Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 144 JANUARY 2008 PART 1

Math. Proc. Camb. Phil. Soc. (2008), **144**, 1
doi:10.1017/S0305004107000539 *Printed in the United Kingdom*
First published online 17 January 2008

Frobenius n-homomorphisms, transfers and branched coverings

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(*Received* 20 *August* 2006; *Revised* 12 *December* 2006)

Abstract

The main purpose is to characterise continuous maps that are n-branched coverings in terms of induced maps on the rings of functions. The special properties of Frobenius n-homomorphisms between two function spaces that correspond to n-branched coverings are determined completely. Several equivalent definitions of a Frobenius n-homomorphism are compared and some of their properties are proved. An axiomatic treatment of n-transfers is given in general and properties of n-branched coverings are studied and compared with those of regular coverings.

1. *Introduction*

In previous papers [**1**, **2** and **3**] we have studied the relationship between Frobenius n-homomorphisms and symmetric products, in particular we showed that Frobenius n-homomorphisms $C(X) \to C(Y)$ correspond precisely to continuous maps $f : Y \to \mathrm{Sym}^n(X)$, thus generalising the classical theorem of Gelfand–Kolmogorov. In this paper we characterise n-branched coverings using transfer maps which we treat as Frobenius n-homomorphisms of a special type. In Section 2 we review and prove the equivalence of a number of definitions of Frobenius n-homomorphisms that were not covered in our previous papers. Section 3 is devoted to showing that the composition of a Frobenius n-homomorphism with a Frobenius m-homomorphism is a Frobenius nm-homomorphism, the proof depends heavily on a previous result from [**1**]. A version of this result appeared in a preliminary preprint of the present paper and D. V. Gugnin has given an improvement of that result. Section 4 reviews the definition of an n-branched covering due to L. Smith [**8**] and A. Dold [**5**] and we prove some new properties and give examples; in particular we make a detailed analysis of n-branched coverings over an interval. Section 5 introduces the notion of an n-transfer for a ring homomorphism as a special kind of Frobenius n-homomorphism. It is shown that the kernel introduced by Gugnin [**6**] for a Frobenius n-homomorphism is trivial for an n-transfer. Section 6 contains the main theorem which characterises n-branched coverings in terms of n-transfers between the relevant function spaces.

2. *Frobenius n-homomorphisms*

In [**1**, **2**] we introduced the concept of a Frobenius n-homomorphism and studied some of its properties (these papers also contain references to related work by other authors). We recall the basic definition.

Consider a linear map $f\colon A \to B$ between two algebras [throughout this paper all algebras will be commutative, associative $\mathbb{C}$ algebras]. The maps $\Phi_n(f)\colon A^{\otimes n} \to B$ are defined as follows:

Each permutation $\sigma \in \Sigma_n$ the symmetric group on n letters, can be decomposed into a product of disjoint cycles of total length n, say $\sigma = \gamma_1\gamma_2\cdots\gamma_r$. If $\gamma = (i_1 \cdots i_m)$ is a cycle, let $f_\gamma(a_1, a_2, \ldots, a_n) = f(a_{i_1}a_{i_2}\cdots a_{i_m})$ then

$$\Phi_n(f)(a_1, a_2, \ldots, a_n) = \sum_{\sigma\in\Sigma_n} \varepsilon_\sigma f_{\gamma_1}(a_1, a_2, \ldots, a_n) f_{\gamma_2}(a_1, a_2, \ldots, a_n)\cdots f_{\gamma_r}(a_1, a_2, \ldots, a_n)$$

where ε_σ is the sign of the permutation σ.

From this definition it is clear that $\Phi_n(f)\colon A^{\otimes n} \to B$ is n-linear and symmetric, one can use polarisation to ease the verification of some of its properties; in other words to prove identities it is enough to consider the values of $\Phi_n(f)(a, a, \ldots, a)$ for all $a \in A$ (we will sometimes abbreviate this to $\Phi_n(f)(a)$).

There is also an inductive definition (first used by Frobenius in the case of group algebras of finite groups) for the $\Phi_n(f)$ starting with $\Phi_1(f) = f$ and, for $n \geqslant 1$,

$$\begin{aligned}\Phi_{n+1}(f)(a_0, a_1, \ldots, a_n) &= f(a_0)\Phi_n(f)(a_1, a_2, \ldots, a_n)\\ &\quad -\sum_{r=1}^{n} \Phi_n(f)(a_1, a_2, \ldots, a_0a_r, \ldots, a_n)\end{aligned}$$

or equivalently, because of the polarisation identities for symmetric multilinear maps,

$$\Phi_{n+1}(f)(a, a, \ldots, a) = f(a)\Phi_n(f)(a, a, \ldots, a) - n\Phi_n(f)(a^2, a, \ldots, a).$$

We will also find it useful to have another equivalent definition: $\Phi_n(f)(a)$ is the determinant of the matrix

$$\begin{pmatrix} f(a) & 1 & 0 & 0 & \ldots & 0\\ f(a^2) & f(a) & 2 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & & \vdots\\ \vdots & & & & & \vdots\\ f(a^{n-1}) & f(a^{n-2}) & f(a^{n-3}) & \ldots & f(a) & n-1\\ f(a^n) & f(a^{n-1}) & f(a^{n-2}) & \ldots & f(a^2) & f(a)\end{pmatrix}.$$

Definition 2·1. A linear map $f: A \rightarrow B$ of algebras is called a *Frobenius n-homomorphism* if it satisfies $f(1) = n$ and $\Phi_{n+1}(f) \equiv 0$.

We note that, by polarisation, the condition $\Phi_{n+1}(f) \equiv 0$ is equivalent to $\Phi_{n+1}(f)(a) = 0$ for all $a \in A$.

Definition 2·2. An algebra A is *connected* if $a(a-1) = 0$ only if $a = 0, 1$.

PROPOSITION 2·1. *If A is a connected algebra then, for each $k \geqslant 1$, the equation $x(x-1)\cdots(x-k) = 0$ has only the obvious $k+1$ solutions.*

Proof. Suppose that $a \in A$ satisfies $a(a-1)\cdots(a-k) = 0$, $a \neq 0, 1, 2, \ldots, k$ and that, for this a, k is the smallest such integer, so that if

$$p(x) = \frac{1}{k!}x(x-1)\cdots(x-k+1)$$

then, $b = p(a) \neq 0$. The polynomial $p(x)^2 - p(x)$ clearly has $0, 1, 2, \ldots, k-1$ as roots, and as $p(k) = 1$ it also has k as a root. Hence, $b^2 - b = 0$ since $x(x-1)\cdots(x-k)$ divides $p(x)$. Now suppose $b = 1$, then $a(a-1)\cdots(a-k+1) = k!$ and multiplying this equation by $a-k$ gives $0 = k!(a-k)$, hence $a = k$, contradicting the hypothesis. We have found a $b \neq 0, 1$ with $b^2 = b$ and so A is not connected.

The following result is easily proved by induction

LEMMA 2·1. *For any linear map f one has*

$$\Phi_{n+1}(f)(a, 1, \ldots, 1) = f(a)(f(1)-1)(f(1)-2)\cdots(f(1)-n).$$

By taking $a = 1$ one obtains.

COROLLARY 2·1. *Let B be connected and $f: A \rightarrow B$ be such that $\Phi_{n+1}(f) \equiv 0$ then $f(1) \in \{0, 1, 2, \ldots, n\}$.*

These results are closely related to those of [**1**, corollary 2·5].

The condition $f(1) = n$ plays a crucial rôle [**1**, proposition 2·7] in the definition of a Frobenius n-homomorphism, as the following indicates.

PROPOSITION 2·2. *If $\Phi_{n+1}(f) \equiv 0$ and $f(1) = k$ then $\Phi_{k+1}(f) \equiv 0$.*

We will use the following result about symmetric polynomials (it is related to some of those in [**3**, **4**]); it helps to explain the determinant expression used above.

PROPOSITION 2·3. *Let $s_k = \beta_1^k + \beta_2^k + \cdots + \beta_n^k$, then the indeterminates β_r $(1 \leqslant r \leqslant n)$ are the roots of the polynomial $d(t)$ given as the determinant of the matrix*

$$\begin{pmatrix} s_1 & 1 & 0 & 0 & \ldots & \ldots & 0 \\ s_2 & s_1 & 2 & 0 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ s_{n-1} & s_{n-2} & s_{n-3} & \ldots & s_1 & n-1 & 0 \\ s_n & s_{n-1} & s_{n-2} & \ldots & s_2 & s_1 & n \\ t^n & t^{n-1} & t^{n-2} & \ldots & t^2 & t & 1 \end{pmatrix}.$$

Moreover, if $f: A \to B$ is linear and $f(a^k) = s_k$ then the determinant $d(t)$ is, up to a non-zero constant multiple, equal to

$$t^n - \Phi_1(f)(a)t^{n-1} + \frac{1}{2}\Phi_2(f)(a)t^{n-2} - \cdots + \frac{(-1)^n}{n}\Phi_n(f)(a).$$

Proof. As usual, let e_r denote the elementary symmetric polynomial of degree r in β_r ($1 \leqslant r \leqslant n$). Denote the columns of the above matrix by $(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n+1})$ and replace the first column by the column vector

$$\mathbf{c}_1 - e_1\mathbf{c}_2 + e_2\mathbf{c}_3 - \cdots + (-1)^n e_n\mathbf{c}_{n+1}.$$

The determinant is unchanged. Using the standard Newton formulae (see [**7**, page 20]),

$$s_r - s_{r-1}e_1 + \cdots + (-1)^{r-1}s_1 e_{r-1} + (-1)^r r e_r = 0$$

we see that the first column of the new matrix has zero entries except for the last entry which equals

$$p(t) = t^n - e_1 t^{n-1} + e_2 t^{n-2} - \cdots + (-1)^n e_n.$$

So $d(t) = (-1)^n\, n!\, p(t)$ and hence the roots of $d(t)$ are the same as those of $p(t)$, namely $\{\beta_1, \beta_2, \ldots, \beta_n\}$.

Finally, to verify the result about $f: A \to B$ it is enough to consider the special case $A = \mathbb{C}[a]$, $B = \mathbb{C}[\beta_1, \beta_2, \ldots, \beta_n]$. Then, considering the appropriate sub-determinant of the above determinant that defines e_n, we see that $\Phi_n(f)(a) = n! e_n$.

This leads us to yet another characterisation of Frobenius n-homomorphisms (which is closely related to some formulae in [**3**] and [**4**]).

PROPOSITION 2·4. *A linear map $f: A \to B$ is a Frobenius n-homomorphism if and only if for each $a \in A$ there is a polynomial $p_a(t) \in B[t]$ of degree n such that*

$$\sum_{q=0}^{\infty} \frac{f(a^q)}{t^{q+1}} = \frac{\mathrm{d}}{\mathrm{d}t} \log p_a(t).$$

Proof. Given the Frobenius n-homomorphism $f: A \to B$ and $a \in A$, consider the polynomial $p_a(t) = \det(M) \in B[t]$ where M is the matrix

$$\begin{pmatrix} f(a) & 1 & 0 & 0 & \cdots & \cdots & 0 \\ f(a^2) & f(a) & 2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ \vdots & & & & & & \vdots \\ f(a^{n-1}) & f(a^{n-2}) & f(a^{n-3}) & \cdots & f(a) & n-1 & 0 \\ f(a^n) & f(a^{n-1}) & f(a^{n-2}) & \cdots & f(a^2) & f(a) & n \\ t^n & t^{n-1} & t^{n-2} & \cdots & t^2 & t & 1 \end{pmatrix}.$$

Choose an extension $\overline{B}$ of B such that $p_a(t)$ factors completely in $\overline{B}[t]$, say $p_a(t) = n!(t - \beta_1)(t - \beta_2)\cdots(t - \beta_n)$ with $\beta_r \in \overline{B}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \log p_a(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\log(t - \beta_1) + \log(t - \beta_2) + \cdots + \log(t - \beta_n)).$$

but

$$\frac{\mathrm{d}}{\mathrm{d}t}\log(t-\beta) = \frac{1}{t-\beta} = \frac{1}{t}\left(1-\frac{1}{\frac{\beta}{t}}\right) = \frac{1}{t} + \frac{\beta}{t^2} + \frac{\beta^2}{t^3} + \cdots .$$

However, by Proposition 2·2, $\beta_1^r + \beta_2^r + \cdots + \beta_n^r = f(a^r)$ and the result is proved.

Conversely, if

$$\sum_{q=0}^{\infty} \frac{f(a^q)}{t^{q+1}} = \frac{\mathrm{d}}{\mathrm{d}t}\log p_a(t)$$

and $p_a(t) = n!(t-\beta_1)(t-\beta_2)\cdots(t-\beta_n)$ then $f(a^r) = \beta_1^r + \beta_2^r + \cdots \beta_n^r$ and, with $r=0$ we have $f(1) = n$; using this and Proposition 2·2 we get that the rth elementary symmetric polynomial in $\beta_1, \beta_2, \ldots, \beta_n$ is $\Phi_r(f)(a)$ and hence that $\Phi_{n+1}(f)(a) = 0$.

As in [**1**] we denote the subalgebra of symmetric tensors in $A^{\otimes n}$ by $\mathcal{S}^n A$.

We will find the following fact very useful and it is easy to prove.

LEMMA 2·2. *The diagonal map $\Delta_n\colon A \to \mathcal{S}^n A$ defined by*

$$\Delta_n(a) = a \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes a \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes a$$

is a Frobenius n-homomorphism.

THEOREM 2·1. *If $f\colon A \to B$ is a Frobenius n-homomorphism, then the map defined by*

$$\frac{\Phi_n(f)}{n!}\colon \mathcal{S}^n A \longrightarrow B$$

is a ring homomorphism. Conversely, if $f\colon A \to B$ is linear, $f(1) = n$ and $\Phi_n(f)/n!$ is a ring homomorphism, then f is a Frobenius n-homomorphism.

Proof. The first statement is [**1**, theorem 2·8] so we only need to prove the (easier) converse part.

Since $\Phi_n(f)/n!$ is a ring homomorphism we have that, for $\mathbf{a}, \mathbf{b} \in \mathcal{S}^n A$

$$\Phi_n(f)(\mathbf{a})\Phi_n(f)(\mathbf{b}) = n!\Phi_n(f)(\mathbf{ab}).$$

But by the inductive definition and the symmetry of $\Phi_{n+1}(f)$,

$$\Phi_{n+1}(f)(a^{\otimes n+1}) = f(a)\Phi_n(f)(a^{\otimes n}) - n\Phi_n(f)(a^2 \otimes a^{\otimes n-1})$$

and $a^2 \otimes a^{\otimes n-1} + a \otimes a^2 \cdots \otimes a + \cdots + a^{\otimes n-1} \otimes a^2 = a^{\otimes n}\Delta_n(a)$. By Lemma 2·1, $\Phi_n(f)(\Delta_n(a)) = n!f(a)$. So

$$\begin{aligned}\Phi_{n+1}(f)(a^{\otimes n+1}) &= f(a)\Phi_n(f)(a^{\otimes n}) - \Phi_n(f)(a^{\otimes n}\Delta_n(a))\\ &= f(a)\Phi_n(f)(a^{\otimes n}) - \Phi_n(f)(a^{\otimes n})\Phi_n(f)(\Delta_n(a))/n!\\ &= \Phi_n(f)(a^{\otimes n})(f(a) - f(a)) = 0.\end{aligned}$$

In an appropriate sense Δ_n is the universal Frobenius n-homomorphism on A as the following shows.

COROLLARY 2·2. *A Frobenius n-homomorphism $f\colon A \to B$ factors uniquely as*

$$\tilde{f}\Delta_n\colon A \longrightarrow \mathcal{S}^n A \longrightarrow B$$

where $\tilde{f}$ is a ring homomorphism. Moreover, $\tilde{f} = \Phi_n(f)/n!$.

Proof. By Lemma 2·1, $(\Phi_n(f)/n!)(\Delta_n(a)) = f(a)$ which proves existence.

It remains to prove that $\tilde{f}$ is unique. This is true because of the fact that, as an algebra, $\mathcal{S}^n A$ is generated by the elements $\{\Delta_n(a) \colon a \in A\}$ since a general element $\sum_{\sigma\in\Sigma_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}$ of $\mathcal{S}^n A$ equals $\Phi_n(\Delta_n)(a_1, a_2, \ldots, a_n)$ which is a polynomial in elements of the form $\Delta_n(a)$ with $a \in A$.

Following an idea due to D. V. Gugnin we introduce the appropriate categorical concept of the kernel of a Frobenius n-homomorphism.

Definition 2·3. For a Frobenius n-homomorphism $f \colon A \to B$ let $K_f = \{a \colon f(ax) = 0 \text{ for all } x \in A\}$, it is clearly a subspace of the kernel of the linear map f.

When B has no nilpotent elements, D. V. Gugnin has shown that $K_f = \{a \colon f(a^r) = 0 \text{ for } 1 \leqslant r \leqslant n\}$.

3. *Compositions*

In [**1**] we showed that the sum of a Frobenius m-homomorphism and a Frobenius n-homomorphism is a Frobenius $m+n$-homomorphism. The main result of this section is to show that they also behave appropriately under composition.

THEOREM 3·1. *Let A, B, C be associative, commutative $\mathbb{C}$−algebras. If $f \colon A \to B$ and $g \colon B \to C$ are Frobenius n- and m-homomorphisms, respectively then $gf \colon A \to C$ is a Frobenius mn-homomorphism.*

Proof. We first note that the theorem is trivial if either n or m equals 1 and that we will make constant use of Corollary 2·2.

The map f factors through its universal Frobenius n-homomorphism as

$$\tilde{f}\Delta_n \colon A \longrightarrow \mathcal{S}^n A \longrightarrow B.$$

Similarly, g factors as

$$\tilde{g}\Delta_m \colon B \longrightarrow \mathcal{S}^m B \longrightarrow C.$$

Since $\tilde{f}$ is a ring homomorphism and Δ_m is a Frobenius m-homomorphism, the composition

$$\Delta_m \tilde{f} \colon \mathcal{S}^n A \longrightarrow B \longrightarrow \mathcal{S}^m B$$

is a Frobenius m-homomorphism, so it factors through

$$\Delta_m \colon \mathcal{S}^n A \longrightarrow \mathcal{S}^m \mathcal{S}^n A.$$

By direct observation, the composition

$$\Delta_m \Delta_n \colon A \longrightarrow \mathcal{S}^n A \longrightarrow \mathcal{S}^m \mathcal{S}^n A$$

is equal to

$$i\Delta_{mn} \colon A \longrightarrow \mathcal{S}^{mn} A \longrightarrow \mathcal{S}^m \mathcal{S}^n A$$

where i is the inclusion map. Hence the composition $gf \colon A \to C$ factors through

$$\Delta_{mn} \colon A \longrightarrow \mathcal{S}^{mn} A$$

and so gf is a Frobenius mn-homomorphism.

4. *Branched coverings*

We consider branched coverings in the sense studied by Smith [**8**] and by Dold [**5**] and elaborate on their properties. When X is a topological space or a variety, we denote its nth *symmetric product* by $\mathrm{Sym}^n(X)$; it is the quotient space of the Cartesian product X^n under the action of the symmetric group Σ_n acting by permuting the factors. We often regard the elements of $\mathrm{Sym}^n(X)$ as n-multisets $[x_1, x_2, \ldots, x_n]$ with $x_i \in X$.

Definition 4·1. An *n-branched covering* $h\colon X \to Y$ is a continuous map between two Hausdorff spaces and a continuous map $t\colon Y \to \mathrm{Sym}^n(X)$ such that:

(i) $x \in th(x)$ *for every* $x \in X$; and

(ii) $\mathrm{Sym}^n(h)(ty) = ny$ *for every* $y \in Y$.

Example 4·1. The map $p\colon \mathbb{C} \to \mathbb{C}$ defined by a polynomial of degree n is the classic example of an n-branched covering. The map t is defined by $t(w) = [z_1, z_2, \ldots, z_n]$ where the z_r are the roots (counted with multiplicities) of the equation $p(z) = w$. It is straightforward to verify the above axioms in this case.

Example 4·2. If G is a finite group acting continuously and effectively on a Hausdorff space X and $h\colon X \to Y = X/G$ is the map to the space of orbits, then h is an n-branched covering where n is the cardinality of G and the map $t\colon Y \to \mathrm{Sym}^n(X)$ is given by $t(y) =$ the points in the orbit defined by y (counted with multiplicities). More generally (see [**5**, example 1·4]), if $H \subset G$ is a subgroup of finite index n and X is an effective G-space, then the quotient map $X/H \to X/G$ is an n-branched covering.

Every 2-branched covering arises from an action of the group with two elements.

PROPOSITION 4·1. *Let* $f\colon X \to Y$, $s\colon Y \to \mathrm{Sym}^n(X)$ *and* $g\colon Y \to Z$, $t\colon Z \to \mathrm{Sym}^m(Y)$ *be n- and m-branched coverings, then the composition* $h = gf\colon X \to Z$ *is an mn-branched covering with* $u\colon Z \to \mathrm{Sym}^{mn}(X)$ *being the composition*

$$i\ \mathrm{Sym}^m(s)\ t\colon Z \longrightarrow \mathrm{Sym}^m(Y) \longrightarrow \mathrm{Sym}^m(\mathrm{Sym}^n(X)) \longrightarrow \mathrm{Sym}^{mn}(X).$$

Proof. The proof is straightforward.

One defines induced n-branched coverings by taking the obvious pullback: let $h\colon X \to Y, t\colon Y \to \mathrm{Sym}^n(X)$ be an n-branched covering and $\phi\colon Z \to Y$ a continuous map. The usual pullback is $\tilde{X} = \{(z, x) | \phi(z) = h(x)\}$, the projection is $\tilde{h}(z, x) = z$ and $\tilde{t}\colon Z \to \mathrm{Sym}^n(\tilde{X})$ is defined by $\tilde{t}(z) = [(z, x_1), (z, x_2), \ldots, (z, x_n)]$ where $t\phi(z) = [x_1, x_2, \ldots, x_n]$. The properties of an n-branched covering are straightforward to check.

We modify the Ehresmann method [**9**] which constructs the principal fibration associated with a locally trivial fibration. As a result we get a method to 'resolve' an n-branched covering.

An *epimorphism* from the set $[n] = \{1, 2, \ldots, n\}$ to an n-multiset is a map which is onto and the counter image of an element with multiplicity m has size m.

If $h\colon X \to Y$ is an n-branched covering, let E be the set of all maps $\psi\colon [n] \to X$ such that $h\psi(1) = h\psi(2) = \cdots = h\psi(n)$ and ψ is an epimorphism onto the multiset $th\psi$. Clearly the symmetric group Σ_n acts on E and the quotient is Y; moreover, $E \times_{\Sigma_n} [n]$ is isomorphic to X and the projection onto the first factor can then be identified with h. The map ψ can be thought of as a 'universal' labeling for the branches of the covering. This proves the following.

THEOREM 4·1 ([**5**, proposition 1·9]). *Every n-branched covering can be described in the form of a projection* $p\colon E \times_{\Sigma_n} [n] \to E/\Sigma_n$ *for some* Σ_n *space* E.

Example 4·3. To illustrate all this we consider in some detail the case where the base space is an interval; this is in contrast to the case of regular coverings over an interval in which case they are trivial. The following discussion describes many n-branched coverings of an interval, including all those with finitely many 'branch points'.

A *set partition* π of a finite set S consists of a family of disjoint non-empty subsets of S whose union is S. We let $n(\pi)$ denote the number of parts of π.

Let P_S denote the set of all set partitions of S. Two set partitions

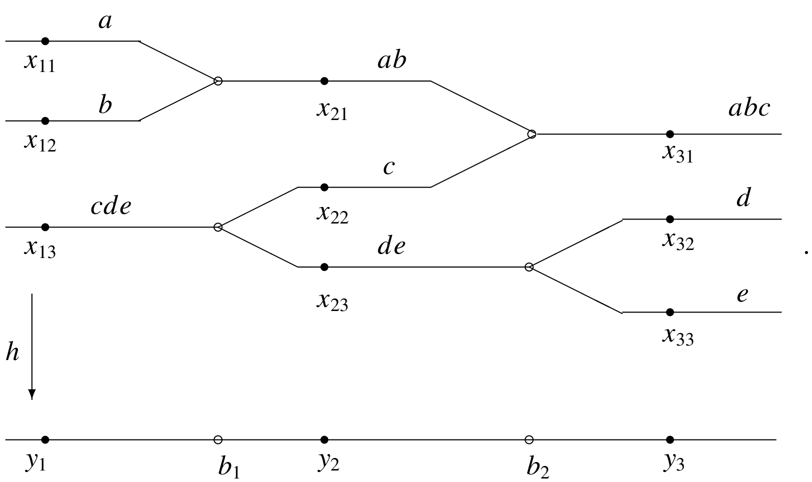
$$\pi_1 = \{A_1, A_2, \ldots, A_p\}, \pi_2 = \{B_1, B_2, \ldots, B_q\}$$

are called *adjacent* if each A_r is either a union of some of the $B's$ or is a subset of one of the $B's$; this relation is symmetric. [For example, if $S = \{u, v, w\}$ then the two partitions $\{u\}, \{v, w\}$ and $\{u, v\}, \{w\}$ are not adjacent but $\{u\}, \{v, w\}$ and $\{u, v, w\}$ are adjacent.] A map $\phi\colon [k+1] \to P_{[n]}$ is *compatible* if the set partitions $\phi(r), \phi(r+1)$ are adjacent for $1 \leqslant r \leqslant k$.

Let $0 = b_0 < b_1 < b_2 < \cdots < b_k < 1 = b_{k+1}$ be a dissection of $I = [0, 1]$. Given a compatible ϕ we can construct an n-branched covering of I that is branched over the points b_r, $1 \leqslant r \leqslant k$ as follows:

Over the interval (b_{r-1}, b_r) there are $n(\phi(r))$ disjoint intervals each labeled by a part of the set partition $\phi(r)$. At the point b_r the two sets of $n(\phi(r))$ and $n(\phi(r+1))$ intervals are joined according to the adjacency between $\phi(r)$ and $\phi(r+1)$.

An illustration with $n = 5$, $k = 2$ and 'branched' over two points b_1, b_2 with $\phi(1) = \{a\}, \{b\}, \{cde\}$, $\phi(2) = \{a, b\}, \{c\}, \{d, e\}$, $\phi(3) = \{a, b, c\}, \{d\}, \{e\}$ is



The map t has the property that each point $x \in h^{-1}y$ appears in the multiset $t(y)$ with multiplicity equal to the size of the parts by which it is labeled, so for the branched covering of the diagram one has $t(y_1) = [x_{11}, x_{12}, x_{13}, x_{13}, x_{13}]$, $t(y_2) = [x_{21}, x_{21}, x_{22}, x_{23}, x_{23}]$ and $t(y_3) = [x_{31}, x_{31}, x_{31}, x_{32}, x_{33}]$. We note that, despite superficial appearances, this example can only be described as an n-branched covering for $n \geqslant 5$.

5. *Frobenius n-homomorphisms and transfer maps*

The aim of this section is to introduce the concept of an n-transfer for a ring homomorphism and to study their algebraic properties as special cases of Frobenius n-homomorphisms.

Definition 5·1. Let A, B be commutative, associative algebras and $f: A \to B$, a ring homomorphism, then a linear map $\tau : B \to A$ is an *n-transfer* for f if:

(i) τ is a Frobenius n-homomorphism;

(ii) $\tau(f(a)b) = a\tau(b)$, that is, τ is a map of A-modules and

(iii) $f\tau : B \to B$ is the sum of the identity and a Frobenius $(n-1)$-homomorphism $g: B \to B$.

We denote the linear subspace $\{b \in B : g(b) = -b\}$ by L.

PROPOSITION 5·1. *If $f: A \to B$ is a ring homomorphism, $\tau : B \to A$ is an n-transfer for f and $g: B \to B$ is as above then*:

(i) *the composition $\tau f: A \to A$ is multiplication by n*;

(ii) *there is a split exact sequence of A-modules*

$$0 \longrightarrow L \longrightarrow B \xrightarrow{\tau} A \longrightarrow 0;$$

(iii) $gf = (n-1)f$ *and* $\tau g = (n-1)\tau$.

Proof. Taking $b = 1$ in Definition 5·1(ii) and because $\tau(1) = n$ the result is immediate.

The map τ is split by f/n. If $\tau(b) = 0$ then $0 = f\tau(b) = (1+g)(b)$ by Definition 5·1(iii) and so the kernel of τ is identified with L.

The equations of (iii) are immediate consequences of the associativity of composition for $f\tau f$ and $\tau f \tau$.

Example 5·1. In the case where A, B are affine algebras, the relations in Proposition 5·1(iii) can be very useful. For example, when $A = B = \mathbb{C}[z]$ and $n = 2$, let $f(z) = p$ and $g(z) = q$ then $gf(z) = p(q(z))$ but this equals $p(z)$ and so g has degree 1.

Example 5·2. This is another application of Proposition 5·1 and we only consider the case $n = 2$. The splitting $B = L \oplus A$ is given by identifying the image of the monomorphism $f: A \to B$ with A. We show that $xy \in A$ for all $x, y \in L$. From the splitting we obtain $xy = a + z$ where $a \in A$, $z \in L$; since $g(\ell) = -\ell$ for all $\ell \in L$ and since $n = 2$, Proposition 5·1(iii) gives that $g(a) = a$. Using the fact that g is a ring homomorphism we get that $g(xy) = g(x)g(y) = (-x)(-y) = xy = a + z$ but also $g(xy) = g(a+z) = a - z$. Hence $z = 0$, showing that $xy \in L$.

PROPOSITION 5·2. *Let $f: A \to B$, $g: B \to C$ be ring homomorphisms and $\tau : B \to A$, $\sigma : C \to B$ be n- and m-transfers for f, g respectively. Then $\sigma\tau : C \to A$ is an nm-transfer for the composition $gf: A \to C$.*

Proof. We check the conditions of Definition 5·1. The first condition is Theorem 3·1 and the other two follow by a direct calculation.

THEOREM 5·1. *If A, B are algebras with no nilpotent elements and $\tau : B \to A$ is an n-transfer for the ring homomorphism $f: A \to B$, then $K_\tau = 0$.*

Proof. Take $b \in K_\tau$, then by Gugnin's result mentioned at the end of Section 2 above, $\tau(b) = \tau(b^2) = \cdots = \tau(b^n) = 0$. By Proposition 5·1(ii), $g(b^r) = -b^r$ for all $1 \leqslant r \leqslant n$.

But since g is a Frobenius $(n-1)$-homomorphism, $\Phi_n(g)(b) = 0$ and by Definition 2·1 this is equivalent to the vanishing of the determinant of the matrix

$$\begin{pmatrix} -b & 1 & 0 & 0 & \dots & 0 \\ -b^2 & -b & 2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & & & & & \vdots \\ -b^{n-1} & -b^{n-2} & -b^{n-3} & \dots & -b & n-1 \\ -b^n & -b^{n-1} & -b^{n-2} & \dots & -b^2 & -b \end{pmatrix}.$$

By adding b times the second column and b^2 times the third column etc to the first column one sees that this determinant equals $(-1)^n\ n!\ b^n$ and since there are no nilpotent elements, we deduce that $b = 0$.

When X is a compact Hausdorff space, $C(X)$ will denote the algebra of continuous functions $X \to \mathbb{C}$ with the supremum norm.

Definition 5·2. The *direct image* $t_!\colon C(X) \to C(Y)$ associated with a continuous map $t\colon Y \to \mathrm{Sym}^n(X)$ is defined by $(t_!\phi)(y) = \sum \phi(x_r)$ and $t(y) = [x_1, x_2, \dots, x_n]$.

THEOREM 5·2. *If X, Y are compact Hausdorff spaces, then the set of all continuous Frobenius n-homomorphisms $C(X) \to C(Y)$ can be identified with the space of continuous maps $Y \to$* $\mathrm{Sym}^n(X)$.

Proof. The map $t_!$ is the sum of n ring homomorphisms and so is a Frobenius n-homomorphism.

Conversely, suppose that $f\colon C(X) \to C(Y)$ be a Frobenius n-homomorphism and let $\mathcal{E}_y\colon C(Y) \to \mathbb{C}$ be evaluation at the point $y \in Y$ then the composition $\mathcal{E}_y f$ is also a Frobenius n-homomorphism and so, by [**1**, theorem 4·1] corresponds to a multi-set $[x_1, x_2, \dots, x_n]$ in X. This defines the required map $t\colon Y \to \mathrm{Sym}^n(X)$.

Remark. In the case $A = C(X)$, the map Δ_n (see Lemma 2·2) corresponds to the identity map on $\mathrm{Sym}^n(X)$ regarded as an n-valued map from $\mathrm{Sym}^n(X)$ to X. If $f\colon A = C(X) \to B = C(Y)$ is a Frobenius n-homomorphism, the ring homomorphism $\Phi_n(f)/n!$ corresponds to $t\colon Y \to \mathrm{Sym}^n(X)$ (see Theorem 2·1).

Example 5·3. The linear map $C(X) \to C(X)$ given by $\phi \to n\phi$ is a Frobenius n-homomorphism and corresponds to the diagonal map $X \to \mathrm{Sym}^n(X)$.

Example 5·4. If $s\colon Y \to \mathrm{Sym}^m(X)$ and $t\colon Y \to \mathrm{Sym}^n(X)$ give rise to the Frobenius m, n-homomorphisms $s_!, t_!\colon C(X) \to C(Y)$ then the composition

$$Y \longrightarrow \mathrm{Sym}^m(X) \times \mathrm{Sym}^n(X) \longrightarrow \mathrm{Sym}^{m+n}(X)$$

corresponds to $s_! + t_!\colon C(X) \to C(Y)$.

Example 5·5. If $s\colon Y \to \mathrm{Sym}^n(X)$ and $t\colon Z \to \mathrm{Sym}^m(Y)$ are continuous and $s_!\colon C(X) \to C(Y)$, $t_!\colon C(Y) \to C(Z)$ are the corresponding n- and m-Frobenius homomorphisms, then the composition

$$Z \longrightarrow \mathrm{Sym}^m(Y) \longrightarrow \mathrm{Sym}^m\mathrm{Sym}^n(X) \longrightarrow \mathrm{Sym}^{mn}(X)$$

corresponds to the composition $t_!s_!\colon C(X) \to C(Z)$ which is a Frobenius mn-homomorphism.

6. *Frobenius n-homomorphisms and n-branched coverings*

The aim of this section is to characterise n-branched coverings in terms of rings of continuous functions and Frobenius n-homomorphisms.

A continuous map $h\colon X \to Y$ induces a ring homomorphism $h^*\colon C(Y) \to C(X)$. If h is an n-branched covering, then as above we have a direct image map $t_!\colon C(X) \to C(Y)$ which is a Frobenius n-homomorphism. We consider properties of $t_!$ which will ensure that h is such a covering.

In Example 4·2 of a finite group action on X, the third property of Definition 5·1 becomes very simple: Let $G = \{e = g_1, g_2, \ldots, g_n\}$ be the group and $h\colon X \to Y = X/G$ is the map to the space of orbits, then the map $h^*t_!\colon C(X) \to C(X)$ corresponds geometrically to the map $X \to X \times \cdots \times X \to X \times \mathrm{Sym}^{n-1}(X)$ given by $x \to (x, g_2x, g_3x, \ldots, g_nx) \to (x, [g_2x, g_3x, \ldots, g_nx])$.

In Theorems 6·1 and 6·2, X, Y will denote compact Hausdorff spaces.

THEOREM 6·1. *Given an n-branched covering $h\colon X \to Y$, $t\colon Y \to \mathrm{Sym}^n(X)$, the direct image $t_!\colon C(X) \to C(Y)$ is an n-transfer for the ring homomorphism $h^*\colon C(Y) \to C(X)$.*

Proof. We check three properties:

As noted in the proof of Theorem 5·2, $t_!$ is a Frobenius n-homomorphism.

By definition, one has that for $\phi \in C(X)$, $t_!(\phi)(y) = \phi(x_1) + \cdots + \phi(x_n)$ where $t(y) = [x_1, x_2, \ldots, x_n]$. Therefore

$$\begin{aligned} t_!(h^*(\psi)\phi)(y) &= \psi(h(x_1))\phi(x_1) + \cdots + \psi(h(x_n))\phi(x_n) \\ &= \psi(y)\phi(x_1) + \cdots + \psi(y)\phi(x_n) = \psi(y)t_!(\phi)(y). \end{aligned}$$

Hence $t_!(h^*(\psi)\phi) = \psi(y)t_!(\phi)$.

The third property follows immediately from Definition 4·1(ii).

The converse of Theorem 6·1 is

THEOREM 6·2. *Given a continuous map $h\colon X \to Y$ and a continuous n-transfer τ for $h^*\colon C(Y) \to C(X)$, then h is an n-branched covering.*

Proof. By the above, a continuous n-transfer $\tau\colon C(X) \to C(Y)$ corresponds to a continuous map $t\colon Y \to \mathrm{Sym}^n(X)$ which is such that $th\colon X \to \mathrm{Sym}^n(X)$ is the diagonal map and $\mathrm{Sym}^n(h)t\colon Y \to \mathrm{Sym}^n(X) \to \mathrm{Sym}^n(Y)$ is of the form $y \to [y_1, y_2, \ldots, y_n]$ with $y_1 = y$.

More generally, we see that the Frobenius n-homomorphism f corresponding to $t\colon Y \to \mathrm{Sym}^n(X)$ is the sum of Frobenius n_1-,n_2-$, \ldots, n_k$-homomorphisms $f_1, f_2, \ldots, f_k$ (where $n = n_1 + n_2 + \cdots + n_k$) if and only if t factors as $Y \to \mathrm{Sym}^{n_1}(X) \times \mathrm{Sym}^{n_2}(X) \times \cdots \times \mathrm{Sym}^{n_k}(X) \to \mathrm{Sym}^n(X)$ where the last map is concatenation.

Using [**1**, theorem 3·4 and corollary 3·6] which consider the relationship between Frobenius n-homomorphisms on affine algebras and symmetric powers of algebraic varieties, one can in a similar way prove.

THEOREM 6·3. *Let A, B be finitely generated commutative algebras and $f\colon A \to B$ a ring homomorphism; let $V(A)$, $V(B)$ be the corresponding varieties and $h\colon V(B) \to V(A)$ the map corresponding to f. Then h is an n-fold branched covering if and only if there is an n-transfer $B \to A$, for h.*

Acknowledgements. The research on which this is based was mainly carried out during visits by VMB to the School of Mathematics, University of Edinburgh supported by the Engineering and Physical Sciences Research Council. We wish to thank the referee for pointing out that the conclusion of Proposition 2·1 holds for all connected algebras.

REFERENCES

[**1**] V. M. BUCHSTABER and E. G. REES. The Gelfand map and symmetric products. *Selecta Mathematica* **8** (2002), 523–535.

[**2**] V. M. BUCHSTABER and E. G. REES. Rings of continuous functions, symmetric products and Frobenius algebras. *Uspekhi Mat. Nauk* **59** (2004), no. 1 (355), 125–144. Translated: *Russian Math. Surveys* **59** (2004), no. 1, 125–146.

[**3**] V. M. BUCHSTABER and E. G. REES. Multivalued groups, their representations and Hopf algebras. *Transform. Groups* **2** (1997), no. 4, 325–349.

[**4**] V. M. BUCHSTABER and E. G. REES. Multivalued groups, n-Hopf algebras and n-ring homomorphisms. Lie groups and Lie algebras. *Math. Appl.* **433** (Kluwer Academic Publsher, 1998), 85–107.

[**5**] A. DOLD. Ramified coverings, orbit projections and symmetric powers. *Math. Proc. Camb. Phil. Soc.* **99** (1986), no. 1, 65–72.

[**6**] D. V. GUGNIN. On continuous and irreducible Frobenius n-homomorphisms. *Uspekhi Mat. Nauk* **60** (2005), no. 5, 181–182. Translated: *Russian Math. Surveys* **60** (2005), no. 5, 967–969.

[**7**] I. G. MACDONALD. *Symmetric Functions and Hall Polynomials* (Clarendon Press, 1979).

[**8**] L. SMITH. Transfer and ramified coverings. *Math. Proc. Camb. Phil. Soc.* **93** (1983), no. 3, 485–493

[**9**] N. E. STEENROD. *The Theory of Fiber Bundles* (Princeton University Press, 1951).