UNSTABLE PARTICLES

IN

MODERN FIELD THEORY

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by

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PREFACE

I would like to express my sincere thanks to Professor N. Kemmer for the hospitality of the Tait Institute and my specially grateful thanks to Dr. G.R. Screaton for introducing me to the subject of this dissertation, for his inspiring encouragement and many valuable discussions.

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I declare the material in the following dissertation to be original except in so far as explicit reference is made.

John McEwan.

CONTENTS

INTRODUCTION

The problem of extending the Quantum Theory of Fields to include a description of decay processes has provoked a rapid growth of interest in recent years. This problem was temporarily by-passed in the early attempts to formulate a relativistic Quantum Field Theory[1)] for the obvious reason that it was simpler initially to ignore decay phenomena and to consider only the collision processes of stable particles. The inadequacy of a field theory of stable particles is evident from the fact that among the sixteen experimentally established particles, and of course their sixteen anti-particles -- although not all particles are distinct from their anti-particles -- only four; the proton, electron, photon and neutrino, are stable.

Let us recall the manner in which the stability restriction was imposed. Among the known interactions between particles three clear subdivisions are observed appropriately named 'strong', 'electromagnetic' and 'weak' according to the magnitudes of their associated coupling constants. In the past the conventional procedure has been to assume that 'nature is most easily described by a sequence of approximations'.[2)] The weak and electromagnetic interactions are successively 'turned off' which has the effect of forbidding all known decay processes exhibited by the observed particles. Thus a simplified field theory describing the strong interactions of all baryons and mesons has evolved and has become one of the main frameworks for theoretical investigations in modern field theory, although it is not clear how serious a distortion of nature is involved. It is certain that a thorough treatment of unstable particles must take account of the role of the 'weaker' interactions.

Let us search the axioms of the field theory of stable particles, which for convenience are listed in detail below, for the critical points where unstable particles are excluded.

I. Quantum Physics: Quantum Field Theory is an extension of Quantum Mechanics to an arbitrary number of degrees of freedom. In particular the vector space formed by the stationary states of the system should be a Hilbert space and all observables are hermitian operators on this space.

II. Field Operators: a set of boson $\phi(x)$ and fermion $\psi(x)$ Heisenberg operators exist to specify the fields associated with the particles of the system. The quantities $\phi(x)$ and $\psi(x)$ are to be interpreted in the sense of operator valued distributions such that the expressions

$$\int_{-\infty}^{\infty} d^4x . g(x) . \phi(x) \quad \text{and} \quad \int_{-\infty}^{\infty} d^4x . g(x) . \psi(x)$$

are operators and give definite results when $g(x)$ is a test-function belonging to the class of all infinitely differentiable functions of compact support in space-time.

III. Relativistic Invariance: If $U(\Lambda, a)$ is a unitary Lorentz operator in the Hilbert space of the state vectors where Λ is a homogeneous Lorentz transformation and a is a translational transformation, then

$$U(\Lambda, a) . \phi(x) . U^{-1}(\Lambda, a) = \phi(\Lambda x + a)$$

$$U(\Lambda,a).\psi(x).U^{-1}(\Lambda,a) = \psi(\Lambda x + a)$$

In particular hermitian displacement operators P_μ exist such that

$$[P_\mu, P_\nu] = 0 \quad \text{and} \quad [P_\mu, F(x)] = i.\frac{\partial F(x)}{\partial x_\mu}$$

where F(x) is an arbitrary Heisenberg operator. In the representation where the P_μ are diagonal we can define the eigenstates of P_μ so

$$P_\mu |p,\alpha\rangle = p_\mu |p,\alpha\rangle$$

The set of such eigenstates $|p,\alpha\rangle$ can be shown to span a Hilbert space which we choose as the Hilbert space of the system.

IV. Energy-Momentum Spectrum: A unique, invariant, normalizable, lowest-energy vacuum state $|0\rangle$ exists and is defined by

$$U(\Lambda,a)|0\rangle = |0\rangle \quad \text{and} \quad P_\mu |0\rangle = 0$$

The eigenvalue p_μ of P_μ has the properties

$$-p^2 = p_0^2 - \underline{p}^2 \geq 0 \quad \text{and} \quad p_0 \geq 0$$

V. Positive Semi-definite Metric: The norms of all vectors in Hilbert space must be greater than or equal to zero.

VI, Microcausality:

$$\left.\begin{array}{l} [\phi(x), \phi(y)] = 0 \\ \{\psi(x), \psi(y)\} = 0 \end{array}\right\} \quad \text{if} \quad (x_0 - y_0)^2 < (\underline{x} - \underline{y})^2$$

VII. Asymptotic Conditions:

$$\lim_{x_0 \to \pm\infty} (\Phi | \phi(x) | \Psi) = (\Phi | \phi_{\substack{out \\ in}}(x) | \Psi)$$

$$\lim_{x_0 \to \pm\infty} (\Phi | \psi(x) | \Psi) = (\Phi | \psi_{\substack{out \\ in}}(x) | \Psi)$$

where Φ and Ψ are arbitrary Heisenberg states and the 'in' and 'out' suffixes indicate free in-going and free out-going Heisenberg field operators.

In addition one could perhaps add an eighth axiom for the restriction to stable particles mentioned earlier.

Even in a field theory of unstable particles we can construct a complete orthonormal system of basic vectors spanning a Hilbert space in the Heisenberg representation from the asymptotic fields of stable particles or from the set of eigenstates of the displacement operator P_μ . Therefore the only axioms which are obviously questionable with respect to unstable particles are III and VII. Firstly violations of invariance under the unitary parity operator P, charge conjugation operator C and the anti-unitary time reversal operator T may be possible among weak interaction phenomena. Hence we should strictly only allow invariance under proper Lorentz transformations in axiom III. We may still assume invariance under the PCT-transformation.[3), 24)] The asymptotic properties of field operators in axiom VII give the theory an interpretation in terms of particles. Unfortunately it may be meaningless to ask for the asymptotic properties of unstable particle field operators since in the infinite time-like limits an unstable particle does not exist physically. In the infinite future only the decay products will be present and in the infinite past only the particles asymptotically associated with the production process to create the unstable particle will be present. We are therefore prevented from interpreting an unstable particle field in terms of a specific particle and in particular from defining a mass to be associated with this operator. In addition some method must be found for defining a lifetime for an unstable particle. Notice that such a mass m and lifetime τ will not be unique since the uncertainty principle predicts a mass distribution with a mean square deviation from the average mass given by $\Delta m \sim \frac{1}{\tau} = \Upsilon$ which also has an uncertainty $\Delta \Upsilon \sim \Upsilon^2/m$. It is clear then that we have to devise a method of defining consistently a mean mass and mean lifetime which will only be reasonably accurate provided Υ is very small or the lifetime large. We shall find in Chapter I that the latter requirement is a physically desirable one.

It is generally believed possible to conceive of single unstable particle states in the Heisenberg representation as approximate eigenstates of P_μ and that the accuracy of the approximation will depend on how long lived the particle is or how nearly stable. Since it should only be possible to interpret a state as a single unstable particle state during the lifetime of the particle itself, such a state seems a rather elusive quantity to define in the Heisenberg representation in which all states are stationary.

A vast literature has accumulated on the definition and treatment of unstable particles in a variety of models using, largely approximate and usually non-rigorous methods. Among the more serious treatments several have aimed at definitions of the mass and lifetime of an unstable particle in field theory. One such attempt has been made by Matthews and Salam[4] who define a mass density closely related to the Lehmann spectral density function.[18] The mean mass and lifetime are defined as the first and second moments of this mass density. Unfortunately it has been observed[6] that such moments do not exist for many physical examples unless the mass density decreases very rapidly for large mass values. But in any event the definitions seem much too artificial.

A specially interesting suggestion made by Peierls[5] appears more natural and has proved popular in later works. Peierls indicates that there may be a pole in the lower half plane of the second Riemann sheet of the propagator and that the real and imaginary parts of the pole serve to define the mass and lifetime of the particle propagated. This has since been verified by Levy[6] for the Lee Model with an unstable particle and he shows further that an exponential decay term is contributed to the time dependence of the propagator by the unphysical pole. This exponential behaviour is believed to correspond to the quantum mechanically well-known exponential decay law of resonance states and receives a thorough discussion in this context by Höhler[7] (there are many earlier references given in this paper). Levy's methods, however, are based on an analytic continuation through the cut in the complex energy plane for the propagator in order to find the unphysical sheet pole and he himself shows that the required analytic properties cannot easily be demonstrated in field theory. For the case where a particle has two or more modes of decay, Levy demonstrates with the Lee Model that it is necessary to continue through a certain restricted region of the cut to obtain the physically correct pole to be associated with an unstable particle. We will examine this interesting conclusion with a more general field theoretic model in Chapter II. Jacob and Sachs[8)] have also discovered a similar unphysical pole in perturbation theory applied to a simplified model of the decay mechanism of an unstable particle. Fairlie and Polkinghorne[9)] using a model based on a separable potential found that unstable states can be associated with unphysical poles. Gunson and Taylor[10)], Oehme[11)] and others have found possible resonance poles on unphysical sheets of a Mandelstam-type representation holding for a two-particle scattering amplitude on the unphysical sheet, by continuing through the elastic region of the physical cut in the energy variable using unitarity. Chew[17)] indicates that if an elementary unstable particle exists in the theory of Mandelstam's double dispersion relations and unitarity, it can be inserted into the theory as a pole of the scattering amplitude on an unphysical sheet. The latter is done somewhat indirectly by introducing a C.D.D. pole* in the denominator function D of the N/D method[17)] at some physical energy which then implies a complex zero in the denominator D itself. The works just mentioned all hint that the

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* Castillejo, Dalitz and Dyson[19)] noticed that poles, now called C.D.D. poles, can be freely added to the denominator function D with two extra arbitrary parameters determining the position and residue.

fundamental ideas involved are sufficiently general to apply to full field theory. This probable generalisation has been dealt with to some extent by Schwinger[12] who takes the view that unstable particle behaviour is already contained in the well-known Lehmann spectral representation[18] if weak interactions are explicitly considered. It is important to note that there is no need for Schwinger to leave the physical sheet in his analysis since he derives the Breit-Wigner resonance formula and the exponential decay law without troubling to look for the unphysical pole. In other words Schwinger finds the effects of the unphysical pole on the physical sheet rather than the pole itself. This has the great asset of avoiding the difficulty with analyticity in field theory discussed by Levy. Schwinger examines also the possibility that the exponential decay law fails after a very long time and concludes that the law is valid in field theory for so long as it is meaningful to identify the state of the system as the single unstable particle state. Hence after a very long time the decay law becomes dependent on the observation and production mechanisms. A very similar conclusion has been reached by Jacob and Sachs[8] in perturbation theory and Newton[14] after examining the same problem in Quantum Mechanics with a time-dependent wave packet formalism. In Chapter II we shall recover some of the main points made by Schwinger in analysing the boson propagator and consider some of the problems to be found in looking for unphysical poles by discussing a field theoretic model of a decay process with analytic properties of a Mandelstam-type. In Chapter III we generalise to unstable fermions the methods used in Chapter II. Moffat[13)] has suggested that Schwinger's work can be applied to single dispersion relations for two-particle scattering amplitudes in field theory. The author too has thought independently along similar lines and has analysed this generalisation in a rather different and more thorough fashion, which will be presented in Chapter IV. Little progress has yet been made to set up an operator formalism to deal with unstable particles. Work by Ida[15)] attempts to justify a conjectured definition of a single unstable particle Heisenberg state. This may yet prove to contain a germ of truth but much is left to be desired when complex masses are arbitrarily introduced and assigned to unstable particles. In spite of the fact that in Chapters II, III, and IV we have a fairly extensive dispersion relation treatment of unstable particles in full field theory, it could be more useful to develop an operator formalism. In view of this we discuss briefly the possible form and properties of a single unstable particle state in Chapter I. We restrict ourselves to very general terms and make no rash claims to have discovered a rigorous treatment. However some aspects of our conjectures appear to have a general validity and throw further light on the results of Chapter II.

CHAPTER I

UNSTABLE PARTICLE STATES

In the introduction we discussed the axiomatic foundation of a quantum field theory applicable to unstable particles. We found it mostly unnecessary to alter the usual axioms for stable particles with the notable exception of the asymptotic conditions. Essentially we have to find a method of defining one particle states for unstable particles without using time-like asymptotic limits. It would clearly be best to look for a method applicable to stable and unstable particles alike.

For stable bosons the usual procedure is to find an operator which will project out the one-particle contribution from the Fourier spectrum of $\phi^{+}(x)\,|0\rangle$ and giving a normalisable state. One arrives at the projection operator*

$$-i \lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{dt}{T}\, \chi(t) \int_{\substack{-\infty \\ x_\mu = \sigma_\mu(t)}}^{\infty} d\sigma^{\mu}(x)\,.\, f_\alpha(x)\,.\, \overleftrightarrow{\frac{\partial}{\partial x_\mu}} \tag{1.1}$$

where $\chi(t)$ is a test-function possessing derivatives of all orders, vanishing faster than any power of t^{-1} outside a region $-2T < t < -T$ and is approximately equal to unity inside this region. The asymptotic condition then ensures that (1.1) applied to $\phi^{+}(x)\,|0\rangle$ produces the same effect as a free field creation operator acting on the vacuum $|0\rangle$ produces namely a one particle

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* General points of notation are contained in Appendix 1.

boson state.

For an unstable boson we must reject the asymptotic limit in (1.1) and assume the particle is created by some external source at a finite time in an infinite region of space-time R. We now choose the test-function $\chi(t)$ to vanish outside the region R and call the time-like extension of R in the direction x_μ, $T = \oint_R dt$. This adjustment is still in accord with axiom II and takes account of the fact that the preparation or detection of a single particle state cannot be accomplished instantaneously and at a geometrical point in space. We make a further plausible conjecture that $f_\alpha(x)$ should be replaced by some similar function $\mathcal{f}_\alpha(x) \approx f_\alpha(x)$ in the region R at least. The function $\mathcal{f}_\alpha(x)$ is to be suitably chosen for the projection of an approximate one-particle state from $\phi^\dagger(x)|0\rangle$ whether $\phi(x)$ refers to a stable or unstable boson. Therefore with these general assumptions we tentatively propose to represent a one-particle state of average energy-momentum k_ν in the form

$$|k,\alpha\rangle = \frac{-i}{N^{1/2}} \int_{-\infty}^{\infty} \frac{dt}{T}\, \chi(t) \underset{x_\mu = \sigma_\mu(t)}{\int_{-\infty}^{\infty} d\sigma^\mu(x)\, \mathcal{f}_\alpha(x) \overleftrightarrow{\frac{\partial}{\partial x_\mu}}\, \phi^\dagger(x)|0\rangle} \tag{1.2}$$

or perhaps we may be permitted to write (1.2) in the simpler form

$$|k,\alpha\rangle = \frac{i}{N^{1/2}T} \int_{-\infty}^{\infty} d^4x\, \phi^\dagger(x)\, n^\mu \overleftrightarrow{\frac{\partial}{\partial x_\mu}}\, \mathcal{F}_\alpha(x)|0\rangle \tag{1.3}$$

where $\mathcal{F}_\alpha(x) = \mathcal{f}_\alpha(x)\,\chi(t)$, n^μ is the unit normal to the space-like surface $\sigma^{\mu}(x)$, and the constant N is chosen such that $| k, \alpha\rangle$ is normalised to unity, so

$$N = -\frac{i}{T}\int_{-\infty}^{\infty} d^4x \left[N^{1/2}\langle 0|\phi(x)|k,\alpha\rangle\right] n^{\mu}\overleftrightarrow{\frac{\partial}{\partial x_{\mu}}}\, f_{\alpha}^{*}(x) \tag{1.4}$$

where

$$\left[N^{1/2}\langle 0|\phi(x)|k,\alpha\rangle\right] = \frac{i}{T}\int_{-\infty}^{\infty} d^4x' \langle 0|\phi(x)\,\phi^{+}(x')|0\rangle n'_{0}\overleftrightarrow{\frac{\partial}{\partial x'_{0}}}\, f_{\alpha}(x')$$

$$= -\frac{1}{T}\int_{-\infty}^{\infty} d^4x' \int_{0}^{\infty} d\kappa^2\, \rho(\kappa^2)\, \Delta^{(+)}(x-x';\kappa^2)\, n'^{0}\overleftrightarrow{\frac{\partial}{\partial x'_{0}}}\, f_{\alpha}(x') \tag{1.5}$$

We have used the usual Lehmann techniques[18] to reduce $\langle 0|\phi(x)\,\phi^{+}(x')|0\rangle$ to the spectral form $i\int_0^{\infty} d\kappa^2 \rho(\kappa^2)\,\Delta^{(+)}(x-x';\kappa^2)$ by inserting a complete set of exact eigenstates of P_{μ} between $\phi(x)$ and $\phi^{+}(x')$. The calculation is identical to the stable particle field operator derivation. That $| k, \alpha\rangle$ is only an approximate eigenstate of P_{μ} follows because $f_{\alpha}(x)$ cannot be an exact solution of the Klein-Gordan equation and can only be represented approximately as a plane wave solution e^{ikx} with $k_0 = \sqrt{\underline{k}^2 + \mu^2}$.

To be certain that $|k, \alpha\rangle$ is a one-particle state our energy measurements must be sufficiently accurate to distinguish $|k, \alpha\rangle$ from many particle states. The uncertainty principle then shows that there is a restriction on the time required to observe $| k, \alpha\rangle$ as a one-particle state. These restrictions on

T were pointed out by Ida[15] and constitute what he has called the particle condition. They are as follows

(a) For a stable particle we must have the indeterminacy of our energy measurement ΔE less than the difference between the energy of a particle of mass μ and momentum $\underline{k}$ and the energy of the threshold state of mass μ_{cl} - i.e. μ_{cl} is the lowest mass value of the continuous mass spectrum - and momentum $\underline{k}$, in the energy spectrum of P_μ.* Hence

$$\Delta E \sim T^{-1} \ll \sqrt{\underline{k}^2 + \mu_{cl}^2} - \sqrt{\underline{k}^2 + \mu^2} \tag{1.6}$$

Also to eliminate negative energy frequencies we must have

$$\Delta E \sim T^{-1} \ll \sqrt{\underline{k}^2 + \mu^2} \tag{1.7}$$

(b) For unstable particles the analogous relations are

$$\Delta E \sim T^{-1} \ll \sqrt{\underline{k}^2 + \mu^2} - \sqrt{\underline{k}^2 + \mu_w^2} \tag{1.8}$$

$$\Delta E \sim T^{-1} \ll \sqrt{\underline{k}^2 + \mu_s^2} - \sqrt{\underline{k}^2 + \mu^2} \tag{1.9}$$

where μ_W and μ_s are the masses associated with the weak and the strong interaction thresholds of the continuous spectrum respectively.

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* This is providing that the discrete one-particle representation in the spectrum is separate from the multiple particle continuum representation. This is not strictly true even for stable particles if we allow electromagnetic interactions.

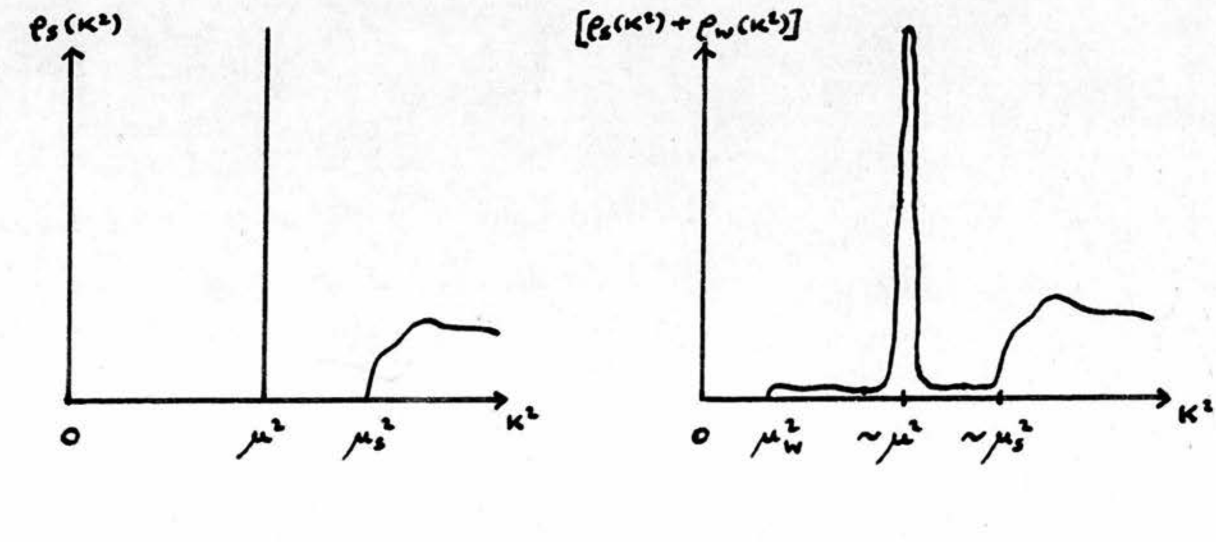
In addition we must have the observation time less than the lifetime to be sure of observing the particle before decay is too far advanced, so

$$T^{-1} \gg \gamma = \tau^{-1} \tag{1.10}$$

In the introduction we mentioned that γ should be very small if we wish to define accurately a mean mass and mean lifetime for an unstable particle. From (1.8) and (1.10) we discover the explicit condition for a narrow distribution of mass for an unstable particle, required for physical reasons, in the form

$$\gamma \lll\ll \sqrt{\underline{k}^2+\mu^2} - \sqrt{\underline{k}^2+\mu_w^2} \le \sqrt{\underline{k}^2+\mu^2} \tag{1.11}$$

We expect the Lehmann density function $\rho(\kappa^2)$ to have a discrete δ-function term expressing a stable particle state under strong interactions only. If this particle becomes unstable under weak interactions we expect the δ-function to spread out into a resonance shape in the continuous spectrum. The forms of $\rho(\kappa^2)$ under these two sets of conditions can be pictured as follows



The reasons for believing these figures are to be found in Chapters II and III.

We also expect the time dependence of a wave packet representing the propagation of an unstable particle to be exponentially decreasing. Such a wave packet or one-particle amplitude can be represented in the Heisenberg representation by

$$g_\alpha(x) = \langle 0|\phi(x)|k,\alpha\rangle$$

$$= \frac{1}{N^{1/2}T}\int_{-\infty}^{\infty} d^4x'\, G(x-x')\, n'^{\mu}\cdot \frac{\overleftrightarrow{\partial}}{\partial x'_\mu}\, \mathcal{F}_\alpha(x') \tag{1.12}$$

if x is in the future of the region of preparation R, where

$$G(x-x') = i\,\langle 0|T[\phi(x)\,\phi^{+}(x)]|0\rangle \tag{1.13}$$

It is only necessary to find the time dependence of $G(x - x')$ due to the single unstable particle contribution. The probability that the particle has not decayed after a time t should have the form, choosing R so that $x = 0 \in R$,

$$|g_\alpha(x)|^2 \approx |g_\alpha(0)|^2\cdot e^{-\gamma t} \tag{1.14}$$

In terms of the Fourier transforms $G(-k'^2)$ and $\tilde{\mathcal{F}}_\alpha(k')$ of $G(x - x')$ and $\mathcal{F}_\alpha(x)$ we can write

$$g_\alpha(x) = \frac{2i}{N^{1/2}T(2\pi)^4}\int_{-\infty}^{\infty} d^4k'\,(n'\cdot k')\,e^{ik'x}\cdot G(-k'^2)\cdot \tilde{\mathcal{F}}_\alpha(k') \tag{1.15}$$

or, since we only wish the time dependence

$$g_\alpha(\underline{k}',t) = \frac{i}{N^{1/2}T\pi}\int_{-\infty}^{\infty} dk_0'\,(n\cdot k')\,e^{-ik_0't}\,G(-k'^2)\,\tilde{\mathfrak{F}}_\alpha(k') \tag{1.16}$$

where

$$\tilde{\mathfrak{F}}_\alpha(k') = \int_{-\infty}^{\infty} d^4x\, e^{-ik'x} f_\alpha(x)\,\chi(x_\mu)$$

$$\approx \begin{cases} (2\pi)^4 & \text{if } k' \approx k \\ 0 & \text{otherwise.} \end{cases} \tag{1.17}$$

since $f_a(x) \approx e^{ikx}$ and $\chi(x_\mu) \approx 1$ for $x_\mu \varepsilon R$.
We can rewrite the energy condition in (1.17), $k_0' \approx k_0$ in the form

$$|E_{k'} - E_k| \lesssim \Delta E \tag{1.18}$$

where ΔE is the precision of our energy determination in (1.8), (1.9) and $E_{k'} = \sqrt{\underline{k}'^2+\mu^2}$, $E_k = \sqrt{\underline{k}^2+\mu^2}$.

The factor $\tilde{\mathfrak{F}}_\alpha(k')$ therefore acts as a kind of mass filter since it only allows the integration over k_0' in (1.16) to run over values consistent with the energies and masses we wish to obtain. We have used rough methods to arrive at (1.16) but the works of Jacob and Sachs[8)], and Newton[14)] indicate that the form and above interpretation of (1.16) is essentially correct. It seems likely that the mass filter introduced by Schwinger* when discussing the unstable particle contribution to $G(x - x')$, can

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* See Chapter II.

be best replaced by some argument similar to that presented here. Further it would appear better, for the sake of easier comparison between theory and experiment, to consider the time dependence of $g_\alpha(x)$ as the one characterising the probability of decay of an unstable particle rather than the time dependence of $G(x - x')$.

Similarly we might conjecture that an unstable fermion state can be represented by

$$|p,\alpha\rangle = \frac{1}{N^{1/2}T}\int_{-\infty}^{\infty} d^4x\, \bar{\psi}(x)\,(n.\gamma)\, w_\alpha(x)\,|0\rangle \tag{1.19}$$

where $w_\alpha(x)$ is an approximate solution of the Dirac equation and vanishes outside an infinite space-time preparation region. A very similar result to (1.16) for the time dependence of the unstable fermion wave-packet is found to be

$$\begin{aligned} s_\alpha(x) &= \langle 0|\psi(x)|p,\alpha\rangle \\ &= -\frac{1}{2N^{1/2}T}\int_{-\infty}^{\infty} d^4x'\, S_F'(x-x')\,(n.\gamma)\, w_\alpha(x') \end{aligned} \tag{1.20}$$

if x is in the future of the preparation region, where

$$S_F'(x-x') = -2\langle 0|T[\psi(x)\bar{\psi}(x')]|0\rangle \tag{1.21}$$

Hence

$$s_\alpha(\underline{k}',t) = -\frac{1}{4\pi N^{1/2}T}\int_{-\infty}^{\infty} dk_0'\, e^{-ik_0't}\, S_F'(k')\,(n.\gamma)\,\tilde{w}_\alpha(k') \tag{1.22}$$

where $\tilde{w}_\alpha(k')$ is the Fourier transform of $w_\alpha(x)$ and has mass filter properties similar to those of (1.17) and (1.18).

The state $|k, \alpha\rangle$ defined by (1.3) is not an exact eigenstate of P_ν but is very nearly identical with an eigenstate of P_ν with real momentum $\underline{k}$ and mass μ since

$$\begin{aligned} P_\nu |k,\alpha\rangle &= \frac{i}{N^{1/2}T}\int_{-\infty}^{\infty} d^4x\,[P_\nu, \phi^\dagger(x)]\, n^\mu \frac{\overleftrightarrow{\partial}}{\partial x_\mu} \mathcal{F}_\alpha(x)|0\rangle \\ &= -\frac{1}{N^{1/2}T}\int_{-\infty}^{\infty} d^4x\, \frac{\partial \phi^\dagger(x)}{\partial x_\nu}\, n^\mu \frac{\overleftrightarrow{\partial}}{\partial x_\mu} \mathcal{F}_\alpha(x)|0\rangle \\ &= -\frac{1}{N^{1/2}T}\left[\int_{-\infty}^{\infty} d\sigma^\nu(x)\,\phi^\dagger(x)\, n^\mu \frac{\overleftrightarrow{\partial}}{\partial x_\mu} \mathcal{F}_\alpha(x)\right]_{x_\nu=-\infty}^{x_\nu=\infty} |0\rangle + \\ &\quad + \frac{1}{N^{1/2}T}\int_{-\infty}^{\infty} d^4x\,\phi^\dagger(x)\, n^\mu \frac{\overleftrightarrow{\partial}}{\partial x_\mu} \frac{\partial \mathcal{F}_\alpha(x)}{\partial x_\nu}|0\rangle \\ &\approx k_\nu |k,\alpha\rangle \end{aligned} \tag{1.23}$$

since $\mathcal{F}_\alpha(x) \longrightarrow 0$ as $|x_\nu| \rightarrow \infty$, x_ν being a time-like vector, in such a way that the surface integral vanishes, and we have used axioms III and IV, and $\mathcal{F}_\alpha(x) \approx e^{ikx}$ for $x_\mu \in R$.

There appears to be no reason why we should not be able to choose $\mathcal{F}_\alpha(x)$ such that $|k, \alpha\rangle$ is an eigenstate of the momentum operator $\underline{P}$ with real momentum $\underline{k}$ as eigenvalue but such that $|k, \alpha\rangle$ is only an approximate eigenstate of the energy operator P_0 with approximate real energy $\sqrt{\underline{k}^2+\mu^2}$ eigenvalue where μ is some mean value for the unstable particle mass. In the stable particle case it would be possible to choose $g_\alpha(x) = \langle 0|\phi(x)|k, \alpha\rangle$ as a representation for $\mathcal{F}_\alpha(x)$ with $x_\mu \in R$ since $G(-k'^2)$ in (1.15) is given by an expression of the form $(-k^2-\mu^2+i\epsilon)^{-1}$ and therefore $g_\alpha(x) = f_\alpha(x)$ defined in Appendix 1. If we can find the analogous expression to $(-k^2-\mu^2+i\epsilon)^{-1}$ for $G(-k'^2)$ in the unstable particle case then we can put it in equation (1.15) and obtain a $g_a(x)$ which appears to be the analogous representation for $\mathfrak{F}_a(x)$ for $x_\mu = R$ in the unstable particle case. In this way we hope that, although $|k, a>$ is not a unique state belonging to the set of eigenstates of P_ν, we may be able to express $|k, a>$ in an approximate way, but sufficiently accurate for experimental purposes, so that it can be treated as if it were a unique one-particle unstable state with a given real momentum and some given energy yet to be defined. In Chapter II we will examine $G(-k'^2)$ in the unstable particle case to find the 'discrete' one-particle contribution from the mass spectrum and therefore derive an expression for $g_a(x)$ or $\mathfrak{F}_a(x)$ for $x_\mu = R$.

The fermion state $|p, a>$ can be treated similarly.

Lastly we should mention that the same general conclusions of this chapter can be drawn from Ida's definition of one-particle unstable states provided this definition is acceptable.

CHAPTER II

THE BOSON PROPAGATOR

We devote this entire chapter to an analysis of the boson propagator for the purpose of obtaining a description of the propagation and allied properties of unstable bosons. We first repeat some of Schwinger's work[12)] but in different detail from the original.

General Properties

The following results due to Lehmann are valid for an unstable particle field operator as well as a stable particle field operator if we avoid Lehmann's use of C, P and T invariance separately, so

$$\langle 0|\phi(x)\phi^{\dagger}(x')|0\rangle = i\int_0^{\infty}\rho(\kappa^2)\Delta^{(+)}(x-x';\kappa^2)\,d\kappa^2 \qquad (2.1)$$

$$\langle 0|\phi^{\dagger}(x')\phi(x)|0\rangle = -i\int_0^{\infty}\rho'(\kappa^2)\Delta^{(-)}(x-x';\kappa^2)\,d\kappa^2 \qquad (2.2)$$

We have only required axioms I to IV with axiom III referring only to proper Lorentz transformations and

$$\rho(-k^2)\theta(-k^2)\theta(k_0) = (2\pi)^3\sum_{\alpha}\langle 0|\phi(0)|k,\alpha\rangle\langle k,\alpha|\phi^{\dagger}(0)|0\rangle \qquad (2.3)$$

$$\rho'(-k^2)\theta(-k^2)\theta(k_0) = (2\pi)^3\sum_{\alpha}\langle 0|\phi^{\dagger}(0)|k,\alpha\rangle\langle k,\alpha|\phi(0)|0\rangle \qquad (2.4)$$

where axiom V shows $\rho(k^2)$ and $\rho'(k^2)$ are real and non-negative.

The use of CPT invariance gives

$$\langle 0|\phi(x)\phi^{\dagger}(x')|0\rangle = \langle 0|\phi^{\dagger}(-x')\phi(-x)|0\rangle \tag{2.5}$$

and this or alternatively axiom VI implies, according to Lovitch and Tomazawa[20], that

$$\rho(\kappa^2) = \rho'(\kappa^2) \tag{2.6}$$

Therefore we can define the boson propagator for stable or unstable particles* as the following function

$$G(x) = i\langle 0|T[\phi(x)\phi^{\dagger}(0)]|0\rangle = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k\, e^{ikx}\, G(-k^2) \tag{2.7}$$

where

$$G(-k^2) = \int_0^{\infty} \frac{\rho(\kappa^2)}{\kappa^2 + k^2 - i\epsilon}\cdot d\kappa^2 \tag{2.8}$$

in which

$$\rho(-k^2)\theta(-k^2)\theta(k_0) = (2\pi)^3 \sum_{\alpha} |\langle 0|\phi(0)|k,\alpha\rangle|^2 \tag{2.9}$$

We can use (2.8) to define $G(-k^2)$ as a function of a complex variable z so

$$G(z) = \int_0^{\infty} \frac{\rho(\kappa^2)\, d\kappa^2}{\kappa^2 - z} = [G(z^*)]^* \tag{2.10}$$

since $\rho(\kappa^2)$ is real and

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* In view of (2.6) it is also clear that the representation (2.8) is true for particles and their anti-particles with the same spectral density function $\rho(\kappa^2)$ for each.

$$\lim_{z \to -k^2 + i\epsilon} G(z) = G(-k^2) \tag{2.11}$$

Formula (2.4) shows that $G(z)$ is an analytic function of z apart from a possible cut from $z = 0$ along the whole positive real axis on the complex z-plane and (2.11) shows that $G(-k^2)$ is the boundary value of $G(z)$ in the real axis. Hence $G(z)$ is an analytic continuation of $G(-k^2)$ into the complex $(-k^2)$-plane. Also a calculation of the discontinuity across the cut in the z-plane using (2.8) gives

$$\mathrm{Im}\, G(-k^2) = \pi \rho(-k^2) \tag{2.12}$$

and so (2.8) is really a dispersion relation for $G(z)$ provided $G(z)$ tends to zero on the infinite circle in the z-plane. We are assuming here that no subtractions in (2.8) are required for convergence. We shall further assume that $G(-k^2)$ has been normalised such that*

$$\int_0^\infty \rho(\kappa^2)\, d\kappa^2 = 1 \tag{2.13}$$

We shall be specially interested in poles or strong variations in $G(z)$ so it will be convenient to have a dispersion relation also for $G^{-1}(z)$ the inverse of $G(z)$. Now $G(z)$ has no complex zeros since, putting $z = -k^2 + iy$

$$\mathrm{Im}\, G(z) = y \int_0^\infty \frac{\rho(\kappa^2)\, d\kappa^2}{(\kappa^2 + k^2)^2 + y^2} \tag{2.14}$$

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* For the case of pseudoscalar mesons with a local Lagrangian, (2.13) can be deduced directly.

and this only vanishes if $y = 0$. Even if $y = 0$, $G(z)$ has no zeros for $(-k^2) \leqslant 0$ but is positive and increasing for $-k^2$ increasing from negative values as can be seen from

$$\frac{d\,G(z)}{dz} = \int_0^\infty \frac{\rho(\kappa^2)\,d\kappa^2}{(\kappa^2-z)^2} \tag{2.15}$$

If we can show

$$\underset{z\to\infty}{Lt}\; z.\,Pf\int_0^\infty \frac{\rho(\kappa^2)\,d\kappa^2}{\kappa^2-z} = -\int_0^\infty \rho(\kappa^2)\,d\kappa^2 = -1 \tag{2.16}$$

where Pf signifies the taking of the principal part of the integral, then it follows that

$$G(z) \approx -\frac{1}{z} \quad \text{as } z \longrightarrow \text{infinite circle} \tag{2.17}$$

A theorem has been proved by Källen[21] and by Ferrari and Jona-Lasinio[22] which asserts that (2.16) is true under very reasonable conditions for $\rho(\kappa^2)$. If $\rho(\kappa^2)$ is integrable for $\kappa^2 \geqslant 0$ (which we have assumed in (2.13)) then the conditions are

$$\left.\begin{aligned} &\underset{\kappa^2\to\infty}{Lt}\; (\kappa^2\log\kappa^2).\,\rho(\kappa^2) = 0 \\ &\underset{\kappa^2\to\infty}{Lt}\; \frac{d\rho(\kappa^2)}{d\kappa^2}.\,\frac{1}{(\kappa^2)^N} = 0 \end{aligned}\right\} \tag{2.18}$$

for all integers $N > N_o > 0$. Even if $\rho(\kappa^2)$ is not integrable the following limit is finite

$$\underset{z\to\infty}{Lt}\; \frac{z.\,Pf\int_0^\infty \frac{\rho(\kappa^2)\,d\kappa^2}{\kappa^2-z}}{\int_0^\infty \rho(\kappa^2)\,d\kappa^2} \tag{2.19}$$

if $\rho(\kappa^2)$ can be written as a sum of an integrable function and a linear combination of functions of the type $(\kappa^2)^{-\gamma}$ $(0 < \gamma \leq 1)$ and $(\kappa^2)^{\gamma}(\log \kappa^2)^n$ where n is a positive integer. These theorems show that (2.17) follows unless $\rho(\kappa^2)$ is a rather badly behaved function which we assume it is not.

Consider the function $\mathcal{G}(z) = (z^{-1}G^{-1}(z) + 1)$ along with the properties of $G(z)$ deduced above. $\mathcal{G}(z)$ has a cut along the positive real axis, no poles except at $z = 0$ and converges to zero on the infinite circle, therefore we can write a dispersion relation

$$\mathcal{G}(z) = \frac{\lambda^2}{z} - \int_0^\infty \frac{s(\kappa^2)\, d\kappa^2}{\kappa^2 - z} \tag{2.20}$$

or

$$G^{-1}(z) = \lambda^2 - z - z\int_0^\infty \frac{s(\kappa^2)\, d\kappa^2}{\kappa^2 - z} \tag{2.21}$$

where

$$s(-k^2) = \frac{1}{2\pi i k^2}\left[G^{-1}(-k^2+i\epsilon) - G^{-1}(-k^2-i\epsilon)\right]$$

$$= \frac{1}{2\pi i k^2}\cdot\frac{\left[G(-k^2-i\epsilon) - G(-k^2+i\epsilon)\right]}{|G(-k^2)|^2}$$

$$= \frac{\rho(-k^2)}{(-k^2)\cdot|G(-k^2)|^2} \geq 0 \tag{2.22}$$

and by comparing $G^{-1}(0)$ with $[G(0)]^{-1}$ we have

$$\lambda^2 = \left[\int_0^\infty \frac{\rho(\kappa^2)d\kappa^2}{\kappa^2}\right]^{-1} > 0 \qquad (2.23)$$

which is certainly finite* because of assumptions (2.13) and (2.18).**

We can get a clear picture of most of the information we have gathered about $G(-k^2)$ and $G^{-1}(-k^2)$ from their graphs below.

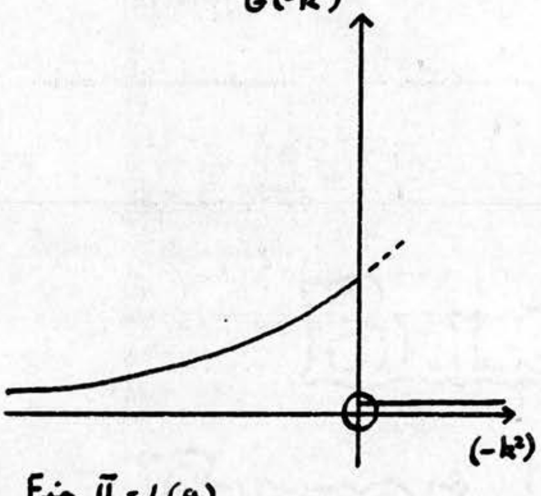

Fig. II - 1(a)

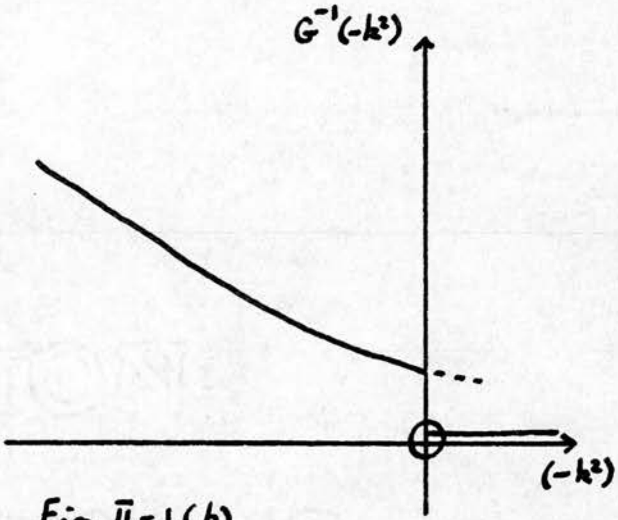

Fig. II - 1(b)

The threshold of the cut in the $(-k^2)$-plane is determined by the lowest energy contribution to $\rho(-k^2)$ in (2.9). If we assume that the threshold occurs at some positive value of $-k^2 = K_o^{\ 2}$, which is true for many physical situations, then we move the threshold away from the origin and consider the likely behaviour of $G(-k^2)$ and $G^{-1}(-k^2)$ for $(-k^2) > 0$. Since $G^{-1}(-k^2)$ is

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* In what follows we shall not include the point $K^2 = 0$ in any spectrum and in fact we shall not explicitly consider the electromagnetic thresholds which appear at the origin for neutral fields and at all poles. We presume their effect does not alter the physical conclusions of our argument.

** See Appendix 2.

a decreasing function for all $(-k^2) < K_o^2$, where K_o^2 is the new position of the threshold, then it is possible that $G^{-1}(-k^2)$ has a zero for $K_o^2 > (-k^2) > 0$ and that $G(-k^2)$ has a pole at this point. It is not at all likely that $G(-k^2)$ has a zero first. A pole in $G(-k^2)$ would correspond to a single particle freely propagating among the various self-energy effects and might well be expected on physical grounds. The correctness of this interpretation follows from an examination of (2.9) with $|k, a\rangle$ as a one-particle state. Then $\rho(-k^2)$ has a δ-function contribution when $-k^2$ is on the mass shell of this particle and a pole term appears in $G(-k^2)$ at this mass value. We now write

$$G(z) = \frac{\rho_0}{\mu^2 - z} + \int_{\kappa_0^2}^{\infty} \frac{\rho(\kappa^2)}{\kappa^2 - z} \cdot d\kappa^2 \qquad (2.24)$$

where

$$\rho_0 + \int_{\kappa_0^2}^{\infty} \rho(\kappa^2) d\kappa^2 = 1 \quad , \quad \rho_0^{-1} = - \left[\frac{d\,G^{-1}(z)}{dz}\right]_{z=\mu^2} > 0$$

$$\lambda^2 = \mu^2 \left[1 + \int_{\kappa_0^2}^{\infty} \frac{s(\kappa^2)}{\kappa^2 - \mu^2} \cdot d\kappa^2\right] = \left[\frac{\rho_0}{\mu^2} + \int_{\kappa_0^2}^{\infty} \frac{\rho(\kappa^2)}{\kappa^2} \cdot d\kappa^2\right]^{-1} > 0$$

$$\mu^2 < \kappa_0^2$$

and we assume $\rho(-k^2)$ is zero in the regions $0 \leq -k^2 < \mu^2$ and $\mu^2 < -k^2 < K_o^2$.

We now picture $G(-k^2)$ and $G^{-1}(-k^2)$ as

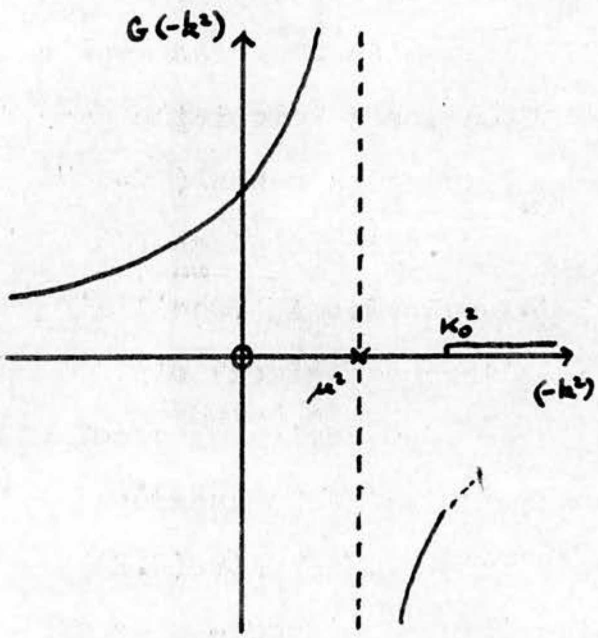

Fig. II-2(a)

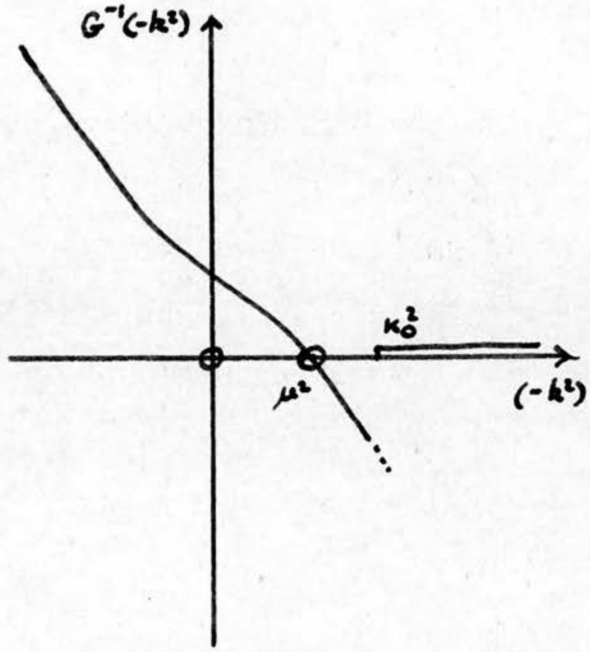

Fig. II-2(b)

The graphs above lead us to suspect and the formula (2.24) confirms that $G(-k^2)$ may now have a zero for $\mu^2 < -k^2 < K_o{}^2$ giving a pole in $G^{-1}(-k^2)$ which in turn may then vanish again for a larger value of $(-k^2)$. Hence repetitions of our first argument applied to $G(-k^2)$ and $G^{-1}(-k^2)$ alternately will produce a possible succession of poles and zeros. In general we have graphically

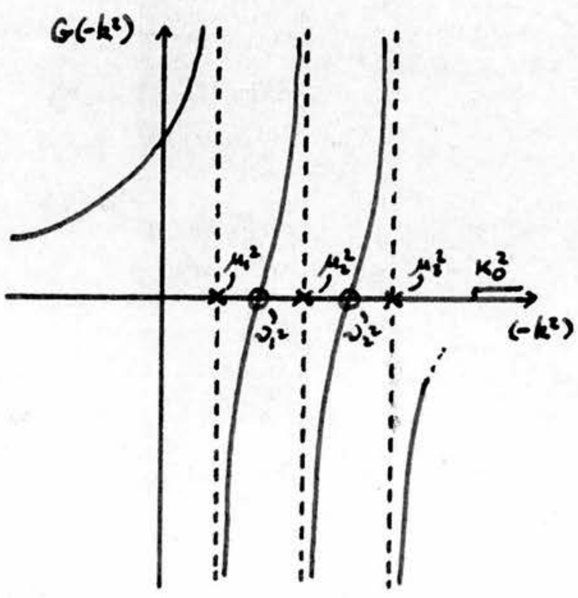

Fig. II-3(a)

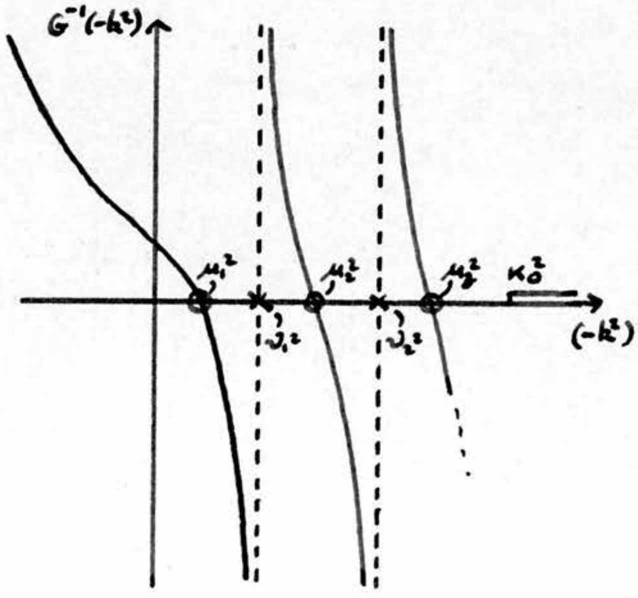

Fig. II-3(b)

Physically we do not expect more than one pole in $G(-k^2)$ since we do not observe two single particles of different masses but identical quantum numbers in nature. Therefore from now on we shall consider only the possibility of one pole and one zero in $G(-k^2)$. Schwinger derives a further spectral representation for $G(-k^2)$ which is useful in later calculations. If $G(-k^2)$ has no poles or zeros but only a cut along the real positive axis starting at K_o^2 then, if $K'^2_o < K_o^2$, from (2.17)

$$(K_o'^2 - z)\,G(z) \approx 1 \quad \text{as } z \longrightarrow \text{infinite circle} \tag{2.25}$$

Also $\left[(K'^2_o - z)\,G(z)\right]$ has no poles, one zero at $z = K'^2_o$ and a cut from K_o^2 to ∞. Hence $\log\left[(K'^2_o - z)G(z)\right]$ has no poles but a cut chosen from K'^2_o to ∞ and converges to zero on the infinite circle. Therefore we can write a dispersion relation of the form

$$\log\left[(K_o'^2 - z)\,G(z)\right] = -\frac{1}{\pi}\int_{K_o'^2}^{\infty} \frac{\phi_o(\kappa^2)\,d\kappa^2}{\kappa^2 - z} \tag{2.26}$$

where

$$\phi_o(-k^2) = -\frac{1}{2i}\left\{\log\left[(K_o'^2 - z)\,G(z)\right]\right\}\Big|_{z=-k^2-i\epsilon}^{z=-k^2+i\epsilon}$$

$$= -\frac{1}{2i}\left[\log\left(\frac{K_o'^2 + k^2 - i\epsilon}{K_o'^2 + k^2 + i\epsilon}\right) + \log\left(\frac{G(-k^2+i\epsilon)}{G(-k^2-i\epsilon)}\right)\right]$$

$$= \pi\,\theta(-k^2 - K_o'^2) - \cot^{-1}\left(\frac{\mathrm{Re}\,G(-k^2+i\epsilon)}{\pi\,\rho(-k^2)}\right) \geq 0 \tag{2.27}$$

in which the branch of the $\cot^{-1}$ must be such that its value lies between 0 and π. It follows that

$$G(z) = (K_o'^2 - z)^{-1} \cdot \exp\left[-\frac{1}{\pi}\int_{K_o'^2}^{\infty} \frac{\phi_o(k^2)\,dk^2}{k^2 - z}\right]$$

$$G^{-1}(z) = (K_o'^2 - z) \cdot \exp\left[\frac{1}{\pi}\int_{K_o'^2}^{\infty} \frac{\phi_o(k^2)\,dk^2}{k^2 - z}\right] \qquad (2.28)$$

In spite of the factor $(K_o'^2 - z)^{-1}$, $G(z)$ does not have a pole since $\phi_o(-k^2)$ has a discontinuity at $K_o'^2$ of π which produces a factor of $(K_o'^2 - z)$ from the exponential. Since $\phi_o(K^2) = \pi$ for $K_o'^2 \leq K \leq K_o^2$ the formulae (2.28) do not imply that $G(z)$ has a cut from $K_o'^2$ to K_o^2 but only a cut from K_o^2 to $+\infty$ is possible. Since $\rho(K_o^2) = 0$ and $Re\, G(K_o^2) > 0$ then $\phi_o(K_o^2) = \pi$. Also $Re\, G(-k^2) \sim \frac{1}{k^2}$ and $\rho(-k^2) \sim \frac{1}{(-k^2)^{1+\delta}}$ as $-k^2 \to \infty$ where δ is a positive number, so that $\phi_o(-k^2) \to 0$ as $-k^2 \to \infty$. It must therefore follow that the integrals in (2.28) converge and that $0 \leq \phi_o \leq \pi$.

Similarly for one pole at $z = \mu^2$ and no zeros of $G(z)$, the forms (2.28) will do if we choose $K_o'^2 < \mu^2$ and

$$\phi_o(-k^2) \to \phi_1(-k^2) = \phi_o(-k^2) - \pi\,\theta(-k^2 - \mu^2) \qquad (2.29)$$

but note here that although $\phi_1(-k^2) \to 0$ as $(-k^2) \to \infty$ as before we now have $\phi_1(K_o^2) = 0$ since $Re\, G(K_o^2) \leq 0$.

For the third case of a pole at $z = \mu^2$ and a zero at $z = \nu^2 > \mu^2$ in $G(z)$ we choose $K_o'^2 < \mu^2 < \nu^2 < K_o^2$ and

$$\phi_o(-k^2) \to \phi_2(-k^2) = \phi_o(-k^2) - \pi\,\theta(-k^2 - \mu^2)\,\theta(\nu^2 + k^2) \qquad (2.30)$$

and here we have $\phi_2(K_o^2) = \pi$ and $\phi_2(-k^2) \to 0$ as $(-k^2) \to \infty$ in the same manner as $\phi_o(-k^2)$.

We picture the three functions ϕ_0, ϕ_1, ϕ_2 as

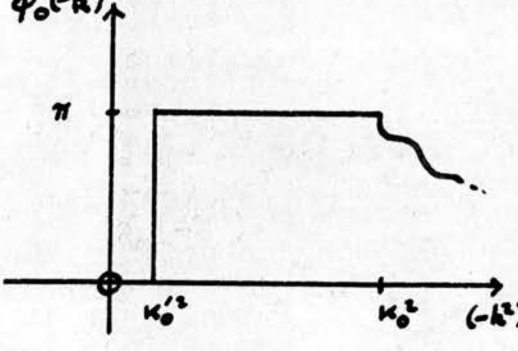

Fig II-4(a)

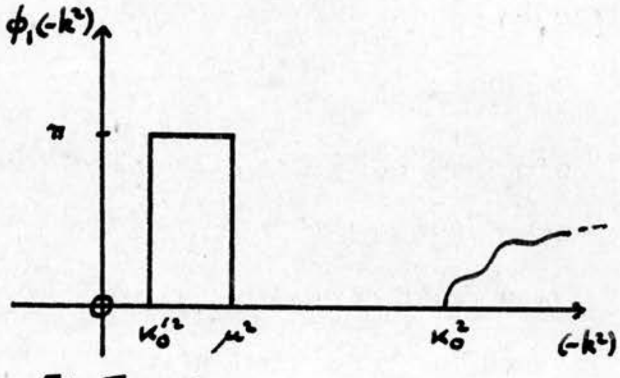

Fig II-4(b)

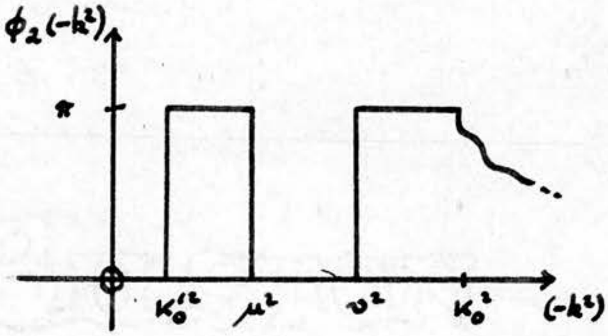

Fig II-4(c)

The generalisation of this spectral representation to cases where $G(z)$ has many poles and zeros is perfectly straightforward.

Stable and Unstable Particle Propagation.

We are now interested in $G(-k^2)$ under two different external conditions

(a) when only strong interactions are allowed and we can apply the usual stable particle theory. This situation is to be thought of as purely hypothetical.

(b) when all possible interactions are allowed (<u>but</u> see note at the foot of page 25) and we have to make reasonable generalisations from the stable particle theory. This situation is the realistic one.

For the case (a) where $G(-k^2)$ has the physically expected pole and no zeros we write (2.24) as

$$G_s(-k^2) = \frac{\rho_0^s}{\mu_s^2 + k^2} + \int_{\kappa_s^2}^{\infty} \frac{\rho^s(\kappa^2)\,d\kappa^2}{\kappa^2 + k^2 - i\epsilon} \tag{2.31}$$

and we also have

$$G_s^{-1}(-k^2) = \lambda_s^2 + k^2 + k^2 \int_{\kappa_s^2}^{\infty} \frac{s^s(\kappa^2)\,d\kappa^2}{\kappa^2 + k^2 - i\epsilon} \tag{2.32}$$

where $G_s^{-1}(\mu_s^2) = 0$ and so $\lambda_s^2 = \mu_s^2 \cdot \left[1 + \int_{\kappa_s^2}^{\infty} \frac{s^s(\kappa^2)\,d\kappa^2}{\kappa^2 - \mu_s^2}\right]$.
Therefore we could write

$$G_s^{-1}(-k^2) = (k^2 + \mu_s^2) \cdot \left[1 + \int_{\kappa_s^2}^{\infty} \frac{\kappa^2 s^s(\kappa^2)\,d\kappa^2}{(\kappa^2 - \mu_s^2)(\kappa^2 + k^2 - i\epsilon)}\right] \tag{2.33}$$

so that

$$(\rho_0^s)^{-1} = 1 + \int_{\kappa_s^2}^{\infty} \frac{\kappa^2 s^s(\kappa^2)\,d\kappa^2}{(\kappa^2 - \mu_s^2)^2} \tag{2.34}$$

and we picture $G_s(-k^2)$ and $G_s^{-1}(-k^2)$ as in Figures II-2(a), (b).

Consider case (b) with the same particle propagating as in $G_s(-k^2)$ above. If the particle is unstable under the introduction of weak or electromagnetic interactions then the threshold of the continuous spectrum will occur closer to $(-k^2) = 0$ than the point $(-k^2) = \mu_s^2$ and we no longer expect physically to have any poles or zeros of the propagators for $(-k^2)$ below the new threshold. Hence it is reasonable to assume in this case that we can represent the propagator $G(-k^2)$ as

$$G(-k^2) = \int_{\kappa_w^2}^{\infty} \frac{\rho(\kappa^2)\,d\kappa^2}{\kappa^2 + k^2 - i\epsilon} \tag{2.35}$$

and picture $G(-k^2)$ as in Figure II-1(a) in which an extra section of the real axis has been cut compared with Figure II-2(a), i.e. the section $\kappa_w^2 \leq (-k^2) \leq \kappa_s^2$. In this region we write $\rho_w(\kappa^2) = \rho(\kappa^2)$ and we expect physically $\rho_w(\kappa^2) << \rho_s(\kappa^2)$ Hence we prefer to write (2.35) in the form

$$G(-k^2) = \int_{\kappa_w^2}^{\infty} \frac{\rho_w(\kappa^2)\,d\kappa^2}{\kappa^2+k^2-i\epsilon} + \int_{\kappa_s^2}^{\infty} \frac{\rho_s(\kappa^2)\,d\kappa^2}{\kappa^2+k^2-i\epsilon} \tag{2.36}$$

where we do not necessarily imply that $\rho_w(\kappa^2) << \rho_s(\kappa^2)$ for $\kappa^2 \geq \kappa_s^2$ We will also write

$$G^{-1}(-k^2) = \lambda_s^2+\lambda_w^2+k^2+k^2\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)\,d\kappa^2}{\kappa^2+k^2-i\epsilon} + k^2\int_{\kappa_s^2}^{\infty} \frac{s_s(\kappa^2)\,d\kappa^2}{\kappa^2+k^2-i\epsilon} \tag{2.37}$$

We now ask the question, what happens to the function $G(-k^2)$ in the region where $G_s(-k^2)$ has a pole? Let us equate real and imaginary parts for $G(-k^2)$ and $G^{-1}(-k^2)$ given by (2.36), (2.37) and by (2.28) for the region of interest $\kappa_w^2 < (-k^2) < \kappa_s^2$. The results are

$$\pi\rho_w(-k^2) = \frac{\sin\phi_0(-k^2)}{(\kappa_w^2+k^2)} \cdot e^{\frac{1}{\pi} P\!f \int_{\kappa_w^2}^{\infty} \frac{\phi_0(\kappa^2)\,d\kappa^2}{\kappa^2+k^2}} \tag{2.38}$$

$$-\pi(-k^2)\,s_w(-k^2) = (\kappa_w^2+k^2)\left[-\sin\phi_0(-k^2)\right] \cdot e^{-\frac{1}{\pi} P\!f \int_{\kappa_w^2}^{\infty} \frac{\phi_0(\kappa^2)\,d\kappa^2}{\kappa^2+k^2}} \tag{2.39}$$

$$\lambda_w^2+\lambda_s^2+k^2+k^2 P\!f\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)\,d\kappa^2}{\kappa^2+k^2} + k^2\int_{\kappa_s^2}^{\infty} \frac{s_s(\kappa^2)\,d\kappa^2}{\kappa^2+k^2}$$
$$= (\kappa_w^2+k^2)\left[\cos\phi_0(-k^2)\right] \cdot e^{-\frac{1}{\pi} P\!f \int_{\kappa_w^2}^{\infty} \frac{\phi_0(\kappa^2)\,d\kappa^2}{\kappa^2+k^2}} \tag{2.40}$$

From (2.38) and (2.39) we get

$$-\pi k^2 s_w(-k^2)\,\rho_w(-k^2) = \left[\sin\phi_0(-k^2)\right]^2 \tag{2.41}$$

Also from (2.39) and (2.40) we get

$$-\pi k^2 s_w(-k^2)\cot\phi_0(-k^2)$$

$$= \lambda_w^2 + \lambda_s^2 + k^2 + k^2\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)d\kappa^2}{\kappa^2+k^2} + k^2\int_{\kappa_s^2}^{\infty} \frac{s_s(\kappa^2)d\kappa^2}{\kappa^2+k^2} \qquad (2.42)$$

Now if $-k^2 \approx \mu_s{}^2$ then

$$G_s^{-1}(-k^2) = \lambda_s^2 + k^2 + k^2\int_{\kappa_s^2}^{\infty} \frac{s_s(\kappa^2)d\kappa^2}{\kappa^2+k^2} \approx (\varrho_0^s)^{-1}(k^2+\mu_s^2) \qquad (2.43)$$

and (2.42) becomes for $-k^2 \approx \mu_s{}^2$

$$-\pi k^2 s_w(-k^2)\cot\phi_0(-k^2) \approx (\varrho_0^s)^{-1}(k^2+\mu_s^2) + \lambda_w^2 + k^2\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)d\kappa^2}{\kappa^2+k^2} \qquad (2.44)$$

By inspection we might expect the right hand side of (2.44) to become small at some point $\mu^2 \approx \mu_s{}^2$. Hence we expand the last term about μ^2, so

$$k^2\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)d\kappa^2}{\kappa^2+k^2} \approx \mu^2\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)d\kappa^2}{\kappa^2-\mu^2} - (k^2+\mu^2)\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{d\kappa^2}{(\kappa^2-\mu^2)}\cdot\frac{d[\kappa^2 s_w(\kappa^2)]}{d\kappa^2}$$

$$(2.45)$$

for $-k^2$ close enough to μ^2. Putting (2.45) into (2.44) we find

$$-\pi k^2 s_w(-k^2)\cot\phi_0(-k^2) \approx (\varrho_0)^{-1}(k^2+\mu^2) \qquad (2.46)$$

for $-k^2 \approx \mu^2 \approx \mu_s{}^2$ and where

$$\mu^2 = \mu_s^2 + \varrho_0^s\cdot\left[\lambda_w^2 - \mu^2\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{s_w(\kappa^2)d\kappa^2}{\kappa^2-\mu^2}\right]$$

$$(2.47)$$

$$(\varrho_0)^{-1} = (\varrho_0^s)^{-1} + P\!\!\int_{\kappa_w^2}^{\infty} \frac{d\kappa^2}{(\kappa^2-\mu^2)}\cdot\frac{d}{d\kappa^2}[\kappa^2 s_w(\kappa^2)]$$

It is clear that the results (2.46) and (2.47) are only valid if the introduction of weak interaction terms has a very small effect in the region of the cut near $-k^2 = \mu_s{}^2$, on $G^{-1}(-k^2)$.

The elimination of $\phi_0(-k^2)$ from (2.41) and (2.46) gives*

$$\rho_w(-k^2) \approx \frac{\rho_0}{\pi} \cdot \frac{\gamma\mu}{(k^2+\mu^2)^2 + (\gamma\mu)^2} \tag{2.48}$$

for $-k^2 \approx \mu^2$ and

$$\gamma = \pi\mu\rho_0 s_w(\mu^2) \tag{2.49}$$

The formula (2.48) has the Breit-Wigner resonance shape we predicted and degenerates to a δ-function $\rho_0\,\delta(k^2+\mu^2)$ when γ cannot be distinguished from zero. Thus (2.48) reduces to a stable particle contribution when the weak interactions are switched off. It seems quite consistent to associate the resonance with a particle which is almost stable, having unique mass μ and lifetime γ.

The Decay Law

The contribution of (2.48) to $G(\underline{k}, t)$ is obtained from

$$G(\underline{k},t) = \int_{-\infty}^{\infty} \frac{dk_0}{\pi} \cdot e^{-ik_0 t} \cdot G(-k^2) \approx \int_{\kappa_w^2}^{\infty} \frac{i \cdot e^{-iE_\kappa |t|}}{2E_\kappa} \cdot \frac{\rho_0}{\pi} \cdot \frac{\gamma\mu}{(\kappa^2-\mu^2)^2 + (\gamma\mu)^2} \cdot d\kappa^2$$

$$\approx \frac{i}{2} \cdot \frac{\rho_0}{E_\mu} \cdot e^{-iE_\mu |t|} \int_{\kappa_w-\mu}^{\infty} d(\kappa-\mu) \cdot \frac{1}{2\pi} \cdot \frac{\gamma}{(\kappa-\mu)^2 + (\frac{1}{2}\gamma)^2} \cdot e^{-i\left(\frac{\mu}{E_\mu}\right)|t|(\kappa-\mu)}$$

(2.50)

---

* It is perhaps not obvious that no further approximation is involved in writing (2.48), (2.49) and more details of the consistency of these results are given in Appendix 2, where we also consider the effect a zero in $G_s(-k^2)$ has on results (2.47), (2.48), (2.49).

and the latter integral can be evaluated by integrating round a complex contour shown below and letting $R \longrightarrow \infty$

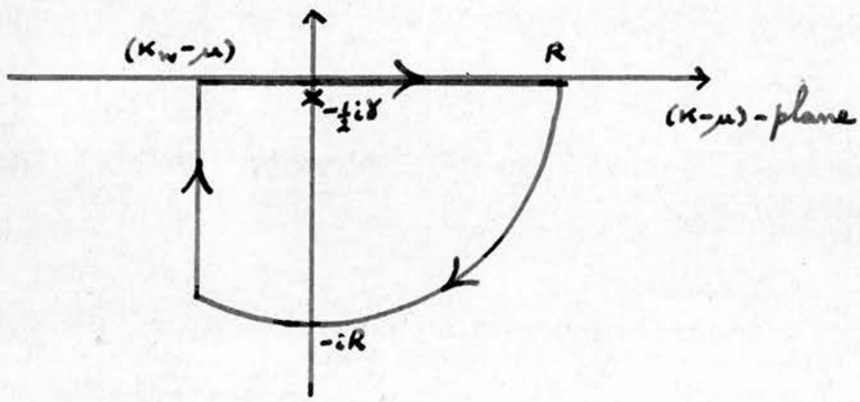

Fig. II-5

$$\oint_{\text{contour}} \frac{d(\kappa-\mu).\,\gamma.\,e^{-i\left(\frac{\mu}{E_\mu}\right)|t|(\kappa-\mu)}}{(\kappa-\mu)^2+(\frac{1}{2}\gamma)^2} = 2\pi\, e^{-\frac{1}{2}\left(\frac{\mu}{E_\mu}\right)\gamma|t|}$$

$$= \int_{\kappa_w-\mu}^{\infty} \frac{d(\kappa-\mu).\,\gamma.\,e^{-i\left(\frac{\mu}{E_\mu}\right)|t|(\kappa-\mu)}}{(\kappa-\mu)^2+(\frac{1}{2}\gamma)^2} - \int_{\kappa_w-\mu}^{\kappa_w-\mu-i\infty} \frac{d(\kappa-\mu).\,\gamma.\,e^{-i\left(\frac{\mu}{E_\mu}\right)|t|(\kappa-\mu)}}{(\kappa-\mu)^2+(\frac{1}{2}\gamma)^2}$$

Therefore, where $E_\mu = \sqrt{\mathbf{k}^2+\mu^2}$ (2.51)

$$G(\mathbf{k},t) \approx \frac{i}{2}\cdot\frac{\rho_0}{E_\mu}\cdot e^{-iE_\mu|t|}\cdot e^{-\frac{1}{2}\left(\frac{\mu}{E_\mu}\right)\gamma|t|} + \frac{i\rho_0 e^{-iE_\mu|t|}}{4\pi E_\mu}\int_{\kappa_w-\mu}^{\kappa_w-\mu-i\infty} \frac{d(\kappa-\mu).\,\gamma.\,e^{-i\left(\frac{\mu}{E_\mu}\right)|t|(\kappa-\mu)}}{(\kappa-\mu)^2+(\frac{1}{2}\gamma)^2}$$

$$\approx \frac{i}{2}\cdot\frac{\rho_0}{E_\mu}\cdot e^{-iE_\mu|t|}\cdot e^{-\frac{1}{2}\left(\frac{\mu}{E_\mu}\right)\gamma|t|} + \frac{\gamma\rho_0 e^{-iE_\mu|t|}}{4\pi\mu(\kappa_w-\mu)^2}\cdot\frac{e^{-i\left(\frac{\mu}{E_\mu}\right)|t|(\kappa_w-\mu)}}{|t|}$$

(2.52)

where we have replaced the integral in (2.52) by the first approximation of an asymptotic power series expansion in $\frac{1}{t}$ for large t and obtain the correction term found by Levy[6)] for the Lee Model and by Matthews and Salam[4)]. Levy was first to notice that the Fourier transform of a function $G(t)$ vanishing below a finite value of its argument behaves for large t like a power of t.

So far as the first and most important term in (2.52) is concerned, the time dependence of the propagator is of a decreasing exponential type as expected. The time dependence of the probability amplitude of an unstable particle wave packet is exhibited in (1.16). It is easy to see that, if $k' \approx k$, $g_a(\underline{k}',t)$ has the same exponential decay term as $G(\underline{k},t)$ apart from a constant factor. We also have

$$|g_a(\underline{k}',t)|^2 \approx \frac{\rho_0^2}{4E_\mu^2}\cdot e^{-\left(\frac{\mu}{E_\mu}\right)\gamma|t|}\cdot\left.\left(\frac{2(2\pi)^3(n!k')^2}{NT^2}\right)\right|_{k_0'=E_\mu} \qquad (2.53)$$

which shows that the particle decays with a lifetime

$$\tau_{\underline{k}} = \frac{(\underline{k}^2+\mu^2)^{1/2}}{\mu}\cdot\gamma^{-1} \qquad (2.54)$$

and correctly gives an elongation of the lifetime for a particle moving relativistically with momentum $\underline{k}$. Like mass, the lifetime is usually quoted for a particle at rest and here the rest lifetime is $\tau = \gamma^{-1}$.

Apart from the correction terms already found in (2.52) other dominant contributions to the propagator at large times will come from terms of definite frequency such as the lowest order thresholds. It can be shown from perturbation theory or from the model we are shortly to discuss that a two-particle threshold at $K^2 = K_W^2$ is characterised by a factor of $[(\kappa^2-\kappa_W^2)/\kappa^2]^{1/2}$ in $\rho(\kappa^2)$. Since we wish $\int_{\kappa_W^2}^{\infty}\rho(\kappa^2)d\kappa^2$ to converge then $\rho(K^2)$ must decrease faster than K^{-2} for large K^2. Therefore we write

$$\rho(\kappa^2) = \frac{\beta}{\kappa^2}\cdot\frac{(\kappa^2-\kappa_W^2)^{1/2}}{(4\pi\kappa^2)^{1/2}}\cdot J(\kappa^2) \qquad (2.55)$$

where β is a constant and $\Upsilon(K^2)$ is a dimensionless 'cut-off' function which tends to zero for large K^2 and is normalised so $\Upsilon(K_W{}^2) = 1$ by choice of β. It has been shown by Schwinger that, for $\rho(\kappa^2) \approx \frac{\beta}{\kappa^2}\left[(\kappa^2-\kappa_W^2)/\kappa^2\right]^{1/2}$ when $\kappa^2 \approx \kappa_W^2$, there is a further correction term to (2.52) of the form

$$\frac{\beta}{E_{\kappa_W}} \cdot \left(\frac{E_{\kappa_W}}{i\kappa_W^2|t|}\right)^{3/2} \cdot e^{-iE_{\kappa_W}|t|} \qquad (2.56)$$

provided $K_W|t| \gg E_{K_W}/K_W$. For the exponential decay term coefficient to dominate the threshold correction we must have

$$\kappa_W|t| \gg \frac{(\beta E_\mu E_{\kappa_W}^{1/2})^{2/3}}{\kappa_W} \qquad (2.57)$$

and the power of the exponential decay term is a large multiple of

$$-\frac{1}{2}\cdot\frac{\mu}{E_\mu}\cdot\gamma\cdot\frac{(\beta E_\mu E_{\kappa_W}^{1/2})^{2/3}}{\kappa_W^2} \quad \text{or} \quad -\frac{1}{2}\cdot\frac{\mu}{E_\mu}\cdot\gamma\cdot\frac{E_{\kappa_W}}{\kappa_W^2}$$

depending on which bound of $|t|$ given above is the larger. Hence if $\gamma \lll K_W$ is small enough there will be an extended time interval when the exponential decay term dominates the threshold contribution. Similarly comparing the coefficients of the two terms in (2.52) we must have $|t| \gg \gamma E_\mu/2\pi\mu(\kappa_W-\mu)^2$ for the decay coefficient to dominate and the power of the decay exponential is a large multiple of $-\gamma^2/4\pi(\kappa_W-\mu)^2$. Hence, if $\gamma \lll \mu - K_W$, the decay exponential dominates for some time interval. It is interesting to note that $\gamma \lll K_W$

and $\gamma <<<< (\mu - K_W)$ are the identical requirements, for a particle at rest, for a resonance to be observable in (1.11). The question is now whether the correction terms with their purely algebraic decrease will overwhelm the exponential decay term after a sufficiently long time. Schwinger's view was to introduce a mass filter to project out a single particle term and not a kinematically equivalent combination of particles. This is to take account of the experimental limitations in measuring devices. Schwinger has

$$\mathfrak{M}\, G(\underline{k}, t) = \int_0^\infty \rho(\kappa^2)\, d\kappa^2 \cdot \frac{i}{2E_\kappa} \cdot e^{-iE_\kappa |t|} \cdot M(\kappa) \tag{2.58}$$

where $M(\kappa) = \begin{cases} 1 & \text{for } |\kappa - \mu| \lesssim \Delta\mu \\ 0 & \text{for } |\kappa - \mu| \gtrsim \Delta\mu \end{cases}$

and $\gamma << \Delta\mu << \mu$, and $\Delta\mu$ is the precision of the mass determination. The similarity of this perhaps artificial introduction of a mass filter and the methods of Chapter I is striking. We have taken such experimental limitations into account in Chapter I in a basic manner by using tempered distributions and the one-particle amplitude instead of the propagator. If we replace $\mathfrak{M}\, G(\underline{k}, t)$ by $g_a(\underline{k}, t)$ and $M(K)$ by $\frac{1}{(2\pi)^4} \tilde{\mathfrak{F}}_a(\underline{k}, E_K)$, then apart from certain overall irrelevant multiplicative factors, the equations (2.58) and (1.16) are identical. From this point on our analysis will fall into line with Schwinger's.[12] Thus like Schwinger we conclude that the exponential law is accurate for $(\Delta E)^{-1} << \left(\frac{E_\mu}{\mu}\right)|t| \lesssim \gamma^{-1}$ but when $\gamma\left(\frac{E_\mu}{\mu}\right)|t| \sim \left(\frac{\Delta E}{\gamma}\right)^a >> 1$ where a is a positive number, the exponential law fails and is no longer independent of observations.

The extension and application of the discussion so far in this chapter to existing particles has been discussed by Schwinger[12] and need not be repeated here.

Let us return to the discussion in Chapter I of the definition of unstable particle states. It can be seen from (2.52) or (2.53) that the one-particle contribution (2.48) from the spectral density function $\rho(K^2)$ to $G(\underline{k}, t)$ or $g_a(\underline{k}, t)$ in plane wave form is, apart from irrelevant factors, $\exp\left[-iE_\mu|t| - \frac{1}{2}\left(\frac{\mu}{E_\mu}\right)\gamma|t|\right]$ and the full wave-packet expression for $g_a(x)$ is, using (1.15), (2.35) and (2.48),

$$g_\alpha(x) = \frac{2i\rho_0}{N^{1/2}T\pi(2\pi)^4}\int_{-\infty}^{\infty} d^4k'\,\theta(k_0').(n.k').e^{ik'x}\frac{\gamma_\mu}{(k'^2+\mu^2)^2+(\gamma_\mu)^2}.\tilde{\mathfrak{F}}_\alpha(k')$$

where $\tilde{\mathfrak{F}}_a(k')$ is given approximately by (1.17). When this wave-packet has a real momentum $\underline{k}'$ the corresponding energy is complex $E_\mu - \frac{1}{2}\left(\frac{\mu}{E_\mu}\right)\gamma \approx E_\mu$ which can be regarded as caused by a complex mass $\mu - \frac{1}{2} i \gamma$. The above wave-packet is very similar to $f_a(x)$ defined in Appendix 1 if the $\delta(k^2 + \mu^2)$ is spread out into a resonance shape. Since $g_a(x)$ appears to have all the necessary properties we identify $g_a(x)$ with $\mathfrak{F}_a(x)$ for $x_\mu \in R$ in order to define the unstable particle state given in Chapter I without ambiguity. We shall not, however, require to make any further reference to this state in what follows.

Analytic Continuation for a Model of a Decay Process

We propose to forge a link between Levy's work with the Lee model and Schwinger's work in full field theory. We shall use a more general model than Levy but we cannot use full field theory because of analyticity difficulties.

We must first carefully define analytic functions for use in the complex $(-k^2)$-plane. We have already defined an analytic continuation of $G(-k^2)$ into the physical sheet by (2.10). In order to continue $G(s)$ into the first unphysical sheet to be reached through the real cut we must define an analytic continuation of $\rho(K^2)$ into the complex K^2-plane. For real K^2 we have defined $\rho(K^2)$ by (2.9). If we restrict ourselves to continuations of $G(s)$ through the lowest energy branch line then we shall require only the lowest energy term or terms in the summation of (2.9). Suppose that the lowest energy contribution to $\rho(K^2)$ comes from two-particle intermediate states only, then consider the following set of reactions

R I	$p \longrightarrow p$	(G)
R II	$k_1 + k_2 \longrightarrow p$	(υ)
R III	$k_1 + k_2 \longrightarrow k_1 + k_2$	(F)

where we have represented particles by their four-momenta. In brackets we have indicated that reaction R I is described by the propagator $G(-k^2)$, reaction R II by an invariant 'vertex' function $\upsilon(-k^2)$ and reaction R III by the s-wave projection $F_0(-k^2)$ of the invariant elastic scattering amplitude

$F(-k^2, \cos\theta)$. The amplitudes G, v and F_o can be considered as elements of

$$\underline{T} = \begin{pmatrix} G & v \\ v & F_o \end{pmatrix} \qquad (2.59)$$

where

$$v[-(k_1+k_2)^2] = \frac{1}{4\pi^{1/2}} \langle 0 | \phi(0) | k_1 k_2 \text{ in} \rangle$$

$$F[-(k_1+k_2)^2, \cos\theta] = \frac{1}{16\pi} \langle k_2 | J(0) | k_1 k_2 \text{ in} \rangle$$

and θ is the scattering angle in the centre of mass system.*
The convenience of (2.59) lies in the conciseness of the unitarity condition which can be written in the region of the two-particle cut

$$\underline{T}(s+io) - [\underline{T}(s+io)]^* = 2i\, \underline{T}(s+io) \begin{pmatrix} 0 & 0 \\ 0 & h(s+io) \end{pmatrix} [\underline{T}(s+io)] \qquad (2.60)$$

where

$$h(z) = \frac{[z-(m_1+m_2)^2]^{1/2}[z-(m_1-m_2)^2]^{1/2}}{z} = -[h(z^*)]^* \qquad (2.61)$$

and m_1, m_2 are the masses of the particles associated with the two-particle branch cut. For complex s or z we choose the branches of the square roots in $h(z)$ such that the real axis is cut except for the region $(m_1 - m_2)^2 < z < (m_1 + m_2)^2$.

We now assume that v and F_o in the physical sheet have only the branch lines and poles associated with physical

---

* Further details of notation and the proof of (2.60) are contained in Appendix 3. Note that we are neglecting iso-spin.

intermediate states of the three reactions R I, R II, R III. This may not be valid in full field theory, but seems plausible from the works of Gunson-Taylor[10], and Oehme[11] using a Mandelstam-type representation; providing there are no anomalous thresholds. These works in particular have shown that the two particle branch points in $\mathcal{V}$ and F_0 have square root characters. The only additional results required are the reality conditions

$$F_0(z) = \left[F_0(z^*)\right]^*, \qquad \mathcal{V}(z) = \left[\mathcal{V}(z^*)\right]^* \tag{2.62}$$

Note that (2.60) automatically defines $\rho(k^2)$ to be

$$\rho_L(k^2) = \frac{1}{\pi} h(k^2+io)\,|\mathcal{V}(k^2+io)|^2 \tag{2.63}$$

From (2.60) we find $\underline{T}(s+io)$ is the boundary value of an analytic function

$$\underline{T}^c(s+io) = \left[1 + 2i\,\underline{T}(s+io)\begin{pmatrix}0 & 0\\ 0 & h(s+io)\end{pmatrix}\right]^{-1}\underline{T}(s+io) \tag{2.64}$$

where

$$\underline{T}^c(s \mp io) = \underline{T}(s \pm io) \tag{2.65}$$

and so $\underline{T}^c$ is an analytic continuation of $\underline{T}$ through the two-particle branch cut and connecting two Riemann sheets. The elements of $\underline{T}$ have the following continuations

$$G^c(z) = G(z) - \frac{2i[\mathcal{V}(z)]^2 h(z)}{1+2i\,h(z)F_0(z)}, \qquad \mathcal{V}^c(z) = \frac{\mathcal{V}(z)}{1+2i\,h(z)F_0(z)},$$

$$F_0^c(z) = \frac{F_0(z)}{1+2i\,h(z)F_0(z)} \tag{2.66}$$

and now we see that the correct analytic definition for $\rho_L(z)$ is

$$\rho_L(z) = \frac{1}{\pi} h(z) \, v(z) \, v^c(z) \xrightarrow[z \to K^2 + i\epsilon]{} \rho_L(K^2) \tag{2.67}$$

where K^2 is in the two-particle cut region. We also have

$$\rho_L(z) = - \left[\rho(z^*)\right]^* \tag{2.68}$$

so $\rho_L(K^2 - i\epsilon) \longrightarrow -\rho_L(K^2)$ as $\epsilon \to 0$. The functions $\rho_L(z)$, $G^c(z)$ have essentially the same cuts as $F_0(z)$ with a square root two-particle branch point. We take as the continuation of $G(z)$ the function

$$G^c(z) = G(z) - 2\pi i \, \rho_L(z) \tag{2.69}$$

If we wish to look for poles of $G^c(z)$ it would seem easier to look for zeros of the inverse $\left[G^c(z)\right]^{-1}$ if we can define such a function.

From (2.69) we obtain

$$\begin{aligned} \left[G^c(z)\right]^{-1} &= G^{-1}(z) + 2\pi i \, \rho_L(z) \, G^{-1}(z) \left[G^c(z)\right]^{-1} \\ &= \frac{G^{-1}(z)}{1 - 2\pi i \, \rho_L(z) \, G^{-1}(z)} \end{aligned} \tag{2.70}$$

Define another function $s_L(z)$ by

$$\left[G^c(z)\right]^{-1} = G^{-1}(z) + 2\pi i \, z \, s_L(z)$$

or

$$s_L(z) = \frac{\rho_L(z) \left[G^{-1}(z)\right]^2}{z \left[1 - 2\pi i \, \rho_L(z) \, G^{-1}(z)\right]} \tag{2.71}$$

Now

$$\left[G^c(K^2-i\epsilon)\right]^{-1} = G^{-1}(K^2+i\epsilon) + \left[G^{-1}(K^2-i\epsilon) - G^{-1}(K^2+i\epsilon) + 2\pi i(K^2-i\epsilon)s_L(K^2-i\epsilon)\right]$$

and so

$$2\pi i(K^2-i\epsilon)s_L(K^2-i\epsilon) \longrightarrow G^{-1}(K^2+i\epsilon) - G^{-1}(K^2-i\epsilon)$$

$$\longrightarrow -2\pi i K^2 s_L(K^2) \quad \text{as } \epsilon \to 0$$

therefore $$s_L(K^2-i\epsilon) \longrightarrow -s_L(K^2) \quad \text{as } \epsilon \to 0 \tag{2.72}$$

and thus (2.71) defines an analytic continuation of $s_L(K^2)$. From (2.71) we can also show

$$s_L(K^2+i\epsilon) = -\left[s_L(K^2-i\epsilon)\right]^* \to s_L(K^2) \quad \text{as } \epsilon \to 0 \tag{2.73}$$

We are now in a position to look for poles of $G^c(z)$ or zeros of

$$\left[G^c(z)\right]^{-1} = \lambda_W^2 + \lambda_S^2 - z - z\int_{K_W^2}^{\infty} \frac{s_W(K^2)dK^2}{K^2-z} - z\int_{K_S^2}^{\infty} \frac{s_S(K^2)dK^2}{K^2-z} + 2\pi i z\, s_L^W(z) \tag{2.74}$$

We now seek zeros in the lower half z-plane of $\left[G^c(z)\right]^{-1}$ close to the real axis in the region between K_W^2 and the next branch point K_1^2. The position of such a zero z_0 and continuation into the first unphysical sheet of $G^{-1}(z)$ exposed by the clockwise rotation of the cut from $K^2 = K_W^2$ is illustrated by Fig. II-6.

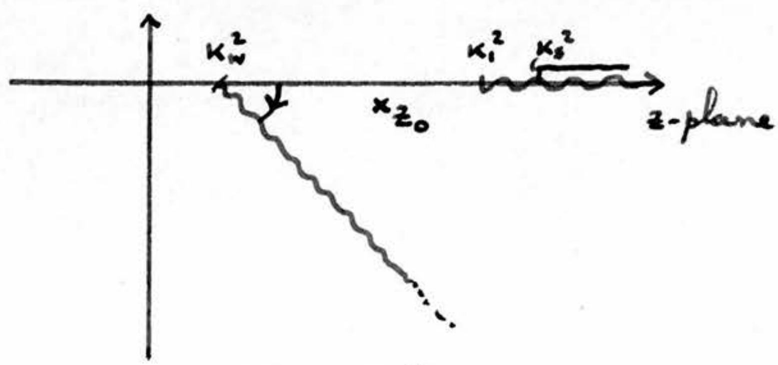

Fig. II-6

Put $z = x - iy$ where y is very small and $K_W^2 < x < K_1^2$ and put $\left[\overset{c}{G}(z)\right]^{-1} = 0$, so taking real and imaginary parts

$$\lambda_s^2 + \lambda_w^2 - x - x\int_{K_w^2}^{\infty} \frac{(\kappa^2 - x)s_w(\kappa^2)d\kappa^2}{(\kappa^2-x)^2 + y^2} + y^2\int_{K_w^2}^{\infty}\frac{s_w(\kappa^2)d\kappa^2}{(\kappa^2-x)^2+y^2} - x\int_{K_s^2}^{\infty}\frac{(\kappa^2-x)s_s(\kappa^2)d\kappa^2}{(\kappa^2-x)^2+y^2}$$

$$+ y^2\int_{K_s^2}^{\infty}\frac{s_s(\kappa^2)d\kappa^2}{(\kappa^2-x)^2+y^2} - 2\pi x \, Im\, s_w^L(x-iy) + 2\pi y \, Re\, s_w^L(x-iy) = 0 \qquad (2.75)$$

$$y + y\int_{K_w^2}^{\infty}\frac{\kappa^2 s_w(\kappa^2)d\kappa^2}{(\kappa^2-x)^2+y^2} + y\int_{K_s^2}^{\infty}\frac{\kappa^2 s_s(\kappa^2)d\kappa^2}{(\kappa^2-x)^2+y^2} + 2\pi x \, Re\, s_w^L(x-iy)$$

$$+ 2\pi y \, Im\, s_w^L(x-iy) = 0 \qquad (2.76)$$

but if y is very small, which is necessary to have a sharp resonance at all, then we approximate (2.75) and (2.76) to get

$$\lambda_s^2 + \lambda_w^2 - x - x\,P\!\!\int_{K_w^2}^{\infty}\frac{s_w(\kappa^2)d\kappa^2}{\kappa^2-x} - x\int_{K_s^2}^{\infty}\frac{s_s(\kappa^2)d\kappa^2}{\kappa^2-x} = 0$$

$$y\left[1 + P\!\!\int_{K_w^2}^{\infty}\frac{\kappa^2 s_w(\kappa^2)d\kappa^2}{(\kappa^2-x)^2} + \int_{K_s^2}^{\infty}\frac{\kappa^2 s_s(\kappa^2)d\kappa^2}{(\kappa^2-x)^2}\right] - \pi x\, s_w^L(x) = 0 \qquad (2.77)$$

where we have used (2.72) and assumed $Im\, s_W^L(x - iy) \lesssim y$ for sufficiently small y. Hence

$$x = \lambda_s^2 + \lambda_w^2 - x\left[P\!\!\int_{K_w^2}^{\infty}\frac{s_w(\kappa^2)d\kappa^2}{(\kappa^2-x)} + \int_{K_s^2}^{\infty}\frac{s_s(\kappa^2)d\kappa^2}{(\kappa^2-x)}\right]$$

$$y = \pi x\, s_w^L(x)\left[1 + P\!\!\int_{K_w^2}^{\infty}\frac{\kappa^2 s_w(\kappa^2)d\kappa^2}{(\kappa^2-x)^2} + \int_{K_s^2}^{\infty}\frac{\kappa^2 s_s(\kappa^2)d\kappa^2}{(\kappa^2-x)^2}\right]^{-1} \qquad (2.78)$$

Therefore we have found a pole of $\overset{c}{G}(z)$ at a point $z = x - iy$ and it is possible to show that $x = \mu^2$, $y = \gamma$ where μ and γ are the mass and lifetime values found by Schwinger and given in (2.47), (2.49). We need only use $G_s^{-1}(\mu_s^2) = 0$, and then the first equation in (2.78) becomes

$$(x-\mu_s^2)\left[1+\int_{K_s^2}^{\infty}\frac{K^2 s_s(K^2)\,dK^2}{(K^2-\mu_s^2)(K^2-x)}\right] = \lambda_w^2 - x\,P\!\!\int_{K_w^2}^{\infty}\frac{s_w(K^2)\,dK^2}{K^2-x} = (\mu^2-\mu_s^2)(\rho_0^s)^{-1} \qquad (2.79)$$

If we use (2.34) and neglect second and higher powers of $(x - \mu_s^2)$ with respect to the first power, we obtain $x = \mu^2$ and $y = \Upsilon$ since

$$\rho_0^{-1} = 1 + P\!\!\int_{K_w^2}^{\infty}\frac{K^2 s_w(K^2)\,dK^2}{(K^2-x)^2} + \int_{K_s}^{\infty}\frac{K^2 s_s(K^2)\,dK^2}{(K^2-x)^2} \qquad (2.80)$$

follows from the definition of ρ_0^{-1} in (2.47) and (2.34).

From the form of $\overset{c}{G}(z)$ this pole at $z = \mu^2 - i\Upsilon$ must occur only in $\rho_L^W(z)$ since $G(z)$ has no complex poles. Now (2.68) implies that $\rho_L^W(z)$ has a pole at the complex conjugate point $z = \mu^2 + i\Upsilon$. These poles at $z = \mu^2 \pm i\Upsilon$ in $\rho_L^W(K^2)$ are present in the $|V(K^2 + io)|^2$ factor in the definition of $\rho_L(K^2)$ in (2.63), and so are independent of the square root branch point at K_W^2. If we exhibit the cuts and singularities of $\rho_L^W(K^2)$ and rotate the cut from $K^2 = K_W^2$ away from the region $K^2 \geq K_W^2$ we find with $z_o = \mu^2 - i\Upsilon$

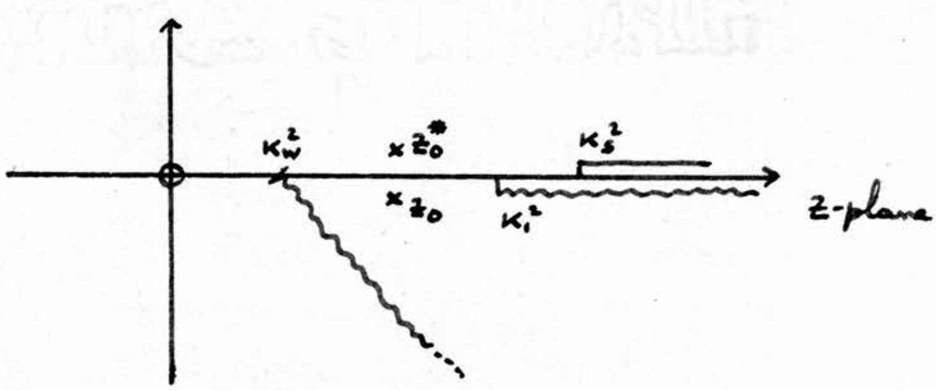

Fig. II-7.

Therefore, for $z \approx \mu^2$, the poles in $\rho_L^W(z)$ exert their effects simultaneously and we can write

$$\rho_L^W(z) \approx \frac{\rho_0}{\pi}\cdot\frac{\gamma_0}{(z-z_0)(z-z_0^*)} = \frac{\rho_0}{2\pi i}\left[\frac{1}{z-z_0^*} - \frac{1}{z-z_0}\right] \qquad (2.81)$$

where $s \approx \mu^2$ and ρ_o contains the factor $h(\mu^2 + io) \approx h(s)$. For $s = K^2 + i\epsilon$ the formula (2.81) is identical with the resonance formula (2.48).

It is also possible to see that $\left[G^c(s)\right]^{-1}$ must vanish at $s = \mu^2 + i\Upsilon$ since instead of using (2.72) in (2.77) we must use (2.73) which involves a change of sign in the value of y in (2.78). Rotating the cut from $K^2 = K_W{}^2$ in Fig. II-6 through nearly 360^o we find the two poles for $G^c(s)$ as shown in Fig. II-8.

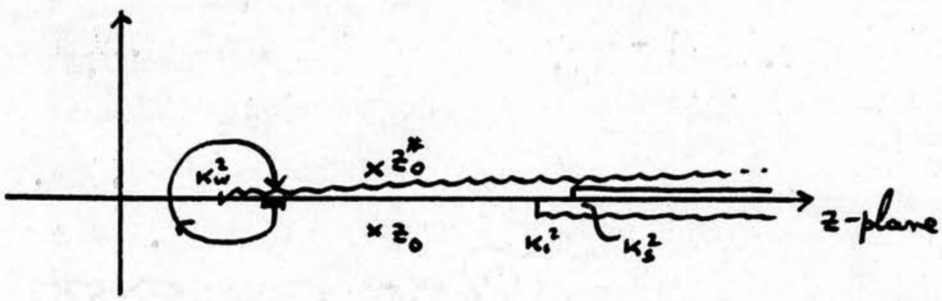

Fig. II-8.

That there may be two complex conjugate poles was first predicted by Gunson and Taylor[10] who reasoned that $\mathcal{V}^c(s)$ has the form given in (2.66). The denominator is a real function and must have complex conjugate zeros if it has zeros at all. We have therefore illustrated the truth of this prediction for a propagator $G(s)$ or vertex function $\mathcal{V}(s)$ or scattering amplitude $F_o(s)$ which have Mandelstam-type analytic properties. Unfortunately insufficient is yet known about the analytic properties of $\mathcal{V}(s)$, $F_o(s)$ in full field theory. Clearly a wider knowledge of such analytic properties will be of great interest, even though Schwinger's methods of finding resonances due to weak interactions manages to avoid such difficulties.

Levy's Ambiguous Poles

We would like to use an extension of the model we introduced in the last section, to examine an ambiguity, found by Levy,[6] in defining the correct pole to assign to an unstable particle. In spite of the fact that there may be two complex conjugate poles for each resonance, there is no confusion in defining the real and imaginary parts of these poles as the mass and lifetime of an unstable particle. When we speak of the pole we shall always infer the pole in the lower half plane of the first unphysical sheet.

Levy considers the case when a particle has two different modes of decay. We shall therefore consider that there are two weak interaction thresholds of a two-particle nature below the strong interaction threshold. We picture $G(-k^2)$ as

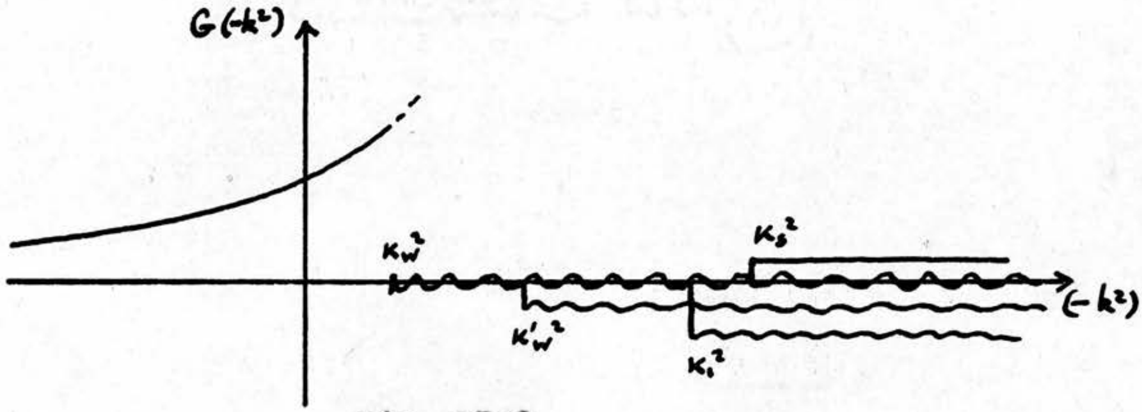

Fig. II-9.

Obviously Schwinger's arguments can be followed through for μ_s^2 in either of the regions $K_W^2 < -k^2 < K'^2_W$ and $K'^2_W < -k^2 < K_s^2$, which we will call region (1) and region (2) respectively for convenience. If the branch points K_W^2, K'^2_W are the lowest branch points and are two-particle branch points, then we can look for poles on the unphysical sheets with a model similar to the one used in the last section. However, the problem is very simply examined by Schwinger's methods. All we wish to determine is the possible lifetimes to be associated with unstable particles with masses in each of the regions (1) and (2). Let $\left[-2\pi i k^2\, s_W(-k^2)\right]$ and $\left[-2\pi i\, k^2\, s'_W(-k^2)\right]$ be the discontinuities of $G^{-1}(-k^2)$ across the cuts in regions (1) and (2) respectively. In region (1) we obtain an inverse lifetime of the form $\left[\pi\mu\rho_o s_W(\mu^2)\right]$ but with a resonance in region (2) we have an inverse lifetime of the form $\left[\pi\mu\rho_o\, s'_W(\mu^2)\right]$. These definitions are independent of poles found by various continuations. But if we assume suitable analytic properties and look for poles by continuing through region (1) as in Fig. II-10

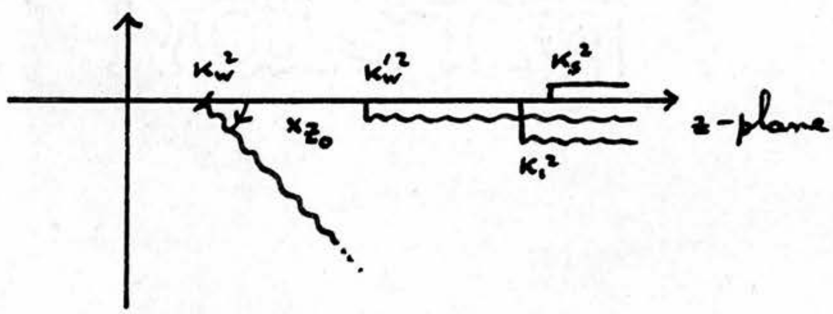

Fig. II-10

we find an imaginary part of the form $\left[\pi\mu\rho_o\, s_W(\mu^2)\right]$ if $\mu^2 < K_W'^2$. On the other hand if $\mu^2 > K_W'^2$, Levy considers two different continuations shown in Fig. II-11 and Fig. II-12.

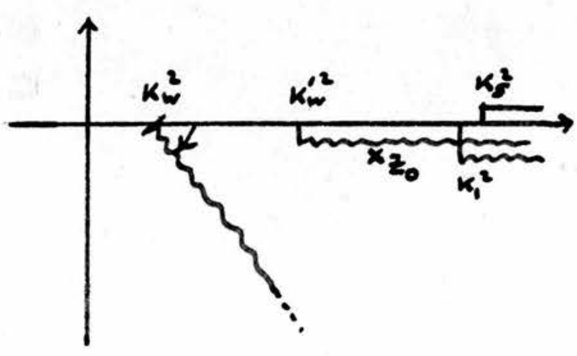

Fig II-11

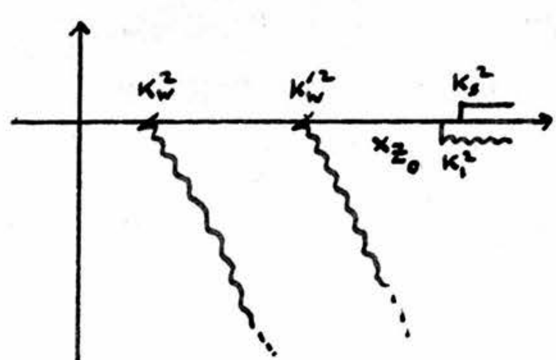

Fig II-12

where we continue through region (1) in Fig. II-11 and find, as did Levy, a pole with imaginary part $[\pi\mu\rho_o(2s_W(\mu^2) - s'_W(\mu^2))]$, while we continue through region (2) in Fig. II-12 and find a pole with imaginary part $[\pi\mu\rho_o s'_W(\mu^2)]$. Levy reasons that, since $s'_W(\mu^2)$ may be larger than $2s_W(\mu^2)$, the first continuation in Fig. II-11 finds a pole with an unphysical negative lifetime and only the second continuation in Fig. II-12 finds the correct pole for $\mu^2 > K'^2_W$. Thus in looking for resonance poles we may have to reject some for physical reasons.* Gunson and Taylor[10)] have said that a more detailed examination of the extended Lee Model shows that the spurious pole actually coincides with the physical pole. The author finds this statement hard to believe in view of the following analysis of the polology of an extension of our analytic model. To do this we add three more reactions to those we gave earlier so that we have two two-particle branch cuts as the lowest energy cuts. The extra reactions required have the forms

R IV	$k_1' + k_2' \rightarrow p$	$(\mathcal{V}')$
R V	$k_1' + k_2' \rightarrow k_1' + k_2'$	(F')
R VI	$k_1 + k_2 \rightarrow k_1' + k_2'$	(H)

where we denote R IV by an invariant 'vertex' function $\mathcal{V}'(z)$, R V by the s-wave projection $F_o'(z)$ of an invariant elastic scattering amplitude $F'(z, \cos\theta)$ and R VI by the s-wave projection $H_o(z)$ of an invariant centre-of-mass amplitude $H(z, \cos\theta)$.**

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* Example given by Levy is the decay of charged pions $\pi \rightarrow \mu + \nu$, $\pi \rightarrow e + \nu$ where $\mathcal{V}_\mu(m_\pi^2) \gg \mathcal{V}_e(m_\pi^2)$.

** Notation and following analysis is very similar to that used by Oehme[11)].

The new matrix of all the reactions can be represented by

$$\underline{T}' = \begin{pmatrix} G & \upsilon & \upsilon' \\ \upsilon & F_0 & H_0 \\ \upsilon' & H_0 & F_0' \end{pmatrix} \tag{2.82}$$

Unitarity gives a relation of the same form as (2.60) for $\underline{T}'$ but in place of the matrix $\begin{pmatrix} 0 & 0 \\ 0 & h(z) \end{pmatrix}$ we must write

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & h(z) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for } (m_1 + m_2)^2 \leqslant z \leqslant (m_1' + m_2')^2$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & h(z) & 0 \\ 0 & 0 & h'(z) \end{pmatrix} \quad \text{for } (m_1' + m_2')^2 \leqslant z \leqslant \text{next threshold.}$$

where

$$h'(z) = \frac{\left[z-(m_1'+m_2')^2\right]^{1/2}\left[z-(m_1'-m_2')^2\right]^{1/2}}{z} = -\left[h'(z^*)\right]^* \tag{2.83}$$

and m_1', m_2' are the masses associated with the higher two-particle branch cut. The above replacements give the corresponding formula to (2.64) for the continuations of $\underline{T}'$ to $\underline{T}'^c$ and $\underline{T}'^{cc}$ depending on whether z is greater or less than $(m_1' + m_2')^2$. Let us list the various continuations of the elements. For region (1) we again obtain the results of (2.66) plus the following

$$\upsilon'^c(z) = \upsilon'(z) - \frac{2i\,h(z)H_0(z)\upsilon(z)}{1+2i\,h(z)F_0(z)}\,, \quad H_0^c(z) = \frac{H_0(z)}{1+2i\,h(z)F_0(z)}$$

$$F_0'^c(z) = F_0'(z) - \frac{2i\,h(z)\left[H_0(z)\right]^2}{1+2i\,h(z)F_0(z)} \tag{2.84}$$

For region (2) below the next threshold

$$G^{cc}(z) = G(z) - \frac{2i}{\Delta}\Big\{ h(z)[\upsilon(z)]^2(1+2ih'(z)F_0'(z)) +$$

$$+ h'(z)[\upsilon'(z)]^2(1+2ih(z)F_0(z)) - 4ih(z)h'(z)\upsilon(z)\upsilon'(z)H_0(z)\Big\}$$

$$\upsilon^{cc}(z) = \frac{1}{\Delta}\Big\{\upsilon(z)(1+2ih'(z)F_0'(z)) - 2ih'(z)\upsilon'(z)H_0(z)\Big\}$$

$$\upsilon'^{cc}(z) = \frac{1}{\Delta}\Big\{\upsilon'(z)(1+2ih(z)F_0(z)) - 2ih(z)\upsilon(z)H_0(z)\Big\}$$

$$F_0^{cc}(z) = \frac{1}{\Delta}\Big\{F_0(z)(1+2ih'(z)F_0'(z)) - 2ih'(z)[H_0(z)]^2\Big\}$$

$$F_0'^{cc}(z) = \frac{1}{\Delta}\Big\{F_0'(z)(1+2ih(z)F_0(z)) - 2ih(z)[H_0(z)]^2\Big\}$$

$$H_0^{cc}(z) = H_0(z)/\Delta$$

$$\Delta = (1+2ih(z)F_0(z))(1+2ih'(z)F_0'(z)) + 4h(z)h'(z)[H_0(z)]^2 \qquad (2.85)$$

For completeness we mention that we can reach only one more Riemann sheet, by continuing through either one of the two two-particle branch cuts and then through the other. The new functions on this sheet have the forms

$$G^{ccc}(z) = G(z) - \frac{2ih'(z)[\upsilon'(z)]^2}{1+2ih'(z)F_0'(z)}\,, \quad \upsilon'^{ccc}(z) = \frac{\upsilon'(z)}{1+2ih'(z)F_0'(z)}$$

$$\upsilon^{ccc}(z) = \upsilon(z) - \frac{2ih'(z)\upsilon'(z)H_0(z)}{1+2ih'(z)F_0'(z)}\,, \quad F_0'^{ccc}(z) = \frac{F_0'(z)}{1+2ih'(z)F_0'(z)}$$

$$F_0^{ccc}(z) = F_0(z) - \frac{2ih'(z)[H_0(z)]^2}{1+2ih'(z)F_0'(z)}\,, \quad H_0^{ccc}(z) = \frac{H_0(z)}{1+2ih'(z)F_0'(z)}$$

$$(2.86)$$

The correct analytic continuation for $\rho(K^2)$ in the upper two particle branch cut region (2) is therefore the function

$$\rho_L'(z) = \frac{1}{\pi\Delta}\Big\{h(z)[\mathcal{V}(z)]^2(1+2ih'(z)F_o'(z)) + h(z)[\mathcal{V}'(z)]^2(1+2ih(z)F_o(z)) - 4ih(z)h'(z)\mathcal{V}(z)\mathcal{V}'(z)H_o(z)\Big\}$$
$$= -\left[\rho_L'(z^*)\right]^* \qquad (2.87)$$

Poles of $G(z)$ or $\mathcal{V}(z)$, $\mathcal{V}'(z)$, $F_o(z)$, $F_o'(z)$, $H_o(z)$ on the first unphysical sheets to be reached through the lower and upper two-particle branch cuts, i.e. regions (1) and (2), occur in complex conjugate pairs and correspond to zeros of $(1 + 2ih(z)F_o(z))$ and Δ respectively. Poles on the only other unphysical sheet are zeros of $(1 + 2ih'(z)F_o'(z))$. The zeros of Δ cannot coincide with zeros of $(1 + 2ih(z)F_o(z))$ or $(1 + 2ih'(z)F_o'(z))$ unless we decouple $F_o(z)$ and $F_o'(z)$ i.e. unless $H_o = 0$ when we have simply $\Delta = (1 + 2ih(z)F_o(z))(1 + 2ih'(z)F_o'(z))$. It is specially interesting to find that ρ_L' has poles at the zeros of $(1 + 2ih(z)F_o(z))$ <u>and</u> $(1 + 2ih'(z)F_o'(z))$ when $H_o = 0$. In fact, if $H_o \approx 0$ we can write to first order in H_o

$$\rho_L'(z) \approx \frac{1}{\pi}\left\{\frac{h(z)[\mathcal{V}(z)]^2}{1+2ih(z)F_o(z)} + \frac{h'(z)[\mathcal{V}'(z)]^2}{1+2ih'(z)F_o'(z)} - \frac{4ih(z)h'(z)\mathcal{V}(z)\mathcal{V}'(z)H_o(z)}{(1+2ih(z)F_o(z))(1+2ih'(z)F_o'(z))}\right. \qquad (2.88)$$

which shows a separation of the two decay modes which therefore could be discussed as independent processes. The two lifetimes obtained by discussing the separate contributions of each decay process to $\rho_L'(z)$ are known as the partial lifetimes. The separation implies that the sum over the set of states in (2.9) can be split into a sum over each of two subsets, one from each of the two decay processes. Because the modes are independent we are allowed to identify the partial lifetimes and discuss them separately and further deducing that an unstable particle and its anti-particle have equal partial lifetimes as well as equal total or physical lifetimes. If H_o is not very small i.e. if the coupling between the two decay modes is not negligible, $\rho_L'(z)$ does not exhibit a separation between the contributions of the two modes. We can no longer discuss any one mode independent of the other and we cannot deduce that the partial lifetimes of a particle and its anti-particle are equal but only that their physical lifetimes are equal. In these conclusions we are in agreement with a very general argument given by Metthews and Salam.[4)]

Let us return to the major discussion of this section. Poles close to region (1) found by continuation of $G(z)$ through region (1) shown in Fig. II-10 have already been discussed and we found imaginary parts* of the form $\left[\pi\mu\rho_o s_W(\mu^2)\right]$. Suppose we continue $G(z)$ through region (2), below the next threshold $K_1{}^2 > K'^2_W > K_W{}^2$, and look for poles near region (1) again as shown in Fig. II-13.

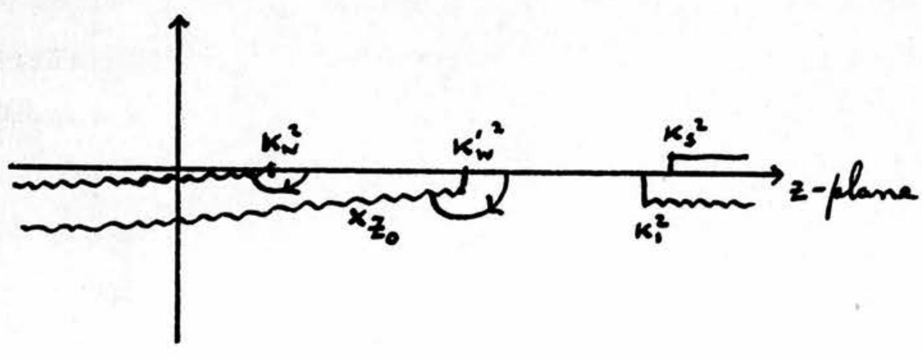

Fig. II-13.

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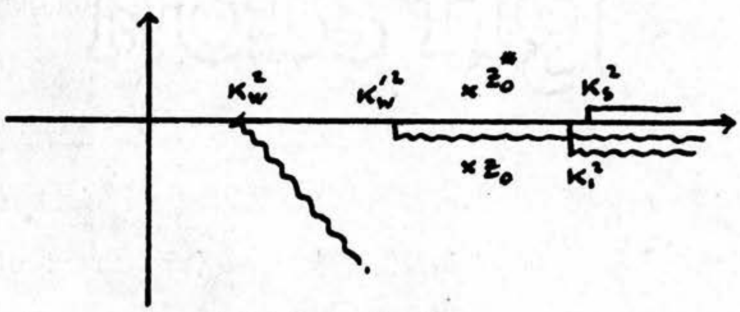
* The real part is obtained from $\mathcal{R}\ell\, G^{-1}(z) = 0$ and is independent of the Riemann sheet we are considering and therefore we quote the imaginary part as being more characteristic of a complex pole.

We find a pole with imaginary part of the form $\left[\pi\mu\rho_o(2s'_W(\mu^2) - s_W(\mu^2))\right]$. Thus we have found a pole near region (1) by continuing to the left of K'^2_W and a pole by continuing to the right of K'^2_W, but although their real parts may coincide their imaginary parts will not coincide unless $s'_W(\mu^2) = s_W(\mu^2)$ which is not true in general. Similarly if we avoid region (1) and continue through region (2) we find that a pole close to region (2) may have an imaginary part of the form $\left[\pi\mu\rho_o s'_W(\mu^2)\right]$. If we decide to continue through region (1) to look for a pole near region (2) we will find an imaginary part of the form $\left[\pi\mu\rho_o(2s_W(\mu^2) - s'_W(\mu^2))\right]$ which is not equal to $\left[\pi\mu\rho_o s'_W(\mu^2)\right]$ in general. However there is in general no ambiguity in choosing the correct pole to associate with an unstable particle. If the pole is close to region (1) it would be illogical to continue through a region above the next branch cut to look for the pole and similarly if the pole is close to region (2) it seems natural to continue through region (2) to find it. The only situation when there may be some ambiguity is when the real part of the pole is close to the threshold at K'^2_W. But in this case it seems reasonable on physical grounds to expect that $s_W(\mu^2) \approx s'_W(\mu^2)$ if $\mu^2 \approx K'^2_W$ and if this is so then all the unphysical sheet poles will be nearly coincident, and any continuation will do. If, as Levy suggested by an example given earlier, $s'_W(\mu^2) > 2s_W(\mu^2)$ then the imaginary part of a possible pole near region (2) on a sheet found by continuing through region (1) will have the wrong sign for the second equation in (2.77) to be true and a pole is no longer possible. There would be no possibility of ambiguity in such a case either. Any poles found by continuing through one region and looking for poles close to another only indicate that resonance poles exist at the same real value on another unphysical sheet. Therefore we believe that although many poles may be associated with an unstable particle only one complex conjugate pair <u>directly</u> produces a resonance in the physical sheet and is found by continuation through a restricted region of the physical cut near the resonance. In the last section we showed in Fig. II-7 how the cut in $\rho_L^W(z)$ can be rotated so that the effects of poles in $\rho_L^W(z)$ can be exhibited by (2.81) as a resonance for $z \approx \mu^2$. If $\rho_L^W(z)$ has poles near region (2) found by continuation through region (1) then we can rotate the cut from K_W^2 out of the way but not the cut from K'^2_W unless we continue through this cut as well. We illustrate the continuation through region (1) by Fig. II-14.



<u>Fig. II-14</u>

showing singularities and cuts of $\left[\rho_W(z) + \rho'_W(z)\right]$. We cannot now write a formula like (2.81) unless we continue through the cut from K'^2_W and thereby finding the correct poles directly producing a resonance. This shows that any pair of complex conjugate poles associated with an unstable particle produces a resonance in Im G(z) either directly or indirectly without any ambiguity.

Vector Bosons

So far we have restricted ourselves to scalar or pseudo-scalar bosons but it is a quite straightforward matter to extend the discussion to vector bosons. A spin 1 boson propagator can be reduced, by choosing a divergenceless field and using the PCT theorem or axiom VI, to

$$D_{\mu\nu}(-k^2) = \int_0^\infty \frac{\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{\kappa^2}\right)\rho(\kappa^2)\,d\kappa^2}{\kappa^2 + k^2 - i\epsilon}$$

$$= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\int_0^\infty \frac{\rho(\kappa^2)\,d\kappa^2}{\kappa^2 + k^2 - i\epsilon} + \frac{k_\mu k_\nu}{k^2}\int_0^\infty \frac{\rho(\kappa^2)\,d\kappa^2}{\kappa^2} \qquad (2.89)$$

where $\rho(K^2)$ is given by an expression of the same form as that in equation (2.9). We could clearly consider $D_{\mu\nu}(-k^2)$ in an almost identical fashion to the spin 0 propagator, $G(-k^2)$.

CHAPTER III

THE FERMION PROPAGATOR

We propose to extend the methods of the last chapter to particles of spin $\frac{1}{2}$. This generalisation is sufficiently different in detail to require a thorough discussion.

If we only use the axioms I to V in the introduction, the fermion propagator has a more complicated character than the familiar result obtained by Kallen and Lehmann.[18] This is because the derivation by Kallen and Lehmann invokes explicitly charge conjugation invariance and implicitly space inversion invariance. Here we shall require only PCT invariance or axiom VI since decay processes have little respect for invariance under transformations with P, C or T separately. We have the following definition of the fermion propagator with $\psi_\alpha(x)$ as the fermion Heisenberg field operator

$$\frac{i}{2} S'_{F\alpha\beta}(x) = -i\langle 0|T[\psi_\alpha(x)\bar{\psi}_\beta(0)]|0\rangle = \frac{i}{(2\pi)^4}\int_{-\infty}^{\infty} d^4p\, e^{ipx} S'_{F\alpha\beta}(p) \qquad (3.1)$$

and the PCT theorem allows the assertion

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(0)|0\rangle = -(\gamma_5)_{\alpha\alpha'}\langle 0|\bar{\psi}_{\beta'}(0)\psi_{\alpha'}(x)|0\rangle(\gamma_5)_{\beta'\beta} \qquad (3.2)^*$$

Now define

$$f_{\alpha\beta}(p) = \sum_n \langle 0|\psi_\alpha(0)|p,n\rangle\langle p,n|\bar{\psi}_\beta(0)|0\rangle \qquad (3.3)$$

then it follows from (3.3) that

---

* We have chosen a hermitian system of γ matrices so that $\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu}$ and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. Further we define

$$\left[f_{\alpha\beta}(p) \right]^{*} = \left[\gamma_4 f(p) \gamma_4 \right]_{\alpha\beta} \tag{3.4}$$

and we may expand (3.3) as a series involving the sixteen linearly independent matrices formed from the γ-matrices and their products. With the help of relativistic invariance properties we have

$$f_{\alpha\beta}(p) = -\frac{1}{(2\pi)^3}\Big[(i\gamma.p - \sqrt{-p^2})_{\alpha\beta}\,\rho_1(-p^2) + \delta_{\alpha\beta}\,\rho_2(-p^2) + i(\gamma_5)_{\alpha\beta}\,\rho_3(-p^2) + $$
$$+ i(\gamma_5\gamma.p)_{\alpha\beta}\,\rho_4(-p^2) + \tfrac{1}{2}(\sigma^{\mu\nu})_{\alpha\beta}\,p_\mu p_\nu\,\rho_5(-p^2)\Big] \tag{3.5}$$

where $\sigma^{\mu\nu} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu$ and therefore $\sigma^{\mu\nu} p_\mu p_\nu \equiv 0$ so we can drop the last term in (3.5). Applying (3.4) to (3.5) gives

$$\left[\rho_j(-p^2)\right]^{*} = \rho_j(-p^2) \quad ; \quad j = 1, 2, 3, 4. \tag{3.6}$$

and from axiom (V) it has clearly been shown by Lovitch and Tomozawa[20] that

$$\sqrt{-p^2}\,\rho_1 \geq \left[(\rho_2 - \sqrt{-p^2}\,\rho_1)^2 + \rho_3^2 + \rho_4^2\right]^{1/2} \tag{3.7}$$

If we now insert a sum over a complete set of Heisenberg states into (3.1) and use (3.2), (3.3) and (3.5) we obtain for the fermion propagator in momentum space

$$\left(S'_F(p)\right)_{\alpha\beta} = \int_0^\infty \frac{\left[(i\gamma.p-\kappa)_{\alpha\beta}\,\rho_1(\kappa^2) + \delta_{\alpha\beta}\,\rho_2(\kappa^2) + i(\gamma_5)_{\alpha\beta}\,\rho_3(\kappa^2) + i(\gamma_5\gamma.p)_{\alpha\beta}\,\rho_4(\kappa^2)\right]d\kappa^2}{p^2+\kappa^2-i\epsilon} \tag{3.8}$$

where $\kappa = \sqrt{\kappa^2} > 0$. Once more we will assume that the spectrum does not contain the origin. In deriving (3.8) we use the well-known Kallen-Lehmann methods based only on the axioms I to V and axiom VI or the PCT theorem (3.2).* As for the boson propagator we assume we can normalise $\rho_1(K^2)$ such that

$$\int_0^\infty \rho_1(\kappa^2)\,d\kappa^2 = 1 \tag{3.9}$$

which certainly follows from using the equal time canonical commutation rules in a field theory derived from a local Lagrangian.

Let us consider the matrix elements of $(S'_F(p))_{\alpha\beta}$ with respect to positive energy spinors u_1 and u_2 such that

$$(i\gamma.p + m)\,u_j = 0 \quad , \quad \bar{u}_j u_j = 1 \quad ; \quad j = 1, 2. \tag{3.10}$$

and we note the results for $-p^2 = m^2$.

$$\left.\begin{aligned} &\bar{u}_j\,(i\gamma.p)\,u_j = -m \\ &\bar{u}_j\,(i\gamma_5)\,u_j = \bar{u}_j\,(i\gamma_5\gamma.p)\,u_j = 0 \end{aligned}\right\} \quad j = 1, 2. \tag{3.11}$$

It is interesting to note that the parity non-conserving terms disappear** and we have

$$\delta(m) = \bar{u}_j S'_F(p)\,u_j = \int_0^\infty \frac{\rho_2(\kappa^2) - (\kappa + m)\,\rho_1(\kappa^2)}{\kappa^2 - m^2 - i\epsilon}\,d\kappa^2 \quad ; \quad j = 1, 2. \tag{3.12}$$

and further $\bar{u}_1 S'_F(p)\,u_2 = \bar{u}_2 S'_F(p)\,u_1 = 0$.

Also with negative energy spinors v_1 and v_2 where

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* Again Lovitch and Tomozawa[20] have shown that the use of the PCT theorem can be replaced by the use of axioms VI.

** This particular point has been made by Ida[15].

$$(i\gamma.p - m)\, v_j = 0 \,, \quad \bar{v}_j v_j = -1 \quad ; \; j = 1, 2. \tag{3.13}$$

and we find

$$\begin{aligned} \bar{v}_j S'_F(p)\, v_j &= -\int_0^\infty \frac{\rho_2(\kappa^2) - (\kappa - m)\rho_1(\kappa^2)}{\kappa^2 - m^2 - i\epsilon}\, d\kappa^2 \quad ; \; j = 1, 2. \\ &= -\mathcal{S}(m) \end{aligned} \tag{3.14}$$

and $\bar{v}_1 S'_F(p)\, v_2 = \bar{v}_2 S'_F(p)\, v_1 = 0.$

Since we have used the PCT theorem then the 'anti-fermion' propagator $S'_F(-p)$ has the same expectation values i.e. $\bar{v}\, S'_F(-p)\, v = \mathcal{S}(m)$ and $\bar{u}\, S'_F(-p)\, u = \mathcal{S}(m)$, Therefore the fermion and anti-fermion propagators have identical properties. We have reduced the discussion of the fermion propagators to an analysis of two functions $\mathcal{S}(m)$ and $\mathcal{S}(-m)$ which we have expressed in terms of two spectral functions ρ_1 and ρ_2. The latter are real and positive and obey the relation obtained from (3.7)

$$0 \leq \rho_2(\kappa^2) \leq 2\kappa\, \rho_1(\kappa^2) \tag{3.15}$$

We can consider $\mathcal{S}(m)$ and $\mathcal{S}(-m)$ as boundary values of an analytic function except for a possible cut from 0 to ∞ on the real axis and a square root cut from 0 to $-\infty$ on the real axis in the complex m^2-plane. Therefore consider the function

$$\mathcal{S}(z) = \int_0^\infty \frac{\rho_2(\kappa^2) - (\kappa + \sqrt{z})\,\rho_1(\kappa^2)}{\kappa^2 - z}\, d\kappa^2 \qquad (3.16)$$

where we do not allow the possibility of a zero mass intermediate state and so the right hand cut will have a threshold at some point greater than zero. Then we have a gap between the cuts and we can continue from upper to lower half planes of the physical sheet. It is straightforward to obtain the results

$$\left[\mathcal{S}(z^*)\right]^* = \mathcal{S}(z) \qquad (3.17)$$

$$\lim_{z \to m^2 + i\epsilon} \mathcal{S}(z) = \mathcal{S}(m) \text{ or } \mathcal{S}(-m) \qquad (3.18)$$

where the latter result depends on which branch of the square root $\sqrt{z}$ we are on. The two sheets of the Riemann surface and the branch cut are defined by $\sqrt{z} \rightarrow m > 0$ as $z \rightarrow m^2 > 0$ and by $\sqrt{z} \rightarrow -m < 0$ as $z \rightarrow m^2 > 0$.

The discontinuity of $\mathcal{S}(z)$ across the right hand cut gives* for $m^2 > 0$ using (3.17), (3.18) and (3.16)

$$\begin{aligned} \operatorname{Im} \mathcal{S}(m) &= \pi\left[\rho_2(m^2) - 2m\,\rho_1(m^2)\right] \\ \operatorname{Im} \mathcal{S}(-m) &= \pi\,\rho_2(m^2) \end{aligned} \qquad (3.19)$$

and the discontinuity across the left hand cut gives for $m^2 < 0$

---

* Note that we chose $\kappa = \sqrt{\kappa^2} > 0$ and that this is independent of the square root branch point in the z-plane.

$$
\begin{aligned}
\operatorname{Im} \mathcal{S}(m) &= -|m| \int_0^\infty \frac{\rho_1(\kappa^2)\, d\kappa^2}{\kappa^2 - m^2} \\
\operatorname{Im} \mathcal{S}(-m) &= |m| \int_0^\infty \frac{\rho_1(\kappa^2)\, d\kappa^2}{\kappa^2 - m^2}
\end{aligned}
\tag{3.20}
$$

Again we have assumed that the integrals we deal with converge. We are once more interested in writing a dispersion relation for the inverse of $\mathcal{S}(z)$ i.e. $\mathcal{S}^{-1}(z)$. Let us first examine $\mathcal{S}(z)$ for complex zeros. Consider $z = re^{i\theta}$ and choose $\sqrt{z} = \pm\, me^{i\theta/2}$ where $-\pi < \theta < \pi$, therefore

$$
\operatorname{Im} \mathcal{S}(z) = -\left(\operatorname{Im}\sqrt{z}\right) \int_0^\infty \frac{\left[\left(\kappa^2 \pm 2m\kappa\cos\frac{\theta}{2} + m^2\right)\rho_1(\kappa^2) \mp 2m\cos\frac{\theta}{2}\cdot\rho_2(\kappa^2)\right]}{|\kappa^2 - z|^2}\, d\kappa^2
\tag{3.21}
$$

and from (3.15) it can easily be shown that the numerator of the integral in (3.21) is always positive. Hence $\operatorname{Im} \mathcal{S}(z) = 0$ only if $\operatorname{Im}\sqrt{z} = 0$ or if $\theta = 0$ or z real and positive. If the behaviour of $\rho_1(K^2)$ and $\rho_2(K^2)$ is such that we can apply the Ferrari and Jona-Lasinio[22)] theorem, then

$$
\mathcal{S}(z) \approx \frac{1}{\sqrt{z}} \quad \text{as } z \rightarrow \text{infinite circle}
\tag{3.22}
$$

provided we avoid the cuts on the real axis.

Hence $\left[\left(\sqrt{z}\right)^{-1}\mathcal{S}^{-1}(z) - 1\right]$ has cuts from 0 to $-\infty$ and $0+$ to $+\infty$ on the real axis in the z-plane, no poles, and converges to zero on the infinite circle. Therefore we can write a dispersion relation in the form

$$\mathcal{S}^{-1}(z) = \sqrt{z}\cdot\left(1 + \int_0^\infty \frac{\tau(\kappa)\,d\kappa^2}{\kappa^2 - z} + \int_0^\infty \frac{\chi(\kappa)\,d\kappa^2}{\kappa^2 + z}\right) \tag{3.23}$$

where for the sheet in which $\sqrt{z} \to m > 0$, $z \to m^2 > 0$ (call $\mathcal{S}(z) = \mathcal{S}^{I}(z)$)

$$\tau(\kappa) = \frac{1}{2\pi i\kappa}\cdot\left\{\left[\mathcal{S}^{I}(\kappa^2 + i\epsilon)\right]^{-1} - \left[\mathcal{S}^{I}(\kappa^2 - i\epsilon)\right]^{-1}\right\}$$

$$= \frac{1}{2\pi i\kappa}\cdot\frac{\left[\mathcal{S}^{I}(\kappa^2 - i\epsilon) - \mathcal{S}^{I}(\kappa^2 + i\epsilon)\right]}{|\mathcal{S}^{I}(\kappa^2)|^2}$$

$$= \frac{2\kappa\rho_1(\kappa^2) - \rho_2(\kappa^2)}{\kappa\,|\mathcal{S}^{I}(\kappa^2)|^2} \geq 0 \tag{3.24}$$

and*

$$\chi(\kappa) = \frac{1}{2\pi i}\left\{\frac{\left[\mathcal{S}^{I}(-\kappa^2 - i\epsilon)\right]^{-1}}{(-\kappa^2 - i\epsilon)^{1/2}} - \frac{\left[\mathcal{S}^{I}(-\kappa^2 + i\epsilon)\right]^{-1}}{(-\kappa^2 + i\epsilon)^{1/2}}\right\}$$

$$= \frac{1}{2\pi\kappa}\cdot\left\{\left[\mathcal{S}^{I}(-\kappa^2 - i\epsilon)\right]^{-1} + \left[\mathcal{S}^{I}(-\kappa^2 + i\epsilon)\right]^{-1}\right\}$$

$$= \frac{1}{2\pi\kappa}\cdot\frac{\left[\mathcal{S}^{I}(-\kappa^2 + i\epsilon) + \mathcal{S}^{I}(-\kappa^2 - i\epsilon)\right]}{|\mathcal{S}^{I}(-\kappa^2)|^2}$$

$$= \frac{1}{\pi\kappa\,|\mathcal{S}^{I}(-\kappa^2)|^2}\cdot\int_0^\infty \frac{\left[\rho_2(\kappa'^2) - \kappa'\rho_1(\kappa'^2)\right]d\kappa'^2}{\kappa'^2 + \kappa^2} \tag{3.25}$$

---

* Note the change in sign of the $\frac{1}{\sqrt{z}}$ factor across the square root cut.

and so the sign of $\chi(K)$ is indefinite in the sense that it is dependent on the dynamics of the problem. Also for the sheet in which $\sqrt{z} \longrightarrow -m$, (call $\mathcal{S}(z) = \mathcal{S}^{II}(z)$) $z \to m^2 > 0$

$$\tau(\kappa) = -\frac{1}{2\pi i\kappa}\left\{\left[\mathcal{S}^{II}(\kappa^2+i\epsilon)\right]^{-1} - \left[\mathcal{S}^{II}(\kappa^2-i\epsilon)\right]^{-1}\right\} = \frac{\rho_2(\kappa^2)}{\kappa\,|\mathcal{S}^{II}(\kappa^2)|^2} \geq 0 \tag{3.26}$$

and

$$\chi(\kappa) = \frac{-1}{\pi\kappa\,|\mathcal{S}^{II}(-\kappa^2)|^2}\cdot\int_0^\infty \frac{\left[\rho_2(\kappa'^2) - \kappa'\rho_1(\kappa'^2)\right]d\kappa'^2}{\kappa'^2+\kappa^2} \tag{3.27}$$

which also has an indefinite sign.

We can obtain more information from the derivative of $\mathcal{S}(z)$

$$\frac{d\,\mathcal{S}(z)}{dz} = \frac{1}{2\sqrt{z}}\int_0^\infty \frac{\left[2\sqrt{z}\,\rho_2(\kappa^2) - (\kappa+\sqrt{z})^2\rho_1(\kappa^2)\right]d\kappa^2}{(\kappa^2-z)^2} \tag{3.28}$$

and so for real, positive z

$$\frac{d\,\mathcal{S}^{I}(z)}{dz} \leq -\frac{1}{2\sqrt{z}}\int_0^\infty \frac{(\kappa-\sqrt{z})^2\rho_1(\kappa^2)\,d\kappa^2}{(\kappa^2-z)^2} \tag{3.29}$$

Thus $\frac{d\,\mathcal{S}^{I}(z)}{dz} \leq 0$ for real, positive z and $\frac{d\,\mathcal{S}^{I}(z)}{dz} \to -\infty$ as $z \to 0+$ but $\mathcal{S}(0) = \int_0^\infty \frac{\left[\rho_2(\kappa^2) - \kappa\rho_1(\kappa^2)\right]d\kappa^2}{\kappa^2}$ which may be positive, negative or zero but is certainly finite since

$$-\int_0^\infty \frac{\rho_1(\kappa^2)\,d\kappa^2}{\kappa^2} \leq \mathcal{S}(0) \leq \int_0^\infty \frac{\rho_1(\kappa^2)\,d\kappa^2}{\kappa^2} \tag{3.30}$$

and in view of assumption (3.9).

Similarly $\frac{d\mathcal{S}^{II}(z)}{dz} \geq 0$ for real, positive z and $\frac{d\mathcal{S}^{II}(z)}{dz} \to +\infty$ as $z \to 0+$. In fact $\mathcal{S}^{II}(z) = \mathcal{S}^{I}(z)$ only if $z \longrightarrow 0+$. If the lower limit of the integrals is $K_o'^2$ and $K_o^2 > K_o'^2$ the first point or threshold where the spectral functions are no longer zero, we can deduce

$$
\begin{aligned}
-\frac{1}{K_o'-m} &< -\int_{K_o'^2}^{\infty} \frac{\rho_1(\kappa^2)d\kappa^2}{\kappa-m} \leq \mathcal{S}(m) \leq \int_{K_o'^2}^{\infty} \frac{\rho_1(\kappa^2)d\kappa^2}{\kappa+m} < \frac{1}{K_o'+m} \\
-\frac{1}{K_o'+m} &< -\int_{K_o'^2}^{\infty} \frac{\rho_1(\kappa^2)d\kappa^2}{\kappa+m} \leq \mathcal{S}(-m) \leq \int_{K_o'^2}^{\infty} \frac{\rho_1(\kappa^2)d\kappa^2}{\kappa-m} < \frac{1}{K_o'-m}
\end{aligned}
\tag{3.31}
$$

All the information gathered about $\mathcal{S}$(m) and $\mathcal{S}$(-m) allows us to picture them roughly as follows in the region between the cuts

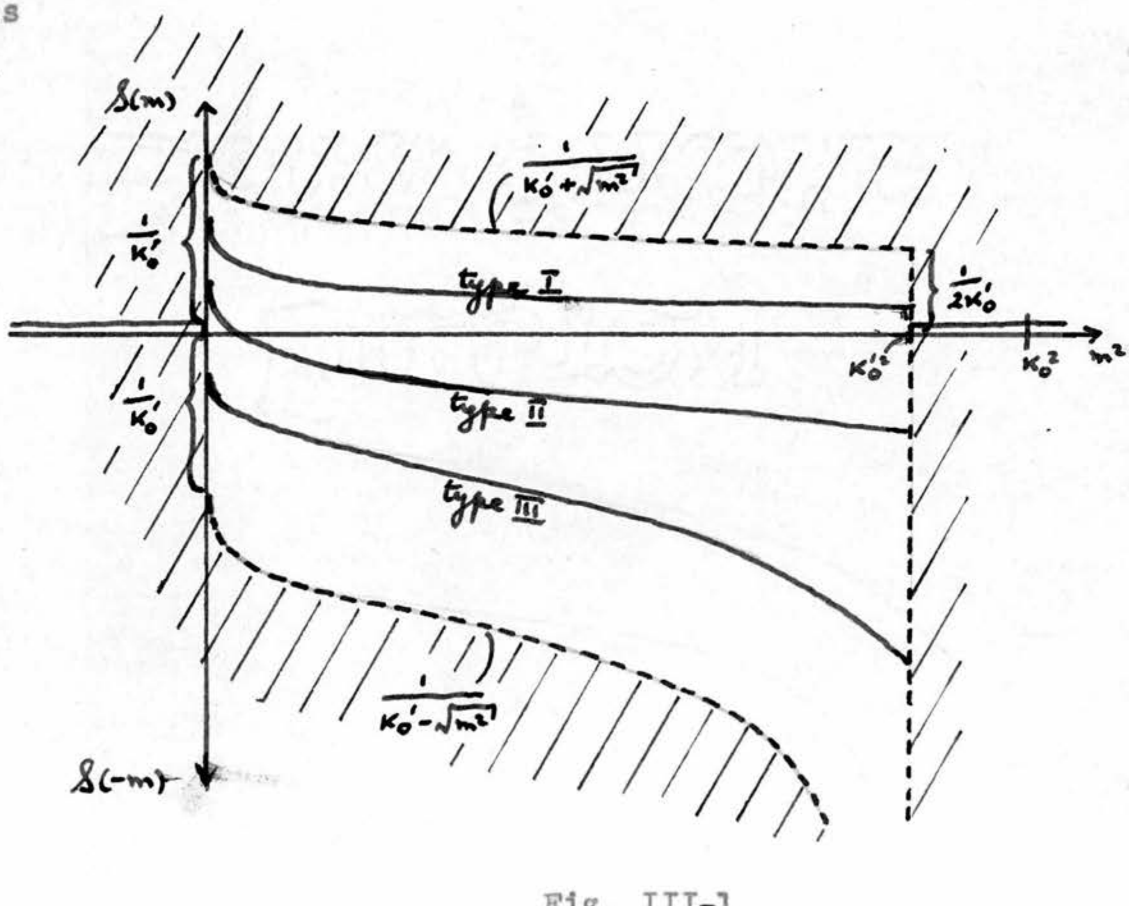

Fig. III-1.

The curved dotted lines are the limits found in (3.31) and since

$\frac{d\mathcal{S}(m)}{dm^2} \leq 0$ and $\frac{d\mathcal{S}(-m)}{dm^2} \geq 0$, there are three types of behaviour

for each of $\mathcal{S}(m)$ and $\mathcal{S}(-m)$ envisaged.

Let us examine each type separately for $\mathcal{S}(m)$ first

Type I : here we have $\mathcal{S}(m) > 0$ for $0 < m^2 < K_0'^2$ and so the rough forms for $\mathcal{S}^{-1}(m)$ in this region, using (3.31) again, is

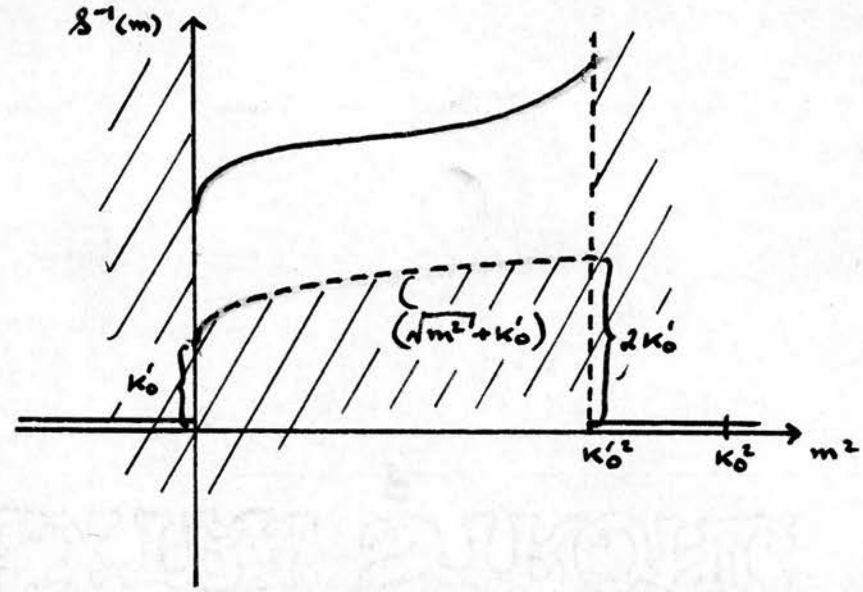

Fig. III-2.

This figure shows quite clearly that $\mathcal{S}^{-1}(m)$ has no zeros and will have no zeros even if we are allowed to move the value of $K_0'^2$ up to K_0^2 . Physically we expect a pole of $\mathcal{S}(m)$ or a zero of $\mathcal{S}^{-1}(m)$ to appear on the real axis to represent the one fermion contribution to the intermediate states in the propagator. Hence this type seems to be uninteresting.

Type III : in this case the form of $\mathcal{S}^{-1}(m)$ is

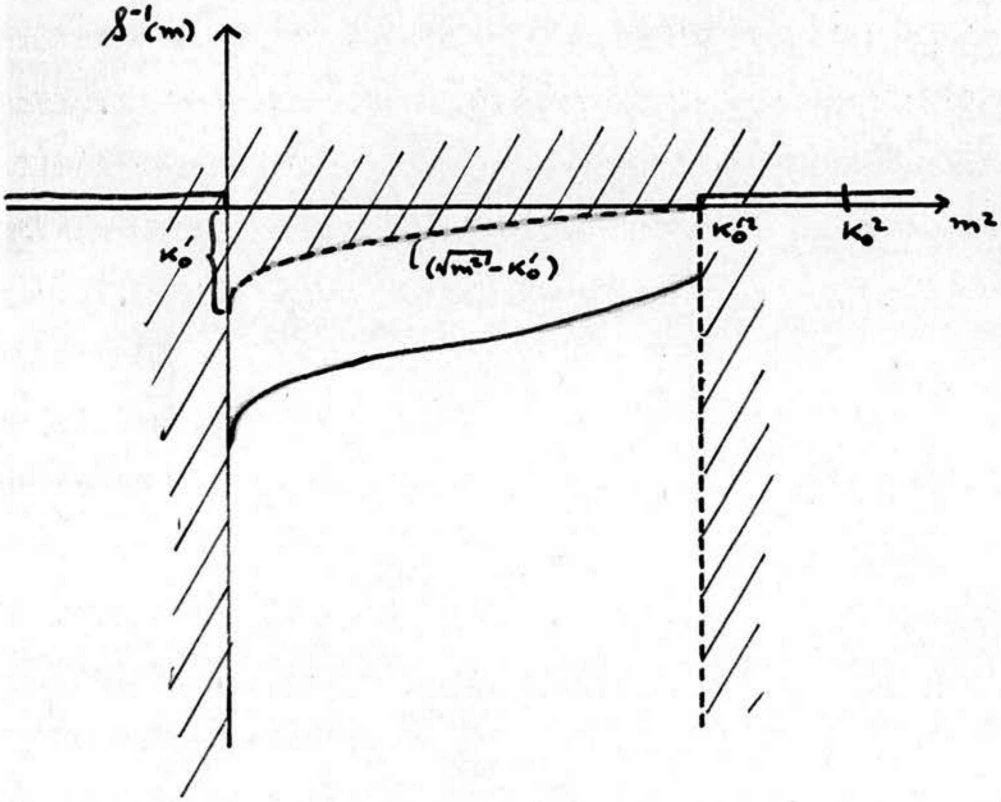

Fig. III-3.

Now we see that $\mathcal{S}^{-1}(m)$ is negative and increasing in the region $0 < m^2 < K_0'^2$ and there is a possibility that, as we increase $K_0'^2$ up to K_0^2, $\mathcal{S}^{-1}(m)$ may cross the real axis. There will be a pole in $\mathcal{S}(m)$ * so we must write

$$\mathcal{S}(m) = \frac{\rho_{10}}{m-M} + \int_{K_0^2}^{\infty} \frac{[\rho_2(\kappa^2)-(\kappa+m)\rho_1(\kappa^2)]\,d\kappa^2}{\kappa^2-m^2-i\epsilon} \qquad (3.32)$$

such that

---

* This is strictly a pole in the variable m only not m^2 but the distinction is no more than academic here.

$$\rho_{10} + \int_{K_0^2}^{\infty} \rho_1(\kappa^2)\,d\kappa^2 = 1 \quad , \quad (\rho_{10})^{-1} = -\left[\frac{d\,\mathcal{S}^I(z)}{dz}\right]_{z=M^2} > 0$$

$$1 + \int_{K_0^2}^{\infty} \frac{\tau(\kappa)\,d\kappa^2}{\kappa^2 - M^2} + \int_{0}^{\infty} \frac{\chi(\kappa)\,d\kappa^2}{\kappa^2 + M^2} = 0$$

$$M^2 < K_0^2$$

where we assume there is a region $M^2 < m^2 < K_o^2$ in which ρ_1 and ρ_2 are zero. The existence of such a pole in (3.32) depends upon $\chi(K)$ being negative and we shall consider the implications of this later. We picture $\mathcal{S}(m)$ and $\mathcal{S}^{-1}(m)$ now as

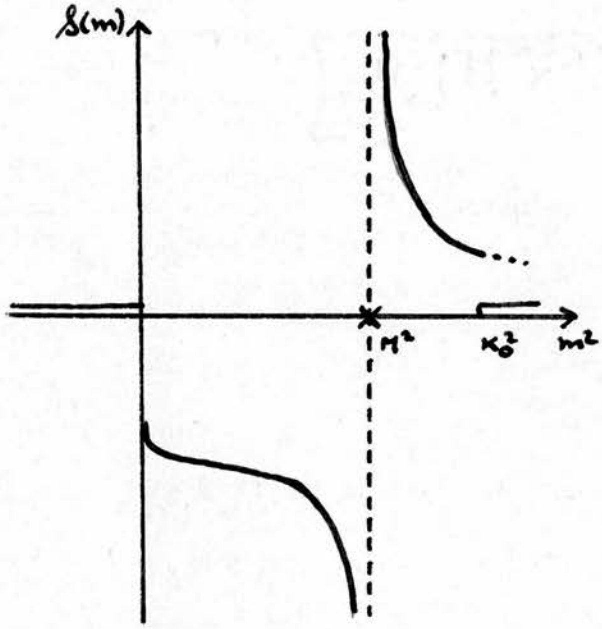

Fig. III-4(a)

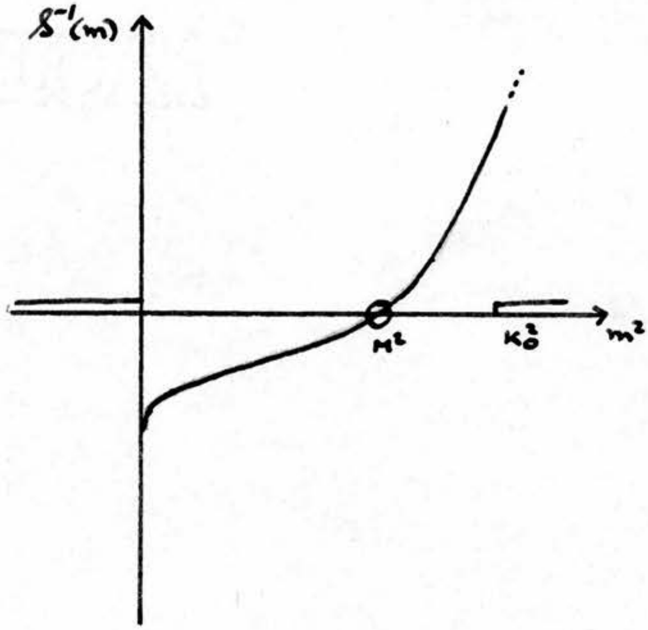

Fig. III-4(b)

We could continue to show that zeros and poles of $\mathcal{S}(m)$ can occur alternately along the real positive m^2-axis as for the boson propagator but since we know of no physical situation in which there is more than one pole, we shall stop at this point with poles. A zero of $\mathcal{S}(m)$ is possible from Figure III-4(a) and this can be discussed in an analogous manner to the case of the boson propagator with a zero. We shall not consider such zeros for this type further.

Type II : The picture of $S^{-1}(m)$ is more complicated to begin with and is roughly shown

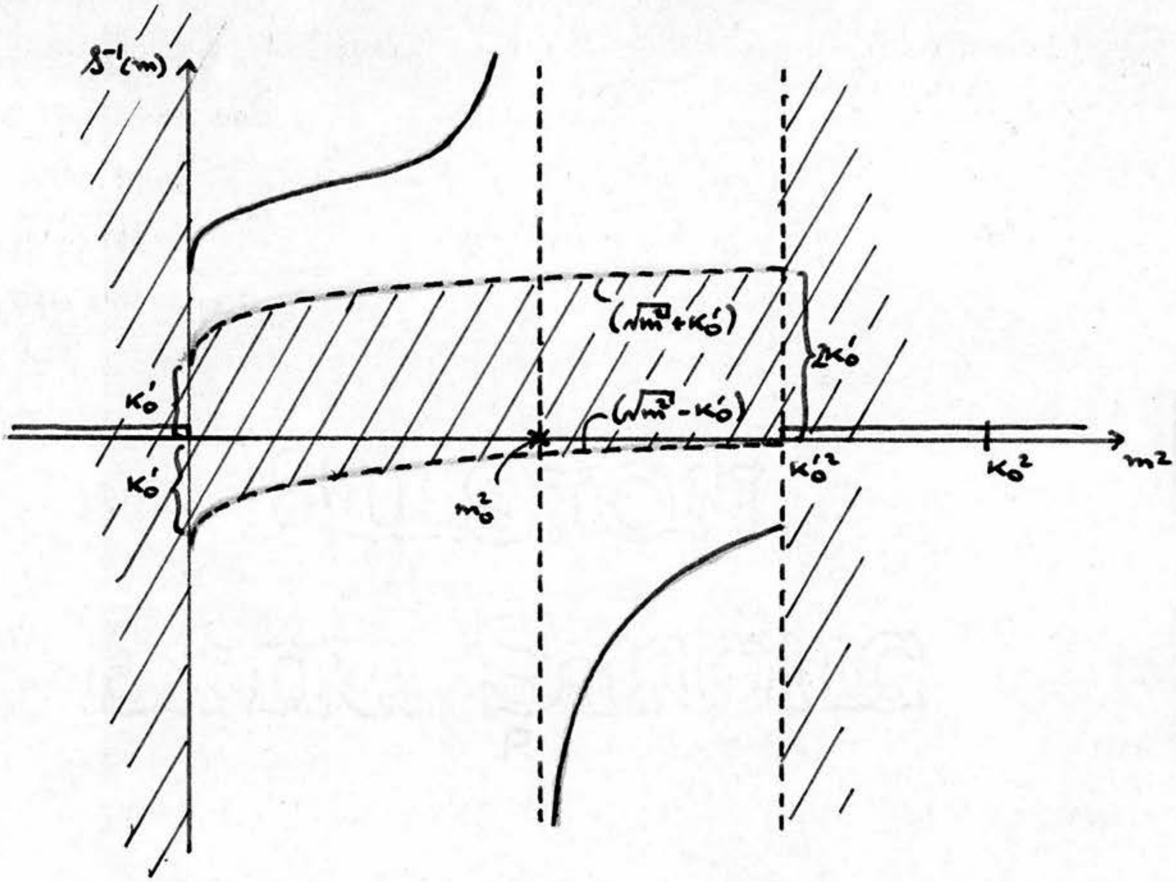

Fig. III-5.

Here we must accommodate the pole of $S^{-1}(m)$ and rewrite (3.23) in the following form

$$S^{-1}(m) = m\left(1 + \int_{K_0'^2}^{\infty}\frac{\tau(\kappa)d\kappa^2}{\kappa^2-m^2-i\epsilon} + \int_0^{\infty}\frac{\chi(\kappa)d\kappa^2}{\kappa^2+m^2+i\epsilon} + \frac{\lambda^2}{m_0^2-m^2}\right) \qquad (3.33)$$

where

$$\int_{K_0'^2}^{\infty} \frac{\left[\rho_2(\kappa^2) - (\kappa+m_0)\,\rho_1(\kappa^2)\right] d\kappa^2}{\kappa^2 - m_0^2} = 0$$

$$\lambda^2 = -\frac{1}{m_0}\cdot \frac{d\,\mathcal{S}(m)}{dm^2}\bigg|_{m^2=m_0^2} = -\frac{1}{2m_0^2}\int_{K_0'^2}^{\infty} \frac{\left[2m_0\,\rho_2(\kappa^2) - (\kappa+m_0)^2\rho_1(\kappa^2)\right]d\kappa^2}{(\kappa^2-m_0^2)^2}$$

$$\geq \frac{1}{2m_0}\cdot\int_{K_0'^2}^{\infty}\frac{\rho_1(\kappa^2)\,d\kappa^2}{(\kappa+m_0)^2} > 0$$

The form of $\mathcal{S}^{-1}(m)$ in the region $m_0^2 < m^2 < K_0'^2$ is very similar to that of type III and we conclude that $\mathcal{S}^{-1}(m)$ may vanish as we increase $K_0'^2$ and so we have (3.32) and (3.33) as valid forms of $\mathcal{S}(m)$ and $\mathcal{S}^{-1}(m)$ for this case with the physically expected pole. The only difference here compared with type III is that the zero of $\mathcal{S}(m)$ must be included. There is no objection to be raised in having a zero of $\mathcal{S}(m)$ at the origin, i.e. $m_0^2 = 0$ since the pole in $\mathcal{S}^{-1}(m)$ would then be incorporated in the left hand cut.

Now let us turn to the consideration of $\mathcal{S}(-m)$. It is clear that type I for $\mathcal{S}(-m)$ is similar to type I for $\mathcal{S}(m)$ in that no poles or zeros can occur. Types II and III for $\mathcal{S}(m)$ and $\mathcal{S}(-m)$ are also analogous. The point, here, is to find exactly what behaviour is physically expected of $\mathcal{S}(-m)$. There is of course no obvious objection to zeros but we must question the possibility of poles. We observe that it is well-known that the one-particle contributions to the intermediate states in the fermion propagator which are physically expected appear to be associated with discrete δ-function terms in

$\rho_1(K^2)$[24] while $\rho_2(K^2) = 0$ at such points and never has a δ-function term.[20] We have already shown by (3.19) that only δ-function terms in $\rho_2(K^2)$ give poles in $S(-m)$. Therefore we do not expect $S(-m)$ to have any poles. If we wish to forbid poles in $S(-m)$ completely then the type of behaviour when $S(-m) < 0$ for $0 < m^2 < K_o^2$ is the most suitable. This would imply $S(0+) < 0$ which further implies from (3.23) that $\chi(K)$ is predominantly negative in (3.25) but positive in (3.27). This is just what we required when we examined Type III of $S(m)$ above for a pole to appear. The most likely behaviour for $S(m)$ would then be type III. However it is still possible for $S(m)$ to have poles and $S(-m)$ not to have poles even if $S(0+) > 0$ so long as $S(0+)$ is not too large since $\chi(K)$ will have to be predominantly negative in (3.25). From now on we shall discuss only the behaviour of $S(m)$.

As in the boson case we can derive a further spectral representation of $S(m)$ for later convenience. If $S(m)$ has no poles or zeros and only cuts from 0 to $-\infty$ and from K_o^2 to $+\infty$ in the m^2-plane, if $K_o'^2 < K_o^2$,

$$(\sqrt{z} - K_o')S(z) \approx 1 \text{ as } z \rightarrow \text{infinite circle} \tag{3.34}$$

Also $\left[(\sqrt{z} - K_o')S(z)\right]$ has no poles, one zero at $K_o'^2$ and cuts 0 to $-\infty$ and K_o^2 to $+\infty$. Hence $\log\left[(\sqrt{z} - K_o')S(z)\right]$ has no poles, a cut from $K_o'^2$ to $+\infty$ and from 0 to $-\infty$ and converges to zero on the infinite circle. Therefore we can write a dispersion relation of the form

$$\log\left[(\sqrt{z}-\kappa_0')\,\mathcal{S}(z)\right] = \frac{1}{\pi}\int_{\kappa_0'^2}^{\infty}\frac{\phi(\kappa)\,d\kappa^2}{\kappa^2-z} + \frac{1}{\pi}\int_0^{\infty}\frac{\eta(\kappa)\,d\kappa^2}{\kappa^2+z} \tag{3.35}$$

where

$$\begin{aligned}\phi(\kappa) &= \frac{1}{2i}\left\{\log\left[\frac{\sqrt{\kappa^2+i\epsilon}-\kappa_0'}{\sqrt{\kappa^2-i\epsilon}-\kappa_0'}\right] + \log\left[\frac{\mathcal{S}(\kappa^2+i\epsilon)}{\mathcal{S}(\kappa^2-i\epsilon)}\right]\right\}\\ &= \pi\,\theta(\kappa-\kappa_0') - \cot^{-1}\left\{\frac{\mathrm{Re}\,\mathcal{S}(\kappa)}{\pi\left[2\kappa\rho_1(\kappa^2)-\rho_2(\kappa^2)\right]}\right\} \geq 0\end{aligned} \tag{3.36}$$

also

$$\eta(\kappa) = -\tan^{-1}\left(\frac{\kappa}{\kappa_0}\right) - \tan^{-1}\left[\kappa\int_0^{\infty}\frac{\rho_1(\kappa'^2)\,d\kappa'^2}{\kappa'^2+\kappa^2}\Big/\mathrm{Re}\,\mathcal{S}(\kappa)\right] \tag{3.37}$$

and it follows that

$$\begin{aligned}\mathcal{S}(z) &= \frac{1}{\sqrt{z}-\kappa_0'}\cdot e^{\frac{1}{\pi}\int_{\kappa_0'^2}^{\infty}\frac{\phi(\kappa)d\kappa^2}{\kappa^2-z}+\frac{1}{\pi}\int_0^{\infty}\frac{\eta(\kappa)d\kappa^2}{\kappa^2+z}}\\ \mathcal{S}^{-1}(z) &= (\sqrt{z}-\kappa_0')\cdot e^{-\frac{1}{\pi}\int_{\kappa_0'^2}^{\infty}\frac{\phi(\kappa)d\kappa^2}{\kappa^2-z}-\frac{1}{\pi}\int_0^{\infty}\frac{\eta(\kappa)d\kappa^2}{\kappa^2+z}}\end{aligned} \tag{3.38}$$

and we can choose the behaviour of $\phi(K)$ in exactly the same manner as in the boson case for the representations of types I and III. For the case of type II where $\mathcal{S}(z)$ must have a zero at $z = m_0^2$ we need only choose $K_0'^2 = m_0^2$ in the above formulae from (3.34) to (3.38).

We now wish to examine $\mathcal{S}(+m)$ under the two different types of interaction (a) and (b) which we introduced in Chapter II. For case (a) when $\mathcal{S}(m)$ has no zeros and one pole, we write (3.32) as

$$\mathcal{S}_s(m) = \frac{\rho_{10}^{s}}{m - M_s} + \int_{K_s^2}^{\infty} \frac{\left[\rho_2^{s}(\kappa^2) - (\kappa + m)\,\rho_1^{s}(\kappa^2)\right] d\kappa^2}{\kappa^2 - m^2 - i\epsilon} \tag{3.39}$$

and (3.23) as

$$\mathcal{S}_s^{-1}(m) = m\left(1 + \int_{K_s^2}^{\infty} \frac{\tau^{s}(\kappa)\, d\kappa^2}{\kappa^2 - m^2 - i\epsilon} + \int_{0}^{\infty} \frac{\chi^{s}(\kappa)\, d\kappa^2}{\kappa^2 + m^2 + i\epsilon}\right) \tag{3.40}$$

where $\mathcal{S}_s^{-1}(M_s) = 0$ and so $1 + \int_{K_s^2}^{\infty} \frac{\tau^{s}(\kappa)\, d\kappa^2}{\kappa^2 - M_s^2} + \int_{0}^{\infty} \frac{\chi^{s}(\kappa)\, d\kappa^2}{\kappa^2 + M_s^2} = 0$

But for $\mathcal{S}(m)$ of type II we must include one zero of $\mathcal{S}(m)$ if $\mathcal{S}(m)$ is to have a pole. We can use the above form (3.39) for $\mathcal{S}(m)$ with the condition $\mathcal{S}(m_o) = 0$ and instead of (3.40)

$$\mathcal{S}_s^{-1}(m) = m\left(1 + \frac{d_s^2}{m_o^2 - m^2} + \int_{K_s^2}^{\infty} \frac{\tau^{s}(\kappa)\, d\kappa^2}{\kappa^2 - m^2 - i\epsilon} + \int_{0}^{\infty} \frac{\chi^{s}(\kappa)\, d\kappa^2}{\kappa^2 + m^2 + i\epsilon}\right) \tag{3.41}$$

with $\mathcal{S}_s^{-1}(M_s) = 0$ so $1 + \frac{d_s^2}{m_o^2 - M_s^2} + \int_{K_s^2}^{\infty} \frac{\tau^{s}(\kappa)\, d\kappa^2}{\kappa^2 - M_s^2} + \int_{0}^{\infty} \frac{\chi^{s}(\kappa)\, d\kappa^2}{\kappa^2 + M_s^2} = 0$

Under interactions of type (b) we consider that the first continuum threshold K_W^2 is nearer the origin than M_s^2 so that $\mathcal{S}(m)$ of type III under strong interactions becomes

$$\mathcal{S}(m) = \int_{K_W^2}^{\infty} \frac{\left[\rho_2^{W}(\kappa^2) - (\kappa + m)\,\rho_1^{W}(\kappa^2)\right] d\kappa^2}{\kappa^2 - m^2 - i\epsilon} + \int_{K_s^2}^{\infty} \frac{\left[\rho_2^{S}(\kappa^2) - (\kappa + m)\,\rho_1^{S}(\kappa^2)\right] d\kappa^2}{\kappa^2 - m^2 - i\epsilon} \tag{3.42}$$

$$\mathcal{S}^{-1}(m) = m\left(1 + \int_{K_W^2}^{\infty} \frac{\tau^{W}(\kappa)\, d\kappa^2}{\kappa^2 - m^2 - i\epsilon} + \int_{K_s^2}^{\infty} \frac{\tau^{s}(\kappa)\, d\kappa^2}{\kappa^2 - m^2 - i\epsilon} + \int_{0}^{\infty} \frac{\chi(\kappa)\, d\kappa^2}{\kappa^2 + m^2 + i\epsilon}\right) \tag{3.43}$$

But when $\mathcal{S}(m)$ is of type II under interaction type (a) there are two possibilities, i.e. that $\mathcal{S}(m)$ may have a zero in the region $0 < m^2 < K_W^{\,2}$ or may not. If $\mathcal{S}(m)$ has no zero and is of type II under interaction type (a) then $\mathcal{S}(m)$ and $\mathcal{S}^{-1}(m)$ are represented by (3.42) and (3.43). If $\mathcal{S}(m)$ has a zero then this zero will not occur at $m_o^{\,2}$ in general. Thus we have in this case the form (3.42) with $\mathcal{S}(m_o') = 0$ and with

$$\mathcal{S}^{-1}(m) = m\left(1 + \frac{\lambda^2}{m_o'^2 - m^2} + \int_{K_W^2}^{\infty} \frac{\tau(\kappa)\,d\kappa^2}{\kappa^2 - m^2 - i\epsilon} + \int_0^{\infty} \frac{\chi(\kappa)\,d\kappa^2}{\kappa^2 + m^2 + i\epsilon}\right) \tag{3.44}$$

In all cases we try to find the behaviour of $\mathcal{S}(m)$ in the region near $m^2 = M_s^{\,2}$ above the threshold $K_W^{\,2}$. We carry out an identification of real and imaginary parts for the two spectral forms of $\mathcal{S}(m)$ in the region $K_W^{\,2} < m^2 < K_s^{\,2}$ similar to the boson case.

Consider first $\mathcal{S}(m)$ of type III and we have

$$\pi\left[\rho_2^W(\kappa^2) - 2\kappa\,\rho_1^W(\kappa^2)\right] = \frac{\sin\phi(\kappa)}{\kappa - K_W}\cdot e^{\frac{1}{\pi}P\int_{K_W^2}^{\infty}\frac{\phi(\kappa')\,d\kappa'^2}{\kappa'^2-\kappa^2} + \frac{1}{\pi}\int_0^{\infty}\frac{\eta(\kappa')\,d\kappa'^2}{\kappa'^2+\kappa^2}} \tag{3.45}$$

$$\pi\kappa\,\tau^W(\kappa) = -(\kappa - K_W)\sin\phi(\kappa)\cdot e^{-\frac{1}{\pi}P\int_{K_W^2}^{\infty}\frac{\phi(\kappa')\,d\kappa'^2}{\kappa'^2-\kappa^2} + \frac{1}{\pi}\int_0^{\infty}\frac{\eta(\kappa')\,d\kappa'^2}{\kappa'^2+\kappa^2}} \tag{3.46}$$

$$\kappa\left(1 + P\int_{K_W^2}^{\infty}\frac{\tau^W(\kappa')\,d\kappa'^2}{\kappa'^2-\kappa^2} + \int_{K_S^2}^{\infty}\frac{\tau^S(\kappa')\,d\kappa'^2}{\kappa'^2-\kappa^2} + \int_0^{\infty}\frac{\chi^W(\kappa')\,d\kappa'^2}{\kappa'^2+\kappa^2} + \int_0^{\infty}\frac{\chi^S(\kappa')\,d\kappa'^2}{\kappa'^2+\kappa^2}\right)$$
$$= (\kappa - K_W)\cos\phi(\kappa)\cdot e^{-\frac{1}{\pi}P\int_{K_W^2}^{\infty}\frac{\phi(\kappa')\,d\kappa'^2}{\kappa'^2-\kappa^2} - \frac{1}{\pi}\int_0^{\infty}\frac{\eta(\kappa')\,d\kappa'^2}{\kappa'^2+\kappa^2}} \tag{3.47}$$

The equations (3.45) and (3.46) give

$$\pi^2 \kappa \tau^W(\kappa)\left[\rho_2^W(\kappa^2) - 2\kappa\rho_1^W(\kappa^2)\right] = -\left[\sin\phi(\kappa)\right]^2 \tag{3.48}$$

and from (3.46) and (3.47)

$$-\pi\kappa\tau^W(\kappa)\cot\phi(\kappa)$$
$$= \kappa\left(1 + P\!\!\int_{\kappa_W^2}^{\infty}\frac{\tau^W(\kappa')d\kappa'^2}{\kappa'^2-\kappa^2} + \int_{\kappa_S^2}^{\infty}\frac{\tau^S(\kappa')d\kappa'^2}{\kappa'^2-\kappa^2} + \int_0^{\infty}\frac{\chi^W(\kappa')d\kappa'^2}{\kappa'^2+\kappa^2} + \int_0^{\infty}\frac{\chi^S(\kappa')d\kappa'^2}{\kappa'^2+\kappa^2}\right) \tag{3.49}$$

If $K^2 \approx M_s^{\ 2}$ then using

$$\mathcal{S}_s^{-1}(\kappa) \approx (\rho_{10}^s)^{-1}(\kappa - M_s) \tag{3.50}$$

we have for (3.49), with (3.39), (3.40) and (3.50)

$$-\pi\kappa\tau^W(\kappa)\cot\phi(\kappa) \approx (\rho_{10}^s)^{-1}(\kappa-M_s) + \kappa P\!\!\int_{\kappa_W^2}^{\infty}\frac{\tau^W(\kappa')d\kappa'^2}{\kappa'^2-\kappa^2} + \kappa\int_0^{\infty}\frac{\chi^W(\kappa')d\kappa'^2}{\kappa'^2+\kappa^2} \tag{3.51}$$

and there will be some point $M^2 \approx M_s^{\ 2}$, if weak interaction contributions are small enough, such that

$$-\pi\kappa\tau^W(\kappa)\cot\phi(\kappa) \approx (\rho_{10})^{-1}(\kappa - M) \tag{3.52}$$

Comparing (3.51) and (3.52) we conclude

$$M = M_s - \rho_{10}^s M\left(P\!\!\int_{\kappa_W^2}^{\infty}\frac{\tau^W(\kappa)d\kappa^2}{\kappa^2-M^2} + \int_0^{\infty}\frac{\chi^W(\kappa)d\kappa^2}{\kappa^2+M^2}\right) \tag{3.53}$$

$$(\rho_{10})^{-1} = (\rho_{10}^s)^{-1} + P\!\!\int_{\kappa_W^2}^{\infty}\frac{\kappa^2\tau^W(\kappa)d\kappa^2}{(\kappa^2-M^2)^2} + \int_0^{\infty}\frac{\kappa^2\chi^W(\kappa)d\kappa^2}{(\kappa^2+M^2)^2}$$

which again are valid if the weak interactions have a small enough effect in the neighbourhood of the cut at $m^2 = \mu_s^2$.

Eliminating ϕ from (3.48) and (3.52) gives

$$\left[2\kappa \rho_1^w(\kappa^2) - \rho_2^w(\kappa^2)\right] \approx \frac{\rho_{10}}{\pi} \cdot \frac{(\gamma/2)}{(\kappa - M)^2 + (\gamma/2)^2} \tag{3.54}$$

where

$$\gamma = 2\pi M \rho_{10} \tau^w(M) \tag{3.55}$$

The contribution of this resonance in $\left[2K\rho_1(K^2) - \rho_2(K^2)\right]$ to the time dependence of the propagator $S_F'(x)$ is obtained from *

$$\bar{u}_j S_F'(p,t) u_j = \bar{u}_j \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \cdot e^{-ip_0 t} S_F'(-p^2) u_j = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \cdot e^{-ip_0 t} \delta(m) \tag{3.56}$$

where $-p^2 = m^2$ and therefore

$$\bar{u}_j S_F'(p,t) u_j = \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \cdot e^{-ip_0 t} \int_0^{\infty} \frac{\left[\rho_2(\kappa^2) - (\kappa + \sqrt{-p^2})\, \rho_1(\kappa^2)\right] d\kappa^2}{\kappa^2 + p^2 - i\epsilon}$$

$$= \frac{i}{2} \int_0^{\infty} \frac{\left[\rho_2(\kappa^2) - 2\kappa \rho_1(\kappa^2)\right]}{\sqrt{p^2 + \kappa^2}} \cdot e^{-i\sqrt{p^2 + \kappa^2}\,|t|} d\kappa^2 \approx \frac{-i \rho_{10}}{2E_M} \cdot e^{-iE_M|t|} \cdot e^{-\left(\frac{M\gamma}{2E_M}\right)|t|} \tag{3.57}$$

---

* From physical arguments, $\rho_2(K^2)$ has no δ-function singularities and there-fore we suspect that $\rho_1^w(K^2)$ is the spectral function possessing the resonance and that we could neglect $\rho_2^w(K^2)$ in (3.54) and have instead

$$\rho_1^w(\kappa^2) \approx \frac{\rho_{10}}{\pi} \cdot \frac{(\gamma M)}{(\kappa^2 - M^2)^2 + (\gamma M)^2}$$

where $E_M = \sqrt{p^2 + M^2}$.

An examination of the above algebra shows that if $\mathcal{S}(m)$ is of type II with a zero under interaction (a) and a zero under interaction (b), the argument remains almost exactly the same. Slight alterations are required to $\mathcal{S}_s^{-1}$ which has the extra term $m\lambda_s^2/(m_0^2 - m^2)$ and to $\mathcal{S}^{-1}(m)$ which has the extra term $m\lambda^2/(m_0'^2 - m^2)$. This means that instead of (3.51) we have

$$-\pi\kappa\tau^w(\kappa)\cot\phi(\kappa) \approx (\rho_{10}^s)^{-1}(\kappa - M_s) + \frac{\kappa\lambda^2}{m_0'^2 - \kappa^2} - \frac{\kappa\lambda_s^2}{m_0^2 - \kappa^2} + \kappa\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{\tau^w(\kappa')d\kappa'^2}{\kappa'^2 - \kappa^2} + \kappa\int_0^{\infty} \frac{\chi^w(\kappa')d\kappa'^2}{\kappa'^2 + \kappa^2} \tag{3.58}$$

which only alters M and $(\rho_{10})^{-1}$ in (3.53) to

$$M = M_s - \rho_{10}^s M\left[\frac{\lambda_s^2}{m_0^2 - M^2} - \frac{\lambda^2}{m_0'^2 - M^2} + P\!\!\int_{\kappa_w^2}^{\infty} \frac{\tau^w(\kappa)d\kappa^2}{\kappa^2 - M^2} + \int_0^{\infty} \frac{\chi^w(\kappa)d\kappa^2}{\kappa^2 + M^2}\right]$$

$$(\rho_{10})^{-1} = (\rho_{10}^s)^{-1} + \frac{m_0'^2\lambda^2}{(m_0'^2 - M^2)^2} - \frac{m_0^2\lambda_s^2}{(m_0^2 - M^2)^2} + P\!\!\int_{\kappa_w^2}^{\infty} \frac{\kappa^2\tau^w(\kappa)d\kappa^2}{(\kappa^2 - M^2)^2} + \int_0^{\infty} \frac{\chi^w(\kappa)\kappa^2 d\kappa^2}{(\kappa^2 + M^2)^2} \tag{3.59}$$

All the other results will have the same form as for type III.

If $\mathcal{S}(m)$ is of type II under interaction (a) but has no zero under interaction (b) then we write instead of (3.51)

$$-\pi\kappa\tau^w(\kappa)\cot\phi(\kappa) \approx (\rho_{10}^s)^{-1}(\kappa - M_s) - \frac{\kappa\lambda_s^2}{m_0^2 - \kappa^2} + \kappa\, P\!\!\int_{\kappa_w^2}^{\infty} \frac{\tau^w(\kappa')d\kappa'^2}{\kappa'^2 - \kappa^2} + \kappa\int_0^{\infty} \frac{\chi^w(\kappa')d\kappa'^2}{\kappa'^2 + \kappa^2} \tag{3.60}$$

giving for (3.53) the results

$$M = M_s + \rho_{10}^{s} M \left(\frac{\lambda_s^2}{m_0^2 - M^2} - P\!\!f \int_{K_W^2}^{\infty} \frac{\tau^W(\kappa)\, d\kappa^2}{\kappa^2 - M^2} - \int_0^{\infty} \frac{\chi^W(\kappa)\, d\kappa^2}{\kappa^2 + M^2} \right) \tag{3.61}$$

$$(\rho_{10})^{-1} = (\rho_{10}^{s})^{-1} - \frac{m_0^2 \lambda_s^2}{(m_0^2 - M^2)^2} + P\!\!f \int_{K_W^2}^{\infty} \frac{\kappa^2 \tau^W(\kappa)\, d\kappa^2}{(\kappa^2 - M^2)^2} + \int_0^{\infty} \frac{\kappa^2 \chi^W(\kappa^2)\, d\kappa^2}{(\kappa^2 + M^2)^2}$$

At first sight it appears that the pole at m_o^2 in $\mathcal{S}_s^{-1}(m)$ has too large a contribution for $M \approx M_s$. However the point m_o^2 where $\mathcal{S}_s^{-1}(m)$ has a pole is a point where we might expect $\mathcal{S}^{-1}(m)$ to have a strong pole-like behaviour although $m_o^2 > K_W^2$. That this is indeed the case follows on the same lines as Appendix 2 in which we showed that near zeros of the boson propagator produce a pole-like behaviour in the inverse propagator when the zero is covered by a weak interaction cut. In this case we put for $K \approx m_o$

$$Re\, \mathcal{S}_s(\kappa) \approx (\tau_0^{s})^{-1} \cdot \left(\frac{m_0^2 - \kappa^2}{m_0} \right) \tag{3.62}$$

and find for $K \approx m_o \approx m_o'$

$$Re\, \mathcal{S}(\kappa) \approx (\tau_0)^{-1} \left(\frac{m_0'^2 - \kappa^2}{m_0'} \right) \tag{3.63}$$

where

$$m_0'^2 = m_0^2 + \tau_0^{s} m_0^2\, P\!\!f \int_{K_W^2}^{\infty} \frac{\left[\rho_2^W(\kappa^2) - (\kappa + m_0')\, \rho_1^W(\kappa^2) \right] d\kappa^2}{\kappa^2 - m_0'^2}$$

$$(m_0' \tau_0)^{-1} = (m_0 \tau_0^{s})^{-1} - P\!\!f \int_{K_W^2}^{\infty} \frac{\left[m_0' \rho_2^W(\kappa^2) - \frac{1}{2} (\kappa - m_0')^2 \rho_1^W(\kappa^2) \right] d\kappa^2}{(\kappa^2 - m_0'^2)^2} \tag{3.64}$$

and we also have

$$\tau^{w}(\kappa) \approx \frac{\tau_0}{\pi} \cdot \frac{\gamma' m_0'}{(\kappa^2 - m_0'^2)^2 + (\gamma' m_0')^2} \tag{3.65}$$

where
$$\gamma' = \pi \tau_0 \left[2 m_0' \rho_1^{w}(m_0'^2) - \rho_2^{w}(m_0'^2) \right] \geq 0 \tag{3.66}$$

This shows that there will be a pole-like contribution to the integrals in (3.61) due to the resonance behaviour of $\tau^{w}(K)$ in (3.65) which compensates for the pole terms near m_o^2.

Thus we conclude that only minor differences in the definitions of mass, lifetime and renormalisation constant to be associated with the unstable fermion, occur for each of the propagator types. The time dependence of the probability amplitude of an unstable fermion wave packet can be obtained in exactly the same manner as for unstable bosons and the same conclusions for very long times can be deduced. It is quite straightforward to adjust the discussion of the model which we introduced in Chapter II so that it applies to fermions. The application to physical particles is not difficult and it follows from the use of the (CPT)-theorem that the masses and lifetimes of particles and their anti-particles are identical.

CHAPTER IV

TWO-PARTICLE SCATTERING AMPLITUDES

We complete the application of the methods used in Chapters II, III of describing resonances in dispersion relation theory with the consideration of dispersion relations for two-particle scattering amplitudes.* We shall restrict ourselves to scattering amplitudes for which dispersion relations have been proved in axiomatic field theory for strong interactions. Therefore we shall consider only the dispersion relations proved by Lehmann[26] for the elastic scattering of the two particle systems $\gamma - e$, $\pi - h$, $\pi - \pi$ and $\pi - \Xi$. To have a definite process in mind which contains interesting features among the above reactions, we shall consider $\pi - h$ scattering. If we denote the momenta of the incoming boson and fermion by k and p and the outgoing by k' and p' with $-k^2 = -k'^2 = \mu^2$, $-p^2 = -p'^2 = M^2$ we have for the invariant amplitude, where $J(x) = (\Box - \mu^2)\phi(x)$ and $\phi(x)$ is the Heisenberg field operator describing a pion field with mass μ,

$$F[-(p+k)^2, \cos\theta] = \frac{1}{16\pi} \langle p' | J(0) | p, k \text{ in} \rangle$$

$$= \frac{1}{16\pi} M[-(p+k)^2, \cos\theta] \qquad (4.1)$$

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* A similar approach has been published by Moffat[13] unknown to the present author until a separate analysis had been worked out. Dispersion relations for the inverse scattering amplitude have previously been written out by Feldman, Matthews and Salam.[25]

where

$$M\left[-(p+k)^2, \cos\theta\right] = i\int_{-\infty}^{\infty} d^4x \, e^{-\frac{1}{2}i(k+k')x} \langle p' | \left[J(\tfrac{x}{2}), J(-\tfrac{x}{2}) \right] | p \rangle \, \theta(x) \tag{4.2}$$

and we have used notations similar to those in Appendix 3. If we take account of spin and iso-spin denoting the charge variables of the initial and final pions by indices λ, λ', M can be written in the general form

$$M_{\lambda'\lambda} = \bar{u}(p')\Big\{ \delta_{\lambda'\lambda} T_1 + \tfrac{1}{2}[\tau_{\lambda'}, \tau_\lambda] T_2 - \tfrac{1}{2}i[\gamma.(k+k')] T_3$$
$$- \tfrac{1}{2}i\,[\gamma.(k+k')][\tau_{\lambda'}, \tau_\lambda] T_4 \Big\} u(p) \tag{4.3}$$

where the τ_λ are the usual Pauli matrices and we have already defined spinors u and $\bar{u}$ in Chapter III.

According to Lehmann the amplitudes T_1 and T_3 satisfy

$$T_{1,3}(\omega, \Delta^2) = \frac{2}{\pi}\int_0^\infty \frac{\omega' \, \mathrm{Im}\, T_{1,3}(\omega', \Delta^2)\, d\omega'}{\omega'^2 - \omega^2 - i\epsilon} \tag{4.4}$$

where $2\omega M = W^2 - 2\Delta^2 - M^2 - \mu^2$ and $W^2 = -(p+k)^2$, $4\Delta^2 = (k'-k)^2$, while the amplitudes T_2 and T_4 satisfy

$$T_{2,4}(\omega, \Delta^2) = \frac{2\omega}{\pi}\int_0^\infty \frac{\mathrm{Im}\, T_{2,4}(\omega', \Delta^2)\, d\omega'}{\omega'^2 - \omega^2 - i\epsilon} \tag{4.5}$$

The relations (4.4) and (4.5) are true for fixed momentum transfer Δ^2 which must be further restricted for pion-nucleon scattering such that

$$0 \leq \Delta^2 < \frac{8}{3}\left(\frac{2M+\mu}{2M-\mu}\right)\mu^2 \qquad (4.6)$$

provided the spectrum allows the matrix elements $\langle p | J(o) | p_n \rangle$ to differ from zero, apart from discrete values of p_n, only for $-p_n{}^2 \geq (M+\mu)^2$. If the mass associated with the threshold of the continuous spectrum is allowed to be lower than $(M+\mu)$, the condition for the validity of the dispersion relations is[27)]

$$W_m^3 - W_m^2(m_1+m_2) - W_m(M^2+\mu^2-2m_1m_2) + (\mu^2-M^2)(m_1-m_2) > 0 \qquad (4.7)$$

where $W_m = \frac{1}{3}\left[(m_1+m_2) + \sqrt{(m_1+m_2)^2 + 3(\mu^2+M^2-2m_1m_2)}\right]$ and m_1 and m_2 are the smallest masses of states $|p_n\rangle$, $|p_n'\rangle$ such that

$$\begin{aligned} &\langle 0 | J(o) | p_n \rangle \langle p_n | J(o) | 0 \rangle \neq 0 \\ &\langle 0 | j(o) | p_n' \rangle \langle p_n' | j(o) | 0 \rangle \neq 0 \end{aligned} \qquad (4.8)$$

where $(i\gamma . p + m)\psi(x) = j(x)$ and $-p_n^2 = m_1^2$, $-p_n'^2 = m_2^2$.

Only the amplitudes T_3 and T_4 have one particle terms due to the physically expected single nucleon intermediate states. Hence let us examine first the amplitude T_3 and ignore the iso-spin flop term. For simplicity denote T_3 by plain T and rewrite (4.4) in the form for strong interactions only

$$T(W^2;\Delta^2) = g^2\left[\frac{1}{W^2-a^2+M^2} - \frac{1}{W^2-M^2}\right] + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dW'^2\, Im\, T(W'^2;\Delta^2).$$

$$\cdot\left[\frac{1}{W'^2-W^2-i\epsilon} + \frac{1}{W'^2-a^2+W^2+i\epsilon}\right] \qquad (4.9)$$

where $a^2 = 4\Delta^2 + 2M^2 + 2\mu^2$, $g^2 = |V(M^2)|^2$ as shown in the Appendix 3. Note that the one-particle poles lie between the cuts if

$$\Delta^2 < \frac{1}{4}\mu(2M - \mu) \tag{4.10}$$

and that this upper limit is essentially the same as that in (4.6) since $M/\mu \approx 7$ for pions and nucleons. Hence the dispersion relation will be valid so long as the poles occur between the strong interaction cuts. The only other relation which involves the absorptive part of T or $Im\,T$ is the unitarity relation which we shall use implicitly. Subtractions cause only trivial modifications to (4.9), and we ignore them.

Now consider $T(W^2, \Delta^2)$ as the boundary value of an analytic function $T(z, \Delta^2)$ given by

$$T(z,\Delta^2) = g^2\left[\frac{1}{z-a^2+M^2} - \frac{1}{z-M^2}\right] + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dW'^2\, Im\,T(W'^2,\Delta^2).$$

$$\cdot\left[\frac{1}{W'^2-z} + \frac{1}{W'^2-a^2+z}\right] \tag{4.11}$$

which has two cuts on the real axis from $(M + \mu)^2$ to $+\infty$ and from $[a^2 - (M + \mu)^2]$ to $-\infty$ and two poles at M^2, $a^2 - M^2$. It is difficult to obtain further information about the behaviour of $T(z, \Delta^2)$ in the z-plane since our knowledge of $Im\,T$ is very sketchy in general. If we assume that the behaviour of $T(W^2, \Delta^2)$ between the cuts is dominated by the pole terms then we have the following picture noting the symmetry with respect to $W^2 = a^2/2$

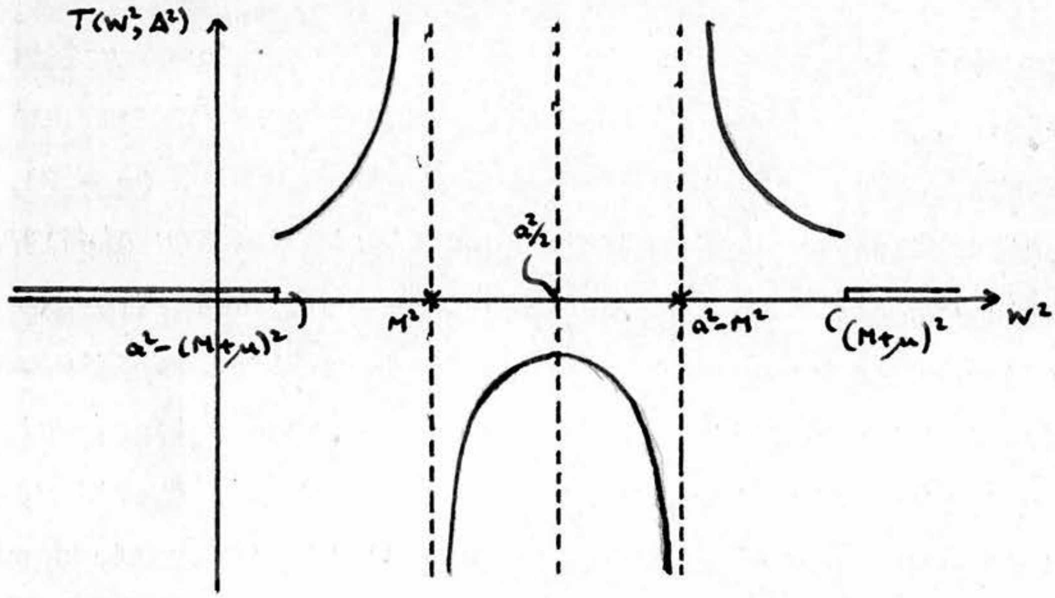

Fig. IV-1.

If we are to obtain a dispersion relation for the inverse of $T(W^2, \Delta^2)$ we must look for complex zeros of $T(W^2, \Delta^2)$. Hence with $z = x + iy$ and putting $R\ell\, T(z, \Delta^2) = 0 = \operatorname{Im} T(z, \Delta^2)$ we have

$$g^2\left[\frac{-(a^2-M^2)}{|z-a^2+M^2|^2} + \frac{M^2}{|z-M^2|^2}\right] + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dW'^2\, Im\, T(W'^2, \Delta^2) \cdot$$

$$\cdot\left[\frac{W'^2}{|W'^2-z|^2} + \frac{W'^2-a^2}{|W'^2-a^2+z|^2}\right] = 0$$

$$g^2\left[\frac{-1}{|z-a^2+M^2|^2} + \frac{1}{|z-M^2|^2}\right] + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} dW'^2\, Im\, T(W'^2, \Delta^2) \cdot \tag{4.12}$$

$$\cdot\left[\frac{1}{|W'^2-z|^2} - \frac{1}{|W'^2-a^2+z|^2}\right] = 0$$

$$\therefore \quad \frac{g^2(x^2+y^2+M^4-a^4/2)}{[(x-M^2)^2+y^2][(x-a^2+M^2)^2+y^2]} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \frac{dW'^2\, Im\, T(W'^2;\Delta^2).\,(x^2+y^2+W'^4-a^4/2)}{[(x-W'^2)^2+y^2][(x-a^2+W'^2)^2+y^2]} \tag{4.13}$$

$$= \left(x-\frac{a^2}{2}\right)\left\{\frac{g^2(a^2-2M^2)}{[(x-M^2)^2+y^2][(x-a^2+M^2)^2+y^2]} - \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{dW'^2\, Im\, T(W'^2;\Delta^2).\,(2W'^2-a^2)}{[(x-W'^2)^2+y^2][(x-a^2+W'^2)^2+y^2]}\right\} = 0$$

where the factor $(x - \frac{a^2}{2})$ does not help us to find any zero for non-zero y. If the pole terms are dominant for some region of z then there are no complex zeros in that region since $a^2 \neq 2M^2$. If there are any zeros in the general expression then they must occur in complex conjugate pairs since y is contained only as y^2. Because of the symmetry with respect to $z = \frac{a^2}{2}$ we conclude that zeros occur in sets of four. There is no reason why we should not have zeros on the real axis occurring in pairs if the contributions of the cuts are strong enough as can be seen from Figure IV-1. For the moment we shall ignore all zeros of $T(z, \Delta^2)$ since they can easily be included in $T^{-1}(z, \Delta^2)$ if they occur and they only obscure the point we wish to make. Assuming that $Im\ T(W^2, \Delta^2)$ does not have too wild a behaviour we take as the behaviour at infinity

$$T(z,\Delta^2) \approx \frac{c}{z^2} \tag{4.14}$$

where we have retained the symmetry with respect to $\frac{a^2}{2}$ and put

$$g^2(a^2-2M^2) + \int_{(M+\mu)^2}^{\infty} (a^2-2W'^2)\, Im\, T(W'^2;\Delta^2)\, dW'^2 = C \tag{4.15}$$

Hence we can write a dispersion relation, choosing some convenient constant M_o^2, for $\left[(z-M_o^2)^{-1}(z-a^2+M_o^2)^{-1}T^{-1}(z,\Delta^2) - C^{-1}\right]$ which has the same cuts as $T(z,\Delta^2)$, converges to zero on the infinite circle, poles at $z = M_o^2$ and $z = a^2 - M_o^2$, and so

$$T^{-1}(z,\Delta^2) = \lambda_1(z-M_o^2) + \lambda_2(z-a^2+M_o^2) + (z-M_o^2)(z-a^2+M_o^2).$$
$$\cdot\left\{C^{-1} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} T(w'^2,\Delta^2)\,dw'^2\left[\frac{1}{w'^2-z} + \frac{1}{w'^2-a^2+z}\right]\right. \tag{4.16}$$

but to retain symmetry with respect to $\frac{a^2}{2}$ we have

$$\lambda_2 = -\lambda_1 = -\lambda \tag{4.17}$$

Therefore we write finally for the inverse amplitude

$$T^{-1}(z,\Delta^2) = \lambda(a^2-2M_o^2) + (z-M_o^2)(z-a^2+M_o^2).$$
$$\cdot\left\{C^{-1} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} T(w'^2,\Delta^2)\,dw'^2\left[\frac{1}{w'^2-z} + \frac{1}{w'^2-a^2+z}\right]\right\} \tag{4.18}$$

and if T has poles at M^2, $a^2 - M^2$ then

$$\lambda(a^2-2M_o^2) = -(M^2-M_o^2)(M^2-a^2+M_o^2).$$
$$\cdot\left\{C^{-1} + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} T(w'^2,\Delta^2)\,dw'^2\left[\frac{1}{w'^2-M^2} + \frac{1}{w'^2-a^2+M^2}\right]\right\} \tag{4.19}$$

which implies that

$$T^{-1}(z,\Delta^2) = (z-M^2)(z-a^2+M^2)\Big\{c^{-1}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \mathcal{T}(w'^2,\Delta^2)\,dw'^2\,.$$

$$\cdot\frac{(w'^2-M_0^2)(w'^2-a^2+M_0^2)}{(w'^2-M^2)(w'^2-a^2+M^2)}\left[\frac{1}{w'^2-z}+\frac{1}{w'^2-a^2+z}\right]\Big\}$$

$$= (z-M^2)(z-a^2+M^2)\Big\{c^{-1}+\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{\mathrm{Im}\,T^{-1}(w'^2,\Delta^2)\,dw'^2}{(w'^2-M^2)(w'^2-a^2+M^2)}\,.$$

$$\cdot\left[\frac{1}{w'^2-z}+\frac{1}{w'^2-a^2+z}\right]\Big\} \qquad (4.20)$$

The behaviour of $T^{-1}(W^2, \Delta^2)$ between the cuts assuming the zeros are the dominant terms is roughly

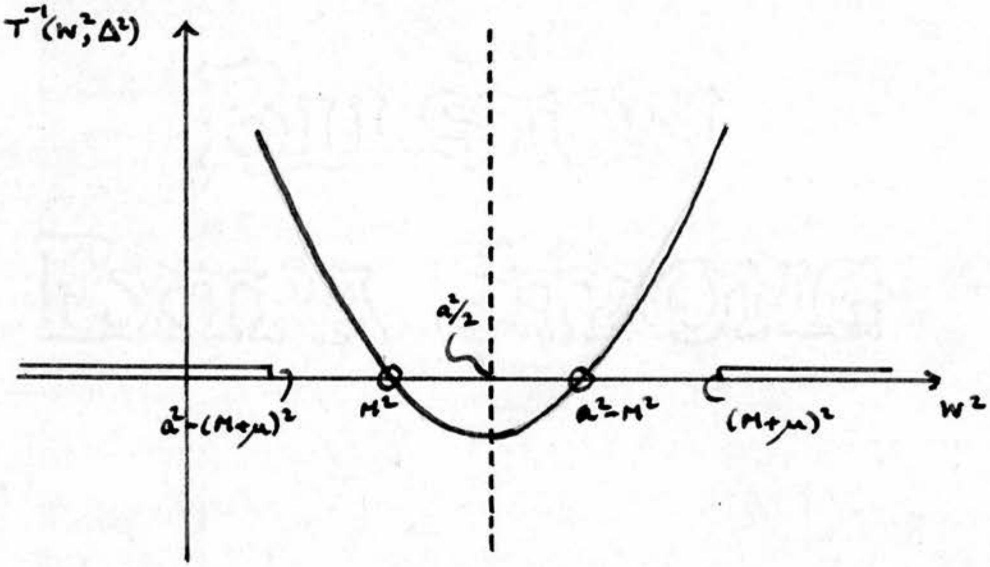

Fig. IV-2.

It has been shown by Minguzzi and Taffara[27)] that Lehmann's dispersion relations have only been proved for stable external particles. Hence we are strictly not allowed to extend the cuts by introducing weak interactions to cover the poles and force the nucleons to be unstable. We therefore do not allow such weak interaction cuts and assume that the nucleons and pions, which are the external particles in our scattering problem, to be stable. The introduction of some additional strong interaction need not introduce any new cuts in the region $\left[a^2 - (M + \mu)^2\right] < W^2 < (M + \mu)^2$ but may introduce resonances above the thresholds of the cuts shown in Figures IV-1 and IV-2. That resonances occur above the strong interaction thresholds for pion-nucleon scattering is now well established experimentally. We shall examine such a point on the physical energy cut where we suspect strongly that there is a sharp resonance.

Once more we can write an exponential spectral representation for $T(W^2, \Delta^2)$ by noting that $\log\left[\frac{(z-M^2)(z-a^2+M^2)T(z,\Delta^2)}{c}\right]$ has no poles, converges to zero on the infinite circle, and can be arranged to have the same cuts on the real axis as $T(z, \Delta^2)$. Hence

$$T(z,\Delta^2) = \frac{c\, e^{\frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \phi(W'^2, \Delta^2)\, dW'^2 \left[\frac{1}{W'^2 - z} + \frac{1}{W'^2 - a^2 + z}\right]}}{(z-M^2)(z-a^2+M^2)}$$

$$= \left[T^{-1}(z,\Delta^2)\right]^{-1} \qquad (4.21)$$

Hence by comparing real or dispersive and imaginary or absorptive parts of the expressions for T and T^{-1} in equations (4.9), (4.20), (4.21) with $W^2 > (M + \mu)^2$

$$[\mathrm{Im}\, T(W^2, \Delta^2)]\,[\mathrm{Im}\, T^{-1}(W^2, \Delta^2)] = -[\sin\phi(W^2, \Delta^2)]^2 \qquad (4.22)$$

$$-[\mathrm{Im}\, T^{-1}(W^2, \Delta^2)] \cot\phi(W^2, \Delta^2) = \mathrm{Re}\, T^{-1}(W^2, \Delta^2) \qquad (4.23)$$

and since we have assumed that W^2 is close to an energy value M_r^2 where there is a sharp resonance expected then we put

$$\mathrm{Re}\, T^{-1}(W^2, \Delta^2) \approx g_r^{-2}(M_r^2 - W^2) \qquad (4.24)$$

If $T(W^2, \Delta^2)$ has real zeros they can be considered by the same adjustments we made for $G(z)$ and $\mathcal{S}(z)$ in Chapters II and III and give poles or resonances in $\mathrm{Im}\, T^{-1}(W^2, \Delta^2)$ which have little interest for us.

If $T(W^2, \Delta^2)$ has complex zeros, say at one point z_0, then we must include a term of the form

$$\frac{\alpha}{z - z_0} + \frac{\alpha^*}{z - z_0} + \frac{\alpha}{z - a^2 + z_0} + \frac{\alpha^*}{z - a^2 + z_0^*}$$

in $T^{-1}(z, \Delta^2)$. If $z_0 = x_0 + iy_0$ and y_0 is very small, i.e. the poles very close to the axis, then these terms produce resonance-like behaviour in $\mathrm{Im}\, T^{-1}$ or pole-like behaviour in T^{-1} which may affect the approximation in (4.24). However if resonances are predicted experimentally then we would expect that these complex zeros are not too important although the resonance shape may well be altered. One wonders whether any points which are candidates for resonances theoretically but are counteracted by complex zeros can ever be found to exist.

Ignoring complex zeros we put (4.24) in (4.23) and eliminate ϕ from (4.22) and (4.23) so giving the resonance formula for the imaginary or absorptive part of $T(W^2, \Delta^2)$ *

$$\mathrm{Im}\, T(W^2; \Delta^2) \approx g_r^2 \frac{M_r \Gamma(\Delta^2)}{(W^2 - M_r^2)^2 + [M_r \Gamma(\Delta^2)]^2} \tag{4.25}$$

where

$$\Gamma(\Delta^2) = \frac{g_r^2}{M_r} \cdot \mathrm{Im}\, T^{-1}(M_r^2, \Delta^2) \tag{4.26}$$

The re-application of this method for finding the resonance shape and lifetime to amplitudes other than the one we have specifically considered is quite simple.

In the case of T_1 which has the same form as T_3 but has no pole terms, we can write the inverse in the form (4.18). To get an exponential representation we must now choose the function $\log \left[\frac{(z - M_o^2)(z - a^2 + M_o^2)}{c'} \cdot T_1(z, \Delta^2)\right]$ such that the region $(a^2 - M_o^2) < W^2 < M_o^2$ is not cut. With these slight adjustments the formulae equivalent to (4.22) to (4.26) for T_1 are identical in form. The shape of T_1 between the cuts would seem to be roughly one of the following curves

* It is possible to use the model introduced in Chapter II to show that a pair of complex conjugate poles can be associated with this resonance. The analysis of $T(W^2, \Delta^2)$ towards this end is very similar to that given for $G(-k^2)$ in Chapter II but is only applicable if the resonance occurs in the elastic scattering or two-particle branch cut region of the physical cut.

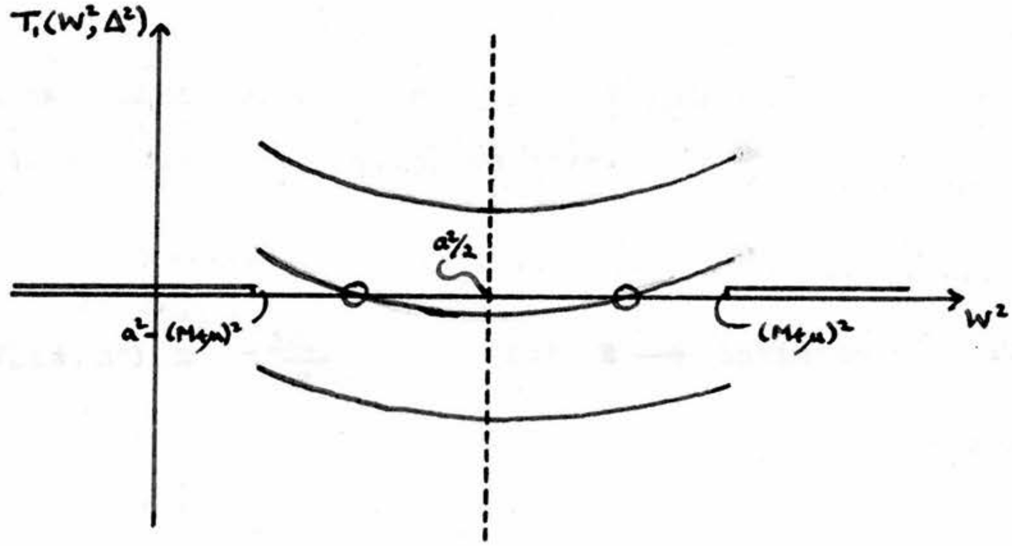

Fig. IV-3

assuming $\operatorname{Im} T_1$ is largely positive, knowing that $\frac{dT_1}{dW^2} = 0$ for $W^2 = \frac{a^2}{2}$ and using the symmetry about $\frac{a^2}{2}$. It is amusing to note the similarity $T_1(W^2, \Delta^2)$ has with $T_3^{-1}(W^2, \Delta^2)$ when each has two zeros.

The dispersion relation T_2 and T_4 can be treated similarly. For T_4 we have

$$T_4(W^2; \Delta^2) = -g^2\left[\frac{1}{W^2-a^2+M^2} + \frac{1}{W^2-M^2}\right] + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} \operatorname{Im} T_4(W'^2; \Delta^2)\, dW'^2 \cdot \left[\frac{1}{W'^2-W^2} - \frac{1}{W'^2-a^2+W^2}\right] \tag{4.27}$$

and picture between the cuts roughly

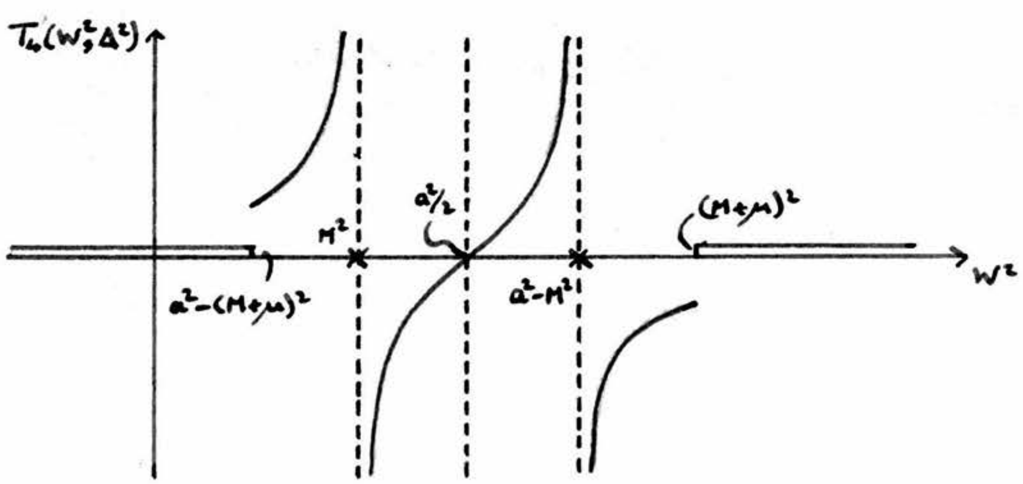

Fig. IV-4

Again one cannot conclusively predict complex or real zeros. Instead of (4.14) and (4.15) we have

$$T_4(z,\Delta^2) \approx -\frac{2c_4}{z} \qquad \text{for } z \longrightarrow \text{ infinite circle}$$

(4.28)

$$c_4 = g^2 + \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty} Im\, T_4(W'^2, \Delta^2)\, dW'^2$$

(4.29)

Hence the function $\left[(2z-a)(z-M_o^2)^{-1}(z-a^2+M_o^2)^{-1}T_4^{-1}(z,\Delta^2)+c_4^{-1}\right]$

is analytic apart from poles at $z = M_o^2$, $a^2 - M_o^2$ and the cuts of T_4 and converges to zero on the infinite circle. Therefore

$$T_4^{-1}(z,\Delta^2) = \frac{1}{(2z-a)}\Big\{\lambda_4(a^2-2M_o^2) - (z-M_o^2)(z-a^2+M_o^2).$$

$$.\left[c_4^{-1} - \frac{1}{\pi}\int_{(M+\mu)^2}^{\infty}\frac{(2W'^2-a^2)\, Im\, T_4^{-1}(W'^2, \Delta^2)\, dW'^2}{(W'^2-M_o^2)(W'^2-a^2+M_o^2)}\left(\frac{1}{W'^2-z}+\frac{-1}{W'^2-a^2+z}\right)\right]\Big\}$$

(4.30)

where we have preserved the anti-symmetry with respect to $z = \frac{a^2}{2}$, and we picture T_4^{-1} between the cuts as

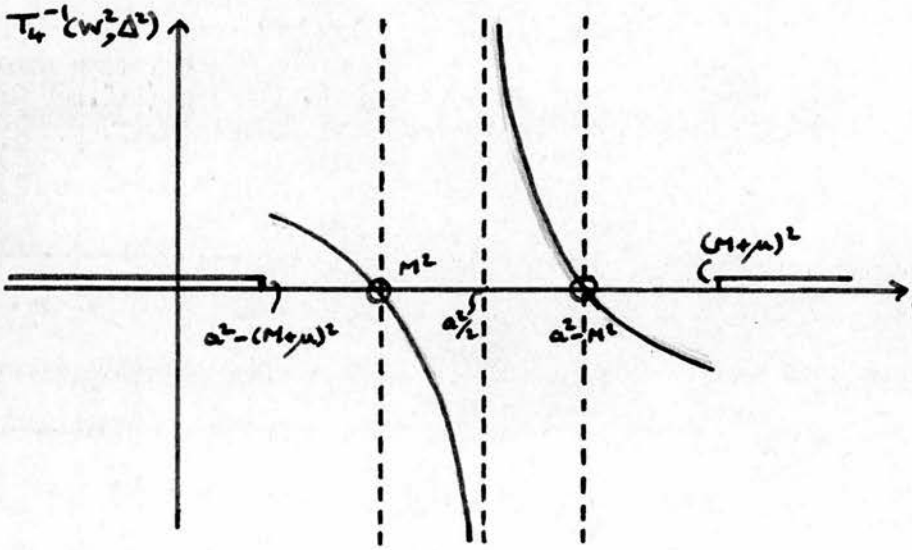

Fig. IV-5.

To obtain an exponential form for T_4 or T_4^{-1} we must consider the function $\log\left[\frac{(z-M^2)(z-a^2+M^2)}{c_4(a^2-2z)}.T_4(z,\Delta^2)\right]$. Once again similar answers to those of (4.22) to (4.26) appear.

For T_2 there are no pole terms and we picture T_2 as

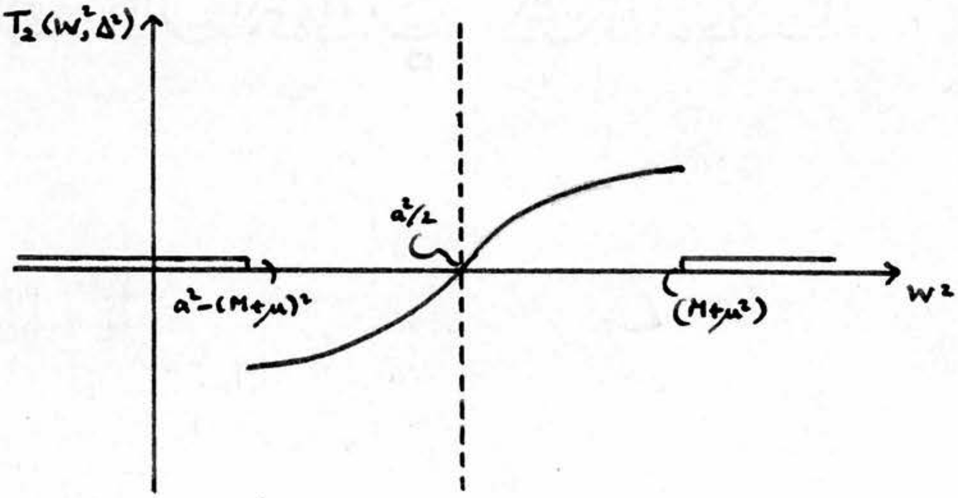

Fig. IV-6.

The form of T_2^{-1} is the same as T_4^{-1} but with a different value of λ_4 in general and to find the exponential form we consider the function $\log\left[\frac{(z-M_0^2)(z-a^2+M_0^2)}{c_2(a^2-2z)}.T_2(z,\Delta^2)\right]$.

The results then (4.25) and (4.26) may apply to T_1, T_2, T_3 and T_4 but in actual fact resonances may not appear in T_1 and T_2 for similar physical reasons that we gave in Chapter III for why we do not expect a resonance in $\mathcal{S}(-m)$. It is simply that T_1 and T_2 have no pole terms so if a resonance is due to a single particle contribution then it should not show up in T_1 or T_2 to give a pole like behaviour.

The application of these techniques to the other two-particle scattering amplitudes that we mentioned earlier is just as straightforward. Calculations using unitarity to find the dominant contributions to $\operatorname{Im} T^{-1}(W^2, \Delta^2)$ will provide a critical test of dispersion relations in predicting resonance peaks.

If (4.25) is approximately correct for all Δ^2 in the range $0 \leq \Delta^2 < \frac{1}{4}\mu(2M - \mu)$ then it makes a contribution to the total cross-section for $W^2 \approx M_r^2$ of

$$\sigma(W^2) = \frac{4}{M_r^2}\int_{-1}^{1} \frac{g_r^2}{|(M_r^2 - W^2) - iM_r\Gamma(\Delta^2)|^2}\, d(\cos\theta).$$

$$= \frac{8g_r^2}{K_r^2 M_r^2}\int_0^{K_r^2} \frac{d\Delta^2}{(W^2 - M_r^2)^2 + [M_r\Gamma(\Delta^2)]^2} \qquad (4.31)$$

where

$$K_r^2 = \frac{[W^2-(M+\mu)^2][W^2-(M-\mu)^2]}{4W^2} \approx \frac{[M_r^2-(M+\mu)^2][M_r^2-(M-\mu)^2]}{4M_r^2}$$

$< \frac{1}{4}\mu(2M-\mu)$ provided $M_r^2 < (M+2\mu)^2$ and $K_r^2 > 0$

since $M_r^2 > (M+\mu)^2 > (M-\mu)^2$.

It is easy to verify that $\sigma(W^2)$ in (4.31) has a maximum at $W^2 = M_r^2$ and is symmetric about $W^2 = M_r^2$. Thus we could write $\sigma(W^2)$ roughly as

$$\sigma(W^2) \approx \frac{8g_r^2}{M_r^2 K_r^2} \cdot \frac{1}{(W^2 - M_r^2)^2 + (M_r \bar{\Gamma})^2} \tag{4.32}$$

where $\bar{\Gamma}$ is the average width given approximately as

$$\bar{\Gamma} = \left\{ \int_0^{K_r^2} \frac{d\Delta^2}{[\Gamma(\Delta^2)]^2} \right\}^{-1/2} \tag{4.33}$$

If (4.32) is to be observable $\bar{\Gamma} <<<< M_r$ so that we have a sharp peak in $\sigma(W^2)$ at $W^2 = M_r^2$. Therefore we have a Breit-Wigner form of resonance in the total cross-section.

Elementary unstable particles or particlesunstable under strong interactions alone and decaying into a pion and a nucleon would therefore contribute a resonance of the form (4.32) to the pion-nucleon scattering cross-section.

A resonance in the unphysical region has a rather different behaviour as a function of W^2. We need not repeat the Schwinger technique again since we noted the symmetry property of the dispersion relations, i.e. symmetric to the replacement of W^2 by $a^2 - W^2$. Hence if we have a resonance in the physical region given by (4.25) then we must also have a resonance in the unphysical region at the symmetric point unless some selection rule forbids it and destroys the symmetry. The resonance in the unphysical region related by symmetry to (4.25) contributes to the absorptive part of the scattering amplitude the term

$$Im\, T(W^2, \Delta^2) \approx g_r^2 \frac{(a^2 - M_r^2)\,\Gamma'(\Delta^2)}{(W^2 - a^2 + M_r^2)^2 + \left[(a^2 - M_r^2)\Gamma'(\Delta^2)\right]^2} \qquad (4.34)$$

where

$$\Gamma'(\Delta^2) = \frac{g_r^2}{(a^2 - M_r^2)} \cdot Im\, T^{-1}(a^2 - M_r^2, \Delta^2)$$

$$= \frac{M_r^2}{(a^2 - M_r^2)} \cdot \Gamma(\Delta^2) \qquad (4.35)$$

This resonance makes a contribution to the total cross-section of the form, for W^2 in the unphysical region,

$$\sigma(W^2) = \frac{4}{W^2} \int_{-1}^{1} \frac{g_r^2}{(W^2 - a^2 + M_r^2)^2 + [M_r \Gamma(\Delta^2)]^2}\, d(\cos\theta) \qquad (4.36)$$

If $\Gamma(\Delta^2)$ is relatively insensitive to variations in Δ^2 and remains very small for $-1 \leq \cos\theta \leq 1$ then we can treat Γ as a constant and integrate (4.36) since $a^2 = 2K_r'^2(1-\cos\theta) + 2M^2 + 2\mu^2$ where

$$K_r'^2 = \frac{[W^2 - (M+\mu)^2][W^2 - (M-\mu)^2]}{4W^2} \qquad (4.37)$$

Or the maximum value of Δ^2 as $\cos\theta$ varies from -1 to +1 so again we have $0 \leq K_r'^2 < \frac{1}{4}\mu(2M-\mu)$ if approximately $(M-2\mu)^2 < W^2 < (M-\mu)^2$.*

---

* Actually the lower limit of W^2 should be slightly higher being roughly $27\mu^2$ for $M = 7\mu$.

Integrating (4.36) gives

$$\sigma(W^2) = \frac{2g_\tau^2}{K_\tau'^2 M_\tau \Gamma W^2} \cdot \tan^{-1}\left[\frac{4K_\tau'^2 M_\tau \Gamma}{M_\tau^2\Gamma^2 + (W^2-2M^2-2\mu^2+M_\tau^2)(W^2-2M^2-2\mu^2+M_\tau^2-4K_\tau'^2)}\right] \tag{4.38}$$

This has a much more complicated shape than that of the simple symmetric Breit-Wigner type. The shape can be seen more clearly if we write (4.38) in the form

$$\sigma(W^2) = \frac{8g_\tau^2}{M_\tau\Gamma(W^2-A^2)(W^2-B^2)} \tan^{-1}\left\{\frac{(W^2-A^2)(W^2-B^2)\Gamma}{M_\tau\left[(W^2-C^2)(W^2-D^2)+\Gamma^2D^2\right]}\right\} \tag{4.39}$$

where $A^2 = (M+\mu)^2$, $B^2 = (M-\mu)^2$, $C^2 = \frac{(AB)^2}{M^2} - \Gamma^2$, $D^2 = 2M^2 + 2\mu^2 - M_\tau^2$ and if we choose $M_\tau \approx 9\mu$, $M \approx 7\mu$ then $A^2 \approx 64\mu^2$, $B^2 \approx 36\mu^2$, $C^2 \approx 29\mu^2$, $D^2 \approx 19\mu^2$ while if $M_\tau \approx 8\mu \approx A$ then $B^2 \approx C^2 \approx D^2 \approx 36\mu^2$. The larger M_τ becomes compared to 9μ the smaller and more separate become C^2 and D^2 if M and μ are fixed. For $M_\tau \approx 10\mu$ in fact $D^2 \approx 0$ and becomes negative for $M_\tau > 10\mu$.

For the case $M_\tau \approx 9\mu$ we find a pronounced peak in $\sigma(W^2)$ between the values $\frac{1}{2}(C^2 + D^2)$ and C^2 i.e. $\approx 26\mu^2$. The shape can be drawn roughly as follows and the sharp variation from the maximum to point B^2 is due to the factors $(W^2 - A^2)^{-1}(W^2 - B^2)^{-1}$ which are relatively ineffective in the region between the origin and D^2.

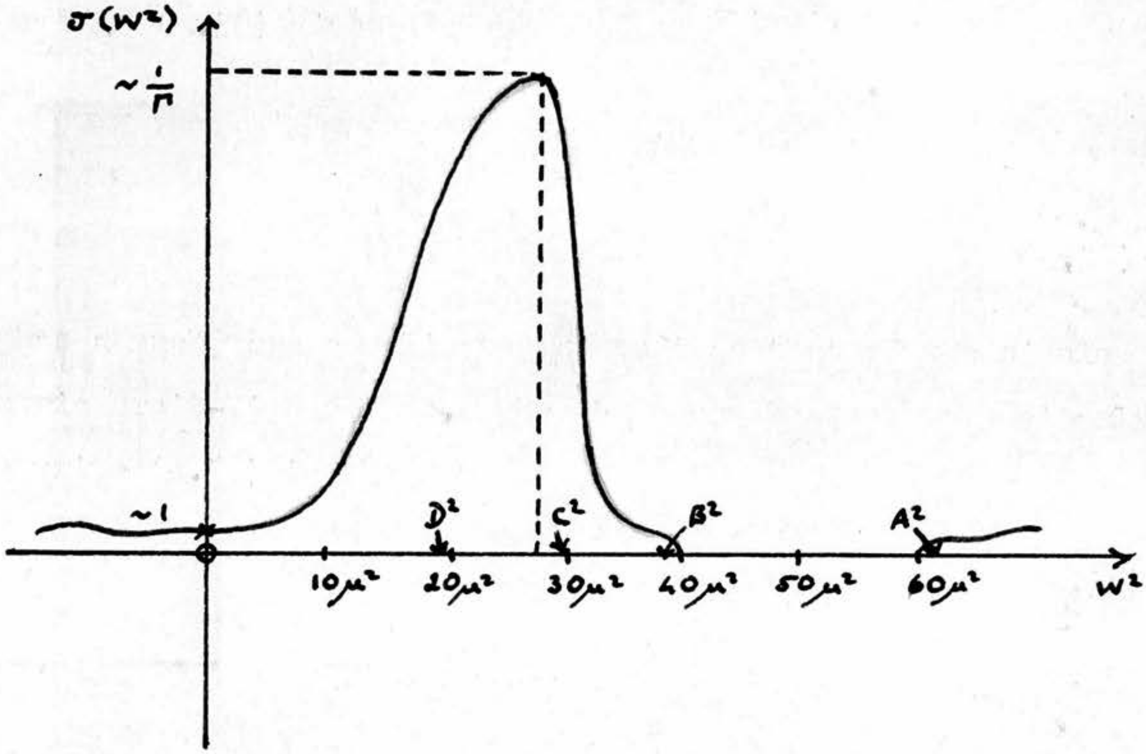

Fig. IV-7.

As M_π approaches $(M + \mu)$ and D^2 and C^2 approach B^2, the peak becomes narrower and narrower and more pole like. In contradistinction as M_π becomes large the resonance moves to the left and spreads more over a larger region of the axis. This curious behaviour can be explained by noticing that the resonance (4.34) is integrated over a range of cos θ which implies a movement of the zero of $R\ell\, T^{-1}$ i.e. the point $a^2 - M_\pi^{\;2}$ is spread over a range given by the range of Δ^2 as cos θ varies from -1 to $+1$. The range of Δ^2 is precisely $K_\pi'^2$ which is zero at $W^2 = (M - \mu)^2$ and so if the peak is close to $(M - \mu)^2$, it becomes concentrated near this point. If the peak is far from $(M - \mu)^2$, the range of Δ^2 is large and the resonance is spread out. The position and shape of the peak therefore depend on M_{γ}^{2} in just the manner we found from (4.39). A similar argument assuming the existence of an unphysical sheet pole has been given by Nauenberg and Pais[28] for a special case.

We now have a basis for inserting resonances due to elementary unstable particles or otherwise into dispersion relations derived from axiomatic field theory. The theory is capable of describing elementary stable and unstable particles on an equal footing apart from one very important aspect. So far we have only been able to discuss the scattering of stable external particles and have allowed unstable particles only in internal processes. No treatment has yet been evolved without some serious assumption or assumptions to deal with the scattering of unstable particles. We needed no wild assumptions in order to derive a spectral representation for unstable particle propagators. Unfortunately the conditions under which dispersion relations for scattering amplitudes have been proved in axiomatic field theory categorically exclude unstable particles in external states.[27]

In the spirit of the techniques we used on propagators the best we can do here is to add extra strong interaction absorptive cuts to the strong interaction cuts in (4.9) for example and assume the dispersion relations are still valid. Let us consider then that we have an excited nucleon of mass M^{*} capable of decaying into a nucleon and a pion under strong interactions but stable if we switch off some of the strong interactions. We take the dispersion relation for the scattering of this excited nucleon with a pion to be of the form

$$T(W^2, \Delta^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} m(W'^2, \Delta^2)\left[\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 - A^2 + W^2 + i\epsilon}\right] dW'^2 \quad (4.40)$$

where from (4.2)

$$m(W^2, \Delta^2) = \int_{-\infty}^{\infty} d^4x \, e^{-\frac{1}{2}i(k+k')x} \langle p' | [J(\tfrac{x}{2}), J(-\tfrac{x}{2})] | p \rangle$$

$$= (2\pi)^4 \sum_n \Big[\langle p' | J(0) | p_n \rangle \langle p_n | J(0) | p \rangle \, \delta(p_n - p - k) - \langle p' | J(0) | p_n \rangle \langle p_n | J(0) | p \rangle \, \delta(p_n - p + k') \Big] \quad (4.41)$$

and

$$Im\, T(W^2, \Delta^2) = \frac{1}{2}\left[m(W^2, \Delta^2) - m(A^2 - W^2, \Delta^2) \right]$$

$$A^2 = 4\Delta^2 + 2M^{*2} + 2\mu^2 \quad (4.42)$$

If we switch off the extra strong interactions causing the instability of the excited nucleon the lowest value of $-p_n^{\,2} = -(p + k)^2 = W^2$ for which $\mathrm{Im}\, T \neq 0$ is M^{*2} which denotes a discrete pole term and the continuous threshold begins at $(M^* + \mu)^2$. In fact we have the same situation as for pion-nucleon scattering but with M^* replacing M.

If we consider all possible strong interactions so that the excited nucleon is unstable, the lowest value of $-p_n^{\,2}$ is M^2 denoting a discrete term and the continuous threshold begins at $(M + \mu)^2$. This implies a system of cuts for $T(W^2, \Delta^2)$ of the form

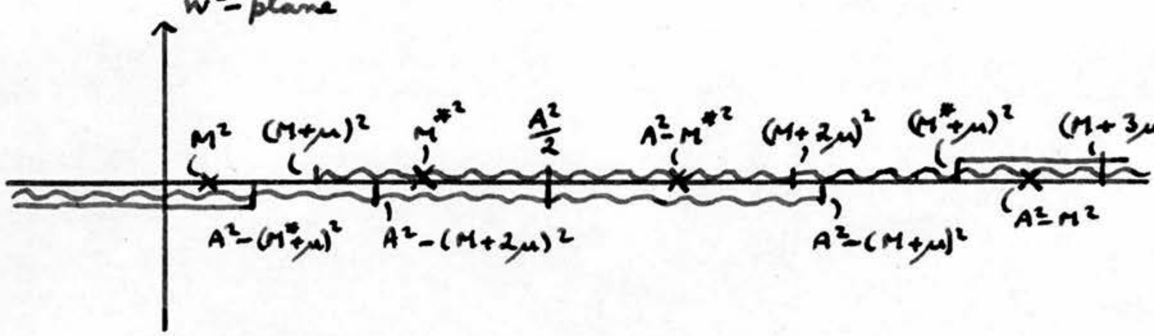

Fig. IV-8.

The question is what happens at the points M^{*2}, $A^2 - M^{*2}$, M^2 and $A^2 - M^2$. The effect of the right (left)-hand cuts on M^{*2} ($A^2 - M^{*2}$) should be just the broadening of a δ-function to a resonance shape in Im T that we are familiar with.

The dispersion relation (4.40) can be written using (4.42) as

$$T(W^2; \Delta^2) = \frac{1}{2\pi}\int_{M^2}^{\infty} dW'^2\, \mathcal{M}(W'^2; \Delta^2)\left[\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 - A^2 + W^2 + i\epsilon}\right]$$

$$= \frac{1}{\pi}\left\{\int_{A^2-M^2}^{\infty} dW'^2\, \mathrm{Im}\, T(W'^2; \Delta^2) + \int_{A^2/2}^{A^2-M^2} dW'^2\, \frac{[\mathcal{M}(W'^2; \Delta^2) - \mathcal{M}(A^2 - W'^2; \Delta^2)]}{2}\right\} \cdot$$

$$\cdot \left[\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 - A^2 + W^2 + i\epsilon}\right]$$

$$= \frac{1}{\pi}\int_{A^2/2}^{\infty} dW'^2\, \mathrm{Im}\, T(W'^2; \Delta^2)\left[\frac{1}{W'^2 - W^2 - i\epsilon} + \frac{1}{W'^2 - A^2 + W^2 + i\epsilon}\right] \quad (4.44)$$

It is straightforward to show that the points M^{*2} and $A^2 - M^{*2}$ are points of resonance similar to the previous sections of this chapter. The peak at $A^2 - M^{*2}$ in the total cross-section now occurs between the analogous points to C^2 and D^2 which are now between the points A^2 and B^2 so a sharp drop to zero can occur on either side of the peak.

The candidates for poles M^2 and $A^2 - M^2$ occur above the extra strong interaction thresholds $A^2 - (M^* + \mu)^2$ and $(M^* + \mu)^2$. If we assume the behaviour $Re\, T^{-1} \approx C_M^{-1}(M^2 - W^2)$ for example for $W^2 \approx M^2$ and C_M is a constant then we may not have a peak since $\mathrm{Im}\, T^{-1}(M^2, \Delta^2)$ is not necessarily small. It follows then that broad resonances appear likely which may or may not be observed. It has been conjectured[29)] that the resonance at $A^2 - M^2$ in the excited nucleon-pion scattering amplitude may produce a resonance in the pion-nucleon scattering amplitude in the same energy region, i.e. above the production threshold at $W^2 = (M + 2\mu)^2$. This can only be settled by a reasonably accurate calculation from the dynamics of the problem of coupling the nucleon-pion amplitude with the excited nucleon-pion amplitude using unitarity. We suggest however that the resonance at $(A^2 - M^{*2})$ in the excited nucleon-pion scattering may have some further resonance effect on the pion-nucleon scattering if a resonance at $(A^2 - M^2)$ exists.

The theory of unstable particles in dispersion relations is also incomplete in another respect. So far we have only been able to build in single unstable particle terms and have largely neglected any modifications to multiple particle thresholds in which one of the particles is unstable.

DISCUSSION

The primary purpose of this dissertation has been to clarify and extend a uniform theory of stable and unstable particles developed by Schwinger. Originally Schwinger based his ideas on the structure of a one-particle Green's function for a spinless boson field. We have carried out the generalisation to one-particle Green's functions for a spin one-half fermion field in detail and indicated the trivial extension to a vector boson field with mass. We have, however, taken care to avoid the use of separate P, C and T invariance but only required (PCT)-invariance when considering weak interaction phenomena. This does not alter the conclusions nor will subtractions even if they are necessary for the dispersion relations we use. The (PCT)-theorem ensures particles and anti-particles have equal masses and lifetimes. We have neglected electromagnetic thresholds which appear at poles and at zero energy since the character of their branch points is unknown, but these also seem unlikely to affect the resonance formulae since their contributions to spectral functions should be zero or small close to such branch points. The Schwinger method is essentially based on an 'intuitively obvious' result for a one-particle Green's function $G(z)$ of a complex variable z

$$\mathrm{Im}\, G(z) = \frac{\mathrm{Im}\, G^{-1}(z)}{[\mathrm{Re}\, G^{-1}(z)]^2 + [\mathrm{Im}\, G^{-1}(z)]^2}$$

whose proof requires great care with analytic properties of $G(z)$. We need only characterise unstable particles of mass μ or resonances in Im G by $Re\, G^{-1}(\mu^2) = 0$ and $Im\, G^{-1}(\mu^2) \neq 0$ but very small. Then $\left[Im\, G^{-1}(\mu^2)\right]^{-1}$ is essentially the lifetime or inverse width $\tau = 1/\gamma$. Also $Re\, G^{-1}(\mu^2) = 0$ may be caused by any singularity or combination of singularities. We have considered the case of strong and weak interactions acting together as the exact one while the case of strong interactions acting alone is purely hypothetical, but, because of the uncertainty in μ and γ , it was unnecessary to work with better than first, non-zero, order terms in γ. We have shown that these approximations are consistent if the condition for a resonance to be observed is true, i.e. $\gamma <<<< \mu$.

Much fundamental work has yet to be done towards formulating an operator algebra for unstable particles in field theory. We have proposed some properties of one-particle unstable states in view of their general validity in Quantum Mechanics and perturbation theory. We have defined one particle states without using the asymptotic condition so that particles are created and annihilated at a finite time. We cannot observe particles with perfect accuracy so a one-particle wave packet need not describe a single particle exactly but must be almost exact to be observed as a single particle at all. Therefore a wave packet may not be observed as an exact one-particle wave-packet due to one or both of the reasons (a) an imperfect experimental set-up, (b) a nonexact one-particle wave packet. The time dependence we observe is that of a one-particle wave packet rather than that of a Green's function. It is the filter effect of the unstable particle wave packet on the Green's function which takes account of the experimental measuring process rather than the artificial mass filter used by Schwinger. In spite of various correction terms to the exponential decay law for very large times, the effect of the experimental limitations is the strongest, unless we cannot avoid the inclusion of a threshold, electromagnetic for example, within the energy range for observing a single particle.

We set up a model of a decay process which may not be far from the true situation in full field theory but we have to assume maximal analyticity to make continuations into unphysical sheets of $G(s)$. Considering two-particle cuts we find not only a pole in the lower half of the first unphysical sheet associated with a resonance but also a complex conjugate pole. In an extended model with two channels we are not in agreement with Levy since we find no ambiguity in associating poles with particles or resonances. There is, however, only one pair of complex conjugate poles on a particular unphysical sheet which <u>directly</u> produce a resonance.

The techniques which we used for propagators is usefully applied to single dispersion relations for two-particle scattering amplitudes similar to work by Moffat. For the sake of rigour, however, it is necessary to consider only reactions initiating and terminating with stable particles. A new feature of such reactions may be the presence of high energy resonances in pion-nucleon scattering occurring in a manner suggested by R.F. Peierls and having a curious unsymmetrical shape.

Perhaps this more or less complete dispersion relation treatment of unstable particles or resonances can be used as a more rigorous basis for calculations of lifetimes even though it will be necessary to resort to perturbation theory to find the absorptive amplitude from unitarity.

In all the foregoing we have been unable to analyse multiple particle thresholds where one of the particles is unstable.

It is yet to be shown that field operators for unstable particles exist and satisfy the usual axioms apart from the asymptotic condition. The possible construction of such field operators provides another interesting problem which has been examined to some extent recently by Hama and Tanaka[16]. We have not tackled such problems here but have assumed that unstable particle field operators exist or can be constructed.

Lastly, it has not been demonstrated that the presence of a resonance in the absorptive part of a scattering amplitude or cross-section implies the presence of a particle. We obviously cannot assign a particle with every little bump discernible in the cross-sections of reactions. Clearly the use of the very concept of 'particle' becomes doubtful when one entertains such thoughts.

APPENDIX I : Notation in General

Any notation not explained here will be defined as it is introduced in the text.

$x_\mu = (x_1, x_2, x_3, ix_0) = (\underline{x}, ix_0)$ represents a four-vector and

$d^4x = dx_1 dx_2 dx_3 dx_0 = d^3\underline{x}\, dx_0$ is the corresponding four dimensional volume element.

$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is the metric.

$\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and zero otherwise is the Kronecker δ-symbol.

$x_\mu y_\mu = x \cdot y = \underline{x} \cdot \underline{y} - x_0 y_0$ is the scalar product of two four-vectors.

$\Box = \nabla^2 - \frac{\partial^2}{\partial x_0^2} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_0^2}$ is the D'Alembertian.

$\delta^4(x) = \delta(x_1)\delta(x_2)\delta(x_3)\delta(x_0) = \delta^3(\underline{x})\delta(x_0)$ where δ is the Dirac symbol.

$\theta(x) = 1$ if $x_0 > 0$, 0 if $x_0 < 0$.

The before a product of operators will symbolise the time ordering of these operators $T\phi(x)\phi(y) = \theta(x-y)\phi(x)\phi(y) + \theta(y-x)\phi(y)\phi(x)$

$$T\psi(x)\psi(y) = \theta(x-y)\psi(x)\psi(y) - \theta(y-x)\psi(y)\psi(x)$$

$[A, B] = AB - BA$ is the commutator of two operators A, B.

$\{A, B\} = AB + BA$ is the anti-commutator of two operators A, B.

Take the natural units $\hbar = c = 1$

Use $*$ for complex conjugate and $\dagger$ for hermitian conjugate.

A neutral, scalar or pseudoscalar, hermitian Heisenberg free field operator $\phi(x)$ satisfying the Klein-Gordon equation $(\Box - \mu^2)\phi(x) = 0$ describing particles of mass μ, has the commutation rules $[\phi(x), \phi(y)] = i\,\Delta(x-y)$ where the invariant function is defined

$$\Delta(x) = \frac{-i}{(2\pi)^3} \int_{-\infty}^{\infty} d^4k\, e^{ikx}\, \delta(k^2+\mu^2)\left[\theta(k_0) - \theta(-k_0)\right]$$

In order to define normalisable states we must use a discrete set of normalisable "wave-packet" solutions of positive energy $\{f_\alpha(x)\}$ of the Klein-Gordon equation so

$$f_\alpha(x) = \int_{-\infty}^{\infty} d^4k\, \theta(k_0)\, \delta(k^2+\mu^2)\, e^{ikx}\, \tilde{f}_\alpha(k)$$

These solutions form a linear vector space which becomes a Hilbert space by defining the scalar product

$$(f_\alpha, f_\beta) = i\int_{-\infty}^{\infty} d\sigma^\mu(x)\, f_\alpha^*(x) \overleftrightarrow{\frac{\partial}{\partial x_\mu}} f_\beta(x) = \delta_{\alpha\beta}$$

where
$$f_\alpha^* \overleftrightarrow{\frac{\partial}{\partial x_\mu}} f_\beta = \frac{\partial f_\alpha^*}{\partial x_\mu}\cdot f_\beta - f_\alpha^* \frac{\partial f_\beta}{\partial x_\mu}$$

and $d\sigma^\mu(x)$ is a space-like surface element with normal n_μ. This result is not dependent on n_μ since f_α and f_β obey the Klein-Gordon equation. The system $\{f_\alpha\}$ is complete and obeys

$$\sum_{\alpha=1}^{\infty} f_\alpha(x)\, f_\alpha^*(y) = i\,\Delta^{(+)}(x-y\,;\mu^2)$$

where

$$\Delta^{(+)}(x-y\,;\mu^2) = \frac{-i}{(2\pi)^3}\int_{-\infty}^{\infty} d^4k\, \theta(k_0)\, \delta(k^2+\mu^2)\, e^{ik(x-y)}$$

$$\Delta^{(-)}(x-y\,;\mu^2) = \frac{i}{(2\pi)^3}\int_{-\infty}^{\infty} d^4k\,\theta(-k_0)\,\delta(k^2+\mu^2)\,e^{ik(x-y)}$$

$$= -\Delta^{(+)}(y-x\,;\mu^2)$$

$$\Delta(x-y\,;\mu^2) = \Delta^{(+)}(x-y\,;\mu^2) + \Delta^{(-)}(x-y\,;\mu^2)$$

Whenever it is permissible and convenient we shall replace the $\{f_\alpha(x)\}$ by a continuous system of plane wave amplitudes $e^{i\underline{k}\cdot\underline{x} - i\sqrt{\underline{k}^2+\mu^2}\,x_0}$ and $\sum_{\alpha=1}^{\infty}$ by $\int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^3}\cdot\theta(k_0)\,\delta(k^2+\mu^2)$. This is an invariant choice of plane waves which implies

$$(f_\alpha, f_{\alpha'}) = 2(2\pi)^3 k_0\,\delta(\underline{k}-\underline{k}')$$

in the limit of plane waves.

We can expand $\phi(x)$ in the form

$$\phi(x) = \sum_{\alpha=1}^{\infty}\left(f_\alpha(x)\,\phi^\alpha(x_\mu) + f_\alpha^*(x)\,\phi^{\alpha+}(x_\mu)\right)$$

where

$$\phi^\alpha(x_\mu) = -i\int_{-\infty}^{\infty} d\sigma^\mu(x)\,\phi(x)\,\overleftrightarrow{\frac{\partial}{\partial x_\mu}}\,f_\alpha^*(x)$$

$$\phi^{\alpha+}(x_\mu) = i\int_{-\infty}^{\infty} d\sigma^\mu(x)\,\phi(x)\,\overleftrightarrow{\frac{\partial}{\partial x_\mu}}\,f_\alpha(x)$$

which are independent of x_μ for free fields.

Further $[\phi^\alpha, \phi^{\beta+}] = \delta_{\alpha\beta}$ and single particle states are formed as follows $|\alpha\rangle = \phi^{\alpha+}|0\rangle$ having unit norm.

Finally we shall often have occasion to use a quantity $\varepsilon \gtrsim 0$ which is to be regarded as a very small positive number and is allowed to tend to zero after all other operations are completed.

APPENDIX 2 : Miscellaneous Topics from Chapter II.

High Energy behaviour of $G(-k^2)$

If $\int_0^\infty s(\kappa^2).d\kappa^2$ is convergent we have the more detailed asymptotic property $G(-k^2) \approx (k^2+\mu_0^2)^{-1}$ where

$$\mu_0^2 = \lambda^2 + s(\infty) = \int_0^\infty \kappa^2 \rho(\kappa^2)\, d\kappa^2$$ and therefore

$$G^{-1}(-k^2) = k^2 + \mu_0^2 - \int_0^\infty \frac{\kappa^2 s(\kappa^2)\, d\kappa^2}{\kappa^2 + k^2 - i\epsilon}$$

The quantity μ_0 defined here is identical with the quantity called the bare mass in pseudoscalar meson theory but the definition above is independent of the type of interaction. Schweber's[24] interpretation of this asymptotic property is that at large values of the momentum or small space-time distances the propagator $G(x)$ is determined by the bare mass, since x is very small the self-coupling has had no time to take effect. Ford,[23] however, is of the opinion that this asymptotic property implies a very sharp resonance peak at very large energy which may be due to some extremely heavy quasi-stable particle with the same quantum numbers as the neutral boson usually associated with the propagator.

Resonance formula derivation

In Schwinger's calculations he appears to set $s_w(-k^2) = s_w(\mu^2)$ in obtaining the approximate equation (2.48). Consider the equations from which (2.48) is to be obtained

$$-\pi k^2 s_w(-k^2) \cot\phi(-k^2) \approx \rho_0^{-1}(k^2+\mu^2)$$

We note here that $s_W(-k^2) > 0$ if $-k^2 > K_W^{\ 2}$ from earlier reasoning so that it is $\cot \phi(-k^2)$ which has the zero or near zero at $-k^2 = \mu^2$. In fact we expect ϕ to have a behaviour which is almost discontinuous, since $-k^2 = \mu^2$ is almost a pole in $G(-k^2)$, and therefore varies rapidly from a value near π to a value near zero close to $-k^2 = \mu^2$, i.e. passes through $\pi/2$ where $\cot \phi = 0$. Hence near $-k^2 = \mu^2$, $\phi(-k^2)$ varies rapidly and the centre of the variation occurs at $-k^2 = \mu^2$. If we now write

$$\pi \left\{ \mu^2 s_W(\mu^2) - (k^2+\mu^2) \left[\frac{d}{d(-k^2)} (-k^2 s_W(-k^2)) \right]\Big|_{-k^2=\mu^2} \right\} . \cot \phi(-k^2)$$

$$\approx \rho_0^{-1} (+k^2+\mu^2)$$

then we can neglect $(k^2+\mu^2).\cot \phi(-k^2) . \left[\frac{d}{d(-k^2)} (-k^2 s_W(-k^2)) \right]\Big|_{-k^2=\mu^2}$

compared to the other terms in this approximate equation provided $-k^2$ is near enough to μ^2. Therefore

$$\pi \mu^2 s_W(\mu^2) . \cot \phi(-k^2) \approx \rho_0^{-1} (k^2+\mu^2)$$

and with

$$-\pi^2 k^2 s_W(-k^2) \rho_W(-k^2) = [\sin \phi(-k^2)]^2$$

we have

$$-\pi^2 k^2 s_W(-k^2) \rho_W(-k^2) \approx \frac{[\pi \rho_0 \mu^2 s_W(\mu^2)]^2}{(k^2+\mu^2)^2 + [\pi \rho_0 \mu^2 s_W(\mu^2)]^2}$$

and since we expect ρ_W to have the large behaviour at $-k^2 = \mu^2$ we can write the left hand side as $\pi \mu^2 s_W(\mu^2) \rho_W(-k^2)$.

We have therefore shown the approximations involved in writing (2.48) and (2.49) to be consistent and valid if $-k^2$ is close enough to μ^2.

Zeros of $G(-k^2)$

If $G_s(-k^2)$ has a zero at $-k^2 = \nu^2$ then a term $k^2 s_0/(\nu^2+k^2)$ must be added to $G_s^{-1}(-k^2)$ in equation (2.32). A term $k^2 s_0'/(\nu'^2+k^2)$ must also be added to $G^{-1}(-k^2)$ in equation (2.37) provided $\nu^2 < K_W^2$ since the introduction of weak interactions to G_s^{-1} may not mean cutting the axis at $-k^2 = \nu^2$ but will move the zero and alter the residue a little at least. These extra terms will then appear in the approximate form of $R\ell\, G^{-1}$ near μ^2 and thereafter alter the formulae (2.47) only so that

$$\mu^2 = \mu_s^2 + \rho_0^s\left[\lambda_W^2 - \frac{\mu^2 s_0'}{\nu'^2-\mu^2} + \frac{\mu^2 s_0}{\nu^2-\mu^2} - \mu^2\, Pf\!\int_{K_W^2}^{\infty} \frac{s_W(\kappa^2)\,d\kappa^2}{\kappa^2-\mu^2}\right]$$

$$(\rho_0)^{-1} = (\rho_0^s)^{-1} + \frac{\nu'^2 s_0'}{(\nu'^2-\mu^2)^2} - \frac{\nu^2 s_0}{(\nu^2-\mu^2)^2} + Pf\!\int_{K_W^2}^{\infty} \frac{d\kappa^2}{\kappa^2-\mu^2}\cdot\frac{d}{d\kappa^2}\left(\kappa^2 s_W(\kappa^2)\right)$$

where the extra terms tend to cancel since we expect $\nu^2 \approx \nu'^2$ and $s_0 \approx s_0'$.

If we have $\nu^2 > K_W^2$ then $G^{-1}(-k^2)$ has no pole at ν^2 but we expect a pole-like behaviour near $-k^2 = \nu^2$ which will produce terms in μ^2 and $(\rho_0)^{-1}$ to almost cancel $\mu^2 s_0 \rho_0^s/(\nu^2-\mu^2)$ and $-\nu^2 s_0/(\nu^2-\mu^2)^2$. In fact, if we put for $-k^2 \approx \nu^2$

$$R\ell\, G_s(-k^2) \approx -(s_0\nu^2)^{-1}(\nu^2+k^2)$$

and find for $-k^2 \approx \nu^2 \approx \nu'^2$

$$R\ell\, G(-k^2) \approx -(s_0'\nu'^2)^{-1}(\nu'^2+k^2)$$

where

$$\nu'^2 = \nu^2 - s_0\nu^2\, Pf\!\int_{K_W^2}^{\infty} \frac{\rho_W(\kappa^2)\,d\kappa^2}{\kappa^2-\nu^2}$$

$$(s_0'\nu'^2)^{-1} = (s_0\nu^2)^{-1} + Pf\!\int_{K_W^2}^{\infty} \frac{\rho_W(\kappa^2)\,d\kappa^2}{(\kappa^2-\nu^2)^2}$$

then we also have

$$s_W(\kappa^2) \approx \frac{s_0'}{\pi} \cdot \frac{(\beta\nu')}{(\kappa^2-\nu'^2)^2 + (\beta\nu')^2}$$

where

$$\beta = \pi s_0' \nu' \rho_W(\nu'^2)$$

This term produces the pole-like behaviour we expected in G^{-1} and indicates what happens to zeros of $G_s(-k^2)$ when covered by a weak interaction cut. They do not appear to have any deep significance and we shall ignore them whenever we can.

Consistency of resonance formula.

If we calculate the contribution the approximate formula for the Breit-Wigner resonance makes to the propagator, we find

$$G(-k^2) = \int_{K_W^2}^{\infty} \frac{d\kappa^2}{\kappa^2 + k^2 - i\epsilon} \cdot \frac{\rho_0}{\pi} \cdot \frac{\gamma\mu}{(\kappa^2-\mu^2)^2 + (\gamma\mu)^2}$$

$$= \frac{\rho_0 \,\Theta(-k^2-K_W^2)}{-k^2-\mu^2-i\gamma\mu} + \frac{\rho_0}{\pi} \cdot \frac{\left[\tan^{-1}\left(\frac{\gamma\mu}{\mu^2-K_W^2}\right) - \gamma\mu \log \frac{|K_W^2+k^2|}{\sqrt{(K_W^2-\mu^2)^2+(\gamma\mu)^2}}\right]}{(\mu^2+k^2)^2 + (\gamma\mu)^2}$$

The first term is the one we expect from the approximations we made in putting $\mathrm{Re}\, G^{-1}(-k^2) \approx \rho_0^{-1}(-k^2-\mu^2)$ with $\mathrm{Im}\, G^{-1}(\mu^2) = -\frac{\gamma\mu}{\rho_0}$ The extra terms will be negligible if $\mu^2 - K_W^2 >>>> \gamma_\mu$ which is satisfied if the resonance is sharp enough to dominate the correction terms to the exponential decay laws. This condition is essentially the result (1.11).

Therefore the approximations made in reaching the resonance formula for $\rho^W(K^2)$ are consistent with one another if we have a sharp enough resonance or more accurately if the particle condition stated in Chapter I for unstable particles is satisfied.

APPENDIX 3 : "Two Particle" Unitarity and Cross-Sections.

Here we show that the unitarity condition can be written concisely in the form (2.60) given in the main text. We also exhibit our conventions for defining the partial wave expansion of amplitudes used in the main text, deriving the one particle contributions to the absorptive part of a scattering amplitude, and defining the differential cross-section. For simplicity we will be considering only the centre of mass frame of reference which is sufficient and convenient for our purposes.

The definition of the imaginary part of the propagator we found in (2.12) to be $\pi\rho(-p^2)$ where from (2.9)

$$\theta(-p^2)\,\theta(p_0)\,\rho(-p^2) = (2\pi)^3 \sum_{\alpha} |\langle 0|\phi(0)|k_1 k_2\, in\rangle|^2$$

and now we consider the case when $|p, \alpha\rangle = |k_1\, k_2 \;\; in\rangle$ a two particle in-going state with $-k_1^{\,2} = m_1^{\,2}$, $-k_2^{\,2} = m_2^{\,2}$ and $p = k_1 + k_2$, therefore choosing the Lorentz frame $\underline{p} = \underline{k}_1 + \underline{k}_2 = 0$

$$\theta(-p^2)\,\theta(p_0)\,\rho(-p^2) = (2\pi)^3 \iint_{-\infty}^{\infty} \frac{d^4k_1\, d^4k_2}{(2\pi)^6}\, \delta(k_1^2+m_1^2)\,\delta(k_2^2+m_2^2)\,\theta(k_{10})\,\theta(k_{20}).$$

$$.\,\delta^4(p-k_1-k_2)\,\delta^3(\underline{k}_1+\underline{k}_2).\, |\langle 0|\phi(0)|\, k_1 k_2\, in\rangle|^2$$

$$= \frac{1}{(2\pi)^3} \iint_{-\infty}^{\infty} \frac{d^3[\frac{1}{2}(\underline{k}_1-\underline{k}_2)]\, d^3(\underline{k}_1+\underline{k}_2)}{4k_{10}k_{20}}\,.\,\delta^3(\underline{k}_1+\underline{k}_2)\,\delta(p_0-k_{10}-k_{20}).\,|\langle 0|\phi(0)|k_1 k_2\, in\rangle|^2$$

$$= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^3\underline{k}}{4k_{10}k_{20}}\, \delta\left(p_0 - \sqrt{\underline{k}^2+m_1^2} - \sqrt{\underline{k}^2+m_2^2}\right).\, |\langle 0|\phi(0)|\, k_1 k_2\, in\rangle|^2$$

$$= \frac{1}{(2\pi)^2} \int_{0}^{\infty} \frac{d\left(\sqrt{\underline{k}^2+m_1^2}+\sqrt{\underline{k}^2+m_2^2}\right)}{2}\,.\,\frac{|\underline{k}|}{(k_{10}+k_{20})}\,.\,\delta\left(p_0-\sqrt{\underline{k}^2+m_1^2}-\sqrt{\underline{k}^2+m_2^2}\right).\,|\langle 0|\phi(0)|k_1 k_2\, in\rangle|^2$$

where we have used the fact that $\langle 0|\phi(0)|k_1 k_2 \text{ in}\rangle$ depends only on $|\underline{k}|$ by invariance and we write

$$\langle 0|\phi(0)|k_1 k_2 \text{ in}\rangle = (16\pi)^{1/2}\, \mathcal{V}[-(k_1+k_2)^2]$$

where $\mathcal{V}$ is closely related to what is known as the invariant 'vertex' function which is defined later. We can finally write as the general result

$$\rho(-p^2) = \frac{1}{\pi}\cdot\frac{\left[(-p^2)-(m_1+m_2)^2\right]^{1/2}\left[(-p^2)-(m_1-m_2)^2\right]^{1/2}}{(-p^2)}\cdot|\mathcal{V}(-p^2)|^2$$

We can derive a similar result for the absorptive part of $\mathcal{V}(W^2)$, which we can obtain by replacing $i\theta(-x_0)$ by $\frac{1}{2}$ in the following form for $\mathcal{V}(W^2)$, where $(\Box - m_1{}^2)\phi_{k_1}(x) = J(x)$, using the usual Lehmann, Symanzik and Zimmermann[1)] techniques

$$\mathcal{V}(W^2) = \frac{i}{2(16\pi)^{1/2}}\int_{-\infty}^{\infty} d^4x\, e^{ik_1x}\langle 0|[\phi(0), J(x)]|k_2\rangle\, \theta(-x_0)$$

and so

$$\mathrm{Abs}\,\mathcal{V}(W^2) = \frac{1}{4(16\pi)^{1/2}}\int_{-\infty}^{\infty} d^4x\, e^{ik_1x}\langle 0|[\phi(0), J(x)]|k_2\rangle$$

$$= \frac{1}{4(16\pi)^{1/2}}\int_{-\infty}^{\infty} d^4x\, e^{ik_1x}\sum_n\Big\{\langle 0|\phi(0)|p',n\rangle\langle p',n|J(x)|k_2\rangle - \langle 0|J(x)|p',n\rangle\langle p',n|\phi(x)|k_2\rangle\Big\}$$

$$= \frac{(2\pi)^4}{4(16\pi)^{1/2}}\sum_n \delta(p'-k_1-k_2)\langle 0|\phi(0)|p',n\rangle\langle p',n|J(0)|k_2\rangle$$

$$= \frac{(2\pi)^4}{4(16\pi)^{1/2}}\iint_{-\infty}^{\infty}\frac{d^4k_1'\,d^4k_2'}{(2\pi)^6}\delta(k_1'^2+m_1^2)\delta(k_2'^2+m_2^2)\theta(k_{1_0}')\theta(k_{2_0}')\delta^3(\underline{k}_1'+\underline{k}_2')\delta^4(k_1'+k_2'-k_1-k_2).$$
$$.\langle 0|\phi(0)|k_1'k_2' \text{ in}\rangle\langle k_1'k_2' \text{ in}|J(0)|k_2\rangle$$

$$= \frac{1}{4(2\pi)^2(16\pi)^{1/2}}\int_0^{\infty} d\left(\sqrt{\underline{k}'^2+m_1^2}+\sqrt{\underline{k}'^2+m_2^2}\right)\frac{|\underline{k}'|}{4(k_{1_0}'+k_{2_0}')}\int_{-1}^{1} d(\cos\theta).\,\delta(k_{1_0}'+k_{2_0}'-k_{1_0}-k_{2_0}).$$
$$.\langle 0|\phi(0)|k_1'k_2' \text{ in}\rangle\langle k_1'k_2' \text{ in}|J(0)|k_2\rangle$$

where we have put $|p', n\rangle = |k_1' k_2' \text{ in}\rangle$ to find the two particle state contribution with $p' = k_1' + k_2'$, $-k_1'^2 = m_1^2$, $-k_2'^2 = m_2^2$ $\underline{k}_1' + \underline{k}_2' = \underline{p}' = 0$. We have also used the fact that $\langle 0|\phi(0)|k_1'k_2' \text{ in}\rangle\langle k_1'k_2' \text{ in}|J(0)|k_2\rangle$ depends only on $|\underline{k}'|$ and $\cos\theta = \underline{k}_1 \cdot \underline{k}_1' / |\underline{k}_1| . |\underline{k}_1'|$ by invariance and we put

$$\langle k_1' k_2' \text{ in}| J(0) |k_2\rangle = 16\pi \left\{ F\left[-(k_1+k_2)^2, \cos\theta\right]\right\}^*$$

where $F(W^2, \cos\theta)$ is the invariant elastic two-particle scattering amplitude and we have $4\Delta^2 = -(k_1 - k_1')^2 = -2|\underline{k}|(1-\cos\theta)$ as the invariant momentum transfer. Therefore in general

$$\begin{aligned} \text{Abs}\, \mathcal{V}(W^2) &= \frac{\left[W^2 - (m_1+m_2)^2\right]^{1/2}\left[W^2 - (m_1-m_2)^2\right]^{1/2}}{W^2} \mathcal{V}(W^2)\frac{1}{2}\int_{-1}^{1} d(\cos\theta)\left[F(W^2, \cos\theta)\right]^* \\ &= \frac{\left[W^2 - (m_1+m_2)^2\right]^{1/2}\left[W^2 - (m_1-m_2)^2\right]^{1/2}}{W^2} \mathcal{V}(W^2)\left[F_0(W^2)\right]^* \end{aligned}$$

where we have used $k_{1_0} = k_{1_0}'$, $k_{2_0} = k_{2_0}'$ since $|\underline{k}| = |\underline{k}'|$ and we are considering an elastic scattering amplitude $F(W^2, \cos\theta)$. We have also defined the partial wave expansion

$$F(W^2, \cos\theta) = \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta)\, F_l(W^2)$$

$$F_l(W^2) = \frac{1}{2}\int_{-1}^{1} d(\cos\theta)\, P_l(\cos\theta)\, F(W^2, \cos\theta)$$

Elastic unitarity for the full amplitude $F(W^2, \cos\theta)$ is obtained by writing

$$\begin{aligned} F(W^2, \cos\theta) &= (16\pi)^{-1}\langle k_2| J(0) |k_1' k_2' \text{ in}\rangle \\ &= \frac{i}{16\pi}\int_{-\infty}^{\infty} d^4x\, e^{-\frac{1}{2}i(k_1+k_1')x}\langle k_2|\left[J(\tfrac{x}{2}), J(-\tfrac{x}{2})\right]|k_2'\rangle\, \theta(x) \end{aligned}$$

and therefore the absorptive part is

$$\mathrm{Abs}\,F(W^2;\cos\theta) = \frac{1}{32\pi}\int_{-\infty}^{\infty} d^4x\, e^{-\frac{1}{2}i(k_1+k_1')x} \langle k_2|[J(\tfrac{x}{2}),J(-\tfrac{x}{2})]|k_2'\rangle$$

$$= \frac{1}{32\pi}\int_{-\infty}^{\infty} d^4x\, e^{-\frac{1}{2}i(k_1+k_1')x} \sum_n \{\langle k_2|J(\tfrac{x}{2})|p'',n\rangle\langle p'',n|J(-\tfrac{x}{2})|k_2'\rangle$$

$$- \langle k_2|J(-\tfrac{x}{2})|p'',n\rangle\langle p'',n|J(\tfrac{x}{2})|k_2'\rangle\}$$

and putting a two-particle in-going state $|k_1'' k_2''\ \mathrm{in}\rangle$ instead of $|p'',n\rangle$ with $-k_1''^2=m_1^2$, $-k_2''^2=m_2^2$, $k_1''+k_2''=p''=k_1+k_2$, $\underline{k}_1''+\underline{k}_2''=0$

$$\mathrm{Abs}\,F(W^2;\cos\theta) = \frac{(2\pi)^3}{16}\sum_n \delta(p''-k_1-k_2)\langle k_2|J(0)|p'',n\rangle\langle p'',n|J(0)|k_2'\rangle$$

$$= \frac{(2\pi)^3}{16}\iint_{-\infty}^{\infty}\frac{d^4k_1''\,d^4k_2''}{(2\pi)^6}\,\delta(k_1''^2+m_1^2)\,\delta(k_2''^2+m_2^2)\,\theta(k_{10}'')\,\theta(k_{20}'')\,\delta(\underline{k}_1''+\underline{k}_2'')\,\delta(k_1''+k_2''-k_1-k_2)\,.$$

$$.\langle k_2|J(0)|k_1k_2\,\mathrm{in}\rangle\langle k_1k_2\,\mathrm{in}|J(0)|k_2'\rangle$$

$$= \frac{[W^2-(m_1+m_2)^2]^{1/2}[W^2-(m_1-m_2)^2]^{1/2}}{W^2}\cdot\frac{1}{4\pi}\int d\Omega(\underline{k}_1'')\,F\left(W^2,\frac{\underline{k}_1''.\underline{k}_1}{|\underline{k}_1''||\underline{k}_1|}\right)\left[F\left(W^2,\frac{\underline{k}_1''.\underline{k}_1'}{|\underline{k}_1''||\underline{k}_1'|}\right)\right]^*$$

where $\int d\Omega(\underline{k}_1'')$ contains the integrations over the angular directions of $\underline{k}_1''$ only.

If we expand $F(W^2;\cos\theta)$, $F\left(W^2,\frac{\underline{k}_1''.\underline{k}_1}{|\underline{k}_1''||\underline{k}_1|}\right)$ and $F\left(W^2,\frac{\underline{k}_1''.\underline{k}_1'}{|\underline{k}_1''||\underline{k}_1'|}\right)$ in partial waves we find that the last equation can be written

$$\mathrm{Abs}\,F_\ell(W^2)$$

$$= \frac{[W^2-(m_1+m_2)^2]^{1/2}[W^2-(m_1-m_2)^2]^{1/2}}{W^2}\sum_{\ell'=0}^{\infty}\sum_{\ell''=0}^{\infty}\frac{1}{2\pi}\cdot(\ell'+\tfrac{1}{2})(\ell''+\tfrac{1}{2})\int_{-1}^{1}d(\cos\theta)\int d\Omega(\underline{k}_1'')\,.$$

$$.\,P_\ell(\cos\theta)\,P_{\ell'}\left(\frac{\underline{k}_1''.\underline{k}_1}{|\underline{k}_1''||\underline{k}_1|}\right)P_{\ell''}\left(\frac{\underline{k}_1''.\underline{k}_1'}{|\underline{k}_1''||\underline{k}_1'|}\right)F_{\ell'}(W^2)\left[F_{\ell''}(W^2)\right]^*$$

$$= \frac{[W^2-(m_1+m_2)^2]^{1/2}[W^2-(m_1-m_2)^2]^{1/2}}{W^2}\cdot|F_\ell(W^2)|^2$$

if

$$\int_{-1}^{1} d(\cos\theta)\int d\Omega(\underline{k}_1'')\, P_\ell(\cos\theta)\, P_{\ell'}\left(\frac{\underline{k}_1''.\underline{k}_1}{|\underline{k}_1''||\underline{k}_1|}\right) P_{\ell''}\left(\frac{\underline{k}_1''.\underline{k}_1'}{|\underline{k}_1''||\underline{k}_1'|}\right) = \frac{2\pi}{(\ell'+\frac{1}{2})(\ell''+\frac{1}{2})}\,\delta_{\ell\ell''}\,\delta_{\ell'\ell''}$$

after carrying out the integration of

$$\frac{1}{2\pi}(\ell'+\tfrac{1}{2})(\ell''+\tfrac{1}{2})\int_{-1}^{1} d(\cos\theta)\int d\Omega(\underline{k}_1'')\, P_\ell(\cos\theta)\, P_{\ell'}\left(\frac{\underline{k}_1''.\underline{k}_1}{|\underline{k}_1''||\underline{k}_1|}\right) P_{\ell''}\left(\frac{\underline{k}_1''.\underline{k}_1'}{|\underline{k}_1''||\underline{k}_1'|}\right)$$

and to do this we put $\underline{k}_1 = (0, 0, 1)$, $\underline{k}_1' = (\sin\theta, 0, \cos\theta)$ and $\underline{k}_1'' = (\sin\psi\cos\phi, \sin\psi\sin\phi, \cos\psi)$ so the integration become

$$\frac{1}{2\pi}(\ell'+\tfrac{1}{2})(\ell''+\tfrac{1}{2})\int_{-1}^{1} d(\cos\theta)\int_{-1}^{1} d(\cos\psi)\int_{0}^{2\pi} d\phi\; P_\ell(\cos\theta)\, P_{\ell'}(\cos\psi)\, P_{\ell''}(\cos\psi\cos\theta+\sin\psi\sin\theta\cos\phi)$$

$$= \delta_{\ell\ell''}\,\delta_{\ell'\ell''}$$

if we use the standard results for Legendre polynomials,

$$P_n(\cos\theta\cos\psi+\sin\theta\sin\psi\cos\phi)$$

$$= P_n(\cos\theta)P_n(\cos\psi) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\psi)\cos m\phi$$

$$(n+\tfrac{1}{2})\int_{-1}^{1} P_m(x)P_n(x)\,dx = \delta_{mn}\,, \quad \int_{-1}^{1} P_m(x)P_n^r(x)\,dx = 0 \quad \text{if } r \neq 0.$$

Thence it follows that we have the required result

$$\text{Abs}\, F_\ell(W^2) = \frac{\left[W^2-(m_1+m_2)\right]^{1/2}\left[W^2-(m_1-m_2)^2\right]^{1/2}}{W^2}\cdot|F_\ell(W^2)|^2$$

The one-particle terms in Abs $F(W^2, \cos\theta)$ can be got from the above results by putting $|p'', n\rangle = |k\rangle$ a one particle state with $-k^2 = m^2$ say and $\underline{k} = 0$, $k = k_1 + k_2 = p''$

$$\text{Abs}\, F(W^2; \cos\theta) = \frac{(2\pi)^3}{16} \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^3} \delta(k^2+m^2)\,\theta(k_0)\,\delta(\underline{k})\,\delta(k-k_1-k_2) .$$

$$. \langle k_2 | J(0) | k \rangle \langle k | J(0) | k_2' \rangle$$

$$= \frac{\delta(k_0 - k_{10} - k_{20})}{2k_0} . \pi . \left| V\left[-(k_1+k_2)^2\right] \right|^2$$

$$= \pi\,\delta(W^2 - m^2) . |V(m^2)|^2 \; = \; \pi g^2 \delta(W^2 - m^2)$$

where g is the coupling constant and V is the invariant 'vertex' function

$$\langle \bar{k}_1 | J(0) | k_2 \rangle = (16\pi)^{1/2}\, V\left[-(k_1+k_2)^2\right]$$

where $\langle \bar{k}_1 |$ is the anti-particle states of $\langle k_1 |$.

The differential cross-section is given in terms of the physical amplitude $f(\theta)$ by

$$\frac{d\sigma}{d\Omega} = \frac{q_f}{q_i} |f(\theta)|^2$$

where q_f, q_i are the final and initial magnitudes of the three momenta and θ the angle between them, i.e. we are in the centre of mass system that we have used throughout this appendix. The invariant amplitude is related to $f(\theta)$ by

$$F(W^2; \cos\theta) = \tfrac{1}{2} W f(\theta)$$

and since we deal largely with elastic scattering, then $q_f = q_i$ and

$$\frac{d\sigma}{d\Omega} = \frac{4}{W^2} . |F(W^2; \cos\theta)|^2$$

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