# Integrable Systems and their Finite-Dimensional Reductions

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To my mother, Louise Hone.

# Abstract

This thesis concerns some of the different ways in which integrable systems admit finite-dimensional reductions. The point is that partial diferential equations which are integrable (usually in the sense of being solvable by inverse scattering) admit certain special classes of solutions, and these special solutions may be viewed as finite-dimensional mechanical systems in their own right.

The first chapter introduces some of the important concepts and structures associated with integrability, and includes a brief overview of some of the applications of integrable systems and their reductions in field theory.

Chapter 2 describes the scaling similarity reductions of the Sawada-Kotera, fifth-order KdV, and Kaup-Kupershmidt equations. Similarity solutions of these evolution equations satisfy certain ODEs which are naturally viewed as fourth-order analogues of the Painlevé transcendents; they may also be written as non-autonomous Hamiltonian systems, which are time-dependent generalizations of the integrable Hénon-Heiles systems. The solutions to these systems are encoded into a tau-function, and Bäcklund transformations are presented which allow the construction of rational solutions and some other special solutions.

The third chapter is concerned with the motion of the poles of singular solutions (especially rational solutions) of the NLS equation. It is demonstrated that the linear problem for NLS admits an analogue of the well-known Crum transformation for Schrödinger operators, leading to the construction of a sequence of rational solutions. The poles and zeros of these rational solutions are found to satisfy constrained Calogero-Moser equations, and some other singular solutions are also considered. Much use is made of Hirota's bilinear formalism, as well as a trilinear form for NLS related to its reduction from the KP hierarchy.

The final chapter deals with soliton solutions of the $A_n^{(1)}$ affine Toda field theories. By writing the soliton tau-functions as determinants of a particular form, these solutions are related to the hyperbolic spin Ruijsenaars-Schneider system. These results generalize the connection between the ordinary (non-spin) Ruijsenaars-Schneider model and the soliton solutions of the sine-Gordon equation.

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# Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Andrew N.W. Hone*)

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# Chapter 1

# Introduction

## 1.1 General Introduction

Whenever one considers the various models available to describe the physical world, one is often struck by "the unreasonable effectiveness of mathematics"[1]. General relativity and quantum mechanics, the two main pillars of twentieth century physics, are particular examples where exact mathematical solutions display a remarkable agreement with experimental observations. Increasingly, however, it has been realized that most realistic models cannot be solved exactly, even in principle. In the latter half of this century, there has been a great deal of interest in systems of both differential and discrete equations exhibiting chaos. Coupled with the rapid development of computer technology, this has meant that many scientists are now largely inclined to explore their mathematical models with approximate numerical experiments. Running contrary to this trend, yet also having some of its origins in work done with computers, the last thirty years has seen a dramatic growth in the study of completely integrable nonlinear differential equations, with whole hierarchies of such equations being discovered. It turns out that these integrable systems are also relevant in a wide variety of physical situations, ranging from the propagation of water waves to quantum gravity.

Given the inhuman accuracy and speed of computer calculations, one might wonder why completely integrable equations should be worthy of consideration. From a mathematical viewpoint, it transpires that such equations have many beautiful algebraic and geometric properties, and thus are of aesthetic interest in their own right. Taking a more pragmatic view, not only are exact solutions extremely useful for testing the accuracy of numerical algorithms, but also many modern physical theories (such as the Standard Model of particle physics, or the neural network approach to brain interactions) require calculations so complicated that modern computer power is tested to its limits, and for this reason it is of

[1] E.P.Wigner.

prime importance to develop integrable models.

The period of renewed interest in integrable systems began in the 1960s, when it was discovered that the Korteweg-deVries (or KdV) equation,

$$u_t = u_{xxx} + 6uu_x,$$

admits exact, stable solutions in the form of superpositions of an arbitrary number of solitary waves. These waves, which were originally observed in numerical studies, have the remarkable property that each one preserves its amplitude and speed after interaction with the others. This was very suggestive of the behaviour of quantum particles, and hence these special solitary waves were named 'solitons'. The reason for the existence of these soliton solutions turned out to be connected to the fact that KdV has an infinite number of conservation laws, which mean that it can be interpreted as an infinite-dimensional integrable Hamiltonian system. Even more fundamental was the discovery that KdV can be viewed as the isospectral deformation of an associated linear eigenvalue problem, which led to its exact solution by the inverse scattering technique.

The original work on KdV has inspired numerous generalizations, so that large classes of integrable evolution equations are now known. In turn, the new methods used to solve these partial differential equations (PDEs) have produced great insights in the understanding of integrable systems of ordinary differential equations (ODEs). Furthermore, as well as the soliton solutions, integrable PDEs admit many sorts of special solutions (such as rational solutions, and similarity solutions), which may themselves be interpreted as finite-dimensional mechanical systems. This thesis is concerned with some particular types of these finite-dimensional reductions. As well as having interesting properties of their own, such special solutions of PDEs are usually the ones which are most important for applications.

The organization of the thesis is as follows. In this introductory chapter we provide a brief review of some of the ideas behind the notion of integrability, while giving examples that are relevant to the other chapters. Section 1.2 describes finite-dimensional Hamiltonian mechanics, taking the (rational) Calogero-Moser model as a prime example of an integrable system in this context. As well as providing a simple illustration of the concept of the Lax pair, Calogero-Moser equations reappear in Chapter 3.

Section 1.3 describes the most well-known example of an integrable hierarchy of PDEs, the KdV hierarchy. Bäcklund transformations and Hirota's bilinear (tau-function) formalism are also introduced, since these are essential tools in the development of the other chapters. Chapter 2 is largely concerned with the

similarity solutions of some fifth-order evolution equations, one of which is a member of the KdV hierarchy.

In Section 1.4 we present a concise description of the KP hierarchy, indicating the way in which many other integrable hierarchies (including KdV) arise as suitable reductions of it. Chapter 3 contains a construction of rational solutions to the NLS equation, which is yet another example of a reduction of KP; we show how the poles and zeros of these rational solutions evolve according to constrained Calogero-Moser equations.

Although integrable systems originally occurred in traditional areas of applied mathematics (such as fluid mechanics), they have interesting applications in theoretical physics which are not quite so well-known. The purpose of Section 1.5 is to illustrate the importance of exact classical-mechanical solutions in certain problems of quantum field theory. We outline the matrix models, used in the discrete approach to 2-D quantum gravity, which have solutions in terms of ODEs of Painlevé type. Such ODEs are naturally written as non-autonomous Hamiltonian systems, and we make use of this idea in Chapter 2. Another area where classical solutions are important is in finding the correct quantum description for the affine Toda field theories; we would hope that our work on the solitons in these theories (Chapter 4) could lead to further insights, at least at the classical level.

## 1.2 Integrable Hamiltonian Systems in Finite Dimensions

Until relatively recently, there were very few systems known in classical mechanics for which the equations of motion could be integrated explicitly or solved by quadratures. Historically the most well-known examples of such systems are those corresponding to motion in a central potential, free motion on spheres or ellipsoids, and special cases of motion of a rigid body with a fixed point (see e.g. [130] for a review). The classical approach, which reached its culmination in the work of Jacobi, involved finding constants of the motion (in involution with respect to Poisson brackets) and then applying the method of separation of variables. Modern developments have required considerable generalizations of the notion of integrability, and it is fair to say that there is no universally accepted definition of what 'integrable' means [155]. However, there has been a huge increase in the knowledge of systems to which this adjective might be applied.

With the advent of soliton theory in the 1960s, it was discovered that certain partial differential equations, such as KdV, could be interpreted as integrable

Hamiltonian systems with infinitely many degrees of freedom [67]. An essential feature of this interpretation is that integrable PDEs like KdV have an infinite number of conserved quantities. The further understanding of integrable infinite-dimensional systems also produced many new insights into systems with a finite number of degrees of freedom, and led to new techniques for analyzing these. Since finite reductions of integrable systems constitute the main subject of this thesis, we present here a brief review of Hamiltonian mechanics in finite dimensions, emphasizing the structures that will be relevant in the generalization to the infinite-dimensional case.

### 1.2.1 Finite-Dimensional Hamiltonian Mechanics

The usual arena for Hamiltonian mechanics is a symplectic manifold, which consists of a manifold $M$ (of finite dimension, $2n$ say) together with a nondegenerate, closed two-form $\omega$ on $M$ (symplectic form). Given any function (Hamiltonian) $H$ on $M$, the symplectic form allows the canonical definition of an associated vector field $X_H$, via

$$dH = -i_{X_H}\omega. \tag{1.2.1}$$

Alternatively, one may think of the symplectic form as defining an isomorphism between vector fields and one forms, and so (1.2.1) is equivalent to

$$X_H = J\,dH, \tag{1.2.2}$$

where $J$ is the Poisson operator. Given any pair of functions $f$, $g$ on $M$, their Poisson bracket is defined by

$$\{f, g\} = \omega(X_f, X_g).$$

By (1.2.2), the Poisson brackets may equivalently be defined using the Poisson operator:

$$\{f, g\} = -df \cdot J\,dg. \tag{1.2.3}$$

This bracket gives a Lie algebra structure to the space of functions on $M$. If $x$ denotes coordinates on $M$, then the flow generated by the Hamiltonian $H$ may be written

$$\dot{x}(t) = \{H, x\}, \tag{1.2.4}$$

where $t$ corresponds to time. In other words

$$\dot{x} = X_H,$$

and the time derivative of any function on $M$ is just given by

$$\dot{f} = \{H, f\}.$$

The most common setup is that $M$ is a cotangent bundle, i.e. $M = T^*Q$ for a configuration space $Q$. Then the positions $q_j$ are the coordinates on the base space, while the momenta $p_j$ are the fibre coordinates, and $M$ has the canonical one-form

$$\alpha = \sum_j p_j dq_j.$$

Thus $M$ is a symplectic manifold with canonical two-form

$$\omega = d\alpha.$$

With these coordinates the Poisson operator is constant, and may be written in the standard block form

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The canonical Poisson brackets between the coordinates are

$$\{q_j, q_k\} = 0 = \{p_j, p_k\}, \qquad \{p_j, q_k\} = \delta_{jk},$$

leading to

$$\{f, g\} = \sum_j \left( \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right)$$

for any functions $f$, $g$. The equations of motion (1.2.4) are then just Hamilton's equations,

$$\begin{aligned} \frac{dq_j}{dt} &= \frac{\partial H}{\partial p_j}, \\ \frac{dp_j}{dt} &= -\frac{\partial H}{\partial q_j}. \end{aligned}$$

Now that we have the basic concepts at hand, we can define what it means for a Hamiltonian system to be Liouville integrable. A Hamiltonian system with Hamiltonian $H$ on a symplectic manifold $M$ of dimension $2n$, corresponding to the equations of motion (1.2.4), is said to be integrable (in the sense of Liouville) if there exist $n$ independent constants of motion in involution. In other words, there is a set $\{H_1 = H, H_2, \ldots, H_n\}$ of independent functions on $M$ satisfying

$$\{H_j, H_k\} = 0.$$

Given these $n$ constants in involution, it is then possible (at least in principle) to solve the equations exactly. It is often convenient (especially for the infinite-dimensional systems which we will be describing shortly) to associate a time $t_j$ to each Hamiltonian $H_j$, and consider the commuting flows generated by all of the $H_j$ simultaneously. Also certain modifications are necessary for non-autonomous (time-dependent) Hamiltonian systems, which will be relevant to Chapter 2. For a more detailed introduction to Hamiltonian mechanics, see [6].

### 1.2.2 Lax Pairs

At this point we introduce an idea which is of great importance in the theory of integrable systems, that of the Lax pair (named after Peter Lax [108]). Given a dynamical system describing the evolution of some quantities $x$ (which in general may lie in an infinite-dimensional space), suppose we have two operators, denoted by $L$, $P$, which are functions of $x$ taking values in some Lie algebra. Then $L$, $P$ are said to constitute a Lax pair if the equations of motion for $x$ are equivalent to

$$\dot{L} = [P, L], \tag{1.2.5}$$

where $[\,,\,]$ denotes the Lie bracket. In the case of a finite-dimensional system these operators usually belong to a finite-dimensional Lie algebra or its corresponding loop algebra, while for systems of partial differential equations $L$ and $P$ are differential operators. The existence of a Lax pair immediately suggests that a system might be integrable, as it provides a means to construct conserved quantities, and may also lead to a way to solve the equations of motion. For instance, the discovery that the KdV equation could be written in Lax form (with $L$ in that case being a Schrödinger operator) was the key to its solution by inverse scattering. The essential feature of (1.2.5) is that it corresponds to an isospectral deformation of the Lax operator $L$. We illustrate this point with a finite-dimensional example, the Calogero-Moser system, which will reappear in Chapter 3. The generality of the Lax technique will become apparent in Section 1.3, when we consider integrable hierarches of PDEs.

### 1.2.3 The Calogero-Moser System

The original Calogero-Moser system has the Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + g^2 \sum_{j\leq k} (q_j - q_k)^{-2}. \tag{1.2.6}$$

It was originally solved in the quantum case by Calogero [34]. This led to the conjecture that the classical version should also be integrable, which was proved by Moser [119]. Hamilton's equations for (1.2.6) imply

$$\ddot{q}_j = 2g^2 \sum_{k\neq j} (q_j - q_k)^{-3}, \tag{1.2.7}$$

the second-order equations of motion for the Calogero-Moser system.

The key to the proof of integrability is the construction of the Lax pair, which consists of a pair of $n \times n$ matrices $L$, $P$ with entries given by

$$L_{jk} \;=\; p_j \delta_{jk} + ig(1 - \delta_{jk})(q_j - q_k)^{-1}, \tag{1.2.8}$$

$$P_{jk} = -ig\delta_{jk}\sum_{l\neq j}(q_j - q_l)^{-2} + ig(1-\delta_{jk})(q_j - q_k)^{-2}. \tag{1.2.9}$$

The Lax equation (1.2.5) for these particular matrices yields the equations of motion (1.2.7) immediately. Notice that, with the conventions chosen here, $L$ is hermitian and $P$ is anti-hermitian, and so up to factors of $i$ we may regard them as being elements of the Lie algebra $u(n)$. Choosing a unitary matrix $U$ as the solution to the differential equation

$$\dot{U} = PU, \qquad U(0) = 1,$$

it is clear that

$$\frac{d}{dt}(U^{-1}LU) = 0,$$

and therefore

$$U^{-1}LU = L(0).$$

Thus we see that the eigenvalues of $L$ are unchanged by this evolution, and so the Lax equation (1.2.5) gives rise to an isospectral deformation. A more convenient set of constants of motion are the traces of powers of $L$,

$$H_m = tr\, L^m,$$

which are just symmetric functions of the eigenvalues. In particular we have

$$H_1 = \sum_j p_j,$$

which is just the total linear momentum, and the Hamiltonian (1.2.6) is given by

$$H = \frac{1}{2}H_2 = \frac{1}{2}tr\, L^2. \tag{1.2.10}$$

It is obvious that the set of integrals $\{H_1, \ldots, H_n\}$ are independent, since

$$H_m = \sum_j p_j^m + O(p^{m-1}).$$

To prove Liouville integrability, it is also necessary to show that these integrals are in involution. There are at least two different ways to do this. Moser [119] considered the asymptotic motion of the particles as $t \to \pm\infty$, where the $q_j$ are well-separated, and showed that in these limits the Poisson brackets of the $H_m$ vanish; as these Poisson brackets are conserved, they must then be zero for all $t$. A more sophisticated technique involves the construction of an r-matrix for the Calogero-Moser system, which may be found in [18, 144]. In general, suppose that a Hamiltonian system has a Lax pair $L$, $P$, taking values in a Lie algebra $\mathcal{G}$. It

has been proved [20] that the eigenvalues of $L$ are in involution if and only if the Poisson brackets of the entries of $L$ may be encoded into the following equation in $\mathcal{G}\otimes\mathcal{G}$:

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2].$$

In the above,

$$L_1 = L \otimes 1, \qquad L_2 = 1 \otimes L,$$

and the (generalized) r-matrix may be written (in a suitable basis $\{X_\mu\}$ for $\mathcal{G}$)

$$r_{12} = \sum_{\mu\nu} r^{\mu\nu} X_\mu \otimes X_\nu$$

(and similarly for $r_{21}$ with the tensor product reversed). The r-matrix structure is very important for quantum integrable systems and solvable models in statistical mechanics (see [13] and references therein). The Jacobi identity for the Poisson structure leads to a consistency condition on the r-matrix. In the case that $r_{12}$ is constant and antisymmetric, this condition becomes what is known as the classical Yang-Baxter equation. However, the Calogero-Moser system provides one of the original examples of a dynamical r-matrix, which is explicitly dependent on the phase space variables. The general theory of dynamical r-matrices is still poorly understood, and is currently an important subject for investigation (see e.g. [54, 78]). R-matrices are also extremely important for understanding the Poisson structures in the Hamiltonian formulation of integrable PDEs [15, 57].

The existence of a Lax pair and conserved quantities does not in itself provide a constructive procedure for integrating the equations of motion. However, it turns out that the Calogero-Moser system with Hamiltonian (1.2.6) can be solved exactly by the so-called projection method [130]. A common approach to trying to solve a mechanical system is to use symmetries to eliminate degrees of freedom, in the hope that the reduced problem will be more tractable. Sometimes quite the opposite can occur, so that starting from simple equations (e.g. geodesic motion) in a large space leads to more complicated dynamics in a reduced space. If the centre of mass motion is removed, then it turns out that the Calogero-Moser equations (1.2.7) may be reduced from free motion on the space $\mathcal{G}$ of traceless Hermitian matrices (i.e. the Lie algebra of the group $SU(n)$ in the fundamental representation, up to a factor of $i$). Solving the motion on the larger space is trivial, and also provides a simple solution to the Calogero-Moser equations. The full geometrical interpretation of this is via the method of orbits. For our present purposes we will merely indicate how this method enables the model to be solved, and refer the reader to [18, 95] and [130] for more detailed discussions.

Following [130], we let the coordinate $X$ denote a traceless Hermitian matrix, with corresponding momentum $\dot{X}$. Then the phase space $T^*\mathcal{G}$ has the standard symplectic form,

$$\omega = tr\,(d\dot{X} \wedge dX),$$

and with the Hamiltonian

$$H = \frac{1}{2} tr\, \dot{X}^2 \tag{1.2.11}$$

the motion is free,

$$\ddot{X} = 0. \tag{1.2.12}$$

Clearly the solution to (1.2.12) is

$$X = At + B, \tag{1.2.13}$$

for some constants $A$, $B$. It is also apparent that the commutator

$$C = [X, \dot{X}] \tag{1.2.14}$$

takes the constant value $[B, A]$. In fact, (1.2.14) is an example of what is known as a moment map. The reduction to the Calogero-Moser system is achieved by choosing a very special value for $C$:

$$C = ig(ee^\dagger - 1), \qquad e = (1, \ldots, 1)^\dagger. \tag{1.2.15}$$

So $C$ has $(n-1)$ eigenvalues the same.

The position coordinates for Calogero-Moser come from diagonalizing the matrix $X$ with a unitary matrix $U$,

$$Q = UXU^{-1}, \qquad Q_{jk} = q_j \delta_{jk},$$

while the Lax matrix $L$ is obtained by applying the same unitary transformation to $\dot{X}$,

$$L = U\dot{X}U^{-1}.$$

If $P$ is given by

$$P = \dot{U}U^{-1}$$

then (using $\ddot{X} = 0$) it is simple to show that $P$, $L$ satisfy the Lax equation (1.2.5). Also, differentiating the definition of $Q$ leads to the equation

$$\dot{Q} = L + [P, Q]. \tag{1.2.16}$$

Taking the natural convention $U(0) = 1$ as before, the constant matrices $A$, $B$ are given by

$$A = L(0), \qquad B = Q(0).$$

If the initial value $L(0)$ is chosen in the form (1.2.8) then $[Q(0), L(0)]$ is indeed given by (1.2.15), and it is possible to take $U$ such that this value of the moment map is preserved, i.e.

$$[Q, L] = C.$$

Under the reduction from variables $X$, $\dot{X}$ to $Q$, $L$ the Hamiltonian (1.2.11) is invariant under gauging by a unitary matrix, and is just given by (1.2.10) (with $L$, $P$ as in (1.2.8),(1.2.9)). Hence we have seen that the solution $Q(t)$ to the Calogero-Moser equations (1.2.7) is found by diagonalizing the matrix

$$X = L(0)\,t + Q(0).$$

Because of the connection with the Lie algebra $su(n)$, the Calogero-Moser system described above is naturally connected with the root system $A_{n-1}$. There are obvious integrable generalizations to other root systems [130], as well as hyperbolic, trigonometric and elliptic versions (with the the potential in (1.2.6) being replaced by a sum of Weierstrass $\wp$-functions or degenerations of this [144]). These systems are intimately related to the pole motion of rational and elliptic solutions to intgrable PDEs such as the KdV equation [8] and the KP equation [103, 142]. In Chapter 3 we construct rational solutions to the nonlinear Schrödinger (NLS) equation, and show that the pole motion is governed by constrained Calogero-Moser equations. A further variation of these equations includes the addition of a set of spin vectors to the dynamics [71]. There are corresponding quantum mechanical versions [35] (often called Calogero-Sutherland models), which are also exactly solvable, and have many interesting connections with symmetric polynomials and matrix models (see [19] and references).

The Calogero-Moser systems have relativistic generalizations, known as the Ruijsenaars-Schneider models [138]. The solution of these (at least in the rational and hyperbolic cases [135]) follows a pattern very similar to that for the Calogero-Moser system described above. In Chapter 4 we study the connection between the spin-generalized versions of these relativistic models and soliton solutions of affine Toda theories. These solitons may be given in terms of a certain matrix $V$, with the position coordinates of a hyperbolic spin Ruijsenaars-Schneider system being determined from the eigenvalues of $V$; the eigenvalues are given by a diagonal matrix $Q$ satisfying an equation very similar to (1.2.16).

Having reviewed some important features of the finite-dimensional case, we must now turn to properties of integrable nonlinear PDEs. We shall see that the Hamiltonian formalism and Lax equations are fundamental in this setting as well. At the risk of being unoriginal, we shall take the KdV equation (and its associated integrable hierarchy) as our canonical example. The KdV equation has a long

and interesting history, and although much of this is extremely well-known, it provides a good illustration of most of the necessary concepts.

## 1.3 The KdV Hierarchy

The prototype example of an integrable nonlinear PDE is the Korteweg-de Vries (KdV) equation,

$$u_t = u_{xxx} + 6uu_x. \tag{1.3.1}$$

This was first derived by Korteweg and de Vries [100] in 1895, as a description of the evolution of long waves in a shallow channel. The exact choice of coefficients in (1.3.1) is inessential, as they may be altered by scale transformations. An important feature of (1.3.1) is that it admits a travelling wave solution,

$$u(x,t) = 2k^2 \operatorname{sech}^2\left(k(x - x_0) + 4k^3 t\right), \tag{1.3.2}$$

which is called the one-soliton solution. The original discovery of this type of wave should really be credited to a Scottish engineer named John Scott Russell. In August 1834, by the side of the Union Canal near Edinburgh, he observed "a large solitary elevation ... which continued its course along the channel apparently without change of form or diminuition of speed". He followed the wave for two miles on horseback, and was afterwards able to recreate this phenomenon in the laboratory, but his results were received with considerable scepticism by some of the leading scientists at the time, such as Airy and Stokes. The modern theory of solitons has entirely validated the work of John Scott Russell, and his important contribution was commemorated at a recent conference [2] with a visit to the Union Canal, where a plaque was unveiled and some solitary waves were reproduced.

It is fair to say that the KdV equation was all but forgotten for the first half of this century, until the work of Zabusky and Kruskal in 1965 [105] concerning the Fermi-Pasta-Ulam nonlinear lattice equations,

$$m\ddot{y}_n = K(y_{n+1} - 2y_n + y_{n-1})\left(1 + \alpha(y_{n+1} - y_{n-1})\right). \tag{1.3.3}$$

KdV arises as the continuum limit of (1.3.3). While doing numerical studies on the periodic case, Zabusky and Kruskal observed that initial conditions given by a cosine function evolved into a series of pulses of the form (1.3.2), and these pulses interacted elastically before continuing with the same amplitude and speed. These particle-like properties led them to coin the name 'soliton' for such special solitary waves. It turns out that KdV has exact multi-soliton solutions, consisting of

[2] Nonlinear Coherent Structures in Physics and Biology, Heriot-Watt University, July 1995.

nonlinear superpositions of waves like (1.3.2) with different speeds, which scatter elastically. The existence and stability of these multi-solitons is one of the main features that distinguishes integrable equations like KdV from other nonlinear equations admitting solitary wave solutions. This is deeply related to the fact that KdV has an infinite sequence of conservation laws, which we shall describe shortly. For a more thorough account of the history of soliton theory, with full references, we refer the reader to e.g. [5, 47, 62].

### 1.3.1 Hamiltonian Formulation of KdV

The numerical studies of Zabusky and Kruskal led to the undertaking of a detailed analytical study of KdV by Gardner, Greene, Kruskal and Miura, which culminated in its solution by the inverse scattering method [66]. One of the first clues to the integrability of KdV was the discovery of an infinite number of conservation laws, given by

$$\frac{\partial \rho_n}{\partial t} + \frac{\partial J_n}{\partial x} = 0 \tag{1.3.4}$$

for suitable functions $\rho_n$, $J_n$ of $u$ and its $x$-derivatives. The first thing to notice is that (1.3.1) is itself already in conservation form, with

$$\rho_0 = u, \qquad J_0 = -u_{xx} - 3u^2.$$

The next two conserved densities and fluxes are given by

$$\rho_1 = \frac{1}{2}u^2, \qquad J_1 = -uu_{xx} + \frac{1}{2}u_x^2 - 2u^3,$$

$$\rho_2 = -\frac{1}{2}u_x^2 + u^3, \qquad J_2 = u_x u_{xxx} - \frac{1}{2}u_{xx}^2 - 3u^2 u_{xx} + 6uu_x^2 - \frac{9}{2}u^4.$$

There are various constructive proofs of the existence of an infinite sequence of such densities and fluxes, but before we mention some of these it is convenient to introduce the Hamiltonian formalism for KdV.

Suppose for simplicity that $u$ is either periodic or rapidly decaying on the real line (i.e. $u(x,t) \to 0$ as $x \to \pm\infty$). Then upon integrating (1.3.4) over a suitable interval, it is apparent that

$$\frac{d}{dt}\int \rho_n \, dx = 0.$$

Thus the sequence of conservation laws yields a sequence of conserved quantities for KdV. Furthermore, for a suitable Poisson structure these conserved quantities turn out to correspond to a sequence of Hamiltonians for the KdV equation (1.3.1) and a whole hierarchy of commuting flows known as the KdV hierarchy.

In order to extend Hamiltonian mechanics to the infinite-dimensional case, one may generalize either the symplectic or the Poisson formalism. Here we take

the latter option (and refer the reader to [57] for examples of the former). Given a functional (Hamiltonian) $\mathcal{H}$ on phase space, the Fréchet derivative is defined as

$$\mathcal{H}'[u]\,v = \frac{d}{d\epsilon}\mathcal{H}[u+\epsilon v]\,|_{\epsilon=0}.$$

Using this, the variational derivative $\delta\mathcal{H}/\delta u$ is given by

$$\mathcal{H}'[u]\,v = \int \frac{\delta\mathcal{H}}{\delta u}\,v\,dx.$$

A given operator $B$ is said to be a Hamiltonian operator (the analogue of the Poisson operator in this context) if Poisson brackets between functionals can be defined by

$$\{\mathcal{F},\mathcal{G}\} = -\int \frac{\delta\mathcal{F}}{\delta u}\,B\,\frac{\delta\mathcal{G}}{\delta u},$$

which is the analogue of (1.2.3). It is apparent that for the brackets to be skew-symmetric, $B$ should be a skew-symmetric operator; the Jacobi identity is a more complicated criterion to check. Then the flow generated by the Hamiltonian $\mathcal{H}$ is just

$$u_t = B\,\frac{\delta\mathcal{H}}{\delta u}. \tag{1.3.5}$$

The simplest functionals to consider are those which may be written as integrals of local Hamiltonian densities, i.e.

$$\mathcal{H}[u] = \int H[u]\,dx$$

with $H[u]$ being a function of $u$ and its $x$-derivatives. In that case it is easy to see that

$$\frac{\delta\mathcal{H}}{\delta u} = \delta_u H,$$

where

$$\delta_u H := \sum_j (-\partial_x)^j \frac{\partial H}{\partial u_{jx}}.$$

In this situation (see Chapter 2) we shall often refer to a local density $H$ as a 'Hamiltonian', although strictly it is a Hamiltonian density.

For the case of KdV, define a sequence of densities by

$$H_0 = \frac{1}{2}\rho_0,$$

and

$$H_n = \rho_n$$

for $n > 0$. If the Hamiltonian is taken to be the integral of the local density $H_2$, then it is immediately apparent that (1.3.1) may be written in the Hamiltonian form (1.3.5), with the Hamiltonian operator

$$B = \partial_x.$$

Explicitly, (1.3.1) may be written as

$$u_t = \partial_x \delta_u H_2.$$

What is less obvious is that there is a second Poisson structure associated with KdV, with the Hamiltonian operator

$$\tilde{B} = \partial_x^3 + 4u\partial_x + 2u_x. \tag{1.3.6}$$

It is then possible to write (1.3.1) in the alternative form

$$u_t = \tilde{B}\delta_u H_1.$$

Thus KdV is said to be bi-Hamiltonian, since it has two Poisson (or Hamiltonian) structures.

Given the sequence of conserved densities, it is natural to define a hierarchy of flows, via

$$u_{t_n} = B\delta_u H_n, \tag{1.3.7}$$

or equivalently

$$u_{t_n} = \tilde{B}\delta_u H_{n-1}. \tag{1.3.8}$$

For example, the first (trivial) flow is

$$u_{t_1} = u_x,$$

while the third (which will reappear in Chapter 2) is

$$u_t = u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x.$$

This sequence of evolution equations is called the KdV hierarchy, and it is possible to show that the corresponding Hamiltonians are mutually in involution (with respect to either Poisson structure) and hence the flows all commute. The conserved densities may be calculated by succesively integrating

$$\partial_x \delta_u H_n = (\partial_x^3 + 4u\partial_x + 2u_x)\delta_u H_{n-1}. \tag{1.3.9}$$

By taking $\delta_u H_0 = \frac{1}{2}$ and setting the constant of integration at each stage to be zero, the relation (1.3.9) serves as a recursive definition for the differential polynomials $\delta_u H_n$ (known as the Gelfand-Dikii polynomials [69]). The existence of the hierarchy is due to the fact that $B$ and $\tilde{B}$ are compatible, in the sense that $B + \tilde{B}$ is also a Hamiltonian operator. It is also possible to obtain each flow from the previous one by using the (integro-differential) recursion operator for the KdV hierarchy,

$$R := \tilde{B}B^{-1} = \partial_x^2 + 4u + 2u_x\partial_x^{-1}.$$

This is probably the most concise way to generate the sequence of KdV flows, in that it does not require the computation of variational derivatives.

Recursion operators also occur in a more general context than Hamiltonian systems [65]. The prime example of this is the Burgers equation,

$$u_t = u_{xx} + 2uu_x, \tag{1.3.10}$$

which may be linearized, by making the Cole-Hopf substitution

$$u = (\log[\phi])_x,$$

to yield the heat equation

$$\phi_t = \phi_{xx}.$$

Although (1.3.10) is not Hamiltonian and does not have soliton solutions, it has an associated recursion operator,

$$R = \partial_x + u + u_x \partial_x^{-1},$$

which allows the construction of a hierarchy of commuting flows:

$$u_{t_n} = K_n[u], \qquad K_1 = u_x, \qquad K_{n+1} = R\,K_n.$$

A general introduction to infinite-dimensional Hamiltonian theory may be found in Chapter 11 of [62], while the references [50, 65] contain detailed treatments of the algebraic aspects of Hamiltonian operators and recursion operators. Rather than dwelling on the Hamiltonian theory, we prefer to introduce the Lax representation of KdV, since this is more fundamental.

### 1.3.2 Lax Formalism and Inverse Scattering

The key to the integrability of KdV is its connection with the theory of linear Schrödinger operators. More precisely, the KdV equation (1.3.1) may be written in the form of the Lax equation (1.2.5), where the Lax operator $L$ is just a Schrödinger operator,

$$L = \partial_x^2 + u, \tag{1.3.11}$$

and the other half of the Lax pair is the third-order operator

$$P = 4\partial_x^3 + 6u\partial_x + 3u_x. \tag{1.3.12}$$

Lax also found an infinite sequence of operators $P_n$, such that the Lax equation

$$L_{t_n} = [P_n, L] \tag{1.3.13}$$

corresponds to the $n$-th flow (1.3.7) of the KdV hierarchy described above [108]. Each operator $P_n$ is of degree $2n-1$, and can be constructed (along with the associated Hamiltonian) entirely from $L$. To see this requires the introduction of pseudo-differential operators, which we postpone until the next section where we discuss the KP hierarchy.

As we have already remarked, each Lax equation (1.3.13) corresponds to an isospectral deformation of the operator $L$, and so the KdV hierarchy is naturally viewed as a commuting family of such deformations. This prompts the investigation of the Schrödinger eigenvalue problem,

$$L\psi = \lambda\psi, \tag{1.3.14}$$

together with the associated time evolution of the eigenfunction,

$$\psi_{t_n} = P_n\psi. \tag{1.3.15}$$

The Lax equation (1.3.13) may be regarded as the consistency condition for (1.3.14,1.3.15), given that the eigenvalue $\lambda$ is time-independent. Conversely, given the eigenvalue equation (1.3.14) and the assumption that the eigenfunction satisfies (1.3.15), it is easy to show that

$$(L_{t_n} - [P_n, L] - \lambda_{t_n})\psi = 0. \tag{1.3.16}$$

Hence if the potential $u$ of the Schrödinger operator $L$ evolves according to one of the equations of the KdV hierarchy (taking the Lax form (1.3.13)), then the spectrum of $L$ is unchanged by this evolution, since from (1.3.16) we see immediately that

$$\lambda_{t_n} = 0.$$

The analysis of linear systems such as (1.3.14,1.3.15) is the key to the solution of the associated nonlinear evolution equations by the inverse scattering technique, which can be understood as a nonlinear version of the solution of linear equations via Fourier transforms. The inverse scattering for the KdV equation (1.3.1) is the simplest to describe, because of the connection with Schrödinger operators. The scattering theory of Schrödinger operators has been widely studied because of its importance in quantum mechanics. Thus it is well-known that (for rapidly decaying potentials $u$ on the real line) the Lax operator $L$ (as in (1.3.11) above) has a discrete spectrum consisting of positive eigenvalues $\lambda_n = k_n^2$,

$$L\psi_n = k_n^2\psi_n,$$

with normalizable eigenfunctions $\psi_n$ (bound states), as well as a continuous spectrum $\lambda = -\kappa^2$ (corresponding to scattering states $\psi_k$).

There are three steps involved in the inverse scattering solution of the KdV equation (1.3.1). First of all, given the potential $u(x,0)$, one may construct the scattering data $S(0)$, which consists of the discrete spectrum (the eigenvalues $\lambda_n$) of $L$ together with the asymptotics of both the bound states and the scattering states. Next, using the time evolution (1.3.15) of the eigenfunctions (which in the KdV case must be modified by adding a constant to the operator $P$ in (1.3.12)), it is simple to show that the scattering data $S(t)$ for the potential $u(x,t)$ evolves linearly with the time $t$; it is also possible to demonstrate that, within the Hamiltonian formalism, the scattering data corresponds to action-angle variables for KdV [57]. The final step is the reconstruction of $u(x,t)$ from the scattering data $S(t)$. It was shown by Gelfand and Levitan [68] that this may be achieved using the solution of a linear integral equation,

$$K(x,y;t) + F(x+y;t) + \int_x^\infty K(x,z;t)F(z+y;t)dz = 0, \tag{1.3.17}$$

called the Gelfand-Levitan-Marchenko (GLM) equation. The potential $u$ (the solution to KdV) of the operator $L$ is found from the integral kernel $K$, via the formula

$$u(x,t) = 2\frac{\partial}{\partial x}K(x,x;t). \tag{1.3.18}$$

Hence although KdV is intrinsically nonlinear, its solution via inverse scattering involves only linear equations at each stage.

We refer the reader to [5] for a full description of inverse scattering. The most interesting case corresponds to the so-called reflectionless potentials, when only the discrete spectrum of the operator $L$ is important. By assuming that the number of discrete eigenvalues is finite, N say, an explicit formula is obtained for the integral kernel $K$, leading to

$$u(x,t) = 2(\log\det(1+C))_{xx}, \tag{1.3.19}$$

where the $N \times N$ matrix $C$ is given by

$$C_{mn} = \frac{c_m c_n}{k_m + k_n}\exp[-(k_m+k_n)x - 4(k_m^3 + k_n^3)t].$$

The potential in (1.3.19) is the $N$-soliton solution to the KdV equation, and the constants $k_j$, $c_j$ are related to the velocity and positions of the solitons respectively. It is easy to see that (1.3.2) follows from (1.3.19) with $N = 1$. In Chapter 4 we present similar formulae for the soliton solutions of affine Toda theories.

The linear system (1.3.14,1.3.15) can clearly be generalized to the case of higher-order differential operators $L$. For example, if we take the Lax operator

$$L = \partial_x^3 + 3u_1\partial_x + 3(u_2 + u_{1,x}), \tag{1.3.20}$$

and

$$P = \partial_x^2 + 2u_1, \tag{1.3.21}$$

then the usual Lax equation (1.2.5) yields the coupled system

$$\begin{aligned} u_{1,t} &= u_{1,xx} + 2u_{2,x}, \\ u_{2,t} + u_{1,xt} &= \frac{1}{3}u_{1,3x} - 2u_1 u_{1,x} + u_{2,xx}. \end{aligned} \tag{1.3.22}$$

Eliminating $u_2$ from (1.3.22) gives

$$u_{1,tt} + \left(\frac{1}{3}u_{1,xx} + 2u_1^2\right)_{xx} = 0,$$

which is known as the Boussinesq equation (see [60], for instance). In Section 1.4 we shall see how higher-order Lax operators arise in reductions of the KP hierarchy.

There is a different linear system associated with KdV, which in many ways is more convenient. Consider the matrix system

$$\Phi_x = F\Phi, \tag{1.3.23}$$

$$\Phi_t = G\Phi, \tag{1.3.24}$$

where

$$\begin{aligned} F &= \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}, \\ G &= \begin{pmatrix} -u_x & 4\lambda + 2u \\ 4\lambda^2 - 2\lambda u - u_{xx} - 2u^2 & u_x \end{pmatrix}, \end{aligned}$$

and $\Phi$ is either a column vector,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{1.3.25}$$

or a $2 \times 2$ fundamental solution matrix. If $\Phi$ is chosen in the form (1.3.25), then (1.3.23) is just

$$\begin{aligned} \phi_{1,x} &= \phi_2, \\ \phi_{2,x} &= (\lambda - u)\phi_1, \end{aligned}$$

which (on elimination of $\phi_2$)

$$L\phi_1 = \lambda\phi_1,$$

with $L$ being the Lax operator (1.3.11) for KdV. Thus the original Lax formalism is contained within the system (1.3.23,1.3.24), and the compatibility condition for this system, the zero curvature equation

$$F_t - G_x + [F, G] = 0,$$

yields the KdV equation (1.3.1). The zero curvature equation may also be written as a Lax equation,

$$\dot{\tilde{L}} = [\tilde{P}, \tilde{L}],$$

by taking

$$\tilde{L} = 1\partial_x + F, \qquad \tilde{P} = G.$$

The matrices $F$, $G$ above clearly belong to the Lie algebra $sl(2)$. Hence it is natural to consider the system (1.3.23,1.3.24) with matrices in other Lie algebras, and this leads to classifications of integrable systems in terms of simple Lie algebras [52, 61]. Alternatively, considering the eigenvalue $\lambda$ as a parameter on the circle leads to a connection between KdV-type equations and loop groups and algebras, as developed by Segal and Wilson [141]. Recently this approach to the KdV has been refined somewhat by Schiff [140], to obtain a deeper understanding both of symmetries and of the Bäcklund transformations, mapping solutions to solutions, which we now introduce.

### 1.3.3 Miura Map and Bäcklund Transformations

One of the remarkable properties of soliton equations like KdV is the existence of nonlinear transformations which allow the construction of a family of solutions to one PDE from a given solution of another (possibly the same) PDE. The general name given to these is Bäcklund transformations, and historically they originally occurred in differential geometry. In particular, the sine-Gordon equation

$$u_{xt} = \sin u \tag{1.3.26}$$

was first studied in relation to surfaces of constant mean curvature; a Bäcklund transformation for (1.3.26) was originally found by Bianchi and used to generate families of surfaces (for further details, and the original references, see [137]). The sine-Gordon equation is the simplest of the $A_n^{(1)}$ affine Toda theories, for which we discuss Bäcklund transformations and soliton solutions in Chapter 4.

One way of generating transformations between integrable equations is to factorize the Lax operator [60]. The original example of this is the KdV case, where the Schrödinger operator can be written as

$$L = (\partial_x + v)(\partial_x - v),$$

which leads to the Miura map for the potential:

$$u = -v_x - v^2 =: M[v]. \tag{1.3.27}$$

This leads to the connection with another well-known integrable hierarchy, namely the modified KdV (mKdV) hierarchy. Indeed, it may be checked directly that if $v$ satisfies the mKdV equation,

$$v_t = v_{xxx} - 6v^2 v_x, \tag{1.3.28}$$

and if $u$ is given by the Miura map (1.3.27), then $u$ must satisfy the KdV equation (1.3.1). It turns out that this is intimately related to the second Hamiltonian structure of the KdV hierarchy, because the second Hamiltonian operator may be written in terms of $v$ as

$$\tilde{B} = M'(-\partial_x)(M')^* \tag{1.3.29}$$

(where $M'$ denotes the Fréchet derivative of the Miura map (1.3.27); $*$ is the real adjoint, i.e. $(a\partial_x^j)^* = (-\partial_x)^j a$ ). The $n$-th flow of the mKdV hierarchy may be written in the Hamiltonian form

$$v_{t_n} = (-\partial_x)\delta_v H_{n-1}[M[v]], \tag{1.3.30}$$

with the sequence of Hamiltonian densities of mKdV being given by

$$H_{n-1}[M[v]] = H_{n-1}[u]|_{u=M[v]}.$$

Using the formula (1.3.29), it is then straightforward to show that for $v$ satisfying the Hamiltonian flows (1.3.30) of the mKdV hierarchy, the Miura-related variable $u$ must satisfy flows (1.3.8) of the KdV hierarchy. Note that this Miura map is not invertible: $u$ being a solution of KdV does not necessarily mean that $v$ is a solution of mKdV. However, in Chapter 2 we shall see that for the scaling similarity solutions of KdV and mKdV the Miura map becomes a one-one correspondence, leading to a method for generating sequences of similarity solutions.

The mKdV equation (1.3.28) and each of the flows in its hierarchy can also be written in zero curvature form, by gauging the linear system (1.3.23,1.3.24) with an element $g$ of the group $SL(2)$. In fact the gauge transformation is defined by

$$g = \frac{1}{\sqrt{2k}}\begin{pmatrix} -v-k & 1 \\ v-k & -1 \end{pmatrix}, \qquad \lambda = k^2,$$

so that

$$F \to \tilde{F} = gFg^{-1} + g_x g^{-1},$$

and similarly for $G$. Explicit calculation shows that

$$\tilde{F} = \begin{pmatrix} -k & v \\ v & k \end{pmatrix},$$

and the formula for the gauged matrices $\tilde{G}$ corresponding to each of the flows of the mKdV hierarchy can be found in Chapter 2.

While it may be useful to use transformations between different integrable equations, an alternative method for generating solutions is via auto-Bäcklund transformations (ABTs), which relate sets of solutions to the same equation. For example, if $\tilde{u}$ is a solution to the KdV equation (1.3.1), and if $\phi(x,t)$ satisfies

$$\begin{aligned} \tilde{u} &= \beta - \frac{(\sqrt{\phi_x})_{xx}}{\sqrt{\phi_x}}, \\ \frac{\phi_t}{\phi_x} &= 6\beta + \left(\frac{\phi_{xxx}}{\phi_x} - \frac{3\phi_{xx}^2}{2\phi_x^2}\right), \end{aligned} \tag{1.3.31}$$

then

$$u = \tilde{u} + 2(\log[\phi])_{xx} \tag{1.3.32}$$

is also a solution of the KdV equation. For example, starting from the trivial (vacuum) solution $\tilde{u} = 0$, it is possible to generate the 1-soliton solution (1.3.2) to KdV. Alternatively, setting

$$\tilde{u} = 0, \qquad \beta = k^2, \qquad \phi = (x - 12k^2t) + \frac{\sinh(2kx + 8k^3t)}{2k}$$

it may be shown [140] that (1.3.31) holds, and this yields a mixed rational-solitonic solution to KdV, given by

$$u = 2(\log[\phi])_{xx}. \tag{1.3.33}$$

Thus (1.3.31,1.3.32) constitute an ABT for KdV. By repeated application of this transformation, it is possible to generate sequences of solutions to KdV (in particular, soliton solutions).

The relations (1.3.31,1.3.32) are referred to in [140] as the Galas transformation, although it seems most likely that they originally occurred in the work of Weiss and others (see [37, 152]) on Painlevé expansions for PDEs. We shall use the latter approach in Chapter 3 to investigate an ABT for the Nonlinear Schrödinger (NLS) equation, and thereby generate a sequence of rational solutions. We should also mention that there are many other ways to generate Bäcklund transformations for an integrable nonlinear PDE, such as applying a Darboux transformation to the Lax operator [116], or by 'dressing' the zero curvature equation (see [23] and references). Our investigation of the Bäcklund transformation for NLS in Chapter 3 is also linked to its zero curvature representation. The book [137] is an excellent introduction to the subject of Bäcklund transformations.

### 1.3.4 Bilinear Form and Hirota's Method

As well as the Bäcklund transformations, one of the most powerful ways of generating exact solutions to integrable PDEs is Hirota's method. This consists of

making a suitable transformation of the dependent variables so that the equations in the new variables take a more transparent form. In the case of the KdV equation (1.3.1) the correct substitution is

$$u = 2(\log[\tau])_{xx}, \tag{1.3.34}$$

and after an integration (subject to suitable boundary conditions) a bilinear equation is obtained for the new dependent variable $\tau$. This bilinear equation may be written concisely as

$$(D_x D_t - D_x^4)\tau \cdot \tau = 0, \tag{1.3.35}$$

by making use of the Hirota derivatives:

$$D_y^j D_z^k g \cdot f := \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^j \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^k g(y,z) f(y',z')|_{y'=y, z'=z}.$$

There are several reasons why the substitution (1.3.34), with its associated bilinear equation (1.3.35), is a judicious one. First of all, notice that for the $N$-soliton solution (1.3.19) the variable $\tau$ (which is called the tau-function) is just a polynomial in simple exponential functions of $x$ and $t$. Hirota's direct method for finding solutions involves writing the tau-function as a truncated expansion in some parameter $\epsilon$,

$$\tau = \tau_0 + \epsilon\tau_1 + \ldots + \epsilon^N \tau_N,$$

and then determining the coefficients by comparing the different powers of $\epsilon$ in the bilinear equation (1.3.35). It turns out that this truncation is consistent, and may be used to obtain the $N$-soliton solution of KdV (setting $\epsilon = 1$ at the end of the analysis). In practice, once the 2-soliton has been found it is natural to conjecture an expression for the $N$-soliton and then prove inductively that it satisfies (1.3.35). Hirota used this technique to construct solitons and other solutions to a wide variety of nonlinear equations, and was also able to define Bäcklund transformations within his bilinear formalism (see the references to Hirota starting with e.g. [79]). Notice that the substitution (1.3.34) also looks appropriate in the context of the ABT (1.3.32). The wide applicability of the tau-function approach was an indication that it should have some deeper significance, but this only became apparent in investigations of the algebraic structure of the KP hierarchy [126], which we discuss in the next section.

Another thing to observe about the substitution (1.3.34) is that the soliton tau-functions may naturally be written as determinants. This is plain to see in the formula (1.3.19) for the $N$-soliton solution of KdV. More generally the tau-function can be related to the inverse scattering approach [127], by means of the GLM equation (1.3.17). Indeed, for reflectionless potentials $u$, one may consider

the Fredholm operator $\hat{F}$, with symmetric kernel $F(x+y;t)$ as in (1.3.17), and then using a Neumann expansion (subject to suitable analytic assumptions) it is possible to show that (1.3.18) leads to

$$u = 2(\log \det(1+\hat{F}))_{xx}$$

The $N$-soliton formula (1.3.19) just corresponds to the case of a finite kernel. The relationship between tau-functions and inverse scattering has also been explored for the affine Toda theories [121], which are discussed in Chapter 4.

Determinantal formulae for tau-functions play a major rôle in the rest of this thesis. Chapter 2 contains a tau-function approach to certain ODEs arising as similarity solutions of integrable PDEs; we are able to write some of these tau-functions as determinants. In Chapter 3 we make much use of bilinear techniques to deal with the NLS equation and some Bäcklund transformations, and construct rational solutions which are written in determinantal form. The results about affine Toda solitons derived in Chapter 4 also depend crucially on the fact that these can be written in terms of determinants.

It is not always the case that the tau-functions of soliton equations satisfy bilinear equations like (1.3.35). Sometimes it is necessary to use trilinear or even multilinear equations [73]. We shall see how the NLS equation is related to a trilinear equation in Chapter 3. This trilinear equation arises [125] from reduction of the bilinear KP hierarchy, which we now introduce.

## 1.4 The KP Hierarchy

Despite the natural interpretation of the KdV hierarchy in terms of an infinite sequence of times, the KdV equation is really only physically relevant in (1+1)-dimensional situations. The original form of the inverse scattering technique was only applicable to problems in two dimensions, but it was found that the method could be extended to solve the genuinely (2+1)-dimensional Kadomtsev-Petviashvili (KP) equation,

$$(4u_t - 12uu_x - u_{xxx})_x - 3u_{yy} = 0. \tag{1.4.1}$$

A full description of the inverse scattering transform for (1.4.1) may be found in Chapter 4 of [5]. For the purposes of this thesis we will only be interested in the formal algebraic description of the KP hierarchy, which requires pseudo-differential Lax operators. We also describe some of the salient features of Sato's approach to KP, where the solutions lie in an infinite-dimensional Grassmanian, and the tau-function appearing in Hirota's bilinear formalism is of central importance.

### 1.4.1 The Lax Operator for KP

In order to define the KP hierarchy in its Lax form, it is convenient to use the pseudo-differential operator

$$L = \partial_x + \sum_{j=1}^{\infty} u_j \partial_x^{-j}. \tag{1.4.2}$$

The $u_j$ are functions of $x$, and $\partial_x^{-1}$ is treated as the formal inverse of $\partial_x$. To make all (positive and negative) powers of $\partial_x$ well-defined requires the Liebniz rule

$$\partial_x^n f(x) = \sum_{r=0}^{\infty} \frac{n(n-1)\dots(n-r+1)}{r!} \frac{\partial^r f}{\partial x^r} \partial_x^{n-r}.$$

This allows the computation of commutators of pseudo-differential operators, giving them a Lie algebra structure. For any pseudo-differential operator, say

$$A = \sum_{j=-\infty}^{m} a_j \partial_x^j$$

with $m \geq 0$, there is a natural splitting into a positive part,

$$A_+ = \sum_{j=0}^{m} a_j \partial_x^j,$$

and a negative part,

$$A_- = \sum_{j<0} a_j \partial_x^j,$$

so that

$$A = A_+ + A_-.$$

This leads to a decomposition of the pseudo-differential operators into a direct sum of the subalgebras of positive and strictly negative operators. Such decompositions are naturally associated with integrability [9, 15].

It is straightforward to calculate positive powers of the Lax operator $L$ defined by (1.4.2), and then by taking their positive parts we obtain a sequence of purely differential operators,

$$B_n = (L^n)_+.$$

For example, the first three in the sequence are

$$B_1 = \partial_x, \quad B_2 = \partial_x^2 + 2u_1, \quad B_3 = \partial_x^3 + 3u_1\partial_x + 3(u_2 + u_{1,x}).$$

If the $u_j$ are now allowed to depend on an infinite set of times $t_1$, $t_2$, $t_3, \dots$, then the KP hierarchy may be defined to be the sequence of commuting Lax flows

$$\partial_{t_n} L = [B_n, L]. \tag{1.4.3}$$

Clearly it is consistent to identify $t_1 \equiv x$, and these Lax equations generate a sequence of flows for the dependent variables $u_j$. It is customary to eliminate all but the variable $u_1$, and then the sequence of equations for $u_1$ is also called the KP hierarchy. Upon setting $u = u_1$, $y = t_2$, $t = t_3$, the KP equation (1.4.1) may be obtained from the $n = 2$ and $n = 3$ cases of (1.4.3) (after elimination of $u_2$ and $u_3$ [126]).

The Lax equations (1.4.3) may also be regarded as the consistency conditions for the linear system

$$\begin{aligned} L\psi &= k\psi, \\ \psi_{t_n} &= B_n\psi, \end{aligned} \tag{1.4.4}$$

with the spectral parameter $k$ being time-independent,

$$k_{t_n} = 0.$$

The eigenfunction $\psi$ is known as the wave-function, or in the context of the algebro-geometric theory [102] (where it must have particular asymptotics) it is called the Baker-Akhiezer function. Another consistency condition for the system (1.4.4) is the sequence of zero-curvature (or Zakharov-Shabat) equations,

$$\partial_{t_n} B_m - \partial_{t_m} B_n + [B_m, B_n] = 0. \tag{1.4.5}$$

It is also quite simple to derive this from the Lax equations (1.4.3), and the ordinary KP equation (1.4.1) follows almost immediately from (1.4.5) with $m = 2$, $n = 3$.

Up to a scaling, it is apparent that $B_2$ is the Schrödinger operator (1.3.11), while $B_3$ is the Boussinesq Lax operator (1.3.20). In fact this is no accident, for the KdV and Boussinesq hierarchies arise as reductions of the KP hierarchy. More precisely, the $p$-reduction of the KP hierarchy (also known as the $p$-KdV hierarchy) is obtained by constraining the $p$-th power of $L$ to be purely differential, so that

$$L^p = (L^p)_+,$$

or equivalently

$$(L^p)_- = 0.$$

This implies that the flows corresponding to times which are multiples of $p$ are all trivial, i.e.

$$L_{t_{np}} = 0.$$

Thus the modified variables $u_j$ are independent of the times $t_p, t_{2p}, t_{3p}, \ldots$, and the zero curvature equations (1.4.5) become a hierarchy of Lax equations for the

Lax operator $\mathcal{L} = B_p$,

$$\partial_{t_m}\mathcal{L} = [B_m, \mathcal{L}]$$

(non-trivial only when $m$ is not a multiple of $p$). In particular, the KdV hierarchy arises as the 2-reduction of KP, with

$$\mathcal{L} = B_2,$$

and the non-trivial flows for the odd times take the form

$$\partial_{t_{2n+1}}\mathcal{L} = [(\mathcal{L}^{\frac{2n+1}{2}})_+, \mathcal{L}]. \tag{1.4.6}$$

Note that this requires a relabelling and rescaling of the times compared with (1.3.13). Similarly, the Boussinesq hierarchy arises as the 3-reduction of KP. This may be easily seen for the lowest members of these hierarchies, since setting $u_y = 0$ in (1.4.1) yields the KdV equation (after an integration), while putting $u_t = 0$ instead gives the Boussinesq equation.

The KP hierarchy may also be written in Hamiltonian form [15]. This hinges on the fact that there is a natural definition of the trace of a pseudo-differential operator,

$$tr\, A := \int res\, A\, dx,$$

with $res\, A = A_{-1}$ being the coefficient of $\partial_x^{-1}$ in $A$. The sequence of conserved quantities (Hamiltonians) for KP are then given by traces of powers of the Lax operator:

$$\mathcal{H}_n = tr\, L^n, \tag{1.4.7}$$

Note that in the $p$-reduction, all the powers $L^{np}$ are purely differential and so have zero trace. Thus in the reduction to the KdV hierarchy, for instance, the usual sequence of Hamiltonians are given by the traces of the odd powers of $L$. The Hamiltonian description of KP can also be understood in terms of r-matrices, but we shall not dwell on this any further.

### 1.4.2 Sato Theory and the Tau-Function of KP

Sato's approach (see [126] and references) provides an alternative way to construct KP which is more fundamental than the Lax formulation. Starting from the dressing operator,

$$W = 1 + \sum_{j=1}^{\infty} w_j \partial_x^{-j},$$

the Lax operator (1.4.2) is obtained by 'dressing' the bare operator $\partial_x$, i.e.

$$L = W \partial_x W^{-1}.$$

Given that the dressing operator evolves according to the Sato equation,

$$\partial_{t_n} W = B_n W - W \partial_x^n, \tag{1.4.8}$$

with

$$B_n = (W \partial_x^n W^{-1})_+,$$

the Lax equation (1.4.3) is an immediate consequence.

If the linear system (1.4.4) is considered, it is apparent that the Baker-Akhiezer function can be given by

$$\psi = \hat{w}(\underline{t}, k) \exp[\xi(\underline{t}, k)], \quad \xi(\underline{t}, k) = \sum_{j=1}^{\infty} t_j k^j, \tag{1.4.9}$$

where $\hat{w}$ is built out of the coefficients of the dressing operator,

$$\hat{w} = 1 + \sum_{j=1}^{\infty} w_j k^{-j}.$$

An asymptotic expansion of $(\log \psi)_x$ in powers of $k$ provides yet another way to generate the sequence of conservation laws for the KP hierarchy (or its reductions [17]), and then it is natural to introduce the tau-function $\tau(\underline{t}) = \tau(t_j)$ as a holomorphic function of the times, so that

$$w_1 = res\, W = -(\log \tau)_x.$$

The other coefficients of $W$ can be determined from the formula

$$\hat{w} = \frac{\tau(t_j - 1/jk^j)}{\tau(t_j)}.$$

In terms of the tau-function, the densities $H_n$ for the conserved quantities (1.4.7) are given by

$$H_n = (\log \tau)_{xt_n},$$

while for the dependent variable $u_1$ there is the usual Hirota substitution

$$u_1 = (\log \tau)_{xx}.$$

With these substitutions, it is trivial to derive the conservation laws

$$\partial_{t_n} u_1 = \partial_x H_n.$$

Also, calculating residues in the Sato equation (1.4.8) leads to the equations for the KP hierarchy in bilinear form,

$$\left(\frac{1}{2} D_1 D_n - p_{n+1}(\tilde{D})\right) \tau \cdot \tau = 0, \tag{1.4.10}$$

with the Schur polynomials defined by the generating function

$$\sum_{j=0}^{\infty} p_j k^j = \exp[\xi(\underline{t}, k)],$$

and $\tilde{D}$ denotes the sequence, $D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots,$ of rescaled Hirota derivatives. Because these derivatives are skew-symmetric, the first non-trivial case of (1.4.10) corresponds to $n = 3$,

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau \cdot \tau = 0,$$

and it is straightforward to check that this is the bilinear form of the KP equation (1.4.1).

With the basic apparatus of the tau-function and the associated Baker-Akhiezer function, it is possible to interpret KP in terms of the Sato's Grassmanian $Gr$ of the Hilbert space $L^2(S^1)$ [46, 141]. This is achieved by taking $k$ to be the coordinate on the circle, and then constructing the wave-function $\psi_W$ lying in some suitable subspace $\tilde{W}$. More precisely, the Hilbert space has a natural decomposition into positive and negative powers of $k$,

$$L^2(S^1) = H_+ \oplus H_-;$$

$Gr$ contains only those subspaces $\tilde{W}$ such that the projection onto $H_+$ is invertible, and as $\psi_W$ evolves according to the KP flows it remains within $\tilde{W}$. The tau-function $\tau_W$ then corresponds to the Plücker coordinates of the subspace $\tilde{W}$, and the bilinear equations (1.4.10) are the Plücker relations of $Gr$. It turns out that there are rational solutions of KP with tau-functions given by Schur functions [126], and these can be connected with finite-dimensional Grassmanians and bispectral operators [94].

The Grassmanian $Gr$ also has a nice physical interpretation as a free fermion Fock space [45], and this leads to connections with infinite-dimensional (Kac-Moody) Lie algebras and the vertex operator constructions of conformal field theory [92]. Nimmo has also found an alternative algebraic construction of the bilinear equations of the KP hierarchy [123], based on the fact that soliton and rational solutions can be written as Wronskians. In Chapter 3 we derive Wronskian formulae for rational solutions of the NLS equation, which occurs as a reduction of KP.

## 1.5 Reductions of Integrable Systems in Field Theory

One of the remarkable features of integrable systems is that not only do they have many attractive properties from a mathematical point of view, but also they are applicable in a wide variety of physical situations. Many of the original examples of soliton equations appeared in fluid dynamics, plasma physics and optics, but for more details of these applications we refer the reader to the comprehensive accounts in [5, 47, 62] and elsewhere in the bibliography. The purpose of this section is to provide a brief review of some of the ways in which integrable systems and their reductions have been found to be of great importance in certain areas of theoretical physics, especially quantum field theory.

### 1.5.1 Solitons and Field Theory

Local quantum field theories currently provide the most successful framework for describing the fundamental particles of nature and their interactions [25]. Nowadays the most common formulation of quantum field theory is in terms of the Feynman path integral, i.e. with the partition function of the theory being expressed as a functional integral over all the classical field configurations weighted by an exponential of the action,

$$Z = \int [d\phi] \exp[-S[\phi]]. \tag{1.5.1}$$

Usually the action $S$ is given as the integral of a local Lagrangian density, and then in principle the correlation functions and scattering amplitudes for the theory may be calculated from functional integrals over suitable combinations of the fields. In principle it should be possible to compute these path integrals exactly, but in practice this is virtually never the case, and hence it is customary to write the Lagrangian as a perturbation of that for a free field theory and expand all amplitudes as a power series in a small parameter, the coupling constant. The terms in this perturbation expansion have a natural representation as a sequence of graphs (Feynman diagrams), and this technique can yield extremely accurate predictions of physical quantities. However, the problem with perturbation theory is that it makes two major assumptions, neither of which may be true in general.

The first inherent assumption of perturbation theory is that the coupling constant is small, so that each term in the expansion is smaller than the preceding one. An important concept in quantum field theory is that of renormalization, whereby the magnitude of all physical quantities depends on the energy (or distance) scale

of the measurements. This is particularly relevant to quantum chromodynamics (QCD), the model of the strong interaction which describes how quarks are combined with gluons to make up the hadrons. At large distances the effective coupling of the theory becomes very small, and so it is possible to do accurate perturbative calculations for lepton-hadron scattering. On the other hand, at small distances the coupling is very strong (leading to quark confinement), and so perturbation theory breaks down.

In order to understand how quarks are bound into the hadrons, essentially different non-perturbative techniques are required. A popular approach, requiring the most up-to-date computer technology, is to simulate QCD numerically as a lattice gauge theory [97]. An alternative approach is to construct exactly solvable models, in the hope that these will give insight into more realistic theories. In this area there has been much recent progress with theories possessing duality [38], in particular the supersymmetric Yang-Mills theories. The latter are naturally related to certain integrable systems (Toda lattice equations), and this connection means that the mass spectrum can be calculated [48, 115].

The other main assumption behind a perturbative treatment of quantum field theory is that none of the relevant quantities are singular in the expansion parameter. However, if the classical theory admits soliton solutions then this is often violated. In the physical literature the word soliton is used to mean a stable, localized classical solution interpolating between two different vacua; this is in contrast with the more precise mathematical notion of a soliton, which is only appropriate for an integrable theory. Prime examples of soliton-type solutions occur in two different (1+1)-dimensional theories, namely the (integrable) sine-Gordon theory and the (non-integrable) $\phi^4$ theory.

Both the sine-Gordon (sG) theory and the $\phi^4$ theory have simple Lagangian densities of the form

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi)$$

with

$$U(\phi) = \frac{m^4}{g^2}\left[1 - \cos\left(\frac{g\phi}{m}\right)\right],$$

for sine-Gordon (sG) and

$$U(\phi) = \frac{m^4}{2g^2}\left[1 - \frac{g^2\phi^2}{m^2}\right]^2$$

for $\phi^4$. The classical vacua are the minima of the potentials, $U'(\phi_0) = 0$,

$$\begin{aligned} sG: &\quad \phi_0 = 2\pi n m/g,\ n \in \mathbb{Z}, \\ \phi^4: &\quad \phi_0 = \pm m/g. \end{aligned}$$

For each theory there is a static kink soliton (using this word in the loose sense),

$$\begin{aligned} sG: &\quad \phi_c(x) = \pm 4(m/g)\tan^{-1}\exp[\pm m(x-x_0)], \\ \phi^4: &\quad \phi_c(x) = \pm(m/g)\tanh[m(x-x_0)], \end{aligned} \tag{1.5.2}$$

and these solutions minimize the energy. For a static solution the energy is just

$$E[\phi] = \int \left(\frac{1}{2}\phi_x^2 + U(\phi)\right) dx,$$

so that

$$\begin{aligned} sG: &\quad E[\phi_c] = 8(m^3/g^2), \\ \phi^4: &\quad E[\phi_c] = \frac{4}{3}(m^3/g^2). \end{aligned} \tag{1.5.3}$$

Clearly the solutions (1.5.2) can be made time-dependent by applying a Lorentz boost.

The $\pm$ signs in (1.5.2) differentiate between kink and anti-kink solutions, and even at the classical level these solutions are localized around $x = x_0$ and may be identified as particles and anti-particles. The main thing to observe is that each solution (1.5.2), as well as its energy (1.5.3) (corresponding to the lowest order approximation to the particle mass in the quantum theory), is singular in the coupling constant $g$, and thus would be completely missed by conventional perturbation theory. So to develop the full quantum theory it is necessary to include a set of states corresponding to the soliton sector of the theory, as well as the usual vacuum sector [89].

The point of the preceding discussion is to emphasize the importance of classical solutions, and especially solitons, in quantum field theory. The $\phi^4$ theory is not integrable, but is a useful toy model which still has some features that are worth studying [114]. The quantum sine-Gordon model is more interesting in that it is one of the prime examples of an integrable quantum field theory, and so the results of semi-classical quantization are exact [24]. The sine-Gordon theory and its multi-component generalizations, the affine Toda theories, have been the subject of much investigation (see e.g. [31]), and we shall return to these in Chapter 4. Classical solutions of more physically realistic (3+1)-dimensional theories of Yang-Mills type have also been studied a great deal [7], and recently there have been many new results concerning monopole solutions [85], which have fundamental connections with the Toda lattice equations [145]. Integrable systems also have a rôle to play in various attempts to construct a consistent quantum theory of gravity, particularly in the context of the random matrix models which we now introduce.

### 1.5.2 Random Matrices, Correlation Functions and the Painlevé Transcendents

Random matrices have been used extensively in nuclear physics to model the Hamiltonians of large nuclei. The rationale behind this is that for a complicated system the Hamiltonian will not be known, and even if it were it would be too difficult to solve. Thus it makes sense to derive a description of the statistical behaviour of the energy levels from the eigenvalues of a large random matrix; only the overall symmetries of the system are needed as constraints. Random matrix models arise in many other contexts, such as in describing the local fluctuation properties of the zeros of the Riemann zeta function (see references in Mehta's book [117]), in determining the spectrum of the Dirac operator in QCD [149], and in the matrix models of 2-D quantum gravity [51, 75] (described in the next subsection).

Original work of Dyson [53] showed that by imposing a simple symmetry constraint (invariance under unitary transformations) and statistical independence of matrix entries, the most general probability measure $d\mu$ on the space of $N \times N$ hermitian matrices (denoted $H$) is given by that for the Gaussian Unitary Ensemble (GUE),

$$d\mu = \exp(-a\, tr\, H^2 + b\, tr\, H + c)\, dH,$$

where $a$, $b$ and $c$ are real constants with $a > 0$, and

$$dH = \prod_j dH_{jj} \prod_{j<k} \Re dH_{jk} \Im dH_{jk}.$$

By diagonalizing $H$ with a unitary matrix and suitably rescaling, the joint probability density for the $N$ eigenvalues is obtained:

$$P_{N\beta}(y_1, ..., y_N) = C_{N\beta} \exp\left(-\frac{1}{2}\sum_{j=1}^{N} y_j^2\right) |\Delta(y)|^\beta \tag{1.5.4}$$

In the expression (1.5.4) $C_{N\beta}$ is a normalization constant, and

$$\Delta(y) = \prod_{j<k}(y_j - y_k)$$

is the Vandermonde determinant. For the GUE it is found that $\beta = 2$, but there are two other types of matrix ensemble corresponding to invariance under the orthogonal and symplectic groups, and for these ensembles the eigenvalue density is given by the formula (1.5.4) with $\beta = 1, 4$ respectively.

The normalization constant $C_{N\beta}$ can be calculated explicitly [117], making use of a corollary of Selberg's integral formula [14],

$$C_{N\beta}^{-1} = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} |\Delta(y)|^\beta \prod_{j=1}^{N} e^{-\frac{1}{2}y_j^2} dy_j = (2\pi)^{\frac{N}{2}} \prod_{j=1}^{N} \frac{\Gamma(1+\frac{j\beta}{2})}{\Gamma(1+\frac{\beta}{2})}.$$

This exact evaluation is helpful because it leads to a thermodynamical model of the eigenvalues, treating them like a gas of $N$ point particles on a line with potential

$$W = \frac{1}{2}\sum_j y_j^2 - \sum_{j<k} \log|y_i - y_j|.$$

The Gibbs measure for the system in equilibrium at inverse temperature $\beta$ is

$$P_{N\beta} = \frac{e^{-\beta W}}{Z(\beta)},$$

with the partition function

$$Z(\beta) = \int \dots \int e^{-\beta W} dy_1 \dots dy_N.$$

With this alternative interpretation in terms of statistical mechanics, a lot of hard probabilistic analysis is avoided, and it is possible to derive asymptotic results about correlations between eigenvalues in the limit $N \to \infty$. The consideration of these correlations then leads to some remarkable connections with integrable systems.

Following the exposition of Tracy and Widom [147], we define the $n$-point correlation functions,

$$R_{n\beta}(y_1, \dots, y_n) := \frac{N!}{(N-n)!} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} P_{N\beta}(y_1, \dots, y_N) dy_{n+1} \dots dy_N,$$

which give the probability of finding a level (eigenvalue) around each of the points $y_1, \dots, y_n$ with the rest being unobserved. Girko [72] has shown that the level density $R_{1\beta}$ for a very wide class of symmetric random matrices ($\beta = 1$) tends to a semi-circle law in the limit $N \to \infty$; after scaling the endpoints, the eigenvalues are distributed in a semi-circle around the mean value. It turns out that the $n$-point correlation functions can be expressed in terms of determinants involving the kernels of certain integral operators. To consider the scaling limit $N \to \infty$, it is necessary to examine the local statistics of the eigenvalues in the neighbourhood of some point $y_0$, and then define new variables

$$\xi_j = R_{1\beta}(y_0)(y_j - y_0),$$

with $\xi_j$ fixed as

$$y_j \to y_0.$$

Henceforth we consider only the GUE case ($\beta = 2$). Expressing everything in the scaled variables, the relevant integral kernel is

$$K(\xi, \eta) = \frac{1}{\pi} \frac{\sin(\pi(\xi - \eta))}{(\xi - \eta)},$$

and the formula for the scaled $n$-point function is

$$R_{n2}(\xi_1, ..., \xi_n) = \det(K(\xi_j, \xi_k))_{j,k=1,...,n}.$$

We should emphasize that this and similar results in terms of an integral kernel $K$ hold true for a large class of hermitian matrix ensembles, such as those constructed from the classical orthogonal polynomials [147], in particular the ensembles relevant to 2-D quantum gravity [75]. A direct connection with an integrable system arises on considering $m$ disjoint open intervals $I_k = (a_{2k-1}, a_{2k})$, $1 \leq k \leq m$.

A quantity of central importance is the Fredholm determinant

$$D(I; \lambda) = \det(1 - \hat{K}), \tag{1.5.5}$$

where the integral operator is given in terms of its kernel by

$$\hat{K} = \sum_{j=1}^{m} \lambda_j K(\xi, \eta) \chi_{I_j}(\eta).$$

Observe that it is possible to write

$$K(\xi, \eta) = \frac{A(\xi)A'(\eta) - A'(\xi)A(\eta)}{\xi - \eta}$$

where $A(\xi) = \frac{1}{\pi}\sin(\pi\xi)$. Considering the set of endpoints (denoted $a$) of the intervals $I_k$ as times, it is then possible to obtain a system of equations in the dynamical variables

$$Q_j(a) := lim_{\xi \to a_j} \sqrt{\lambda_j}(1 - \tilde{K})^{-1} A(\xi),$$

$$P_j(a) := lim_{\xi \to a_j} \sqrt{\lambda_j}(1 - \tilde{K})^{-1} A'(\xi)$$

(where we have made a slight alteration to the notation of [147]). The resulting integrable system, known as the Jimbo-Miwa-Mori-Sato (JMMS) equations [90], has a natural Hamiltonian structure in terms of canonically conjugate variables $p_j$, $q_j$ defined by

$$q_{2j} := 2iQ_{2j}, \qquad q_{2j+1} := 2Q_{2j+1},$$

$$p_{2j} := iP_{2j}, \qquad q_{2j+1} := Q_{2j+1}.$$

The Hamiltonian corresponding to the time $a_j$ is given by

$$H_j := \frac{\pi^2}{4} q_j^2 + p_j^2 - \frac{1}{4} \sum_{j \neq k} \frac{(q_j p_k - q_k p_j)^2}{a_j - a_k}. \tag{1.5.6}$$

If the $a_j$ are regarded as fixed constants then these Hamiltonians yield the well-known integrable Neumann systems.

The Hamiltonians for the JMMS equations are Poisson-commuting, and are given by the exact one-form

$$\Theta = \sum_{j=1}^{2m} H_j da_j = d_a \log \tau,$$

with the tau-function $\tau$ being given by the Fredholm determinant (1.5.5). We have already seen that such determinants naturally yield tau-functions of KdV [127], and Tracy and Widom have used their matrix model approach to derive similar solutions to other integrable PDEs [148]. The appearance of the times $a_j$ in (1.5.6) indicates that the JMMS equations constitute a non-autonomous Hamiltonian system; they also have an interpretation in terms of loop algebras [76]. An alternative way to view them is as differential equations of Painlevé type, meaning that their solutions can have only poles as movable singularities. In particular by restricting to the case of one interval and looking at dependence on the interval length $z$, the JMMS equations reduce to a form of the Painlevé V equation,

$$\begin{aligned} \frac{d^2w}{dz^2} &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} \\ &\quad + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma}{z}w + \delta\frac{w(w+1)}{w-1} \end{aligned} \tag{1.5.7}$$

($\alpha$, $\beta$, $\gamma$, $\delta$ are parameters).

The Painlevé equations were discovered as a result of the work of Paul Painlevé (see references in Chapter 7 of [5]), who was studying second-order ODEs of the form

$$\frac{d^2w}{dz^2} = \mathcal{F}\left(\frac{dw}{dz}, w, z\right), \tag{1.5.8}$$

with $\mathcal{F}$ being a rational in $w$ and $\frac{dw}{dz}$ and analytic in $z$. More specifically, Painlevé and his co-workers succeeded in classifying ODEs (1.5.8) such that the solutions have no movable singularities other than poles. It was found that (after suitable changes of variables) all these ODEs had general solutions in terms of classical special functions, except for six special equations which are now known as Painlevé I-VI (or just PI-VI). The equation (1.5.7) is the fifth of these, and its general solution (or sometimes the equation itself) may be referred to as a Painlevé transcendent, since it essentially defines a new transcendental function.

Painlevé transcendents naturally arise as similarity reductions of certain integrable PDEs [1]. For example, the first Painlevé transcendent (PI),

$$\frac{d^2w}{dz^2} = 6w^2 + z, \tag{1.5.9}$$

produces similarity solutions of the KdV equation (1.3.1), via the substitution

$$u(x,t) = -2\left(w(z(x,t)) + t\right), \qquad z = x - 6t^2.$$

The connection between integrable nonlinear evolution equations and the Painlevé transcendents has led to a direct test for integrability using local expansions around a singular manifold [30]; the method is usually referred to as Painlevé analysis, and we make use of it in Chapters 2 and 3. In Chapter 2 we study some fourth-order ODEs obtained as similarity reductions of fifth-order KdV-type equations; these ODEs may also be written as non-autonomous Hamiltonian systems.

The equations PI-VI (as well as some of their higher order analogues) have also arisen in various 2-D field theories (see e.g. [21, 56, 88] and below), where they give exact formulas for correlation functions. In this context it is essential to have a good understanding of the structure and asymptotics of the solutions to these ODEs, as only certain solutions are relevant for physical applications. These considerations are particularly important for the Painlevé-type equations occurring in the matrix models of 2-D quantum gravity, which we briefly review.

### 1.5.3 Matrix Models of 2-D Quantum Gravity

One of the main unsolved problems in theoretical physics is to find a consistent framework for a quantum theory of gravity. Although Einstein's theory of general relativity seems to be the correct geometrical description of the large-scale structure of the universe, it breaks down at very small distances, where gravity (and hence space-time itself) should be subject to quantum fluctuations. If the space-time metric is treated as a quantum field, then the usual perturbative methods using path-integrals (1.5.1) fail due to irremovable divergences. Thus it is necessary to find some alternative description. In recent years one of the most popular approaches has been that of string theory, which essentially replaces the point particles of ordinary field theory by one-dimensional strings. While there are considerable conceptual difficulties with string theory, it has produced many physical insights as well as providing the inspiration for whole new areas of mathematics. For an introduction to the main concepts of string theory we refer the reader to Polyakov's book [133]. Here we merely wish to emphasize some of the connections with integrable hierarchies and matrix models.

The simplest and most successful models of quantum gravity are those in two dimensions. As well as being of independent interest, such models are important in string theory because they provide a description of the world-sheet swept out by a string as it evolves in space-time. In the continuum approach advocated by

Polyakov, the quantum picture of the string requires a suitably weighted sum (or path integral) over two-dimensional random surfaces corresponding to the possible configurations of the world-sheet. It is also necessary to allow for variations in the topology of these surfaces, and in two dimensions this just depends on the number of handles (genus). Hence if the fluctuations in the metric are ignored, so that the space-time background is fixed, then this approach leads to the study of conformal field theories on Riemann surfaces, governed by suitable extensions of the Virasoro algebra

$$[l_m, l_n] = (m-n)l_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{1.5.10}$$

It turns out that this algebra is related to the KdV hierarchy and certain classes of its solutions, and this connection can be understood in relation to matrix models.

The matrix models constitute a non-perturbative, discrete approach to 2-D quantum gravity, whereby surfaces are constructed from random triangulations (or polygonal configurations). Generally these models describe conformal matter coupled to topological gravity [46], but we shall largely concentrate on the simplest case of pure gravity, which can be derived from the matrix integral

$$Z_N(\underline{t}) = \int \exp[-tr(\sum_{j=0}^{\infty} t_j H^j)]\, dH. \tag{1.5.11}$$

The partition function (1.5.11) is given as an integral over $N \times N$ hermitian matrices (as for the GUE described above), and defines the one-matrix model. It may also be regarded as a discrete analogue of the path integral (1.5.1), and has a natural expansion in terms of planar diagrams [75, 151]. Each diagram corresponds to a surface built out of a finite number of polygons, and by taking $N \to \infty$ and suitably scaling the couplings $t_j$ (in what is known as a double-scaling limit) different sorts of continuum theory can be recovered.

There are many subtleties to the way in which the partition function is scaled. However, it is a remarkable fact that, even before the double-scaling limit is taken, the sequence of partition functions are naturally related to an integrable system - the Toda lattice equations. Indeed, after integrating out the angular variables (as for the GUE) the expression (1.5.11) is reduced to an integral over the eigenvalues of $H$,

$$Z_N = \int |\Delta(\lambda)|^2 \prod_{k=1}^{N} \exp[-\sum_{j=0}^{\infty} t_j \lambda_k^j]\, d\lambda_k$$

(where an overall constant prefactor has been removed), and then using a certain Vandermonde identity [151] (which can be derived from Plücker relations), it is found that

$$Z_N \frac{\partial^2 Z_N}{\partial t_1^2} - \left(\frac{\partial Z_N}{\partial t_1}\right)^2 = Z_{N-1} Z_{N+1}. \tag{1.5.12}$$

Thus the $Z_N$ are seen to be tau-functions of the Toda lattice, since with the substitution $\phi_N = \log[Z_N/Z_{N-1}]$ the sequence of bilinear relations (1.5.12) yields the usual Toda equations,

$$\frac{\partial^2 \phi_N}{\partial t_1^2} = \exp[\phi_{N+1} - \phi_N] - \exp[\phi_N - \phi_{N-1}].$$

Recently Aratyn and others have shown that more general matrix models satisfy equations of Toda lattice type, within the framework of Darboux-Bäcklund transformations for reductions of the KP hierarchy (see [17] and references).

The next remarkable property of the partition function (1.5.11) is that under suitable adjustments of the parameters $t_j$, it has double-scaling limits corresponding to the flows (1.4.6) of the KdV hierarchy subject to a constraint,

$$[\mathcal{L}, \mathrm{P}] = 1, \tag{1.5.13}$$

known as the string equation. For the string equation it is found that

$$P = (\mathcal{L}^{\frac{2k+1}{2}})_+$$

for some integer $k$, with each choice of $k$ giving a different conformal model coupled to gravity.

The simplest non-trivial case is $k = 1$, corresponding to pure gravity, and it is easy to check that for this model (after an integration) the string equation is just the first Painlevé transcendent (1.5.9). Gross and Migdal [75] have given a thorough discussion of the relevant asymptotics for solutions of this string equation, and these considerations have received more attention recently [124]. In the double-scaling limit the partition function also becomes the square of a KdV tau-function,

$$Z_N \to \tau^2,$$

with the potential of the Schrödinger operator $\mathcal{L}$ given by the usual formula,

$$u = 2(\log \tau)_{xx}.$$

Using orthogonal polynomial techniques, Douglas [51] has also shown that more general $(p-1)$-matrix models, with partition functions

$$Z = \int \exp[tr(-\sum_{j=1}^{p-1} V_j(H_j) + \sum_{j=1}^{p-2} g_j H_j H_{j+1})] \prod_{j=1}^{p-1} dH_j,$$

have double-scaling limits corresponding to the $p$-reduction of the KP hierarchy, constrained by a string equation of the form (1.5.13) (where the operator $\mathcal{L}$ is of order $p$).

The second Hamiltonian structure of KdV provides another direct link with conformal field theory. A short calculation with the second Hamiltonian operator (1.3.6) shows that

$$\{u(x), u(y)\}_2 = -\delta'''(x-y) - 2(u(x)+u(y))\delta'(x-y).$$

Taking periodic functions $u(x)$ (with period $2\pi$), define the Fourier modes by

$$u_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-inx}dx - \frac{1}{4}\delta_{n0},$$

and then it is easy to compute their Poisson brackets,

$$\{u_m, u_n\}_2 = \frac{1}{2\pi i}\left(2(m-n)u_{m+n} - (m^3-m)\delta_{m+n,0}\right),$$

which are equivalent to the Virasoro algebra (1.5.10) after a simple rescaling. It turns out that the second Poisson structures of the $p$-KdV hierarchies are similarly related to some other extended conformal algebras ($W$-algebras). Further connections between integrable hierarchies and these algebras are explored in the thesis of de Vos [151]. Approaching these theories from a different direction, Adler and van Moerbeke have shown [10] that if the flows of the $p$-KdV hierarchy are subject to a constraint (1.5.13) then the tau-function can be given as a limit of a matrix integral; equivalently such tau-functions can be characterized as vacuum vectors of the Virasoro algebra,

$$l_n\tau = 0,$$

or a suitable extension. The matrix integrals of these authors are of a slightly different type to the ones previously considered, and are naturally related to the interpretation of 2-D gravity as a topological field theory [46]. There remain many unanswered questions concerning these matrix models, but so far the connections with integrable hierarchies seem to provide the most fruitful lines of enquiry.

For the sake of clarity we review the contents of the other chapters once more. Chapter 2 concerns similarity reductions of some integrable fifth-order evolution equations. These reductions are naturally viewed as ODEs of Painlevé type, having an interpretation as non-autonomous Hamiltonian systems; it is clear from the above that such equations are important for computing correlation functions in certain field theories. For the systems considered we construct Bäcklund transformations and identify the rôle of the tau-function.

Another way to obtain finite-dimensional mechanical systems from integrable PDEs is to look at the motion of the poles of certain classes of solutions. In Chapter 3 we look at some singular solutions of the NLS equation. With the use of a particular type of Bäcklund transformation (an analogue of the Crum

transformation appearing in the theory of KdV) a sequence of rational solutions is constructed. It is straightforward to demonstrate that the poles and zeros of these rational solutions evolve according to constrained Calogero-Moser equations. These results are quite natural in the light of the fact that NLS occurs as a reduction of KP, since rational solutions of KP are also related to the Calogero-Moser system (without constraints).

Finally, Chapter 4 aims to give a dynamical description of the soliton solutions of the affine Toda field theories. It is shown that the solitons in the $A_n^{(1)}$ case are related to the Ruijsenaars-Schneider models with spin, generalizing the connection between the sine-Gordon solitons and non-spin Ruijsenaars-Schneider models. We make use of the tau-function formalism for the affine Toda theories, and give an expression for the $N$-soliton tau-functions in terms of the positions and spins of the $N$ particles in a Ruijsenaars-Schneider model.

# Chapter 2

# Non-autonomous Hénon-Heiles Systems

In this chapter we consider scaling similarity solutions of three integrable PDEs, namely the Sawada-Kotera, fifth-order KdV and Kaup-Kupershmidt equations. We show that the resulting ODEs may be written as non-autonomous Hamiltonian equations, which are time-dependent generalizations of the well-known integrable Hénon-Heiles systems. The original inspiration behind this was Fordy's discovery that stationary flows of the same three PDEs yield the usual (autonomous) Hénon-Heiles systems. Since these PDEs all arise as reductions of the KP hierarchy, they each have an associated tau-function, and this tau-function is inherited by the scaling similarity solutions. It turns out that the (time-dependent) Hamiltonians are given by logarithmic derivatives of the tau-functions. The ODEs for the similarity solutions also have inherited Bäcklund transformations, which may be used to generate sequences of rational solutions as well as other special solutions. We exhibit some of these solutions explicitly. These results on non-autonomous Hamiltonians are an extension of the approach used by Okamoto in his description of the Painlevé transcendents PI-VI. Some other examples indicate that this approach should be applicable in a more general setting.

## 2.1 Introduction

The six Painlevé transcendents have received a considerable amount of attention in recent years, and have been studied from many different points of view. Their original discovery came about from Painlevé's classification of second-order ODEs having no movable critical points. They have also been approached by way of isomonodromic deformation of linear differential equations [58], or via abelian integrals and algebraic geometry [124, 113]. Furthermore, they have found numerous physical applications. In the matrix models of 2-D gravity [51, 75], the

first Painlevé transcendent (PI) is the simplest equation arising from a Heisenberg relation for two linear differential operators,

$$[L, P] = 1, \tag{2.1.1}$$

and in this context it is referred to as the string equation. The scaling limit of the Ising model (which describes free fermions) has correlation functions which are governed by Painlevé III (see [21] and references therein). The fifth Painlevé transcendent (PV) and some systems which generalize it were found in connection to correlation functions for the spectrum of random matrices (see [76, 90, 147]), and these same equations drive correlation functions for the quantum nonlinear Schrödinger equation [88]. A new development has shown that PV is also related to the correlation functions of the XXZ spin chain in the phase which describes interacting fermions [56].

The classification programme of Painlevé only applied to second-order ODEs of a particular form. As the order of the equations increases, the problem of classifying those which are of Painlevé type (i.e. having no movable singularities other than poles) becomes more and more difficult. For example, with third-order equations there is the possibility of natural boundaries beyond which solutions cannot be analytically continued (as in the case of the Chazy equation [5]). It would be extremely useful to have some general classification techniques for this type of equation, independent of the order. For instance, higher order equations of Painlevé type occur in the matrix models as the order of the operators in (2.1.1) increases. Based on the idea of deformations of Riemann surfaces, Novikov [124] has determined the asymptotics of some of these equations. However, as pointed out in [124], these techniques and the related isomonodromic methods are ineffective at revealing general results; detailed analysis is required for each particular equation considered.

Another context in which ODEs of Painlevé type arise naturally is as similarity reductions of integrable PDEs. Given a PDE in 1+1 dimensions with independent variables $x$, $t$ and dependent variable $u(x,t)$, the problem of finding solutions is somewhat simplified if we seek a solution in the similarity form. That means we have

$$u(x,t) = U(w(z), x, t), \tag{2.1.2}$$

where

$$z = z(x,t)$$

is the similarity variable, and on substituting $U(w,x,t)$ into the PDE, an ODE for $w(z)$ is obtained. There are various ways of finding similarity forms, the most

common being the classical Lie symmetry approach (although this method does not yield all possible similarity solutions; see [43] for a case where it fails, as well as references to the other techniques). Now if the original PDE has lots of nice properties (such as solvability by inverse scattering) then the resulting ODE should be correspondingly manageable. This is expressed more precisely in the conjecture of Ablowitz, Ramani and Segur (ARS [1]), which states that all similarity reductions of integrable PDEs are of Painlevé type. A more detailed discussion, as well as some theorems which support the ARS conjecture, may be found in Chapter 7 of [5]. The main thing to observe is that similarity reductions of soliton equations inherit much of the stucture associated with integrability, such as Bäcklund transformations and solutions in terms of special functions.

The simplest sort of similarity solution for a PDE is just the stationary flow, which corresponds to taking

$$U = w(z)$$

with $z = x$ in (2.1.2). Stationary flows of integrable nonlinear evolution equations naturally lead to integrable finite-dimensional Hamiltonian systems. Indeed, given the zero curvature representation of the evolution equation,

$$F_t - G_x + [F, G] = 0,$$

the restriction to the stationary manifold automatically yields a Lax equation:

$$G_x = [F, G].$$

Hence $G$ becomes the Lax matrix for the stationary flow, and traces of powers of $G$ yield the Hamiltonian and the other constants of motion. Some particular examples of this, relevant to the rest of this chapter, may be found in [26, 63]. The general description of the reduction to stationary flows is given by Fordy in [64].

The other sort of similarity solution most commonly considered is the scaling similarity solution. For comparison with what follows it is worth looking at a well-known example. If we start with the modified Korteweg-deVries (mKdV) equation

$$v_t = v_{xxx} - 6v^2 v_x, \tag{2.1.3}$$

and notice that it has a scaling symmetry, then this gives us its scaling similarity solutions. More explicitly, (2.1.3) is invariant under

$$x \to \beta x, \qquad t \to \beta^3 t, \qquad v \to \beta^{-1} v,$$

and so this implies that there is a similarity solution

$$v(x,t) = (-3t)^{-\frac{1}{3}} y(z(x,t)), \tag{2.1.4}$$

with the similarity variable $z = (-3t)^{-\frac{1}{3}} x$. Substituting this form into (2.1.3) we find that $y$ satisfies

$$y''' = 6y^2 y' + zy' + y \tag{2.1.5}$$

($'$ denotes $\frac{d}{dz}$), which may be integrated once to give

$$y'' = 2y^3 + zy + \alpha, \tag{2.1.6}$$

for some constant $\alpha$. The resulting equation (2.1.6) is the second Painlevé equation (PII).

The example of PII is extremely instructive, in that it is closely related to some of the equations we shall be studying in the rest of the chapter. The first thing to observe is that it may be obtained from the Hamiltonian system

$$\begin{aligned} \frac{dy}{dz} &= \frac{\partial h}{\partial p}, \\ \frac{dp}{dz} &= -\frac{\partial h}{\partial y}, \end{aligned}$$

with the polynomial Hamiltonian

$$h = \frac{1}{2}p^2 - (y^2 + \frac{z}{2})p - (\alpha + \frac{1}{2})y. \tag{2.1.7}$$

Note that this Hamiltonian is non-autonomous: it has explicit dependence on the time $z$, and hence is no longer a constant of motion. Instead of a Lax equation (as for the stationary flows), the zero curvature representation of mKdV (2.1.3) yields a zero curvature representation for PII, which we derive in Section 2.3. It turns out that each of the equations PI-VI may be written as a non-autonomous Hamiltonian system.

In the work of Manin [113], the following Hamiltonian for PVI is presented:

$$h = \frac{1}{2}P^2 - \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp(Q + \frac{T_j}{2}, \omega). \tag{2.1.8}$$

In the above, the Weierstrass $\wp$-function has periods 1 and $\omega$, with $\omega$ being the time, the $\alpha_j$ are parameters, and $(T_0, ..., T_3) = (0, 1, \omega, 1+\omega)$. A substitution (originally due to Fuchs) is required to convert this to the usual form of PVI, from which the other Painlevé equations PI-V may be obtained by a suitable limiting process. While Manin's approach is very elegant and uses the powerful

machinery of algebraic geometry, we do not see a simple way to extend it to higher order equations. We prefer to develop the alternative methods of Okamoto [128], who uses polynomial Hamiltonians for the Painlevé equations, and defines a tau-function for each of them by

$$h = \frac{d}{dz}\log[\tau(z)].$$

In the case of PII, this tau-function essentially coincides with the tau-function of mKdV/KdV (after a simple scaling). Hence the Hamiltonian $h$ is a very natural object from the viewpoint of the original evolution equation. Since the Painlevé equations can all be derived as reductions of higher-dimensional integrable equations (such as self-dual Yang-Mills [5]), we expect that the tau-functions should occur very naturally in these reductions.

Another feature of PII (also inherited from mKdV) is its Bäcklund transformation, which takes a solution $y_\alpha$ of (2.1.6) to another solution $y_{\alpha+1}$ for parameter value $\alpha+1$. It is given explicitly by

$$y_{\alpha+1} = -y_\alpha - \frac{2\alpha+1}{2y'_\alpha + 2y_\alpha^2 + z}.$$

This may be viewed as a canonical transformation in the Hamiltonian framework. The rational solutions to the mKdV/KdV hierarchy (constructed by Adler and Moser in [8]) reduce to give a rational solution to PII for each integer value of $\alpha$, in a sequence related by the Bäcklund transformation. All these solutions may be generated by applying this transformation starting from the solution $y_0 = 0$. For half-integer values of $\alpha$ there is a different sequence of solutions which may be expressed in terms of Airy functions. These results have been derived many times in different ways (e.g. compare the methods in [3, 93] and [128]), and the other Painlevé equations display the same sort of structure.

The majority of this chapter is devoted to the the scaling similarity solutions of three different integrable, fifth-order evolution equations, which are known as the Sawada-Kotera, fifth-order KdV and Kaup-Kupershmidt equations. We show that the ODEs for these similarity solutions may be written as non-autonomous Hamiltonian systems (referred to in the text as the systems $\mathcal{H}_{(i)-(iii)}$), which inherit Bäcklund transformations and tau-functions from the original PDEs. Each of these PDEs belongs to an integrable hierarchy of commuting flows, and we wish to illustrate some of the general features of similarity reductions of the equations in these hierarchies. Hence, in Section 2.2, we consider the three relevant integrable hierarchies of evolution equations (the Sawada-Kotera, Korteweg-deVries and Kaup-Kupershmidt hierarchies), and develop a general formalism to describe

the scaling similarity forms of each of their flows. Each hierarchy has a corresponding modified hierarchy related to it by a Miura map, and this Miura map extends to the similarity solutions. As a particular example of the general formalism, we explicitly describe the one-one correspondence between the scaling similarity solutions of the ordinary (third-order) KdV equation and those of the mKdV equation (i.e. solutions to PII). We also describe this correspondence for the case of the fifth-order evolution equation in each hierarchy. To make contact with isomonodromy ideas, Section 2.3 contains a derivation of the zero curvature representation for each of the ODEs arising as similarity reductions of the flows of the modified hierarchies. For most of the rest of the chapter, we concentrate on the specific example of the scaling similarity solutions of the fifth-order evolution equations.

Section 2.4 concerns some completely integrable finite dimensional Hamiltonian systems, known as the Hénon-Heiles systems, and their non-autonomous generalizations. We describe how (as discovered by Fordy [63]) they are related to stationary flows of the fifth-order PDEs previously introduced, and relate the similarity solutions of these PDEs to non-autonomous versions of the Hénon-Heiles systems. For example, stationary flows of the fifth-order KdV equation,

$$u_t = u_{5x} + 10uu_{3x} + 20u_x u_{xx} + 30u^2 u_x, \tag{2.1.9}$$

may be written as a Hamiltonian system, with the Hamiltonian

$$h = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}q_1 q_2^2 + q_1^3 - \frac{\lambda^2}{2}q_2^{-2},$$

by making the identification $u = q_1$ ($x$ is the time, and $\lambda$ is a constant of integration). The particular ratio of terms in the potential of the above Hamiltonian corresponds to one of the integrable cases of the Hénon-Heiles system. By a slight modification of Fordy's approach, we are able to relate the scaling similarity solutions of (2.1.9),

$$u = (-5t)^{-\frac{2}{5}} w(z), \qquad z = (-5t)^{-\frac{1}{5}} x,$$

to a system with Hamiltonian

$$h = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}q_1 q_2^2 + q_1^3 - \frac{\lambda^2}{2}q_2^{-2} - \frac{1}{2}zq_1. \tag{2.1.10}$$

To do this we must identify $w = q_1$, and the time is now denoted by $z$. Thus it is apparent that the Hamiltonian (2.1.10) is time-dependent; there are similar Hamiltonians for the scaling similarity solutions of the Sawada-Kotera and Kaup-Kupershmidt equations.

It is helpful to define tau-functions for these non-autonomous Hamiltonians, so that e.g. for (2.1.10) we have

$$h = -\frac{d}{dz}\log[\tau(z)].$$

We then find Bäcklund transformations which are seen to be canonical transformations in the Hamiltonian setting. In order to derive these Bäcklund transformations, we make use of the Miura map for the similarity solutions. In particular, (2.1.9) is related to the fifth-order equation in the mKdV hierarchy,

$$v_t = \left(v_{4x} - 10(v^2 v_{xx} + v v_x^2) + 6v^5\right)_x, \tag{2.1.11}$$

by the Miura map,

$$u = -v_x - v^2.$$

This Miura map leads to a one-one correspondence between scaling similarity solutions of (2.1.11),

$$v = (-5t)^{-\frac{1}{5}} y(z), \qquad z = (-5t)^{-\frac{1}{5}} x,$$

and solutions of the system with Hamiltonian (2.1.10). Repeated application of the Bäcklund transformations may be used to generate sequences of solutions to the Hamiltonian systems. Hence (in Section 2.5) we present sequences of rational solutions, as well as special solutions which may be expressed in terms of solutions to PI. This is in contrast to the special sequences of solutions to the usual Painlevé equations, which are all given in terms of classical functions. Section 2.5 also contains a brief discussion of Painlevé analysis for these systems. Finally we indicate some other applications of these techniques, and suggest ways in which they might be developed. Some of this work has already appeared in [87].

## 2.2 Scaling Similarity Solutions in the Sawada-Kotera, KdV and Kaup-Kupershmidt Hierarchies

Although most of this chapter is concerned with the scaling similarity solutions of three particular fifth-order evolution equations (one of which is the fifth-order KdV equation (2.1.9)) and their associated modified equations (such as (2.1.11)), these fifth-order equations are only particular flows of certain integrable hierarchies, which we will refer to as SK, KdV and KK for short. For example, the KdV equation ((2.2.13) below) is the first non-trivial flow in the KdV hierarchy,

while (2.1.9) is the next flow in this hierarchy. In this section we develop a concise notation to describe the scaling similarity solutions of any one of the flows of SK, KdV or KK. We then apply it to some particular examples, including the similarity solutions of fifth-order equations which we study in detail in Sections 2.4 and 2.5.

### 2.2.1 General Description of Scaling Similarity Solutions

Before looking at the particular non-autonomous systems which are the main subject of this chapter, we will consider some aspects of three different hierarchies of PDEs, known as the Sawada-Kotera (SK), KdV and Kaup-Kupershmidt (KK) hierarchies, that are needed in what follows. Each hierarchy is a sequence of evolution equations or flows with respect to times $t_n$ ($n = 1, 2, 3, ...$), which can all be put into Hamiltonian form. SK and KK have only one Hamiltonian structure, but KdV is bi-Hamiltonian, and here we will be using the second Hamiltonian structure. Following Fordy [63], we are able to consider all three hierarchies at once. The $n$-th flow in each of the hierarchies can be written as

$$\frac{\partial u}{\partial t_n} = (\partial_x^3 + 8au\partial_x + 4au_x)\delta_u H_n[u], \tag{2.2.1}$$

where $a = 1/2$ for SK and KdV, $a = 1/4$ for KK, and $H_n$ is the $n$-th Hamiltonian for the hierarchy in question. For the purposes of computing variational derivatives, we make no distinction between a Hamiltonian and its corresponding Hamiltonian density. For more details on these hierarchies and ways of calculating the sequence of Hamiltonians, see e.g. [60, 69].

There is also a Miura map from the modified versions of the hierarchies, given by

$$u = -v_x - 2av^2 =: M[v].$$

Then re-writing the Hamiltonian in terms of $v$ and derivatives, the $n$-th modified flow may be expressed as

$$\frac{\partial v}{\partial t_n} = (-\partial_x)\delta_v H_n[M[v]]. \tag{2.2.2}$$

The Miura map means that given $v$ satisfying (2.2.2) for each $n$, the corresponding $u = -v_x - 2av^2$ satisfies (2.2.1).

The $n$-th flow of the hierarchy is unchanged by the scaling

$$x \to \beta x, \qquad t_n \to \beta^m t_n, \qquad u \to \beta^{-2} u,$$

where $m = m(n)$ is a scale weight dependent on the hierarchy and on which flow is being considered. It is easy to show that each flow has a scaling symmetry by

looking at its Lax representation in terms of differential operators. Similarly the modified flow is invariant under the same scaling but with

$$v \to \beta^{-1} v.$$

Hence there are scaling similarity solutions looking like $u = t^{-\frac{2}{m}}\, w(x/t^{\frac{1}{m}})$ (up to rescaling of $w$ and the similarity variable $z$). For convenience in what follows we scale the similarity variable so that

$$u(x, t_n) = \theta^2(t_n) w(z),$$

where

$$z = x\theta(t_n), \qquad \frac{d\theta}{dt_n} = \theta^{m+1}.$$

The corresponding similarity solution for the modified flow is

$$v = \theta(t_n) y(z),$$

with the scaled Miura map giving

$$w = -y' - 2ay^2$$

($'$ denotes $\frac{d}{dz}$ throughout).

In the context of an integrable hierarchy, it is customary to think of the dependent variable as a function of all the times,

$$u = u(t_1 = x, t_2, t_3, \ldots).$$

When we consider the scaling similarity solutions to the $n$-th flow, it is better to drop dependence on anything other than $x$ and $t_n$. In fact, because of the way that the variables must scale, it appears to be inconsistent to consider the other flows simultaneously. We shall see a particular manifestation of this in Section 5, when we come to consider the rational solutions. It appears that the only way to incorporate the other flows is to allow the similarity variable to depend on some of the other times (as can be done to get similarity solutions of the KP equation [136]). Henceforth we will drop the suffix $n$, bearing in mind that the actual form of the Hamiltonian depends on which particular flow we have chosen.

Substituting the similarity forms into the equations of motion (2.2.1) and (2.2.2) (and cancelling out powers of $\theta$ on either side) yields the ODEs for $w$ and $y$. If we let $\tilde{H}$ denote the scaled Hamiltonian (expressed in terms of $w$ with powers of $\theta$ divided out) then we obtain the equations for the similarity solutions in the following form:

$$(\partial^3 + 8aw\partial + 4aw')\left(\delta_w \tilde{H} - \frac{1}{4a} z\right) = 0, \tag{2.2.3}$$

$$\partial(\delta_y \tilde{H} + zy) = 0. \tag{2.2.4}$$

The symbol $\partial$ denotes derivatives with respect to $z$. It is worth describing how this scaling process works in slightly more detail. For the scaling similarity solutions, the left-hand side of (2.2.1) is

$$\begin{aligned} u_t &= 2\theta\frac{d\theta}{dt}w(z) + \theta^2\frac{\partial z}{\partial t}w'(z), \\ &= \theta^{m+2}(2w + zw'), \\ &= \theta^{m+2}(\partial^3 + 8aw\partial + 4aw')\left(\frac{1}{4a}z\right). \end{aligned} \tag{2.2.5}$$

The crucial step is the last line (2.2.5), where the $t$ derivative is rewritten in terms of the Poisson operator appearing on the right-hand side of (2.2.1). Since this right-hand side must also scale correctly, we have

$$\delta_u H = \theta^{m-1}\delta_w \tilde{H}.$$

To obtain $\tilde{H}$ it is necessary to replace every $x$ derivative $u_{kx}$ in $H$ by the corresponding (rescaled) $z$ derivative $w^{(k)}$. Upon making use of (2.2.5) and the scaled Hamiltonian $\tilde{H}$, (2.2.3) follows directly from (2.2.1). A similar calculation leads to (2.2.4) from (2.2.2).

Both equations (2.2.3,2.2.4) can be integrated once, and are conveniently written in terms of

$$f := \delta_w \tilde{H} - \frac{1}{4a}z. \tag{2.2.6}$$

Integration of (2.2.3) yields immediately

$$\frac{d^2 f}{dz^2} + 4awf + \frac{\lambda^2 - (\frac{df}{dz})^2}{2f} = 0, \tag{2.2.7}$$

with $\lambda$ being a constant of integration. For (2.2.4), note that

$$\delta_v H = (M')^* \delta_u H = (\partial_x - 4av)\delta_u H, \tag{2.2.8}$$

where $M'$ is the Fréchet derivative of $M$. The scaled similarity form of this relation (involving $y$ and $\delta_w \tilde{H}$) allows (2.2.4) to be written in terms of the quantity $f$ and integrated to

$$\frac{df}{dz} - 4ayf + \lambda = 0. \tag{2.2.9}$$

A more obvious direct integration of (2.2.4) would be

$$\delta_y \tilde{H} + zy + \alpha = 0, \tag{2.2.10}$$

and in fact this is exactly equivalent to (2.2.9), with the constant $\alpha$ given in terms of $\lambda$ by

$$\alpha = \lambda - \frac{1}{4a}.$$

Indeed, the scaled version of (2.2.8) implies

$$\delta_y \tilde{H} = (\partial - 4ay)\left(f + \frac{1}{4a}z\right),$$

which shows how (2.2.9) and (2.2.10) are related. The ODEs for the similarity solutions are completely specified by (2.2.7,2.2.9), together with the definition (2.2.6) and the scaled Miura map. We remark that the equations for the stationary flows are simply obtained by removing the $-\frac{1}{4a}z$ terms from $f$.

In (2.2.7) $f$ is to be thought of as a function of $w$ and its derivatives, while in (2.2.9) it is expressed instead in terms of $y$ and derivatives of $y$ (replacing each $w$ by $-y' - 2ay^2$). The constant of integration $\lambda$ is the same in both cases, as the Miura map becomes a one-one correspondence between the two equations. The form of the equations makes it particularly simple to see the relationship between them. We have the scaled Miura map,

$$w = -y' - 2ay^2, \tag{2.2.11}$$

and it has an inverse given by

$$y = \frac{f' + \lambda}{4af}. \tag{2.2.12}$$

In (2.2.12) we regard $f$ as being a function of $w$ and its derivatives, and it is necessary to assume $f \not\equiv 0$ since otherwise this equation breaks down. Now suppose that we have a solution $w$ of (2.2.7), and we define the modified variable $y$ by the inverse Miura map. Then we may calculate directly

$$\begin{aligned} -y' - 2ay^2 &= \frac{1}{4af}\left(-f'' + \frac{f'^2 - \lambda^2}{2f}\right), \\ &= w, \end{aligned}$$

where the last line follows at once on rearranging (2.2.7). Hence the inverse Miura map (2.2.12) together with (2.2.7) implies the Miura map (2.2.11). This in turn means that $f$ can be reinterpreted as a function of $y$ and its derivatives, and then (2.2.12) may be rearranged to yield the ODE (2.2.9) for $y$. The converse follows by a reversal of this argument (or immediately upon scaling the usual Miura map for the PDEs). To make things more concrete, it is worth looking at some particular cases.

### 2.2.2 Similarity Solutions of KdV and PII

The first example to consider is the scaling similarity solutions of the ordinary KdV equation. Putting $H = \frac{1}{2}u^2$ (the first non-trivial Hamiltonian in the KdV hierarchy) into (2.2.1) with $a = 1/2$ we obtain KdV:

$$u_t = u_{xxx} + 6uu_x. \tag{2.2.13}$$

The scaling similarity solutions are given by

$$u(x,t) = (-3t)^{-\frac{2}{3}}w(z),$$

with the similarity variable $z = (-3t)^{-\frac{1}{3}}x$. After substituting into KdV and integrating once we find the ODE for $w$:

$$w'' + 2w^2 - zw + \frac{\alpha(\alpha+1) + w' - (w')^2}{2w - z} = 0. \tag{2.2.14}$$

Using the scaled Hamiltonian $\tilde{H} = \frac{1}{2}w^2$ we find

$$f = w - \frac{z}{2},$$

and substituting into (2.2.7) with this $f$ and $a = \frac{1}{2}$ does indeed give the equation (2.2.14) on setting $\lambda = \alpha + \frac{1}{2}$. Also the Miura map $u = -v_x - v^2$ goes from mKdV to KdV. For the scaling similarity solutions (2.1.4) of mKdV (2.1.3) we find that there is a one-one correspondence between solutions of PII (equation (2.1.6) of the previous section) and (2.2.14), given by

$$w = -y' - y^2$$

(the scaled Miura map) and

$$y = \frac{w' + \alpha}{2w - z},$$

the latter being a particular case of the inverse Miura formula (2.2.12). Also note that, in terms of $y$, we have

$$f = -y' - y^2 - \frac{z}{2},$$

and on putting this into (2.2.9) with $a = \frac{1}{2}$, PII results. This example is also considered in [3], for instance.

### 2.2.3 Similarity Solutions of Fifth-Order Equations

Our second example constitutes the main subject of this chapter. We take the fifth-order equations in each of the hierarchies, which (following [63]) may be written as

$$u_t = (u_{xxxx} + (8a - 2b)uu_{xx} - 2(a+b)u_x^2 - \frac{20}{3}abu^3)_x, \tag{2.2.15}$$

where we have three cases (i),(ii),(iii), corresponding to

$$a = \frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{4}, \qquad b = -\frac{1}{2}, \quad -3, \quad -4,$$

respectively. These are the only values of $a, b$ for which an equation of the form (2.2.15) is integrable [63], and (i) and (iii) are respectively the Sawada-Kotera [101] and Kaup-Kupershmidt [96] equations, while (ii) is the fifth-order KdV equation (2.1.9). The expression (2.2.15) may be obtained from the Hamiltonian formalism described above (2.2.1), by taking the Hamiltonian to be

$$H = -\frac{1}{2}u_x^2 - \frac{1}{3}bu^3.$$

When specializing to one of the three hierarchies, it is necessary to take the correct values of $a$ and $b$ in each case. While some of the calculations we present are valid for arbitrary $a$, $b$, all properties relevant to the integrability of the equations are lost in general. The scaling similarity solutions of (2.2.15) take the form

$$u(x,t) = (-5t)^{-\frac{2}{5}}w(z),$$

where now $z = (-5t)^{-\frac{1}{5}}x$. We find

$$f = w'' - bw^2 - \frac{z}{4a},$$

and putting this into (2.2.7) we get a fourth-order ODE for $w$, which we prefer not to present in gory detail.

It is helpful to have the explicit forms of the fourth-order ODEs for the scaling similarity solutions of the associated modified equations,

$$v(x,t) = (-5t)^{-\frac{1}{5}}y(z).$$

Using the scaled Miura map, we may express $f$ in terms of the modified variable $y$:

$$f = -y''' - 4ayy'' - (4a+b)(y')^2 - 4aby^2y' - 4a^2by^4 - \frac{z}{4a}. \tag{2.2.16}$$

Then the ODE for $y$ is (from (2.2.9))

$$y^{(iv)} = -2(6a+b)y'y'' + 4a(4a-b)(y^2y'' + y(y')^2) + 16a^3by^5 + zy + \alpha, \tag{2.2.17}$$

with

$$\alpha = \lambda - \frac{1}{4a}.$$

Given a solution to (2.2.17) we can then obtain a solution to the fourth-order ODE for $w$, via $w = -y' - 2ay^2$. The equation for $w$ is rather unwieldy when written

out in full, so rather than giving it explicitly we will write it as a Hamiltonian system in Section 2.4.

On substituting in the relevant values of $a$ and $b$ into the equation (2.2.17) we find

$$y^{(iv)} = -5y'y'' + 5(y^2y'' + y(y')^2) - y^5 + zy + \alpha, \tag{2.2.18}$$

$$y^{(iv)} = 10(y^2y'' + y(y')^2) - 6y^5 + zy + \alpha, \tag{2.2.19}$$

$$y^{(iv)} = 5y'y'' + 5(y^2y'' + y(y')^2) - y^5 + zy + \alpha, \tag{2.2.20}$$

for (i),(ii),(iii) respectively. So (2.2.19) is the ODE for the scaling similarity solutions of the fifth-order mKdV equation (2.1.11). Notice that (2.2.18) and (2.2.20) differ only by a sign in the even ($y'y''$) terms. Hence if $y_{(i)}$ is as solution to (2.2.17) for case (i), then $y_{(iii)} = -y_{(i)}$ will be a solution to that equation for case (iii) with $\alpha$ replaced by $-\alpha$. This is because the modified hierarchies in these two cases are essentially the same. In fact both SK and KK have a third-order Lax operator, which is factorized to yield the Miura map (see [60]). We shall use this connection between case (i) and case (iii) to derive the Bäcklund transformation for the scaling similarity equations. It is no longer necessary to consider (2.2.20) separately.

We should like to view (2.2.18) and (2.2.19) as fourth-order analogues of PII. In particular, their general solutions should not be expressible in terms of classical special functions, and should therefore define new transcendents. The majority of our results concerning these equations (2.2.18,2.2.19) involve using the Miura map to relate them to some non-autonomous Hamiltonian systems, as we describe in Section 2.4. Another important feature shared by (2.2.18) and (2.2.19) is that they have associated linear systems and zero curvature representations, which they inherit from the PDEs. Although we do not make use of the linear systems elsewhere, they allow an interpretation in terms of isomonodromic deformations, and so we present them in the next section for completeness.

## 2.3 Zero Curvature Equations

Given that the equations (2.2.18) and (2.2.19) have arisen as similarity reductions of integrable PDEs, we would expect them to be of Painlevé type [1]. In other words, all their solutions should be globally meromorphic. The usual method of testing for this is Painlevé analysis, which consists of finding all possible formal expansions of a solution around an arbitrary point $z_0$. However, while this test can show that an ODE is not of Painlevé type (if a noninteger power or logarithmic term appears in the expansion), it is a local test, and hence is not sufficient

for proving that an equation with well-defined formal expansions is of this type. Painlevé's original proof that the six transcendents PI-VI have no movable singularities other than poles is extremely laborious and notoriously unclear, requiring detailed analysis and special transformations for each equation. Recently Joshi and Kruskal [91] have given a concise proof which deals with all six equations on an equal footing. Essentially their method examines solutions in the neighbourhood of singularities, converting each ODE into an integral equation (by integrating the dominant terms) and then showing that the singularities must all be isolated poles. Unfortunately we have been unable to extend this method to deal with the fourth-order equations (2.2.18,2.2.19). Nonetheless, Painlevé analysis does provide some useful information, and we present the results of this in Section 2.5.

Another way of understanding the solutions of Painlevé equations is through the concept of isomonodromic deformation of a linear system. An important feature of the equations we consider is that the zero curvature representation of the PDEs in the modified hierarchy scales to give the corresponding representation for their similarity solutions. Starting from a linear system,

$$\begin{aligned} \Phi_x &= F\Phi, \\ \Phi_{t_n} &= G\Phi, \end{aligned}$$

the $n$-th flow of a hierarchy is obtained from the compatibility condition,

$$F_{t_n} - G_x + [F, G] = 0.$$

The matrices $F$,$G$ will depend on a spectral parameter $k$ as well as on $x$ and $t_n$. To get a linear system for the scaling similarity solutions, it is necessary to allow derivatives with respect to a rescaled spectral parameter $\zeta$, yielding a new zero curvature equation,

$$U_\zeta - V_z + [U, V] = 0. \tag{2.3.1}$$

We illustrate this method in the case of the mKdV hierarchy, which for these purposes we write as

$$\frac{\partial v}{\partial t_n} = (-\partial_x)\delta_v H_{n-1}[M[v]],$$

where we have shifted the labels on the Hamiltonians compared with (2.2.2), so that the corresponding (bi-Hamiltonian) flows for KdV are written as

$$\begin{aligned} \frac{\partial u}{\partial t_n} &= (\partial_x^3 + 8au\partial_x + 4au_x)\delta_u H_{n-1}[u], \\ &= \partial_x \delta_u H_n[u]. \end{aligned}$$

The zero curvature representation of the $n$-th flow in the mKdV hierarchy comes from a linear system,

$$\begin{aligned}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x &= \begin{pmatrix}-k & v\\ v & k\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},\\ \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_{t_n} &= \begin{pmatrix}A & B-C\\ B+C & -A\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}.\end{aligned}$$

In the above, $k$ is the spectral parameter, and we have

$$A = 2k(\partial_x S_{n-1} - T_n), \qquad B = -S_n, \qquad C = -2k\partial_x S_{n-1},$$

where

$$\begin{aligned}S_n &= \sum_{j=0}^{n-1}(4k^2)^j \delta_v H_{n-1-j},\\ T_n &= \sum_{j=0}^{n-1}(4k^2)^j \delta_u H_{n-1-j}.\end{aligned}$$

(The sequence of Hamiltonians starts $H_0{=}\frac{1}{2}u$, $H_1{=}\frac{1}{2}u^2$, $H_2{=}-\frac{1}{2}u_x^2+u^3$, ...) Now for the similarity solutions we express everything in terms of the new variables,

$$z = x\theta(t_n), \qquad k = \zeta\theta(t_n), \qquad v(x,t_n) = \theta(t_n)y(z), \qquad \phi_j(x,t_n;k) = \chi_j(z;\zeta).$$

(For the $n$-th flow of mKdV we have $\frac{d\theta}{dt_n} = \theta^{2n}$.) When everything is written in terms of these similarity variables, certain powers of $\theta$ can be divided out, and $\delta_y \tilde{H}_{n-1}$ may be eliminated using (2.2.10), to yield the new scaled linear system

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_z = \begin{pmatrix}-\zeta & y\\ y & \zeta\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{2.3.2}$$

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}_\zeta = \begin{pmatrix}\Xi & \Gamma-\Delta\\ \Gamma+\Delta & -\Xi\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \tag{2.3.3}$$

with

$$\Xi = -2\partial\tilde{S}_{n-1} + 2\tilde{T}_n - z, \qquad \Gamma = 4\zeta\tilde{S}_{n-1} - \frac{\alpha}{\zeta}, \qquad \Delta = 2\partial\tilde{S}_{n-1}.$$

Again we have used the convention that a quantity with a tilde is written in terms of the similarity variables with powers of $\theta$ scaled out.

To make this more concrete, we refer back to our earlier examples. The $n$-th similarity equation in the mKdV hierarchy is the compatibility condition (2.3.1) arising from the linear system (2.3.2,2.3.3). Hence the matrix

$$U = \begin{pmatrix}-\zeta & y\\ y & \zeta\end{pmatrix}$$

is the same for all $n$, while

$$V = \begin{pmatrix} \Xi & \Gamma - \Delta \\ \Gamma + \Delta & -\Xi \end{pmatrix}$$

depends on which similarity equation is being considered. PII (2.1.6) corresponds to the case $n = 2$, and the entries of $V$ are given by

$$\Xi = 4\zeta^2 - 2y^2 - z, \qquad \Gamma = -4\zeta y - \frac{\alpha}{\zeta}, \qquad \Delta = -2y'.$$

The fourth order ODE (2.2.19) comes from the third flow of the mKdV hierarchy, and in that case $V$ has entries

$$\Xi = 16\zeta^4 - 8\zeta^2 y^2 - 4yy'' + 2(y')^2 + 6y^4 - z,$$

$$\Gamma = -16\zeta^3 y - 4\zeta(y'' - 2y^3) - \frac{\alpha}{\zeta}, \qquad \Delta = -8\zeta^2 y' - 2y''' + 12y^2 y'.$$

The zero curvature representation of the flows in the Sawada-Kotera and Kaup-Kupershmidt hierarchies, as well as in their common modified hierarchy, requires 3×3 matrices [60]. Hence to represent the equation (2.2.18) as the zero curvature condition (2.3.1) we must take

$$U = \begin{pmatrix} 0 & \zeta & 0 \\ 0 & y & \zeta \\ \zeta & 0 & -y \end{pmatrix},$$

$$V = \begin{pmatrix} -2\zeta^2 L & 9\zeta^3 y + J - K & 9\zeta^4 + \zeta L' \\ 9\zeta^4 - \zeta L' & \zeta^2 L - \frac{\alpha}{\zeta} & J + 2K \\ -9\zeta^3 y + J - K & 9\zeta^4 & \zeta^2 L + \frac{\alpha}{\zeta} \end{pmatrix},$$

where

$$J = 2yy'' - (y')^2 + 2y^2 y' - y^4 + z,$$

$$K = y''' + yy'' + 2(y')^2 - 2y^2 y', \qquad L = 3(y' + y^2).$$

Given the zero curvature form, the initial value problem for $y$ can be reduced to an inverse monodromy problem, which is solved in terms of a system of singular integral equations or a Riemann-Hilbert problem. This approach constitutes an ODE analogue of the inverse scattering transform, and has been applied in detail to PII by Flaschka and Newell [58], and also by Fokas and Ablowitz [4]. The other similarity equations in the mKdV hierarchy are also discussed in [58], where they are referred to as the Painlevé II Family, and the same inverse monodromy scheme is outlined for the whole Family (which of course includes the equation (2.2.19) as its second member). However, as the order of the ODEs increases, the problem becomes much more complicated. Rather than trying to apply this scheme any further to the fourth order equations (2.2.18,2.2.19), in the next section we proceed to develop the Hamiltonian formalism of Okamoto, while at the same time generalizing some results about stationary flows due to Fordy.

## 2.4 Hénon-Heiles Systems

### 2.4.1 Stationary Flows and Integrable Hénon-Heiles

The original Hénon-Heiles system is given by a Hamiltonian with two degrees of freedom:

$$h = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{1}{3}bq_1^3, \tag{2.4.1}$$

The equations of motion are just Hamilton's equations

$$\frac{dq_j}{dz} = \frac{\partial h}{\partial p_j}, \tag{2.4.2}$$

$$\frac{dp_j}{dz} = -\frac{\partial h}{\partial q_j}. \tag{2.4.3}$$

(We are denoting the time by $z$ here to make connection with our other results.) It has been known for some time from Painlevé analysis [40] that this system is integrable for only three values of the ratio $r = a/b$ (because of a scaling symmetry of the equations the integrability only depends on this ratio), namely

$$r = -1, \quad -1/6, \quad -1/16.$$

More recently, Fordy [63] has shown that for these integrable cases the equations of motion are just disguised versions of the stationary flows of some fifth-order soliton equations - the Sawada-Kotera, fifth-order Korteweg-deVries and Kaup-Kupershmidt equations (all particular cases of the equation (2.2.15)). Thus the choice of values for $a$ and $b$ as given in Section 2.2 gives the right values for the ratio $r$ in the cases (i),(ii),(iii) respectively. The zero curvature form of these PDEs yields a matrix Lax representation of the stationary flows, and then traces of powers of the Lax matrix give the Hamiltonian and the second constant of motion (which shows that these systems are indeed Liouville integrable). It was subsequently shown that all three systems are completely separable in suitable coordinates, and may be integrated in terms of theta functions of genus one (cases (i) & (iii)) or genus two (case (ii))[33].

### 2.4.2 Non-autonomous Hamiltonians for Scaling Similarity Solutions

Instead of looking at the stationary flows of these three fifth-order PDEs (all of the form (2.2.15)), we take the equations for their scaling similarity solutions, and rewrite them in Hamiltonian form. These similarity equations are most

conveniently described using the notation of Section 2.2. We have the definition of $f$ in terms of $w$,

$$f = w'' - bw^2 - \frac{z}{4a}, \tag{2.4.4}$$

and on substituting this into (2.2.7) we obtain the full fourth-order ODE for $w$. As we stated previously, we prefer not to present this ODE explicitly, as it is not very instructive to do so. Instead, if we set

$$w = q_1, \qquad f = -aq_2^2,$$

then we may rewrite (2.2.7,2.4.4) as a coupled system for $q_1, q_2$:

$$q_1'' = bq_1^2 - aq_2^2 + \frac{z}{4a}, \tag{2.4.5}$$

$$q_2'' = -2aq_1q_2 - \frac{\lambda^2}{4a^2q_2^3}. \tag{2.4.6}$$

This coupled system just follows from Hamilton's equations (2.4.2,2.4.3), where now the Hamiltonian is given by

$$h = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{1}{3}bq_1^3 - \frac{\lambda^2}{8a^2}q_2^{-2} - \frac{1}{4a}zq_1. \tag{2.4.7}$$

Compared with (2.4.1), this has an extra inverse square term and a non-autonomous (time-dependent) term in the potential.

Thus we have shown that the equations for the similarity solutions introduced at the end of Section 2.2 may be viewed as non-autonomous Hénon-Heiles systems. Because of the explicit time-dependence, the Hamiltonian is no longer a constant of motion, and there is no matrix Lax representation as in the autonomous case. However, the Hamiltonian theory of the six Painlevé transcendents, as developed by Okamoto [128], can be extended completely analogously to the Hamiltonian system defined by (2.4.7), for the three special values of the ratio $r$. In particular these three special systems, which we denote by $\mathcal{H}_{(i)}$, $\mathcal{H}_{(ii)}$, $\mathcal{H}_{(iii)}$, have Bäcklund transformations which can be viewed as canonical transformations in the variables $q_j, p_j, z$. More precisely, we will present transformations between the Hamiltonian systems,

$$(q_j, p_j, z) \longrightarrow (\tilde{q}_j, \tilde{p}_j, \tilde{z}),$$

such that the two-form

$$\omega = \sum_{j=1,2} dp_j \wedge dq_j - dh \wedge dz$$

is preserved (i.e. $\omega = \tilde{\omega}$). We have also found that each of $\mathcal{H}_{(i)-(iii)}$ admit a second independent quantity, $c$, in involution with the Hamiltonian:

$$\{c, h\} = 0.$$

However, $c$ is also time-dependent.

The construction of the Bäcklund transformations is possible due to the existence of the Miura maps. For the system $\mathcal{H}_{(ii)}$ we have (by the general arguments of Section 2.2) a one-one correspondence between its solutions and the solutions of the modified similarity equation (2.2.19), while both $\mathcal{H}_{(i)}$ and $\mathcal{H}_{(iii)}$ have similar correspondences with (2.2.18). Thus the Hamiltonian approach elucidates the solution structure of the modified equations considerably. We are also able to define a tau-function for the Hamiltonians, via

$$h(q_j(z), p_j(z), z) \propto (\log[\tau(z)])'$$

(for some constant of proportionality, which is dependent on which of the three cases is being considered). As well as simplifying many of the derivations, this tau-function is naturally inherited from the original PDEs.

Before dealing with these issues, we first show that the time-dependent Hamiltonian $h$ must also satisfy a fourth-order ODE. At this point it is worth recalling that when we discussed the similarity solutions in Section 2.2, the relationship between the Miura map and its inverse was most easily seen by considering $f$ alternately either as a function of $w$ and derivatives (given by (2.4.4) in this case) or as a function of $y$ and its derivatives (given by (2.2.16)). For the proof of the following proposition, it is convenient to further abuse our notation and think of $f$ as a function of $h$ and its derivatives.

**Proposition 2.1.** *If the quantity $f$ is rewritten in terms of the Hamiltonian (2.4.7) by*

$$f = -4ah''' - 16a^2b(h')^2 - \frac{z}{4a},$$

*then $h$ satisfies the fourth order nonlinear ODE*

$$(f')^2 - 32a^2h'f^2 - 8af\left(8a^2(h'')^2 + \frac{64}{3}ba^3(h')^3 + zh' - h\right) - \lambda^2 = 0. \qquad (2.4.8)$$

*Define a generic solution of (2.4.8) to be one for which*

$$f' \not\equiv 0.$$

*Conversely, if $h$ is a generic solution to (2.4.8), then the functions $(q_j(z), p_j(z))$ defined by*

$$q_1 = -4ah', \quad p_1 = -4ah'', \quad q_2 = \left(4h''' + 16ab(h')^2 + \frac{z}{4a^2}\right)^{\frac{1}{2}},$$

$$p_2 = \left(4h''' + 16ab(h')^2 + \frac{z}{4a^2}\right)^{-\frac{1}{2}}\left(2h^{(iv)} + 16abh'h'' + \frac{1}{8a^2}\right) \qquad (2.4.9)$$

*satisfy Hamilton's equations (2.4.2,2.4.3) for the Hamiltonian $h$.*

**Proof.** Given the (Hamiltonian) equations of motion, it is immediately apparent that

$$\frac{dh}{dz} = \frac{\partial h}{\partial z} = -\frac{1}{4a}q_1.$$

It is then a simple matter to express $(q_j, p_j)$ in terms of $z$, $h$ and its derivatives. The formulae (2.4.9) result, but it is simpler to work with $f$, $f'$ instead of $q_2$, $p_2$ (to avoid square roots). Substituting these expressions into (2.4.7) and rearranging yields (2.4.8). As for the converse, all of Hamilton's equations are direct consequences of (2.4.9), apart from

$$\frac{dp_2}{dz} = -2aq_1q_2 - \frac{\lambda^2}{4a^2q_2^3}.$$

In fact, this is more easily written in terms of $f$ and derivatives, and is equivalent to (2.2.7). Noticing that (2.4.8) contains the quantity

$$g = 8a^2(h'')^2 + \frac{64}{3}ba^3(h')^3 + zh' - h,$$

satisfying

$$g' = -4ah''f,$$

it is easy to see that differentiating (2.4.8) gives

$$2f'\left(f'' - 32a^2h'f - 4ag\right) = 0. \tag{2.4.10}$$

Assuming that $h$ is generic ($f' \not\equiv 0$) implies that the bracketed expression in (2.4.10) vanishes, and using (2.4.8) again to substitute for $g$ finally leads to (2.2.7), as required. In fact we shall see in Section 2.5 that there are non-generic solutions to (2.4.8), corresponding to $f \equiv 0$ (with $\lambda = 0$), which give degenerate solutions to the Hamiltonian system. □

It is interesting to make comparison with Okamoto's equivalent result about the Hamiltonian (2.1.7) for PII, which satisfies a second-order ODE. The condition for a generic solution in that case is

$$\frac{d^2h}{dz^2} \not\equiv 0.$$

Using our notation of Section 2.2, this turns out to be the same condition $f' \not\equiv 0$, for we have

$$h' = \frac{1}{2}f,$$

where

$$f = -y' - y^2 - \frac{z}{2}.$$

Also the special case $f \equiv 0$ (with $\lambda = 0$) corresponds to the special solutions of PII in terms of Airy functions. With the substitution

$$y = (\log[\tau(z)])',$$

the vanishing of $f$ leads to

$$\frac{d^2\tau}{dz^2} + \frac{1}{2}z\tau = 0,$$

whence e.g. we may take

$$\tau = Ai(-2^{-\frac{1}{3}}z).$$

The analogues of these solutions for the fourth-order equations (2.2.18,2.2.19) are related to PI, and we consider them with the rational solutions in Section 2.5.

It is also interesting to note that all of the Hénon-Heiles systems (irrespective of the values of $a$ and $b$) are invariant under the transformation

$$q_j \to \rho^3 q_j, \quad p_j \to \rho^2 p_j, \quad z \to \rho z, \quad h \to \rho^4 h, \tag{2.4.11}$$

where $\rho$ is a fifth root of unity.

Hitherto we have left $a$ and $b$ arbitrary, but from now on we treat the special cases $\mathcal{H}_{(i)-(iii)}$ (corresponding to the scaling similarity solutions of the three integrable fifth-order PDEs) separately. We start by describing the Bäcklund transformation and tau-function for the system $\mathcal{H}_{(ii)}$, as this is perhaps the simplest case, having the most in common with PII.

### 2.4.3 The Hamiltonian System $\mathcal{H}_{(ii)}$

The Hamiltonian for $\mathcal{H}_{(ii)}$ is given explicitly by

$$h_\lambda = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}q_1 q_2^2 + q_1^3 - \frac{\lambda^2}{2}q_2^{-2} - \frac{1}{2}zq_1. \tag{2.4.12}$$

We will henceforth put suffixes on all quantities to denote their dependence on $\lambda$, as the Bäcklund transformation relates the same quantities for different values of this parameter. There is also the alternative parameter $\alpha = \lambda - \frac{1}{2}$ appearing in (2.2.19), but $\lambda$ is the more natural one in that (2.4.12) depends only on $\lambda^2$, i.e.

$$h_\lambda = h_{-\lambda},$$

and thus the same is true for the solutions $(q_j(z), p_j(z))$ to the system with Hamiltonian $h_\lambda$. As in Section 2.2, the most convenient variables to use for this system are

$$w_\lambda = q_1, \qquad f_\lambda = -\frac{1}{2}q_2^2,$$

where (by (2.4.4) with $a = 1/2$, $b = -3$) we have

$$f_\lambda = w_\lambda'' + 3w_\lambda^2 - \frac{z}{2}. \tag{2.4.13}$$

If we now use the Miura map, we have a one-one correspondence with solutions to (2.2.19). In other words, given a solution to the Hamiltonian system, we find a solution $y = y_\lambda$ to equation (2.2.19) (with $\alpha = \lambda - \frac{1}{2}$) via the formula (2.2.12),

$$y_\lambda = \frac{f_\lambda' + \lambda}{2f_\lambda}. \tag{2.4.14}$$

We also have the usual Miura expression,

$$w_\lambda = -y_\lambda' - y_\lambda^2, \tag{2.4.15}$$

which means that conversely a solution to (2.2.19) determines a solution to the system $\mathcal{H}_{(ii)}$. Before presenting the Bäcklund transformation, it is helpful to introduce the tau-function.

**Definition 2.1.** *For a solution $(q_j(z), p_j(z))$ of the system $\mathcal{H}_{(ii)}$, we have the Hamiltonian $h_\lambda(z) = h_\lambda(q_j(z), p_j(z), z)$. The tau-function associated with this solution is given by*

$$h_\lambda(z) = -\frac{d}{dz}\log[\tau_\lambda(z)].$$

The above definition is chosen to be consistent with the tau-function of the KdV hierarchy, where the dependent variable $u$ is expressed as

$$u(x,t) = 2(\log[\tau(x,t)])_{xx}. \tag{2.4.16}$$

For scaling similarity solutions we require that $\tau$ depends on $x$ and $t$ only through the combination $z = x\theta(t)$, which means we must have

$$w_\lambda = 2(\log[\tau(z)])''.$$

This agrees with our definition, for differentiating the Hamiltonian gives

$$-(\log[\tau(z)])'' = h_\lambda' = -\frac{1}{2}q_1 = -\frac{1}{2}w_\lambda.$$

Alternatively, if one had no a priori knowledge that the system $\mathcal{H}_{(ii)}$ was a reduction of fifth-order KdV, one would choose to define the tau-function in this way to fit in with the pole structure of $w_\lambda$ found from Painlevé analysis, which may be seen very clearly from the rational solutions (see Section 2.5). We are now able to demonstrate:

**Proposition 2.2.** *The Bäcklund transformation for the equation (2.2.19) may be written in the form*

$$y_{\lambda+1} = -y_\lambda + \frac{\lambda}{f_\lambda}. \tag{2.4.17}$$

*Moreover, this induces a canonical transformation from the system $\mathcal{H}_{(ii)}$ with parameter $\lambda$ to the same system with parameter $\lambda + 1$:*

$$q_j \to \tilde{q}_j, \quad p_j \to \tilde{p}_j, \quad z \to z, \quad h_\lambda \to h_{\lambda+1}.$$

*The modified variable $y_\lambda$ may also be written in terms of the two tau-functions related by this transformation:*

$$y_\lambda = (\log[\tau_{\lambda-1}/\tau_\lambda])'. \tag{2.4.18}$$

**Proof.** The first thing to observe is that $w_\lambda$ is related to two different modified variables by the Miura map (2.4.15):

$$w_\lambda = -y'_\lambda - y_\lambda^2 = -y'_{-\lambda} - y_{-\lambda}^2.$$

This is a well-known property of Miura maps, but can be seen directly from the fact that the fourth-order ODE for $w_\lambda$ (or equivalently $h_\lambda$) just depends on $\lambda^2$, so

$$w_\lambda = w_{-\lambda}.$$

The inverse Miura map (2.4.14) gives

$$y_{\pm\lambda} = \frac{f'_\lambda \pm \lambda}{2f_\lambda}, \tag{2.4.19}$$

the solutions to (2.2.19) for parameter $\alpha = \pm\lambda - \frac{1}{2}$. Looking at the modified equation (2.2.19), we see that it is unchanged on sending

$$y \to -y, \qquad \alpha \to -\alpha.$$

Hence $-y_{-\lambda}$ will be a solution to this equation for $\alpha = \lambda + \frac{1}{2} = (\lambda + 1) - \frac{1}{2}$, or in other words

$$y_{\lambda+1} = -y_{-\lambda}.$$

It is then straightforward to derive (2.4.17) using the inverse Miura formula. This same argument also works for PII and the rest of the PII Family. Note that $f_\lambda$ in the right hand side of the Bäcklund transformation may be written in terms of $y_\lambda$ and its derivatives.

To define the induced transformation of the Hamiltonian system $\mathcal{H}_{(ii)}$, first of all it is necessary to write $y_{\lambda+1}$ in terms of the variables appearing in $h_\lambda$:

$$y_{\lambda+1} = -\frac{p_2 q_2 + \lambda}{q_2^2}. \tag{2.4.20}$$

The Miura map produces a new solution to the fourth order equation (2.2.7) at parameter value $\lambda+1$,

$$w_{\lambda+1} = -y'_{\lambda+1} - y^2_{\lambda+1}.$$

Now we can define the new variables $(\tilde{q}_j, \tilde{p}_j)$ via

$$\tilde{q}_1 = w_{\lambda+1}, \quad \tilde{p}_1 = w'_{\lambda+1}, \quad \tilde{q}_2 = (-2f_{\lambda+1})^{\frac{1}{2}}, \quad \tilde{p}_2 = -(-2f_{\lambda+1})^{-\frac{1}{2}} f'_{\lambda+1},$$

and these will clearly satisfy the system $\mathcal{H}_{(ii)}$ with Hamiltonian $h_{\lambda+1}$. It is simple to write the new variables in terms of the old, and we present the formulae here for completeness:

$$\begin{aligned} \tilde{q}_1 &= -q_1 - 2y^2_{\lambda+1}, \\ \tilde{p}_1 &= -p_1 - 4y_{\lambda+1}(q_1 + y^2_{\lambda+1}), \\ \tilde{q}_2 &= \Upsilon^{\frac{1}{2}}, \\ \tilde{p}_2 &= y_{\lambda+1}\Upsilon^{\frac{1}{2}} + (\lambda+1)\Upsilon^{-\frac{1}{2}}, \end{aligned} \tag{2.4.21}$$

where

$$\Upsilon = -q_2^2 + 8y_{\lambda+1}p_1 + 8y^2_{\lambda+1}q_1 - 4q_1^2 + 2z,$$

and $y_{\lambda+1}$ is to be interpreted as the function of $p_2$ and $q_2$ given by (2.4.20). For the sake of clarity, we present the (invertible) transformations for $w_\lambda$ and $y_\lambda$ diagramatically:

$$\begin{tikzcd} y_\lambda \arrow[r, "(2.4.17)"] \arrow[d, "(2.4.15)"'] & y_{\lambda+1} \arrow[d, "(2.4.15)"] \\ w_\lambda \arrow[r, "(2.4.21)"] & w_{\lambda+1} \end{tikzcd}$$

A similar calculation yields the inverse of the transformation (2.4.21), but we spare the details here. It is obvious that this is a canonical transformation, since the equations for both sets of variables are Hamiltonian. Alternatively this is shown directly by considering the canonical one-form,

$$\psi = \sum_{j=1,2} p_j dq_j - h_\lambda dz$$

(such that $\omega = d\psi$), and its image under the transformation. We find

$$\tilde{\psi} - \psi = \sum_{j=1,2} (\tilde{p}_j d\tilde{q}_j - p_j dq_j) + (h_\lambda - h_{\lambda+1})dz = d\chi,$$

with

$$\chi = 4y_{\lambda+1}p_1 + 8y^3_{\lambda+1}q_1 + \frac{16}{5}y^5_{\lambda+1} + \lambda\log[q_2] + (\lambda+1)\log[\tilde{q}_2].$$

The derivation if this is much facilitated by first calculating the difference in the Hamiltonians as

$$h_{\lambda+1} - h_\lambda = y_{\lambda+1}, \tag{2.4.22}$$

which also yields (2.4.18) immediately.

In deriving (2.4.22), the working is simplified by replacing the variables $p_2$,$q_2$ in $h_\lambda$ by $f'_\lambda$,$f_\lambda$, to give

$$h_\lambda = \frac{1}{2}p_1^2 + q_1(q_1^2 - f_\lambda - \frac{1}{2}z) + \frac{\lambda^2 - f_\lambda'^2}{4f_\lambda}.$$

Now we may use the inverse Miura formula (2.4.14) (or equivalently (2.4.20)) to substitute

$$f'_\lambda = \lambda - 2f_\lambda y_{\lambda+1},$$

and then rearranging gives

$$h_\lambda - \lambda y_{\lambda+1} = E(p_1, q_1, f_\lambda, y_{\lambda+1}), \tag{2.4.23}$$

with

$$E(p, q, f, y) := \frac{1}{2}p^2 + q(q^2 - f - \frac{1}{2}z) - fy^2.$$

A similar calculation gives

$$h_{\lambda+1} - (\lambda+1)y_{\lambda+1} = E(\tilde{p}_1, \tilde{q}_1, f_{\lambda+1}, y_{\lambda+1}). \tag{2.4.24}$$

Taking the difference of (2.4.24) and (2.4.23) and substituting for the quantities with tildes from the formulae (2.4.21) for the canonical transformation, we find that the right-hand side is

$$E(p_1, q_1, f_\lambda, y_{\lambda+1}) - E(\tilde{p}_1, \tilde{q}_1, f_{\lambda+1}, y_{\lambda+1}) = 0,$$

and thus (2.4.22) follows. This completes the proof. □

The key to the integrability of the usual autonomous Hénon-Heiles systems is the existence of a second conserved quantity that Poisson commutes with the Hamiltonian. Although we do not have constants of motion for the non-autonomous systems, it is straightforward to modify the results of [63] and find a second independent quantity in involution with the Hamiltonian (with respect to the standard Poisson brackets). However, we note that unlike the systems considered in [76, 147], this involutive quantity does not define a second flow commuting with the flow generated by $h$. By direct calculation we have the following:

**Proposition 2.3.** *If $h_\lambda$ denotes the Hamiltonian (2.4.12) for the system $\mathcal{H}_{(ii)}$, then there is an independent quantity $c_\lambda$, given by*

$$c_\lambda = q_2p_1p_2 - q_1p_2^2 + \frac{1}{8}q_2^4 + \frac{1}{2}q_1^2q_2^2 + \lambda^2q_1q_2^{-2} - \frac{1}{4}zq_2^2, \tag{2.4.25}$$

*that Poisson commutes with $h_\lambda$. Moreover, on removing the $z$ term, $c_\lambda$ reduces to the second constant of motion for case (ii) of the autonomous Hénon-Heiles system.*

We now present the analogous results for both $\mathcal{H}_{(i)}$ and $\mathcal{H}_{(iii)}$ together, since they are related by the Miura map to the same modified equation (2.2.18).

### 2.4.4 The Hamiltonian Systems $\mathcal{H}_{(i)}$ and $\mathcal{H}_{(iii)}$

The Hamiltonian for $\mathcal{H}_{(i)}$ is

$$h_\lambda = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}q_1 q_2^2 + \frac{1}{6}q_1^3 - \frac{\lambda^2}{2}q_2^{-2} - \frac{1}{2}zq_1, \tag{2.4.26}$$

while for $\mathcal{H}_{(iii)}$ it is

$$H_\lambda = \frac{1}{2}(P_1^2 + P_2^2) + \frac{1}{4}Q_1 Q_2^2 + \frac{4}{3}Q_1^3 - 2\lambda^2 Q_2^{-2} - zQ_1. \tag{2.4.27}$$

To avoid confusion between the two, we use lower/upper case letters for the variables of the systems $\mathcal{H}_{(i)}/\mathcal{H}_{(iii)}$ respectively. Again we find it convenient to use alternative variables,

$$w_\lambda = q_1, \qquad f_\lambda = -\frac{1}{2}q_2^2,$$

$$W_\lambda = Q_1, \qquad F_\lambda = -\frac{1}{4}Q_2^2.$$

Hence we have (using (2.4.4) with the appropriate values of $a$ and $b$)

$$\begin{aligned} f_\lambda &= w_\lambda'' + \frac{1}{2}w_\lambda^2 - \frac{z}{2}, \\ F_\lambda &= W_\lambda'' + 4W_\lambda^2 - z. \end{aligned}$$

The Miura map for case (i) is given by

$$w_\lambda = -y_\lambda' - y_\lambda^2, \tag{2.4.28}$$

with the inverse,

$$y_\lambda = \frac{f_\lambda' + \lambda}{2f_\lambda},$$

giving a solution to (2.2.18) for $\alpha = \lambda - \frac{1}{2}$. Although case (iii) is related to the same modified equation, it will be helpful for deriving the Bäcklund transformation to return to the original formalism of Section 2.2, where the case (iii) Miura map (with $a = 1/4$) is

$$W_\lambda = -Y_\lambda' - \frac{1}{2}Y_\lambda^2, \tag{2.4.29}$$

and the inverse,

$$Y_\lambda = \frac{F'_\lambda + \lambda}{F_\lambda},$$

gives a solution to (2.2.20) for $\alpha = \lambda - 1$. The derivation of the Bäcklund transformation for the equation (2.2.18) (or equivalently (2.2.20)) is most easily achieved with the tau-functions, naturally related to the tau-functions of the SK/KK hierarchies (see [101, 137] for definitions of these).

**Definition 2.2.** *For a solution $(q_j(z), p_j(z))$ of the system $\mathcal{H}_{(i)}$, with the Hamiltonian $h_\lambda(z) = h_\lambda(q_j(z), p_j(z), z)$, the associated tau-function is given by*

$$h_\lambda(z) = -3\frac{d}{dz}\log[\tau_\lambda(z)].$$

**Definition 2.3.** *For a solution $(Q_j(z), P_j(z))$ of the system $\mathcal{H}_{(iii)}$, with the Hamiltonian $H_\lambda(z) = H_\lambda(Q_j(z), P_j(z), z)$, the associated tau-function is given by*

$$H_\lambda(z) = -\frac{3}{2}\frac{d}{dz}\log[\tilde{\tau}_\lambda(z)].$$

Now we may show:

**Proposition 2.4.** *The Bäcklund transformation for the equation (2.2.18) may be written in the form*

$$y_{\lambda+3} = y_\lambda - \frac{\lambda}{f_\lambda} + \frac{2(\lambda + \frac{3}{2})}{F_{\lambda+\frac{3}{2}}}. \tag{2.4.30}$$

*This is related to a canonical transformation from the system $\mathcal{H}_{(i)}$ with parameter $\lambda$ to the system $\mathcal{H}_{(iii)}$ with parameter $\lambda - \frac{3}{2}$:*

$$q_j \to Q_j, \quad p_j \to P_j, \quad z \to z, \quad h_\lambda \to H_{\lambda-\frac{3}{2}}.$$

*The modified variable $y_\lambda$ may also be written in terms of the two tau-functions related by this transformation:*

$$y_\lambda = (\log[\tilde{\tau}_{\lambda-\frac{3}{2}}/\tau_\lambda^2])'. \tag{2.4.31}$$

**Proof.** As for case (ii), $w_\lambda$ (and also $W_\lambda$) may be related to two different modified variables by the Miura map. Hence we see that

$$y_\lambda - y_{-\lambda} = \frac{\lambda}{f_\lambda}, \tag{2.4.32}$$

$$Y_\lambda - Y_{-\lambda} = \frac{2\lambda}{F_\lambda}. \tag{2.4.33}$$

Clearly (2.4.32) constitutes a Bäcklund transformation for (2.2.18), as it relates two solutions for different parameter values. However, it is not very useful because

it cannot be iterated to obtain a sequence of solutions. To achieve this requires a canonical tranformation from $\mathcal{H}_{(i)}$ to $\mathcal{H}_{(iii)}$, and then from $\mathcal{H}_{(iii)}$ back to $\mathcal{H}_{(i)}$ with the overall change $\lambda \to \lambda + 3$. First of all observe that on comparing the parameters $\alpha$ in (2.2.18) and (2.2.20), it is apparent that we may make the identification

$$y_\lambda = -Y_{-\lambda+\frac{3}{2}},$$

and so the Miura map for case (iii) implies

$$W_{-\lambda+\frac{3}{2}} = y'_\lambda - \frac{1}{2}y_\lambda^2.$$

Since $W_{-\lambda+\frac{3}{2}}$ may be found from $w_\lambda$, via the formula

$$W_{-\lambda+\frac{3}{2}} = -w_\lambda - \frac{3}{2}y_\lambda^2, \tag{2.4.34}$$

it is obvious that there is an induced canonical transformation from the system with Hamiltonian $h_\lambda$ to the system with Hamiltonian $H_{-\lambda+\frac{3}{2}} = H_{\lambda-\frac{3}{2}}$. This transformation and its inverse may be calculated explicitly in terms of coordinates, e.g. we have

$$\begin{aligned}
Q_1 &= -q_1 - \frac{3}{2}y_\lambda^2, \\
P_1 &= -p_1 + 3y_\lambda(q_1 + y_\lambda^2), \\
Q_2 &= \tilde{\Upsilon}^{\frac{1}{2}}, \\
P_2 &= -\frac{1}{2}y_\lambda\tilde{\Upsilon}^{\frac{1}{2}} - 2(\lambda - \frac{3}{2})\tilde{\Upsilon}^{-\frac{1}{2}},
\end{aligned}$$

where

$$\tilde{\Upsilon} = -2q_2^2 - 12y_\lambda p_1 - 6q_1^2 + 6z,$$

and in the above $y_\lambda$ is to be interpreted as a function of $p_2$ and $q_2$:

$$y_\lambda = \frac{p_2 q_2 - \lambda}{q_2^2}.$$

This is the analogue of the transformation between autonomous Hénon-Heiles systems considered in [55, 27]. We have also calculated explicitly:

$$\sum_{j=1,2}(P_j dQ_j - p_j dq_j) + (h_\lambda - H_{\lambda-\frac{3}{2}})dz = d\tilde{\chi},$$

$$\tilde{\chi} = 3y_\lambda^2 p_1 - 3y_\lambda^3 q_1 - \frac{9}{5}y_\lambda^5 - 2(\lambda - \frac{3}{2})\log[Q_2] - \lambda\log[q_2].$$

The tau-function formula (2.4.31) follows directly from a calculation of the difference of the two Hamiltonians:

$$h_\lambda - H_{\lambda-\frac{3}{2}} = \frac{3}{2}y_\lambda.$$

Then in terms of tau-functions, we have

$$y_{\lambda+3} - y_\lambda = \frac{d}{dz}\left(\log[\tilde{\tau}_{\lambda+\frac{3}{2}}/\tilde{\tau}_{\lambda-\frac{3}{2}}] + 2\log[\tau_\lambda/\tau_{\lambda+3}]\right).$$

On using the formulae (2.4.32,2.4.33), the Bäcklund transformation (2.4.30) follows. Thus overall there is an induced canonical transformation from $\mathcal{H}_{(i)}$ to itself, with the parameter $\lambda \to \lambda + 3$. This is most easily understood with a diagram:

$$\begin{array}{ccccccccc}
y_\lambda & \xrightarrow{(2.4.32)} & y_{-\lambda} & = & -Y_{\lambda+\frac{3}{2}} & \xrightarrow{(2.4.33)} & Y_{-\lambda-\frac{3}{2}} & = & -y_{\lambda+3} \\
\downarrow{\scriptstyle (2.4.28)} & & \downarrow{\scriptstyle (2.4.28)} & & \downarrow{\scriptstyle (2.4.29)} & & \downarrow{\scriptstyle (2.4.29)} & & \downarrow{\scriptstyle (2.4.28)} \\
w_\lambda & = & w_{-\lambda} & \xrightarrow[(2.4.34)]{} & W_{\lambda+\frac{3}{2}} & = & W_{-\lambda-\frac{3}{2}} & \xrightarrow[(2.4.34)]{} & w_{\lambda+3}
\end{array}$$

Note that in the right hand side of (2.4.30), $f_\lambda$ may be determined entirely in terms of $y_\lambda$ and its derivatives, and similarly for $F_{\lambda+\frac{3}{2}}$. □

We have also calculated the analogues of the quantity (2.4.25):

**Proposition 2.5.** *The Hamiltonian $h_\lambda$ (given by (2.4.26)) for the system $\mathcal{H}_{(i)}$ Poisson commutes with the independent quantity*

$$c_\lambda = \left(p_1p_2 + \frac{1}{6}q_2^3 + \frac{1}{2}q_1^2q_2 - \frac{1}{2}zq_2\right)^2 - \lambda^2\left(p_1^2q_2^{-2} + \frac{2}{3}q_1^2\right).$$

*The corresponding quantity for the Hamiltonian $H_\lambda$ (2.4.27) of $\mathcal{H}_{(iii)}$ is*

$$C_\lambda = 3\left(p_2^2 + \frac{1}{2}q_1q_2^2 - 4\lambda^2q_2^{-2}\right)^2 - q_2^3p_1p_2 - \frac{1}{24}q_2^2 - q_1^2q_2^4 + 8\lambda^2q_1 + \frac{1}{4}zq_2^4.$$

### 2.4.5 Analogues of the Toda Lattice for Sequences of Tau-Functions

For many purposes, the tau-functions provide the most convenient and concise expressions for solutions to integrable equations. For our Hamiltonian systems $\mathcal{H}_{(i)-(iii)}$ we have seen that the tau-functions related by Bäcklund transformations allow one to determine both the Hamiltonians $h_\lambda$ and the associated modified variable $y_\lambda$ satisfying one of the equations (2.2.18,2.2.19). Hence it is useful to work with the tau-functions directly, as this provides an efficient way to compute sequences of solutions to the modified equations and their associated Hamiltonian systems. A particular application of this is the computation of rational solutions, presented in Section 2.5.

It is well known [93, 128] that for PII, the tau-functions related by the Bäcklund transformation (after rescaling) satisfy the Toda lattice equation,

$$D_z^2\tau_\lambda \cdot \tau_\lambda = \tau_{\lambda-1}\tau_{\lambda+1}. \tag{2.4.35}$$

The derivation [128] of the lattice equation (2.4.35) provides a pattern for deriving analogous lattice equations for any of the similarity equations decribed in the formalism of Section 2.2. Here we give full details for a bilinear lattice equation which is the analogue of (2.4.35) for the equation (2.2.19).

For the system $\mathcal{H}_{(ii)}$, we have already seen that the definition of the tau-function implies

$$q_1 = w_\lambda = 2(\log[\tau_\lambda])''.$$

Substituting this into (2.4.13) yields

$$f_\lambda = \tau_\lambda^{-2}(D_z^4\tau_\lambda \cdot \tau_\lambda) - \frac{z}{2}. \tag{2.4.36}$$

Now we have on the one hand (from (2.4.18))

$$y_\lambda + y_{-\lambda} = (\log[\tau_{\lambda+1}\tau_{\lambda-1}/\tau_\lambda^2])',$$

while on the other hand (using (2.4.19))

$$y_\lambda + y_{-\lambda} = (\log[f_\lambda])'.$$

Hence we find

$$(\log[f_\lambda])' = (\log[\tau_{\lambda+1}\tau_{\lambda-1}/\tau_\lambda^2])',$$

which we integrate and then substitute from (2.4.36) to obtain the bilinear form

$$2D_z^4\tau_\lambda \cdot \tau_\lambda - z\tau_\lambda^2 \;=\; k_\lambda\tau_{\lambda-1}\tau_{\lambda+1}. \tag{2.4.37}$$

The equation (2.4.37) is the analogue of the Toda lattice equation (2.4.35). Note that the constant of integration $k_\lambda$ may be rescaled arbitrarily, since the tau-functions can always be rescaled (without affecting $h_\lambda$,$y_\lambda$). Here we take the convention $k_\lambda = -1$, which ensures that the tau-functions for the rational solutions are monic polynomials in $z$. However, the above derivation has made the generic assumption $f_\lambda \not\equiv 0$. This may be violated for $\lambda = 0$, in which case the constant in (2.4.37) vanishes (i.e. $k_0 = 0$).

Almost identical arguments lead to the following equations for the tau-functions of the systems $\mathcal{H}_{(i)}$ and $\mathcal{H}_{(iii)}$:

$$z\tau_\lambda^4 - 6\tau_\lambda^2 D_z^4\tau_\lambda \cdot \tau_\lambda + 9(D_z^2\tau_\lambda \cdot \tau_\lambda)^2 \;=\; \tilde{\tau}_{\lambda-\frac{3}{2}}\tilde{\tau}_{\lambda+\frac{3}{2}}, \tag{2.4.38}$$

$$z\tilde{\tau}_\lambda^2 - \frac{3}{4}D_z^4\tilde{\tau}_\lambda \cdot \tilde{\tau}_\lambda \;=\; \tau_{\lambda-\frac{3}{2}}\tau_{\lambda+\frac{3}{2}}\tilde{\tau}_\lambda. \tag{2.4.39}$$

We have used the same conventions and genericity assumptions as for $\mathcal{H}_{(ii)}$. It is interesting to observe that these equations are no longer bilinear, and also that it

is consistent to assign weight one to $\tau$ and weight two to $\tilde{\tau}$ in the two equations. However, if we consider the Miura map corresponding to the equation (2.2.18),

$$w_\lambda = -y'_{\pm\lambda} - y^2_{\pm\lambda},$$

and use the relevant tau-function formulae, given by (2.4.31) and

$$w_\lambda = 6(\log[\tau_\lambda])'',$$

then we do find a bilinear equation,

$$4\tau''_\lambda \tilde{\tau}_{\lambda\pm\frac{3}{2}} - 4\tau'_\lambda \tilde{\tau}'_{\lambda\pm\frac{3}{2}} + \tau_\lambda \tilde{\tau}''_{\lambda\pm\frac{3}{2}} = 0. \tag{2.4.40}$$

Since it contains only second derivatives, (2.4.40) may be used to relate the tau-functions to sequences of second-order operators.

## 2.5 Painlevé Analysis and Special Solutions

### 2.5.1 The Painlevé Test

Before looking at rational solutions and some other special solutions to the equations (2.2.18,2.2.19) and the Hamiltonian systems $\mathcal{H}_{(i)-(iii)}$ related to them by Miura maps, we briefly discuss the application of the Painlevé test to these systems. Since they are reductions of soliton equations, we expect them to pass this test (because of the ARS conjecture), and indeed this is the case. To apply the test, it is necessary to expand $q_1$ and $q_2$ as power series in $Z = z - z_0$, where the constant $z_0$ is the (movable) location of a pole. In fact, rather than using the coupled system (2.4.5,2.4.6) coming from the Hamiltonian equations of motion, it is simpler to use the pair of equations (2.2.7,2.4.4) in $w = q_1$ and $f = -aq_2^2$, and this avoids powers of $Z^{\frac{1}{2}}$. First of all it is necessary to look for the leading order behaviour, $w \sim \gamma Z^\mu$, $f \sim \delta Z^\nu$, as $Z \to 0$. As is to be expected for equations with the Painlevé property, for each of the cases (i)-(iii) we find certain types of balances with a formal series solution corresponding to each. The balances may be classified by their resonances, i.e. the places where arbitrary constants can be introduced into the Laurent series. For the Painlevé test to be satisfied there must be a principal balance, containing the same number of arbitrary constants as the order of the system. We classify the possible balances, according to which terms are dominant in (2.2.7,2.4.4), as follows:

**Type 1:** $\mu = -2$, $\gamma = 6/b$ for (ii) and (iii). $\nu = -2$ for (ii), $\nu = -1$ for (iii), with $\delta$ arbitrary. There is no Type 1 behaviour for (i).

**Type 2:** $\mu = -2$, $\nu = -4$, $\gamma = -3/a$, $\delta = \gamma(6 - b\gamma)$.

Type 3: There are two sub-cases depending on the value of $\lambda$, with $\mu = -2$, $\gamma = 6/b$.

(a) $\lambda = 0$: $\nu = 8, 4, 3$ for (i),(ii),(iii) respectively, with $\delta$ arbitrary.

(b) $\lambda \neq 0$: $\nu = 1$, with $\delta^2/\lambda^2 = \frac{1}{49}, \frac{1}{9}, \frac{1}{4}$ for (i),(ii),(iii) respectively.

We present the list of resonances for each type of balance in Table 2.1. Note that Type 1 is the principal balance for (ii) and (iii), while for (i) it is Type 2. Also note the presence of negative resonances, as discussed in [44]. The explicit time-dependence doesn't affect the resonances and, as the results are almost identical to those for the autonomous Hénon-Heiles systems (see [30, 40]), we present no further details of the Painlevé analysis. The only real difference introduced by the time-dependent term is that the coefficients in the power series contain $z_0$ explicitly. For example, for the principal balance in case (ii) (Type 1) we have

$$\begin{aligned} w &= -\frac{2}{Z^2} + c_0 + (\frac{3}{2}c_0^2 - \frac{1}{20}z_0)Z^2 + c_1 Z^3 + c_2 Z^4 + O(Z^5), \\ f &= -\frac{12c_0}{Z^2} - 12c_0^2 - (\frac{1}{2} + 6c_1)Z + c_0(9c_0^2 - \frac{3}{10}z_0)Z^2 - \frac{1}{5}c_0 Z^3 + O(Z^4), \end{aligned}$$

where $z_0, c_0, c_1, c_2$ are the four arbitrary constants.

| | Type 1 | Type 2 | Type 3(a) | Type 3(b) |
|---|---|---|---|---|
| (i) | none | -1,2,3,6 | -7,-1,0,6 | -7,-1,6,7 |
| (ii) | -1,0,3,6 | -3,-1,6,8 | -3,-1,0,6 | -3,-1,3,6 |
| (iii) | -1,0,2,6 | -7,-1,6,12 | -2,-1,0,6 | -2,-1,2,6 |

**Table 2.1. List of resonances.**

### 2.5.2 Rational Solutions

Painlevé analysis is also useful for finding rational solutions. A rational solution must be meromorphic at $z = \infty$, and doing an expansion about this point for the equations (2.2.18) and (2.2.19) it is easy to see that there can be at most one such solution for any $\alpha$, and the expansion takes the form

$$y = -\alpha z^{-1}\left(1 + \sum_{j=1}^{\infty} \eta_j z^{-5j}\right) \tag{2.5.1}$$

for both these equations. The form of this expansion reflects the scaling symmetry (2.4.11) noted previously. Also, for both equations there is the trivial rational solution $y_{\frac{1}{2}} = 0$ for $\alpha = 0$, and (2.5.1) can be truncated after the first term for precisely four non-zero values of $\alpha$, namely

$$\alpha = -4, \quad -1, \quad 2, \quad 3$$

for (2.2.18), and

$$\alpha = -2, \quad -1, \quad 1, \quad 2$$

for (2.2.19).

The corresponding rational solutions for each of the parameter values above are naturally related to each other by the Bäcklund transformations constructed in the previous section (although we prefer to label the solutions with the parameter $\lambda = \alpha + \frac{1}{2}$). Since the Bäcklund transformations only involve differentiations and algebraic operations, repeated application of them produces a sequence of rational solutions. For (2.2.18), if we have a solution $y_\lambda$ we can immediately find $y_{-\lambda}$ (using the Miura map), and we can also use the Bäcklund transformation (2.4.30) to find $y_{\lambda+3}$. Continuing to apply this we essentially have two related sequences of rational solutions, with every third integer value of $\alpha$ being missed out. We present these, with the corresponding Miura-related solutions to the systems $\mathcal{H}_{(i)}$ and $\mathcal{H}_{(iii)}$, in Table 2.2; Table 2.3 contains the corresponding tau-functions.

| $\lambda$ | $-7/2$ | $-5/2$ | $-1/2$ | $1/2$ | $5/2$ | $7/2$ | $11/2$ |
|---|---|---|---|---|---|---|---|
| $y_\lambda$ | $\frac{4}{z}$ | $\frac{3(z^5-24)}{z(z^5+36)}$ | $\frac{1}{z}$ | $0$ | $-\frac{2}{z}$ | $-\frac{3}{z}$ | $\frac{-5z^4(z^5+216)}{(z^5+36)(z^5-144)}$ |
| $w_\lambda$ | $-\frac{12}{z^2}$ | $-\frac{6}{z^2}$ | $0$ | $0$ | $-\frac{6}{z^2}$ | $-\frac{12}{z^2}$ | $\frac{-30z^3(z^5+576)}{(z^5-144)^2}$ |
| $W_{\lambda-\frac{3}{2}}$ | $-\frac{12}{z^2}$ | $-\frac{15z^3(z^5-144)}{2(z^5+36)^2}$ | $-\frac{3}{2z^2}$ | $0$ | $0$ | $-\frac{3}{2z^2}$ | $-\frac{12}{z^2}$ |

**Table 2.2. Rational solutions for cases (i) and (iii).**

From an algorithmic point of view, (2.4.30) is very inconvenient, and it is better to use the bilinear equations (2.4.38,2.4.39) for the tau-functions. These two equations can be solved iteratively, obtaining a new tau-function at each step.

| $\lambda$ | $1/2$ | $5/2$ | $7/2$ | $11/2$ |
|---|---|---|---|---|
| $\tau_\lambda$ | $1$ | $z$ | $z^2$ | $z^5 - 144$ |
| $\tilde{\tau}_{\lambda-\frac{3}{2}}$ | $1$ | $1$ | $z$ | $z^5 + 36$ |
| $\tilde{\tau}_{\lambda+\frac{3}{2}}$ | $z$ | $z^5 + 36$ | $z^8$ | $z^{16} - 2^7.3^2z^{11} + 2^{11}.3^4.11z^6 + 2^{14}.3^6.11z$ |

**Table 2.3. Polynomial tau-functions for cases (i) and (iii).**

Rational solutions of (2.2.19) can be obtained in a similar fashion, applying the Bäcklund transformation (2.4.17) repeatedly starting with the solution $y_{\frac{1}{2}} = 0$, or more conveniently by iteratively solving (2.4.37) to obtain the associated sequence of tau-functions. We present a few of the rational solutions to (2.2.19) in Table 2.4. However, perhaps an easier method (which proves their uniqueness) is to derive them from the rational solutions of the mKdV/KdV hierarchy, which were constructed in [8]. These rational solutions are obtained from a sequence of polynomial tau-functions $\pi_k$ (for $k$=0,1,2,...) depending on $x = t_1$ and a sequence of parameters $t_2$, $t_3$,..., which after suitable scaling may be identified with the times of the hierarchy. The sequence of rational solutions to mKdV is given by the standard tau-function substitution,

$$v_k = (\log[\pi_k/\pi_{k+1}])_x.$$

(Our $\pi_k$, $t_j$ are denoted $\theta_k$, $\tau_j$ in [8].) The sequence of polynomials has a homogeneity property,

$$\pi_k(\beta t_1, \beta^3 t_2, ..., \beta^{2k-1} t_k) = \beta^{\frac{1}{2}k(k+1)} \pi_k(t_1, t_2, ..., t_k),$$

and so the requirement for similarity solutions that $\pi_k$ should be proportional to a function of $z = x\theta$ puts very strong constraints on the values that the parameters $t_j$ may take. In particular for the $n$=2 flow of mKdV, to get polynomial tau-functions of PII requires $t_2$=$4\theta^{-3}$ and all higher $t_j$ must be zero; while for $n$=3 (corresponding to (2.2.19) we have $t_2$=0, $t_3 = -144\theta^{-5}$, and all higher $t_j$ are zero. In Table 2.5 we present some of the $\pi_k$ with the corresponding polynomial tau-functions $\tau_\lambda(z)$ of PII and (2.2.19).

| $\lambda$ | 1/2 | 3/2 | 5/2 | 7/2 | 9/2 |
|---|---|---|---|---|---|
| $y_\lambda$ | 0 | $-\frac{1}{z}$ | $-\frac{2}{z}$ | $-\frac{3(z^5+96)}{z(z^5-144)}$ | $-\frac{4(z^{15}-72z^{10}+217728z^5-1741824)}{z(z^{15}-1152z^{10}+96768z^5+6967296)}$ |
| $w_\lambda$ | 0 | $-\frac{2}{z^2}$ | $-\frac{6}{z^2}$ | $-\frac{12(z^{10}+432z^5+3456)}{z^2(z^5-144)^2}$ | $-\frac{20z^3(z^{15}+1008z^{10}+943488z^5-47542144)}{(z^{10}-1008(z^5+48))^2}$ |

**Table 2.4. Rational solutions for case (ii).**

| $k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\pi_k$ | 1 | $x$ | $x^3+t_2$ | $x^6+5t_2x^3+t_3x-5t_2^2$ | $x^{10}+15t_2x^7+7t_3x^5$ $+35t_2t_3x^2+175t_2^3x-\frac{7}{3}t_3^2$ $+t_4x^3+t_4t_2$ |
| $\tau_{\lambda=k+\frac{1}{2}}$ $(n=2)$ | 1 | $z$ | $z^3+4$ | $z^6+20z^3-80$ | $z^{10}+60z^7+11200z$ |
| $\tau_{\lambda=k+\frac{1}{2}}$ $(n=3)$ | 1 | $z$ | $z^3$ | $z^6-144z$ | $z^{10}-1008(z^5+48)$ |

**Table 2.5. Polynomial tau-functions for mKdV and similarity reductions.**

The restriction of rational solutions of the mKdV hierarchy to the similarity solutions has been used in [93] to derive determinantal forms for the polynomial tau-functions of PII. We are able to extend this approach to the rational solutions of (2.2.19). The polynomial tau-functions of mKdV, $\pi_k$ in Table 2.5, can be written as Wronskians of Schur polynomials. The formula given in [8] is

$$\pi_k \propto [\psi_1, \ldots, \psi_k],$$

with $[\ldots]$ denoting Wronskian [1], and the polynomials $\psi_j$ are defined recursively by

$$\psi_{j,xx} = \psi_{j-1}, \qquad \psi_1 = x, \qquad \psi_{j,t_n} = \mu_n \psi_{j,(2n-1)x},$$

for suitable constants $\mu_n$. This recursive definition identifies the $\psi_j$ as the sequence of odd Schur polynomials. If we now restrict to the scaling similarity solutions, we must scale these $\psi_j$ so that they depend only on the similarity variable $z$. Thus for the solutions of (2.2.19) we have $\mu_3 = 16$, while all other $\mu_n$ are zero. Then the polynomial tau-functions are given by

$$\tau_{k+\frac{1}{2}} \propto [\psi_1(z), \ldots, \psi_k(z)], \tag{2.5.2}$$

where the sequence of scaled polynomials may be defined by

$$\psi_j'' = \psi_{j-1}, \qquad \psi_1 = z, \qquad z\psi_j' - (2j-1)\psi_j = 16\psi_j^{(v)}.$$

[1] i.e.

$$[\psi_1, \ldots, \psi_k] = \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_k \\ \psi_{1,x} & \psi_{2,x} & \cdots & \psi_{k,x} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{1,(k-1)x} & \psi_{2,(k-1)x} & \cdots & \psi_{k,(k-1)x} \end{vmatrix}.$$

The first terms in this sequence are

$$\psi_1 = z, \quad \psi_2 = \frac{1}{6}z^3, \quad \psi_3 = \frac{1}{5!}(z^5 - 384), \quad \psi_4 = \frac{1}{7!}(z^7 - 8064z^2).$$

It is simple to check that (up to overall scale factors) the tau-functions in Table 2.5 are given by the determinants (2.5.2).

### 2.5.3 Solutions Related to PI

We have also found special solutions related to the first Painlevé transcendent (PI). If we consider the system $\mathcal{H}_{(i)}$ in the case $\lambda = 0$, the same substitution that works in the ordinary (autonomous) system causes the equations of motion to separate. Putting

$$Q_\pm = q_1 \pm q_2$$

into Hamilton's equations for $h_0$, we find

$$Q''_\pm = -\frac{1}{2}Q_\pm^2 + \frac{z}{2},$$

which (up to a scaling) is just two separate copies of PI. The corresponding solution to (2.2.18) is

$$y_0 = (\log[Q_+ - Q_-])',$$

where we assume that $Q_+$ and $Q_-$ are not equal. So applying the Bäcklund transformation to this we get the general solution to the system $\mathcal{H}_{(i)}$ for $\lambda = 3j$, and to $\mathcal{H}_{(iii)}$ for $\lambda = 3(j + \frac{1}{2})$, for all integers $j$. However, there is also the degenerate case $Q_+ = Q_-$, for which

$$f_0 \equiv 0.$$

This implies that the inverse Miura map and the Bäcklund transformation both break down. However, it is still possible to obtain a sequence of special solutions for the same parameter values, and they are also related to PI. Similarly, starting from a degenerate solution $Y_0$, corresponding to

$$F_0 \equiv 0,$$

the Bäcklund transformation (2.4.30) gives a sequence of special solutions to the system $\mathcal{H}_{(i)}$ for $\lambda = 3(j+\frac{1}{2})$, and to $\mathcal{H}_{(iii)}$ for $\lambda = 3j$. We explain these degenerate solutions in more detail for case (ii).

The degenerate case for all three systems is $\lambda = 0$, $f_0 \equiv 0$. So in case (ii) $w = w_0$ must satisfy

$$w'' + 3w^2 - \frac{z}{2} = 0,$$

which is equivalent to PI (after rescaling $w$ and $z$). The inverse Miura map breaks down in this case. However, the ordinary Miura map means that $y = y_0$ satisfies the Riccati equation

$$y' + y^2 + w_0 = 0.$$

This is linearized by setting $y = (\log[\tau])'$, giving

$$\tau'' + w_0\tau = 0.$$

Thus a solution $y_0$ to (2.2.19) is found from an eigenfunction $\tau$ of a Schrödinger operator with a PI potential. Because $f_0 \equiv 0$, the Bäcklund transformation (2.4.17) breaks down for the solution $y_0$. However, we still have

$$y_{\lambda+1} = -y_{-\lambda},$$

and so we can safely apply the Bäcklund transformation to $y_1 = -y_0$ to obtain a sequence of solutions for all integers $\lambda$, as well as corresponding solutions to the system $\mathcal{H}_{(ii)}$.

It is interesting to note that in Okamoto's work the parameters of each of PII-VI are embedded into a root space, and the application of Bäcklund transformations is identified with the action of the affine Weyl group. Also the classical solutions are all observed to lie in the walls of the Weyl chambers. In the case of PII, the relevant root space is $A_1$, and $\lambda$ is the natural parameter in this root space. The walls of the Weyl chambers correspond to both the integer values of $\lambda$ (where there are solutions in terms of Airy functions) and the half-integer values (where there are rational solutions), and the Bäcklund transformation gives a shift $\lambda \to \lambda + 1$ in the root space. The equation (2.2.19) is related to $A_1$ in precisely the same way, except that the special solutions for integer values of $\lambda$ are not given in terms of classical special functions, but are instead expressed in terms of solutions to the first Painlevé transcendent PI. The equation (2.2.18) is more complicated, because there are the two different Hamiltonian systems $\mathcal{H}_{(i),(iii)}$ (with their respective tau-functions) associated to it. Nevertheless, we would hope to be able to view it in a similar way. In particular we observe that (in Section 2.3) the zero curvature representation of (2.2.19) involves matrices in the fundamental representation of $sl(2)$, corresponding to the root space $A_1$. Thus we conjecture that the special solutions of (2.2.18) should correspond to distinguished points in the root space $A_2$, since it has an $sl(3)$ zero curvature representation. As yet we have not pursued this idea further.

## 2.6 Generalizations

### 2.6.1 A Conjecture

One of the main motivations for our approach has been the observation that similarity reductions of integrable PDEs provide more general examples of finite-dimensional integrable Hamiltonian systems than purely stationary flows. We make the conjecture that given an integrable evolution equation with a stationary flow that can be written as a natural Hamiltonian system, all of its similarity reductions will be written as (possibly non-autonomous) integrable generalizations of this system. Thus the approach of Fordy [63] should extend to all such similarity reductions. The central step in this procedure is to take the evolution equation in the Hamiltonian form,

$$\underline{u}_t = B_{\underline{u}} \delta_{\underline{u}} H, \tag{2.6.1}$$

with Poisson operator $B_{\underline{u}}$, and show that all similarity reductions may be written in the form

$$B_{\underline{w}} \underline{f} = 0, \tag{2.6.2}$$

with $\underline{w}$ representing the similarity variables, and $\underline{f}$ containing the variational derivative of the Hamiltonian plus the extra terms arising from $\underline{u}_t$ in (2.6.1). We present two further examples where this works, and thus far we have been unable to find a counterexample.

### 2.6.2 More Hénon-Heiles Systems

First we consider "travelling wave" similarity solutions of the fifth-order equations of the form (2.2.15). Then we have

$$u = w(z) + k_1, \qquad z = x + k_2 t,$$

for constants $k_1$, $k_2$. Substituting into (2.2.1) gives

$$k_2 w' = (\partial^3 + 8a(w + k_1)\partial + 4aw')(w'' - b(w + k_1)^2),$$

and we find that for this to be in the required form (2.6.2) we must take

$$k_2 = -4abk_1^2,$$

and set

$$f = w'' - b(w^2 + 2k_1 w).$$

The analogue of the equation (2.2.7) of Section 2.2 is then

$$\frac{d^2 f}{dz^2} + 4a(w + k_1)f + \frac{\lambda^2 - (\frac{df}{dz})^2}{2f} = 0,$$

and the same substitutions $w = q_1$, $f = -aq_2^2$ lead to a system with Hamiltonian

$$h = \frac{1}{2}(p_1^2 + p_2^2 + c_1 q_1^2 + c_2 q_2^2) + a q_1 q_2^2 - \frac{1}{3} b q_1^3 - \frac{\lambda^2}{8a^2} q_2^{-2}, \qquad (2.6.3)$$

where the constants in front of the quadratic terms are

$$\begin{aligned} c_1 &= -2bk_1, \\ c_2 &= 2ak_1. \end{aligned}$$

Thus we have $c_1 = -\frac{b}{a} c_2$. As for ordinary Hénon-Heiles, the Hamiltonian (2.6.3) can again only be integrable for the same three values of the ratio $r = \frac{b}{a}$, and there are further restrictions on the quadratic terms. In the case (i) (reduction of the SK equation) this gives $c_1 = c_2$, while in case (iii) (reduction of the KK equation) $c_1 = 16c_2$, and these are the only integrable cases isolated by Painlevé analysis [40]. For the case (ii) corresponding to fifth-order KdV, the analysis of [40] shows that (2.6.3) is integrable for arbitrary $c_1$, $c_2$, but we have not found similarity reductions which lead to a Hamiltonian system of this form.

### 2.6.3 Scaling Similarity Solutions to the Hirota-Satsuma System

As our second example, we take PDEs whose stationary flows lead to integrable quartic potentials. These stationary flows are considered in [26, 27]. One of the PDEs considered is the Hirota-Satsuma system [82],

$$u_t = \frac{1}{2} u_{3x} + 3uu_x - 6\phi\phi_x, \qquad (2.6.4)$$

$$\phi_t = -\phi_{3x} - 3u\phi_x, \qquad (2.6.5)$$

while the other one is related to it by a gauge transformation. Scaling similarity solutions of (2.6.4,2.6.5) are given by

$$u = \theta^2 w(z), \qquad \phi = \theta^2 \xi(z),$$

for $z = x\theta$, $\frac{d\theta}{dt} = \theta^4$. On using the scaled Hamiltonian version of (2.6.4,2.6.5) we obtain an equation of the form (2.6.2), with

$$B_{\underline{w}} = \begin{pmatrix} \frac{1}{2}\partial^3 + 2w\partial + w' & 2\xi\partial + \xi' \\ 2\xi\partial + \xi' & \frac{1}{2}\partial^3 + 2w\partial + w' \end{pmatrix},$$

$$\underline{f} = \begin{pmatrix} \delta_w \tilde{H} - z \\ \delta_\xi \tilde{H} \end{pmatrix},$$

$$\tilde{H} = \frac{1}{2}w^2 - \xi^2.$$

On making a substitution,

$$\underline{f} = \begin{pmatrix} q_2^2 + q_1^2 \\ q_2^2 - q_1^2 \end{pmatrix},$$

we are led to a system with Hamiltonian

$$h_1 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{8}(q_1^4 + 6q_1^2q_2^2 + q_2^4) + \frac{1}{2}(lq_1^{-2} + kq_2^{-2}) + \frac{1}{2}z(q_1^2 + q_2^2).$$

This is a non-autonomous generalization of one of the integrable Hamiltonians with quartic potentials derived in [26]. Applying the same procedure to scaling similarity solutions of the other PDE there considered, we arrive at

$$h_2 = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{16}(Q_1^4 + 6Q_1^2Q_2^2 + 8Q_2^4) - KQ_2 + \frac{L}{2}Q_1^{-2} + z(\frac{1}{2}Q_1^2 + Q_2^2).$$

The two PDEs are related to the same equation in a modified hierarchy, so there are canonical transformations between these two Hamiltonian systems, just as for cases (i) and (iii) of non-autonomous Hénon-Heiles. We notice that for $k = l = 0$, the equations of motion for $h_1$ are separable in coordinates $q_1 \pm q_2$, and (up to a rescaling) give two copies of PII for parameter $\alpha = 0$. Also we observe that if we define a tau-function by

$$h_1(z) = (\log[\tau(z)])',$$

then this gives the (similarity solution) tau-function of the Hirota-Satsuma equation after scaling $\tau$ by a factor of $\exp[z^3/12]$.

## 2.7 Conclusion

We have considered the scaling similarity solutions of the Sawada-Kotera, fifth-order KdV and Kaup-Kupershmidt equations, and have shown that they may be understood as solutions to non-autonomous Hamiltonian systems $\mathcal{H}_{(i)-(iii)}$, which are time-dependent generalizations of the well-known integrable Hénon-Heiles systems. We have also used the Miura maps for each of the PDEs (relating them to a PDE in a modified hierarchy) to give Miura maps for these similarity solutions, which can in fact be inverted. More precisely, we have seen that solutions of the fourth-order ODE (2.2.18) are in one-one correspondence with solutions of both $\mathcal{H}_{(i)}$ and $\mathcal{H}_{(iii)}$, while there is also a one-one correspondence between the solutions of (2.2.19) and $\mathcal{H}_{(ii)}$. These correspondences have led to natural derivations of Bäcklund transformations, resulting in the generation of special sequences of

solutions. The tau-function for each of these non-autonomous Hamiltonians has also appeared very naturally, providing a concise way to encode and generate solutions.

Since the ODEs we have studied are similarity reductions of integrable PDEs, they should be viewed as fourth-order analogues of the Painlevé transcendents. In the light of our results, we believe that approaching such ODEs by way of associated Hamiltonian systems is extremely useful. The examples in Section 2.6 indicate that the techniques of this chapter are quite general. We expect that they should apply to similarity solutions of all PDEs occurring as reductions of the KP hierarchy. For example, higher order stationary flows of coupled KdV equations lead to some of the integrable polynomial Hamiltonian systems of [54], and thus there should be associated non-autonomous systems. We intend to develop these ideas further in the future.

**Note.** After this thesis was submitted, we were made aware of some recent results of Kudryashov concerning similarity solutions of the mKdV and KdV hierarchies [106]. This work may be viewed as complementary to ours, and although it does not describe the Hamiltonian formalism for these equations, it overlaps in several places.

# Chapter 3

# Singular Solutions of the Nonlinear Schrödinger Equation and their Pole Dynamics

The linear problem of NLS admits an analogue of the Crum transformation for linear Schrödinger operators. This leads to the construction of a sequence of singular rational solutions, which may be written in terms of Wronskians of Schur polynomials. Bilinear methods provide a straightforward way to show that the poles and zeros of the rational solutions evolve according to constrained Calogero-Moser equations. NLS also has a trilinear form which is related to its reduction from the KP hierarchy. Some other singular solutions also appear to have interesting pole dynamics.

## 3.1 Introduction

It is well known that for many nonlinear PDEs solvable by inverse scattering, the motion of the poles of rational solutions is determined by finite-dimensional integrable Hamiltonian systems. Perhaps the canonical example is the KP hierarchy [103, 142], where the pole motion with respect to each of the times is governed by a corresponding Calogero-Moser flow. Similarly, rational solutions of the KdV and Boussinesq equations [11, 130], as well as the Burgers equation [16], have poles whose equations of motion are just those of constrained Calogero-Moser systems, but essentially this occurs because these PDEs arise via reduction of the KP hierarchy.

### 3.1.1 KP and Calogero-Moser

To clarify these ideas, we briefly review the case of the KP hierarchy and its reduction to KdV. We follow Shiota's exposition in [142]. A polynomial tau-

function for KP may be written in the form

$$\tau(x,\underline{t}) = \prod_{j=1}^{N}(x - x_j(\underline{t})), \tag{3.1.1}$$

where $\underline{t} = (t_2, t_3, ...)$ are the times of the hierarchy. The dependent variable in the original form of the KP equation is related to the tau-function by

$$u(x,\underline{t}) = (\log[\tau])_{xx}, \tag{3.1.2}$$

and so taking the polynomial tau-function (3.1.1) gives rational solutions $u$ decaying at infinity, i.e.

$$u = -\sum_{j=1}^{N}(x - x_j(\underline{t}))^{-2}.$$

The first bilinear equation in the hierarchy is

$$(D_1^4 + 3D_2^2 - 4D_1D_3)\tau \cdot \tau = 0. \tag{3.1.3}$$

($D_m$ denotes the derivative with respect to $t_m$, and we identify $t_1 \equiv x$.) The usual form of the KP equation follows from (3.1.3) on dividing through by $\tau^2$, differentiating twice with respect to $x$ and rewriting the resulting expression in terms of $u$:

$$(4u_{t_3} - u_{3x} - 12uu_x)_x - 3u_{t_2t_2} = 0. \tag{3.1.4}$$

Clearly the zeros of the polynomial tau-function (3.1.1) are just the poles of the rational function $u$. Then the main result of [142] (generalizing the observation of Krichever [103]) is that a polynomial tau-function satisfies the KP hierarchy if and only if for each of the times $t_m$ ($m$=2,3,...,$N$) its zeros evolve according to

$$\frac{\partial}{\partial t_m}\begin{pmatrix} x_j \\ \xi_j \end{pmatrix} = (-)^m \begin{pmatrix} \partial H_m/\partial \xi_j \\ -\partial H_m/\partial x_j \end{pmatrix}, \tag{3.1.5}$$

where $\xi_j = \frac{1}{2}\partial x_j/\partial t_2$, $H_m = trY^m$, and $Y$ is the $N \times N$ Moser matrix with entries

$$Y_{jk} = \xi_j\delta_{jk} + \frac{1-\delta_{jk}}{x_j - x_k}. \tag{3.1.6}$$

The equations (3.1.5) are those of the Calogero-Moser hierarchy, which is completely integrable [130]. The Hamiltonians $H_1, \ldots H_N$ are $N$ independent conserved quantities in involution with respect to the standard Poisson brackets.

To see what happens when we reduce from the KP hierarchy to KdV, consider the first ($m = 2$) flow given by (3.1.5). The equations of motion for the $x_j$, $j = 1, ..., N$, are

$$\frac{\partial^2 x_j}{\partial t_2^2} = -8\sum_{k\neq j}(x_j - x_k)^{-3}, \tag{3.1.7}$$

which are just the original Calogero-Moser equations for the Hamiltonian

$$H_2 = \sum_{j=1}^{N} \xi_j^2 - \sum_{j,k}(x_j - x_k)^{-2}.$$

Now the KdV hierarchy is obtained from KP by making all the flows with respect to the even times stationary. So in particular stationarity of the $t_2$ flow gives the constraint

$$\sum_{k\neq j}(x_j - x_k)^{-3} = 0, \tag{3.1.8}$$

as well as $\xi_j = 0$, for $j = 1, ..., N$, and the $t_3$ flow becomes simply

$$\frac{\partial x_j}{\partial t_3} = 3\sum_{k\neq j}(x_j - x_k)^{-2}. \tag{3.1.9}$$

A more general form of tau-function for KP which yields interesting pole motion is the elliptic polynomial

$$\tau(x, \underline{t}) = \exp(\alpha(\underline{t})x + \beta(\underline{t}))\prod_{j=1}^{n}\sigma(x - x_j(\underline{t})), \tag{3.1.10}$$

where $\sigma$ is the Weierstrass sigma function. This gives

$$u(x, \underline{t}) \;=\; -\sum_{j=1}^{n}\wp(x - x_j(\underline{t})),$$

with $\wp$ being the usual Weierstrass elliptic function[1]. The poles evolve according to the elliptic Calogero-Moser system, and in the reduction to KdV the constraint involves a sum of derivatives of $\wp$-functions. In fact the rational solutions may be obtained from the elliptic ones by letting the periods tend to infinity (and there is an intermediate case where only one period goes to infinity, which gives Calogero-Moser with an inverse sine-squared or sinh-squared potential). For further details, and the corresponding result for the Boussinesq equation, see [11, 130].

### 3.1.2 Pole Motion for NLS

In this chapter we consider the Nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} - 2|\psi|^2\psi = 0, \tag{3.1.11}$$

[1]The exponential prefactor in (3.1.10) does not alter the form of the physical variable $u$ but is necessary to ensure that the bilinear equation (3.1.3) is satisfied.

and show various ways to produce singular solutions and describe the motion of their poles. Our approach makes much use of Bäcklund transformations and bilinear methods. We present a construction of rational solutions of NLS which has an exact analogue in the case of the KdV equation. These rational solutions of NLS are given by

$$\psi = \frac{g}{f},$$

with $g$ and $f$ both being polynomial tau-functions of the form (3.1.1) (up to a constant prefactor). It is then possible to demonstrate that the poles and zeros of such rational solutions evolve according to constrained Calogero-Moser equations. We also give a description of a few other singular solutions.

An outline of the chapter is as follows. First of all, in the next section, we apply the singular manifold method used in [120, 152] to (3.1.11), and are thus able to obtain the standard auto-Bäcklund transformation (ABT) and associated linear problem (zero curvature representation). In Section 3.3 we derive the Hirota bilinear form of NLS, present the ABT in bilinear form, and show how this may be used to produce a sequence of singular rational solutions. Rational solutions of the KdV hierarchy were studied by Adler and Moser [8] from the point of view of the Crum transformation for (linear) Schrödinger operators. We proceed to show in Section 3.4 that there is a Crum-type transformation for the linear problem of NLS, which provides a direct construction of the rational solutions (rather than repeated application of the ABT, which is laborious). The Crum transformation leads to Wronskian formulae for the rational solutions, which generalize some similarity solutions found by Hirota and Nakamura [83, 84] via a connection with the classical Boussinesq equation. Section 3.5 contains a direct derivation of the trilinear form for NLS, and we explain how this is connected to its reduction from the KP hierarchy. Following this we indicate how use of the trilinear form shows that the rational solutions have poles which move according to constrained Calogero-Moser equations, while using the bilinear form demonstrates that in fact the zeros of the rational solutions satisfy constrained Calogero-Moser equations as well (Section 3.6). We briefly discuss similarity solutions of NLS, plus some other sorts of singular solutions and their pole motions, in Section 3.7. Finally we discuss how these methods might be applied further in our Conclusion.

## 3.2 NLS and the Singular Manifold Method

The Nonlinear Schrödinger (NLS) equation

$$i\psi_t + \psi_{xx} + 2\delta|\psi|^2\psi = 0, \tag{3.2.1}$$

(where $\delta = \pm 1$), is one of the most ubiquitous examples of an integrable nonlinear equation. It is an important equation in nonlinear optics [118], as well as describing the modulation of a sinusoidal wavetrain in an isotropic elastic medium [131]; also it has an interesting correspondence with the Localized-Induction-Approximation equations

$$\underline{r}_t = \underline{r}_x \wedge \underline{r}_{xx},$$

$$\underline{r}_x . \underline{r}_x = 1,$$

which approximate the motion of a thin vortex filament (see [146]). In fact NLS is an appropriate first approximation for the evolution of any slowly-varying wave-envelope in a weakly nonlinear system (as is shown by multiple-scales analysis in [47], Chapter 8). A thorough discussion of the physical applications of NLS may be found in Gibbon's survey article ([62], Chapter 6), which contains a full list of references.

### 3.2.1 NLS and AKNS

In the form (3.2.1) NLS is really two different equations describing different physical behaviours: the focussing- and nonfocussing-NLS equations, corresponding to $\delta = +1$ and $\delta = -1$ respectively. Both these cases may be obtained from the AKNS system

$$q_{t_2} = q_{xx} + 2q^2 r, \tag{3.2.2}$$

$$r_{t_2} = -r_{xx} - 2qr^2, \tag{3.2.3}$$

on setting $t_2 = it$, $q = \psi$ and $r = \delta\overline{\psi}$ (for real $x, t$, with the bar denoting complex conjugate). We denote the AKNS time variable by $t_2$ to identify it with the second time of the KP hierarchy, since AKNS is a reduction of KP (we will return to this point when we come to derive the trilinear form of NLS later). So the two different NLS equations give solutions to AKNS with particular reality conditions. In [134], Previato derived the hyperelliptic quasiperiodic solutions of AKNS using algebraic geometry, and then studied the reality conditions corresponding to $\delta = \pm 1$, showing how certain limits of these solutions gave the N-soliton formulae found by Hirota [80] using his bilinear formalism. A simple check of leading-order behaviour in (3.2.1) shows that singular-type solutions are admitted only in the case $\delta = -1$, and so because we are interested in finding pole motions we will henceforth consider only NLS in the form (3.1.11).

### 3.2.2 Inverse Scattering and the ABT

To make a comparison with later results we note here that the usual inverse scattering scheme for (3.1.11) is the $su(1,1)$ zero curvature representation

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} -ik & \psi \\ \overline{\psi} & ik \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{3.2.4}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} -i(|\psi|^2 + 2k^2) & i\psi_x + 2k\psi \\ -i\overline{\psi}_x + 2k\overline{\psi} & i(|\psi|^2 + 2k^2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{3.2.5}$$

For the case $\delta = +1$ there is a similar $su(2)$ spectral problem, originally found by Zakharov and Shabat [154]. Their solution of NLS by the inverse scattering method was one of the first indications of the remarkable generality of this technique (which had previously only been applied to the KdV equation).

For future reference, we also present the standard auto-Bäcklund transformation (ABT) of NLS (studied by Boiti and Pempinelli in [29]):

$$(\psi - \tilde{\psi})_x = \frac{ic}{2}(\psi - \tilde{\psi}) - (\psi + \tilde{\psi})\sqrt{|\psi - \tilde{\psi}|^2 + \sigma^2}, \tag{3.2.6}$$

$$i(\psi - \tilde{\psi})_t = -\frac{ic}{2}(\psi - \tilde{\psi})_x + (\psi + \tilde{\psi})_x\sqrt{|\psi - \tilde{\psi}|^2 + \sigma^2} + (\psi - \tilde{\psi})(|\psi|^2 + |\tilde{\psi}|^2). \tag{3.2.7}$$

It is easy to verify that if $\tilde{\psi}$ is any solution of (3.1.11), then given (3.2.6) and (3.2.7), $\psi$ must also satisfy (3.1.11). For example, starting from the vacuum solution $\tilde{\psi} = 0$, we apply (3.2.6) and (3.2.7), and find the singular 1-soliton

$$\psi = \frac{\sigma \exp[i(\frac{cx}{2} + (\sigma^2 - \frac{c^2}{4})t)]}{\sinh[\sigma(x - ct)]}.$$

(We ignore the arbitrary constant shifts in $x$, $t$ and the phase of $\psi$, which are always possible.) At this point it is worth observing that the NLS equation has a Galilean symmetry: given any solution $\psi$ of (3.1.11), then another solution is given by

$$\Psi(x,t) = \exp\left[i\left(\frac{cx}{2} - \frac{c^2 t}{4}\right)\right]\psi(x',t'),$$

where

$$x' = x - ct, \qquad t' = t.$$

It is easy to check that if two solutions $\psi$ and $\tilde{\psi}$ are related by the ABT (3.2.6) and (3.2.7) with the parameter $c$ set to zero, then the Galilean-boosted solutions $\Psi$ and $\tilde{\Psi}$ are related by (3.2.6) and (3.2.7) with the parameter $c$ reinserted. Also we remark that the 1-pole rational solution

$$\psi = \frac{1}{x}$$

may be found from the 1-soliton in the limit $\sigma \to 0$, $c \to 0$ (or equivalently by applying (3.2.6) and (3.2.7) with $\sigma = c = 0$ to the vacuum solution). This rational solution is the first in a sequence which we derive later.

### 3.2.3 Application of the Singular Manifold Method

In the rest of this section we apply a branch of Painlevé analysis pioneered by Weiss (see [120, 152] and references in [30]), which we will refer to as the singular manifold method. The usual Painlevé test for PDEs [37] involves substituting a Painlevé expansion of the form

$$u(x,t) = \phi^{-K} \sum_{n=0}^{\infty} u_n(x,t)\phi^n$$

into the PDE, with $\phi = \phi(x,t)$ being an arbitrary (non-characteristic) function defining the singularity manifold $\phi = 0$. For an integrable PDE $K$ must normally be a positive integer (unless it is an example of the "weak Painlevé" property [30]), and we may then proceed with the singular manifold method.

The first step is to truncate the expansion at the "constant" level:

$$u(x,t) = \phi^{-K} \sum_{n=0}^{K-1} u_n(x,t)\phi^n + u_K(x,t).$$

While substituting the full expansion into the PDE yields an infinite set of equations for $\phi$ and the $u_j$, the truncated expansion should give only a finite number, at least for an evolution equation of the form

$$u_t = F[u]$$

(where $F$ is a polynomial in $u$ and its $x$-derivatives). Then the last of these equations is

$$u_{K,t} = F[u_K],$$

which means that $u_K$ satisfies the same PDE as $u$, and so the truncated expansion constitutes an ABT. Further analysis of the equations for $\phi$ and the $u_j$ ($j = 0, 1, ..., K$) gives a better characterization of the ABT, and may be used to derive both the inverse scattering formalism and the Hirota bilinear form for the PDE in question. Thus the technique is of great practical value, as well as highlighting the intimate relationships between bilinear forms, Bäcklund transformations and inverse scattering (see for example [70, 99]). There are some subtleties as to when this truncation procedure works properly [132], but they will not concern us. Rather than trying to describe the general situation in any more detail, we shall proceed to apply the singular manifold method to the NLS equation. Essentially our analysis reproduces the results on the AKNS system found in [120].

Looking at (3.1.11), we see immediately that the leading order behaviour is just a simple pole in terms of the singularity manifold function $\phi$ (which we will

assume to be real). More precisely we find

$$\psi \sim \frac{u_0}{\phi},$$

where

$$|u_0|^2 = \phi_x^2.$$

Upon setting $u_1 = \tilde{\psi}$, our truncated expansion is just

$$\psi = \frac{u_0}{\phi} + \tilde{\psi}, \tag{3.2.8}$$

and we substitute this into (3.1.11) and set the terms at each order in $\phi$ to zero. We find the following four equations (the singular manifold equations):

$$\begin{aligned} |u_0|^2 - \phi_x^2 &= 0, && (3.2.9)\\ i\phi_t + 2\phi_x(\log[u_0])_x + \phi_{xx} + 2u_0\overline{\tilde{\psi}} + 4\overline{u_0}\tilde{\psi} &= 0, && (3.2.10)\\ iu_{0,t} + u_{0,xx} - 4u_0|\tilde{\psi}|^2 - 2\overline{u_0}\tilde{\psi}^2 &= 0, && (3.2.11)\\ i\tilde{\psi}_t + \tilde{\psi}_{xx} - 2|\tilde{\psi}|^2\tilde{\psi} &= 0. && (3.2.12) \end{aligned}$$

So (3.2.9) (the coefficient of $\phi^{-3}$) just yields the leading order behaviour, while the "constant" (order $\phi^0$) term (3.2.12) means that the truncated expansion constitutes an auto-Bäcklund transformation for NLS, provided that these equations are all consistent. In fact the consistency is shown directly by deriving the zero curvature representation of NLS, which we do below.

There are some consequences of (3.2.9-3.2.12) which lead to simpler formulae. The derivations are made simpler by using the two equations found from (3.2.10) on taking real and imaginary parts:

$$\begin{aligned} \phi_{xx} + \overline{u_0}\tilde{\psi} + u_0\overline{\tilde{\psi}} &= 0,\\ i\phi_t + \phi_x(\log[u_0/\overline{u_0}])_x - u_0\overline{\tilde{\psi}} + \overline{u_0}\tilde{\psi} &= 0. \end{aligned}$$

After further calculation, we find that we must have

$$\begin{aligned} u_{0,x} &= -2iku_0 - 2\phi_x\tilde{\psi}, && (3.2.13)\\ iu_{0,t} &= u_0(4k^2 + 2|\tilde{\psi}|^2) + \phi_x(-4ik\tilde{\psi} + 2\tilde{\psi}_x), && (3.2.14) \end{aligned}$$

where $k$ is a real constant. The manipulations required to derive (3.2.13) and (3.2.14) are not very instructive, so we present them separately in Appendix A.

Now the standard ABT for NLS follows immediately. For if we rearrange the equation (3.2.8) then we have

$$u_0 = \phi(\psi - \tilde{\psi}),$$

which gives

$$\phi_x^2 = \phi^2|\psi - \tilde{\psi}|^2$$

on substitution into (3.2.9). Then substituting for $u_0$ and $\phi_x$ in (3.2.13) gives

$$(\psi - \tilde{\psi})_x = -2ik(\psi - \tilde{\psi}) - (\psi + \tilde{\psi})|\psi - \tilde{\psi}|,$$

which is just (3.2.6) for $\sigma = 0$ and $c = -4k$. Similarly the corresponding time part (3.2.7) of the auto-Bäcklund transformation may be found by making these same substitutions in (3.2.14). Note, however, that this truncation has not provided us with the full transformation (for non-zero $\sigma$) needed to produce a singular 1-soliton solution from the vacuum. For this reason we think of this transformation (with $\sigma = 0$) as the "rational ABT". The standard ABT with $\sigma \neq 0$ (the "solitonic ABT") could presumably be obtained by truncating the Painlevé expansion at some level higher than $\phi^0$, as has been found by Pickering for some other PDEs [132]. Since we are primarily interested in the rational solutions of NLS, this will not be important.

The zero curvature form of NLS follows if we now make the "squared eigenfunction" substitution (see [120])

$$u_0 = -v^2.$$

Then from (3.2.9) we must have (up to a sign, which we fix)

$$\phi_x = |v|^2.$$

So in terms of $v$ and $\overline{v}$, (3.2.13) and (3.2.14) give

$$v_x = -ikv + \tilde{\psi}\overline{v}, \tag{3.2.15}$$

$$v_t = -i(|\tilde{\psi}|^2 + 2k^2)v + (i\tilde{\psi}_x + 2k\tilde{\psi})\overline{v}. \tag{3.2.16}$$

This is the same as the standard NLS spectral problem (3.2.4,3.2.5) if we replace $\psi$ by $\tilde{\psi}$, set

$$v_1 = v,$$

and make the consistent choice

$$v_2 = \overline{v}.$$

Since we already know that the consistency condition for the spectral problem (3.2.15,3.2.16) is just the NLS equation for $\tilde{\psi}$, this implies that the singularity manifold equations (3.2.9-3.2.12) are also consistent. We have shown how the singular manifold method applied to NLS can be used to derive the ABT (at least for rational solutions) and inverse scattering scheme. We note that the

choice of complex conjugate components $v_2 = \overline{v_1}$ is not the usual one made in applications of the spectral problem (3.2.4,3.2.5) (for instance when finding soliton solutions), but the equation (3.2.15) will be important when we introduce a Crum-type transformation in Section 3.4. At this stage the analysis is much simplified by writing the NLS equation and its ABT in bilinear form, which we do in the next section.

## 3.3 NLS and its Auto-Bäcklund Transformation in Bilinear Form

We have seen how the singular manifold method may be used to derive both an inverse scattering scheme and a Bäcklund transformation for an integrable evolution equation. Yet another application of this method is in finding the Hirota bilinear form for a PDE by truncating the Painlevé expansion before the "constant" term (i.e. at order $\phi^{-1}$). For example, for the KP hierarchy (and also for its reduction to KdV [70]), the most general type of Painlevé expansion obtained by substitution into (3.1.4) has a double pole in $\phi$:

$$u = \frac{u_0}{\phi^2} + \frac{u_1}{\phi} + u_2 + \ldots$$

It is easily found that

$$\begin{aligned} u_0 &= -\phi_x^2, \\ u_1 &= \phi_{xx}, \end{aligned}$$

so that when we truncate in the usual way we find that the Bäcklund transformation relating $u$ and $u_2$ has

$$u = (\log[\phi])_{xx} + u_2.$$

Now if we make the further truncation (i.e. set $u_2 = 0$) and make the identification $\phi = \tau$, then we have

$$u = (\log[\tau])_{xx},$$

which leads to the bilinear form (3.1.3).

### 3.3.1 The Bilinear Form

The truncation method may be used to obtain the bilinear form of NLS, but it is necessary to have two tau-functions to get bilinear equations. With only one tau-function we find a trilinear equation instead, which will be discussed later. In

the truncation (3.2.8) we set $\tilde{\psi} = 0$, $u_0 = g$, $\phi = f$, and then the NLS equation (3.1.11) becomes

$$\frac{1}{f^2}\left((iD_t + D_x^2)g \cdot f\right) - \frac{g}{f^3}\left(D_x^2 f \cdot f + 2|g|^2\right) = 0.$$

This means that if the tau-functions $f, g$ satisfy

$$(iD_t + D_x^2)g \cdot f = 0, \tag{3.3.1}$$

$$D_x^2 f \cdot f + 2|g|^2 = 0, \tag{3.3.2}$$

then $\psi = g/f$ satisfies (3.1.11). This is the usual Hirota bilinear form for NLS, as used in [80] to obtain soliton solutions. The second of these bilinears has an immediate consequence which we will make use of throughout the rest of the chapter, namely

$$|\psi|^2 = -(\log[f])_{xx}. \tag{3.3.3}$$

Another consequence of (3.3.1,3.3.2) is the bilinear equation

$$iD_x D_t f \cdot f - 2D_x g \cdot \overline{g} = i\gamma f^2. \tag{3.3.4}$$

In the above $\gamma$ is a real constant, which may be set to zero without loss of generality (i.e. by rescaling both $f$ and $g$ by $\exp[\gamma x t/2]$). In Section 3.5 we will show that using (3.1.11) and the substitution (3.3.3) it is possible to obtain a trilinear equation for $f$. We note here that this trilinear equation also follows from the bilinears (3.3.1,3.3.2) and (3.3.4) with $\gamma = 0$.

### 3.3.2 Bäcklund Transformations in Bilinear Form

Given a bilinear form for a PDE it is often convenient to express its Bäcklund transformations in terms of the tau-functions. Also, given an ABT in bilinear form one may generate a PDE in a new dependent variable by choosing some suitable combination of the tau-functions. A very common example [137] is the KdV equation,

$$U_t = U_{3x} + 6UU_x.$$

It is straightforward to obtain this from the bilinear equation

$$D_x(D_x^3 - D_t)\tau \cdot \tau = 0, \tag{3.3.5}$$

where the dependent variable is given in terms of the tau-function by

$$U = 2(\log[\tau])_{xx}.$$

The bilinear ABT for (3.3.5) is

$$(D_t - 3\beta D_x - D_x^3)\tau \cdot \tilde{\tau} = 0, \tag{3.3.6}$$

$$(D_x^2 - \beta)\tau \cdot \tilde{\tau} = 0. \tag{3.3.7}$$

This means that given two tau-functions $\tilde{\tau}$, $\tau$ related by (3.3.6,3.3.7), if $\tilde{\tau}$ satisfies (3.3.5) then so must $\tau$ (and vice-versa). The Bäcklund parameter $\beta$ is arbitrary.

Applying the transformation starting from the vacuum $\tilde{\tau} = 0$ gives a 1-soliton solution for $\beta \neq 0$ and a (1-pole) rational solution for $\beta = 0$, which suggests that $\beta$ plays the same rôle as the parameter $\sigma$ in the ABT (3.2.6,3.2.7) for NLS. Now if we consider the bilinear equations (3.3.6,3.3.7) with the Bäcklund parameter set to zero, we may define a new dependent variable

$$V = (\log[\tau/\tilde{\tau}])_x.$$

Then it is simple to demonstrate that this bilinear ABT with $\beta = 0$ implies that $V$ satisfies the *modified* KdV (mKdV) equation:

$$V_t = V_{3x} - 6V^2 V_x.$$

Further examples of this may be found in [79, 81], but more relevant to our discussion is the work [83] of Hirota and Nakamura concerning the classical Boussinesq system.

### 3.3.3 Classical Boussinesq and NLS

The classical Boussinesq equation for the dependent variable $u$ is derived from the system

$$\begin{aligned} u_t &= ((1+u)v - v_{xx})_x, \\ v_t &= (u + \frac{1}{2}v^2)_x. \end{aligned}$$

In [83] it was shown that this system has a bilinear form,

$$\begin{aligned} (iD_t + D_x^2)\overline{F} \cdot F &= 0, \\ (iD_t D_x + D_x^3)\overline{F} \cdot F &= 0. \end{aligned}$$

To make the dependent variables $u$, $v$ real (for real $x$ and $t$), the tau-functions must be taken as a conjugate pair $F, \overline{F}$, with

$$\begin{aligned} u &= -1 - 2(\log[F\overline{F}])_{xx}, \\ v &= 2i(\log[F/\overline{F}])_x. \end{aligned}$$

Using the bilinear form, Hirota and Nakamura found an ABT which allowed the construction of a sequence of rational solutions known as "explode-decay" solitons. In fact these solutions all depend on the similarity variable

$$z = \frac{x}{2t^{\frac{1}{2}}}$$

(we have rescaled $z$ compared with [83] to be consistent with other results below), and the tau-functions are written in terms of Wronskian determinants of Hermite polynomials.

Another result proved in [83] is that, using the tau-functions related via the bilinear ABT for classical Boussinesq, one may construct new tau-functions $f, g$ satisfying the bilinear equations (3.3.1,3.3.2) for NLS. More precisely, given a conjugate pair of classical Boussinesq tau-functions $F, \overline{F}$ which is related to another conjugate pair $F', \overline{F}'$ by the bilinear ABT, the NLS tau-functions are found from

$$f^2 = \frac{1}{2}(F\overline{F}' + F'\overline{F}), \tag{3.3.8}$$

$$gf = \frac{1}{2}(D_x\overline{F} \cdot \overline{F}'). \tag{3.3.9}$$

The corresponding solutions to NLS found in this way are rational, and since they depend essentially just on the variable $z$ they are also similarity solutions to NLS. The first three of these were calculated by Hirota and Nakamura, and we present them in Table 3.1 below.

| $n$ | 1 | 2 | 3 |
|---|---|---|---|
| $f_n$ | $x$ | $x^4 - 12t^2$ | $x^9 - 72x^5t^2 - 2160xt^4$ |
| $g_n$ | 1 | $-2x^3 + 12ixt$ | $3x^8 - 48ix^6t - 360x^4t^2 + 2160t^4$ |

**Table 3.1. Polynomial tau-functions for NLS similarity solutions.**

The scaling similarity solutions of NLS have been considered in some detail by Boiti and Pempinelli [29]; they depend on the variable $z$ and are in one-one correspondence with a particular case of the fourth Painlevé transcendent (PIV). We will return to these similarity solutions in Section 3.7, but for the moment we simply remark that the sequence of rational solutions found by Hirota and Nakamura is the same as that obtained by repeated application of the Boiti-Pempinelli ABT for similarity solutions, starting from the trivial similarity solution $\psi = 0$. The Hirota-Nakamura formulae, giving the explode-decay soliton solutions of the classical Boussinesq system in terms of Wronskians of Hermite polynomials, are related to certain classes of special solutions to PIV [74, 111]. These Wronskian determinants are special cases of formulae for rational solutions which we derive in the next section.

### 3.3.4 Bilinear ABT

The method of Hirota and Nakamura is an extremely indirect way to construct rational similarity solutions of NLS, because it requires repeated application of the ABT for classical Boussinesq, as well as the substitutions (3.3.8) and (3.3.9) (which, as observed in [83], do not ensure that $f^2$ is positive definite). Instead, one could apply the ABT (3.2.6,3.2.7) for NLS (with $\sigma = c = 0$ to get purely rational solutions). This too is a somewhat laborious task; substituting in the 1-pole solution and trying to find the next rational solution from it is extremely difficult by this method. Hence after considering the bilinear approach of [83] we were led to the discovery of the following bilinear ABT for NLS (which as far as we know is original[2]):

$$(D_x - ic/2)(g \cdot \tilde{f} - \tilde{g} \cdot f) = 0, \tag{3.3.10}$$

$$(iD_t + \sigma^2 - c^2/4)g \cdot \tilde{f} + (D_x^2 + c^2/4)\tilde{g} \cdot f = 0, \tag{3.3.11}$$

$$(iD_t + \sigma^2 - c^2/4)\tilde{g} \cdot f + (D_x^2 + c^2/4)g \cdot \tilde{f} = 0, \tag{3.3.12}$$

$$(iD_t + icD_x)f \cdot \tilde{f} = \overline{g}\tilde{g} - g\overline{\tilde{g}}, \tag{3.3.13}$$

$$D_x f \cdot \tilde{f} = \sqrt{|g\tilde{f} - \tilde{g}f|^2 + \sigma^2 f^2 \tilde{f}^2}. \tag{3.3.14}$$

The bilinear relations (3.3.10-3.3.14) are such that if the pair of tau-functions $\tilde{g}$, $\tilde{f}$ satisfy the NLS bilinears (3.3.1,3.3.2), then so do $g$, $f$. We prove this in Appendix B.

In deriving the above we were led to consider its relationship with the singular manifold method. The connections between Painlevé analysis and the Hirota formalism has been considered for a number of different PDEs (including NLS) in the paper [70] of Gibbon et al. In particular, they show that the singular manifold equations (3.2.9-3.2.11) imply a Bäcklund transformation for the NLS bilinear equations (3.3.1,3.3.2). This means that given a pair of tau-functions $\tilde{g}$, $\tilde{f}$ satisfying the NLS bilinears, another pair $g$, $f$ may be constructed from the relation

$$\frac{g}{f} = \frac{u_0}{\phi} + \frac{\tilde{g}}{\tilde{f}},$$

with $\phi = f/\tilde{f}$, provided that the singular manifold equations (3.2.9-3.2.11) hold (where $\tilde{\psi}$ is replaced by $\tilde{g}/\tilde{f}$ throughout). We have shown independently that (3.3.10-3.3.14) constitutes an ABT for the bilinear form of NLS (see Appendix B). So this means that the bilinear version of the ABT must be equivalent to the singular manifold equations (but only for the non-solitonic case $\sigma = 0$). Rather

[2]Shortly before completing this work we became aware of the reference [122] in which Nimmo presents a very similar set of bilinear equations. See note in Appendix B.

than check this, we prefer to apply the bilinear ABT to show how it generates rational solutions.

It is easy to check that the bilinear ABT with $\sigma \neq 0$ produces the singular 1-soliton,

$$g = \sigma \exp\left[i\left(\frac{cx}{2} + (\sigma^2 - \frac{c^2}{4})t\right)\right], \quad f = \sinh[\sigma(x - ct)]$$

when applied to the vacuum,

$$\tilde{g} = 0, \qquad \tilde{f} = 1.$$

(As before we always neglect shifts in $x$, $t$ and the constant phase shift in $g$.) Since we want to generate purely rational solutions we set $\sigma = c = 0$, and then as expected the vacuum produces the 1-pole rational solution

$$g = 1, \qquad f = x.$$

The inclusion of the parameter $c$ would just give this a Galilean boost; this is discussed in Appendix B.

Note that the bilinear ABT is not completely symmetric under interchange of the tau-functions with tildes and those without: the square root sign in (3.3.14) introduces this asymmetry (which is clearly essential to get anything new from repeated application of the transformation). As with our singular manifold equations, we stick to the convention that the old quantities have tildes, while the sought-after new quantities do not. So, applying the rational bilinear ABT ($\sigma = c = 0$) to the one-pole solution, we obtain the 4-pole solution

$$g = -2x^3 + 12ixt + \tau_3, \quad f = x^4 + \tau_3 x - 12t^2,$$

where $\tau_3$ is an arbitrary constant (and again we have neglected an arbitrary shift in $t$). This solution clearly reduces to the 4-pole similarity solution $f_2, g_2$ of Hirota and Nakamura (as in Table 3.1) in the special case $\tau_3 = 0$.

Looking at the table of similarity solutions, we see that we would expect the next rational solution to have nine poles (i.e. nine zeros in $f$). This is indeed the case, with more and more arbitrary constants appearing each time we apply the ABT (and the similarity solutions arising on setting all arbitrary constants to zero). By analogy with the well-known results about rational solutions of KdV [8], we would expect the arbitrary constants to correspond to the higher times in the NLS hierarchy. For the 4-pole solution we compute this directly. The equation for the next flow in the hierarchy (which is just a restriction of the corresponding flow for AKNS [41]) is

$$\psi_{t_3} = \psi_{3x} + 6|\psi|^2\psi_x,$$

which leads to the bilinear equation

$$D_{t_3} g \cdot f = D_x^3 g \cdot f$$

(where (3.3.1,3.3.2) still hold). Substituting the 4-pole solution into this bilinear equation, we find that we must make the identification

$$\tau_3 = -12 t_3.$$

The bilinear ABT is still a very inefficient way to generate rational solutions, and it is not very clear how the times for the higher flows arise as constants of integration.

In the next section we reconsider the linear problem for NLS (as derived by the singular manifold method), and find that it admits an analogue of the Crum transformation for linear Schrödinger operators. By repeated application of this transformation, which we call the NLS Crum transformation (NCT) we are led to an algorithmic way in which to compute the sequence of rational solutions. We find that the bilinear variables $g$, $f$ lead to a very concise description of this, as well as making more contact with the results of Gibbon et al. At the same time, the Crum transformation provides a natural derivation of the substitutions (3.3.8,3.3.9) in terms of classical Boussinesq tau-functions. Also we are able to write the rational solutions in terms of Wronskian determinants of Schur polynomials, which generalize Hirota and Nakamura's formulae for the similarity solutions.

## 3.4 The NLS Crum Transformation and Rational Solutions

### 3.4.1 The Crum Transformation for KdV

The construction of the polynomial tau-functions for KdV was shown [8] to be most easily achieved by considering the factorization of a second order operator into two first order operators:

$$\tilde{L} - \lambda 1 = -A^* A.$$

In the above we have the Schrödinger operator

$$\tilde{L} = \partial_x^2 + \tilde{U},$$

the first order operator

$$A = \partial_x - V,$$

and its adjoint $A^*$. The operator $\tilde{L}$ is just half of the Lax pair for the KdV equation (or for each of the flows in the KdV hierarchy). It is straightforward to obtain $\tilde{U}$ in terms of $V$, and for $\lambda = 0$ this recovers the well-known Miura map between a solution $V$ of mKdV and $\tilde{U}$ satisfying KdV.

Starting from the eigenvalue equation

$$(\tilde{L} - \lambda)\phi = 0, \tag{3.4.1}$$

and a solution $\phi \neq 0$, the factorization of $\tilde{L}$ can be reversed to yield a new Schrödinger operator

$$L = \partial_x^2 + U,$$

where

$$L - \lambda 1 = -AA^*.$$

This is simply because the eigenfunction $\phi$ yields the factorization via the formula

$$A = \phi \partial_x \phi^{-1},$$

and then it is straightforward to see that $\phi^{-1}$ is an eigenfunction for the operator $L$, i.e.

$$(L - \lambda)\phi^{-1} = 0.$$

This is the Crum transformation. The rational solutions of KdV are the sequence of potentials of the Schrödinger operators generated by repeated application of this transformation, with the special choice of eigenvalue $\lambda = 0$, beginning with the potential $U_0 = 0$. If we apply the same transformation to the operator $L$ above, with the eigenfunction $\phi^{-1}$ providing the factorization, then the Crum transformation will just lead back to the original operator $\tilde{L}$. Hence to generate a new potential, at each stage another eigenfunction must be found (such that it is linearly independent with $\phi^{-1}$).

### 3.4.2 NLS Crum Transformation

If we consider the $x$ part of the linear problem for NLS, we find that it admits an analogue of the usual Crum transformation for the Schrödinger spectral problem (3.4.1). We start from the equation (3.2.15) found using the singular manifold method:

$$v_x = -ikv + \tilde{\psi}\bar{v}.$$

This is the analogue of the eigenvalue problem for the Schrödinger operators; it may also be considered as a second order problem for the real or imaginary parts of $v$. We shall refer to the functions $v$ throughout as "eigenfunctions", although

strictly the vector with $v$ and $\overline{v}$ as its components is the eigenfunction for the matrix spectral problem (3.2.4); similarly we shall refer to the functions $\tilde{\psi}$ as the "potentials". Using the results of Section 3.2, we define the singular manifold function $\phi$ via

$$\phi_x = |v|^2. \tag{3.4.2}$$

Now we can define a new eigenfunction

$$v^* = \frac{v}{\phi}.$$

It is easy to see that $V = v^*$ is a solution to the eigenvalue equation

$$V_x = -ikV + \psi\overline{V}, \tag{3.4.3}$$

with the new potential

$$\psi = \tilde{\psi} - \frac{v^2}{\phi}.$$

The above transformation from $\tilde{\psi}$ to $\psi$ constitutes the NLS Crum transformation (NCT). We can of course define a new singular manifold function $\phi^*$ by

$$\phi^*_x = |v^*|^2,$$

but up to a constant we must have

$$\phi^* = -\phi^{-1},$$

and so applying the same transformation again just leads back to the old potential $\tilde{\psi}$. So in order to get anything new we need to find a new eigenfunction $v'$ satisfying (3.4.3), such that $v'$ and $v^*$ are linearly independent (over the real numbers). It is well known that two independent solutions to a Schrödinger eigenvalue problem have a Wronskian which is a non-zero constant. In this case the analogue of the Wronskian is the quantity

$$W[v^*, v'] := (2i)^{-1}(v^*\overline{v'} - \overline{v^*}v'). \tag{3.4.4}$$

It can be checked directly that for any two solutions $v^*$, $v'$ of (3.4.3), $W[v^*, v']$ is a constant, which will be non-zero when they are independent. In fact $W$ arises naturally as a determinant back in the matrix formulation (3.2.4).

### 3.4.3 A Sequence of Rational Functions

We proceed to construct a sequence of rational functions $\psi_n$ for $n = 0, 1, 2, \ldots$ by repeated application of the NCT, with the eigenvalue $k = 0$, starting from the

vacuum $\psi_0 = 0$. These rational functions turn out to be solutions to the NLS equation, but we need some preliminary results before we can prove this. All the manipulations involved are much easier when carried out in bilinear form. Hence we are lead to the following equations which must be solved successively (starting from $f_0 = 1$, $h_0 = i$):

$$\begin{aligned} D_x h_n \cdot f_n &= \overline{h_n} g_n, && (3.4.5)\\ W[h_n, h_{n-1}] &= \sqrt{(2n-1)(2n+1)} f_n^2, && (3.4.6)\\ D_x f_{n+1} \cdot f_n &= |h_n|^2, && (3.4.7)\\ g_{n+1} f_n - g_n f_{n+1} &= -h_n^2; && (3.4.8) \end{aligned}$$

$W$ is as defined in (3.4.4). It is a simple matter to show that solving these equations is equivalent to applying the NCT repeatedly.

**Proposition 3.1.** *If $h_n$, $g_n$ and $f_n$ are solutions to (3.4.5-3.4.8) with the initial conditions $f_0 = 1$, $h_0 = i$, then the most general potential $\psi_n$ obtained by $n$ applications of the NCT (with eigenvalue $k = 0$) to the vacuum $\psi_0 = 0$ is of the form*

$$\psi_n = \frac{g_n}{f_n}, \tag{3.4.9}$$

*where*

$$|\psi_n|^2 = -(\log[f_n])_{xx}. \tag{3.4.10}$$

*The new eigenfunctions at each stage are given by*

$$v_n = \frac{h_n}{f_n}, \tag{3.4.11}$$

*so that*

$$v_{n,x} = \psi_n \overline{v_n}, \tag{3.4.12}$$

*and the singular manifold function is obtained from the formula*

$$\phi_n = \frac{f_{n+1}}{f_n}. \tag{3.4.13}$$

**Proof.** First we consider $n = 0$. We may substitute for $f_0$ and $h_0$ in (3.4.5) immediately to get $g_0 = 0$, which gives $\psi_0 = 0$. Since $f_0$ is a constant (3.4.10) also holds, and clearly $v_0 = h_0/f_0$ is just a constant and satisfies $v_{0,x} = \psi_0 \overline{v_0} = 0$. The equation (3.4.2) for the singular manifold function gives $\phi_{0,x} = 1$, and thus $\phi_0 = f_1/f_0 = x + \tau_1$, with $\tau_1$ constant. As usual we neglect this translation in $x$ and find $f_1 = x$, which is just what we find on solving (3.4.7). Finally (3.4.8) yields $g_1 = 1$, and so $\psi_1 = 1/x$ as expected.

We may proceed by induction. Assume $f_0, ..., f_n, g_0, ..., g_n$ and $h_0, ..., h_{n-1}$ have been found. To apply the NCT we require a solution $V$ of

$$V_x = \psi_n \overline{V}. \tag{3.4.14}$$

After a little manipulation of (3.4.5-3.4.8) we find

$$D_x h_{n-1} \cdot f_n = \overline{h_{n-1}} g_n, \tag{3.4.15}$$

which implies that $V = h_{n-1}/f_n = v_{n-1}/\phi_{n-1}$ is a solution of (3.4.14). However, essentially this gives the singular manifold function $-\phi_{n-1}^{-1}$, and so applying the NCT with this $V$ just leads back to $\psi_{n-1}$. Hence we need a new solution $v_n$ to (3.4.14) such that the quantity $W[v_n, v_{n-1}/\phi_{n-1}]$ is a non-zero constant. We find this by solving (3.4.5) with the normalization condition (3.4.6) (this normalization is chosen to ensure that each of the $f_n$ is a monic polynomial). On dividing through (3.4.5) by $f_n^2$ it is simple to check that the new eigenfunction $v_n$ is given by (3.4.11). Similarly dividing through (3.4.7) by the same factor produces the equation for the singular manifold function, that is

$$\phi_{n,x} = |v_n|^2,$$

with $\phi_n$ given by (3.4.13). If we then divide (3.4.8) by $f_n f_{n+1}$ on both sides then we find the correct equation (3.4.9) for the new potential:

$$\psi_{n+1} = \frac{g_{n+1}}{f_{n+1}} = \psi_n - \frac{v_n^2}{\phi_n}.$$

The only thing still to verify is that the modulus of the new potential satisfies (3.4.10). By the inductive hypothesis we have

$$(\log[f_{n+1}])_{xx} = (\log[\phi_n])_{xx} - |\psi_n|^2. \tag{3.4.16}$$

Because we have constructed the linear problem (3.4.14) via the singular manifold equations (3.2.9-3.2.12), it is obvious that their purely $x$-dependent parts must be consequences of the equations for the Crum transformation. So for $\phi = \phi_n$ and $u_0 = \phi_n(\psi_{n+1} - \psi_n)$, the equation (3.2.9) gives

$$\phi_{n,x}^2 = \phi_n^2 |\psi_{n+1} - \psi_n|^2,$$

while the real part of (3.2.10) implies

$$\phi_{n,xx} = \phi_n(|\psi_{n+1} - \psi_n|^2 + |\psi_n|^2 - |\psi_{n+1}|^2).$$

Alternatively these derivatives of $\phi_n$ may be computed directly using (3.4.5-3.4.8), and then substituting into (3.4.16) yields

$$|\psi_{n+1}|^2 = -(\log[f_{n+1}])_{xx},$$

as required. This completes the proof. □

We are now able to show that the application of the Crum transformation is purely algorithmic, in the sense that the new potential is obtained by performing two integrations and then solving one algebraic equation. At this stage we also fix a convention for the constants of integration.

**Proposition 3.2.** *Given $h_{n-1}$, $f_n$, $g_n$ found by $n$ applications of the NCT in the form (3.4.5-3.4.8), the new potential $\psi_{n+1} = g_{n+1}/f_{n+1}$ is obtained via the following three steps. First, integrate*

$$\left(\frac{h_n}{h_{n-1}}\right)_x = -2i\sqrt{(2n+1)(2n-1)}\frac{f_n g_n}{h_{n-1}^2}, \tag{3.4.17}$$

*to find $h_n$, and denote the constant of integration by $\sqrt{(2n+1)(2n-1)}\,\tau_{2n}$. Next, use $h_n$ to find $f_{n+1}$ by integrating*

$$\left(\frac{f_{n+1}}{f_n}\right)_x = \frac{|h_n|^2}{f_n^2}, \tag{3.4.18}$$

*where the second constant of integration is denoted by $\tau_{2n+1}$. Finally, solve for $g_{n+1}$ by rearranging the algebraic relation (3.4.8) to give*

$$g_{n+1} = f_n^{-1}(g_n f_{n+1} - h_n^2).$$

*The constants of integration are real.*

**Proof.** To find a solution $h_n$ to (3.4.5,3.4.6) (which generates a new eigenfunction $V$ for the linear problem (3.4.14)), observe that

$$f_n(D_x h_n \cdot h_{n-1}) = h_{n-1}(D_x h_n \cdot f_n) - h_n(D_x h_{n-1} \cdot f_n).$$

Substituting for the bracketed expressions on the right hand side from (3.4.5) and (3.4.15), and using the normalization condition (3.4.6) yields

$$D_x h_n \cdot h_{n-1} = -2i\sqrt{(2n+1)(2n-1)}f_n g_n, \tag{3.4.19}$$

from which (3.4.17) follows instantly. Note that we can always add on any real multiple of $h_{n-1}$ to $h_n$ and it will still satisfy (3.4.5, 3.4.6). We choose this multiple to be $\sqrt{(2n+1)(2n-1)}\,\tau_{2n}$. The equation (3.4.18) is just the expression for the derivative of the singular manifold function $\phi_n$ (written as the ratio (3.4.13) of two tau-functions). Hence the constant $\tau_{2n+1}$ is just the arbitrary multiple of $f_n$ that may be added to $f_{n+1}$. □

It is interesting to observe that if we multiply either $h_{n-1}$ or $h_n$ by $i$ and remove the normalization factor $\sqrt{(2n+1)(2n-1)}$ then (3.4.6) and (3.4.19) are identical

to the substitutions (3.3.8, 3.3.9), when we identify the classical Boussinesq tau-functions $\overline{F}$, $\overline{F}'$ with the rescaled $h_n$, $h_{n-1}$. Hence we are lead to the conjecture that, up to scaling, $h_n$ is a tau-function for the classical Boussinesq equation, for all $n$. We postpone consideration of this matter for the moment, but henceforth refer to the $h_n$ as tau-functions.

The first few NLS tau-functions found by applying the NCT are

$$f_0 = 1, \qquad g_0 = 0,$$

$$f_1 = x, \qquad g_1 = 1,$$

$$f_2 = x^4 + \tau_3 x - 3\tau_2^2, \qquad g_2 = -2x^3 + 6i\tau_2 x + \tau_3,$$

$$\begin{aligned} f_3 &= x^9 + 6\tau_3 x^6 - 18\tau_2^2 x^5 + \tau_5 x^4 - 60\tau_2\tau_4 x^3 + 90\tau_2^2\tau_3 x^2 \\ &\quad +(\tau_3\tau_5 - 135\tau_2^4 - 15\tau_4^2)x + 30\tau_2\tau_3\tau_4 - 5\tau_3^2 - 3\tau_2^2\tau_5, \end{aligned}$$

$$\begin{aligned} g_3 &= 3x^8 - 24i\tau_2 x^6 + 6\tau_3 x^5 - 30(3\tau_2^2 + i\tau_4)x^4 - 2(\tau_5 - 30i\tau_2\tau_3)x^3 \\ &\quad +30\tau_3^2 x^2 + (-90\tau_2^2\tau_3 + i(6\tau_2\tau_5 - 30\tau_3\tau_4))x + 135\tau_2^4 \\ &\quad +\tau_3\tau_5 - 15\tau_4^2 + 30i(\tau_2\tau_3^2 - 3\tau_2^2\tau_4). \end{aligned}$$

In computing the above the first few $h_n$ are also needed:

$$h_0 = i,$$

$$h_1 = -\sqrt{3}(x^2 - i\tau_2),$$

$$h_2 = -i\sqrt{5}\begin{pmatrix} x^6 - 3i\tau_2 x^4 + 2\tau_3 x^3 - (9\tau_2^2 + 3i\tau_4)x^2 \\ +6i\tau_2\tau_3 x + \tau_3^2 - 3\tau_2\tau_4 - 9i\tau_2^3 \end{pmatrix}.$$

We note that, after setting $\tau_2 = 2t$ and $\tau_j = 0$ for $j \geq 3$, these $f_n$, $g_n$ are the same as the similarity solutions found by Hirota and Nakamura (as in [83] and Table 3.1), and also the $h_n$ correspond to the tau-functions which provide similarity solutions to the classical Boussinesq equation.

We have shown that the application of the NCT is purely algorithmic, and generates a sequence of eigenfunctions $v_n$ and potentials $\psi_n$ satisfying the purely $x$-dependent part (3.4.12) of the NLS linear problem. However, although the first few terms in this sequence are clearly rational, we have yet to prove this in general. The proof of this follows from the fact (proved in the next subsection) that all the tau-functions $f_n$, $g_n$, $h_n$ may be written as Wronskian determinants of Schur polynomials, and hence are themselves polynomials. Other Wronskian identities show that the tau-functions $g_n$, $f_n$ also satisfy the $t$ part (3.3.1) of the NLS bilinears, and hence the sequence of potentials $\psi_n$ really do provide rational

solutions of the NLS equation (3.1.11)(on identifying $\tau_2 = 2t$). More generally, the constants $\tau_j$ should correspond to the times of the NLS hierarchy. In fact, the flows of the AKNS hierarchy can be given recursively in bilinear form [120],

$$D_{t_{j+1}} g \cdot f = D_{t_j} D_x g \cdot f, \tag{3.4.20}$$

and on reducing to the NLS hierarchy all the odd times $t_{2n+1}$ must be real while the even times $t_{2n}$ must be purely imaginary. We postpone further discussion of the higher bilinears (3.4.20) for the moment.

### 3.4.4 Wronskian Formulae

At this point we make use of the sequence of Schur polynomials $p_j$ for $j$=0,1,2,..., defined by

$$\exp[\xi(\underline{t}, \nu)] = \sum_{j=0}^{\infty} p_j(\underline{t})\nu^j, \qquad \xi(\underline{t}, \nu) = \sum_{j=1}^{\infty} t_j \nu^j, \qquad t_1 = x.$$

From this definition it is simple to show the following identities:

$$\frac{\partial^k p_j}{\partial x^k} = p_{j-k}, \tag{3.4.21}$$

$$\frac{\partial p_j}{\partial t_k} = \frac{\partial^k p_j}{\partial x^k}. \tag{3.4.22}$$

The first five Schur polynomials are

$$p_0 = 1, \qquad p_1 = x, \qquad p_2 = \frac{1}{2}x^2 + t_2,$$

$$p_3 = \frac{1}{6}x^3 + t_2 x + t_3, \quad p_4 = \frac{1}{24}x^4 + \frac{1}{2}t_2 x^2 + t_3 x + \frac{1}{2}t_2^2 + t_4.$$

We are able to demonstrate that up to scale factors, all of the $f_n$, $g_n$, $h_n$ found by applying the NCT are just given by double Wronskians of Schur polynomials, which implies immediately that these tau-functions are themselves polynomials in $x$. To make this identification requires the constant of integration $\tau_j$ to be proportional to $t_j$, for each $j$.

In what follows, we use the following notation for the Wronskian of $n$ functions $a_1, a_2, \ldots, a_n$:

$$[a_1, a_2, \ldots, a_n] := \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_{1,x} & a_{2,x} & \cdots & a_{n,x} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,(n-1)x} & a_{2,(n-1)x} & \cdots & a_{n,(n-1)x} \end{vmatrix}.$$

With this notation we introduce the quantities

$$\begin{aligned} F_n &= [p_{2n-1}, \ldots, p_{2n-1,(n-1)x}], \\ G_n &= [p_{2n-1}, \ldots, p_{2n-1,nx}], \\ H_n &= [p_{2n}, \ldots, p_{2n,nx}]. \end{aligned}$$

These expressions are all double Wronskians of Schur polynomials, and it turns out that they are proportional to $f_n$, $g_n$, and $h_n$ respectively. We also make use of

$$\begin{aligned} \overline{G}_n &= [p_{2n-1}, \ldots, p_{2n-1,(n-2)x}], \\ \overline{H}_n &= [p_{2n}, \ldots, p_{2n,(n-1)x}]. \end{aligned}$$

Strictly speaking, the bars above do not denote complex conjugate, although it will turn out that the complex conjugate of $G_n$ (repectively $H_n$) is equal to $\overline{G}_n$ (respectively $\overline{H}_n$) up to a minus sign.

The key to proving that the tau-functions generated by the NCT are proportional to these Wronskians is showing that, up to scale factors, the Wronskians satisfy all of the bilinears (3.4.5-3.4.8). We shall see that all these bilinears may be reduced to Laplace expansions of certain determinants. It is also necessary to show that the form of these Wronskians is compatible with the algorithmic procedure of Proposition 3.2, by identifying the constants of integration $\tau_j$ with the $t_j$ appearing in the Schur polynomials. To demonstrate this we require a few preliminary results.

**Proposition 3.3.** *The double Wronskians $F_n$, $G_n$, $H_n$, $\overline{G}_n$, $\overline{H}_n$ satisfy analogues of (3.4.5,3.4.7) and the equation (3.4.19), which are given by*

$$D_x H_n \cdot F_n = \overline{H}_n G_n, \tag{3.4.23}$$

$$D_x F_{n+1} \cdot F_n = H_n \overline{H}_n, \tag{3.4.24}$$

$$D_x H_n \cdot H_{n-1} = F_n G_n. \tag{3.4.25}$$

*They also satisfy the "conjugates" of these equations, obtained by swapping $G_n$ with $\overline{G}_n$, and $H_n$ with $\overline{H}_n$.*

**Proof.** We will give the full details for (3.4.25), since this is (the bilinear form of) one of the steps in the algorithm of Proposition 3.2. We are able to show that (3.4.25) is equivalent to the Laplace expansion of a certain $(2n+1) \times (2n+1)$ determinant. Since (3.4.23,3.4.24) are equivalent to essentially the same sort of Laplace expansion, we do not consider them separately.

To prove (3.4.25), we let $(0)$ denote $p_{2n}$, and for any positive integer $k$ we let $(k)$ denote $p_{2n-k}$. So using the property (3.4.21) we may write e.g.

$$H_n = [(0), \ldots, (n)],$$

and

$$H_{n-1,x} = [(2), \ldots, (n), (n+2)].$$

In this notation, (3.4.25) is equivalent to

$$\begin{aligned} &[(0), \ldots, (n-1), (n+1)] \cdot [(2), \ldots, (n+1)] \\ &\quad -[(0), \ldots, (n)] \cdot [(2), \ldots, (n), (n+2)] \\ &\quad\quad -[(1), \ldots, (n+1)] \cdot [(1), \ldots, (n)] \;=\; 0. \end{aligned} \tag{3.4.26}$$

To see why (3.4.26) must hold, observe that it is just the Laplace expansion in the first $(n+1)$ rows of the determinant

$$\left|\begin{array}{c|ccc|c|c|ccc|} (0) & (1) & \cdots & (n-1) & (n) & (n+1) & & & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & 0 & \\ (n) & (n+1) & \cdots & (2n-1) & (2n) & (2n+1) & & & \\ \hline (1) & & & & (n+1) & (n+2) & (2) & \cdots & (n) \\ \vdots & & 0 & & \vdots & \vdots & \vdots & \ddots & \vdots \\ (n) & & & & (2n) & (2n+1) & (n+1) & \cdots & (2n-1) \end{array}\right|. \tag{3.4.27}$$

It is straightforward to show that (3.4.27) vanishes, and thus (3.4.25) is proved. The proofs for (3.4.23,3.4.24) and the "conjugates" are almost identical. □

We present the analogous versions of (3.4.6,3.4.8) separately, as they require a slightly different type of Laplace expansion.

**Proposition 3.4.** *The following analogues of (3.4.6,3.4.8) may be shown to hold:*

$$H_n \overline{H}_{n-1} - \overline{H}_n H_{n-1} \;=\; -F_n^2, \tag{3.4.28}$$

$$G_{n+1} F_n - G_n F_{n+1} \;=\; -H_n^2. \tag{3.4.29}$$

*The "conjugates" of these are also satisfied.*

**Proof.** As before we will use the notation $(k) = p_{2n-k}$. Then (3.4.28) just becomes

$$\begin{aligned} &[(0), \ldots, (n)] \cdot [(2), \ldots, (n)] \\ &\quad -[(0), \ldots, (n-1)] \cdot [(2), \ldots, (n+1)] \\ &\quad\quad +[(1), \ldots, (n)] \cdot [(1), \ldots, (n)] \;=\; 0. \end{aligned} \tag{3.4.30}$$

While the three terms in (3.4.26) all consisted of products of an $n \times n$ with an $(n+1) \times (n+1)$ determinant, (3.4.30) has instead two pairs of determinants of order $n$, as well as one of order $n+1$ with another of order $n-1$. Thus (3.4.30) occurs as the Laplace expansion of the $(2n+1) \times (2n+1)$ determinant

$$\left|\begin{array}{c|ccc|c|c|ccc}
(0) & (1) & \dots & (n-1) & (n) & 0 & & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & 0 & \\
(n-1) & (n) & \dots & (2n-2) & (2n-1) & 0 & & & \\
(n) & (n+1) & \dots & (2n-1) & (2n) & 1 & & & \\
\hline
(1) & & & & (n+1) & 0 & (2) & \dots & (n) \\
\vdots & & 0 & & \vdots & \vdots & \vdots & \ddots & \vdots \\
(n-1) & & & & (2n-1) & 0 & (n) & \dots & (2n-2) \\
(n) & & & & (2n) & 1 & (n+1) & \dots & (2n-1)
\end{array}\right|, \tag{3.4.31}$$

which clearly vanishes. The equation (3.4.29) and the "conjugates" follow from essentially the same Laplace expansion. □

Having proved these determinantal identities, it is now obvious that the tau-functions found via the NCT must be proportional to the double Wronskians defined above. It remains to determine the scale factors and identify the $\tau_j$ in terms of the $t_j$. Note that the original scaling was chosen to make all of the $f_n$ monic polynomials, at the expense of introducing square roots into the normalization condition (3.4.6).

**Proposition 3.5.** *The tau-functions $f_n$, $g_n$, $h_n$ found from the application of the NCT, as well as their complex conjugates, are all polynomials. They may be written in terms of the Wronskian determinants of Schur polynomials,*

$$\begin{aligned}
f_n &= (-)^{[\frac{n}{2}]} \frac{n!}{(2n)!} c(n) F_n, \\
g_n &= (-)^{n+1+[\frac{n+1}{2}]} \frac{(2n-1)!}{(n-1)!} c(n-1) G_n, \\
\overline{g}_n &= (-)^{n+1+[\frac{n-1}{2}]} \frac{(2n-1)!}{(n-1)!} c(n-1) \overline{G}_n, \\
h_n &= (-)^{[\frac{n+1}{2}]} i^{n+1} \sqrt{2n+1}\, c(n) H_n, \\
\overline{h}_n &= (-)^{[\frac{n}{2}]} (-i)^{n+1} \sqrt{2n+1}\, c(n) \overline{H}_n,
\end{aligned} \tag{3.4.32}$$

*where*

$$c(n) := \prod_{j=0}^{n} \frac{(n+j)!}{j!}.$$

*The $t_j$ are related to the constants of integration $\tau_j$ by*

$$t_{2n} = (-)^{n+1} \frac{(n-1)!n!}{(2n-2)!(2n)!} i \tau_{2n}, \quad t_{2n+1} = (-)^n \frac{(n!)^2}{(2n)!(2n+1)!} \tau_{2n+1}.$$

**Proof.** Essentially this follows immediately by induction, on comparing the algorithmic steps of the NCT, equivalent to the bilinears (3.4.7,3.4.8) and (3.4.19), with their counterparts written in terms of the Wronskians. By Proposition 3.2, the tau-functions and their complex conjugates are uniquely determined at each step, up to the constants of integration. The formulae relating the $\tau_j$ to the $t_j$ are found by comparing the conventions chosen in Proposition 3.2,

$$h_{n,\tau_{2n}} = \sqrt{(2n-1)(2n+1)}\, h_{n-1}, \qquad f_{n+1,\tau_{2n+1}} = f_n,$$

with the corresponding expressions obtained by differentiating the Wronskians and using the property (3.4.22) of Schur polynomials,

$$H_{n,t_{2n}} = H_{n-1}, \qquad F_{n+1,t_{2n+1}} = F_n.$$

Note that the formulae for $g_n$ and $\overline{g}_n$ may differ by an overall sign in front (and similarly for $h_n$ and $\overline{h}_n$). □

The Wronskian machinery also provides an easy proof that the tau-functions $g_n$, $f_n$ satisfy the NLS bilinear equation (3.3.1), and hence with (3.4.10) this implies that each $\psi_n$ is a solution of the NLS equation (3.1.11).

**Proposition 3.6.** *The tau-functions $g_n$, $f_n$ satisfy*

$$(iD_t + D_x^2)g_n \cdot f_n = 0.$$

**Proof.** Using the scaling properties and the fact that $t = \frac{1}{2}\tau_2 = -it_2$, the proposition is equivalent to

$$D_{t_2}G_n \cdot F_n = D_x^2 G_n \cdot F_n. \tag{3.4.33}$$

By the properties (3.4.21,3.4.22), it is apparent that (3.4.33) may be expanded out using the same notation as in (3.4.26,3.4.30). After a few cancellations the resulting expression,

$$\begin{aligned} &[(0),\ldots,(n-2),(n),(n+1)]\cdot[(0),\ldots,(n-1)] \\ -&[(0),\ldots,(n-1),(n+1)]\cdot[(0),\ldots,(n-2),(n)] \\ &+[(0),\ldots,(n)]\cdot[(0),\ldots,(n-2),(n+1)] \;=\; 0, \end{aligned}$$

just corresponds to the Laplace expansion of a determinant of the same form as (3.4.27). □

An almost identical argument shows that $h_n$, $\overline{h}_n$ satisfy the bilinear equations of the classical Boussinesq system, as has been proved for the rational similarity

solutions by Hirota [84]. We have attempted to extend this Wronskian approach to deal with the whole NLS (or AKNS) bilinear hierarchy (3.4.20), but the expansions become too complicated to analyze directly. A full treatment would require an NLS analogue of the KP bilinear recursion operators developed by Nimmo [123]. Although we have not found a general proof that (3.4.20) is satisfied by the whole sequence of Wronskians, it is straightforward to see that

$$D_{t_{2j+1}} G_n \cdot F_n = D_{t_{2j}} D_x G_n \cdot F_n$$

when $n \leq j$, because these Wronskians are independent of $t_{2j}$ and $t_{2j+1}$, and a short calculation shows that it is also satisfied for $n = j + 1$. In the next section we shall see how NLS arises as a reduction of the KP hierarchy, with the times given by the $t_j$.

## 3.5 The Trilinear Form of NLS

In this section, rather than using our formulae found by application of an auto-Bäcklund transformation, we go back to the NLS equation (3.1.11) and find that the complex amplitude $\psi$ can be (almost) completely determined by a single real tau-function $f$ which satisfies a trilinear equation, rather than as a ratio of two tau-functions satisfying the coupled bilinear equations (3.3.1,3.3.2). This is not really surprising when the connection is made with the way the AKNS hierarchy (and hence NLS) arises as a reduction of the KP hierarchy, as in the work of Cheng and Strammp et al [41, 125]. Rather than employing the results of these authors immediately, we prefer to go back to first principles to derive the trilinear form, since this is the route we originally took. The use of a trilinear equation may appear to be unnecessary when we have already discovered so much about the rational solutions from the auto-Bäcklund transformation. However, the trilinear form is useful in that it provides a simple way to determine equations of motion for the poles of the rational solutions.

### 3.5.1 Direct Derivation of Trilinear Form

We begin from the coupled equations for the amplitude and phase of $\psi$. Writing $\psi = w^{\frac{1}{2}} \exp(i\chi)$, the NLS equation (3.1.11) is equivalent to the system

$$\chi_t + \chi_x^2 - \frac{1}{2}\frac{w_{xx}}{w} + \frac{1}{4}\left(\frac{w_x}{w}\right)^2 + 2w = 0, \tag{3.5.1}$$

$$w_t + 2(w\chi_x)_x = 0. \tag{3.5.2}$$

Next we define $w_1 = w$, $w_2 = \chi_x$. On differentiating (3.5.1) with respect to $x$ and expressing both the resulting equation and (3.5.2) in terms of $w_j$, $j = 1, 2$, we obtain the Hamiltonian form

$$\underline{w}_t := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}_t = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x & 0 \end{pmatrix} \delta_{\underline{w}} H, \tag{3.5.3}$$

where

$$H = w_1 w_2^2 + w_1^2 + \frac{w_{1,x}^2}{4w_1}.$$

(This is not the standard Hamiltonian form for NLS, but is closely related to it. See [57].) If we define the new dependent variable $\eta = -2w_1 w_2$, then the system (3.5.3) implies the following two evolution equations for $w = w_1$ and $\eta$:

$$w_t = \eta_x, \tag{3.5.4}$$

$$\eta_t = \left(2w^2 - w_{xx} + \frac{w_x^2 + \eta^2}{w}\right)_x. \tag{3.5.5}$$

At that this stage we are ready to express everything concisely in terms of the single tau-function $f$. First we define

$$\Lambda = \log[f].$$

The second bilinear equation (3.3.2) for NLS implies immediately

$$w = -\Lambda_{xx}.$$

On substituting this into (3.5.4) and integrating once with respect to $x$ we find

$$\eta = -\Lambda_{xt},$$

where the arbitrary function of time that arises can always be absorbed into $f$. Then after substituting for $\eta$ and $w$ in (3.5.5) and performing another integration with respect to $x$ we obtain a PDE for $\Lambda$:

$$\Lambda_{tt}\Lambda_{xx} - \Lambda_{xt}^2 - \Lambda_{3x}^2 + 2\Lambda_{xx}^3 + \Lambda_{xx}\Lambda_{4x} = 0. \tag{3.5.6}$$

Once again the arbitrary function of time from the integration has been set to zero, as it can also be absorbed into the tau-function $f$. The equation (3.5.6) may now be rewritten in terms of $f$. There are many terms which cancel, and then after multiplying through by $f^3$ we find that, remarkably enough, the remaining terms may be written as a sum of two determinants:

$$\begin{vmatrix} f & f_x & f_t \\ f_x & f_{xx} & f_{xt} \\ f_{xx} & f_{xt} & f_{tt} \end{vmatrix} + \begin{vmatrix} f & f_x & f_{xx} \\ f_x & f_{xx} & f_{3x} \\ f_{xx} & f_{3x} & f_{4x} \end{vmatrix} = 0. \tag{3.5.7}$$

The equation (3.5.7) is the trilinear form of NLS.

### 3.5.2 AKNS as a Reduction of KP

Trilinear equations, and more generally multilinear equations, provide a natural extension of Hirota's bilinear formalism. Grammaticos et al [73] have developed a useful notation for multilinear operators, and have provided a partial classification of integrable trilinear equations of low order. In particular, the determinants in (3.5.7) appear in their work as particular examples of integrable trilinears. Multilinear equations also arise naturally as the equations for the tau-function of the KP hierarchy under the so-called generalized $k$-constraint, which has been studied in detail by Cheng [41] and Strammp et al [125]. These $k$-constraints have analogues for other PDEs, leading to finite-dimensional integrable Hamiltonian systems [109].

Recall that the flows of the KP hierarchy in terms of the Lax operator $L$ are given by

$$\partial_{t_n} L = [(L^n)_+, L],$$

this being the compatibility condition for the linear system

$$\begin{aligned} Lq &= \lambda q, \\ q_{t_n} &= (L^n)_+ q, \end{aligned} \tag{3.5.8}$$

where $q$ is the wave-function. Similarly the adjoint wave-function $r$ satisfies

$$r_{t_n} = -(L^{*n})_+ r.$$

All the flows of the KP hierarchy commute, i.e.

$$[\partial_{t_n}, \partial_{t_m}] = 0,$$

but there is another vector field $\partial_G$ (known as the ghost symmetry) which acts on $L$ by

$$\partial_G L = [L, q\partial_x^{-1} r]$$

and also commutes with all the flows:

$$[\partial_{t_n}, \partial_G] = 0.$$

The most common reduction of KP is the $k$-reduction,

$$(L^k)_- = 0.$$

For example, the KdV hierarchy results from the case $k = 2$. Because of the ghost symmetry, it is possible to make the more general compatible constraint,

$$(L^k)_- = q\partial_x^{-1} r,$$

which is known as the generalized $k$-constraint.

We consider in detail the case of the generalized 1-constraint, which is relevant to our trilinear equation (3.5.7). If we write the Lax operator as

$$L = \partial_x + u\partial_x^{-1} + ...,$$

then the linear equation (3.5.8) for the second time flow is just a time-dependent Schrödinger equation

$$q_{t_2} = q_{xx} + 2uq. \tag{3.5.9}$$

Imposing the 1-constraint, we set

$$L = \partial_x + q\partial_x^{-1}r,$$

and find immediately that

$$u = qr.$$

Hence the equation (3.5.9) just becomes the first half (3.2.2) of the usual AKNS system. Similarly the adjoint to (3.5.9) becomes (3.2.3), and Cheng has shown [41] that in fact the 1-constraint yields the whole AKNS hierarchy.

At this point it is useful to mention the Kaup-Broer system

$$\begin{aligned} H_{t_2} &= (H_x + 2X^*H)_x, \\ X^*_{t_2} &= (-X^*_x + X^{*2} + 2H)_x. \end{aligned}$$

This may be found from the AKNS system (3.2.2, 3.2.3) via

$$H = qr, \tag{3.5.10}$$

$$X^* = -(\log[r])_x. \tag{3.5.11}$$

On making the ansatz

$$H = (\log[\tau])_{xx}, \tag{3.5.12}$$

$$HX^* = \frac{1}{2}\left((\log[\tau])_{xt_2} - (\log[\tau])_{xxx}\right), \tag{3.5.13}$$

the Kaup-Broer system leads to the following trilinear equation for $\tau$:

$$\begin{vmatrix} p_0^+p_0^-(\tau) & p_0^+p_1^-(\tau) & p_0^+p_2^-(\tau) \\ p_1^+p_0^-(\tau) & p_1^+p_1^-(\tau) & p_1^+p_2^-(\tau) \\ p_2^+p_0^-(\tau) & p_2^+p_1^-(\tau) & p_2^+p_2^-(\tau) \end{vmatrix} = 0. \tag{3.5.14}$$

In the above we have used the operators

$$p_j^{\pm} = p_j(\pm\tilde{\partial}), \qquad \tilde{\partial} = (\partial_x, \frac{1}{2}\partial_{t_2}, \frac{1}{3}\partial_{t_3}, ...),$$

which are written in terms of the Schur polynomials $p_j(\underline{t})$, defined in the previous section. There exist numerous generalizations of the determinant (3.5.14); they correspond to multilinear equations for the higher flows of the AKNS hierarchy, or for the flows of the other $k$-constraint hierarchies arising as reductions of KP.

Now we can make the connection with the trilinear (3.5.7) derived from NLS. We have seen we get NLS from the AKNS system by setting $t_2 = it$, $q = \psi$, $r = -\overline{\psi}$. Hence, if we put $\psi = w^{\frac{1}{2}} \exp(i\chi)$ as before, the Kaup-Broer variables defined by (3.5.10, 3.5.11) become

$$\begin{aligned} H &= -w, \\ X^* &= -\frac{1}{2}(\log[w])_x + i\chi_x. \end{aligned}$$

When we also identify $\tau = f$, then the ansatz (3.5.12, 3.5.13) gives

$$w = -\Lambda_{xx}, \tag{3.5.15}$$

$$\chi_x = -\frac{\Lambda_{xt}}{2\Lambda_{xx}}, \tag{3.5.16}$$

agreeing with our previous formulae. Finally it is a simple matter to check that for imaginary time ($t_2 = it$) the determinant (3.5.14) is just the sum of the two determinants in (3.5.7).

The solutions of trilinear equations have been studied by Satsuma and others (see [77] and references therein), and take the form of double Wronskians,

$$\tau = \begin{vmatrix} \Delta & \Delta_x & \cdots & \Delta_{(N-1)x} \\ \Delta_x & \Delta_{xx} & \cdots & \Delta_{Nx} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{(N-1)x} & \Delta_{Nx} & \cdots & \Delta_{2(N-1)x} \end{vmatrix}, \tag{3.5.17}$$

with

$$\Delta_{t_2} = \Delta_{xx}.$$

The polynomial tau-functions found in the previous section are particular examples of these determinants. It is a result originally due to Sato that all Wronskians of Schur polynomials (or in other words, Schur functions corresponding to arbitrary Young diagrams) satisfy the bilinear equations of the KP hierarchy [126]. Hence all of the tau-functions $f_n$, $g_n$, $h_n$ found in the previous section satisfy KP; we shall come back to this point in the next section.

We have seen that the trilinear form provides an alternative way to consider the solutions of NLS, in terms of a single tau-function. For a solution $f$ of (3.5.7) to correspond to a bona-fide solution of NLS, there is the additional requirement that $-(\log[f])_{xx}$ should be non-negative definite, since (from (3.5.15)) this gives

the modulus squared of the amplitude $\psi$. Also, to find the argument $\chi$ of $\psi$ it is necessary to perform the integral of the right hand side of (3.5.16), which only determines the phase up to a function of $t$. For example, the Galilean-boosted one-pole solution has

$$f = x - ct,$$

giving

$$\psi = \frac{\exp[i(\frac{cx}{2} + \tilde{\chi}(t))]}{x - ct},$$

for some function $\tilde{\chi}(t)$. To determine this function of $t$ it is necessary to substitute either directly into (3.1.11) or into the bilinear form (3.3.1). In general, integrating the quantity $\frac{\Lambda_{xt}}{\Lambda_{xx}}$ explicitly may not be easy, so it is useful to have the bilinear methods as well. In the next section we look at the rational solutions of NLS, and use both the trilinear form and the bilinear form to derive equations and constraints on the motion of the poles.

## 3.6 Dynamics of the Poles and Zeros of Rational Solutions

In Section 3.4 we constructed a sequence of singular rational solutions to NLS, with the amplitude in the form of a ratio of two polynomial tau-functions,

$$\psi = \frac{g}{f}.$$

The poles of $\psi$ are just the zeros of $f$, so rather than considering the bilinear equations (3.3.1,3.3.2) which involve both $f$ and $g$, the motion of the poles may be studied directly from the trilinear equation (3.5.7) which involves $f$ alone. Hence we take $f$ in the form of a polynomial of degree $N$,

$$f = \prod_{j=1}^{N} \Phi_j(x,t), \tag{3.6.1}$$

with $\Phi_j = x - x_j(t)$. Then the modulus squared of $\psi$ is given by

$$|\psi|^2 = -(\log[f])_{xx} = \sum_{j=1}^{N} \frac{1}{(x - x_j(t))^2}.$$

This is the same as the form of the rational solutions to KP, under the constraint $u = -|\psi|^2$ which we described in the previous section. The poles of $\psi$ are at $x = x_j(t)$. On expanding a solution to the NLS equation about a pole (which we allow to be complex, in contrast to the real singular manifold function $\phi$ of Section 4.2) the leading order behaviour shows that the pole positions $x_j(t)$ must either be real or in complex conjugate pairs.

### 3.6.1 Calogero-Moser via the Trilinear Form

Substituting the expression (3.6.1) for $f$ into the trilinear equation (3.5.7) gives a polynomial $P$ of degree $3(N-1)$ in $x$. The equations for the pole motion may be found from the requirement that $P$, $P_x$ and $P_{xx}$ should vanish at $x = x_j$, for $j = 1, ..., N$. In fact, since $P$ is of degree $3(N-1)$, showing that $P$ and its first and second derivatives vanish at any $N-1$ of the $x_j$ is sufficient to show that $P$ is identically zero. This would suggest that the equations found for, say, the first $N-1$ of the $x_j$ must imply the corresponding equations for $x_N$. Instead of using the trilinear equation (3.5.7) directly, we find that the calculations are made easier by considering the equivalent equation (3.5.6) and expanding about each of the (possibly complex) singular manifolds $\Phi_j$. The highest order singularity is a triple pole, and their are also simple and double pole terms.

After some calculation, the equation (3.5.6) (which is just the trilinear (3.5.7) divided through by $f^3$) yields

$$\sum_{j=1}^{N} A_j(t)\Phi_j^{-3} + B_j(t)\Phi_j^{-2} + C_j(t)\Phi_j^{-1} = 0,$$

and for this to hold we require that the coefficients at each order in $\Phi_j$ should vanish (for $j = 1, ..., N$ ). Explicitly we find

$$A_j = \ddot{x}_j - 8\sum_{k\neq j}\Delta_{jk}^{-3}, \tag{3.6.2}$$

$$\begin{aligned} B_j &= \sum_{k\neq j}\left(\ddot{x}_k\Delta_{jk}^{-1} + (\dot{x}_j - \dot{x}_k)^2\Delta_{jk}^{-2} + 24\Delta_{jk}^{-4}\right) \\ &\quad +12\sum_{k<l}{}' \left(2\Delta_{kl}^{-3}(\Delta_{jk}^{-1} - \Delta_{jl}^{-1}) - \Delta_{kl}^{-2}(\Delta_{jk}^{-2} + \Delta_{jl}^{-2})\right), \end{aligned} \tag{3.6.3}$$

$$\begin{aligned} C_j &= \sum_{k\neq j}\left((\ddot{x}_j - \ddot{x}_k)\Delta_{jk}^{-2} - 2(\dot{x}_j - \dot{x}_k)^2\Delta_{jk}^{-3} - 48\Delta_{jk}^{-5}\right) \\ &\quad +24\sum_{k<l}{}' \left(\Delta_{kl}^{-2}(\Delta_{jk}^{-3} + \Delta_{jl}^{-3}) - \Delta_{kl}^{-3}(\Delta_{jk}^{-2} - \Delta_{jl}^{-2})\right), \end{aligned} \tag{3.6.4}$$

where the dot stands for $\frac{d}{dt}$, $\Delta_{jk} = x_j - x_k$, and $\sum_{k<l}'$ denotes a sum over $k < l$ with $k \neq j \neq l$.

We see that the vanishing of the leading order terms $A_j$ given by (3.6.2) just yields the equations of the ordinary Calogero-Moser system,

$$\ddot{x}_j = 8\sum_{k\neq j}(x_j - x_k)^{-3}. \tag{3.6.5}$$

This differs in sign from the equation (3.1.7) for pole motion of solutions to the KP equation, but this is because we have set $t_2 = it$ in making the reduction

to NLS. The vanishing of the quantities $B_j$, $C_j$ given in (3.6.3,3.6.4) take the form of constraints on the motion, but do not appear to come from Hamiltonian reduction (as is the case for KdV pole motion discussed in the Introduction).

It is interesting to see how the constraints are satisfied by the 4-pole solution, for which we have the tau-function

$$f = x^4 + \tau_3 x - 12t^2.$$

This has real coefficients, and the coefficients of $x^3$ and $x^2$ vanish, it is clear that the roots must satisfy

$$\sum_{j=1}^{4} x_j = 0 = \sum_{j=1}^{4} x_j^2.$$

Hence it has either two complex conjugate pairs of roots or one complex conjugate pair and two real roots. The condition for two distinct real roots is

$$16t^2 + \left(\frac{\tau_3}{4}\right)^{\frac{1}{3}} > 0.$$

In particular this condition holds for the 4-pole similarity solution (when $\tau_3 = 0$, $t \neq 0$), where the roots are

$$x_j = 3^{\frac{1}{4}} \exp[(j-1)\pi/2](2|t|)^{\frac{1}{2}}.$$

These lie at the vertices of a square in the complex plane, and move towards the origin until they coalesce at $t = 0$, and then repeat their motion in reverse. This pattern appears to be repeated for the 9-pole and 16-pole similarity solutions, with the poles lying at the vertices of squares of different sizes (and an extra pole at the origin in the case of the 9-pole). It would be interesting to make a further study of the patterns of the roots of these solutions (as has been done in the KdV case [11]), but we have not pursued this.

In order to gain a better understanding of the constraints on the $x_j$, it is instructive to use the bilinear equations once more. We are thus able to demonstrate that not only the poles but also the zeros of rational solutions to NLS satisfy Calogero-Moser equations, and there is an interesting coupling between the motion of the poles and the zeros, as well as some further constraints.

### 3.6.2 Coupled Equations for Poles and Zeros

In order to consider rational solutions we start with the ansatz that the tau-functions $g$ and $f$ are coprime polynomials in $x$, and we may assume that $f$ is monic. On substituting into (3.3.2), and comparing coefficients at leading order, it is at once apparent that $g$ must be of degree one less than $f$. The leading

order coefficient of $g$ is also essentially fixed (up to multiplication by a constant of modulus one). More precisely we have

$$f = \prod_{j=1}^{N}(x - x_j(t)),$$

as before, and

$$g = \sqrt{N}\prod_{J=1}^{N-1}(x - y_J(t)).$$

As expected, (up to inessential minus signs in the definition of $g$) this is precisely the form of the polynomial tau-functions that we found in Section 3.4 by applying the Crum transformation. In fact we only found rational solutions with $N = n^2$, for integer $n = 1, 2, \ldots$ labelling the sequence of solutions.

With the top coefficients fixed, we substitute $f$ and $g$ as above into (3.3.2) and find

$$N\prod_{J=1}^{N-1}(x - y_J)(x - \overline{y_J}) = \sum_{j=1}^{N}\prod_{k\neq j}(x - x_k)^2. \tag{3.6.6}$$

Hence we may regard the $y_J$ (and their conjugates) as being determined by symmetric functions of the $x_j$. We may also derive equations for the $t$ evolution of the zeros $y_J$ and poles $x_j$, by substituting into (3.3.2). If we set

$$M = \log[g],$$

and $\Lambda = \log[f]$ as before, then (3.3.2) is equivalent to

$$i(M_t - \Lambda_t) + M_{xx} + \Lambda_{xx} + (M_x - \Lambda_x)^2 = 0. \tag{3.6.7}$$

Putting the polynomial ansatz into (3.6.7) and calculating the residues at each simple pole in the resulting expression gives, for each $j$ and $J$,

$$i\dot{x}_j = 2\left(\sum_{k\neq j}(x_j - x_k)^{-1} - \sum_{J}(x_j - y_J)^{-1}\right), \tag{3.6.8}$$

$$i\dot{y}_J = 2\left(-\sum_{K\neq J}(y_J - y_K)^{-1} + \sum_{j}(y_J - x_j)^{-1}\right). \tag{3.6.9}$$

Differentiating (3.6.8) leads to the Calogero-Moser equations (3.6.5), and similarly the $y_J$ must satisfy

$$\ddot{y}_J = 8\sum_{K\neq J}(y_J - y_K)^{-3}.$$

We remark that the calculation of these second-order equations is identical to that for rational solutions of the Benjamin-Ono equation [39], where the poles

$x_j$ evolve according to the equation (3.6.8) with the $y_J$ replaced by the complex conjugates $\overline{x_j}$.

We have shown that both the poles and the zeros of rational solutions to NLS evolve according to Calogero-Moser equations, and that these poles and zeros are coupled by the differential constraints (3.6.8,3.6.9), as well as the further constraints (3.6.6). As mentioned earlier, the bilinear equations (3.3.1,3.3.2) imply the trilinear equation (3.5.7). Essentially this is because to derive the trilinear equation requires that the modulus and argument of $\psi$ are related to the tau-function $f$ by (3.5.15,3.5.16). The relation (3.5.15) is equivalent to (3.3.2), while (3.5.16) is equivalent to the bilinear equation (3.3.4) for $\gamma = 0$. So it is apparent that the vanishing of the quantities $B_j$, $C_j$, which were found via the trilinear formalism, must be consequences of (3.6.8,3.6.9) and (3.6.6), although we have not checked this directly. Thus in some sense the bilinear approach is more fundamental, and certainly the form of the constraints arising in this way is more tractable.

Given polynomial tau-functions in the form of Wronskians of Schur polynomials, we know that they satisfy the bilinear equations of the KP hierarchy [126], and hence by Shiota's result [142] their zeros must evolve according to the equations of the Calogero-Moser hierarchy with respect to the times $t_j$. The tau-functions $f_n$, $g_n$, $h_n$ found in Section 3.4 all have this Wronskian form, but they also satisfy certain constraints corresponding to the reduction from KP. We have not tried to solve the NLS-constrained Calogero-Moser system in general, but it seems most likely that the solutions generated by the NCT method in Section 3.4 are the only ones allowed. In fact we are able to outline an argument for this when we consider the similarity solutions in the next section.

## 3.7 Similarity Solutions and other Singular Solutions

In this section we give a brief discussion of some other singular solutions of NLS, to illustrate the great variety of these. Our first example concerns the scaling similarity solutions, which were studied in detail by Boiti and Pempinelli [29]. The ODE for the similarity solutions turns out to be in one-one correspondence with a particular case of the fourth Painlevé transcendent (PIV). PIV has a Bäcklund transformation, which in this case may be used to generate two sequences of rational solutions. In this way, two sequences of similarity solutions for NLS are generated. The first just corresponds to (scaled versions of) the sequence of rational solutions obtained in Section 3.4. The second sequence does not provide

solutions to NLS which are strictly rational, but it is simple to separate out the non-rational part and show that the motion of the poles is governed by (rescaled) Calogero-Moser equations. As our other example, we consider the 2-soliton solution of NLS, and show how a certain limiting process leads to a solution which we refer to as a "singular 2-lump".

### 3.7.1 Scaling Similarity Solutions

Boiti and Pempinelli [29] showed that scaling similarity solutions of the NLS equation (3.1.11) are of the form

$$\psi = |\psi| \exp[i\chi],$$

where the modulus and argument of $\psi$ are both given in terms of a function $Y(z)$,

$$|\psi|^2 = \frac{1}{2t} Y'(z), \tag{3.7.1}$$

$$\chi'(z) = z + \frac{Y(z)}{Y'(z)}. \tag{3.7.2}$$

We have the convention that a dash denotes differentiation with respect to the similarity variable $z$, where

$$z = x\theta(t), \qquad \theta(t) = \frac{1}{2} t^{-\frac{1}{2}}.$$

The coupled equations for the modulus and argument of $\psi$ are equivalent to a second-order ODE for $Y$, which is in one-one correspondence with the equation

$$WW'' = \frac{1}{2}(W')^2 - 6W^4 + 8zW^3 - 2z^2W^2 - \frac{1}{2}(\mu - 1)^2, \tag{3.7.3}$$

with $\mu$ a constant. After suitable rescaling of $W$ and $z$, (3.7.3) is just a particular case of PIV. Note that PIV has two parameters, while in the above there is the single parameter $\mu$. $Y$ is given in terms of $W$ by

$$Y = \frac{1}{2} W(W - z)^2 + \frac{1}{8W}\left((W')^2 - 2W' - \mu^2 + 1\right). \tag{3.7.4}$$

For further details of this correspondence, we refer the reader to the original source. Henceforth we shall translate the results of [29] into a form more compatible with our previous notation.

The first thing to observe is that the substitutions (3.7.1,3.7.2) arise naturally in our trilinear approach of Section 3.5, if we require that (possibly after scaling by suitable powers of $\theta$) the tau-function $f$ should only depend on the similarity variable $z$. Then we may assume that

$$\Lambda(x,t) = \lambda(z(x,t)) + m \log[\theta(t)], \tag{3.7.5}$$

where $\Lambda(x,t) = \log[f(x,t)]$, and we have included potential powers of $\theta$. If we now set

$$Y(z) = -\frac{1}{2}\lambda'(z),$$

then (3.7.1) and (3.7.2) follow immediately from (3.5.15) and (3.5.16) respectively. The second-order ODE for $Y$, as derived by Boiti and Pempinelli, is equivalent to the following ODE for $\lambda$:

$$E := (\lambda''')^2 + 4\left((z\lambda'' - \lambda')^2 + (\lambda'')^3 + \mu^2\lambda''\right) = 0. \tag{3.7.6}$$

The trilinear form of NLS requires that $\Lambda(x,t)$, as given by (3.7.5), should satisfy (3.5.6). Using this we obtain another equation for $\lambda$,

$$\tilde{E} := \lambda''\lambda^{(iv)} - (\lambda''')^2 + 2(\lambda'')^3 - 4(\lambda')^2 + 4z\lambda'\lambda'' + 8m\lambda'' = 0.$$

In fact, provided that we identify

$$m = -\frac{\mu^2}{4},$$

this turns out to be a direct consequence of (3.7.6), for we have

$$\tilde{E} = \frac{(\lambda'')^3}{2\lambda'''}\frac{d}{dz}\left[\frac{E}{(\lambda'')^2}\right].$$

It was shown in [29] that the ABT for NLS naturally leads to a Bäcklund transformation for the similarity solutions. In terms of solutions to (3.7.6), this is given by

$$\tilde{\lambda}' = \lambda' + \frac{(1 \pm \mu)\left((\lambda'')^2 + \mu^2\right)}{\mp\frac{\mu}{2}\lambda''' + \lambda'\lambda'' + \mu^2 z}, \tag{3.7.7}$$

where $\tilde{\lambda}$ is a solution to (3.7.6) for $\mu$ replaced by $\mu \pm 2$. This is equivalent to the well-known Bäcklund transformation for PIV, using the one-one correspondence mentioned above. For the particular case of PIV corresponding to (3.7.3), there are two families of rational solutions [74, 111]. The first family corresponds to even integer values of $\mu$, and may be generated by applying the Bäcklund transformation to the parent solution

$$W = z,$$

which is the rational solution to (3.7.3) for $\mu = 2$.

Making use of the one-one correspondence, the rational solution to (3.7.6) for $\mu = 2$ is found from (3.7.4) to be

$$\lambda' = (\log[z])'.$$

After rescaling, this leads to the tau-function

$$f_1 = x.$$

We can apply the Bäcklund transformation (3.7.7) in two directions, either increasing or decreasing $\mu$ by two at each stage. The solution for $\mu = 0$ is just

$$\lambda' = 0,$$

with the tau-function

$$f_0 = 1.$$

The equation (3.7.6) is invariant under $\mu \to -\mu$, and so for this sequence of even integers nothing new is gained by considering the negative values of $\mu$ separately.

It is apparent that repeated application of (3.7.7) leads to a sequence of rational solutions for $\mu = 2n$, in the form

$$\lambda(z) = \log\left[[\theta(t)]^{n^2} f_n(x,t)\right],$$

where $f_n$ is a polyomial of degree $n^2$ in $x$, which scales correctly so that $\lambda$ depends on $x$ and $t$ through the combination $z$ alone. In particular we find

$$f_2 = x^4 - 12t^2, \qquad f_3 = x^9 - 72x^5t^2 - 2160xt^4,$$

just as in Table 3.1. These are the tau-functions of the rational similarity solutions to NLS, which are special cases of the rational solutions obtained via the NCT in Section 3.4. Since only these similarity solutions can lead to rational solutions of NLS (essentially by the uniqueness of the rational solutions to PIV), this implies that the rational solutions found using the NCT should be the only such solutions. A further consequence of this would then be that the most general solutions to the constrained Calogero-Moser systems of Section 3.6 are given by the poles and zeros of the rational solutions found in Section 3.4.

There is a second family of rational solutions to PIV, which comes from applying the Bäcklund transformation to the parent solution

$$W = \frac{1}{3}z,$$

corresponding to $\mu = \frac{2}{3}$. By using (3.7.4), we find

$$\lambda' = \left(-\frac{z^4}{27}\right)',$$

with the associated tau-function being given by

$$f_{\frac{1}{3}} = \theta^{-\frac{1}{9}} \exp[-z^4/27].$$

The solution to NLS which this yields is not purely rational, but is given by

$$\psi = \frac{2}{3}\theta^2 x \exp\left[\frac{2}{3}i\theta^2 x^2\right].$$

This is clearly not a singular solution. However, the transformation (3.7.7) may be applied in both directions to give a sequence of solutions to (3.7.6) for $\mu = 2(n+\frac{1}{3})$, for all integers $n$. We denote the corresponding tau-functions by $f_{n+\frac{1}{3}}$, and find that they are of the form

$$f_{n+\frac{1}{3}} = \theta^{M(n)} \exp[-z^4/27] \prod_{j=1}^{N(n)} (x - x_j(t)), \tag{3.7.8}$$

where $N(n) = n(3n+2)$, $M(n) = N(n) - (n+\frac{1}{3})^2$. These tau-functions clearly give singular solutions to NLS, with $N(n)$ poles. After scaling away the powers of $\theta$ and the exponential piece, these tau-functions may simply be characterized by polynomials $P_{n+\frac{1}{3}}(z)$. We list a few of these below:

$$P_{-\frac{2}{3}} = z, \qquad P_{\frac{1}{3}} = 1, \qquad P_{\frac{4}{3}} = z^5 + \frac{45}{4}z,$$

$$P_{\frac{7}{3}} = z^{16} + 135z^{12} + \frac{22275}{8}z^8 + \frac{1002375}{16}z^4 - \frac{9021375}{256}.$$

By the scaling property of the similarity solutions, we know that each pole position $x_j$ appearing in (3.7.8) may be written as

$$x_j = x_j^{(0)} t^{\frac{1}{2}},$$

for some constant $x_j^{(0)}$. So by substituting (3.7.8) into (3.5.6) and expanding around each pole, we find that the leading order term gives

$$\ddot{x}_j = 72 \sum_{k \neq j} (x_j - x_k)^{-3}.$$

Thus the motion of poles for these singular solutions is also governed by rescaled Calogero-Moser equations. There are also constraints on the poles, coming from the other terms in the pole expansion. We have not explored these constraints any further, largely because this method would not apply to possible non-similarity generalizations of the solutions (3.7.8). Actually we would expect that NLS should admit solutions with tau-functions of the slightly more general form

$$f = \exp[p(x, \underline{t})] \prod_{j=1}^{N} (x - x_j(\underline{t})),$$

where $p$ is a quartic polynomial in $x$, since Veselov has shown [150] that the KP equation has non-decreasing rational solutions which may be written

$$u = ax^2 + bx + x - \sum_{j=1}^{N}(x - x_j)^{-2}.$$

The $x_j$ satisfy equations of Calogero-Moser type, and these involve $a$, $b$ and $c$, which are functions of the times satisfying certain constraints. It would be interesting to see how these solutions would need to be restricted in order to satisfy NLS.

### 3.7.2 Two-Lump Solution

Having considered some of the similarity solutions to NLS, we now provide an example of a singular solution of a rather different character. It has been shown [2] that by applying a certain limiting procedure to the soliton solutions of KdV, the rational solutions may be obtained. Similarly, the same sort of limits for solitons of other equations, such as KP, lead to interesting solutions known as "lumps". We proceed to apply this method to the 2-soliton of NLS, and obtain a singular 2-lump solution. As far as we are aware, the lump solutions to NLS have not been studied.

The 2-soliton solution to (3.1.11) may (after making a slight adaptation of Hirota's formulae [80]) be given by the tau-functions,

$$g = 2\left(\begin{array}{c}\sigma_1 \exp[\eta_1] + \sigma_2 \exp[\eta_2] - \left(\frac{\alpha_1-\alpha_2}{\overline{\alpha}_1+\alpha_2}\right)^2 \sigma_2 \exp[\eta_1 + \overline{\eta}_1 + \eta_2] \\ - \left(\frac{\alpha_1-\alpha_2}{\alpha_1+\overline{\alpha}_2}\right)^2 \sigma_1 \exp[\eta_1 + \eta_2 + \overline{\eta}_2]\end{array}\right) \quad (3.7.9)$$

$$\begin{aligned} f &= 1 - \exp[\eta_1 + \overline{\eta}_1] - \exp[\eta_2 + \overline{\eta}_2] \\ &\quad - \left(4\frac{\sigma_1\sigma_2}{(\alpha_1 + \overline{\alpha}_2)^2}\exp[\eta_1 + \overline{\eta}_2] + c.c.\right) \\ &\quad + \left|\frac{\alpha_1 - \alpha_2}{\overline{\alpha}_1 + \alpha_2}\right|^4 \exp[\eta_1 + \overline{\eta}_1 + \eta_2 + \overline{\eta}_2], \end{aligned} \quad (3.7.10)$$

where

$$\eta_j = \alpha_j(x + i\alpha_j t) + \eta_j^{(0)}, \qquad \alpha_j = \sigma_j + i\rho_j.$$

The real parameters $\sigma_j$, $\rho_j$ correspond respectively to the amplitude and speed of the $j$th soliton ($j = 1, 2$). Each $\eta_j^{(0)}$ is an arbitrary complex phase, which we will henceforth set to zero.

To get the 2-lump solution, we must take the limit

$$\sigma_j \to 0,$$

for $j = 1, 2$. We regard $\sigma_1$ and $\sigma_2$ as being of the same order of magnitude ($\sigma$, say). It is convenient to introduce the notation

$$a_j = x - 2\rho_j t, \qquad b_j = \rho_j(x - \rho_j t).$$

Expanding out the terms in (3.7.9), we find e.g.

$$\begin{aligned}
\exp[\eta_j] &= \exp[ib_j]\{1 + \sigma_j a_j + O(\sigma^2)\}, \\
\exp[\eta_1 + \overline{\eta}_1 + \eta_2] &= \exp[ib_2]\{1 + 2\sigma_1 a_1 + \sigma_2 a_2 + O(\sigma^2)\}, \\
\left(\frac{\alpha_1 - \alpha_2}{\overline{\alpha}_1 + \alpha_2}\right)^2 &= 1 - \frac{4i\sigma_1}{\rho_1 - \rho_2} + O(\sigma^2).
\end{aligned}$$

This leads to

$$g = 4\sigma_1\sigma_2\left(-\{a_2 + 2i(\rho_1 - \rho_2)^{-1}\}\exp[ib_1] - \{a_1 - 2i(\rho_1 - \rho_2)^{-1}\}\exp[ib_2]\right) + O(\sigma^3).$$

Similarly, expanding the terms in (3.7.10) it is seen that the zero order and first order terms all cancel, as well as the $\sigma_j^2$ terms, and then

$$f = 8\sigma_1\sigma_2\left(a_1 a_2 - 2(\rho_1 - \rho_2)^{-2}\sin^2[\tfrac{1}{2}(b_1 - b_2)]\right) + O(\sigma^3).$$

Thus if both $g$ and $f$ are rescaled by the factor $8\sigma_1\sigma_2$, then in the limit $\sigma_j \to 0$ we obtain the singular 2-lump in the form

$$\begin{aligned}
g &= -\left(\frac{1}{2}a_2 + i(\rho_1 - \rho_2)^{-1}\right)\exp[ib_1] - \left(\frac{1}{2}a_1 - i(\rho_1 - \rho_2)^{-1}\right)\exp[ib_2], \\
f &= a_1 a_2 - 2(\rho_1 - \rho_2)^{-2}\sin^2[\tfrac{1}{2}(b_1 - b_2)]. \qquad (3.7.11)
\end{aligned}$$

It is interesting to consider the pole dynamics of the 2-lump solution. To do so, we need only consider $f$ given by (3.7.11). The description is much simplified by removing the "centre of mass" motion with a Galilean boost,

$$x \to x + (\rho_1 + \rho_2)t,$$

and using the coordinates

$$X = (\rho_1 - \rho_2)x, \qquad T = (\rho_1 - \rho_2)^2 t.$$

After a simple rescaling, $f$ may then be written as

$$f(X, T) = X^2 - T^2 - 2\sin^2[X/2].$$

Thus the pole positions are the two solutions $X_j(T)$ $(j = 1, 2)$ of

$$f(X(T), T) = 0.$$

By using the Galilean boost, we have arranged the poles so that $X_2 = -X_1$. The scattering image of the two poles is asymptotic to the lines $X = \pm T$, and is shown in Figure 3.1. Notice that the poles coalesce at $X(0) = 0$, when $g$ also vanishes. Ideally it would be desirable to have a formula for the $N$-lump, and obtain a dynamical description of the pole motion. This could provide the basis for a more detailed study.

## 3.8 Conclusions

We have constructed a sequence of rational solutions of NLS, and shown that they correspond to constrained Calogero-Moser systems. The construction is a direct analogue of the Crum transformation used by Adler and Moser [8] to produce rational solutions of KdV. At the same time we have extended the work [83] of Hirota and Nakamura on the connection between some explode-decay solutions of the classical Boussinesq system and rational similarity solutions of NLS. It is worth noting that such rational solutions are mentioned (for the AKNS hierarchy) in [120], but an explicit formula is given only for the one-pole solution.

Throughout the chapter we have found that direct methods, especially Bäcklund transformations and Hirota's bilinear formalism, provide the simplest means to generate solutions. We have also tried to indicate how these methods are intimately related to Painlevé analysis (via the singular manifold method) and the inverse scattering formalism. At the same time, we have seen that NLS (or AKNS) is naturally viewed as a reduction of the KP hierarchy, and essentially the solutions may be encoded into a single tau-function satisfying a trilinear equation. Although these methods have provided a great deal of information about some of the singular solutions of NLS, there remain many unanswered questions.

It would be interesting to make a more detailed study of the system (3.6.8,3.6.9) with the constraint (3.6.6), which correspond to a coupling of the Calogero-Moser systems for the zeros and poles of the rational solutions. The configurations of the poles could be studied along the lines of [11]. At the same time it should be possible to consider the constraints of the whole Calogero-Moser hierarchy corresponding to the higher flows (3.4.20), but at present we lack a direct proof that the rational solutions constructed in Section 3.4 satisfy all these flows. We have also attempted a study of elliptic solutions to NLS corresponding to tau-functions of the form (3.1.10). These would generalize the one-pole elliptic solution (the

general stationary solution [29]) given by

$$\psi = 2\exp[-\zeta(\kappa)x + 3i\wp(\kappa)t]\frac{\sigma(x+\kappa)}{\sigma(x)\sigma(\kappa)}, \tag{3.8.1}$$

where $\zeta$ is the Weierstrass zeta-function and $\kappa$ is an imaginary constant. The $\wp$-function satisfies the differential equation

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3,$$

where the constants $g_2$, $g_3$ (the Eisenstein series) are taken to be real so that $\wp(\kappa)$ is real and $\zeta(\kappa)$ is imaginary (see [153] for an introduction to elliptic functions). In the trilinear formalism of Section 3.5, the solution (3.8.1) can be encoded into the single tau-function $f$ satisfying (3.5.7),

$$f = \sigma(x)\exp\left[\frac{1}{2}\wp(\kappa)x^2 + \alpha_0 tx + \beta_0 t^2\right],$$

where the real constants $\alpha_0$ and $\beta_0$ are given by

$$\alpha_0 = -i\wp'(\kappa),$$

$$\beta_0 = \frac{1}{4}g_2 - 3\wp(\kappa)^2.$$

$N$-pole elliptic solutions of NLS should have pole motion corresponding to constrained elliptic Calogero-Moser systems, and could presumably be derived as suitable degenerations of the hyperelliptic solutions in [134]. Preliminary results suggest that there are some problems in using the ansatz (3.1.10) for the tau-functions of such solutions.

The singular solutions of nonlinear PDEs have received much attention recently. In particular, the Darboux transformation approach of Matveev [116] has been used to generate new classes of solutions of KdV which are written in Wronskian form, some of which are singular and have interesting pole dynamics (notably the negaton solutions [98]). The dynamics of the $N$-lump solutions of NLS, generalizing the 2-lump solution presented in Section 3.7, are worthy of investigation. Also the second sequence of scaling similarity solutions considered in that section should be particular solutions to another type of constrained Calogero-Moser system, and this could be explored further. We remark that the Darboux transformation method applies to most of the known exact solutions to integrable evolution equations, but we have not been able to use this to generate our sequence of rational solutions of NLS in a straightforward manner.

The results of this chapter indicate that NLS has a rich variety of singular solutions, and there is still much work that could be done to gain a better understanding of their pole dynamics.

**Note.** After this thesis was submitted, we were made aware of the work [139] of Sachs, where pole motions for rational solutions the classical Boussinesq system are considered, and these are related to solutions of AKNS. As yet we have not studied this work, but we intend to consider how it relates to our results in a forthcoming article.

## 3.9 Appendix A

Here we start from the singular manifold equations of NLS in the form

$$
\begin{aligned}
|u_0|^2 - \phi_x^2 &= 0, && (3.9.1)\\
\phi_{xx} + \overline{u_0}\tilde{\psi} + u_0\overline{\tilde{\psi}} &= 0, && (3.9.2)\\
i\phi_t + \phi_x(\log[u_0/\overline{u_0}])_x - u_0\overline{\tilde{\psi}} + \overline{u_0}\tilde{\psi} &= 0, && (3.9.3)\\
iu_{0,t} + u_{0,xx} - 4u_0|\tilde{\psi}|^2 - 2\overline{u_0}\tilde{\psi}^2 &= 0, && (3.9.4)
\end{aligned}
$$

and show that they have the simpler consequences (3.2.13,3.2.14). Using (3.9.1) and (3.9.2) we find

$$(\log[u_0] + \log[\overline{u_0}])_{xx} = -2\left(\frac{(\overline{u_0}\tilde{\psi} + u_0\overline{\tilde{\psi}})_x}{\phi_x} + \frac{(\overline{u_0}\tilde{\psi} + u_0\overline{\tilde{\psi}})^2}{\phi_x^2}\right). \qquad (3.9.5)$$

Differentiating (3.9.3) with respect to $x$ gives

$$(\log[u_0/\overline{u_0}])_{xx} = \phi_x^{-1}(-i\phi_{xt} + (u_0\overline{\tilde{\psi}} - \overline{u_0}\tilde{\psi})_x) - \phi_{xx}\phi_x^{-2}(-i\phi_t + u_0\overline{\tilde{\psi}} - \overline{u_0}\tilde{\psi}). \qquad (3.9.6)$$

Now differentiating (3.9.1) with respect to $t$ we have

$$u_{0,t}\overline{u_0} + u_0\overline{u_{0,t}} = 2\phi_x\phi_{xt},$$

and then from (3.9.1) and (3.9.4) with its complex conjugate we obtain

$$\phi_{xt} = i(\phi_{xx}(\log[u_0/\overline{u_0}])_x + \frac{1}{2}\phi_x(\log[u_0/\overline{u_0}])_{xx} - \phi_x^{-1}(u_0^2\tilde{\psi}^2 - u_0^2\overline{\tilde{\psi}}^2).$$

Now in (3.9.6) we may substitute for $\phi_{xt}$ as above and for $\phi_t$ from (3.9.3), and after some rearrangement we find

$$(\log[u_0] - \log[\overline{u_0}])_{xx} = 2\left(\frac{(\overline{u_0}\tilde{\psi} - u_0\overline{\tilde{\psi}})_x}{\phi_x} - \frac{(\overline{u_0}^2\tilde{\psi}^2 - u_0^2\overline{\tilde{\psi}}^2)}{\phi_x^2}\right). \qquad (3.9.7)$$

Adding (3.9.5) and (3.9.7), and making use of (3.9.2) once more, yields

$$(\log[u_0])_{xx} = -2\left(\frac{\overline{u_0}\tilde{\psi}}{\phi_x}\right)_x,$$

which we can integrate immediately to give

$$(\log[u_0])_x + 2\left(\frac{\overline{u_0}\tilde{\psi}}{\phi_x}\right) + 2ik = 0, \tag{3.9.8}$$

for $k$ a constant. If we use (3.9.1) and (3.9.2), then we find immediately that

$$(\log[u_0] + \log[u_0])_x = -2\frac{(\overline{u_0}\tilde{\psi} + u_0\overline{\tilde{\psi}})}{\phi_x},$$

and on comparing this with (3.9.8) plus its complex conjugate it is apparent that $k$ must be real. We rearrange (3.9.8) after substituting $\overline{u_0} = u_0^{-1}\phi_x^2$ (from (3.9.1)), and get

$$u_{0,x} = -2iku_0 - 2\phi_x\tilde{\psi}.$$

The above is just our previous equation (3.2.13), and it provides a new expression for $u_{0,xx}$, which when put into (3.9.4) gives

$$iu_{0,t} = u_0(4k^2 + 2|\tilde{\psi}|^2) + \phi_x(-4ik\tilde{\psi} + 2\tilde{\psi}_x).$$

The latter is the equation (3.2.14), as required. □

## 3.10 Appendix B

In this appendix we consider NLS in the bilinear form (3.3.2,3.3.1), and prove that it has a bilinear auto-Bäcklund transformation (ABT) given by

$$(D_x - ic/2)(g \cdot \tilde{f} - \tilde{g} \cdot f) = 0, \tag{3.10.1}$$

$$(iD_t + D_x^2 + \sigma^2)(\tilde{g} \cdot f + g \cdot \tilde{f}) = 0, \tag{3.10.2}$$

$$(iD_t + icD_x)f \cdot \tilde{f} = \overline{g}\tilde{g} - g\overline{\tilde{g}}, \tag{3.10.3}$$

$$D_x f \cdot \tilde{f} = \sqrt{|g\tilde{f} - \tilde{g}f|^2 + \sigma^2 f^2 \tilde{f}^2}. \tag{3.10.4}$$

More precisely, this means that if the pair of tau-functions $\tilde{g}$, $\tilde{f}$ are solutions to the NLS bilinears (3.3.2,3.3.1), then the pair $g$, $f$ will be solutions to the same bilinear equations provided that (3.10.1-3.10.4) are satisfied. Notice that here we take equation (3.10.2), obtained by adding (3.3.11) and (3.3.12), as part of the bilinear ABT, since the proof below only requires that this sum should vanish. However, (3.3.11) and (3.3.12) are both satisfied separately by all the solutions considered in the main body of the chapter (in particular the rational solutions).

The proof that (3.10.1-3.10.4) constitute an ABT is considerably simplified by observing that, without loss of generality, we can set $c = 0$ from the start. This

follows from the fact that (as noted previously) NLS is invariant under a Galilean transformation, which in terms of the tau-functions can be written as

$$g(x,t) \to \exp\left[i\left(\frac{cx}{2} - \frac{c^2 t}{4}\right)\right] g(x',t'), \quad f(x,t) \to f(x',t'),$$

with

$$x' = x - ct, \qquad t' = t.$$

On applying this transformation to both the pairs $\tilde{g}$, $\tilde{f}$ and $g$, $f$ in (3.10.1-3.10.4), the constant $c$ may be removed. So henceforth we will assume that $\tilde{g}$, $\tilde{f}$ and $g$, $f$ are related by the equations (3.10.1-3.10.4) with $c = 0$.

The most straightforward part of the proof is showing that if $\tilde{g}$, $\tilde{f}$ satisfy (3.3.2) then so do $g$, $f$ (and vice-versa by symmetry). It is helpful to make use of the NLS amplitude,

$$\psi = \frac{g}{f}$$

(and similarly $\tilde{\psi}$), as well as the singular manifold function,

$$\phi = \frac{f}{\tilde{f}}.$$

We have already seen that, in terms of the NLS amplitude, (3.3.2) is completely equivalent to the equation

$$|\psi|^2 = -(\log[f])_{xx}. \tag{3.10.5}$$

If we now re-write (3.10.1) and (3.10.4) in terms of the NLS amplitudes and the singular manifold function, we find

$$(\psi - \tilde{\psi})_x = -(\psi + \tilde{\psi})(\log[\phi])_x, \tag{3.10.6}$$

$$(\log[\phi])_x^2 = |\psi - \tilde{\psi}|^2 + \sigma^2. \tag{3.10.7}$$

An immediate consequence of the equations (3.10.6,3.10.7) is the purely $x$-dependent part (3.2.6) of the usual ABT for NLS (for $c = 0$). On differentiating (3.10.7) with respect to $x$, we find

$$(\log[\phi])_x(\log[\phi])_{xx} = \frac{1}{2}(\psi - \tilde{\psi})_x(\psi - \tilde{\psi}) + c.c.$$

Then we may substitute for $(\psi - \tilde{\psi})_x$ from (3.10.6), and after cancelling out $(\log[\phi])_x$ (which we may assume to be non-zero) from both sides we have

$$(\log[f])_{xx} - (\log[\tilde{f}])_{xx} = |\tilde{\psi}|^2 - |\psi|^2.$$

Thus it is apparent that (3.10.5) is satisfied if and only if

$$|\tilde{\psi}|^2 = -(\log[\tilde{f}])_{xx}.$$

This is the necessary result which ensures that the $x$ part (3.3.2) of the NLS bilinears holds.

To show how the ABT yields the $t$ part (3.3.1) of NLS in bilinear form, we consider the quantity

$$Q := (D_x f \cdot \tilde{f}) \left[ \left( (iD_t + D_x^2) g \cdot f \right) \tilde{f}^2 + f^2 \left( (iD_t + D_x^2) \tilde{g} \cdot \tilde{f} \right) \right].$$

Provided that $Q$ vanishes, if $\tilde{g}$, $\tilde{f}$ satisfy (3.3.1) then so must $g$, $f$, and vice-versa. By direct manipulation we find

$$\begin{aligned} Q &= (D_x f \cdot \tilde{f}) \begin{bmatrix} \left( iD_t(g \cdot \tilde{f} + \tilde{g} \cdot f) + D_x^2(g \cdot \tilde{f} + \tilde{g} \cdot f) \right) f\tilde{f} \\ -(g\tilde{f} - \tilde{g}f)(iD_t f \cdot \tilde{f} + f\tilde{f}_{xx} - f_{xx}\tilde{f}) \\ -2(g_x\tilde{f} - \tilde{g}_x f)(D_x f \cdot \tilde{f}) \end{bmatrix} \\ &= -\sigma^2 (D_x f \cdot \tilde{f}) f\tilde{f}(g\tilde{f} + \tilde{g}f) \\ &\quad -(g\tilde{f} - \tilde{g}f)(D_x f \cdot \tilde{f}) \left( \overline{g}\tilde{g} - g\overline{\tilde{g}} - (D_x f \cdot \tilde{f})_x \right) \\ &\quad -2(D_x f \cdot \tilde{f})^2 \left( D_x(g \cdot \tilde{f} - \tilde{g} \cdot f) + g\tilde{f}_x - \tilde{g}f_x \right) \qquad (3.10.8) \\ &= -\sigma^2 f\tilde{f} \left( (D_x f \cdot \tilde{f})(g\tilde{f} + \tilde{g}f) + 2f\tilde{f}(g\tilde{f}_x - \tilde{g}f_x) \right) \\ &\quad +(g\tilde{f} - \tilde{g}f) \begin{pmatrix} (D_x f \cdot \tilde{f})(g\overline{\tilde{g}} - \overline{g}\tilde{g}) + \frac{1}{2}[(D_x f \cdot \tilde{f})^2]_x \\ -2(\overline{g}\tilde{f} - \overline{\tilde{g}}f)(g\tilde{f}_x - \tilde{g}f_x) \end{pmatrix}. \qquad (3.10.9) \end{aligned}$$

The second line (3.10.8) above is obtained using (3.10.2) and (3.10.3), while to get (3.10.9) it is necessary to use (3.10.1) and (3.10.4). Making use of (3.10.4) and then (3.10.1) once more, we see that

$$\begin{aligned} \frac{1}{2}[(D_x f \cdot \tilde{f})^2]_x &= \sigma^2 f\tilde{f}(f_x\tilde{f} + f\tilde{f}_x) \\ &\quad + \frac{1}{2} \left[ \left( D_x(g \cdot \tilde{f} - \tilde{g} \cdot f) + 2(g\tilde{f}_x - \tilde{g}f_x) \right) (\overline{g}\tilde{f} - \overline{\tilde{g}}f) + \text{c.c.} \right] \\ &= \sigma^2 f\tilde{f}(f_x\tilde{f} + f\tilde{f}_x) + \left[ (g\tilde{f}_x - \tilde{g}f_x)(\overline{g}\tilde{f} - \overline{\tilde{g}}f) + \text{c.c.} \right]. \end{aligned}$$

Finally, substituting this expression for $\frac{1}{2}[(D_x f \cdot \tilde{f})^2]_x$ into (3.10.9) leads to

$$Q = 0,$$

as required. □

**Note.** In the reference [122] (and in Chapter 4 of [62]), Nimmo presents a bilinear ABT for NLS (strictly for the focussing-NLS equation (3.2.1) with $\delta = +1$). This is almost identical to the one given here, except that (3.10.3) and (3.10.4) are effectively replaced by the single equation

$$(iD_t + icD_x + D_x^2 - \sigma^2) f \cdot \tilde{f} = -2g\overline{\tilde{g}}. \qquad (3.10.10)$$

The imaginary part of this equation is clearly just (3.10.3), while the real part is a consequence of (3.10.4) (provided that (3.3.2) is satisfied as well). Although the equation (3.10.4) is not strictly bilinear in the usual sense, it corresponds to the similarity manifold equation (3.9.1), and is more fundamental than the real part of (3.10.10) (which is not sufficient with (3.10.1) to show the $x$ part of the NLS bilinears).

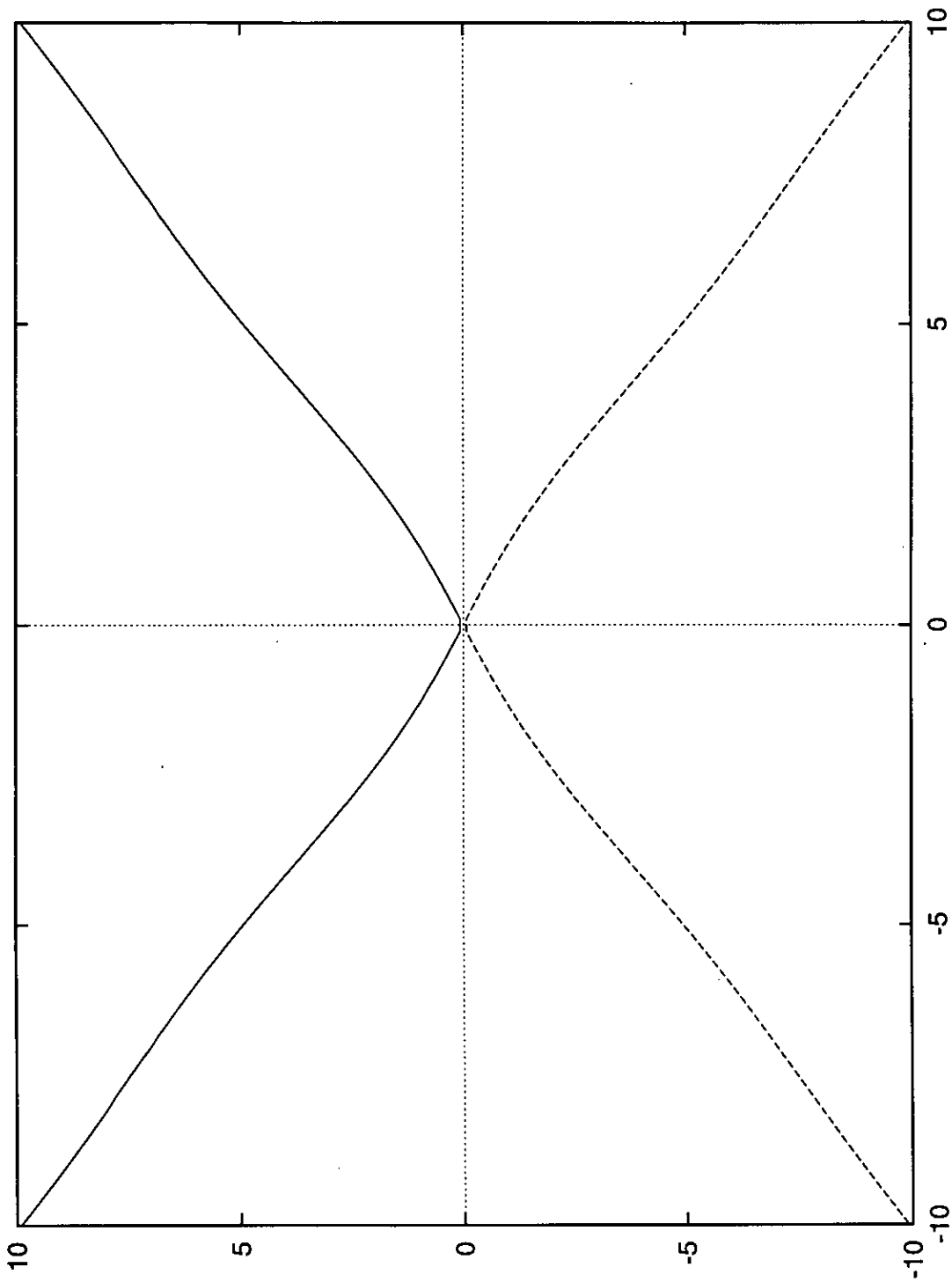


Figure 3.1: The collision of poles in the two-lump solution. Dotted/undotted lines denote before/after collision.

# Chapter 4

# Affine Toda Solitons and Ruijsenaars-Schneider Systems with Spin

The aim of this chapter is to relate the solitons of the affine Toda theories to the spin extensions of the Ruijsenaars-Schneider model, the latter also being known as the relativistic Calogero-Moser model. We succeed in obtaining a generalization of the known connection between the solitons of the sine-Gordon equation and the (non-spin) Ruijsenaars-Schneider model. The $N$-soliton tau-functions of the $A_n^{(1)}$ affine Toda theory are written as determinants involving a certain matrix $V$, and by diagonalizing $V$ both the positions and spins of a hyperbolic Ruijsenaars-Schneider model are found.

## 4.1 Introduction

We have been exploring the general phenomenon that classes of solutions of integrable PDEs may be identified with finite-dimensional mechanical systems. For example, the pole solutions of the KP equation [103, 142] and its reductions (such as KdV [11] and NLS in the previous chapter) are related to the non-relativistic Calogero-Moser model, while the sine-Gordon solitons are related to its relativistic counterpart, the Ruijsenaars-Schneider model (see [22, 138]). Other more recent examples are the peakon solutions appearing in fibre-optics and shallow water waves which have associated mechanical systems [35, 36]. There is currently a great deal of interest in field theoretic models possessing duality [48, 115], and finite-dimensional integrable systems have also arisen in this context. Since the dynamics of the finite-dimensional systems are often easier to understand (or simulate) than the equations of motion for the full field theory, this approach gives qualitative information about field theories by reducing the number of degrees

of freedom. Recently Babelon, Bernard and Smirnov [24] have taken this correspondence between field theories and mechanical systems beyond the classical to include the quantum regime as well, focussing on a particular $N$-particle sector of the Hilbert space for the quantum sine-Gordon model. It would be interesting to see whether this approach would work for the other affine Toda theories. The results of this chapter might be relevant to this problem.

The original work of Ruijsenaars and Schneider [138] showed that the soliton solutions of a variety of equations are related to dynamics built from the Hamiltonians (with canonically conjugate variables $q_j$, $p_j$)

$$H_\pm = \sum_j^N e^{\pm p_j} \prod_{k\neq j}^N \coth\Big(\frac{q_j - q_k}{2}\Big),$$

giving the equations of motion (for either $H_\pm$)

$$\ddot{q}_j = 2\sum_{k\neq j} \frac{\dot{q}_j \dot{q}_k}{\sinh(q_j - q_k)}. \tag{4.1.1}$$

In particular, the eigenvalues $i\,e^{q_j}$ of an $N \times N$ matrix associated with the tau-function describing an $N$-soliton solution of the sine-Gordon equation evolve according to (4.1.1). The variables $q_j$, $p_j$ may, at least when they are well separated, be related to the positions and rapidities of $N$ constituent single solitons; the dynamics of the system encodes the various soliton phase shifts. Thus the system governed by $H_+$ describes how the space-time trajectories of the $N$ constituent solitons interact. Of course the same system has an alternative description via the inverse scattering transform, which leads to action-angle variables for the solitons [57]. The point of the Ruijsenaars-Schneider approach is that it provides a dynamical description, thus making greater contact with the particle description of the soliton.

The Ruijsenaars-Schneider (RS) models can be seen as relativistic versions of the ordinary Calogero-Moser models. For instance, by considering the Hamiltonians given above, an appropriate ("non-relativistic") scaling limit of $H = H_+ + H_-$ yields the Calogero-Moser system with a hyperbolic potential. RS models are integrable finite-dimensional Hamiltonian systems, and have a dynamical r-matrix [107], which in a certain gauge turns out to be the same as the Calogero-Moser r-matrix [144]. A new development is Krichever and Zabrodin's construction of the spin-generalization of the RS models, to describe the pole motion of solutions of the non-abelian Toda lattice. These spin RS models are the relativistic counterpart of Gibbons and Hermsen's spin-generalized Calogero-Moser models [71]. In [104] the elliptic spin RS models were shown to be exactly solvable in terms of theta functions. A simpler solution for the rational and hyperbolic versions was

given in [135], where integrable discretizations were also constructed. However, the correct Hamiltonian formulation of the spin RS models apparently remains elusive.

The sine-Gordon model may be viewed as the $A_1^{(1)}$ affine Toda theory with imaginary coupling. Thus it is natural to wonder whether the Ruijsenaars-Schneider description of sine-Gordon solitons can be generalized to the other affine Toda theories, briefly reviewed in Section 4.2. The imaginary coupling case of these theories is of particular interest, because the solitons have real energy and momentum, despite the fact that the fields are complex-valued (except in the sine-Gordon case). This suggests that when the theory is quantized, the solitonic sector should be well-defined, although the theory as a whole violates unitarity [129]. Spence and Underwood [143] have recently used a vertex operator approach to obtain the symplectic form on the space of affine Toda solitons, and have proceeded to construct a sort of semi-classical theory for these solitons. Yet a dynamical description generalizing the sine-Gordon/Ruijsenaars-Schneider correspondence has proved elusive. In this chapter we provide such a generalization, describing the dynamics of the affine Toda solitons in terms of spin RS models. One new feature we have found in our correspondence is the appearance of new degrees of freedom, the internal spins of the model. The tau-functions for the affine Toda solitons are given by determinants involving a certain matrix $V$, and the spins are required to diagonalize this matrix. It does not seem possible to remove them from the description, as can be done in the sine-Gordon case. We will come back to this point in our Conclusion.

In Section 4.2 we outline a few salient features of affine Toda theories, before reviewing the construction of solitons in the following section. Section 4.4 concerns the symplectic form on the reduced phase-space of the $N$-soliton solution. We are then in a position to relate the affine Toda solitons to the hyperbolic spin RS model (Section 4.5). Our discussion is limited to the $A_n^{(1)}$ case, both for simplicity and to make clear the generalization of the sine-Gordon/Ruijsenaars-Schneider correspondence. We discuss further generalizations and unresolved problems in the Conclusion. Most of this work has already appeared in [32].

## 4.2 Affine Toda Field Theories

The affine Toda theories are a family of massive relativistic 2D field theories with Lagrangian

$$L = \frac{1}{2}(\partial_\nu \phi, \partial^\nu \phi) - \frac{m^2}{\beta^2} \sum_{\alpha \in \bar{\Delta}} n_\alpha e^{\beta(\alpha,\phi)}.$$

Here $m$ is a real mass parameter, $\bar{\Delta} = \Delta \cup \{\alpha_0\}$ is the set of simple roots of a Lie algebra $g$ together with $\alpha_0 = -\sum_{\alpha\in\Delta} n_\alpha \alpha$ (minus the highest root), the $n_\alpha$ are positive integers and by convention $n_{\alpha_0} = 1$ . In light cone coordinates $x_\pm = \frac{1}{\sqrt{2}}(t \pm x)$ the equations of motion become

$$\partial_+\partial_-\phi + \frac{m^2}{2\beta}\sum_{\alpha\in\bar{\Delta}} n_\alpha \alpha e^{\beta(\alpha,\phi)} = 0.$$

The classical and quantum versions of these theories have been extensively studied in recent years both for real and imaginary couplings. In the real coupling regime a beautiful structure was uncovered and exact S-matrices have been conjectured [31, 42, 49]. The imaginary coupling regime has also been investigated and classically the solitons have real energy-momentum although the Lagrangian is complex (see [129] and references therein).

For what follows we will be concerned only with the $A_n^{(1)}$ theories. In this case the elements of $\bar{\Delta}$ are given by

$$\alpha_j = e_j - e_{j+1}$$

for $j = 0, 1, ..., n$, where $e_0, e_1, ..., e_n$ are a basis for $\mathbb{C}^{n+1}$ (orthonormal with respect to the bilinear form), and all indices are read modulo $n+1$ where necessary. All our expressions will be in terms of the field components

$$\phi_j = (e_j, \phi),$$

for which the equations of motion read

$$\partial_+\partial_-\phi_j + \frac{m^2}{2\beta}(e^{\beta(\phi_j-\phi_{j+1})} - e^{\beta(\phi_{j-1}-\phi_j)}) = 0, \tag{4.2.1}$$

$j = 0, 1, ..., n$.

A few remarks are in order here. If we define the vector $e = \sum_{j=0}^{n} e_j$, then it is easy to observe from (4.2.1) that $(e, \phi)$ is a free field. Hence this part of $\phi$ is often discarded, and only the part of $\phi$ lying in $< e >^\perp$ (i.e. the root space) is considered. Then the equations may be written instead in terms of the $n$ fields $w_j$ defined by

$$w_j = (\lambda_j, \phi) = \left(\sum_{k=1}^{j} e_k - \frac{j}{n+1}e, \phi\right),$$

with $\lambda_j$ being the fundamental dominant weights of $sl(n+1)$. These fields determine the solitonic sector of the theory, and the soliton solutions are often expressed in terms of them. It has been shown [129] that when the coupling $\beta$ is purely imaginary, the energy and momentum of these solitons is real. In the

simplest case of $A_1^{(1)}$, we just obtain the sine-Gordon theory, and all the soliton solutions may be taken to be real. However, for $n > 1$ the soliton fields are intrinsically complex, and it is curious that they should nevertheless have real energy and momentum. In the following we hope to gain a better understanding of this by considering the reduced phase space of the soliton solutions, and viewing the motion of the solitons in terms of a finite-dimensional dynamical system.

## 4.3 The $A_n^{(1)}$ Affine Toda Solitons

We wish to look at the $A_n^{(1)}$ affine Toda theory with imaginary coupling, so we send $\beta \to i\beta$ in (4.2.1), and then the equations of motion become

$$\partial_+\partial_-\phi_j + \frac{m^2}{2i\beta}(e^{i\beta(\phi_j-\phi_{j+1})} - e^{i\beta(\phi_{j-1}-\phi_j)}) = 0, \tag{4.3.1}$$

$j = 0, 1, ..., n$. Indices on the components of the field $\phi$ are read modulo $(n+1)$ where necessary. As already mentioned above, in the solitonic sector of the theory $\sum_{j=0}^{n} \phi_j = 0$.

### 4.3.1 Soliton Tau-Functions

There are various ways to construct and parametrize soliton solutions to (4.3.1). Perhaps the simplest methods to implement from a practical point of view are the application of the auto-Bäcklund transformation (ABT) derived by Fordy and Gibbons [59] or the bilinear formalism developed by Hirota [79]. The inverse scattering approach to affine Toda theories also appears to have some unusual features. There are also the powerful vertex operator techniques which make full use of the representation theory of the $A_n^{(1)}$ algebra [129]. While the latter approach is currently the most popular, we wish to make contact with the original work of Ruijsenaars and Schneider [138], which made much reference to the soliton formulae of Hirota. Hence we would like to employ the form of N-soliton solution for (4.3.1) derived by Hollowood [86] via Hirota's direct method. The $l$-th component of the field $\phi$ is given by

$$e^{i\beta\phi_l} = \frac{\tau_{l-1}}{\tau_l},$$

leading to the bilinear equations

$$D_+D_-\tau_j \cdot \tau_j + m^2(\tau_j^2 - \tau_{j-1}\tau_{j+1}) = 0,$$

and the soliton tau-function $\tau_l$ takes the form

$$\tau_l = \sum_{\epsilon} \exp\left(\sum_{j<k} \epsilon_j\epsilon_k B_{jk} + \sum_j \epsilon_j\zeta_{j,l}(x_+, x_-)\right). \tag{4.3.2}$$

In the above the $\epsilon$ indicates a summation over all possible combinations of $\epsilon_j$ taking the values 0 or 1, and the indices $j$ and $k$ take values in $\{1, ..., N\}$.

We will explain shortly what the various terms in (4.3.2) mean, and how we have parametrized the $A_n^{(1)}$ affine Toda solitons. For the moment we would like to comment that expression (4.3.2) is a rather generic form of the soliton tau-function for an integrable PDE, the precise nature of $B_{jk}$ and $\zeta_{j,l}$ depending on the particular PDE being considered; it may be viewed as a degeneration of the theta function solutions of the PDE given via algebraic geometry in which the $\epsilon_j$'s run over all of the integers. Ruijsenaars and Schneider succeeded in making the connection between their relativistic Calogero-Moser systems and soliton solutions of the sine-Gordon and KdV equations, among others, by showing a direct correspondence between the coordinates of the N-particle system and the parameters of the N-soliton solution. An important part of the correspondence was that all the tau-functions of form (4.3.2) being considered in [138] could be written in terms of determinants like

$$\det(1 + M)$$

for suitable matrices $M$. In what follows we express the $N$-soliton solutions of the $A_n^{(1)}$ affine Toda theory in this way, and thereby obtain a relation to spin-generalized Ruijsenaars-Schneider (spin RS) systems. Recently Beggs and Johnson [28] have used a type of dressing method to find more complicated sorts of one-soliton solutions than were previously known, but we shall not be concerned with these solutions here.

### 4.3.2 ABT and Soliton Determinants

In Hollowood's original treatment [86], the tau-functions were not actually written explicitly as determinants. Since we want to make use of the determinantal form, we shall start from the formulae of Olive, Turok and Liao [110] and demonstrate that they lead to (4.3.2). By repeated application of the Toda ABT [59],

$$\begin{aligned}
\partial_+(\phi_j - \tilde{\phi}_j) &= \frac{m}{\sqrt{2}\beta} A \left[ e^{i\beta(\tilde{\phi}_j - \phi_{j+1})} - e^{i\beta(\tilde{\phi}_{j-1} - \phi_j)} \right], \\
\partial_-(\phi_j - \tilde{\phi}_{j-1}) &= \frac{m}{\sqrt{2}\beta} A^{-1} \left[ e^{i\beta(\phi_j - \tilde{\phi}_j)} - e^{i\beta(\phi_{j-1} - \tilde{\phi}_{j-1})} \right],
\end{aligned}$$

the authors of [110] showed that $N$-soliton solutions could be obtained, with the $l$th component of $\phi$ given by

$$e^{i\beta\phi_l} = A_1 A_2 ... A_N \frac{\det T^{1,2,...,N}_{l-1,...,l-N}}{\det T^{1,2,...,N}_{l,...,l-N+1}}. \tag{4.3.3}$$

The $A_j$ are Bäcklund parameters, and the $T$s are $N$-by-$N$ matrices. Before describing all the matrix elements of these, it is useful to introduce the notation used in [110] to parametrize the solitons. Each individual soliton making up the $N$-soliton solution has a rapidity denoted by $\eta_j$, a position parameter denoted by $Q_j$, and a discrete parameter $\theta_j$ taking values in $\{\frac{2\pi k}{(n+1)}|k = 1, 2, ..., n\}$ (so that $\exp(i\theta_j)$ is an $(n+1)$th root of unity). The different values of the discrete parameters $\theta_j$ give $n$ different species of soliton, with masses

$$m_j = 2m\sin(\theta_j/2). \tag{4.3.4}$$

For what follows we will also need to define

$$\mu_j = \exp[\eta_j], \qquad \mu_j^{\pm} = \exp\left[\eta_j \pm \frac{1}{2}i\theta_j\right].$$

In terms of these parameters the matrices appearing in (4.3.3) are given by

$$(T^{1,2,...,N}_{l,...,l-N+1})_{jk} = T^j_{l-k+1},$$

where

$$T^j_l = (\mu_j^+)^{-l}\left[e^{il\theta_j}Q_j e^{m_j(x\cosh(\eta_j)-t\sinh(\eta_j))} - 1\right].$$

For our purposes we choose the Bäcklund parameters as $A_j = (\mu_j^+)^{-1}$ (so that $\prod_{j=0}^n \exp(i\beta\phi_j) = 1$), and after absorbing all these factors into the numerator of the right-hand side of (4.3.3), and then multiplying the $j$th row of the matrices appearing in numerator and denominator by $-(\mu_j^+)^l$ (for each $j$), we find that it may be written as a slightly different ratio

$$\frac{\det \tilde{T}_{l-1}}{\det \tilde{T}_l}.$$

In the above expression we have

$$\tilde{T}_l = M_1 - M_{2,l},$$

where (using light cone coordinates) the matrices $M_1$ and $M_{2,l}$ are given by

$$(M_1)_{jk} = (\mu_j^+)^{k-1},$$

$$(M_{2,l})_{jk} = Q_j(\mu_j^-)^{k-1}\exp\left(m_j((\mu_j)^{-1}x_+ - \mu_j x_-)/\sqrt{2} + il\theta_j\right).$$

Now we may expand

$$\det \tilde{T}_l = \sum_{j=0}^{N}(-)^j \sum_{(j)} D_{(j)}(M_1, M_{2,l}), \tag{4.3.5}$$

where $(j)$ corresponds to a distinct choice of $j$ rows, and $D_{(j)}$ corresponds to the determinant formed by replacing the corresponding rows in $M_1$ by those in $M_{2,l}$. After scaling numerator and denominator of (4.3.5) by the Vandermonde determinant $D_{(0)} = \det M_1$, we find

$$e^{i\beta\phi_l} = \frac{\tau_{l-1}}{\tau_l}, \tag{4.3.6}$$

which is a ratio of two tau-fuctions of the form (4.3.2). The terms in the Hirota sum (4.3.2) are given by

$$B_{jk} = \log\left(\frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)}\right),$$

$$\zeta_{j,l}(x_+, x_-) = \log\left[a_j \exp\left(m_j((\mu_j)^{-1}x_+ - \mu_j x_-)/\sqrt{2} + il\theta_j\right)\right],$$

where we have introduced the more convenient position parameters

$$a_j = -Q_j \prod_{k\neq j}\left(\frac{\mu_j^- - \mu_k^+}{\mu_j^+ - \mu_k^+}\right).$$

Henceforth we will use $a_j$ for the positions, and the rapidities $\eta_j$ as our parameters. The rapidities are all real, while the $a_j$ are pure imaginary for solitons; there are different reality conditions for other types of solution (e.g. for breathers it is necessary to take complex conjugate pairs $a_j, \overline{a_j}$). Also there are the different values of the discrete parameters $\theta_j$ giving the $n$ different species of soliton. To make a comparison with the vertex operator formulae, we note that the notation of reference [143] has $X_{j,k} = \exp[B_{jk}]$, $Q_j = a_j$.

It is a simple matter to write the tau-functions $\tau_l$ as determinants. We set

$$X_j = a_j(\mu_j^+ - \mu_j^-)\exp\left(m_j(e^{-\eta_j}x_+ - e^{\eta_j}x_-)/\sqrt{2}\right),$$

and define $N$-by-$N$ matrices $V$, $\Theta$ by

$$V_{jk} = \frac{\sqrt{X_j X_k}}{\mu_j^+ - \mu_k^-}, \tag{4.3.7}$$

and

$$\Theta = \mathrm{diag}(\theta_1, \theta_2, ..., \theta_N).$$

Then we find that

$$\tau_l = \det\left(1 + e^{il\Theta/2} V e^{il\Theta/2}\right). \tag{4.3.8}$$

To verify (4.3.8) it is necessary to expand the determinant on the right-hand side in terms of the principal cofactors of $V$, and then use Cauchy's identity:

$$\det\left(\frac{1}{\mu_j^+ - \mu_k^-}\right)_{j,k} = \prod_j \frac{1}{\mu_j^+ - \mu_j^-} \prod_{j<k} \frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^+ - \mu_k^-)(\mu_j^- - \mu_k^+)}.$$

Writing the right-hand side of (4.3.2) in terms of these parameters and comparing with the cofactor expansion gives the required result. Note that in the $A_1^{(1)}$ (sine-Gordon) case the $\theta_j$ must all take the value $\pi$, which means that the matrix exponentials appearing in (4.3.8) are multiples of the identity, and we reproduce the standard result

$$e^{i\beta\phi_0} = e^{-i\beta\phi_1} = \frac{\det\,(1-V)}{\det\,(1+V)}.$$

## 4.4 The Reduced Symplectic Form

In this section we describe the phase space of the $N$-soliton solution in terms of its symplectic form, before describing how spin-generalized Ruijsenaars-Schneider systems arise in the following section. The phase space of the affine Toda system has the standard symplectic form

$$\Omega = \int_{-\infty}^{\infty} (\delta\phi_t \wedge, \delta\phi)\, dx. \tag{4.4.1}$$

On substitution of the $N$-soliton solution into (4.4.1), one obtains (after an integration) the reduced symplectic form on the $N$-soliton phase space. In practice it is not possible to perform the integration for anything other than the one-soliton solution [143] (except for the sine-Gordon case, where Babelon and Bernard succeeded in showing that the integrand could be written as an exact derivative for both the one- and two-soliton [22]). For the one-soliton phase space, the reduced symplectic form is (up to an irrelevant numerical factor independent of $\theta$)

$$\omega^{(1)} = \frac{da}{a} \wedge d\eta.$$

The intractability of the integral (4.4.1) for the general $N$-soliton solution does not matter, as it is a standard result that as $t \to \pm\infty$ (the *out/in* limits) the $N$-soliton decomposes into a superposition of $N$ one-solitons with a shift of the parameters. So the symplectic form may just be written

$$\omega^{(N)} = \sum_j \frac{da_j^{out}}{a_j^{out}} \wedge d\eta_j^{out} = \sum_j \frac{da_j^{in}}{a_j^{in}} \wedge d\eta_j^{in}. \tag{4.4.2}$$

By direct calculation using the formula (4.3.8) for the tau-functions of the $N$-soliton solution, we find the relations between the *out/in* parameters and the standard ones:

$$\eta_j^{in} = \eta_j^{out} = \eta_j,$$

$$a_j^{in} = a_j \prod_{k>j} \frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)} = a_j \prod_{k>j} \exp(B_{jk})$$

(and similarly for $a_j^{out}$ with the inequality reversed). This agrees with the formulae of Spence and Underwood [143] obtained via vertex operator arguments, where for their notation it is necessary to replace $a_j$ by $Q_j$ and $\exp(B_{jk})$ by $X_{j,k}$. So substituting for the *in* parameters into (4.4.2), we obtain the $N$-soliton symplectic form as

$$\omega^{(N)} = \sum_j \frac{da_j}{a_j} \wedge d\eta_j + \sum_{j<k} E_{jk}(\eta) \sinh(\eta_j - \eta_k) d\eta_j \wedge d\eta_k, \tag{4.4.3}$$

where

$$E_{jk}(\eta) = \frac{1}{\cosh(\eta_j - \eta_k) - \cos((\theta_j - \theta_k)/2)} - \frac{1}{\cosh(\eta_j - \eta_k) - \cos((\theta_j + \theta_k)/2)}.$$

We observe that $\omega^{(N)}$ is clearly real if we choose the $\eta_j$ to be real and the $a_j$ to be pure imaginary (which in the $A_1^{(1)}$ case coincides with the condition on $a_j$ for sine-Gordon solitons given in [22]). This means that the matrix $V$ defined in (4.3.7) is anti-hermitian, which will be important in the next section when we look at the dynamics of the eigenvalues of $V$.

## 4.5 Ruijsenaars-Schneider Systems

### 4.5.1 Spin RS Model

The spin-generalized Ruijsenaars-Schneider (spin RS) model was introduced by Krichever and Zabrodin in [104]. It is defined in terms of $N$ particle positions $x_j$ and their internal degrees of freedom (spins) given by $l$-dimensional vectors $a_j$ and $l$-dimensional covectors $b_j^\dagger$, subject to the equations of motion

$$\ddot{x}_j = \sum_{k \neq j} (b_j^\dagger a_k)(b_k^\dagger a_j)(\mathcal{V}(x_j - x_k) - \mathcal{V}(x_k - x_j)), \tag{4.5.1}$$

$$\dot{a}_j = \sum_{k \neq j} a_k (b_k^\dagger a_j) \mathcal{V}(x_j - x_k), \tag{4.5.2}$$

$$\dot{b}_j^\dagger = -\sum_{k \neq j} b_k^\dagger (b_j^\dagger a_k) \mathcal{V}(x_k - x_j). \tag{4.5.3}$$

The potential $\mathcal{V}$ is expressed in terms of the Weierstrass zeta function,

$$\mathcal{V}(x) = \zeta(x) - \zeta(x + \gamma),$$

or its rational or hyperbolic limits, which are

$$\mathcal{V}(x) = \frac{1}{x} - \frac{1}{x + \gamma}$$

and

$$\mathcal{V}(x) = \coth(x) - \coth(x + \gamma)$$

respectively. The case relevant to the affine Toda solitons is the hyperbolic limit with parameter $\gamma = \frac{i\pi}{2}$ (after a suitable scaling of the variables). The equation (4.5.1) is a natural generalization of the equation (4.1.1) in the ordinary RS model.

Before making the connection with solitons, we observe some other properties of spin RS models. The equations (4.5.1,4.5.2,4.5.3) have the scaling symmetry

$$a_j \to \alpha_j a_j, \quad b_j^\dagger \to \frac{1}{\alpha_j} b_j^\dagger.$$

The corresponding integrals of motion are $\dot{x}_j - b_j^\dagger a_j$, and setting them to zero yields

$$\dot{x}_j = b_j^\dagger a_j. \tag{4.5.4}$$

It is customary (see [135]) to impose this constraint from the start, and take (4.5.2,4.5.3,4.5.4) as the equations of motion. These equations have a Lax pair, and the Lax matrix given in [135] has entries

$$L_{jk} = \frac{\exp(x_k - x_j)}{\cosh(x_j - x_k)} b_j^\dagger a_k \tag{4.5.5}$$

in the hyperbolic case relevant to our discussion; we find that the Toda solitons naturally yield a different Lax matrix. There is also a gauge freedom in the spins, which means that by rescaling $a_j$, $b_j^\dagger$ suitably, it is possible to insert the term $\mathcal{W}_j a_j$ into the right-hand side of (4.5.2), and the term $-\mathcal{W}_j b_j^\dagger$ into the right-hand side of (4.5.3), for arbitrary functions $\mathcal{W}_j(t)$. As we shall see, this corresponds to a freedom in choosing the diagonal entries of the matrix $M$ in the Lax pair. Bearing all this in mind, we can now show the relationship with the soliton formulae of Section 4.3.

### 4.5.2 Spin RS Equations from Toda Solitons

We proceed to consider how the eigenvalues of the matrix $V$ defined by (4.3.7) evolve with respect to each of the light cone coordinates, and find that a particular sort of spin RS model results. Since $V$ is anti-hermitian, it may be diagonalized with a unitary matrix $U$:

$$Q := UVU^\dagger = \mathrm{diag}(i\exp(q_1), ..., i\exp(q_N)).$$

If we let a dot denote $\frac{d}{dx_\pm}$, then $V$ satisfies

$$\dot{V} = \frac{1}{2}(\Lambda V + V\Lambda), \tag{4.5.6}$$

for the constant diagonal matrix $\Lambda$ defined by

$$\Lambda = \pm\frac{1}{\sqrt{2}}\,\mathrm{diag}(m_1\exp(\mp\eta_1), ..., m_N\exp(\mp\eta_N)),$$

with the masses $m_j$ as given in (4.3.4). Now let $u_j$ denote the $j$th row of $U$ (considered as a column vector, so that the $u_j$ are the left eigenvectors of $V$). Define the Lax matrix $L$ by

$$L_{jk} = u_k^\dagger \Lambda u_j,$$

implying

$$L = U\Lambda U^\dagger.$$

Then $L$ satisfies the Lax equation

$$\dot{L} = [M, L], \tag{4.5.7}$$

for $M = \dot{U}U^\dagger$. Differentiating the definition of $Q$ gives

$$\dot{Q} = [M, Q] + U\dot{V}U^\dagger,$$

and after substituting for $\dot{V}$ from (4.5.6) and using the definition of $L$ we find the identity

$$LQ + QL = 2(\dot{Q} + [Q, M]). \tag{4.5.8}$$

Rewriting everything in components, (4.5.8) reads

$$L_{jk}(e^{q_j} + e^{q_k}) = 2(\dot{q}_j e^{q_j}\delta_{jk} + M_{jk}(e^{q_j} - e^{q_k})),$$

which yields

$$L_{jj} = \dot{q}_j \tag{4.5.9}$$

and

$$M_{jk} = \frac{1}{2}\coth((q_j - q_k)/2)L_{jk},$$

(for $j \neq k$).

When $V$ is diagonalized we may always choose the phases of the left eigenvectors $u_j$ so that $M_{jj} = u_j^\dagger \dot{u}_j = 0$. Finally, substituting for the entries of $M$ in the Lax equation (4.5.7) produces the equations of motion:

$$\dot{L}_{jj} = \ddot{q}_j = \sum_{k\neq j} \coth((q_j - q_k)/2)L_{jk}L_{kj}, \tag{4.5.10}$$

$$\begin{aligned}\dot{L}_{jk} = {} & \tfrac{1}{2}\coth((q_j - q_k)/2)(\dot{q}_k - \dot{q}_j)L_{jk} \\ & + \textstyle\sum_{l\neq j,k} \tfrac{1}{2}(\coth((q_j - q_l)/2) - \coth((q_l - q_k)/2))L_{jl}L_{lk}\end{aligned} \tag{4.5.11}$$

($j \neq k$). These equations follow from the spin RS equations with certain constraints, although to see this requires comparison with the formulae (4.5.1-4.5.3) of Krichever and Zabrodin.

To make contact with our equations we set $x_j = q_j$ and choose the hyperbolic potential

$$\mathcal{V}(q_j - q_k) = \frac{1}{2}\coth((q_j - q_k)/2).$$

In [104] the spin degrees of freedom were real, but here we allow them to be complex, and identify them with the eigenvectors of $V$ by setting

$$b_j^\dagger = u_j^\dagger, \quad a_j = \Lambda u_j.$$

So we have taken $l = N$, and in fact our spins are expressed entirely in terms of the eigenvectors of $V$ and the constant matrix $\Lambda$; in particular the $b_j^\dagger$ must form an orthonormal basis. In the notation of [104] the components of this Lax matrix are given by

$$L_{jk} = b_k^\dagger a_j.$$

We observe that this is a non-standard choice for the components of the Lax matrix compared with [104, 135], where $L$ has entries of the form (4.5.5). The equation (4.5.9) is immediately seen to be equivalent to the usual constraint (4.5.4) imposed on RS models. Also (4.5.2,4.5.3) follow from our definition of the spins in terms of the eigenvectors of $V$, and may be seen directly from the equations

$$(U\Lambda)^{\cdot} = MU\Lambda, \qquad \dot{U}^\dagger = -U^\dagger M.$$

In a sense these equations are more fundamental than the Lax equation (4.5.7). From the definition of $L$ in terms of the spins we can compute $\dot{L}_{jk}$. So for $j = k$ (4.5.1) is equivalent to (4.5.10), while for $j \neq k$ (4.5.11) is a consequence of (4.5.2) and (4.5.3). Note that we have also exploited the gauge freedom of the spins to choose $M_{jj} = 0$.

### 4.5.3 Sine-Gordon and Spinless RS

To make the correspondence between the solitons and the many-body system clearer, it is worth considering the sine-Gordon case in more detail and comparing it with the general situation. The results about sine-Gordon solitons are explained in detail in [22], and we have kept our notation as similar to this reference as possible to make comparison easier. The first thing to observe is that in the $A_1^{(1)}$ case only knowledge of the $q_j$ is required to specify the field components, as we have

$$e^{i\beta\phi_0} = e^{-i\beta\phi_1} = \prod_{j=1}^{N} \left( \frac{1 - i\exp(q_j)}{1 + i\exp(q_j)} \right).$$

In the general case the presence of the matrix $e^{il\Theta/2}$ in the expression for the tau-functions (4.3.8) means that knowledge of both the spin vectors $u_j$ (which make up the matrix $U$) and the $q_j$ is required to evaluate these determinants.

The essential difference is that for sine-Gordon there is only one soliton species, while in the $A_n^{(1)}$ case there are $n$ different species corresponding to the different

allowed values of $\theta_j$. This difference is also apparent at the level of the equations of motion. In fact when we differentiate the matrix $V$, in the case of sine-Gordon we find from (4.5.6) that

$$\dot{V} = i(ee^\dagger)$$

for a certain vector $e$. But then conjugating the equation (4.5.6) with $U$ we obtain

$$i\tilde{e}\tilde{e}^\dagger = \frac{1}{2}(LQ + QL),$$

where $\tilde{e} = Ue$. Actually $\tilde{e}$ is a real vector, and in terms of its components $\tilde{e}_j$ we have

$$L_{jk} = 2\,\frac{\tilde{e}_j\tilde{e}_k}{\exp(q_j) + \exp(q_k)}.$$

Since we know the diagonal elements of $L$ explicitly in terms of the $q_j$ (from (4.5.9)) the above formula means that we then know all the $\tilde{e}_j$ and hence the off-diagonal elements of $L$ are found to be

$$L_{jk} = \frac{\sqrt{\dot{q}_j\dot{q}_k}}{\cosh((q_j - q_k)/2)}.$$

This may then be substituted into (4.5.10,4.5.11) to give the ordinary (non-spin) RS equations. In this case (4.5.10) yields (4.1.1) and (4.5.11) is a consequence. Babelon and Bernard have shown [22] that there is a canonical transformation between the soliton parameters and the dynamical variables $q_j, \dot{q}_j$. We discuss how this could possibly be extended to the $A_n^{(1)}$ case in our Conclusion.

## 4.6 Conclusion

We have shown the connection between the spin-generalized Ruijsenaars-Schneider systems and $A_n^{(1)}$ affine Toda solitons. The $N$-soliton tau-functions are determined by the positions $q_j$ of $N$ particles on the line as well as an orthonormal set of $N$-dimensional spin vectors $u_j$, which are together subject to the equations of the hyperbolic spin RS model. This extends the known result for the sine-Gordon equation, where the spins are no longer part of the dynamics and there is a canonical transformation between the positions and momenta of the particles and the parameters of the solitons. For the general case such a transformation is no longer apparent, although we note that the $N$-soliton phase space is still of dimension $2N$, and so it is worth exploring exactly how the extra spin degrees of freedom are absorbed in the transition from the dynamical variables to the soliton parameters.

Concerning the relationship between the soliton parameters and the spin RS variables, we observe that in [135] the solution of the hyperbolic spin RS model

involves gauging the Lax matrix,

$$L = UL_0U^{-1},$$

although in that case $L$ has entries given by the formula (4.5.5). The positions $x_j$ of the model are found to be the eigenvalues of a matrix

$$X_0 e^{2tL_0},$$

and the corresponding solutions for the matrices of the spin vectors are

$$A = UA_0, \qquad B = B_0U^{-1}.$$

The degenerate case of the ordinary RS model corresponds to a special choice of initial conditions,

$$A_0 = B_0 = 1.$$

Although the form of Lax pair we have used is slightly different, it would appear that the affine Toda solitons might also be understood as corresponding to special initial conditions of the form

$$L_0 = A_0 = \Lambda, \qquad B_0 = 1.$$

Hence there might be a better way to understand the spin degrees of freedom. Also it would be interesting to see what rôle the spins might play in the quantum theory.

We would like to extend this work to the soliton solutions of the Toda systems corresponding to the other affine algebras [112], and at the same time elucidate the connections to the vertex operator constructions used in [129, 143]. It might also be worthwhile considering the more general sorts of solitons found by Beggs and Johnson [28], which have extra degrees of freedom. We intend to pursue these points in the future.

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# Affine Toda solitons and systems of Calogero-Moser type

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**Abstract**

The solitons of affine Toda field theory are related to the spin-generalised Ruijsenaars-Schneider (or relativistic Calogero-Moser) models. This provides the sought after extension of the correspondence between the sine-Gordon solitons and the Ruijsenaars-Schneider model.

## 1. Introduction

The purpose of this letter is to relate the solitons of the affine Toda system to the spin extensions of the Ruijsenaars-Schneider model, the latter also being known as the relativistic Calogero-Moser model. This work generalises the known connection between the solitons of the sine-Gordon equation and the (non-spin) Ruijsenaars-Schneider model. The connection made here should be viewed as part of a larger programme that seeks to identify classes of solutions of PDEs with finite dimensional mechanical systems, whereby the evolution of the solutions to the PDE is expressed as a dynamical system on the (finite-dimensional) moduli space of solutions. Thus, for example, the pole solutions of the KP equation [1,2] and its reductions (such as KdV [3]) are related to the non-relativistic Calogero-Moser model, while the sine-Gordon solitons are related to its relativistic counterpart, the Ruijsenaars-Schneider model (see [4,5]). This programme also extends to include the peakon solutions appearing in fibre-optics and shallow water waves which have associated mechanical systems [6,7]. A similar connection may well underlie the appearance of finite dimensional mechanical systems in the study of various models possessing duality [8,9]. As the dynamics of mechanical systems are often easier to understand (or simulate) than the equations of motion for a field theory, such a programme aims at giving qualitative information about field theories by an appropriate reduction of degrees of freedom. The recent work of Babelon, Bernard and Smirnov [10] may be viewed as taking this correspondence between field theories and mechanical systems beyond the classical to include the quantum regime as well, though the ability to focus attention solely on a fixed $N$-particle sector of the full quantum Hilbert space appears to depend crucially on the model. Our work will reveal further new features in such correspondences, as well as provide a sought after generalisation of known results about the sine-Gordon model to the case of affine Toda solitons.

Ruijsenaars and Schneider's seminal work [5] showed that the soliton solutions of a variety of equations were related to dynamics built from the Hamiltonians (with canonically conjugate variables $q_j$, $p_j$)

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$$_{\pm} = \sum_{j}^{N} e^{\pm p_j} \prod_{k\neq j}^{N} \coth\left(\frac{q_j - q_k}{2}\right),$$

d equations of motion (for either $H_{\pm}$)

$$= 2\sum_{k\neq j} \frac{\dot{q}_j \dot{q}_k}{\sinh(q_j - q_k)}. \tag{1}$$

n appropriate scaling limit of $H = H_+ + H_-$ yields system of Calogero-Moser type. In particular, the genvalues $i\,e^{q_j}$ of an $N \times N$ matrix associated with e tau function describing an $N$-soliton solution of the ne-Gordon equation evolve according to (1). The 's and $p_j$'s may, at least when they are well separated, be related to the positions and rapidities of $N$ nstituent single-solitons; the dynamics of the system encodes the various soliton phase shifts. (More tails of this will be given below.) Thus the system overned by $H_+$ describes how the space-time trajecries of the 'constituent' solitons interact. Of course e same system may be described via the inverse scatring transform by a free system with linearly evolvg data: the point of the Ruijsenaars-Schneider deription is to make greater contact with the particle escription of the soliton.

Viewing the sine-Gordon model as the $A_1^{(1)}$ affine oda system (with imaginary coupling) a natural uestion to ask is how the above results generalise other affine Toda systems. These systems have een extensively studied in recent years both for real d imaginary couplings. In the real coupling regime beautiful structure was uncovered and exact $S$-atrices have been conjectured for the theories (see 1–13] and references therein). The imaginary couing regime has also been investigated and classically e solitons have real energy-momentum although e Lagrangian is complex ([14] and references erein). Spence and Underwood [15] have recently sed this work to obtain the symplectic form on the ace of affine Toda solitons but a dynamical deription generalising the sine-Gordon/Ruijsenaarschneider correspondence has proved elusive. The urpose of the present letter is to give this generisation. Just as the affine Toda systems generalise e sine-Gordon model, there are spin-generalisations f the Ruijsenaars-Schneider systems, and it is these stems which describe the dynamics of the affine oda solitons. These models (which have been most studied in the $A_n$ setting) are the relativistic extension of Gibbon and Hermsen's spin generalisation of the original Calogero-Moser model [16]. One new feature we have found in our correspondence is the appearance of new degrees of freedom, the internal spins of the model. Although not needed to describe the solitons of the affine Toda system, these spins determine the matrix that diagonalises the Lax pair. We will comment further on this later in the letter.

An outline of the letter is as follows. First we will review the construction of affine Toda solitons, and then in Section 3 consider the reduced symplectic form of the theory. We are then in a position to relate the affine Toda solitons to the spin-generalised Ruijsenaars-Schneider model in Section 4. For the purposes of this letter we shall limit our discussion to the $A_n^{(1)}$ case, both for simplicity and to make clear the generalisation of the sine-Gordon/Ruijsenaars-Schneider correspondence.

## 2. The $A_n^{(1)}$ affine Toda solitons

For the $A_n^{(1)}$ affine Toda theory with imaginary coupling, the equations of motion read

$$\partial_+\partial_-\phi_j + \frac{m^2}{2i\beta}\left(e^{i\beta(\phi_j-\phi_{j+1})} - e^{i\beta(\phi_{j-1}-\phi_j)}\right) = 0, \tag{2}$$

$j = 0, 1, \ldots, n$. Here $\pm$ denotes differentiation with respect to light-cone coordinates $x_{\pm} = 1/\sqrt{2}(t \pm x)$, and the indices on the components of the field $\phi$ are read modulo $(n+1)$ where necessary. We shall be considering the solitonic sector of the theory, which means assuming $\sum_{j=0}^{n} \phi_j = 0$ (in other words, discarding the free field part of $\phi$).

There are various ways to construct and parametrise soliton solutions to (2). Perhaps the simplest methods to implement from a practical point of view are the application of the Bäcklund transformation derived by Fordy and Gibbons [17] or the bilinear formalism developed by Hirota [18]. There are also the powerful vertex operator techniques which make full use of the representation theory of the $A_n^{(1)}$ algebra [14]. While the latter approach is currently the most popular, we wish to make contact with the original work of Ruijsenaars and Schneider [5], which made much reference to the soliton formulae of Hirota. Hence we

choose to start from the form of the $N$-soliton solution of (2) derived by Hollowood [19] via Hirota's direct method. The $l$-th component of the field $\phi$ is given by

$$e^{i\beta\phi_l} = \frac{\tau_{l-1}}{\tau_l}, \tag{3}$$

where the tau function $\tau_l$ is of the form

$$\tau_l = \sum_{\epsilon} \exp\left(\sum_{j<k} \epsilon_j \epsilon_k B_{jk} + \sum_j \epsilon_j \zeta_{j,l}(x_+, x_-)\right). \tag{4}$$

In the above the $\epsilon$ indicates a summation over all possible combinations of $\epsilon_j$ taking the values 0 or 1, and the indices $j$ and $k$ take values in $\{1, \ldots, N\}$. We will explain shortly what the various terms in (4) mean, and how we have parametrised the $A_n^{(1)}$ affine Toda solitons. For the moment we would like to comment that expression (4) is a rather generic form of the soliton tau function for an integrable PDE, the precise nature of $B_{jk}$ and $\zeta_{j,l}$ depending on the particular PDE being considered; it may be viewed as a degeneration of the theta function solutions of the PDE given via algebraic geometry in which the $\epsilon_j$'s run over all of the integers. Ruijsenaars and Schneider succeeded in making the connection between their relativistic Calogero-Moser systems and soliton solutions of the sine-Gordon and KdV equations, among others, by showing a direct correspondence between the coordinates of the $N$-particle system and the parameters of the $N$-soliton solution. An important part of the correspondence was that all the tau functions of form (4) being considered in [5] could be written in terms of determinants like

$$\det(1 + M)$$

for suitable matrices $M$. In what follows we express all the $N$-soliton solutions of the $A_n^{(1)}$ affine Toda theory in this way, and thereby obtain a relation to spin-generalised Ruijsenaars-Schneider systems.

First of all we should explain the parameters of the Toda $N$-soliton which appear in (4). Each soliton has a rapidity denoted by $\eta_j$, a position parameter denoted by $a_j$, and a discrete parameter $\theta_j$ taking values in $\{2\pi k/(n+1) | k = 1, 2, \ldots, n\}$ (so that $\exp(i\theta_j)$ is an $(n+1)$th root of unity). The rapidities are all real, while the $a_j$ are pure imaginary for solitons (there are different reality conditions for other types of solution e.g. breathers). The different values of $\theta_j$ give $n$ di ferent species of soliton in the $A_n^{(1)}$ affine Toda th ory whose masses are $2m\sin(\theta_j/2)$. We also need define

$$\mu_j^{\pm} = \exp(\eta_j \pm \tfrac{1}{2} i\theta_j).$$

With this choice of parameters, the terms in the su (4) are given by

$$B_{jk} = \log\left(\frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)}\right),$$

$$\zeta_{j,l}(x_+, x_-) = \log\left(a_j \exp\left(\sqrt{2}m(e^{-\eta_j}x_+ - e^{+\eta_j}x_- \right.\right.$$
$$\left.\left. \times \sin(\theta_j/2) + il\theta_j\right)\right).$$

(To make a comparison with the vertex operat formulae, we note that in terms of the notation Ref. [15], we have $B_{jk} = \log(X_{j,k})$, $a_j = Q_j$. We wi deal with the general formalism elsewhere.)

We are now ready to write the tau functions as d terminants. In fact Olive, Turok and Liao [20] four that determinants naturally arose when they derive the $N$-soliton solution by the Bäcklund transform tion, but the matrices involved are not of the rig form for our purposes. Instead we set $X_j = a_j(\mu_j^+ - \mu_j^-)\exp\left(\sqrt{2}m(e^{-\eta_j}x_+ - e^{\eta_j}x_-)\sin(\theta_j/2)\right)$, an define $N$-by-$N$ matrices $V$, $\Theta$ by

$$V_{jk} = \frac{\sqrt{X_j X_k}}{\mu_j^+ - \mu_k^-}, \tag{5}$$

and

$$\Theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_N).$$

Then we find that

$$\tau_l = \det\left(1 + e^{il\Theta/2} V e^{il\Theta/2}\right). \tag{6}$$

To verify (6) it is necessary to expand the determ nant on the right-hand side in terms of the princip cofactors of $V$, and then use Cauchy's identity:

$$\det\left(\frac{1}{\mu_j^+ - \mu_k^-}\right)_{j,k} = \prod_j \frac{1}{\mu_j^+ - \mu_j^-} \times \prod_{j<k} \frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^+ - \mu_k^-)(\mu_j^- - \mu_k^+)}.$$

Writing the right-hand side of (4) in terms of these new parameters and comparing with the cofactor expansion gives the required result. Note that in the $A_1^{(1)}$ (sine-Gordon) case the $\theta_j$ must all take the value $\pi$, which means that the matrix exponentials appearing in (6) are multiples of the identity, and we reproduce the standard result

$$e^{i\beta\phi_0} = e^{-i\beta\phi_1} = \frac{\det\,(1-V)}{\det\,(1+V)}.$$

## 3. The reduced symplectic form

In this section we describe the phase space of the $N$-soliton solution in terms of its symplectic form, before describing how spin-generalised Ruijsenaars-Schneider systems arise in the following section. The phase space of the affine Toda system has the standard symplectic form

$$\Omega = \int_{-\infty}^{\infty} (\delta\phi_t \wedge \delta\phi)\; dx. \tag{7}$$

On substitution of the $N$-soliton solution into (7), one obtains (after an integration) the reduced symplectic form on the $N$-soliton phase space. In practice it is not possible to perform the integration for anything other than the one-soliton solution [15] (except for the sine-Gordon case, where Babelon and Bernard succeeded in showing that the integrand could be written as an exact derivative for both the one- and two-soliton [4]). For the one-soliton phase space, the reduced symplectic form is (up to an irrelevant numerical factor independent of $\theta$)

$$\omega^{(1)} = \frac{da}{a} \wedge d\eta.$$

The intractability of the integral (7) for the general $N$-soliton solution does not matter, as it is a standard result that as $t \to \pm\infty$ (the out/in limits) the $N$-soliton decomposes into a superposition of $N$ one solitons with a shift of the parameters. So the symplectic form may just be written

$$\omega^{(N)} = \sum_j \frac{da_j^{\text{out}}}{a_j^{\text{out}}} \wedge d\eta_j^{\text{out}} = \sum_j \frac{da_j^{\text{in}}}{a_j^{\text{in}}} \wedge d\eta_j^{\text{in}}. \tag{8}$$

By direct calculation using the formula (4) for the tau functions of the $N$-soliton solution, we find the relations between the out/in parameters and the standard ones:

$$\eta_j^{\text{in}} = \eta_j^{\text{out}} = \eta_j,$$

$$a_j^{\text{in}} = a_j \prod_{k>j} \frac{(\mu_j^+ - \mu_k^+)(\mu_j^- - \mu_k^-)}{(\mu_j^- - \mu_k^+)(\mu_j^+ - \mu_k^-)} = a_j \prod_{k>j} \exp(B_{jk})$$

(and similarly for $a_j^{\text{out}}$ with the inequality reversed). This agrees with the formulae of Spence and Underwood [15] obtained via vertex operator arguments, where in their notation $a_j = Q_j$ and $\exp(B_{jk}) = X_{j,k}$. So substituting for the *in* parameters into (8), we obtain the $N$-soliton symplectic form as

$$\omega^{(N)} = \sum_j \frac{da_j}{a_j} \wedge d\eta_j + \sum_{j<k} E_{jk}(\eta)\sinh(\eta_j - \eta_k)\, d\eta_j \wedge d\eta_k, \tag{9}$$

where

$$E_{jk}(\eta) = \frac{1}{\cosh(\eta_j - \eta_k) - \cos((\theta_j - \theta_k)/2)} - \frac{1}{\cosh(\eta_j - \eta_k) - \cos((\theta_j + \theta_k)/2)}.$$

We observe that $\omega^{(N)}$ is clearly real if we choose the $\eta_j$ to be real and the $a_j$ to be pure imaginary (which in the $A_1^{(1)}$ case coincides with the condition on $a_j$ for sine-Gordon solitons given in [4]). This means that the matrix $V$ defined in (5) is anti-hermitian, which will be important in the next section when we look at the dynamics of the eigenvalues of $V$.

## 4. Ruijsenaars-Schneider systems

Here we consider how the eigenvalues of the matrix $V$ evolve with respect to each of the light cone coordinates, and find that spin-generalised Ruijsenaars-Schneider equations result. Since $V$ is anti-hermitian, it may be diagonalised with a unitary matrix $U$:

$$Q := UVU^\dagger = \mathrm{diag}(i\exp(q_1), \ldots, i\exp(q_N)).$$

If we let a dot denote $d/dx_\pm$, then $V$ satisfies

$$\dot{V} = \tfrac{1}{2}(\Lambda V + V\Lambda), \tag{10}$$

for the constant diagonal matrix $\Lambda$ given by

$$\Lambda = \mathrm{diag}(\pm\sqrt{2}m\exp(\mp\eta_1)\sin(\theta_1/2), \ldots, \pm\sqrt{2}m\exp(\mp\eta_N)\sin(\theta_N/2)).$$

Now let $u_j$ denote the $j$th row of $U$ (considered as a column vector, so that the $u_j$ are the left eigenvectors of $V$). Define the Lax matrix $L$ by

$$L_{jk} = u_k^\dagger \Lambda u_j,$$

so that

$$L = U\Lambda U^\dagger.$$

Then $L$ satisfies the Lax equation

$$\dot{L} = [M, L], \tag{11}$$

for $M = \dot{U}U^\dagger$. Differentiating the definition of $Q$ gives

$$\dot{Q} = [M, Q] + U\dot{V}U^\dagger,$$

and after substituting for $\dot{V}$ from (10) and using the definition of $L$ we find the identity

$$LQ + QL = 2(\dot{Q} + [Q, M]). \tag{12}$$

Upon setting $Q_j = ie^{q_j}$, then (12) in components reads

$$L_{jk}(Q_j + Q_k) = 2(\dot{Q}_j\delta_{jk} + M_{jk}(Q_j - Q_k)),$$

which yields

$$L_{jj} = \dot{q}_j \tag{13}$$

and

$$M_{jk} = \frac{1}{2}\left(\frac{Q_j + Q_k}{Q_j - Q_k}\right)L_{jk} = \tfrac{1}{2}\coth((q_j - q_k)/2)L_{jk},$$

(for $j \neq k$). When $V$ is diagonalised we may always choose the phases of the left eigenvectors $u_j$ so that $M_{jj} = u_j^\dagger \dot{u}_j = 0$. Substituting these into the Lax Eq. (11) produces the equations of motion:

$$\dot{L}_{jj} = \ddot{q}_j = \sum_{k\neq j}\coth((q_j - q_k)/2)L_{jk}L_{kj}, \tag{14}$$

$$\begin{aligned}\dot{L}_{jk} &= \tfrac{1}{2}\coth((q_j - q_k)/2)(\dot{q}_k - \dot{q}_j)L_{jk} \\ &\quad + \sum_{l\neq j,k}\tfrac{1}{2}(\coth((q_j - q_l)/2) \\ &\quad - \coth((q_l - q_k)/2))L_{jl}L_{lk}\end{aligned} \tag{15}$$

($j \neq k$). These are in fact the spin-generalised Ruijsenaars-Schneider equations with certain constraints, although to see this requires comparison with the formulae of Krichever and Zabrodin [21].

In [21] the generalised Ruijsenaars-Schneider model is defined in terms of $N$-particle positions $x_j$ and their internal degrees of freedom (spins) given by $l$-dimensional vectors $a_j$ and $l$-dimensional covectors $b_j^\dagger$, subject to the equations of motion

$$\ddot{x}_j = \sum_{k\neq j}(b_j^\dagger a_k)(b_k^\dagger a_j)(\mathcal{V}(x_j - x_k) - \mathcal{V}(x_k - x_j)), \tag{16}$$

$$\dot{a}_j = \sum_{k\neq j} a_k(b_k^\dagger a_j)\mathcal{V}(x_j - x_k), \tag{17}$$

$$\dot{b}_j^\dagger = -\sum_{k\neq j} b_k^\dagger(b_j^\dagger a_k)\mathcal{V}(x_k - x_j). \tag{18}$$

The potential $\mathcal{V}$ is expressed in terms of the Weierstrass zeta function or its rational or hyperbolic limits. To make contact with our equations we set $x_j = q_j$ and choose the hyperbolic potential

$$\mathcal{V}(q_j - q_k) = \tfrac{1}{2}\coth((q_j - q_k)/2).$$

Then (16) generalises (1). In [21] the spin degrees of freedom were real, but here we allow them to be complex, and identify them with the eigenvectors of $V$ by setting

$$b_j^\dagger = u_j^\dagger, \quad a_j = \Lambda u_j.$$

So we have taken $l = N$, and in fact our spins are expressed entirely in terms of the eigenvectors of $V$

and the constant matrix $\Lambda$; in particular the $b_j^\dagger$ must form an orthonormal basis. In the notation of [21] the components of the Lax matrix are given by

$$L_{jk} = b_k^\dagger a_j.$$

There are various other constraints that we have imposed on our system. First the Eqs. (16), (17) and (18) have the scaling symmetry

$$a_j \to \alpha_j a_j, \quad b_j^\dagger \to \frac{1}{\alpha_j} b_j^\dagger.$$

The corresponding integrals of motion are $\dot{x}_j - b_j^\dagger a_j$, and setting them to zero and rewriting them in terms of our coordinates shows that this is equivalent to Eq. (13). Similarly our requirement that $M_{jj} = 0$ is another constraint on the system. Now given these constraints we find that from the definition of $L$ in terms of the spins we can compute $\dot{L}_{jk}$. So for $j = k$ (16) is equivalent to (14), while for $j \neq k$ (17) and (18) yield (15).

To make the correspondence between the solitons and the many-body system clearer, it is worth considering the sine-Gordon case in more detail and comparing it with the general situation. The results about sine-Gordon solitons are explained in detail in [4], and we have kept our notation as similar to this reference as possible to make comparison easier. The first thing to observe is that in the $A_1^{(1)}$ case only knowledge of the $q_j$ is required to specify the field components, as we have

$$e^{i\beta\phi_0} = e^{-i\beta\phi_1} = \prod_{j=1}^{N} \left( \frac{1 - i\exp(q_j)}{1 + i\exp(q_j)} \right).$$

In the general case the presence of the matrix $e^{il\Theta/2}$ in the expression for the tau functions (6) means that knowledge of both the spin vectors $u_j$ (which make up the matrix $U$) and the $q_j$ is required to evaluate these determinants. The essential difference is that for sine-Gordon there is only one soliton species, while in the $A_n^{(1)}$ case there are $n$ different species corresponding to the different allowed values of $\theta_j$. This difference is also apparent at the level of the equations of motion. In fact when we differentiate the matrix $V$, in the case of sine-Gordon we find from (10) that

$$\dot{V} = i(ee^\dagger)$$

for a certain vector $e$. But then conjugating the Eq. (10) with $U$ we obtain

$$i\tilde{e}\tilde{e}^\dagger = \tfrac{1}{2}(LQ + QL),$$

where $\tilde{e} = Ue$. Actually $\tilde{e}$ is a real vector, and in terms of its components $\tilde{e}_j$ we have

$$L_{jk} = 2\,\frac{\tilde{e}_j \tilde{e}_k}{\exp(q_j) + \exp(q_k)}.$$

Since we know the diagonal elements of $L$ explicitly in terms of the $q_j$ (from (13)) the above formula means that we then know all the $\tilde{e}_j$ and hence the off-diagonal elements of $L$ are found to be

$$L_{jk} = \frac{\sqrt{\dot{q}_j \dot{q}_k}}{\cosh((q_j - q_k)/2)}.$$

This may then be substituted into (14), (15) to give the ordinary (non-spin) Ruijsenaars-Schneider equations. In this case (14) yields (1) and (15) is a consequence. Babelon and Bernard have shown [4] that there is a canonical transformation between the soliton parameters and the dynamical variables $q_j, \dot{q}_j$ (more precisely, they formulate this in terms of the variables $Q_j = i\ \exp(q_j)$). We discuss how this could possibly be extended to the $A_n^{(1)}$ case in our Conclusion.

## 5. Conclusion

We have shown the connection between spin-generalised Ruijsenaars-Schneider systems and $A_n^{(1)}$ affine Toda solitons. The soliton tau functions are determined by the positions $q_j$ of $n$ particles on the line as well as an orthonormal set of $n$-dimensional spin vectors $u_j$, which are together subject to the equations of a constrained spin-generalised Ruijsenaars-Schneider model. This extends the known result for the sine-Gordon equation, where the spins are no longer part of the dynamics and there is a canonical transformation between the positions and momenta of the particles and the parameters of the solitons. For the general case such a transformation is no longer apparent, although we note that the $N$-soliton phase space is still of dimension $2N$, and so it is worth exploring exactly how the extra spin degrees of freedom are absorbed in the transition from the dynamical variables to the soliton parameters. Also it

would be interesting to see what rôle the spins might play in the quantum theory. Finally there remains the extension to the other affine algebras and elucidating the connections to the vertex operator constructions mentioned at various points in the text. We intend to pursue these points in the future.

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# Non-Autonomous Hénon-Heiles Systems

Andrew N.W. Hone

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**Abstract**

We obtain non-autonomous generalizations of the Hénon-Heiles system by considering scaling similarity solutions of certain fifth-order nonlinear PDEs. The resulting equations are integrable in the sense of having the Painlevé property, and we exhibit Bäcklund transformations for them and produce some rational solutions as well as others related to the first Painlevé transcendent.

## 1 Introduction

In the following we will be looking at some similarity reductions of infinite dimensional integrable systems, which yield ODEs with the Painlevé property. One might wonder why such reductions of PDEs should be worthy of interest. There are various reasons to consider them. The first point to note is that, as described in Ragnisco's lecture (appearing elsewhere in this volume), if an ODE is the stationary flow of a soliton equation then it inherits some of the integrable structure of that soliton equation e.g. conserved quantities, Bäcklund transformations (see [1],[2] for examples of stationary flows related to the systems described below). More generally, given a PDE in 1+1 dimensions with independent variables $x$, $t$ and dependent variable $u(x,t)$, the problem of finding solutions is considerably simplified if we seek a solution in the similarity form, that is

$$u(x,t) = U(w(z), x, t), \tag{1}$$

where $z = z(x,t)$ is the similarity variable, and on substituting $U(w,x,t)$ into the PDE, an ODE for $w(z)$ is obtained. There are various ways of finding similarity forms, the most common being the classical Lie symmetry approach (although this method does not yield all possible similarity solutions; see [3] for a case where it fails, as well as references to the other techniques). Obviously a stationary flow is just a special sort of similarity solution. One may similarly find similarity solutions to systems of equations, and for PDEs with more independent variables, but we will not be interested in such complications here.

Once we have a similarity form, we are left with an ODE to solve. The fact is that if we start with a PDE which has lots of nice properties (such as solvability by inverse scattering) then the resulting ODE should be correspondingly manageable. This is expressed more precisely in the conjecture of Ablowitz, Ramani and Segur (ARS), which states that all similarity reductions of integrable PDEs have the Painlevé property (although due to the fact that no universally accepted definition exists, we are being deliberately imprecise about what we mean by "integrable"). For a good account of the Painlevé property, refer to Goldstein's talk (in these Proceedings), and for further details as well as some theorems which support the ARS conjecture, see [4].

For comparison with what follows it is worth looking at a well-known example. If we start with the modified Korteweg-deVries (mKdV) equation

$$v_t = v_{xxx} - 6v^2 v_x, \tag{2}$$

and notice that it has a scaling symmetry, then this gives us its scaling similarity solutions. More explicitly, (2) is invariant under $x \to \beta x$, $t \to \beta^3 t$, $v \to \beta^{-1} v$, and so this implies that there is a similarity solution

$$v(x,t) = (-3t)^{-\frac{1}{3}} y(z(x,t)), \tag{3}$$

with the similarity variable $z = (-3t)^{-\frac{1}{3}} x$. Substituting this form into (2) we find that $y$ satisfies

$$y''' = 6y^2 y' + zy' + y \tag{4}$$

($'$ denotes $\frac{d}{dz}$), which may be integrated once to give

$$y'' = 2y^3 + zy + \alpha, \tag{5}$$

for some constant $\alpha$. (5) is known as the second Painlevé equation (P2), being the second exceptional equation in Paul Painlevé's classification of second order equations having no movable singularities other than poles (exceptional in the sense that its general solution cannot be expressed in terms of classical transcendental functions).

In the next section we will look at three integrable hierarchies of evolution equations (the Sawada-Kotera, Korteweg-deVries and Kaup-Kupershmidt hierarchies), and derive their scaling similarity forms. The third section concerns some completely integrable finite dimensional Hamiltonian systems known as the Hénon-Heiles systems. We describe how they are related to stationary flows of some of the PDEs looked at in the previous section, and relate the similarity solutions of these PDEs to non-autonomous versions of the Hénon-Heiles systems. In the last section we derive Bäcklund transformations for the non-autonomous equations, and show how they may be used to generate special families of solutions.

## 2 Some properties of the KdV, Sawada-Kotera and Kaup-Kupershmidt Hierarchies

Before looking at the non-autonomous systems of the title, we will consider some aspects of three different hierarchies of PDEs, known as the KdV, Sawada-Kotera (SK) and Kaup-Kupershmidt (KK) hierarchies, that are needed in what follows. Each hierarchy is a sequence of evolution equations or flows with respect to times $t_n$ $(n = 1, 2, 3, ...)$, which can all be put into Hamiltonian form. SK and KK have only one Hamiltonian structure, but KdV is bi-Hamiltonian, and here we will be using the second Hamiltonian structure. The $n$-th flow in each of the hierarchies can be written as

$$\frac{\partial u}{\partial t_n} = (\partial_x^3 + 8au\partial_x + 4au_x)\delta_u H_n[u], \tag{6}$$

where $a = 1/2$ for SK and KdV, $a = 1/4$ for KK, and $H_n$ is the $n$-th Hamiltonian for the hierarchy in question. For the purposes of computing variational derivatives, we make no distinction between a Hamiltonian and its corresponding Hamiltonian density. For more details on these hierarchies and ways of calculating the sequence of Hamiltonians, see e.g. [7],[8].

There is also a Miura map from the modified versions of the hierarchies, given by

$$u = -v_x - 2av^2 =: M[v].$$

Then re-writing the Hamiltonian in terms of $v$ and derivatives, the $n$-th modified flow may be expressed as

$$\frac{\partial v}{\partial t_n} = (-\partial_x)\delta_v H_n[M[v]]. \tag{7}$$

The Miura map means that given $v$ satisfying (7) for each $n$, the corresponding $u = -v_x - 2av^2$ satisfies (6).

The $n$-th flow of the hierarchy is unchanged by the scaling

$$x \to \beta x,$$

$$t_n \to \beta^m t_n,$$

$$u \to \beta^{-2} u,$$

where $m = m(n)$ is a scale weight dependent on the hierarchy. Similarly the modified flow is invariant under the same scaling but with

$$v \to \beta^{-1} v.$$

Hence there are scaling similarity solutions looking like $u = t^{-\frac{2}{m}}\, w(\frac{x}{t^{\frac{1}{m}}})$. For convenience in what follows we scale the similarity variable so that

$$u(x, t_n) = \theta^2(t_n) w(z),$$

where $z = x\theta(t_n)$ and $\frac{d\theta}{dt_n} = \theta^{m+1}$. The corresponding similarity solution for the modified flow is

$$v = \theta(t_n)y(z),$$

with the scaled Miura map giving $w = -y' - 2ay^2$ ($'$ denotes $\frac{d}{dz}$ throughout).

Henceforth we will drop the suffix $n$. Substituting the similarity forms into the equations of motion (6) and (7) (and cancelling out powers of $\theta$ on either side) yields the ODEs for $w$ and $y$. If we let $\tilde{H}$ denote the scaled Hamiltonian (expressed in terms of $w$ with powers of $\theta$ divided out) then we have simply

$$(\partial^3 + 8aw\partial + 4aw')(\delta_w\tilde{H} - \frac{1}{4a}z) = 0, \tag{8}$$

$$\partial(\delta_y\tilde{H} + zy) = 0. \tag{9}$$

Both of these equations can be integrated once, and are conveniently written in terms of

$$f := \delta_w\tilde{H} - \frac{1}{4a}z.$$

Integration of (8) yields immediately

$$\frac{d^2f}{dz^2} + 4awf + \frac{\lambda^2 - (\frac{df}{dz})^2}{2f} = 0. \tag{10}$$

For (9), note that

$$\delta_v H = (M')^\dagger \delta_u H = (\partial_x - 4av)\delta_u H, \tag{11}$$

where $M'$ is the Fréchet derivative of $M$. The scaled similarity form of this relation (involving $y$ and $\delta_w\tilde{H}$) allows (9) to be written in terms of the quantity $f$ and integrated to

$$\frac{df}{dz} - 4ayf + \lambda = 0. \tag{12}$$

In (10) $f$ is to be thought of as a function of $w$ and its derivatives, while in (12) it is expressed instead in terms of $y$ and derivatives of $y$ (replacing each $w$ by $-y' - 2ay^2$). $\lambda$ is a constant of integration, and is in fact the same constant in both cases, as the Miura map becomes a one-one correspondence between the two equations. Note that the equations for the stationary flows are simply obtained by removing the $-\frac{1}{4a}z$ terms from $f$. The form of the equations makes it particularly simple to see the relationship between them. The scaled Miura map means that if $y$ satisfies (12) then $w$ must satisfy (10), and this is obtained directly by substituting

$$y = \frac{f' + \lambda}{4af}$$

into $w = -y' - 2ay^2$. Conversely, given $w$ satisfying (10), then the same substitution for $y$ (with $f$ expressed in terms of $w$) rearranges to give (12).

To make things more concrete, it is worth looking at some particular cases. The first example to consider is the scaling similarity solutions of the ordinary KdV equation. Putting $H = \frac{1}{2}u^2$ (the first non-trivial Hamiltonian in the KdV hierarchy) into (6) with $a = \frac{1}{2}$ we obtain KdV:

$$u_t = u_{xxx} + 6uu_x.$$

The scaling similarity solutions are given by

$$u(x,t) = (-3t)^{-\frac{2}{3}} w(z),$$

with the similarity variable $z = (-3t)^{-\frac{1}{3}}x$. After substituting into KdV and integrating once we find the ODE for w:

$$w'' + 2w^2 - zw + \frac{\alpha(\alpha+1) + w' - (w')^2}{2w - z} = 0. \tag{13}$$

Using the scaled Hamiltonian $\tilde{H} = \frac{1}{2}w^2$ we find

$$f = w - \frac{z}{2},$$

and substituting into (10) with this $f$ and $a = \frac{1}{2}$ does indeed give the equation (13) on setting $\lambda = \alpha + \frac{1}{2}$. Also the Miura map $u = -v_x - v^2$ goes from mKdV to KdV. For the scaling similarity solutions of mKdV (3) we find that there is a one-one correspondence between solutions of P2 (equation (5) of the previous section) and (13), given by

$$w = -y' - y^2$$

(the scaled Miura map) and

$$y = \frac{w' + \alpha}{2w - z}.$$

Note that in terms of $y$, we have

$$f = -y' - y^2 - \frac{z}{2},$$

and on putting this into (12) with $a = \frac{1}{2}$, P2 results. This particular case is considered in [9] for example.

As our second example we take the fifth order equations in each of the hierarchies, which following [1] may be written as

$$u_t = (u_{xxxx} + (8a - 2b)uu_{xx} - 2(a+b)u_x^2 - \frac{20}{3}abu^3)_x, \tag{14}$$

where we have three cases (i),(ii),(iii) corresponding to $a = \frac{1}{2}, \frac{1}{2}, \frac{1}{4}$ and $b = -\frac{1}{2}, -3, -4$ respectively. These are the only values of $a, b$ for which an equation of

the form (14) is integrable, and (i) and (iii) are respectively the Sawada-Kotera and Kaup-Kupershmidt equations (see references in [8]), while (ii) is the fifth order KdV equation. (14) may be obtained from the Hamiltonian formalism described above, by taking

$$H = -\frac{1}{2}(u_x)^2 - \frac{1}{3}bu^3$$

with the correct values of $a$ and $b$ for each case. Then the scaling similarity form is

$$u(x,t) = (-5t)^{-\frac{2}{5}}w(z),$$

where now $z = (-5t)^{-\frac{1}{5}}x$. We find

$$f = w'' - bw^2 - \frac{z}{4a},$$

and putting this into (10) we find a fourth order ODE for w. Using the scaled Miura map, we may express $f$ in terms of $y$:

$$f = -y''' - 4ayy'' - (4a+b)(y')^2 - 4aby^2y' - 4a^2by^4 - \frac{z}{4a}.$$

Then the ODE for $y$ is (from (12))

$$y^{(iv)} = -2(6a+b)y'y'' + 4a(4a-b)(y^2y'' + y(y')^2) + 16a^3by^5 + zy + \alpha, \quad (15)$$

with $\alpha = \lambda - \frac{1}{4a}$. Given a solution to (15) we can then obtain a solution to the fourth order ODE for $w$ via $w = -y' - 2ay^2$. Notice that on substituting in the relevant values of $a$ and $b$ the resulting equations (15) for cases (i) and (iii) differ only by a sign in the even ($y'y''$) terms. So if $y_\alpha$ is a solution to the equation in case (i), then $Y_{-\alpha} = -y_\alpha$ is a solution to the equation in case (iii) at parameter value $-\alpha$. This is because the modified hierarchies in these two cases are essentially the same. We shall be using this property in the final section to derive the Bäcklund transformation.

# 3 Hénon-Heiles Systems

The original Hénon-Heiles system is given by a Hamiltonian with two degrees of freedom:

$$h = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{1}{3}bq_1^3, \quad (16)$$

The equations of motion are just Hamilton's equations

$$\frac{dq_j}{dz} = \frac{\partial h}{\partial p_j}, \quad (17)$$

$$\frac{dp_j}{dz} = -\frac{\partial h}{\partial x_j} \tag{18}$$

(we are denoting the time by $z$ here to make connection with results which appear below). It was known for some time from Painlevé analysis [5] that this system is integrable for three values of the ratio $r = a/b$ (because of a scaling symmetry of the equations the integrability only depends on this ratio), i.e. $r = -1, -1/6, -1/16$.

More recently, Fordy [1] showed that for these integrable cases the equations of motion were just disguised versions of the stationary flows of some fifth order soliton equations - the Sawada-Kotera, fifth order Korteweg-deVries and Kaup-Kupershmidt equations (hence the choice of values for $a$ and $b$ as in the previous section gives the right values for the ratio $r$ in the cases (i),(ii),(iii)) . The zero curvature form of these PDEs yields a matrix Lax representation of the stationary flows, and then traces of powers of the Lax matrix give the Hamiltonian and the second constant of motion (which shows that these systems are indeed Liouville integrable). It was subsequently shown that all three systems are completely separable in suitable coordinates, and may be integrated in terms of theta functions of genus one (cases (i) & (iii)) or genus two (case (ii))[6].

Instead of looking at the stationary flows of these three PDEs, we will take the equations for the scaling similarity solutions, and rewrite them in Hamiltonian form. So we have $f = w'' - bw^2 - \frac{z}{4a}$, and $f$ must satisfy the equation (10). On setting $w = q_1$, $f = -aq_2^2$, we find

$$q_1'' = bq_1^2 - aq_2^2 + \frac{z}{4a},$$

$$q_2'' = -2aq_1q_2 - \frac{\lambda^2}{4a^2q_2^3}. \tag{19}$$

These are just Hamilton's equations for the system with Hamiltonian

$$h = \frac{1}{2}(p_1^2 + p_2^2) + aq_1q_2^2 - \frac{1}{3}bq_1^3 - \frac{\lambda^2}{8a^2}q_2^{-2} - \frac{1}{4a}zq_1,$$

which is just (16) with an extra inverse square term and a non-autonomous (time-dependent) term in the potential. So the similarity solutions introduced at the end of Section 2 may be viewed as non-autonomous Hénon-Heiles systems. Because of the explicit time-dependence, the Hamiltonian is no longer a constant of motion, and we don't get a matrix Lax representation any more. Below we discuss the integrability properties of these equations, as well as Bäcklund transformations which generate some special families of solutions.

# 4 Integrability, Bäcklund transformations and some special solutions

Given the zero curvature representation of the fifth order PDEs considered above, it is straightforward to obtain the matrix Lax representation of the equations for the stationary flows. It is no longer possible to do this for the corresponding non-autonomous systems, so we have to use other methods to test their integrability. The most obvious thing to try is the Painlevé test, and indeed this does find principal balances (with four arbitrary constants in the power series solutions) in each of the cases (i),(ii),(iii) (for more details see [10]). A more useful approach from the point of view of finding exact solutions is to look at the similarity equations coming form the modified hierarchy.

The variable $w$ appearing as $q_1$ is in one-one correspondence with a solution $y$ of (15), and the latter is a lot more tractable for various reasons. The zero curvature representation of the PDEs in the modified hierarchy scales nicely, so that on restricting it to the scaling similarity forms it produces a zero curvature representation of (15). This then means that the initial value problem for $y$ can be solved in terms of an inverse monodromy problem (an ODE analogue of inverse scattering). This method has been pursued in detail for the second Painlevé transcendent P2 (see [11],[12]). In [11] the scheme is also outlined for the scaling similarity solutions of the higher order equations in the modified KdV hierarchy (referred to as the Painlevé II Family). The equation (15) in case (ii) is the next equation up from P2 in this family.

A simpler approach to finding solutions is to derive the Bäcklund transformations for the equation (15) for each of the three cases. We consider (ii) first, as it is the simplest case and the Bäcklund transformation takes the same form as for P2 (and indeed for the whole P2 family [13]). Everything is most conveniently expressed in the notation of Section 2. So we take the equation for $y$ in the form (12). Now let $y_\alpha$ denote a solution to this at parameter value $\alpha$, and define

$$w_\alpha = -y'_\alpha - y_\alpha^2 \tag{20}$$

to be the corresponding solution to (10), with $\lambda = \alpha + \frac{1}{2}$, $a = \frac{1}{2}$. Also take

$$f_\alpha = w''_\alpha + 3w_\alpha^2 - \frac{z}{2}.$$

Now to find the Bäcklund transformation we just need to use two facts. The first is that given a solution $y_\alpha$ for parameter value $\alpha$, we have that $y_{-\alpha} = -y_\alpha$ is also a solution for parameter value $-\alpha$. The second thing to notice is that under the scaled Miura map (20), two solutions $y_\alpha$ and $y_{-(\alpha+1)}$ correspond to the same value of $\lambda^2$ and hence the same solution of (10), we obtain

$$y_{\alpha+1} = -y_\alpha + \frac{2\alpha+1}{2f_\alpha}. \tag{21}$$

This gives the standard transformation for P2 [9], on putting $f_\alpha = w_\alpha - \frac{z}{2}$. To obtain the sequence of rational solutions to (15) in case (ii), we apply (21) repeatedly starting from $y_0 = 0$. Some of these solutions (together with the corresponding $w_\alpha$) are given in a table below.

| $\alpha$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $y_\alpha$ | 0 | $-\frac{1}{z}$ | $-\frac{2}{z}$ | $\frac{2}{z} - \frac{5z^4}{z^5-144}$ |
| $w_\alpha$ | 0 | $-\frac{2}{z^2}$ | $-\frac{6}{z^2}$ | $-\frac{2}{z^2} - \frac{10z^3(z^5+576)}{(z^5-144)^2}$ |

These solutions may also be obtained from the sequence of polynomials [14] which give the rational solutions of the KdV hierarchy; this is discussed in more detail in [10].

Since, as mentioned in Section 2, the equations (15) differ only by a minus sign for (i) and (iii), these two cases may be dealt with together. We employ the same notation as before, with lower case and upper case letter y corresponding to solutions to (15) in (i) and (iii) respectively. Also we introduce $w_\alpha = -y'_\alpha - y_\alpha^2$, $f_\alpha = w''_\alpha + \frac{1}{2}w_\alpha^2 - \frac{1}{2}z$, and similarly $W_\alpha = -Y'_\alpha - \frac{1}{2}Y_\alpha^2$, $F_\alpha = W''_\alpha + 4W_\alpha^2 - z$. Using $Y_{-\alpha} = -y_\alpha$ as well as the fact that (corresponding to the same value of $\lambda^2$ in each case) $w_\alpha = w_{-\alpha-1}$ and $W_\alpha = W_{-\alpha-2}$, we find a slightly different sort of Bäcklund transformation:

$$y_{\alpha+3} = y_\alpha - \frac{2\alpha+1}{2f_\alpha} + \frac{2(\alpha+2)}{F_{\alpha+1}}.$$

Again we can find a sequence of rational solutions by applying this starting at $y_0 = 0$, but now we miss out integer values in between. In fact we can find the solutions only for every third integer. We present a few of these in a table, with the corresponding $w_\alpha$ and $W_{-\alpha} = y'_\alpha - \frac{1}{2}y_\alpha^2$, thus:

| $\alpha$ | -4 | -3 | -1 | 0 | 2 | 3 |
|---|---|---|---|---|---|---|
| $y_\alpha$ | $\frac{4}{z}$ | $-\frac{2}{z} + \frac{5z^4}{z^5+36}$ | $\frac{1}{z}$ | 0 | $-\frac{2}{z}$ | $-\frac{3}{z}$ |
| $w_\alpha$ | $-\frac{12}{z^2}$ | $-\frac{6}{z^2}$ | 0 | 0 | $-\frac{6}{z^2}$ | $-\frac{12}{z^2}$ |
| $W_{-\alpha}$ | $-\frac{12}{z^2}$ | $-\frac{15z^3(z^5-144)}{2(z^5+36)^2}$ | $-\frac{3}{2z^2}$ | 0 | 0 | $-\frac{3}{2z^2}$ |

One might wonder if there was any advantage in writing the equation (10) for $w$ as a non-autonomous Hamiltonian system. In fact in case (i) when $\alpha = -\frac{1}{2}$

($\lambda = 0$) the same substitution that works in the ordinary (autonomous) system causes the equations of motion to separate. Putting

$$Q_{\pm} = q_1 \pm q_2$$

into (19), we find

$$Q''_{\pm} = -\frac{1}{2}Q_{\pm}^2 + \frac{z}{2},$$

which (up to a scaling) is just two separate copies of the first Painlevé equation. The corresponding solution to (15) is

$$y_{-\frac{1}{2}} = (log(Q_+ - Q_-))',$$

where we assume that $Q_+$ and $Q_-$ are not equal. So plugging this into the Bäcklund transformation we get a whole sequence of solutions in terms of the first Painlevé transcendent. For further details and results see [10].

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