THE ENDOMORPHISM NEAR-RINGS OF THE SYMMETRIC GROUPS

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All the work in this thesis is my own , except where specific acknowledgement is made in the text . This thesis has been composed by myself and does not contain work submitted for any other degree or qualification .

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ABSTRACT

Let $(G,+)$ be a group , $Inn(G)$, $Aut(G)$ and $End(G)$ the semigroups of all inner automorphisms , automorphisms and endomorphisms of G respectively.These semigroups generate the d.g. near-rings $I(G)$, $A(G)$ and $E(G)$ respectively . This dissertation is mainly concerned with the detailed structure of $E(S_n)$ for $n \geq 4$ where S_n is the symmetric group on n symbols .

It is already known that $E(S_n) = A(S_n) = I(S_n)$ for $n \geq 5$ and some results about the structure of these near-rings have been determined (see J.D.P. Meldrum [15]) . In Part two of this dissertation , we determine the precise additive and multiplicative structure of these near-rings and list all right , left and two-sided ideals of $E(S_n)$ where $n \geq 5$. Besides we determine all the possible monogenic right $E(S_n)$-subgroups and left $E(S_n)$-subgroups.

The case $n = 4$ has not been studied before . In Part three , we determine the structure of $E(S_4)$ whose order is $2^{35}3^{3}$. Besides we determine the precise algebraic structure of this near-ring by writing down its precise tables of addition and multiplication and find its radical and all its maximal right ideals .

In Part four of this dissertation , we present a chapter on inverse semigroups of endomorphisms . Those newly established theorems , concerning the semi-direct decompositions of an arbitrary group G associated with idempotent endomorphisms of an inverse semigroup $S \subseteq End(G)$, are expected to be powerful tools in tackling the structure of endomorphism near-rings of an arbitrary group which is a direct sum of n copies of isomorphic finite groups .

CONTENTS

PART ONE

INTRODUCTION

It is a well-known fact that the set of all endomorphisms of an abelian group is a ring with identity . H. Fitting [6] was the first mathematician to investigate near-rings generated by endomorphisms in his study of normal endomorphisms of a non-abelian group. In 1958 , A. Fröhlich [7,8,9] laid the foundation stone in the study of distributively generated near-rings (in short , d.g. near-rings) . Some years later , J. J. Malone and his students emphasised a special class of d. g. near-rings , i.e. the endomorphism near-rings . Now , in this dissertation , it is our main purpose to investigate a particular class of endomorphism near-rings , i.e. the endomorphism near-rings of the symmetric groups . Here S_n denotes the symmetric group of degree n . In 1968 , C. G. Lyons [13] gave a full description of the structure of the endomorphism near-ring of S_3 . There was then a gap until 1977 when J. D. P. Meldrum [15] gave a beautiful result on the structure of morphism near-rings and we then know some detailed information about the structure of the endomorphism near-rings of S_n where $n \geq 5$. So the only gap that remains in this line is the endomorphism near-ring of S_4 . In Part three of this dissertation , we aim to give a full description of how to build up the algebraic structure of this monster , the endomorphism near-ring of S_4 , denoted by $E(S_4)$. Besides , we also study the radical of $E(S_4)$ and all its maximal right ideals . In Part two we are going to determine the exact algebraic structure of the endomorphism near-rings of S_n , denoted by $E(S_n)$, where $n \geq 5$ and the structure of their ideals and $E(S_n)$-subgroups . In Part four , we shall give a chapter on inverse semigroups of endomorphisms . In this chapter , those newly established theorems are expected to be powerful tools in tackling the unsolved problem , the structure of endomorphism near-rings of an arbitrary group which is a direct sum of n copies of isomorphic groups .

Chapter 1

Some basic results and definitions of near-rings

The concept of near-rings arises very naturally from the study of an algebraic system of group mappings with two binary operations , say addition and multiplication . If we let $T(G) = \{ f ; f : G \longrightarrow G \}$ where G is an arbitrary group (not necessarily abelian) and define the product $f \cdot g$ of the two mappings f , g in $T(G)$ by the rule $x(f \cdot g) = (xf)g$ for all x in G and the sum $f + g$ by $x(f + g) = xf + xg$ for all x in G , then $(T(G) , + , \cdot)$

satisfies all the ring axioms except possibly the right distributive law and the commutative law of addition . With this motivation , we then have the definition of near-ring in the following .

Definition 1.1. A near-ring is an algebraic system R with two binary operations " + " and " . " such that

(a) (R , +) is a not necessarily commutative group with identity o .

(b) (R , ∘) is a semigroup .

(c) $x(y + z) = xy + xz$ for all x , y , z in R .

A near-ring (R , + , •) is said to be zero-symmetric if $0x = 0$ for all x in R .

Example 1. Let $T_0(G) = \{ f \in T(G) ; (o)f = o \}$. Then ($T_0(G)$, + , •) is a zero-symmetric near-ring .

In the following we are going to give some general definitions and basic results of near-rings .

Definition 1.2. Let R be a near-ring . A subset H of R is called a sub-near-ring of R if it is a subgroup of the additive group of R and if it is closed under multiplication .

As in the case of rings , the intersection of an arbitrary number of sub-near-rings of R is a sub-near-ring of R .

Now we turn our attention on d. g. near-rings . It is a well-known fact that all the endomorphisms of an additive group G form a subset of the transformation near-ring $T(G)$ and are in fact a multiplicative semigroup . As might be expected , we are only interested in the sub-near-rings of $T(G)$ which are generated additively by the subset $End(G)$ the set of all the endomorphisms of G .

Before pursuing these sub-near-rings of $T(G)$ in further detail , we have

Definition 1.3. An element s of an arbitrary near-ring R is said to be right distributive if and only if $(r + t)s = rs + ts$ for all r , t in R .

It is a known fact that an element s of $T(G)$ is right distributive if and only if s is an endomorphism of G (see A. Fröhlich [7]) . Now let $End(G)$ denote the multiplicative semigroup of all the endomorphisms of G and $E(G)$ the endomorphism near-ring which is additively generated by $End(G)$. A routine check shows that $E(G)$ is a sub-near-ring of $T(G)$ and $E(G)$ is in fact zero-symmetric . This motivates the following definition.

Definition 1.4. A near-ring R is said to be distributively generated or a d. g. near-ring if R contains a multiplicative semigroup S of right distributive elements that generates the additive group of R .

Remark : All d. g. near-rings are zero-symmetric .

The concepts of ideal of a near-ring and quotient near-ring generalize the similar notions for a ring. Since the additive group of a near-ring is non-abelian, these concepts will be given in terms of normal subgroups.

Definition 1.5. Let R be a near-ring and I a subset of R. Then

(a) I is called a left ideal of R if I is a normal subgroup of the additive group R and $RI \subseteq I$.

(b) I is called a right ideal of R if I is a normal subgroup of R and $(r + i)t - rt \in I$ for all $r, t \in R$, $i \in I$.

(c) I is said to be a two-sided ideal (or ideal) of R if I is a right ideal and as well a left ideal.

In the sequel, we simply call a two-sided ideal of R an ideal of R. As in the case of rings, the intersection of any arbitrary collection of ideals (right, left) of R is again an (a right, a left) ideal of R. Here we would like to point out that if R is a d. g. near-ring, then a right ideal of R is simply a normal subgroup of R such that $IR \subseteq I$.

Again it is easy to see that if I is an ideal of R, then the quotient group R/I can be made into a near-ring which is in fact a homomorphic image of R and R/I is called a quotient near-ring .

Definition 1.6. A mapping θ of a near-ring R into a near-ring S is called a near-ring homomorphism if

$$(x + y)\theta = x\theta + y\theta$$

and

$$(x \cdot y)\theta = (x\theta)(y\theta) \qquad \text{for all } x , y \in R .$$

Thus a near-ring homomorphism is a homomorphism of (R , +) into (S , +) that preserves multiplication . If the near-ring homomorphism θ is a one-to-one mapping , then θ is called a near-ring monomorphism . In the sequel , we use homomorphism instead of near-ring homomorphism (similarly for isomorphisms and monomorphisms) . Readers should have no confusion in using such terminologies .

The near-rings R and S are said to be isomorphic , denoted by $R \cong S$, if θ is a monomorphism of R onto S . As in the case of rings , we have $R\theta$ is a sub-near-ring of S .

We now turn our attention to the definition of near-ring modules .

Definition 1.7. Let (G , +) be a group , (R , + , •) a near-ring . Then G is called an R-module or a near-ring module over R , denoted by G_R , if there is a homomorphism

$$\theta : (R , + , \cdot) \longrightarrow (T(G) , + , \cdot) .$$

Such a homomorphism is called a representation of R .

In general, we write gr for $g(r\theta)$ where $g \in G$, $r \in R$. Thus

$$g(r_1 + r_2) = gr_1 + gr_2 \quad \text{and} \quad g(r_1 r_2) = (gr_1)r_2$$

for all $r_1, r_2 \in R$, $g \in G$. These equations are sufficient to define an R-module structure. Moreover, if R contains an identity 1 and $x \cdot 1 = x$ for all x in R, then G_R is said to be unital. Any near-ring can be considered as a near-ring module over itself, denoted by R_R, under the right regular representation $r(t\theta) = rt$. In particular, the near-rings $T(G)$ and $E(G)$ are near-ring modules over themselves. Besides, G can be considered as an $T(G)$-module as well as an $E(G)$-module.

A representation is faithful if it is a monomorphism. Again it is a well-known fact that every near-ring has a faithful representation.

Definition 1.8. Let G be an R-module. An R-subgroup of G is a subgroup H of G such that $HR \subseteq H$. An R-submodule of G is a normal subgroup H of G such that $(g + h)r - gr \in H$ for all $g \in G$, $h \in H$, $r \in R$.

In general, an R-submodule is not necessarily an R-subgroup. For if we take R as a Z-near-ring, i.e. a near-ring $(R, +, \cdot)$ with $xy = y$ for all x, y in R, then any normal subgroup of $(R, +)$ is an R-submodule of R_R but R_R only has R as an R-subgroup . If R is a zero-symmetric near-ring then every R-submodule does also appear as an R-subgroup .

Definition 1.9. Let G and H be R-modules . An R-homomorphism $\theta : G \longrightarrow H$ is a group homomorphism such that

$$(gr)\theta = (g\theta)r \quad \text{for all } g \in G \ , \ r \in R .$$

Thus the kernels of R-homomorphisms are R-submodules and every R-submodule is the kernel of a suitable R-homomorphism . Furthermore , $G\theta$ is an R-subgroup of H , $G/_{\ker\theta}$ is R-isomorphic to $G\theta$.

Definition 1.10. Let R be a near-ring . Then a left R-subgroup of R is a subgroup S of R such that $RS \subseteq S$. A right R-subgroup is an R-subgroup of R_R .

Now we write a d.g. near-ring as (R , S) . Here S is a multiplicative semigroup that generates R as an additive group , but need not be the semigroup of all distributive elements of R .

Definition 1.11. A homomorphism $\theta : (R , S) \longrightarrow (E(G) , End(G))$ that maps S into End(G) is called a d. g. near-ring representation . Then we call G an (R , S)-module .

Thus (R , +) is an (R , S)-module if (R , S) is a d. g. near-ring under right regular representation .

Definition 1.12. Let X be a subset of an R-module G . The annihilator of X in R , denoted by Ann(X) , is the set

$\{ r \in R \; ; \; xr = o \text{ for all } x \text{ in } X \}$.

It is easy to see that $Ann(X)$ is always a right ideal . If $XR \subseteq X$ then $Ann(X)$ is an ideal . From the definitions , we see that $Ann(G)$ is the kernel of the representation of R on G . So we can say that a representation of R on G is faithful if and only if $Ann(G) = \{ o \}$.

Definition 1.13. (a) Let R be a near-ring . An R-module G is R-simple if $GR \neq \{ o \}$ and G has no non-trivial proper R-submodules . (b) R is semi-primitive if R has a faithful representation on an R-simple R-module .

In particular a near-ring R is simple if R has no non-trivial ideals .

Example 2. Let $T_0(G) = \{ f \in T(G) \; ; \; Of = 0 \}$. Then $(T_0(G) , + , \cdot)$ is a simple near-ring whenever $|G| > 2$ (see G. Berman - R. J. Silverman [2] and W. Nöbauer - W. Philipp [18]) .

Some authors have characterized the radical of a near-ring (see G. Betsch [4] and J. C. Beidleman [1]) . The radical here is the analogue of the Jacobson radical of ring theory . Here we give the definition of a radical under the restriction of d. g. near-rings .

Definition 1.14. A radical of a d. g. near-ring R , denoted by $J(R)$, is defined to be the intersection of all the maximal right ideals of R .

The next theorem is due to J. C. Beidleman [1] .

Theorem 1.15. If R is a finite d. g. near-ring with identity whose additive group $(R , +)$ is solvable then $J(R)$ is nilpotent and the quotient near-ring $R/_{J(R)}$ is a ring .

By the work given by J. J. Malone and C. G. Lyons [14] , we know that $E(S_3)$ is a finite d. g. near-ring with identity whose additive group $(E(S_3) , +)$ is solvable . Thus $E(S_3)/_{J(E(S_3))} \cong Z_2 \oplus Z_3$ is trivially a ring and $J(E(S_3))^2 = \{ 0 \}$. In Part three of this dissertation we shall show that in the case of $E(S_4)$, we also have a quotient near-ring

$$E(S_4)/_{J(E(S_4))} \cong Z_2 + Z_3 + M_2(Z_2)$$

where $M_2(Z_2)$ denotes the ring of 2×2 matrices over Z_2 and $J(E(S_4))^3 = \{ 0 \}$. This provides a new example for Beidleman's Theorem .

Definition 1.16. An element e of a near-ring R is called an idempotent element of R if $e^2 = e$.

As we know from the definitions , $Ann(e)$ is always a right ideal of R . We state the next theorem which is due to G. Berman and R. J. Silverman [3] .

Theorem 1.17. Let R be a near-ring . If $e \in R$ such that

$e^2 = e$, then we get a Peirce Decomposition of the near-ring R ,

i.e.

$$R = \text{Ann}(e) + eR$$

where

$$\text{Ann}(e) = \{ r - er ; r \in R \} ,$$

$$eR = \{ er ; r \in R \}$$

and

$$\text{Ann}(e) \cap eR = \{ 0 \} .$$

But Theorem 1.17 is too general to be of any real use . Fortunately we have a more advanced form of the above theorem that was given by J. J. Malone and C. G. Lyons [14] if we know something about the generating set of the additive group of the near-ring .

Theorem 1.18. Let (R , S) be a d. g. near-ring such that $(R , +) = gp\langle S , + \rangle$. Then $R = \text{Ann}(e) + eR$ where e is an idempotent element of R , Ann(e) is the normal subgroup generated by $\{ s - es ; s \in S \}$, eR is the subgroup generated by $\{ es ; s \in S \}$ and $\text{Ann}(e) \cap eR = \{ 0 \}$.

With the inspiration of Theorem 1.18 , in Part three , we study the structure of $E(S_4)$ first by examining the generating set $\text{End}(S_4)$ of $E(S_4)$ which consists of 58 endomorphisms of S_4 . If we pick a suitable idempotent element e of $\text{End}(S_4)$, then we have a semi-direct decomposition of $E(S_4)$,i.e.

$$E(S_4) = \text{Ann}(e) + eE(S_4)$$

where $\text{Ann}(e)$ = normal closure of $\{s - es ; s \in \text{End}(S_4) \}$,

$eE(S_4) = gp\langle es \; ; \; s \in End(S_4) \rangle$ and $Ann(e) \cap eE(S_4) = \{ 0 \}$. Here difficulties arise because of the huge sizes of the two summands of $E(S_4)$. Fortunately , new light was shed on $E(S_4)$ since the discovery of a general structure theorem for morphism near-rings that is due to J. D. P. Meldrum [15] . This has been done by using the connection between the structure of a near-ring and that of the group on which it acts faithfully . Meldrum's Theorem does guarantee the existence of a non-trivial ideal N of $E(S_4)$ which is again a faithful annihilating near-ring and the quotient near-ring $E(S_4)/_N$ is in fact a subdirect sum of semi-primitive near-rings . By studying the connections between the properties of the nilpotent ideal N and that of the two summands Ann(e) and $eE(S_4)$ of $E(S_4)$ respectively , we then obtain the exact algebraic structure of N . Thus $E(S_4)$ is at hand !

PART TWO

THE STRUCTURE OF ENDOMORPHISM NEAR-RINGS $E(S_n)$ WHERE $n \geq 5$

The goal of this section is to investigate some further properties of the structure of the endomorphism near-rings $E(S_n)$ where $n \geq 5$, S_n denotes the symmetric group of degree n , which have been studied by J.D.P.Meldrum [15]. Here we would like to quote the exact algebraic structure of $E(S_n)$ by writing down the precise tables of addition and multiplication of $E(S_n)$. Then by making use of these tables we can figure out the structures of all possible ideals and $E(S_n)$-subgroups of $E(S_n)$.

Chapter 2

The algebraic structure of $E(S_n)$

Here and throughout , let S_n denote the symmetric group of degree n , the subgroup A_n of S_n the alternating group of degree n .

The following theorem is due to J.D.P.Meldrum [15] .

Theorem 2.1. Let $n \geq 5$. Then

$$I(S_n)=A(S_n)=E(S_n) ,$$

and $E(S_n)$ has an ideal N such that

$$N^2=\{0\} ,$$

$$E(S_n)/N \cong T_o(A_n) \oplus Z_2 ,$$

N consists of all maps from $S_n - A_n$ to A_n , annihilating A_n .

Here $I(S_n)$, $A(S_n)$, $E(S_n)$ denote d.g. near-rings generated by the multiplicative semigroup of inner automorphisms $Inn(S_n)$, automorphisms $Aut(S_n)$ and endomorphisms $End(S_n)$ of S_n respectively .

Since $T_o(A_n)$ is the set of all maps from A_n into itself which leave 0 fixed , the order of $T_o(A_n)$ is equal to $(n!/2)^{(n!/2)-1}$. H.E. Heatherly [10] proved the following theorem .

Theorem 2.2. For any group $(G , +)$, the group $(T(G),+)$ and $(T_o(G),+)$ are the unrestricted direct sum of $|G|$ and $|G|-1$ copies of $(G,+)$ respectively .

According to this theorem , we have

$$T_o(A_n) \cong \sum_{\frac{n!}{2}-1} (A_n , +) \quad .$$

In the sequel , we simply make a routine check and give a new expression for $T_o(A_n)$.

Let $A_n = \{ g_o , g_1 , g_2 , \ldots\ldots\ldots\ldots , g_m \}$ where $m + 1 = n!/2$, $g_o = o$ the identity of the alternating group A_n . Take $e_i \in T_o(A_n)$ such that

$$g_j e_i = \begin{cases} g_i & \text{if } j = i \quad \text{for all } g_j \in A_n \\ o & \text{if } j \neq i \end{cases}$$

where $1 \leqslant i \leqslant m$.

It is easy to see that $e_iT_o(A_n) \cap e_jT_o(A_n) = \{o\}$ for all $i \neq j$, and that $e_iT_o(A_n) = Ann (A_n - \{g_i\})$ and so is a normal subgroup of $T_o(A_n)$ for all i , $1 \leq i \leq m$.

Here we rewrite

$$e_i T_o(A_n) = \{ \epsilon_{ij} \; ; \; o \leq j \leq m \}$$

where

$$\epsilon_{ij} : g_k \longrightarrow o \quad \text{if } k \neq i$$
$$g_k \longrightarrow g_j \quad \text{if } k = i \; .$$

Then

$$\epsilon_{ij} + \epsilon_{ik} = \epsilon_{il}$$

where

$$g_l = g_j + g_k$$

as can easily be checked . So $e_i T_o(A_n) \cong A_n$ as an additive group. Furthermore , for every $f \in T_o(A_n)$, f can be written in the form of

$$f = \alpha_1 + \alpha_2 + \dots\dots\dots + \alpha_m$$

where

$$\alpha_i \in e_i T_o(A_n) \qquad \text{for } 1 \leq i \leq m \; .$$

Hence

$$T_o(A_n) = e_1 T_o(A_n) \oplus e_2 T_o(A_n) \oplus \dots\dots\dots \oplus e_m T_o(A_n) \; .$$

Therefore

$$T_o(A_n) \cong \sum_m (A_n , +) \; .$$

Hence we have

$$E(S_n)/N \cong \sum_m (A_n , +) \oplus Z_2 \; .$$

From Theorem 2.1 , we can build up the nilpotent ideal N of $E(S_n)$ as follows .

Let $S_n - A_n = \{ g_{m+1} , g_{m+2} , \ldots\ldots\ldots , g_{2m+1} \}$

where $g_{m+1+i} = g_i + (12)$ for $o \leq i \leq m$.

Then N , the set of all maps from $S_n - A_n$ to A_n annihilating A_n can be described in much the same way as $T_o(A_n)$ was above.

If we take $d_i \in N$ such that

$$g_j d_i = \begin{cases} g_1 & \text{if} \quad j = m + i \\ o & \text{if} \quad j \neq m + i \end{cases}$$

for $1 \leq i \leq m + 1$. Then

$$d_i T_o(A_n) \cap d_j T_o(A_n) = \{o\} \qquad \text{for all} \quad i \neq j$$

and

$$d_i T_o(A_n) = N \cap \text{Ann} \left(S_n - \{ g_{m+i} \} \right)$$

and so is a normal subgroup of N for all i , $1 \leq i \leq m+1$.

Here we rewrite

$$d_i T_o(A_n) = \{ \eta_{ij} \ ; \ o \leq j \leq m \}$$

for $1 \leq i \leq m+ 1$, where

$$\eta_{ij} : \begin{array}{l} g_{m+k} \longrightarrow o \qquad \text{if} \quad k \neq i \\ g_{m+i} \longrightarrow g_j \ . \end{array}$$

Then

$$\eta_{ij} + \eta_{ik} = \eta_{il}$$

where

$$g_l = g_j + g_k$$

as can easily be checked . So

$$d_i T_o(A_n) \cong A_n$$

as an additive group . Furthermore , for every $g \in N$, g can be written in the form of

$$g = \beta_1 + \beta_2 + \beta_3 + \cdots\cdots + \beta_{m+1}$$

where

$$\beta_i \in d_i T_o(A_n) \quad , \quad \text{for } 1 \leq i \leq m+1 .$$

Hence

$$N = d_1 T_o(A_n) \oplus d_2 T_o(A_n) \oplus \cdots\cdots \oplus d_{m+1} T_o(A_n) .$$

Therefore

$$N \cong \sum_{m+1} (A_n , +) .$$

Multiplication in N is determined by Theorem 2.1 since $N^2 = \{o\}$.

Since $\{0\} \lhd A_n \lhd S_n$ and $S_n/A_n \cong Z_2$, $E(S_n)$ acts on A_n giving $T_o(A_n)$ and on S_n / A_n giving a sub-near-ring which is isomorphic to Z_2 . Thus $Z_2 \cong gp\langle\theta\rangle$ where

$$\theta : \begin{array}{l} A_n \longrightarrow o \\ S_n - A_n \longrightarrow (12) \end{array} \qquad (12) \in S_n - A_n, \left|(12)\right| = 2 .$$

Since $Z_2 \cong gp\langle\theta\rangle \subseteq E(S_n)$ and $E(S_n) / N \cong T_o(A_n) \oplus Z_2$, we then choose an element α in $E(S_n)$ such that

$$N + \alpha = x + y$$

where $x \in T_o(A_n)$, $y \in gp\langle\theta\rangle$. Since N contains all maps from $S_n - A_n$ to A_n , we can find an element β in $E(S_n)$ which annihilates $S_n - A_n$ and behaves in a prescribed manner on A_n . Also we can take an element θ in $E(S_n)$ which maps $S_n - A_n$ to (12) and A_n to o . So an arbitrary element of $E(S_n)$ can be written in the form

$$\eta + \beta + \alpha \quad \text{or} \; (\eta , \beta , \alpha)$$

where $\eta \in N$, $\beta \in T_o(A_n)$ annihilates $S_n - A_n$, $\alpha \in \{ o , \theta \}$.

Let $\eta \in N$, $\alpha \in \{ o , \theta \}$. Then denote η^α the element of N which is defined by

$$g\eta^\alpha = \begin{cases} g\eta & \text{if} \quad \alpha = o \\ -(12) + g\eta + (12) & \text{if} \quad \alpha = \theta . \end{cases}$$

In this way θ induces an automorphism of (N , +) of order two , which maps each $d_i T_o(A_n)$ into itself .

Thus we have the following lemma .

Lemma 2.3. Let $gp \langle d_i T_o(A_n) , \theta \rangle$ denote the group which is generated additively by the elements of $d_i T_o(A_n)$ and θ . Then for every $i \in \{ 1,2,\ldots\ldots,m+1 \}$

$$gp \langle d_i T_o(A_n) , \theta \rangle = (S_n , +)$$

under the correspondence

$$\eta_{ij} \longmapsto g_j \quad , \quad \theta \longmapsto (12) \; .$$

Proof : Immediate .

We prove now

Lemma 2.4. Let (η , β , α), $(\eta', \beta', \alpha') \in E(S_n)$.

Then

$$(\eta , \beta , \alpha) + (\eta', \beta', \alpha') = (\eta + \eta'^{\alpha} , \beta + \beta', \alpha + \alpha') .$$

Proof : Let $g \in A_n$. Then

$$g((\eta , \beta , \alpha) + (\eta' ,\beta' ,\alpha')) = g(\eta , \beta , \alpha) + g(\eta' ,\beta' ,\alpha')$$

$$= g\beta + g\beta'$$

$$= g(\ \beta + \beta')$$

$$= g(\ \eta + \eta'^{\alpha},\ \beta + \beta'\ ,\ \alpha\ + \alpha'\ ).$$

Let $g \in S_n - A_n$. Then

$$g(\ (\ \eta\ ,\ \beta\ ,\ \alpha\ ) + (\ \eta',\ \beta',\ \alpha')\ ) = g(\ \eta\ ,\ \beta\ ,\ \alpha\ ) + g(\ \eta',\ \beta',\ \alpha')$$

$$= g\eta + g\alpha + g\eta' + g\alpha'$$

$$= g\eta + g\alpha + g\eta' - g\alpha + g\alpha\ + g\alpha'$$

$$= g\eta + g\eta'^{\alpha} + g\alpha + g\alpha'$$

$$= g(\ \eta + \eta'^{\alpha}) + g(\ \alpha + \alpha'\ )$$

$$= g(\ \eta + \eta'^{\alpha},\ \beta + \beta'\ ,\ \alpha + \alpha'\ )\ .$$

Hence result.

The corresponding result for products has to be separated into two cases.

Lemma 2.5. $(\sum_{i=1}^{m+1} d_i r_i\ ,\ \beta\ ,\ 0\ )(\ \eta\ ,\ \beta',\ \alpha\ ) = (\sum_{i=1}^{m+1} d_i r_i \beta'\ , \beta\beta', 0).$

Proof : Let $g \in A_n$. Then

$$g(\ (\sum_{i=1}^{m+1} d_i r_i\ ,\ \beta\ ,\ 0\ )(\ \eta\ ,\ \beta'\ ,\ \alpha\ )\ ) = (g(\sum_{i=1}^{m+1} d_i r_i\ ,\ \beta\ , 0))(\eta, \beta', \alpha)$$

$$= (g\beta)(\ \eta\ ,\ \beta'\ ,\ \alpha\ )$$

$$= g\beta\beta'$$

$$= g(\sum_{i=1}^{m+1} d_i r_i \beta',\ \beta\beta',\ 0\ )\ .$$

Let $g \in S_n - A_n$. Then

$$g(\ (\sum_{i=1}^{m+1} d_i r_i\ ,\ \beta\ ,\ 0)(\ \eta\ ,\ \beta',\ \alpha\ )\ ) = (g(\sum_{i=1}^{m+1} d_i r_i\ ))\ (\ \eta\ ,\ \beta',\ \alpha\ )$$

$$= (g_1 r_j)(\eta , \beta' , \alpha)$$

where $g = g_{j+m}$

$$= g_1 r_j \beta'$$

$$= g(\sum_{i=1}^{m+1} d_i r_i \beta' , \beta\beta', 0) .$$

Hence result .

Here we write an element of N as $\sum_{i=1}^{m+1} d_i r_i$ where $r_i \in T_o(A_n)$ as determined above .

Lemma 2.6. $(\sum_{i=1}^{m+1} \eta_{ij(i)} , \beta, \theta)(\sum_{i=1}^{m+1} d_i r_i , \beta' , \alpha)$

$$= (\sum_{i=1}^{m+1} d_i r_{j(i)+1} , \beta\beta' , \alpha) .$$

Proof : Let $g \in A_n$. Then

$$g((\sum_{i=1}^{m+1} \eta_{ij(i)} , \beta , \theta)(\sum_{i=1}^{m+1} d_i r_i , \beta' , \alpha))$$

$$= (g\beta)\beta'$$

$$= g(\sum_{i=1}^{m+1} d_i r_{j(i)+1} , \beta\beta' , \alpha) .$$

Let $g \in S_n - A_n$. Then

$g = g_{m+i}$ for some i , $1 \leq i \leq m+1$.

$$g((\sum_{i=1}^{m+1} \eta_{ij(i)} , \beta , \theta)(\sum_{i=1}^{m+1} d_i r_i , \beta' , \alpha))$$

$$= (g\eta_{ij(i)} + (12))(\sum_{i=1}^{m+1} d_i r_i , \beta' , \alpha)$$

$$= (g_{j(i)} + (12)) (\sum_{i=1}^{m+1} d_i r_i , \beta' , \alpha)$$

$$= g_{m+j(i)+1} (\sum_{i=1}^{m+1} d_i r_i , \beta' , \alpha)$$

$$= g_{m+j(i)+1} d_{j(i)+1} r_{j(i)+1} + g_{m+j(i)+1} \alpha$$

$$= g_{m+i} (\sum_{i=1}^{m+1} d_i r_{j(i)+1} , \beta\beta' , \alpha) ,$$

Since $g_{m+j(i)+1} \in S_n - A_n$ and

$$g_{m+i} d_i r_{j(i)+1} = g_1 r_{j(i)+1} = g_{m+j(i)+1} d_{j(i)+1} r_{j(i)+1} .$$

Hence result .

Chapter 3

The ideal structure of $E(S_n)$

With the help of the addition and multiplication tables of Lemma 2.4 , 2.5 and 2.6 , we now turn to the ideal structure of $E(S_n)$. From the previous work we have

$$E(S_n) = (N \oplus T_o(A_n)) + gp\langle \theta \rangle$$

where θ has order two and induces an automorphism of order two in each of the direct factors of N . Also $(N , +) \cong \sum_{m+1} (A_n , +)$ and $gp\langle A_n , \theta \rangle \cong S_n$.

Now we use the following notation

$$T_o(A_n) = \sum_{i=1}^{m} e_i T_o(A_n) ,$$

$$N = \sum_{j=1}^{m+1} d_j T_o(A_n)$$

and each of the factors are simple non-abelian groups , being isomorphic to A_n .

Let H be a normal subgroup of $E(S_n)$. If an element of the form $(\eta , \beta , \theta) \in H$ where $\theta : A_n \longrightarrow o$, $S_n - A_n \longrightarrow (12)$ then by the properties of the symmetric group S_n , we deduce

$$N \subseteq H .$$

By using the elementary group property (see W.R.Scott [20]) , any normal subgroup of $\sum_{j=1}^{m+1} d_j T_o(A_n) + \sum_{i=1}^{m} e_i T_o(A_n)$ is of the form

$$\sum_{j\epsilon J} d_j T_o(A_n) + \sum_{i\epsilon I} e_i T_o(A_n) \qquad (1)$$

where $J \subset \{ 1, 2, \ldots\ldots, m+1 \}$ and $I \subset \{ 1, 2, \ldots\ldots, m \}$.

All other normal subgroups of $E(S_n)$ are of the form

$$\sum_{i\epsilon I} e_i T_o(A_n) + N + gp\langle \theta \rangle \qquad (2)$$

Theorem 3.1. All normal subgroups of $(E(S_n), +)$ as listed in (1) and (2) are right ideals and they are all annihilators of suitable subsets of S_n, except for $T_o(A_n) \oplus N$.

Proof : This follows immediately from the multiplication tables of Lemma 2.5 and 2.6.

Here we use the following result which is due to H.E.Heatherly [11].

Theorem 3.2. The only left ideals of $T_o(A_n)$ are $\{o\}$ and $T_o(A_n)$.

This enables us to prove the following result.

Theorem 3.3. The following is a complete list of left ideals of $E(S_n)$:

$\{o\}$, N, $N + gp\langle\theta\rangle$, $T_o(A_n) \oplus N$ and $E(S_n)$.

Proof : From Lemma 2.5 and 2.6, and the remarks about normal subgroups above, these are left ideals.

Let K be a left ideal of $E(S_n)$. Then $K \cap T_o(A_n) > \{ o \}$ forces $T_o(A_n) \subseteq K$ by Theorem 3.2. Lemma 2.6 shows that if J in (1) is not empty, then $K \supseteq N$, since K is a normal subgroup of $E(S_n)$, and so must be of the form (1) or (2).

Finally we need to show that $T_o(A_n)$ is not a left ideal . To do this , we only need to show $E(S_n)T_o(A_n) \not\subset T_o(A_n)$. Choose two non-zero elements $(\sum_{i=1}^{m+1} d_i r_i , o , o)$, (o , β , o) of $E(S_n)$ and $T_o(A_n)$ respectively . By Lemma 2.5 , we have

$$(\sum_{i=1}^{m+1} d_i r_i , o , o)(o , \beta , o) = (\sum_{i=1}^{m+1} d_i r_i \beta , o , o)$$

which is generally not in $T_o(A_n)$. Thus $T_o(A_n)$ is not a left ideal of $E(S_n)$. This suffices to prove the result .

Thus the next theorem is immediate .

Theorem 3.4. The complete list of ideals of $E(S_n)$ is the same as the list of left ideals.

Proof : Since all the left ideals as determined in Theorem 3.3 are also right ideals , the result follows .

Chapter 4

The $E(S_n)$-subgroups

In this chapter we are going to determine the right $E(S_n)$-subgroups and as well the left $E(S_n)$-subgroups of $E(S_n)$. Here and throughout, let us agree that the term $E(S_n)$-subgroups unmodified will always mean right $E(S_n)$-subgroups of $E(S_n)$. It is a well-known fact that all the right (left) ideals in $E(S_n)$ are again (left) $E(S_n)$-subgroups of $E(S_n)$. So all the right (left) ideals as listed in Theorems 3.1 and 3.3 are (left) $E(S_n)$-subgroups but this list of right (left) ideals does not exhaust all the (left) $E(S_n)$-subgroups of $E(S_n)$.

Before we start our investigation, we would like to remark that all (left) $E(S_n)$-subgroups are sub-near-rings of $E(S_n)$ since for any (left) right R-subgroup H of an arbitrary near-ring R, $HR \subseteq H \Rightarrow HH \subseteq H$ and $RH \subseteq H \Longrightarrow HH \subseteq H$.

As from the definition of R-subgroups we immediately know that gR is an R-subgroup of R for every $g \in R$. So we need to look at $xE(S_n)$ for every $x \in E(S_n)$. Since $E(S_n) = (N \oplus T_o(A_n)) + gp\langle\theta\rangle$, every element of $E(S_n)$ can be written in the form of (η, β, α), where $\eta \in N$, $\beta \in T_o(A_n)$ annihilates $S_n - A_n$ and $\alpha \in gp\langle\theta\rangle$. Without loss of generality we can obtain all the monogenic $E(S_n)$-subgroups through the following seven separate cases :

$(\eta, o, o)E(S_n)$, $(o, \beta, o)E(S_n)$, $(o, o, \theta)E(S_n)$,

$(\eta, o, \theta)E(S_n)$, $(\eta, \beta, o)E(S_n)$, $(o, \beta, \theta)E(S_n)$,

and $(\eta, \beta, \theta)E(S_n)$.

Here we call an $E(S_n)$-subgroup H monogenic if $H = gE(S_n)$ for some $g \in E(S_n)$.

Now we start our work by first looking at $(\eta, o, o)E(S_n)$.

Let $\eta \in N$, $\eta = d_{j_1}\alpha_{j_1} + d_{j_2}\alpha_{j_2} + \dots\dots + d_{j_r}\alpha_{j_r}$, $1 \leq r \leq m+1$, where

$$d_{j_i} : g_{m+j_i} \longrightarrow g_1$$
$$\text{others} \longrightarrow o \ ,$$

$\alpha_{j_i} \in T_o(A_n)$ such that $g_1\alpha_{j_i} \neq o$ for each $i \in \{1, 2, \dots, r\}$.

Now denote $C = \{j_1, j_2, j_3, \dots\dots, j_r\}$. Then we can define an equivalence relation $\sim$ on C as follows.

If j_{i_s} , $j_{i_t} \in C$ where i_s , $i_t \in \{1, 2, \dots\dots, r\}$

then $j_{i_s} \sim j_{i_t}$ means that

$$g_1\alpha_{j_{i_s}} = g_1\alpha_{j_{i_t}} = g_i$$

where $g_i \in A_n - \{o\}$, $i \in \{1, 2, \dots, m\}$, $m+1 = n!/2$.

The verification that this is an equivalence relation is almost trivial . If $j_{i_1} \in C$, then the equivalence class of j_{i_1} is equal to $\{ x \in C ; j_{i_1} \sim x \}$ which is usually expressed as $[j_{i_1}] = C_1$. Here we write $C_t = [j_{k_t}]$ where $g_1 \alpha_{j_{k_t}} = g_k$, $g_k \in A_n - \{ o , g_i \}_{i \le k-1}$. Then there exists a unique positive integer p ($1 \le p \le r$) such that

$$C = C_1 \cup C_2 \cup \ldots\ldots\ldots\cup C_p$$

where $C_i \cap C_j = \phi$ if $i \neq j$, $i , j \in \{ 1 , 2 , \ldots\ldots , p \}$.
Now if we pick an element $\beta_1 \in T_o(A_n)$ such that

$$\beta_1 : \begin{array}{l} g_i \longrightarrow g_i \\ A_n - \{g_i\} \longrightarrow o \end{array} ,$$

then

$$\eta\beta_1 : \begin{array}{l} g_{m+x} \longrightarrow g_i \\ S_n - \{ g_{m+x} ; x \in C_1 \} \longrightarrow o \end{array}$$

where $x \in C_1$. Here it is easy to see that

$$\eta\beta_1 T_o(A_n) = \eta\beta_1 E(S_n) \cong (A_n , +)$$

and is as well an $E(S_n)$-subgroup . A routine check immediately gives rise to the following theorem .

Theorem 4.1. With notation as above , $\eta E(S_n)$ is an $E(S_n)$-subgroup which is group isomorphic to $\sum_p (A_n , +)$.

Proof : It is trivial that $\eta E(S_n)$ is an $E(S_n)$-subgroup . So it suffices to show that $\eta E(S_n) \cong \sum_p (A_n , +)$.

Let $\eta = d_{j_1} \alpha_{j_1} + d_{j_2} \alpha_{j_2} + \ldots\ldots\ldots\ldots + d_{j_r} \alpha_{j_r}$ be given as above . So η can be rewritten as

$$\eta\beta_1 + \eta\beta_2 + \ldots\ldots\ldots\ldots + \eta\beta_p$$

where

$$\beta_q : g_1\alpha_{j_w} \longrightarrow g_1\alpha_{j_w} \qquad (j_w \in C_q, 1 \leq q \leq p)$$

$$\text{others} \longrightarrow o .$$

Then it is easy to see that

$$\eta\beta_q T_o(A_n) = \eta\beta_q E(S_n) \cong (A_n, +)$$

for each $q \in \{1, 2, \ldots\ldots\ldots, p\}$.

Hence $\eta E(S_n) \cong \sum_p (A_n, +)$.

Remarks : One can easily see that if the order of any one of the subsets C_i $(1 \leq i \leq p)$ of C is greater than 1, then $\eta E(S_n)$ is obviously not normal in $E(S_n)$. For if $j, k \in C_i$ where $j \neq k$, pick $\beta = d_j\alpha'$ where $\alpha' \neq o$ such that $g_{m+j}d_j\alpha'$ does not commute with $g_{m+j}\eta$. Then we have

$$g_{m+j}(-\eta - \beta + \eta + \beta) \neq o ,$$

$$g_{m+k}(-\eta - \beta + \eta + \beta) = o .$$

Thus $-\eta - \beta + \eta + \beta \notin \eta E(S_n)$. Again we like to point out that if every member of the partition $C_1, C_2, \ldots\ldots, C_p$ contains a single element then $\eta E(S_n)$ would reduce to a special case in list (1) of Theorem 3.1 as a right ideal of $E(S_n)$ and is in fact equal to $d_{j_1}T_o(A_n) \oplus d_{j_2}T_o(A_n) \oplus \ldots\ldots\ldots\ldots \oplus d_{j_r}T_o(A_n)$.

As in the case of $(o, \beta, o)E(S_n)$, now we pick an element $\beta \in T_o(A_n)$ which annihilates $S_n - A_n$, such that

$$\beta = e_{\mu_1}\alpha_{\mu_1} + e_{\mu_2}\alpha_{\mu_2} + \dots\dots\dots\dots + e_{\mu_s}\alpha_{\mu_s}$$

where $1 \leq \mu_1 < \mu_2 < \dots\dots\dots < \mu_s \leq m$, $m+1 = n!/2$ and

$$e_{\mu_i} : g_{\mu_i} \longrightarrow g_{\mu_i}$$
$$\text{others} \longrightarrow o$$

$\mu_i \in \{ 1 , 2 ,\dots\dots, m \}$. Now let $P = \{ \mu_1 , \mu_2 ,\dots\dots, \mu_s \}$.

Then we can define an equivalence relation $\sim$ on P .

If $\mu_{i_r} , \mu_{i_t} \in P$ where $i_r , i_t \in \{ 1 , 2 ,\dots\dots, s \}$

then $\mu_{i_r} \sim \mu_{i_t}$ means that

$$g_{\mu_{i_r}}\alpha_{\mu_{i_r}} = g_{\mu_{i_t}}\alpha_{\mu_{i_t}} = g_i$$

where $g_i \in A_n - \{ o \}$, $i \in \{ 1 , 2 ,\dots\dots, m \}$, $m+1 = n!/2$.

If $\mu_{i_1} \in P$, then the equivalence class of μ_{i_1} , denoted by

$[\mu_{i_1}]$ or P_1 , is equal to the set $\{ x \in P ; \mu_{i_1} \sim x \}$ where

$g_{\mu_{i_1}}\alpha_{\mu_{i_1}} = g_x\alpha_x = g_i$. So we can write $P_t = [\mu_{w_t}]$ where

$g_{\mu_{w_t}}\alpha_{\mu_{w_t}} = g_w$, $g_w \in A_n - \{ o , g_i \}_{i<w}$. Thus there exists a positive

integer v ($1 \leq y \leq s$) such that

$$P = P_1 \cup P_2 \cup \dots\dots\dots \cup P_v$$

where $P_i \cap P_j = \phi$ if $i \neq j$, $i , j \in \{ 1 , 2 ,\dots\dots, v \}$.

Analogously to the previous work for η in N if we choose an

element $\beta_1 \in T_o(A_n)$ such that

$$\beta_1 : g_i \longrightarrow g_i$$

$$A_n - \{g_i\} \longrightarrow o \quad ,$$

then

$$\beta\beta_1 : g_x \longrightarrow g_i \qquad \text{if} \quad x \in P_1$$

$$S_n - \{ g_x \; ; \; x \in P_1 \} \longrightarrow o$$

Again it is easy to see that

$$\beta\beta_1 T_o(A_n) = \beta\beta_1 E(S_n) \cong (A_n , +)$$

and is as well an $E(S_n)$-subgroup .

Thus we have the following theorem .

Theorem 4.2. With the notation as above , $\beta E(S_n)$ is an $E(S_n)$-subgroup which is group isomorphic to $\sum\limits_v (A_n , +)$.

Proof : The proof goes much the same way as in Theorem 4.1. Here we just want to point out that

$$\beta = \beta\beta_1 + \beta\beta_2 + \ldots\ldots\ldots\ldots + \beta\beta_v$$

where

$$\beta_t : g_x\alpha_x \longrightarrow g_x\alpha_x \qquad \text{if} \quad x \in P_t , \; 1 \leqslant t \leqslant v$$

$$\text{others} \longrightarrow o \quad .$$

Thus $\beta\beta_t T_o(A_n) = \beta\beta_t E(S_n) \cong (A_n , +)$

for each $t \in \{ 1 , 2 , \ldots\ldots\ldots\ldots , v \}$.

Hence $\beta E(S_n) \cong \sum\limits_v (A_n , +)$.

Remarks : If the order of any one of the subsets P_i ($1 \leqslant i \leqslant v$) of P is greater than 1 , then $\beta E(S_n)$ is no longer a normal subgroup of $E(S_n)$. For if $k , j \in P_t$ ($k \neq j$) , then pick $\omega = e_k\alpha'$ where $\alpha' \neq o$ such that $g_k e_k \alpha'$ does not commute with $g_k\beta$. Then we have

$$g_k(- \beta - \omega + \beta + \omega) \neq o$$

$$g_j(- \beta - \omega + \beta + \omega) = o$$

Thus $- \beta - \omega + \beta + \omega \notin \beta E(S_n)$. On the other hand , if $|P_i| = 1$ for each $i \in \{ 1 , 2 , \ldots\ldots\ldots , v \}$ then $\beta E(S_n) = e_{\mu_1} T_o(A_n) \oplus e_{\mu_2} T_o(A_n) \oplus \ldots\ldots\ldots\ldots\ldots \oplus e_{\mu_s} T_o(A_n)$ and is in fact a right ideal of $E(S_n)$ as in list (1) of Theorem 3.1.

Let $\alpha \in gp\langle \theta \rangle = \{ o , \theta \}$ where

$$\theta : A_n \longrightarrow o$$

$$S_n - A_n \longrightarrow x .$$

Here $x \in S_n - A_n$ and $|x| = 2$. Without loss of generality we can fix $x = (1\ 2)$. Thus the following theorem is immediate .

Theorem 4.3. Let $\alpha = \theta$. Then $\alpha E(S_n)$ is an $E(S_n)$-subgroup which is group isomorphic to $(S_n , +)$. Moreover $\alpha E(S_n) = \{ \theta_x ; x \in S_n \}$ where θ_x sends A_n to zero , $S_n - A_n$ to x. In fact $\alpha E(S_n)$ is a sub-near-ring of $E(S_n)$ with addition and multiplication given as follows :

$$\theta_x + \theta_y = \theta_{x+y}$$

$$\theta_x\theta_y = \begin{cases} o & \text{if } x \in A_n \end{cases}$$

$$\begin{cases} \theta_y & \text{if} \quad x \notin A_n . \end{cases}$$

Proof : It suffices to show that $\alpha E(S_n) \cong (S_n , +)$ and the rest is immediate from definitions . Since $\alpha N = \{ \theta_x ; x \in A_n \} \cong (A_n , +)$ where $\theta_x : A_n \longrightarrow \theta$, $S_n - A_n \longrightarrow x$, $gp\langle \alpha , \alpha N \rangle = \{ \theta_x ; x \in S_n \} \cong (S_n , +)$. But $gp\langle \alpha , \alpha N \rangle = \alpha E(S_n)$. Hence result .

As a consequence of Theorems 4.2 and 4.3 , the following theorem is immediate .

Theorem 4.4. Let β and θ be the maps described in Theorems 4.2 and 4.3 . Then $(o , \beta , \theta)E(S_n)$ is an $E(S_n)$-subgroup which is group isomorphic to $\sum_v (A_n , +) + (S_n , +)$.

Proof : It is only a routine check by applying Theorem 4.2 and 4.3. Since $E(S_n) = (N \oplus T_o(A_n)) + gp\langle \theta \rangle = T_o(A_n) + N + gp\langle \theta \rangle$, by using the left distributive property ,we have

$$(o , \beta , \theta)E(S_n) = (o , \beta , \theta)(T_o(A_n) + N + gp\langle \theta \rangle)$$

$$= (o , \beta , \theta)T_o(A_n) + (o , \beta , \theta)N + (o , \beta , \theta)gp\langle \theta \rangle .$$

A routine check shows

$$(o , \beta , \theta)N = (o , o , \theta)N ,$$

$$(o , \beta , \theta)T_o(A_n) = (o , \beta , o)T_o(A_n) \cong \sum_v (A_n , +)$$

by Theorem 4.2 , and

$$(o , \beta , \theta)gp\langle \theta \rangle = gp\langle \theta \rangle .$$

From Theorem 4.3 , we have shown that

$$(o , o , \theta)N + gp\langle\theta\rangle = gp\langle\theta , \theta N\rangle \cong (S_n , +) .$$

Thus $(o , \beta , \theta)E(S_n) =$

$$(o , \beta , o)T_o(A_n) + (o , o , \theta)N + gp\langle\theta\rangle ,$$

and is group isomorphic to $\sum_v (A_n , +) + (S_n , +)$.

Hence result .

Without loss of generality , now let η and β be the maps that have been described as in the previous theorems. Then the map (η , β , o) can be written in the form of

$$\eta + \beta = d_{j_1}\alpha_{j_1} + d_{j_2}\alpha_{j_2} + \cdots\cdots + d_{j_r}\alpha_{j_r} + e_{\mu_1}\alpha_{\mu_1} + \cdots\cdots + e_{\mu_s}\alpha_{\mu_s} .$$

Here let $W = \{ j_1 , j_2 ,\ldots\ldots, j_r, \mu_1 , \mu_2 ,\ldots\ldots, \mu_s \}$.

As we can realize immediately from Theorems 4.1 and 4.2 , the maps η and β both send elements of $S_n - A_n$ and A_n to A_n respectively . So it is easy to see that there are equivalence classes in C and P that give the same image by the action of maps η and β . Say , if there are κ equivalence classes in C and P giving the same images under the action of the maps η and β , then we can induce an equivalence relation on W by having $p + v - \kappa$ equivalence classes . Here we can guarantee that each equivalence class in W gives rise to a distinct image in A_n by the action of $\eta + \beta$. Note that

$$p, v \leqslant p + v - \kappa \leqslant m + 1 \quad \text{where} \quad m + 1 = n!/2 .$$

Then we have the next theorem immediately .

Theorem 4.5. Let $(\eta , \beta , o) \in E(S_n)$ as given above . Then $(\eta , \beta , o)E(S_n)$ is an $E(S_n)$-subgroup which is group isomorphic to $\sum_{p+v-\kappa} (A_n , +)$.

Proof : The proof is immediate by applying Theorems 4.1 and 4.2 and the remarks above .

Remarks : If $|C_i| = 1$ for each $i \in \{ 1 , 2 , \ldots\ldots , p \}$, $|P_t| = 1$ for each $t \in \{ 1 , 2 , \ldots\ldots , v \}$ and no equivalence class in C and P gives the same image in A_n by the actions of maps η and β , then $(\eta , \beta , o)E(S_n)$ is in fact a right ideal as in list (1) of Theorem 3.1 which is equal to

$$\sum_{i=1}^{r} d_{j_i} T_o(A_n) + \sum_{i=1}^{s} e_{\mu_i} T_o(A_n) .$$

Without loss of generality , now we pick an element $(\eta, o, \theta) \in E(S_n)$ where η , θ are the maps described in Theorems 4.1 and 4.3 respectively . According to the multiplication table in Lemma 2.6 , we have

$$(\eta , o , \theta)(o , o , \theta) = (o , o , \theta) ,$$

$$(\eta , o , \theta) (o , \beta , o) = (o , o , o) \quad \forall \beta \in T_o(A_n) .$$

Here we write

$$\eta = \sum_{i=1}^{m+1} \eta_{ij(i)} \quad \text{and} \quad \eta' = \sum_{i=1}^{m+1} d_i r_i$$

where $r_i \in T_o(A_n)$, then

$$(\eta , o , \theta)(\eta' , o , o) = (\sum_{i=1}^{m+1} d_i r_{j(i)+1} , o , o) .$$

Since $E(S_n) = (N \oplus T_o(A_n)) + gp < \theta >$,

$$(\eta , o , \theta)E(S_n) = (\eta , o , \theta)(N + T_o(A_n) + gp< \theta >)$$

$$= (\eta , o , \theta)N + (\eta , o,\theta)T_o(A_n) + (\eta , o , \theta)gp < \theta > .$$

Therefore

$$(\eta , o , \theta)E(S_n) = (\eta , o , \theta)N + gp< \theta > .$$

Before proceeding any further , we pause for a while to examine the structure of the map (η , o , θ) . Since η is the map that sends elements of distinct equivalence classes to distinct non-zero elements of A_n , η can be rewritten as follows : let $\overline{C}_i = \{ g_{m+j_i} ; j_i \in C_i \}$, $\overline{C} = \dot{\cup} \overline{C}_i$

$$\begin{array}{rcl} \eta : \overline{C}_1 & \longrightarrow & g_{i_1} \\ \overline{C}_2 & \longrightarrow & g_{i_2} \\ \vdots & & \vdots \\ \overline{C}_p & \longrightarrow & g_{i_p} \\ \text{others} & \longrightarrow & o \end{array}$$

where $g_{i_\omega} \in A_n - \{ o \}$, $1 \leqslant \omega \leqslant p$ and $g_{i_\omega} \neq g_{i_\lambda}$ if $\omega \neq \lambda$. Now we denote

$$(S_n - A_n) - \overline{C} = \overline{C}_{p+1}$$

where $\overline{C} = \dot{\bigcup}_{1 \leq i \leq p} \overline{C}_i$. Then (η , o , θ) is the map that sends

$$
\begin{aligned}
A_n &\longrightarrow o \\
\overline{C}_1 &\longrightarrow g_{i_1} + (12) = g_{m+1+i_1} \\
\overline{C}_2 &\longrightarrow g_{i_2} + (12) = g_{m+1+i_2} \\
&\vdots \\
\overline{C}_p &\longrightarrow g_{i_p} + (12) = g_{m+1+i_p} \\
\overline{C}_{p+1} &\longrightarrow o + (12) = g_{m+1}
\end{aligned}
$$

where g_{m+1} , $g_{m+1+i_\omega} \in S_n - A_n$, $1 \leq \omega \leq p$ and

g_{m+1} , g_{m+1+i_1} , g_{m+1+i_2} ,..............., g_{m+1+i_p} are all distinct.

Hence

$$(\eta , o , \theta)N \cong \sum_{p+1} (A_n , +) .$$

Since $gp\langle \theta \rangle = \{ o , \theta \}$ where $\theta : S_n - A_n \longrightarrow g_{m+1} = (12)$,

$A_n \longrightarrow o$ and $gp\langle \theta \rangle \cong (Z_2 , +)$,

$$
\begin{aligned}
\Omega &= -(o , o , \theta) + (\eta , o , \theta) + (o , o , \theta) \\
&= -(o , o , \theta) + (\eta , o , o) \\
&= (\eta^{\theta} , o , \theta) .
\end{aligned}
$$

So Ω is the map that sends

$$
\begin{aligned}
A_n &\longrightarrow o \\
\overline{C}_1 &\longrightarrow (12) + g_{i_1} \\
\overline{C}_2 &\longrightarrow (12) + g_{i_2} \\
&\vdots
\end{aligned}
$$

$$\begin{array}{ccc} \vdots & & \vdots \\ \overline{C}_p & \longrightarrow & (12) + g_{i_p} \\ \overline{C}_{p+1} & \longrightarrow & (12) + o \end{array}$$

where $(12) + g_{i_1}, \ldots\ldots\ldots, (12) + g_{i_p}, (12) \in S_n - A_n$

and they are all distinct . Then we have

$$\Omega N \cong \sum_{p+1} (A_n , +) .$$

So it is easy to see that

$$(\eta , o , \theta)N + gp\langle\theta\rangle \cong (\sum_{p+1} (A_n , +)) + (Z_2 , +) .$$

Thus the next theorem is immediate .

Theorem 4.6. Let $(\eta , o , \theta) \in E(S_n)$ where η , θ are are the maps described in Theorem 4.1 and 4.3 respectively . Then $(\eta , o , \theta)E(S_n)$ is an $E(S_n)$-subgroup and is group isomorphic to $(\sum_{p+1} (A_n , +)) + (Z_2 , +)$ where $(Z_2 , +)$ is a cyclic group of order two .

Remark : In the above theorem , by using Lemma 2.3 , we know that the cyclic group $(Z_2 , +)$ acts on every single factor $(A_n , +)$ of $\sum_{p+1} (A_n , +)$ giving $(S_n , +)$.

Now we prove

Theorem 4.7. Let $(\eta , \beta , \theta) \in E(S_n)$ where η , β , θ are those maps described in Theorems 4.1 , 4.2 and 4.3 respectively.

Then $(\eta, \beta, \theta)E(S_n)$ is an $E(S_n)$-subgroup which is group isomorphic to $\sum_{v}(A_n, +) + ((\sum_{p+1}(A_n, +)) + (Z_2, +))$.

Proof : Since $(\eta, \beta, \theta)E(S_n) =$

$(\eta, \beta, \theta)(T_o(A_n) + N + gp<\theta>)$

$= (\eta, \beta, \theta)T_o(A_n) + (\eta, \beta, \theta)N + (\eta, \beta, \theta)gp<\theta>$.

A little calculation shows

$$(\eta, \beta, \theta)T_o(A_n) = (o, \beta, o)T_o(A_n) ,$$

$$(\eta, \beta, \theta)N = (\eta, o, \theta)N \quad \text{and}$$

$$(\eta, \beta, \theta)gp<\theta> = gp<\theta> .$$

Therefore

$$(\eta, \beta, \theta)E(S_n)$$

$= (o, \beta, o)T_o(A_n) + (\eta, o, \theta)N + gp<\theta>$.

Thus the proof is immediate by applying Theorems 4.2 and 4.6.

Up to now we have finished all the monogenic $E(S_n)$-subgroups of $E(S_n)$. The next step is going to be to determine the left $E(S_n)$-subgroups of $E(S_n)$ which do not appear as left ideals of $E(S_n)$.

From Theorem 4.3 we know that $\theta E(S_n) = \{ \theta_x ; x \in S_n \}$ where $\theta_x : A_n \longrightarrow o$, $S_n - A_n \longrightarrow x$ with addition and multiplication given as follows

$$\left.\begin{aligned} \theta_x + \theta_y &= \theta_{x+y} && \text{for all } x , y \in S_n \\ \theta_x\theta_y &= o && \text{if } x \in A_n \\ &= \theta_y && \text{if } x \notin A_n \end{aligned}\right\} \quad (A)$$

So $\theta E(S_n)$ is an $E(S_n)$-subgroup and as well a sub-near-ring of $E(S_n)$ which is group isomorphic to $(S_n, +)$ via $\theta_x \longrightarrow x$. Now, let $P = \theta E(S_n)$. A little calculation shows that

$$E(S_n)P \subseteq P .$$

For if $(\eta, \beta, \alpha) \in E(S_n)$, $\theta_x \in P$, then $(\eta, \beta, \alpha)\theta_x$ is the map that sends

$$g \in A_n \longrightarrow o$$

$$g \in S_n - A_n \longrightarrow (g\eta + g\alpha)\theta_x = o \quad \text{if} \quad \alpha = o$$

$$= x \quad \text{if} \quad \alpha = \theta$$

i.e. $(\eta, \beta, \alpha)\theta_x = o$ if $\alpha = o$

$= \theta_x$ if $\alpha = \theta$ — (B)

for all $(\eta, \beta, \alpha) \in E(S_n)$.

Thus $E(S_n)P \subseteq P$ and P is a left $E(S_n)$-subgroup.

Now we want to examine how do the additive subgroups of $(P,+)$ behave under the action of $E(S_n)$ on the left. Since $(P, +) \cong (S_n, +)$ via $\theta_x \longrightarrow x$, we then have a one-to-one correspondence between subgroups of $(P, +)$ and subgroups of $(S_n, +)$.

Thus the next theorem is immediate.

Theorem 4.8. With the notation as above, let F be the set that consists of all subgroups of $(P, +)$ and T the set of all subgroups of $(S_n, +)$. Then there exists a one-to-one correspondence between F and T . Moreover every member in F is a sub-near-ring of $E(S_n)$ and is again a left $E(S_n)$-subgroup of $E(S_n)$.

Proof : The proof of the first part is trivial by the remarks given above . From the addition and multiplication tables of (A) and (B) , we have the rest .

Definition 4.9. An $E(S_n)$-subgroup which is also a left $E(S_n)$-subgroup is called an invariant $E(S_n)$-subgroup of $E(S_n)$.

Thus P is an invariant $E(S_n)$-subgroup . Analogously to the restriction of S_n to A_n , if we take $Q = \{ \theta_x \in P ; x \in A_n \}$ then $(Q , + , .)$ is obviously a sub-near-ring of $E(S_n)$ and is group isomorphic to A_n . Besides Q is also an invariant $E(S_n)$-subgroup . For if we choose a particular element $\eta \in N$ such that

$$\eta ; \quad A_n \longrightarrow o$$
$$S_n - A_n \longrightarrow g$$

where $g \in A_n - \{ o \}$, then

$$\eta E(S_n) = \eta T_o(A_n) \cong (A_n , +) .$$

In fact $\eta E(S_n) = Q$, so is again an $E(S_n)$-subgroup (by Theorem 4.1) . Hence Q is an invariant $E(S_n)$-subgroup .

PART THREE

THE ENDOMORPHISM NEAR-RING OF S_4

Let $\mathrm{End}(S_4)$ denote the multiplicative semigroup of all the endomorphisms of the symmetric group S_4 and $E(S_4)$ the endomorphism near-ring which is generated additively by the elements of $\mathrm{End}(S_4)$ (not necessarily the whole of $\mathrm{End}(S_4)$) . As in Theorem 1.18 , we know that by picking a suitable idempotent element e which lies in $\mathrm{End}(S_4)$, we can obtain a structure theorem for $E(S_4)$ through the Peirce Decomposition which was proved by G. Berman and R. J. Silverman [3] .

In Chapter 5, we determine the endomorphisms of S_4 , and the multiplicative structure of $\mathrm{End}(S_4)$. With the help of this multiplication table , in Chapter 6 , we then have a complete description of how to build up $E(S_4)$ by having a semi-direct sum of sub-near-rings of $E(S_4)$. Besides we also determine the size of $E(S_4)$ which is equal to $2^{35}3^{3}$. In Chapter 7 we give an exact algebraic structure of $E(S_4)$ by writing down the precise tables of addition and multiplication . In Chapter 8 , we deal with the concept of the near-ring radical of $E(S_4)$. Thus the structure of all maximal right ideals is an immediate consequence once the radical is known .

Chapter 5

The generating set End(S_4)

Here and throughout we shall write the symmetric group S_4 additively . Though we already know a lot about the group structure of S_4 , we still need to give a full description of how to determine a complete list of normal subgroups of S_4 and all the endomorphisms of S_4 . It is because up to now , there still exists no formal text or paper giving such a record . The main concept in finding a complete list of normal subgroups of S_4 is based on an application of the following theorems which can usually be found in any standard text of elementary group theory (e. g. see W. R. Scott [20]) .

Theorem 5.1. The non-empty subgroup C of the group G is a normal subgroup if and only if it is the union of complete conjugacy classes of the elements of G , including the identity .

Theorem 5.2. Two permutations are conjugate if and only if they have the same type of cycle pattern .

Here we write o as the identity element of S_4 and

S_4 = { o , (12) , (13) , (14) , (23) , (24) , (34) , (123) , (132) , (124) , (142) , (134) , (143) , (234) , (243) , (1234) , (1432) , (1243) , (1342) , (1324) , (1423), (12)+(34) , (13)+(24) , (14)+(23) } .

According to Theorem 5.2 , there are five different conjugacy classes , say :

Table 5A

$C_1 = \{ o \}$, $C_2 = \{ (12) , (13) , (14) , (23) , (24) , (34) \}$,

$C_3 = \{ (123) , (132) , (124) , (142) , (134) , (143) , (234) , (243) \}$,

$C_4 = \{ (12)+(34) , (13)+(24) , (14)+(23) \}$

$C_5 = \{ (1234) , (1432) , (1243) , (1342) , (1324) , (1423) \}$.

Thus the following theorem is immediate .

Theorem 5.3. The following is a complete list of normal subgroups of the symmetric group S_4 :

$\{ o \}$, V_4 , A_4 , S_4 .

Proof : By the application of Lagrange's Theorem , we know that the only possible subgroups of S_4 which have to be formed by the union of any arbitrary set of conjugacy classes shown in Table 5A , are

$$C_1 \ , \ C_1 \cup C_4 \ , \ C_1 \cup C_3 \cup C_4 \text{ and } \bigcup_{i=1}^{5} C_i \ .$$

Here we denote $\{ o \} = C_1$, $V_4 = C_1 \cup C_4$, $A_4 = C_1 \cup C_3 \cup C_4$ and $S_4 = \bigcup_{i=1}^{5} C_i$. Thus the proof follows immediately from Theorem 5.1. Hence result .

In the following we are going to determine all the endomorphisms of S_4 by studying all the quotient groups of S_4 . From

Theorem 5.3 , we have only four quotient groups of S_4 , namely , S_4 / S_4 , $S_4 / \{ o \}$, S_4 / A_4 , S_4 / V_4 . Now we study the four cases separately .

In the first place , since $S_4 / S_4 \cong \{ o \}$, we then have a zero map , denoted by 0 , which sends the whole S_4 to the identity element of S_4 , i.e. $0 : S_4 \longrightarrow S_4$ via

$$g0 = o \qquad \forall g \in S_4 .$$

It is clear enough that there will be no confusion in using the same symbol for the identity element of S_4 and the zero endomorphism of S_4 .

In the second place , we have $S_4 / \{ o \} \cong S_4$. Therefore we need to determine all the automorphisms of S_4 . Before we proceed any further , we pause for a while to give a well-known result (that can be found in W. R. Scott [20]) in the following .

Theorem 5.4. Let Aut(S_n) denote the automorphism group of the symmetric group S_n for all n where n is a positive integer. If $n \geq 3$ and $n \neq 6$ (n may be infinite) , then $Aut(S_n) \cong S_n$.

It is a well-known fact that the group of all inner automorphisms of S_n , denoted by $Inn(S_n)$, is a normal subgroup of $Aut(S_n)$ which is isomorphic to S_n . This together with Theorem 5.4. forces

$$\mathrm{Inn}(S_4) = \mathrm{Aut}(S_4) \quad .$$

Thus we have 24 automorphisms of S_4 which are in fact the inner automorphisms of S_4. They are

$$\rho_x : S_4 \longrightarrow S_4 \qquad \text{where } x \in S_4$$

defined by $g\rho_x = g^x = -x + g + x \qquad g \in S_4$.

In the third place, since $S_4 / A_4 \cong Z_2$ where Z_2 is the cyclic group of order two, we then know that there are only nine such endomorphisms of S_4. It is because there are only nine subgroups of S_4 which are of order two. They are

$$\{ o , y \}$$

where $y \in \{ (12) , (13) , (14) , (23) , (24) , (34) , (12)+(34) , (13)+(24) , (14)+(23) \}$.

Thus we have

$$\varphi_y : A_4 \longrightarrow o$$
$$S_4 - A_4 \longrightarrow y$$

where $y \in S_4$, $|y| = 2$.

In the fourth place, we have $S_4 / V_4 \cong S_3$. Now we first need to find all the possible subgroups of S_4 which are isomorphic to S_3. Before we proceed any further, we shall prove the following preliminary lemma.

Lemma 5.5. Let $H_i = \{ g \in S_4 ; \ ig = i \}$ where $i \in \{1,2,3,4\}$.

Then the H_i's are subgroups of S_4 which are isomorphic to S_3. Besides these are the only four such subgroups of S_4.

Proof : The first part of the proof is immediate from elementary group theory. Now we show that those H_i's are the only four such subgroups.

Supposing that H is a subgroup of S_4 such that $H \cong S_3$ and $H \neq H_i$ for all $i \in \{1, 2, 3, 4\}$.

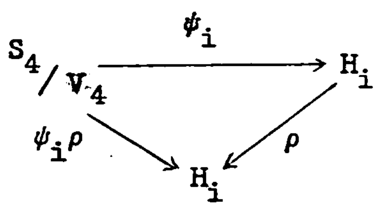
Therefore H must contain elements of order two and three. Now we can choose two distinct elements ρ, γ of H such that $|\rho| = 2$, $|\gamma| = 3$ and ρ, γ do not lie in the same H_i. Otherwise $gp\langle \rho, \gamma \rangle = H_i$ for some $i \in \{1, 2, 3, 4\}$. Thus it can be easily checked that the order of the sum $\rho + \gamma$ (or $\gamma + \rho$) is 4 or an element of order 3 not $\pm \gamma$. This contradicts the assumption on H. Hence result.

Having obtained Lemma 5.5, we can exhaust all the rest of the endomorphisms of S_4 in the following steps.

Since S_4 can be written in the form of a semi-direct sum of V_4 and $S_{\{2,3,4\}}$ where $S_{\{2,3,4\}}$ denotes the symmetric group on integers 2, 3, 4, we then write

$$S_4 / V_4 = \{ V_4 + x \; ; \; x \in S_{\{2,3,4\}} \} .$$

Thus we have the following diagram :



where ψ_i is the endomorphism of S_4, $\rho \in \mathrm{Aut}(H_i)$ for all i, $i \in \{1, 2, 3, 4\}$.

Since $H_i \cong S_3$ $\forall i$, $1 \leq i \leq 4$ and $\mathrm{Aut}(S_3) \cong S_3$ by Theorem 5.4, $\mathrm{Aut}(H_i) = \mathrm{Inn}(H_i)$. It is easy to see that each $\psi_i\rho$ is again an endomorphism of S_4 since the composition of two endomorphisms is an endomorphism. Thus we have 24 endomorphisms in this case.

Without loss of generality, we define

$$\psi_1 : S_4 \longrightarrow S_4$$

via

$$g\psi_1 = x$$

where $g \in V_4 + x$, $x \in S_{\{2,3,4\}}$.

ψ_1 is an endomorphism of S_4 as can be easily checked. From elementary group theory we know that

$$H_i = H_j^{\,x}$$

for some $x \in S_4$, $i, j \in \{1, 2, 3, 4\}$ since every H_i is a stabilizer of the subset $\{i\}$ of $\{1, 2, 3, 4\}$. Thus every $\psi_i\rho$ can be uniquely written in the form $\psi_1\rho_x$ where $\rho_x \in \mathrm{Inn}(S_4)$, $x \in S_4$. Here we denote $\psi_1\rho_x = \psi_x$.

Thus

$$\psi_x : S_4 \longrightarrow S_4 \qquad \text{where } x \in S_4$$

via

$$g\psi_x = \omega^x \quad \text{if } g \in V_4 + \omega,\ \omega \in S_{\{2,3,4\}}.$$

Note : $\psi_o = \psi_1 \rho_o = \psi_1$ since $g\psi_o = (g\psi_1)\rho_o = - o + g\psi_1 + o$ $= g\psi_1$.

In the following , we use the symbol ψ_1 instead of ψ_o .

Hence we have proved the following theorem .

Theorem 5.6. Let $End(S_4)$ be the set that consists of all the endomorphisms of S_4 . Then

$$End(S_4) = \{ 0 , \varphi_y , \psi_x , \rho_x ; x , y \in S_4 , |y| = 2 \}$$

where

$0 : S_4 \longrightarrow S_4$ via $g0 = o$ $\forall g \in S_4$,

$\varphi_y : S_4 \longrightarrow S_4$ via $A_4\varphi_y = o$, $(S_4 - A_4)\varphi_y = y$

where $y \in S_4$, $|y| = 2$,

$\psi_x : S_4 \longrightarrow S_4$ via $g\psi_x = \omega^x$

where $g \in V_4 + \omega$, $\omega \in S_{\{2,3,4,\}}$, $x \in S_4$ and

$\rho_x : S_4 \longrightarrow S_4$ where $x \in S_4$ via

$g\rho_x = g^x$ $\forall g \in S_4$.

Having obtained Theorem 5.6 , we now prove

Theorem 5.7. With the above notation , $End(S_4)$ is a multiplicative semigroup . Besides we have the following multiplication table :

Table 5B

(1) $0\beta = \beta 0 = 0$ $\forall \beta \in End(S_4)$

(2) $\varphi_x \varphi_y = 0$ if $x \in A_4$

(2) $\varphi_x \varphi_y = \varphi_y$ if $x \notin A_4$

(3) $\rho_x \rho_y = \rho_{x+y}$

(4) $\psi_x \psi_y = \psi_{(x\psi_1)+y}$

(5) $\varphi_x \rho_y = \varphi_{x^y}$

(6) $\rho_y \varphi_x = \varphi_x$

(7) $\varphi_x \psi_y = \varphi_{x\psi_y}$

(8) $\psi_x \varphi_y = \varphi_y$

(9) $\psi_x \rho_y = \psi_{x+y}$

(10) $\rho_x \psi_y = \psi_{(x\psi_1)+y}$

Proof : Here it suffices to show the multiplication table from (2) to (10) .

(2) If $g \in A_4$, then

$$g\varphi_x \varphi_y = o\varphi_y = o \ .$$

If $g \notin A_4$, then

$$g\varphi_x \varphi_y = x\varphi_y = \begin{cases} o & \text{if} \quad x \in A_4 \\ y & \text{if} \quad x \notin A_4 \end{cases}$$

Hence result . $\varphi_x \varphi_y = \begin{cases} o & \text{if} \quad x \in A_4 \\ \varphi_y & \text{if} \quad x \notin A_4 \end{cases}$

(3) $\forall \ g \in S_4$,

$$g\rho_x \rho_y = (- x + g + x)\rho_y$$

$$= - y + (- x + g + x) + y$$

$$= - (x + y) + g + (x + y)$$

$$= g\rho_{x+y} \ .$$

Hence $\rho_x \rho_y = \rho_{x+y}$.

(4) $\forall g \in S_4$, $g = v + s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$,

then

$$g\psi_x\psi_y = (v + s)\psi_1\rho_x\psi_y \qquad \text{since} \quad \psi_x = \psi_1\rho_x$$

$$= (v\psi_1 + s\psi_1)\rho_x\psi_y$$

$$= (o + s)\rho_x\psi_y$$

$$= (- x + s + x)\psi_1\rho_y$$

$$= (-(x\psi_1) + s + (x\psi_1))\rho_y$$

$$= - y - (x\psi_1) + s + (x\psi_1) + y$$

$$= - (x\psi_1 + y) + s + (x\psi_1 + y)$$

$$= s\rho_{x\psi_1 + y}$$

$$= (v + s)\psi_1\rho_{x\psi_1 + y}$$

$$= g\psi_{x\psi_1 + y} \quad .$$

Hence $\psi_x\psi_y = \psi_{x\psi_1 + y}$.

(5) If $g \in A_4$, then

$$g\varphi_x\rho_y = o\rho_y = o \quad .$$

If $g \notin A_4$, then

$$g\varphi_x\rho_y = x\rho_y = x^y = g\varphi_{x^y} \quad .$$

Hence $\varphi_x\rho_y = \varphi_{x^y}$

(6) If $g \in A_4$, then

$$g\rho_y\varphi_x = (g^y)\varphi_x$$

$$= o \qquad \text{since} \quad g^y \in A_4 \quad .$$

If $g \notin A_4$, then

$$g\rho_y\varphi_x = (-y + g + y)\varphi_x$$

$$= -(y\varphi_x) + x + (y\varphi_x)$$

$$= \begin{cases} x & \text{if } y \in A_4 \\ -x + x + x = x & \text{if } y \notin A_4 \end{cases} .$$

Hence $\rho_y\varphi_x = \varphi_x$.

(7) If $g \in A_4$, then

$$g\varphi_x\psi_y = o\psi_y = o .$$

If $g \notin A_4$, then

$$g\varphi_x\psi_y = x\psi_y$$

$$= g\varphi_{x\psi_y} .$$

Hence $\varphi_x\psi_y = \varphi_{x\psi_y}$.

(8) $\forall g \in S_4$, $g = v + s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$.

If $g \in A_4$, then $s \in A_{\{2,3,4\}}$. So $s^x \in A_4$ for every x in S_4 . If $g \notin A_4$, then $s \in S_{\{2,3,4\}} - A_{\{2,3,4\}}$. So $s^x \notin A_4$ for every x in S_4 . Then

$$g\psi_x\varphi_y = (v + s)\psi_1\rho_x\varphi_y$$

$$= s^x\varphi_y = \begin{cases} o & \text{if } g \in A_4 \\ y & \text{if } g \notin A_4 \end{cases}$$

Hence $\psi_x\varphi_y = \varphi_y$.

(9) $\psi_x\rho_y = (\psi_1\rho_x)\rho_y = \psi_1(\rho_x\rho_y) = \psi_1\rho_{x+y} = \psi_{x+y}$.

(10) $\forall\ g \in S_4$, $g = v + s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$.

Then

$$g\rho_x\psi_y = (\,-x + v + s + x\,)\psi_1\rho_y$$
$$= (\,-(x\psi_1) + o + s + (x\psi_1)\,)\rho_y$$
$$= -\,(x\psi_1)\rho_y + s^y + (x\psi_1)\rho_y$$
$$= -y - x\psi_1 + y - y + s + y - y + x\psi_1 + y$$
$$= -\,(\,x\psi_1 + y\,) + s + (\,x\psi_1 + y\,)$$
$$= s\rho_{x\psi_1+y}$$
$$= (\,v + s\,)\psi_1\rho_{x\psi_1+y}$$
$$= g\psi_{x\psi_1+y}$$

Hence $\rho_x\psi_y = \psi_{x\psi_1+y}$.

From Table 5B , we would like to point out the following facts which are important for later use :

$$\varphi_y^2 = \varphi_y \qquad \text{if} \quad y \in S_4 - A_4 \ , \ |y| = 2 \ ,$$

$$\psi_1^2 = \psi_1 \qquad \text{and} \quad \psi_x = \psi_1\rho_x \quad \forall \quad x \in S_4 \ .$$

With the help of Theorems 5.6 and 5.7 , we now turn to the structure of $E(S_4)$ in the next chapter .

Chapter 6

The Structure of $E(S_4)$

As in Theorem 1.18 , if we choose a suitable idempotent element e of $E(S_4)$, we would have a semi-direct decomposition of $E(S_4)$ into the two summands $\mathrm{Ann}(e)$ and $eE(S_4)$. Now we choose ψ_1 as our idempotent element . Thus the following theorem is immediate .

Theorem 6.1. Let $\psi_1 \in \mathrm{End}(S_4)$. Then

$$E(S_4) = \mathrm{Ann}(\psi_1) + \psi_1 E(S_4)$$

where

$\mathrm{Ann}(\psi_1)$ = normal closure of $r - \psi_1 r$ for all $r \in \mathrm{End}(S_4)$,

$\psi_1 E(S_4)$ = group generated by $\psi_1 r$ for all $r \in \mathrm{End}(S_4)$

and

$$\mathrm{Ann}(\psi_1) \cap \psi_1 E(S_4) = \{ 0 \} .$$

Moreover $\mathrm{Ann}(\psi_1) = gp< -\alpha + (1 - \psi_1)\rho_x + \alpha \ ; \alpha \in E(S_4) \ , x \in S_4 >$ and $\psi_1 E(S_4) = gp< \psi_x , \varphi_y \ ; \ x , y \in S_4 \ , |y| = 2 >$.

Proof : The first part of the proof is an immediate consequence of Theorem 1.18 . From the multiplication table (5B) of $E(S_4)$, we have

$$0 - \psi_1 0 = 0 \quad , \quad \varphi_y - \psi_1 \varphi_y = 0 \ ,$$

$$\psi_x - \psi_1 \psi_x = 0 \quad , \quad \rho_x - \psi_1 \rho_x = (1 - \psi_1)\rho_x$$

and $\psi_1 0 = 0$, $\psi_1 \varphi_y = \varphi_y$, $\psi_1 \psi_x = \psi_x$, $\psi_1 \rho_x = \psi_x$ for all $\varphi_y , \psi_x , \rho_x \in \mathrm{End}(S\ )$, $x , y \in S_4$, $|y| = 2$.

Hence result .

In the following, first let us turn our attention to $Ann(\psi_1)$ and determine all the possible maps. Since S_4 can be decomposed into a semi-direct sum of summands V_4 and $S_{\{2,3,4\}}$, so for all $g \in S_4$, g can be written uniquely as $v + s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$. As we shall see $(1-\psi_1)$ acts on $g = v + s$, gives v and the elements of $Ann(\psi_1)$ are those that can be written as a sum of conjugates of $(1-\psi_1)\rho_x$ $(x \in S_4)$ by the elements of $E(S_4)$. So $(1-\psi_1)\rho_x$ has the same effect on $v+s$ as it does on v. Since every map in $Ann(\psi_1)$ is of the form

$$\sum (- \alpha_i + (1 - \psi_1)\rho_x + \alpha_i)$$

where $x \in S_4$, $\alpha_i \in E(S_4)$, we then have

$$\begin{aligned}(v + s)\sum (- \alpha_i + (1 - \psi_i)\rho_x + \alpha_i) \\ = \sum (- (v+s)\alpha_i + (v+s)(1-\psi_1)\rho_x + (v+s)\alpha_i) \\ = \sum (- (v+s)\alpha_i + v\rho_x + (v+s)\alpha_i) \\ = \sum v^{x+(v+s)\alpha_i} \in V_4 .\end{aligned}$$

Thus $Ann(\psi_1)$ consists of maps of $E(S_4)$ which send S_4 into V_4.

It is a well-known fact that the set $\{ \rho_x ; x \in S_4 \}$ acting on V_4 gives the whole ring of endomorphisms of V_4, denoted by $R(V_4)$, which is isomorphic to $M_2(Z_2)$, the ring of 2×2 matrices over Z_2. Thus $R(V_4)$ consists of 16 elements. A routine calculation shows that

$$R(V_4) = gp\langle (1-\psi_1)(1+\rho_{(12)})\rho_x,\ (1-\psi_1)(1+\rho_{(13)})\rho_x,$$
$$(1-\psi_1)(1+\rho_{(23)})\rho_x\rangle .$$

which is a sub-near-ring of $E(S_4)$ and is contained in $Ann(\psi_1)$.

Since $S_4 = V_4 + S_{\{2,3,4\}}$, $g \in S_4$ can be written in the form of $v+s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$. Thus

$$v + s \xrightarrow{1 - \psi_1} v \xrightarrow{1 + \rho_x} v + v^x ,$$

since $(v+s)(1-\psi_1) = (v+s) - (v\psi_1 + s\psi_1) = (v+s)-(o+s) = v$

Therefore

$$(1-\psi_1)(1+\rho_x) : o + S_{\{2,3,4\}} \longrightarrow o$$
$$v + S_{\{2,3,4\}} \longrightarrow v + v^x$$

where $v \in V_4-\{o\}$, $x \in S_4$.

So

$$(1-\psi_1)(1+\rho_{(12)}) : o + S_{\{2,3,4\}} \longrightarrow o$$
$$((12)+(34)) + S_{\{2,3,4\}} \longrightarrow o$$
$$((13)+(24)) + S_{\{2,3,4\}} \longrightarrow (12)+(34)$$
$$((14)+(23)) + S_{\{2,3,4\}} \longrightarrow (12)+(34)$$

$$(1-\psi_1)(1+\rho_{(13)}) : o + S_{\{2,3,4\}} \longrightarrow o$$
$$((12)+(34)) + S_{\{2,3,4\}} \longrightarrow (13)+(24)$$
$$((13)+(24)) + S_{\{2,3,4\}} \longrightarrow o$$
$$((14)+(23)) + S_{\{2,3,4\}} \longrightarrow (13)+(24)$$

$$(1-\psi_1)(1+\rho_{(23)}) : o + S_{\{2,3,4\}} \longrightarrow o$$
$$((12)+(34)) + S_{\{2,3,4\}} \longrightarrow (14)+(23)$$
$$((13)+(24)) + S_{\{2,3,4\}} \longrightarrow (14)+(23)$$
$$((14)+(23)) + S_{\{2,3,4\}} \longrightarrow o$$

Here we denote $a = (12)+(34)$, $b = (13)+(24)$, $c = (14)+(23)$, we then have $2a = 2b = 2c = o$ and $a+b = b+a = c$. Since each of the above functions sends $o + S_{\{2,3,4\}}$ to o , each function can then be represented by a 3-tuple: the first co-ordinate being the image of $a + S_{\{2,3,4\}}$, the second the image of $b + S_{\{2,3,4\}}$ and the third the image of $c + S_{\{2,3,4\}}$. Thus $(1-\psi_1)(1+\rho_{(12)}) = (o , a , a)$, $(1-\psi_1)(1+\rho_{(13)}) = (b , o , b)$ and $(1-\psi_1)(1+\rho_{(23)}) = (c , c , o)$. Hence

$gp< (1-\psi_1)(1+\rho_x)\rho_y ; y, x \in \{ (12) , (13) , (23) \} >$

$= gp< (o , x , x) , (x , o , x) , (x , x , o) >$

$= \{ (o , o , o) , (a , b , c) , (a , c , b) , (b,a,c) ,$

$(b , c , a) , (c , a , b) , (c , b , a) , (o , a , a),$

$(o , b , b) , (o , c , c) , (a , o , a) , (b , o , b),$

$(c , o , c) , (a , a , o) , (b , b , o) , (c , c , o) >.$

Hence we obtain the following theorem .

Theorem 6.2. The action of the endomorphism near-ring $E(S_4)$ restricted to V_4 gives a ring of order 16 which is in fact the ring of all endomorphisms of V_4 , denoted by $R(V_4)$, and is isomorphic to $M_2(Z_2)$.

Remark : By what we have already shown in Theorem 6.2 , $R(V_4)$ can be expressed in the following two versions for future use , i.e.

$R(V_4) = gp< (1-\psi_1)(1+\rho_x)\rho_y ; y,x \in \{(12),(13),(23)\} >$ (6.2,A)

$$= gp\langle (1+\rho_x)\rho_y \,;\, y, x \in \{(12),(13),(23)\} \rangle \qquad (6.2,B)$$

Notice that here we are abusing notation . Readers should make sure that (6.2,A) consists of those maps which act on the whole group S_4 while (6.2,B) consists of those maps that act on V_4 only !

Now we pause for a while and turn our attention to the structure of the sub-near-ring $\psi_1 E(S_4)$. Since we have already shown that $\psi_1 E(S_4)$ contains a generating set $\{ \varphi_x , \psi_y \; ; \; x, y \in S_4 , |x| = 2 \}$ we can examine $\psi_1 E(S_4)$ through the following steps .

In the first place let us look at the elements φ_x where $x \in S_4$, $|x| = 2$. Since

$$\varphi_x + \varphi_y : \begin{array}{l} g \in A_4 \longrightarrow 0 \\ g \notin A_4 \longrightarrow x + y \end{array} ,$$

therefore we have

$$\varphi_x + \varphi_y = \varphi_{x+y} \quad .$$

Thus elements φ_x where $x \in S_4$ and $|x| = 2$ generate additively a sub-near-ring of $E(S_4)$ of order 24 , i.e.

$$gp\langle \varphi_x \; ; \; x \in S_4 , |x| = 2 \rangle = \{ \varphi_x \; ; \; x \in S_4 \} ,$$

which is obviously contained in $\psi_1 E(S_4)$ and is group isomorphic to $(S_4 , +)$. Secondly , if we examine those elements of the form ψ_x where $x \in S_4$, we immediately gain the idea that ψ_x , $x \in S_4$, has the same effect on $v+s$ as it does on s . Since $S_4 = V_4 + S_{\{2,3,4\}}$, ψ_x is the map that sends

$$g = v+s \longrightarrow (v+s)\psi_x = (v+s)\psi_1\rho_x = ((v\psi_1)+(s\psi_1))\rho_x = s\rho_x$$

where $v \in V_4$, $s \in S_{\{2,3,4\}}$. So by restricting the elements of $\{ \psi_x \; ; \; x \in S_{\{2,3,4\}} \}$ to $S_{\{2,3,4\}}$, we know that ψ_x is simply an inner automorphism of $S_{\{2,3,4\}}$ ($\cong S_3$) . This shows that $\{ \psi_x \; ; x \in S_{\{2,3,4\}} \}$ generates a sub-near-ring of $E(S_4)$ which is isomorphic to the endomorphism near-ring $E(S_3)$ that has been given by J. J. Malone and C.G. Lyons [14] .Hence we have enough to say that $E(S_4)$ acts on S_4 / V_4 to give a sub-near-ring of $E(S_4)$ which is isomorphic to $E(S_3)$. Thus we have proved the following theorem .

Theorem 6.3. The action of $E(S_4)$ on the quotient group S_4 / V_4 gives a sub-near-ring of $E(S_4)$ which is isomorphic to $E(S_3)$.

Before we proceed any further , we give some definitions and general results in group theory .

Definition 6.4. A group G is said to be a subdirect product of groups G_i if G is a subgroup of the direct product $\prod G_i$ and for all elements $g_j \in G_j$, there exists at least one element $g \in G$ which has g_j as its j <u>th</u> component .

Thus the following lemma is immediate.

Lemma 6.5. If G is a subdirect product of $H + K$ where H , K are groups and $H+0 \subseteq G$ ($0+K \subseteq G$) then $G = H + K$.

Definition 6.6. An R-module G has an R-series

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots\cdots\cdots \rhd G_r = \{o\} \qquad (1)$$

if it satisfies the following conditions

(a) $G_i \lhd_R G_{i-1}$ $\quad \forall\, i\ ,\ 1 \leq i \leq r$

(b) $G - \{o\} = \bigcup_{i=1}^{r} (G_{i-1} - G_i)$.

Here the symbol $H \lhd_R G$ denotes that H is an R-submodule of G . Such a series of length r is said to be of type r . The series is said to be invariant if , in addition , $G_i \lhd_R G$ for all $i \in \{1, 2, 3, \ldots\ldots, r\}$.

Definition 6.7. A near-ring is called an annihilating near-ring if there exists an R-module G with an R-series as in (1) such that $G_{i-1}R \leq G_i$ for all $i \in \{1, 2, \ldots\ldots, r\}$.

The following two theorems are due to J. D. P. Meldrum [15] .

Theorem 6.8. Let R be a near-ring and G an R-module with an R-series of type n which is annihilated by R . Then $R^n \leq \mathrm{Ann}(G)$.

Theorem 6.9. Let R be a near-ring with a faithful representation on an R-module G . Then there exists an ideal N of R which is a faithful annihilating near-ring and R/N is a subdirect product of semi-primitive near-rings .

It is easy to see that the symmetric group S_4 is an $E(S_4)$-module having an $E(S_4)$-series $\{o\} \lhd V_4 \lhd S_4$ of type 2 . Thus the following lemma is immediate .

Lemma 6.10. Let $N = \{ \theta \in E(S_4) ; \theta : V_4 \longrightarrow o , S_4 - V_4 \longrightarrow V_4 \}$. Then S_4 is an N-module , N is an annihilating near-ring of the N-series $\{o\} \lhd V_4 \lhd S_4$ and $N^2 = \{ o \}$.

Proof : The proof is immediate from Definition 6.7 and Theorem 6.8 .

Since V_4 is abelian and S_4/V_4 is isomorphic to S_3 , so by Theorem 6.2 and 6.3 $E(S_4)$ acts on V_4 giving $R(V_4)$ and on S_4/V_4 giving $E(S_3)$. We then have the next theorem .

Theorem 6.11. With the notation as above , we have

$$E(S_4)/_N \lesssim E(S_3) + M_2(Z_2)$$

and $N^2 = \{ o \}$.

Here $\lesssim$ means " is a subdirect product of " .

Proof : It is immediate from Lemma 6.10 , Theorem 6.9 and the remarks above .

Corollary 6.12. As in Theorem 6.11 , we in fact have

$$E(S_4)/_N \cong E(S_3) + M_2(Z_2) .$$

Proof : By what we have stated in Lemma 6.5 , it suffices to show that either $E(S_3) + 0 \subseteq E(S_4)/_N$ or $0 + M_2(Z_2) \subseteq E(S_4)/_N$.

Since $M_2(Z_2) \cong R(V_4) \subseteq E(S_4)$, we only need to show there exists elements of $E(S_4)$ which generate $R(V_4)$. By what we have proved above , those elements of the form $\sum (1-\psi_1)\rho_x$ ($x \in S_4$) in $E(S_4)$ do generate $R(V_4)$. Again the map $(1-\psi_1)\rho_x$ ($x \in S_4$) does send S_4 into V_4 . This shows $0 + R(V_4) \subseteq E(S_4)/_N$.

Hence result .

Now we need the structure of N . The following lemmas do give some description of N but not necessarily the whole of N .

Lemma 6.13. Let $\psi_1 E(S_4) = gp\langle \psi_x , \varphi_y ; x , y \in S_4 , |y| = 2 \rangle$. Then there exists a map $\theta \in \psi_1 E(S_4)$ such that

$$\theta : V_4 + x \longrightarrow y$$

$$S_4 - (V_4 + x) \longrightarrow o$$

where $x \in S_{\{2,3,4\}}$, $x \neq o$ and $y \in V_4$, $y \neq o$.

Proof : Without loss of generality , if we take

$$\theta = \psi_1 (1+1)(1+\rho_{(13)})(\rho_{(23)}+1)$$

where 1 denotes the identity element of $E(S_4)$, a little calculation shows that θ does send $V_4 + (234)$ to $(14)+(23)$ and the rest $S_4 - (V_4 + (234))$ to zero . Again if we take

$$\beta = \psi_1 (1+1+1)(\rho_{(12)}+\rho_{(34)})(\rho_{(23)}+1)$$

then

$$\beta : V_4 + (23) \longrightarrow (14)+(23)$$

$$S_4 - (V_4 + (23)) \longrightarrow o \quad .$$

Hence result .

From Lemma 6.13 , a problem arises whether it is possible or not that there are maps in $E(S_4)$ which send V_4 to zero and two distinct elements of one coset of V_4 into distinct elements in V_4 . Unfortunately , the answer is yes . For if we take

$$\beta = \sum (1-\psi_1)\rho_x = (1-\psi_1)\sum \rho_x \in \mathrm{Ann}(\psi_1)$$

then for each $g = v+s \in S_4$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$,

we have

$$(v + s)\beta = v\beta \in V_4 .$$

Since $-1+\beta+1 : v \longrightarrow v(-1+\beta+1) = -v+v\beta+v = v\beta$

$$v+s \longrightarrow (v+s)(-1+\beta+1) = -(v+s)+v\beta+(v+s) = -s+v\beta+s ,$$

therefore $-\beta-1+\beta+1 : v \longrightarrow o$

$$v+s \longrightarrow -v\beta-s+v\beta+s .$$

In particular if we take $v_1\beta = o$, $v_2\beta = v_2$ where $v_1 , v_2 \in V_4$ and v_1 , v_2 are distinct , then

$$(v_1 + s)(- \beta - 1 + \beta + 1) = o$$

$$(v_2 + s)(- \beta - 1 + \beta + 1) = - v_2 - s + v_2 + s$$

where $- v_2 - s + v_2 + s$ in general does not equal to zero . Here we would like to give an example by taking $\beta = (1-\psi_1)(\rho_{(23)}+\rho_{(132)})$, $v_1 = (12)+(34)$, $v_2 = (13)+(24)$ and $s = (23)$.

Then we have

$$(v_1 + s)(- \beta - 1 + \beta + 1) = o$$

$$(v_2 + s)(- \beta - 1 + \beta + 1) = (14)+(23)$$

Thus we have proved the following lemma .

Lemma 6.14. In $E(S_4)$, there are maps which send V_4 to zero and two distinct elements of one coset of V_4 ($\neq V_4$) to two distinct elements in V_4 .

Theorem 6.15. Let $x \in E(S_4)$. Then $x = y + z$ where $y \in Ann(\psi_1)$, $z \in \psi_1 E(S_4)$. If $x \in N$ (the same notation as in Lemma 6.10) then y , $z \in N$.

Proof : As we know from Theorem 6.1 , every element $x \in E(S_4)$, x can be written as a unique sum in the form of $x = y+z$ where $y \in Ann(\psi_1)$ and $z \in \psi_1 E(S_4)$. If $x \in N$ then $V_4 x = o$. Therefore

$$v(y + z) = o \qquad \forall \; v \in V_4 \; .$$

This implies $vy + vz = o$. But $vz = o \; \forall \; v \in V_4$ (since ψ_1 maps V_4 to zero and hence so does $\psi_1 E(S_4)$) . Therefore

$$vy = o \qquad \forall \; v \in V_4 \; .$$

This implies

$$y \in Ann(V_4) \cap Ann(\psi_1) \subseteq N \cap Ann(\psi_1) \; ,$$

since we have shown that $Ann(\psi_1)$ maps S_4 into V_4 .

Therefore

$$y \in N \; .$$

Since $z = - y + x \Longrightarrow z \in N$. Hence result .

Corollary 6.16. The nilpotent ideal N of $E(S_4)$ is a semi-direct sum of the two intersections $Ann(\psi_1) \cap N$ and $\psi_1 E(S_4) \cap N$, i.e. $N = Ann(\psi_1) \cap N + \psi_1 E(S_4) \cap N$.

Proof : It is immediate from Theorem 6.15 .

Now our main work is going to be to determine the structure of these two intersections $Ann(\psi_1) \cap N$ and $\psi_1 E(S_4) \cap N$.

Since $\psi_1 E(S_4)$ is a sub-near-ring of $E(S_4)$ which is generated additively by those endomorphisms of the form φ_x , ψ_y where x , $y \in S_4$, $|x| = 2$, and φ_x , ψ_y are being constructed by using the normal subgroups A_4 and V_4 of S_4 as their kernels respectively , these do guarantee that all the maps in the sub-near-ring $\psi_1 E(S_4)$ send V_4 to zero . Thus , from this remark and Lemma 6.13 , we have the next theorem immediately .

Theorem 6.17. Let $\psi_1 E(S_4)$ and N be as described in Theorem 6.1 and Lemma 6.10 respectively . Then $\psi_1 E(S_4) \cap N$ is a sub-near-ring of $E(S_4)$ which consists of the maps that send V_4 to zero and the coset $V_4 + x$ where $x \in S_{\{2,3,4\}} - \{ o \}$ into an element of V_4 . Moreover the size of this sub-near-ring is equal to 4^5 , i.e. $|\psi_1 E(S_4) \cap N| = 4^5$.

Proof : From Lemma 6.13 , we know that $\forall\ x \in S_{\{2,3,4\}} - \{o\}$, there exists map θ_x such that θ_x sends $V_4 + x$ to a non-zero element of V_4 and the rest to zero . So it is easy to see that $\theta_x E(S_4) \cong (V_4 , +)$, $\forall\ x \in S_{\{2,3,4\}} - \{ o \}$. Thus

$$\psi_1 E(S_4) \cap N \cong \sum_x (\theta_x E(S_4))$$

where $x \in S_{\{2,3,4\}} - \{ o \}$. Hence result .

We have already shown that $Ann(\psi_1) = gp< -\alpha+(1-\psi_1)\rho_x+\alpha \ ; \alpha \in E(S_4) \ , \ x \in S_4 >$. Since $(1-\psi_1)\rho_x$ ($x \in S_4$) acts on V_4 as ρ_x does , so the maps of the form $-\alpha+(1-\psi_1)\rho_x+\alpha$ ($x \in S_4$, $\alpha \in E(S_4)$) which send $v \in V_4$ to $v\rho_x$ are in fact not equal to zero in general . So any element y that lies in $Ann(\psi_1)$ which also lies in N must consist of the sum of at least two elements of the form $-\alpha+(1-\psi_1)\rho_x+\alpha$ where $\alpha \in E(S_4)$, $x \in S_4$. To determine the intersection $Ann(\psi_1) \cap N$, we need to find all the maps that lie in $Ann(\psi_1)$ and send V_4 to the identity . Before we proceed any further , let us first examine the action of the map $-\alpha+(1-\psi_1)\rho_x+\alpha$ more closely . Here we have

$$v(-\alpha+(1-\psi_1)\rho_x+\alpha) = -(v\alpha)+v\rho_x+ (v\alpha) \qquad \text{where } v \in V_4$$

$$= v\rho_x$$

as $v\alpha \in V_4$ and V_4 is abelian , and

$$(v+s)(-\alpha+(1-\psi_1)\rho_x+\alpha) = -(v+s)\alpha+v\rho_x+(v+s)\alpha$$

where $v \in V_4$, $s \in S_{\{2,3,4\}}$.

Generally speaking $(v+s)\alpha$ does not commute with elements of V_4 ; it does not unless $(v+s)\alpha \in V_4$. But we have enough to say that $(v+s)\alpha$ can be split as a semi-direct sum $v_\alpha + s_\alpha$ where $v_\alpha \in V_4$, $s_\alpha \in S_{\{2,3,4\}}$, i.e. $(v + s)\alpha = v_\alpha + s_\alpha$. It is obvious that any element $\alpha \in E(S_4)$ that acts on S_4/V_4 is simply a map that sends $V_4 + s$ to $V_4 + s\alpha$. Therefore

$$s_\alpha = v' + s\alpha \qquad \text{where } v' \in V_4$$

This shows that

$$(v+s)(-\alpha+(1-\psi_1)\rho_x+\alpha) = -s\alpha + v\rho_x + s\alpha$$

for all $\alpha \in E(S_4)$, $x \in S_4$.

With this powerful tool , the whole picture of the intersection $N \cap Ann(\psi_1)$ is at hand . Since for any element $g \in S_4$,

g can be written as $g = v + s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$, so any element $s \in S_{\{2,3,4\}}$ can also be written as $s = o + s$ where $o \in V_4$, $s \in S_{\{2,3,4\}}$. Therefore it is easy to see that those maps in $\mathrm{Ann}(\psi_1)$ of the form

$$-\alpha+(1-\psi_1)\rho_x+\alpha-\beta+(1-\psi_1)\rho_x+\beta \quad \text{where } x \in S_4 , \ \alpha , \beta \in E(S_4)$$

that send V_4 to zero , also send $S_{\{2,3,4\}}$ to zero . For if $s \in S_{\{2,3,4\}}$,

$$\begin{aligned} s(-\alpha+(1-\psi_1)\rho_x+\alpha-\beta+(1-\psi_1)\rho_x+\beta) &= (o+s)(-\alpha+(1-\psi_1)\rho_x+\alpha-\beta+(1-\psi_1)\rho_x+\beta) \\ &= -(s\alpha)+o\rho_x+(s\alpha)-(s\beta)+o\rho_x+(s\beta) \\ &= o \quad . \end{aligned}$$

So far we have shown that elements of the form

$$-\alpha+(1-\psi_1)\rho_x+\alpha-\beta+(1-\psi_1)\rho_x+\beta \qquad (\ x \in S_4 \ , \ \alpha , \beta \in E(S_4) \ )$$

send V_4 to zero and $S_4 - V_4$ to V_4 . Again from the previous remark , we know that all these maps do send the representatives of the cosets of V_4 to zero . Now the remaining problem is how do the rest of the elements in each coset , i.e. $V_4 + x$, $x \in S_{\{2,3,4\}} - \{ o \}$, other than the representatives behave . In the following we give some routine steps in finding certain basic elements of $\mathrm{Ann}(\psi_1) \cap N$ which generate a sub-near-ring of $\mathrm{Ann}(\psi_1)$ that contains elements of N . Here let

$$\theta = (-\rho_{(234)} + (\alpha+\omega+\mu+\phi+\eta+\xi) + \rho_{(234)}) + \phi + \beta$$

where

$$\left.\begin{aligned}
\beta &= -\rho_{(1324)}+(1-\psi_1)\rho_{(13)}+\rho_{(1324)}-\rho_{(14)}+(1-\psi_1)\rho_{(13)}+\rho_{(14)}\\
\phi &= -\psi_1+(1-\psi_1)\rho_{(12)}+\psi_1-\varphi_{(14)}+(1-\psi_1)\rho_{(12)}+\varphi_{(14)}\\
\alpha &= -\rho_{(12)}+(1-\psi_1)\rho_{(34)}+\rho_{(12)}+(1-\psi_1)\rho_{(34)}\\
\mu &= -\rho_{(14)}+(-1+\alpha+\gamma+1)+\rho_{(14)}\\
\gamma &= -\rho_{(23)}+(1-\psi_1)\rho_{(12)}+\rho_{(23)}+(1-\psi_1)\rho_{(12)}\\
\omega &= -1+(1-\psi_1)\rho_{(12)}+1+(1-\psi_1)\rho_{(12)}\\
\xi &= -\rho_{(1324)}+\left[-\rho_{(12)}+\beta+\rho_{(12)}+(-1+\alpha+\gamma+1)\right]+\rho_{(1324)}\\
\eta &= -\rho_{(14)}+\left[-\rho_{(12)}+\beta+\rho_{(12)}+(-1+\alpha+\gamma+1)\right]+\rho_{(14)}\ .
\end{aligned}\right\} \quad (6\text{A})$$

Thus θ is the map that sends

$$V_4 + (23) = \begin{bmatrix} o + (23) \\ ((12)+(34))+(23) \\ ((13)+(24))+(23) \\ ((14)+(23))+(23) \end{bmatrix} \begin{matrix} \longrightarrow o \\ \longrightarrow (13)+(24) \\ \longrightarrow (14)+(23) \\ \longrightarrow (12)+(34) \end{matrix}$$

$$S_4 - (V_4 + (23)) \longrightarrow o$$

Hence $\theta \in \mathrm{Ann}(\psi_1) \cap N$. For details of the map θ , see Appendix A .

Here and throughout , we write

$$V_4 + x \longrightarrow \begin{cases} o \\ w \\ y \\ z \end{cases}$$

to denote the correspondence as follows :

$$\begin{aligned} o \quad + x &\longrightarrow o \\ ((12)+(34)) + x &\longrightarrow w \\ ((13)+(24)) + x &\longrightarrow y \\ ((14)+(23)) + x &\longrightarrow z \end{aligned}$$.

In particular $V_4 + x \longrightarrow o$, represents the correspondence that sends the whole coset $V_4 + x$ to o .

Thus the next lemma is immediate .

Lemma 6.18. Let $\theta \in \mathrm{Ann}(\psi_1) \cap N$ be as described above . Then $\theta E(S_4) \cong M_2(Z_2)$ where $M_2(Z_2)$ is the ring of 2×2 matrices over Z_2 and is in fact a right ideal of $E(S_4)$.

Proof : Since θ is the map that sends

$$V_4 + (23) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$S_4 - (V_4 + (23)) \longrightarrow o$$,

so

$$\theta\rho_{(243)} : V_4 + (23) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$S_4 - (V_4 + (23)) \longrightarrow o$$

Hence $\theta E(S_4) = \theta\rho_{(243)} E(S_4)$. By Theorem 6.2 , $\theta\rho_{(243)} E(S_4) \cong M_2(Z_2)$. Thus we have proved the first part of the theorem .

$\theta E(S_4)$ is a right ideal of $E(S_4)$ since $\theta E(S_4) = N \cap \mathrm{Ann}(S_4 - (V_4 + (23))) \cap \mathrm{Ann}(\phi_1)$.

Hence result .

Analogously we can do the same thing to the other cosets V_4+x where $x \in S_{\{2,3,4\}}$, $|x| = 2$. Now the only remaining problem is whether we can do the same job to the cosets $V_4 + x$ for $|x| = 3$, $x \in S_{\{2,3,4\}}$ or not. Fortunately, the answer is yes. In the following we give routine steps in finding such an element. Now let

$$a = (1-\psi_1)\rho_{(134)} - \rho_{(142)} + (1-\psi_1)\rho_{(134)} + \rho_{(142)} .$$

Then

$$a : V_4 \longrightarrow 0 , \qquad V_4+(34) \searrow \begin{cases} 0 \\ 0 \\ (14)+(23) \\ (14)+(23) \end{cases}$$

$$V_4+(23) \searrow \begin{cases} 0 \\ (13)+(24) \\ (13)+(24) \\ 0 \end{cases} , \qquad V_4+(234) \searrow \begin{cases} 0 \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$V_4+(24) \searrow \begin{cases} 0 \\ (12)+(34) \\ 0 \\ (12)+(34) \end{cases} , \qquad V_4+(243) \searrow \begin{cases} 0 \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) . \end{cases}$$

For details of map a, see Appendix A.

Lemma 6.18 gaurantees that there exists a map a' in $N \cap \text{Ann}(\psi_1)$ such that map a' sends cosets $V_4 + (23)$, $V_4 + (24)$ and $V_4 + (34)$ to the same images as map a does, sending the rest of the elements in S_4 to zero. So we have the map

$$a + a' : \quad V_4 + (234) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$: \quad V_4 + (243) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases} \qquad (6B)$$

$$S_4 - \Big((V_4+(234)) \cup (V_4+(243)) \Big) \longrightarrow 0 \ .$$

Now if we take

$$p = -\rho_{(234)} + (\alpha+\omega+\mu+\phi+\eta+\xi) + \rho_{(234)} + (a + a')$$

where α , ω , μ , ϕ , η , ξ and $a+a'$ are the same as in (6A) and (6B) respectively , then p is the map that sends

$$V_4 + (234) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (12)+(34) \\ 0 \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} 0 \\ (14)+(23) \\ (14)+(23) \\ 0 \end{cases}$$

$$S_4 - \Big((V_4+(234)) \cup (V_4+(243)) \Big) \longrightarrow 0 \ .$$

Remark : The map p can easily be checked by using the result of (6B) and that of the Appendix A .

Again if we take $q_1 = \alpha + \omega + \mu$ where α , ω , μ are the same as in (6A) . Then

$$q_1 : \quad V_4 + (34) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$\vdots$$

$$\vdots$$

$$q_1 : V_4 + (234) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases}$$

$$V_4 \cup (V_4+(23)) \cup (V_4+(24)) \longrightarrow o \quad .$$

For details of q_1 , see Appendix A .

Lemma 6.18 again guarantees that there exists a map q_2 in $\mathrm{Ann}(\psi_1) \cap N$ such that $q_1 + q_2$ sends $S_4-((V_4+(234)) \cup (V_4+(243)))$ to zero and the cosets $V_4 + (234)$, $V_4 + (243)$ to the same images as q_1 does . Here we call this map q_3 . Since $\mathrm{Ann}(\psi_1) \cap N$ is a right ideal of $E(S_4)$, $q_3 \rho_{(234)} \in \mathrm{Ann}(\psi_1) \cap N$ where $\rho_{(234)} \in E(S_4)$. Now we denote

$$q = (a + a') + q_3 \rho_{(234)}$$

where $(a + a')$, q_3 are the same as given above .

Thus

$$q : V_4 + (234) \longrightarrow \begin{cases} o \\ o \\ (13)+(24) \\ (13)+(24) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (14)+(23) \\ o \end{cases}$$

$$S_4 - ((V_4+(234)) \cup (V_4+(243))) \longrightarrow o \quad .$$

Hence

$$p + q \;:\; V_4 + (234) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases}$$

$$S_4 - (\, V_4 + (234)\,) \longrightarrow 0 \;.$$

By what we have just shown and together with Lemma6.18 , the following theorem is immediate .

Theorem 6.19. Let $\mathrm{Ann}(\psi_1) \cap N = \{\, \gamma \in \mathrm{Ann}(\psi_1) \;;\; V_4\gamma = 0 \,\}$. Then there exists a map θ in $\mathrm{Ann}(\psi_1) \cap N$ such that θ sends $V_4 + x$ into V_4 for each $x \in S_{\{2,3,4\}} - \{0\}$, sending the representative x to zero , and $S_4 - (\, V_4 + x\,)$ to zero . Moreover $\theta E(S_4)$ is a right ideal of $E(S_4)$ and is in fact isomorphic to $M_2(Z_2)$. Thus we also have $\mathrm{Ann}(\psi_1) \cap N \cong \sum_5 M_2(Z_2)$.

Proof : It is an immediate consequence of Lemma 6.18 and the remarks above .

From the last statement of Theorem 6.19 , we know that the size of the right ideal $\mathrm{Ann}(\psi_1) \cap N$ is 4^{10} . As we have already shown in Theorem 6.17, the size of $\psi_1 E(S_4) \cap N$ is 4^5 . Thus the size of N is at hand . Since the nilpotent ideal N of $E(S_4)$, known by Corollary 6.16 , is a semi-direct sum of $\mathrm{Ann}(\psi_1) \cap N$ and $\psi_1 E(S_4) \cap N$, we then have

$$N \cong (\, \psi_1 E(S_4) \cap N \,) + \sum_5 M_2(Z_2) \;.$$

And the size of N is equal to 4^{15}.

Again from Corollary 6.12, we have

$$E(S_4) / _N \cong E(S_3) \oplus M_2(Z_2)$$

Thus

$$|E(S_4)| = 2^{35}3^3 \quad (= 927,712,935,936)$$

Since $|E(S_3)| = 54$ (see J. J. Malone and C. G. Lyons [14]), $|M_2(Z_2)| = 16$ and $|N| = 4^{15}$.

Chapter 7

The algebraic structure of $E(S_4)$

In this chapter, our main goal is to determine the exact structure of $E(S_4)$ by putting down its precise tables of addition and multiplication.

Let A denote the set of all inner automorphisms of the symmetric group $S_{\{2,3,4\}}$. Then $gp\langle A \rangle = gp\langle \rho_x ; x \in S_{\{2,3,4\}} \rangle$ is a near-ring which is isomorphic to $E(S_3)$. Now if we let $G = gp\langle \rho_x ; x \in S_{\{2,3,4\}} \rangle$, then $\psi_1 G$ is again isomorphic to $E(S_3)$ and is in fact a sub-near-ring of $E(S_4)$. Hence $\psi_1 G \subseteq E(S_4)$. We have already shown that $M_2(Z_2) \cong R(V_4)$ and $R(V_4)$ is a sub-near-ring of $E(S_4)$. Thus $R(V_4) \subseteq E(S_4)$. According to Corollary 6.12, we have

$$E(S_4)/_N \cong E(S_3) \oplus M_2(Z_2) .$$

Then by the application of the theory of group extension, $E(S_4)$ can be immediately written in the form of

$$N + R(V_4) + \psi_1 G .$$

Here we denote $\underline{R}(V_4)$ as the ring of 2×2 matrices which acts on V_4 like $R(V_4)$ and is zero on $S_4 - V_4$. Then

$$\underline{R}(V_4) \cong R(V_4) .$$

By (6.2,A), $R(V_4)$ sends S_4 into V_4, since $R(V_4) \subseteq Ann(\psi_1)$.

So given $x \in R(V_4)$, we can choose $n_x \in N$ such that

$$gx = gn_x \qquad \text{for all} \quad g \in S_4 - V_4 \ .$$

then $\underline{R}(V_4) = \{ (x + n_x) ; x \in R(V_4) \}$ acts like $M_2(Z_2)$ on V_4 and annihilates $S_4 - V_4$.

Since $N \cong \sum_5 R(V_4) + \psi_1 E(S_4) \cap N$, we then have

$$N + R(V_4) = N + \underline{R}(V_4) \ .$$

Thus $E(S_4)$ can be rewritten in the form

$$N + \underline{R}(V_4) + \psi_1 G \ .$$

In the sequel we write an arbitrary element in $E(S_4)$ as

$$(\eta , \gamma , \beta) \quad \text{or} \quad \eta + \gamma + \beta$$

where $\eta \in N$, $\gamma \in \underline{R}(V_4)$, $\beta \in \psi_1 G$, and the map $-\beta + \gamma + \beta$ as γ^β .

Before we proceed any further with the structure of $E(S_4)$, we first take a look at the following lemmas .

Lemma 7.1. Every element in $\underline{R}(V_4)$ commutes additively with each element in N ,i.e. $\eta + \gamma = \gamma + \eta \quad \forall \eta \in N , \ \gamma \in \underline{R}(V_4)$.

Proof : $\forall \ g \in S_4$, $\gamma \in \underline{R}(V_4)$, $\eta \in N$,

$$\begin{aligned} g(\eta + \gamma) &= g\eta + g\gamma \\ &= g\gamma + g\eta \qquad \text{since } g\gamma , g\eta \in V_4 \\ &= g(\gamma + \eta) \ . \end{aligned}$$

Hence result .

Lemma 7.2. Let $N \cdot \psi_1 G = \{ \eta\beta ; \eta \in N , \ \beta \in \psi_1 G \}$. Then $N \cdot \psi_1 G = \{ o \}$.

Proof : Since $N = \{ \eta \in E(S_4) \ ; \ V_4\eta = o \ , \ (S_4-V_4)\eta \subseteq V_4 \}$ and $V_4\beta = o$ for all $\beta \in \psi_1 G$, we then have

$\eta\beta = 0$ for all $\eta \in N$, $\beta \in \psi_1 G$.

Thus $N\cdot\psi_1 G = \{ 0 \}$.

Lemma 7.3. For every $\gamma \in \underline{R}(V_4)$, $\beta \in \psi_1 G$, we have

(a) $\gamma + \beta = \beta + \gamma$

(b) $\gamma \cdot \beta = \beta \cdot \gamma = 0$.

Proof : Every element in $\psi_1 G$ acts on $v+s$ (where $v \in V_4$, $s \in S_{\{2,3,4\}}$) as it does on s , i.e. $v\beta = o$, $(v+s)\beta = s\beta$ for all $\beta \in \psi_1 G$.

(a) $\forall \ g \in S_4$, $g = v + s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$, $\gamma \in \underline{R}(V_4)$, $\beta \in \psi_1 G$.

If $s = o$, then

$$v(\gamma + \beta) = v\gamma + o = o + v\gamma = v(\beta + \gamma) .$$

If $s \neq o$, then

$$(v+s)(\gamma + \beta) = o + s\beta = s\beta + o = (v+s)(\beta + \gamma) .$$

(b) $\forall \ g \in S_4$, $g\gamma \in V_4$, $g\beta \in S_{\{2,3,4\}}$ where $\gamma \in \underline{R}(V_4)$, $\beta \in \psi_1 G$. Therefore

$$g(\gamma\cdot\beta) = (g\gamma)\beta = o = (g\beta)\gamma = g(\beta\cdot\gamma) .$$

Hence result .

Lemma 7.4. $(\eta + \gamma)\eta' = 0 \quad \forall \ \eta , \eta' \in N$, $\gamma \in \underline{R}(V_4)$.

Proof : $\forall \ \gamma \in \underline{R}(V_4)$, $\eta , \eta' \in N$, $g \in S_4$, we have

$$g(\eta + \gamma)\eta' = (v+s)(\eta + \gamma)\eta' = ((v+s)\eta + (v+s)\gamma)\eta'$$

where $v \in V_4$, $s \in S_{\{2,3,4\}}$.

If $s = o$, then

$$g(\eta + \gamma)\eta' = (o + v\gamma)\eta' = o \qquad (\text{since } v\gamma \in V_4)$$

If $s \neq o$, then

$$g(\eta + \gamma)\eta' = ((v+s)\eta + o)\eta' = (v+s)(\eta\eta') = o .$$

Since $N^2 = \{ 0 \}$, $\eta\eta' = 0$ the zero map .

Thus $(\eta + \gamma)\eta' = 0$

From the above lemmas , we have enough to put down the precise additive and multiplicative tables for the structure of $E(S_4)$ in the following theorem .

Theorem 7.5. Let $E(S_4) = \{ (\eta , \gamma , \beta) ; \eta \in N , \gamma \in \underline{R}(V_4) , \beta \in \psi_1 G \}$. Then for any two elements (η , γ , β) , $(\eta' , \gamma' , \beta') \in E(S_4)$, we have

$$(\eta , \gamma , \beta) + (\eta' , \gamma' , \beta') = (\eta + \eta'^{-\beta} , \gamma + \gamma' , \beta + \beta')$$

and $(\eta , \gamma , \beta)(\eta' , \gamma' , \beta') = ((\eta + \beta)(\eta' + \gamma') , \gamma\gamma' , \beta\beta')$.

Proof : Given any (η , γ , β) , $(\eta' , \gamma' , \beta') \in E(S_4)$, we have

$$\begin{aligned} (\eta , \gamma , \beta) + (\eta' , \gamma' , \beta') &= (\eta + \gamma + \beta) + (\eta' + \gamma' + \beta') \\ &= \eta + \gamma + \eta'^{-\beta} + \beta + \gamma' + \beta' \\ (\text{by Lemma 7.1 and 7.3(a)}) \quad &= \eta + \eta'^{-\beta} + \gamma + \gamma' + \beta + \beta' \\ &= (\eta + \eta'^{-\beta} , \gamma + \gamma' , \beta + \beta') . \end{aligned}$$

In the following we write an element $g \in S_4$ as $v+s$ where $v \in V_4$, $s \in S_{\{2,3,4\}}$. Since $(\eta, \gamma, \beta)(\eta', \gamma', \beta') = (\eta+\gamma+\beta)\eta' + (\eta+\gamma+\beta)\gamma' + (\eta+\gamma+\beta)\beta'$, we consider the three summands separately as follows :

(1) If $s = o$, then

$$v(\eta+\gamma+\beta)\beta' = (o + v\gamma + o)\beta'$$

$$= o \quad (\text{since } v\gamma \in V_4 \ , \ (v\gamma)\beta' = o)$$

$$= v(\beta\beta') \ .$$

If $s \neq o$, then

$$(v+s)(\eta+\gamma+\beta)\beta' = ((v+s)\eta + o + s\beta)\beta'$$

$$= (s\beta)\beta' \quad (\text{since } (v+s)\eta \in V_4 \ , \ s\beta \in S_{\{2,3,4\}})$$

$$= (v+s)(\beta\beta') \ .$$

Hence $(\eta+\gamma+\beta)\beta' = \beta\beta'$.

(2) If $s = o$, then

$$v(\eta+\gamma+\beta)\gamma' = (o + v\gamma + o)\gamma'$$

$$= v(\gamma\gamma')$$

$$= v((\eta+\beta)\gamma' + \gamma\gamma') \ .$$

Since $v(\eta+\beta)\gamma' = 0$.

If $s \neq o$, then

$$(v+s)(\eta+\gamma+\beta)\gamma' = ((v+s)\eta + o + (v+s)\beta)\gamma'$$

$$= (v+s)((\eta+\beta)\gamma')$$

$$= (v+s)((\eta+\beta)\gamma' + \gamma\gamma') \ .$$

Since $(v+s)(\gamma\gamma') = o$. Hence $(\eta+\gamma+\beta)\gamma' = (\eta+\beta)\gamma'+\gamma\gamma'$.

(3) If $s = o$, then

$$(v+s)(\eta + \gamma + \beta)\eta' = (o + v\gamma + o)\eta'$$

$$= o$$

$$= v(\eta + \beta)\eta' .$$

If $s \neq o$, then

$$(v+s)(\eta + \gamma + \beta)\eta' = ((v+s)\eta + o + (v+s)\beta)\eta'$$

$$= (v+s)((\eta + \beta)\eta') .$$

Hence $(\eta + \gamma + \beta)\eta' = (\eta + \beta)\eta'$.

Thus

$$(\eta , \gamma , \beta)(\eta' , \gamma' , \beta') = (\eta+\beta)\eta' + (\eta+\beta)\gamma' + \gamma\gamma' + \beta\beta'$$

$$= (\eta+\beta)(\eta'+\gamma') + \gamma\gamma' + \beta\beta'$$

$$= ((\eta+\beta)(\eta'+\gamma') , \gamma\gamma', \beta\beta') .$$

Remark : $(\eta + \beta)\gamma' = (\eta + \beta)\gamma' - \beta\gamma' \in N$ since $\beta\gamma' = 0$ (by Lemma 7.3(b)) .

Chapter 8

The radical and maximal right ideals of $E(S_4)$

We shall be getting more familiar with the structure of $E(S_4)$ by studying its radical . From now on , we use J to denote the radical of $E(S_4)$. In the following we look at the radical J in two different aspects . Before we start our investigation , we give the following definition .

Definition 8.1. A variety of groups is the class of all groups that satisfies a given set of laws or words .

Example : The variety of abelian groups is the class of all groups that satisfies the law $[x, y] = -x - y + x + y = 0$.

Let V be a variety of groups and (R, S) a d. g. near-ring . Then we define the variety of d. g. near-rings by $(R, S) \in V$ if $(R, +) \in V$. Note that there will be no confusion in using the same symbol for a variety of groups and a variety of d. g. near-rings .

The next theorem is due to J. D. P. Meldrum [16] .

Theorem 8.2. Let (R, S) be a d. g. near-ring with a faithful representation on the (R, S)-module G . Let $G \in V$, a variety of groups . Then $(R, S) \in V$.

Thus the next lemma is immediate .

Lemma 8.3. The additive group $(E(S_4) , +)$ of the d. g. near-ring $E(S_4)$ is solvable .

Proof : It follows immediately from Theorem 8.2 since $(S_4 , +)$ is solvable .

Now we have enough to say that $E(S_4)$ is a finite d. g. near-ring with identity whose additive group $(E(S_4) , +)$ is solvable . Then , by Theorem 1.15 , the radical J of $E(S_4)$ is nilpotent and the quotient near-ring $E(S_4)/J$ is a ring . This does provide us with a rough idea of what the radical J is . But this is not the end ! One can get the exact algebraic structure of the quotient near-ring $E(S_4)/J$ by applying the powerful structure theorem of Theorem 6.9 and the remarks in Chapter 6 .

Since the symmetric group S_4 has a maximal $E(S_4)$-series

$$\{ o \} \lhd V_4 \lhd A_4 \lhd S_4$$

of type 3 , the next theorem follows in the same way as Corollary 6.12 .

Theorem 8.4. Let J be the radical of $E(S_4)$. Then $E(S_4)$ has J as its nilpotent ideal such that

$$J^3 = \{ o \}$$

and

$$E(S_4)/J \cong Z_2 \oplus Z_3 \oplus M_2(Z_2) .$$

Proof : Since the symmetric group S_4 is an $E(S_4)$-module and has a maximal $E(S_4)$-series $\{ o \} \lhd V_4 \lhd A_4 \lhd S_4$ of type 3 , $E(S_4)$ acting on S_4/A_4 , A_4/V_4 , V_4 gives rise to the images of $E(S_4)$ which are isomorphic to the rings Z_2 , Z_3 and $M_2(Z_2)$ respectively . Thus the proof follows immediately from [14] ,Theorem 6.9 and the remarks in Chapter 6 . Hence result .

Since J is the sum of all the nilpotent ideals of $E(S_4)$, we then have

$$N \subseteq J \qquad (8A)$$

Here we pause for a while to give some definitions and preliminary theorems .

Definition 8.5. Let G be an R-module .

(a) G is called of type 0 if and only if $GR \neq \{ 0 \}$, G is monogenic and has only the trivial R-submodules $\{ 0 \}$ and G .

(b) G is called of type 1 if and only if $GR \neq \{ 0 \}$, G is of type 0 and $gR = \{ 0 \}$ or $gR = G$ for all $g \in G$.

(c) G is called of type 2 if and only if $GR \neq \{ 0 \}$ and G has only the trivial R-subgroups $\{ 0 \}$ and G .

Remark : If R has an identity it is immediate that type 1 and type 2 modules coincide .

In the following we are going to define three radicals for a near-ring R .

Definition 8.6. Let R be a near-ring , $i \in \{ 0 , 1 , 2 \}$.

$$J_i(R) := \cap (\mathrm{Ann}(G) ; G \text{ is an } R\text{-module of type } i)$$

is called the i - radical of R . Here the symbol := means ' is defined to be ' .

Here we denote N(R) to be the sum of all nilpotent ideals of R . Then

$$N(R) \leq J_0(R) \leq J_1(R) \leq J_2(R) .$$

The first of these inequalities can be proved in a straightforward way ; the other two inequalities are obvious from the definitions.

The next theorem is due to J. D. P. Meldrum and C. G. Lyons [17] .

Theorem 8.7. Let G be a finite group , (R , S) have a faithful d. g. representation θ on G such that $S\theta \supseteq \mathrm{Inn}(G)$. Then

$$J_2(R) = J_0(R) = N(R) .$$

The following theorem can be found in G. Pilz's book [19] .

Theorem 8.8. Let R be a zero-symmetric near-ring . If $R = \sum_{\lambda\in\Lambda} I_\lambda$ where I_λ is a direct summand of R , then

$$J_2(R) = \sum_{\lambda \in \Lambda} J_2(I_\lambda) .$$

With the help of Theorem 8.7 and 8.8 , we then prove

Theorem 8.9. Let J be the radical of $E(S_4)$. Then

$$J/_N \cong J(M_2(Z_2)) \oplus J(E(S_3))$$

where $J(M_2(Z_2))$ is the radical of $M_2(Z_2)$ and $J(E(S_3))$ is the radical of $E(S_3)$. Moreover

$$J = N + J(\psi_1 G) .$$

Proof : From Corollary 6.12 , we have

$$E(S_4)/_N \cong M_2(Z_2) \oplus E(S_3) .$$

By Theorem 8.7 and 8.8 , the radical of $M_2(Z_2) \oplus E(S_3)$, denoted by $J(M_2(Z_2) \oplus E(S_3))$, is equal to $J(M_2(Z_2)) \oplus J(E(S_3))$,i.e.

$$J(M_2(Z_2) \oplus E(S_3)) = J(M_2(Z_2)) \oplus J(E(S_3)) .$$

Since $N \subseteq J$ (8A) , we then have

$$J(E(S_4) / N) = J / N$$

Hence

$$J/_N \cong J(M_2(Z_2)) \oplus J(E(S_3)) .$$

It is a well-known fact that $M_2(Z_2)$ is a semi-simple ring , so $J(M_2(Z_2)) = \{ o \}$. Thus

$$J = N + J(\psi_1 G) ,$$

since $E(S_3) \cong \psi_1 G \subseteq E(S_4)$, $J(M_2(Z_2)) = \{ o \}$ and $N \subseteq E(S_4)$.

Hence we have completed the proof .

In the sequel we are going to determine the algebraic structure of $J(\psi_1 G)$. By the result of J. J. Malone and C. G. Lyons [14] , we know that

$$J(E(S_3)) = gp\langle \theta' \rangle \oplus gp\langle \phi' \rangle ,$$

where

$$\theta' : (12) \longmapsto (123)$$
$$(13) \longmapsto (132)$$
$$S_3 - \{ (12) , (13) \} \longrightarrow o ,$$

and

$$\phi' : (12) \longmapsto (123)$$
$$(23) \longmapsto (132)$$
$$S_3 - \{ (12) , (23) \} \longrightarrow o .$$

Since $E(S_3) \cong \psi_1 G$, a routine calculation shows that

$$J(\psi_1 G) = gp\langle \theta \rangle \oplus gp\langle \phi \rangle$$

where

$$\theta : V_4 + (23) \longrightarrow (234)$$
$$V_4 + (24) \longrightarrow (243)$$
$$S_4 - \{(V_4+(23)) \cup (V_4+(24))\} \longrightarrow o$$

and

$$\phi : V_4 + (23) \longrightarrow (234)$$
$$V_4 + (34) \longrightarrow (243)$$
$$S_4 - \{(V_4+(23)) \cup (V_4+(34))\} \longrightarrow o .$$

Thus $J(E(S_3)) \cong J(\psi_1 G)$ under the correspondence

$$\theta' \longrightarrow \theta \quad , \quad \phi' \longrightarrow \phi$$

Hence we have proved the following theorem .

Theorem 8.10. With the notation as above , we have

$$J = N + (gp\langle \theta \rangle \oplus gp\langle \phi \rangle)$$

and

$$| J | = 2^{30} \cdot 3^2 .$$

Proof : The proof of the first part of the theorem is immediate from Theorem 8.9 and the remarks above . The second part follows since $|N| = 4^{15}$ and $|gp\langle \theta \rangle| = |gp\langle \phi \rangle| = 3$.

Analogously to what we have done in Theorem 6.2 , here we let $\underline{R}(V_4) = gp\langle (o, x, x), (x, o, x), (x, x, o) \rangle$ where x is one of a=(12)+(34) , b=(13)+(24) , c=(14)+(23). Since each map in $\underline{R}(V_4)$ sends $S_4 - V_4$ to zero , each map can then be represented by a 3-tuple : the first co-ordinate being the image of a , the second the image of b and the third the image of c . Again it is a well-known fact that there are only three maximal right ideals of $\underline{R}(V_4)$. They are

$$I_1 = \{ (x, x, o) ; x \in V_4 \} ,$$

$$I_2 = \{ (o, x, x) ; x \in V_4 \}$$

and

$$I_3 = \{ (x, o, x) ; x \in V_4 \} . \qquad (8B)$$

Thus we have the following lemma .

Lemma 8.11. The above (8B) is a complete list of maximal right ideals of $\underline{R}(V_4)$.

Before we deal with the maximal right ideals of $E(S_4)$, we need to give a new exact algebraic structure of $E(S_4)$ in terms of its radical and sub-near-rings . According to Theorem 8.4 , we have

$$E(S_4)/_J \cong Z_2 \oplus Z_3 \oplus M_2(Z_2) .$$

Since $M_2(Z_2) \cong \underline{R}(V_4) \subseteq E(S_4)$, we only need to determine two sub-near-rings of $E(S_4)$ which are group isomorphic to $(Z_2, +)$ and $(Z_3, +)$ respectively. From Theorem 5.7 $End(S_4) = \{ \varphi_y, \phi_x, \rho_x, 0 \ ; \ x, y \in S_4, |y| = 2 \}$, without loss of generality if we choose $\varphi_{(12)} \in End(S_4)$, then

$$gp\langle \varphi_{(12)} \rangle = \{ 0, \varphi_{(12)} \} \subseteq E(S_4)$$

and

$$gp\langle \varphi_{(12)} \rangle \cong (Z_2, +).$$

Also we can choose $\phi \in E(S_4)$ such that in $E(S_4)/J$

$$gp\langle \phi + J \rangle \cong (Z_3, +)$$

since $\quad E(S_4)/J \cong Z_2 + Z_3 + M_2(Z_2)$.

Thus we have proved the following theorem.

Theorem 8.12. With the notation as above, we have

$$E(S_4) = J + gp\langle \varphi_{(12)} \rangle + gp\langle \phi \rangle + \underline{R}(V_4).$$

Thus we have

Theorem 8. 13. The following is a complete list of maximal right ideals of $E(S_4)$:

$$J + gp\langle \varphi_{(12)} \rangle + \underline{R}(V_4) \quad , \quad J + gp\langle \phi \rangle + \underline{R}(V_4)$$

and $\quad J + gp\langle \varphi_{(12)} \rangle + gp\langle \phi \rangle + I_i$

where I_i are the maximal right ideals of $\underline{R}(V_4)$, $i \in \{1, 2, 3\}$. Moreover the factors of $E(S_4)$ by these maximal right ideals of $E(S_4)$ as listed above are simply the $E(S_4)$-modules which are isomorphic to Z_3, Z_2 and V_4 (three times) respectively.

Proof : It is immediate from Lemma 8.11 , Theorem 8.12 and the remarks above .

Comments

Now we know the basic structure of the endomorphism near-rings of the symmetric groups . But the following questions are still of great interest .

(1) Besides those monogenic $E(S_n)$-subgroups of $E(S_n)$, where $n \geq 5$, as shown in Chapter 4 , how do the rest behave ?

(2) In the case of $E(S_4)$, what does the complete list of left , right ideals and $E(S_4)$-subgroups look like ?

It is hoped that with the help of Lemma 2.4 , 2.5 , 2.6 and Theorem 7.5 , we can solve these interesting problems in the near future .

PART FOUR

INVERSE SEMIGROUPS OF ENDOMORPHISMS

Here we present a chapter on inverse semigroups of endomorphisms. Those newly developed theorems in this chapter are expected to be powerful tools in tackling the structure of the endomorphism near-rings of an arbitrary group which is the direct sum of groups G_i , $1 \leq i \leq n$ where $G_i \cong G \; \forall i$ and G_i is finite .

Chapter 9

Some theorems on inverse semigroups of endomorphisms

Before we proceed any further , we are going to give some general definitions and basic results of semigroups.

Definition 9.1. An element a of a semigroup S is called regular if there exists an element x in S such that axa = a (note that x is far from unique) . A semigroup S is called regular if every element of S is regular .

Definition 9.2. Two elements a and b of a semigroup S are said to be inverses of each other if aba = a and bab = b .

If an element a of a semigroup S has an inverse in S , then a is evidently regular . The converse (Lemma 9.3) was due toGThierrin [21] . Thus a regular semigroup is one in which every

every element has at least one inverse .

Lemma 9.3. If a is a regular element of S, say $axa = a$ with x in S, then a has at least one inverse in S, in particular xax.

Definition 9.4. By an inverse semigroup we mean a semigroup in which every element has a unique inverse .

The next theorem can be found in any standard text of semigroup theory .

Theorem 9.5. The following two conditions on a semigroup S are equivalent :

(1) S is regular and any two idempotent elements of S commute with each other ;

(2) S is an inverse semigroup .

Now we prove

Theorem 9.6. Let G be a group , θ be an idempotent endomorphism of G. Then G is a semi-direct sum of $\mathrm{Ker}\theta$ and $\mathrm{Im}\theta$, where

$$\mathrm{Ker}\theta = \{ g - g\theta \ ; g \in G \} \ , \ \mathrm{Im}\theta = G\theta$$

and

$$\mathrm{Ker}\theta \cap \mathrm{Im}\theta = \{ 0 \} .$$

Proof : For every $g \in G$, g can be uniquely written as $g = (g - g\theta) + g\theta$ where $g - g\theta \in \mathrm{Ker}\theta$ and $g\theta \in \mathrm{Im}\theta = G\theta$. It is trivial that $\mathrm{Ker}\theta \lhd G$ and $\mathrm{Im}\theta \leq G$. Suppose $g \in G$, then

g can be written in the form

$$g = k + h = k' + h'$$

for some $k , k' \in \mathrm{Ker}\theta$, $h , h' \in \mathrm{Im}\theta$. Then

$$- k' + k = h' + (- h) \in \mathrm{Ker}\theta \cap \mathrm{Im}\theta \ .$$

Let $x \in \mathrm{Ker}\theta \cap \mathrm{Im}\theta$ then $x \in \mathrm{Ker}\theta$ and $x \in \mathrm{Im}\theta$, i.e. $x\theta = o$ and there exists $y \in G$ such that $x = y\theta$. Therefore

$$o = x\theta = (y\theta)\theta = y\theta^2 = y\theta = x \quad ,$$

since θ is an idempotent endomorphism . So we have

$$\mathrm{Ker}\theta \cap \mathrm{Im}\theta = \{ o \} \ .$$

So $- k' + k = o$ and $h' + (-h) = o$ implies $k = k'$ and $h = h'$.

Hence G is a semi-direct sum of $\mathrm{Ker}\theta$ and $\mathrm{Im}\theta$.

If a , b are elements of an inverse semigroup S , we can define a partial order relation $\leq$ on the elements of S by $a \leq b$ if there exists an idempotent element e in S such that $a = eb$. In particular , if both elements e and f are idempotents of S , then we have

$$e \leq f \text{ if and only if } ef = e \ .$$

Here , let e, f be two idempotent elements of an inverse semigroup S which is contained in the set of all endomorphisms of G , denoted by End(G) . So to each pair of idempotent elements e , f of G , there corresponds a semi-direct decomposition $G = K_e + H_e$ and $G = K_f + H_f$ respectively . Here and throughout we denote $K_e = Ker(e)$ (kernel of e) , $K_f = Ker(f)$, $H_e = Im(e) = Ge$, $H_f = Im(f) = Gf$.

In the following, we are going to prove some results about the structure of G .

Theorem 9.7. Let e , f be any two idempotent elements of an inverse semigroup S which is contained in $End(G)$. Then we have

$e \leq f$ if and only if $G = K_e + H_e = K_f + H_f$ with $K_f \subseteq K_e$ and $H_e \subseteq H_f$.

Proof : Assume $e \leq f$ holds . That is $ef = e$.

By Theorem 9.6 , it is trivial that we have the following semi-direct decomposition

$$G = K_e + H_e = K_f + H_f$$

where $K_e = \{ g \in G ; ge = o \}$, $H_e = Ge$, $K_e \cap H_e = \{ o \}$, $K_f = \{ g \in G ; gf = o \}$, $H_f = Gf$ and $K_f \cap H_f = \{ o \}$. For every $g \in K_f$ we have $(g)f = o$. Since

$$(g)e = g(ef) = g(fe) = (gf)e = (o)e = o$$

(for $e \leq f \Longleftrightarrow e = ef = fe$ since any two idempotents of an inverse semigroup commute) this implies $g \in Ker(e) = K_e$

Hence $K_f \subseteq K_e$.

Analogously , we have

$\forall g \in H_e = Ge \Longrightarrow \exists g' \in G$ such that $g = (g')e$. Since

$$g = (g')e = g'(ef) = (g'e)f = (g)f \quad ,$$

this implies $g \in Gf = H_f$. Hence $H_e \subseteq H_f$.

Conversely , let e , f be any two idempotent elements of an inverse semigroup S . Assume $G = K_e + H_e = K_f + H_f$ and $K_f \subseteq K_e$ holds . Then $\forall\ g \in G$, we have $g - gf \in K_f$ since

$$(g - gf)f = gf - gf^2 = gf - gf = o \quad .$$

This implies $g - gf \in K_e$ for $K_f \subseteq K_e$.

Therefore $\quad (g-gf)e = o$

i.e. $\quad ge - gfe = o$

i.e. $\quad ge = gfe = gef \quad$ for all g in G .

Hence $e = ef$, i.e. $e \leq f$.

Here we present an example to show that equality does not hold in general .

Example 1. Let $G = G_1 \times G_2 \times \dots\dots\dots \times G_n$,

$$e : (g_1 , g_2 , \dots\dots\dots , g_n) \longrightarrow (g_1 , 1 , \dots\dots , 1)$$

and $$f : (g_1 , g_2 , \dots\dots\dots , g_n) \longrightarrow (g_1 , g_2 , 1 , \dots , 1)$$

It is obvious that $e^2 = e$ and $f^2 = f$. Also $ef = fe = e$.

Then $\{ e , f \}$ forms an inverse semigroup under the composition of mappings .

(Semi-lattice)

$$K_e = \{ (1, g_2, g_3, \ldots\ldots, g_n) \ ; g_i \in G_i , 2 \leq i \leq n \} ,$$

$$H_e = \{ (g_1, 1, \ldots\ldots\ldots\ldots, 1) \ ; \ g_1 \in G_1 \} ,$$

$$K_f = \{ (1, 1, g_3, g_4, \ldots, g_n) \ ; g_i \in G_i , 3 \leq i \leq n \}$$

and

$$H_f = \{ (g_1, g_2, 1, 1, \ldots\ldots, 1) ; \ g_1 \in G_1 , g_2 \in G \} .$$

So we have

$$K_f \subsetneq K_e \quad \text{and} \quad H_e \subsetneq H_f \ .$$

In general, we can restrict the condition $e \leq f$ stated in Theorem 9.7 to $e < f$ and the set inclusion $\subseteq$ to $\subsetneq$, then we have the following corollary.

Corollary 9.8. The hypotheses are the same as for Theorem 9.7. Then

$$e < f \text{ if and only if } G = K_e + H_e = K_f + H_f \text{ with } K_f \subsetneq K_e$$

and $H_e \subsetneq H_f$.

Proof : Theorem 9.7 shows $e \leq f \Longleftrightarrow K_f \subseteq K_e$, $H_e \subseteq H_f$.

If $e < f$ then $f \leq e$ is false, hence so is $K_e \subseteq K_f$, $H_f \subseteq H_e$. Hence $K_f \subset K_e$ and $H_e \subset H_f$.

Conversely, if $K_f \subset K_e$, $H_e \subset H_f$ then $K_e \subseteq K_f$ and $H_f \subseteq H_e$ are false $\Longrightarrow f \leq e$ false. Hence $e < f$.

If we examine Theorem 9.7 closely, we can observe that the normal subgroup K_e can be further decomposed into a semi-direct sum of K_f and $K_e \cap H_f$.

That is

$$K_e = K_f + (K_e \cap H_f) .$$

For $K_f \subseteq K_e$, K_f , $K_e \lhd G \Longrightarrow K_f \lhd K_e$,

$K_e \cap H_f \subseteq K_e$, $K_e \cap H_f \leqslant G \Longrightarrow K_e \cap H_f \leqslant K_e$,

$K_f \cap (K_e \cap H_f) = (K_f \cap K_e) \cap H_f = K_f \cap H_f = \{ 0 \}$.

Here it is easy to see that $K_f + (K_e \cap H_f) \subseteq K_e$. Now we need to show $K_e \subseteq K_f + (K_e \cap H_f)$. $\forall x \in K_e$

$$x = (x - xf) + xf \in K_f + (K_e \cap H_f) .$$

Since $(x - xf)f = xf - xf^2 = xf - xf = 0 \Longrightarrow x - xf \in K_f$ and $(xf)e = x(fe) = x(ef) = xe = 0 \Longrightarrow xf \in K_e$. But $xf \in H_f$. This implies $xf \in K_e \cap H_f$. Hence $K_e \subseteq K_f + (K_e \cap H_f)$.

Analogously , the subgroup H_f again can further be decomposed into a semi-direct sum of H_e and $K_e \cap H_f$. That is

$$H_f = H_e + (K_e \cap H_f) .$$

Here we can rewrite the semi-direct decomposition of the group G in Theorem 9.7 as

$$G = K_f + (K_e \cap H_f) + H_e \qquad (1)$$

Moreover , if $e_i \in S$ (an inverse semigroup which is contained in End(G)) with $e_i^2 = e_i$ for all $i \in \{ 1 , 2 ,..., n \}$ and $e_i \leqslant e_{i+1}$ where $i = 1 , 2 , 3 ,.........., n - 1$ then we obtain two chains as follows :

$$K_{e_n} \subseteq K_{e_{n-1}} \subseteq \cdots\cdots \subseteq K_{e_1}$$

and

$$H_{e_1} \subseteq H_{e_2} \subseteq \cdots\cdots \subseteq H_{e_n}$$

where $K_{e_i} = K_{e_{i+1}} + (K_{e_i} \cap H_{e_{i+1}})$ and

$$H_{e_{i+1}} = H_{e_i} + (K_{e_i} \cap H_{e_{i+1}})$$

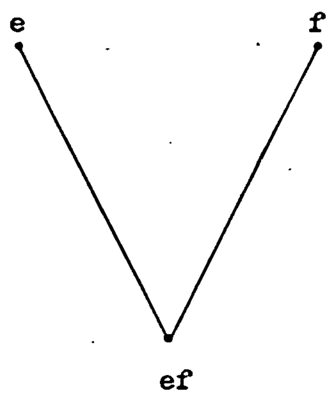
for all $i \in \{ 1 , 2 , 3 , \ldots\ldots\ldots\ldots , n - 1 \}$.

By using (1) and induction on n , we have established

Theorem 9.9. Let S be an inverse semigroup which is contained in End(G) and $e_i \in S$ with $e_i^2 = e_i$ and $e_i \leq e_{i+1}$ where $i = 1 , 2 , \ldots\ldots , n - 1$. Then G has a semi-direct decomposition

$$G = K_{e_n} + \sum_{i=1}^{n-1} (K_{e_{n-i}} \cap H_{e_{n-i+1}}) + H_{e_1} .$$

If e , f are idempotent elements in an inverse semigroup $S \subseteq$ End(G) , then $ef = fe$. It is a well-know fact that we have the following small semi-lattice .



$ef \leq e$
$\leq f$

From Theorem 9.7 , we certainly have K_e , $K_f \subseteq K_{ef}$ and $H_{ef} \subseteq H_e$, H_f . Unfortunately , we cannot get any nice relation among the kernels K_e , K_f and the images H_e , H_f . For

$$G = K_f + (K_{ef} \cap H_f) + H_{ef}$$

or $= K_e + (K_{ef} \cap H_e) + H_{ef}$

where

$$K_{ef} = K_f + (K_{ef} \cap H_f) = K_e + (K_{ef} \cap H_e) ,$$

$$H_f = (K_{ef} \cap H_f) + H_{ef}$$

and

$$H_e = (K_{ef} \cap H_e) + H_{ef} .$$

From the above decomposition , it is obvious that we cannot impose any condition on K_e , K_f , H_e , H_f .

Here we shall consistently write E for the set of idempotent elements of the inverse semigroup S which is contained in End(G). It is a subsemigroup of S , for if $e , f \in E$ then $(ef)^2 = ef$. So ef is again an idempotent element and therefore belongs to E. Indeed it is a commutative semigroup of idempotents and so it forms a lower semilattice . It is a known fact that E is also an inverse semigroup .

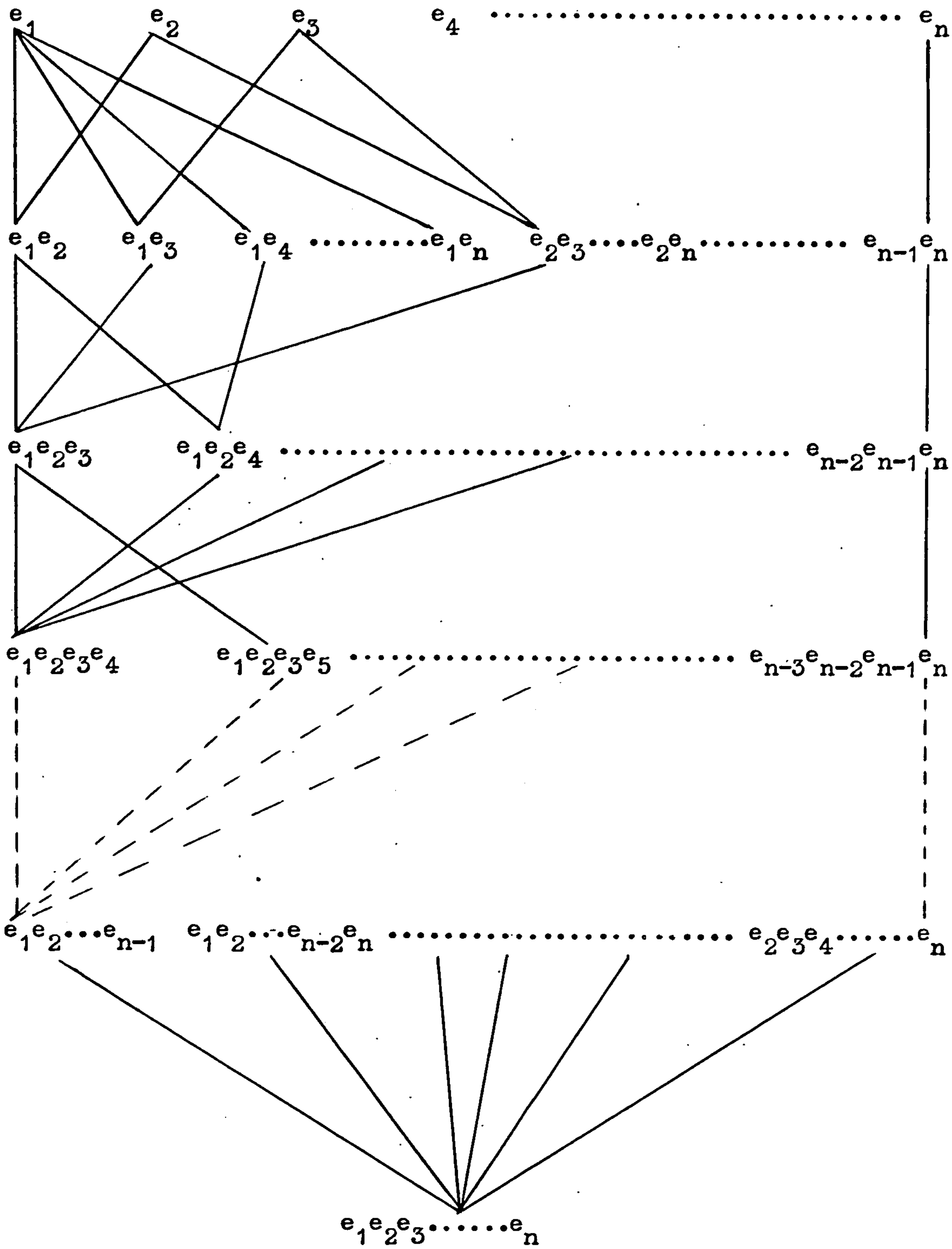
Let J be a finite index set . Suppose $e_i \in E$ with $i \in J$, then the product of any given m terms of $e_i \in E$ with $i \in J$ is the meet of the products of the (m - 1) terms out of the previous given m terms of e_i in E . That is to say if we have any product

$$e_1 e_2 e_3 \cdots\cdots\cdots e_m$$

then

$$e_1 e_2 \cdots\cdots\cdots e_i e_{i+2} e_{i+3} \cdots\cdots e_m \geq e_1 e_2 \cdots\cdots e_m$$

for all i . In the following , we thus build up a Hasse diagram for n given idempotent elements $e_1 , e_2 , \ldots\ldots , e_n$ in E .



It is obvious that the elements in the Hasse diagram may not all be distinct . Hence we have at most $n!$ distinct semi-direct decompositions of the given group G . Without loss of generality , we only consider the following semi-direct decomposition of the group G by using the particular chain

$$e_1e_2\cdots e_n \leq e_1e_2\cdots e_{n-1} \leq \cdots\cdots \leq e_1e_2e_3 \leq e_1e_2 \leq e_1 .$$

By applying the same argument as given for Theorem 9.9, we have

$$K_{e_1} \subseteq K_{e_1e_2} \subseteq K_{e_1e_2e_3} \subseteq \cdots\cdots \subseteq K_{\prod_{i=1}^{n-1} e_i} \subseteq K_{\prod_{i=1}^{n} e_i}$$

and

$$H_{\prod_{i=1}^{n} e_i} \subseteq H_{\prod_{i=1}^{n-1} e_i} \subseteq \cdots\cdots \subseteq H_{e_1e_2e_3} \subseteq H_{e_1e_2} \subseteq H_{e_1}$$

where $\quad K_{\prod_{i=1}^{m} e_i} = K_{\prod_{i=1}^{m-1} e_i} + \left(K_{\prod_{i=1}^{m} e_i} \cap H_{\prod_{i=1}^{m-1} e_i} \right)$

and $\quad H_{\prod_{i=1}^{m-1} e_i} = H_{\prod_{i=1}^{m} e_i} + \left(K_{\prod_{i=1}^{m} e_i} \cap H_{\prod_{i=1}^{m-1} e_i} \right)$

for $m \in \{ 2, 3, \ldots\ldots, n \}$. So by the same argument as in Theorem 9.9, we can decompose the group G as follows :

$$G = K_{e_1} + \sum_{m=2}^{n} \left(K_{\prod_{i=1}^{m} e_i} \cap H_{\prod_{i=1}^{m-1} e_i} \right) + H_{\prod_{i=1}^{n} e_i} .$$

Hence we have established

Theorem 9.10. Let E be the set of idempotent elements of the inverse semigroup $S \subseteq End(G)$. If there are n distinct idempotent elements in E, then we have at most $n!$ distinct semi-direct decompositions of the given group G associated with idempotents of S.

In the sequel we are going to prove some very nice properties of certain kernels of endomorphisms that are contained in an inverse semigroup . Let S be an inverse semigroup that is contained in End(G) . From the definition of an inverse semigroup , if $a \in S$ then there exists a unique element (inverse) $b \in S$ such that $aba = a$ and $bab = b$. Again it is a well-known fact that ab and ba are idempotent elements of S .

Here we denote by K_a , K_b , K_{ab} and K_{ba} the kernels of a , b , ab and ba respectively . Now we claim

$$K_a = K_{ab} \quad .$$

For if $g \in K_a$ then we have $ga = o$. Since

$$g(ab) = (ga)b = ob = o ,$$

then

$$g \in K_{ab} ,$$

that is

$$K_a \subseteq K_{ab} \quad .$$

Again we have

$$K_{ab} \subseteq K_{aba} = K_a \quad ,$$

since $aba = a$. Hence $K_a = K_{ab}$.

Analogously , we have

$$K_b \subseteq K_{ba} \subseteq K_{bab} = K_b \quad .$$

Therefore

$$K_b = K_{ba} \quad .$$

So we have proved

Theorem 9.11. If a is an element of an inverse semigroup $S \subseteq End(G)$ with $aba = a$ and $bab = b$ (where b is a unique inverse of a) then

$$K_a = K_{ab} \text{ and } K_b = K_{ba} .$$

The next corollary is immediate from Lemma 9.3 and Theorem 9.11.

Corollary 9.12. Let a be a regular element of the semigroup $S \subseteq End(G)$. Then there exists an element b in S such that $aba = a$ and $bab = b$ (b is far from unique) ; we have

$$K_a = K_{ab} \text{ and } K_b = K_{ba} .$$

Theorem 9.13. Let $e , f \in End(G)$, $e^2 = e$, $f^2 = f$. Then

$$ef = e \Longleftrightarrow H_e \subseteq H_f .$$

Proof : Assume $ef = e$ holds . Then

$\forall\, y \in H_e \;\exists\, x \in G$ such that $xe = y$. Therefore

$$y = xe = x(ef) = (xe)f = yf \in H_f .$$

Since $e \in End(G) \Longrightarrow y \in G$. Hence $H_e \subseteq H_f$.

Conversely if $H_e \subseteq H_f$, then

$\forall\, x \in G$, $x(ef) = (xe)f = (zf)f$.

Since $xe \in H_e \subseteq H_f \Longrightarrow \exists\, z \in G$ such that $zf = xe$.

Therefore $x(ef) = zf^2 = zf = xe$, since $f^2 = f$.

Hence $ef = e$.

Theorem 9.14. Let $e , f \in End(G)$, $e^2 = e$, $f^2 = f$.

Then $fe = e \Longleftrightarrow K_f \subseteq K_e$.

Proof : Assume $fe = e$ holds . Then

$$\forall x \in K_f , \quad xe = x(fe) \qquad (\text{ since } fe = e)$$
$$= (xf)e$$
$$= oe \qquad (\text{ since } x \in K_f , xf = o)$$
$$= o \qquad (\text{ since } e \in End(G) , oe = o).$$

This implies $x \in K_e$. Hence $K_f \subseteq K_e$.

Conversely if $K_f \subseteq K_e$, then

$\forall g \in G$, we have $g - gf \in K_f$. Since

$$(g - gf)f = gf - gf^2 = gf - gf = o .$$

Since $K_f \subseteq K_e$, $(g - gf)e = o$.

Therefore

$$ge - gfe = o .$$

Thus we have

$$ge = gfe \qquad \forall g \in G .$$

Hence $fe = e$.

Thus we have the next corollary .

Corollary 9.15. Let $e , f \in End(G)$, $e^2 = e$, $f^2 = f$.

Then $K_f \subseteq K_e$ and $H_e \subseteq H_f \Longleftrightarrow ef = fe = e$.

The following example shows that in general the converse of Corollary 9.15 does not hold if we only have $ef = fe$.

Example 2 . Let G_i be an arbitrary group where $i \in \{ 1 , 2 , 3 , 4 , 5 \}$ and $G = G_1 \times G_2 \times G_3 \times G_4 \times G_5$ a group direct sum in the usual sense . Define

$$e , f : G \longrightarrow G \qquad \text{via}$$

$$e : (g_1, g_2, g_3, g_4, g_5) \longmapsto (g_1, 1, 1, 1, g_5) \quad ,$$

$$f : (g_1, g_2, g_3, g_4, g_5) \longmapsto (g_1, g_2, 1, g_4, 1)$$

It is obvious that $e^2 = e$, $f^2 = f$, e , $f \in End(G)$. Again we have

$$ef = fe : (g_1, g_2, g_3, g_4, g_5) \longmapsto (g_1, 1, 1, 1, 1) .$$

But we have

$$K_e = \{ (1, g_2, g_3, g_4, 1) \; ; \; g_i \in G_i \; , \; 2 \leq i \leq 4 \} ,$$

$$K_f = \{ (1, 1, g_3, 1, g_5) \; ; \; g_i \in G_i \; , \; i = 3 , 5 \} ,$$

$$H_e = \{ (g_1, 1, 1, 1, g_5) \; ; \; g_i \in G_i \; , \; i = 1 , 5 \}$$

and

$$H_f = \{ (g_1, g_2, 1, g_4, 1) \; ; \; g_i \in G_i \; , i = 1 , 2 , 4 \} .$$

Thus we cannot impose any relation among the kernels K_e , K_f and the images H_e , H_f even when we have $ef = fe$.

APPENDIX A

In the sequel , we shall give details of how to determine the map θ given in Lemma 6.18 . Throughout the rest , we write

$$V_4 + x \searrow \begin{cases} o \\ w \\ y \\ z \end{cases}$$

to denote the correspondence as follows :

$$\begin{aligned} o + x &\longrightarrow o \\ ((12)+(34)) + x &\longrightarrow w \\ ((13)+(24)) + x &\longrightarrow y \\ ((14)+(23)) + x &\longrightarrow z \end{aligned}$$

In particular $V_4 + x \longrightarrow o$, represents the correspondence that sends the whole coset $V_4 + x$ to o . Furthermore we write

$$V_4'(1-\psi_1)\rho_x = \begin{matrix} ((12)+(34))^x \\ ((13)+(24))^x \\ ((14)+(23))^x \end{matrix}$$

$$S'_{\{2,3,4\}}\rho_x = \begin{matrix} (23)^x \\ (24)^x \\ (34)^x \\ (234)^x \\ (243)^x \end{matrix}$$

(1). Let $\beta = -\rho_{(1324)} + (1-\psi_1)\rho_{(13)} + \rho_{(1324)} - \rho_{(14)} + (1-\psi_1)\rho_{(13)} + \rho_{(14)}$.

$$\text{Since}\quad V_4' \, (1-\psi_1)\rho_{(13)} = \begin{array}{l} (13)+((12)+(34))+(13) = (14)+(23) \\ (13)+((13)+(24))+(13) = (13)+(24) \\ (13)+((14)+(23))+(13) = (12)+(34) \end{array} ,$$

$$S'_{\{2,3,4\}}\rho_{(1324)} = \begin{array}{ll} (1423)+(23)+(1324) & = (24) \\ \qquad\qquad (24) & = (14) \\ \qquad\qquad (34) & = (12) \\ \qquad\qquad (234) & = (142) \\ \qquad\qquad (243) & = (124) \end{array} ,$$

$$S'_{\{2,3,4\}}\rho_{(14)} = \begin{array}{ll} (14)+(23)+(14) & = (23) \\ \qquad (24) & = (12) \\ \qquad (34) & = (13) \\ \qquad (234) & = (123) \\ \qquad (243) & = (132) \end{array} , \quad \text{we then have}$$

$$\begin{array}{lll} (24)+((14)+(23))+(24) & +\ (23)+((14)+(23))+(23) & = (13)+(24) \\ \qquad (13)+(24) & \qquad (13)+(24) & = (14)+(23) \\ \qquad (12)+(34) & \qquad (12)+(34) & = (12)+(34) \end{array}$$

$$\begin{array}{lll} (14)+((14)+(23))+(14) & +\ (12)+((14)+(23))+(12) & = (12)+(34) \\ \qquad (13)+(24) & \qquad (13)+(24) & = (13)+(24) \\ \qquad (12)+(34) & \qquad (12)+(34) & = (14)+(23) \end{array}$$

$$\begin{array}{lll} (12)+((14)+(23))+(12) & +\ (13)+((14)+(23))+(13) & = (14)+(23) \\ \qquad (13)+(24) & \qquad (13)+(24) & = (12)+(34) \\ \qquad (12)+(34) & \qquad (12)+(34) & = (13)+(24) \end{array}$$

$$\begin{array}{lll} (124)+((14)+(23))+(142) & +\ (132)+((14)+(23))+(123) & = 0 \\ \qquad (13)+(24) & \qquad (13)+(24) & = 0 \\ \qquad (12)+(34) & \qquad (12)+(34) & = 0 \end{array}$$

$$\begin{array}{lll} (142)+((14)+(23))+(124) & +\ (123)+((14)+(23))+(132) & = 0 \\ \qquad (13)+(24) & \qquad (13)+(24) & = 0 \\ \qquad (12)+(34) & \qquad (12)+(34) & = 0 \end{array} .$$

$$\text{Thus}\quad \beta : \begin{array}{l} V_4 \longrightarrow 0 \\ V_4 + (234) \longrightarrow 0 \\ V_4 + (243) \longrightarrow 0 \end{array}$$

$$\beta : V_4 + (23) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$V_4 + (24) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (12)+(34) \\ (13)+(24) \end{cases} .$$

(2). Let $\phi = -\psi_1+(1-\psi_1)\rho_{(12)}+\psi_1-\varphi_{(14)}+(1-\psi_1)\rho_{(12)}+\varphi_{(14)}$.

Since
$$V_4' \, (1-\psi_1)\rho_{(12)} = \begin{array}{lcl} (12) + ((12)+(34)) + (12) & = & (12)+(34) \\ (13)+(24) & = & (14)+(23) \\ (14)+(23) & = & (13)+(24) \end{array} ,$$

$$S'_{\{2,3,4\}}\psi_1 = \begin{array}{l} (23) \\ (24) \\ (34) \\ (234) \\ (243) \end{array} , \qquad S'_{\{2,3,4\}}\varphi_{(14)} = \begin{array}{l} (14) \\ (14) \\ (14) \\ o \\ o \end{array} ,$$ we then have

$$\begin{array}{lll} (23)+((12)+(34))+(23) & + \; (14)+((12)+(34))+(14) & = o \\ (14)+(23) & \quad (14)+(23) & = o \\ (13)+(24) & \quad (13)+(24) & = o \end{array}$$

$$\begin{array}{lll} (24)+((12)+(34))+(24) & + \; (14)+((12)+(34))+(14) & = (12)+(34) \\ (14)+(23) & \quad (14)+(23) & = (13)+(24) \\ (13)+(24) & \quad (13)+(24) & = (14)+(23) \end{array}$$

$$\begin{array}{lll} (34)+((12)+(34))+(34) & + \; (14)+((12)+(34))+(14) & = (14)+(23) \\ (14)+(23) & \quad (14)+(23) & = (12)+(34) \\ (13)+(24) & \quad (13)+(24) & = (13)+(24) \end{array}$$

$$\begin{array}{lll} (243)+((12)+(34))+(234) & +\ o+((12)+(34))+o & = (14)+(23) \\ (14)+(23) & (14)+(23) & = (13)+(24) \\ (13)+(24) & (13)+(24) & = (12)+(34) \end{array}$$

$$\begin{array}{lll} (234)+((12)+(34))+(243) & +\ o+((12)+(34))+o & = (13)+(24) \\ (14)+(23) & (14)+(23) & = (12)+(34) \\ (13)+(24) & (13)+(24) & = (14)+(23) \ . \end{array}$$

Thus $\phi : V_4 \longrightarrow o \ , \quad V_4 + (23) \longrightarrow o \ ,$

$$V_4 + (24) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}, \qquad V_4 + (34) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (12)+(34) \\ (13)+(24) \end{cases},$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases}, \qquad V_4 + (243) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}.$$

(3). $\phi + \beta : V_4 \longrightarrow o$

$$V_4 + (23) \quad \begin{cases} o \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$V_4 + (24) \longrightarrow o$$

$$V_4 + (34) \longrightarrow o$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases} .$$

(4). Let $\alpha = -\rho_{(12)} + (1-\psi_1)\rho_{(34)} + \rho_{(12)} + (1-\psi_1)\rho_{(34)}$.

Since

$$
\begin{array}{llll}
V_4' (1-\psi_1)\rho_{(34)} = & (34) + ((12)+(34)) + (34) & = & (12)+(34) \\
 & (13)+(24) & = & (14)+(23) \\
 & (14)+(23) & = & (13)+(24)
\end{array}
$$

$$
\begin{array}{llll}
S'_{\{2,3,4\}}\rho_{(12)} = & (12)+(23)+(12) & = & (13) \\
 & (24) & = & (14) \\
 & (34) & = & (34) \\
 & (234) & = & (134) \\
 & (243) & = & (143)\ ,
\end{array}
$$

we then have

$$
\begin{array}{lll}
(13)+((12)+(34))+(13) + ((12)+(34)) & = & (13)+(24) \\
\qquad (14)+(23) \qquad\qquad (14)+(23) & = & (13)+(24) \\
\qquad (13)+(24) \qquad\qquad (13)+(24) & = & 0
\end{array}
$$

$$
\begin{array}{lll}
(14)+((12)+(34))+(14) + ((12)+(34)) & = & (14)+(23) \\
\qquad (14)+(23) \qquad\qquad (14)+(23) & = & 0 \\
\qquad (13)+(24) \qquad\qquad (13)+(24) & = & (14)+(23)
\end{array}
$$

$$
\begin{array}{lll}
(34)+((12)+(34))+(34) + ((12)+(34)) & = & 0 \\
\qquad (14)+(23) \qquad\qquad (14)+(23) & = & (12)+(34) \\
\qquad (13)+(24) \qquad\qquad (13)+(24) & = & (12)+(34)
\end{array}
$$

$$
\begin{array}{lll}
(143)+((12)+(34))+(134) + ((12)+(34)) & = & (13)+(24) \\
\qquad (14)+(23) \qquad\qquad (14)+(23) & = & (12)+(34) \\
\qquad (13)+(24) \qquad\qquad (13)+(24) & = & (14)+(23)
\end{array}
$$

$$
\begin{array}{lll}
(134)+((12)+(34))+(143) + ((12)+(34)) & = & (14)+(23) \\
\qquad (14)+(23) \qquad\qquad (14)+(23) & = & (13)+(24) \\
\qquad (13)+(24) \qquad\qquad + (13)+(24) & = & (12)+(34)\ .
\end{array}
$$

Thus $\alpha : V_4 \longrightarrow 0$

$$
V_4 + (23) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (13)+(24) \\ 0 \\ \vdots \end{cases}
$$

$$\alpha \;:\; V_4 + (24) \longrightarrow \begin{cases} \vdots \\ o \\ (14)+(23) \\ o \\ (14)+(23) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} o \\ o \\ (12)+(34) \\ (12)+(34) \end{cases}$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases} .$$

(5). Let $y = -\rho_{(23)} + (1-\psi_1)\rho_{(12)} + \rho_{(23)} + (1-\psi_1)\rho_{(12)}$.

Since $(1-\psi_1)\rho_{(12)} \;:\; V_4 \longrightarrow \begin{cases} 0 & (\text{ by } (2)\ ) \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) , \end{cases}$

and $S'_{\{2,3,4\}}\rho_{(23)} = \begin{array}{ll} (23)+(23)+(23) & = (23) \\ (24) & = (34) \\ (34) & = (24) \\ (234) & = (243) \\ (243) & = (234) , \end{array}$ we then have

$$\begin{array}{llll} (23)+((12)+(34))+(23) & + & ((12)+(34)) & = (14)+(23) \\ \quad (14)+(23) & & (14)+(23) & = o \\ \quad (13)+(24) & & (13)+(24) & = (14)+(23) \end{array}$$

$$\begin{array}{llll} (34)+((12)+(34))+(34) & + & ((12)+(34)) & = o \\ \quad (14)+(23) & & (14)+(23) & = (12)+(34) \\ \quad (13)+(24) & & (13)+(24) & = (12)+(34) \end{array}$$

$$\begin{array}{llcll} (24)+((12)+(34))+(24) & + & ((12)+(34)) & = & (13)+(24) \\ \quad (14)+(23) & & (14)+(23) & = & (13)+(24) \\ \quad (13)+(24) & & (13)+(24) & = & 0 \end{array}$$

$$\begin{array}{llcll} (234)+((12)+(34))+(243) & + & ((12)+(34)) & = & (13)+(24) \\ \quad (14)+(23) & & (14)+(23) & = & (12)+(34) \\ \quad (13)+(24) & & (13)+(24) & = & (14)+(23) \end{array}$$

$$\begin{array}{llcll} (243)+((12)+(34))+(234) & + & ((12)+(34)) & = & (14)+(23) \\ \quad (14)+(23) & & (14)+(23) & = & (13)+(24) \\ \quad (13)+(24) & & (13)+(24) & = & (12)+(34) \end{array} .$$

Thus $\gamma \; : \; V_4 \longrightarrow 0$

$$V_4 + (23) \longrightarrow \begin{cases} 0 \\ (14)+(23) \\ 0 \\ (14)+(23) \end{cases}$$

$$V_4 + (24) \longrightarrow \begin{cases} 0 \\ 0 \\ (12)+(34) \\ (12)+(34) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (13)+(24) \\ 0 \end{cases}$$

$$V_4 + (234) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} 0 \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases} .$$

(6). $\alpha + \gamma$:

$$V_4 \longrightarrow 0$$

$$V_4 + (23) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$V_4 + (24) \longrightarrow \begin{cases} 0 \\ (14)+(23) \\ (12)+(34) \\ (13)+(24) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$V_4 + (234) \longrightarrow 0$$

$$V_4 + (243) \longrightarrow 0$$

(7). $-1+\alpha+\gamma+1$:

$$V_4 \longrightarrow 0$$

$$V_4 + (23) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}$$

$$V_4 + (24) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} 0 \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases}$$

$$V_4 + (234) \longrightarrow 0$$

$$V_4 + (243) \longrightarrow 0 \quad .$$

(8). Let $\mu = -\rho_{(14)} + (- 1 + \alpha + \gamma + 1) + \rho_{(14)}$. Since

$$S'_{\{2,3,4\}}\rho_{(14)} = \begin{array}{l}(23)\\(12)\\(13)\\(123)\\(132)\end{array}$$

(see (1)) and

$$\begin{aligned}
(23)+(13)+(24)+(23) &= (12)+(34)\\
(12)+(34) &= (13)+(24)\\
(14)+(23) &= (14)+(23)\\
\\
(12)+(12)+(34)+(12) &= (12)+(34)\\
(14)+(23) &= (13)+(24)\\
(13)+(24) &= (14)+(23)\\
\\
(13)+(14)+(23)+(13) &= (12)+(34)\\
(13)+(24) &= (13)+(24)\\
(12)+(34) &= (14)+(23)\ ,
\end{aligned}$$

$$\mu \ : \ V_4 \longrightarrow 0$$

$$\left.\begin{array}{l} V_4 + (23)\\ V_4 + (24)\\ V_4 + (34)\end{array}\right\} \longrightarrow \begin{cases} 0\\ (12)+(34)\\ (13)+(24)\\ (14)+(23)\end{cases}$$

$$V_4 + (234) \longrightarrow 0$$

$$V_4 + (243) \longrightarrow 0 \ .$$

(9). Let $\omega = - 1 + (1-\psi_1)\rho_{(12)} + 1 + (1-\psi_1)\rho_{(12)}$

Since $(1-\psi_1)\rho_{(12)} \ : \ V_4 \longrightarrow \begin{cases} 0\\ (12)+(34)\\ \vdots \end{cases}$

$$\vdots$$

$$\left\lfloor \begin{array}{l}(14)+(23)\\(13)+(24)\end{array}\right.$$

and

$$\begin{array}{llll}
(23)+(12)+(34)+(23)+ & (12)+(34) & = & (14)+(23)\\
\qquad (14)+(23) & (14)+(23) & = & 0\\
\qquad (13)+(24) & (13)+(24) & = & (14)+(23)
\end{array}$$

$$\begin{array}{llll}
(24)+(12)+(34)+(24)+ & (12)+(34) & = & (13)+(24)\\
\qquad (14)+(23) & (14)+(23) & = & (13)+(24)\\
\qquad (13)+(24) & (13)+(24) & = & 0
\end{array}$$

$$\begin{array}{llll}
(34)+(12)+(34)+(34)+ & (12)+(34) & = & 0\\
\qquad (14)+(23) & (14)+(23) & = & (12)+(34)\\
\qquad (13)+(24) & (13)+(24) & = & (12)+(34)
\end{array}$$

$$\begin{array}{llll}
(243)+(12)+(34)+(234)+ & (12)+(34) & = & (14)+(23)\\
\qquad (14)+(23) & (14)+(23) & = & (13)+(24)\\
\qquad (13)+(24) & (13)+(24) & = & (12)+(34)
\end{array}$$

$$\begin{array}{llll}
(234)+(12)+(34)+(243)+ & (12)+(34) & = & (13)+(24)\\
\qquad (14)+(23) & (14)+(23) & = & (12)+(34)\\
\qquad (13)+(24) & (13)+(24) & = & (14)+(23)
\end{array},$$

$$\omega \;:\; V_4 \longrightarrow 0$$

$$V_4 + (23) \longrightarrow \begin{cases} 0\\ (14)+(23)\\ 0\\ (14)+(23)\end{cases}$$

$$V_4 + (24) \longrightarrow \begin{cases} 0\\ (13)+(24)\\ (13)+(24)\\ 0\end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} 0\\ 0\\ (12)+(34)\\ (12)+(34)\end{cases}$$

$$\omega \quad : \quad V_4 + (234) \longrightarrow \text{o} \left\{\begin{array}{l}(14)+(23)\\(13)+(24)\\(12)+(34)\end{array}\right.$$

$$V_4 + (243) \longrightarrow \text{o} \left\{\begin{array}{l}(13)+(24)\\(12)+(34)\\(14)+(23)\end{array}\right. .$$

(10). We have already shown β in (1) .

$$-\rho_{(12)} + \beta + \rho_{(12)} \quad : \quad V_4 \longrightarrow \text{o}$$

$$V_4 + (23) \longrightarrow \text{o} \left\{\begin{array}{l}(13)+(24)\\(12)+(34)\\(14)+(23)\end{array}\right.$$

$$V_4 + (24) \nearrow (13)+(24)$$

$$V_4 + (34) \nearrow (14)+(23)$$

$$V_4 + (234) \longrightarrow \text{o}$$

$$V_4 + (243) \longrightarrow \text{o}$$

Since $(23)\rho_{(12)} = (13)$, $(24)\rho_{(12)} = (14)$, $(34)\rho_{(12)} = (34)$ and

$$\begin{array}{rl}(13)+(13)+(24)+(13) & = (13)+(24)\\ (14)+(23) & = (12)+(34)\\ (12)+(34) & = (14)+(23)\end{array}$$

$$\begin{array}{rl}(14)+(12)+(34)+(14) & = (13)+(24)\\ (13)+(24) & = (12)+(34)\\ (14)+(23) & = (14)+(23)\end{array}$$

$$\begin{array}{rl}(34)+(14)+(23)+(34) & = (13)+(24)\\ (12)+(34) & = (12)+(34)\\ (13)+(24) & = (14)+(23)\end{array} .$$

(11). See (7) and (10) . We have

$$-\rho_{(12)} + \beta + \rho_{(12)} + (-1 + \alpha + \gamma + 1) \quad :$$

$$V_4 \longrightarrow o$$

$$V_4 + (23) \longrightarrow o$$

$$V_4 + (24) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases}$$

$$V_4 + (234) \longrightarrow o$$

$$V_4 + (243) \longrightarrow o \quad .$$

Let $\xi = -\rho_{(1324)} + \left(-\rho_{(12)} + \beta + \rho_{(12)} + (-1+\alpha+\gamma+1)\right) + \rho_{(1324)}$.

$$\xi \quad : \quad V_4 \longrightarrow o$$

$$V_4 + (23) \longrightarrow o$$

$$V_4 + (24) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (12)+(34) \\ (13)+(24) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$V_4 + (234) \longrightarrow o$$

$$V_4 + (243) \longrightarrow o \quad .$$

Since $(24)\rho_{(1324)} = (14)$, $(34)\rho_{(1324)} = (12)$ and

$$\begin{aligned}
(14)+(14)+(23)+(14) &= (14)+(23)\\
(13)+(24) &= (12)+(34)\\
(12)+(34) &= (13)+(24)
\end{aligned}$$

$$\begin{aligned}
(12)+(12)+(34)+(12) &= (12)+(34)\\
(14)+(23) &= (13)+(24)\\
(13)+(24) &= (14)+(23) .
\end{aligned}$$

(12). Let $\eta = -\rho_{(14)} + \left(-\rho_{(12)} + \beta + \rho_{(12)} + (-1+\alpha+\gamma+1)\right) + \rho_{(14)}$.

$$\begin{aligned}
\eta \; : \; V_4 &\longrightarrow o\\
V_4 + (23) &\longrightarrow o\\
V_4 + (24) &\longrightarrow \begin{cases} o \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}\\
V_4 + (34) &\longrightarrow \begin{cases} o \\ (14)+(23) \\ (12)+(34) \\ (13)+(24) \end{cases}\\
V_4 + (234) &\longrightarrow o\\
V_4 + (243) &\longrightarrow o .
\end{aligned}$$

Since $(24)\rho_{(14)} = (12)$, $(34)\rho_{(14)} = (13)$ and (by (11))

$$\begin{aligned}
(12)+(14)+(23)+(12) &= (13)+(24)\\
(13)+(24) &= (14)+(23)\\
(12)+(34) &= (12)+(34)
\end{aligned}$$

$$\begin{aligned}
(13)+(12)+(34)+(13) &= (14)+(23)\\
(14)+(23) &= (12)+(34)\\
(13)+(24) &= (13)+(24) .
\end{aligned}$$

(13). By (2) , (11) and (12) , we have

$$\phi + \eta + \xi : \; V_4 \longrightarrow o$$

$$V_4 + (23) \longrightarrow o$$

$$V_4 + (24) \longrightarrow o$$

$$V_4 + (34) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}$$

(14). By (4) , (8) and (9) , we have

$$\alpha + \omega + \mu : \; V_4 \longrightarrow o$$

$$V_4 + (23) \longrightarrow o$$

$$V_4 + (24) \longrightarrow o$$

$$V_4 + (34) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases}$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (12)+(34) \\ (14)+(23) \\ (13)+(24) \end{cases} .$$

(15). By (13) and (14), we have

$$\alpha + \omega + \mu + \phi + \eta + \xi : \quad V_4 \longrightarrow o$$

$$V_4 + (23) \longrightarrow o$$

$$V_4 + (24) \longrightarrow o$$

$$V_4 + (34) \longrightarrow o$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases} .$$

(16). By (15), we have

$$-\rho_{(234)} + (\alpha+\omega+\mu+\phi+\eta+\xi) + \rho_{(234)} :$$

$$V_4 \longrightarrow o$$

$$V_4 + (23) \longrightarrow o$$

$$V_4 + (24) \longrightarrow o$$

$$V_4 + (34) \longrightarrow o$$

$$V_4 + (234) \longrightarrow \begin{cases} o \\ (14)+(23) \\ (13)+(24) \\ (12)+(34) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} o \\ (13)+(24) \\ (12)+(34) \\ (14)+(23) \end{cases}$$

Since $(234)\rho_{(234)} = (234)$, $(243)\rho_{(243)} = (243)$ and

$$\begin{array}{lcl} (243)+(13)+(24)+(234) & = & (14)+(23) \\ (12)+(34) & = & (13)+(24) \\ (14)+(23) & & (12)+(34) \end{array}$$

$$\begin{array}{lcl} (234)+(14)+(23)+(243) & = & (13)+(24) \\ (13)+(24) & & (12)+(34) \\ (12)+(34) & & (14)+(23) \ . \end{array}$$

(17). By (16) and (3) , we have

$$\theta = -\rho_{(234)} + (\alpha+\omega+\mu+\phi+\eta+\xi) + \rho_{(234)} + \phi + \beta \ :$$

$$V_4 + (23) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$S_4 - (\ V_4 + (23)\ ) \longrightarrow 0 \ .$$

(18). Let $a = (1-\psi_1)\rho_{(134)} - \rho_{(142)} + (1-\psi_1)\rho_{(134)} + \rho_{(142)}$.

Since

$$V_4' \ (1-\psi_1)\rho_{(134)} \ \begin{array}{lcl} = (143)+(12)+(34)+(134) & = & (14)+(23) \\ (13)+(24) & = & (12)+(34) \\ (14)+(23) & = & (13)+(24) \ , \end{array}$$

$$S'_{\{2,3,4\}}\rho_{(142)} \ \begin{array}{lcl} = (124)+(23)+(142) & = & (13) \\ (24) & = & (12) \\ (34) & = & (23) \\ (234) & = & (132) \\ (243) & = & (123) \end{array}$$

and

$$\begin{array}{llcl} (14)+(23)+(13)+ & (14)+(23)+(13) & = & (13)+(24) \\ (12)+(34) & (12)+(34) & = & (13)+(24) \\ (13)+(24) & (13)+(24) & = & 0 \end{array}$$

$$\begin{array}{llcl} (14)+(23)+(12)+ & (14)+(23)+(12) & = & (12)+(34) \\ (12)+(34) & (12)+(34) & = & 0 \\ (13)+(24) & (13)+(24) & = & (12)+(34) \end{array}$$

$$\begin{array}{llcl}
(14)+(23)+(23)+ & (14)+(23)+(23) & = & 0 \\
(12)+(34) & (12)+(34) & = & (14)+(23) \\
(13)+(24) & (13)+(24) & = & (14)+(23)
\end{array}$$

$$\begin{array}{llcl}
(14)+(23)+(123)+ & (14)+(23)+(132) & = & (13)+(24) \\
(12)+(34) & (12)+(34) & = & (14)+(23) \\
(13)+(24) & (13)+(24) & = & (12)+(34)
\end{array}$$

$$\begin{array}{llcl}
(14)+(23)+(132)+ & (14)+(23)+(123) & = & (12)+(34) \\
(12)+(34) & (12)+(34) & = & (13)+(24) \\
(13)+(24) & (13)+(24) & = & (14)+(23)
\end{array} ,$$

we then have

$$a : V_4 \longrightarrow 0$$

$$V_4 + (23) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (13)+(24) \\ 0 \end{cases}$$

$$V_4 + (24) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ 0 \\ (12)+(34) \end{cases}$$

$$V_4 + (34) \longrightarrow \begin{cases} 0 \\ 0 \\ (14)+(23) \\ (14)+(23) \end{cases}$$

$$V_4 + (234) \longrightarrow \begin{cases} 0 \\ (13)+(24) \\ (14)+(23) \\ (12)+(34) \end{cases}$$

$$V_4 + (243) \longrightarrow \begin{cases} 0 \\ (12)+(34) \\ (13)+(24) \\ (14)+(23) \end{cases} .$$

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