GLOBAL STRUCTURE OF QUANTUM FIELD THEORY

Thesis

Submitted by

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DECLARATION

This thesis and the material presented in it are my own, except where otherwise indicated. This thesis is partially based on work which is contained in the following papers: The Gribov Ambiguity in Gauge Theories on the 4-torus (Edinburgh Preprint 1984 and Phys. Lett. 138B (1984) 87), Non-linear σ-models on Compact Riemann Surfaces (Edinburgh Preprint 1984) and Global Aspects of Fixing the Gauge in the Polyakov String and Einstein Gravity (Edinburgh Preprint 1984).

He is quick, thinking in clear images;
I am slow, thinking in broken images.

He becomes dull, trusting to his clear images;
I become sharp, mistrusting my broken images.

Trusting his images, he assumes their relevance;
Mistrusting my images, I question their relevance.

Assuming their relevance, he assumes the fact;
Questioning their relevance, I question the fact.

When the fact fails him, he questions his senses;
When the fact fails me, I approve my senses.

He continues quick and dull in his clear images;
I continue slow and sharp in my broken images -

He in a new confusion of his understanding;
I in a new understanding of my confusion.

In _Broken Images_ by Robert Graves

ABSTRACT

In recent years it has become clear that the global structure of field theories has important physical consequences. In this thesis we examine the global structure of some of the most important field theories in physics.

The geometrical formulation of non-Abelian gauge theories is reviewed to serve as an introduction to latter parts of the thesis and also to allow a comparison to be made with certain results obtained in subsequent chapters.

A generalisation of the O(3) non-linear σ-model is introduced which retains all the interesting features of the O(3) model. The instanton solutions of this generalised model are studied and the number of independent self-dual solutions is calculated. The classifying space for the inequivalent quantisations of the generalised σ-model is obtained and the topology of the space of instanton fields is discussed.

The topological structure of gauge theories on compact 4-manifolds is considered. It is shown that the topologically non-trivial nature of the group of gauge transformations results in no continuous global gauge existing and also in the existence of inequivalent quantisations of gauge theories. The existence of inequivalent quantisations of coupled gauge theories is also considered, as is the existence of global anomalies.

Finally, the problem of globally fixing the gauge in the Polyakov string theory and four dimensional Einstein gravity is considered. It is shown that in many circumstances no global gauge choice exists.

CONTENTS

C O N T E N T S (Contd.)

CHAPTER 1

CLASSICAL GAUGE THEORIES

1. Introduction

Since their introduction by Yang and Mills in 1954 (Yang and Mills (1954)) non-Abelian gauge theories have assumed a dominant position in theoretical physics. The notion of a gauge theory was first introduced into physics by Weyl (Weyl (1918)) in an attempt to reformulate Maxwell's theory of electromagnetism. The gauge concept was however limited to the study of electromagnetic interactions until the work of Yang and Mills, although Klein had considered a non-Abelian gauge theory 15 years earlier (Klein (1939)).

A non-Abelian gauge theory is a generalisation of electromagnetism in which the Abelian group U(1) is replaced by some non-Abelian group such as SU(2) or SU(3). This seemingly straightforward modification results in non-Abelian gauge theories possessing many remarkable properties which are not present in electromagnetism. These differences manifest themselves at both the classical and the quantum level. The development of the Higgs mechanism (Higgs (1966)) in non-Abelian gauge theory led to the to the successful model of electro-weak interactions (Weinberg (1967) and Salam (1968)) by allowing the gauge fields to acquire mass. A non-Abelian gauge theory (Q.C.D.) is also the most successful candidate for a quantum theory of the strong interaction.

In this chapter the geometrical formulation of classical non-Abelian gauge theory is described. Section 2 introduces the idea of a connection and its curvature. The Yang-Mills equations are obtained in Section 3 and the notion of gauge invariance is introduced. Section 4 considers self-dual Yang-Mills potentials which are important because they yield an absolute lower bound on the Yang-Mills action. Finally, the coupled theory of Yang-Mills fields and matter fields is discussed in Section 5. Good expositions of classical gauge theory from the point of view adopted here are Bourguignon and Lawson (1982) and Parker (1982). More specialised results can be found in Atiyah and Bott (1982) and Atiyah, Hitchin and Singer (1978).

2. Connections and Curvature

To describe Yang-Mills theory in terms of differential geometry one introduces a compact Lie group G and a compact Riemannian manifold M. The group G is called the gauge group and M represents the space-time. We then fix a principal G-bundle over M

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

with canonical projection π (see Kobayashi and Nomuzu (1963) and Steenrod (1951). Associated to P by the adjoint action of G on itself is the bundle of groups $\mathrm{Ad}\, P$

$$\mathrm{Ad}\, P = P \times_G G$$

and the bundle associated to P by the adjoint action of G on its Lie algebra $\mathfrak{g}$ is

$$\mathrm{ad}\, P = P \times_G \mathfrak{g} \quad .$$

Alternatively, the bundle ad P can be thought of as the pull-back of the tangent bundle to Ad P in the fibre direction, $T_F \mathrm{Ad}\, P$ by the canonical section s_e of Ad P which sends each point in M to the identity $e \in G$. Thus, we have the diagram

$$\begin{array}{ccc} s_e^*(T_F \mathrm{Ad}\, P) & \longrightarrow & T_F\, \mathrm{Ad}\, P \\ \downarrow & & \downarrow \\ M & \xrightarrow[s_e]{} & \mathrm{Ad}\, P \end{array}$$

and the identification

$$\mathrm{ad}\, P = s_e^*(T_F\, \mathrm{Ad}\, P) \quad .$$

It is now possible to define the space of p-forms on M with values in ad P

$$\Omega^p(M;\mathrm{ad}\, P) = \Gamma(\Lambda^p T^* M \otimes \mathrm{ad}\, P)$$

and the space of all ad P-valued forms on M

$$\Omega^*(M;\mathrm{ad}\, P) = \Gamma(\Lambda^* T^* M \otimes \mathrm{ad}\, P)$$

where $\Lambda^* T^* M = \bigoplus_{p \geq 0} \Lambda^p T^* M$ is the exterior algebra bundle. The space $\Omega^*(M;\mathrm{ad}\, P)$ is a graded Lie algebra with the following operations (see Atiyah and Bott (1982) and Parker (1982)).

(i) The Lie bracket on $\mathfrak{g}$ together with exterior multiplication give a map

$$\Omega^p(M;\text{ad } P) \otimes \Omega^q(M;\text{ad } P) \rightarrow \Omega^{p+q}(M;\text{ad } P)$$

$$(\alpha \otimes A) \otimes (\beta \otimes B) \rightarrow [A,B]\, \alpha \wedge \beta$$

for $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ and $A, B \in \Gamma$ ad P,

denoted by $\omega \otimes \theta \rightarrow [\omega,\theta]$. For $\omega \in \Omega^p(M;\text{ad } P)$ $\theta \in \Omega^q(M;\text{ad } P)$ this operation satisfies

$$[\omega,\theta] = (-1)^{pq+1} [\theta,\omega]$$

and the Jacobi identity (for $\eta \in \Omega^r(M;\text{ad } P)$)

$$(-1)^{pr}[\omega,[\theta,\eta]] + (-1)^{pq}[\theta,[\eta,\omega]] + (-1)^{qr}[\eta,[\omega,\theta]] = 0 .$$

(ii) A compact Lie group G always admits a positive definite inner product on its Lie algebra $\mathfrak{g}$, given by minus the Killing form, which is invariant under the adjoint action. Such an inner product on $\mathfrak{g}$ induces a Riemannian metric on ad P and gives a map

$$\Omega^p(M;\text{ad } P) \otimes \Omega^q(m;\text{ad } P) \rightarrow \Omega^{p+q}(M)$$

by

$$(\alpha \otimes A) \otimes (\beta \otimes B) \rightarrow (A, B)\alpha \wedge \beta$$

for $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ and $A, B \in \Gamma$ ad P,

denoted by $\omega \otimes \theta \rightarrow \omega \wedge \theta$. This operation satisfies

$$[\omega,\theta] \wedge \eta = \omega \wedge [\theta,\eta] .$$

Finally, the space $\Omega^*(M;\text{ad } P)$ can be given a natural inner product structure. If M is oriented then it possesses a unique volume element μ of unit length in the orientation of M. The corresponding Hodge duality operator $* \in \Gamma \operatorname{End}(\Omega^*(M))$ is characterised by

$$\omega \wedge *\omega = \langle \omega, \omega \rangle_M \, \mu$$

for $\omega \in \Omega^p(M)$, where $\langle \, , \, \rangle_M$ denotes the natural Riemannian structure on $\Omega^p(M)$. Then the inner product on $\mathfrak{g}$, and the Riemannian metric of M combine to give a natural inner product on $\Omega^*(M;\mathrm{ad}\,P)$

$$\langle \omega, \theta \rangle = \int_M \omega \wedge *\theta \qquad (2.1)$$

for $\omega, \theta \in \Omega^p(M;\mathrm{ad}\,P)$. Note that we will use the notation $\|\omega\|^2 = \langle \omega, \omega \rangle$ and $|\omega|^2 = \omega \wedge *\omega$.

We now introduce the concept of a connection and its curvature. If we let $T_F P$ denote the tangent bundle along the fibres of P and $\pi^* TM$ is the pull-back of the tangent bundle of M via the canonical projection $\pi\colon TM \to M$, then a connection A for P is a G-invariant splitting of the natural exact sequence of vector bundles over P

$$0 \to T_F P \to TP \to \pi^* TM \to 0 \quad .$$

The bundle $T_F P$ is often called the vertical bundle, and its complement $T_A P$, is called the horizontal bundle of A. Thus the splitting A is equivalent to a G-invariant direct sum decomposition

$$TP = T_F P \oplus T_A P \quad .$$

A dual way of thinking of a connection is as a $\mathfrak{g}$-valued 1-form A on P which

(i) has horizontal kernel:

$$A(i_*X) = X, \quad \text{for } X \in \mathfrak{g} \quad \text{and}$$

$i_*: \mathfrak{g} \hookrightarrow TP$ the natural inclusion of $\mathfrak{g}$ in the vertical subspace;

(ii) is G-equivariant: $R_g^* A = (\mathrm{Ad}\, g^{-1})A$ for

$R_g: G \to G$ the right action on G given by

$$R_g g' = g'g, \quad \text{for} \quad g, g' \in G.$$

Note that it follows from property (i) that $T_A P = \ker A$. If A and A' are two connections on P, then their difference $A - A'$ has the following properties

(i) $R_g^*(A - A') = (\mathrm{Ad}\, g^{-1})(A - A')$;

(ii) $(A - A')(i_*X) = 0, \quad \text{for } X \in \mathfrak{g}$,

i.e., $A - A'$ vanishes on vertical vectors.

Hence, $A - A'$ pulls-down to M as a 1-form with values in $\mathrm{ad}\, P$, i.e. $A - A' \in \Omega^1(M; \mathrm{ad}\, P)$. Therefore, the space of connections $\mathscr{C}(P)$ on P is naturally an affine space, with associated vector space $\Omega^1(M; \mathrm{ad}\, P)$.

The <u>curvature</u> of a connection A on P is the $\mathfrak{g}$-valued 2-form on P defined by

$$F_A(X,Y) = dA(hX, hY), \tag{2.2}$$

for X, Y ε ΓTP, where h is the projection onto the horizontal subspace of A. From the structure equation the curvature F_A can be written as

$$F_A = dA + \tfrac{1}{2}[A,A] \quad . \tag{2.3}$$

It follows from (2.2) and (2.3) that F_A is an adjoint invariant 2-form on P which vanishes on vertical vectors. Thus, F_A pulls-down to M as a 2-form with values in ad P, i.e. $F_A \; \varepsilon \; \Omega^2(M;\text{ad } P)$.

3. The Yang-Mills Equations

The Yang-Mills action S[A] on the space of connections $\mathscr{C}(P)$ is defined to be

$$\begin{aligned} S[A] &= \tfrac{1}{2}\| F_A\|^2 \\ &= \tfrac{1}{2}\int_M |F_A|^2 \end{aligned} \tag{3.1}$$

where $\|\cdot\|$ is the norm on $\Omega^2(M;\text{ad } P)$ defined by (2.1). A critical point A ε $\mathscr{C}(P)$ of the Yang-Mills action S is called a Yang-Mills connection (potential) and the curvature F_A is called a Yang-Mills field. The variational equations of (3.1) are called the Yang-Mills equations and to obtain these equations it is necessary to introduce the Yang-Mills operator.

To define this operator recall that a connection A on P induces a natural covariant derivative ∇_A^X on all the associated vector bundles of P. Thus if

$$E = P \times_G V$$

is associated to P by a representation

$$\rho : G \to GL(V)$$

then A gives a way of differentiationg sections $s \in \Gamma E$, along a vector field $X \in \Gamma TM$, by the following procedure. The sections $s \in \Gamma E$ can be identified with G-equivariant maps from P to V. Hence $s \in \Gamma E$ corresponds to a G-equivariant map

$$\tilde{s} : P \to V, \qquad \tilde{s}(pg) = \rho(g^{-1})\tilde{s}(p) \quad .$$

Now given a vector field X on M, its horizontal A-lift $\tilde{X}$ to P is a well defined G-invariant vector field. Hence

$$\tilde{X}\cdot\tilde{s} : P \to V$$

is also G-invariant and corresponds to the section $\nabla_A^X s \in \Gamma E$. This covariant derivative dually corresponds to a differential operator

$$\nabla_A : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

given by

$$\nabla_A : s \to \nabla_A^X s$$

for $s \in \Gamma E$. This differential operator then extends to a differential operator

$$D_A : \Gamma(\Lambda^p T^*M \otimes E) \to \Gamma(\Lambda^{p+1} T^*M \otimes E)$$

defined by

$$D_A(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \otimes \nabla_A s ,$$

where $\alpha \in \Gamma(\Lambda^p T^* M)$ and $s \in \Gamma E$, and extended to any $\omega \in \Gamma(\Lambda^p T^* M \otimes E)$ by linearity.

In our case $E = \mathrm{ad}\, P$ is the vector bundle associated to P by the action of G on $\mathfrak{g}$ and we obtain differential operators

$$\nabla_A : \Omega^0(M;\mathrm{ad}\, P) \to \Omega^1(M;\mathrm{ad}\, P)$$

and

$$D_A : \Omega^*(M;\mathrm{ad}\, P) \to \Omega^{*+1}(M;\mathrm{ad}\, P).$$

In particular D_A acts on $\Omega^*(M;\mathrm{ad}\, P)$ as a derivation relative to both the $[\,,\,]$ and $\wedge$ operations (Atiyah and Bott (1982) and Parker (1982)):

$$D_A[\omega,\theta] = [D_A\omega,\theta] + (-1)^p[\omega,D_A\theta]$$

$$d(\omega\wedge\theta) = D_A\omega \wedge \theta + (-1)^p \omega \wedge D_A\theta$$

for $\omega \in \Omega^p(M;\mathrm{ad}\, P)$, $\theta \in \Omega^*(M;\mathrm{ad}\, P)$.

The adjoint D_A^* of D_A relative to the inner product (2.1) is defined by

$$\langle D_A\omega,\theta\rangle = \langle \omega, D_A^*\theta\rangle$$

for $\omega \in \Omega^p(M;\mathrm{ad}\, P)$ and $\theta \in \Omega^{p+1}(M;\mathrm{ad}\, P)$. The adjoint D_A^* is called the Yang-Mills operator. Therefore, given a connection $A \in \mathcal{C}(P)$ we have a sequence of operators

$$\Omega^0(M;\mathrm{ad}\, P) \underset{D_A^*}{\overset{D_A}{\rightleftarrows}} \Omega^1(M;\mathrm{ad}\, P) \underset{D_A^*}{\overset{D_A}{\rightleftarrows}} \Omega^2(M;\mathrm{ad}\, P) \underset{D_A^*}{\overset{D_A}{\rightleftarrows}} \cdots$$

We now obtain the Yang-Mills equations as the variational equations of (3.1). Fix a connection $A \in \mathscr{C}(P)$, because $\mathscr{C}(P)$ is an affine space, it suffices to vary A along lines

$$A_t = A + t\eta \quad , \qquad \eta \in \Omega^1(M;\text{ad } P) .$$

The curvature F_t of A_t is given by

$$F_t = F + t\, D_A\eta + \tfrac{1}{2}t^2[\eta,\eta] \tag{3.2}$$

where F is the curvature of A. The derivative of the Yang-Mills action is

$$\frac{d}{dt} S[A_t]\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\{S[A_t] - S[A]\}$$

$$= \lim_{t\to 0} \frac{1}{t}\, \tfrac{1}{2}\{\|F_t\|^2 - \|F\|^2\} .$$

From (3.2) we have that

$$\|F_t\|^2 = \|F\|^2 + 2t\langle D_A\eta,F\rangle + t^2\{\|D_A\eta\|^2 + \langle F,[\eta,\eta]\rangle\} + \text{higher terms.}$$

Thus

$$\frac{d}{dt} S[A_t]\Big|_{t=0} = \langle D_A\eta,F\rangle \quad .$$

Hence, A is a critical point of S if

$$0 = \frac{d}{dt} S[A_t]\Big|_{t=0} = \langle D_A\eta,F\rangle$$

$$= \langle \eta \, , \, D_A^* F\rangle$$

for all $\eta \in \Omega^1(M;\text{ad } P)$. Therefore, $A \in \mathscr{C}(P)$ is a Yang-Mills connection if and only if

$$D_A^* F_A = 0 . \qquad (3.3)$$

This is the Yang-Mills equation and it can also be written in terms of the Hodge * operator which relates D_A and D_A^* by

$$D_A^* = (-1)^{p+1} * D_A *$$

on $\Omega^p(M; \text{ad } P)$. Hence, the Yang-Mills equation (3.3) is equivalent to

$$D_A * F_A = 0 .$$

It is also possible to write the Yang-Mills equation in one other form by introducing the Hodge-de Rham Laplacian on $\Omega^p(M; \text{ad } P)$

$$\Delta_A = D_A D_A^* + D_A^* D_A .$$

From the Bianchi identity

$$D_A F_A = 0,$$

for all A, and the compactness of M we see that (3.3) is equivalent to the equation

$$\Delta_A F_A = 0 .$$

Thus, Yang-Mills connections are connections with harmonic curvature.

We will now consider the invariance of the Yang-Mills action under gauge transformations. The total space P of the principal G-bundle is a manifold with a free right action of G defined on it. An automorphism of P is a diffeomorphism $f: P \to P$ which preserves this structure, i.e., $f(pg) = f(p)g$, for all $p \in P$ and $g \in G$. Any automorphism f of P induces a diffeomorphism $\pi(f)$ of M and the group of all bundle automorphisms such that $\pi(f) : M \to M$ is orientation preserving is denoted by Aut P. The subgroup of Aut P which induces the identity transformation on M is denoted by $\mathrm{Aut}_o\, P$. There is an exact sequence

$$0 \to \mathrm{Aut}_o P \to \mathrm{Aut}\, P \to \mathrm{Diff}\, M$$

where Diff M is the group of orientation preserving diffeomorphisms of M. An automorphism $f \in \mathrm{Aut}_o P$ is called a gauge transformation and the group of all gauge transformations is denoted by

$$\mathcal{G}(P) = \mathrm{Aut}_o\, P.$$

The group of gauge transformations $\mathcal{G}(P)$ can also be identified with the space of sections of the bundle of groups Ad P, i.e.,

$$\mathcal{G}(P) \simeq \Gamma \mathrm{Ad}\, P$$

which forms a group under pointwise multiplication. This can be seen as follows. The sections of Ad P can be identified with maps $f : P \to G$ satisfying

$$f(pg) = g^{-1} f(p) g \quad . \tag{3.4}$$

Given a section of Ad P represented by $f : P \to G$ we can define $\hat{f} : P \to P$ by

$$\hat{f}(p) = pf(p) \quad .$$

It then follows from (3.4) that $\hat{f}$ is G-equivariant and hence $\hat{f} \in \mathrm{Aut}_o\, P$. Conversely, given an automorphism $\hat{f} \in \mathrm{Aut}_o\, P$, $\hat{f}$

defines a unique map $f : P \to G$ such that

$$\hat{f}(p) = p\, f(p)$$

and G-equivariance of $\hat{f}$ requires that f satisfies the relation (3.4), i.e., f can be identified with a section of Ad P. This establishes the correspondence $\mathcal{G}(P) \simeq \Gamma \text{Ad } P$.

The group of gauge transformations $\mathcal{G}(P)$ acts naturally on the space of connections $\mathcal{C}(P)$ on P. If $A \in \mathcal{C}(P)$ is a connection and $f \in \mathcal{G}(P)$ a gauge transformation then f transforms A to f^*A, the pull-back of A by f. The action of f on A is given locally by

$$f^*A = (\text{Ad } f^{-1})A + f^{-1}\, df \tag{3.5}$$

and the curvature of f^*A is

$$F_{f^*A} = (\text{Ad } f^{-1})F_A \quad . \tag{3.6}$$

When G is a matrix group these expressions become

$$f^*A = f^{-1}Af + f^{-1}\, df$$

and

$$F_{f^*A} = f^{-1}F_A\, f \quad .$$

It is clear that the Yang-Mills action (3.1) is invariant under the transformation in F given by (3.6), i.e.,

$$S[f^*A] = S[A]$$

for all $A \in \mathcal{C}(P)$ and $f \in \mathcal{G}(P)$. Thus we have seen that the Yang-Mills action is gauge invariant.

4. Self-Dual Yang-Mills Connections

On four dimensional manifolds the Hodge star operator is an involution of $\Lambda^2 T^* M$, giving rise to the decomposition

$$\Lambda^2 T^* M \quad = \quad \Lambda^2_+ T^* M \oplus \Lambda^2_- T^* M \tag{4.1}$$

where $\Lambda^2_\pm T^* M$ are the ± 1 eigenspaces of $*$. Relative to this decomposition the Riemannian curvature tensor R has irreducible components $\{s, B, W_\pm\}$, where s is the scalar curvature, B is the traceless Ricci tensor and $W_\pm$ together give the conformally invariant Weyl tensor $W = W_+ + W_-$. Note that the metric is Einstein if and only if $B = 0$ and the metric is conformally flat if and only if $W = 0$. An oriented Riemannian 4-manifold is self-dual if its Weyl tensor $W = W_+$, i.e., if $W_- = 0$.

It follows from the decomposition (4.1) that $\Omega^2(M;\text{ad } P)$ decomposes as

$$\Omega^2(M;\text{ad } P) \quad = \quad \Omega^2_+(M;\text{ad } P) \oplus \Omega^2_-(M;\text{ad } P) \ .$$

A connection $A \in \mathcal{C}(P)$ on a 4-manifold M is said to be self-dual if its curvature $F_A \in \Omega^2_+(M;\text{ad } P)$ and anti-self-dual if $F_A \in \Omega^2_-(M;\text{ad } P)$. Every $\omega \in \Omega^2(M;\text{ad } P)$ and can be written uniquely as $\omega = \omega_+ + \omega_-$ where

$$\omega_\pm \quad = \quad \tfrac{1}{2}(1 \pm *)\omega \ .$$

Hence, a connection A is self-dual if $F_A = *F_A$ and anti-self-dual if $F_A = -*F_A$. The importance of the (anti-)self-dual connections is that they give the absolute minima of the Yang-Mills action. To see this recall that principal G-bundles, with G a compact simply connected simple Lie group, over a compact 4-manifold M possess a characteristic class, which lies in dimension 4; this is the second Chern class $c_2(P) \in H^4(M; \mathbb{Z})$. In terms of the (anti-)self-dual components $F_\pm$ of $F = F_+ + F_-$ we have

$$c_2(P) = -\frac{1}{8\pi^2}\{\|F_+\|^2 - \|F_-\|^2\} \quad .$$

As $\|F\|^2 = \|F_+\|^2 + \|F_-\|^2$ we have that

$$\|F_+\|^2 + \|F_-\|^2 \geqslant \left| \|F_+\|^2 - \|F_-\|^2 \right| \qquad (4.2)$$

and thus the Yang-Mills action satisfies

$$S \geqslant 4\pi^2 |c_2(P)| \quad . \qquad (4.3)$$

Hence, the Yang-Mills action is bounded below by a topological invariant of P. It is clear from (4.2) that those connections which realise this absolute lower bound are the (anti-)self-dual connections.

As the (anti-)self-dual connections give the absolute minima of the Yang-Mills action it is interesting to know whether such connections exist. When M is self-dual then there is a correspondence due to Ward (see Atiyah and Ward (1977)) between self-dual connections on a bundle over M and holomorphic bundles on a complex manifold. When $M = S^4$, the Ward correspondence has led to the construction of all self-dual connections on G-bundles over S^4 (see Atiyah, Hitchin, Drinfeld and Manin (1978)). It has also been shown by Atiyah, Hitchin and Singer (1978) that the space of self-dual connections modulo gauge transformations, defined on a principal

G-bundle over a self-dual manifold M with positive scalar curvature, is either empty or a manifold of dimension

$$p_1(\text{ad } P) - \tfrac{1}{2} \dim G(\chi - \tau) \qquad (4.4)$$

where $p_1(\text{ad } P)$ is the first Pontrjagin class of ad P, χ is the Euler characteristic of M, and τ is the signature of M. The question of the existence of self-dual connections has been answered in greater generality by Taubes (1982). He has shown that a sufficient condition for there to exist principal G-bundles P over M which admit self-dual connections is that the second de Rham cohomology group $H^2_{DR}(M)$ of M should have no anti-self-dual elements.

Finally, it is of interest to know whether all the critical points of the Yang-Mills action are (anti-)self-dual. There is still an open problem in general, however, it has been shown (Bourguignon and Lawson (1981)) that any weakly stable (i.e. non-negative second variation) Yang-Mills connection over S^4 with group SU(2) or SU(3) is (anti-)self-dual.

5. The Coupled Yang-Mills Equations

Let $\pi : P \to M$ be a principal G-bundle over an oriented 4-manifold M with compact structure group G. Suppose that

$$\rho : G \to GL(V)$$

is a representation of G and

$$E = P \times_G V$$

is the associated vector bundle. Let W be any bundle associated to the frame bundle of M. The coupled field equations will specify a connection A from the space $\mathscr{C}(P)$ of connections on P and a section Φ from the space $\mathcal{E}(P) = \Gamma(E \otimes W)$ of E-valued fields on M. We shall assume that these equations arise as the stationary points of an action integral

$$S[A, \Phi] = \int_M \mathscr{L}(A, \Phi) \tag{5.1}$$

where the Lagrangian $\mathscr{L}$ is a 4-form constructed from A and Φ. The group of gauge transformations acts on $\Phi \varepsilon \Gamma(E \otimes W)$ by pull-back. We require that the action (4.1) be gauge invariant, i.e., for $f \varepsilon \mathscr{G}(P)$, $A \varepsilon \mathscr{C}(P)$ and $\Phi \varepsilon \mathcal{E}(P)$

$$S[f^*A, f^*\Phi] = S[A, \Phi] \quad .$$

The specific form of the action depends upon the nature of the field being described. For example, the action of a massless fermi field is defined as follows. Let M be a spin 4-manifold (i.e. its second Stiefel-Whitney class $w_2(M) \varepsilon H^2(M; \mathbb{Z}_2)$ vanishes) with Riemannian connection ∇. ∇ induces a connection ∇^S on the spin bundle S over M. A connection A on P induces a connection ∇^E on any associated bundle $E = P \times_G V$ in the manner described in Section 3. The connections ∇^S and ∇^E induce a connection $\nabla = \nabla^S \otimes 1 + 1 \otimes \nabla^E$ on the bundle $S \otimes E$. The covariant derivative ∇ is a map

$$\Gamma(S \otimes E) \xrightarrow{\nabla} \Gamma(S \otimes E \otimes T^* M)$$

and Clifford multiplication gives a map

$$\Gamma(S \otimes E \otimes T^*M) \xrightarrow{C} \Gamma(S \otimes E).$$

The Dirac operator D on E-valued spinors $\psi \in \Gamma(S \otimes E)$ on M is defined to be the composition of ∇ and C, i.e.,

$$D : \Gamma(S \otimes E) \rightarrow \Gamma(S \otimes E)$$

is given by

$$D = C \circ \nabla .$$

Now the massless fermion action on E-valued spinors $\psi \in \Gamma(S \otimes E)$ is defined as

$$S[A, \psi] = \int_M [\tfrac{1}{2}|F_A|^2 + \langle\psi, D\psi\rangle \mu(g)]$$

where $\langle \, , \, \rangle$ is the inner product on $S \otimes E$ and $\mu(g)$ is the measure associated to the metric g on M.

It is also possible to write down an action for boson fields and to obtain the variational equations of the fermi and boson actions (see Parker (1982)). We will not consider this further because the specific form of the action will not be required in later chapters.

CHAPTER 2

NON-LINEAR σ-MODELS ON COMPACT RIEMANN SURFACES

1. Introduction

The main reason for studying the classical O(3) non-linear σ-model in two dimensions is its similarities with pure Yang-Mills theory in four dimensions. The O(3) model (Belavin and Polyakov (1975)) is a theory of a smooth three component real field $\underline{\phi} = (\phi^a)$ (a = 1,2,3) defined on $\mathbb{R}^2$, i.e. $\underline{\phi} : \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth map. The action of the theory is

$$S[\underline{\phi}] = \frac{1}{2}\int_{\mathbb{R}^2} \partial_\mu \underline{\phi}\cdot\partial^\mu\underline{\phi}\, d^2x$$

$$= \frac{1}{2}\int_{\mathbb{R}^2} \delta^{\mu\nu}\, \partial_\mu \phi^a\, \partial_\mu \phi^a\, d^2x \tag{1.1}$$

where $\delta^{\mu\nu}$ is the Euclidean metric on $\mathbb{R}^2$. The field $\underline{\phi}$ is subject to the constraint

$$\underline{\phi}^2 \equiv \phi^a\phi^a = 1 \quad . \tag{1.2}$$

The action (1.1) is invariant under a conformal change in the metric

$$g_{\mu\nu} = \Omega^2\, \delta_{\mu\nu} \tag{1.3}$$

for Ω a smooth real valued function on $\mathbb{R}^2$. Taking

$$\Omega = 2/(1 + x^2) \tag{1.4}$$

for $x = (x_1,x_2) \in \mathbb{R}^2$, and assuming that the field $\underline{\phi}$ obeys the boundary condition

$$\underline{\phi}(x) \to \phi_\infty \quad \text{as} \quad |x| \to \infty \tag{1.5}$$

where ϕ_∞ is a constant, shows that the field defines a smooth map $\phi: S^2 \to S^2$, from the conformally compactified Euclidean 2-space to the unit 2-sphere in $\mathbb{R}^3$. The maps from S^2 to S^2 are partitioned into homotopy classes which form a group $\pi_2(S^2) \simeq \mathbb{Z}$, this isomorphism is given by the degree of the map. Associated with each homotopy class of maps is a topological charge

$$Q[\underline{\phi}] = \frac{1}{8\pi} \int_{\mathbb{R}^2} \varepsilon_{ab}\, \underline{\phi} \cdot (\partial_a \underline{\phi} \times \partial_b \underline{\phi}) d^2x \tag{1.6}$$

and it follows from the inequality

$$\int_{\mathbb{R}} (\partial_a \underline{\phi} \pm \varepsilon_{ab}\, \underline{\phi} \times \partial^b \underline{\phi}) \cdot (\partial^a \underline{\phi} \pm \varepsilon^{ac}\, \underline{\phi} \times \partial_c \underline{\phi}) d^2x \geq 0 \tag{1.7}$$

that

$$S \geq 4\pi\, |Q| \, . \tag{1.8}$$

The equality in (1.8) will hold if and only if

$$\partial_a \underline{\phi} \pm \varepsilon_{ab}\, \underline{\phi} \times \partial^b \underline{\phi} = 0 \tag{1.9}$$

and such a field is said to be (anti-) self-dual. In discussing the solutions of (1.9) it is important to remember that the 2-sphere S^2 has a unique complex structure. This arises when it is regarded as the complex projective line $\mathbb{P}^2$. Under this identification the (anti-) self-dual fields correspond to (anti-) holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}^1$.

In this chapter a generalisation of the O(3) model is considered in which the field ϕ is a smooth map from a compact Riemann surface M into a compact Kähler manifold V. Using techniques from the theory of harmonic maps it is shown in section 2 that the action of this theory is bounded below by a topological charge and that the fields which realise this absolute lower bound are the (anti-) holomorphic maps from M to V. For suitable choices of M and V this model coincides with the classical $O(3)$, CP^N and complex Grassmannian models (see, for example, Belavin and Polyakov (1975), Eichenherr (1978), Din and Zakrzewski (1980), (1981)). In section 3 the case when $V = \mathbb{P}^N$ (the N dimensional complex projective space) is discussed. In particular, the dimension of the space of self-dual fields from M to $\mathbb{P}^N$ of degree n is calculated in terms of N, n and the genus g of M. This result gives, for example, the number of independent instanton solutions (of a given degree) of the $O(3)$ or CP^N model. The existence of holomorphic maps from a compact Riemann surface to the complex Grassmannian $G_k(C^n)$ is also briefly discussed. The topology of the configuration space $\mathcal{Q}$ of maps from M to V is considered in section 4. The homotopy groups of the configuration space are calculated in terms of the homotopy groups of V and the genus g of M. The first homotopy group of $\mathcal{Q}$ is related to the existence of inequivalent quantisations of the theory and the classifying space for these quantisations is calculated. Finally, the relationship between the topology of the space of self-dual fields and the topology of the space of all fields is considered. It is shown, for example, that the space of self-dual fields, of degree greater than one, in the $O(3)$ model is not simply connected.

2. Generalised Non-linear σ-Model.

Let M be a compact Riemann surface with metric g and V a compact simply connected n-dimensional Riemannian manifold with metric h. Given the Riemannian metric $g \in \Gamma(TM \otimes TM)^*$, we write $\langle u,v\rangle$ for $g_x(u,v)$, $x \in M$, $u,v \in T_xM$, $\|u\|^2 = \langle u,v\rangle$, and similarly for $h \in \Gamma(TV \otimes TV)^*$. If $\phi: M \to V$ is a smooth map then the differential of ϕ at $x \in M$ is a linear map

$$d\phi(x): \quad T_xM \to T_{\phi(x)}V \tag{2.1}$$

and hence $d\phi(x) \in T_x^*M \otimes T_{\phi(x)}V$. The norm $\|d\phi(x)\|$ is defined using the metric induced on $T_x^*M \otimes T_{\phi(x)}V$ from the Riemannian structures on M and V. The generalisation of the $O(3)$ model is a theory of smooth fields $\phi: M \to V$ with the action given by the "energy" of the field. The Lagrangian density $\mathcal{L}(\phi): M \to \mathbb{R}^{\geq 0}$ is defined (see Eells and Lemaire (1978))

$$\mathcal{L}(\phi)(x) \quad = \quad \tfrac{1}{2}\,\|d\phi(x)\|^2 \tag{2.2}$$

and the action is

$$S[\phi] \quad = \quad \tfrac{1}{2}\int_M \|d\phi(x)\|^2 \;\; d\mu(g) \tag{2.3}$$

where $d\mu(g)$ is the canonical volume measure associated with g. In local coordinates

$$\mathcal{L}(\phi) \;=\; \tfrac{1}{2}\, g^{\mu\nu}\,\frac{\partial\phi^a}{\partial x^\mu}\,\frac{\partial\phi^b}{\partial x^\nu}\,h_{ab} \tag{2.4}$$

and the correspondence between (2.3) and (1.1) is clearly seen.

An important feature of the $O(3)$ model is that the range S^2 has a complex structure, $S^2 \simeq \mathbb{P}^1$. To incorporate this aspect of the O(3) model into this generalisation it will be assumed that V has a complex structure.

An (almost) complex structure on the manifold V is a section $J_V \varepsilon \Gamma$ End TV such that $J_V^2 = -\mathrm{id}$, similarly $J_M \varepsilon \Gamma$ End TM such that $J_M^2 = -\mathrm{id}$ is an (almost) complex structure on M (see Kobayashi and Nomizu (1963), (1969) for further details). It will be assumed here that these almost complex structures are integrable and hence define complex structures. A map $\phi: M \to V$ is holomorphic if its differential $d\phi$ commutes with the complex structures on M and V, i.e.

$$d\phi \cdot J_M = J_V \cdot d\phi \quad . \tag{2.5}$$

A Hermitean metric on V is a Riemannian metric h such that

$$\langle u,v\rangle = \langle J_V u, J_V v\rangle \tag{2.6}$$

for all $u, v \,\varepsilon\, T_pV$, $p \,\varepsilon\, V$. The Kähler form $\omega \,\varepsilon\, \Gamma(\Lambda^2 T^* V)$ is defined by

$$\omega(u,v) = \langle u, J_V v\rangle \quad . \tag{2.7}$$

If ω is closed then V is a Kähler manifold. The complexification of TM is $T^{\mathbb{C}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ and J_M may be extended by complex linearity to $J_M^{\mathbb{C}} \,\varepsilon\, \Gamma$ End $T^{\mathbb{C}}M$. Since $(J_M^{\mathbb{C}})^2 = -\mathrm{id}$, there is a direct sum decomposition $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$ and $T^{0,1}M$ are the eigenbundles corresponding to the eigenvalues $+i$ and $-i$ of $J_M^{\mathbb{C}}$, respectively. The differential of any map $\phi: M \to V$ can be extended by complex linearity to $d^{\mathbb{C}}\phi: T^{\mathbb{C}}M \to T^{\mathbb{C}}V$, with the canonical decomposition $d^{\mathbb{C}}\phi = \partial\phi + \bar{\partial}\phi$, where

$$\begin{aligned} \partial\phi: \quad & T^{1,0}M \to T^{1,0}V \\ \bar{\partial}\phi: \quad & T^{1,0}M \to T^{0,1}V \end{aligned} \tag{2.8}$$

are defined to be the composition of $d^{\mathbb{C}}$ followed by projection in $T^{\mathbb{C}}V$. A map $\phi: M \to V$ is (anti-) holomorphic if and only if $(\partial\phi = 0)\bar{\partial}\phi = 0$.

Using the derivatives given in (2.8) we can define the (1,0) and (0,1) Lagrangian densities by

$$\begin{aligned} \mathcal{L}^{(1,0)}(\phi)(x) &= \|\partial\phi(x)\|^2 \\ \mathcal{L}^{(0,1)}(\phi)(x) &= \|\bar{\partial}\phi(x)\|^2 \end{aligned} \tag{2.9}$$

with the corresponding actions

$$\begin{aligned} S^{(1,0)}[\phi] &= \int_M \|\partial\phi(x)\|^2 \, d\mu(g) \\ S^{(0,1)}[\phi] &= \int_M \|\bar{\partial}\phi(x)\|^2 \, d\mu(g) \quad . \end{aligned} \tag{2.10}$$

The natural decomposition of the Lagrangian density

$$\mathcal{L}^{(\phi)} = \mathcal{L}^{(1,0)}(\phi) + \mathcal{L}^{(0,1)}(\phi) \tag{2.11}$$

induces the decomposition

$$S[\phi] = S^{(1,0)}[\phi] + S^{(0,1)}[\phi] \tag{2.12}$$

of the action.

To obtain a lower bound on the action we introduce the topological charge $Q[\phi]$ of the field $\phi: M \to V$ given by

$$Q[\phi] = \tfrac{1}{2} \int_M \phi^* \omega \tag{2.13}$$

where ω is the Kähler form of V. Then a direct calculation (Lichnerowicz (1970)) shows that

$$\int_M \phi^*\omega = \int_M [\|\partial\phi(x)\|^2 - \|\bar{\partial}\phi(x)\|^2]\, d\mu(g)$$

$$= S^{(1,0)}[\phi] - S^{(0,1)}[\phi] \; . \qquad (2.14)$$

Thus the inequality

$$S^{(1,0)}[\phi] + S^{(0,1)}[\phi] \geqslant |S^{(1,0)}[\phi] - S^{(0,1)}[\phi]| \quad (2.15)$$

is the equivalent to the inequality

$$S \geqslant 2|Q| \qquad (2.16)$$

and we see that the action is bounded below by a multiple of the absolute value of the topological charge, just as in the O(3) model. In general, the topological charge defined by (2.13) is not invariant under continuous deformations of the field ϕ and thus does not define an absolute lower bound on the action in each homotopy class of maps from M to V. This defect can be remedied by requiring V to be a Kähler manifold. Let $\phi_1, \phi_2: M \to V$ be homotopic (denoted by $\phi_1 \sim \phi_2$) and let $\Phi: M \times [0,1] \to V$ be a homotopy of ϕ_1 and ϕ_2. Then

$$\int_M \phi_1^*\omega - \int_M \phi_2^*\omega = \int_{M\times\{0\}} \Phi^*\omega - \int_{M\times\{1\}} \Phi^*\omega$$

$$= \int_{\partial(M\times[0,1])} \Phi^*\omega = \int_{M\times[0,1]} \Phi^*(d\omega) \quad (2.17)$$

and thus

$$Q[\phi_1] - Q[\phi_2] = \int_{M\times[0,1]} \Phi^*(d\omega) \quad . \tag{2.18}$$

If V is a Kähler manifold then $d\omega = 0$ and the topological charge Q defined by (2.13) is a homotopy invariant. Henceforth it will be assumed that V has a Kähler structure. Note that the topological lower bound on the action of the theory is exactly analogous to the topological lower bound on the Yang-Mills action obtained in section 3 of Chapter 1.

The space of maps $\phi: M \to V$ (which will be assumed to be base-point preserving) are partitioned into homotopy classes, the set of which is denoted by $[M;V]_*$. The manifold V is simply connected and thus by the Hopf classification theorem (Whitehead (1978))

$$\begin{aligned}[M;V]_* &\simeq H^2(M;\pi_2(V)) \\ &\simeq \pi_2(V) \quad . \end{aligned} \tag{2.19}$$

Thus non-trivial topological classes of maps will exist for those spaces V which have a non-trivial second homotopy group. In each of these homotopy classes the action of the model will be bounded below by twice the absolute value of the topological charge Q. Those fields which reasise this absolute lower bound are called instanton solutions of the model. It is clear from (2.15) that an instanton field satisfies either

$$\bar{\partial}\phi = 0$$

or (2.20)

$$\partial\phi = 0$$

and hence is either holomorphic (self-dual) or anti-holomorphic (anti-self-dual). For certain choices of V such maps exist. The case when $V = \mathbb{P}^N$, the N-dimensional complex projective space, is discussed in the next section.

3. The Space of Self-dual Maps from M to $\mathbb{P}^N$

The complex projective space $\mathbb{P}^N$ with the Fubini-Study metric is a compact simply connected Kähler manifold with $\pi_2(\mathbb{P}^N) \simeq \mathbb{Z}$, for all $N \geq 1$. Thus, by (2.19) there exist non-trivial topological classes of maps from any compact Riemann surface M to $\mathbb{P}^N$. For $V = \mathbb{P}^N$ it is possible to write the topological charge (2.13) in terms of $\deg \phi$, the degree of the map $\phi: M \to \mathbb{P}^N$, and Q is given by (Wood (1979))

$$Q[\phi] = 2\pi \deg \phi \quad . \tag{3.1}$$

There is a bijective correspondence between $\deg \phi$ and the elements of $\pi_2(\mathbb{P}^N) \simeq \mathbb{Z}$ and thus within each homotopy class of maps of a given degree the action is minimised by the (anti-) holomorphic maps. If we denote the space of all maps from M to V by Map(M;V) and the space of all holomorphic maps by Hol(M;V) then $\mathrm{Map}(M; \mathbb{P}^N)_n$ and $\mathrm{Hol}(M; \mathbb{P}^N)_n$ denote the component of $\mathrm{Map}(M; \mathbb{P}^N)$ and $\mathrm{Hol}(M; \mathbb{P}^N)$ of degree n, respectively. In this section we calculate the dimension of $\mathrm{Hol}(M; \mathbb{P}^N)_n$ which is the number of independent self-dual fields from M to $\mathbb{P}^N$ of degree n.

To calculate the dimension of $\mathrm{Hol}(M; \mathbb{P}^N)_n$ it is necessary to introduce a correspondence between holomorphic maps from M to $\mathbb{P}^N$ and holomorphic line bundles over M. Before explaining this correspondence we will first recall some notions from algebraic geometry (see Griffiths and Harris (1978)).

A <u>divisor</u> D on a compact Riemann surface is a finite sum

$$D = \Sigma n_i x_i$$

of points $x_i \in M$ with multiplicities n_i. The set of divisors on M forms an additive group, denoted Div M. If $n_i \geq 0$, for all i, then D is called <u>effective</u>. In terms of sheaves, a divisor D on M is a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$, where $\mathcal{M}^*$ denotes the multiplicative sheaf of non-zero meromorphic functions on M and $\mathcal{O}^*$ the subsheaf of non-zero holomorphic functions on M. Thus we have the identification

$$\text{Div } M = H^o(M; \mathcal{M}^*/\mathcal{O}^*) \quad . \tag{3.3}$$

Let $\pi: L \to M$ be a holomorphic line bundle over M, fcr an open cover $\{U_\alpha\}$ of M there are trivialisations

$$\psi_\alpha : L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$$

of $L|_{U_\alpha} = \pi^{-1}(U_\alpha)$ and transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \mathbb{C}^*$ for L given by

$$g_{\alpha\beta}(x) = (\psi_\alpha \cdot \psi_\beta^{-1})|_{L_x} \in \mathbb{C}^* .$$

The transition functions $g_{\alpha\beta}$ are holomorphic, non-vanishing and satisfy the standard cocycle condition. Given a holomorphic line bundle $L \to M$ with trivialisation $\{\psi_\alpha\}$ and transition functions $\{g_{\alpha\beta}\}$, then for any collection of non-zero holomorphic functions on U_α, $f_\alpha \in \mathcal{O}^*(U_\alpha)$, we can define a new trivialisation over $\{U_\alpha\}$ by

$$\psi'_\alpha = f_\alpha \cdot \psi_\alpha$$

and new transition functions

$$g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta} \quad . \tag{3.4}$$

As any trivialisation of L over $\{U_\alpha\}$ can be obtained in this way, the collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition functions define the same holomorphic line bundle if and only if there exist functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying (3.4). In terms of sheaves the transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ represent a Cech cocycle and two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same line bundle if and only if their difference $\{g_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta}\}$ is a Cech coboundary. Thus, the set of all line bundles L over M is $H^1(M;\mathcal{O}^*)$. The set of all line bundles over M has a group structure with multiplication given by tensor product and inverses given by dual bundles. This group structure coincides with the group structure of $H^1(M;\mathcal{O}^*)$ and is called the Picard group of M, denoted by Pic M.

The exact exponential sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0 \tag{3.5}$$

induces in cohomology the boundary map

$$H^1(M;\mathcal{O}^*) \xrightarrow{\delta} H^2(M;\mathbb{Z}) \quad . \tag{3.6}$$

For a line bundle $L \in \text{Pic}\, M = H^1(M;\mathcal{O}^*)$ the first Chern class $c_1(L)$ is defined to be $\delta(L) \in H^2(M;\mathbb{Z})$. The degree $\deg L$ of the line bundle L is defined to be $c_1(L)$. The set of all holomorphic line bundles $L \in \text{Pic}\, M$ of degree n is denoted by $\text{Pic}^n M$.

Let $L \to M$ be a holomorphic line bundle with trivialisation $\psi_\alpha \colon L|_{U_\alpha} \to U_\alpha \times \mathbb{C}$ over $\{U_\alpha\}$ and with transition functions $\{g_{\alpha\beta}\}$

relative to $\{\psi_\alpha\}$. The trivialisations ψ_α induce isomorphisms

$$\psi_\alpha^*: \mathcal{O}(L)(U_\alpha) \rightarrow \mathcal{O}(U_\alpha)$$

and from the correspondence

$$s \in \mathcal{O}(L)(U) \rightarrow \{s_\alpha = \psi_\alpha^*(s) \in \mathcal{O}(U \cap U_\alpha)\}$$

it is clear that a holomorphic section s of L over $U \subset M$ is equivalent to a collection of functions $s_\alpha \in \mathcal{O}(U \cap U_\alpha)$ satisfying

$$s_\alpha = g_{\alpha\beta} \cdot s_\beta$$

in $U \cap U_\alpha \cap U_\beta$. Similarly, a meromorphic section s of L over U is given by a collection of meromorphic functions $s_\alpha \in \mathcal{M}(U \cap U_\alpha)$ which satisfy $s_\alpha = g_{\alpha\beta} \cdot s_\beta$ in $U \cap U_\alpha \cap U_\beta$. If s is a global meromorphic section of L then the order of s is independent of $\{\psi_\alpha\}$ and we may define the divisor (s) of s to be

$$(s) = \Sigma \, \mathrm{ord}_{x_i}(s) \cdot x_i \quad .$$

The section s is holomorphic if and only if (s) is effective and the space of holomorphic sections of L over M is $\Gamma(L) = H^o(M; \mathcal{O}(L))$.

We now describe the correspondence between holomorphic maps from M to $\mathbb{P}^N$ and holomorphic line bundles L over M. Associated to any subspace E of the vector space $\Gamma(L)$ is the <u>linear system</u> $|E|$ of effective divisors corresponding to the sections in E, i.e.

$$|E| = \{(s)\}_{s \in E} \subset \mathrm{Div}\, M.$$

As M is compact $(s) = (s')$ only if $s = \lambda s'$, for $\lambda \in \mathbb{C}^*$, thus $|E|$ is parametrised by $\mathbb{P}(E)$, the projectivisation of E. The linear system $|E|$ is said to have no base points if not all the sections $s \in E$ vanish at any $x \in M$. In this case the set of sections $s \in E$ which vanish at $x \in M$ define a hyperplane $\bar{H}_x \subset E$. Equivalently, the set of divisors $D \in |E|$ which contain x forms a hyperplane $H_x \subset \mathbb{P}(E)$. Thus, one can define a map from M to the dual projective space $\mathbb{P}(E)^*$ ($\mathbb{P}(E)^*$ is the set of hyperplanes in $\mathbb{P}(E)$)

$$f_E\colon\ M \to \mathbb{P}(E)^*$$

by sending a point $x \in M$ to the hyperplane $H_x \in \mathbb{P}(E)^*$.

This map can be described more explicitly by letting $E \subset \Gamma(L)$ be $N+1$ dimensional with a basis $s_o, \ldots, s_N$. For any trivilisation $\{\psi_\alpha\}$ of L over $U \subset M$ let $s_{i,\alpha} = \psi_\alpha^*(s_i) \in \mathcal{O}(U)$, then the point $[s_{o,\alpha}(x), \ldots, s_{N,\alpha}(x)] \in \mathbb{P}^N$ is independent of the trivialisation $\{\psi_\alpha\}$ and can be written as $[s_o(x), \ldots, s_N(x)]$. The map $f_E\colon M \to \mathbb{P}(E)^* \simeq \mathbb{P}^N$ is then defined by

$$f_E(x) = [s_o(x), \ldots, s_N(x)]$$

for $x \in M$, and f_E is seen to be holomorphic. Thus a subspace E of the space of holomorphic sections of a line bundle $L \to M$ determines a holomorphic map to $\mathbb{P}^N$. Conversely, let $f_E\colon M \to \mathbb{P}^N$ be a holomorphic map and let H be the hyperplane bundle on $\mathbb{P}^N$, then $L = f_E^* H$ and any section $s \in E$ is the pull-back of a section of H on $\mathbb{P}^N$, i.e.,

$$E = f_E^* H^o(\mathbb{P}^N; \mathcal{O}(H)) \subset H^o(M; \mathcal{O}(L)) \quad .$$

Thus, the map $f_E\colon M \to \mathbb{P}^N$ determines both the line bundle L and the subspace $E \subset \Gamma(L)$. This results in the following

<u>Correspondence</u> Holomorphic maps $f: M \to \mathbb{P}^N$, modulo projective automorphisms $\leftrightarrow$ holomorphic line hundles $L \to M$ with $E \subset \Gamma(L)$ such that $|E|$ has no base points.

Note that the maps f are only determined up to automorphisms of $\mathbb{P}^N$ because a different choice of basis $s_o, \ldots, s_N$ for E gives different homogeneous coordinates on $\mathbb{P}^N$. Also note that maps $f: M \to \mathbb{P}^N$ of degree n correspond to $E \subset \Gamma(L)$ for line bundles L of degree n.

To obtain the dimension of $Hol(M; \mathbb{P}^N)_n$ we need the following result.

<u>Lemma 3.1</u> Let L be a holomorphic line bundle of degree n over a compact Riemann surface of genus g. Then for $n \geq 2g$ the complete linear system $|\Gamma(L)|$ has no base points.

<u>Proof</u> For any $x \in M$, we have the short exact sequence of sheaves

$$0 \to \mathcal{O}(L-x) \to \mathcal{O}(L) \to L_x \to 0 \qquad (3.7)$$

which gives rise in cohomlogy to the sequence

$$\ldots \to H^o(M; \mathcal{O}(L)) \overset{r_x}{\to} H^o(M; L_x) \to H^1(M; \mathcal{O}(L-x)) \to \ldots \qquad (3.8)$$

where r_x is evaluation at x. Let K_M be the canonical bundle of M and L_1 any line bundle over M, then it follows from the Kodaira vanishing theorem that if $\deg L > \deg K_M$ then $H^1(M; \mathcal{O}(L)) = 0$. On a Riemann surface of genus g the degree of K_M is given by the Riemann-Hurwitz formula to be $\deg K_M = 2g - 2$. Applying this to the line bundle $L-x$ we obtain that if $\deg(L-x) = \deg L-1 > 2g-2$ then $H^1(M; \mathcal{O}(L-x)) = 0$. Thus, for $\deg L \geq 2g$ the exact sequence (3.8) reduces to

$$\cdots \to H^{o}(M;\mathcal{O}(L)) \xrightarrow{r_x} L_x \to 0 \quad .$$

Hence, the evaluation map r_x is surjective and not all the section $s \in \Gamma(L)$ can vanish at x.

We now calculate the dimension of $\mathrm{Hol}(M; \mathbb{P}^N)_n$.

<u>Theorem 3.2</u> Let M be a compact Riemann surface of genus g, then for $n \geq 2g$ the dimension of $\mathrm{Hol}(M; \mathbb{P}^N)_n$ is given by

$$\dim \mathrm{Hol}(M; \mathbb{P}^N)_n = (N+1)n - N(g-1) \quad .$$

<u>Remarks</u> (i) If L is a line bundle of negative degree over M then $H^o(M; \mathcal{O}(L)) = 0$. Thus, by the correspondence introduced above there are no holomorphic maps from M to $\mathbb{P}^N$ of negative degree.

(ii) If L is a line bundle of degree n over a compact Riemann surface M of genus g then for $n \geq 2g-1$ the dimension of $\Gamma(L) = H^o(M; \mathcal{O}(L))$ is given by the Riemann-Roch theorem to be $n-g+1$. Thus, the dimension of $|\Gamma(L)| = \dim \mathbb{P}(\Gamma(L)) = n-g$.

<u>Proof</u> Consider the short exact sequence (which follows from Lemma 3.1)

$$0 \to K_x \to \Gamma(L) \xrightarrow{r_x} L_x \to 0$$

where $K_x = \ker r_x$ and thus $\dim K_x = n-g$. The point $x \in M$ is a base point of $|E|$ if and only if all the sections $s \in E$ vanish at x. Thus, if x is a base point, the map

$$r_x|_E : E \subset \Gamma(L) \to L_x$$

obtained by restricting r_x to E, which takes a section $s \in E$

to $s(x) \in \mathbb{C}$, is zero, i.e. $E = \ker r_x|_E \subset \ker r_x$. Thus, x is a base point of $|E|$ if and only if $E \subset K_x$, and conversely, $|E|$ has no base points if and only if $E \not\subset K_x$, for all $x \in M$. For a given $x \in M$, $K_x = \ker r_x$ gives a hyperplane in the projective space $\mathbb{P}(\Gamma(L))$ parametrising $\Gamma(L)$, and thus $K_x \in \mathbb{P}(\Gamma(L))^*$, the dual projective space. For a fixed $K_x \in \mathbb{P}(\Gamma(L))^*$ we have the Grassmannian $G_{N+1}(K_x)$ of N+1 dimensional spaces E in the n-g dimensional space K_x. This Grassmanian is the fibre over K_x of the fibre bundle

$$\begin{array}{ccc} G_{N+1}(K) & \longrightarrow & \mathcal{F} \\ & & \big\downarrow \mathrm{pr}_1 \\ & & \mathbb{P}(\Gamma(L))^* \end{array}$$

where $\mathcal{F}$ is the flag manifold consisting of pairs (K,E) with $E \subset K \subset \Gamma(L)$ and $\dim E = N+1$, $\dim K = n-g$. The total space $\mathcal{F}$ has two canonical projections $\mathrm{pr}_1(K,E) = K \in \mathbb{P}(\Gamma(L))^*$ and $\mathrm{pr}_2(K,E) = E \in G_{N+1}(\Gamma(L))$. By Lemma 3.1, if $\deg L \geq 2g$ then $|\Gamma(L)|$ has no base points and there is a well defined map $f\colon M \to \mathbb{P}(\Gamma(L))^*$ given by the correspondence introduced earlier. Thus we have the diagram

$$\begin{array}{ccccc} G_{N+1}(K) & & G_{N+1}(K) & & \\ \big\downarrow & & \big\downarrow & & \\ f^*\mathcal{F} & \xrightarrow{\ \hat{f}\ } & \mathcal{F} & \xrightarrow{\mathrm{pr}_2} & G_{N+1}(\Gamma(L)) \\ \big\downarrow & & \big\downarrow \mathrm{pr}_1 & & \\ M & \xrightarrow[\ f\ ]{} & \mathbb{P}(\Gamma(L))^* & & \end{array}$$

$E \subset K_x$, for some $x \in M$, if and only if $E \in \operatorname{im} \mathrm{pr}_2 \circ \hat{f}$, thus there is no $x \in M$ such that $E \subset K_x$ if and only if $E \notin \operatorname{im} \mathrm{pr}_2 \circ \hat{f}$.

Hence, $\mathrm{im}\ pr_2 \circ \hat{f}$ consists of exactly those E for which $|E|$ has a base point. The dimension of $G_{N+1}(\Gamma(L))$ is $(N+1)[n-g+1-(N+1)]$ $= (N+1)(n-g)+(N+1)-(N+1)^2$ and $\dim(\mathrm{im}\ pr_2 \circ \hat{f}) \leqslant \dim f^*\mathcal{F} =$ $1+(N+1)(n-g)-(N+1)^2$. Thus, $\dim(\mathrm{im}\ pr_2 \circ \hat{f}) < \dim G_{N+1}(\Gamma(L))$ if $N \geqslant 1$ and hence $pr_2 \circ \hat{f}$ is not surjective. $\mathrm{Im}\ pr_2 \circ \hat{f}$ is a closed subvariety in $G_{N+1}(\Gamma(L))$. The complement $G_{N+1}(\Gamma(L))\backslash \mathrm{im}\ pr_2 \circ \hat{f}$ is open and consists of those E's with no base points. The Grassmannian $G_{N+1}(\Gamma(L))$ can be considered as the fibre over $L \in \mathrm{Pic}^n(M)$ of the fibre bundle

$$\begin{array}{ccc} G_{N+1}(\Gamma(L)) & \longrightarrow & \mathcal{G}_{N+1}(M) \\ & & \Big\downarrow \mathcal{R} \\ & & \mathrm{Pic}^n(M) \end{array}$$

where the total space $\mathcal{G}_{N+1}(M)$ consists of pairs (L,E), $E \subset \Gamma(L)$, and $\mathcal{R}(L,E) = L \in \mathrm{Pic}^n(M)$. From the above argument those E's for which $|E|$ has no base points form a Zariski open set in $\mathcal{G}_{N+1}(M)$ which is the complement of a subvariety in $\mathcal{G}_{N+1}(M)$. Thus the dimension of the space of holomorphic maps from M to $\mathbb{P}^N$, modulo projective automorphism, is equal to $\dim \mathcal{G}_{N+1}(M) = g + (N+1)(n-g) - N(N+1)$. Finally, the dimension of $\mathrm{Hol}(M; \mathbb{P}^N)_n$ is equal to $\dim \mathcal{G}_{N+1}(M)$ plus the dimension of $PGL_{N+1}(\mathbb{C})$, the group of automorphisms of $\mathbb{P}^N$. Hence

$$\begin{aligned} \dim \mathrm{Hol}(M; \mathbb{P}^N)_n &= \dim \mathcal{G}_{N+1}(M) + \dim PGL_{N+1}(\mathbb{C}) \\ &= g + (N+1)(n-g) - N(N+1) + (N+1)^2 - 1 \\ &= (N+1)n - N(g-1) \quad . \end{aligned}$$

This formula for the dimension of $Hol(M: \mathbb{P}^N)_n$ is analogous to the expression for the dimension of the moduli space of instantons in Yang-Mills theory given in Chapter 1, section 3.

An application of this result is to calculate the number of independent self-dual solutions, of degree n, of the classical CP^N model. This corresponds to calculating the dimension of $Hol(S^2; \mathbb{P}^N)_n$ Recall from the remark made earlier that there are no holomorphic maps from S^2 to $\mathbb{P}^N$ of negative degree and therefore there are no self-dual fields of negative topological charge. As S^2 has $g = 0$ we have for all $n \geqslant 0$ that

$$\dim Hol(S^2; \mathbb{P}^N)_n \quad = \quad (N+1)n+N \quad . \qquad (3.9)$$

The classical $O(3)$ model corresponds to the CP^1 model and hence, for all $n \geqslant 0$,

$$\dim Hol(S^2; \mathbb{P}^N)_n \quad = \quad 2n + 1 \qquad (3.10)$$

which agrees with the number of independent parameters in the general, explicitly known, self-dual solution of degree n.

To conclude this section we note that a theory of maps from M to the complex Grassmannian $G_K(\mathbb{C}^m)$ generalises the complex Grassmannian model (see Din and Zakrzewski (1981)). The Grassmannian $G_K(\mathbb{C}^m)$ is a simply connected Kähler manifold and thus the self-dual fields from M to $G_K(\mathbb{C}^m)$ are given by the holomorphic maps $Hol(M;G_K(\mathbb{C}^m))$. Although the analogue of theorem 3.2 for the dimension of $Hol(M;G_K(\mathbb{C}^m))$ is not known, certain holomorphic maps from M to $G_K(\mathbb{C}^m)$ do exist. For example, if M is holomorphically immersed in $\mathbb{P}^N$ then the Gauss map (see Griffiths and Harris (1978))

$$\gamma: M \to G_2(\mathbb{C}^{N+1})$$

is holomorphic (see Eells and Lemaire (1978), for example).

4. Topology of the Configuration Space

An interesting feature of field theories with non simply connected configuration spaces is that they can possess inequivalent quantisations. If $\mathcal{Q}$ is the configuration space of the theory in question then the inequivalent quantisations are classified by (Dowker (1980) and Isham (1981))

$$\Theta \ = \ \mathrm{Hom}(\pi_1(\mathcal{Q}), \ U(1)). \qquad (4.1)$$

In fact, the arguments leading to this result are not quite complete as they ignore the possibility of the theory possessing a Wess-Zumino type term. This problem can be seen most clearly from the canonical view-point. Let $\mathcal{Q}$ be the configuration space of the theory and the cotangent bundle $\pi\colon T^*\mathcal{Q} \to \mathcal{Q}$ is the phase space, this carries a canonical non-degenerate symplectic 2-form Ω_o, defining the natural Hamiltonian structure. In canonical quantisation we choose a complex line bundle $\mathcal{L} \to \mathcal{Q}$; the Hilbert space $\mathcal{H}$ of states of the quantised theory is the space of sections of $\mathcal{L}$ and the equations of motion of the theory are implemented as operator equations on $\mathcal{H}$. If the canonical symplectic structure on $T^*\mathcal{Q}$ defined by Ω_o can be changed by adding a curvature term pulled-back from $\mathcal{Q}$, then the equations of motion defined by this new symplectic structure will differ from those defined by Ω_o. An example of such a change in the symplectic structure occurs when one considers the motion of a charged particle in the field of a magnetic monopole. The quantisation of the magnetic charge of the monopole is a consequence of the modification in the symplectic structure. A second important example of such a modification in the equation of motion of a physical system is the addition of the Wess-Zumino term in the SU(3) non-linear σ-model. It is the presence of this term in the model that is responsible for the important consequences discovered by Witten (Witten (1983)). The way in which the Wess-Zumino term arises in the SU(3) σ-model by changing the symplectic structure has been investigated by Ramadas (Ramadas (1984)). If, however, we consider a theory which has no Wess-Zumino type term then to eliminate the possibility of altering the canonical symplectic structure we can require that the complex line bundle $\mathcal{L} \to \mathcal{Q}$ must be flat. Then it is well known that the flat complex line bundles over $\mathcal{Q}$ are classified by $\mathrm{Hom}(\pi_1(\mathcal{Q}),\ U(1))$, which gives (4.1). As there are no Wess-Zumino type terms in the non-linear σ-models being considered here the classification (4.1) is valid.

For the classical O(3) model $\mathcal{Q} = \mathrm{Map}_*(S^2;S^2)$ and $\pi_1(\mathcal{Q}) \simeq \pi_1(\Omega^2 S^2) \simeq \pi_3(S^2) \simeq \mathbb{Z}$, thus*

$$\begin{aligned} \Theta &= \mathrm{Hom}(\mathbb{Z}, U(1)) \\ &\simeq U(1) \quad . \end{aligned} \qquad (4.2)$$

For a generalised non-linear σ-model discussed in section 2, $\mathcal{Q} = \mathrm{Map}_*(M;V)$ and the homotopy groups $\pi_q(\mathcal{Q})$ are given by the following theorem (the space $\mathrm{Map}_*(M;V)$ is assumed to have the compact - open topology (see Whitehead (1978)).

<u>Theorem 4.1</u> Let M be a compact Riemann surface of genus g and V a compact topological space. The homotopy groups of $\mathrm{Map}_*(M;V)$ are given by

$$\pi_q(\mathrm{Map}_*(M;V) \simeq [\pi_{q+1}(V)]^{2g} \oplus \pi_{q+2}(V)$$

for $q \geq 1$.

* It should be noted that in this section Θ classifies the inequivalent quantisations of a non-linear σ-model in 2 space + 1 time dimensions.

<u>Proof</u> Recall that $\pi_1(M)$ = free group on $a_1b_1a_2b_2 \ldots a_gb_g$ subject to the relation $a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1} = 1$. M can be obtained from the wedge product of $2g$ circles by attaching a cell in dimension two via the map $a = a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1}$, i.e.,

$$M \simeq \bigvee_{2g} S^1 \cup_a e^2 \quad .$$

Now

$$a \in \pi_1(\bigvee_{2g} S^1)$$

and its suspension

$$Sa \in \pi_2(\bigvee_{2g} S^2)$$

is null-homotopic $(Sa \simeq 0)$ because π_2 is Abelian. Thus, the suspension of M is

$$SM \simeq \bigvee_{2g} S^2 \cup_{Sa} e^3$$

$$\simeq \bigvee_{2g} S^2 \vee S^3 \quad .$$

Suspending this $q-1$ times gives

$$S^qM \simeq \bigvee_{2g} S^{q+1} \vee S^{q+2}$$

and the homotopy groups of $Map_*(M;V)$ are given by

$$\pi_q\ Map_*(M;V) \simeq [S^qM;V]_*$$

$$\simeq [S^{q+1};V]_*^{2g} \oplus [S^{q+2};V]_*$$

$$\simeq [\pi_{q+1}(V)]^{2g} \oplus {}_{q+2}(V) \quad .$$

We also have the following consequences:

Corollary 4.2

$$\pi_1 \operatorname{Map}_*(M; \mathbb{P}^N) \simeq \begin{cases} \mathbb{Z}^{2g} \oplus \mathbb{Z}, & \text{for } N = 1 \\ \mathbb{Z}^{2g}, & \text{for } N \geq 2 \end{cases}$$

Proof. This follows from the homotopy groups of $\mathbb{P}^N$ which are obtained from the exact homotopy sequence of the Hopf fibration

$$U(1) \to S^{2N+1} \to \mathbb{P}^N .$$

Corollary 4.3 For $m \geq k+2$

$$\pi_1 \operatorname{Map}_*(M; G_k(\mathbb{C}^m)) \simeq \mathbb{Z}^{2g}.$$

Proof. This follows from the homotopy result (see appendix)

$$\pi_q(G_k(\mathbb{C}^m)) \simeq \pi_{q-1}(U(k)) \quad \text{for} \quad q < 2(m-k) .$$

Thus, the classifying space for inequivalent quantisations for $V = \mathbb{P}^N$ is

$$\Theta = \begin{cases} \operatorname{Hom}(\mathbb{Z}^{2g} \oplus \mathbb{Z}, U(1)), & \text{for } N = 1 \\ \operatorname{Hom}(\mathbb{Z}^{2g}, U(1)), & \text{for } N \geq 2 \end{cases}$$

and for $V = G_k(\mathbb{C}^m)$ is

$$\Theta = \operatorname{Hom}(\mathbb{Z}^{2g}, U(1)) \quad \text{for} \quad m > k+1 .$$

Note that for $M = S^2$ (i.e., $g = 0$) both the complex Grasmannian model and the CP^N $(N \geq 2)$ model have a unique quantisation. Only the O(3) model has a non-trivial Θ for $g = 0$, which is given by (4.2). The related issue of the existence of inequivalent quantisation of gauge theories will be considered in Chapter 3.

To conclude, we briefly consider the relationship between the topology of the space of self-dual fields $\mathrm{Hol}_*(M;V)$ and the topology of the space of all fields $\mathrm{Map}_*(M;V)$. For $M = S^2$ and $V = \mathbb{P}^N$ this problem has been solved by a theorem of Segal's (Segal (1979)). This theorem states that the inclusion $\mathrm{Hol}_*(S^2; \mathbb{P}^N)_n \hookrightarrow \mathrm{Map}_*(S^2; \mathbb{P}^N)_n$ is a homotopy equivalence up to dimension $n(2N-1)$. For example, when $N = 1$,

$$\pi_q\ \mathrm{Hol}_*(S^2;S^2)_n \simeq \pi_q\ \mathrm{Map}_*(S^2;S^2)_n \simeq \pi_{q+2}(S^2)$$

for $q < n$. For $q = 1$, we obtain

$$\pi_1\ \mathrm{Hol}_*(S^2;S^2)_n \simeq \pi_3(S^2) \simeq \mathbb{Z}$$

for $n > 1$, and hence the space of self-dual fields of degree greater than 1 in the O(3) model is not simply connected.

Appendix

We prove here the formula for the stable homotopy of $G_k(\mathbb{C}^m)$ used in corollary 4.3, namely

$$\pi_q(G_k(\mathbb{C}^m)) \simeq \pi_{q-1}(U(k)), \tag{A1}$$

for $q < 2(m-k)$.

First recall that as a homogeneous space

$$G_k(\mathbb{C}^m) = \frac{U(m)}{U(k) \times U(m-k)} \tag{A2}$$

We know that $U(m+1)/U(m) = S^{2m+1}$ and from the homotopy exact sequence of the fibration

$$\begin{array}{ccc} U(m) & \to & U(m+1) \\ & & \downarrow \\ & & S^{2m+1} \end{array}$$

we see that the inclusion $U(m) \hookrightarrow U(m+1)$ is a homotopy equivalence up to dimension $2m$, i.e., $\pi_q(U(m)) \simeq \pi_q(U(m+1))$, for $q < 2m$. Applying this result to the inclusion $U(m-k) \hookrightarrow U(m)$ gives

$$\pi_q(U(m-k)) \simeq \pi_q(U(m)), \tag{A3}$$

for $q < 2(m-k)$. The homotopy exact sequence of the fibration

$$\begin{array}{ccc} U(m-k) & \to & U(m) \\ & & \downarrow \\ & & U(m)/U(m-k) \end{array}$$

together with (A3) result in

$$\pi_q(U(m)/U(m-k)) = 0, \tag{A4}$$

for $q < 2(m-k)$. Finally, from the expression (A2) for $G_k(\mathbb{C}^m)$ as a homogeneous space it is clear that we have a fibration

$$\begin{array}{ccc} U(k) & \to & U(m)/U(m-k) \\ & & \downarrow \\ & & G_k(C^m) \end{array}$$

and the homotopy exact sequence of this together with (A4) results in the desired formula (A1).

CHAPTER 3

THE TOPOLOGY OF GAUGE THEORIES ON COMPACT 4-MANIFOLDS

1. Introduction

It is well known that certain topological aspects play an important role in non-Abelian gauge theories defined on compact orientable 4-dimensional manifolds. Recall from Chapter 1, for example, that if $P \to M$ is a principal G-bundle over a compact 4-manifold M then the Yang-Mills action, defined on the space of connections on P, is bounded below by a topological invariant of P (i.e., minus the second Chern class of P). The existence of such a topological lower bound on the action of a Yang-Mills system is crucially important for the existence of instanton solutions of the Yang-Mills equations. These instanton solutions have been used in several interesting ways in the study of non-Abelian gauge theories (e.g., the semi-classical analysis of Yang-Mills systems and the resolution of the U(1) problem).

In this chapter we will be concerned with certain topological aspects of gauge theories defined on compact 4-manifolds which are of a different nature to those which lead to the instanton phenomenon. Whereas the lower bound on the Yang-Mills action is a consequence of the topological properties of the principal bundle $P \to M$, we will be concerned here with the topological properties of the group $\mathcal{G}(P)$ of gauge transformations of P. The topology of $\mathcal{G}(P)$ is interesting in its own right and also leads to important physical consequences. One of the most immediate consequences of the topologically non-trivial nature of $\mathcal{G}(P)$ is that it is impossible to fix globally the gauge in a gauge theory defined on any of a large class of 4-manifolds. The topology of $\mathcal{G}(P)$ is also related to the topology of the gauge orbit space $\mathcal{C}(P)/\mathcal{G}(P)$. A knowledge of the topology of $\mathcal{G}(P)$, and hence of $\mathcal{C}(P)/\mathcal{G}(P)$, allows one to examine the possibility of there existing inequivalent quantisations of gauge theories on 4-manifolds. The analogous task for non-linear σ-models on Riemann surfaces was carried out in Chapter 2. Finally, the tolopogically non-trivial nature of $\mathcal{G}(P)$ may result in the gauge theory coupled to left-handed fermions being inconsistent. If the gauge group $G = SU(n)$, for instance, then it is shown that such an inconsistency can occur only when $n = 2$.

2. The Geometry of the Orbit Space $\mathcal{C}/\mathcal{G}$

In this section we will show that, with suitable restrictions, the action of $\mathcal{G}$ on $\mathcal{C}$ results in a principal fibre bundle $\mathcal{C} \to \mathcal{C}/\mathcal{G}$.

To do this we first show that there exists a local slice through any connection $A \in \mathcal{C}$ which intersects each orbit once and only once in a neighbourhood of A. Recall from Chapter 1 that the group $\mathcal{G}(P)$ of gauge transformations of P is

$$\begin{aligned}\mathcal{G}(P) &= \Gamma \text{ Ad } P \\ &= \Gamma(P \times_G G) \quad .\end{aligned}$$

The space of sections of $\text{ad } P = P \times_G \mathfrak{g}$ has the natural structure of a Lie algebra induced by the Lie algebra structure on each fibre.

Hence the space

$$\mathfrak{G}(P) = \Gamma \text{ ad } P$$

$$= \Gamma(P \times_G \mathfrak{g})$$

plays the role of the Lie algebra of the group $\mathcal{G}(P)$ of gauge transformations. The exponential map

$$\exp: \mathfrak{g} \to G$$

induces a natural map

$$\text{Exp}: \mathfrak{G}(P) \to \mathcal{G}(P)$$

defined by

$$\text{Exp}(\sigma)(p) = \exp(\sigma(p))$$

where $\sigma \in \mathfrak{G}(P)$ and $p \in P$. Given $\sigma \in \mathfrak{G}(P)$ the smooth map $t \longrightarrow \text{Exp}(t\sigma)$ defines a 1-parameter subgroup of $\mathcal{G}(P)$ which satisfies (Bleeker (1981))

$$\frac{d}{dt} \text{Exp}(t\sigma)(p) = \sigma(p) \quad .$$

If we consider the curve $g_t = \text{Exp}(t\sigma)$ in $\mathcal{G}(P)$ then for $A \in \mathcal{C}(P)$ we have that (Bleecker (1981))

$$\frac{d}{dt} g_t^* A \Big|_{t=0} = D_A \sigma \ . \tag{2.1}$$

$\mathcal{C}(P)$ is an affine space with $\Omega^1(M;\text{ ad } P)$ as its vector group, therefore there is a natural identification

$$T_A \mathcal{C} \simeq \Omega^1(M;\text{ ad } P)$$

for the tangent space to $\mathcal{C}$ at $A \in \mathcal{C}$. From (2.1) the tangent space to the orbit of $\mathcal{G}$ at $A \in \mathcal{C}$ is exactly the image of the map

$$D_A : \Omega^0(M; \text{ad } P) \to \Omega^1(M; \text{ad } P) \quad .$$

Given $\eta \in \Omega^1(M; \text{ad } P) \simeq T_A\mathcal{C}$, then relative to the norm on $\Omega^1(M; \text{ad } P)$ defined in Chapter 1, η lies in the orthogonal complement of $\text{im } D_A$ if

$$0 = \langle \eta, D_A \sigma \rangle = \langle D_A^* \eta, \sigma \rangle$$

for all $\sigma \in \Omega^0(M; \text{ad } P)$. Hence

$$D_A^* \eta = 0 \quad .$$

The orthogonal complement of $\text{im } D_A$ is precisely the kernel of D_A^*. For $A \in \mathcal{C}$ the transversal slice at A across the orbits of $\mathcal{G}$ is given by

$$\mathcal{S}_A = \{A + \eta \mid \eta \in \Omega^1(M; \text{ad } P) \text{ and } D_A^*\eta = 0\}. \tag{2.2}$$

It may be shown (Mitter and Viallet (1981)) that these local slices are globally effective, i.e., that in a sufficiently small neighbourhood of $A \in \mathcal{C}$ they intersect each orbit once and only once. Infinitesimal variations of connections in the direction of $\text{im } D_A$ are infinitesimal variations through gauge equivalent connections. Infinitesimal variations in the direction of $\ker D_A^*$ are infinitesimal deformations of the connection A. It follows from the above argument that the tangent space to $A \in \mathcal{C}$ has the direct sum decomposition

$$T_A \mathscr{C} = \text{im } D_A \oplus \ker D_A^* . \qquad (2.3)$$

The local slice $\mathscr{S}_A$ provides a good local gauge around the connection A.

The Yang-Mills action S is a $\mathscr{G}$-invariant functional defined on the space $\mathscr{C}$ of connections. Hence, S pulls-down to give a well-defined function on the gauge orbit space $\mathscr{C}/\mathscr{G}$. Unfortunately, the quotient space $\mathscr{C}/\mathscr{G}$ is not a manifold because the action of $\mathscr{G}$ on $\mathscr{C}$ is not free, i.e., there are gauge transformations $f \in \mathscr{G}$, $f \neq id$, such that $f^*A = A$. It is however possible to obtain a free group action by restricting the action of $\mathscr{G}$ on $\mathscr{C}$ in either one of the following two ways. The action of $f \in \mathscr{G}$ on $A \in \mathscr{C}$ is given by

$$\begin{aligned} f \cdot A &= f^{-1} A f + f^{-1} df \\ &= A + f^{-1} D_A f. \end{aligned}$$

If A is a fixed point of $f \in \mathscr{G}$ then $f \cdot A = A$, which implies that $D_A f = 0$. We define the group $\tilde{\mathscr{G}}$ of effective gauge transformations by

$$\tilde{\mathscr{G}} = \mathscr{G}/Z \qquad (2.4)$$

where Z is the centre of G. Also let $\tilde{\mathscr{C}}$ be the space of irreducible connections on P. Then $\tilde{\mathscr{G}}$ acts freely on $\tilde{\mathscr{C}}$ and the local slices give $\tilde{\mathscr{C}}/\tilde{\mathscr{G}}$ a (Hilbert) manifold structure (see Mitter and Viallet (1981)). There is also a principal $\tilde{\mathscr{G}}$-bundle over $\tilde{\mathscr{C}}/\tilde{\mathscr{G}}$

$$\begin{array}{ccc} \tilde{\mathscr{G}} & \longrightarrow & \tilde{\mathscr{C}} \\ & & \downarrow \\ & & \tilde{\mathscr{C}}/\tilde{\mathscr{G}} \end{array} \qquad (2.5)$$

Alternatively, we can choose a base-point $x_o \varepsilon M$ and consider the restricted group of gauge transformations $\mathcal{G}_*$ equal to the identity on P_{x_o}, the fibre of P over $x_o \varepsilon M$, i.e.,

$$\mathcal{G}_*(P) = \{f \varepsilon \mathcal{G}(P) | f(p_o) = p_o, \text{ for } p_o \varepsilon P_{x_o}\}. \quad (2.6)$$

If $f \varepsilon \mathcal{G}$ fixes $A \varepsilon \mathcal{C}$ then $D_A f = 0$. Hence f is covariantly constant and f preserves the parallel transport defined by $A \varepsilon \mathcal{C}$. Since $f = id$ at p_o, parallel transport makes it the identity everywhere. Therefore if $f \cdot A = A$ then $f = id$ and the action of $\mathcal{G}_*$ on $\mathcal{C}$ is free. The free action of $\mathcal{G}_*$ on $\mathcal{C}$ together with the existence of local slices results in $\mathcal{C}/\mathcal{G}_*$ having a manifold structure modelled on a Hilbert space and in a principal $\mathcal{G}_*$-bundle over $\mathcal{C}/\mathcal{G}_*$ (see Mitter and Viallet (1981))

$$\begin{array}{ccc} \mathcal{G}_* & \longrightarrow & \mathcal{C} \\ & & \downarrow \\ & & \mathcal{C}/\mathcal{G}_* \end{array} \quad (2.7)$$

We will now discuss the relationship between the bundles (2.5) and (2.7) and the problem of globally fixing the gauge in a gauge theory.

In the Feynman path integral approach to quantising gauge theories one is interested in quantities of the form

$$Z = \int_{\mathcal{C}} \mathfrak{D}A \exp - S[A]$$

where S[A] is the Yang-Mills action. The difficulty with the integral is that the $\mathcal{G}$-orbits have infinite measure. To remove these infinities one should integrate over the orbit space $\mathscr{C}/\mathscr{G}$, which is, however, intractible. To circumvent this difficulty one attempts to fix the gauge, that is, to choose in a continuous manner one gauge potential on each $\mathscr{G}$-orbit. Therefore, the choice of gauge is a continuous map $s: \mathscr{C}/\mathscr{G} \to \mathscr{C}$ such that $p \circ s = \text{id}$ where $p: \mathscr{C} \to \mathscr{C}/\mathscr{G}$ is the canonical projection. The functional integral is then evaluated over $s(\mathscr{C}/\mathscr{G})$ with a weight factor given by the Jacobian of $p: s(\mathscr{C}/\mathscr{G}) \to \mathscr{C}/\mathscr{G}$.

In electrodynamics on $\mathbb{R}^4$ the Coulomb gauge is frequently chosen. If A_μ ($\mu = 1,\ldots, 4$) are the components of the vector potential the Coulomb gauge condition is that

$$\partial_i A_i = 0 \tag{2.8}$$

for $i = 1,2,3$. Under a gauge transformation A_μ becomes

$$A'_\mu = A_\mu + \partial_\mu \Lambda \quad .$$

Thus any vector potential A'_μ can be transformed into a potential A_μ which satisfies the Coulomb gauge condition if

$$\nabla^2\Lambda = -\partial_i A'_i \tag{2.9}$$

where $\nabla^2 = \partial_i\partial_i$ is the spatial Laplacian. If Λ is regular everywhere and finite at infinity, then (2.9) will have a unique solution if the boundary conditions which are imposed are such that there are no non-trivial solutions of the equation

$$\nabla^2\Lambda = 0 \; .$$

Therefore, under these assumptions, the Coulomb gauge is a good gauge fixing condition for electrodynamics defined on $\mathbb{R}^4$.

For a non-Abelian theory on $\mathbb{R}^4$ it would seem reasonable to attempt to fix the gauge by imposing the three dimensional transversality condition (2.8) on the gauge potential A_μ. Under a non-Abelian gauge transformation g, A_μ will transform to

$$A_\mu' = g^{-1} A_\mu g + g^{-1} \partial_\mu g$$

and (2.8) will be satisfied if

$$\partial_i A_i + [D_i, \partial_i g \cdot g^{-1}] = 0 \qquad (2.10)$$

where $D_i = \partial_i + A_i$ is the spatial covariant derivative. If (2.10) possesses a unique solution under the assumption of suitable boundary conditions at infinity, then the Coulomb gauge will be a good gauge fixing condition in a non-Abelian theory. The existence of a unique solution of equation (2.10) was considered by Gribov (1978) who showed that, for large enough fields, (2.10) has more than one solution. Therefore, the Coulomb gauge "fixing" condition does not fix the gauge uniquely in such a theory. Motivated by this result, Singer (1978) showed that it was impossible to find a continuous gauge fixing condition for any SU(n) gauge theory defined on a space-time which is the 4-sphere S^4 (which amounts to studying gauge fields on $\mathbb{R}^4$ with certain asymptotic behaviour). Singer proved this result by studying the global geometry of the gauge theory concerned and showing that there existed a topological obstruction to the existence of a global gauge fixing condition. The same idea will be used here to show that a non-Abelian theory defined on any of a large class of 4-manifolds must possess a Gribov ambiguity (i.e., that it is impossible to continuously fix the gauge in such a theory).

The geometrical part of the argument leading to this conclusion involves the fibre bundles (2.5) and (2.7) introduced earlier in this section. The proof is completed as a consequence of the information obtained in the next section concerning the topological structure of the group of gauge transformations.

The space $\tilde{\mathscr{C}}$ of irreducible connections is open and dense in $\mathscr{C}$ (Singer (1978)). Therefore if a continuous gauge $s: \mathscr{C}/\mathscr{G} \to \mathscr{C}$ were to exist then the restriction $s|_{\tilde{\mathscr{C}}/\tilde{\mathscr{G}}}: \tilde{\mathscr{C}}/\tilde{\mathscr{G}} \to \tilde{\mathscr{C}}$ would give a global section of the bundle $\tilde{\mathscr{C}} \to \tilde{\mathscr{C}}/\tilde{\mathscr{G}}$. Such a global section exists if and only if the bundle is trivial, i.e.,

$$\tilde{\mathscr{C}} = \tilde{\mathscr{C}}/\tilde{\mathscr{G}} \times \tilde{\mathscr{G}} .$$

Applying $\pi_q(\cdot)$ to this gives

$$\pi_q(\tilde{\mathscr{C}}) \simeq \pi_q(\tilde{\mathscr{C}}/\tilde{\mathscr{G}}) \oplus \pi_q(\tilde{\mathscr{G}})$$

for all $q \geq 0$. The space $\tilde{\mathscr{C}}$ is, in fact, contractible (Singer (1978)) and thus

$$\pi_q(\tilde{\mathscr{C}}/\tilde{\mathscr{G}}) \oplus \pi_q(\tilde{\mathscr{G}}) \simeq 0$$

for all $q \geq 0$. Therefore, if $\pi_q(\tilde{\mathscr{G}}) \neq 0$, for some $q \geq 0$, then no continuous gauge $s: \mathscr{C}/\mathscr{G} \to \mathscr{C}$ exists. It will be shown in the next section that $\tilde{\mathscr{G}}$ is homotopically non-trivial for a large class of 4-manifolds.

A second variant of the gauge fixing problem involves the restricted group of gauge transformations $\mathscr{G}_*$. In many circumstances in a gauge theory on $\mathbb{R}^4$ it is natural to work with those gauge transformations which are the identity at infinity. When $\mathbb{R}^4$ is conformally compactified to give S^4, with the base-point $x_0 \in S^4$ corresponding to infinity, the gauge transformations which are the identity at infinity correspond to the gauge transformations which preserve x_o. In this framework a choice of gauge is precisely a global section of the bundle $\mathcal{C} \to \mathcal{C}/\mathcal{G}_*$. As before such a section exists if and only if the bundle is trivial, i.e.,

$$\mathcal{C} \simeq \mathcal{C}/\mathcal{G}_* \times \mathcal{G}_*$$

and hence

$$\pi_q(\mathcal{C}) \simeq \pi_q(\mathcal{C}/\mathcal{G}_*) \oplus \pi_q(\mathcal{G}_*)$$

for all $q \geq 0$. Recalling that $\mathcal{C}$ is an affine space and thus contractible gives

$$\pi_q(\mathcal{C}/\mathcal{G}_*) \oplus \pi_q(\mathcal{G}_*) \simeq 0$$

for all $q \geq 0$. Hence, no global gauge (i.e., no global section) exists if $\mathcal{G}_*$ is homotopically non-trivial. In section 3 it will be shown that $\mathcal{G}_*$ is topologically non-trivial on any closed compact orientable 4-manifold.

Although it is often impossible to fix the gauge globally in a Yang-Mills theory there always exist local gauges which are given by the slices $\mathcal{S}_A$. If $\mathcal{U}$ is a sufficiently small neighbourhood of a connection $A \in \mathcal{C}$ then $\mathcal{S}_A$ is a good gauge in $\mathcal{U}$. If the local path integral we wish to evaluate is

$$Z = \int_{\mathcal{U}} \mathfrak{D}A \exp - S[A]$$

and $\mathcal{P}: \mathcal{C} \to \mathcal{C}/\mathcal{G}_*$ is the canonical projection with the restriction $\mathcal{P}|_{\mathcal{S}_A} : \mathcal{S}_A \longrightarrow \mathcal{C}/\mathcal{G}_*$ then we can write

$$Z = \int_{\mathcal{R}|_{\mathcal{S}_A}(\mathcal{S}_A)} \mathfrak{D}A \exp - S[A]$$

$$= \int_{\mathcal{S}_A} \mathcal{R}|^{*}_{\mathcal{S}_A} \mathfrak{D}A \exp - S[\mathcal{R}|^{*}_{\mathcal{S}_A} A]$$

$$= \int_{\mathcal{S}_A} \det(\mathcal{R}|_{\mathcal{S}_A}) \mathfrak{D}\hat{A} \exp - S[\hat{A}]$$

The Jacobian determinant $\det(\mathcal{R}|_{\mathcal{S}_A})$ of $\mathcal{R}|_{\mathcal{S}_A}$ is the Fadeev-Popov determinant associated with the local gauge $\mathcal{S}_A$.

3. The Topology of $\mathcal{G}$ on Compact 4-Manifolds

We have seen in section 2 that the topological non-triviality of $\tilde{\mathcal{G}}$ or $\mathcal{G}_*$ represents the obstruction to globally fixing the gauge. Also recall from section 4 of Chapter 2 that the inequivalent quantisations of the non-linear σ-model were classified in terms of the fundamental group of the configuration space of the theory. For a Yang-Mills theory the configuration space can be taken to be $\mathcal{C}/\mathcal{G}_*$. A knowledge of the homotopy groups of $\mathcal{G}_*$ is equivalent to a knowledge of the homotopy groups of $\mathcal{C}/\mathcal{G}_*$. Thus the classifying space for the inequivalent quantisations of a Yang-Mills theory can be determined from the topological nature of $\mathcal{G}_*$. We will also see in section 4 that the topology of $\mathcal{G}_*$ can be such that a global anomaly is present when the gauge theory is coupled to left-handed fermions. In this section we will discuss the topology of the groups $\tilde{\mathcal{G}}$ and $\mathcal{G}_*$ on compact 4-manifolds. In particular, it is shown that the topology of $\mathcal{G}_*$ is essentially completely determined on any compact simply connected spin 4-manifolds.

For a given principal G-bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \\ & & M \end{array}$$

over M we have the group $\mathcal{G}(P)$ of gauge transformations of P. Together with $\mathcal{G}(P)$ there are the two subgroups $\tilde{\mathcal{G}}(P)$ and $\mathcal{G}_*(P)$, defined in section 2. Thus for each principal G-bundle there are the groups $\mathcal{G}(P)$, $\tilde{\mathcal{G}}(P)$ and $\mathcal{G}_*(P)$. The isomorphism classes of princupal G-bundles over M are classified by (Steenrod (1951))

$$\mathcal{B}_G(M) \quad = \quad [M; BG]$$

where BG is the universal classifying space of G. When G is simply connected, which is the case we will concentrate on, $\mathcal{B}_G(M)$ is given by the following result.

<u>Theorem 3.1</u> Let M be a closed compact orientable 4-manifold and G a compact simply connected semi-simple Lie group. If the Lie algebra $\mathfrak{g}$ of G has a decomposition

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots\cdots \oplus \mathfrak{g}_\ell$$

into ℓ non-trivial simple ideals $\mathfrak{g}_i$ $(1 \leqslant i \leqslant \ell)$ then

$$\mathcal{B}_G(M) \simeq \mathbb{Z}^{\ell} .$$

Proof. If G is a compact simply connected semi-simple Lie group then the 5-skeleton of BG is homotopically a wedge product of ℓ 4-spheres (Bott (1956)). Thus

$$\begin{aligned} \mathcal{B}_G(M) &= [M; BG] \\ &\simeq [M; \bigvee_{\ell} S^4] \\ &\simeq \bigoplus_{\ell} [M; S^4] \end{aligned}$$

By the Hopf classification theorem (Whitehead (1978))

$$\begin{aligned} [M; S^4] &\simeq H^4(M; \mathbb{Z}) \\ &\simeq \mathbb{Z} . \end{aligned}$$

Hence

$$\mathcal{B}_G(M) \simeq \mathbb{Z}^{\ell} .$$

Thus, for G any compact simply connected semi-simple Lie group, there are a countably infinite number of inequivalent principal G-bundles over any compact 4-manifold M. One might expect that the homotopy type of the groups $\mathcal{G}(P)$, $\tilde{\mathcal{G}}(P)$ and $\mathcal{G}_*(P)$ would depend on the isomorphism class of P. The next theorem, proved by Singer (Singer (1978)) for $G = SU(n)$, asserts that the homotopy type of $\mathcal{G}_*(P)$ is independent of P.

Theorem 3.2. Let M be a closed compact orientable 4-manifold and G a compact simply connected Lie group. If $\mathcal{G}_*(P)$ is the group of restricted gauge transformations on P and $Map_*(M;G)$ the group of base-point preserving maps from M to G then there is the (weak) homotopy equivalence

$$\mathcal{G}_*(P) \sim Map_*(M;G)$$

Proof. Let $x_o \in M$ be the base-point and D a disc centred on x_o. Note that $c_1(P) = 0$ and $c_2(P)$ can be localised. P is trivial over D. The complement $M_o = M \setminus D$ is homotopy equivalent to a 3-complex and since G is simply connected P is also trivial over M_o. Let $P_1 = P|_D \simeq D \times G$ and $P_2 = P|_{M_o} \simeq M_o \times G$. Then P is determined by a patching map $\phi\colon S^3 \to G$ where $S^3 \simeq \partial D$. If $f \in \mathcal{G}$ then $f = \{f_1, f_2\}$ where $f_1\colon D \to G$ and $f_2\colon M_o \to G$. On the boundary S^3 of D f satisfies $f_2 = \phi f_1 \phi^{-1}$. If $f \in \mathcal{G}_*$ then $f(x_o) = e$. Let

$$\beta : \mathcal{G}_* \to \{f_1\colon (D, x_o) \to (G, e)\} = \mathcal{H}$$

be given by $f \longrightarrow f_1$. β is a homomorphism with kernel

$$K = \{f \in \mathcal{G}_* \mid f_1\colon D \to e\}$$

$$\simeq \{f_2\colon (M_o, S^3) \to (G,e)\} .$$

Thus we have the fibration

$$0 \longrightarrow K \longrightarrow \mathcal{G}_* \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$$

Since $\mathcal{H}$ is contractible there is a weak homotopy equivalence $K \sim \mathcal{G}_*$. However, $(M_o, S^3) \sim (M, x_o)$ and hence

$$\mathcal{G}_* \sim \{f_2:(M,x_o) \to (G,e)\} \ .$$

Using this result we can now investigate the topological structure of $\mathcal{G}_*$. The main case we will consider will be when M is simply connected.

Theorem 3.3. Let M be a closed compact orientable simply connected 4-manifold and G a compact simply connected Lie group. Also let $r = \text{rank } H^2(M)$. Then we have the long exact sequence

$$0 \leftarrow \pi_o(\mathcal{G}_*) \leftarrow \pi_4(G) \leftarrow [\pi_3(G)]^r \leftarrow \pi_1(\mathcal{G}_*) \leftarrow \pi_5(G)$$

$$\leftarrow [\pi_4(G)]^r \leftarrow \ldots \leftarrow [\pi_{j+2}(G)]^r \leftarrow \pi_j(\mathcal{G}_*) \leftarrow \pi_{j+4}(G) \leftarrow \ldots \ .$$

Proof. The 4-manifold M is simply connected so it is, homotopically, a wedge product of r 2-spheres with a 4-cell attached (Milnor (1958)) That is, the 2-skeleton $M^{(2)}$ of M is

$$M^{(2)} \sim \bigvee_r S^2 \ .$$

This implies that there is a cofibration (Switzer (1975), Whitehead (1978))

$$S^3 \xrightarrow{\alpha} \bigvee_r S^2 \longrightarrow M \longrightarrow S^4 \xrightarrow{S\alpha} \bigvee_r S^3 \longrightarrow SM \longrightarrow S^5 \xrightarrow{S^2\alpha} \bigvee_r S^4 \to \ldots$$

$$\ldots \to \bigvee_r S^{j+2} \longrightarrow S^jM \longrightarrow S^{j+4} \xrightarrow{S^{j+1}\alpha} \bigvee_r S^{j+3} \to \ldots$$

where $\alpha: S^3 \to M^{(2)} \simeq \bigvee_r S^2$ is the attaching map for the top cell, $S^j\alpha$ is the j'th suspension of α and S^jM is the j'th suspension of M (see Switzer (1975), Whitehead (1978). Mapping this cofibration into the group G gives the long exact sequence

$$[S^3;G]_* \leftarrow \bigoplus_r [S^2;G]_* \leftarrow [M;G]_* \leftarrow [S^4;G]_*$$

$$\leftarrow \bigoplus_r [S^3;G]_* \leftarrow [SM;G]_* \leftarrow [S^5;G] \rightarrow \bigoplus_r [S^4;G] \leftarrow \dots$$

$$\dots \leftarrow \bigoplus_r [S^{j+2};G]_* \leftarrow [S^jM;G]_* \leftarrow [S^{j+4};G] \leftarrow \bigoplus_r [S^{j+3};G]_* \leftarrow \dots$$

Recalling that

$$\pi_o \operatorname{Map}_*(M;G) \simeq [M;G]_*$$

$$\pi_1 \operatorname{Map}_*(M;G) \simeq [SM;G]_*$$

$$\vdots$$

$$\pi_j \operatorname{Map}_*(M;G) \simeq [S^jM;G]_*$$

$$\vdots$$

and that $\pi_2(G) = 0$ yields

$$0 \leftarrow \pi_o \operatorname{Map}_*(M;G) \leftarrow \pi_4(G) \leftarrow [\pi_3(G)]^r \leftarrow \pi_1 \operatorname{Map}_*(M;G)$$

$$\leftarrow \pi_5(G) \leftarrow [\pi_4(G)]^r \leftarrow \dots \leftarrow [\pi_{j+2}(G)]^r \leftarrow \pi_j \operatorname{Map}_*(M;G) \leftarrow \pi_{j+4}(G) \leftarrow \dots$$

Finally, using the homotopy equivalence of theorem 3.2 completes the proof.

We can now show that $\mathcal{G}_*$ is homotopically non-trivial on a compact simply connected 4-manifold. First we give a useful

Lemma 3.4. The even dimensional rational homotopy groups of a compact simply connected Lie group G vanish, i.e.,

$$\pi_{2k}(G) \otimes \mathbb{Q} \simeq 0$$

Proof. $H^*(G; \mathbb{Q})$ is an exterior algebra on odd dimensional generators (Spanier (1966). Hence G is rationally homotopy equivalent to a product of odd dimensional spheres. Odd dimensional spheres have no even dimensional rational homotopy groups.

We now have the following

Theorem 3.5. If M is a closed compact orientable simply connected 4-manifold and G a compact simply connected Lie group then

$$\pi_j(\mathcal{G}_*) \neq \{0\}$$

for some $j \geq 0$.

Proof. Assume that $\pi_j(\mathcal{G}_*) = \{0\}$, for all $j \geq 0$. Then from theorem 3.3 we obtain that

$$\pi_4(G) \simeq [\pi_3(G)]^r .$$

Tensoring with $\mathbb{Q}$ gives

$$\pi_4(G) \otimes \mathbb{Q} \simeq [\pi_3(G)]^r \otimes \mathbb{Q} .$$

Then lemma 3.4 implies that

$$[\pi_3(G)]^r \otimes \mathbb{Q} \simeq 0$$

which is a contradiction. Hence, either $\pi_o(\mathcal{G}_*) \neq \{0\}$ or $\pi_1(\mathcal{G}_*) \neq \{0\}$.

From the discussion in section 2 we know that the topological non-triviality of $\mathcal{G}_*$ is the obstruction to the existence of a global section $s: \mathcal{C}/\mathcal{G}_* \to \mathcal{C}$. Therefore, theorem 3.5 implies that no global section exists. Hence, relative to the group of restricted gauge transformations $\mathcal{G}_*$, there is no continuous global gauge in any non-Abelian gauge theory defined on a compact simply connected 4-manifold, with any compact simply connected gauge group.

Given an additional restriction on the topology of M it is possible to determine the topology of $\mathcal{G}_*$ essentially completely. We have the following result.

Theorem 3.6. Let M be a closed compact orientable simply connected spin 4-manifold and G a compact simply connected Lie group. Then we have the following relationships for the homotopy groups of $\mathcal{G}_*$

$$\pi_o(\mathcal{G}_*) \simeq \pi_4(G)$$

and

$$0 \leftarrow [\pi_{j+2}(G)]^r \leftarrow \pi_j(\mathcal{G}_*) \longleftarrow \pi_{j+4}(G) \longleftarrow 0$$

for all $j \geq 1$.

Proof. We have the cofibration

$$S^3 \xrightarrow{\alpha} \bigvee_r S^2 \to M \to S^4 \xrightarrow{S\alpha} \bigvee_r S^3 \to SM \to S^4 \xrightarrow{S^2\alpha} \bigvee_r S^4 \to \cdots$$

$$\cdots \to \bigvee_r S^{j+2} \to S^jM \to S^{j+4} \xrightarrow{S^{j+1}\alpha} \bigvee_r S^{j+3} \to \cdots$$

where $\alpha: S^3 \to \bigvee_r S^2$ is the attaching map for the 4-cell e^4. Let $\beta = S\alpha \in \pi_4(\bigvee_r S^3) \simeq \bigoplus_r \pi_4(S^3)$. Thus $\beta = \beta_1 \oplus \beta_2 \oplus \cdots \oplus \beta_r$ where $\beta_i \in \pi_4(S^3) \simeq \mathbb{Z}_2$. Let $\gamma = \beta_i$ for $1 \leq i \leq r$. Then $\gamma \in \mathbb{Z}_2$ and for $\gamma \simeq 0$ the mapping cone C_γ of γ is $C_\gamma \simeq S^3 \vee S^5$;

for $\gamma \not\simeq 0$, $C_\gamma \simeq S^3 \bigcup_\gamma e^5 \simeq S\,\mathbb{P}^2$, the suspension of the complex projective plane $\mathbb{P}^2$.

In general if $X = S^n \bigcup_\gamma e^{n+2}$, for $n \geq 3$, then $X \simeq S^n \vee S^{n+2}$ if and only if $Sq^2 \colon H^n(X; \mathbb{Z}_2) \to H^{n+2}(X; \mathbb{Z}_2)$ is zero (Spanier 1966)). If C_β is the mapping cone for $\beta = \beta_1 \oplus \beta_2 \oplus \ldots \oplus \beta_r$ (i.e. $C_\beta \simeq SM$ is the cofibre of the cofibration given above) then $H^3(C_\beta; \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \ldots \oplus \mathbb{Z}_2$ (r copies) and $\beta_i = 0$ if and only if $Sq^2\, x_i = 0$, for x_i the generator of the i'th copy of $\mathbb{Z}_2$. Hence, $\beta \simeq 0$ if and only if $Sq^2 x = 0$, for all $x \in H^2(M;\mathbb{Z}_2)$, if and only if $x.v_2 = 0 \mod 2$, for all $x \in H^2(M;\mathbb{Z}_2)$, by the definition of the Wu class v_2. On an orientable 4-manifold $v_2 = w_2$ where w_2 is the second Stiefel-Whitney class. Thus, $\beta = S\alpha \simeq 0$ if and only if $w_2 = 0$. Hence, if M is spin $S^j\alpha \simeq 0$, for all $j \geq 1$. Mapping the cofibration of theorem 3.3 into the group G and using this result proves the theorem.

The rational homotopy groups of $\mathcal{G}_*$ are even more completely determined.

<u>Theorem 3.7</u>. If M is a closed compact orientable simply connected spin 4-manifold and G is a compact simply connected Lie group then

$$\pi_{2j}(\mathcal{G}_*) \otimes \mathbb{Q} \simeq 0$$

and

$$\pi_{2j+1}(\mathcal{G}_*) \otimes \mathbb{Q} \simeq [\pi_{2j+3}(G)]^r \otimes \mathbb{Q} \oplus \pi_{2j+5}(G) \otimes \mathbb{Q} \quad .$$

<u>Proof</u>. The first statement follows from applying lemma 3.4 to the results of theorem 3.6. The second statement follows from theorem 3.6 because any short exact sequence of rational vector

spaces splits.

Using theorem 3.6 it is now possible to prove that if M is spin then $\tilde{\mathcal{G}}$ is homotopically non-trivial.

Theorem 3.8. If M is a compact simply connected spin 4-manifold and G a compact simply connected Lie group then $\tilde{\mathcal{G}}$ is homotopically non-trivial, i.e.

$$\Pi_j(\tilde{\mathcal{G}}) \neq \{0\}$$

for some $j \geq 0$.

Proof. The groups $\mathcal{G}$, $\tilde{\mathcal{G}}$ and $\mathcal{G}_*$ are related by the fibrations (Singer (1978))

$$0 \longrightarrow Z \longrightarrow \mathcal{G} \longrightarrow \tilde{\mathcal{G}} \longrightarrow 0 \qquad (3.1)$$

and

$$0 \longrightarrow \mathcal{G}_* \longrightarrow \mathcal{G} \longrightarrow G \longrightarrow 0 \qquad (3.2)$$

where Z is the centre of G. It follows from the exact homotopy sequence (3.1) that

$$\pi_j(\mathcal{G}) \simeq \pi_j(\tilde{\mathcal{G}})$$

for all $j \geq 2$. Now assume that

$$\pi_j(\tilde{\mathcal{G}}) = \{0\}$$

for all $j \geq 0$. Then from the homotopy exact sequence of (3.2) we have that

$$\pi_j(\mathcal{G}_*) \simeq \pi_{j+1}(G)$$

for all $j \geq 2$. Inserting this in the exact sequence of theorem 3.6 gives

$$0 \longleftarrow [\pi_{j+2}(G)]^r \longleftarrow \pi_{j+1}(G) \longleftarrow \pi_{j+4}(G) \longleftarrow 0$$

for all $j \geq 2$. Taking $j = 2$ and tensoring with $\mathbb{Q}$ yields the short exact sequence

$$0 \longleftarrow [\pi_4(G)]^r \otimes \mathbb{Q} \longleftarrow \pi_3(G) \otimes \mathbb{Q} \longleftarrow \pi_6(G) \otimes \mathbb{Q} \longleftarrow 0.$$

Lemma 3.4 implies that

$$\pi_3(G) \otimes \mathbb{Q} \simeq 0$$

which is a contradiction.

Therefore, from the discussion of section 2, theorem 3.8 implies that no continuous global gauge $s: \mathcal{C}/\mathcal{G} \longrightarrow \mathcal{C}$ exists in a non-Abelian gauge theory (with simply connected gauge group) defined on a compact simply connected spin 4-manifold.

In the later discussion of the existence of inequivalent quantisations of gauge theories and global anomalies the homotopy group $\pi_o(\mathcal{G}_*)$ will be crucial. It was proved in theorem 3.6 that

$$\pi_o(\mathcal{G}_*) \simeq \pi_4(G) \qquad (3.3)$$

if M is simply connected and spin. One might wonder whether the assumption that M is spin is necessary to obtain (3.3). In fact it is, as can be seen from the following

Example. Let M be the complex projective plane $\mathbb{P}^2$. Recall that $\mathbb{P}^2$ is a simply connected 4-manifold which is not spin. Then if $G = SU(2)$ we have that

$$\pi_o(\mathcal{G}_*) \simeq 0 .$$

This is because, by theorem 3.2,

$$\pi_o(\mathcal{G}_*) \simeq \pi_o \, \mathrm{Map}_* (\mathbb{P}^2; SU(2))$$

$$\simeq [\mathbb{P}^2; SU(2)]_*$$

$$\simeq 0 .$$

However, $\pi_4(SU(2)) \simeq \mathbb{Z}_2$. Thus, theorem 3.6 does not hold when M is not spin.

We now establish a relationship between the homotopy types of the orbit space $\tilde{\mathcal{C}} / \tilde{\mathcal{G}}$ (respectively $\mathcal{C}/\mathcal{G}_*$) and the group of gauge transformations $\tilde{\mathcal{G}}$ (respectively $\mathcal{G}_*$). These relationships are given in the next theorem (related issues concerning the homotopy type of $\mathcal{G}$ are also discussed in Atiyah and Bott (1982)).

<u>Theorem 3.9</u> For M a compact 4-manifold and G a compact Lie group there are homotopy equivalences

$$\tilde{\mathcal{C}}(P) / \tilde{\mathcal{G}}(P) \sim \mathrm{Map}(M; BG)_P$$

and

$$\mathcal{C}(P) / \mathcal{G}_*(P) \sim \mathrm{Map}_*(M; BG)_P$$

where $\mathrm{Map}(M; BG)_P$ (respectively $\mathrm{Map}_*(M; BG)_P$) denotes the component of Map(M; BG) (respectively $\mathrm{Map}_*(M;BG)$) which induces the bundle P.

<u>Proof</u> The fibre bundle

$$\begin{array}{ccc} \tilde{\mathcal{G}}(P) & \longrightarrow & \tilde{\mathcal{C}}(P) \\ & & \downarrow \\ & & \tilde{\mathcal{C}}(P)/\tilde{\mathcal{G}}(P) \end{array}$$

has contractible total space. Hence this bundle is universal for $\tilde{\mathcal{G}}(P)$ and

$$B\tilde{\mathcal{G}}(P) \sim \tilde{\mathcal{C}}(P)/\tilde{\mathcal{G}}(P) .$$

Let

$$\begin{array}{ccc} G & \longrightarrow & EG \\ & & \downarrow \\ & & BG \end{array}$$

be a universal bundle for G. Let $\mathrm{Map}(P;EG)_G$ be the space of G-equivariant maps from P to EG. The group $\tilde{\mathcal{G}}(P)$ acts naturally on $\mathrm{Map}(P;EG)_G$ by composition. This results in the principal $\tilde{\mathcal{G}}(P)$ - bundle

$$\begin{array}{ccc} \tilde{\mathcal{G}}(P) & \longrightarrow & \mathrm{Map}(P;EG)_G \\ & & \downarrow \\ & & \mathrm{Map}(M;BG)_P \end{array}$$

The total space $\mathrm{Map}(P;EG)_G$ is contractible so this is a universal bundle for $\tilde{\mathcal{G}}(P)$. Hence

$$B\tilde{\mathcal{G}}(P) \sim \mathrm{Map}(M;\ BG)_P$$

and thus

$$\tilde{\mathcal{C}}(P)/\tilde{\mathcal{G}}(P) \sim \mathrm{Map}(M;\ BG)_P .$$

Similarly, for the second case, the bundle

$$\begin{array}{ccc} \mathcal{G}_*(P) & \longrightarrow & \mathcal{C}(P) \\ & & \downarrow \\ & & \mathcal{C}(P)/\mathcal{G}_*(P) \end{array}$$

is universal for $\mathcal{G}_*(P)$. Hence

$$B\,\mathcal{G}_*(P) \sim \mathcal{C}(P)/\mathcal{G}_*(P)$$

The group $\mathcal{G}_*(P)$ acts naturally on the space $\mathrm{Map}_*(P;\ EG)_G$, of base-point preserving G-equivariant maps from P to EG, by composition. This yields a principal $\mathcal{G}_*(P)$ - bundle

$$\begin{array}{ccc} \mathcal{G}_*(P) & \longrightarrow & \mathrm{Map}_*(P;EG)_G \\ & & \downarrow \\ & & \mathrm{Map}_*(M;\ BG)_P \end{array}$$

$\mathrm{Map}_*(P;EG)_G$ is contractible and hence

$$B\,\mathcal{G}_*(P) \sim \mathrm{Map}_*(M;\ BG)_P \quad .$$

Thus

$$\mathcal{C}(P)/\mathcal{G}_*(P) \sim \mathrm{Map}_*(M;\ BG)_P \quad .$$

The independence of the homotopy type of $\mathcal{G}_*(P)$ on the isomorphism class of P (theorem 3.2), when G is simply connected, is equivalent to the independence of the homotopy type of the component $\mathrm{Map}_*(M;BG)_P$ on the isomorphism class of P. Thus

<u>Theorem 3.10</u> For M a closed compact orientable 4-manifold and G a compact simply connected Lie group we have that

$$\mathcal{C}(P)/\mathcal{G}_*(P) \sim \mathrm{Map}_*(M;\ BG)$$

Proof. Directly from theorems 3.2 and 3.9.

For $M = S^4$, theorem 3.10 results in

$$\begin{aligned} \mathscr{C}(P)/\mathscr{G}_*(P) &\sim \mathrm{Map}_*(S^4; BG) \\ &\sim \Omega^4(BG) \\ &\sim \Omega^3(G) \quad . \end{aligned}$$

This reproduces a result of Atiyah and Jones (1978).

When M is simply connected and spin the homotopy groups of $\mathscr{C}(P)/\mathscr{G}_*(P)$ are given by the following

Theorem 3.11. Let M be a closed compact orientable simply connected spin 4-manifold and G a compact simply connected Lie group. Then

$$\pi_1(\mathscr{C}/\mathscr{G}_*) \simeq \pi_4(G)$$

and

$$0 \leftarrow [\pi_{j+1}(G)]^r \longleftarrow \pi_j(\mathscr{C}/\mathscr{G}_*) \longleftarrow \pi_{j+3}(G) \longleftarrow 0$$

for all $j \geqslant 2$.

Proof. Inserting $\pi_j(\mathscr{C}/\mathscr{G}_*) \simeq \pi_j(B\mathscr{G}_*) \simeq \pi_{j-1}(\mathscr{G}_*)$, for all $j \geqslant 1$, into the expressions given in theorem 3.6 gives this result.

The point has now been reached where we can discuss the inequivalent quantisations of Yang-Mills theories. The notion of inequivalent quantisations of a field theory was discussed in Chapter 2. This phenomenon has been discussed in connection with canonical quantisation by Isham (1981) and from a path integral point of view by Dowker (1980). As we saw in Chapter 2, if $\mathcal{Q}$ is the configuration space of the field theory under consideration and the theory possesses no Wess-Zumino type terms in its action then the inequivalent quantisations of the theory are classified by

$$\Theta \quad = \quad \mathrm{Hom}(\pi_1(\mathcal{Q}), U(1)) \quad . \tag{3.4}$$

As Yang-Mills theory has no Wess-Zumino type terms this classification is valid.

If we consider the Yang-Mills action to be a $\mathcal{G}_*$-invariant functional on the space $\mathcal{C}$ of connections then the configuration space is

$$\mathcal{Q} = \mathcal{C}/\mathcal{G}_* \ .$$

We know from theorem 3.11 that if M is a closed compact orientable simply connected spin 4-manifold and G is a compact simply connected Lie group then $\pi_1(\mathcal{C}/\mathcal{G}_*) \simeq \pi_4(G)$. Hence, for such M and G the classifying space for inequivalent quantisations of a Yang-Mills theory is*

$$\Theta \quad = \quad \mathrm{Hom}(\pi_4(G), U(1)) \quad .$$

<u>Examples</u>. (i) For G = SU(n) we have that

$$\pi_4(SU(n)) \quad \simeq \quad \begin{cases} \mathbb{Z}_2 & \text{for} \quad n = 2 \\ 0 & \text{for} \quad n \geq 3 \end{cases} \quad .$$

* It should be noted that here Θ classifies the inequivalent quantisations of a pure gauge theory in 4 space + 1 time dimensions.

Thus SU(n) Yang-Mills theories possess a unique quantisation for all $n \geq 3$. For $n = 2$

$$\begin{aligned}\Theta &= \mathrm{Hom}(\mathbb{Z}_2, U(1)) \\ &\simeq \mathbb{Z}_2 .\end{aligned}$$

(ii) For $G = Sp(n)$ we have $\pi_4(Sp(n)) \simeq \mathbb{Z}_2$, for all $n \geq 1$. Hence, Sp(n) gauge theories have inequivalent quantisations for all n; these are classified by

$$\begin{aligned}\Theta &= \mathrm{Hom}(\mathbb{Z}_2, U(1)) \\ &\simeq \mathbb{Z}_2 .\end{aligned}$$

So far the discussion of the inequivalent quantisations has been limited to pure gauge theories. We will now consider gauge theories coupled to matter fields. Recall from Chapter 1 that a matter field Φ is a section from the space $\mathcal{E} = \Gamma(E \otimes W)$, where E is a vector bundle associated to P by some representation of G and W is any bundle associated to the frame bundle of M. The group of gauge transformations acts on $\mathcal{E}$ by pull-back. To discuss the inequivalent quantisations of coupled gauge theories it is necessary to understand the geometry of the configuration space involved. The group $\mathcal{G}_*$ of restricted gauge transformations acts freely on $\mathcal{C}$ and hence acts freely on $\mathcal{C} \times \mathcal{E}$. Furthermore, it has been proved by Parker (1982) that there exist globally effective local slices for this action. The free group action together with the globally effective local slices result in the quotient $\mathcal{C} \times \mathcal{E} / \mathcal{G}_*$ being a smooth manifold and in a smooth principal $\mathcal{G}_*$ - bundle

$$\begin{array}{ccc} \mathcal{G}_* & \longrightarrow & \mathcal{C} \times \mathcal{E} \\ & & \downarrow \\ & & \mathcal{C} \times \mathcal{E} / \mathcal{G}_* \end{array} \qquad (3.5)$$

over $\mathcal{C} \times \mathcal{E} / \mathcal{G}_*$.

If we consider the action for a coupled gauge theory to be a $\mathcal{G}_*$-invariant functional defined on the space $\mathcal{C} \times \mathcal{E}$ of fields then the configuration space of the theory is

$$\mathcal{Q} = \mathcal{C} \times \mathcal{E} / \mathcal{G}_* .$$

The space $\mathcal{E}$ of sections of $E \otimes W$ is a vector space. Hence, $\mathcal{E}$ is contractible. $\mathcal{C}$ is also contractible and thus $\mathcal{C} \times \mathcal{E}$ is contractible. Therefore the bundle (3.5) is a universal bundle for $\mathcal{G}_*$ and we have the homotopy equivalence

$$\mathcal{C} \times \mathcal{E} / \mathcal{G}_* \sim B\mathcal{G}_* .$$

However, we know from the proof of theorem 3.9 that

$$\mathcal{C} / \mathcal{G}_* \sim B\mathcal{G}_* .$$

Hence there is a homotopy equivalence

$$\mathcal{C} \times \mathcal{E} / \mathcal{G}_* \sim \mathcal{C} / \mathcal{G}_*$$

between the configuration spaces for pure and coupled gauge theories.

One important consequence of this homotopy equivalence is that the classifying spaces for the inequivalent quantisations of pure and coupled gauge theories are identical. Thus, an SU(n) gauge theory coupled to fermionic or bosonic matter fields has a non-trivial quantisation only for $n = 2$; in this case

$$\Theta = \text{Hom}(\mathbb{Z}_2\,,\,U(1))$$

$$\simeq \mathbb{Z}_2$$

as for the pure theory.*

So far we have discussed the topology of the group $\mathcal{G}_*$ of restricted gauge transformations on a simply connected manifold in some detail. Under the additional assumption that M is spin the topology of $\mathcal{G}_*$ can be determined essentially completely. In this section we will briefly consider the topological nature of $\mathcal{G}_*$ when M is not simply connected.

We know from theorem 3.2 that for M a compact 4-manifold and G a compact simply connected Lie group the group $\mathcal{G}_*$ is homotopically equivalent to $\text{Map}_*(M;G)$. Thus

$$\pi_o(\mathcal{G}_*) \simeq \pi_o\,\text{Map}_*(M;G)$$

$$\simeq [M;G]_* \quad .$$

For certain compact 4-manifolds M it is easy to show that $\pi_o(\mathcal{G}_*) \neq \{0\}$. For example, if $M = T^4$ (the 4-torus) then $\pi_o(\mathcal{G}_*) \neq \{0\}$ may be shown as follows. First note that in general (Whitehead (1978)) if

$$\Gamma = [X;G]_*$$

and X is a product space of the form

$$X = S^{n_1} \times S^{n_2} \times \ldots\ldots \times S^{n_k}$$

where S^{n_i} is the n_i-sphere, then the group Γ has a central chain of length k

$$\Gamma = \Gamma_o \supset \Gamma_1 \supset \Gamma_2 \supset \ldots\ldots \supset \Gamma_k = \{0\} \qquad (3.6)$$

* It should be noted that here Θ classifies the inequivalent quantisations of a coupled gauge theory in 4 space + 1 time

with

$$\Gamma_{i-1}/\Gamma_i \simeq \prod_{|\alpha|=i} \pi_{n(\alpha)}(G) \quad . \tag{3.7}$$

In this last expression $\prod_{|\alpha|=i}$ denotes the direct product of the homotopy groups $\pi_{n(\alpha)}(G)$ over those subsets $\alpha \subset \{1,2, \ldots, k\}$ which have exactly i members. The number $n(\alpha)$ is defined to be

$$n(\alpha) = \sum_{i\varepsilon\alpha} n_i \quad .$$

Specialising to the case of $X = T^4 \equiv S^1 \times S^1 \times S^1 \times S^1$, the subgroups Γ_i in (3.6) give rise to the central chain

$$\Gamma = \Gamma_o \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \Gamma_4 = \{0\}$$

in which the Γ_i's satisfy

$$\Gamma_3/\Gamma_4 \simeq \pi_4(G)$$

$$\Gamma_2/\Gamma_3 \simeq \pi_3(G) \oplus \pi_3(G) \oplus \pi_3(G) \oplus \pi_3(G)$$

$$\Gamma_1/\Gamma_2 \simeq 0$$

$$\Gamma_o/\Gamma_1 \simeq \pi_1(G) \oplus \pi_1(G) \oplus \pi_1(G) \oplus \pi_1(G)$$

For G a compact semi-simple Lie group $\pi_3(G) \simeq \mathbb{Z}$ and hence

$$\Gamma_2/\Gamma_3 \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \quad .$$

Therefore the group $\Gamma = [T^4;G]_*$ has a non-trivial subgroup and

$$\pi_o(\mathcal{G}_*) \neq \{0\} \ . \tag{3.8}$$

A similar argument also shows that if $M = S^1 \times S^3$ then

$\pi_o(\mathcal{G}_*) \neq \{0\}$.

It is a consequence of (3.8) that it is impossible to fix a global gauge in a theory defined on the 4-torus. It also follows from (3.8) that

$$\pi_1(\mathcal{C}/\mathcal{G}_*) \neq \{0\}$$

and hence there may exist inequivalent quantisations of gauge theories on T^4.

As far as the gauge fixing problem is concerned the obstruction to globally fixing the gauge comes from the non-vanishing of any of the homotopy groups of $\mathcal{G}_*$. It is possible to give a more involved argument involving rational homotopy groups which proves that $\mathcal{G}_*$ is homotopically non-trivial on any compact 4-manifold. A detailed proof of this result is in Killingback and Rees (1984) and here we will only state the result and give an idea of the proof.

Theorem 3.12. Let M be a closed compact orientable 4-manifold and G a compact simply connected semi-simple Lie group. Then

$$\pi_j(\mathcal{G}_*) \neq \{0\}$$

for some $j \geq 0$.

Outline of Proof. By theorem 3.2 it suffices to prove that $\pi_j \text{Map}_*(M;G) \neq \{0\}$ for some $j \geq 0$. That is there exists a $j \geq 0$ such that

$$[M;\ \Omega^j G]_* \neq \{0\} .$$

Then assuming that

$$\dim([M;\ \Omega^j G]_* \otimes \mathbb{Q}) = 0$$

for all $j \geq 0$ and considering several cofibrations for the r-skeleton $(1 \leq r \leq 4)$ of M allows one to obtain an inductive relationship between the dimensions of the rational homotopy groups $\pi_{2k+1}(G) \otimes \mathbb{Q}$. This inductive relationship may then be used to obtain a contradiction.

As a consequence of this theorem it is impossible to globally fix the gauge in a theory defined on any closed compact orientable 4-manifold with a compact simply connected gauge group.

4. Global Anomalies on Compact 4-Manifolds

One of the most interesting consdquences of the topologically non-trivial nature of $\mathcal{G}_*$ is that a gauge theory coupled to a single doublet of left-handed fermions may be inconsistent. We will only give a brief discussion of this phenomenon, concentrating on the topological property of $\mathcal{G}_*$ which is necessary for the inconsistency to arise.

Consider a gauge theory coupled to a left-handed fermion doublet. Let M be a compact oriented spin 4-manifold and P a principal G-bundle over M, where G is a compact simply connected Lie group. Let ρ be a representation of G on a vector space V and E the associated vector bundle

$$E = P \times_G V \ .$$

The spin bundle S over M decomposes to give $S = S^+ \oplus S^-$ where S^+ and S^- are the bundles of positive and negative chirality, respectively. Each $A \in \mathcal{C}(P)$ gives a Dirac operator, as described in Chapter 1, which will now be written as

$$\not{D}_A : \quad \Gamma(S^+ \otimes E) \longrightarrow \Gamma(S^- \otimes E) \quad .$$

Now let $G = SU(2)$, the action of a left-handed fermion doublet coupled to the gauge field is given by

$$S = \int_M [\tfrac{1}{2}|F_A|^2 + \bar{\psi} \not{D}_A \psi \, \mu(g)]$$

where $\psi \in \Gamma(S^+ \otimes E)$ and $\mu(g)$ is the volume measure on M. The partition function of this theory is then given by

$$Z = \int \mathfrak{D}A \, \mathfrak{D}\psi \, \mathfrak{D}\bar{\psi} \, \exp(-S) \quad .$$

To discuss the effective theory one would like to integrate out the fermions. Integrating over $\mathfrak{D}\psi \, \mathfrak{D}\bar{\psi}$ gives

$$\int \mathfrak{D}\psi \, \mathfrak{D}\bar{\psi} \, \exp[-\int_M \bar{\psi} \not{D}_A \psi \, \mu(g)] = (\det \not{D}_A)^{\frac{1}{2}} \ .$$

The quantity $\det \not{D}_A$ is formally the infinite product of all the eigenvalues of $\not{D}_A$. Since $\not{D}_A$ anticommutes with γ_5 the eigenvalues of $\not{D}_A$ occur in pairs $(+\lambda, -\lambda)$. To define $(\det \not{D}_A)^{\frac{1}{2}}$ for a particular connection A we may choose to use either $+\lambda$ or $-\lambda$ for each eigenvalue. Once this choice has been made there is no further freedom because $(\det \not{D}_A)^{\frac{1}{2}}$ must vary smoothly with A

in order to satisfy the Schwinger-Dyson equations. If we define $(\det \not{D}_A)^{\frac{1}{2}}$ to be the product of all the positive eigenvalues $+\lambda$, say, then $(\det \not{D}_A)^{\frac{1}{2}}$ is invariant under infinitesimal gauge transformations. However, $(\det \not{D}_A)^{\frac{1}{2}}$ may not be invariant under gauge transformations which are disconnected from the identity. Let $f \in \mathcal{G}_*$ be a gauge transformation disconnected from the identity. If the fermion integral changes sign under f, i.e.,

$$(\det \not{D}_{f \cdot A})^{\frac{1}{2}} = -(\det \not{D}_A)^{\frac{1}{2}}$$

then the theory will be inconsistent. This is for the following reason (see Witten (1982)). The partition function is

$$Z = \int \mathcal{D}A \, (\det \not{D}_A)^{\frac{1}{2}} \exp(-\|F_A\|^2) .$$

This vanishes identically because the contribution of any gauge potential A is exactly cancelled by the equal and opposite contribution of $f \cdot A$. The same is true for the path integral Z_X. with any gauge invariant insertion X. Therefore the expectation values

$$< X > = \frac{Z_X}{Z} = \frac{0}{0}$$

are indeterminate. Hence, the theory is inconsistent.

Suppose that for a particular connection A $(\det \not{D}_A)^{\frac{1}{2}}$ is defined to be the product of the positive eigenvalues $+\lambda$. Now let $f \in \mathcal{G}_*$ be a topologically non-trivial gauge transformation and consider the connection

$$A_t = (1-t)A + t\, f \cdot A \tag{4.1}$$

with t varied smoothly from 0 to 1. As t varies the eigenvalues may rearrange themselves. If the number of positive eigenvalues that become negative as t varies from 0 to 1 is odd then the fermion integral will change sign. For $M = S^4$ the group of gauge transformations is $\mathcal{G}_* \sim \Omega^4(SU(2))$ and hence

$$\pi_0(\mathcal{G}_*) \simeq \pi_4(SU(2))$$

$$\simeq \mathbb{Z}_2 \quad .$$

Taking $f \in \mathcal{G}_*$ to be disconnected from the identity then it may be shown (Witten (1982)) that for A_t given by (4.1) $(\det \not{D}_A)^{\frac{1}{2}}$ changes sign as t varies from 0 to 1. To demonstrate this requires the use of the mod 2 index theorem for a five dimensional Dirac operator (Atiyah and Singer (1971) and Atiyah, Patodi and Singer (1976)). It is the non-triviality of $\pi_0(\mathcal{G}_*)$ that is the necessary topological condition for this global anomaly to exist. For $G = SU(n)$ $(n \geqslant 3)$ we have that

$$\pi_0(\mathcal{G}_*) \simeq \pi_4(SU(n)) \simeq 0$$

and hence no anomaly can occur for SU(3), for example.

We now return to a theory with a compact simply connected gauge group G defined on a compact simply connected spin 4-manifold M. From theorem (3.6) for such M and G

$$\pi_0(\mathcal{G}_*) \simeq \pi_4(G) \quad .$$

Hence, the topological condition for the global anomaly to occur is identical to that when $M = S^4$. In particular for $G = SU(n)$, the anomaly can only occur for $n = 2$. The sufficient conditions for the existence of a global anomaly depend on index theorem arguments. Further details of this can be found in Atiyah and Singer (1984), Lott (1984) and Singer (1982).

CHAPTER 4

GLOBAL ASPECTS OF FIXING THE GAUGE IN THE POLYAKOV STRING AND EINSTEIN GRAVITY

1. Introduction

The Feynman path integral approach to quantising gauge theories appears to be the best method available at present. It has been applied with considerable success to the quantisation of Yang-Mills theories and QCD. In the Euclidean path integral approach to Yang-Mills theories one considers functional integrals of the form

$$Z = \int_{\mathcal{C}} \mathfrak{D}A \exp - S[A]$$

where $\mathfrak{D}A$ is a measure on the space $\mathcal{C}$ of all gauge potentials A. S[A] is the Yang-Mills action of A and the functional integral is taken over all gauge potentials which satisfy some suitable boundary condition. However, as we have seen in Chapter 3, there is a problem in evaluating this path integral which results from the gauge invariance of the action S[A]. Recall that if $f \in \mathcal{G}$ is a gauge transformation then $S[f \cdot A] = S[A]$, for $A \in \mathcal{C}$. The difficulty arises because the orbits of $\mathcal{G}$ are expected to have infinite measure. The functional integral should really be carried out over the gauge orbit space $\mathcal{C}/\mathcal{G}$. However, $\mathcal{C}/\mathcal{G}$ is an intractable space. The idea of fixing the gauge is intended to circumvent this difficulty. We choose, in a continuous way, one gauge potential on each $\mathcal{G}$-orbit, i.e., we choose a continuous map $s: \mathcal{C}/\mathcal{G} \to \mathcal{C}$ such that $p \circ s = \text{id}$, where $p: \mathcal{C} \to \mathcal{C}/\mathcal{G}$ is the canonical projection. The functional integral is then evaluated over $s(\mathcal{C}/\mathcal{G})$, weighted by the Jacobian of $p: s(\mathcal{C}/\mathcal{G}) \to \mathcal{C}/\mathcal{G}$. This yields the Fadeev-Popov determinant. As was shown in Chapter 3, the topologically non-trivial nature of the group $\mathcal{G}$ of gauge transformations results in the non-existence of a continuous global gauge for theories on many 4-manifolds. In this chapter we consider the possibility of choosing a global gauge fixing condition for theories which possess an invariance under the group of diffeomorphisms of a manifold M. The two theories of this type which we concentrate on are the Polyakov String and four dimensional Einstein gravity. It is shown that in both these theories there exists a topological obstruction to globally fixing the gauge. This obstruction comes from the topologically non-trivial nature of the group of diffeomorphisms of M. It is completely analogous to the obstruction to fixing the gauge in Yang-Mills theories discussed in Chapter 3. It should be noted that similar ideas to those used here have also been discussed by Isham (1981) in connection with the canonical quantisation of gravity.

The remainder of this chapter is organized as follows. In section 2 the formulation of the Polyakov String theory and four dimensional Euclidean gravity is recalled. Section 3 introduces the local gauge slices and the geometric structure of the orbit space of Riemannian metrics modulo diffeomorphisms. The obstruction to globally fixing the gauge in these two theories is then proved in section 4.

2. Formulation of the Polyakov String and Euclidean Gravity

The dynamics of the Polyakov string is described in terms of the world sheet swept out by the string as it evolves in D dimensional (Euclidean) space-time. The world sheet will be assumed to be a closed compact orientable 2 dimensional Riemannian manifold M^2. This 2 dimensional surface can be described by an immersion $x: M^2 \to \mathbb{R}^D$, for $D \geq 3$. Local charts on M^2 are (U_α, ξ_α), where U_α is an open subset of M^2 and ξ_α a homeomorphism of U_α onto an open subset of $\mathbb{R}^2$. The immersion $x: M^2 \to \mathbb{R}^D$ induces a Riemannian metric h on M^2. Now let g by a new Riemannian metric on M^2 independent of h. The local 2-form defined in the chart (U_α,ξ_α) by

$$\begin{aligned}\Omega_{(\alpha)} &= \tfrac{1}{2}\mathrm{Tr}(g^{-1}_{(\alpha)}\, h_{(\alpha)})(\det g_{(\alpha)})^{\frac{1}{2}}\, d\xi^1_{(\alpha)} \wedge d\xi^2_{(\alpha)} \\ &= \tfrac{1}{2}\mathrm{Tr}(g^{-1}_{(\alpha)}\, h_{(\alpha)})\mu(g) \end{aligned} \tag{2.1}$$

where $\mu(g)$ is the volume element associated to g, satisfies $\Omega_{(\alpha)} = \Omega_{(\beta)}$ in $U_\alpha \cap U_\beta$ and hence defines a global 2-form on M^2. The action of the string theory is defined to be (see Onofri and Virasora (1982))

$$\begin{aligned} W[x,g] &= \int_{M^2} \Omega \\ &= \tfrac{1}{2}\int_{M^2} \mathrm{Tr}(g^{-1}h)\mu(g) \quad . \end{aligned} \tag{2.2}$$

This is the action introduced by Polyakov (1981). The partition function is then given by the path integral

$$Z = \int \mathcal{D}g\, \mathcal{D}x \exp - W[x,g] \quad . \tag{2.3}$$

The Polyakov action is invariant under diffeomorphisms of M^2. Let $\mathcal{D}(M^2)$ denote the group of (orientation preserving) diffeomorphisms of M^2 and let $\mathcal{M}(M^2)$ denote the space of Riemannian metrics on M^2. $\mathcal{D}(M^2)$ acts naturally on $\mathcal{M}(M^2)$ by pull-back i.e., for $\pi \varepsilon \mathcal{D}(M^2)$ and $g \varepsilon \mathcal{M}(M^2)$ we have $g^\pi = \pi^* g \varepsilon \mathcal{M}(M^2)$. The diffeomorphism group of M^2 also acts on the immersion $x: M^2 \to \mathbb{R}^D$ to give $x^\pi = x \circ \pi: M^2 \to \mathbb{R}^D$. The metric induced on M^2 by the immersion x^π is $h^\pi = \pi^* h$. The action $W[x,g]$ has the invariance (Polyakov (1981))

$$W[x^\pi, g^\pi] = W[x,g] \quad . \tag{2.4}$$

Therefore, two metrics on the same $\mathcal{D}(M^2)$-orbit correspond to the same geometry, but represented in two different sets of local charts. The space of inequivalent 2-geometries is given by the orbit space $\mathcal{M}(M^2) / \mathcal{D}(M^2)$.

A remarkable feature of the Polyakov string theory is that the partition function (2.3) can be evaluated explicitly (see Polyakov (1981)). To carry out the path integral over $\mathcal{D}g$ it is necessary to fix the gauge by choosing a representative metric of each orbit of $\mathcal{M}(M^2) / \mathcal{D}(M^2)$. Polyakov's choice was the conformal gauge, i.e., to find a representative of each orbit of the form (in a given local chart)

$$g_{ab}(\xi) = e^{\phi(\xi)}\, \delta_{ab} \tag{2.5}$$

which is always possible for 2-dimensional surfaces. This choice will not uniquely specify the gauge if M^2 admits a non-trivial group $C(M^2)$ of conformal transformations (also see Onorfri and Virasoro (1982)) $C(M^2) = \{\pi \in \mathfrak{D}(M^2) \mid$ there is a smooth $\rho: M^2 \to \mathbb{R}$ such that

$$\pi^* g = e^{\rho} g, \quad \text{for all} \quad g \in \mathcal{M}(M^2)\} .$$

For $M^2 = S^2$, $C(S^2) = SL(2,\mathbb{C})$; for $M^2 = T^2$, $C(T^2) = O(2) \times O(2)$; for surfaces of higher genus $C(M^2)$ is discrete (see Goldberg (1962)).

Using the conformal gauge the functional integral over $\mathfrak{D}g$ can be evaluated to obtain the partition function as a functional integral over ϕ of the Liouville action

$$L[\phi] = \int_{M^2} [\tfrac{1}{2}(\partial_a\phi)^2 + \mu^2 e^{\phi}]d^2\xi \quad . \tag{2.6}$$

For this path integral over $\mathfrak{D}\phi$ to be well defined it would seem to be necessary for the gauge choice (2.5) to be continuous. It is not clear that the conformal gauge satisfies this requirement. In fact, it will be shown later that for many surfaces M^2 there is no continuous global gauge fixing condition. This is a purely global result and there always exists a well defined local gauge.

Four dimensional Einstein gravity shares with the Polyakov string theory the property of being invariant under the diffeomorphism group of a compact Riemannian manifold. Let the (Euclidean) space-time be represented by a compact 4-dimensional Riemannian manifold M^4. The Euclidean action is (see Gibbons, Hawking and Perry (1978))

$$S[g] = -\frac{1}{16\pi G} \int_{M^4} R(\det g)^{\frac{1}{2}}\, d^4x \; + \; \text{boundary term} \tag{2.7}$$

where R is the scalar curvature of the metric g on M^4. The manifolds with which we will be concerned will be closed and without boundary. Hence, we will neglect the boundary term in (2.7). If $\mathcal{M}(M^4)$ represents the space of Riemannian metrics on M^4 and $\mathfrak{D}(M^4)$ the group of (orientation preserving) diffeomorphisms of M^4 then for $\pi \in \mathfrak{D}(M^4)$ and $g \in \mathcal{M}(M^4)$.

$$S[\pi^* g] = S[g] \quad . \tag{2.8}$$

Thus, the action is constant on orbits of $\mathfrak{D}(M^4)$.

The partition function is then defined to be

$$Z = \int \mathfrak{D}g \exp - S[g] \quad . \tag{2.9}$$

There are, of course, many problems in attempting to evaluate this functional integral (see Gibbons, Hawking and Perry (1978)). In addition to these one might anticipate that the invariance of the action under the diffeomorphism of M^4 would lead to a similar problem in evaluating the path integral as occurs in gauge theories, i.e., that the orbits of $\mathfrak{D}(M^4)$ would have infinite measure. The path integral should be carried out over the orbit space $\mathcal{M}(M^4)/\mathfrak{D}(M^4)$, which is, however, intractable. This problem could be overcome by fixing the gauge (i.e., choosing in a continuous fashion a metric on each $\mathfrak{D}(M^4)$-orbit) and then proceeding to evaluate the Fadeev-Popov determinant. We shall show later that for many 4-manifolds M^4 such a continuous global gauge does not exist. Local gauges, however, always exist and may be used to define the Fadeev-Popov determinant in a neighbourhood of a given metric $g \in \mathcal{M}(M^4)$

3. The Local Gauge Slices

In this section we will consider the action of the group of diffeomorphisms of a compact manifold on the space of Riemannian metrics defined on the manifold. This topic has been investigated in detail by Ebin (1968) and it has also been studied in connection with the theory of Wheeler-de Witt superspace (Fischer (1967)). Further details of the ideas mentioned here can be found in these references.

Let M be a compact orientable n-dimensional manifold without boundary. TM will be its tangent bundle, T^*M its cotangent bundle and S^2T^*M the bundle of symmetric covariant 2-tensors. The space of Riemannian metrics $\mathcal{M}(M)$ on M is defined to be the space of all smooth sections of S^2T^*M which induce a positive definite inner product on each tangent space T_xM, $x \in M$. Then $\mathcal{M}(M)$ is a positive open cone in $\Gamma(S^2T^*M)$, i.e., $\mathcal{M}(M)$ is open in $\Gamma(S^2T^*M)$ and if $\lambda, \mu > 0$, $g, h \in \mathcal{M}(M)$, then $\lambda g + \mu h \in \mathcal{M}(M)$ Let $\mathfrak{D}(M)$ be the group of orientation preserving diffeomorphisms of M. Then $\mathfrak{D}(M)$ acts on $\Gamma(S^2T^*M)$ as follows: if $\pi \in \mathfrak{D}(M)$, $g \in \Gamma(S^2T^*M)$, and $X, Y \in T_xM$, $x \in M$, then

$$(\pi^*g)_x(X,Y) = g_{\pi(x)}(T\pi\, X, T\pi\, Y) .$$

This action can be written as a map $A: \mathfrak{D}(M) \times \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$. It is clear that $\mathcal{M}(M)$ is invariant under the action, so we can write $A: \mathfrak{D} \times \mathcal{M} \to \mathcal{M}$. Note that when the manifold M is not in question we will simplify $\mathcal{M}(M)$ and $\mathfrak{D}(M)$ to $\mathcal{M}$ and $\mathfrak{D}$, respectively.

The space of metrics has a (weak) Riemannian structure defined on it as follows. Each $h \in \mathcal{M}$ is a Riemannian structure on TM.

It therefore induces a Riemannian structure on T^*M and S^2T^*M. Let $(\, , \,)^h$ be this structure on S^2T^*M. Also $h \in \mathcal{M}$ induces a volume element μ on M. For $\omega, \theta \in \Gamma(S^2T^*M)$ we define $\langle\omega, \theta\rangle_h$ by

$$\langle\omega, \theta\rangle_h = \int_M (\omega, \theta)^h \mu$$

$\langle \, , \, \rangle_h$ is a positive definite bilinear form on $\Gamma(S^2T^*M)$. Since $\mathcal{M}$ is open in $\Gamma(S^2T^*M)$, $\mathcal{M}$ is a manifold whose tangent space at each point is canonically identified with $\Gamma(S^2T^*M)$. Thus for each $h \in \mathcal{M}$, $\langle \, , \, \rangle_h$ on $\Gamma(S^2T^*M)$ defines a Riemannian structure on $\mathcal{M}$.

The most important property of $\langle \, , \, \rangle_h$ is that it is invariant under the natural action of $\mathfrak{D}$ on $\mathcal{M}$, i.e., $\mathfrak{D}$ acts by isometry. To see this first note that (Ebin (1968)), for $\pi \in \mathfrak{D}$, $g \in \mathcal{M}$, $\zeta, \xi \in T_g\mathcal{M} \simeq \Gamma(S^2T^*M)$, $x \in M$

$$(\pi^*\zeta, \pi^*\xi)_x^{\pi^*g} = (\zeta, \xi)_{\pi(x)}^g \quad .$$

The diffeomorphism $\pi \in \mathfrak{D}$ also acts on the set of volume elements of M by pull-back. If μ is the volume element of g then $\pi^*\mu$ is the volume element of π^*g. Hence

$$\int_M (\pi^*\zeta, \pi^*\xi)^{\pi^*g} \, \pi^*\mu = \int_M (\zeta, \xi)^g \mu$$

or

$$\langle\pi^*\zeta, \pi^*\xi\rangle_{\pi^*g} = \langle\zeta, \xi\rangle_g \quad . \qquad (3.2)$$

Therefore, $\pi \in \mathfrak{D}$ is an isometry.

It is now possible to define the local gauge slice through a metric $g \in \mathcal{M}$ as the orthogonal complement of the tangent space to the $\mathfrak{D}$-orbit through g, relative to the inner product

$\langle\,,\,\rangle_g$. The orbit of $\mathfrak{D}$ through $g \in \mathcal{M}$ is defined by the map

$$\lambda_g : \mathfrak{D} \to \mathcal{M} \tag{3.3}$$

given by $\lambda_g(\pi) = \pi^* g$. The derivative of this map at the identity in $\mathfrak{D}$ is

$$T_{id}\,\lambda_g : T_{id}\mathfrak{D} \to T_g\mathcal{M}.$$

Recalling the identifications $T_{id}\mathfrak{D} \simeq \Gamma(TM)$ and $T_g\mathcal{M} \simeq \Gamma(S^2T^*M)$, we have that

$$T_{id}\,\lambda_g : \Gamma(TM) \to \Gamma(S^2T^*M) \quad . \tag{3.4}$$

To compute $T_{id}\,\lambda_g$, let $X \in \Gamma(TM)$ be a smooth vector field on M. X generates a 1-parameter group of diffeomorphisms $\{\pi_t\}$. The smooth curve $C: \mathbb{R} \to \mathfrak{D}$ in $\mathfrak{D}$ given by $C(t) = \pi_t$, with $\pi_o = id_M$, has tangent vector at $t = 0$. given by $X = (d/dt)C(t)\big|_{t=0}$. $T_{id}\,\lambda_g$ maps X to the tangent vector to the curve $(\lambda_g \circ C)(t)$ in $\mathcal{M}$. Thus

$$\begin{aligned}(T_{id}\,\lambda_g)(X) &= \frac{d}{dt}(\lambda_g \circ C)(t)\Big|_{t=0}\\ &= \frac{d}{dt}\lambda_g(\pi_t)\Big|_{t=0}\\ &= \frac{d}{dt}\pi_t^*\,g\Big|_{t=0}\\ &= \mathcal{L}_X\,g\end{aligned}$$

where $\mathcal{L}_X\,g$ is the Lie derivative of g with respect to the vector field X. Therefore the map $T_{id}\,\lambda_g : \Gamma(TM) \to \Gamma(S^2T^*M)$ is given by the first order differential operator $\delta_g : \Gamma(TM) \to \Gamma(S^2T^*M)$ where

$$\delta_g X = \mathcal{L}_X g$$

for $X \in \Gamma(TM)$ and $g \in \mathcal{M}$. Note that $\ker \delta_g$ = space of Killing vectors on the Riemannian manifold (M,g).

It follows from standard properties of differential operators (Palais (1965)) that the image $\mathrm{im}\, \delta_g$ is closed and has closed complement in $\Gamma(S^2T^*M)$. Thus the tangent space to the orbit through $g \in \mathcal{M}$ is given by $\mathrm{im}\, \delta_g$ and the local slice to this orbit at g is the orthogonal complement of $\mathrm{im}\, \delta_g$ relative to the inner product $< , >_g$. Hence, the local slice at $g \in \mathcal{M}$ is given by those $g + s \in \mathcal{M}$, $s \in \Gamma(S^2T^*M)$, for which

$$0 = \langle s, \delta_g X\rangle_g = \langle \delta_g^* s, X\rangle_g \qquad (3.5)$$

for all $X \in \Gamma(T^*M)$. δ_g^* is the $< , >_g$ - adjoint of δ_g. The local slice through $g \in \mathcal{M}$ is written as

$$\mathcal{S}_g = \{g + s \mid s \in \Gamma(S^2T^*M) \text{ and } \delta_g^* s = 0\} . \qquad (3.6)$$

In a sufficiently small neighbourhood of a given metric $g \in \mathcal{M}$ the slice $\mathcal{S}_g$ intersects each orbit once and only once (Ebin (1968)). Therefore this slice defines a good local gauge around g. We also have as a consequence of (3.5) the direct sum decomposition

$$\Gamma(S^2T^*M) \simeq \mathrm{im}\, \delta_g \oplus \ker \delta_g^* . \qquad (3.7)$$

The geometrical structure of the orbit space $\mathcal{M}/\mathfrak{D}$ will now be considered. The space $\mathcal{M}/\mathfrak{D}$ is not a manifold because the action of $\mathfrak{D}$ on $\mathcal{M}$ is not free. If we define the isometry group of $g \in \mathcal{M}$ to be

$$I_g = \{\pi \in \mathfrak{D} \mid \pi^* g = g\}$$

then $\mathcal{D}$ has g as a fixed point if $I_g \neq \{0\}$. There are, however, two ways to obtain a free action. If we restrict our attention to the space of metrics $\tilde{\mathcal{M}} \subset \mathcal{M}$ which have trivial isometry group, i.e.,

$$\tilde{\mathcal{M}} = \{g \in \mathcal{M} | I_g = \{0\}\}$$

then the action of $\mathcal{D}$ on $\tilde{\mathcal{M}}$ is free. The space $\tilde{\mathcal{M}}$ is open and dense in $\tilde{\mathcal{M}}$ (Ebin (1968)). The globally effective local slices $\mathcal{S}_g$ and the free $\mathcal{D}$-action on $\tilde{\mathcal{M}}$ results in a principal $\mathcal{D}$-bundle

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \tilde{\mathcal{M}} \\ & & \downarrow \\ & & \tilde{\mathcal{M}}/\mathcal{D} \end{array} \qquad (3.8)$$

over $\tilde{\mathcal{M}}/\mathcal{D}$ which is now a smooth manifold.

Alternatively, we can restrict to the group $\mathcal{D}_*$ of diffeomorphisms which leave a point $x_o \in M$ fixed and also leave the frame at x_o fixed, i.e.

$$\mathcal{D}_* = \{\pi \in \mathcal{D} \mid \pi(x_o) = x_o \text{ and } T_{x_o}\pi = id_{T_{x_o}M}\} .$$

Now if $\pi \in \mathcal{D}_*$ is an isometry for some $g \in \mathcal{M}$, i.e., if $\pi^* g = g$, then $\pi = id_M$ (see Helgason (1962)). Therefore, $\mathcal{D}_*$ acts freely on $\mathcal{M}$. Again the orbit space $\mathcal{M}/\mathcal{D}_*$ is a manifold and we have a principal $\mathcal{D}_*$ - bundle over $\mathcal{M}/\mathcal{D}_*$

$$\begin{array}{ccc} \mathcal{D}_* & \longrightarrow & \mathcal{M} \\ & & \downarrow \\ & & \mathcal{M}/\mathcal{D}_* \end{array} \qquad (3.9)$$

To conclude this section we make some statements of a geometrical and topological nature.

<u>Proposition 3.1</u>. The manifolds $\tilde{\mathcal{M}}/\mathfrak{D}$ and $\mathcal{M}/\mathfrak{D}_*$ have a natural Riemannian structure.

<u>Proof</u>. It follows from the invariance of $<\, ,\, >_g$ under the action of $\mathfrak{D}$ on $\mathcal{M}$ (equation (3.2)) that $<\, ,\, >_g$ projects to give a well defined metric on $\tilde{\mathcal{M}}/\mathfrak{D}$ and $\mathcal{M}/\mathfrak{D}_*$.

<u>Proposition 3.2</u>. The space

$$\mathcal{S}_g = \{g + s \mid s \in (S^2T^*M) \text{ and } \delta_g^* s = 0\}$$

is the horizontal space at $g \in \tilde{\mathcal{M}}$ or $g \in \mathcal{M}$ of a connection on the bundle (3.8) or (3.9), respectively.

<u>Proof</u>. According to (3.7) the space $\mathcal{S}_g$ is complementary to the tangent space to the fibre at $g \in \mathcal{M}$. Since the Riemannian structure of $\mathcal{M}$ is preserved by the action of $\mathfrak{D}$ we have that $\ker \delta^*_{\pi^* g} = \pi^*(\ker \delta_g^*)$, for all $\pi \in \mathfrak{D}$. Therefore, $\mathcal{S}_{\pi^* g} = \pi^* \mathcal{S}_g$ and the slice $\mathcal{S}_g$ is the horizontal space of a connection.

<u>Proposition 3.3</u>. There are homotopy equivalences

$$B\mathfrak{D} \sim \tilde{\mathcal{M}}/\mathfrak{D}$$

and

$$B\mathfrak{D}_* \sim \mathcal{M}/\mathfrak{D}_* \quad .$$

Proof. These results follow from the contractibility of the total spaces of the bundles (3.8) and (3.9). The space of metrics $\mathcal{M}$ is convex and hence contractible The space $\tilde{\mathcal{M}}$ is also contractible. This can be proved in analogy with Singer's proof (Singer (1978)) that the space of irreducible connections on a principal SU(n)-bundle over a compact manifold is contractible.

Proposition 3.4. The groups $\mathfrak{D}$ and $\mathfrak{D}_*$ are related by the fibration

$$0 \to \mathfrak{D}_* \to \mathfrak{D} \xrightarrow{\pi} F_+(TM) \to 0 \qquad (3.10)$$

where $F_+(TM)$ is the principal $GL_+(n, \mathbb{R})$ - bundle of frames on M with a given orientation.

Proof. The projection π is given by evaluation at the base-point. It is clear that the fibre of π is $\mathfrak{D}_*$ and that (3.10) has the homotopy lifting property.

4. The Obstruction to Globally Fixing the Gauge

We will now use the principal fibre bundles introduced in section 3 to discuss the possibility of globally fixing the gauge in the Polyakov string theory and in four dimensional Euclidean gravity. Suppose that we are considering a theory defined on a compact orientable n-manifold M^n, with an action which is a $\mathfrak{D}(M^n)$ - invariant functional on $\mathcal{M}(M^n)$. Then a global gauge is a continuous map $s: \mathcal{M}/\mathfrak{D} \to \mathcal{M}$ such that $p \circ s = id_{\mathcal{M}/\mathfrak{D}}$, where $p: \mathcal{M} \to \mathcal{M}/\mathfrak{D}$ is the canonical projection. If such a gauge were to exist then the restriction

$$s|_{\tilde{\mathcal{M}}/\mathfrak{D}} : \tilde{\mathcal{M}}/\mathfrak{D} \to \tilde{\mathcal{M}}$$

would give a global section of the principal $\mathfrak{D}$ -bundle (3.8). Such a global section exists if and only if (3.8) is trivial, i.e.,

$$\tilde{\mathcal{M}} \simeq \mathfrak{D} \times \tilde{\mathcal{M}}/\mathfrak{D}$$

Applying $\pi_q(\cdot)$ to this expression and recalling that $\tilde{\mathcal{M}}$ is contractible (see proof of proposition 3.3), i.e. $\pi_q(\tilde{\mathcal{M}}) = 0$, for all $q \geq 0$, gives

$$\pi_q(\mathfrak{D}) \oplus \pi_q(\tilde{\mathcal{M}}/\mathfrak{D}) \simeq 0$$

for all $q \geq 0$. Thus, the obstruction to the existence of a global gauge is the non-vanishing of any of the homotopy groups of $\mathfrak{D}$.

It should be noted that the bundle (3.8) has been used here purely as an auxillary device and has little direct physical significance. This is for the following reason. To obtain (3.8) it was necessary to restrict attention to those metrics with trivial isometry groups. However, it is known that many classical solutions of both the Polyakov string theory and four dimensional Euclidean gravity have non-trivial isometry groups. Therefore, to restrict attention to only those metrics in $\tilde{\mathcal{M}}$ eliminates many classical solutions which may be important in understanding the full theory. For example, such solutions may be required in order to undertake a semiclassical analysis of the theory.

In contradistinction to the unphysical nature of (3.8) the bundle (3.9) does have physical significance. The restriction to those diffeomorphisms of M^n which leave both a point $x_o \in M^n$ fixed and the frame at x_o fixed appears quite acceptable. For example, consider M^n to be the one point compactification of a non-compact manifold $\hat{M}^n$, with $x_o \in M^n$ corresponding to the point at infinity in $\hat{M}^n$. Then the diffeomorphisms in $\mathfrak{D}_*(M^n)$ correspond to the diffeomorphisms of $\hat{M}^n$ which are the identity at infinity and also have their derivative equal to the identity at infinity. This type of restriction on the diffeomorphisms of $\hat{M}^n$ is physically acceptable.

If we now consider fixing the gauge in a theory with a $\mathfrak{D}_*(M^n)$ - invariant action defined on $\mathcal{M}(M^n)$ then the bundle (3.9) may be used directly. In this case a global gauge choice is a global section of (3.9), which exists if and only if the bundle $\mathcal{M} \to \mathcal{M}/\mathfrak{D}_*$ is trivial. By the same reasoning as used earlier, the obstruction to such a global section is the non-vanishing of any of the homotopy groups of $\mathfrak{D}_*(M^n)$.

Taking the theory under consideration to be the Polyakov string, with M^2 a compact Riemann surface of genus p, invariant under either $\mathfrak{D}(M^2)$ or $\mathfrak{D}_*(M^2)$ yields the following results.

Theorem 4.1. There exists no global gauge $s: \mathcal{M}/\mathfrak{D} \to \mathcal{M}$ for M^2 of genus $p = 0$ or 1. For $p > 1$ there is no topological obstruction to the existence of such a gauge.

Proof. This follows directly from the homotopy type of $\mathfrak{D}(M^2)$ (Earle and Eeels (1967)), namely

$$\mathcal{D}(S^2) \sim SO(3) \quad \text{for } p = 0$$

$$\mathcal{D}(T^2) \sim SO(2) \times SO(2) \quad \text{for } p = 1$$

$$\mathcal{D}(M^2) \sim \{0\} \quad \text{for } p > 1 \ .$$

<u>Theorem 4.2</u>. There is no global section of $\mathcal{M} \to \mathcal{M}/\mathcal{D}_*$ for M^2 of genus $p > 0$. For $p = 0$ there is no obstruction to such a section.

<u>Proof</u>. The homotopy groups of $\mathcal{D}_*(M^2)$ and $\mathcal{D}(M^2)$ are related by the exact homotopy sequence of the fibration (3.10)

$$\ldots \to \pi_q(\mathcal{D}_*(M^2)) \to \pi_q(\mathcal{D}(M^2)) \to \pi_q(F_+(TM^2)) \to \pi_{q-1}(\mathcal{D}_*(M^2)) \to \ldots$$

Recall that $F_+(TM^2) \simeq \mathbb{R}^3 \times O(TM^2)$, where $O(TM^2)$ is the principal $SO(2)$ - bundle of orthonormal frames of M. Hence, $F_+(TM^2)$ has the same homotopy type as $O(TM^2)$. For $p = 0$, $\mathcal{D}(S^2) \sim SO(3)$ and $F_+(TS^2) \sim SO(3)$ are isomorphic, hence

$$\pi_q(\mathcal{D}_*(M^2)) \simeq 0$$

for all $q \geq 0$. This gives the last sentence of theorem 4.2. For $p > 0$, assume that

$$\pi_q(\mathcal{D}_*(M^2)) \simeq 0$$

for all $q \geq 0$. It follows that

$$\pi_q(\mathcal{D}(M^2)) \simeq \pi_q(F_+(TM^2)) \qquad (4.1)$$

for all $q \geq 1$. Hence, for $p = 1$ and $q = 1$, (4.1) implies that

$$\pi_1(SO(2) \times SO(2)) \simeq \pi_1(SO(2) \times SO(2) \times SO(2))$$

i.e., $\mathbb{Z} \oplus \mathbb{Z} \simeq \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, which is a contradiction. For $p > 1$, (4.1) implies that

$$\pi_q(F_+(TM^2)) \simeq 0 \qquad (4.2)$$

for all $q \geq 1$. But from the defining fibration of $F_+(TM^2)$

$$\begin{array}{ccc} GL_+(2, \mathbb{R}) & \longrightarrow & F_+(TM^2) \\ & & \downarrow \\ & & M^2 \end{array}$$

and (4.2) it follows that $\pi_1(M^2) \simeq 0$, which is a contradiction. Thus, $\mathcal{D}_*(M^2)$ is non-contractable for M^2 of genus $p > 0$.

For four-dimensional Euclidean gravity, invariant under either $\mathcal{D}(M^4)$ or $\mathcal{D}_*(M^4)$, the obstruction to globally fixing the gauge is the non-contractability of $\mathcal{D}(M^4)$ or $\mathcal{D}_*(M^4)$, respectively. It is probable that for any compact 4-manifold M^4 the groups $\mathcal{D}(M^4)$ and $\mathcal{D}_*(M^4)$ will be homotopically non-trivial. For certain classes of compact 4-manifolds it is possible to show that $\pi_0(\mathcal{D}(M^4)) \neq \{0\}$ and $\pi_0(\mathcal{D}_*(M^4)) \neq \{0\}$. Note that it follows from the exact homotopy sequence of the fibration (3.10) that if $\pi_0(\mathcal{D}(M^4)) \neq \{0\}$ then $\pi_0(\mathcal{D}_*(M^4)) \neq \{0\}$.

The first class of compact 4-manifolds M^4 for which $\pi_0(\mathcal{D}(M^4)) \neq \{0\}$ are product manifolds.

<u>Theorem 4.3</u>. Let $M^4 = N_1 \times N_2$ (where $\dim N_1 = \dim N_2 = 2$) be an oriented product manifold. Then

$$\pi_o(\mathfrak{D}(M^4)) \neq \{0\} .$$

<u>Proof</u>. Let π_1 and π_2 be orientation reversing diffeomorphisms of N_1 and N_2 respectively. Let $[N_1] \in H^2(N_1)$ and $[N_1] \in H^2(N_2)$ be the 2 dimensional cohomology classes of N_1 and N_2, respectively. Then $\pi_1^*[N_1] = -[N_1]$ and $\pi_2^*[N_2] = -[N_2]$. Thus

$$\begin{aligned}(\pi_1 \times \pi_2)^*[N_1 \times N_2] &= \pi_1^*[N_1] \times \pi_2^*[N_2] \\ &= -[N_1] \times -[N_2] \\ &= [N_1] \times [N_2] = [N_1 \times N_2]\end{aligned}$$

and $\pi_1 \times \pi_2$ is an orientation preserving diffeomorphism of $N_1 \times N_2$. However, $(\pi_1 \times \pi_2)^*[N_1] \times 1 = -[N_1] \times 1$ and therefore $\pi_1 \times \pi_2$ is not homotopic to the identity.

The second class of compact 4-manifold which have a disconnected diffeomorphism group are smooth submanifolds of the complex projective space $\mathbb{P}^3$.

<u>Theorem 4.4</u>. Let $M^4 \subset \mathbb{P}^3$ be a smooth compact 4-dimensional submanifold of $\mathbb{P}^3$. Then

$$\pi_o(\mathfrak{D}(M^4)) \neq \{0\} .$$

<u>Proof</u>. Let $V \subset \mathbb{P}^3$ be a smooth 4-dimensional submanifold of $\mathbb{P}^3$

defined by the zero set of a polynomial f with real coefficients

$$V = \{x \in \mathbb{P}^3 | f(x) = 0\} .$$

Let $c: \mathbb{P}^3 \to \mathbb{P}^3$ given by $x \to \bar{x}$ be complex conjugation. Then c sends V to itself. If $a \in H^2(\mathbb{P}^3)$ is the positive generator of $H^2(\mathbb{P}^3)$ then $c^*: H^2(\mathbb{P}^3) \to H^2(\mathbb{P}^3)$ is given by $a \to -a$. Therefore, if $i: V \hookrightarrow \mathbb{P}^3$ is the inclusion of V in $\mathbb{P}^3$, $(c|_V)^*$ sends i^*a to $-i^*a$. Since $2i^*a \neq 0$ it follows that $c|_V$ is not homotopic to the identity. Since any $M^4 \subset \mathbb{P}^3$ is diffeomorphic to a surface $V \subset \mathbb{P}^3$ it follows that

$$\pi_0(\mathfrak{D}(M^4)) \neq \{0\} .$$

In the Euclidean approach to quantum gravity the compact 4-manifolds $S^2 \times S^2$, $\mathbb{P}^2$ and a K3 surface are important as compact gravitational instantons. It follows from theorems 4.3 and 4.4 that the group of diffeomorphisms of these three manifolds is disconnected. Hence we have the following.

<u>Theorem 4.5</u>. Let M^4 be any one of the compact 4-manifolds $S^2 \times S^2$, $\mathbb{P}^2$ or a K3 surface. Then

$$\pi_0(\mathfrak{D}(M^4)) \neq \{0\} .$$

<u>Proof</u>, For $M^4 = S^2 \times S^2$ the result follows directly from theorem 4.3. For $M^4 = \mathbb{P}^2$ the proof of theorem 4.4 implies that $c: \mathbb{P}^2 \to \mathbb{P}^2$ (complex conjugation) is not homotopic to the identity. The model of a K3 surface is the quartic surface in $\mathbb{P}^3$ defined by

$$\hat{M} = \{[x_1,x_2,x_3,x_4] \in \mathbb{P}^3 | \; x_1{}^4 + x_2{}^4 + x_3{}^4 + x_4{}^4 = 0\} \quad .$$

As $\hat{M}$ is defined by a polynomial with real coefficients, theorem 4.4 implies that $\pi_o(\mathfrak{D}(M^4)) \neq \{0\}$. As any K3 surface M^4 is diffeomorphic to the surface $\hat{M}$ it follows that $\pi_o(\mathfrak{D}(M^4)) \neq \{0\}$.

Furthermore, it follows from theorem 4.5 that for $M^4 = S^2 \times S^2$ $\mathbb{P}^2$ or a K3 surface

$$\pi_o(\mathfrak{D}_*(M^4)) \neq \{0\} \; .$$

Therefore, it is impossible to globally fix the gauge in Euclidean gravity, with either a $\mathfrak{D}(M^4)$ or $\mathfrak{D}_*(M^4)$ invariance, defined on $S^2 \times S^2$, $\mathbb{P}^2$ or a K3 surface.

Although, as we have seen, in general it is not possible to define a global gauge in four dimensional Euclidean gravity, there always exist local gauges given by the gauge slices $\mathcal{S}_g$. These local gauges can be used to define the path integral in some sufficiently small neighbourhood $\mathcal{U}$ of $g \in \mathcal{M}$ in which $\mathcal{S}_g$ is a good gauge. We wish to evaluate

$$Z = \int_{\mathcal{U}} \mathfrak{D}g \; \exp - S[g] \quad .$$

If $\mathcal{R} : \mathcal{M} \to \mathcal{M}/\mathfrak{D}_*$ is the canonical projection with the restriction $\mathcal{R}|_{\mathcal{S}_g} : \mathcal{S}_g \to \mathcal{M}/\mathfrak{D}_*$ then we can write

$$Z = \int_{\mathcal{R}|_{\mathcal{S}_g}(\mathcal{S}_g)} \mathfrak{D}g \;\; \exp - S[g]$$

$$= \int_{\mathcal{S}_g} \mathcal{R}|^{*}_{\mathcal{S}_g} \mathfrak{D} g \exp - S[\mathcal{R}|^{*}_{\mathcal{S}_g} g]$$

$$= \int_{\mathcal{S}_g} \det(\mathcal{R}|_{\mathcal{S}_g}) \mathfrak{D} \hat{g} \exp - S[\hat{g}] \quad .$$

The Jacobian determinant $\det(\mathcal{R}|_{\mathcal{S}_g})$ of $\mathcal{R}|_{\mathcal{S}_g}$ is the Fadeev-Popov determinant associated with the local gauge $\mathcal{S}_g$.

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