 0 | .3268 | .7004 | .6750( 5) | .8205( 1) | 01234 |
| 40 | 7 | 6 | 2 | .3597 | .7508 | .7246( 7) | .8460( 3) | 01234 |
| 41 | 11 | 4 | 3 | .1347 | .5555 | .5332(20) | .7491( 7) | 01567 |
| 41 | 9 | 5 | 4 | .2238 | .6555 | .6306(14) | .7991( 5) | 01234 |
| 41 | 7 | 6 | 1 | .3652 | .7574 | .7317( 6) | .8499( 2) | 01235 |
| 42 | 11 | 4 | 2 | .1496 | .5702 | .5485(19) | .7557( 5) | 01256 |
| 42 | 9 | 5 | 3 | .2254 | .6653 | .6408(11) | .8042( 4) | 01234 |
| 42 | 7 | 6 | 0 | .4167 | .7637 | .7386( 3) | .8537( 0) | 01234 |
| 43 | 11 | 4 | 1 | .1793 | .5846 | .5634(17) | .7617( 5) | 01235 |
| 43 | 9 | 5 | 2 | .2389 | .6741 | .6501(10) | .8092( 4) | 01236 |
| 43 | 8 | 6 | 5 | .2743 | .7205 | .6955(10) | .8331( 4) | 01256 |
| 43 | 7 | 7 | 6 | .4167 | .7696 | .7448( 5) | .8571( 1) | 06123 |
| 44 | 11 | 4 | 0 | .2295 | .5973 | .5766( 8) | .7674( 1) | 01234 |
| 44 | 9 | 5 | 1 | .2652 | .6827 | .6591(10) | .8139( 3) | 01234 |
| 44 | 8 | 6 | 4 | .2944 | .7278 | .7033(10) | .8370( 4) | 01235 |
| 44 | 7 | 7 | 5 | .4167 | .7752 | .7509( 6) | .8604( 2) | 01623 |
| 45 | 12 | 4 | 3 | .2207 | .5672 | .5478(14) | .7495( 6) | 01237 |
| 45 | 9 | 5 | 0 | .3000 | .6906 | .6676( 5) | .8182( 1) | 01234 |
| 45 | 8 | 6 | 3 | .3000 | .7345 | .7104( 8) | .8407( 3) | 01234 |
| 45 | 7 | 7 | 4 | .4167 | .7806 | .7567( 6) | .8636( 2) | 01263 |
| 46 | 12 | 4 | 2 | .2196 | .5764 | .5572(15) | .7554( 7) | 01234 |
| 46 | 10 | 5 | 4 | .1981 | .6501 | .6275(12) | .7991( 5) | 01234 |
| 46 | 8 | 6 | 2 | .3015 | .7408 | .7171( 7) | .8443( 3) | 01253 |
| 46 | 7 | 7 | 3 | .4167 | .7858 | .7623( 6) | .8666( 2) | 01236 |
| 47 | 12 | 4 | 1 | .2193 | .5854 | .5664(15) | .7608( 4) | 01234 |
| 47 | 10 | 5 | 3 | .2024 | .6590 | .6368(12) | .8038( 4) | 01263 |
| 47 | 8 | 6 | 1 | .3139 | .7468 | .7236( 7) | .8478( 2) | 01235 |
| 47 | 7 | 7 | 2 | .4167 | .7908 | .7677( 6) | .8695( 2) | 01234 |
| 48 | 12 | 4 | 0 | .2500 | .5942 | .5754( 8) | .7660( 1) | 01234 |
| 48 | 10 | 5 | 2 | .2167 | .6677 | .6459(11) | .8082( 4) | 01236 |
| 48 | 8 | 6 | 0 | .3460 | .7527 | .7300( 4) | .8511( 0) | 01234 |
| 48 | 7 | 7 | 1 | .4286 | .7955 | .7728( 3) | .8723( 1) | 01234 |
| 49 | 13 | 4 | 3 | .1901 | .5649 | .5470(11) | .7496( 5) | 01234 |
| 49 | 10 | 5 | 1 | .2420 | .6761 | .6547(10) | .8124( 3) | 01234 |
| 49 | 9 | 6 | 5 | .2503 | .7144 | .6918(11) | .8329( 4) | 01236 |
| 49 | 7 | 7 | 0 | .5000 | .8000 | .7778( 0) | .8750( 0) | 01234 |
| 50 | 13 | 4 | 2 | .2009 | .5743 | .5566(11) | .7549( 5) | 01234 |
| 50 | 10 | 5 | 0 | .2851 | .6839 | .6630( 5) | .8163( 1) | 01234 |
| 50 | 9 | 6 | 4 | .2718 | .7213 | .6991( 8) | .8363( 3) | 01234 |
| 50 | 8 | 7 | 6 | .3487 | .7633 | .7411( 6) | .8571( 2) | 01672 |
| 51 | 13 | 4 | 1 | .2121 | .5835 | .5660(10) | .7599( 3) | 01234 |
| 51 | 11 | 5 | 4 | .2409 | .6525 | .6324(10) | .7994( 4) | 01234 |
| 51 | 9 | 6 | 3 | .2720 | .7275 | .7057( 8) | .8397( 3) | 01234 |
| 51 | 8 | 7 | 5 | .3539 | .7683 | .7464( 7) | .8599( 2) | 01267 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 52 | 13 | 4 | 0 | .2835 | .5925 | .5753( 3) | .7647( 0) | 01234 |
| 52 | 11 | 5 | 3 | .2438 | .6596 | .6397( 9) | .8035( 3) | 01237 |
| 52 | 9 | 6 | 2 | .2799 | .7334 | .7119( 7) | .8430( 2) | 01236 |
| 52 | 8 | 7 | 4 | .3620 | .7731 | .7515( 7) | .8626( 2) | 01236 |
| 53 | 14 | 4 | 3 | .1855 | .5604 | .5437(13) | .7498( 6) | 01267 |
| 53 | 11 | 5 | 2 | .2563 | .6665 | .6469( 9) | .8075( 3) | 01234 |
| 53 | 9 | 6 | 1 | .2998 | .7391 | .7180( 7) | .8461( 2) | 01234 |
| 53 | 8 | 7 | 3 | .3760 | .7777 | .7565( 7) | .8653( 2) | 01234 |
| 54 | 14 | 4 | 2 | .1844 | .5695 | .5530(11) | .7545( 5) | 01236 |
| 54 | 11 | 5 | 1 | .2697 | .6732 | .6539( 9) | .8113( 3) | 01234 |
| 54 | 9 | 6 | 0 | .3359 | .7445 | .7239( 4) | .8491( 0) | 01234 |
| 54 | 8 | 7 | 2 | .3783 | .7821 | .7612( 5) | .8679( 2) | 01234 |
| 55 | 14 | 4 | 1 | .1999 | .5782 | .5618(11) | .7592( 3) | 01234 |
| 55 | 11 | 5 | 0 | .3268 | .6796 | .6606( 4) | .8148( 0) | 01234 |
| 55 | 10 | 6 | 5 | .2805 | .7126 | .6924(10) | .8331( 4) | 01234 |
| 55 | 8 | 7 | 1 | .3844 | .7863 | .7658( 4) | .8704( 1) | 01234 |
| 56 | 14 | 4 | 0 | .2500 | .5866 | .5704( 4) | .7636( 0) | 01234 |
| 56 | 12 | 5 | 4 | .2190 | .6490 | .6305(11) | .7995( 4) | 01273 |
| 56 | 10 | 6 | 4 | .2885 | .7181 | .6982( 8) | .8361( 3) | 01234 |
| 56 | 8 | 7 | 0 | .4286 | .7904 | .7702( 2) | .8727( 0) | 01234 |
| 57 | 15 | 4 | 3 | .1647 | .5574 | .5417(12) | .7497( 5) | 01237 |
| 57 | 12 | 5 | 3 | .2245 | .6554 | .6371(10) | .8032( 3) | 01237 |
| 57 | 10 | 6 | 3 | .2896 | .7233 | .7036( 8) | .8392( 3) | 01263 |
| 57 | 9 | 7 | 6 | .2981 | .7578 | .7376( 8) | .8571( 3) | 01267 |
| 57 | 8 | 8 | 7 | .4286 | .7943 | .7743( 4) | .8750( 1) | 07123 |
| 58 | 15 | 4 | 2 | .1771 | .5658 | .5503(11) | .7542( 5) | 01234 |
| 58 | 12 | 5 | 2 | .2391 | .6617 | .6436( 9) | .8069( 3) | 01234 |
| 58 | 10 | 6 | 2 | .2983 | .7283 | .7088( 7) | .8421( 3) | 01236 |
| 58 | 9 | 7 | 5 | .3103 | .7624 | .7425( 7) | .8595( 3) | 01236 |
| 58 | 8 | 8 | 6 | .4286 | .7981 | .7783( 5) | .8772( 2) | 01723 |
| 59 | 15 | 4 | 1 | .1952 | .5740 | .5586(11) | .7586( 3) | 01234 |
| 59 | 12 | 5 | 1 | .2591 | .6677 | .6499( 9) | .8103( 3) | 01234 |
| 59 | 10 | 6 | 1 | .3067 | .7331 | .7139( 7) | .8448( 2) | 01234 |
| 59 | 9 | 7 | 4 | .3267 | .7668 | .7472( 7) | .8620( 3) | 01234 |
| 59 | 8 | 8 | 5 | .4286 | .8017 | .7822( 5) | .8793( 2) | 01273 |
| 60 | 15 | 4 | 0 | .2500 | .5818 | .5667( 4) | .7627( 0) | 01234 |
| 60 | 12 | 5 | 0 | .3000 | .6735 | .6559( 5) | .8136( 1) | 01234 |
| 60 | 10 | 6 | 0 | .3209 | .7379 | .7190( 4) | .8475( 0) | 01234 |
| 60 | 9 | 7 | 3 | .3247 | .7710 | .7516( 6) | .8643( 2) | 01234 |
| 60 | 8 | 8 | 4 | .4286 | .8052 | .7860( 5) | .8814( 2) | 01237 |
| 61 | 13 | 5 | 4 | .1962 | .6445 | .6272(12) | .7996( 5) | 01278 |
| 61 | 11 | 6 | 5 | .2880 | .7102 | .6919( 9) | .8330( 3) | 01234 |
| 61 | 9 | 7 | 2 | .3276 | .7750 | .7559( 5) | .8666( 2) | 01236 |
| 61 | 8 | 8 | 3 | .4286 | .8086 | .7897( 5) | .8834( 2) | 01234 |
| 62 | 13 | 5 | 3 | .2024 | .6508 | .6337(11) | .8030( 3) | 01237 |
| 62 | 11 | 6 | 4 | .2940 | .7151 | .6969( 8) | .8358( 3) | 01234 |
| 62 | 9 | 7 | 1 | .3392 | .7789 | .7601( 5) | .8689( 1) | 01234 |
| 62 | 8 | 8 | 2 | .4286 | .8119 | .7933( 5) | .8853( 2) | 01234 |
| 63 | 13 | 5 | 2 | .2174 | .6570 | .6401(11) | .8063( 3) | 01234 |
| 63 | 11 | 6 | 3 | .2925 | .7198 | .7018( 7) | .8385( 3) | 01237 |
| 63 | 9 | 7 | 0 | .3658 | .7827 | .7642( 3) | .8710( 0) | 01234 |
| 63 | 8 | 8 | 1 | .4375 | .8151 | .7968( 3) | .8872( 1) | 01234 |
| 64 | 13 | 5 | 1 | .2366 | .6630 | .6463(10) | .8095( 3) | 01234 |
| 64 | 11 | 6 | 2 | .2948 | .7244 | .7067( 6) | .8412( 2) | 01234 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 64 | 10 | 7 | 6 | .2671 | .7531 | .7346( 8) | .8569( 3) | 01237 |
| 64 | 8 | 8 | 0 | .5000 | .8182 | .8000( 0) | .8889( 0) | 01234 |
| 65 | 13 | 5 | 0 | .2718 | .6687 | .6523( 6) | .8125( 1) | 01234 |
| 65 | 11 | 6 | 1 | .3081 | .7289 | .7114( 6) | .8438( 2) | 01234 |
| 65 | 10 | 7 | 5 | .2844 | .7575 | .7393( 8) | .8592( 3) | 01234 |
| 65 | 9 | 8 | 7 | .3673 | .7898 | .7716( 5) | .8750( 2) | 01782 |
| 66 | 14 | 5 | 4 | .1966 | .6430 | .6270(11) | .7996( 4) | 01234 |
| 66 | 11 | 6 | 0 | .3557 | .7333 | .7160( 3) | .8462( 0) | 01234 |
| 66 | 10 | 7 | 4 | .2925 | .7617 | .7437( 7) | .8614( 2) | 01234 |
| 66 | 9 | 8 | 6 | .3702 | .7932 | .7752( 5) | .8769( 2) | 01278 |
| 67 | 14 | 5 | 3 | .2061 | .6487 | .6328(11) | .8028( 3) | 01234 |
| 67 | 12 | 6 | 5 | .2432 | .7055 | .6884(10) | .8329( 5) | 01234 |
| 67 | 10 | 7 | 3 | .2942 | .7656 | .7479( 6) | .8635( 2) | 01234 |
| 67 | 9 | 8 | 5 | .3746 | .7964 | .7787( 5) | .8788( 2) | 01237 |
| 68 | 14 | 5 | 2 | .2213 | .6542 | .6385(11) | .8059( 3) | 01234 |
| 68 | 12 | 6 | 4 | .2446 | .7103 | .6934( 8) | .8355( 3) | 01234 |
| 68 | 10 | 7 | 2 | .3022 | .7694 | .7519( 5) | .8656( 2) | 01234 |
| 68 | 9 | 8 | 4 | .3815 | .7996 | .7821( 5) | .8806( 2) | 01234 |
| 69 | 14 | 5 | 1 | .2428 | .6596 | .6440(10) | .8088( 3) | 01234 |
| 69 | 12 | 6 | 3 | .2503 | .7149 | .6982( 8) | .8380( 2) | 01234 |
| 69 | 10 | 7 | 1 | .3192 | .7731 | .7559( 5) | .8677( 1) | 01234 |
| 69 | 9 | 8 | 3 | .3933 | .8027 | .7854( 5) | .8823( 2) | 01234 |
| 70 | 14 | 5 | 0 | .2614 | .6647 | .6493( 6) | .8116( 1) | 01234 |
| 70 | 12 | 6 | 2 | .2628 | .7195 | .7030( 8) | .8405( 2) | 01234 |
| 70 | 10 | 7 | 0 | .3465 | .7767 | .7597( 3) | .8696( 0) | 01234 |
| 70 | 9 | 8 | 2 | .3927 | .8057 | .7886( 4) | .8840( 1) | 01234 |
| 71 | 15 | 5 | 4 | .1828 | .6400 | .6250(12) | .7996( 4) | 01234 |
| 71 | 12 | 6 | 1 | .2780 | .7239 | .7076( 7) | .8429( 2) | 01234 |
| 71 | 11 | 7 | 6 | .2854 | .7512 | .7344( 8) | .8570( 3) | 01234 |
| 71 | 9 | 8 | 1 | .3989 | .8085 | .7917( 4) | .8857( 1) | 01234 |
| 72 | 15 | 5 | 3 | .1922 | .6457 | .6308(12) | .8026( 3) | 01234 |
| 72 | 12 | 6 | 0 | .3137 | .7282 | .7121( 4) | .8451( 0) | 01234 |
| 72 | 11 | 7 | 5 | .2991 | .7550 | .7384( 8) | .8590( 3) | 01234 |
| 72 | 9 | 8 | 0 | .4375 | .8114 | .7947( 2) | .8873( 0) | 01234 |
| 73 | 15 | 5 | 2 | .2084 | .6512 | .6365(11) | .8055( 3) | 01234 |
| 73 | 13 | 6 | 5 | .2490 | .7040 | .6882(10) | .8331( 4) | 01234 |
| 73 | 11 | 7 | 4 | .2985 | .7586 | .7422( 6) | .8610( 2) | 01234 |
| 73 | 10 | 8 | 7 | .3186 | .7858 | .7692( 6) | .8750( 2) | 01278 |
| 73 | 9 | 9 | 8 | .4375 | .8141 | .7975( 3) | .8889( 1) | 08123 |
| 74 | 15 | 5 | 1 | .2275 | .6566 | .6420(11) | .8082( 3) | 01234 |
| 74 | 13 | 6 | 4 | .2495 | .7083 | .6927( 8) | .8354( 3) | 01234 |
| 74 | 11 | 7 | 3 | .3013 | .7620 | .7458( 6) | .8630( 2) | 01237 |
| 74 | 10 | 8 | 6 | .3262 | .7889 | .7725( 6) | .8767( 2) | 01237 |
| 74 | 9 | 9 | 7 | .4375 | .8167 | .8003( 4) | .8904( 1) | 01823 |
| 75 | 15 | 5 | 0 | .2603 | .6617 | .6473( 6) | .8108( 1) | 01234 |
| 75 | 13 | 6 | 3 | .2543 | .7124 | .6969( 8) | .8377( 3) | 01234 |
| 75 | 11 | 7 | 2 | .3097 | .7654 | .7493( 6) | .8649( 2) | 01234 |
| 75 | 10 | 8 | 5 | .3366 | .7920 | .7757( 6) | .8783( 2) | 01234 |
| 75 | 9 | 9 | 6 | .4375 | .8193 | .8031( 4) | .8919( 1) | 01283 |
| 76 | 13 | 6 | 2 | .2638 | .7164 | .7011( 7) | .8400( 2) | 01234 |
| 76 | 11 | 7 | 1 | .3205 | .7687 | .7528( 5) | .8667( 1) | 01234 |
| 76 | 10 | 8 | 4 | .3451 | .7949 | .7788( 6) | .8800( 2) | 01234 |
| 76 | 9 | 9 | 5 | .4375 | .8218 | .8057( 4) | .8933( 1) | 01238 |
| 77 | 13 | 6 | 1 | .2795 | .7203 | .7052( 7) | .8421( 2) | 01234 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 77 | 11 | 7 | 0 | .3454 | .7719 | .7562( 3) | .8685( 0) | 01234 |
| 77 | 10 | 8 | 3 | .3443 | .7977 | .7818( 4) | .8816( 2) | 01234 |
| 77 | 9 | 9 | 4 | .4375 | .8242 | .8084( 4) | .8947( 1) | 01234 |
| 78 | 13 | 6 | 0 | .3081 | .7241 | .7091( 4) | .8442( 0) | 01234 |
| 78 | 12 | 7 | 6 | .2914 | .7490 | .7336( 7) | .8570( 2) | 01234 |
| 78 | 10 | 8 | 2 | .3478 | .8004 | .7847( 4) | .8831( 1) | 01234 |
| 78 | 9 | 9 | 3 | .4375 | .8266 | .8109( 4) | .8961( 1) | 01234 |
| 79 | 14 | 6 | 5 | .2385 | .7022 | .6875(10) | .8330( 4) | 01234 |
| 79 | 12 | 7 | 5 | .3026 | .7524 | .7371( 7) | .8588( 2) | 01234 |
| 79 | 10 | 8 | 1 | .3585 | .8031 | .7876( 4) | .8847( 1) | 01234 |
| 79 | 9 | 9 | 2 | .4375 | .8289 | .8134( 4) | .8974( 1) | 01234 |
| 80 | 14 | 6 | 4 | .2406 | .7062 | .6917( 8) | .8352( 3) | 01234 |
| 80 | 12 | 7 | 4 | .3027 | .7556 | .7405( 6) | .8607( 2) | 01234 |
| 80 | 10 | 8 | 0 | .3811 | .8057 | .7904( 2) | .8861( 0) | 01234 |
| 80 | 9 | 9 | 1 | .4444 | .8311 | .8158( 2) | .8987( 1) | 01234 |
| 81 | 14 | 6 | 3 | .2475 | .7101 | .6957( 8) | .8374( 3) | 01234 |
| 81 | 12 | 7 | 3 | .3050 | .7588 | .7437( 6) | .8624( 2) | 01234 |
| 81 | 11 | 8 | 7 | .2846 | .7823 | .7670( 6) | .8749( 2) | 01238 |
| 81 | 9 | 9 | 0 | .5000 | .8333 | .8182( 0) | .9000( 0) | 01234 |
| 82 | 14 | 6 | 2 | .2583 | .7138 | .6996( 8) | .8395( 2) | 01234 |
| 82 | 12 | 7 | 2 | .3103 | .7618 | .7470( 5) | .8642( 2) | 01234 |
| 82 | 11 | 8 | 6 | .2960 | .7853 | .7701( 6) | .8764( 2) | 01234 |
| 82 | 10 | 9 | 8 | .3821 | .8107 | .7956( 4) | .8889( 1) | 01892 |
| 83 | 14 | 6 | 1 | .2765 | .7175 | .7034( 7) | .8415( 2) | 01234 |
| 83 | 12 | 7 | 1 | .3236 | .7649 | .7502( 5) | .8659( 1) | 01234 |
| 83 | 11 | 8 | 5 | .3107 | .7882 | .7732( 6) | .8780( 2) | 01234 |
| 83 | 10 | 9 | 7 | .3838 | .8131 | .7981( 4) | .8903( 1) | 01289 |
| 84 | 14 | 6 | 0 | .3012 | .7211 | .7071( 4) | .8434( 0) | 01234 |
| 84 | 12 | 7 | 0 | .3571 | .7678 | .7533( 3) | .8676( 0) | 01234 |
| 84 | 11 | 8 | 4 | .3109 | .7909 | .7761( 5) | .8794( 2) | 01234 |
| 84 | 10 | 9 | 6 | .3864 | .8154 | .8006( 4) | .8916( 1) | 01238 |
| 85 | 15 | 6 | 5 | .2275 | .7003 | .6867( 9) | .8330( 3) | 01234 |
| 85 | 13 | 7 | 6 | .2767 | .7464 | .7321( 7) | .8569( 3) | 01234 |
| 85 | 11 | 8 | 3 | .3134 | .7936 | .7789( 5) | .8809( 2) | 01234 |
| 85 | 10 | 9 | 5 | .3902 | .8177 | .8030( 4) | .8929( 1) | 01234 |
| 86 | 15 | 6 | 4 | .2308 | .7042 | .6907( 8) | .8351( 3) | 01234 |
| 86 | 13 | 7 | 5 | .2779 | .7496 | .7354( 7) | .8586( 2) | 01234 |
| 86 | 11 | 8 | 2 | .3210 | .7962 | .7816( 4) | .8824( 1) | 01234 |
| 86 | 10 | 9 | 4 | .3962 | .8199 | .8054( 4) | .8941( 1) | 01234 |
| 87 | 15 | 6 | 3 | .2383 | .7079 | .6946( 8) | .8371( 3) | 01234 |
| 87 | 13 | 7 | 4 | .2791 | .7526 | .7386( 6) | .8603( 2) | 01234 |
| 87 | 11 | 8 | 1 | .3356 | .7987 | .7843( 4) | .8838( 1) | 01234 |
| 87 | 10 | 9 | 3 | .4048 | .8221 | .8077( 4) | .8954( 1) | 01234 |
| 88 | 15 | 6 | 2 | .2493 | .7116 | .6984( 8) | .8391( 2) | 01234 |
| 88 | 13 | 7 | 3 | .2839 | .7556 | .7417( 6) | .8620( 2) | 01234 |
| 88 | 11 | 8 | 0 | .3571 | .8012 | .7870( 2) | .8851( 0) | 01234 |
| 88 | 10 | 9 | 2 | .4040 | .8242 | .8100( 3) | .8966( 1) | 01234 |
| 89 | 15 | 6 | 1 | .2676 | .7152 | .7022( 7) | .8409( 2) | 01234 |
| 89 | 13 | 7 | 2 | .2931 | .7585 | .7448( 6) | .8636( 2) | 01234 |
| 89 | 12 | 8 | 7 | .3121 | .7809 | .7669( 6) | .8750( 2) | 01234 |
| 89 | 10 | 9 | 1 | .4101 | .8263 | .8122( 3) | .8978( 1) | 01234 |
| 90 | 15 | 6 | 0 | .2988 | .7187 | .7058( 4) | .8427( 0) | 01234 |
| 90 | 13 | 7 | 1 | .3084 | .7614 | .7478( 5) | .8652( 1) | 01234 |
| 90 | 12 | 8 | 6 | .3163 | .7835 | .7696( 6) | .8764( 2) | 01234 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 90 | 10 | 9 | 0 | .4444 | .8283 | .8144( 2) | .8989( 0) | 01234 |
| 91 | 13 | 7 | 0 | .3355 | .7642 | .7507( 3) | .8667( 0) | 01234 |
| 91 | 12 | 8 | 5 | .3247 | .7860 | .7723( 6) | .8777( 2) | 01234 |
| 91 | 11 | 9 | 8 | .3359 | .8077 | .7939( 5) | .8889( 1) | 01289 |
| 91 | 10 | 10 | 9 | .4444 | .8303 | .8165( 2) | .9001( 1) | 09123 |
| 92 | 14 | 7 | 6 | .2518 | .7437 | .7304( 7) | .8569( 3) | 01234 |
| 92 | 12 | 8 | 4 | .3299 | .7884 | .7748( 6) | .8791( 2) | 01234 |
| 92 | 11 | 9 | 7 | .3409 | .8100 | .7962( 5) | .8901( 2) | 01238 |
| 92 | 10 | 10 | 8 | .4444 | .8322 | .8185( 3) | .9012( 1) | 01923 |
| 93 | 14 | 7 | 5 | .2523 | .7467 | .7336( 7) | .8585( 2) | 01234 |
| 93 | 12 | 8 | 3 | .3405 | .7908 | .7774( 5) | .8804( 2) | 01234 |
| 93 | 11 | 9 | 6 | .3475 | .8121 | .7986( 5) | .8913( 1) | 01234 |
| 93 | 10 | 10 | 7 | .4444 | .8341 | .8205( 3) | .9023( 1) | 01293 |
| 94 | 14 | 7 | 4 | .2545 | .7497 | .7366( 6) | .8601( 2) | 01234 |
| 94 | 12 | 8 | 2 | .3498 | .7931 | .7798( 5) | .8817( 2) | 01234 |
| 94 | 11 | 9 | 5 | .3565 | .8142 | .8008( 5) | .8925( 1) | 01234 |
| 94 | 10 | 10 | 6 | .4444 | .8359 | .8225( 3) | .9033( 1) | 01239 |
| 95 | 14 | 7 | 3 | .2601 | .7526 | .7397( 6) | .8616( 2) | 01234 |
| 95 | 12 | 8 | 1 | .3535 | .7954 | .7822( 4) | .8830( 1) | 01234 |
| 95 | 11 | 9 | 4 | .3602 | .8163 | .8030( 4) | .8936( 1) | 01234 |
| 95 | 10 | 10 | 5 | .4444 | .8377 | .8244( 3) | .9044( 1) | 01234 |
| 96 | 14 | 7 | 2 | .2698 | .7555 | .7427( 6) | .8632( 2) | 01234 |
| 96 | 12 | 8 | 0 | .3750 | .7976 | .7846( 2) | .8843( 0) | 01234 |
| 96 | 11 | 9 | 3 | .3601 | .8183 | .8052( 4) | .8948( 1) | 01234 |
| 96 | 10 | 10 | 4 | .4444 | .8395 | .8263( 3) | .9054( 1) | 01234 |
| 97 | 14 | 7 | 1 | .2847 | .7583 | .7457( 6) | .8646( 2) | 01234 |
| 97 | 13 | 8 | 7 | .2961 | .7785 | .7656( 6) | .8749( 2) | 01234 |
| 97 | 11 | 9 | 2 | .3638 | .8203 | .8072( 3) | .8959( 1) | 01234 |
| 97 | 10 | 10 | 3 | .4444 | .8412 | .8282( 3) | .9063( 1) | 01234 |
| 98 | 14 | 7 | 0 | .3077 | .7610 | .7486( 4) | .8660( 1) | 01234 |
| 98 | 13 | 8 | 6 | .3074 | .7809 | .7682( 6) | .8762( 2) | 01234 |
| 98 | 11 | 9 | 1 | .3738 | .8222 | .8093( 3) | .8970( 1) | 01234 |
| 98 | 10 | 10 | 2 | .4444 | .8429 | .8300( 3) | .9073( 1) | 01234 |
| 99 | 15 | 7 | 6 | .2928 | .7440 | .7318( 7) | .8570( 3) | 01234 |
| 99 | 13 | 8 | 5 | .3111 | .7833 | .7707( 5) | .8775( 2) | 01234 |
| 99 | 11 | 9 | 0 | .3934 | .8241 | .8113( 2) | .8981( 0) | 01234 |
| 99 | 10 | 10 | 1 | .4500 | .8445 | .8318( 2) | .9083( 1) | 01234 |
| 100 | 15 | 7 | 5 | .2982 | .7467 | .7346( 7) | .8585( 2) | 01234 |
| 100 | 13 | 8 | 4 | .3121 | .7856 | .7731( 5) | .8788( 2) | 01234 |
| 100 | 12 | 9 | 8 | .3284 | .8056 | .7930( 5) | .8889( 2) | 01239 |
| 100 | 10 | 10 | 0 | .5000 | .8462 | .8333( 0) | .9091( 0) | 01234 |

TABLE B.2

PROPERTIES OF ≈(0,1)-DESIGNS , R=3 , 13≤V≤100

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 13 | 4 | 4 | 3 | .5556 | .7073 | .6566( 6) | .7498( 4) | 0132 |
| 14 | 4 | 4 | 2 | .5556 | .7310 | .6824( 7) | .7686( 4) | 0123 |
| 15 | 4 | 4 | 1 | .5833 | .7515 | .7063( 4) | .7851( 4) | 1023 |
| 16 | 4 | 4 | 0 | .6667 | .7692 | .7273( 0) | .8000( 0) | 0123 |
| 17 | 5 | 4 | 3 | .4630 | .6934 | .6509(10) | .7501( 8) | 01243 |
| 18 | 5 | 4 | 2 | .4759 | .7125 | .6720( 9) | .7645( 7) | 10234 |
| 19 | 5 | 4 | 1 | .5145 | .7296 | .6913( 5) | .7774( 5) | 12034 |
| 20 | 5 | 4 | 0 | .5833 | .7447 | .7085( 1) | .7895( 2) | 01234 |
| 21 | 6 | 4 | 3 | .4015 | .6820 | .6453(12) | .7499( 8) | 01524 |
| 21 | 5 | 5 | 4 | .5833 | .7578 | .7219( 4) | .8000( 4) | 01423 |
| 22 | 6 | 4 | 2 | .4174 | .6985 | .6633( 8) | .7617( 8) | 01253 |
| 22 | 5 | 5 | 3 | .5833 | .7698 | .7349( 5) | .8093( 6) | 01243 |
| 23 | 6 | 4 | 1 | .4213 | .7129 | .6793( 7) | .7724( 6) | 01234 |
| 23 | 5 | 5 | 2 | .5833 | .7808 | .7472( 5) | .8179( 6) | 10234 |
| 24 | 6 | 4 | 0 | .4625 | .7265 | .6945( 2) | .7826( 3) | 01234 |
| 24 | 5 | 5 | 1 | .6000 | .7908 | .7587( 2) | .8259( 3) | 12034 |
| 25 | 7 | 4 | 3 | .4314 | .6830 | .6543( 9) | .7499( 8) | 01562 |
| 25 | 5 | 5 | 0 | .6667 | .8000 | .7692( 0) | .8333( 0) | 01234 |
| 26 | 7 | 4 | 2 | .4446 | .6960 | .6680( 9) | .7598( 6) | 01245 |
| 26 | 6 | 5 | 4 | .4693 | .7488 | .7174( 7) | .7999( 5) | 05142 |
| 27 | 7 | 4 | 1 | .4728 | .7079 | .6809( 6) | .7691( 5) | 01253 |
| 27 | 6 | 5 | 3 | .4913 | .7587 | .7282( 7) | .8075( 6) | 01524 |
| 28 | 7 | 4 | 0 | .5174 | .7190 | .6928( 1) | .7778( 1) | 01234 |
| 28 | 6 | 5 | 2 | .4932 | .7679 | .7382( 6) | .8145( 5) | 01253 |
| 29 | 8 | 4 | 3 | .4213 | .6776 | .6523(10) | .7498( 7) | 34125 |
| 29 | 6 | 5 | 1 | .5003 | .7764 | .7476( 5) | .8213( 4) | 01234 |
| 30 | 8 | 4 | 2 | .4261 | .6890 | .6643( 9) | .7583( 7) | 34501 |
| 30 | 6 | 5 | 0 | .5333 | .7843 | .7565( 3) | .8276( 1) | 01234 |
| 31 | 8 | 4 | 1 | .4308 | .6996 | .6756( 7) | .7665( 5) | 13456 |
| 31 | 7 | 5 | 4 | .4525 | .7429 | .7157( 8) | .7998( 5) | 05612 |
| 32 | 8 | 4 | 0 | .4310 | .7095 | .6861( 3) | .7742( 2) | 01234 |
| 32 | 7 | 5 | 3 | .4662 | .7516 | .7252( 7) | .8062( 6) | 01562 |
| 33 | 9 | 4 | 3 | .4006 | .6761 | .6541(10) | .7494( 6) | 01234 |
| 33 | 7 | 5 | 2 | .4717 | .7598 | .7340( 7) | .8123( 4) | 01245 |
| 34 | 9 | 4 | 2 | .4253 | .6864 | .6649( 8) | .7572( 4) | 03612 |
| 34 | 7 | 5 | 1 | .4894 | .7675 | .7425( 5) | .8181( 4) | 01253 |
| 35 | 9 | 4 | 1 | .4403 | .6960 | .6751( 6) | .7646( 2) | 01346 |
| 35 | 7 | 5 | 0 | .5333 | .7747 | .7504( 2) | .8235( 1) | 01234 |
| 36 | 9 | 4 | 0 | .5000 | .7052 | .6849( 2) | .7714( 1) | 01234 |
| 36 | 8 | 5 | 4 | .4133 | .7386 | .7147( 8) | .7997( 6) | 01234 |
| 37 | 10 | 4 | 3 | .3559 | .6713 | .6510(12) | .7499( 6) | 01234 |
| 37 | 8 | 5 | 3 | .4293 | .7465 | .7232( 7) | .8052( 6) | 10234 |
| 37 | 7 | 6 | 5 | .5333 | .7875 | .7639( 5) | .8333( 3) | 01263 |
| 38 | 10 | 4 | 2 | .3736 | .6809 | .6611(11) | .7566( 5) | 01472 |
| 38 | 8 | 5 | 2 | .4449 | .7539 | .7312( 7) | .8106( 5) | 12503 |
| 38 | 7 | 6 | 4 | .5405 | .7934 | .7702( 5) | .8378( 4) | 10236 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 39 | 10 | 4 | 1 | .3996 | .6898 | .6706( 7) | .7631( 3) | 01245 |
| 39 | 8 | 5 | 1 | .4591 | .7608 | .7387( 6) | .8157( 4) | 12356 |
| 39 | 7 | 6 | 3 | .5459 | .7991 | .7764( 5) | .8421( 4) | 12034 |
| 40 | 10 | 4 | 0 | .4310 | .6982 | .6794( 2) | .7692( 1) | 01234 |
| 40 | 8 | 5 | 0 | .4866 | .7673 | .7458( 3) | .8205( 1) | 01234 |
| 40 | 7 | 6 | 2 | .5497 | .8044 | .7823( 4) | .8461( 3) | 12304 |
| 41 | 11 | 4 | 3 | .3538 | .6680 | .6494(12) | .7500( 5) | 01523 |
| 41 | 9 | 5 | 4 | .4537 | .7375 | .7169( 7) | .7996( 6) | 43567 |
| 41 | 7 | 6 | 1 | .5673 | .8095 | .7879( 3) | .8499( 2) | 12340 |
| 42 | 11 | 4 | 2 | .3719 | .6766 | .6584(11) | .7560( 5) | 01256 |
| 42 | 9 | 5 | 3 | .4591 | .7440 | .7237( 7) | .8046( 6) | 45036 |
| 42 | 7 | 6 | 0 | .6111 | .8143 | .7933( 1) | .8537( 1) | 01234 |
| 43 | 11 | 4 | 1 | .3840 | .6847 | .6669( 7) | .7618( 3) | 01235 |
| 43 | 9 | 5 | 2 | .4684 | .7501 | .7302( 7) | .8094( 5) | 04561 |
| 43 | 8 | 6 | 5 | .4877 | .7841 | .7634( 6) | .8333( 4) | 01236 |
| 43 | 7 | 7 | 6 | .6111 | .8188 | .7981( 2) | .8572( 2) | 01623 |
| 44 | 11 | 4 | 0 | .4007 | .6923 | .6749( 3) | .7674( 1) | 01234 |
| 44 | 9 | 5 | 1 | .4710 | .7560 | .7364( 5) | .8139( 4) | 01456 |
| 44 | 8 | 6 | 4 | .5000 | .7893 | .7690( 6) | .8371( 4) | 10234 |
| 44 | 7 | 7 | 5 | .6111 | .8231 | .8027( 3) | .8605( 3) | 01263 |
| 45 | 12 | 4 | 3 | .3634 | .6651 | .6478(12) | .7500( 6) | 01234 |
| 45 | 9 | 5 | 0 | .4816 | .7615 | .7423( 3) | .8182( 2) | 01234 |
| 45 | 8 | 6 | 3 | .5000 | .7943 | .7744( 6) | .8408( 4) | 12034 |
| 45 | 7 | 7 | 4 | .6111 | .8273 | .8072( 3) | .8636( 3) | 10236 |
| 46 | 12 | 4 | 2 | .3695 | .6728 | .6559(10) | .7555( 4) | 01236 |
| 46 | 10 | 5 | 4 | .4083 | .7339 | .7151( 7) | .7998( 6) | 01235 |
| 46 | 8 | 6 | 2 | .5122 | .7990 | .7795( 6) | .8444( 3) | 12304 |
| 46 | 7 | 7 | 3 | .6111 | .8312 | .8115( 3) | .8666( 3) | 12034 |
| 47 | 12 | 4 | 1 | .3824 | .6802 | .6637( 7) | .7608( 3) | 01234 |
| 47 | 10 | 5 | 3 | .4196 | .7402 | .7217( 7) | .8042( 5) | 01234 |
| 47 | 8 | 6 | 1 | .5307 | .8036 | .7845( 4) | .8478( 3) | 12346 |
| 47 | 7 | 7 | 2 | .6111 | .8350 | .8157( 3) | .8695( 3) | 12304 |
| 48 | 12 | 4 | 0 | .3970 | .6873 | .6710( 3) | .7660( 1) | 01234 |
| 48 | 10 | 5 | 2 | .4313 | .7461 | .7280( 7) | .8084( 5) | 01253 |
| 48 | 8 | 6 | 0 | .5556 | .8079 | .7892( 2) | .8511( 1) | 01234 |
| 48 | 7 | 7 | 1 | .6190 | .8386 | .8197( 1) | .8723( 1) | 12340 |
| 49 | 13 | 4 | 3 | .3138 | .6619 | .6458(11) | .7497( 5) | 01267 |
| 49 | 10 | 5 | 1 | .4525 | .7517 | .7340( 5) | .8124( 4) | 01235 |
| 49 | 9 | 6 | 5 | .4660 | .7814 | .7629( 6) | .8335( 4) | 12034 |
| 49 | 7 | 7 | 0 | .6667 | .8421 | .8235( 0) | .8750( 0) | 01234 |
| 50 | 13 | 4 | 2 | .3293 | .6692 | .6534( 9) | .7550( 7) | 01236 |
| 50 | 10 | 5 | 0 | .4849 | .7572 | .7398( 2) | .8163( 2) | 01234 |
| 50 | 9 | 6 | 4 | .4759 | .7860 | .7679( 6) | .8368( 4) | 12304 |
| 50 | 8 | 7 | 6 | .5555 | .8151 | .7965( 4) | .8572( 3) | 01236 |
| 51 | 13 | 4 | 1 | .3513 | .6761 | .6606( 8) | .7599( 3) | 01234 |
| 51 | 11 | 5 | 4 | .3879 | .7306 | .7131( 9) | .7999( 5) | 06157 |
| 51 | 9 | 6 | 3 | .4839 | .7905 | .7726( 5) | .8401( 4) | 12340 |
| 51 | 8 | 7 | 5 | .5555 | .8189 | .8006( 4) | .8600( 3) | 02134 |
| 52 | 13 | 4 | 0 | .3744 | .6828 | .6676( 4) | .7647( 1) | 01234 |
| 52 | 11 | 5 | 3 | .3909 | .7364 | .7192( 9) | .8039( 5) | 01672 |
| 52 | 9 | 6 | 2 | .4925 | .7947 | .7772( 4) | .8431( 4) | 12345 |
| 52 | 8 | 7 | 4 | .5555 | .8225 | .8046( 4) | .8627( 3) | 01234 |
| 53 | 14 | 4 | 3 | .3132 | .6589 | .6437(11) | .7501( 6) | 01234 |
| 53 | 11 | 5 | 2 | .4030 | .7420 | .7252( 8) | .8076( 4) | 01267 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 53 | 9 | 6 | 1 | .5047 | .7988 | .7816( 4) | .8461( 3) | 12345 |
| 53 | 8 | 7 | 3 | .5600 | .8261 | .8084( 4) | .8653( 3) | 20134 |
| 54 | 14 | 4 | 2 | .3230 | .6659 | .6510(10) | .7547( 5) | 01234 |
| 54 | 11 | 5 | 1 | .4212 | .7474 | .7311( 6) | .8113( 3) | 01236 |
| 54 | 9 | 6 | 0 | .5260 | .8027 | .7858( 2) | .8491( 1) | 01234 |
| 54 | 8 | 7 | 2 | .5680 | .8294 | .8121( 3) | .8679( 2) | 23014 |
| 55 | 14 | 4 | 1 | .3403 | .6725 | .6579( 8) | .7592( 3) | 01234 |
| 55 | 11 | 5 | 0 | .4472 | .7525 | .7366( 3) | .8148( 2) | 01234 |
| 55 | 10 | 6 | 5 | .4606 | .7788 | .7621( 6) | .8332( 4) | 04678 |
| 55 | 8 | 7 | 1 | .5819 | .8327 | .8156( 2) | .8704( 2) | 02346 |
| 56 | 14 | 4 | 0 | .3589 | .6788 | .6644( 4) | .7636( 1) | 01234 |
| 56 | 12 | 5 | 4 | .3922 | .7296 | .7137( 8) | .7999( 5) | 01234 |
| 56 | 10 | 6 | 4 | .4668 | .7830 | .7666( 6) | .8363( 4) | 01456 |
| 56 | 8 | 7 | 0 | .6190 | .8358 | .8191( 1) | .8728( 0) | 01234 |
| 57 | 15 | 4 | 3 | .3355 | .6585 | .6445( 9) | .7499( 6) | 01237 |
| 57 | 12 | 5 | 3 | .3965 | .7348 | .7192( 8) | .8035( 5) | 01234 |
| 57 | 10 | 6 | 3 | .4740 | .7870 | .7709( 5) | .8392( 4) | 06124 |
| 57 | 9 | 7 | 6 | .5297 | .8130 | .7966( 4) | .8571( 3) | 20134 |
| 57 | 8 | 8 | 7 | .6190 | .8388 | .8222( 2) | .8750( 1) | 02613 |
| 58 | 15 | 4 | 2 | .3451 | .6647 | .6509( 8) | .7543( 4) | 01234 |
| 58 | 12 | 5 | 2 | .4036 | .7398 | .7245( 8) | .8070( 4) | 01234 |
| 58 | 10 | 6 | 2 | .4887 | .7909 | .7750( 4) | .8421( 3) | 01672 |
| 58 | 9 | 7 | 5 | .5369 | .8162 | .8000( 4) | .8596( 3) | 02314 |
| 58 | 8 | 8 | 6 | .6190 | .8417 | .8253( 3) | .8772( 2) | 01236 |
| 59 | 15 | 4 | 1 | .3580 | .6707 | .6571( 6) | .7586( 3) | 01234 |
| 59 | 12 | 5 | 1 | .4129 | .7447 | .7297( 6) | .8103( 3) | 01234 |
| 59 | 10 | 6 | 1 | .4988 | .7947 | .7790( 4) | .8448( 3) | 01246 |
| 59 | 9 | 7 | 4 | .5450 | .8194 | .8033( 3) | .8620( 3) | 01234 |
| 59 | 8 | 8 | 5 | .6190 | .8445 | .8284( 3) | .8793( 2) | 02134 |
| 60 | 15 | 4 | 0 | .3849 | .6765 | .6631( 3) | .7627( 1) | 01234 |
| 60 | 12 | 5 | 0 | .4257 | .7493 | .7345( 3) | .8136( 1) | 01234 |
| 60 | 10 | 6 | 0 | .5212 | .7983 | .7829( 2) | .8475( 1) | 01234 |
| 60 | 9 | 7 | 3 | .5436 | .8224 | .8065( 3) | .8643( 2) | 20134 |
| 60 | 8 | 8 | 4 | .6190 | .8472 | .8313( 3) | .8814( 2) | 01234 |
| 61 | 13 | 5 | 4 | .4054 | .7295 | .7151( 7) | .7999( 6) | 01234 |
| 61 | 11 | 6 | 5 | .4705 | .7775 | .7624( 5) | .8334( 4) | 23014 |
| 61 | 9 | 7 | 2 | .5432 | .8252 | .8096( 3) | .8667( 2) | 23601 |
| 61 | 8 | 8 | 3 | .6190 | .8498 | .8342( 3) | .8833( 2) | 20134 |
| 62 | 13 | 5 | 3 | .4068 | .7341 | .7199( 7) | .8032( 5) | 01234 |
| 62 | 11 | 6 | 4 | .4795 | .7812 | .7663( 5) | .8361( 4) | 23401 |
| 62 | 9 | 7 | 1 | .5469 | .8280 | .8125( 3) | .8689( 2) | 02346 |
| 62 | 8 | 8 | 2 | .6190 | .8523 | .8369( 2) | .8853( 1) | 23014 |
| 63 | 13 | 5 | 2 | .4183 | .7386 | .7246( 7) | .8064( 4) | 01234 |
| 63 | 11 | 6 | 3 | .4821 | .7847 | .7700( 5) | .8387( 3) | 23450 |
| 63 | 9 | 7 | 0 | .5714 | .8308 | .8155( 2) | .8710( 1) | 01234 |
| 63 | 8 | 8 | 1 | .6250 | .8548 | .8396( 1) | .8872( 1) | 02346 |
| 64 | 13 | 5 | 1 | .4265 | .7429 | .7292( 5) | .8095( 3) | 01234 |
| 64 | 11 | 6 | 2 | .4845 | .7882 | .7736( 4) | .8413( 3) | 23456 |
| 64 | 10 | 7 | 6 | .4414 | .8093 | .7939( 5) | .8571( 3) | 07891 |
| 64 | 8 | 8 | 0 | .6667 | .8571 | .8421( 0) | .8889( 0) | 01234 |
| 65 | 13 | 5 | 0 | .4509 | .7471 | .7336( 3) | .8125( 1) | 01234 |
| 65 | 11 | 6 | 1 | .4921 | .7915 | .7771( 4) | .8438( 3) | 02345 |
| 65 | 10 | 7 | 5 | .4531 | .8125 | .7974( 5) | .8594( 3) | 01789 |
| 65 | 9 | 8 | 7 | .5714 | .8362 | .8212( 3) | .8751( 2) | 01283 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 66 | 14 | 5 | 4 | .3965 | .7273 | .7138(10) | .8000( 4) | 01234 |
| 66 | 11 | 6 | 0 | .5048 | .7947 | .7805( 2) | .8462( 1) | 01234 |
| 66 | 10 | 7 | 4 | .4585 | .8156 | .8007( 4) | .8615( 3) | 01278 |
| 66 | 9 | 8 | 6 | .5714 | .8387 | .8239( 3) | .8770( 2) | 10238 |
| 67 | 14 | 5 | 3 | .4040 | .7317 | .7183( 9) | .8030( 4) | 0123$\overline{10}$ |
| 67 | 12 | 6 | 5 | .4294 | .7749 | .7608( 6) | .8335( 4) | 01234 |
| 67 | 10 | 7 | 3 | .4655 | .8186 | .8039( 4) | .8636( 3) | 01236 |
| 67 | 9 | 8 | 5 | .5714 | .8412 | .8266( 3) | .8788( 2) | 12034 |
| 68 | 14 | 5 | 2 | .4105 | .7358 | .7227( 7) | .8060( 3) | 01234 |
| 68 | 12 | 6 | 4 | .4405 | .7784 | .7645( 6) | .8360( 4) | 01234 |
| 68 | 10 | 7 | 2 | .4739 | .8214 | .8070( 4) | .8657( 2) | 01237 |
| 68 | 9 | 8 | 4 | .5714 | .8437 | .8292( 3) | .8806( 2) | 12304 |
| 69 | 14 | 5 | 1 | .4182 | .7399 | .7269( 6) | .8088( 3) | 01234 |
| 69 | 12 | 6 | 3 | .4417 | .7818 | .7680( 6) | .8383( 4) | 01234 |
| 69 | 10 | 7 | 1 | .4854 | .8242 | .8100( 3) | .8677( 2) | 01234 |
| 69 | 9 | 8 | 3 | .5763 | .8460 | .8318( 3) | .8824( 2) | 12340 |
| 70 | 14 | 5 | 0 | .4331 | .7438 | .7310( 3) | .8116( 1) | 01234 |
| 70 | 12 | 6 | 2 | .4532 | .7850 | .7714( 5) | .8406( 4) | 01234 |
| 70 | 10 | 7 | 0 | .5006 | .8269 | .8130( 2) | .8697( 1) | 01234 |
| 70 | 9 | 8 | 2 | .5865 | .8483 | .8343( 3) | .8841( 2) | 12345 |
| 71 | 15 | 5 | 4 | .3584 | .7255 | .7128( 8) | .8001( 5) | 01234 |
| 71 | 12 | 6 | 1 | .4563 | .7881 | .7748( 4) | .8429( 3) | 01234 |
| 71 | 11 | 7 | 6 | .4750 | .8083 | .7945( 5) | .8572( 3) | 0129$\overline{10}$ |
| 71 | 9 | 8 | 1 | .5928 | .8505 | .8367( 2) | .8858( 1) | 12345 |
| 72 | 15 | 5 | 3 | .3675 | .7295 | .7170( 8) | .8029( 4) | 01234 |
| 72 | 12 | 6 | 0 | .4742 | .7912 | .7780( 2) | .8451( 1) | 01234 |
| 72 | 11 | 7 | 5 | .4778 | .8110 | .7973( 5) | .8592( 3) | 01239 |
| 72 | 9 | 8 | 0 | .6250 | .8527 | .8390( 1) | .8874( 1) | 01234 |
| 73 | 15 | 5 | 2 | .3749 | .7334 | .7210( 7) | .8056( 4) | 01234 |
| 73 | 13 | 6 | 5 | .4354 | .7733 | .7602( 7) | .8334( 4) | 01234 |
| 73 | 11 | 7 | 4 | .4792 | .8137 | .8002( 5) | .8612( 3) | 01234 |
| 73 | 10 | 8 | 7 | .4897 | .8338 | .8201( 3) | .8750( 2) | 20134 |
| 73 | 9 | 9 | 8 | .6250 | .8548 | .8412( 2) | .8890( 1) | 01823 |
| 74 | 15 | 5 | 1 | .3836 | .7371 | .7249( 6) | .8082( 3) | 01234 |
| 74 | 13 | 6 | 4 | .4363 | .7764 | .7635( 6) | .8357( 4) | 01234 |
| 74 | 11 | 7 | 3 | .4828 | .8163 | .8029( 4) | .8631( 3) | 01234 |
| 74 | 10 | 8 | 6 | .4954 | .8361 | .8225( 3) | .8767( 2) | 02314 |
| 74 | 9 | 9 | 7 | .6250 | .8568 | .8434( 2) | .8905( 2) | 01283 |
| 75 | 15 | 5 | 0 | .4056 | .7408 | .7287( 3) | .8109( 1) | 01234 |
| 75 | 13 | 6 | 3 | .4423 | .7795 | .7668( 5) | .8379( 4) | 01234 |
| 75 | 11 | 7 | 2 | .4892 | .8188 | .8056( 4) | .8649( 3) | 01234 |
| 75 | 10 | 8 | 5 | .5022 | .8383 | .8249( 4) | .8784( 2) | 01234 |
| 75 | 9 | 9 | 6 | .6250 | .8588 | .8455( 2) | .8919( 2) | 10238 |
| 76 | 13 | 6 | 2 | .4456 | .7825 | .7699( 5) | .8400( 3) | 01234 |
| 76 | 11 | 7 | 1 | .4912 | .8213 | .8083( 3) | .8667( 2) | 01234 |
| 76 | 10 | 8 | 4 | .5060 | .8404 | .8272( 3) | .8800( 2) | 20134 |
| 76 | 9 | 9 | 5 | .6250 | .8607 | .8476( 2) | .8934( 2) | 12034 |
| 77 | 13 | 6 | 1 | .4480 | .7854 | .7730( 4) | .8421( 3) | 01234 |
| 77 | 11 | 7 | 0 | .5022 | .8237 | .8108( 2) | .8685( 1) | 01234 |
| 77 | 10 | 8 | 3 | .5064 | .8425 | .8294( 3) | .8816( 2) | 23014 |
| 77 | 9 | 9 | 4 | .6250 | .8626 | .8496( 2) | .8948( 2) | 12304 |
| 78 | 13 | 6 | 0 | .4592 | .7883 | .7760( 2) | .8442( 1) | 01234 |
| 78 | 12 | 7 | 6 | .4869 | .8070 | .7944( 4) | .8571( 3) | 0123$\overline{10}$ |
| 78 | 10 | 8 | 2 | .5084 | .8446 | .8316( 3) | .8832( 2) | 23470 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 78 | 9 | 9 | 3 | .6250 | .8644 | .8516( 2) | .8961( 2) | 12340 |
| 79 | 14 | 6 | 5 | .3966 | .7705 | .7581( 7) | .8333( 4) | 01234 |
| 79 | 12 | 7 | 5 | .4873 | .8094 | .7968( 4) | .8590( 3) | 01234 |
| 79 | 10 | 8 | 1 | .5151 | .8466 | .8337( 2) | .8847( 2) | 02345 |
| 79 | 9 | 9 | 2 | .6250 | .8661 | .8536( 2) | .8974( 2) | 12345 |
| 80 | 14 | 6 | 4 | .4028 | .7735 | .7612( 6) | .8354( 4) | 05123 |
| 80 | 12 | 7 | 4 | .4873 | .8117 | .7992( 5) | .8608( 3) | 01234 |
| 80 | 10 | 8 | 0 | .5294 | .8485 | .8358( 2) | .8862( 1) | 01234 |
| 80 | 9 | 9 | 1 | .6296 | .8679 | .8555( 1) | .8987( 1) | 12345 |
| 81 | 14 | 6 | 3 | .4029 | .7765 | .7643( 6) | .8375( 4) | 0156$\overline{10}$ |
| 81 | 12 | 7 | 3 | .4878 | .8139 | .8016( 5) | .8625( 3) | 01234 |
| 81 | 11 | 8 | 7 | .5022 | .8329 | .8206( 3) | .8751( 2) | 78012 |
| 81 | 9 | 9 | 0 | .6667 | .8696 | .8571( 0) | .9000( 0) | 01234 |
| 82 | 14 | 6 | 2 | .4042 | .7793 | .7673( 6) | .8396( 4) | 01256 |
| 82 | 12 | 7 | 2 | .4884 | .8161 | .8039( 4) | .8642( 2) | 01234 |
| 82 | 11 | 8 | 6 | .5090 | .8350 | .8228( 4) | .8766( 2) | 07891 |
| 82 | 10 | 9 | 8 | .5549 | .8528 | .8404( 2) | .8890( 2) | 01458 |
| 83 | 14 | 6 | 1 | .3997 | .7821 | .7703( 4) | .8415( 3) | 01235 |
| 83 | 12 | 7 | 1 | .4879 | .8183 | .8061( 4) | .8659( 2) | 01234 |
| 83 | 11 | 8 | 5 | .5183 | .8370 | .8249( 3) | .8781( 2) | 01578 |
| 83 | 10 | 9 | 7 | .5570 | .8546 | .8424( 3) | .8903( 2) | 01245 |
| 84 | 14 | 6 | 0 | .4017 | .7849 | .7731( 3) | .8435( 1) | 01234 |
| 84 | 12 | 7 | 0 | .4870 | .8204 | .8083( 2) | .8676( 1) | 01234 |
| 84 | 11 | 8 | 4 | .5200 | .8390 | .8270( 3) | .8796( 2) | 01256 |
| 84 | 10 | 9 | 6 | .5611 | .8563 | .8442( 3) | .8916( 2) | 01234 |
| 85 | 15 | 6 | 5 | .3828 | .7699 | .7583( 7) | .8334( 4) | 01234 |
| 85 | 13 | 7 | 6 | .4530 | .8055 | .7937( 4) | .8571( 3) | 01234 |
| 85 | 11 | 8 | 3 | .5238 | .8409 | .8291( 3) | .8810( 2) | 01723 |
| 85 | 10 | 9 | 5 | .5674 | .8581 | .8461( 3) | .8929( 2) | 01248 |
| 86 | 15 | 6 | 4 | .3925 | .7728 | .7613( 6) | .8354( 4) | 01234 |
| 86 | 13 | 7 | 5 | .4554 | .8077 | .7960( 4) | .8588( 3) | 40123 |
| 86 | 11 | 8 | 2 | .5303 | .8428 | .8311( 3) | .8824( 2) | 01278 |
| 86 | 10 | 9 | 4 | .5722 | .8598 | .8479( 3) | .8942( 2) | 01234 |
| 87 | 15 | 6 | 3 | .3973 | .7755 | .7643( 6) | .8373( 3) | 01234 |
| 87 | 13 | 7 | 4 | .4556 | .8099 | .7983( 4) | .8605( 3) | 45012 |
| 87 | 11 | 8 | 1 | .5405 | .8447 | .8331( 2) | .8838( 2) | 01235 |
| 87 | 10 | 9 | 3 | .5721 | .8614 | .8497( 2) | .8954( 2) | 04123 |
| 88 | 15 | 6 | 2 | .4033 | .7782 | .7671( 6) | .8391( 3) | 01234 |
| 88 | 13 | 7 | 3 | .4572 | .8120 | .8005( 4) | .8621( 3) | 04561 |
| 88 | 11 | 8 | 0 | .5549 | .8465 | .8350( 1) | .8852( 1) | 01234 |
| 88 | 10 | 9 | 2 | .5729 | .8630 | .8514( 2) | .8966( 1) | 01452 |
| 89 | 15 | 6 | 1 | .4138 | .7809 | .7699( 4) | .8410( 3) | 01234 |
| 89 | 13 | 7 | 2 | .4609 | .8141 | .8027( 4) | .8637( 2) | 01456 |
| 89 | 12 | 8 | 7 | .4462 | .8305 | .8189( 4) | .8751( 2) | 01423 |
| 89 | 10 | 9 | 1 | .5772 | .8646 | .8531( 2) | .8978( 1) | 01245 |
| 90 | 15 | 6 | 0 | .4286 | .7835 | .7726( 3) | .8428( 1) | 01234 |
| 90 | 13 | 7 | 1 | .4698 | .8161 | .8048( 3) | .8652( 2) | 01245 |
| 90 | 12 | 8 | 6 | .4512 | .8325 | .8210( 4) | .8765( 3) | 01245 |
| 90 | 10 | 9 | 0 | .5926 | .8661 | .8548( 1) | .8990( 0) | 01234 |
| 91 | 13 | 7 | 0 | .4834 | .8181 | .8069( 2) | .8667( 1) | 01234 |
| 91 | 12 | 8 | 5 | .4554 | .8344 | .8231( 4) | .8778( 3) | 01234 |
| 91 | 11 | 9 | 8 | .5562 | .8515 | .8403( 2) | .8890( 2) | 20134 |
| 92 | 14 | 7 | 6 | .4529 | .8046 | .7937( 4) | .8571( 3) | 01256 |
| 92 | 12 | 8 | 4 | .4608 | .8363 | .8251( 4) | .8792( 2) | 40123 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 92 | 11 | 9 | 7 | .5591 | .8532 | .8420( 3) | .8902( 2) | 02314 |
| 93 | 14 | 7 | 5 | .4583 | .8067 | .7959( 4) | .8586( 3) | 01235 |
| 93 | 12 | 8 | 3 | .4663 | .8382 | .8271( 4) | .8805( 2) | 04512 |
| 93 | 11 | 9 | 6 | .5630 | .8547 | .8437( 2) | .8913( 2) | 01234 |
| 94 | 14 | 7 | 4 | .4580 | .8087 | .7979( 4) | .8602( 3) | 09123 |
| 94 | 12 | 8 | 2 | .4697 | .8400 | .8290( 3) | .8818( 2) | 01456 |
| 94 | 11 | 9 | 5 | .5685 | .8563 | .8453( 2) | .8925( 2) | 20134 |
| 95 | 14 | 7 | 3 | .4622 | .8106 | .8000( 4) | .8617( 3) | 0159$\overline{10}$ |
| 95 | 12 | 8 | 1 | .4753 | .8417 | .8309( 3) | .8831( 2) | 01245 |
| 95 | 11 | 9 | 4 | .5704 | .8578 | .8469( 2) | .8937( 2) | 23014 |
| 96 | 14 | 7 | 2 | .4661 | .8125 | .8020( 3) | .8632( 2) | 01256 |
| 96 | 12 | 8 | 0 | .4823 | .8435 | .8327( 2) | .8843( 1) | 01234 |
| 96 | 11 | 9 | 3 | .5694 | .8593 | .8485( 2) | .8948( 2) | 23401 |
| 97 | 14 | 7 | 1 | .4675 | .8144 | .8039( 3) | .8647( 2) | 01235 |
| 97 | 13 | 8 | 7 | .4818 | .8296 | .8189( 4) | .8751( 3) | 01234 |
| 97 | 11 | 9 | 2 | .5702 | .8607 | .8500( 2) | .8959( 1) | 23458 |
| 98 | 14 | 7 | 0 | .4774 | .8163 | .8059( 2) | .8661( 1) | 01234 |
| 98 | 13 | 8 | 6 | .4844 | .8314 | .8208( 4) | .8764( 3) | 01234 |
| 98 | 11 | 9 | 1 | .5738 | .8621 | .8516( 2) | .8970( 1) | 02345 |
| 99 | 15 | 7 | 6 | .3943 | .8017 | .7912( 6) | .8572( 3) | 01234 |
| 99 | 13 | 8 | 5 | .4875 | .8331 | .8226( 4) | .8776( 3) | 01234 |
| 99 | 11 | 9 | 0 | .5926 | .8635 | .8530( 1) | .8981( 1) | 01234 |
| 100 | 15 | 7 | 5 | .3965 | .8038 | .7934( 6) | .8587( 3) | 01234 |
| 100 | 13 | 8 | 4 | .4883 | .8348 | .8244( 4) | .8789( 2) | 01234 |
| 100 | 12 | 9 | 8 | .4941 | .8495 | .8390( 3) | .8890( 2) | 01234 |

# TABLE B.3

## PROPERTIES OF α(0,1)-DESIGNS , R=4 , 13≤V≤100

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}(0)$ | $\bar{E}(1)$ | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 13 | 4 | 4 | 3 | .6667 | .7342 | .6863( 0) | .7499( 4) | 0123 |
| 14 | 4 | 4 | 2 | .6667 | .7553 | .7089( 4) | .7686( 3) | 0123 |
| 15 | 4 | 4 | 1 | .6875 | .7736 | .7310( 2) | .7851( 2) | 0123 |
| 16 | 4 | 4 | 0 | .7500 | .7895 | .7500( 0) | .8000( 0) | 0123 |
| 17 | 5 | 4 | 3 | .5833 | .7229 | .6833( 8) | .7498( 4) | 12304 |
| 18 | 5 | 4 | 2 | .5945 | .7399 | .7023( 5) | .7643( 5) | 23014 |
| 19 | 5 | 4 | 1 | .6292 | .7551 | .7197( 3) | .7774( 3) | 30124 |
| 20 | 5 | 4 | 0 | .6875 | .7686 | .7352( 0) | .7895( 0) | 01234 |
| 21 | 6 | 4 | 3 | .4826 | .7144 | .6806( 9) | .7494( 7) | 01245 |
| 21 | 5 | 5 | 4 | .6875 | .7804 | .7468( 3) | .8000( 3) | 01234 |
| 22 | 6 | 4 | 2 | .5061 | .7286 | .6963( 7) | .7613( 5) | 01253 |
| 22 | 5 | 5 | 3 | .6875 | .7911 | .7583( 4) | .8093( 3) | 12304 |
| 23 | 6 | 4 | 1 | .5389 | .7416 | .7107( 4) | .7724( 4) | 02134 |
| 23 | 5 | 5 | 2 | .6875 | .8010 | .7695( 3) | .8179( 3) | 23014 |
| 24 | 6 | 4 | 0 | .5732 | .7533 | .7238( 1) | .7826( 1) | 01234 |
| 24 | 5 | 5 | 1 | .7000 | .8099 | .7799( 1) | .8259( 2) | 30124 |
| 25 | 7 | 4 | 3 | .5059 | .7100 | .6817( 7) | .7496( 7) | 01623 |
| 25 | 5 | 5 | 0 | .7500 | .8182 | .7895( 0) | .8333( 0) | 01234 |
| 26 | 7 | 4 | 2 | .5199 | .7220 | .6946( 6) | .7596( 6) | 01236 |
| 26 | 6 | 5 | 4 | .5472 | .7721 | .7416( 6) | .8000( 4) | 04513 |
| 27 | 7 | 4 | 1 | .5301 | .7330 | .7065( 4) | .7690( 4) | 01324 |
| 27 | 6 | 5 | 3 | .5588 | .7811 | .7515( 5) | .8076( 4) | 05142 |
| 28 | 7 | 4 | 0 | .5754 | .7432 | .7176( 1) | .7778( 1) | 01234 |
| 28 | 6 | 5 | 2 | .5663 | .7895 | .7609( 5) | .8146( 4) | 01523 |
| 29 | 8 | 4 | 3 | .4561 | .7041 | .6781( 9) | .7502( 7) | 01273 |
| 29 | 6 | 5 | 1 | .5780 | .7973 | .7697( 4) | .8213( 3) | 01234 |
| 30 | 8 | 4 | 2 | .4649 | .7148 | .6898( 7) | .7585( 6) | 01237 |
| 30 | 6 | 5 | 0 | .6000 | .8046 | .7780( 2) | .8276( 1) | 01234 |
| 31 | 8 | 4 | 1 | .4857 | .7248 | .7007( 4) | .7665( 4) | 01234 |
| 31 | 7 | 5 | 4 | .5660 | .7689 | .7439( 5) | .8000( 5) | 06152 |
| 32 | 8 | 4 | 0 | .5173 | .7340 | .7107( 2) | .7742( 1) | 01234 |
| 32 | 7 | 5 | 3 | .5762 | .7765 | .7521( 5) | .8063( 5) | 01625 |
| 33 | 9 | 4 | 3 | .3942 | .6994 | .6758( 9) | .7501( 7) | 01238 |
| 33 | 7 | 5 | 2 | .5790 | .7836 | .7598( 4) | .8124( 4) | 01263 |
| 34 | 9 | 4 | 2 | .4029 | .7092 | .6865( 8) | .7574( 6) | 01234 |
| 34 | 7 | 5 | 1 | .5939 | .7903 | .7672( 3) | .8181( 3) | 01234 |
| 35 | 9 | 4 | 1 | .4224 | .7184 | .6965( 5) | .7646( 5) | 01234 |
| 35 | 7 | 5 | 0 | .6198 | .7966 | .7741( 1) | .8235( 1) | 01234 |
| 36 | 9 | 4 | 0 | .4500 | .7269 | .7057( 3) | .7714( 2) | 01234 |
| 36 | 8 | 5 | 4 | .5279 | .7640 | .7412( 6) | .7999( 5) | 01726 |
| 37 | 10 | 4 | 3 | .4412 | .7000 | .6802( 9) | .7498( 6) | 34012 |
| 37 | 8 | 5 | 3 | .5386 | .7709 | .7488( 6) | .8054( 4) | 01273 |
| 37 | 7 | 6 | 5 | .6500 | .8082 | .7863( 4) | .8333( 3) | 05614 |
| 38 | 10 | 4 | 2 | .4549 | .7085 | .6892( 7) | .7565( 5) | 03451 |
| 38 | 8 | 5 | 2 | .5521 | .7775 | .7560( 5) | .8107( 4) | 01234 |
| 38 | 7 | 6 | 4 | .6500 | .8135 | .7920( 4) | .8377( 3) | 06152 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 39 | 10 | 4 | 1 | .4732 | .7165 | .6977( 5) | .7631( 4) | 01345 |
| 39 | 8 | 5 | 1 | .5660 | .7836 | .7628( 3) | .8157( 3) | 01243 |
| 39 | 7 | 6 | 3 | .6500 | .8185 | .7975( 3) | .8420( 3) | 01625 |
| 40 | 10 | 4 | 0 | .5000 | .7240 | .7057( 3) | .7692( 1) | 01234 |
| 40 | 8 | 5 | 0 | .5941 | .7895 | .7693( 1) | .8205( 1) | 01234 |
| 40 | 7 | 6 | 2 | .6552 | .8233 | .8028( 3) | .8460( 3) | 01263 |
| 41 | 11 | 4 | 3 | .4284 | .6962 | .6776( 9) | .7500( 7) | 12340 |
| 41 | 9 | 5 | 4 | .4955 | .7607 | .7402( 7) | .8001( 4) | 01283 |
| 41 | 7 | 6 | 1 | .6718 | .8278 | .8079( 2) | .8499( 2) | 01234 |
| 42 | 11 | 4 | 2 | .4359 | .7039 | .6858( 7) | .7560( 5) | 12345 |
| 42 | 9 | 5 | 3 | .5067 | .7668 | .7467( 6) | .8048( 4) | 01238 |
| 42 | 7 | 6 | 0 | .7083 | .8322 | .8127( 1) | .8537( 0) | 01234 |
| 43 | 11 | 4 | 1 | .4428 | .7112 | .6935( 6) | .7618( 4) | 12345 |
| 43 | 9 | 5 | 2 | .5083 | .7725 | .7530( 5) | .8095( 4) | 01234 |
| 43 | 8 | 6 | 5 | .5516 | .8040 | .7841( 5) | .8333( 4) | 01672 |
| 43 | 7 | 7 | 6 | .7083 | .8362 | .8170( 2) | .8571( 1) | 04561 |
| 44 | 11 | 4 | 0 | .4569 | .7181 | .7008( 3) | .7674( 2) | 01234 |
| 44 | 9 | 5 | 1 | .5135 | .7780 | .7590( 4) | .8139( 3) | 01234 |
| 44 | 8 | 6 | 4 | .5616 | .8088 | .7893( 4) | .8371( 3) | 01726 |
| 44 | 7 | 7 | 5 | .7083 | .8401 | .8212( 3) | .8605( 2) | 05614 |
| 45 | 12 | 4 | 3 | .3910 | .6915 | .6736(10) | .7502( 6) | 12345 |
| 45 | 9 | 5 | 0 | .5285 | .7833 | .7647( 2) | .8182( 2) | 01234 |
| 45 | 8 | 6 | 3 | .5694 | .8133 | .7942( 4) | .8408( 3) | 01273 |
| 45 | 7 | 7 | 4 | .7083 | .8438 | .8252( 3) | .8636( 2) | 06152 |
| 46 | 12 | 4 | 2 | .4013 | .6988 | .6814( 7) | .7555( 6) | 12345 |
| 46 | 10 | 5 | 4 | .4614 | .7597 | .7417( 6) | .8001( 5) | 01239 |
| 46 | 8 | 6 | 2 | .5788 | .8176 | .7990( 3) | .8444( 3) | 01234 |
| 46 | 7 | 7 | 3 | .7083 | .8474 | .8291( 2) | .8666( 2) | 01625 |
| 47 | 12 | 4 | 1 | .4086 | .7057 | .6887( 6) | .7608( 5) | 12345 |
| 47 | 10 | 5 | 3 | .4722 | .7651 | .7475( 6) | .8043( 5) | 01234 |
| 47 | 8 | 6 | 1 | .5906 | .8218 | .8036( 3) | .8478( 2) | 01243 |
| 47 | 7 | 7 | 2 | .7083 | .8508 | .8329( 2) | .8695( 2) | 01263 |
| 48 | 12 | 4 | 0 | .4193 | .7122 | .6955( 3) | .7660( 2) | 01234 |
| 48 | 10 | 5 | 2 | .4810 | .7702 | .7531( 5) | .8085( 4) | 01234 |
| 48 | 8 | 6 | 0 | .6059 | .8257 | .8080( 1) | .8511( 1) | 01234 |
| 48 | 7 | 7 | 1 | .7143 | .8540 | .8366( 1) | .8723( 1) | 01234 |
| 49 | 13 | 4 | 3 | .3852 | .6910 | .6748(10) | .7500( 6) | 12345 |
| 49 | 10 | 5 | 1 | .4944 | .7752 | .7584( 4) | .8125( 3) | 01234 |
| 49 | 9 | 6 | 5 | .5368 | .8017 | .7840( 5) | .8334( 4) | 01827 |
| 49 | 7 | 7 | 0 | .7500 | .8571 | .8400( 0) | .8750( 0) | 01234 |
| 50 | 13 | 4 | 2 | .4026 | .6976 | .6818( 7) | .7550( 5) | 12345 |
| 50 | 10 | 5 | 0 | .5097 | .7799 | .7635( 2) | .8163( 1) | 01234 |
| 50 | 9 | 6 | 4 | .5449 | .8060 | .7886( 5) | .8368( 4) | 01283 |
| 50 | 8 | 7 | 6 | .6157 | .8333 | .8162( 3) | .8572( 2) | 07145 |
| 51 | 13 | 4 | 1 | .4130 | .7040 | .6885( 6) | .7599( 5) | 12345 |
| 51 | 11 | 5 | 4 | .4799 | .7564 | .7398( 5) | .7998( 5) | 10236 |
| 51 | 9 | 6 | 3 | .5524 | .8100 | .7930( 4) | .8400( 4) | 01238 |
| 51 | 8 | 7 | 5 | .6198 | .8366 | .8197( 3) | .8600( 2) | 01672 |
| 52 | 13 | 4 | 0 | .4357 | .7100 | .6948( 3) | .7647( 2) | 01234 |
| 52 | 11 | 5 | 3 | .4852 | .7613 | .7448( 5) | .8038( 4) | 12034 |
| 52 | 9 | 6 | 2 | .5670 | .8139 | .7972( 3) | .8431( 3) | 01234 |
| 52 | 8 | 7 | 4 | .6267 | .8398 | .8231( 3) | .8628( 3) | 01726 |
| 53 | 14 | 4 | 3 | .3795 | .6867 | .6711( 8) | .7499( 8) | 178910 |
| 53 | 11 | 5 | 2 | .4906 | .7659 | .7497( 5) | .8076( 4) | 12380 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 53 | 9 | 6 | 1 | .5703 | .8176 | .8013( 3) | .8461( 2) | 01234 |
| 53 | 8 | 7 | 3 | .6270 | .8429 | .8265( 3) | .8654( 2) | 01273 |
| 54 | 14 | 4 | 2 | .3896 | .6928 | .6774( 7) | .7547( 6) | 12789 |
| 54 | 11 | 5 | 1 | .4929 | .7704 | .7545( 4) | .8113( 3) | 12346 |
| 54 | 9 | 6 | 0 | .5907 | .8211 | .8052( 1) | .8491( 1) | 01234 |
| 54 | 8 | 7 | 2 | .6274 | .8459 | .8297( 2) | .8679( 2) | 01234 |
| 55 | 14 | 4 | 1 | .4006 | .6986 | .6835( 5) | .7592( 5) | 12357 |
| 55 | 11 | 5 | 0 | .5025 | .7747 | .7591( 2) | .8148( 2) | 01234 |
| 55 | 10 | 6 | 5 | .5131 | .7993 | .7831( 5) | .8334( 4) | 01293 |
| 55 | 8 | 7 | 1 | .6301 | .8487 | .8328( 2) | .8704( 1) | 01243 |
| 56 | 14 | 4 | 0 | .4207 | .7043 | .6895( 3) | .7636( 2) | 01234 |
| 56 | 12 | 5 | 4 | .4516 | .7541 | .7386( 6) | .7999( 5) | 12034 |
| 56 | 10 | 6 | 4 | .5194 | .8031 | .7872( 5) | .8364( 4) | 01239 |
| 56 | 8 | 7 | 0 | .6428 | .8515 | .8358( 1) | .8728( 1) | 01234 |
| 57 | 15 | 4 | 3 | .3891 | .6882 | .6742( 8) | .7499( 6) | 12678 |
| 57 | 12 | 5 | 3 | .4560 | .7586 | .7433( 5) | .8035( 4) | 12304 |
| 57 | 10 | 6 | 3 | .5278 | .8067 | .7911( 4) | .8393( 4) | 01234 |
| 57 | 9 | 7 | 6 | .6108 | .8309 | .8155( 4) | .8572( 3) | 01782 |
| 58 | 15 | 4 | 2 | .4007 | .6938 | .6800( 7) | .7543( 8) | 12367 |
| 58 | 12 | 5 | 2 | .4621 | .7630 | .7479( 6) | .8070( 4) | 12347 |
| 58 | 10 | 6 | 2 | .5384 | .8102 | .7949( 4) | .8421( 3) | 01234 |
| 58 | 9 | 7 | 5 | .6134 | .8339 | .8187( 4) | .8597( 3) | 01827 |
| 59 | 15 | 4 | 1 | .4071 | .6992 | .6856( 6) | .7586( 5) | 12346 |
| 59 | 12 | 5 | 1 | .4633 | .7672 | .7524( 4) | .8103( 3) | 12345 |
| 59 | 10 | 6 | 1 | .5526 | .8136 | .7986( 3) | .8448( 2) | 01234 |
| 59 | 9 | 7 | 4 | .6193 | .8368 | .8218( 3) | .8621( 2) | 01283 |
| 60 | 15 | 4 | 0 | .4298 | .7045 | .6911( 3) | .7627( 3) | 01234 |
| 60 | 12 | 5 | 0 | .4674 | .7712 | .7567( 3) | .8136( 2) | 01234 |
| 60 | 10 | 6 | 0 | .5636 | .8168 | .8021( 2) | .8475( 1) | 01234 |
| 60 | 9 | 7 | 3 | .6217 | .8396 | .8248( 3) | .8645( 2) | 01238 |
| 61 | 13 | 5 | 4 | .4277 | .7515 | .7369( 6) | .7999( 5) | 18023 |
| 61 | 11 | 6 | 5 | .5262 | .7982 | .7838( 4) | .8332( 4) | 01342 |
| 61 | 9 | 7 | 2 | .6267 | .8423 | .8277( 2) | .8667( 2) | 01234 |
| 62 | 13 | 5 | 3 | .4300 | .7558 | .7414( 6) | .8032( 5) | 12890 |
| 62 | 11 | 6 | 4 | .5314 | .8016 | .7874( 4) | .8360( 4) | 01234 |
| 62 | 9 | 7 | 1 | .6298 | .8449 | .8305( 2) | .8689( 2) | 01235 |
| 63 | 13 | 5 | 2 | .4430 | .7600 | .7458( 5) | .8064( 2) | 12389 |
| 63 | 11 | 6 | 3 | .5381 | .8048 | .7908( 4) | .8387( 3) | 30124 |
| 63 | 9 | 7 | 0 | .6335 | .8474 | .8333( 1) | .8711( 1) | 01234 |
| 64 | 13 | 5 | 1 | .4453 | .7640 | .7501( 5) | .8095( 3) | 12346 |
| 64 | 11 | 6 | 2 | .5455 | .8079 | .7941( 3) | .8413( 3) | 03471 |
| 64 | 10 | 7 | 6 | .5725 | .8287 | .8145( 4) | .8573( 3) | 01928 |
| 65 | 13 | 5 | 0 | .4548 | .7679 | .7542( 3) | .8125( 2) | 01234 |
| 65 | 11 | 6 | 1 | .5554 | .8110 | .7974( 3) | .8438( 2) | 01345 |
| 65 | 10 | 7 | 5 | .5770 | .8314 | .8174( 4) | .8595( 3) | 01293 |
| 66 | 14 | 5 | 4 | .4320 | .7519 | .7387( 6) | .8000( 5) | 12903 |
| 66 | 11 | 6 | 0 | .5716 | .8139 | .8005( 1) | .8462( 1) | 01234 |
| 66 | 10 | 7 | 4 | .5817 | .8340 | .8202( 3) | .8617( 3) | 01239 |
| 67 | 14 | 5 | 3 | .4401 | .7557 | .7427( 5) | .8030( 5) | 1239$\overline{10}$ |
| 67 | 12 | 6 | 5 | .5015 | .7959 | .7824( 5) | .8333( 4) | 014$\overline{10}$2 |
| 67 | 10 | 7 | 3 | .5873 | .8366 | .8230( 3) | .8637( 3) | 01234 |
| 68 | 14 | 5 | 2 | .4440 | .7594 | .7466( 5) | .8060( 4) | 12349 |
| 68 | 12 | 6 | 4 | .5093 | .7991 | .7858( 5) | .8358( 4) | 01245 |
| 68 | 10 | 7 | 2 | .5891 | .8390 | .8256( 3) | .8657( 3) | 01234 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 69 | 14 | 5 | 1 | .4473 | .7631 | .7504( 4) | .8088( 3) | 12345 |
| 69 | 12 | 6 | 3 | .5173 | .8022 | .7891( 4) | .8382( 4) | 01234 |
| 69 | 10 | 7 | 1 | .5893 | .8414 | .8282( 2) | .8677( 2) | 01234 |
| 70 | 14 | 5 | 0 | .4517 | .7666 | .7541( 2) | .8116( 2) | 01234 |
| 70 | 12 | 6 | 2 | .5243 | .8052 | .7923( 4) | .8406( 3) | 01472 |
| 70 | 10 | 7 | 0 | .5951 | .8438 | .8307( 1) | .8697( 1) | 01234 |
| 71 | 15 | 5 | 4 | .3792 | .7493 | .7366( 6) | .7999( 4) | 23401 |
| 71 | 12 | 6 | 1 | .5348 | .8081 | .7954( 3) | .8429( 3) | 01245 |
| 71 | 11 | 7 | 6 | .4930 | .8259 | .8126( 4) | .8571( 3) | 02134 |
| 72 | 15 | 5 | 3 | .3884 | .7531 | .7406( 6) | .8028( 4) | 2345$\overline{11}$ |
| 72 | 12 | 6 | 0 | .5481 | .8109 | .7984( 2) | .8451( 1) | 01234 |
| 72 | 11 | 7 | 5 | .5007 | .8285 | .8153( 4) | .8591( 3) | 01234 |
| 73 | 15 | 5 | 2 | .4009 | .7568 | .7446( 6) | .8055( 4) | 23456 |
| 73 | 13 | 6 | 5 | .5082 | .7943 | .7817( 5) | .8333( 4) | 10237 |
| 73 | 11 | 7 | 4 | .5041 | .8310 | .8180( 3) | .8611( 3) | 20134 |
| 73 | 10 | 8 | 7 | .6010 | .8504 | .8378( 3) | .8751( 2) | 01892 |
| 74 | 15 | 5 | 1 | .4158 | .7604 | .7484( 4) | .8082( 3) | 02345 |
| 74 | 13 | 6 | 4 | .5122 | .7973 | .7848( 5) | .8356( 4) | 12034 |
| 74 | 11 | 7 | 3 | .5089 | .8334 | .8206( 3) | .8630( 2) | 23014 |
| 74 | 10 | 8 | 6 | .6051 | .8525 | .8400( 3) | .8768( 2) | 01928 |
| 75 | 15 | 5 | 0 | .4344 | .7639 | .7520( 2) | .8109( 2) | 01234 |
| 75 | 13 | 6 | 3 | .5150 | .8001 | .7879( 4) | .8378( 3) | 12390 |
| 75 | 11 | 7 | 2 | .5159 | .8358 | .8232( 3) | .8649( 3) | 23401 |
| 75 | 10 | 8 | 5 | .6097 | .8545 | .8422( 3) | .8785( 2) | 01293 |
| 76 | 13 | 6 | 2 | .5224 | .8029 | .7909( 4) | .8400( 3) | 12349 |
| 76 | 11 | 7 | 1 | .5258 | .8380 | .8256( 2) | .8667( 2) | 02345 |
| 76 | 10 | 8 | 4 | .6114 | .8565 | .8443( 3) | .8801( 2) | 01239 |
| 77 | 13 | 6 | 1 | .5307 | .8057 | .7938( 3) | .8422( 2) | 12345 |
| 77 | 11 | 7 | 0 | .5380 | .8403 | .8280( 2) | .8685( 1) | 01234 |
| 77 | 10 | 8 | 3 | .6146 | .8584 | .8464( 3) | .8817( 2) | 01234 |
| 78 | 13 | 6 | 0 | .5396 | .8083 | .7966( 2) | .8442( 1) | 01234 |
| 78 | 12 | 7 | 6 | .4729 | .8244 | .8120( 4) | .8572( 3) | 01234 |
| 78 | 10 | 8 | 2 | .6186 | .8603 | .8484( 2) | .8832( 2) | 01234 |
| 79 | 14 | 6 | 5 | .4607 | .7923 | .7804( 5) | .8333( 3) | 01237 |
| 79 | 12 | 7 | 5 | .4775 | .8268 | .8146( 4) | .8590( 3) | 10234 |
| 79 | 10 | 8 | 1 | .6236 | .8621 | .8504( 1) | .8847( 1) | 01234 |
| 80 | 14 | 6 | 4 | .4646 | .7951 | .7833( 5) | .8355( 4) | 29013 |
| 80 | 12 | 7 | 4 | .4825 | .8291 | .8171( 3) | .8608( 3) | 12034 |
| 80 | 10 | 8 | 0 | .6311 | .8639 | .8523( 1) | .8862( 0) | 01234 |
| 81 | 14 | 6 | 3 | .4703 | .7978 | .7862( 5) | .8375( 3) | 239$\overline{10}$0 |
| 81 | 12 | 7 | 3 | .4885 | .8313 | .8195( 3) | .8625( 3) | 12304 |
| 81 | 11 | 8 | 7 | .6044 | .8490 | .8375( 3) | .8751( 2) | 0123$\overline{10}$ |
| 82 | 14 | 6 | 2 | .4781 | .8005 | .7890( 4) | .8396( 3) | 2349$\overline{10}$ |
| 82 | 12 | 7 | 2 | .4957 | .8335 | .8219( 3) | .8643( 2) | 12349 |
| 82 | 11 | 8 | 6 | .6053 | .8509 | .8396( 2) | .8766( 2) | 01324 |
| 82 | 10 | 9 | 8 | .6438 | .8674 | .8561( 2) | .8890( 1) | 08917 |
| 83 | 14 | 6 | 1 | .4878 | .8031 | .7918( 3) | .8415( 2) | 02345 |
| 83 | 12 | 7 | 1 | .5037 | .8357 | .8242( 3) | .8659( 2) | 12345 |
| 83 | 11 | 8 | 5 | .6065 | .8528 | .8415( 3) | .8781( 2) | 01234 |
| 83 | 10 | 9 | 7 | .6455 | .8690 | .8578( 2) | .8904( 2) | 09182 |
| 84 | 14 | 6 | 0 | .5015 | .8056 | .7944( 2) | .8435( 2) | 01234 |
| 84 | 12 | 7 | 0 | .5136 | .8378 | .8264( 2) | .8676( 1) | 01234 |
| 84 | 11 | 8 | 4 | .6077 | .8546 | .8434( 3) | .8796( 2) | 30124 |
| 84 | 10 | 9 | 6 | .6472 | .8706 | .8595( 2) | .8917( 2) | 01928 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | CONTROLS |
|---|---|---|---|---|---|---|---|---|
| 85 | 15 | 6 | 5 | .4254 | .7918 | .7808( 5) | .8335( 4) | 23456 |
| 85 | 13 | 7 | 6 | .4707 | .8235 | .8122( 4) | .8571( 3) | 01234 |
| 85 | 11 | 8 | 3 | .6112 | .8563 | .8453( 2) | .8810( 2) | 34012 |
| 85 | 10 | 9 | 5 | .6507 | .8722 | .8612( 2) | .8930( 2) | 01293 |
| 86 | 15 | 6 | 4 | .4252 | .7944 | .7834( 5) | .8354( 4) | 23456 |
| 86 | 13 | 7 | 5 | .4730 | .8257 | .8146( 4) | .8588( 3) | 10237 |
| 86 | 11 | 8 | 2 | .6139 | .8581 | .8472( 2) | .8825( 2) | 03451 |
| 86 | 10 | 9 | 4 | .6512 | .8737 | .8628( 2) | .8942( 2) | 01239 |
| 87 | 15 | 6 | 3 | .4269 | .7968 | .7860( 5) | .8373( 4) | 23456 |
| 87 | 13 | 7 | 4 | .4763 | .8279 | .8168( 3) | .8605( 3) | 12034 |
| 87 | 11 | 8 | 1 | .6153 | .8597 | .8490( 2) | .8838( 1) | 01345 |
| 87 | 10 | 9 | 3 | .6537 | .8752 | .8644( 2) | .8955( 1) | 01234 |
| 88 | 15 | 6 | 2 | .4315 | .7993 | .7886( 4) | .8392( 3) | 23456 |
| 88 | 13 | 7 | 3 | .4814 | .8299 | .8190( 3) | .8621( 2) | 12390 |
| 88 | 11 | 8 | 0 | .6197 | .8614 | .8507( 1) | .8852( 1) | 01234 |
| 88 | 10 | 9 | 2 | .6549 | .8766 | .8660( 1) | .8967( 1) | 01234 |
| 89 | 15 | 6 | 1 | .4375 | .8016 | .7910( 4) | .8410( 3) | 02345 |
| 89 | 13 | 7 | 2 | .4890 | .8320 | .8212( 3) | .8637( 2) | 12349 |
| 89 | 12 | 8 | 7 | .5490 | .8477 | .8370( 3) | .8751( 2) | 01234 |
| 89 | 10 | 9 | 1 | .6572 | .8780 | .8675( 1) | .8978( 1) | 01234 |
| 90 | 15 | 6 | 0 | .4446 | .8039 | .7935( 2) | .8428( 2) | 01234 |
| 90 | 13 | 7 | 1 | .4984 | .8340 | .8233( 2) | .8653( 2) | 12345 |
| 90 | 12 | 8 | 6 | .5563 | .8494 | .8389( 3) | .8765( 2) | 01324 |
| 90 | 10 | 9 | 0 | .6667 | .8794 | .8690( 1) | .8990( 0) | 01234 |
| 91 | 13 | 7 | 0 | .5102 | .8359 | .8254( 2) | .8668( 1) | 01234 |
| 91 | 12 | 8 | 5 | .5597 | .8512 | .8407( 3) | .8779( 2) | 01234 |
| 91 | 11 | 9 | 8 | .6121 | .8660 | .8556( 2) | .8890( 2) | 01$\overline{10}$24 |
| 92 | 14 | 7 | 6 | .4413 | .8220 | .8113( 4) | .8573( 3) | 24567 |
| 92 | 12 | 8 | 4 | .5620 | .8528 | .8425( 3) | .8792( 2) | 30124 |
| 92 | 11 | 9 | 7 | .6162 | .8675 | .8572( 2) | .8902( 2) | 012$\overline{10}$3 |
| 93 | 14 | 7 | 5 | .4425 | .8240 | .8134( 4) | .8588( 3) | 23456 |
| 93 | 12 | 8 | 3 | .5656 | .8544 | .8442( 3) | .8805( 2) | 34012 |
| 93 | 11 | 9 | 6 | .6216 | .8690 | .8588( 2) | .8914( 2) | 01234 |
| 94 | 14 | 7 | 4 | .4445 | .8260 | .8155( 4) | .8603( 3) | 49235 |
| 94 | 12 | 8 | 2 | .5711 | .8560 | .8459( 2) | .8818( 2) | 03451 |
| 94 | 11 | 9 | 5 | .6263 | .8704 | .8603( 2) | .8926( 2) | 01234 |
| 95 | 14 | 7 | 3 | .4481 | .8279 | .8176( 4) | .8618( 3) | 459$\overline{10}$0 |
| 95 | 12 | 8 | 1 | .5784 | .8576 | .8476( 2) | .8831( 2) | 01345 |
| 95 | 11 | 9 | 4 | .6282 | .8718 | .8618( 2) | .8937( 2) | 40123 |
| 96 | 14 | 7 | 2 | .4532 | .8298 | .8196( 3) | .8633( 2) | 4569$\overline{10}$ |
| 96 | 12 | 8 | 0 | .5876 | .8591 | .8492( 1) | .8843( 1) | 01234 |
| 96 | 11 | 9 | 3 | .6310 | .8732 | .8633( 2) | .8949( 2) | 04512 |
| 97 | 14 | 7 | 1 | .4598 | .8317 | .8215( 3) | .8647( 2) | 02456 |
| 97 | 13 | 8 | 7 | .5247 | .8462 | .8362( 3) | .8751( 2) | 43501 |
| 97 | 11 | 9 | 2 | .6351 | .8745 | .8647( 2) | .8960( 1) | 01456 |
| 98 | 14 | 7 | 0 | .4677 | .8335 | .8235( 2) | .8661( 1) | 01234 |
| 98 | 13 | 8 | 6 | .5302 | .8479 | .8379( 3) | .8764( 2) | 45360 |
| 98 | 11 | 9 | 1 | .6404 | .8758 | .8661( 1) | .8970( 1) | 01245 |
| 99 | 15 | 7 | 6 | .5000 | .8224 | .8127( 4) | .8573( 3) | 01256 |
| 99 | 13 | 8 | 5 | .5320 | .8495 | .8397( 3) | .8776( 2) | 45637 |
| 99 | 11 | 9 | 0 | .6488 | .8771 | .8675( 1) | .8981( 0) | 01234 |
| 100 | 15 | 7 | 5 | .5015 | .8242 | .8147( 4) | .8587( 3) | 01235 |
| 100 | 13 | 8 | 4 | .5344 | .8511 | .8414( 3) | .8789( 2) | 45673 |
| 100 | 12 | 9 | 8 | .5792 | .8648 | .8551( 2) | .8890( 2) | 20135 |

## TABLE C.1

## $\alpha$(0,1,2)-DESIGNS , R=2 , $2 \leq S \leq 9$ , $3 \leq K \leq 20$

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 2 | 3 | .5455 | 0 1 1 |
| 2 | 4 | .6000 | 0 1 0 1 |
| 3 | 4 | .5245 | 0 1 2 2 |
| 3 | 5 | .5357 | 0 1 2 1 2 |
| 3 | 6 | .5556 | 0 1 2 0 1 2 |
| 3 | 7 | .5455 | 0 1 2 0 1 2 2 |
| 3 | 8 | .5478 | 0 1 2 0 1 2 1 2 |
| 3 | 9 | .5556 | 0 1 2 0 1 2 0 1 2 |
| 4 | 5 | .5185 | 0 1 2 3 3 |
| 4 | 6 | .5196 | 0 1 2 3 2 3 |
| 4 | 7 | .5283 | 0 1 2 3 1 2 3 |
| 4 | 8 | .5385 | 0 1 2 3 0 1 2 3 |
| 4 | 9 | .5323 | 0 1 2 3 0 1 2 3 3 |
| 4 | 10 | .5318 | 0 1 2 3 0 1 2 3 2 3 |
| 4 | 11 | .5344 | 0 1 2 3 0 1 2 3 1 2 3 |
| 4 | 12 | .5385 | 0 1 2 3 0 1 2 3 0 1 2 3 |
| 4 | 13 | .5355 | 0 1 2 3 0 1 2 3 0 1 2 3 3 |
| 4 | 14 | .5351 | 0 1 2 3 0 1 2 3 0 1 2 3 2 3 |
| 4 | 15 | .5363 | 0 1 2 3 0 1 2 3 0 1 2 3 1 2 3 |
| 4 | 16 | .5385 | 0 1 2 3 0 1 2 3 0 1 2 3 0 1 2 3 |
| 5 | 6 | .5155 | 0 1 2 3 4 4 |
| 5 | 7 | .5139 | 0 1 2 3 4 3 4 |
| 5 | 8 | .5176 | 0 1 2 3 4 2 3 4 |
| 5 | 9 | .5233 | 0 1 2 3 4 1 2 3 4 |
| 5 | 10 | .5294 | 0 1 2 3 4 0 1 2 3 4 |
| 5 | 11 | .5253 | 0 1 2 3 4 0 1 2 3 4 4 |
| 5 | 12 | .5242 | 0 1 2 3 4 0 1 2 3 4 3 4 |
| 5 | 13 | .5250 | 0 1 2 3 4 0 1 2 3 4 2 3 4 |
| 5 | 14 | .5269 | 0 1 2 3 4 0 1 2 3 4 1 2 3 4 |
| 5 | 15 | .5294 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 |
| 5 | 16 | .5275 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 4 |
| 5 | 17 | .5268 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 3 4 |
| 5 | 18 | .5271 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 2 3 4 |
| 5 | 19 | .5280 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 1 2 3 4 |
| 5 | 20 | .5294 | 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 |
| 6 | 7 | .5136 | 0 1 2 3 4 5 5 |
| 6 | 8 | .5111 | 0 1 2 3 4 5 4 5 |
| 6 | 9 | .5126 | 0 1 2 3 4 5 2 3 5 |
| 6 | 10 | .5157 | 0 1 2 3 4 5 2 3 4 5 |
| 6 | 11 | .5197 | 0 1 2 3 4 5 1 2 3 4 5 |
| 6 | 12 | .5238 | 0 1 2 3 4 5 0 1 2 3 4 5 |
| 6 | 13 | .5209 | 0 1 2 3 4 5 0 1 2 3 4 5 5 |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 6 | 14 | .5197 | 0 1 2 3 4 5 0 1 2 3 4 5 4 5 |
| 6 | 15 | .5198 | 0 1 2 3 4 5 0 1 2 3 4 5 2 3 5 |
| 6 | 16 | .5207 | 0 1 2 3 4 5 0 1 2 3 4 5 2 3 4 5 |
| 6 | 17 | .5221 | 0 1 2 3 4 5 0 1 2 3 4 5 1 2 3 4 5 |
| 7 | 8 | .5122 | 0 1 2 3 4 5 6 6 |
| 7 | 9 | .5096 | 0 1 2 3 4 5 6 5 6 |
| 7 | 10 | .5100 | 0 1 2 3 4 5 6 3 4 6 |
| 7 | 11 | .5117 | 0 1 2 3 4 5 6 2 3 4 6 |
| 7 | 12 | .5142 | 0 1 2 3 4 5 6 2 3 4 5 6 |
| 7 | 13 | .5170 | 0 1 2 3 4 5 6 1 2 3 4 5 6 |
| 7 | 14 | .5200 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6 |
| 7 | 15 | .5178 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6 6 |
| 8 | 9 | .5111 | 0 1 2 3 4 5 6 7 7 |
| 8 | 10 | .5086 | 0 1 2 3 4 5 6 7 6 7 |
| 8 | 11 | .5084 | 0 1 2 3 4 5 6 7 4 5 7 |
| 8 | 12 | .5093 | 0 1 2 3 4 5 6 7 3 4 5 7 |
| 8 | 13 | .5109 | 0 1 2 3 4 5 6 7 2 3 4 5 7 |
| 9 | 10 | .5102 | 0 1 2 3 4 5 6 7 8 8 |
| 9 | 11 | .5079 | 0 1 2 3 4 5 6 7 8 7 8 |
| 9 | 12 | .5073 | 0 1 2 3 4 5 6 7 8 5 6 8 |

# TABLE C.2

## $\alpha(0,1,2)$-DESIGNS , R=3 , $2 \leq S \leq 9$ , $3 \leq K \leq 20$

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 2 | 3 | .7435 | 0 1 0 |
| | | | 0 0 1 |
| 2 | 4 | .7692 | 0 1 0 1 |
| | | | 0 1 1 0 |
| 3 | 4 | .7059 | 0 2 1 0 |
| | | | 0 1 2 1 |
| 3 | 5 | .7164 | 0 0 2 1 1 |
| | | | 0 1 1 2 0 |
| 3 | 6 | .7273 | 0 2 0 1 2 1 |
| | | | 0 2 1 2 1 0 |
| 3 | 7 | .7207 | 0 1 2 0 1 2 0 |
| | | | 0 1 0 2 2 1 1 |
| 3 | 8 | .7229 | 0 2 0 1 2 1 2 1 |
| | | | 0 2 2 1 0 2 1 0 |
| 3 | 9 | .7273 | 0 2 0 1 2 0 1 2 1 |
| | | | 0 2 1 1 0 2 2 1 0 |
| 4 | 5 | .6976 | 0 2 3 1 2 |
| | | | 0 1 2 3 3 |
| 4 | 6 | .6981 | 0 2 3 1 2 0 |
| | | | 0 3 2 3 1 2 |
| 4 | 7 | .7038 | 0 2 3 1 2 0 3 |
| | | | 0 3 2 3 1 2 0 |
| 4 | 8 | .7097 | 0 1 2 3 1 2 0 3 |
| | | | 0 1 3 2 3 1 2 0 |
| 4 | 9 | .7056 | 0 1 2 3 0 1 2 3 0 |
| | | | 0 3 2 1 1 2 0 2 3 |
| 4 | 10 | .7055 | 0 3 3 0 0 1 1 2 2 3 |
| | | | 0 1 0 3 2 2 1 1 3 2 |
| 4 | 11 | .7071 | 0 3 3 3 0 1 1 1 2 2 0 |
| | | | 0 2 1 0 2 2 1 3 3 1 3 |
| * 4 | 12 | .7097 | 0 1 2 3 1 2 3 0 1 2 3 0 |
| | | | 0 2 3 1 3 2 0 3 1 0 2 1 |
| 4 | 13 | .7079 | 0 3 0 2 1 2 3 1 2 3 0 1 1 |
| | | | 0 2 3 1 1 2 0 2 3 1 2 3 0 |
| 4 | 14 | .7076 | 0 2 3 0 1 2 0 1 2 3 1 2 3 3 |
| | | | 0 2 3 1 2 3 2 3 0 1 0 1 2 0 |
| 4 | 15 | .7083 | 0 2 3 1 2 3 0 1 2 3 0 1 2 3 0 |
| | | | 0 2 3 2 3 0 1 3 0 1 2 0 1 2 3 |
| 4 | 16 | .7097 | 0 1 2 3 1 2 3 0 1 2 3 0 1 2 3 0 |
| | | | 0 1 2 3 2 3 0 1 3 0 1 2 0 1 2 3 |
| 5 | 6 | .6904 | 0 1 1 2 3 4 |
| | | | 0 4 3 1 2 0 |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 5 | 7 | .6911 | 0 2 1 2 3 4 0 |
| | | | 0 4 4 3 2 2 1 |
| 5 | 8 | .6927 | 0 4 2 3 4 1 2 0 |
| | | | 0 2 3 2 1 0 4 1 |
| 5 | 9 | .6963 | 0 3 0 1 2 3 4 1 4 |
| | | | 0 1 4 3 3 2 1 4 0 |
| 5 | 10 | .7000 | 0 2 3 0 1 2 3 4 1 4 |
| | | | 0 2 1 4 3 3 2 1 4 0 |
| 5 | 11 | .6973 | 0 3 4 1 2 3 4 0 1 2 3 |
| | | | 0 2 1 4 3 3 2 1 0 4 4 |
| 5 | 12 | .6969 | 0 1 2 3 1 2 3 4 0 1 2 4 |
| | | | 0 3 2 1 4 3 2 2 1 0 4 1 |
| 5 | 13 | .6972 | 0 3 1 2 3 4 1 2 3 4 0 1 4 |
| | | | 0 1 4 3 2 1 2 1 0 4 4 3 0 |
| 5 | 14 | .6984 | 0 1 2 3 1 2 3 4 0 1 2 3 4 4 |
| | | | 0 3 2 1 4 3 2 1 1 0 4 3 2 0 |
| 5 | 15 | .7000 | 0 1 2 3 4 1 2 3 4 0 1 2 3 4 0 |
| | | | 0 3 2 1 0 4 3 2 1 1 0 4 3 2 4 |
| 5 | 16 | .6987 | 0 4 0 1 2 3 4 0 1 2 3 4 1 2 3 4 |
| | | | 0 4 3 2 1 0 0 4 3 2 1 1 4 3 2 3 |
| 5 | 17 | .6984 | 0 1 2 3 4 2 3 4 0 1 2 3 4 1 2 3 0 |
| | | | 0 4 3 2 1 0 4 3 3 2 1 0 4 3 2 1 4 |
| 5 | 18 | .6985 | 0 1 2 3 4 1 2 3 4 0 1 2 3 4 0 1 3 0 |
| | | | 0 3 3 2 1 4 4 3 2 1 0 0 4 3 2 1 1 4 |
| 5 | 19 | .6991 | 0 1 2 3 4 1 2 3 4 0 1 2 3 4 0 1 2 3 0 |
| | | | 0 3 2 1 1 4 3 2 2 1 0 4 3 3 2 1 0 4 4 |
| 5 | 20 | .7000 | 0 1 2 3 4 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 |
| | | | 0 3 2 1 0 4 3 2 1 1 0 4 3 2 2 1 0 4 3 4 |
| 6 | 7 | .6874 | 0 5 1 2 3 4 4 |
| | | | 0 3 4 1 5 2 5 |
| 6 | 8 | .6863 | 0 3 3 4 4 5 1 2 |
| | | | 0 5 4 3 2 1 4 1 |
| 6 | 9 | .6873 | 0 0 1 1 2 2 3 4 5 |
| | | | 0 2 4 5 1 3 5 1 0 |
| 6 | 10 | .6889 | 0 1 2 3 4 5 1 2 4 5 |
| | | | 0 1 4 2 5 3 4 1 2 0 |
| 6 | 11 | .6914 | 0 2 3 4 5 1 2 3 4 0 5 |
| | | | 0 4 2 5 3 4 1 5 2 3 0 |
| 6 | 12 | .6939 | 0 5 1 2 3 4 0 1 2 3 4 5 |
| | | | 0 3 4 1 5 2 3 1 4 2 5 0 |
| 6 | 13 | .6919 | 0 5 5 0 1 1 2 2 3 3 4 4 5 |
| | | | 0 5 2 3 3 0 1 4 4 1 2 5 4 |
| 6 | 14 | .6914 | 0 4 4 5 5 5 0 1 1 2 2 3 3 4 |
| | | | 0 4 3 3 2 1 1 0 5 5 4 2 1 5 |
| 6 | 15 | .6913 | 0 0 1 1 1 2 2 2 3 3 4 4 5 5 0 |
| | | | 0 2 0 5 4 4 3 2 2 1 1 5 4 3 1 |
| 6 | 16 | .6919 | 0 5 5 0 0 1 1 1 2 2 2 3 3 4 4 5 |
| | | | 0 3 2 2 1 0 5 4 4 3 2 1 5 3 1 4 |
| 6 | 17 | .6928 | 0 1 2 3 4 5 0 2 3 4 5 1 2 4 5 3 1 |
| | | | 0 3 1 4 2 5 3 4 1 5 2 2 0 1 4 3 0 |

| S | K | Ē' | ARRAY α' |
|---|---|---|---|
| 7 | 8 | .6843 | 0 5 6 1 2 3 4 6 |
| | | | 0 2 1 6 5 4 3 5 |
| 7 | 9 | .6828 | 0 4 6 1 2 3 4 5 5 |
| | | | 0 2 1 6 5 4 3 2 1 |
| 7 | 10 | .6833 | 0 1 2 2 3 3 4 5 6 1 |
| | | | 0 5 4 3 2 1 0 6 1 6 |
| 7 | 11 | .6845 | 0 2 3 4 6 1 2 3 4 5 5 |
| | | | 0 4 3 2 1 6 5 4 3 2 1 |
| 7 | 12 | .6860 | 0 3 4 5 1 2 3 4 5 6 1 6 |
| | | | 0 3 2 1 6 5 4 3 2 1 5 0 |
| 7 | 13 | .6878 | 0 1 2 3 4 5 1 2 3 4 5 6 6 |
| | | | 0 5 4 3 2 1 6 5 4 3 2 1 0 |
| 7 | 14 | .6897 | 0 0 1 2 3 4 5 1 2 3 4 5 6 6 |
| | | | 0 6 5 4 3 2 1 6 5 4 3 2 1 0 |
| 7 | 15 | .6882 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6 6 |
| | | | 0 6 5 4 3 2 1 1 0 6 5 4 3 2 5 |
| 8 | 9 | .6826 | 0 7 1 2 3 4 5 6 6 |
| | | | 0 4 5 1 6 2 7 3 7 |
| 8 | 10 | .6812 | 0 6 6 7 1 2 3 4 5 7 |
| | | | 0 4 3 2 7 6 5 3 6 1 |
| 8 | 11 | .6810 | 0 5 6 7 1 2 3 4 5 2 3 |
| | | | 0 2 7 3 4 1 5 2 6 5 1 |
| 8 | 12 | .6814 | 0 1 2 2 3 3 4 4 5 6 7 1 |
| | | | 0 5 4 3 2 1 0 7 6 5 4 7 |
| 8 | 13 | .6826 | 0 2 3 4 5 6 7 1 2 3 4 6 5 |
| | | | 0 5 2 6 3 7 4 5 1 6 2 3 7 |
| 9 | 10 | .6808 | 0 8 1 2 3 4 5 6 7 7 |
| | | | 0 1 8 7 6 5 4 3 2 1 |
| 9 | 11 | .6793 | 0 7 7 8 1 2 3 4 5 6 8 |
| | | | 0 4 3 2 8 7 6 5 4 1 1 |
| 9 | 12 | .6786 | 0 6 6 7 7 8 1 2 3 4 5 8 |
| | | | 0 5 3 4 2 1 8 7 6 5 4 6 |

## TABLE C.3

### α(0,1,2)-DESIGNS , R=4 , 2≤S≤9 , 3≤K≤20

| S | K | Ē' | ARRAY α' |
|---|---|---|---|
| 2 | 3 | .8235 | 0 1 0 |
| | | | 0 0 1 |
| | | | 0 1 1 |
| 2 | 4 | .8400 | 0 1 0 1 |
| | | | 0 0 1 1 |
| | | | 0 1 1 0 |
| 3 | 4 | .7891 | 0 0 1 2 |
| | | | 0 2 1 0 |
| | | | 0 1 2 1 |
| 3 | 5 | .7975 | 0 2 0 1 2 |
| | | | 0 2 1 2 1 |
| | | | 0 1 2 2 0 |
| 3 | 6 | .8049 | 0 1 2 0 1 2 |
| | | | 0 2 1 2 1 0 |
| | | | 0 0 1 1 2 2 |
| 3 | 7 | .7986 | 0 2 1 2 0 1 1 |
| | | | 0 1 2 2 1 0 1 |
| | | | 0 0 0 1 1 1 2 |
| 3 | 8 | .8019 | 0 1 2 1 2 0 1 0 |
| | | | 0 1 1 2 2 1 0 2 |
| | | | 0 2 0 0 1 1 1 2 |
| 3 | 9 | .8049 | 0 1 2 0 1 2 0 1 2 |
| | | | 0 2 1 1 0 2 2 1 0 |
| | | | 0 0 0 1 1 1 2 2 2 |
| 4 | 5 | .7808 | 0 1 1 2 3 |
| | | | 0 3 2 1 0 |
| | | | 0 2 3 1 2 |
| 4 | 6 | .7812 | 0 1 2 3 2 0 |
| | | | 0 3 1 0 3 2 |
| | | | 0 2 3 1 1 2 |
| 4 | 7 | .7853 | 0 1 2 3 1 2 3 |
| | | | 0 0 3 1 2 1 3 |
| | | | 0 3 1 2 1 3 0 |
| 4 | 8 | .7895 | 0 1 2 0 1 2 3 3 |
| | | | 0 2 1 2 0 3 1 3 |
| | | | 0 1 3 2 3 1 2 0 |
| 4 | 9 | .7865 | 0 3 0 1 2 3 1 2 3 |
| | | | 0 0 2 1 3 2 3 1 1 |
| | | | 0 1 2 0 1 3 2 3 2 |
| 4 | 10 | .7865 | 0 0 1 1 1 2 2 3 3 2 |
| | | | 0 1 3 2 1 0 1 3 2 3 |
| | | | 0 3 1 3 2 1 0 2 0 3 |

| S | K | Ē' | ARRAY $\underset{\sim}{\alpha}'$ |
|---|---|---|---|
| 4 | 11 | .7876 | 0 2 2 2 3 3 3 0 1 1 0 |
| | | | 0 1 0 3 2 1 0 3 2 1 2 |
| | | | 0 2 3 0 1 3 2 1 2 0 3 |
| * 4 | 12 | .7895 | 0 1 2 3 1 2 3 0 1 2 3 0 |
| | | | 0 2 3 1 3 2 0 3 1 0 2 1 |
| | | | 0 3 1 2 2 0 3 3 0 2 1 1 |
| * 4 | 13 | .7881 | 0 1 2 3 0 1 2 3 0 1 2 3 2 |
| | | | 0 2 3 1 1 3 2 0 3 1 0 2 1 |
| | | | 0 3 1 2 1 2 0 3 3 0 2 1 3 |
| 4 | 14 | .7880 | 0 1 2 1 2 3 0 1 3 0 1 2 3 3 |
| | | | 0 0 2 1 3 0 1 2 1 2 3 1 2 3 |
| | | | 0 3 0 0 1 1 1 1 2 2 2 3 3 0 |
| 4 | 15 | .7885 | 0 3 1 2 3 0 1 2 3 0 1 2 3 1 0 |
| | | | 0 3 1 3 0 1 2 0 1 2 3 1 2 0 3 |
| | | | 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 4 | 16 | .7895 | 0 1 2 3 0 1 2 3 0 1 2 3 1 2 3 0 |
| | | | 0 1 2 3 1 2 3 0 2 3 0 1 0 1 2 3 |
| | | | 0 0 0 0 1 1 1 1 2 2 2 2 3 3 3 3 |
| 5 | 6 | .7737 | 0 2 3 4 1 1 |
| | | | 0 3 2 1 4 3 |
| | | | 0 1 4 2 3 2 |
| 5 | 7 | .7736 | 0 1 2 3 4 1 0 |
| | | | 0 3 3 2 1 4 4 |
| | | | 0 2 1 4 2 3 4 |
| 5 | 8 | .7756 | 0 1 2 3 4 1 2 0 |
| | | | 0 4 3 2 2 0 4 1 |
| | | | 0 3 1 4 3 4 2 1 |
| 5 | 9 | .7782 | 0 3 4 1 2 3 4 1 0 |
| | | | 0 1 0 3 3 2 1 4 4 |
| | | | 0 3 1 2 1 4 2 3 4 |
| 5 | 10 | .7808 | 0 1 2 3 4 1 2 3 4 0 |
| | | | 0 3 2 1 0 4 3 2 1 4 |
| | | | 0 2 0 3 1 3 1 4 2 4 |
| 5 | 11 | .7788 | 0 2 3 4 1 2 3 4 0 1 1 |
| | | | 0 3 2 1 4 4 3 2 1 0 3 |
| | | | 0 1 4 2 3 2 0 3 1 4 2 |
| 5 | 12 | .7784 | 0 0 1 2 4 1 2 3 4 1 3 2 |
| | | | 0 1 0 4 2 3 2 1 0 4 2 3 |
| | | | 0 1 4 2 3 2 0 3 1 3 4 1 |
| 5 | 13 | .7788 | 0 4 1 3 0 1 2 3 4 0 1 2 2 |
| | | | 0 1 4 2 3 2 1 0 0 4 3 2 3 |
| | | | 0 2 3 4 3 1 4 2 1 4 2 0 1 |
| 5 | 14 | .7797 | 0 3 4 1 2 3 4 1 2 3 4 0 1 0 |
| | | | 0 1 0 3 3 2 1 4 4 3 2 1 0 4 |
| | | | 0 3 1 2 1 4 2 3 2 0 3 1 4 4 |
| 5 | 15 | .7808 | 0 1 2 3 4 1 2 3 4 0 1 2 3 4 0 |
| | | | 0 3 2 1 0 4 3 2 1 1 0 4 3 2 4 |
| | | | 0 2 0 3 1 3 1 4 2 1 4 2 0 3 4 |
| 5 | 16 | .7799 | 0 2 3 4 1 2 3 4 0 1 2 3 4 0 1 1 |
| | | | 0 3 2 1 4 4 3 2 1 0 0 4 3 2 1 3 |
| | | | 0 1 4 2 3 2 0 3 1 4 3 1 4 2 0 2 |

| S | K | Ē' | ARRAY α' |
|---|---|---|---|
| 5 | 17 | .7796 | 0 2 3 4 0 1 2 3 4 1 2 3 4 0 1 2 1 |
| | | | 0 1 1 0 4 3 2 2 1 4 3 3 2 1 0 4 2 |
| | | | 0 4 3 1 4 2 0 4 2 3 1 0 3 1 4 2 1 |
| 5 | 18 | .7798 | 0 3 1 2 3 4 0 1 2 3 4 0 1 2 3 4 1 4 |
| | | | 0 1 4 3 2 1 1 0 4 3 2 2 1 0 4 3 3 0 |
| | | | 0 3 3 1 4 2 1 4 2 0 3 2 0 3 1 4 2 1 |
| 5 | 19 | .7802 | 0 2 3 4 0 1 2 3 4 0 1 2 3 4 1 2 3 4 1 |
| | | | 0 0 4 3 3 2 1 0 4 4 3 2 1 0 4 3 2 1 1 |
| | | | 0 3 1 4 3 1 4 2 0 4 2 0 3 1 3 1 4 2 0 |
| 5 | 20 | .7808 | 0 1 2 3 4 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 |
| | | | 0 3 2 1 0 4 3 2 1 1 0 4 3 2 2 1 0 4 3 4 |
| | | | 0 2 0 3 1 3 1 4 2 1 4 2 0 3 2 0 3 1 4 4 |
| 6 | 7 | .7706 | 0 3 4 5 1 2 3 |
| | | | 0 4 5 2 3 1 1 |
| | | | 0 2 1 3 2 4 5 |
| 6 | 8 | .7692 | 0 2 3 4 5 1 4 1 |
| | | | 0 4 1 5 2 3 2 0 |
| | | | 0 1 5 1 3 2 4 5 |
| 6 | 9 | .7701 | 0 4 5 1 2 3 4 1 3 |
| | | | 0 5 3 4 1 5 2 1 2 |
| | | | 0 0 2 2 4 1 3 5 4 |
| 6 | 10 | .7717 | 0 2 3 4 5 1 2 4 5 3 |
| | | | 0 4 1 5 2 3 1 2 5 4 |
| | | | 0 3 5 2 4 4 0 5 1 2 |
| 6 | 11 | .7735 | 0 2 3 4 5 1 2 3 4 5 1 |
| | | | 0 4 1 5 2 3 1 4 2 5 0 |
| | | | 0 3 5 2 4 4 0 2 5 1 1 |
| 6 | 12 | .7753 | 0 1 2 3 4 5 0 2 3 4 5 1 |
| | | | 0 3 1 4 2 5 3 4 1 5 2 0 |
| | | | 0 4 0 2 5 1 3 3 5 2 4 1 |
| 6 | 13 | .7738 | 0 3 4 5 0 1 2 3 4 5 1 2 2 |
| | | | 0 5 2 0 3 1 4 2 5 3 4 1 3 |
| | | | 0 1 5 1 3 0 2 4 2 4 3 5 1 |
| 6 | 14 | .7733 | 0 1 2 3 4 5 0 1 2 3 4 5 5 1 |
| | | | 0 3 1 4 2 5 3 0 4 1 5 2 4 2 |
| | | | 0 4 0 2 5 1 3 1 3 5 2 4 0 3 |
| 6 | 15 | .7734 | 0 2 3 4 5 0 1 2 3 4 5 1 3 4 1 |
| | | | 0 1 4 2 5 3 0 4 1 5 2 3 3 1 5 |
| | | | 0 4 2 4 0 3 5 1 5 1 3 2 1 3 4 |
| 6 | 16 | .7738 | 0 3 4 5 1 2 3 4 5 0 1 2 4 5 2 1 |
| | | | 0 2 5 3 4 1 5 2 0 3 1 4 4 2 0 3 |
| | | | 0 4 2 4 3 5 1 5 1 3 0 2 1 3 4 2 |
| 6 | 17 | .7745 | 0 1 2 3 4 5 0 2 3 4 5 1 2 4 5 3 1 |
| | | | 0 3 1 4 2 5 3 4 1 5 2 2 0 1 4 3 0 |
| | | | 0 4 0 2 5 1 3 3 5 2 4 3 5 4 0 1 1 |
| 7 | 8 | .7674 | 0 5 6 1 2 3 4 4 |
| | | | 0 2 1 6 5 4 3 2 |
| | | | 0 6 3 4 1 5 2 1 |
| 7 | 9 | .7663 | 0 3 5 6 1 2 3 4 4 |
| | | | 0 3 2 1 6 5 4 3 2 |
| | | | 0 4 6 3 4 1 5 2 1 |

| S | K | $\bar{E}'$ | ARRAY $\alpha'$ |
|---|---|---|---|
| 7 | 10 | .7667 | 0 3 5 6 1 2 3 4 1 4 |
| | | | 0 3 2 1 6 5 4 3 5 2 |
| | | | 0 4 6 3 4 1 5 2 3 1 |
| 7 | 11 | .7676 | 0 1 2 3 5 6 1 2 3 4 4 |
| | | | 0 5 4 3 2 1 6 5 4 3 2 |
| | | | 0 3 0 4 6 3 4 1 5 2 1 |
| 7 | 12 | .7688 | 0 1 2 3 5 6 1 2 3 4 5 4 |
| | | | 0 5 4 3 1 1 6 5 4 3 2 2 |
| | | | 0 3 0 4 5 3 4 1 5 2 6 1 |
| 7 | 13 | .7701 | 0 1 2 3 5 6 1 2 3 4 5 6 4 |
| | | | 0 5 4 3 1 0 6 5 4 3 2 1 2 |
| | | | 0 3 0 4 5 2 4 1 5 2 6 3 1 |
| 7 | 14 | .7714 | 0 1 2 3 4 5 6 1 2 3 4 5 6 0 |
| | | | 0 5 4 3 2 1 0 6 5 4 3 2 1 6 |
| | | | 0 3 0 4 1 5 2 4 1 5 2 6 3 6 |
| 7 | 15 | .7703 | 0 1 2 3 4 5 6 0 1 2 3 4 5 6 4 |
| | | | 0 6 5 4 3 2 1 1 0 6 5 4 3 2 2 |
| | | | 0 4 1 5 2 6 3 1 5 2 6 3 0 4 1 |
| 8 | 9 | .7657 | 0 5 6 7 1 2 3 4 5 |
| | | | 0 2 7 3 4 1 5 2 6 |
| | | | 0 3 4 6 2 5 7 1 7 |
| 8 | 10 | .7644 | 0 4 4 5 6 7 1 2 3 5 |
| | | | 0 2 6 3 7 4 5 1 6 4 |
| | | | 0 5 1 2 4 6 3 5 7 3 |
| 8 | 11 | .7643 | 0 4 5 6 7 1 2 3 4 2 5 |
| | | | 0 6 2 7 3 4 1 5 2 5 6 |
| | | | 0 2 5 1 4 3 7 2 6 3 1 |
| 8 | 12 | .7649 | 0 3 4 5 6 7 1 2 3 4 6 5 |
| | | | 0 2 6 3 7 4 5 1 6 2 3 7 |
| | | | 0 7 2 6 1 5 4 7 3 6 5 2 |
| 8 | 13 | .7656 | 0 2 3 4 5 6 7 1 2 3 4 6 5 |
| | | | 0 5 2 6 3 7 4 5 1 6 2 3 7 |
| | | | 0 3 7 2 6 1 5 4 7 3 6 5 2 |
| 9 | 10 | .7639 | 0 1 8 1 2 3 4 5 6 7 |
| | | | 0 2 1 8 7 6 5 4 3 2 |
| | | | 0 5 6 3 8 4 7 1 2 6 |
| 9 | 11 | .7629 | 0 4 4 5 6 7 8 1 2 3 5 |
| | | | 0 7 5 4 3 2 1 8 7 6 6 |
| | | | 0 2 1 2 5 8 4 3 6 7 8 |
| 9 | 12 | .7624 | 0 6 4 4 5 6 7 8 1 2 3 5 |
| | | | 0 8 7 5 4 3 2 1 8 7 6 6 |
| | | | 0 0 2 1 2 5 8 4 3 6 7 8 |

## TABLE D.1

## PROPERTIES OF $\underset{\sim}{\alpha}$(0,1,2)-DESIGNS , R=2 , 5≤V≤100

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | $\bar{E}$(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 3 | 1 | .4167 | .6542 | .5250( 8) | .7000( 0) | 1.000( 0) | 01 |
| 6 | 2 | 3 | 0 | .3333 | .6667 | .5263(17) | .7273( 0) | 1.000( 0) | 01 |
| 7 | 2 | 4 | 1 | .4167 | .7394 | .6122( 8) | .7692( 0) | 1.000( 0) | 01 |
| 8 | 2 | 4 | 0 | .5000 | .7778 | .6667( 0) | .8000( 0) | 1.000( 0) | 01 |
| 10 | 3 | 4 | 2 | .4167 | .6863 | .6144( 6) | .7649( 2) | 1.000( 0) | 021 |
| 11 | 3 | 4 | 1 | .3750 | .6996 | .6251(11) | .7778( 2) | 1.000( 0) | 012 |
| 12 | 3 | 4 | 0 | .3750 | .7082 | .6322(11) | .7895( 0) | 1.000( 0) | 012 |
| 13 | 3 | 5 | 2 | .3587 | .7309 | .6557(12) | .8041( 2) | 1.000( 0) | 021 |
| 14 | 3 | 5 | 1 | .3681 | .7492 | .6767(11) | .8166( 2) | 1.000( 0) | 012 |
| 15 | 3 | 5 | 0 | .4000 | .7636 | .6939( 8) | .8276( 0) | 1.000( 0) | 012 |
| 16 | 3 | 6 | 2 | .4000 | .7819 | .7152( 8) | .8387( 2) | 1.000( 0) | 021 |
| 17 | 4 | 5 | 3 | .4250 | .7242 | .6754( 4) | .8078( 1) | 1.000( 0) | 0312 |
| 17 | 3 | 6 | 1 | .4333 | .7971 | .7341( 4) | .8484( 1) | 1.000( 0) | 012 |
| 18 | 4 | 5 | 2 | .4000 | .7325 | .6830( 8) | .8149( 3) | 1.000( 0) | 0132 |
| 18 | 3 | 6 | 0 | .5000 | .8095 | .7500( 0) | .8571( 0) | 1.000( 0) | 012 |
| 19 | 4 | 5 | 1 | .4000 | .7394 | .6896( 8) | .8215( 1) | 1.000( 0) | 0123 |
| 19 | 3 | 7 | 2 | .4524 | .8168 | .7574( 3) | .8628( 1) | 1.000( 0) | 021 |
| 20 | 4 | 5 | 0 | .4000 | .7451 | .6951( 8) | .8276( 0) | 1.000( 0) | 0123 |
| 20 | 3 | 7 | 1 | .4286 | .8228 | .7637( 5) | .8679( 1) | 1.000( 0) | 012 |
| 21 | 4 | 6 | 3 | .3785 | .7557 | .7060(10) | .8346( 2) | 1.000( 0) | 0312 |
| 21 | 3 | 7 | 0 | .4286 | .8276 | .7689( 5) | .8727( 0) | 1.000( 0) | 012 |
| 22 | 4 | 6 | 2 | .3856 | .7650 | .7160( 9) | .8411( 3) | 1.000( 0) | 0132 |
| 22 | 3 | 8 | 2 | .4170 | .8354 | .7782( 5) | .8780( 1) | 1.000( 0) | 021 |
| 23 | 4 | 6 | 1 | .3688 | .7732 | .7252( 9) | .8470( 1) | 1.000( 0) | 0123 |
| 23 | 3 | 8 | 1 | .4203 | .8421 | .7865( 5) | .8829( 1) | 1.000( 0) | 012 |
| 24 | 4 | 6 | 0 | .3821 | .7804 | .7333( 7) | .8525( 0) | 1.000( 0) | 0123 |
| 24 | 3 | 8 | 0 | .4375 | .8479 | .7937( 4) | .8873( 0) | 1.000( 0) | 012 |
| 25 | 4 | 7 | 3 | .3764 | .7896 | .7435( 8) | .8581( 1) | 1.000( 0) | 0312 |
| 25 | 3 | 9 | 2 | .4375 | .8550 | .8028( 4) | .8919( 1) | 1.000( 0) | 021 |
| 26 | 5 | 6 | 4 | .4333 | .7559 | .7199( 3) | .8379( 1) | 1.000( 0) | 04123 |
| 26 | 4 | 7 | 2 | .4030 | .7981 | .7531( 7) | .8633( 2) | 1.000( 0) | 0132 |
| 26 | 3 | 9 | 1 | .4583 | .8612 | .8110( 2) | .8961( 1) | 1.000( 0) | 012 |
| 27 | 5 | 6 | 3 | .4167 | .7612 | .7251( 6) | .8422( 2) | 1.000( 0) | 01423 |
| 27 | 4 | 7 | 1 | .3986 | .8054 | .7616( 7) | .8682( 1) | 1.000( 0) | 0123 |
| 27 | 3 | 9 | 0 | .5000 | .8667 | .8182( 0) | .9000( 0) | 1.000( 0) | 012 |
| 28 | 5 | 6 | 2 | .4167 | .7660 | .7299( 6) | .8462( 2) | 1.000( 0) | 01243 |
| 28 | 4 | 7 | 0 | .4286 | .8120 | .7695( 5) | .8727( 0) | 1.000( 0) | 0123 |
| 29 | 5 | 6 | 1 | .4167 | .7703 | .7342( 6) | .8500( 1) | 1.000( 0) | 01234 |
| 29 | 4 | 8 | 3 | .4286 | .8194 | .7780( 5) | .8772( 1) | 1.000( 0) | 0312 |
| 30 | 5 | 6 | 0 | .4167 | .7742 | .7382( 6) | .8537( 0) | 1.000( 0) | 01234 |
| 30 | 4 | 8 | 2 | .4286 | .8261 | .7859( 5) | .8813( 2) | 1.000( 0) | 0132 |
| 31 | 5 | 7 | 4 | .3938 | .7802 | .7444( 7) | .8577( 1) | 1.000( 0) | 04123 |
| 31 | 4 | 8 | 1 | .4464 | .8323 | .7933( 2) | .8852( 1) | 1.000( 0) | 0123 |
| 32 | 5 | 7 | 3 | .3965 | .7856 | .7501( 8) | .8615( 2) | 1.000( 0) | 01423 |
| 32 | 4 | 8 | 0 | .5000 | .8378 | .8000( 0) | .8889( 0) | 1.000( 0) | 0123 |
| 33 | 5 | 7 | 2 | .3880 | .7907 | .7556( 8) | .8650( 2) | 1.000( 0) | 01243 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 33 | 4 | 9 | 3 | .4583 | .8415 | .8039( 2) | .8916( 1) | 1.000( 0) | 0312 |
| 34 | 5 | 7 | 1 | .3781 | .7954 | .7608( 7) | .8683( 1) | 1.000( 0) | 01234 |
| 34 | 4 | 9 | 2 | .4444 | .8448 | .8075( 3) | .8941( 1) | 1.000( 0) | 0132 |
| 35 | 5 | 7 | 0 | .3844 | .7997 | .7655( 5) | .8716( 0) | 1.000( 0) | 01234 |
| 35 | 4 | 9 | 1 | .4444 | .8478 | .8109( 3) | .8966( 0) | 1.000( 0) | 0123 |
| 36 | 5 | 8 | 4 | .3755 | .8050 | .7712( 6) | .8749( 1) | 1.000( 0) | 04123 |
| 36 | 4 | 9 | 0 | .4444 | .8505 | .8139( 3) | .8989( 0) | 1.000( 0) | 0123 |
| 37 | 6 | 7 | 5 | .4405 | .7816 | .7538( 2) | .8600( 1) | 1.000( 0) | 05123 |
| 37 | 5 | 8 | 3 | .3814 | .8100 | .7767( 6) | .8780( 2) | 1.000( 0) | 01423 |
| 37 | 4 | 10 | 3 | .4302 | .8542 | .8181( 4) | .9013( 1) | 1.000( 0) | 0312 |
| 38 | 6 | 7 | 4 | .4286 | .7852 | .7575( 5) | .8628( 2) | 1.000( 0) | 01523 |
| 38 | 5 | 8 | 2 | .3857 | .8146 | .7819( 6) | .8810( 1) | 1.000( 0) | 01243 |
| 38 | 4 | 10 | 2 | .4347 | .8576 | .8219( 4) | .9037( 1) | 1.000( 0) | 0132 |
| 39 | 6 | 7 | 3 | .4286 | .7886 | .7609( 5) | .8654( 2) | 1.000( 0) | 01253 |
| 39 | 5 | 8 | 1 | .3829 | .8189 | .7868( 6) | .8838( 1) | 1.000( 0) | 01234 |
| 39 | 4 | 10 | 1 | .4225 | .8607 | .8256( 4) | .9059( 0) | 1.000( 0) | 0123 |
| 40 | 6 | 7 | 2 | .4286 | .7917 | .7641( 5) | .8679( 2) | 1.000( 0) | 01235 |
| 40 | 5 | 8 | 0 | .3989 | .8230 | .7914( 4) | .8865( 0) | 1.000( 0) | 01234 |
| 40 | 4 | 10 | 0 | .4293 | .8635 | .8290( 3) | .9080( 0) | 1.000( 0) | 0123 |
| 41 | 6 | 7 | 1 | .4286 | .7947 | .7671( 5) | .8704( 1) | 1.000( 0) | 01234 |
| 41 | 5 | 9 | 4 | .3963 | .8275 | .7964( 5) | .8893( 1) | 1.000( 0) | 04123 |
| 41 | 4 | 11 | 3 | .4252 | .8669 | .8330( 3) | .9102( 2) | 1.000( 0) | 0312 |
| 42 | 6 | 7 | 0 | .4286 | .7974 | .7700( 5) | .8727( 0) | 1.000( 0) | 01234 |
| 42 | 5 | 9 | 3 | .4067 | .8317 | .8012( 5) | .8918( 1) | 1.000( 0) | 01423 |
| 42 | 4 | 11 | 2 | .4407 | .8701 | .8368( 3) | .9122( 1) | 1.000( 0) | 0132 |
| 43 | 6 | 8 | 5 | .4058 | .8012 | .7739( 6) | .8753( 1) | 1.000( 0) | 05123 |
| 43 | 5 | 9 | 2 | .4199 | .8357 | .8057( 5) | .8943( 1) | 1.000( 0) | 01243 |
| 43 | 4 | 11 | 1 | .4373 | .8730 | .8403( 3) | .9142( 0) | 1.000( 0) | 0123 |
| 44 | 6 | 8 | 4 | .4061 | .8047 | .7776( 6) | .8777( 1) | 1.000( 0) | 01523 |
| 44 | 5 | 9 | 1 | .4183 | .8394 | .8100( 5) | .8966( 1) | 1.000( 0) | 01234 |
| 44 | 4 | 11 | 0 | .4545 | .8758 | .8437( 2) | .9160( 0) | 1.000( 0) | 0123 |
| 45 | 6 | 8 | 3 | .4023 | .8080 | .7812( 7) | .8800( 1) | 1.000( 0) | 01253 |
| 45 | 5 | 9 | 0 | .4444 | .8429 | .8141( 3) | .8989( 0) | 1.000( 0) | 01234 |
| 45 | 4 | 12 | 3 | .4545 | .8788 | .8473( 2) | .9179( 0) | 1.000( 0) | 0312 |
| 46 | 6 | 8 | 2 | .3935 | .8112 | .7846( 6) | .8822( 1) | 1.000( 0) | 01235 |
| 46 | 5 | 10 | 4 | .4444 | .8466 | .8183( 3) | .9011( 1) | 1.000( 0) | 04123 |
| 46 | 4 | 12 | 2 | .4545 | .8816 | .8508( 2) | .9198( 1) | 1.000( 0) | 0132 |
| 47 | 6 | 8 | 1 | .3884 | .8143 | .7879( 5) | .8843( 1) | 1.000( 0) | 01234 |
| 47 | 5 | 10 | 3 | .4444 | .8502 | .8223( 3) | .9033( 1) | 1.000( 0) | 01423 |
| 47 | 4 | 12 | 1 | .4659 | .8843 | .8541( 1) | .9215( 0) | 1.000( 0) | 0123 |
| 48 | 6 | 8 | 0 | .3917 | .8171 | .7910( 4) | .8864( 0) | 1.000( 0) | 01234 |
| 48 | 5 | 10 | 2 | .4444 | .8535 | .8262( 3) | .9053( 1) | 1.000( 0) | 01243 |
| 48 | 4 | 12 | 0 | .5000 | .8868 | .8571( 0) | .9231( 0) | 1.000( 0) | 0123 |
| 49 | 6 | 9 | 5 | .3899 | .8205 | .7947( 5) | .8885( 1) | 1.000( 0) | 05123 |
| 49 | 5 | 10 | 1 | .4556 | .8567 | .8298( 2) | .9073( 0) | 1.000( 0) | 01234 |
| 49 | 4 | 13 | 3 | .4712 | .8887 | .8593( 1) | .9245( 0) | 1.000( 0) | 0312 |
| 50 | 7 | 8 | 6 | .4464 | .8027 | .7805( 2) | .8770( 1) | 1.000( 0) | 06123 |
| 50 | 6 | 9 | 4 | .3956 | .8238 | .7982( 6) | .8905( 1) | 1.000( 0) | 01523 |
| 50 | 5 | 10 | 0 | .5000 | .8596 | .8333( 0) | .9091( 0) | 1.000( 0) | 01234 |
| 50 | 4 | 13 | 2 | .4615 | .8904 | .8614( 2) | .9258( 1) | 1.000( 0) | 0132 |
| 51 | 7 | 8 | 5 | .4375 | .8052 | .7831( 4) | .8788( 1) | 1.000( 0) | 01623 |
| 51 | 6 | 9 | 3 | .4027 | .8269 | .8016( 5) | .8925( 1) | 1.000( 0) | 01253 |
| 51 | 5 | 11 | 4 | .4636 | .8618 | .8357( 1) | .9106( 0) | 1.000( 0) | 04123 |
| 51 | 4 | 13 | 1 | .4615 | .8921 | .8633( 2) | .9271( 0) | 1.000( 0) | 0123 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 52 | 7 | 8 | 4 | .4375 | .8076 | .7856( 4) | .8806( 1) | 1.000( 0) | 01263 |
| 52 | 6 | 9 | 2 | .4030 | .8298 | .8048( 5) | .8944( 1) | 1.000( 0) | 01235 |
| 52 | 5 | 11 | 3 | .4545 | .8637 | .8378( 2) | .9120( 1) | 1.000( 0) | 01423 |
| 52 | 4 | 13 | 0 | .4615 | .8936 | .8651( 2) | .9282( 0) | 1.000( 0) | 0123 |
| 53 | 7 | 8 | 3 | .4375 | .8099 | .7880( 4) | .8824( 1) | 1.000( 0) | 01236 |
| 53 | 6 | 9 | 1 | .4000 | .8326 | .8079( 5) | .8962( 1) | 1.000( 0) | 01234 |
| 53 | 5 | 11 | 2 | .4545 | .8656 | .8399( 2) | .9134( 1) | 1.000( 0) | 01243 |
| 53 | 4 | 14 | 3 | .4510 | .8955 | .8673( 2) | .9295( 0) | 1.000( 0) | 0312 |
| 54 | 7 | 8 | 2 | .4375 | .8121 | .7903( 4) | .8841( 1) | 1.000( 0) | 01234 |
| 54 | 6 | 9 | 0 | .4038 | .8352 | .8108( 4) | .8980( 0) | 1.000( 0) | 01234 |
| 54 | 5 | 11 | 1 | .4545 | .8674 | .8419( 2) | .9148( 0) | 1.000( 0) | 01234 |
| 54 | 4 | 14 | 2 | .4543 | .8972 | .8694( 2) | .9307( 0) | 1.000( 0) | 0132 |
| 55 | 7 | 8 | 1 | .4375 | .8142 | .7925( 4) | .8857( 1) | 1.000( 0) | 01234 |
| 55 | 6 | 10 | 5 | .3845 | .8380 | .8139( 4) | .8998( 1) | 1.000( 0) | 05123 |
| 55 | 5 | 11 | 0 | .4545 | .8691 | .8438( 2) | .9160( 0) | 1.000( 0) | 01234 |
| 55 | 4 | 14 | 1 | .4450 | .8989 | .8714( 2) | .9318( 0) | 1.000( 0) | 0123 |
| 56 | 7 | 8 | 0 | .4375 | .8163 | .7946( 4) | .8873( 0) | 1.000( 0) | 01234 |
| 56 | 6 | 10 | 4 | .3876 | .8409 | .8170( 5) | .9015( 1) | 1.000( 0) | 01523 |
| 56 | 5 | 12 | 4 | .4401 | .8711 | .8461( 3) | .9174( 0) | 1.000( 0) | 04123 |
| 56 | 4 | 14 | 0 | .4495 | .9004 | .8733( 2) | .9329( 0) | 1.000( 0) | 0123 |
| 57 | 7 | 9 | 6 | .4155 | .8188 | .7973( 5) | .8890( 1) | 1.000( 0) | 06123 |
| 57 | 6 | 10 | 3 | .4001 | .8436 | .8201( 4) | .9031( 1) | 1.000( 0) | 01253 |
| 57 | 5 | 12 | 3 | .4416 | .8731 | .8483( 3) | .9188( 1) | 1.000( 0) | 01423 |
| 57 | 4 | 15 | 3 | .4463 | .9022 | .8755( 2) | .9341( 0) | 1.000( 0) | 0312 |
| 58 | 7 | 9 | 5 | .4143 | .8212 | .7998( 5) | .8907( 1) | 1.000( 0) | 01623 |
| 58 | 6 | 10 | 2 | .3968 | .8462 | .8230( 4) | .9047( 1) | 1.000( 0) | 01235 |
| 58 | 5 | 12 | 2 | .4355 | .8749 | .8504( 3) | .9200( 1) | 1.000( 0) | 01243 |
| 58 | 4 | 15 | 2 | .4573 | .9039 | .8775( 2) | .9352( 0) | 1.000( 0) | 0132 |
| 59 | 7 | 9 | 4 | .4133 | .8236 | .8023( 5) | .8923( 1) | 1.000( 0) | 01263 |
| 59 | 6 | 10 | 1 | .3990 | .8486 | .8257( 4) | .9063( 1) | 1.000( 0) | 01234 |
| 59 | 5 | 12 | 1 | .4295 | .8767 | .8524( 3) | .9212( 0) | 1.000( 0) | 01234 |
| 59 | 4 | 15 | 1 | .4546 | .9054 | .8795( 2) | .9363( 0) | 1.000( 0) | 0123 |
| 60 | 7 | 9 | 3 | .4055 | .8258 | .8047( 5) | .8938( 1) | 1.000( 0) | 01236 |
| 60 | 6 | 10 | 0 | .4134 | .8510 | .8284( 3) | .9078( 0) | 1.000( 0) | 01234 |
| 60 | 5 | 12 | 0 | .4326 | .8784 | .8543( 2) | .9224( 0) | 1.000( 0) | 01234 |
| 60 | 4 | 15 | 0 | .4667 | .9069 | .8814( 1) | .9372( 0) | 1.000( 0) | 0123 |
| 61 | 7 | 9 | 2 | .4009 | .8280 | .8070( 5) | .8953( 1) | 1.000( 0) | 01234 |
| 61 | 6 | 11 | 5 | .4121 | .8535 | .8312( 3) | .9093( 1) | 1.000( 0) | 05123 |
| 61 | 5 | 13 | 4 | .4264 | .8803 | .8565( 3) | .9236( 0) | 1.000( 0) | 04123 |
| 61 | 4 | 16 | 3 | .4667 | .9086 | .8834( 1) | .9384( 0) | 1.000( 0) | 0312 |
| 62 | 7 | 9 | 1 | .3980 | .8301 | .8093( 4) | .8968( 1) | 1.000( 0) | 01234 |
| 62 | 6 | 11 | 4 | .4168 | .8560 | .8339( 4) | .9107( 1) | 1.000( 0) | 01523 |
| 62 | 5 | 13 | 3 | .4296 | .8821 | .8586( 2) | .9247( 1) | 1.000( 0) | 01423 |
| 62 | 4 | 16 | 2 | .4667 | .9101 | .8854( 1) | .9394( 0) | 1.000( 0) | 0132 |
| 63 | 7 | 9 | 0 | .3999 | .8321 | .8115( 3) | .8982( 0) | 1.000( 0) | 01234 |
| 63 | 6 | 11 | 3 | .4250 | .8583 | .8366( 4) | .9121( 1) | 1.000( 0) | 01253 |
| 63 | 5 | 13 | 2 | .4311 | .8838 | .8606( 2) | .9258( 1) | 1.000( 0) | 01243 |
| 63 | 4 | 16 | 1 | .4750 | .9116 | .8873( 1) | .9404( 0) | 1.000( 0) | 0123 |
| 64 | 7 | 10 | 6 | .3961 | .8344 | .8139( 4) | .8996( 1) | 1.000( 0) | 06123 |
| 64 | 6 | 11 | 2 | .4311 | .8605 | .8391( 4) | .9135( 1) | 1.000( 0) | 01235 |
| 64 | 5 | 13 | 1 | .4290 | .8855 | .8625( 2) | .9270( 0) | 1.000( 0) | 01234 |
| 64 | 4 | 16 | 0 | .5000 | .9130 | .8889( 0) | .9412( 0) | 1.000( 0) | 0123 |
| 65 | 8 | 9 | 7 | .4514 | .8201 | .8021( 2) | .8903( 0) | 1.000( 0) | 07123 |
| 65 | 7 | 10 | 5 | .3970 | .8367 | .8164( 5) | .9010( 1) | 1.000( 0) | 01623 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 65 | 6 | 11 | 1 | .4318 | .8627 | .8415( 3) | .9148( 0) | 1.000( 0) | 01234 |
| 65 | 5 | 13 | 0 | .4378 | .8871 | .8644( 2) | .9281( 0) | 1.000( 0) | 01234 |
| 66 | 8 | 9 | 6 | .4444 | .8220 | .8040( 3) | .8916( 1) | 1.000( 0) | 01723 |
| 66 | 7 | 10 | 4 | .4014 | .8388 | .8187( 4) | .9024( 1) | 1.000( 0) | 01263 |
| 66 | 6 | 11 | 0 | .4545 | .8647 | .8439( 2) | .9160( 0) | 1.000( 0) | 01234 |
| 66 | 5 | 14 | 4 | .4359 | .8889 | .8664( 2) | .9291( 0) | 1.000( 0) | 04123 |
| 67 | 8 | 9 | 5 | .4444 | .8238 | .8058( 3) | .8929( 1) | 1.000( 0) | 01273 |
| 67 | 7 | 10 | 3 | .4055 | .8409 | .8210( 4) | .9037( 1) | 1.000( 0) | 01236 |
| 67 | 6 | 12 | 5 | .4545 | .8669 | .8463( 2) | .9173( 0) | 1.000( 0) | 05123 |
| 67 | 5 | 14 | 3 | .4422 | .8905 | .8684( 2) | .9301( 1) | 1.000( 0) | 01423 |
| 68 | 8 | 9 | 4 | .4444 | .8255 | .8076( 3) | .8942( 1) | 1.000( 0) | 01237 |
| 68 | 7 | 10 | 2 | .4072 | .8429 | .8232( 4) | .9050( 1) | 1.000( 0) | 01234 |
| 68 | 6 | 12 | 4 | .4545 | .8690 | .8486( 2) | .9186( 1) | 1.000( 0) | 01523 |
| 68 | 5 | 14 | 2 | .4498 | .8921 | .8702( 2) | .9311( 0) | 1.000( 0) | 01243 |
| 69 | 8 | 9 | 3 | .4444 | .8272 | .8094( 3) | .8954( 1) | 1.000( 0) | 01234 |
| 69 | 7 | 10 | 1 | .4116 | .8449 | .8253( 4) | .9063( 1) | 1.000( 0) | 01234 |
| 69 | 6 | 12 | 3 | .4545 | .8710 | .8509( 2) | .9197( 1) | 1.000( 0) | 01253 |
| 69 | 5 | 14 | 1 | .4487 | .8937 | .8720( 2) | .9321( 0) | 1.000( 0) | 01234 |
| 70 | 8 | 9 | 2 | .4444 | .8288 | .8111( 3) | .8966( 1) | 1.000( 0) | 01234 |
| 70 | 7 | 10 | 0 | .4293 | .8467 | .8273( 3) | .9074( 0) | 1.000( 0) | 01234 |
| 70 | 6 | 12 | 2 | .4545 | .8729 | .8531( 2) | .9209( 1) | 1.000( 0) | 01235 |
| 70 | 5 | 14 | 0 | .4643 | .8951 | .8738( 2) | .9330( 0) | 1.000( 0) | 01234 |
| 71 | 8 | 9 | 1 | .4444 | .8304 | .8127( 3) | .8978( 0) | 1.000( 0) | 01234 |
| 71 | 7 | 11 | 6 | .3869 | .8487 | .8294( 4) | .9087( 1) | 1.000( 0) | 06123 |
| 71 | 6 | 12 | 1 | .4621 | .8747 | .8552( 1) | .9220( 0) | 1.000( 0) | 01234 |
| 71 | 5 | 15 | 4 | .4643 | .8967 | .8756( 2) | .9340( 0) | 1.000( 0) | 04123 |
| 72 | 8 | 9 | 0 | .4444 | .8319 | .8143( 3) | .8989( 0) | 1.000( 0) | 01234 |
| 72 | 7 | 11 | 5 | .3905 | .8507 | .8316( 4) | .9099( 1) | 1.000( 0) | 01623 |
| 72 | 6 | 12 | 0 | .5000 | .8765 | .8571( 0) | .9231( 0) | 1.000( 0) | 01234 |
| 72 | 5 | 15 | 3 | .4643 | .8982 | .8774( 2) | .9349( 0) | 1.000( 0) | 01423 |
| 73 | 8 | 10 | 7 | .4233 | .8337 | .8162( 4) | .9002( 1) | 1.000( 0) | 07123 |
| 73 | 7 | 11 | 4 | .4004 | .8526 | .8337( 4) | .9111( 1) | 1.000( 0) | 01263 |
| 73 | 6 | 13 | 5 | .4679 | .8779 | .8587( 1) | .9241( 0) | 1.000( 0) | 05123 |
| 73 | 5 | 15 | 2 | .4643 | .8997 | .8791( 2) | .9358( 0) | 1.000( 0) | 01243 |
| 74 | 8 | 10 | 6 | .4214 | .8354 | .8180( 4) | .9013( 1) | 1.000( 0) | 01723 |
| 74 | 7 | 11 | 3 | .4101 | .8545 | .8358( 4) | .9122( 1) | 1.000( 0) | 01236 |
| 74 | 6 | 13 | 4 | .4615 | .8792 | .8600( 2) | .9250( 1) | 1.000( 0) | 01523 |
| 74 | 5 | 15 | 1 | .4714 | .9011 | .8808( 1) | .9367( 0) | 1.000( 0) | 01234 |
| 75 | 8 | 10 | 5 | .4221 | .8372 | .8198( 5) | .9024( 1) | 1.000( 0) | 01273 |
| 75 | 7 | 11 | 2 | .4105 | .8563 | .8377( 3) | .9133( 1) | 1.000( 0) | 01234 |
| 75 | 6 | 13 | 3 | .4615 | .8804 | .8614( 2) | .9258( 1) | 1.000( 0) | 01253 |
| 75 | 5 | 15 | 0 | .5000 | .9024 | .8824( 0) | .9375( 0) | 1.000( 0) | 01234 |
| 76 | 8 | 10 | 4 | .4150 | .8388 | .8216( 5) | .9035( 1) | 1.000( 0) | 01237 |
| 76 | 7 | 11 | 1 | .4168 | .8580 | .8397( 3) | .9144( 0) | 1.000( 0) | 01234 |
| 76 | 6 | 13 | 2 | .4615 | .8816 | .8627( 2) | .9267( 1) | 1.000( 0) | 01235 |
| 76 | 5 | 16 | 4 | .4750 | .9035 | .8836( 1) | .9383( 0) | 1.000( 0) | 04123 |
| 77 | 8 | 10 | 3 | .4109 | .8404 | .8233( 5) | .9046( 1) | 1.000( 0) | 01234 |
| 77 | 7 | 11 | 0 | .4357 | .8597 | .8415( 2) | .9154( 0) | 1.000( 0) | 01234 |
| 77 | 6 | 13 | 1 | .4615 | .8827 | .8640( 2) | .9275( 0) | 1.000( 0) | 01234 |
| 77 | 5 | 16 | 3 | .4687 | .9045 | .8848( 1) | .9390( 0) | 1.000( 0) | 01423 |
| 78 | 8 | 10 | 2 | .4083 | .8420 | .8250( 4) | .9057( 1) | 1.000( 0) | 01234 |
| 78 | 7 | 12 | 6 | .3954 | .8614 | .8434( 3) | .9165( 0) | 1.000( 0) | 06123 |
| 78 | 6 | 13 | 0 | .4615 | .8838 | .8653( 2) | .9282( 0) | 1.000( 0) | 01234 |
| 78 | 5 | 16 | 2 | .4687 | .9055 | .8859( 1) | .9398( 0) | 1.000( 0) | 01243 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | $\bar{E}$(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 79 | 8 | 10 | 1 | .4065 | .8435 | .8266( 4) | .9067( 0) | 1.000( 0) | 01234 |
| 79 | 7 | 12 | 5 | .3972 | .8631 | .8453( 3) | .9176( 1) | 1.000( 0) | 01623 |
| 79 | 6 | 14 | 5 | .4476 | .8851 | .8666( 2) | .9291( 0) | 1.000( 0) | 05123 |
| 79 | 5 | 16 | 1 | .4687 | .9065 | .8870( 1) | .9404( 0) | 1.000( 0) | 01234 |
| 80 | 8 | 10 | 0 | .4076 | .8450 | .8282( 3) | .9077( 0) | 1.000( 0) | 01234 |
| 80 | 7 | 12 | 4 | .4047 | .8648 | .8472( 3) | .9185( 1) | 1.000( 0) | 01263 |
| 80 | 6 | 14 | 4 | .4476 | .8863 | .8680( 2) | .9299( 0) | 1.000( 0) | 01523 |
| 80 | 5 | 16 | 0 | .4687 | .9074 | .8881( 1) | .9410( 0) | 1.000( 0) | 01234 |
| 81 | 8 | 11 | 7 | .4028 | .8466 | .8300( 3) | .9088( 0) | 1.000( 0) | 07123 |
| 81 | 7 | 12 | 3 | .4102 | .8665 | .8490( 3) | .9195( 1) | 1.000( 0) | 01236 |
| 81 | 6 | 14 | 3 | .4448 | .8875 | .8693( 2) | .9306( 0) | 1.000( 0) | 01253 |
| 81 | 5 | 17 | 4 | .4582 | .9084 | .8893( 1) | .9417( 0) | 1.000( 0) | 04123 |
| 82 | 9 | 10 | 8 | .4556 | .8348 | .8197( 1) | .9010( 0) | 1.000( 0) | 08123 |
| 82 | 8 | 11 | 6 | .4019 | .8483 | .8317( 4) | .9098( 1) | 1.000( 0) | 01723 |
| 82 | 7 | 12 | 2 | .4089 | .8681 | .8507( 3) | .9205( 1) | 1.000( 0) | 01234 |
| 82 | 6 | 14 | 2 | .4396 | .8887 | .8706( 2) | .9314( 0) | 1.000( 0) | 01235 |
| 82 | 5 | 17 | 3 | .4593 | .9094 | .8905( 2) | .9424( 0) | 1.000( 0) | 01423 |
| 83 | 9 | 10 | 7 | .4500 | .8362 | .8212( 3) | .9020( 1) | 1.000( 0) | 01823 |
| 83 | 8 | 11 | 5 | .4038 | .8498 | .8334( 4) | .9107( 1) | 1.000( 0) | 01273 |
| 83 | 7 | 12 | 1 | .4126 | .8696 | .8524( 3) | .9214( 0) | 1.000( 0) | 01234 |
| 83 | 6 | 14 | 1 | .4365 | .8898 | .8719( 2) | .9321( 0) | 1.000( 0) | 01234 |
| 83 | 5 | 17 | 2 | .4547 | .9104 | .8916( 2) | .9431( 0) | 1.000( 0) | 01243 |
| 84 | 9 | 10 | 6 | .4500 | .8375 | .8226( 3) | .9030( 1) | 1.000( 0) | 01283 |
| 84 | 8 | 11 | 4 | .4074 | .8514 | .8350( 4) | .9117( 1) | 1.000( 0) | 01237 |
| 84 | 7 | 12 | 0 | .4249 | .8711 | .8541( 2) | .9223( 0) | 1.000( 0) | 01234 |
| 84 | 6 | 14 | 0 | .4381 | .8909 | .8731( 2) | .9329( 0) | 1.000( 0) | 01234 |
| 84 | 5 | 17 | 1 | .4504 | .9113 | .8927( 1) | .9437( 0) | 1.000( 0) | 01234 |
| 85 | 9 | 10 | 5 | .4500 | .8389 | .8240( 3) | .9039( 1) | 1.000( 0) | 01238 |
| 85 | 8 | 11 | 3 | .4081 | .8529 | .8366( 4) | .9127( 1) | 1.000( 0) | 01234 |
| 85 | 7 | 13 | 6 | .4242 | .8726 | .8558( 3) | .9232( 0) | 1.000( 0) | 06123 |
| 85 | 6 | 15 | 5 | .4369 | .8921 | .8744( 2) | .9336( 0) | 1.000( 0) | 05123 |
| 85 | 5 | 17 | 0 | .4524 | .9122 | .8937( 1) | .9443( 0) | 1.000( 0) | 01234 |
| 86 | 9 | 10 | 4 | .4500 | .8402 | .8254( 3) | .9048( 1) | 1.000( 0) | 01234 |
| 86 | 8 | 11 | 2 | .4074 | .8543 | .8382( 3) | .9136( 1) | 1.000( 0) | 01234 |
| 86 | 7 | 13 | 5 | .4267 | .8742 | .8575( 3) | .9241( 1) | 1.000( 0) | 01623 |
| 86 | 6 | 15 | 4 | .4399 | .8932 | .8757( 2) | .9344( 0) | 1.000( 0) | 01523 |
| 86 | 5 | 18 | 4 | .4477 | .9132 | .8949( 1) | .9449( 0) | 1.000( 0) | 04123 |
| 87 | 9 | 10 | 3 | .4500 | .8414 | .8267( 3) | .9057( 1) | 1.000( 0) | 01234 |
| 87 | 8 | 11 | 1 | .4119 | .8557 | .8397( 3) | .9145( 0) | 1.000( 0) | 01234 |
| 87 | 7 | 13 | 4 | .4306 | .8756 | .8591( 3) | .9250( 1) | 1.000( 0) | 01263 |
| 87 | 6 | 15 | 3 | .4430 | .8944 | .8770( 2) | .9351( 0) | 1.000( 0) | 01253 |
| 87 | 5 | 18 | 3 | .4499 | .9141 | .8960( 1) | .9455( 0) | 1.000( 0) | 01423 |
| 88 | 9 | 10 | 2 | .4500 | .8427 | .8280( 3) | .9066( 1) | 1.000( 0) | 01234 |
| 88 | 8 | 11 | 0 | .4213 | .8571 | .8412( 3) | .9154( 0) | 1.000( 0) | 01234 |
| 88 | 7 | 13 | 3 | .4373 | .8771 | .8607( 3) | .9258( 1) | 1.000( 0) | 01236 |
| 88 | 6 | 15 | 2 | .4424 | .8954 | .8782( 2) | .9358( 0) | 1.000( 0) | 01235 |
| 88 | 5 | 18 | 2 | .4507 | .9151 | .8971( 1) | .9460( 0) | 1.000( 0) | 01243 |
| 89 | 9 | 10 | 1 | .4500 | .8439 | .8292( 3) | .9075( 0) | 1.000( 0) | 01234 |
| 89 | 8 | 12 | 7 | .3885 | .8585 | .8427( 3) | .9163( 0) | 1.000( 0) | 07123 |
| 89 | 7 | 13 | 2 | .4398 | .8785 | .8623( 3) | .9266( 1) | 1.000( 0) | 01234 |
| 89 | 6 | 15 | 1 | .4403 | .8965 | .8794( 2) | .9365( 0) | 1.000( 0) | 01234 |
| 89 | 5 | 18 | 1 | .4491 | .9160 | .8981( 1) | .9467( 0) | 1.000( 0) | 01234 |
| 90 | 9 | 10 | 0 | .4500 | .8450 | .8304( 3) | .9083( 0) | 1.000( 0) | 01234 |
| 90 | 8 | 12 | 6 | .3897 | .8600 | .8443( 4) | .9171( 1) | 1.000( 0) | 01723 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 7 | 13 | 1 | .4415 | .8798 | .8638( 2) | .9275( 0) | 1.000( 0) | 01234 |
| 90 | 6 | 15 | 0 | .4423 | .8975 | .8806( 1) | .9371( 0) | 1.000( 0) | 01234 |
| 90 | 5 | 18 | 0 | .4551 | .9168 | .8992( 1) | .9472( 0) | 1.000( 0) | 01234 |
| 91 | 9 | 11 | 8 | .4299 | .8464 | .8319( 3) | .9092( 0) | 1.000( 0) | 08123 |
| 91 | 8 | 12 | 5 | .3945 | .8614 | .8458( 3) | .9180( 1) | 1.000( 0) | 01273 |
| 91 | 7 | 13 | 0 | .4615 | .8811 | .8653( 2) | .9282( 0) | 1.000( 0) | 01234 |
| 91 | 6 | 16 | 5 | .4302 | .8986 | .8818( 2) | .9378( 0) | 1.000( 0) | 05123 |
| 91 | 5 | 19 | 4 | .4537 | .9177 | .9002( 1) | .9478( 0) | 1.000( 0) | 04123 |
| 92 | 9 | 11 | 7 | .4275 | .8477 | .8332( 4) | .9101( 1) | 1.000( 0) | 01823 |
| 92 | 8 | 12 | 4 | .4028 | .8628 | .8474( 3) | .9188( 1) | 1.000( 0) | 01237 |
| 92 | 7 | 14 | 6 | .4615 | .8825 | .8668( 2) | .9291( 0) | 1.000( 0) | 06123 |
| 92 | 6 | 16 | 4 | .4319 | .8997 | .8830( 2) | .9384( 0) | 1.000( 0) | 01523 |
| 92 | 5 | 19 | 3 | .4581 | .9186 | .9013( 1) | .9484( 0) | 1.000( 0) | 01423 |
| 93 | 9 | 11 | 6 | .4287 | .8490 | .8346( 4) | .9109( 1) | 1.000( 0) | 01283 |
| 93 | 8 | 12 | 3 | .4038 | .8641 | .8488( 3) | .9196( 1) | 1.000( 0) | 01234 |
| 93 | 7 | 14 | 5 | .4615 | .8838 | .8682( 2) | .9299( 1) | 1.000( 0) | 01623 |
| 93 | 6 | 16 | 3 | .4391 | .9007 | .8842( 2) | .9390( 0) | 1.000( 0) | 01253 |
| 93 | 5 | 19 | 2 | .4635 | .9195 | .9023( 1) | .9489( 0) | 1.000( 0) | 01243 |
| 94 | 9 | 11 | 5 | .4228 | .8502 | .8359( 4) | .9117( 1) | 1.000( 0) | 01238 |
| 94 | 8 | 12 | 2 | .4046 | .8654 | .8502( 3) | .9204( 1) | 1.000( 0) | 01234 |
| 94 | 7 | 14 | 4 | .4615 | .8851 | .8697( 2) | .9306( 1) | 1.000( 0) | 01263 |
| 94 | 6 | 16 | 2 | .4366 | .9017 | .8854( 2) | .9396( 0) | 1.000( 0) | 01235 |
| 94 | 5 | 19 | 1 | .4626 | .9203 | .9033( 1) | .9494( 0) | 1.000( 0) | 01234 |
| 95 | 9 | 11 | 4 | .4190 | .8515 | .8373( 4) | .9125( 1) | 1.000( 0) | 01234 |
| 95 | 8 | 12 | 1 | .4108 | .8667 | .8516( 3) | .9212( 0) | 1.000( 0) | 01234 |
| 95 | 7 | 14 | 3 | .4615 | .8863 | .8711( 2) | .9313( 1) | 1.000( 0) | 01236 |
| 95 | 6 | 16 | 1 | .4377 | .9027 | .8865( 2) | .9403( 0) | 1.000( 0) | 01234 |
| 95 | 5 | 19 | 0 | .4737 | .9212 | .9043( 1) | .9499( 0) | 1.000( 0) | 01234 |
| 96 | 9 | 11 | 3 | .4166 | .8527 | .9386( 4) | .9133( 1) | 1.000( 0) | 01234 |
| 96 | 8 | 12 | 0 | .4167 | .8679 | .8530( 2) | .9220( 0) | 1.000( 0) | 01234 |
| 96 | 7 | 14 | 2 | .4615 | .8876 | .8724( 2) | .9320( 1) | 1.000( 0) | 01234 |
| 96 | 6 | 16 | 0 | .4459 | .9037 | .8876( 1) | .9408( 0) | 1.000( 0) | 01234 |
| 96 | 5 | 20 | 4 | .4737 | .9220 | .9054( 1) | .9504( 0) | 1.000( 0) | 04123 |
| 97 | 9 | 11 | 2 | .4150 | .8538 | .8399( 3) | .9141( 1) | 1.000( 0) | 01234 |
| 97 | 8 | 13 | 7 | .3907 | .8692 | .8544( 3) | .9228( 0) | 1.000( 0) | 07123 |
| 97 | 7 | 14 | 1 | .4670 | .8887 | .8737( 1) | .9327( 0) | 1.000( 0) | 01234 |
| 97 | 6 | 17 | 5 | .4449 | .9047 | .8888( 2) | .9415( 0) | 1.000( 0) | 05123 |
| 97 | 5 | 20 | 3 | .4737 | .9229 | .9064( 1) | .9510( 0) | 1.000( 0) | 01423 |
| 98 | 9 | 11 | 1 | .4138 | .8550 | .8411( 3) | .9149( 0) | 1.000( 0) | 01234 |
| 98 | 8 | 13 | 6 | .3929 | .8705 | .8558( 3) | .9235( 1) | 1.000( 0) | 01723 |
| 98 | 7 | 14 | 0 | .5000 | .8899 | .8750( 0) | .9333( 0) | 1.000( 0) | 01234 |
| 98 | 6 | 17 | 4 | .4479 | .9057 | .8899( 2) | .9421( 0) | 1.000( 0) | 01523 |
| 98 | 5 | 20 | 2 | .4737 | .9237 | .9073( 1) | .9515( 0) | 1.000( 0) | 01234 |
| 99 | 9 | 11 | 0 | .4146 | .8561 | .8423( 2) | .9156( 0) | 1.000( 0) | 01234 |
| 99 | 8 | 13 | 5 | .3990 | .8718 | .8572( 3) | .9243( 1) | 1.000( 0) | 01273 |
| 99 | 7 | 15 | 6 | .4714 | .8908 | .8761( 1) | .9340( 0) | 1.000( 0) | 06123 |
| 99 | 6 | 17 | 3 | .4528 | .9066 | .8910( 2) | .9427( 0) | 1.000( 0) | 01253 |
| 99 | 5 | 20 | 1 | .4789 | .9245 | .9082( 0) | .9520( 0) | 1.000( 0) | 01234 |
| 100 | 9 | 12 | 8 | .4094 | .8573 | .8436( 3) | .9164( 0) | 1.000( 0) | 08123 |
| 100 | 8 | 13 | 4 | .4089 | .8730 | .8586( 3) | .9250( 1) | 1.000( 0) | 01237 |
| 100 | 7 | 15 | 5 | .4667 | .8917 | .8771( 1) | .9346( 0) | 1.000( 0) | 01623 |
| 100 | 6 | 17 | 2 | .4564 | .9076 | .8921( 2) | .9433( 0) | 1.000( 0) | 01235 |
| 100 | 5 | 20 | 0 | .5000 | .9252 | .9091( 0) | .9524( 0) | 1.000( 0) | 01234 |

## TABLE D.2

### PROPERTIES OF $\alpha(0,1,2)$-DESIGNS , R=3 , $5 \leq V \leq 100$

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}(0)$ | $\bar{E}(1)$ | $\bar{E}(2)$ | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 3 | 1 | .4444 | .6777 | .5448( 0) | .6458( 6) | .8178( 2) | 01 |
| 6 | 2 | 3 | 0 | .5556 | .7435 | .6349( 0) | .7143( 0) | .8511( 0) | 01 |
| 7 | 2 | 4 | 1 | .6111 | .7938 | .6822( 0) | .7674( 3) | .8713( 0) | 01 |
| 8 | 2 | 4 | 0 | .6667 | .8235 | .7273( 0) | .8000( 0) | .8889( 0) | 01 |
| 10 | 3 | 4 | 2 | .5556 | .7457 | .6797( 4) | .7643( 3) | .8715( 0) | 012 |
| 11 | 3 | 4 | 1 | .5000 | .7585 | .6895( 4) | .7770( 6) | .8783( 1) | 102 |
| 12 | 3 | 4 | 0 | .5000 | .7674 | .6961( 5) | .7883( 5) | .8852( 0) | 012 |
| 13 | 3 | 5 | 2 | .5310 | .7874 | .7204( 6) | .8028( 4) | .8939( 1) | 012 |
| 14 | 3 | 5 | 1 | .5641 | .8032 | .7410( 4) | .8154( 4) | .9011( 1) | 102 |
| 15 | 3 | 5 | 0 | .6000 | .8155 | .7572( 2) | .8265( 2) | .9076( 0) | 012 |
| 16 | 3 | 6 | 2 | .6000 | .8290 | .7728( 3) | .8382( 3) | .9132( 1) | 012 |
| 17 | 4 | 5 | 3 | .5000 | .7677 | .7204( 7) | .8007( 5) | .8944( 1) | 0231 |
| 17 | 3 | 6 | 1 | .6222 | .8404 | .7874( 2) | .8482( 2) | .9183( 1) | 102 |
| 18 | 4 | 5 | 2 | .5396 | .7800 | .7342( 5) | .8103( 6) | .8984( 2) | 0312 |
| 18 | 3 | 6 | 0 | .6667 | .8500 | .8000( 0) | .8571( 0) | .9231( 0) | 012 |
| 19 | 4 | 5 | 1 | .5451 | .7903 | .7460( 5) | .8191( 5) | .9033( 1) | 0123 |
| 19 | 3 | 7 | 2 | .6032 | .8563 | .8065( 1) | .8627( 1) | .9266( 0) | 012 |
| 20 | 4 | 5 | 0 | .5333 | .7994 | .7564( 4) | .8272( 3) | .9074( 0) | 0123 |
| 20 | 3 | 7 | 1 | .5714 | .8615 | .8121( 3) | .8677( 2) | .9297( 0) | 102 |
| 21 | 4 | 6 | 3 | .5367 | .8081 | .7656( 5) | .8342( 4) | .9117( 1) | 0213 |
| 21 | 3 | 7 | 0 | .5714 | .8658 | .8168( 3) | .8723( 2) | .9326( 0) | 012 |
| 22 | 4 | 6 | 2 | .5326 | .8158 | .7742( 4) | .8406( 4) | .9153( 1) | 0123 |
| 22 | 3 | 8 | 2 | .5877 | .8722 | .8249( 3) | .8776( 2) | .9356( 0) | 012 |
| 23 | 4 | 6 | 1 | .5382 | .8226 | .7819( 4) | .8466( 4) | .9184( 1) | 0213 |
| 23 | 3 | 8 | 1 | .6055 | .8776 | .8321( 2) | .8825( 2) | .9383( 0) | 102 |
| 24 | 4 | 6 | 0 | .5556 | .8286 | .7888( 4) | .8521( 2) | .9214( 0) | 0123 |
| 24 | 3 | 8 | 0 | .6250 | .8824 | .8384( 1) | .8869( 1) | .9408( 0) | 012 |
| 25 | 4 | 7 | 3 | .5556 | .8359 | .7971( 4) | .8578( 3) | .9246( 1) | 0312 |
| 25 | 3 | 9 | 2 | .6250 | .8877 | .8452( 2) | .8917( 1) | .9431( 0) | 012 |
| 26 | 5 | 6 | 4 | .4859 | .7923 | .7572( 5) | .8301( 5) | .9106( 1) | 04123 |
| 26 | 4 | 7 | 2 | .5693 | .8424 | .8049( 4) | .8630( 3) | .9275( 1) | 0132 |
| 26 | 3 | 9 | 1 | .6389 | .8924 | .8514( 1) | .8960( 1) | .9453( 0) | 102 |
| 27 | 5 | 6 | 3 | .4969 | .8007 | .7666( 5) | .8365( 6) | .9140( 1) | 01432 |
| 27 | 4 | 7 | 1 | .5904 | .8483 | .8120( 3) | .8679( 2) | .9302( 1) | 0123 |
| 27 | 3 | 9 | 0 | .6667 | .8966 | .8571( 0) | .9000( 0) | .9474( 0) | 012 |
| 28 | 5 | 6 | 2 | .5108 | .8084 | .7753( 5) | .8425( 5) | .9168( 1) | 01243 |
| 28 | 4 | 7 | 0 | .6190 | .8536 | .8185( 2) | .8725( 1) | .9327( 0) | 0123 |
| 29 | 5 | 6 | 1 | .5138 | .8155 | .7834( 4) | .8482( 4) | .9193( 1) | 01234 |
| 29 | 4 | 8 | 3 | .6190 | .8592 | .8250( 2) | .8770( 2) | .9350( 0) | 0312 |
| 30 | 5 | 6 | 0 | .5556 | .8221 | .7911( 1) | .8535( 2) | .9220( 0) | 01234 |
| 30 | 4 | 8 | 2 | .6190 | .8643 | .8311( 2) | .8812( 2) | .9372( 1) | 0132 |
| 31 | 5 | 7 | 4 | .5263 | .8238 | .7925( 4) | .8550( 3) | .9239( 1) | 01234 |
| 31 | 4 | 8 | 1 | .6310 | .8689 | .8368( 1) | .8852( 1) | .9392( 0) | 0123 |
| 32 | 5 | 7 | 3 | .5341 | .8296 | .7991( 5) | .8594( 4) | .9262( 1) | 10234 |
| 32 | 4 | 8 | 0 | .6667 | .8732 | .8421( 0) | .8889( 0) | .9412( 0) | 0123 |
| 33 | 5 | 7 | 2 | .5413 | .8350 | .8053( 4) | .8636( 3) | .9283( 1) | 12034 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 33 | 4 | 9 | 3 | .5555 | .8729 | .8403( 3) | .8893( 2) | .9422( 0) | 0312 |
| 34 | 5 | 7 | 1 | .5561 | .8399 | .8110( 3) | .8676( 3) | .9304( 1) | 12304 |
| 34 | 4 | 9 | 2 | .5598 | .8769 | .8452( 3) | .8926( 3) | .9439( 1) | 0132 |
| 35 | 5 | 7 | 0 | .5896 | .8446 | .8165( 1) | .8714( 1) | .9324( 0) | 01234 |
| 35 | 4 | 9 | 1 | .5692 | .8806 | .8498( 2) | .8958( 2) | .9455( 0) | 0123 |
| 36 | 5 | 8 | 4 | .5714 | .8487 | .8209( 3) | .8747( 2) | .9342( 0) | 01234 |
| 36 | 4 | 9 | 0 | .5926 | .8841 | .8541( 1) | .8988( 1) | .9470( 0) | 0123 |
| 37 | 6 | 7 | 5 | .5556 | .8230 | .7984( 4) | .8577( 3) | .9253( 0) | 04512 |
| 37 | 5 | 8 | 3 | .5635 | .8525 | .8252( 3) | .8778( 2) | .9360( 1) | 10234 |
| 37 | 4 | 10 | 3 | .5844 | .8870 | .8576( 2) | .9012( 1) | .9484( 0) | 0231 |
| 38 | 6 | 7 | 4 | .5556 | .8271 | .8029( 5) | .8609( 3) | .9272( 1) | 05142 |
| 38 | 5 | 8 | 2 | .5644 | .8561 | .8292( 3) | .8808( 2) | .9376( 1) | 12034 |
| 38 | 4 | 10 | 2 | .5851 | .8898 | .8608( 2) | .9035( 1) | .9497( 0) | 0312 |
| 39 | 6 | 7 | 3 | .5804 | .8310 | .8071( 4) | .8640( 3) | .9284( 1) | 01524 |
| 39 | 5 | 8 | 1 | .5693 | .8595 | .8331( 3) | .8836( 2) | .9390( 0) | 12304 |
| 39 | 4 | 10 | 1 | .5930 | .8923 | .8639( 2) | .9057( 1) | .9509( 0) | 0123 |
| 40 | 6 | 7 | 2 | .5785 | .8347 | .8111( 4) | .8670( 3) | .9301( 1) | 01253 |
| 40 | 5 | 8 | 0 | .5833 | .8626 | .8367( 2) | .8864( 1) | .9405( 0) | 01234 |
| 40 | 4 | 10 | 0 | .6000 | .8947 | .8668( 2) | .9078( 1) | .9521( 0) | 0123 |
| 41 | 6 | 7 | 1 | .5714 | .8381 | .8149( 3) | .8699( 2) | .9314( 1) | 01234 |
| 41 | 5 | 9 | 4 | .5833 | .8661 | .8407( 3) | .8891( 2) | .9420( 0) | 04123 |
| 41 | 4 | 11 | 3 | .5934 | .8973 | .8699( 2) | .9100( 1) | .9532( 0) | 0312 |
| 42 | 6 | 7 | 0 | .5714 | .8414 | .8185( 2) | .8727( 1) | .9327( 0) | 01234 |
| 42 | 5 | 9 | 3 | .5893 | .8694 | .8444( 3) | .8917( 2) | .9434( 1) | 01432 |
| 42 | 4 | 11 | 2 | .6015 | .8997 | .8729( 2) | .9121( 1) | .9543( 0) | 0132 |
| 43 | 6 | 8 | 5 | .5594 | .8421 | .8190( 3) | .8736( 2) | .9338( 0) | 01234 |
| 43 | 5 | 9 | 2 | .5933 | .8725 | .8480( 3) | .8942( 2) | .9447( 1) | 01243 |
| 43 | 4 | 11 | 1 | .5960 | .9020 | .8757( 2) | .9141( 1) | .9554( 0) | 0123 |
| 44 | 6 | 8 | 4 | .5633 | .8457 | .8228( 3) | .8763( 3) | .9352( 1) | 12034 |
| 44 | 5 | 9 | 1 | .6061 | .8754 | .8514( 2) | .8965( 1) | .9459( 0) | 01234 |
| 44 | 4 | 11 | 0 | .6061 | .9042 | .8784( 2) | .9160( 1) | .9563( 0) | 0123 |
| 45 | 6 | 8 | 3 | .5656 | .8490 | .8266( 3) | .8790( 3) | .9365( 1) | 21304 |
| 45 | 5 | 9 | 0 | .6296 | .8781 | .8547( 1) | .8988( 1) | .9471( 0) | 01234 |
| 45 | 4 | 12 | 3 | .6364 | .9065 | .8813( 1) | .9179( 1) | .9573( 0) | 0123 |
| 46 | 6 | 8 | 2 | .5686 | .8522 | .8301( 3) | .8815( 3) | .9379( 1) | 23140 |
| 46 | 5 | 10 | 4 | .6296 | .8810 | .8579( 2) | .9011( 1) | .9482( 0) | 04123 |
| 46 | 4 | 12 | 2 | .6364 | .9087 | .8840( 1) | .9197( 1) | .9583( 0) | 0123 |
| 47 | 6 | 8 | 1 | .5678 | .8552 | .8335( 3) | .8839( 2) | .9392( 0) | 23401 |
| 47 | 5 | 10 | 3 | .6296 | .8836 | .8610( 2) | .9032( 1) | .9493( 1) | 01432 |
| 47 | 4 | 12 | 1 | .6439 | .9107 | .8865( 1) | .9215( 0) | .9591( 0) | 0123 |
| 48 | 6 | 8 | 0 | .5678 | .8581 | .8368( 2) | .8863( 1) | .9404( 0) | 01234 |
| 48 | 5 | 10 | 2 | .6296 | .8862 | .8640( 2) | .9053( 1) | .9504( 1) | 01243 |
| 48 | 4 | 12 | 0 | .6667 | .9126 | .8889( 0) | .9231( 0) | .9600( 0) | 0123 |
| 49 | 6 | 9 | 5 | .5431 | .8589 | .8374( 3) | .8873( 2) | .9413( 0) | 05123 |
| 49 | 5 | 10 | 1 | .6370 | .8886 | .8669( 1) | .9073( 1) | .9514( 0) | 01234 |
| 49 | 4 | 13 | 3 | .6111 | .9136 | .8899( 2) | .9241( 1) | .9607( 0) | 0123 |
| 50 | 7 | 8 | 6 | .5952 | .8444 | .8260( 1) | .8769( 1) | .9370( 0) | 05612 |
| 50 | 6 | 9 | 4 | .5474 | .8619 | .8407( 3) | .8896( 2) | .9425( 1) | 01523 |
| 50 | 5 | 10 | 0 | .6667 | .8909 | .8696( 0) | .9091( 0) | .9524( 0) | 01234 |
| 50 | 4 | 13 | 2 | .6227 | .9153 | .8919( 1) | .9255( 1) | .9614( 0) | 1023 |
| 51 | 7 | 8 | 5 | .5833 | .8466 | .8282( 2) | .8788( 2) | .9370( 0) | 06152 |
| 51 | 6 | 9 | 3 | .5617 | .8648 | .8440( 3) | .8918( 2) | .9436( 1) | 01253 |
| 51 | 5 | 11 | 4 | .6182 | .8927 | .8715( 1) | .9106( 1) | .9533( 0) | 03412 |
| 51 | 4 | 13 | 1 | .6209 | .9168 | .8938( 1) | .9269( 1) | .9622( 0) | 1203 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 52 | 7 | 8 | 4 | .5833 | .8488 | .8304( 2) | .8806( 2) | .9374( 0) | 01625 |
| 52 | 6 | 9 | 2 | .5640 | .8675 | .8471( 2) | .8939( 2) | .9447( 1) | 01235 |
| 52 | 5 | 11 | 3 | .6061 | .8943 | .8734( 1) | .9121( 1) | .9541( 0) | 04132 |
| 52 | 4 | 13 | 0 | .6154 | .9183 | .8956( 1) | .9282( 1) | .9629( 0) | 0123 |
| 53 | 7 | 8 | 3 | .5833 | .8508 | .8325( 2) | .8824( 2) | .9383( 0) | 01263 |
| 53 | 6 | 9 | 1 | .5688 | .8702 | .8501( 2) | .8960( 2) | .9457( 0) | 01234 |
| 53 | 5 | 11 | 2 | .6061 | .8959 | .8751( 1) | .9134( 1) | .9549( 0) | 01423 |
| 53 | 4 | 14 | 3 | .6149 | .9197 | .8974( 1) | .9294( 1) | .9635( 0) | 0312 |
| 54 | 7 | 8 | 2 | .5833 | .8527 | .8346( 2) | .8841( 2) | .9392( 0) | 01236 |
| 54 | 6 | 9 | 0 | .5926 | .8727 | .8530( 1) | .8979( 1) | .9468( 0) | 01234 |
| 54 | 5 | 11 | 1 | .6061 | .8974 | .8768( 1) | .9148( 1) | .9556( 0) | 01234 |
| 54 | 4 | 14 | 2 | .6117 | .9211 | .8991( 1) | .9306( 1) | .9642( 0) | 0132 |
| 55 | 7 | 8 | 1 | .5833 | .8546 | .8365( 2) | .8857( 2) | .9401( 0) | 01234 |
| 55 | 6 | 10 | 5 | .5687 | .8746 | .8550( 3) | .8997( 1) | .9478( 0) | 05123 |
| 55 | 5 | 11 | 0 | .6061 | .8988 | .8784( 1) | .9161( 1) | .9563( 0) | 01234 |
| 55 | 4 | 14 | 1 | .6133 | .9224 | .9007( 1) | .9318( 1) | .9648( 0) | 0123 |
| 56 | 7 | 8 | 0 | .5833 | .8563 | .8384( 2) | .8874( 1) | .9407( 0) | 01234 |
| 56 | 6 | 10 | 4 | .5699 | .8768 | .8575( 3) | .9014( 2) | .9487( 1) | 01523 |
| 56 | 5 | 12 | 4 | .5873 | .8998 | .8795( 2) | .9169( 1) | .9570( 0) | 01423 |
| 56 | 4 | 14 | 0 | .6190 | .9237 | .9022( 1) | .9329( 0) | .9654( 0) | 0123 |
| 57 | 7 | 9 | 6 | .5639 | .8585 | .8407( 3) | .8890( 2) | .9419( 0) | 01523 |
| 57 | 6 | 10 | 3 | .5726 | .8790 | .8599( 3) | .9031( 2) | .9496( 1) | 01253 |
| 57 | 5 | 12 | 3 | .5944 | .9016 | .8816( 2) | .9183( 2) | .9577( 0) | 01243 |
| 57 | 4 | 15 | 3 | .6190 | .9250 | .9039( 1) | .9341( 1) | .9660( 0) | 0312 |
| 58 | 7 | 9 | 5 | .5620 | .8605 | .8429( 3) | .8906( 2) | .9430( 1) | 10256 |
| 58 | 6 | 10 | 2 | .5780 | .8810 | .8622( 3) | .9047( 2) | .9504( 1) | 01235 |
| 58 | 5 | 12 | 2 | .6032 | .9033 | .8835( 2) | .9197( 1) | .9585( 0) | 10234 |
| 58 | 4 | 15 | 2 | .6245 | .9263 | .9055( 1) | .9352( 1) | .9666( 0) | 0132 |
| 59 | 7 | 9 | 4 | .5646 | .8625 | .8450( 4) | .8922( 2) | .9439( 1) | 01235 |
| 59 | 6 | 10 | 1 | .5842 | .8830 | .8645( 2) | .9062( 1) | .9512( 0) | 01234 |
| 59 | 5 | 12 | 1 | .6056 | .9049 | .8855( 2) | .9211( 1) | .9592( 0) | 12034 |
| 59 | 4 | 15 | 1 | .6328 | .9275 | .9070( 1) | .9362( 1) | .9671( 0) | 0123 |
| 60 | 7 | 9 | 3 | .5537 | .8644 | .8470( 3) | .8938( 2) | .9447( 1) | 10234 |
| 60 | 6 | 10 | 0 | .6000 | .8849 | .8666( 1) | .9078( 1) | .9520( 0) | 01234 |
| 60 | 5 | 12 | 0 | .6217 | .9064 | .8873( 1) | .9223( 1) | .9598( 0) | 01234 |
| 60 | 4 | 15 | 0 | .6444 | .9287 | .9085( 0) | .9372( 0) | .9677( 0) | 0123 |
| 61 | 7 | 9 | 2 | .5454 | .8662 | .8490( 3) | .8952( 2) | .9455( 1) | 12034 |
| 61 | 6 | 11 | 5 | .6000 | .8868 | .8688( 2) | .9093( 1) | .9528( 0) | 05123 |
| 61 | 5 | 13 | 4 | .6098 | .9079 | .8890( 2) | .9235( 1) | .9605( 0) | 04123 |
| 61 | 4 | 16 | 3 | .6444 | .9299 | .9101( 1) | .9383( 0) | .9682( 0) | 0312 |
| 62 | 7 | 9 | 1 | .5439 | .8680 | .8509( 3) | .8967( 2) | .9461( 0) | 12350 |
| 62 | 6 | 11 | 4 | .6000 | .8887 | .8709( 2) | .9107( 1) | .9536( 0) | 01523 |
| 62 | 5 | 13 | 3 | .6047 | .9093 | .8906( 2) | .9247( 1) | .9611( 0) | 01432 |
| 62 | 4 | 16 | 2 | .6444 | .9311 | .9116( 1) | .9393( 1) | .9687( 0) | 0132 |
| 63 | 7 | 9 | 0 | .5464 | .8697 | .8528( 2) | .8981( 1) | .9468( 0) | 01234 |
| 63 | 6 | 11 | 3 | .6000 | .8904 | .8729( 2) | .9121( 1) | .9543( 0) | 01253 |
| 63 | 5 | 13 | 2 | .6052 | .9107 | .8922( 2) | .9258( 1) | .9617( 0) | 01243 |
| 63 | 4 | 16 | 1 | .6500 | .9323 | .9130( 0) | .9403( 0) | .9692( 0) | 0123 |
| 64 | 7 | 10 | 6 | .5695 | .8718 | .8552( 2) | .8996( 1) | .9477( 0) | 01623 |
| 64 | 6 | 11 | 2 | .6064 | .8922 | .8749( 2) | .9135( 1) | .9550( 0) | 01235 |
| 64 | 5 | 13 | 1 | .6078 | .9120 | .8937( 1) | .9269( 1) | .9622( 0) | 01234 |
| 64 | 4 | 16 | 0 | .6667 | .9333 | .9143( 0) | .9412( 0) | .9697( 0) | 0123 |
| 65 | 8 | 9 | 7 | .5833 | .8569 | .8416( 2) | .8893( 1) | .9423( 0) | 06712 |
| 65 | 7 | 10 | 5 | .5715 | .8735 | .8571( 3) | .9010( 1) | .9485( 1) | 01263 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}(0)$ | $\bar{E}(1)$ | $\bar{E}(2)$ | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 65 | 6 | 11 | 1 | .6166 | .8938 | .8768( 1) | .9148( 1) | .9557( 0) | 01234 |
| 65 | 5 | 13 | 0 | .6154 | .9132 | .8952( 1) | .9280( 1) | .9628( 0) | 01234 |
| 66 | 8 | 9 | 6 | .5833 | .8588 | .8437( 3) | .8908( 2) | .9433( 1) | 07162 |
| 66 | 7 | 10 | 4 | .5685 | .8753 | .8590( 3) | .9023( 2) | .9493( 1) | 10236 |
| 66 | 6 | 11 | 0 | .6364 | .8954 | .8787( 1) | .9161( 0) | .9564( 0) | 01234 |
| 66 | 5 | 14 | 4 | .6154 | .9146 | .8968( 1) | .9291( 1) | .9634( 0) | 04123 |
| 67 | 8 | 9 | 5 | .5833 | .8607 | .8457( 3) | .8922( 2) | .9440( 1) | 01726 |
| 67 | 7 | 10 | 3 | .5667 | .8769 | .8608( 3) | .9036( 2) | .9500( 0) | 12034 |
| 67 | 6 | 12 | 5 | .6364 | .8971 | .8805( 1) | .9174( 1) | .9570( 0) | 05123 |
| 67 | 5 | 14 | 3 | .6154 | .9159 | .8983( 1) | .9301( 1) | .9639( 0) | 01432 |
| 68 | 8 | 9 | 4 | .6014 | .8625 | .8476( 3) | .8936( 2) | .9446( 1) | 01273 |
| 68 | 7 | 10 | 2 | .5677 | .8785 | .8626( 3) | .9049( 2) | .9506( 0) | 12304 |
| 68 | 6 | 12 | 4 | .6364 | .8987 | .8823( 1) | .9186( 1) | .9577( 0) | 01523 |
| 68 | 5 | 14 | 2 | .6207 | .9171 | .8998( 1) | .9311( 1) | .9644( 0) | 01243 |
| 69 | 8 | 9 | 3 | .5969 | .8642 | .8495( 3) | .8950( 2) | .9453( 1) | 01237 |
| 69 | 7 | 10 | 1 | .5671 | .8801 | .8643( 2) | .9062( 1) | .9513( 0) | 12340 |
| 69 | 6 | 12 | 3 | .6364 | .9002 | .8841( 1) | .9198( 1) | .9583( 0) | 01253 |
| 69 | 5 | 14 | 1 | .6288 | .9183 | .9012( 1) | .9321( 1) | .9649( 0) | 01234 |
| 70 | 8 | 9 | 2 | .5926 | .8659 | .8513( 3) | .8963( 2) | .9460( 1) | 01234 |
| 70 | 7 | 10 | 0 | .5733 | .8816 | .8659( 2) | .9074( 1) | .9519( 0) | 01234 |
| 70 | 6 | 12 | 2 | .6364 | .9017 | .8858( 1) | .9209( 1) | .9589( 0) | 01235 |
| 70 | 5 | 14 | 0 | .6429 | .9195 | .9026( 0) | .9331( 0) | .9654( 0) | 01234 |
| 71 | 8 | 9 | 1 | .5926 | .8675 | .8530( 2) | .8977( 1) | .9465( 0) | 01234 |
| 71 | 7 | 11 | 6 | .5527 | .8830 | .8674( 2) | .9087( 1) | .9525( 0) | 01523 |
| 71 | 6 | 12 | 1 | .6414 | .9031 | .8874( 1) | .9221( 0) | .9594( 0) | 01234 |
| 71 | 5 | 15 | 4 | .6429 | .9206 | .9040( 1) | .9341( 1) | .9659( 0) | 04123 |
| 72 | 8 | 9 | 0 | .5926 | .8691 | .8547( 2) | .8990( 1) | .9471( 0) | 01234 |
| 72 | 7 | 11 | 5 | .5561 | .8846 | .8692( 2) | .9099( 1) | .9532( 0) | 10256 |
| 72 | 6 | 12 | 0 | .6667 | .9045 | .8889( 0) | .9231( 0) | .9600( 0) | 01234 |
| 72 | 5 | 15 | 3 | .6429 | .9218 | .9053( 1) | .9350( 1) | .9664( 0) | 01432 |
| 73 | 8 | 10 | 7 | .5729 | .8692 | .8547( 2) | .8994( 2) | .9476( 0) | 01723 |
| 73 | 7 | 11 | 4 | .5619 | .8861 | .8709( 2) | .9110( 1) | .9538( 0) | 01235 |
| 73 | 6 | 13 | 5 | .6239 | .9056 | .8903( 0) | .9241( 0) | .9605( 0) | 04512 |
| 73 | 5 | 15 | 2 | .6429 | .9229 | .9066( 1) | .9358( 1) | .9669( 0) | 01243 |
| 74 | 8 | 10 | 6 | .5730 | .8709 | .8565( 3) | .9006( 2) | .9484( 1) | 01273 |
| 74 | 7 | 11 | 3 | .5648 | .8876 | .8726( 2) | .9122( 2) | .9544( 0) | 10234 |
| 74 | 6 | 13 | 4 | .6154 | .9066 | .8914( 1) | .9250( 1) | .9610( 0) | 05142 |
| 74 | 5 | 15 | 1 | .6476 | .9240 | .9079( 0) | .9367( 0) | .9673( 0) | 01234 |
| 75 | 8 | 10 | 5 | .5721 | .8725 | .8583( 3) | .9019( 2) | .9490( 1) | 10237 |
| 75 | 7 | 11 | 2 | .5665 | .8891 | .8742( 2) | .9133( 1) | .9550( 0) | 12034 |
| 75 | 6 | 13 | 3 | .6154 | .9076 | .8925( 1) | .9259( 1) | .9615( 0) | 01524 |
| 75 | 5 | 15 | 0 | .6667 | .9250 | .9091( 0) | .9375( 0) | .9677( 0) | 01234 |
| 76 | 8 | 10 | 4 | .5740 | .8741 | .8600( 3) | .9031( 2) | .9496( 1) | 12034 |
| 76 | 7 | 11 | 1 | .5709 | .8905 | .8757( 2) | .9144( 1) | .9556( 0) | 12350 |
| 76 | 6 | 13 | 2 | .6154 | .9086 | .8936( 1) | .9267( 1) | .9620( 0) | 01253 |
| 76 | 5 | 16 | 4 | .6333 | .9259 | .9103( 0) | .9384( 0) | .9682( 0) | 03412 |
| 77 | 8 | 10 | 3 | .5772 | .8757 | .8617( 3) | .9043( 2) | .9503( 1) | 12304 |
| 77 | 7 | 11 | 0 | .5784 | .8918 | .8773( 1) | .9154( 1) | .9561( 0) | 01234 |
| 77 | 6 | 13 | 1 | .6154 | .9096 | .8947( 1) | .9275( 1) | .9624( 0) | 01234 |
| 77 | 5 | 16 | 3 | .6250 | .9267 | .9112( 1) | .9391( 1) | .9685( 0) | 04132 |
| 78 | 8 | 10 | 2 | .5816 | .8772 | .8634( 2) | .9054( 2) | .9509( 1) | 12340 |
| 78 | 7 | 12 | 6 | .5847 | .8931 | .8786( 2) | .9165( 1) | .9567( 0) | 06123 |
| 78 | 6 | 13 | 0 | .6154 | .9105 | .8957( 1) | .9283( 1) | .9629( 0) | 01234 |
| 78 | 5 | 16 | 2 | .6250 | .9275 | .9121( 1) | .9397( 1) | .9689( 0) | 01423 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 79 | 8 | 10 | 1 | .5825 | .8787 | .8650( 2) | .9066( 2) | .9514( 0) | 12345 |
| 79 | 7 | 12 | 5 | .5847 | .8944 | .8801( 2) | .9175( 1) | .9573( 0) | 01623 |
| 79 | 6 | 14 | 5 | .5960 | .9111 | .8963( 2) | .9288( 1) | .9633( 0) | 04512 |
| 79 | 5 | 16 | 1 | .6250 | .9283 | .9130( 1) | .9404( 0) | .9693( 0) | 01234 |
| 80 | 8 | 10 | 0 | .6000 | .8801 | .8666( 1) | .9077( 1) | .9520( 0) | 01234 |
| 80 | 7 | 12 | 4 | .5849 | .8958 | .8816( 2) | .9185( 1) | .9578( 0) | 01263 |
| 80 | 6 | 14 | 4 | .5982 | .9122 | .8975( 2) | .9297( 1) | .9637( 0) | 05142 |
| 80 | 5 | 16 | 0 | .6250 | .9290 | .9139( 1) | .9411( 0) | .9696( 0) | 01234 |
| 81 | 8 | 11 | 7 | .5666 | .8813 | .8679( 3) | .9087( 1) | .9526( 0) | 01324 |
| 81 | 7 | 12 | 3 | .5862 | .8970 | .8830( 2) | .9195( 1) | .9583( 0) | 01236 |
| 81 | 6 | 14 | 3 | .6042 | .9132 | .8987( 2) | .9305( 1) | .9641( 0) | 01524 |
| 81 | 5 | 17 | 4 | .6121 | .9296 | .9145( 1) | .9416( 1) | .9700( 0) | 04123 |
| 82 | 9 | 10 | 8 | .6074 | .8708 | .8586( 1) | .9011( 1) | .9498( 0) | 01723 |
| 82 | 8 | 11 | 6 | .5667 | .8826 | .8693( 3) | .9097( 1) | .9532( 0) | 10234 |
| 82 | 7 | 12 | 2 | .5910 | .8983 | .8844( 2) | .9205( 1) | .9588( 0) | 01234 |
| 82 | 6 | 14 | 2 | .6071 | .9143 | .8999( 1) | .9313( 1) | .9646( 0) | 01253 |
| 82 | 5 | 17 | 3 | .6168 | .9305 | .9155( 1) | .9423( 1) | .9703( 0) | 01432 |
| 83 | 9 | 10 | 7 | .6000 | .8720 | .8598( 1) | .9020( 1) | .9506( 0) | 10278 |
| 83 | 8 | 11 | 5 | .5685 | .8839 | .8707( 3) | .9107( 1) | .9538( 0) | 12304 |
| 83 | 7 | 12 | 1 | .5990 | .8995 | .8858( 1) | .9214( 1) | .9593( 0) | 01234 |
| 83 | 6 | 14 | 1 | .6079 | .9153 | .9011( 1) | .9321( 1) | .9650( 0) | 01234 |
| 83 | 5 | 17 | 2 | .6227 | .9313 | .9165( 1) | .9429( 1) | .9707( 0) | 01243 |
| 84 | 9 | 10 | 6 | .6000 | .8731 | .8610( 2) | .9030( 1) | .9508( 0) | 01237 |
| 84 | 8 | 11 | 4 | .5721 | .8852 | .8720( 3) | .9117( 2) | .9543( 0) | 31240 |
| 84 | 7 | 12 | 0 | .6111 | .9007 | .8871( 1) | .9223( 1) | .9597( 0) | 01234 |
| 84 | 6 | 14 | 0 | .6190 | .9162 | .9022( 1) | .9329( 0) | .9654( 0) | 01234 |
| 84 | 5 | 17 | 1 | .6242 | .9321 | .9175( 1) | .9436( 1) | .9710( 0) | 01234 |
| 85 | 9 | 10 | 5 | .6000 | .8743 | .8622( 2) | .9039( 1) | .9510( 0) | 10234 |
| 85 | 8 | 11 | 3 | .5728 | .8864 | .8734( 3) | .9126( 2) | .9548( 0) | 34125 |
| 85 | 7 | 13 | 6 | .6111 | .9019 | .8885( 1) | .9232( 1) | .9603( 0) | 06123 |
| 85 | 6 | 15 | 5 | .6120 | .9172 | .9032( 1) | .9336( 1) | .9658( 0) | 01234 |
| 85 | 5 | 17 | 0 | .6349 | .9329 | .9184( 1) | .9443( 0) | .9714( 0) | 01234 |
| 86 | 9 | 10 | 4 | .6000 | .8754 | .8633( 2) | .9048( 1) | .9512( 0) | 12034 |
| 86 | 8 | 11 | 2 | .5741 | .8875 | .8746( 3) | .9136( 1) | .9553( 0) | 34501 |
| 86 | 7 | 13 | 5 | .6111 | .9030 | .8898( 1) | .9241( 1) | .9607( 0) | 01623 |
| 86 | 6 | 15 | 4 | .6137 | .9181 | .9043( 1) | .9344( 1) | .9662( 0) | 10234 |
| 86 | 5 | 18 | 4 | .6169 | .9336 | .9193( 1) | .9449( 0) | .9717( 0) | 04123 |
| 87 | 9 | 10 | 3 | .6000 | .8765 | .8645( 2) | .9057( 1) | .9515( 0) | 12304 |
| 87 | 8 | 11 | 1 | .5785 | .8887 | .8759( 2) | .9145( 1) | .9557( 0) | 13456 |
| 87 | 7 | 13 | 4 | .6111 | .9041 | .8910( 1) | .9250( 1) | .9612( 0) | 01263 |
| 87 | 6 | 15 | 3 | .6099 | .9189 | .9052( 1) | .9351( 1) | .9666( 0) | 12034 |
| 87 | 5 | 18 | 3 | .6192 | .9343 | .9201( 1) | .9455( 1) | .9720( 0) | 01432 |
| 88 | 9 | 10 | 2 | .6000 | .8775 | .8656( 2) | .9066( 1) | .9517( 0) | 12340 |
| 88 | 8 | 11 | 0 | .5848 | .8898 | .8771( 2) | .9154( 1) | .9562( 0) | 01234 |
| 88 | 7 | 13 | 3 | .6111 | .9052 | .8923( 1) | .9258( 1) | .9616( 0) | 01236 |
| 88 | 6 | 15 | 2 | .6102 | .9198 | .9062( 1) | .9358( 1) | .9669( 0) | 12304 |
| 88 | 5 | 18 | 2 | .6220 | .9351 | .9210( 1) | .9461( 1) | .9723( 0) | 01243 |
| 89 | 9 | 10 | 1 | .6000 | .8785 | .8667( 2) | .9075( 1) | .9518( 0) | 12345 |
| 89 | 8 | 12 | 7 | .5753 | .8909 | .8783( 2) | .9163( 1) | .9566( 0) | 01723 |
| 89 | 7 | 13 | 2 | .6159 | .9063 | .8935( 1) | .9267( 1) | .9621( 0) | 01234 |
| 89 | 6 | 15 | 1 | .6109 | .9206 | .9072( 1) | .9365( 1) | .9673( 0) | 12340 |
| 89 | 5 | 18 | 1 | .6228 | .9358 | .9218( 1) | .9467( 0) | .9726( 0) | 01234 |
| 90 | 9 | 10 | 0 | .6000 | .8795 | .8677( 2) | .9083( 1) | .9521( 0) | 01234 |
| 90 | 8 | 12 | 6 | .5728 | .8920 | .8795( 2) | .9171( 1) | .9571( 0) | 01273 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}$(0) | $\bar{E}$(1) | $\bar{E}$(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 7 | 13 | 1 | .6239 | .9074 | .8946( 1) | .9275( 1) | .9625( 0) | 01234 |
| 90 | 6 | 15 | 0 | .6122 | .9214 | .9081( 1) | .9371( 0) | .9676( 0) | 01234 |
| 90 | 5 | 18 | 0 | .6296 | .9364 | .9227( 1) | .9473( 0) | .9729( 0) | 01234 |
| 91 | 9 | 11 | 8 | .5683 | .8797 | .8678( 2) | .9087( 1) | .9526( 0) | 01823 |
| 91 | 8 | 12 | 5 | .5747 | .8932 | .8807( 2) | .9180( 1) | .9575( 0) | 10237 |
| 91 | 7 | 13 | 0 | .6410 | .9084 | .8958( 1) | .9282( 0) | .9629( 0) | 01234 |
| 91 | 6 | 16 | 5 | .5955 | .9223 | .9091( 1) | .9378( 1) | .9680( 0) | 04512 |
| 91 | 5 | 19 | 4 | .6296 | .9371 | .9235( 1) | .9479( 0) | .9732( 0) | 04123 |
| 92 | 9 | 11 | 7 | .5678 | .8809 | .8691( 2) | .9096( 2) | .9531( 0) | 01283 |
| 92 | 8 | 12 | 4 | .5771 | .8942 | .8819( 2) | .9188( 1) | .9580( 0) | 12034 |
| 92 | 7 | 14 | 6 | .6410 | .9094 | .8970( 1) | .9290( 1) | .9633( 0) | 06123 |
| 92 | 6 | 16 | 4 | .5969 | .9231 | .9100( 1) | .9384( 1) | .9683( 0) | 05142 |
| 92 | 5 | 19 | 3 | .6296 | .9378 | .9243( 1) | .9484( 0) | .9735( 0) | 01432 |
| 93 | 9 | 11 | 6 | .5677 | .8821 | .8703( 2) | .9105( 2) | .9537( 0) | 10238 |
| 93 | 8 | 12 | 3 | .5698 | .8953 | .8831( 2) | .9196( 1) | .9584( 0) | 12304 |
| 93 | 7 | 14 | 5 | .6410 | .9104 | .8981( 1) | .9298( 1) | .9637( 0) | 01623 |
| 93 | 6 | 16 | 3 | .6002 | .9239 | .9109( 1) | .9391( 1) | .9687( 0) | 01524 |
| 93 | 5 | 19 | 2 | .6334 | .9385 | .9251( 1) | .9490( 1) | .9738( 0) | 01243 |
| 94 | 9 | 11 | 5 | .5696 | .8832 | .8716( 2) | .9114( 2) | .9542( 0) | 12034 |
| 94 | 8 | 12 | 2 | .5673 | .8963 | .8842( 2) | .9204( 1) | .9588( 0) | 12340 |
| 94 | 7 | 14 | 4 | .6410 | .9114 | .8992( 1) | .9306( 1) | .9641( 0) | 01263 |
| 94 | 6 | 16 | 2 | .5980 | .9247 | .9118( 1) | .9397( 1) | .9690( 0) | 01253 |
| 94 | 5 | 19 | 1 | .6391 | .9391 | .9259( 0) | .9495( 0) | .9741( 0) | 01234 |
| 95 | 9 | 11 | 4 | .5713 | .8844 | .8728( 2) | .9123( 1) | .9546( 1) | 12304 |
| 95 | 8 | 12 | 1 | .5684 | .8973 | .8853( 2) | .9212( 1) | .9593( 0) | 12345 |
| 95 | 7 | 14 | 3 | .6410 | .9124 | .9003( 1) | .9313( 1) | .9644( 0) | 01236 |
| 95 | 6 | 16 | 1 | .5989 | .9254 | .9127( 1) | .9403( 1) | .9693( 0) | 01234 |
| 95 | 5 | 19 | 0 | .6491 | .9398 | .9266( 0) | .9500( 0) | .9743( 0) | 01234 |
| 96 | 9 | 11 | 3 | .5697 | .8855 | .8740( 2) | .9131( 2) | .9550( 1) | 12340 |
| 96 | 8 | 12 | 0 | .5718 | .8983 | .8864( 1) | .9220( 1) | .9597( 0) | 01234 |
| 96 | 7 | 14 | 2 | .6410 | .9133 | .9013( 1) | .9321( 1) | .9648( 0) | 01234 |
| 96 | 6 | 16 | 0 | .6042 | .9262 | .9136( 1) | .9409( 0) | .9696( 0) | 01234 |
| 96 | 5 | 20 | 4 | .6491 | .9404 | .9275( 0) | .9506( 0) | .9746( 0) | 04123 |
| 97 | 9 | 11 | 2 | .5712 | .8866 | .8752( 2) | .9140( 2) | .9554( 1) | 12345 |
| 97 | 8 | 13 | 7 | .5864 | .8994 | .8876( 2) | .9228( 1) | .9602( 0) | 05712 |
| 97 | 7 | 14 | 1 | .6447 | .9142 | .9023( 0) | .9328( 0) | .9652( 0) | 01234 |
| 97 | 6 | 17 | 5 | .6140 | .9270 | .9145( 1) | .9415( 1) | .9699( 0) | 01234 |
| 97 | 5 | 20 | 3 | .6491 | .9411 | .9282( 1) | .9511( 0) | .9749( 0) | 01432 |
| 98 | 9 | 11 | 1 | .5709 | .8877 | .8764( 2) | .9148( 1) | .9558( 0) | 12345 |
| 98 | 8 | 13 | 6 | .5872 | .9004 | .8887( 2) | .9235( 1) | .9606( 0) | 01567 |
| 98 | 7 | 14 | 0 | .6667 | .9151 | .9032( 0) | .9333( 0) | .9655( 0) | 01234 |
| 98 | 6 | 17 | 4 | .6160 | .9277 | .9153( 1) | .9421( 1) | .9702( 0) | 10234 |
| 98 | 5 | 20 | 2 | .6491 | .9417 | .9289( 0) | .9515( 0) | .9752( 0) | 01243 |
| 99 | 9 | 11 | 0 | .5767 | .8887 | .8775( 1) | .9156( 1) | .9562( 0) | 01234 |
| 99 | 8 | 13 | 5 | .5876 | .9014 | .8898( 2) | .9243( 1) | .9610( 0) | 01725 |
| 99 | 7 | 15 | 6 | .6286 | .9158 | .9041( 0) | .9341( 0) | .9659( 0) | 05612 |
| 99 | 6 | 17 | 3 | .6202 | .9284 | .9162( 1) | .9427( 1) | .9705( 0) | 12034 |
| 99 | 5 | 20 | 1 | .6526 | .9423 | .9296( 0) | .9519( 0) | .9754( 0) | 01234 |
| 100 | 9 | 12 | 8 | .5483 | .8897 | .8786( 2) | .9164( 1) | .9567( 0) | 06812 |
| 100 | 8 | 13 | 4 | .5885 | .9023 | .8908( 2) | .9250( 1) | .9614( 0) | 01273 |
| 100 | 7 | 15 | 5 | .6222 | .9165 | .9049( 1) | .9346( 1) | .9662( 0) | 06152 |
| 100 | 6 | 17 | 2 | .6197 | .9292 | .9170( 1) | .9433( 1) | .9708( 0) | 12304 |
| 100 | 5 | 20 | 0 | .6667 | .9429 | .9302( 0) | .9524( 0) | .9756( 0) | 01234 |

TABLE D.3

PROPERTIES OF ≃(0,1,2)-DESIGNS , R=4 , 5≤V≤100

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 5 | 2 | 3 | 1 | .5833 | .7107 | .5833( 3) | | .7517( 3) | 01 |
| 6 | 2 | 3 | 0 | .6667 | .7692 | .6667( 0) | | .8000( 0) | 01 |
| 7 | 2 | 4 | 1 | .7083 | .8132 | .7083( 0) | | .8338( 1) | 01 |
| 8 | 2 | 4 | 0 | .7500 | .8400 | .7500( 0) | | .8571( 0) | 01 |
| 10 | 3 | 4 | 2 | .4908 | .7400 | .6713( 2) | .7257( 5) | .8373( 3) | 021 |
| 11 | 3 | 4 | 1 | .5121 | .7673 | .7036( 4) | .7523( 4) | .8530( 2) | 012 |
| 12 | 3 | 4 | 0 | .5625 | .7891 | .7313( 0) | .7748( 0) | .8667( 0) | 012 |
| 13 | 3 | 5 | 2 | .6250 | .8052 | .7447( 4) | .7970( 3) | .8607( 2) | 021 |
| 14 | 3 | 5 | 1 | .6500 | .8209 | .7641( 3) | .8127( 2) | .8704( 3) | 012 |
| 15 | 3 | 5 | 0 | .7000 | .8336 | .7805( 1) | .8262( 0) | .8791( 1) | 012 |
| 16 | 3 | 6 | 2 | .7000 | .8457 | .7936( 2) | .8380( 2) | .8869( 2) | 201 |
| 17 | 4 | 5 | 3 | .6250 | .7897 | .7462( 4) | .7988( 2) | .8691( 1) | 0231 |
| 17 | 3 | 6 | 1 | .7167 | .8558 | .8067( 2) | .8482( 1) | .8938( 1) | 012 |
| 18 | 4 | 5 | 2 | .6326 | .8008 | .7589( 3) | .8088( 3) | .8722( 1) | 0132 |
| 18 | 3 | 6 | 0 | .7500 | .8644 | .8182( 0) | .8571( 0) | .9000( 0) | 012 |
| 19 | 4 | 5 | 1 | .6181 | .8103 | .7696( 3) | .8178( 3) | .8771( 1) | 0123 |
| 19 | 3 | 7 | 2 | .6250 | .8657 | .8165( 2) | .8578( 2) | .9020( 3) | 102 |
| 20 | 4 | 5 | 0 | .6000 | .8186 | .7788( 2) | .8264( 2) | .8813( 1) | 0123 |
| 20 | 3 | 7 | 1 | .6429 | .8724 | .8253( 2) | .8645( 2) | .9067( 2) | 120 |
| 21 | 4 | 6 | 3 | .6048 | .8236 | .7819( 4) | .8318( 3) | .8836( 2) | 0213 |
| 21 | 3 | 7 | 0 | .6786 | .8782 | .8330( 1) | .8708( 1) | .9109( 1) | 012 |
| 22 | 4 | 6 | 2 | .6170 | .8318 | .7920( 4) | .8388( 3) | .8891( 2) | 0123 |
| 22 | 3 | 8 | 2 | .6786 | .8843 | .8407( 2) | .8766( 1) | .9150( 2) | 201 |
| 23 | 4 | 6 | 1 | .6319 | .8391 | .8010( 4) | .8453( 2) | .8939( 2) | 0213 |
| 23 | 3 | 8 | 1 | .6920 | .8896 | .8478( 1) | .8820( 1) | .9187( 1) | 012 |
| 24 | 4 | 6 | 0 | .6667 | .8456 | .8089( 2) | .8516( 1) | .8980( 0) | 0123 |
| 24 | 3 | 8 | 0 | .7187 | .8943 | .8542( 0) | .8870( 0) | .9221( 1) | 012 |
| 25 | 4 | 7 | 3 | .6667 | .8521 | .8159( 4) | .8580( 2) | .9009( 1) | 0312 |
| 25 | 3 | 9 | 2 | .7187 | .8990 | .8602( 1) | .8917( 1) | .9253( 1) | 201 |
| 26 | 5 | 6 | 4 | .6500 | .8232 | .7943( 2) | .8378( 1) | .8879( 0) | 01234 |
| 26 | 4 | 7 | 2 | .6732 | .8580 | .8232( 3) | .8631( 2) | .9048( 1) | 0132 |
| 26 | 3 | 9 | 1 | .7292 | .9032 | .8658( 1) | .8960( 1) | .9283( 1) | 012 |
| 27 | 5 | 6 | 3 | .6250 | .8278 | .7986( 2) | .8421( 2) | .8895( 1) | 12304 |
| 27 | 4 | 7 | 1 | .6896 | .8633 | .8299( 3) | .8680( 2) | .9084( 1) | 0123 |
| 27 | 3 | 9 | 0 | .7500 | .9070 | .8710( 0) | .9000( 0) | .9310( 0) | 012 |
| 28 | 5 | 6 | 2 | .6250 | .8319 | .8026( 3) | .8463( 2) | .8911( 1) | 23014 |
| 28 | 4 | 7 | 0 | .7143 | .8682 | .8359( 1) | .8726( 1) | .9116( 1) | 0123 |
| 29 | 5 | 6 | 1 | .6250 | .8357 | .8061( 3) | .8504( 2) | .8927( 1) | 30124 |
| 29 | 4 | 8 | 3 | .7143 | .8731 | .8416( 2) | .8771( 1) | .9148( 1) | 0312 |
| 30 | 5 | 6 | 0 | .6250 | .8392 | .8092( 3) | .8544( 2) | .8944( 0) | 01234 |
| 30 | 4 | 8 | 2 | .7143 | .8777 | .8472( 2) | .8812( 1) | .9178( 1) | 0132 |
| 31 | 5 | 7 | 4 | .6163 | .8439 | .8146( 4) | .8581( 2) | .8987( 1) | 04123 |
| 31 | 4 | 8 | 1 | .7232 | .8819 | .8523( 1) | .8852( 1) | .9205( 1) | 0123 |
| 32 | 5 | 7 | 3 | .6184 | .8482 | .8197( 4) | .8617( 2) | .9021( 2) | 01423 |
| 32 | 4 | 8 | 0 | .7500 | .8857 | .8571( 0) | .8889( 0) | .9231( 0) | 0123 |
| 33 | 5 | 7 | 2 | .6271 | .8523 | .8243( 4) | .8652( 2) | .9050( 1) | 01243 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 33 | 4 | 9 | 3 | .6875 | .8886 | .8602( 1) | .8915( 1) | .9251( 1) | 0123 |
| 34 | 5 | 7 | 1 | .6355 | .8560 | .8286( 3) | .8686( 2) | .9075( 1) | 01234 |
| 34 | 4 | 9 | 2 | .6667 | .8912 | .8630( 2) | .8941( 1) | .9269( 1) | 0123 |
| 35 | 5 | 7 | 0 | .6429 | .8594 | .8325( 3) | .8718( 1) | .9098( 0) | 01234 |
| 35 | 4 | 9 | 1 | .6667 | .8936 | .8656( 2) | .8967( 1) | .9285( 1) | 0123 |
| 36 | 5 | 8 | 4 | .6429 | .8625 | .8364( 3) | .8737( 2) | .9134( 1) | 01234 |
| 36 | 4 | 9 | 0 | .6667 | .8958 | .8679( 2) | .8991( 1) | .9300( 1) | 0123 |
| 37 | 6 | 7 | 5 | .6667 | .8400 | .8171( 3) | .8578( 2) | .9004( 1) | 01352 |
| 37 | 5 | 8 | 3 | .6464 | .8663 | .8409( 3) | .8770( 2) | .9158( 1) | 10234 |
| 37 | 4 | 10 | 3 | .6631 | .8980 | .8707( 2) | .9009( 2) | .9318( 1) | 3021 |
| 38 | 6 | 7 | 4 | .6667 | .8438 | .8212( 3) | .8610( 2) | .9030( 1) | 01234 |
| 38 | 5 | 8 | 2 | .6501 | .8699 | .8451( 2) | .8802( 2) | .9181( 1) | 12034 |
| 38 | 4 | 10 | 2 | .6726 | .9007 | .8740( 2) | .9034( 2) | .9335( 1) | 0312 |
| 39 | 6 | 7 | 3 | .6667 | .8474 | .8251( 3) | .8642( 2) | .9047( 1) | 10235 |
| 39 | 5 | 8 | 1 | .6618 | .8733 | .8490( 2) | .8832( 2) | .9203( 1) | 12304 |
| 39 | 4 | 10 | 1 | .6686 | .9032 | .8770( 1) | .9057( 1) | .9352( 1) | 0132 |
| 40 | 6 | 7 | 2 | .6520 | .8507 | .8288( 3) | .8672( 2) | .9066( 1) | 01234 |
| 40 | 5 | 8 | 0 | .6875 | .8765 | .8528( 1) | .8862( 1) | .9223( 0) | 01234 |
| 40 | 4 | 10 | 0 | .6793 | .9055 | .8799( 1) | .9079( 1) | .9368( 0) | 0123 |
| 41 | 6 | 7 | 1 | .6429 | .8539 | .8322( 3) | .8702( 2) | .9081( 1) | 13024 |
| 41 | 5 | 9 | 4 | .6875 | .8796 | .8561( 2) | .8892( 1) | .9233( 1) | 04123 |
| 41 | 4 | 11 | 3 | .6723 | .9078 | .8828( 1) | .9099( 1) | .9384( 1) | 0231 |
| 42 | 6 | 7 | 0 | .6428 | .8569 | .8355( 2) | .8730( 1) | .9095( 0) | 01234 |
| 42 | 5 | 9 | 3 | .6875 | .8825 | .8595( 2) | .8918( 2) | .9253( 1) | 01423 |
| 42 | 4 | 11 | 2 | .6805 | .9100 | .8855( 1) | .9120( 1) | .9399( 1) | 0132 |
| 43 | 6 | 8 | 5 | .6192 | .8585 | .8369( 3) | .8746( 2) | .9123( 1) | 01523 |
| 43 | 5 | 9 | 2 | .6911 | .8853 | .8628( 2) | .8942( 2) | .9271( 1) | 01243 |
| 43 | 4 | 11 | 1 | .6731 | .9121 | .8881( 1) | .9140( 1) | .9412( 1) | 0123 |
| 44 | 6 | 8 | 4 | .6235 | .8615 | .8402( 3) | .8771( 2) | .9142( 1) | 01253 |
| 44 | 5 | 9 | 1 | .7023 | .8879 | .8660( 2) | .8966( 1) | .9287( 1) | 01234 |
| 44 | 4 | 11 | 0 | .6818 | .9140 | .8905( 1) | .9159( 1) | .9425( 0) | 0123 |
| 45 | 6 | 8 | 3 | .6277 | .8643 | .8433( 3) | .8796( 2) | .9160( 1) | 10235 |
| 45 | 5 | 9 | 0 | .7222 | .8904 | .8689( 1) | .8989( 0) | .9303( 0) | 01234 |
| 45 | 4 | 12 | 3 | .7273 | .9161 | .8931( 1) | .9179( 1) | .9438( 0) | 0123 |
| 46 | 6 | 8 | 2 | .6215 | .8670 | .8463( 3) | .8820( 2) | .9177( 1) | 12034 |
| 46 | 5 | 10 | 4 | .7222 | .8929 | .8718( 1) | .9011( 1) | .9319( 1) | 04123 |
| 46 | 4 | 12 | 2 | .7273 | .9180 | .8955( 1) | .9197( 0) | .9450( 1) | 0123 |
| 47 | 6 | 8 | 1 | .6193 | .8695 | .8492( 3) | .8843( 2) | .9193( 1) | 12304 |
| 47 | 5 | 10 | 3 | .7222 | .8953 | .8746( 1) | .9032( 1) | .9334( 1) | 01423 |
| 47 | 4 | 12 | 1 | .7330 | .9199 | .8978( 1) | .9214( 0) | .9462( 0) | 0123 |
| 48 | 6 | 8 | 0 | .6250 | .8719 | .8519( 2) | .8865( 1) | .9208( 0) | 01234 |
| 48 | 5 | 10 | 2 | .7222 | .8976 | .8773( 1) | .9053( 1) | .9348( 1) | 01243 |
| 48 | 4 | 12 | 0 | .7500 | .9216 | .9000( 0) | .9231( 0) | .9474( 0) | 0123 |
| 49 | 6 | 9 | 5 | .6277 | .8744 | .8547( 2) | .8886( 2) | .9226( 1) | 02341 |
| 49 | 5 | 10 | 1 | .7278 | .8998 | .8799( 1) | .9072( 1) | .9362( 0) | 01234 |
| 49 | 4 | 13 | 3 | .7067 | .9230 | .9016( 0) | .9244( 0) | .9483( 0) | 0123 |
| 50 | 7 | 8 | 6 | .6696 | .8593 | .8423( 1) | .8770( 1) | .9162( 0) | 01243 |
| 50 | 6 | 9 | 4 | .6330 | .8768 | .8574( 3) | .8906( 2) | .9242( 1) | 34012 |
| 50 | 5 | 10 | 0 | .7500 | .9018 | .8824( 0) | .9091( 0) | .9375( 0) | 01234 |
| 50 | 4 | 13 | 2 | .6923 | .9243 | .9032( 1) | .9257( 0) | .9492( 0) | 0123 |
| 51 | 7 | 8 | 5 | .6562 | .8614 | .8444( 2) | .8788( 1) | .9172( 0) | 12034 |
| 51 | 6 | 9 | 3 | .6407 | .8791 | .8599( 2) | .8925( 2) | .9256( 1) | 40235 |
| 51 | 5 | 11 | 4 | .6955 | .9035 | .8841( 1) | .9106( 0) | .9386( 0) | 01234 |
| 51 | 4 | 13 | 1 | .6923 | .9255 | .9046( 1) | .9270( 0) | .9501( 0) | 0123 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 52 | 7 | 8 | 4 | .6562 | .8634 | .8465( 2) | .8806( 1) | .9179( 0) | 21340 |
| 52 | 6 | 9 | 2 | .6467 | .8813 | .8624( 2) | .8944( 2) | .9270( 1) | 03451 |
| 52 | 5 | 11 | 3 | .6818 | .9050 | .8858( 1) | .9121( 1) | .9396( 1) | 12304 |
| 52 | 4 | 13 | 0 | .6923 | .9267 | .9060( 1) | .9283( 0) | .9509( 0) | 0123 |
| 53 | 7 | 8 | 3 | .6562 | .8653 | .8484( 2) | .8824( 1) | .9185( 0) | 23401 |
| 53 | 6 | 9 | 1 | .6549 | .8833 | .8648( 2) | .8963( 1) | .9283( 1) | 04123 |
| 53 | 5 | 11 | 2 | .6818 | .9064 | .8875( 1) | .9135( 1) | .9406( 1) | 23014 |
| 53 | 4 | 14 | 3 | .6938 | .9279 | .9074( 1) | .9293( 1) | .9518( 1) | 0312 |
| 54 | 7 | 8 | 2 | .6562 | .8671 | .8503( 2) | .8841( 1) | .9192( 0) | 41235 |
| 54 | 6 | 9 | 0 | .6667 | .8853 | .8670( 1) | .8981( 1) | .9295( 0) | 01234 |
| 54 | 5 | 11 | 1 | .6818 | .9078 | .8890( 1) | .9149( 1) | .9415( 1) | 30124 |
| 54 | 4 | 14 | 2 | .6978 | .9292 | .9091( 1) | .9306( 1) | .9526( 1) | 0132 |
| 55 | 7 | 8 | 1 | .6562 | .8688 | .8521( 2) | .8859( 1) | .9198( 0) | 24501 |
| 55 | 6 | 10 | 5 | .6667 | .8874 | .8694( 2) | .8998( 1) | .9308( 0) | 02341 |
| 55 | 5 | 11 | 0 | .6818 | .9092 | .8904( 1) | .9162( 0) | .9423( 0) | 01234 |
| 55 | 4 | 14 | 1 | .7027 | .9304 | .9106( 1) | .9318( 1) | .9535( 0) | 0123 |
| 56 | 7 | 8 | 0 | .6562 | .8705 | .8538( 2) | .8875( 1) | .9205( 0) | 01234 |
| 56 | 6 | 10 | 4 | .6667 | .8894 | .8716( 2) | .9015( 1) | .9320( 1) | 34012 |
| 56 | 5 | 12 | 4 | .6745 | .9101 | .8916( 1) | .9168( 1) | .9434( 1) | 01234 |
| 56 | 4 | 14 | 0 | .7143 | .9316 | .9121( 1) | .9329( 0) | .9543( 0) | 0123 |
| 57 | 7 | 9 | 6 | .6416 | .8725 | .8559( 3) | .8892( 1) | .9222( 1) | 01243 |
| 57 | 6 | 10 | 3 | .6694 | .8913 | .8738( 2) | .9032( 2) | .9331( 1) | 40235 |
| 57 | 5 | 12 | 3 | .6738 | .9117 | .8934( 1) | .9182( 1) | .9444( 1) | 12304 |
| 57 | 4 | 15 | 3 | .7143 | .9328 | .9137( 1) | .9341( 1) | .9551( 0) | 0312 |
| 58 | 7 | 9 | 5 | .6398 | .8744 | .8580( 3) | .8908( 2) | .9236( 1) | 12034 |
| 58 | 6 | 10 | 2 | .6767 | .8931 | .8759( 2) | .9048( 1) | .9342( 1) | 03451 |
| 58 | 5 | 12 | 2 | .6756 | .9132 | .8952( 1) | .9196( 1) | .9454( 1) | 23014 |
| 58 | 4 | 15 | 2 | .7168 | .9340 | .9151( 1) | .9352( 1) | .9558( 0) | 0132 |
| 59 | 7 | 9 | 4 | .6421 | .8762 | .8600( 3) | .8924( 2) | .9249( 1) | 21340 |
| 59 | 6 | 10 | 1 | .6839 | .8949 | .8779( 2) | .9064( 1) | .9353( 1) | 04123 |
| 59 | 5 | 12 | 1 | .6803 | .9146 | .8969( 1) | .9209( 1) | .9462( 1) | 30124 |
| 59 | 4 | 15 | 1 | .7233 | .9351 | .9165( 1) | .9363( 0) | .9566( 0) | 0123 |
| 60 | 7 | 9 | 3 | .6430 | .8779 | .8619( 3) | .8939( 2) | .9260( 1) | 23401 |
| 60 | 6 | 10 | 0 | .7000 | .8966 | .8799( 1) | .9079( 1) | .9363( 0) | 01234 |
| 60 | 5 | 12 | 0 | .6875 | .9160 | .8986( 1) | .9223( 0) | .9471( 0) | 01234 |
| 60 | 4 | 15 | 0 | .7333 | .9362 | .9179( 0) | .9373( 0) | .9573( 0) | 0123 |
| 61 | 7 | 9 | 2 | .6452 | .8796 | .8637( 3) | .8954( 2) | .9270( 1) | 41235 |
| 61 | 6 | 11 | 5 | .7000 | .8983 | .8818( 2) | .9094( 1) | .9374( 1) | 01234 |
| 61 | 5 | 13 | 4 | .6875 | .9174 | .9001( 1) | .9235( 1) | .9479( 0) | 01234 |
| 61 | 4 | 16 | 3 | .7333 | .9373 | .9193( 0) | .9383( 0) | .9580( 0) | 0312 |
| 62 | 7 | 9 | 1 | .6504 | .8812 | .8655( 2) | .8969( 2) | .9280( 1) | 24501 |
| 62 | 6 | 11 | 4 | .7000 | .9000 | .8838( 2) | .9108( 1) | .9385( 1) | 10234 |
| 62 | 5 | 13 | 3 | .6913 | .9187 | .9017( 1) | .9246( 1) | .9487( 1) | 12304 |
| 62 | 4 | 16 | 2 | .7333 | .9383 | .9206( 1) | .9394( 0) | .9587( 0) | 0132 |
| 63 | 7 | 9 | 0 | .6552 | .8828 | .8672( 2) | .8984( 1) | .9289( 0) | 01234 |
| 63 | 6 | 11 | 3 | .7000 | .9016 | .8856( 2) | .9122( 1) | .9395( 1) | 12034 |
| 63 | 5 | 13 | 2 | .6958 | .9199 | .9031( 1) | .9258( 1) | .9495( 1) | 23014 |
| 63 | 4 | 16 | 1 | .7375 | .9393 | .9219( 0) | .9403( 0) | .9594( 0) | 0123 |
| 64 | 7 | 10 | 6 | .6552 | .8845 | .8691( 2) | .8997( 2) | .9305( 1) | 01243 |
| 64 | 6 | 11 | 2 | .7021 | .9032 | .8874( 1) | .9136( 1) | .9405( 1) | 12304 |
| 64 | 5 | 13 | 1 | .7013 | .9211 | .9045( 1) | .9269( 1) | .9502( 0) | 30124 |
| 64 | 4 | 16 | 0 | .7500 | .9403 | .9231( 0) | .9412( 0) | .9600( 0) | 0123 |
| 65 | 8 | 9 | 7 | .6875 | .8710 | .8569( 2) | .8894( 1) | .9225( 1) | 05671 |
| 65 | 7 | 10 | 5 | .6554 | .8862 | .8710( 3) | .9011( 2) | .9317( 1) | 12034 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}(0)$ | $\bar{E}(1)$ | $\bar{E}(2)$ | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 65 | 6 | 11 | 1 | .7107 | .9047 | .8892( 1) | .9149( 1) | .9415( 1) | 12340 |
| 65 | 5 | 13 | 0 | .7115 | .9222 | .9059( 1) | .9280( 0) | .9509( 0) | 01234 |
| 66 | 8 | 9 | 6 | .6854 | .8727 | .8588( 2) | .8908( 2) | .9239( 1) | 06715 |
| 66 | 7 | 10 | 4 | .6588 | .8878 | .8727( 3) | .9024( 2) | .9327( 1) | 21340 |
| 66 | 6 | 11 | 0 | .7273 | .9061 | .8908( 1) | .9161( 0) | .9424( 0) | 01234 |
| 66 | 5 | 14 | 4 | .7115 | .9234 | .9073( 1) | .9291( 1) | .9516( 0) | 04123 |
| 67 | 8 | 9 | 5 | .6854 | .8744 | .8606( 2) | .8923( 2) | .9250( 1) | 07162 |
| 67 | 7 | 10 | 3 | .6622 | .8893 | .8745( 3) | .9037( 2) | .9336( 1) | 23401 |
| 67 | 6 | 12 | 5 | .7273 | .9076 | .8925( 1) | .9174( 1) | .9433( 0) | 01234 |
| 67 | 5 | 14 | 3 | .7115 | .9246 | .9087( 1) | .9302( 1) | .9523( 1) | 01423 |
| 68 | 8 | 9 | 4 | .6854 | .8761 | .8624( 2) | .8937( 2) | .9258( 1) | 01726 |
| 68 | 7 | 10 | 2 | .6671 | .8908 | .8761( 2) | .9050( 2) | .9346( 1) | 41235 |
| 68 | 6 | 12 | 4 | .7273 | .9090 | .8941( 1) | .9186( 1) | .9442( 1) | 10234 |
| 68 | 5 | 14 | 2 | .7136 | .9257 | .9100( 1) | .9312( 1) | .9530( 1) | 01243 |
| 69 | 8 | 9 | 3 | .6721 | .8777 | .8641( 2) | .8951( 1) | .9269( 1) | 01273 |
| 69 | 7 | 10 | 1 | .6729 | .8922 | .8777( 2) | .9063( 1) | .9354( 1) | 24501 |
| 69 | 6 | 12 | 3 | .7273 | .9103 | .8957( 1) | .9198( 1) | .9450( 1) | 12034 |
| 69 | 5 | 14 | 1 | .7203 | .9268 | .9113( 1) | .9322( 0) | .9537( 0) | 01234 |
| 70 | 8 | 9 | 2 | .6667 | .8792 | .8658( 2) | .8964( 1) | .9278( 1) | 01237 |
| 70 | 7 | 10 | 0 | .6802 | .8936 | .8793( 1) | .9075( 1) | .9362( 0) | 01234 |
| 70 | 6 | 12 | 2 | .7273 | .9117 | .8972( 1) | .9210( 1) | .9458( 1) | 12304 |
| 70 | 5 | 14 | 0 | .7321 | .9278 | .9125( 0) | .9331( 0) | .9544( 0) | 01234 |
| 71 | 8 | 9 | 1 | .6667 | .8807 | .8674( 1) | .8978( 1) | .9286( 1) | 01234 |
| 71 | 7 | 11 | 6 | .6580 | .8950 | .8807( 2) | .9088( 1) | .9369( 1) | 01243 |
| 71 | 6 | 12 | 1 | .7311 | .9129 | .8987( 1) | .9221( 0) | .9466( 0) | 12340 |
| 71 | 5 | 15 | 4 | .7321 | .9289 | .9138( 1) | .9341( 0) | .9550( 0) | 04123 |
| 72 | 8 | 9 | 0 | .6667 | .8821 | .8690( 1) | .8991( 1) | .9293( 0) | 01234 |
| 72 | 7 | 11 | 5 | .6591 | .8964 | .8823( 2) | .9100( 1) | .9379( 1) | 12034 |
| 72 | 6 | 12 | 0 | .7500 | .9142 | .9000( 0) | .9231( 0) | .9474( 0) | 01234 |
| 72 | 5 | 15 | 3 | .7321 | .9299 | .9150( 1) | .9350( 0) | .9557( 0) | 01423 |
| 73 | 8 | 10 | 7 | .6562 | .8835 | .8704( 2) | .9002( 1) | .9305( 1) | 03451 |
| 73 | 7 | 11 | 4 | .6620 | .8978 | .8839( 2) | .9112( 1) | .9387( 1) | 21340 |
| 73 | 6 | 13 | 5 | .7019 | .9152 | .9012( 0) | .9241( 0) | .9480( 0) | 01234 |
| 73 | 5 | 15 | 2 | .7321 | .9309 | .9162( 1) | .9359( 0) | .9563( 0) | 01243 |
| 74 | 8 | 10 | 6 | .6538 | .8848 | .8719( 2) | .9014( 1) | .9316( 1) | 45013 |
| 74 | 7 | 11 | 3 | .6630 | .8991 | .8854( 2) | .9123( 1) | .9396( 1) | 23401 |
| 74 | 6 | 13 | 4 | .6923 | .9161 | .9023( 1) | .9250( 1) | .9486( 0) | 12304 |
| 74 | 5 | 15 | 1 | .7357 | .9318 | .9173( 0) | .9368( 0) | .9569( 0) | 01234 |
| 75 | 8 | 10 | 5 | .6558 | .8861 | .8733( 2) | .9025( 2) | .9325( 1) | 54601 |
| 75 | 7 | 11 | 2 | .6673 | .9004 | .8868( 2) | .9134( 1) | .9404( 1) | 41235 |
| 75 | 6 | 13 | 3 | .6923 | .9171 | .9033( 1) | .9259( 1) | .9492( 0) | 23140 |
| 75 | 5 | 15 | 0 | .7500 | .9328 | .9184( 0) | .9375( 0) | .9574( 0) | 01234 |
| 76 | 8 | 10 | 4 | .6516 | .8874 | .8746( 2) | .9036( 1) | .9333( 1) | 56034 |
| 76 | 7 | 11 | 1 | .6727 | .9017 | .8882( 1) | .9145( 1) | .9411( 0) | 24501 |
| 76 | 6 | 13 | 2 | .6923 | .9180 | .9043( 1) | .9267( 1) | .9498( 0) | 32401 |
| 76 | 5 | 16 | 4 | .7125 | .9336 | .9194( 0) | .9383( 0) | .9580( 0) | 01234 |
| 77 | 8 | 10 | 3 | .6508 | .8886 | .8759( 2) | .9047( 1) | .9341( 1) | 04567 |
| 77 | 7 | 11 | 0 | .6818 | .9029 | .8896( 1) | .9156( 1) | .9419( 0) | 01234 |
| 77 | 6 | 13 | 1 | .6923 | .9189 | .9053( 1) | .9275( 0) | .9503( 0) | 34012 |
| 77 | 5 | 16 | 3 | .7031 | .9343 | .9203( 1) | .9391( 0) | .9585( 0) | 12304 |
| 78 | 8 | 10 | 2 | .6495 | .8898 | .8772( 2) | .9058( 1) | .9348( 1) | 05146 |
| 78 | 7 | 12 | 6 | .6818 | .9041 | .8909( 1) | .9165( 1) | .9428( 0) | 01243 |
| 78 | 6 | 13 | 0 | .6923 | .9197 | .9062( 1) | .9283( 0) | .9508( 0) | 01234 |
| 78 | 5 | 16 | 2 | .7031 | .9350 | .9211( 1) | .9398( 0) | .9589( 0) | 23014 |

| V | S | K | P | E(MIN) | $\bar{E}$ | $\bar{E}(0)$ | $\bar{E}(1)$ | $\bar{E}(2)$ | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 79 | 8 | 10 | 1 | .6492 | .8910 | .8785( 2) | .9068( 1) | .9354( 0) | 01562 |
| 79 | 7 | 12 | 5 | .6818 | .9053 | .8923( 2) | .9176( 1) | .9435( 1) | 12034 |
| 79 | 6 | 14 | 5 | .6843 | .9206 | .9073( 1) | .9292( 1) | .9514( 0) | 01234 |
| 79 | 5 | 16 | 1 | .7031 | .9358 | .9219( 1) | .9404( 0) | .9594( 0) | 30124 |
| 80 | 8 | 10 | 0 | .6500 | .8921 | .8797( 2) | .9079( 1) | .9361( 0) | 01234 |
| 80 | 7 | 12 | 4 | .6838 | .9065 | .8936( 1) | .9186( 1) | .9442( 1) | 21340 |
| 80 | 6 | 14 | 4 | .6806 | .9215 | .9083( 1) | .9300( 1) | .9520( 0) | 12304 |
| 80 | 5 | 16 | 0 | .7031 | .9364 | .9227( 1) | .9410( 0) | .9599( 0) | 01234 |
| 81 | 8 | 11 | 7 | .6457 | .8929 | .8806( 2) | .9085( 1) | .9371( 1) | 01562 |
| 81 | 7 | 12 | 3 | .6864 | .9076 | .8949( 1) | .9196( 1) | .9449( 1) | 23401 |
| 81 | 6 | 14 | 3 | .6802 | .9224 | .9093( 1) | .9308( 1) | .9526( 1) | 23140 |
| 81 | 5 | 17 | 4 | .6989 | .9372 | .9236( 1) | .9418( 0) | .9603( 0) | 10234 |
| 82 | 9 | 10 | 8 | .6592 | .8818 | .8702( 1) | .9000( 1) | .9318( 0) | 02671 |
| 82 | 8 | 11 | 6 | .6457 | .8942 | .8819( 2) | .9096( 1) | .9379( 1) | 16025 |
| 82 | 7 | 12 | 2 | .6902 | .9087 | .8962( 1) | .9205( 1) | .9456( 1) | 41235 |
| 82 | 6 | 14 | 2 | .6789 | .9233 | .9103( 1) | .9315( 1) | .9531( 0) | 32401 |
| 82 | 5 | 17 | 3 | .6985 | .9379 | .9244( 1) | .9424( 1) | .9608( 0) | 12034 |
| 83 | 9 | 10 | 7 | .6576 | .8831 | .8717( 2) | .9011( 1) | .9327( 1) | 70123 |
| 83 | 8 | 11 | 5 | .6475 | .8954 | .8833( 2) | .9106( 1) | .9387( 1) | 01267 |
| 83 | 7 | 12 | 1 | .6953 | .9098 | .8974( 1) | .9215( 1) | .9462( 0) | 24501 |
| 83 | 6 | 14 | 1 | .6791 | .9241 | .9112( 1) | .9323( 1) | .9536( 0) | 34012 |
| 83 | 5 | 17 | 2 | .7011 | .9386 | .9253( 1) | .9431( 1) | .9613( 0) | 21304 |
| 84 | 9 | 10 | 6 | .6576 | .8844 | .8731( 2) | .9022( 1) | .9335( 1) | 02781 |
| 84 | 8 | 11 | 4 | .6492 | .8966 | .8846( 2) | .9116( 1) | .9394( 1) | 01236 |
| 84 | 7 | 12 | 0 | .7083 | .9109 | .8986( 1) | .9224( 1) | .9468( 0) | 01234 |
| 84 | 6 | 14 | 0 | .6786 | .9249 | .9121( 1) | .9330( 1) | .9541( 0) | 01234 |
| 84 | 5 | 17 | 1 | .7036 | .9393 | .9261( 1) | .9437( 0) | .9617( 0) | 23140 |
| 85 | 9 | 10 | 5 | .6576 | .8857 | .8744( 2) | .9033( 1) | .9343( 1) | 01237 |
| 85 | 8 | 11 | 3 | .6448 | .8977 | .8858( 2) | .9126( 1) | .9401( 1) | 10234 |
| 85 | 7 | 13 | 6 | .7083 | .9119 | .8998( 1) | .9233( 1) | .9476( 0) | 01243 |
| 85 | 6 | 15 | 5 | .6796 | .9257 | .9131( 1) | .9337( 1) | .9548( 0) | 01452 |
| 85 | 5 | 17 | 0 | .7059 | .9399 | .9268( 1) | .9443( 0) | .9621( 0) | 01234 |
| 86 | 9 | 10 | 4 | .6581 | .8869 | .8758( 2) | .9043( 1) | .9349( 1) | 01234 |
| 86 | 8 | 11 | 2 | .6432 | .8989 | .8871( 2) | .9135( 1) | .9408( 1) | 12035 |
| 86 | 7 | 13 | 5 | .7083 | .9130 | .9010( 1) | .9242( 1) | .9482( 1) | 12034 |
| 86 | 6 | 15 | 4 | .6800 | .9265 | .9140( 1) | .9344( 1) | .9553( 0) | 01524 |
| 86 | 5 | 18 | 4 | .7059 | .9406 | .9276( 1) | .9449( 0) | .9626( 0) | 01423 |
| 87 | 9 | 10 | 3 | .6611 | .8881 | .8771( 1) | .9054( 2) | .9355( 0) | 20137 |
| 87 | 8 | 11 | 1 | .6432 | .9000 | .8883( 2) | .9145( 1) | .9414( 1) | 12360 |
| 87 | 7 | 13 | 4 | .7083 | .9140 | .9021( 1) | .9250( 1) | .9488( 1) | 21340 |
| 87 | 6 | 15 | 3 | .6819 | .9273 | .9150( 1) | .9351( 1) | .9558( 0) | 01253 |
| 87 | 5 | 18 | 3 | .7084 | .9413 | .9284( 1) | .9455( 0) | .9630( 0) | 01243 |
| 88 | 9 | 10 | 2 | .6611 | .8893 | .8784( 2) | .9064( 1) | .9362( 0) | 23014 |
| 88 | 8 | 11 | 0 | .6472 | .9010 | .8895( 1) | .9154( 1) | .9420( 0) | 01234 |
| 88 | 7 | 13 | 3 | .7083 | .9150 | .9032( 1) | .9259( 1) | .9494( 1) | 23401 |
| 88 | 6 | 15 | 2 | .6827 | .9281 | .9159( 1) | .9358( 1) | .9563( 0) | 10234 |
| 88 | 5 | 18 | 2 | .7115 | .9419 | .9292( 1) | .9461( 0) | .9634( 0) | 10234 |
| 89 | 9 | 10 | 1 | .6618 | .8905 | .8796( 1) | .9074( 1) | .9367( 0) | 02347 |
| 89 | 8 | 12 | 7 | .6474 | .9021 | .8906( 1) | .9163( 1) | .9426( 1) | 02571 |
| 89 | 7 | 13 | 2 | .7115 | .9160 | .9043( 1) | .9267( 1) | .9499( 0) | 41235 |
| 89 | 6 | 15 | 1 | .6845 | .9289 | .9167( 1) | .9365( 1) | .9567( 0) | 12034 |
| 89 | 5 | 18 | 1 | .7152 | .9425 | .9300( 1) | .9467( 0) | .9638( 0) | 12034 |
| 90 | 9 | 10 | 0 | .6750 | .8916 | .8808( 1) | .9084( 1) | .9373( 0) | 01234 |
| 90 | 8 | 12 | 6 | .6494 | .9031 | .8917( 1) | .9172( 1) | .9433( 1) | 01235 |

| V | S | K | P | E(MIN) | Ē | Ē(0) | Ē(1) | Ē(2) | CONTROLS |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 7 | 13 | 1 | .7165 | .9169 | .9054( 1) | .9275( 1) | .9505( 0) | 24501 |
| 90 | 6 | 15 | 0 | .6862 | .9296 | .9176( 1) | .9372( 0) | .9572( 0) | 01234 |
| 90 | 5 | 18 | 0 | .7222 | .9431 | .9307( 0) | .9473( 0) | .9642( 0) | 01234 |
| 91 | 9 | 11 | 8 | .6570 | .8923 | .8815( 2) | .9090( 1) | .9372( 0) | 05681 |
| 91 | 8 | 12 | 5 | .6541 | .9041 | .8929( 2) | .9180( 1) | .9439( 1) | 01273 |
| 91 | 7 | 13 | 0 | .7308 | .9178 | .9064( 0) | .9283( 0) | .9510( 0) | 01234 |
| 91 | 6 | 16 | 5 | .6873 | .9304 | .9184( 1) | .9378( 1) | .9576( 0) | 01234 |
| 91 | 5 | 19 | 4 | .7222 | .9438 | .9315( 1) | .9479( 0) | .9646( 0) | 01234 |
| 92 | 9 | 11 | 7 | .6539 | .8933 | .8826( 2) | .9099( 1) | .9379( 1) | 06157 |
| 92 | 8 | 12 | 4 | .6607 | .9051 | .8940( 2) | .9189( 1) | .9444( 1) | 02135 |
| 92 | 7 | 14 | 6 | .7308 | .9187 | .9074( 1) | .9291( 0) | .9516( 0) | 06123 |
| 92 | 6 | 16 | 4 | .6878 | .9311 | .9193( 1) | .9385( 1) | .9581( 0) | 12304 |
| 92 | 5 | 19 | 3 | .7222 | .9444 | .9322( 1) | .9484( 0) | .9650( 0) | 10234 |
| 93 | 9 | 11 | 6 | .6541 | .8943 | .8837( 2) | .9108( 1) | .9386( 1) | 01678 |
| 93 | 8 | 12 | 3 | .6619 | .9061 | .8950( 2) | .9197( 1) | .9450( 1) | 01234 |
| 93 | 7 | 14 | 5 | .7308 | .9196 | .9085( 1) | .9299( 1) | .9521( 0) | 01623 |
| 93 | 6 | 16 | 3 | .6896 | .9318 | .9201( 1) | .9391( 1) | .9585( 0) | 23140 |
| 93 | 5 | 19 | 2 | .7237 | .9450 | .9329( 1) | .9490( 0) | .9654( 0) | 12034 |
| 94 | 9 | 11 | 5 | .6558 | .8954 | .8848( 2) | .9116( 1) | .9393( 1) | 08126 |
| 94 | 8 | 12 | 2 | .6637 | .9070 | .8961( 2) | .9205( 1) | .9455( 1) | 20134 |
| 94 | 7 | 14 | 4 | .7308 | .9205 | .9094( 1) | .9306( 1) | .9526( 0) | 01263 |
| 94 | 6 | 16 | 2 | .6911 | .9326 | .9210( 1) | .9398( 1) | .9590( 0) | 32401 |
| 94 | 5 | 19 | 1 | .7284 | .9456 | .9336( 0) | .9495( 0) | .9658( 0) | 12304 |
| 95 | 9 | 11 | 4 | .6487 | .8963 | .8859( 2) | .9125( 1) | .9399( 1) | 01823 |
| 95 | 8 | 12 | 1 | .6672 | .9079 | .8971( 1) | .9213( 1) | .9460( 0) | 02351 |
| 95 | 7 | 14 | 3 | .7308 | .9214 | .9104( 1) | .9313( 1) | .9531( 0) | 01236 |
| 95 | 6 | 16 | 1 | .6935 | .9332 | .9218( 1) | .9404( 1) | .9594( 0) | 34012 |
| 95 | 5 | 19 | 0 | .7368 | .9461 | .9343( 0) | .9500( 0) | .9662( 0) | 01234 |
| 96 | 9 | 11 | 3 | .6432 | .8973 | .8869( 2) | .9133( 1) | .9405( 1) | 01283 |
| 96 | 8 | 12 | 0 | .6695 | .9088 | .8981( 1) | .9221( 1) | .9465( 0) | 01234 |
| 96 | 7 | 14 | 2 | .7308 | .9222 | .9114( 1) | .9320( 1) | .9536( 0) | 01234 |
| 96 | 6 | 16 | 0 | .6959 | .9339 | .9226( 1) | .9410( 0) | .9598( 0) | 01234 |
| 96 | 5 | 20 | 4 | .7368 | .9467 | .9350( 0) | .9505( 0) | .9665( 0) | 04123 |
| 97 | 9 | 11 | 2 | .6410 | .8982 | .8879( 2) | .9141( 1) | .9411( 1) | 01235 |
| 97 | 8 | 13 | 7 | .6687 | .9097 | .8990( 1) | .9229( 1) | .9472( 0) | 02571 |
| 97 | 7 | 14 | 1 | .7335 | .9230 | .9123( 0) | .9327( 0) | .9541( 0) | 01234 |
| 97 | 6 | 17 | 5 | .6914 | .9346 | .9233( 1) | .9416( 0) | .9603( 0) | 01234 |
| 97 | 5 | 20 | 3 | .7368 | .9473 | .9357( 0) | .9511( 0) | .9668( 0) | 01423 |
| 98 | 9 | 11 | 1 | .6410 | .8992 | .8889( 2) | .9149( 1) | .9416( 0) | 01236 |
| 98 | 8 | 13 | 6 | .6710 | .9106 | .9000( 1) | .9236( 1) | .9477( 0) | 01235 |
| 98 | 7 | 14 | 0 | .7500 | .9238 | .9130( 0) | .9333( 0) | .9545( 0) | 01234 |
| 98 | 6 | 17 | 4 | .6936 | .9353 | .9241( 1) | .9422( 1) | .9607( 0) | 10234 |
| 98 | 5 | 20 | 2 | .7368 | .9478 | .9364( 0) | .9516( 0) | .9671( 0) | 01243 |
| 99 | 9 | 11 | 0 | .6432 | .9001 | .8899( 1) | .9157( 1) | .9421( 0) | 01234 |
| 99 | 8 | 13 | 5 | .6742 | .9115 | .9010( 1) | .9244( 1) | .9482( 1) | 01273 |
| 99 | 7 | 15 | 6 | .7071 | .9245 | .9139( 0) | .9341( 0) | .9550( 0) | 01243 |
| 99 | 6 | 17 | 3 | .6965 | .9359 | .9249( 1) | .9427( 1) | .9611( 0) | 12034 |
| 99 | 5 | 20 | 1 | .7395 | .9484 | .9371( 0) | .9521( 0) | .9675( 0) | 01234 |
| 100 | 9 | 12 | 8 | .6313 | .9010 | .8909( 1) | .9164( 1) | .9430( 0) | 05681 |
| 100 | 8 | 13 | 4 | .6770 | .9124 | .9019( 1) | .9251( 1) | .9487( 1) | 02135 |
| 100 | 7 | 15 | 5 | .7000 | .9251 | .9147( 1) | .9346( 0) | .9554( 0) | 12034 |
| 100 | 6 | 17 | 2 | .6992 | .9366 | .9256( 1) | .9433( 0) | .9615( 0) | 12304 |
| 100 | 5 | 20 | 0 | .7500 | .9489 | .9375( 0) | .9524( 0) | .9677( 0) | 01234 |