

# Optimising the Value of Assets and Financial Contracts in the Energy Industries

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## Abstract

The UK energy industries have been through a number of changes in recent years, leading to a need for market participants to adapt to the new environment. The new market structure is a topical research area due to the large number of energy products now being traded. As a result, the valuation of both simple and more complex products which are now appearing, is of high priority to market participants. This research considers three major types of contract found in energy markets; gas storage, take or pay contracts and tolling deals.

Gas storage enables participants to purchase gas when market prices are deemed low, to be used during periods of high market prices, but are reliant on accurate pricing models. Take or pay contracts were developed from the gas storage problem; these contracts allow the holder to vary the amount of gas taken on each day within specified daily and annual limits. Tolling deals are a method of converting fuel into energy without actually having to own a power station. These contracts also describe certain constraints that must be satisfied, such as environmental limits on the amount of harmful gases released by the process.

In collaboration with Innogy plc, techniques were developed to give an indication of the value which can be placed on holding the above types of contract. The contracts are modelled in a variety of ways. Linear programming models are first used to give simple approximations to the problems. The energy prices present in such models are stochastic but can be approximated by using expected prices which are predicted from market data. Models were developed to include stochastic prices by constructing a tree of expected future prices. This approach lends itself to a stochastic dynamic programming approach as this is an efficient way of traversing a tree structure. Stochastic dynamic programming is simpler and faster for this type of problem than other techniques such as stochastic linear programming.

Data from the energy industries has been obtained and used to construct a realistic test set of problems. The models provide a framework for companies to

investigate the consequences of a variety of different situations. Insights into the way in which the assets and contracts function have also been obtained.

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## Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

*(Corrie L Aitken)*

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# Chapter 1

## Introduction and Research Objectives

### 1.1 Introduction

The energy industries have undergone a number of changes in recent years. These changes have brought new situations, which require extensive research. Models can be developed to describe the new structure of both the industries and complex contracts which are being developed and traded. The changes have led to greater interest in the physical nature of assets as generators must now try to maximise all possible profits. These assets can be either owned or leased. The complex contracts have been developed in order to take account of this new emphasis.

### 1.2 Research Objectives

The research was undertaken in conjunction with Innogy plc. The aim of the research is to identify the areas which were of specific interest as a result of the new industry structure and to develop models for use in each of these areas. These models place a value on various contracts and assets present in each of the areas previously identified and represent an improvement on those models currently being used. This will aid energy companies when trading. The thesis demonstrates the benefits of using mathematical programming techniques in these



areas.

After consultation with Innogy plc., the following areas were identified:

- **Gas Storage**

This is a method by which gas can be bought when market prices are deemed to be low and injected into the storage facility. During periods of high market prices, the gas can be extracted from the facility in order to avoid buying requirements at high prices.

- **Take or Pay Contracts**

Take or pay contracts were developed from the gas storage problem. When gas is released from a storage facility, physically the flow cannot be stopped, only adjusted whilst ensuring it still lies between certain limits. This type of contract reflects these constraints by specifying maximum and minimum daily and annual amounts of gas which can be taken. The holder of such a contract can take any amount of gas on each day provided the constraints are satisfied.

- **Tolling Deals**

Tolling deals are a method of converting fuel into energy without actually having to own a power station. Included in these contracts are certain constraints which must be satisfied, such as environmental limits on the amount of harmful gases released by the process.

A stochastic dynamic programming model was developed for each of these areas in order to provide energy companies with an improved, detailed model with which to place a value on contracts and assets. Each model was developed in a number of stages, allowing the earlier stages to be used to check the model as it incorporates additional features. This results in a complex, validated model which can be considered to be accurate as a result of the checks which have taken place. In order to incorporate uncertain energy prices, these models must include stochastic parameters. This increases the complexity of the models and can also lead to much larger solution times.

### 1.3 Structure of Thesis

An overview of the energy industries is given in Chapter 2, which focuses on the electricity and gas industries with which the research is concerned. The aim of this chapter is to provide a background on the recent restructuring of these industries which has given rise to the need for the development of new types of model. A background on various types of power plants and fuels is given, together with a broad outline of the way in which the electricity and gas industries function. The section also includes details of the aims of the European Commission to liberalise the industries to enable market participants to trade in any country.

Chapter 3 gives a background of financial definitions, which focuses on terms which are commonly found when modelling in the energy industries. The financial terms and features described within this chapter will be widely used in the remainder of the thesis. Black and Scholes [9], developed a model for valuing an option. Their model has been extended by Black [8], to price options on futures and also by Margrabe [43], to determine a value for exchanging one risky asset for another. A description of the three models is given. Energy prices can be of two types; spot prices and forward prices. A spot price is the price at which energy can be sold today, whereas a forward price is the price at which energy can be sold at some point in the future. When constructing models to value financial contracts, it is very important to have a true representation of forward prices for each day in the time period. Various methods are described for estimating these forward prices.

Chapter 4 describes different methods for solving problems which arise in energy markets. This section is written to be accessible by non-specialists and introduces the techniques which are employed in the construction of the models, detailed in Chapters 6, 7 and 8. It begins with a description of common mathematical programming solution methods, which include the techniques of linear and integer programming and stochastic dynamic programming, all of which are employed in subsequent chapters. The major advantages of using mathematical programming techniques are that they are well established and have a strong mathematical basis. Much research has taken place into the techniques,

including the various solution methods, and there are a variety of texts available on the subject; see Winston [70], and Taha [61]. A major advantage of using such techniques is the ease with which models can be implemented and results interpreted.

One method of incorporating uncertain or stochastic variables into problems is by means of a tree structure of possible values which the variables can take. Two different methods by which these trees can be generated are discussed. The first of these describes a method for making the values at each node of the tree consistent with values from a forward curve which can be predicted by the market. The second approach involves constructing the tree by considering statistical values which can be obtained from studying historical data.

Mathematical programming solution methods have traditionally been applied in a variety of areas in the energy industries, which was one of the main reasons for applying these methods to the problems discussed in this research. Recently, interest in the structure of the industries after restructuring has emerged and various approaches are detailed in Chapter 5. Literature in the area of portfolio optimisation is also considered. This area is becoming increasingly more important due to the impact of the changes on the attitudes of market participants. An overview of literature in the areas of changed industry structure and portfolio optimisation is presented, leading to the construction of a model which aims to optimise the value of the assets and contracts which it holds under the new market structure. This was seen to be the main area of interest after privatisation.

The model developed is reasonably complex and therefore unlikely to be understood by non-specialists. The model represents a general approach for representing the aims of market participants. In order for models to be implemented in industry, the way in which they work and the benefits which they provide should be clear. Investigating the main contracts and assets which are present will give more information. A selection of assets and contracts have been identified to be important in the new situation. The assets include gas storage facilities which are investigated in Chapter 6. The most important types

of contract identified were take or pay contracts, detailed in Chapter 7, and, to a lesser extent, tolling deals, modelled in Chapter 8. The models which have been developed to represent each of these situations can be seen to be components of this larger model.

The gas storage problem described in Chapter 6, gives a brief description of storage in the UK and the different types of storage facility available. Various models were developed to give estimates of the value of an amount of storage which can be either owned or leased. The solution to the models give indications of when to inject gas into and extract gas from the facility in order to maximise the resulting value. The simpler models were extended to include stochastic variables by constructing a tree structure of possible gas prices using the method described by Hull and White [34], and Clewlow and Strickland [19]. This resulted in a stochastic dynamic programming model to describe the situation. Gas storage is important as it can provide gas when inaccuracies have occurred in predicting supply and demand, such as large forecasting errors, breakdowns or large changes in pressure in the pipeline. Energy has periods during the year in which demand is high and periods in which it is low. Gas can be stored during periods of low demand for use in periods when demand is higher.

Take or pay contracts (also known as swing contracts) allow the holder to vary the amount of commodity bought on a particular day within daily and annual limits. Failing to take the required quantity of gas leads to very high penalty charges. Take or pay contracts are very common in energy markets as they provide a flexible way of buying gas to satisfy unknown demands. Chapter 7 describes the properties of such contracts. Models were developed which aim to maximise the value of such a contract. The solutions to these models give indications of the amount of gas which should be bought on each day throughout the time period. Stochastic variables have been incorporated in a similar way to that used in Chapter 6, when the gas storage problem was considered. The resulting stochastic dynamic programming model can be used to value various take or pay contracts.

Tolling deals allow the holder of the contract to exchange fuel for power, either in a physical or financial context. These contracts usually have various constraints associated with them. These could be environmental constraints on the amount of harmful gases released during the power generation process or, alternatively, limits on the number of times in which the power plant can change generation level. This type of contract is described in Chapter 8. Models have been developed to optimise the value of holding contracts such as these by determining the optimal generating level at every stage. This corresponds to the amount of power taken in each period. Stochastic variables have been incorporated by means of a tree structure for two linked variables; gas price and power price. This was achieved by extending an approach described by Hull and White [35]. This produces a stochastic dynamic programming model. It is fairly easy to adapt this model to value tolling deals which include a variety of different constraints.

Solutions obtained from mathematical models such as these can never hope to represent the future exactly, but they can give good indications of what may happen and also insights into properties of such contracts. The values obtained from the solution of these models can be used to give guidance about the best way to maximise the value in each case. The modelling approaches for each of the different types of contract considered are compared. Conclusions are made in Chapter 9 and areas in which further research could be undertaken and developed are also proposed.

# Chapter 2

## The Energy Industries

### 2.1 Introduction

Significant changes in both the electricity and gas industries have occurred since 1990 when the move towards privatisation began. This chapter aims to highlight new or modified operational issues which need to be addressed. This chapter gives a background about the recent changes which have occurred in the energy industries. This restructuring has led to a need for new types of model to be developed to represent the changing situations. Much of the background contained in this chapter provides context to the industries, allowing the reader to become familiar with the features which are present. It begins with an overview of the various types of generation plant which exist and the reasons behind using specific fuels. Environmental concerns have led to stringent regulations on emissions produced when generating power, which can impact on the choice of fuel and type of plant used.

The recent move towards privatisation of the energy industries in many countries around the world is discussed next. It is believed that the industries will become more efficient as a result. The continual development of the industries is demonstrated by a description of the aims of the recent European Commission directive to promote liberalisation of the industries in Europe. The ultimate aim is to create energy industries throughout Europe in which everyone has the right to trade freely. A background on the complete restructuring of both the electricity



and gas industries in the UK is given. This background focuses on England and Wales since this is the area for which most of the research was developed.

## 2.2 Various Types of Power Plant

The primary way to meet the growing demand for electricity is to build power stations. The technology chosen will generally be the one which can supply energy at least cost. Power stations which burn fossil fuels are the most popular, but prices of such fuels are constantly changing which can impact on long-term generating costs. The three main fossil fuels used for power generation are coal, oil and natural gas. Of these, in general, coal is the cheapest and oil the most expensive, although transportation costs may also need to be considered. If coal is available near a proposed project, it will tend to be the cheaper option. However, if it has to be transported far, costs can rise very quickly.

Breeze [12], suggests that coal is probably the most important and widely used fuel for electricity generation and that the major attraction of coal is that it can be found in most parts of the world. The main drawback is that when it is burned, it produces large amounts of toxic emissions and is responsible for some of the worst environmental damage.

Breeze [12], also cites the major problem with coal-fired power stations to be the release of  $\text{CO}_2$ , one of the main greenhouse gases contributing to the slow warming of the atmosphere of the earth. High temperature combustion produces nitrogen oxides ( $\text{NO}_x$ ), both from nitrogen contained in the coal and also that found in the atmosphere. All natural coal contains some sulphur which emerges as sulphur dioxide during the process. This is converted to acid in the atmosphere, causing acid rain. Some minerals can also escape with these gases into the atmosphere. These contain trace metals which are potentially harmful.

Modern developments have sought to make coal combustion as environmentally friendly as possible. The only known ways to reduce the amount of sulphur dioxide produced are either to clean the coal before combustion takes place or alternatively, to capture the sulphur after the coal has been burnt by using a

chemical reagent such as lime or limestone. Controlling the temperature at which combustion takes place and the amount of oxygen available during the process can help to control the quantity of nitrogen oxides generated. It is also possible to capture nitrogen oxides after combustion by using ammonia gas or urea. The release of carbon dioxide cannot yet be controlled.

The perceived inefficiency of coal-fired power stations led to the development of the combined cycle gas turbine (CCGT) power plant. This was virtually the only type of large baseload station to be constructed during the 1990s in the UK. The designs for such plants have grown out of those used for aviation engines. Breeze [12] gives the major advantages over conventional coal-fired stations as:

- improved efficiency, i.e. the ratio of energy produced to fuel used is increased
- faster construction times
- less environmental impact
- lower capital costs

In order to achieve increased efficiency, CCGT plants combine both gas and steam turbines. The electricity is produced at two stages. Initially, natural gas is passed through a gas turbine, generating electricity. The exhaust gases from this process are then passed into a heat recovery system which produces steam. This steam is then passed into the steam turbine to produce additional electricity. This results in a significantly higher level of power being produced than can be obtained by using conventional combustion turbines.

Most gas turbine power plants are extremely sensitive to low levels of impurities in the fuel, therefore fuel must be extensively cleaned before it can be burnt. Gas turbines can produce significant quantities of nitrogen oxides, some carbon monoxide and small amounts of hydrocarbons. The amount of nitrogen oxides produced is directly related to the temperature at which the combustion takes place. The higher the temperature, the more nitrogen oxides are produced. Designers are pushing for even higher temperatures in order to increase the efficiency, therefore, the problem of generation of nitrogen oxides has become



more acute. Gas turbines also produce  $\text{CO}_2$  but in much smaller quantities than a coal-fired plant of the same size, since natural gas contains less carbon than an equivalent amount of coal.

Where gas supplies are limited or non-existent, importing liquefied natural gas (LNG) is a possibility. Breeze [12], notes that LNG costs more than piped gas when the cost of liquefaction, transportation and regasification are taken into account. The cost of development of new gas fields and the cost of transportation from ever more different regions produce significant upward pressure on gas prices. On the other hand, increased competition between gas suppliers tend to push natural gas prices down. The European Interconnector between the UK and Belgium allows surplus gas from the UK North Sea fields to be sold in mainland Europe, bringing down European prices.

Power plants which use renewable sources of energy can also be found. Hydropower plants generate energy from flowing water which in most cases costs nothing, but a constant supply is needed. Wind turbines harness wind and are only constructed in areas which are deemed to have enough wind to make such turbines economically viable. Solar plants turn sunlight into electricity. In each of these cases the energy source is free, however, these power stations can not normally provide fixed output, day and night, all year round.

The supply of energy is a matter of strategic interest. Daniel [21] states that power must flow in adequate quantities to ensure that economies flourish, to preserve or improve standards of living and to maintain security. Political decisions play a significant role in the energy sector due to the key role that power plays in the economy. Government intervention is most likely to be seen in the choice of fuel used to generate power. Decisions are normally based on market factors but the market can easily be distorted by government intervention to promote one type of generation or inhibit another.

In recent years, legislation has emerged to encourage renewable energy technologies such as wind and hydropower. Daniel [21], states that in the UK, France, Germany and Poland, coal subsidies have distorted the market in favour

of coal. Developing countries are under pressure to use local fuel reserves and natural resources rather than importing fuel. This helps to preserve valuable foreign currency reserves. Some countries with natural resources well in excess of their own needs such as Laos, Nepal and Iceland, are planning to export both fuel and power to their neighbours.

## 2.3 The Move Towards Privatisation

Energy industries in the UK have recently undergone vast restructuring as a result of privatisation. Privatisation is the transfer of assets from public sector to private sector control. Barnett [4] describes the number of forms which this may take to be:

- the flotation of the entire company on the stock exchange in one step.
- the periodic flotation of pieces of the company, typically of around 20% each.
- the flotation of a 49% stake in the company, the government retaining majority ownership and thus control.
- the flotation of the great majority of the company, whilst the government retains a “golden share” allowing it to veto contentious decisions.
- the sale of the company to a single or several private sector companies.

Under state ownership, the energy industries in the UK were operated as monopolies. The basic structure and the reason for their monopoly positions is the existence of national networks for electricity transmission and gas transportation. These networks have been run as monopolies since it is uneconomic for competing networks to be built. Undue exploitation of power in these areas was not considered a major problem other than in allowing inefficiency to persist.

This state ownership lasted until the late 1980s/early 1990s when the situation changed dramatically, due to the system being perceived as inefficient. The gas industry was sold to the private sector in 1986 and electricity in 1990.

Barnett [4] suggests that the main rationale for these privatisations was to improve performance and service provided to consumers. This was expected to happen when the industries were exposed to market forces, such as competition and demand fluctuations. Financial factors also had an important influence.

The 1989 Electricity Act established the Office of Electricity Regulation (OFFER), headed by the director general of electricity supply, as an independent body responsible for regulating the electricity supply. OFFER has recently merged with the Office of Gas Regulation (OFGAS) to form the Office of Gas and Electricity Markets (OFGEM), headed by a single director general. Further information can be obtained from the OFGEM website [47]. This move reflects the increasing convergence of the two markets.

Barnett [4] states that major success has been achieved by encouraging improved operating efficiency amongst market participants. This has resulted in large profits for companies as they have surpassed any expectation of the regulator. Critics have argued that the benefits of this enforced efficiency should be shared with customers in the form of price cuts rather than increasing shareholder gains. At the same time, companies have been criticised for falling service standards, observed by the increased number of complaints. The regulators have attempted to control standards by introducing quality levels to some aspects of the service.

Existing power utilities are commonly faced with home markets which have limited growth for energy demand, stringent regulation and/or intensifying competition. All of these factors contribute to limited prospects for growth of the company and restricted profits. The potential in some underdeveloped overseas markets, which usually have less restrictions is seen to be large in comparison. A common way to expand into countries in which the market is underdeveloped is by buying an existing company.

Privatisation allows smaller market players to become more prominent. Despite this, such companies often do not pose a significant threat to the larger, well established companies. In order to overcome this, smaller companies may form

alliances with each other. Companies may also form alliances in order to expand overseas. Even the largest companies have difficulty in developing expertise over a wide range of countries, fuels and technologies. By pooling resources and experiences within consortia, the efficiency of operations may be raised. As both resources and efforts are not duplicated, but are coordinated, consumers may be better served or may receive service at a better cost. A growing number of companies are using strategic alliances as a vehicle for entering new international markets and diversifying geographically.

### 2.3.1 Liberalisation of Energy Markets Across Europe

Daniel [21], describes the policy of the European Commission to encourage competition between energy companies throughout Europe; their aim being to increase efficiency and levels of customer service. This represents the continually changing nature of these industries. Creation of internal energy markets will allow a company to supply energy anywhere within the European Union. This policy encourages competition within national boundaries and also across country borders. The ultimate aim of the policy is to build trans-European energy networks and to liberalise the energy markets of all member states.

Regulation of the conduct of companies is largely concerned with placing restrictions upon competitive practices of a company which has monopoly power. Liberalisation is the removal of many of the rules and regulations governing the market. Regulation of prices will continue until it is evident that there is no longer any distortion of the market through excessive market power, and that the industry is strong enough to be left to market forces. The removal of regulatory controls is known as deregulation. In a deregulated market, no company should be able to distort prices. Once deregulation occurs in energy markets, regulatory bodies are required to cover only the remaining monopoly activities such as transmission and distribution.

Daniel [21], suggests that liberalising the energy markets involves allowing easy entry, and if necessary, exit of competing companies to every country's energy market. Governments or regulators must introduce new frameworks

to allow competitive markets, if market forces are to be effective. There is also a requirement to allow market entrants access to the transportation and distribution pipeline networks which are operated as monopolies in many countries. In supporting this, the European Commission has sought to prevent practices such as price or access discrimination. Construction of new capacity, such as for transportation and storage, should be coordinated between countries in order to avoid inefficiency. Sufficient investment in this new capacity must also take place.

The ultimate aim of the Commission is for there to be effectively a single liberalised market for energy, with companies able to produce, transport and supply across all European Union member states. The energy policy of the EU has an important impact on each of the member states, as those countries are bound to comply with directives of which they are signatories. Progress towards liberalisation in the rest of Europe will vary considerably between countries, therefore single European energy markets will take time to emerge.

## 2.4 The Electricity Industry

The UK Electricity System [62], describes the three separate electricity systems which are present in the UK. There are separate systems in England and Wales, Scotland and Northern Ireland. A direct current link exists between England and France, allowing French companies to buy and sell electricity in the UK and vice versa. The research is focused on the industry in England and Wales as this is the largest and more mature of the three. The structure of this market is continually changing, most recently with the introduction of the New Electricity Trading Arrangements. Information for this section has also been obtained from Enron [25].

Following privatisation in April 1990, electricity companies became private sector enterprises, whose main objective was to maximise profit. In pursuit of this goal, it is possible that companies may ignore public interest by acting in a manner which is optimal for them. Monopolistic or oligopolistic behaviour can be observed if one or a number of companies can influence pricing levels. Aside

from the increase in profits which may result from such action, the influence of such companies in domestic and international markets may grow as a result.

There must be sufficient competitors or regulations in a market in order to restrict this type of power. Customers must have a reasonable choice of companies from which to purchase power. In the utility sector, the structure of the industry is such that transmission and distribution will always be operated as monopolies due to the high capital requirement of another network. Furthermore, supplying electricity efficiently requires optimisation which is best achieved from a centrally coordinated network.

Privatisation led to the electricity industry in England and Wales being restructured. Reasons for this restructuring included the introduction of competition into the generation and supply aspects of the industry and financial independence from the Government. The electricity industry has undergone the most extensive restructuring of all the privatised UK utilities. The previous structure was dominated by one large generation and transmission company, the Central Electricity Generating Board (CEGB), which sold electricity to twelve distribution boards, each of which served a closed, regional supply area.

After privatisation, the generation assets were split between three companies. The fossil fired generators were divided between National Power and Power Gen and the nuclear power stations were transferred to Nuclear Electric. The transmission became the remit of the newly created National Grid Company (NGC), whose task was to distribute electricity and facilitate a competitive supply environment. The National Grid Company also initially held the pump storage stations which were used to meet peak demand. Twelve regional electricity companies were formed from the twelve area distribution boards. As of May 1999, these companies no longer have a guaranteed supply area. Every customer can now choose their supplier to be any one of the twelve, or alternatively a new supplier, such as British Gas.

The Central Electricity Generating Board successors are not the only players in the wholesale electricity market in England and Wales. After privatisation,



regulations changed so as to require both new and existing companies to apply for generation and supply licenses. Scottish electricity companies, Electricité de France and a growing number of new entrants are all now generating electricity for sale in the market in England and Wales. By 1997, more than ten independent gas-fired generators had entered the market. However, generators no longer have any obligation to supply, nor have any assured demand; they must compete for their share of an increasingly competitive market. Figure 2.1 gives an overview of the structure, both before and after privatisation.

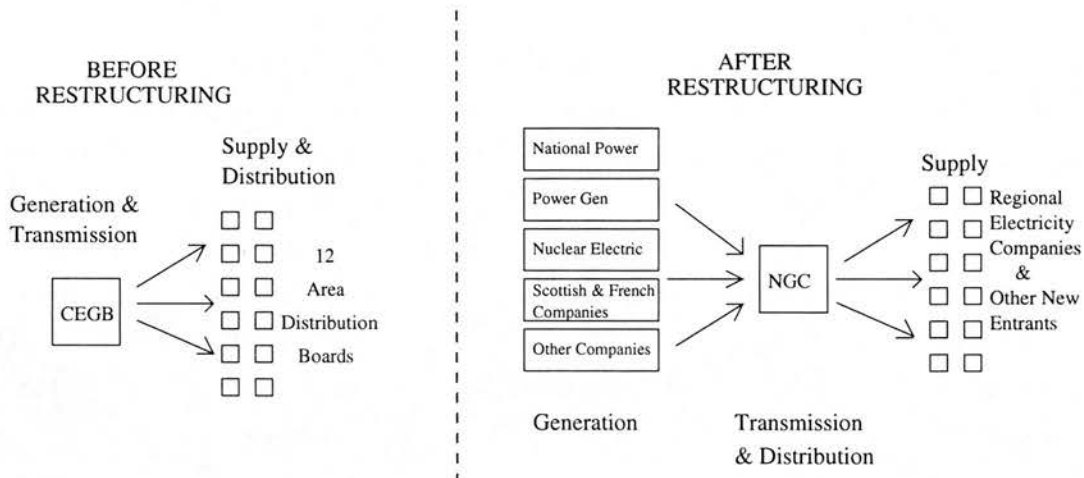


Figure 2.1: Changes to Electricity Industry Structure Following Privatisation

Privatisation and restructuring have brought significant changes to the industry since 1990. The change process is continuing today with efforts to improve regulation and remove market distortions.

### 2.4.1 New Electricity Trading Arrangements (NETA)

The following section gives a brief introduction to the New Electricity Trading Arrangements which were introduced in Spring 2001. Haigh [31], describes this in more detail. The idea behind this new approach is to introduce a short term physical market in place of the previous pool structure. The main aims of the New Electricity Trading Arrangements are:

- To encourage more cost-reflective pricing in the market.
- To increase demand-side participation.
- To make market participants responsible for any imbalance costs they impose on the system.

The intention of the New Electricity Trading Arrangements is to enable generators and suppliers to enter into freely negotiated contracts. Depending upon their requirements, they can do this either in the long-term forward market or the short-term bilateral market. The purpose of the new arrangements is to provide mechanisms to balance the amount of physical electricity flowing into and out of the National Grid and to settle differences between physical and contractual positions of traders. The **System Operator** will determine the actions which need to be taken in order to maintain the required national and local balances of generation and consumption.

Companies may make notifications to the System Operator at any time up to a year in advance, of how much electricity they are willing to generate for each and every half hour of every day. These bids can be altered at any point up until **Gate Closure** which occurs at three and a half hours ahead of the beginning of the half hour period.

When notifying their proposed operating level, electricity generators can also, if they wish, indicate a willingness to deviate from these operating levels. In exchange for payment, generators may be willing to increase or decrease the output of their operating units. A company can make an **offer**, which indicates a willingness to increase the level of generation or reduce the level of demand. Similarly, a company can make a **bid**, which indicates a willingness to decrease the level of generation or increase the level of demand. The operator may accept particular offers and bids placed by the generators in order to balance both local and national demand and ensure the security of the system. It will pay for accepted offers at Offer Price and charge for accepted bids at Bid Price. This system is called the **Balancing Mechanism**.



In addition to the balancing mechanism, there exists another system which settles imbalances between the actual and contractual positions of parties. Generators and suppliers are responsible for producing enough and consuming no more electricity than the amounts stated in their contractual obligations. Each company has two energy imbalance accounts, a "Production" account and a "Consumption" account. The level of energy imbalance is calculated as the difference between the amount of electricity produced and the corresponding contracted amount for the Production account and, similarly the amount of electricity consumed and the contracted amount for the Consumption account. Gas has to be bought and sold by the System Operator in order to settle the resulting imbalances. Companies are charged a very high price for any deficit in the Production account and paid a very low price (which can sometimes be negative) for any excess. Similarly, they are paid a low price (again, sometimes negative) for any deficit in the Consumption account and charged a very high price for any excess.

## 2.5 The Gas Industry

Over the last ten years, the UK natural gas market has developed from an integrated monopoly business to a competitive market, thereby leading the way for gas markets across Europe. All gas users can choose their supplier of gas. In the same period, gas demand has increased by almost 50%, predominantly due to a surge in the use of gas for power generation. The regulator has promoted competition by extending the competitive market to smaller commercial companies. A gas connector joins England to Belgium to enable gas to flow between the two countries in either direction. Information for this section was obtained from Enron [26], and the Network Code [63].

According to the Network Code [63], the Gas Industry has been reorganised into five separate businesses:

- Public Gas Supply
- Contract Trading

- Transportation and Storage
- Servicing and Installation
- Retailing

Barnett [3], states the most important use for natural gas is generation of electricity. Demand from the electricity sector is generally growing strongly, in many cases being the major driver in an upward trend in overall gas use. The use of gas-fired generators by Europe's electricity companies is encouraged by two key factors: efficiency and environmental requirements. Other major uses for gas are as a source of heating fuel and for petrochemicals. Minor uses include transportation fuel and air conditioning.

The structure of the gas industry is shown in Figure 2.2. Of the stages, only transportation and distribution (operating the network) are natural monopolies. As with all utilities which operate a network, once the network has been constructed (usually at very high cost), the infrastructure effectively has no resale value. It is therefore inefficient to develop alternative networks. Retail has few fixed assets. The only requirement for retail suppliers is to organise contracts with the producers and with customers. Therefore, barriers to entry to the retail supply market are low, allowing a large number of retail suppliers to compete. This means that competition is largely on price, with any differentiation consisting merely of financial arrangements in the terms of the contract.

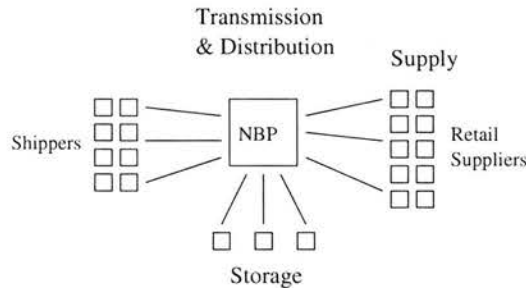


Figure 2.2: Structure of Gas Industry

Transco, the transportation division of British Gas, owns and maintains a network of pipes, called the National Transmission System, which operates at high pressure to move gas between terminals, storage facilities and local distribution sites around the UK. Companies who wish to buy and sell gas must first obtain a license from OFGEM (the office of gas and electricity markets) in order to transport gas around the country.

Transco needs to know how much gas each company wishes to transport on a certain day in order to schedule its daily operations. It finds this out by means of **gas nominations**, which each company makes for every day it wants to use the network. Each company can make their nominations up to one month in advance and can alter these at any time up until 1pm in the afternoon on the day before the day to which they apply. Entry and exit capacity must also be booked on the system. This represents the amount of gas which can be put into and taken out of the system on each day.

The network has been constructed so as to allow gas to move between two points by a variety of routes. This is essential since certain parts of the system may not be available due to maintenance, leaks or for other reasons. In some cases, it may not be possible to transport all the gas. The National Transmission System has a maximum operating capacity which will vary according to the percentage of the system being operated. In such situations, rules have been agreed upon to decide which of the nominations to curtail.

The **On-the-day Commodity Market** was introduced in 1999. This new market replaced the old system in which there was no incentive for Transco to buy and sell gas at the most economical price. The aim of these new trading arrangements was to provide a more liquid and competitive wholesale market that brings gas trading into line with normal commodity trading. The market can be used by companies to maintain their balance of gas flowing into and out of the system. It can also be used from a purely financial point of view, with no gas actually changing hands.

The On-the-day Commodity Market is operated by EnMO, a joint venture between the National Grid Group, the UK based electricity transmission and market services company and Altra Energy Technologies, the market-leading US electronic energy trading services company. The key players in the new system are Transco, who own and operate the National Transmission system, and shippers who can trade with Transco or directly with each other.

The Network Code [63], describes the three types of trade which can be made on the On-the-day Commodity Market:

- **Standard Trade**

One party agrees to give a certain amount of gas to another for an agreed price. This trade can either be physical or purely financial, with no gas actually changing hands.

- **Physical Trade**

One party agrees to give a certain amount of gas to another but the trade must actually be physical. The extra gas must be available for transportation to the second party.

- **Locational Trade**

If a company has a shortfall of gas at a specific location around the country, another company which has excess gas at the same location can be involved in the trade.

When the transportation is complete, Transco must calculate how much of the gas belonged to each company in order to make the appropriate charges. The charges are based on the cost of all trades which are done on the On-the-day Commodity Market in order to make up the shortfall or sell off the excess gas. Companies can use this as a trading mechanism. If they predict that the system marginal price will be low then they can deliberately leave themselves short of gas so that they can buy their requirements at the low price. Likewise if they expect it to be high then gas can be sold at the high price.

There are strong incentives to ensure that companies book sufficient entry and exit capacity on the system. For example, if more gas is delivered than the

company has booked space for, then a penalty charge equivalent to twelve months capacity at the premium rate is incurred. If a company finds itself with more gas than it has booked space for, then in order to avoid the charges, capacity can be booked from another company which has more space than it requires. In this way, the company with spare capacity can recover some of its costs and the company with insufficient capacity can avoid the high charges.

It is important for companies to avoid these high charges, therefore there is a greater need for accurate storage, production and usage models. These models can help to limit errors which may occur when estimating production and the amount required for transportation.

# Chapter 3

## Modelling in Energy Markets

### 3.1 Introduction

In recent years, contracts for the exchange of energy, both financial and physical, have become increasingly more common. Options and forward contracts are now traded extensively by both energy companies and financial institutions. Complex contracts have also been created to model specific situations. This chapter aims to give a broad overview of the financial definitions and concepts which have been used in the remainder of the thesis.

An explanation of financial terms is given, together with a description of the most common type of contracts found in energy markets. In order to obtain a fair price when buying and selling contracts, accurate valuation methods must be used. The most famous of these is the Black Scholes model. Various extensions of this model exist, which include the Black model and the Margrabe model. A description of each of these models is given. These models form the basis of the most commonly found valuation methods.

In order to value contracts accurately, the underlying price process must be modelled. There is an element of randomness to the movements of energy prices from one day to the next. These fluctuations can be approximated by stochastic processes. Brief descriptions of such processes are given. A number of characteristics are also present in a series of energy prices, such as mean reversion,

seasonality and large price jumps. A description of these factors is given, together with a number of approaches to modelling such series.

## 3.2 Financial Definitions

Valuing contracts for the exchange of energy has become more important as the volumes of such traded contracts increase. A number of different types of contract exist, the most basic of which are options and forwards. There is an increased need to develop more complex contracts which have been tailored to meet specific situations. The financial definitions in this section are based mainly on Hull [36], and Wilmott [69].

An **option** gives the holder the right to buy or sell a given quantity of an underlying asset at a fixed price, called the **strike price** at a certain time in the future. A **call option** gives the holder the right to buy something, whereas a **put option** gives the holder the right to sell something. An option protects the holder from downside risk. The holder is not obliged to buy or sell anything. **American options** may be exercised at any time during their lifetime, whereas **European options** may only be exercised at the expiration date. The **payoff** from an option is the maximum amount which can be obtained at the expiry date of the option. The payoff can be zero if it is not worthwhile to exercise the option.

The payoff from a call option to buy a specific asset is:

$$\max(S_T - K, 0),$$

where  $S_T$  is the final underlying price of the asset and  $K$  is the strike price of the option. This is represented in Figure 3.1(a). The payoff from a call option to sell a specific asset is:

$$\min(K - S_T, 0).$$

This is shown in Figure 3.1(b).

The payoff from a put option to buy a specific asset is:

$$\max(K - S_T, 0),$$

where  $S_T$  is the final underlying price of the asset and  $K$  is the strike price of the

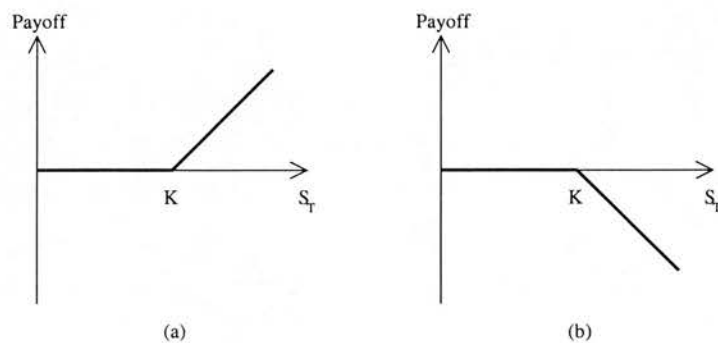


Figure 3.1: Payoff for Call Options

option. This is represented in Figure 3.2(a). The payoff from a put option to sell a specific asset is:

$$\min(S_T - K, 0).$$

This is shown in Figure 3.2(b).

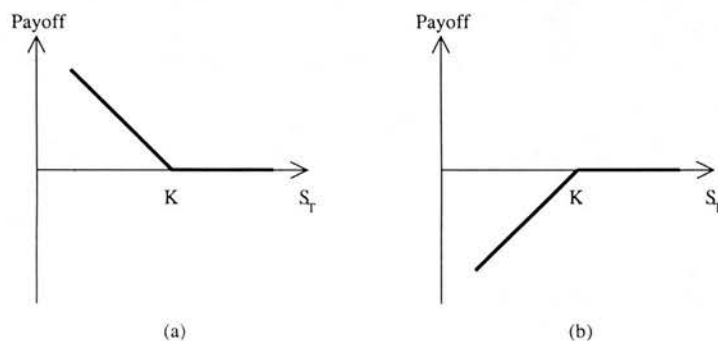


Figure 3.2: Payoff for Put Options

A **forward contract** is a binding agreement between two parties for one to supply a given quantity of a certain asset to the other at a certain price at a certain time in the future. A forward contract is very similar to a bet on the future price of an asset. A **futures contract** is similar to a forward contract, the major difference being that futures contracts are standardised and traded through clearinghouses which act as intermediaries.

Futures contracts are usually cash settled. In order to protect itself from the risk associated with this process, the clearinghouse employs a process called



**margin**ing. This process sets up what is known as a margin account. A contract is bought at a certain price which is known as the **initial margin**. At the end of every day, the net present value of the contract with respect to the market price on that day is calculated. This value represents the current value of the contract on that day. The value of the contract on the previous day is now subtracted from the current value of the contract to give the **mark to market** value of the contract.

The amount in the margin account is adjusted to reflect the difference between this mark to market value and the value of the margin account on the previous day. If the mark to market amount is greater than the margin account, the holder of the contract pays the difference between the two values into the margin account. If the mark to market value is less than the margin account, the holder of the contract is paid the difference between the two. The amount paid or received is called the **variation margin**.

A contract may be risky in that if it does not perform as well as is expected, large losses may be incurred. Some contracts have a limit on the amount which can be lost, whereas others can have unlimited losses. **Hedging** is the process by which the risk associated with a contract is reduced or eliminated. In order to hedge a forward or futures contract, a forward is bought for the opposite direction. For example, if someone holds a forward contract to buy a stock, then they could also buy a forward contract to sell the same amount of stock at a higher price. In order to hedge an option, a forward contract or another option can be bought in the opposite direction. This has the effect of eliminating large jumps in price which are often found in power and gas prices. This ensures that losses can be restricted to within known boundaries.

The **spot price** is the price at which a commodity, such as energy, can currently be bought or sold in the market. The **forward price** is the price at which a commodity is bought or sold now for delivery at a future point in time. The **long term equilibrium price** is an estimate of the forward price as it tends to infinity. Curves can be constructed which show how these parameters vary against time, an example of which is given in Figure 3.3.

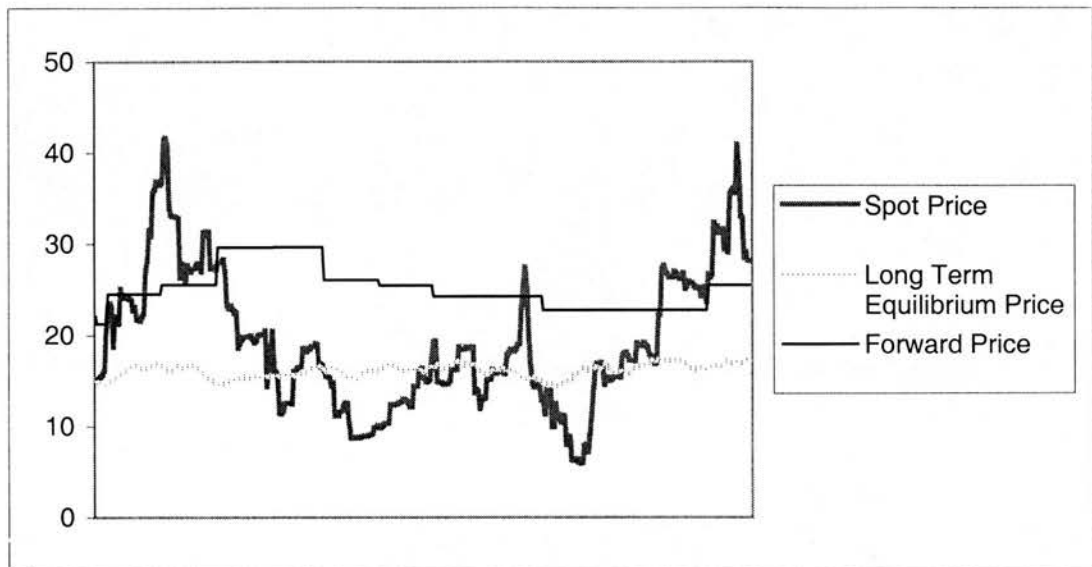


Figure 3.3: Examples of Price Curves

The **bid price** is the price at which the market will buy from a seller and the **offer price** is the price at which the market will sell to a buyer. The offer price always lies above the bid price. The **mid price** is the average of the bid price and the offer price. A bid curve can be constructed for a series of bid prices over a certain time period. Offer and mid curves can be constructed in the same way. The mid curve is often used in energy models in order to reduce the number of stochastic variables by using one set of data rather than two.

The **bid-offer spread** is the difference between the bid and offer prices. For energy prices, it has a similar shape to that shown in Figure 3.4. This implies that the difference between the two values decreases in the short term but increases in the longer term as shown in Figure 3.5. This difference is a reflection of the liquidity of the market. A market in which large quantities of a commodity, such as gas or power, are being bought or sold, thereby making trading straightforward, is called a **liquid market**. A liquid market has the effect of reducing the spread

between the bid and offer prices. The increase in the bid offer spread corresponds to an illiquid market. It is not particularly easy to estimate the price on a particular day in such a market.

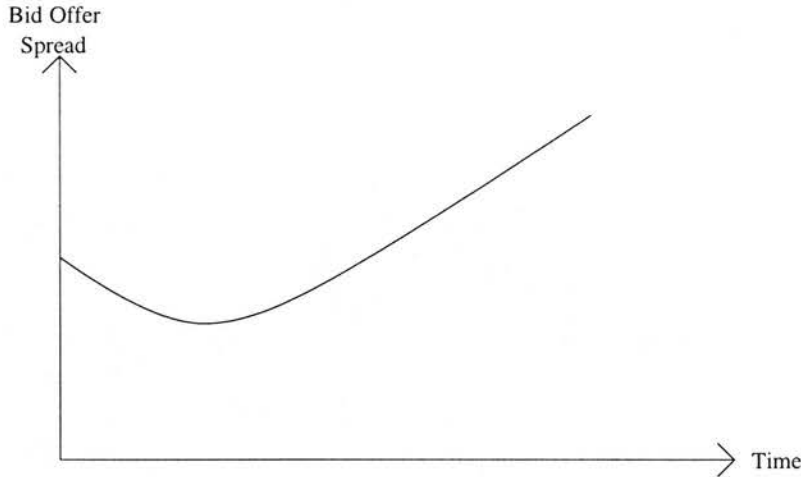


Figure 3.4: Bid-Offer Spread

A variety of complex contracts are commonly found in energy markets. These can include:

- **Asian Options**

The payoff of this type of option is defined in terms of the average value of the underlying asset during a certain time period, rather than in terms of its final value.

- **Caps and Floors**

These options restrict the payoff by creating upper and lower bounds on the price. The payoff from a cap is usually the highest price of the underlying asset over a certain time period, which does not exceed a set upper limit. Similarly, the payoff from a floor is usually the lowest price reached, which can not be lower than a set minimum value.

- **Spark Spread Options**

These options have a payoff which is dependent on the difference between the price of power and the price of the fuel used to generate it.

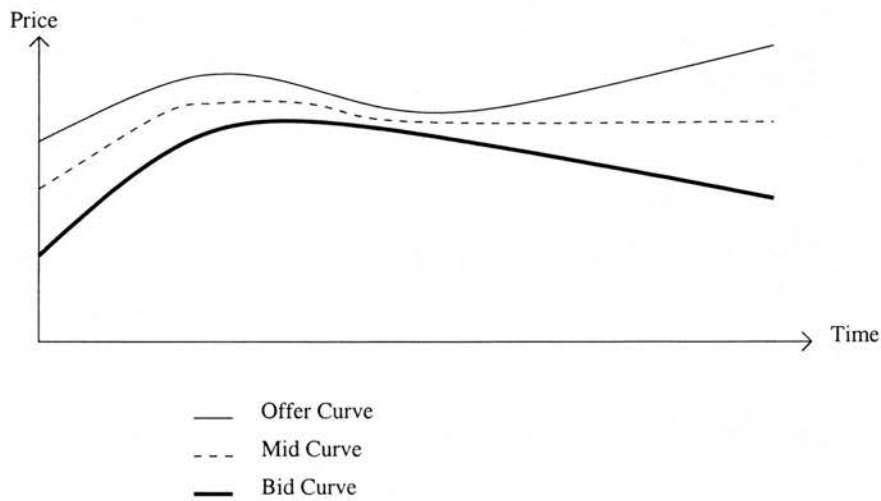


Figure 3.5: Bid and Offer Curves

- **Take or Pay Contracts**

These options allow the holder to vary the amount of energy taken on a specific day, within both daily and annual limits. These contracts are also known as swing contracts. The valuation of take or pay contracts is considered in depth in Chapter 7.

- **Tolling Deals**

These contracts give the holder the right to convert the price of fuel into the price of power subject to certain constraints. A physical tolling deal, in which the power produced actually changes hands, is also known as a Virtual Power Station. The valuation of different types of tolling deal are discussed in Chapter 8.

The value of contracts must be estimated, both when they are bought and sold, and also when they are actually held. This is essential in order to try to predict how the contract will act in the future. Valuation is relatively straightforward for simple contracts but becomes much more difficult when they increase in complexity. A number of variables impact on the value of contracts present in energy markets. These include:

- **The stock and strike prices**

The payoff from a contract will be the amount by which the stock price exceeds the strike price. Payoffs from contracts can be negative.

- **The time to expiration**

This is the amount of time remaining until the contract expires. This is an important factor when valuing some types of contract as, for example, American options become less valuable as the time to expiration decreases.

- **The volatility of the stock price**

Roughly speaking, the volatility is a measure of how uncertain we are about future stock price movements. As the volatility increases, the probability that a stock will perform either very well or very poorly increases.

- **The risk free rate of interest**

This is the continuously compounded rate of interest which would be earned on a riskless asset. As interest rates increase, the expected growth rate of a stock price tends to increase. However, the present value of any future cash flows received, decreases.

Dividends can be important in other types of market, but do not usually affect contracts present in energy markets.

### 3.3 Stochastic Processes

Any variable whose value changes over time in an uncertain way, such as energy price, is said to follow a **stochastic process**. Stochastic processes can be classified as discrete or continuous time processes. A **discrete time** stochastic process is one where the value of the variable can change only at certain fixed points in time, whereas a **continuous time** stochastic process is one where changes can take place at any time. A number of stochastic processes relevant to the research are described.

A **Poisson process**,  $dq$  is a type of stochastic process commonly found when modelling jumps present in power and gas spot and forward curves. It is defined by:

$$dq = \begin{cases} 0 & \text{with probability } (1 - \lambda) dt \\ 1 & \text{with probability } \lambda dt \end{cases}$$

The probability of a movement in the process in the timestep  $dt$  is  $\lambda dt$ . The parameter  $\lambda$  is called the intensity of the Poisson process. This process can be used to describe the probability of there being a jump in the price in any time period. Jumps which occur in series of energy prices are described in more detail in Section 3.6.1.

A **Markov process** is a type of stochastic process in which a future event does not depend upon any event which has preceded it. This is sometimes known as the “lack of memory” property. Energy prices tend to exhibit this kind of behaviour. A **Markov chain** describes the transitional behaviour of a process over equally spaced intervals of time. A **transition probability**,  $p_{ij}$ , is the conditional probability of the process moving to state  $j$  at stage  $n$ , given it was in state  $i$  at stage  $n - 1$ . The transition probabilities form a **transition matrix** if the following conditions are satisfied:

- $p_{ij} \geq 0, \forall i, j.$
- $\sum_j p_{ij} = 1, \forall i.$

When problems have a Markovian property, they have a simpler structure. When the process is in a particular state, the future looks the same regardless of which states the process has been in at previous stages.

**Brownian motion** is a type of Markov process. It is also often known as a Wiener process. A variable,  $z$ , must have two properties for it to be Brownian motion:

- $dz$  must be related to  $dt$  by the equation:

$$dz = \varepsilon\sqrt{dt},$$

where  $dt$  is a small interval of time,  $dz$  is the change in  $z$  during  $dt$ , and  $\varepsilon$  is a random variable from a standardised normal distribution (i.e. a normal distribution with a mean of zero and a standard deviation of one).

- The values of  $dz$  for any two different short intervals of time  $dt$  must be independent.

The second of these properties must be true for  $z$  to follow a Markov process.

**Geometric Brownian motion** is denoted by:

$$dS = \mu S dt + \sigma S dz,$$

where  $dt$  is a small interval of time,  $dS$  is a small change in the spot price,  $\mu$  is the expected rate of return of the process and  $\sigma$  is the stock price volatility. This process has instantaneous expected drift rate  $\mu S$  and instantaneous variance rate  $\sigma^2 S^2$ . Geometric Brownian motion ensures that the price distribution at the expiry of the option is lognormal, rather than normal and the prospect of the asset price becoming negative is eliminated. Energy prices are often modelled by using a geometric Brownian motion process.

Ito [37], generalised Brownian motion to give an **Ito process** denoted by:

$$dx = a(x, t)dt + b(x, t)dz,$$

where  $dz$  is Brownian motion and  $a(x, t)$  and  $b(x, t)$  are functions of the underlying variable  $x$  and the time  $t$ . An Ito process is an example of a stochastic differential equation. **Ito's Lemma** was developed to provide rules for computing the Ito process given by a function of Ito processes. If a variable  $x$  follows an Ito process,

then Ito's Lemma shows that a function  $G(x, t)$  follows the process:

$$dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz.$$

Thus  $G$  also follows an Ito process with a drift rate of:

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2,$$

and a variance rate of:

$$\left( \frac{\partial G}{\partial x} \right)^2 b^2 dz.$$

### 3.4 Volatility

Volatility is a very important factor which must be considered when constructing energy models. Energy prices are governed by a variety of factors and random movements. The volatility measures the amount of uncertainty in future stock price movements. As the volatility increases, the chance that the stock will do either very well or very poorly increases. The derivation of the volatility equations for energy markets which are described below are given in Putney [52].

The **instantaneous volatility** is the volatility at a given point in time. This type of volatility fluctuates from day to day. One consequence of mean reversion in spot prices is that the instantaneous volatility of the forward price increases exponentially as the maturity date is approached. The instantaneous volatility can also be influenced by seasonal dependencies.

In the majority of financial models, such as the Black Scholes, Black and Margrabe models described in Section 3.5, the volatility is assumed to be constant. Therefore, in order to apply these formulae, a constant volatility must be derived. This volatility must have the same terminal price distribution as the instantaneous volatility. An "average" volatility can be obtained by integrating the instantaneous variance ( $\sigma^2$ ) between the valuation and exercise dates. The resulting formula is:

$$\sigma_A = \sqrt{\frac{1}{t_1 - t_2} \int_{t_2}^{t_1} \sigma^2(t) dt} ,$$



where  $t_1$  is the valuation date,  $t_2$  is the exercise date and  $\sigma^2(t)$  is the square of the instantaneous volatility. The value obtained from this equation can then be used as an approximation in many financial models.

## 3.5 Financial Models

A number of financial models have been developed to try to place a value on certain types of financial contracts. The following models can be found in more detail in Hull [36].

### 3.5.1 The Black and Scholes Model

Black and Scholes [9], developed formulae for pricing European call and put options, assuming that the stock price,  $S$ , follows geometric brownian motion:

$$dS = \mu S dt + \sigma S dz,$$

where  $\mu$  is the expected rate of return of the stock and  $\sigma$  is volatility of the stock price.

Suppose that  $f$  is the price of a derivative dependent on  $S$ . From Ito's Lemma:

$$df = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz.$$

It can be shown that this leads to the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf,$$

where  $\sigma$  is the average volatility and  $r$  is the risk free rate of interest.

The Black Scholes pricing formulae for call and put options are given by finding solutions to this partial differential equation. The solutions are shown to be:

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1),$$

where:

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The notation used is:

- $K$  — exercise price for the contract
- $r$  — risk free rate of interest
- $\sigma$  — average volatility
- $T - t$  — time to expiry
- $N(x)$  — cumulative probability distribution function for a variable that is normally distributed with a mean of zero and a standard deviation of one, i.e. the probability that such a variable will be less than  $x$

The Black Scholes pricing formula for a call option can be rewritten as:

$$c = e^{-r(T-t)} [Se^{r(T-t)}N(d_1) - KN(d_2)].$$

This shows that the value of the call option is the present value of its expected terminal value. The expected asset price at the option's expiry is  $Se^{r(T-t)}$ . The expression  $Se^{r(T-t)}N(d_1)$  is the expected asset price, conditional on the asset price exceeding the exercise price at expiry of the option. The expected exercise cost is the exercise price multiplied by the probability that the option will be exercised and is denoted by  $KN(d_2)$ . The formula for a put option can be rewritten in a similar way.

Black and Scholes made a number of assumption when constructing their classic model. Their assumptions include:

- Delta hedging is continuous, costs nothing and eliminates all risk. Delta hedging is described in Section 3.5.2.
- The underlying asset path is continuous and unaffected by trade in the option.
- Volatility, interest rates and dividends are known constants or known deterministic functions.

Black and Scholes also discovered that as long as a risk-free hedge can be formed between an option and its underlying asset, the value of an option relative to the asset will be the same for all investors regardless of their risk preferences. This implies that the payoff of a European call can be identically duplicated by a portfolio consisting of the asset and risk-free bonds. Therefore, the value of the option does not depend on the expected return of the asset and can be thought of as being free of risk. Options can be valued in a risk-neutral world where expected asset returns and expected option returns all equal the risk-free rate of interest.

The Black Scholes model is not appropriate for pricing energy options because the model assumes that the underlying spot price follows a geometric Brownian motion process. This is not usually the case since energy spot prices tend to have strong mean reverting tendencies, random jumps and are affected by seasonality. The Black Scholes model also assumes that the options can be hedged with a position in the underlying asset but the fact that electricity cannot be stored creates a problem when considering this. Many energy options are written on an underlying forward contract, the behaviour of which can be reasonably approximated by geometric Brownian motion. These options can be priced by applying Black's Model, which is described in the next section.

### 3.5.2 The Black Model

Black [8], developed a model for pricing options on futures. These formulae can be applied when constructing energy models since the differences between forward and futures prices in these markets are sufficiently small enough to be

ignored. Thus, forward and futures prices are valued in exactly the same manner and the terms can be used interchangeably as they both reflect the same value.

The model assumes that the underlying futures or forward price,  $F$ , follows geometric Brownian motion:

$$dF = \mu F dt + \sigma F dz,$$

where  $\mu$  is the drift of  $F$  and  $\sigma$  is the volatility. The European call and put prices for an option on a forward or futures contract are given by the following formulae:

$$c = e^{-r(T-t)}[FN(d_1) - KN(d_2)],$$

$$p = e^{-r(T-t)}[KN(-d_2) - FN(-d_1)],$$

where:

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The notation used is as previously, with  $F$  as the forward or futures price.

Option traders need sophisticated hedging techniques in order to eliminate a large proportion of the risk associated with options. In order to calculate this risk, they can use what are known as “The Greeks”. The most important of these is the delta which can be calculated using formulae derived from Black’s equations.

The **delta**,  $\Delta$ , of an option is the rate of change of the option price with respect to the asset price. In order to hedge the option, we need to buy or sell  $\Delta$  shares of the underlying asset. The value of this purchase or sale is  $\Delta$  times the market value of the underlying asset. The delta of a call option,  $\Delta^C$ , and the delta of a

put option,  $\Delta^P$ , can be calculated using the following formulae:

$$\Delta^C = e^{-r(T-t)}N(d_1)$$

$$\Delta^P = e^{-r(T-t)}(N(d_1) - 1),$$

where:

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

The notation used is as previously.

The **gamma**,  $\Gamma$ , of an option is the rate of change of delta with respect to the asset price. It gives a measure of how frequently a position must be rehedge in order to maintain a delta neutral position. The remaining greeks are less common. The **theta**,  $\theta$ , of an option is the rate of change of the option price with respect to time. The **rho**,  $\rho$ , of an option is the rate of change of the option price with respect to the interest rate. The **vega** of an option is the rate of change of the option price with respect to the volatility.

Sometimes the payoff of an option is sufficiently straight forward that the value of the option can be found from an analytic formula. Mostly, however, the payoff from the option is so complex that it is not possible to find analytic solutions. In these cases, the same risk-neutral theory continues to apply but the values of the options must be calculated numerically.

### 3.5.3 The Margrabe Model

Margrabe [43], developed a model for valuing a European option with a strike price of zero to buy the difference between the prices of two assets. His model is also an extension of the Black Scholes model. Since the strike price is zero, this can be thought of as an option to exchange the price of one asset for the other. This model is normally used to value spark spread options in the energy industries. These are options to exchange the price of fuel for the price of power produced from that fuel.

Margrabe's model assumes that the spot prices of both assets,  $s_1$  and  $s_2$ , are lognormally distributed and follow geometric Brownian motion with volatilities

$\sigma_1$  and  $\sigma_2$  respectively. The instantaneous correlation between  $s_1$  and  $s_2$  is  $\rho$ . The value of a call option is given by:

$$c = [s_2 N(d_1) - s_1 N(d_2)] e^{-r(T-t)},$$

where:

$$d_1 = \frac{\ln(s_2/s_1) + (\sigma^2/2)(T-t)}{\sigma\sqrt{(T-t)}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2},$$

where  $\sigma$  is the volatility of  $s_1/s_2$ .

The above formulae are independent of the risk free rate of interest,  $r$ . This is because, as  $r$  increases, the growth rate of both asset prices in a risk neutral world increases. This is offset by an increase in the discount rate.

Margrabe's method can be generalised for forward prices. Assuming both forward prices,  $F_1$  and  $F_2$ , are lognormally distributed, the formula for a call becomes:

$$c = [F_1 N(d_1) - F_2 N(d_2)] e^{-r(T-t)},$$

where:

$$d_1 = \frac{\ln(F_1/F_2) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

$$\sigma = \left[ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) dt \right]^{1/2}.$$

$\sigma$  is the average volatility between  $t_1$  and  $t_2$ .

### 3.6 Modelling Price Processes

Energy prices are the main source of uncertainty to be considered when developing models for these types of markets. Including stochastic variables in models can often make them large, complex and reasonably difficult to solve.

### 3.6.1 Characteristics of Energy Prices

Energy markets share some of the characteristics of other commodity markets. There is a spot market for immediate delivery and a forward market for delivery at some specified time in the future. Spot prices are typically very volatile, whereas forward prices are much less so. As with other commodities, there is a cost associated with storage of energy for future use or delivery. This storage could simply be a matter of physically holding the commodity in a specific location. Often, as with electricity, storage is not possible or complex storage arrangements must be made such as pumping water into a reservoir for release at a later date.

A number of factors can be observed when examining a series of energy prices:

- **Mean reversion**

This is the ability of stock prices to return to the same long term average level over time. Mean reversion is an important factor when modelling power and gas prices. When the risk free interest rate,  $r$ , is high, mean reversion tends to have a negative drift, whereas when  $r$  is low, a positive drift is more common.

- **Seasonality**

Prices have a strong seasonal component which is clearly seen in the forward price curve. The most expensive prices occur in winter months when more heating is used, but there is also a smaller peak in the height of summer due to the use of air conditioning.

- **Large price jumps**

Forward price curves will also show large price movements for expiration dates which are fairly close. Longer times to expiry have a much smoother curve. The reason for these sudden price jumps in power prices stems from lack of storability. If power could easily be stored, then rapid fluctuations in demand would not necessarily lead to large jumps in the price. Natural gas storage does exist but the capacity is not large enough to diminish these price jumps.

### 3.6.2 Spot Price Curves

Geometric Brownian motion is often used as a basis for generating spot price curves. Geometric Brownian motion is described in Section 3.3 to be:

$$dS = \mu S dt + \sigma S dz,$$

where  $dt$  is a small interval of time,  $dS$  is a small change in the spot price,  $\mu$  is the expected rate of return of the process and  $\sigma$  is the stock price volatility.

Clewlow and Strickland [18], show that this process can be discretised as follows:

$$S_t = S_{t-1} e^{v\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t},$$

where  $\Delta t = T/N$  and  $v = r - \frac{1}{2}\sigma^2$ . The following notation is used:

- $t$  — index of time periods ( $t = 1, \dots, T$ )
- $N$  — number of time steps
- $S_t$  — spot price at time  $t$
- $\sigma$  — volatility
- $\Delta t$  — small change in time
- $\varepsilon_t$  — random variable from a standardised normal distribution
- $r$  — risk free rate of interest

Since the drift and volatility terms do not depend on the variables  $S$  and  $t$  then the discretisation is correct for any time step chosen.

The geometric Brownian motion process was found not to be a good enough approximation for energy spot price behaviour. One property observed in spot price processes, which is not encompassed by geometric Brownian motion, is the tendency of prices to revert to the long-term equilibrium price. This is known as **mean reversion**. A mean reverting process in which terms fluctuate around a mean of zero is called an **Ornstein-Uhlenbeck process**. The following Ornstein-Uhlenbeck process for modelling energy spot prices is proposed by Schwartz [56]:

$$dS = \alpha(\mu - \ln S)S dt + \sigma S dz,$$

where  $\alpha$  is the rate of mean reversion,  $\mu$  is the expected rate of return and  $\sigma$



is the volatility. The spot price can be shown to mean revert to the long term equilibrium level,  $e^\mu$ , at a speed  $\alpha$ .

Clewlow and Strickland [18] show that this process can be discretised to become:

$$\Delta x = \left[ \alpha(\mu - x) - \frac{1}{2}\sigma^2 \right] \Delta t + \sigma\sqrt{\Delta t}\varepsilon_t.$$

Since the drift term depends upon the variable  $x$ , the discretisation is only correct as the limit of the time step tends towards zero. Thus, the time steps must be small in relation to the speed of mean reversion,  $\alpha$ .

Large jumps in the spot price are another feature commonly observed in energy markets. These are caused by various factors, including demand fluctuations and transmission shortages. Merton [45], proposed the following model which incorporates these jumps into the spot price process:

$$dS = (r - \phi\bar{\kappa})Sdt + \sigma Sdz + \kappa S dq,$$

where  $\kappa$  denotes a jump,  $\bar{\kappa}$  is the mean jump size,  $\phi$  is the annualised frequency of the jumps and  $\Pr(dq = 1) = \phi dt$ . The discretisation given by Clewlow and Strickland [18], is:

$$\Delta x = \left( r - \phi\bar{\kappa} - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma\sqrt{\Delta t}\varepsilon_1 + (\bar{\kappa} + \gamma\varepsilon_2)(u < \phi\Delta t),$$

where  $\gamma$  is the jump volatility and  $(u < \phi\Delta t)$  equals one if the condition is true and zero otherwise.  $u$  denotes a random number from a uniform distribution lying between 0 and 1. This is shown to generate jumps randomly at the correct frequency. When a jump occurs, its size is equal to  $\bar{\kappa}$  plus a normally distributed random amount which has a standard deviation of  $\gamma$ .

A strong seasonality effect is seen by the existence of peaks in energy price curves. An opportunity for arbitrage exists for buying when prices are low and storing, if possible, for use at the next and subsequent peaks of the curve. When an arbitrage opportunity exists, it usually disappears quickly due to the large number of people wishing to take advantage of it. This should, in turn lead to a flatter energy price curve. Clewlow and Strickland [18], note that this does not

happen with energy prices and peaks continue to exist for a number of reasons including:

- In order to exploit arbitrage opportunities the player needs a presence in both the physical and financial markets.
- The ability to store significant quantities of energy is needed.
- Arbitrage strategies require an advanced level of expertise in trading and risk management.

Seasonality is defined as a pattern that repeats itself over fixed intervals of time. A process which produces high values in the winter and low values in the summer indicates a twelve month seasonal pattern. Energy spot price processes can be thought of as a function of the underlying spot price plus seasonality effects. Pilopović [49], denotes the annual seasonal parameter in her model as:

$$\beta \cos(2\pi(t - t_A)),$$

where  $\beta$  is the annual seasonal parameter and  $t_A$  is the time at which the annual peak occurs. Semi-annual peaks can be modelled in a similar way.

### 3.6.3 Forward Price Curves

The forward price can be related to the spot price by using arbitrage arguments. If the forward price is greater than the cost of buying the spot price energy and storing it then a riskless profit can be made by selling energy at the forward price, buying energy at the spot price and storing it. Similarly, if the forward price is less than the cost of buying the spot price energy and storing it then a riskless profit can be made by buying the forward and selling the spot price energy. The forward price should be equal to the spot price plus the cost of carrying the spot price energy to the forward date:

$$F(t, T) = S(t)e^{r(T-t)},$$

where  $F(t, T)$  is the forward price at time  $t$  expiring at time  $T$  and  $r$  is the risk free rate of interest.

The forward price usually tends towards the long term equilibrium price following an exponential distribution. If a forward curve is upward sloping, it is said to be in **contango**. If it is downward sloping, it is said to be in **backwardation**. Examples of this are shown in Figure 3.6. These properties do not necessarily occur separately. Curves may exhibit contango in the short term and backwardation in the long term or vice versa. If the whole of a forward curve is in contango or alternatively if the whole of a forward curve is in backwardation, arbitrage opportunities exist. Energy curves will often exhibit these features as arbitrage can not be excluded completely from these markets.

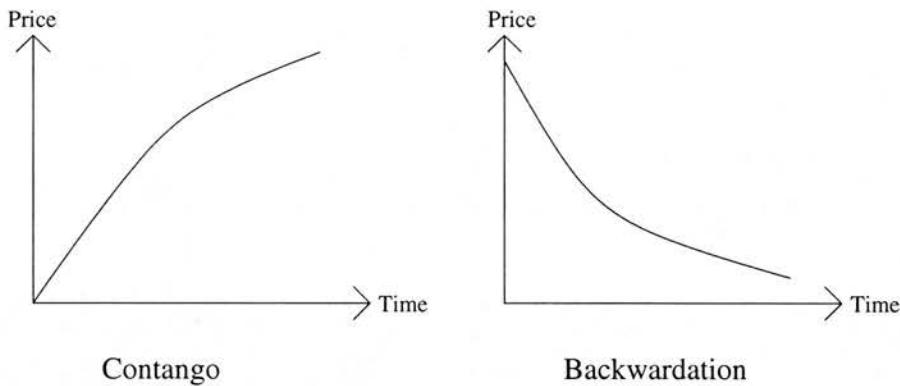


Figure 3.6: Contango and Backwardation

One technique which can be used to construct forward curves is known as **Principal Component Analysis**. Further details can be found in Clewlow and Strickland [18]. This is a mathematical procedure that transforms a number of correlated variables into a number of uncorrelated variables called principal components. The first principal component accounts for as much of the variability in the data as possible, and each succeeding component accounts for as much of the remaining variability as possible.

When applied to energy markets, this approach considers various factors or volatility functions associated with the source of risk. Clewlow and Strickland [18], suggest that the first factor typically represents a parallel shift which causes the forward curve to move parallel to the x-axis. Their findings show that this factor often contributes to approximately 80% of the curve. The second

most important factor that they found is the tilt factor which causes the short and long term maturity ends of the forward curve to move in opposite directions. This factor was shown to contribute to about 10% of the forward curve. The third factor was seen to be a bending factor which causes the short and long ends of the curve to move in the opposite direction to the middle section. Other factors were observed but were thought to account for a reasonably small amount of the forward curve.

The forward curve is then given by:

$$\frac{dF(t, T)}{F(t, T)} = \sigma_1(t, T) dz_1(t) + \sigma_2(t, T) dz_2(t) + \sigma_3(t, T) dz_3(t),$$

where  $\sigma_1(t, T)$ ,  $\sigma_2(t, T)$  and  $\sigma_3(t, T)$  are the first, second and third factors respectively and  $dz_1(t)$ ,  $dz_2(t)$  and  $dz_3(t)$  are independent Wiener processes. This approach can be extended to any number of factors but it was shown that these three factors explain the majority of movement in the forward curve and thus give a reasonable approximation.

Clewlow and Strickland [18], extend this idea by showing how a covariance matrix for the forward prices can be computed. The eigenvalues of this matrix give an indication of how important each of the various factors is, whilst the eigenvectors of the matrix give the discretised volatility functions  $\sigma_i(t, t + dt)$ .

### 3.6.4 Real Life Approach

The forward price curve is constructed for a particular day by considering data obtained for natural gas from the Heren European Spot Gas Markets Report and the Argus European Natural Gas Report and for power from the Heren European Daily Electricity Markets Report and the Platts European Power Daily Report (which are all published every working day). These reports give the forward prices at which brokers are currently trading. The main values obtained are for day ahead prices, remainder of the month prices, the six subsequent months and the next three years. Other values can also be incorporated if they are available. These values are used to construct graphs similar to the one shown in Figure 3.7.

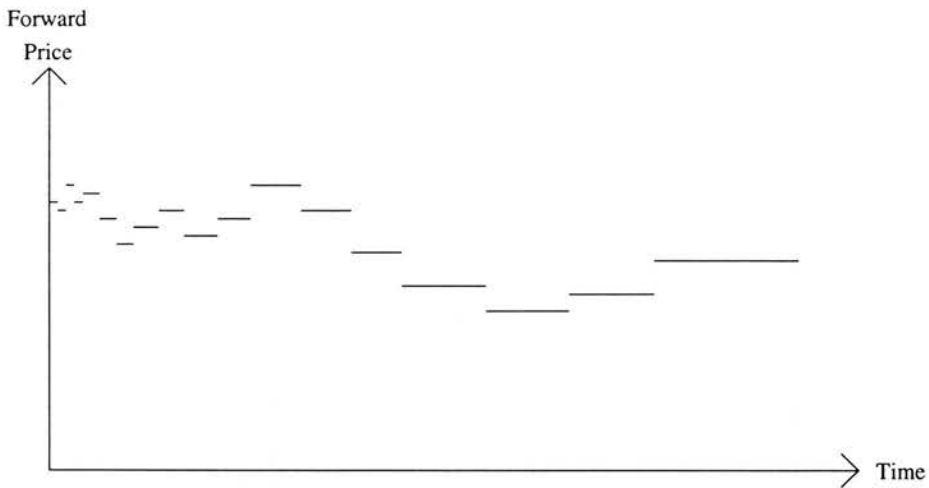


Figure 3.7: Construction of Forward Price Curve

The method of least squares is used to approximate a curve to these data. This gives a good approximation of the forward curve when various parameters are introduced to enable the curve to be altered by eye. These parameters may include the amplitude of the peaks and troughs, the width of the peaks and troughs and the speed of decay of the spot price towards the long term equilibrium price. A parameter can also be included to move the forward curve parallel to the x-axis. Values can be estimated for these variables on a daily basis in order to represent actual market trends.

# Chapter 4

## Solution Techniques for Energy Models

### 4.1 Introduction

Mathematical models can be formulated to represent features and characteristics of some object or situation. Algebraic constraints denote relationships, such as limited time or resources, which must be taken into account when modelling certain situations. This chapter aims to give a broad overview of the solution techniques which are used in the remainder of the thesis. The techniques described can be readily applied when modelling in the energy industries.

In this chapter, mathematical programming techniques are defined and discussed in some depth. The structure of many contracts found in the energy industries is especially suited to techniques of this type. This chapter has been written to be accessible to someone with little knowledge of this type of technique. Mathematical programming modelling approaches have been applied in Chapters 5, 6, 7 and 8. Problems which include uncertain parameters can often be modelled by generating and including a number of possible scenarios. Tree structures provide a useful way of incorporating these scenarios into such problems. Two different approaches for developing such structures are compared; the first approach uses a forward curve obtained from the market to give an estimation of movements in the spot price in the future and the second uses

statistics calculated from historical data to approximate the behaviour of such a curve.

## 4.2 Mathematical Programming Techniques

Mathematical programming models are generally constructed for situations in which decisions must be made about the course of action to follow or, alternatively, when an indication of the expected profit or loss is required. The models generally have a number of distinct features:

- Most mathematical programming models involve **optimisation**. They either wish to maximise or minimise some function such as profit or cost.
- The quantity which is being optimised is known as the **objective function**.
- They contain a number of **parameters** whose values may be known or unknown. A **deterministic** problem is one in which the values of all parameters are known with certainty throughout the time period. A **stochastic** problem contains parameters whose values are uncertain and must be approximated.
- A set of **decision variables** is present, which completely describe the decisions to be made.
- The values taken by the decision variables must satisfy a set of **constraints**, which are either represented by equalities or inequalities. These constraints represent limits on the decision variables.
- The **feasible region** is the set of points satisfying all the constraints. The feasible region contains all possible solutions to the problem.
- An **optimal solution** is a point in the feasible region with the highest objective function value for a maximisation problem and the lowest value for a minimisation.

Mathematical programming models are generally built to find the optimal value of the objective function, the optimal decision which should be made at each period or both. There may be more than one optimal solution, i.e. different combinations of values for the decision variables which lead to the same objective function value. Sensitivity analysis can be performed on the solution to give indications of how the solution of the model will change when the original model is varied. This is described in more depth in Section 4.2.2.

Constructing mathematical models is useful for a number of reasons, which include:

- Models can often reveal relationships which may not be apparent on first consideration. This can lead to greater understanding of the problem.
- Mathematical analysis can be undertaken on the model which may suggest alternative courses of action.
- It allows experimentation of different situations without the threat of serious consequences.

Mathematical models can never hope to represent real-life situations exactly. It is usually impossible to satisfactorily quantify some of the required parameters. Problems can also result from inaccurate data or the incorporation of uncertain parameters into the model. A mathematical model should therefore be used as one of a number of tools in order to make decisions about the situation. By successive questioning of the answers and altering the model as necessary, it should be possible to determine alternative courses of action which can lead to a greater understanding of what may be possible. Problems can often be modelled in more than one way. Comparing and contrasting results from different types of model can sometimes give valuable information, besides being a useful check.

### 4.2.1 Linear and Integer Programming

A **linear program** is an optimisation problem in which the objective function is a linear function of the decision variables, i.e. all terms are of first order.



The constraints must also only contain linear terms. There are a number of assumptions regarding linear programming models:

- **The Proportionality Assumption**

The contribution to the objective function from each decision variable is proportional to the value of the decision variable.

- **The Additivity Assumption**

The contribution to the objective function for any variable is independent of the values of the other decision variables.

- **The Divisibility Assumption**

Each decision variable is allowed to assume fractional values.

- **The Certainty Assumption**

Each parameter is known with certainty.

It can be shown that the feasible region for a linear program is always convex. A region is said to be **convex** if a line joining any two points within the region lies entirely within the region. An **extreme point** of a convex region is a point which lies at the “corners” or vertices. This is important since it can be shown that the optimal solution can always be found at an extreme point of the feasible region. This reduces the number of possible solutions drastically and it is relatively easy to find a solution for a linear program, if one exists.

A general maximisation formulation is:

$$\begin{aligned} \max \quad & c_1x_1 + \dots + c_nx_n \\ \text{subject to:} \quad & \\ & a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\ & a_{21}x_1 + \dots + a_{2n}x_n \geq b_2 \\ & \vdots \qquad \qquad \qquad \vdots \\ & a_{m1}x_1 + \dots + a_{mn}x_n = b_m \\ & x_j \geq 0, \qquad \qquad \forall j = 1, \dots, n, \end{aligned}$$

where  $x_1, \dots, x_n$  are the decision variables.

When the solution to a linear program is found, one of the following four cases will apply:

- There is a unique solution.
- There are (an infinite number of) alternative optimal solutions.
- The linear program is infeasible, i.e. there is no feasible solution.
- The linear program is unbounded, i.e. there are points within the feasible region with infinitely large objective function values.

**Multi-stage linear programs** can be used when decisions must be made at a number of points in time. The formulation consists of a series of inter-dependent linear programs, one for each stage. The decisions made at each time period affect the availability of resources at future stages and thus decisions are dependent on what has happened in previous time periods. The objective is to minimise the total cost or maximise the total profit over all periods.

A linear program in which some or all of the variables must be non-negative integers is called an **integer program**. An integer program in which all variables are required to be integers is called a **pure integer program**. A **mixed integer program** requires only some of the variables to be integers. The linear program obtained by loosening the requirement that variables should take integer values is called the **linear program relaxation** of the integer program. The feasible region for an integer program is a subset of the feasible region for its linear program relaxation, but the feasible region for an integer program is not normally convex. In this situation, the optimal solution does not necessarily lie at an extreme point, therefore integer programs are usually harder to solve.

Further details on the formulation and solution of linear and integer programming problems can be found in Taha [62], and Winston [71].

### Linear Programming Example

An energy company owns a reservoir and a hydro plant for generating power. It must decide how much water to release,  $x$ , in order to satisfy the demand,  $d$ .



It costs  $c$  to release each unit of water. The company wishes to minimise the cost of meeting the demand.

The problem is formulated as:

$$\begin{array}{ll} \min & cx \\ \text{subject to:} & \\ & x \geq d. \end{array}$$

### 4.2.2 Sensitivity Analysis

**Sensitivity analysis** (also known as post-optimal analysis) is concerned with studying how the optimal solution of a linear program changes when changes are made to the original model. These changes may result in:

- The current solution becoming infeasible. Changes in resource availability or the addition of new constraints may affect feasibility.
- The current solution becoming non-optimal. Changes in the objective function or changes in resource usage of each activity may affect optimality.

Sensitivity analysis can also identify whether a constraint is binding or non-binding. A constraint is said to be **binding** if equality holds when the decision variables are replaced by their optimal values. In a maximisation problem, a binding constraint corresponds to a scarce resource. A constraint is said to be **non-binding** if equality does not hold when the optimal values are included. In a maximisation problem, this corresponds to an abundant resource.

The following results are important when performing sensitivity analysis:

- The addition of a new constraint can never improve the value of the objective function.
- The addition of a new variable can never worsen the value of the objective function.

### 4.2.3 Non-linear Programming

A **non-linear program** allows non-linear terms to appear in the objective function and constraints. This means that the decision variables can now be multiplied together, divided by each other or raised to a power. The divisibility and certainty assumptions for linear programming problems also hold for those which are non-linear. The proportionality and additivity assumptions do not hold in this case.

A **global solution** is a feasible solution whose objective function value is as good as or better than all other feasible solutions to the model. The feasible region for a non-linear programming problem is not normally convex, and even if this is the case, the optimal solution does not necessarily lie at an extreme point. Therefore, it is not particularly easy to find the global solution to a non-linear program.

A **local solution** has an objective function value which is as good as or better than all other feasible solutions in the immediate neighbourhood, although better solutions may exist which lie some distance away. Non-linear optimisation models may have several solutions which are locally optimal. Algorithms for general non-linear programming problems work by finding a solution and then testing values near this solution to see if a better objective function value can be discovered. There is a strong possibility that solution algorithms will end up with a local solution due to the way in which they advance their search. The difference between local and global solutions is shown in Figure 4.1. Certain problems have specific structures for which techniques have been developed to find an optimal solution or alternatively, to verify that a particular solution is optimal.

More information on non-linear programming problem formulation and solution methods can be found in Taha [61], and Winston [70].

#### Non-Linear Programming Example

An energy company is trying to determine where they should locate a gas terminal. The positions (in the x-y plane) of their four largest customers and the

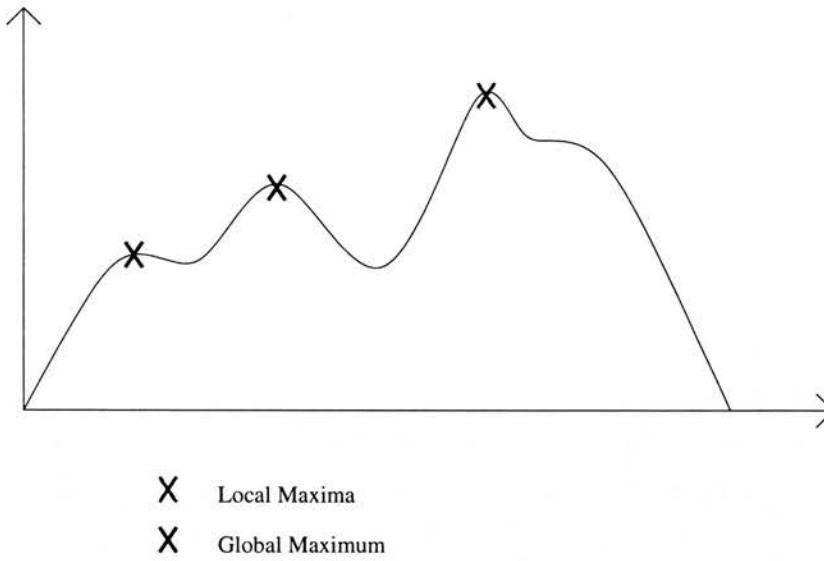


Figure 4.1: Local and Global Maxima

number of shipments made each year to each customer are:

Customer	x coordinate	y coordinate	No. of Shipments
1	$x_1$	$y_1$	$s_1$
2	$x_2$	$y_2$	$s_2$
3	$x_3$	$y_3$	$s_3$
4	$x_4$	$y_4$	$s_4$

$X$  is defined to be the x coordinate of the gas terminal and  $Y$  is defined to be the y coordinate. The non-linear program is formulated as:

$$\min s_1\sqrt{(X - x_1)^2 + (Y - y_1)^2} + s_2\sqrt{(X - x_2)^2 + (Y - y_2)^2} + s_3\sqrt{(X - x_3)^2 + (Y - y_3)^2} + s_4\sqrt{(X - x_4)^2 + (Y - y_4)^2},$$

$$X, Y \geq 0.$$

#### 4.2.4 Dynamic Programming

Dynamic programming is a technique which can be used to solve a variety of optimisation problems. Dynamic models are usually sequential or multi-stage problems where decisions are made at more than one point in time. In such a model, decisions made during the current period influence decisions in future

periods. The solution method works by breaking up a large, complicated problem into a series of smaller, more tractable ones and consecutively solving each stage.

Dynamic programming problems have the following characteristics:

- A number of **stages** with a decision to be made at each stage.
- Each stage has a number of **states** associated with it. These states describe all possible situations which could occur at a certain stage.
- The decision made at each stage determines the starting state for the next stage. This is described by a **transition function**.
- The **reward** or **cost** obtained at each stage depends upon the state and the decision made.
- Given the current state, the optimal decision for each of the remaining stages must not depend on decisions made until that point in time. This is called the **Principle of Optimality**, more details of which can be found in Bellman [5].

Dynamic programming is a more efficient technique than explicit enumeration of all possible decisions that can be made during all stages. For this reason, it lends itself particularly well to determining the optimal route through a network. Other advantages include the ability to solve stochastic problems and the fact that non-linearities can be incorporated into the formulation.

Unfortunately, many practical applications of dynamic programming involve very large state spaces. The dimension of a dynamic program is equal to the number of state variables which exist at each stage. For continuous state spaces, parameters must be discretised which can increase the problem size dramatically. This, in turn, increases solution time. Therefore, considerable computational effort is required to solve the resulting large problems. This is known as the “Curse of Dimensionality”.

A dynamic programming problem can be solved in two ways; by using either forward or backward recursion. Some types of problems favour one approach over the other. The backward recursion can be defined as:

$$f_i(t) = \min_{j \in S(t+1)} \{c_{ij}(t) + f_j(t+1)\},$$

$$\forall i \in S(t) \text{ and } 1 \leq t \leq T,$$

where  $f_i(T+1) = 0, \forall i$ .

The stages in such a problem are denoted by  $t = 1, \dots, T$ ,  $i$  denotes a possible state at stage  $t$  and  $j$  denotes a possible state at stage  $t+1$ . The above formulation attempts to minimise the cost incurred over all stages of the problem.  $f_i(t)$  is the minimum cost incurred during stages  $t, \dots, T$  given that the state at stage  $t$  is  $i$ ,  $c_{ij}(t)$  is the cost incurred by moving from state  $i$  at stage  $t$  to state  $j$  at stage  $t+1$  and  $S(t)$  is the state space at stage  $t$ .

The forward recursion for the same problem can be defined as:

$$f_i(t) = \min_{k \in S(t-1)} \{c_{ki}(t-1) + f_k(t-1)\},$$

$$\forall i \in S(t) \text{ and } 1 \leq t \leq T,$$

where  $f_i(0) = 0, \forall i$ .

In this case,  $k$  denotes a possible state at stage  $t-1$ ,  $f_i(t)$  is the minimum cost incurred during stages  $1, \dots, t$  and  $c_{ki}$  is the cost of moving from state  $k$  at stage  $t-1$  to state  $i$  at stage  $t$ .

More information on dynamic programming problem formulation and solution can be found in Sniedovich [60], Taha [61], and Winston [70].

### Dynamic Programming Example

A gas company wishes to find the shortest route by which to transport gas between gas terminals 1 and gas terminal 6. A number of gas terminals exist and are linked as shown in Figure 4.2. The cost  $c_{ij}$  of moving from terminal  $i$  to terminal  $j$  are shown on the arcs linking the nodes (which represent the terminals) in the network.

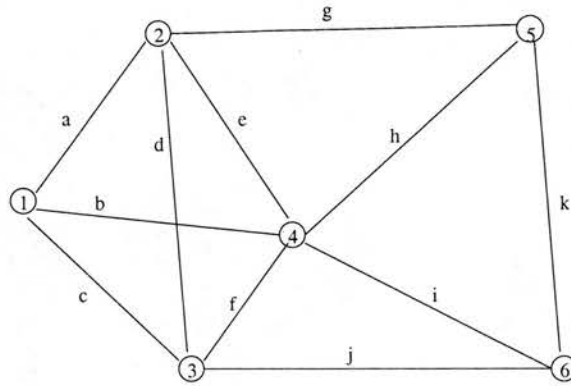


Figure 4.2: Network of Gas Terminals

The network can be represented by a cost matrix, where the matrix elements represent the cost of moving directly from one node to another:

From Terminal	To Terminal					
	1	2	3	4	5	6
1	0	a	c	b	$\infty$	$\infty$
2	a	0	d	e	g	$\infty$
3	c	d	0	f	$\infty$	j
4	b	e	f	0	h	i
5	$\infty$	g	$\infty$	h	0	k
6	$\infty$	$\infty$	j	i	k	0

Define  $f_n(i)$  as the cost of moving from terminal  $i$  to terminal 6 in at most  $n$  moves using an optimal policy and  $s_n(i)$  as the terminal to which the next move is made in an optimal path of at most  $n$  moves from terminal  $i$  to terminal 6. The dynamic program recurrences are formulated to be:

$$f_n(i) = \min_j [c_{ij} + f_{n-1}(j)],$$

with  $f_1(i) = c_{i6}$ .

### 4.2.5 Stochastic Linear Programming

A **stochastic linear program** is a linear program in which some of the parameters have unknown values. Such problems can again be solved to give



the optimal value of the objective function or, alternatively, to give an indication of the decisions which should be made at each time period. One method of solving such problems is to replace any uncertain parameters with a value for their mean. This is known as an **expected value problem**. The resulting value of the objective function for this type of problem is normally a conservative estimate to the value of the actual stochastic problem. If the original problem has many different stochastic parameters, then the difference between the stochastic solution and the solution of the expected value problem can be large as shown by Birge [6]. Multi-stage stochastic linear programs can capture both the dynamic and uncertain nature of the programming problem.

Stochastic linear problems can be roughly classified into two groups:

- **Chance Constrained Problems**

These problems contain one or more constraints that need only be satisfied with a certain probability. In order to solve this type of problem, the formulation can be transformed into a deterministic problem by requiring that each constraint must be satisfied. If it is possible to solve the resulting problem, then this will provide a lower or upper bound on the original problem.

- **Recourse Problems**

This type of problem has a number of stages. A decision is made at the first stage and then the expected profit or costs are optimised over the remaining stages. Constraints exist which describe the links between the first stage decisions and those still to be made. Each constraint must hold for the decision made at every time period.

A chance constrained problem has the general formulation:



generated is of the form:

$$x^{k+1} = x^k + A(x^k),$$

where  $A$  is specified by its approximation to some underlying function to be optimised.

More information on stochastic linear programming problem formulation and solution can be found in Kall and Wallace [40], and Birge and Louveaux [7].

### Stochastic Linear Programming Example

An electricity supplier buys  $x$  units of electricity at a price  $c$  per unit. This number has an upper limit  $u$ , representing either the purchase power of the supplier, or a limit set by the seller. As many units of electricity as possible are sold to customers at a selling price  $q$ . Since electricity cannot be stored, it is assumed that unsold units can be returned to the network operator at a price  $r$ , with  $r < c$ .

The aim of the supplier is to decide how many units of electricity to buy on each day. The decision must be made before the demand on each day becomes known. The demand varies over days and is described by a random variable,  $\xi$ . To describe the profit,  $y$  is defined to be the number of sales and  $w$  as the amount of units remaining at the end of the day.

The problem is formulated as:

$$\begin{aligned} \min \quad & c^T x + Q(x) \\ & 0 \leq x \leq u \end{aligned}$$

where  $Q(x) = E_\xi Q(x, \xi)$  and:

$$\begin{aligned} Q(x, \xi) = \min \quad & (-qy(\xi) - rw(\xi)) \\ \text{subject to:} \quad & \\ & y(\xi) \leq \xi \\ & y(\xi) + w(\xi) \leq x \\ & y(\xi), w(\xi) \geq 0. \end{aligned}$$

$E_\xi$  denotes the expectation with respect to  $\xi$ .

The expected profit on sales and returns is given by  $-Q(x)$ , while  $-Q(x, \xi)$  is the profit on sales and returns if the demand is  $\xi$ .

### 4.2.6 Stochastic Dynamic Programming

Dynamic programming is concerned with solving deterministic sequential decision problems over a finite horizon. Stochastic dynamic programming evolved from this subject to enable modelling of stochastic systems where probabilities depend upon the stage, state and action taken. This type of problem can also be used to model time horizons of infinite length.

A problem to maximise expected return can be formulated as a backward recursion:

$$f_i(t) = \max_{k \in D_i(t)} \left\{ q_i^k(t) + \sum_{j \in S(t+1)} p_{ij}^k(t) f_j(t+1) \right\},$$

$$\forall i \in S(t) \text{ and } 1 \leq t \leq T$$

where  $f_i(T+1) = 0, \forall i$ .

In this problem, the stages are again denoted by  $t = 1, \dots, T$ , the state at stage  $t$  is denoted by  $i$  and the state at stage  $t+1$  is denoted by  $j$ . A number of possible actions are available, denoted by  $k$ . These belong to the set of all possible actions at stage  $t$  and state  $i$ ,  $D_i(t)$ .  $f_i(t)$  is the maximum expected return over the remaining  $t$  stages given the state is  $i$ ,  $p_{ij}^k(t)$  is the probability of moving from state  $i$  at stage  $t$  to state  $j$  at stage  $t+1$  under action  $k$  and  $q_i^k(t)$  is the expected return in stage  $t$  under action  $k$  given the state is  $i$ .

A **stationary** policy is one that does not depend upon time; the decision taken depends only upon the current state of the system. Planning horizons of infinite length can be modelled in situations where a stationary policy exists. The following conditions define a stationary model:

- The transition probabilities and the expected transition returns do not change with time.
- If returns are discounted, then the discount is the same for all time periods.
- The set of decisions and states available is the same for all time periods.

Stochastic dynamic programming models can be adapted for a time period of infinite horizon. This type of problem can be formulated by letting  $f_i$  be the maximum expected return over an infinite horizon given the current state is  $i$ . Then:

$$f_i = \lim_{t \rightarrow \infty} f_i(t).$$

The recurrence relation becomes:

$$f_i = \max_{k \in D_i} \left\{ q_i^k + \sum_{j \in S} p_{ij}^k f_j \right\},$$

$$\forall i \in S.$$

More information about Stochastic Dynamic Programming can be found in Ross [54] and Puterman [50].

### Stochastic Dynamic Programming Example

A European call option to buy power has an exercise price of  $K$ . The return on the option can take one of two values:  $u$  with probability  $p$  or  $d$  with probability  $1 - p$ . There are  $n$  periods until maturity, the current share price is  $s$  and the risk free rate of interest is  $r$ .

$f_i(n)$  is the price of the European call option when there are  $n$  periods until maturity,  $i$  periods since the start of the time horizon and the current share price is  $s(1+u)^i(1+d)^{T-n-i}$ . The terminal values are:

$$f_i(0) = \max\{s(1+u)^i(1+d)^{T-i} - K, 0\}.$$

The optimality equation can be defined as:

$$f_i(n) = \frac{pf_{i+1}(n-1) + (1-p)f_i(n-1)}{1+r}.$$

### 4.3 Generation of Scenario Trees

One of the most common approaches for modelling uncertain variables is to use a scenario tree which describes possible values which can occur over a certain time period. A **scenario** is a sequence of uncertain outcomes, starting from an initial known value. A collection of scenarios can be represented by a tree structure, as shown in Figure 4.3, where each scenario is a path through this tree which starts at the initial node. At each time stage, the nodes of the tree correspond to one or more uncertain variables which can occur at that time, for example, each node of a tree could correspond to a possible value for the price and also to a possible value for the demand.

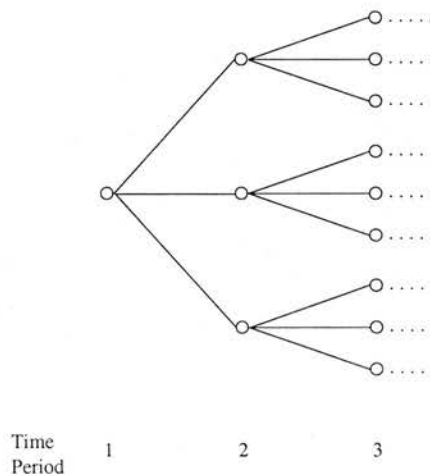


Figure 4.3: Example of Scenario Tree

The most common types of tree used in practice are binomial and trinomial trees, although trees with mixed structures are also fairly common. A **binomial** tree has two possible moves from each node; often a move to a higher value and a move to a lower value. Each move is denoted by a line, known as a branch. **Trinomial** trees, an example of which is shown in Figure 4.3, have three possible moves; often higher, lower and no change in value. The number of nodes in a tree increases exponentially with the number of branches from a node. Therefore, multi-nomial trees with large numbers of branches are less common. Binomial and trinomial models can be used to model the same problems, using the same

underlying stochastic process. This process is approximated in a different way in each situation.

Trees whose branches recombine over time can be used to limit the number of possible nodes at a certain time stage. An example of a tree of this type is shown in Figure 4.4. In such a tree, decisions that are taken are not dependent on the decisions which have been taken to reach that node. The deeper into the tree a node occurs, the more possibilities there are of how that node could have been reached.

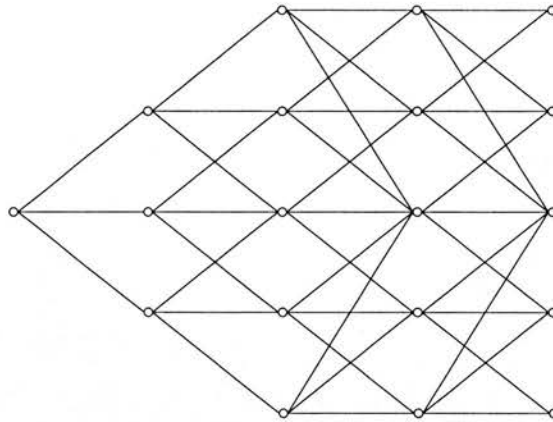


Figure 4.4: Tree with Recombining Branches

A tree structure causes the number of possible scenarios to increase exponentially. Despite the power of computers in recent years increasing substantially and the recent developments in the areas of parallel and distributed computing, it is still impossible to solve very complicated stochastic problems accurately. Generalisations must be made in order to reduce stochastic problems to a manageable size.

One of the main generalisations which can be made is a restriction on the number of time periods. Variable length time periods can be used, for example a three year period could be covered using six time periods by setting  $t_1 = 1$  month,  $t_2 = 2$  months,  $t_3 = 3$  months,  $t_4 = 6$  months,  $t_5 = 1$  year and  $t_6 = 1$  year. This would give most detail in early time periods which seems reasonable since the

near future can be predicted more accurately than further ahead in time. It is likely that models will be used to predict outcomes or decisions for the following period. When the values for this period become known, the model can be resolved using these values to predict for the next period.

Sampling techniques can be used to give approximations of the uncertainty present in stochastic problems, by selecting a small set of all possible future scenarios. These sampled paths can give an estimate of the actual uncertainty. The statistical properties of this estimate can be calculated and used as a guide of whether to re-sample, increasing the number of samples used or whether to stop.

Sampling techniques can be incorporated into solution algorithms both externally and internally. External sampling selects a sample of possible future values before the algorithm is employed. This sample is then used to form a simplified version of the actual problem. Solving the resulting problem can give an approximate answer to the original problem. The process can be repeated with a different sample and the results used to improve the estimates of the solution. Internal sampling is performed within the algorithm. Each iteration provides a new sample which can, but does not necessarily, depend on what has occurred in the previous iteration.

Tree approaches are widely used when modelling asset prices and are especially useful for valuing options. They replace the assumption present in the Black Scholes formulae that the asset price moves smoothly and continuously through time, with an assumption that the asset price moves in discrete jumps over discrete time intervals during the life of the option. Tree approaches determine the option price by discounting the value of the expected option payoff.

Cox et al. [20], developed a binomial option pricing method, in which the asset price moves up or down by a fixed proportion at each time step over the life of the option. The values of a move in the up direction,  $u$ , and a move in the down direction,  $d$ , and their respective probabilities,  $p_u$  and  $p_d$ , are determined in such a manner that the mean and variance of the distribution of the tree are equal to



the mean and variance of the price distribution for the asset. The probabilities of an up move and the corresponding down move must sum to one. The following formulae were shown to satisfy these properties:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = \frac{1}{u}, \quad p_u = \frac{e^{r\Delta t} - d}{u - d} \quad \text{and} \quad p_d = 1 - p_u,$$

where  $\sigma$  is the volatility of the stock price,  $r$  is the risk free rate of interest and  $\Delta t$  is a small change in time.

Boyle [10], developed a method for constructing a trinomial tree. Trinomial tree approaches are useful for incorporating mean reversion, an important feature of energy prices, which is detailed in Section 3.6.1. They give more accurate results than binomial trees since they include an additional branch at each node. Hull [36], shows that using a trinomial tree is equivalent to applying the explicit finite difference method, a method which values a financial contract by solving the differential equation which the contract satisfies. Boyle [11], has adapted his process to produce a trinomial tree for two underlying state variables that do not depend on the value taken by the other.

Hull and White [33], extend the Cox et al. [20], method described previously to show how to value an American option from a tree which was constructed to value a European option. The approach is shown to produce a considerable improvement over the basic tree approach for estimating the value of an American option. The method can easily be extended to other options for which an analytic solution is not readily apparent by comparing the structure of the option under consideration to others for which solutions are known.

### 4.3.1 Fitting Forward Prices

The stochastic models which have been constructed for the different types of problem considered in Chapters 6, 7 and 8 have spot prices as their uncertain variables. The spot price processes for each of the models are represented by a recombining trinomial tree. The method for constructing a trinomial tree of this type is described by Hull and White [34]. The resulting tree has a structure similar to that shown in Figure 4.3.

Hull and White [34], set a maximum number of price nodes which can occur at each time period. The maximum number of price levels above the central level,  $j_{\max}$ , is set to be equal to the smallest integer greater than:

$$\frac{c}{e^{-\alpha\Delta t} - 1} ,$$

where  $\alpha$  is the rate of mean reversion of the spot price,  $\Delta t$  is the size of time step between periods and  $c$  has the suggested value of 0.184. The estimates of the probabilities are shown to always be positive if this equation is used. This means the maximum number of price levels at any time period can not exceed:

$$j \leq 2 j_{\max} + 1.$$

At the first and last nodes for each stage in the tree, the branching changes so that the tree will recombine. This prevents the tree from growing too large. This leads to three different branching patterns which are shown in Figure 4.5. Hull and White [34], show that by using these branching types, the possibility of the probabilities becoming negative is avoided.

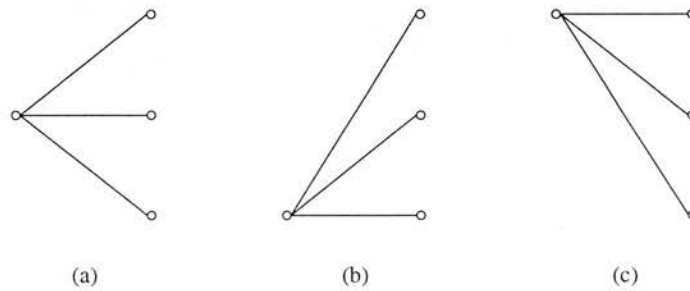


Figure 4.5: Types of Branching

Three different sets of formulae are used to calculate the probabilities in each of the branching variations. These probabilities are chosen to match the expected change and variance in the spot price,  $x$ , over the next time step. The expected change is given by:

$$E[dx] = xM,$$

where  $M = e^{-\alpha\Delta t} - 1$ ,  $\alpha$  is the rate of mean reversion of the spot price and  $\Delta t$  is the size of the time step between periods. The variance of this change is given

by:

$$\text{Var}(x) = \frac{\sigma_s^2(1 - e^{-2\alpha\Delta t})}{2\alpha},$$

where  $\sigma_s$  is the volatility of the spot price,  $\alpha$  is the rate of mean reversion of the spot price and  $\Delta t$  is the size of time step between periods.

Hull and White [34], give formulae for calculating the probabilities for each of the different branching types. For normal branching, as shown in Figure 4.5(a), the probabilities of the three branches are given by:

$$p_u = \frac{1}{6} + \frac{M^2 j^2 \Delta t^2 - M j \Delta t}{2}$$

$$p_m = \frac{2}{3} - M^2 j^2 \Delta t^2$$

$$p_d = \frac{1}{6} + \frac{M^2 j^2 \Delta t^2 + M j \Delta t}{2}$$

$p_u$  denotes the probability of an upwards jump,  $p_m$  denotes the probability of a middle jump and  $p_d$  is the probability of a downwards jump. If branching is as shown in Figure 4.5(b), then the probabilities used are:

$$p_{2u} = \frac{1}{6} + \frac{M^2 j^2 \Delta t^2 + M j \Delta t}{2}$$

$$p_{1u} = -\frac{1}{3} - M^2 j^2 \Delta t^2 - 2M j \Delta t$$

$$p_m = \frac{7}{6} + \frac{M^2 j^2 \Delta t^2 + 3M j \Delta t}{2}$$

In this case,  $p_{2u}$  is the probability of an upwards jump by two nodes,  $p_{1u}$  is the probability of an upwards jump. If the branching occurs as shown in Figure 4.5(c),

the probabilities are given by:

$$p_m = \frac{7}{6} + \frac{M^2 j^2 \Delta t^2 - 3Mj\Delta t}{2}$$

$$p_{1d} = -\frac{1}{3} - M^2 j^2 \Delta t^2 + 2Mj\Delta t$$

$$p_{2d} = \frac{1}{6} + \frac{M^2 j^2 \Delta t^2 - Mj\Delta t}{2}.$$

Similarly,  $p_{1d}$  is the probability of a downwards jump and  $p_{2d}$  denotes the probability of a downwards jump by two nodes. In the above formulae,  $M = e^{-\alpha\Delta t} - 1$ , as before and  $j$  denotes the price level. The above formulae satisfy the requirement that the sum of the three probabilities at any node is equal to one.

The Black Model, described in Section 3.5.2, is used in energy markets to value options on futures. In order to apply this model, it must be assumed that forward prices follow a geometric Brownian motion process of the form:

$$dF = \mu F dt + \sigma_F F dz,$$

where  $\mu$  is the drift of the forward price process,  $\sigma_F$  is the volatility of the forward price process and  $dz$  is a Wiener process. Hull [36], shows that the expected growth rate or drift of a futures price is zero. Since there is little or no difference between forward and futures prices in energy markets, the drift of this process can be set to zero. The model becomes:

$$dF = \sigma_F F dz.$$

Clewlow and Strickland [19], use the above result to describe a method for constructing a trinomial spot price tree which is consistent with the observed market volatility and a given forward price curve. In order to construct the spot price tree the following information must be obtained:

- A forward curve which is obtained from market data,  $F(t, T)$ , where  $t$  is the index of time periods. Forward curves are described in Section 3.6.3
- The volatility of the spot price,  $\sigma_s$ . This is discussed in Section 3.4.
- The rate of mean reversion of the spot price,  $\alpha$ , described in Section 3.6.1.
- The risk free rate of interest,  $r$ . This is described in Section 3.2.

A two parameter volatility term structure is introduced:

$$\sigma_F = \sigma_s e^{-\alpha(T-t)},$$

where  $\sigma_s$  is the volatility of the spot price process and  $\alpha$  is the rate of mean reversion of the spot price or the rate at which the forward prices decline.

Clewlow and Strickland [19], show that the above forward price process and volatility term structure imply a spot price process of the form:

$$ds_t = \alpha [\mu(t) - \ln s_t] s_t dt + \sigma_s s_t dz$$

where:

$$\mu(t) = \frac{1}{\alpha} \frac{\partial \ln F(0, t)}{\partial t} + \ln F(0, t) + \frac{\sigma_s^2}{4\alpha} (1 - e^{-2\alpha t}).$$

These equations can be shown to be consistent with an initial given forward curve. The forward price at any time is shown to be a function of the spot price at that time, the initial forward curve and the volatility parameters. This allows the payoff of options to be evaluated analytically when using tree approaches, rather than estimated using numerical procedures.

By setting  $x(t) = \ln s_t$  and applying Ito's Lemma, Clewlow and Strickland [19], show that the spot price process may be written as:

$$dx(t) = [\theta(t) - \alpha x(t)] dt + \sigma_s dz(t),$$

where:

$$\theta(t) = \left[ \frac{\partial \ln F(0, t)}{\partial t} + \alpha \ln F(0, t) + \frac{\sigma_s^2}{4} (1 - e^{-2\alpha t}) - \frac{1}{2} \sigma_s^2 \right].$$

Assuming  $\theta(t) = 0$ , the above spot price process becomes:

$$dx(t) = -\alpha x(t)dt + \sigma_s dz(t).$$

The spot price tree is constructed using the results from Hull and White [34], for the actual structure of the tree and the results from Clewlow and Strickland [19], for the values at each of the nodes. The construction takes place in two phases. The first phase involves constructing the structure of the tree by determining the appropriate moves in price between each time period. The second phase adds the appropriate drift values to each node to make the spot prices consistent with the forward price curve.

Using the spot price process derived above, a tree is constructed for  $x^*(t) = \ln s_t$ . The value of the initial node is assumed to be zero. The tree is constructed with constant time steps,  $\Delta t$ , and constant price steps,  $\Delta x^*$ .  $\Delta t$  is given by dividing the length of the time horizon by the number of time steps which are required. Hull and White [34], suggest setting:

$$\Delta x^* = \sqrt{3 \text{Var}(x)} = \sqrt{\frac{3 \sigma_s^2 (1 - e^{-2\alpha \Delta t})}{2\alpha}},$$

in order to minimise errors. In this way, the basic structure of the tree is formed.

The second phase of the process is concerned with displacing the nodes in order to add the proper drift and to be consistent with the observed forward prices. The nodes can be displaced by ensuring that the given forward price curve matches the expected spot price. Clewlow and Strickland [19], show that this is achieved when:

$$F(0, t) = e^{rt\Delta t} \sum_{t,j} \lambda(t, j) s_t^j.$$

$F(0, t)$  denotes the initial forward curve price at time  $t$ ,  $r$  is the risk free rate of interest,  $\lambda(t, j)$  is the Arrow-Debreu state price at time  $t$  for price level  $j$  and  $s_t^j$  denotes the spot price at node  $(t, j)$ .

The drift can then be worked out using the following formula:

$$a_t = \ln \left( \frac{F(0, t)e^{-rt\Delta t}}{\sum_j \lambda(t, j)e^{x^*(t, j)}} \right),$$

where  $x^*(t, j)$  is the initial spot price value at time  $t$  for price level  $j$ . These formulae require the use of Arrow-Debreu state prices.

Arrow [2], and Debreu [22], describe a method for calculating the state price of each node. The **Arrow-Debreu state price** of a node in a tree is the probability of reaching that node, discounted by a factor of  $e^{-rt\Delta t}$ , where  $r$  is the risk free interest rate and  $\Delta t$  is the size of the time step between periods. Calculating Arrow-Debreu state prices becomes more complicated for nodes which occur far into the tree, since the method involves enumeration of all possible paths which lead to that node. The number of paths increases exponentially the further into the tree the node occurs. It is possible to compute Arrow-Debreu state prices iteratively which overcomes this problem.

Figure 4.6 gives an indication of how to calculate iterative state prices for the Arrow-Debreu method. Each node in the tree is denoted by  $(t, j)$ , where  $t$  is the point in time and  $j$  is the price level. The Arrow-Debreu state price at a node is denoted by  $\lambda(t, j)$  and the probability of an upwards jump by  $p_u(j)$ , the probability of a middle jump by  $p_m(j)$ , and the probability of a downwards jump by  $p_d(j)$ .

The iterative Arrow-Debreu state prices can be calculated using the following formula:

$$\lambda(t, j) = e^{-r\Delta t} [p_d(j)\lambda(t-1, j) + p_m(j-1)\lambda(t-1, j-1) + p_u(j-2)\lambda(t-1, j-2)].$$

In a recombining tree, this formula must be adapted to cover all possible branching possibilities.

If  $j = 1$  then:

$$\lambda(t, 1) = e^{-r\Delta t} p_d(1)\lambda(t-1, 1).$$

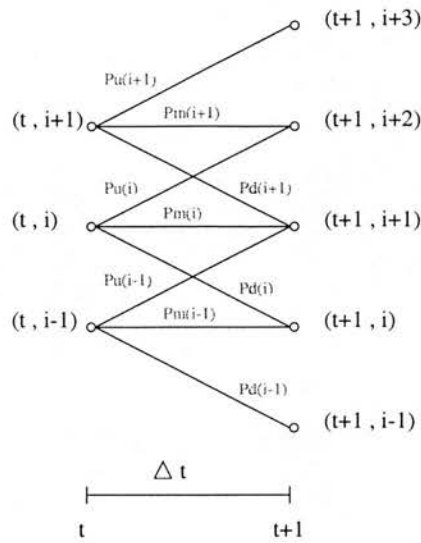


Figure 4.6: Calculation of Iterative Arrow-Debreu Prices

If  $j = 2$  then:

$$\lambda(t, 2) = e^{-r\Delta t} [p_m(1)\lambda(t - 1, 1) + p_d(2)\lambda(t - 1, 2)].$$

If  $j = 2t - 1$  then:

$$\lambda(t, 2t - 1) = e^{-r\Delta t} p_u(2t - 3)\lambda(t - 1, 2t - 3).$$

If  $j = 2t - 2$  then:

$$\lambda(t, 2t - 2) = e^{-r\Delta t} [p_m(2t - 3)\lambda(t - 1, 2t - 3) + p_u(2t - 4)\lambda(t - 1, 2t - 4)].$$

The tree was constructed to be consistent with the process where  $x(t) = \ln s_t$ . The final values of  $x(t, j)$  can be calculated by adding together the values from the first and second stages:

$$\ln s_t^j = x(t, j) = x^*(t, j) + a_t,$$

where  $x^*(t, j)$  is the value at node  $(t, j)$  for the first phase and  $a_t$  is the value for the drift, calculated in the second phase. The actual spot values for each node in the tree can be calculated by taking the exponential of this value.

The above approach is used to construct spot price trees for the models described in Chapters 6, 7 and 8. This approach is chosen as forward curves



are readily available for use and the method is consistent with other pricing techniques currently used in the market.

### 4.3.2 Moment Matching

The following method is presented as an alternative to that described in the previous section. It provides an alternative procedure for calculating the values at each node of a scenario tree. The methods presented are not implementable due to lack of historical data available. The type of problems considered in this research have daily resolution and a time period which spans one year. In order to implement this approach, many years of data would be needed.

The method described by Høyland and Wallace [32], uses historical data to generate values for a trinomial tree. The process consists of calculating various statistics for a given set of data and using the resulting values in a non-linear program to calculate values for the nodes of the tree. The non-linear program works by minimising the square distance between the statistical properties and the values at the nodes.

The first stage in the process is to select the statistics which are to be matched when constructing the tree. The most common variables used are:

- The **maximum** value of the data set.
- The **minimum** value of the data set.
- The **expected value** or **mean** which calculates the average value of the data set:

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n}.$$

- The **variance** which measures the spread of the data set:

$$\text{Var}(X) = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}.$$

- The **skewness** which measures how symmetrical the data set is:

$$\text{Skew}(X) = \frac{\sum_{i=1}^n (X_i - \bar{X})^3}{n}.$$

- The **kurtosis** which measures the degree of peakedness of the distribution:

$$\text{Kurt}(X) = \frac{\sum_{i=1}^n (X_i - \bar{X})^4}{n}.$$

The maximum and minimum values are included to ensure that extreme events are captured.

Some or all of the above values can be calculated depending upon the data set under consideration. The non-linear program to generate variables for one time period of the trinomial tree using all the above statistical values is given below:

$$\begin{aligned} \min \quad & (x_1 p_1 + x_2 p_2 + x_3 p_3 - \bar{X})^2 + \\ & \left( (x_1 - \bar{X})^2 p_1 + (x_2 - \bar{X})^2 p_2 + (x_3 - \bar{X})^2 p_3 - \text{Var}(X) \right)^2 + \\ & \left( (x_1 - \bar{X})^3 p_1 + (x_2 - \bar{X})^3 p_2 + (x_3 - \bar{X})^3 p_3 - \text{Skew}(X) \right)^2 + \\ & \left( (x_1 - \bar{X})^4 p_1 + (x_2 - \bar{X})^4 p_2 + (x_3 - \bar{X})^4 p_3 - \text{Kurt}(X) \right)^2 \end{aligned}$$

subject to:

$$x_1, x_2, x_3 \leq \text{Max}(X)$$

$$x_1, x_2, x_3 \geq \text{Min}(X)$$

$$p_1 + p_2 + p_3 = 1$$

$$p_1, p_2, p_3 \geq 0.$$

$x_1$ ,  $x_2$  and  $x_3$  denote the values at each of the nodes and  $p_1$ ,  $p_2$  and  $p_3$  are the corresponding probabilities associated with each branch, as shown in Figure 4.7.  $\bar{X}$ ,  $\text{Var}(X)$ ,  $\text{Skew}(X)$ ,  $\text{Kurt}(X)$ ,  $\text{Max}(X)$  and  $\text{Min}(X)$  are the mean, variance, skewness, kurtosis, maximum and minimum values respectively. These are constants which have previously been calculated from a given data set.

Since it is relatively difficult to solve non-linear programs, any solution with distribution properties close to or equal to the specifications is satisfactory,

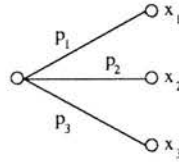


Figure 4.7: Values Generated by Non-Linear Program

although better solutions may exist. An objective value close to or equal to zero indicates that the distribution of the scenarios has a good or perfect match with the specifications.

Extending this approach to more than one period may complicate the method by implying that dependencies between stages exist and must be considered. In order to extend this approach to more than one stage, conditional distributions can be specified for the second period. The same approach can be used to construct the values at each of the three nodes, but the probabilities must be adjusted to reflect the dependency on the first stage. Figure 4.8, represents the values to be calculated in the second stage at the three nodes,  $y_1$ ,  $y_2$  and  $y_3$ , and also the structure of the tree.

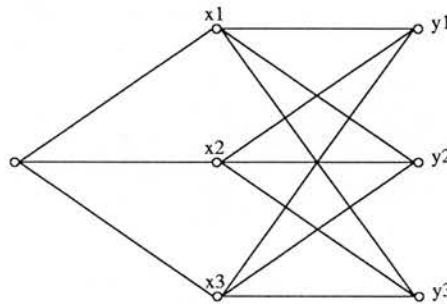


Figure 4.8: Scenario Tree for Two Periods

The probabilities can be adjusted as follows. The following equations must hold:

$$\begin{aligned}
\Pr(y_1 | x_1)\Pr(x_1) + \Pr(y_1 | x_2)\Pr(x_2) + \Pr(y_1 | x_3)\Pr(x_3) &= \Pr(y_1) \\
\Pr(y_2 | x_1)\Pr(x_1) + \Pr(y_2 | x_2)\Pr(x_2) + \Pr(y_2 | x_3)\Pr(x_3) &= \Pr(y_2) \\
\Pr(y_3 | x_1)\Pr(x_1) + \Pr(y_3 | x_2)\Pr(x_2) + \Pr(y_3 | x_3)\Pr(x_3) &= \Pr(y_3) \\
\Pr(y_1 | x_1) + \Pr(y_2 | x_1) + \Pr(y_3 | x_1) &= 1 \\
\Pr(y_1 | x_2) + \Pr(y_2 | x_2) + \Pr(y_3 | x_2) &= 1 \\
\Pr(y_1 | x_3) + \Pr(y_2 | x_3) + \Pr(y_3 | x_3) &= 1,
\end{aligned}$$

where  $x_1$ ,  $x_2$  and  $x_3$  represent the values calculated at time period 1 and  $y_1$ ,  $y_2$  and  $y_3$  represent the values calculated at time period 2.  $\Pr(y_1)$ ,  $\Pr(y_2)$  and  $\Pr(y_3)$  are the probability values given by solving the non-linear program.

This is a total of six equations and nine unknowns therefore there is an amount of freedom in the choice that these values can take. This can be overcome by performing analysis on the original data set to give some idea about the approximate values of the unknowns. It was thought that the values of  $\Pr(y_1 | x_1)$ ,  $\Pr(y_2 | x_2)$  and  $\Pr(y_3 | x_3)$  should be reasonably large as the probability of remaining in the same state is usually high. This can be checked by analysing the data set.

In order to confirm this, the values obtained for the first period can be used. The values corresponding to the first period in the time set are classified as low, mid or high values. If a value is less than:

$$\frac{x_2 - x_1}{2} + x_1,$$

then it is classified as a low value. If a value is greater than:

$$\frac{x_3 - x_2}{2} + x_2,$$

then it is classified as a high value. Otherwise, it is classified as a mid value.

Techniques such as Monte Carlo simulation can also be used when constructing models for situations which arise in energy industries. The focus of this thesis is on mathematical programming solution methods and therefore this method has not been described or applied. Details of this technique can be found in Hull [36], and Daily [30].



## Chapter 5

# Mathematical Programming in the Energy Industries

### 5.1 Introduction

Privatisation and deregulation have brought many changes to the structure of the energy industries. Prior to restructuring, generating companies were responsible for meeting demand for power and wished to achieve this at the lowest possible cost. Their objectives have now changed as they no longer have an assured market and must compete for customers. Energy must now be provided at low cost in order to increase market share. Energy companies wish ultimately to maximise profits. Traditional models must be adapted or alternative models developed in order to represent the new structure accurately.

This chapter aims to give an overview of various approaches that have been developed in order to model the structure of the privatised industries. The first stage of privatisation in many countries was the development of a pool through which all market participants were required to trade. Literature which describes models which were developed to optimise under this type of structure is described first, the majority of which focuses on specific countries. These areas have been heavily researched and represent ways in which mathematical programming techniques have been extensively applied in energy industries. The introduction of the New Electricity Trading Arrangements which are detailed in Section 2.4.1,

means that this type of model can no longer be applied in the UK, although it is still widely applicable in other countries of the world.

The new trading arrangements have altered the industry by shifting the focus to place more emphasis on coordination of contracts and assets. Energy market participants tend to hold a portfolio of such contracts and assets in order to have an assured income. These portfolios can be coordinated and optimised in order to maximise profits and the position of the participant in the market. This is another area in which mathematical programming techniques have historically been extensively applied. A review of literature examining this area is detailed next.

Finally, a model was developed, drawing on the research described in these areas. It aims to optimise the profits resulting from a series of contracts and assets in a deregulated market. The most common types of assets found in energy industries are power generation and storage facilities. The gas storage problem is described in Chapter 6. Examples of contracts which may be found are take or pay contracts and tolling deals. Take or pay contracts represent over 80% of the contract market. Tolling deals are becoming more common as their popularity increases. Details of these contracts can be found in Chapters 7 and 8.

## 5.2 Review of Literature

### 5.2.1 Restructured Markets

Extensive research into the competitive behaviour of market players has led to a greater understanding of the effects of privatisation. The structure of energy industries in most countries is centred around a pool. The research described details different approaches to modelling this type of structure.

Egeland et al. [24], describe a mathematical model for use in a power pool where electricity is provided by hydro electric generation plants. The aim is to minimise the expected variable cost of the entire interconnected system whilst satisfying the given constraints. The model incorporates stochastic variables and

because of this can only be applied to systems with small amounts of reservoirs. Real life generation networks must be considerably simplified before the dynamic programming solution methods can be used. Ruusunen et al. [55], describe a similar stochastic approach for a group of interconnected utilities, again operating in an electricity power pool. This model aims to minimise the total operating cost of the pool subject to operational constraints before distributing any savings fairly between the participants. The model also allows for the exchange of electricity directly between utilities.

Vlahos et al. [64], have developed a simulation platform which allows the modeller to represent the industry in terms of the main players, their strategies and the way in which they interact. Each player can be viewed as an object with specific attributes and methods, represented by decision rules. Complex decision rules can be implemented as external models which are run during the simulation. The platform aims to model the evolution of the pool structure and contract markets, competition, investments and regulation. The paper focuses on the interaction between the electricity pool and the contract market and also the relationship between the gas and electricity markets. The extent to which generators can manipulate the pool through different bidding strategies has also been investigated.

Ferrero and Shahidehpour [27], present an optimisation approach which has a non-linear objective function and linear constraints. The aim is to optimise the scheduling of power transactions in deregulated electricity markets. The research is centred on the US electricity market in which federal regulations require that third parties are allowed access to the grids. The paper details two different situations. The first of these describes a centrally operated power system in which independent utilities are linked by a network representing a power pool. The second is a deregulated environment in which independent utilities are linked by a transmission network, each defining its own operation policy.

Wangensteen et al. [65], give an overview of the new structure of the Norwegian electricity market. Norway takes approximately 99% of its electricity from hydro power. A power pool has been in existence in Norway for over 20 years



and deregulation has eliminated all formal restrictions on participation and the contracts which can be exchanged. A model is described which aims to maximise the expected profit resulting from the sale of power to the pool. The optimisation problem described has been designed explicitly to take risk aversion into account. Stochastic dynamic programming was used to solve the model.

Scott and Read [58], describe a simulation model for hydro-electric reservoir operation in a deregulated market. The model was constructed for New Zealand, where approximately 80% of electricity comes from hydro power. Generating firms must coordinate the use of hydro and thermal stations throughout the time horizon. The model examines the trade off between using water held in reservoirs at the present time with the value from holding it in storage until later periods when it may be more valuable. It also aims to quantify the extent of losses which can occur due to manipulation of prices by station operators; the resulting losses are compared with gains which may result in other areas due to increased efficiency levels. It is noted that the objectives of generators may change substantially when contracts are held. Dual dynamic programming is used to solve this problem.

### 5.2.2 Portfolio Optimisation

A series of liability payments (financial obligations which are made to satisfy contractual terms) are known to occur in the future. An investor wishes to construct a portfolio of securities which allows him to meet these liabilities under a variety of plausible scenarios. From all feasible portfolios, he wishes to select the one that optimises some optimality criterion such as minimum cost or maximum profit. The size of the liability payments and returns from securities may depend upon events which happen in the future, such as fluctuations in price.

Klaasen [42], presents aggregation methods which can be used in combination with financial asset pricing models to reduce the number of states and time periods in the event tree whilst preserving the desired properties. These methods give an arbitrage-free description of the uncertainty which is shown to be consistent with observed market prices. An iterative solution approach can

be developed with the resulting aggregated discrete time event tree structure being concise enough to be used in a stochastic linear programming model. These techniques are applied to an asset management model which aims to meet liabilities at future points in time in an optimal way. The model includes the possibility of rebalancing in response to new information that becomes available.

Gautier and Granot [29], detail an asset allocation model which is general in that no specific asset categories are defined. It is formulated as a non-linear parametric network flow problem with an additional linear constraint. The chosen set of assets can be redistributed at certain points so as to achieve optimal performance over all time periods in the model. The problem can be formulated either to maximise the expected return or to minimise some risk function. Mulvey and Ziemba [46], tackle a similar problem of asset allocation by formulating a multi-stage stochastic program. Scenarios are generated which aim to give a general representation of all possible outcomes. A set of these scenarios, thought to give a good approximation to all possibilities, is determined. Both optimistic and pessimistic views are incorporated into the selection.

Fleten et al. [28], describe an asset/liability model for a generator operating in a deregulated electricity market. The model has been developed with Scandinavian markets in mind, where the majority of electricity comes from hydro power. The paper focuses on the coordination of physical generation resources (i.e. reservoirs) with financial contracts for electricity (i.e. forwards and futures) in order to lessen the risk associated with uncertain variables, such as price and inflow. It has been formulated as a multi-stage stochastic program with the objective being to maximise profit. It may be necessary to restrict the number of scenarios and realisations of random variables in order to solve this problem.

### 5.3 Model to Optimise the Value of Contracts and Assets

The idea for the following model was obtained from Fleten et al. [28], and Mulvey and Ziemba [46]. A description of both papers is included in Section 5.2.

The described model is an extension of the one proposed in Archibald and Thomas [1], and was developed to optimise the value of a selection of contracts and assets. The majority of energy is currently bought and sold through financial contracts, which enable market players to evaluate and control both their position and the risk to which they are exposed.

The aim of the model was to provide flexibility in the coordination of a selection of contracts and assets. It was designed to cope with contracts specific to energy markets and allows the incorporation of stochastic parameters. The model was constructed as a result of the rapidly changing trading environment of energy markets. The need for accurate pricing models for the more complex and innovative contracts which are now emerging has grown drastically. The model aims to improve on simplified methods commonly used in practice.

One of the main aims of electricity generators is to try to coordinate their use of assets and contracts in order to obtain greater profits. The model was formulated for a number of different assets and a number of different contracts. Typical assets can include power generation units and gas storage facilities. The generating units can be of different types, such as hydro-electric or combined cycle gas turbine, as can the storage facilities. Generators must make decisions about how much power to generate or how much gas to extract from storage in order to meet the requirements of the contracts that they have entered into. The model also includes the flexibility of obtaining new contracts or assets at any of the time periods throughout the horizon.

The most common types of contract found in energy markets are take or pay contracts, although tolling deals are becoming increasingly more common. Take or pay contracts allow the holder to vary the amount of energy bought each day within specified daily and annual limits. More details of this type of contract can be found in Chapter 7. Tolling deals allow the holder to exchange the price of fuel with the price of power subject to certain constraints. This type of contract is described in Chapter 8.

There are a number of time periods in the planning horizon and also a number of uncertain variables. The uncertain variables could be financial (such as price), structural (such as amount produced) or could relate to the climate. In each of the time periods there are a number of possible values which the uncertain variable can take. A tree of scenarios can be constructed to represent this situation. Either of the methods described in Section 4.3, could be used for this purpose depending upon the specific properties of the problem under consideration and the amount and type of data available.

The following notation is used in the construction of the model:

$i$	—	index of power generation units ( $i = 1, \dots, I$ )
$k$	—	index of scenarios ( $k = 1, \dots, K$ )
$t$	—	index of time periods ( $t = 1, \dots, T$ )
$m$	—	index of histories up to and including time $t$ , i.e. path through scenario tree which has been followed up until this point
$j$	—	index of contracts ( $j = 1, \dots, J$ )
$\pi(k)$	—	probability of scenario $k$ occurring
$K_m^t$	—	set of all scenarios having uncertain outcomes in common up to and including period $t$
$x_i^t(k)$	—	amount of power generated by unit $i$ in period $t$ under scenario $k$
$c_i^t(x)$	—	cost of generating $x$ units of power from unit $i$ in period $t$
$g_i^t(k)$	—	amount of resource available to unit $i$ at the beginning of period $t$ under scenario $k$
$q_i^t(x)$	—	amount of resource needed to generate $x$ units of power by unit $i$ in period $t$
$d_i^t(k)$	—	amount of resource obtained by unit $i$ in period $t$ under scenario $k$
$y_j^{t,r-t}(k)$	—	amount of energy required to satisfy contract $j$ bought in time $t$ which reaches maturity in $r - t$ periods under scenario $k$
$s_j^{t,r-t}(k)$	—	“strike” price per unit for contract $j$ bought in time $t$ which reaches maturity in $r - t$ periods under scenario $k$
$V_i^T(k, g_i^t(k))$	—	value given to resource (i.e. water or fuel) kept for use in the next and subsequent time periods

The decision variables for this problem are the amount of power produced in each period, denoted by  $x_i^t(k)$ . The functions  $c_i^t(x_i^t(k))$ , the cost of producing the power and  $q_i^t(x_i^t(k))$ , the amount of resource needed to generate the power both depend upon the amount of power being produced. It is assumed that the amount of resource obtained in each period,  $d_i^t(k)$ , is stochastic. The remainder of the variables in the model are deterministic, the values of which are known within the time frame.

A value is attributed to the fuel retained for the next and subsequent periods. This will depend on values from the previous period. For example, in the case of a problem which features hydro stations, if there was little rain in the previous period then it is probably reasonable to expect low rainfall in this period, and thus it may be beneficial to limit the electricity produced in the previous period in order to be able to satisfy the demand in this period.

The linear programming problem with stochastic variables is formulated as follows:

$$\max \sum_{k=1}^K \pi(k) \left( \sum_{t=1}^T \left( \sum_{r=t}^T \sum_{j=1}^J s_j^{t,r-t}(k) y_j^{t,r-t}(k) - \sum_{i=1}^I c_i^t(x_i^t(k)) \right) + \sum_{i=1}^I V_i^T(k, g_i^{T+1}(k)) \right)$$

subject to:

$$\sum_{i=1}^I x_i^t(k) \geq \sum_{r=1}^t \sum_{j=1}^J y_j^{r,t-r}(k) \quad \forall t = 1, \dots, T, \quad \forall k = 1, \dots, K$$

$$g_i^{t+1}(k) = g_i^t(k) - q_i^t(x_i^t(k)) + d_i^t(k) \geq 0 \quad \forall i = 1, \dots, I, \quad \forall t = 1, \dots, T, \quad \forall k = 1, \dots, K$$

$$x_i^t(k) = x_i^t(k') \quad \forall k' \in \{K_m^t\}$$

The first constraint indicates that the amount of power produced must always satisfy the contractual demand. The second constraint shows that the resource

level at the end of a period must equal the resource level at the beginning of the period, minus the amount of resource used to generate power in the period, plus the amount of resource obtained during the period. The greater than or equal to zero part of the constraint indicates that the amount of power produced cannot exceed the amount of resource available for the generation. The last constraint is for non-anticipativity, which means that any decision made cannot depend on what will happen in the future.

Non-anticipativity can be explained with the aid of the diagram in Figure 5.1. The decision made at node (a) must be the same no matter which of the nodes (b), (c) or (d) is applicable for the next time period. It must be assumed that the value of the stochastic variable is unknown further into the scenario tree. Therefore, the decision taken must depend only upon what has happened in the past.

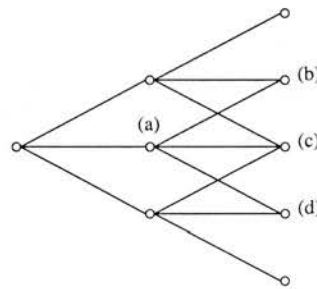


Figure 5.1: Non-anticipativity

The above problem has not been implemented as it is reasonably complex and more benefits can be obtained from investigating the individual contracts and assets present within the portfolio. The model represents a general approach for depicting the situation, but in real life the problem will almost always be extremely large due to the number of contracts and assets held by energy companies. This will prevent this model being of much use in an actual industry setting.



# Chapter 6

## Gas Storage

### 6.1 Introduction

The benefits of using natural gas are rapidly being discovered by an increasing number and variety of users, such as power producers or petrochemical companies. The major uses of gas are as a source of power and heating and as a fuel for power generation. Future demand for gas is expected to grow rapidly, leading to depletion of natural gas reserves. Although new reserves will probably be discovered, these will typically be smaller and more costly to exploit than those currently being used. Gas reserves must be operated efficiently in order to preserve the remaining supplies.

One of the major benefits of gas is that it can be stored. In this chapter, the main reasons as to why gas is stored are given, along with a brief description of the mechanics of storage. Various different types of storage facility exist throughout the UK and these are described next. When space is available in such a facility, a decision must be made about whether to inject or extract gas from storage on a particular day. Mathematical models can help when making these decisions by considering relevant variables and giving indications of the effect of any actions taken. Research in this area is relatively new. A review of some recent approaches to modelling the gas storage problem is given here which leads to the construction of the models in the next section.



The aim of the chapter is to develop a model to value a gas storage facility. An amount of gas can either be injected into or extracted from the facility on each day, or alternatively, the facility can remain as it is with no gas being injected or extracted. The gas price on each day is stochastic and these stochastic variables must be incorporated into the model. The model is developed in a number of stages, beginning with a description of an intuitive approach. This type of approach is used initially as it is easy to understand and easy to implement. This leads to the construction of a linear programming approach for the same problem. The linear program offers more benefits than the intuitive model in that it can be easily extended to incorporate additional features.

The stochastic prices can be incorporated into the formulation by using the method described in Section 4.3.1. The resulting spot price tree is checked for accuracy by using it to price a European call option and comparing the outcome with that given by Black's formulae. A stochastic dynamic programming model was developed to describe the gas storage problem, which incorporates this tree structure.

## 6.2 Storage in the UK

Natural gas usage has increased as it is cheaper to build gas-fired power stations than coal-fired ones. Natural gas also has the useful property that it can be stored. Regulatory controls have been removed which make gas politically and economically more attractive. The recent restructuring of the industry has led to a greater need for companies to try to maximise their profits as they now have no assured customer base. Money can be made by buying gas when the market price is deemed to be low, placing it in storage and then removing it to sell when the price is deemed high. Storage can also be used for risk management as it is a way of knowing for certain that gas is available on a certain day. The Network Code [63], has provided much of the information for this section.

There are two long-term uses of storage:

- **To smooth out seasonal variations in gas demand.**

A large proportion of gas usage in Britain is affected by the weather. Production is levelled out by storing gas when there is low demand, for example, in summer months, for use at times of high demand.

- **To ensure security of supply.**

If the gas pressure in the pipeline drops below a certain level, the network could become unsafe. It is very difficult, not to mention expensive, to shut off and reconnect supplies, so this is only an option in exceptional circumstances. Stored gas can be used to cope with problems such as large forecasting errors, breakdowns or problems with the pipeline which can lead to a decrease in pressure.

Gas storage has restrictions on both capacity and the rate at which gas can be withdrawn (deliverability). The capacity is the total quantity of gas which the facility can store at any one time. Deliverability is the maximum amount that can be extracted from the facility on any one day. A company can book an amount of deliverability in a facility to have a firm entitlement to withdraw its gas up to a stated limit on any day it chooses. Booking capacity allows gas to be injected into the facility up to a set limit. It is possible to book only capacity, or both capacity and deliverability. If only capacity is booked, it will cost less to withdraw gas but this can only be done on days when unused withdrawal capacity is available. Both capacity and deliverability can be obtained by a direct sale with the owner of the facility or, alternatively, by trading with other companies.

The following facilities for gas storage are operated in the UK:

- **LNG (Liquefied Natural Gas)**

At five sites around the UK, gas is cooled until it becomes liquid and then stored in insulated metal tanks. The LNG facilities have high deliverability but the space available is relatively small. They are mainly used to ensure that peak demand is met in all areas of the network.

- **Rough**

A partially depleted gas field has been converted into a storage facility. Natural gas is stored deep underground at high pressure. It is reasonably slow to inject and extract from this type of facility, therefore its main use is to provide a source of seasonal storage.

- **Salt Cavity**

Nine large cavities have been created far below ground by dissolving layers of salt. Their capacity is much less than that found in a Rough facility, but more than an individual LNG facility. It takes longer to extract gas from these cavities than from an LNG site. They are used primarily to cope with peak demand but can also be used as a trading mechanism.

Storage has the advantage for market participants that, once booked, its availability and cost is certain. Buying any additional gas which is required may be subject to market forces on the day. Therefore, for many power supply companies, booking storage capacity and filling it when gas is plentiful (and cheaper) is an effective way to manage their gas requirements. This is an application of risk management.

## 6.3 Review of Literature

Optimising the value of gas kept in storage is a relatively new area of research. The increase in interest has resulted from the need for companies to maximise their profits. This is due to the effects of privatisation, which have opened the market to increased competition. Literature in this area is fairly sparse. The following papers describe recent approaches for modelling specific gas storage problems. The first approach is similar to the problem under investigation,

whereas the others provide examples of different methods for modelling storage problems and issues.

In Scott et al. [57], a model is developed to investigate various aspects of gas storage. The model aims to maximise the total revenue over the time horizon and can be solved by using stochastic dual dynamic programming. The formulation is extended to encompass stochastic price variables by considering various factors such as mean price, volatility and speed of mean reversion. These parameters can be used to produce a marginal value surface, from which the optimal decision for any allowable storage level at any time period can be determined. The marginal value of a specified amount of gas in storage on any day can also be determined, together with the optimal amount to store on a particular day, given the market price.

There are differences between the model described and the one developed in the later parts of the chapter. The model described here uses a stochastic dual dynamic programming approach which has a different format than the stochastic dynamic programming approach used for the model developed. The stochastic spot price variables in each case were determined and incorporated in different ways. Also, the model constructed towards the end of the chapter gives a more detailed description of the problem under consideration. The formulation given in the paper makes a number of assumptions and approximations.

Sinayuc et al. [59], developed a linear programming approach for extracting gas from an underground storage reservoir with ten wells. This gas must then be used to supply a variable demand. The model aims to maximise the total amount withdrawn, subject to pressure constraints. These constraints are important as the pressure of a gas storage field must not drop below a certain level, otherwise it will become unsafe. When solved, the model will give the optimal amount of gas to extract in a certain time period. The model developed towards the end of this chapter is distinct in that it deals with a different problem.

Butler and Dyer [15], developed a method to quantify the value of gas held in a storage facility, using a linear programming model. The paper investigates

the benefits of holding various contracts and the optimal operation of the storage facility. The model aims to minimise the total cost of gas used by the system. The solution of the model describes what is thought to be the optimal portfolio of gas contracts that should be held. The optimal portfolio is shown to be highly sensitive to both demand and price. The model incorporates three scenarios, representing low, normal and high demand years. The low and high demand scenarios were selected to represent the tails of the expected scenario distribution in order that the model be applicable in all situations which may occur. This model was developed to value a storage facility by valuing various contracts which are held. This is, again, a different problem from the one under consideration in the latter parts of this chapter.

## 6.4 Deterministic Models

Gas storage models are important since storage capacity must be booked in advance. Operators need an indication of how much gas to inject or extract from storage on a particular day in order to decide how much storage to book. The models which are developed in this section determine the best way to use the facility over a specified time period, subject to the given forward curve. These models give indications about the expected injection and extraction values for each day in the planning horizon, together with the amount of gas which should be left in storage for subsequent periods. The deterministic models use a given market forward curve for the gas prices over all time periods. These models were shown to be scenario dependent. This prompted the development of stochastic models, in which the gas prices were included by means of a scenario tree, which was constructed to be consistent with the given market forward prices.

Throughout the chapter,  $t$  will denote the index of time periods ( $t = 1, \dots, T$ ) and  $r$  will be the risk free rate of interest.

### 6.4.1 Intuitive Model

The gas storage problem can be modelled in an intuitive way. The model indicates injection and extraction timing, along with the amounts of gas involved.

Moreover, the remaining storage usage can also be predicted. A value is placed on holding and operating a certain amount of storage.

The major benefits from considering an approach such as this are that it is easily understood and intuitively appealing. The ease of understanding makes it more accessible to people with no prior knowledge of the area or any mathematical techniques which could be used. The intuitive model works by extracting from storage on days when the cost of gas is relatively high and injecting into storage on days when cost is relatively low. The simplicity of this approach leads to a model which can be treated as a benchmark by which to compare other approaches. By demonstrating that alternative models give significantly higher results and improve on operating policies, the likelihood of these approaches being adopted is increased.

The intuitive model works in the following way:

1. A forward curve of gas prices for every day in the time period is obtained, which is believed to be a reasonable approximation to the behaviour of the market.
2. The average price over the time period is calculated. The days are considered in order beginning with the first.
3. If the price on that day is greater than the average price, then the maximum amount of gas possible is injected into the storage facility, ensuring that daily or capacity constraints are not violated.
4. If the price on that day is less than the average price, provided that there is gas present in storage, the maximum amount of gas possible is extracted, ensuring the level does not fall below zero.
5. If the price on that day is equal to the average price, then do nothing.
6. Calculate the value of the action on each day by multiplying the price on that day by the amount which has been injected or extracted.
7. Calculate the value of the storage facility over the specified time period by summing the values for each day.

### 6.4.2 Linear Programming Approach

The gas storage problem was extended to a linear programming approach, which aims to maximise the profit obtainable, subject to constraints on the level of gas in the storage facility and the maximum injection and extraction rates. The model again determines a policy for gas storage that specifies the days on which to inject into or extract from storage and the corresponding amounts.

The model is formulated as follows:

$$\max \sum_{t=1}^T (p_t x_t - s_t y_t - c_t x_t - d_t y_t) e^{-rt}$$

subject to:

$$g_{t+1} = g_t - x_t + y_t \quad \forall t = 1, \dots, T-1$$

$$x_t \leq E \quad \forall t = 1, \dots, T$$

$$y_t \leq I \quad \forall t = 1, \dots, T$$

$$0 \leq g_t \leq S \quad \forall t = 1, \dots, T$$

$$x_t, y_t \geq 0 \quad \forall t = 1, \dots, T$$

The following notation is used:

- $t$  — index of time periods ( $t = 1, \dots, T$ )
- $x_t$  — amount of gas sold in period  $t$
- $y_t$  — amount of gas bought in period  $t$
- $g_t$  — amount of gas in storage at the beginning of period  $t$
- $p_t$  — revenue from selling a unit of gas in period  $t$
- $s_t$  — cost of buying a unit of gas in period  $t$
- $c_t$  — cost of extracting one unit of gas from storage in period  $t$
- $d_t$  — cost of injecting one unit of gas into storage in period  $t$



In the above formulation,  $I$  represents the maximum amount of gas which can be injected into the storage facility in any time period.  $E$  represents the maximum amount of gas which can be extracted from the storage facility in any time period. These constants are set by the owner of the storage facility and do not depend on the amount of gas in storage at any time period.  $S$  denotes the capacity of the amount of storage available in the facility which has been obtained by purchasing directly from the owner or by trading.

The decision variables in the above problem are the amount of gas sold in period  $t$ ,  $x_t$ , and the amount of gas bought in period  $t$ ,  $y_t$ . The above model is deterministic in that the uncertain prices for buying and selling the gas,  $s_t$  and  $p_t$ , have been approximated by values obtained from the gas forward curve.  $e^{-rt}$  represents a discount factor. The value at each time period should be discounted back to the beginning of the time horizon in order to give an accurate valuation of the storage facility.

### 6.4.3 Case Study

The following constants were obtained from the BG Storage website, [13], for the Hornsea salt cavity storage facility:



$I = 737,020$  therms/day (maximum injection rate)

$E = 6,653,650$  therms/day (maximum extraction rate)

$S = 119,424,500$  therms (storage capacity)

$c_t = 0.2$  p/therm (cost of extracting one unit of gas)

$d_t = 0.7$  p/therm (cost of injecting one unit of gas)

These values can be substituted into both the intuitive model and the linear programming model to value this storage facility over a time period of one year with daily resolution. This problem is computationally large, incorporating 730 decision variables and over 1500 constraints.

The models were constructed using Visual Basic for Applications. This package was chosen as it has a practical graphical user interface in the form of Excel, which is easy to use for anyone wishing to run the models. Another advantage is that Visual Basic itself is relatively simple to learn and the models developed can readily be employed in industry settings. The main disadvantage when using Visual Basic for Applications is the limitations on the size of the problem which can be solved. Faster packages exist but speed was seen to be less of a priority in this research.

Both the intuitive and linear programming models were solved for a variety of different forward curves. Decisions about whether to inject, extract or do nothing are made daily. The results from one such curve are:

Intuitive Approach = 2,455,683.

Linear Programming Approach = 19,718,050.

These results are shown graphically in Figures 6.1 and 6.2.

The linear program was seen to give a higher result. This is because the linear programming approach allows the storage facility to be emptied and refilled within the time horizon, whereas the intuitive approach does not allow this to happen. The linear programming approach also has the advantage that it can be easily adapted to incorporate other features or characteristics, such as constraints on the usage of the storage facility.

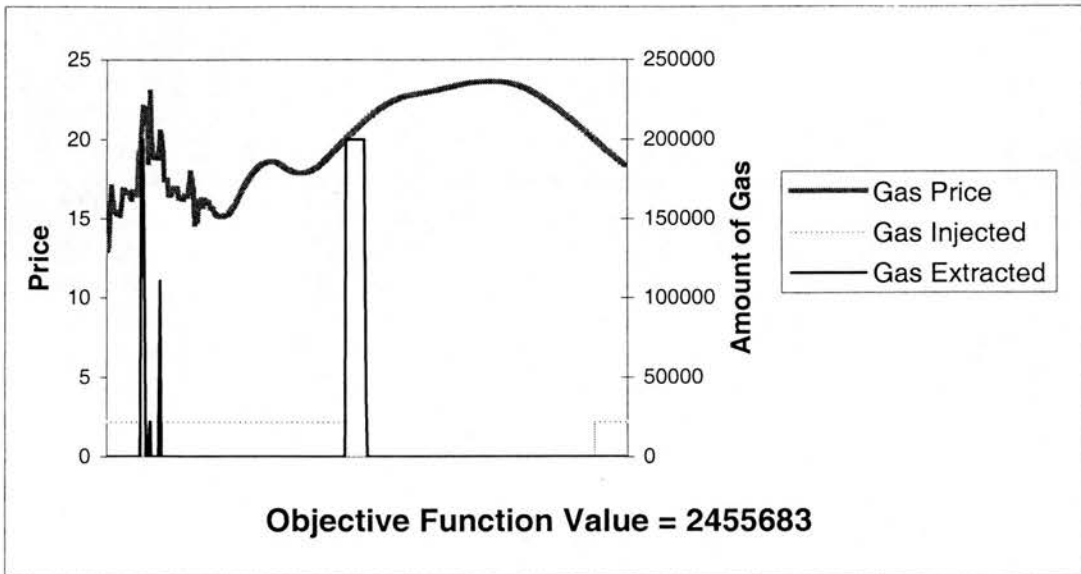


Figure 6.1: Results from Intuitive Approach

The sensitivity of the linear programming model to different scenarios was tested by using different forward curves which could occur. One of these forward curves is in contango and the other is in backwardation. The curves used are shown in Figure 6.3. These curves represent two extreme scenarios which could occur. The same starting value is used in each case. The results from this comparison are:

$$\text{Contango} = 16,987,358$$

$$\text{Backwardation} = 7,389,338.$$

The results are shown graphically in Figures 6.4 and 6.5.

The large difference in values obtained when solving each case implies that the linear program is sensitive to different scenarios. This motivates the development of a stochastic model which will encompass a variety of possible situations. This stochastic approach uses a scenario tree which encompasses a range of prices at each time period. This allows the unknown prices to be modelled more accurately

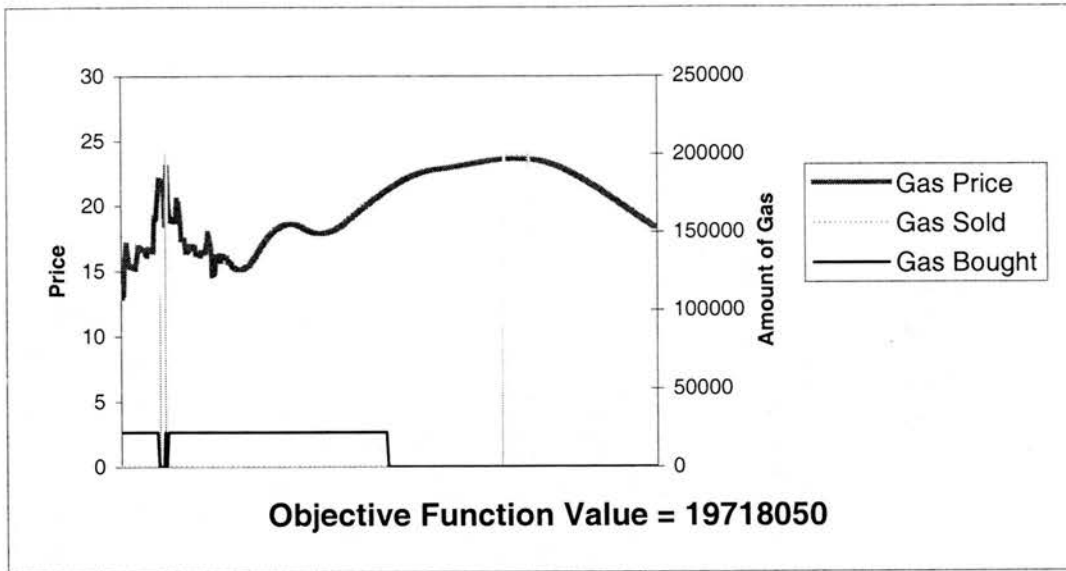


Figure 6.2: Results from Linear Programming Approach

as there is now more than one possibility for the value taken at any time period. This will help to minimise the large discrepancies in value between different scenarios.

## 6.5 Stochastic Model

A stochastic dynamic program can be constructed to solve the same problem, in which the gas price on each day is uncertain. These stochastic prices can be incorporated into the problem by means of a trinomial spot price tree.

### 6.5.1 Spot Price Tree Construction

The spot price tree is constructed using the method described by Hull and White [34], and can be fitted to a given market forward curve using the approach in Clewlow and Strickland [19]. The combined method produces a spot price tree which is consistent with observed market features; a lognormal forward price and

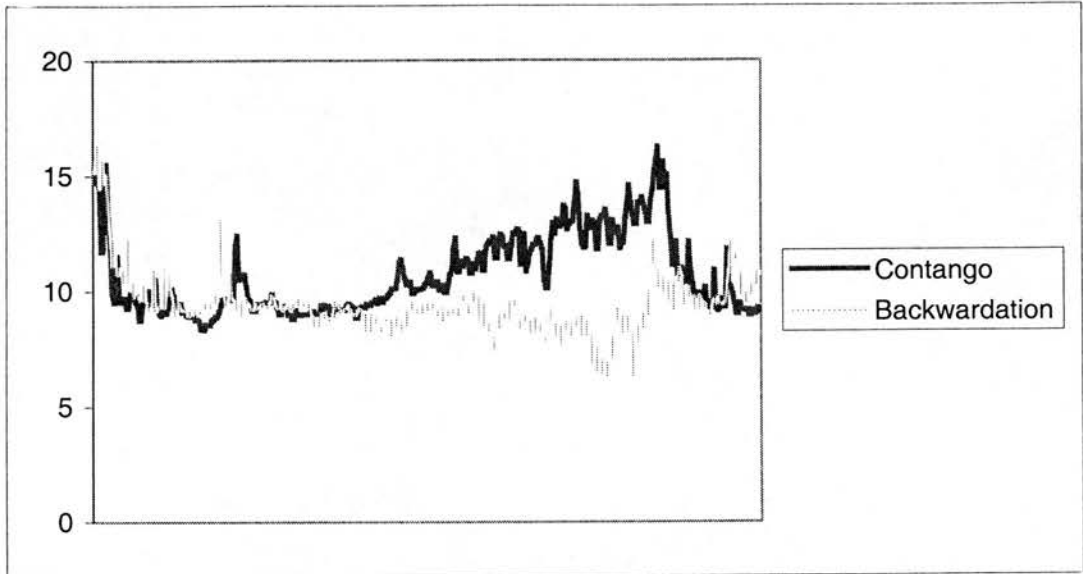


Figure 6.3: Contango and Backwardation Curves

a volatility which decays over time. An outline of the procedure used is given in Section 4.3.1.

The tree must be constructed in order to value a storage facility for the same problem as that described in Section 6.4. This is necessary in order that comparisons can be made between the model being developed and those described previously. The tree was constructed for a time period of one year with daily resolution time steps. Thus:

$$\Delta t = \frac{1}{365}.$$

From any node, the price can move up by a certain amount, stay the same or move down by a certain amount. Assuming  $\sigma = 100\%$  and  $\alpha = 3$ , the following

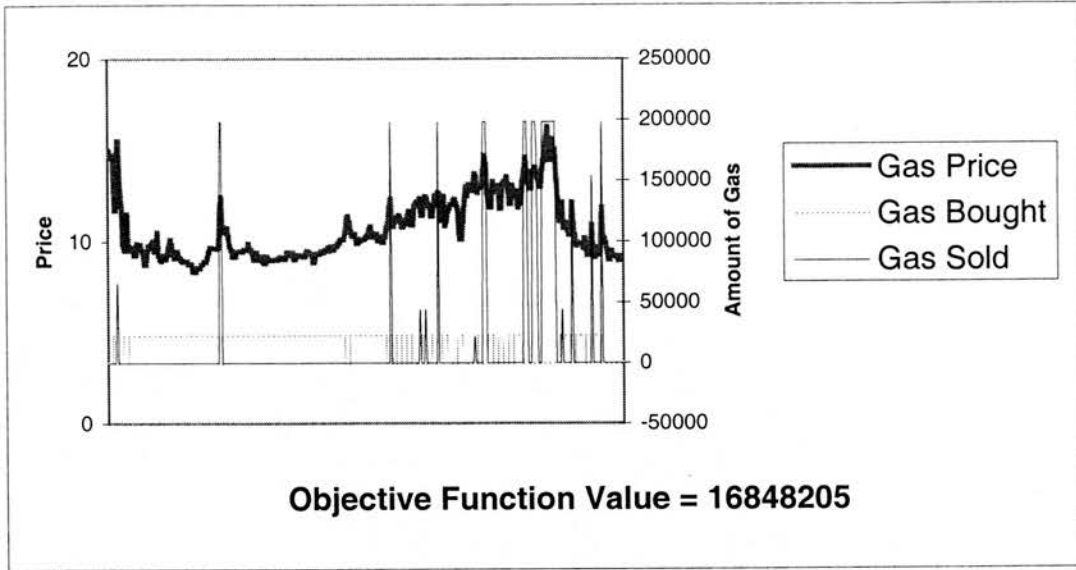


Figure 6.4: Results from Curve in Contango

formula can be used:

$$\Delta x = \sqrt{\frac{3\sigma^2(1 - e^{-2\alpha\Delta t})}{2\alpha}} = 0.09.$$

The maximum number of price levels,  $j_{\max}$ , can also be obtained from the previously given formula:

$$j_{\max} = \left\lceil \frac{-0.184}{e^{-\alpha\Delta t} - 1} \right\rceil + 1 = 23.$$

### Tree Validation Example

Tree approaches are often used to value options. The prices obtained from the tree building process can be used to value an option. This value can be compared with the answer given by the Black formulae for pricing the same option. The Black formulae are used to price options on futures, and also forwards in energy markets. The Black value for a call option can be worked out using the formula

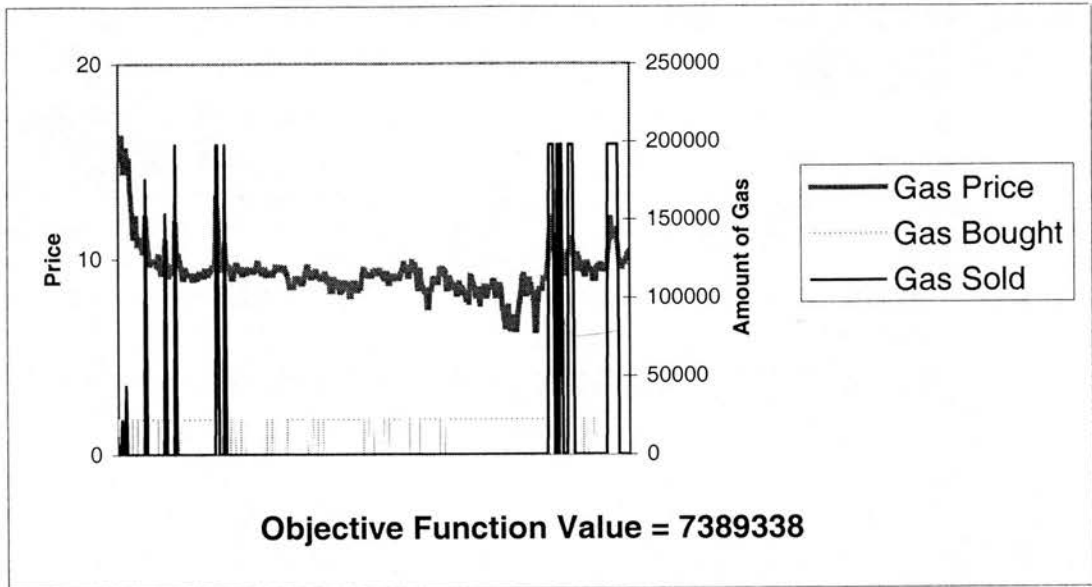


Figure 6.5: Results from Curve in Backwardation

described in Section 3.5.2:

$$c = e^{-r(T-t)} [FN(d_1) - KN(d_2)],$$

where:

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(F/K) - (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

The tree constructed in the previous section can be used to value a call option which expires in the final period. The probabilities on each branch can be used to calculate the expected value of the option at the valuation date. In order to achieve this, the value must be discounted back to the first period. At the end of each branch of the final period of the tree, the spot price for each node is compared with the strike price of the option. If the spot price is above the strike price, then the value at that node is equal to the spot price minus the strike price.

Otherwise, the value is equal to zero. The values of nodes at previous periods can be calculated by determining the discounted expected value of the option at each of those nodes. This process can be repeated by working backwards until the first node is reached. The value at this node can be taken to be the value of the option being priced.

The following values for the constants were assumed:

$$r = 0.06 \text{ (risk free rate of interest)}$$

$$T = 365/365 \text{ (day on which the option expires)}$$

$$t = 1/365 \text{ (day on which option is valued)}$$

$$F = 23.02 \text{ (forward price for day on which option expires)}$$

$$K = 24 \text{ (strike price of option).}$$

These values can be used to price a call option by applying the techniques described. The results of such a valuation are:

$$\text{Tree Value} = 3.142503$$

$$\text{Black Value} = 3.134162$$

$$\text{Percentage Difference} = 0.2\%.$$

As an additional check, the tree can be considered as a strip of call options. This strip contains call options which expire at every time period. The tree can again be used to value call options expiring in every time period using the method described previously. These values can be added together to give the value of the strip. Similarly, Black's formula can be used to value each of these options and the results summed to give the value of this strip of options. The results of valuing this strip are:

$$\text{Sum of Tree Values} = 1488.092056$$

$$\text{Sum of Black Values} = 1467.63325$$

$$\text{Percentage Difference} = 1.375\%.$$

As is expected, the difference between the value obtained using the spot price tree and the value using the Black model is relatively small. This proves that the values calculated for the spot price tree are accurate enough to be used in stochastic models.

### 6.5.2 Stochastic Dynamic Programming Approach

The spot price tree can be incorporated into a stochastic dynamic programming model to represent a variety of possible scenarios. The stochastic dynamic programming model can be constructed by either working forwards or by working backwards. The gas storage problem lends itself to a backward recursion approach. In finite horizon problems, initial values must be specified in order to begin the recursions. In a backward recursion approach, values for the  $T + 1$ th period are specified. These can be chosen to be zero as all of the gas should normally be extracted by the end of the planning horizon. The initial values can easily be altered to match the requirements of the user in other cases.

The stages in the stochastic dynamic program are denoted by the time period,  $t$ . The states are the price level,  $j$ , and the amount of gas in storage,  $g$ . The maximum expected value of the swing contract in time periods  $t, \dots, T$ , given the price level is  $j$  and the amount of gas in storage is  $g$  is denoted by  $f_t(j, g)$ . The aim is to maximise the expected value of the storage facility by finding the value of:

$$f_1(1, 0),$$

which gives the value of the storage facility in time period 1 and at price level 1 when there is no gas in storage.

It is realistic to assume that there is no gas in storage at the beginning of the time horizon, as companies need to book capacity for a certain length of time. If a company has no capacity previously, then they cannot have any gas in storage at the beginning of the time horizon. Companies cannot be certain that they will have any storage in the next period and so they usually aim to have no gas remaining at the end of the time horizon.



The recursions are defined to be:

$$f_t(j, g) = \max \begin{cases} (s_t(j) + d_t)x + e^{-r\Delta t} \sum_k p_t(j, k) f_{t+1}(k, g + x) \\ (s_t(j) - c_t)y + e^{-r\Delta t} \sum_k p_t(j, k) f_{t+1}(k, g - y) \\ e^{-r\Delta t} \sum_k p_t(j, k) f_{t+1}(k, g) \end{cases} ,$$

$$f_{T+1}(j, g) = 0, \forall j \text{ and } g.$$

The following notation is used in the formulation of this problem:

- $j, k$  — indices of price levels
- $x$  — amount of gas which can be injected
- $y$  — amount of gas which can be extracted
- $d_t$  — cost of injecting one unit of gas in time  $t$
- $c_t$  — cost of extracting one unit of gas in time  $t$
- $p_t(j, k)$  — transition probability of going from price level  $j$  at time  $t$  to price level  $k$  at time  $t + 1$
- $s_t(j)$  — spot price of gas at price level  $j$  and time  $t$

The following constraints must also be satisfied:

$$\begin{aligned} 0 &\leq g \leq S \\ 0 &\leq x \leq E \\ 0 &\leq y \leq I, \end{aligned}$$

where  $S$  is the capacity of the storage facility,  $I$  is the maximum amount which can be injected in any time period and  $E$  is the maximum amount which can be extracted in any time period.

In order to implement the stochastic dynamic program, it is necessary to discretise the possible storage levels and the amount of gas which can be injected or extracted at each time period. This is necessary in order to restrict the amount of parameters in the model, allowing a solution to be obtained reasonably quickly. Large numbers of parameters lead to models which can be impossible to solve or which have very slow solution times. The disadvantage of reducing the number of parameters in this way is that the model can become less accurate. Intuitively,

when it is optimal to inject into the facility, it is usually optimal to inject as much as possible:

$$\max(I, S - g).$$

Similarly, when it is optimal to extract from the facility, then it is optimal to extract as much as possible:

$$\max(E, g).$$

This provides a valid method of reducing the number of variables.

### 6.5.3 Case Study

The constants obtained from the BG Storage website, [13], for the Hornsea salt cavity storage facility which were used in Section 6.4.3 are again employed in the stochastic dynamic programming model. As in the deterministic case, the problem is formulated over a time period of one year with daily resolution. There are 730 decision variables and over 35,000 constraints to be considered. The stochastic dynamic program is again constructed using Visual Basic for Applications.

The formulation of this model can be checked by setting the volatility equal to zero. This creates a dynamic program in which the price does not vary at each time period. The value obtained from solving this problem can be compared with that obtained from the linear program. The results of this comparison are:

$$\begin{aligned}\text{Linear Programming Approach} &= 19,718,050 \\ \text{Dynamic Programming Approach} &= 20,021,745 \\ \text{Percentage Difference} &= 1.517\%.\end{aligned}$$

There is little difference between these values, therefore it can be assumed that the stochastic dynamic program has been formulated correctly.

The stochastic dynamic programming model can now be solved by setting the volatility equal to 100%. This allows the entire spot price tree to be used when solving the model, providing more than one possible price at each time period. The values obtained from solving all models to value the same gas storage facility

are:

Intuitive Approach = 19,718,050

Linear Programming Approach = 19,718,050

Stochastic Dynamic Programming Approach = 26,785,234.

This comparison is shown graphically in Figure 6.6.

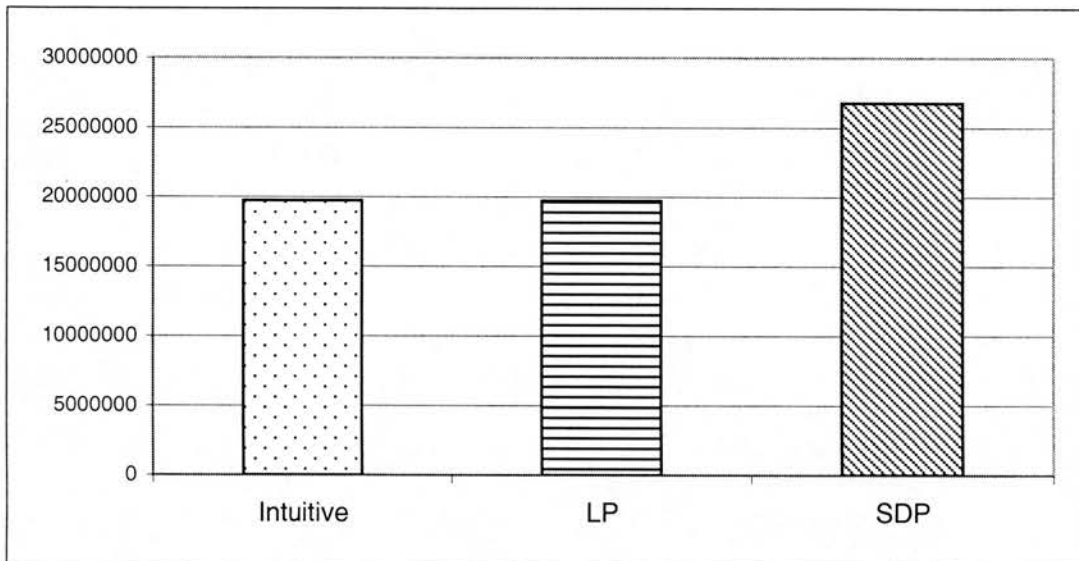


Figure 6.6: Comparison of Approaches

The stochastic dynamic programming approach is seen to give a higher value for the valuation of an identical gas storage facility. This is assumed to be the case since it takes into account a larger number of prices at each time period.

The gas storage models presented in this chapter give indications of the value of a storage facility and also of the operating decisions to be made at each time period. These models are an improvement on those used previously and are appealing as they are based on strong mathematical theory. The techniques

developed in this section can now be applied to take or pay contracts in the next chapter.



# Chapter 7

## Take or Pay Contracts

### 7.1 Introduction

Take or pay contracts allow the holder to vary the amount of commodity they buy within specified limits. As the name suggests, the holder of such a contract must take a minimum amount over the time period or pay penalty charges. The penalty charges are normally so high that this is only an option in exceptional circumstances. This type of contract is often known as a swing contract as the holder can “swing”, by altering the amount taken on any day in the time period.

Take or pay contracts were first developed to model the constraints present in gas storage facilities. When gas is released from a storage facility, it cannot be turned on and off, only varied within certain limits. Take or pay contracts take account of the minimum and maximum amounts which can be taken. They have been found in natural gas markets for a number of years and are becoming increasingly more common in other commodity markets as participants look for opportunities to exploit situations in order to make more money.

The following section gives a description of take or pay contracts and their applications to gas markets, although they are now being introduced into other types of commodity market. The aim of the chapter is to develop a model to value a take or pay contract. Such a contract can be used to vary the amount of gas taken on each day in the time horizon between both daily and annual limits.

In a similar way to that described in Chapter 6, the model is developed in a number of stages, beginning with various deterministic models. An intuitive approach is described. This type of approach is used initially as it is easy to understand and easy to implement. This leads to a linear programming formulation of the same problem. The linear program offers more benefits than the intuitive model in that it can be easily extended to include additional features. The optionality present in such contracts is the difference between the value of a forward contract and the value of contracts which consist of or include options. This optionality allows a variable amount of commodity to be taken on each day and can be taken into account by adapting the linear programming model. This results in a semi-analytic model. Such models have been developed to combine numerical methods with formal proof. This model aims to place a more accurate value on this type of contract by taking such optionality into account.

The energy price on each day is stochastic and can be incorporated into the model using the method described in Section 4.3.1. The spot price tree can be incorporated into a stochastic dynamic programming formulation. The resulting model aims to represent the uncertain values present, thus providing a more accurate valuation for the take or pay contract.

## 7.2 Properties of Take or Pay Contracts

Take or pay contracts are options; the holder can choose, but is not contractually obliged, to swing (change the requested amount) on a particular day. Exercising a swing on a particular day may require that more or less gas is taken on days further into the contract. This type of option is American in character, i.e. the exercise dates are not predetermined and can be selected at will by the exercising party. A take or pay contract can be thought of as a series of American options whose exercise is mutually dependent; once this type of option has been bought, the optimal exercise strategy must be decided upon in response to the changing market conditions. The price to be paid per unit of the commodity is fixed or can be worked out using a series of indices. This value is called the **contract price**.

Take or pay contracts present in gas markets have a daily amount attached to them, called the daily contract quantity (DCQ). This is the amount which is taken if the holder decides not to exercise a swing on a particular day. The amount taken on any day must lie between two limits - the minimum daily quantity (LDQ) and the maximum daily quantity (MDQ). Both the minimum and maximum daily quantities are defined in the contract as a percentage of the daily contract quantity. Similarly, each contract has an annual amount attached to it, called the annual contract quantity (ACQ). The amount taken over the course of one year must also lie between two limits - the minimum annual quantity, also called take or pay (TOP) and the maximum annual quantity (MAQ). Both the minimum and maximum annual quantities are defined as a percentage of the annual contract quantity.

If the holder of this type of contract wishes to purchase an amount which is less than or greater than the daily contract quantity then sufficient notice must be given to the seller of the contract. Each contract has a different notice period associated with it, but generally, 24 hours notice is required for a change greater than 50% of the contracted amount, otherwise 12 hours notice is required. This produces a need for models which can accurately value such contracts in advance.

Given the values of LDQ, MDQ, TOP and MAQ for a take or pay contract, the value of the contract together with the corresponding decisions that should be taken at each time period can be calculated. In order to exercise the contract in the most beneficial way, the holder needs to examine the contract price and the market price; if the market price is below the contract price, it is preferable to buy gas from the market, although the minimum daily quantity must be taken on each day in order to satisfy the terms of the contract and avoid the high penalty costs. When the market price is higher than the contract price, it is preferable to buy gas from the contract and therefore take the largest amount possible from the contract, the maximum daily quantity. The amount of gas required on any day must be greater than the maximum daily quantity in order for this to hold. Otherwise, an amount of gas equal to the demand should be taken.



### 7.3 Review of Literature

Methods for determining the value of take or pay contracts, together with the corresponding decisions to be made at each stage is a relatively recent area of research. The privatisation of the energy industries has led to a need for market participants to maximise the profits which can be obtained from holding a variety of contracts and assets. A selection of relevant literature is detailed, each of which describes a method for valuing this type of contract.

Pilopović and Wengler [48], describe specifications and features of take or pay contracts. This type of contract can be thought of as a set of co-dependent American options and therefore can be valued using a tree structure. If there are no minimum and maximum bounds associated with the contract, then a closed form solution can be obtained for the daily call option price. A tree with four time periods is used to value a contract which has two opportunities to exercise a swing. The method works by using the tree structure to value both options by keeping track of the two values at every node in the tree. The tree is built to represent the predicted forward price and the options can be valued by traversing the tree backwards. The pricing method does not include discounting.

Jaillet et al. [39] and [38], also give a brief description of types of take or pay contract which are available and describe the valuation of such options and their corresponding exercise boundaries. A stochastic process is specified to model the movement of the spot price. Geometric Brownian motion is used, although it is noted that this model is unrealistic in the case of energy prices but can nevertheless give a useful insight into the behaviour of the contract. An example is given, in which a three-step binomial forest is used to find the value of a take or pay contract. A **binomial forest** is a binomial tree which has been extended to three dimensions. Binomial trees are described in Section 4.3. The papers also investigate the effect on the value of a take or pay contract when penalty and volatility levels are varied.

Kaminski et al. [41], develop a procedure for valuing swing options by extending a binomial tree approach. Again, it is assumed that energy prices follow geometric

Brownian motion and binomial trees are used to approximate the spot prices. This is similar to the approach described by Jaillet et al. [39]. The approach is extended to price swing ratchet options. This type of option requires that the daily amount taken should lie between some minimum and maximum values and cannot change by more than a specified number of units from one day to the next.

The models described in the literature differ from those developed in the remainder of the chapter in a number of ways. The most important of these differences is the incorporation of a market forward curve in order to make the pricing methods consistent with others found in energy markets. The models developed also include discounting in order to determine the value of such a contract on the valuation date. This value will change as the expiration date approaches. Take or pay contracts are usually valued over a number of years and therefore a longer time length was required to value such contracts. It is easy to extend the models described in the remainder of the chapter to value longer time periods.

## 7.4 Deterministic Models

Models for take or pay contracts are important as the value of the contract must be determined in order that it is bought or sold at a fair price, as with any financial contract. The correct valuation of a take or pay contract depends upon:

- the determination of the risk neutral market value. It is assumed that the option is fully hedged, although this is usually impossible due to illiquid portions of the forward curve.
- the level of optionality exhibited. The inclusion of optionality into the model allows the amount of commodity bought at each time period to vary. Thus, a greater amount can be bought when the price is low and less when the price is high, provided the amount taken lies within the specified limits.
- the determination of the decision which should be taken at each stage in order to maximise the profit from the contract.

Models to value such contracts are constructed from the view of the holder or buyer of a take or pay contract, and give indications about the amount which should be taken on each day in the planning horizon. Deterministic models are those in which the values of all variables are known with certainty throughout the time period. A given market forward curve of gas prices over all time periods is used to determine these values.

In the following models, the price paid for the contract is assumed to be constant. This is a reasonable assumption since the volatility of the contract price is much less than the volatility of the forward price.

### 7.4.1 Intuitive Approach

An intuitive approach can be used to value such a contract by allocating the amount of gas to be taken according to the highest mark to market values. The intuitive approach ignores the embedded optionality and is equivalent to assuming that there is no volatility in the prices. The model gives indications of the expected value of the contract, together with the amount of gas which should be taken at each time period. This type of model is useful as it is easily understood and intuitively appealing. The ease of understanding makes it more accessible to people with no prior knowledge of mathematical programming methods. This approach works in the following way:

1. A forward curve of gas prices is obtained, which is believed to be a good approximation to the behaviour of the market.
2. For each day remaining in the time period the mark to market value of the contract is calculated.
3. The limits on the amount which can be taken on each day are specified. The volume of gas taken so far in the current year is also noted.
4. The minimum and maximum amounts of gas available for the remainder of the year are calculated using the minimum and maximum annual quantities.
5. The mark to market value of the contract for each day in the time period is calculated by subtracting the contract price from the commodity price on

that day and discounting to the valuation date. These values are placed in order starting with the maximum.

6. The minimum daily quantity is allocated to each day.
7. The difference between the maximum daily quantity and the minimum daily quantity is added to the days with the highest mark to market values until the minimum annual quantity is reached.
8. The difference between the maximum daily quantity and the minimum daily quantity is added to remaining days until the mark to market values become negative or the maximum annual quantity is reached.

### 7.4.2 Linear Programming Approach

The take or pay problem was formulated using a linear programming approach. This model aims to maximise the value of the contract by exercising the swing to take more commodity from the contract when the market price is high and less when it is low. The model again determines the amount of commodity to take at each time period in order to obtain the maximum possible value from the contract. The daily and annual constraints must be satisfied throughout the time period.

The model is formulated as follows:

$$\begin{aligned} & \max \sum_{t=1}^T m^t y^t \\ & \text{subject to:} \\ & \quad \text{LDQ} \leq y^t \leq \text{MDQ} \\ & \quad \quad \quad \forall t = 1, \dots, T \\ & \quad \text{TOP} \leq \sum_{t=1}^T y^t \leq \text{MAQ}, \end{aligned}$$

where  $m^t = (s^t - c^t)e^{-r(t-t_0)/n}$

The following notation is used:

- $t$  — index of time periods ( $t = 1, \dots, T$ )
- $t_0$  — valuation date
- $y^t$  — amount of gas bought from the contract in period  $t$
- $s^t$  — cost of buying a unit of gas in period  $t$
- $c^t$  — contract price of a unit of gas in period  $t$
- $m^t$  — mark to market value of the contract at time  $t$
- $r$  — risk free rate of interest
- $n$  — number of days in the year

The decision variable in the above problem is  $y^t$ , the amount of gas taken in period  $t$ . The above model is deterministic (the values of all variables are known with certainty throughout the time period) in that the uncertain prices for buying a unit of the gas,  $s^t$ , have been approximated by values obtained from the gas forward curve.

### 7.4.3 Semi-Analytic Linear Programming Approach

A semi-analytic linear programming approach was developed to extend the linear programming model to take account of some of the optionality present in take or pay contracts. The optionality is the difference between the value of a forward contract and the value of contracts which consist of or include options. The modelling approach uses the properties of the Black model which is described in Section 3.5.2. Therefore, it must be assumed that the forward price,  $F$ , follows geometric Brownian motion:

$$dF = \mu F dt + \sigma F dz,$$

where  $\mu$  is the expected growth rate in  $F$  and  $\sigma$  is its volatility.

Putney [51], gives details of some semi-analytic methods. Each model describes a different way of splitting the contract into forwards and options and methods are also described to place values on the resulting parts. These models determine how much of a commodity should be taken on each day for a given forward contract. Once this has been determined, there exists options on each day to buy or sell the commodity provided that the daily and annual constraints are still satisfied.

The models described each employ a different method for obtaining the initial amounts to be taken, for example determining the optimal take or assuming that the daily contract quantity is taken on each day.

This semi-analytic modelling approach was developed in order to maximise the value of a take or pay contract. This type of contract can be decomposed into a series of daily forward contracts and corresponding call and put options to take account of the optionality. The forward contracts require a certain amount of gas to be taken on each day, the daily contract quantity. The call options allow an amount of gas to be bought on the same day provided the amount taken is less than MDQ, and the put options allow an amount of gas to be sold, whilst ensuring that the amount of gas taken is greater than LDQ.

The mark to market value of a call option to purchase gas at the contract price on day  $t$  is denoted by  $M_t^C$ . Similarly, the mark to market value of a put option to sell gas at the contract price on day  $t$  is denoted by  $M_t^P$ . By defining the options in this way, their values can be calculated using the Black model. The semi-analytic model works by calculating the expected mark to market value of the contract on each day, together with the corresponding expected amounts of gas to be taken.

The model is formulated as:

$$\max \sum_{t=1}^T (V_t^F M_t^F + V_t^C M_t^C + V_t^P M_t^P)$$

subject to:

$$\sum_{t=1}^T (V_t^F + V_t^C) \leq \text{MAQ}$$

$$\sum_{t=1}^T (V_t^F - V_t^P) \geq \text{TOP}$$

$$V_t^C \leq \text{MDQ} - V_t^F \quad \forall t = 1, \dots, T$$

$$V_t^P \leq V_t^F - \text{LDQ} \quad \forall t = 1, \dots, T$$

$$V_t^F, V_t^C, V_t^P \geq 0 \quad \forall t = 1, \dots, T,$$

where  $M_t^F$ ,  $M_t^C$  and  $M_t^P$  are the mark to market values of one unit of the forward contract, the call option and the put option, respectively.  $V_t^F$  denotes the volume bought from the forward contract on day  $t$ ,  $V_t^C$  denotes the volume bought from the call option on day  $t$  and  $V_t^P$  is the volume sold from the put option on day  $t$ .

The first constraint ensures that the amount taken does not exceed the maximum annual quantity, whereas the second constraint ensures that the amount taken does not exceed the minimum annual quantity. The next constraint ensures that the amount taken from the call option and the forward contract does not exceed the maximum daily quantity. The following constraint ensures that the minimum daily quantity is always satisfied.

The mark to market values of the forward contract,  $M_t^F$ , and the call and put options,  $M_t^C$  and  $M_t^P$ , are calculated by using Black's formulae:

$$\begin{aligned} M_t^F &= (f_t - c_t) e^{-r(t-t_0)} \\ M_t^C &= [f_t N(d_1) - c_t N(d_2)] e^{-r(t-t_0)} \\ M_t^P &= [c_t N(-d_2) - f_t N(-d_1)] e^{-r(t-t_0)}, \end{aligned}$$

where  $d_1 = \frac{\ln(f_t/c_t) + (\sigma_t^2/2)(t-t_0)}{\sigma_t\sqrt{t-t_0}}$  and  $d_2 = \frac{\ln(f_t/c_t) - (\sigma_t^2/2)(t-t_0)}{\sigma_t\sqrt{t-t_0}}$ .

$f_t$  is the forward price on day  $t$ ,  $c_t$  is the contract price on day  $t$ ,  $\sigma_t$  is the average volatility on day  $t$  and  $N(x)$  denotes the cumulative probability distribution function for a variable that is normally distributed with a mean of zero and a standard deviation of one, i.e. the probability that such a variable will be less than  $x$ .

Semi-analytic methods such as these have been developed to take account of the optionality present in such contracts. These approaches neglect a portion of the optionality since, if a put option is exercised on a specific day, then an extra call option should be available in the future, and vice versa. The above model makes no provision for this, as incorporation would lead to a more complex model.



#### 7.4.4 Case Study

The following data were devised to represent a typical take or pay contract over a time period of one year:

$$r = 6\%$$

$$\text{ACQ} = 38,000,000 \text{ therms (annual contract quantity)}$$

$$\text{TOP} = 85\% \text{ of ACQ (minimum annual quantity)}$$

$$\text{MAQ} = 100\% \text{ of ACQ (maximum annual quantity)}$$

$$\text{DCQ} = \text{ACQ} / \text{number of days in contract (daily contract quantity)}$$

$$\text{LDQ} = 0\% \text{ of DCQ (minimum daily quantity)}$$

$$\text{MDQ} = 125\% \text{ of DCQ (maximum daily quantity)}$$

These values can be substituted into the deterministic models described previously.

The intuitive, linear programming and semi-analytic linear programming models were modelled using Visual Basic for Applications. This package has the advantage of being easy to use and has a graphical user interface, Excel, with which the majority of people who would wish to run the models are comfortable. The main disadvantage is limitations on the size of the problems which can be solved and the speed with which problems can be solved. The models were solved using actual forward curves and the results from each of the models were compared.

The models developed are large as each is modelled over a time period of one year with daily resolution. This leads to problems which are large in size and reasonably complex. They each contain 365 decision variables and over 1,400 constraints. There is also a number of calculations which must be carried out at each stage, such as the mark to market value for that time period.

The graphs of results from using one such curve are given in Figures 7.1, 7.2 and 7.3. The results from all models show that it is optimal to take either the minimum or maximum daily contract quantities on each day throughout the time period, except for one day when the amount taken may lie between these two values in order to satisfy an annual constraint. As is expected, more gas is



taken from the contract when the market price is deemed to be high and less when it is deemed low.

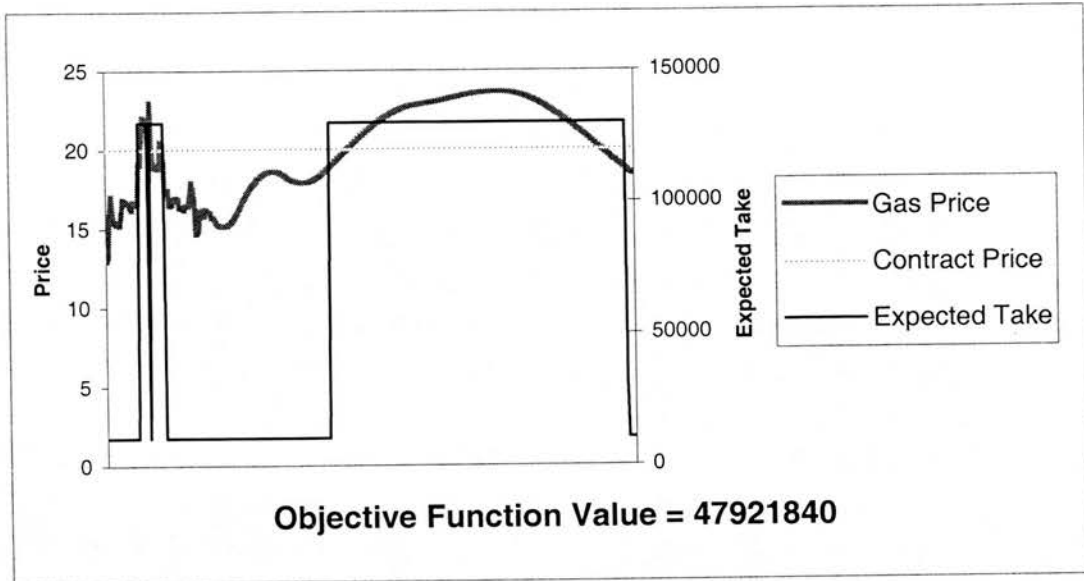


Figure 7.1: Results from Intuitive Approach

The values of the objective function in each case are:

Intuitive Approach = 47,921,840

Linear Programming Approach = 47,921,840

Semi-Analytic Linear Programming Approach = 65,755,137.

When comparing the results, the semi-analytic linear programming approach gives a higher objective function value. This could be due to the fact that this approach takes account of the optionality which is present in the problem. The intuitive and linear programming approaches have approximately the same objective function value. The linear and semi-analytic linear programming approaches also have the advantage that they have been constructed using a

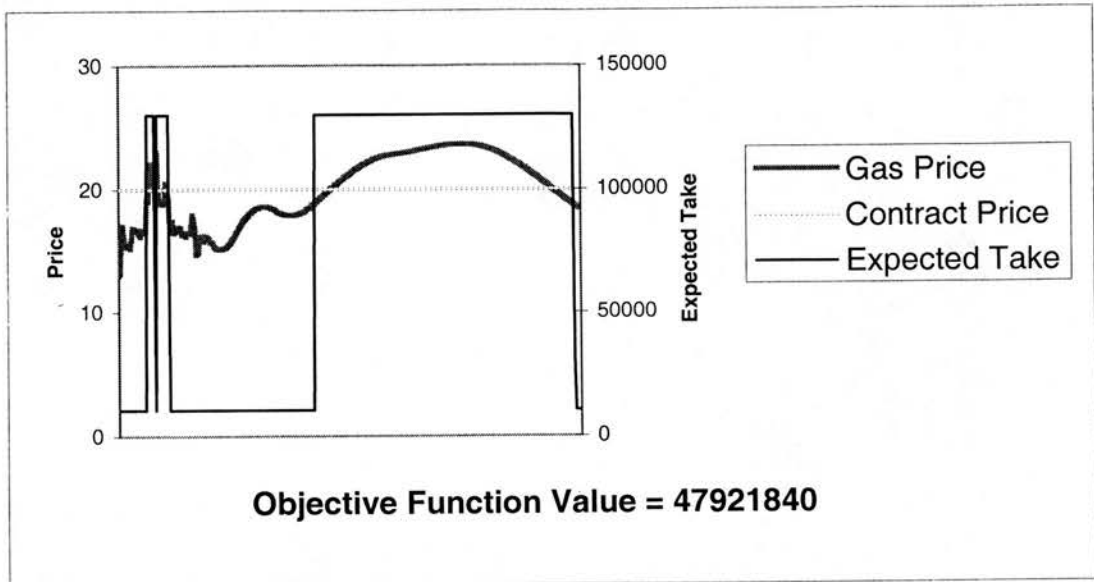


Figure 7.2: Results from Linear Programming Approach

solid mathematical background and it is fairly easy to adapt each to incorporate other features or characteristics which may arise.

The sensitivity of the linear programming model to different scenarios was tested by using a forward curve which is in contango and another which is in backwardation. These curves were selected from actual data to have the same starting point and each represents a possible future scenario which could occur. The results from each case are shown in Figure 7.4 and 7.5.

The objective function values are:

$$\text{Contango} = 45,408,720$$

$$\text{Backwardation} = -3,200,378.$$

These results show that the linear program is sensitive to the scenario which occurs. This motivates the development of a stochastic model which will encompass a variety of possible scenarios, including both high and low values.

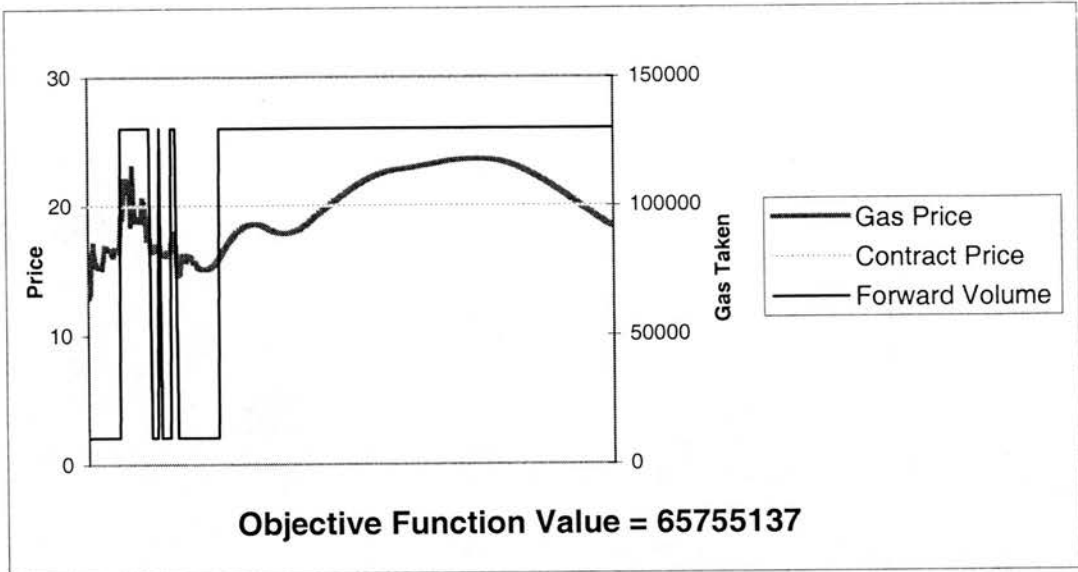


Figure 7.3: Results from Semi-Analytic Linear Programming Approach

This will provide a more accurate valuation of the contract.

## 7.5 Stochastic Model

A stochastic dynamic program can be constructed to solve the problem of valuing a take or pay contract with gas prices which vary. As demonstrated when modelling the gas storage problem in Section 6.5.1, these stochastic prices can be incorporated into the problem by means of a trinomial spot price tree. The construction of the tree builds upon the method described by Hull and White [34], and can be fitted to a given market forward curve using the approach detailed by Clewlow and Strickland [19]. An outline of this tree building procedure is given in Section 4.3.1. The combined method produces a spot price tree which is consistent with observed market features; a lognormal forward price and a volatility which decays over time.

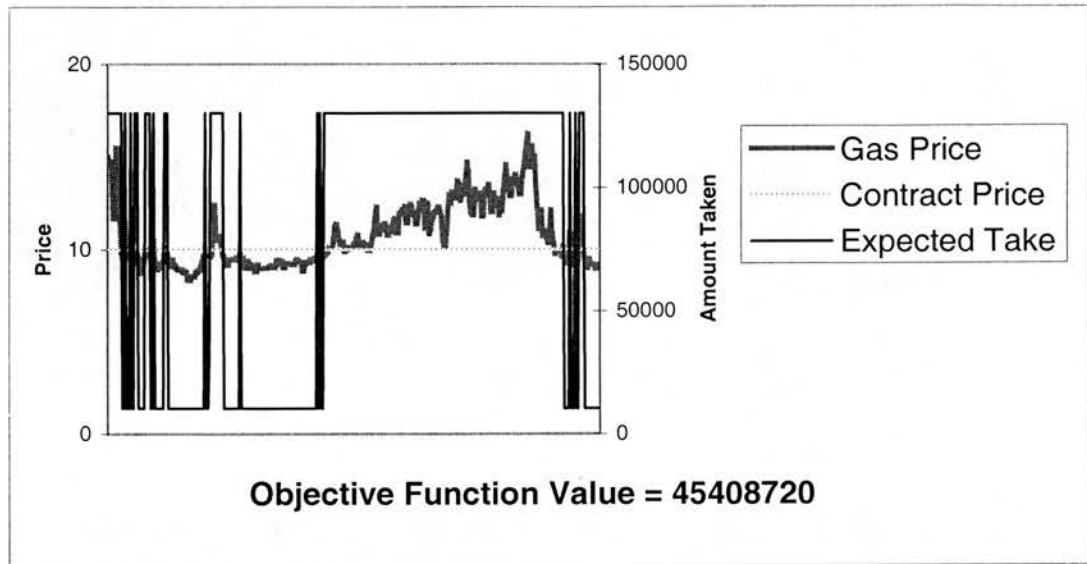


Figure 7.4: Results from Curve in Contango

### 7.5.1 Spot Price Tree Construction

The tree must be constructed to value a take or pay contract which has the same properties as those described in Section 7.4. This is necessary in order that comparisons can be made between the model developed and those described previously.

The tree was constructed for a time period of one year with daily resolution time steps:

$$\Delta t = \frac{1}{365}.$$

From any node, the price can move up by a certain amount, stay the same or move down by a certain amount. As described in the gas storage problem, values for constants are assumed. When  $\sigma = 100\%$  and  $\alpha = 3$ , the change in price level

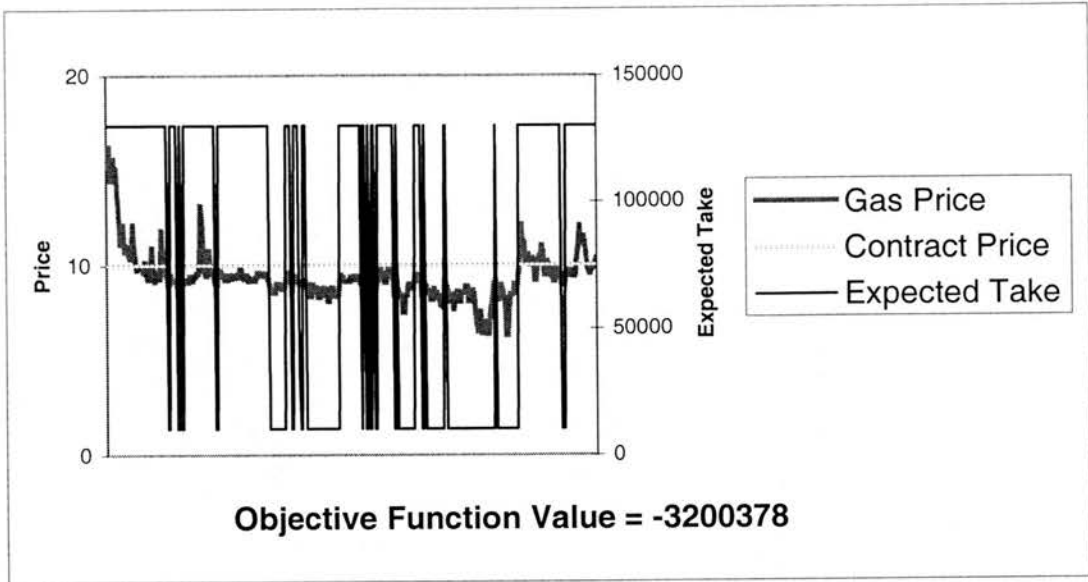


Figure 7.5: Results from Curve in Backwardation

becomes:

$$\Delta x = \sqrt{\frac{3\sigma^2(1 - e^{-2\alpha\Delta t})}{2\alpha}} = 0.09.$$

The maximum number of price levels,  $j_{\max}$ , is also obtained from the formula given previously:

$$j_{\max} = \left\lceil \frac{-0.184}{e^{-\alpha\Delta t} - 1} \right\rceil + 1 = 23.$$

### 7.5.2 Stochastic Dynamic Programming Approach

The spot price tree can be incorporated into a stochastic dynamic programming model to represent a variety of possible gas prices. The take or pay model lends itself to a backward recursion approach. In order to simplify the problem, it can be assumed that it is optimal to take either the minimum daily quantity or the maximum daily quantity on every day, depending upon whether the market price is above or below the contract price. This is a valid assumption since

Mavroidis [44], proves that this property holds in continuous time.

In this problem, the take or pay contract has a minimum and maximum daily quantity and a minimum and maximum annual quantity. It can be assumed that the minimum daily quantity is taken on every day. In order to satisfy the annual constraints, the maximum daily quantity must be taken on a certain number of days. The minimum and maximum number of days on which the maximum daily quantity can be taken without violating any annual constraint are worked out from the following formulae:

$$k_{\min} = \frac{(\text{TOP} - 365 \times \text{LDQ})}{(\text{MDQ} - \text{LDQ})},$$

$$k_{\max} = \frac{(\text{MAQ} - 365 \times \text{LDQ})}{(\text{MDQ} - \text{LDQ})}.$$

$k_{\min}$  denotes the minimum number of days on which the maximum daily quantity is taken and  $k_{\max}$  is the maximum number of days on which the maximum daily quantity is taken. In this particular problem, a **swing** is a day on which the maximum daily quantity is taken. If it is decided to exercise a swing on a particular day, then there will be one less swing available at the next day.

Published literature focuses more predominantly and frequently on binomial forests for the valuation of take or pay contracts. This approach is extended to incorporate a trinomial tree, resulting in a trinomial forest, and can be adapted to model the problem under consideration in this research. A stochastic dynamic programming approach lends itself well to this problem. A trinomial forest has three dimensions to the state space. In this problem, the first two dimensions relate to the time and price dimensions of the trinomial tree constructed in Section 7.5.1. The third dimension can be thought of as layers of these trinomial trees, each layer corresponding to the number of swings taken so far. The numbering of the layers will run from 1 to  $k_{\max}$ , where  $k_{\max}$  can be worked out using the formulae given above.

The stage in the stochastic dynamic program is the time period,  $t$ . The state is the price level,  $j$ , and the number of swings remaining to be exercised,  $d$ . The

maximum expected value of the swing contract in time periods  $t, \dots, T$ , given the price level is  $j$  and there are  $d$  units of swing remaining is denoted by  $f_t(k, d)$ .

The problem can now be formulated using the following backward recursion equations:

$$f_t(j, d) = \max \begin{cases} r_t^j(\text{LDQ}) + e^{-r\Delta t} \sum_k p_t(j, k) f_{t+1}(k, d) \\ r_t^j(\text{MDQ}) + e^{-r\Delta t} \sum_k p_t(j, k) f_{t+1}(k, d - 1) \end{cases}$$

$$f_{T+1}(j, k_{\max}) = 0, \quad \forall j.$$

The following notation is used:

- $j, k$  — index of price levels
- $p_t(j, k)$  — transition probability of going from price level  $j$  at time  $t$  to price level  $k$  at time  $t + 1$
- $r_t^j(x)$  — profit from taking  $x$  units of gas at time  $t$  and price level  $j$

The valuation is performed by working backwards and downwards through the layers. The aim is to maximise the expected value of the take or pay contract by finding the value of:

$$\max_{0 \leq k \leq k_{\max}} f_1(1, k).$$

The value of  $k$  which gives the maximum profit denotes the number of days on which the maximum daily quantity should be taken. The reward function is given by:

$$r_t^j(x) = (s_t(j) - c_t(j))x,$$

where  $s_t(j)$  is the spot price at time  $t$  and price level  $j$ , the value of which can be obtained from the spot price tree and  $c_t(j)$  is the contract price at time  $t$  and price level  $j$ .

There are two choices which can be made at each node in the problem; the value of exercising a swing by taking the maximum daily quantity and having one less swing for the remainder of the time period is compared with the value

obtained from not exercising a swing by taking the minimum daily quantity. The maximum of these two values is assigned to that node.

Only some states are valid in the dynamic program and this must be taken into account when formulating the problem. For example, it is not possible to take LDQ on every day since the minimum annual quantity is not then satisfied. Valid states are shown in Figure 7.6. This diagram shows that the number of days on which MDQ should be taken must lie between  $k_{\min}$  and  $k_{\max}$ . Restrictions on the states which are possible ensure that the annual limits are always satisfied.

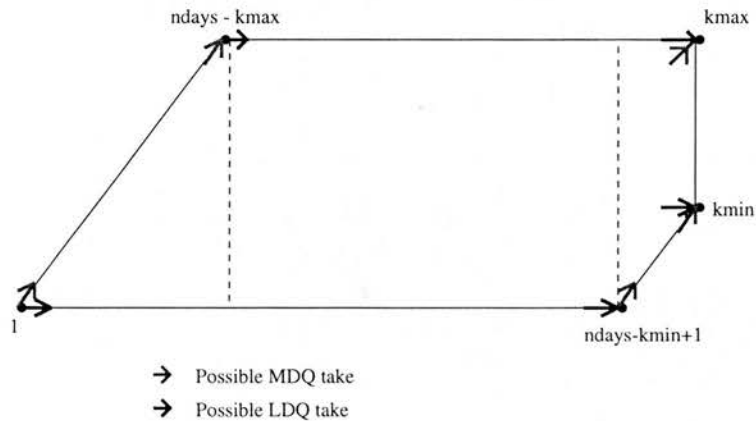


Figure 7.6: Valid States in Stochastic Dynamic Program

### 7.5.3 Case Study

The constants used in the construction of the deterministic models in Section 7.5.1 were again employed to solve the stochastic dynamic programming model. This allows the models to be compared. The model is complex as it is large in size with many interdependencies and was again constructed using Visual Basic for Applications. The stochastic dynamic program can be compared to the linear program by setting the volatility equal to zero. This has the effect of preventing the price from varying at each time period. This creates, in effect, a dynamic program as the prices are constant. As both the linear and dynamic programs are optimised against the same forward curve, they should give approximately the same objective function value.



The results of this comparison are:

Linear Programming Approach = 47,921,840

Dynamic Programming Approach = 47,948,540

Percentage Difference = 0.056%.

The difference between the two approaches is small, therefore it can be assumed that the stochastic dynamic program has been correctly constructed.

In order to solve the stochastic dynamic programming model, the volatility can now be set to 100%. This allows the entire spot price tree to be used in the valuation. The objective function values from the models are:

Intuitive Approach = 47,921,840

Linear Programming Approach = 47,921,840

Semi-Analytic Linear Programming Approach = 65,755,137

Stochastic Dynamic Programming Approach = 92,817,787.

This comparison is shown graphically in Figure 7.7.

The stochastic dynamic programming approach is seen to give a higher value for the valuation of an identical take or pay contract. This is assumed to be the case since it takes into account more of the optionality which is present within the contract.

The models developed to demonstrate the properties of take or pay contracts in this chapter can be used to determine an accurate value for such contracts. This is necessary as traders wish to be able to buy and sell contracts for a fair price. The models are flexible and can be easily adapted to incorporate other features which may be present, such as late delivery of the commodity. The techniques used for these valuations can be extended to model tolling deals in the next chapter.

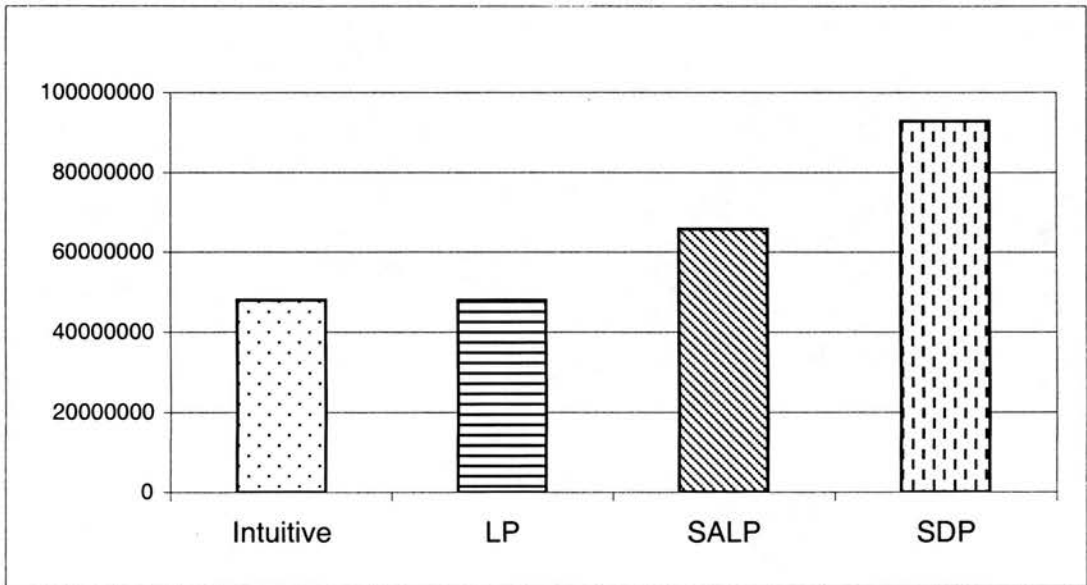


Figure 7.7: Comparison of Approaches



# Chapter 8

## Tolling Deals

### 8.1 Introduction

Tolling deals are a relatively new and topical feature of energy markets. The first tolling deal was traded in the UK in 1996. Since then, they have become increasingly more common. The need for contracts such as these has arisen from the changes which have occurred from restructuring the energy industries in recent years. Tolling deals were first designed to enable owners of generating plants to lease generating capacity to others for a fee. Such deals of the physical variety are also known as virtual power stations. Recently, similar contracts of a purely financial nature have emerged to be used solely as trading mechanisms.

The two main types of tolling deal present in energy markets today are physical and financial. Definitions of these are given, together with various properties and features exhibited by this type of contract. Tolling deals can be loosely related to spark-spread options and this relationship is examined. A number of features can be found in this type of contract. These can include:

- non-zero strike price.
- a limit on the total volume produced to satisfy environmental constraints.
- different efficiencies when running at different generating levels.
- cost of switching between different generating levels.

- a limited number of starts that can be made.
- ramp rates.
- minimum on and off times.

The aim of the chapter is to develop a model to value a tolling deal with stochastic prices, which incorporates some of the features mentioned above. The most important were deemed to be non-zero strike price, environmental constraints, different efficiencies, costs and the limited number of starts that can be made. The models are developed in a number of incremental stages in a similar way to previous chapters. Each stage can be used to validate the next. This is seen to be essential when developing complicated models. The resulting complex models were thought to give a more accurate representation of the contract.

In order to incorporate stochastic variables into the formulation, the method used previously for constructing the spot price tree must be adapted to include two correlated variables; fuel and power prices. The resulting correlated spot price tree can then be used in stochastic dynamic programming models of this problem.

The first stage in the development of models is the construction of an intuitive approach which is easy to understand and to implement. A linear program can then be developed to model the same problem. Integer constraints can be added to model the limit which is placed on the number of starts which are possible. The stochastic dynamic programming model is built in a number of stages to allow validation at each stage. The first stage develops a stochastic dynamic programming model with no constraints. Each feature can be incorporated in turn, starting with that which describes the different efficiencies which are present at each operating level.

## 8.2 Properties of Tolling Deals

A **tolling deal** gives the buyer the right to convert fuel into energy. In return for these conversion rights, the buyer pays a fixed charge over the term

of the contract, and a variable charge depending upon the amount of electricity produced. Tolling deals are popular, perhaps in part, due to the liberalisation of the electricity and gas industries. As a result of the changes which have been made, many power purchasers now have access to both fuel and power resources. Tolling deals can be either physical or financial.

A **physical tolling deal** gives the buyer the right to exchange physical fuel for physical power at a predetermined conversion factor. This type of contract can be used to provide the same function and flexibility as a generation plant without the need to own one. The owner of a generating plant can sell a physical tolling contract in order to lessen the risk from exposure resulting from fluctuating fuel and power prices. In doing so, a constant revenue stream is obtained, which can help to finance the plant.

A **financial tolling deal** is a contract which does not involve the physical commodities, fuel and power. Instead, the contract gives the buyer the right to exchange the price of fuel for the price of power. Charges are made for this contract which are analogous to those paid for a physical contract. This type of contract can be used by the owners of generating plants to hedge against the risk resulting from fuel and power fluctuations.

Various tolling deals are feasible, but the generator is normally paid a fee for each unit of power produced. The buyer of such a contract must abide by a minimum generation level to cover the capital and fixed costs of the plant. For a power generator, tolling deals offer the prospect of higher profit margins in return for acceptance of dual-commodity (fuel and power) market risks. The mechanism can also offer a long-term and guaranteed outlet for the power produced by the plants. From the view of the buyer, tolling can allow for substantial risk management, as part of the commodity risk is taken on by the owner of the plant.

Both the conversion factor between fuel and power, and the fuel and power market indices to be used, are agreed between the buyer and seller and stated in the contract. Other variables which are present in tolling deals include:

- limits on the volume which can be produced in any time period.
- the number of times the power station can be switched on over the period of the contract.
- the efficiency of the plant.
- the time by which nominations about how much power to take must be made.

Power stations have lead times since changing between generating levels is not instant. It takes time to move from one level to another. The tolling contract contains a limit on the number of starts that can be made. This is the number of times which the generating plant can start producing again once it has been switched off. This process of changing levels can be extremely costly.

The number of starts that can be made are divided into a number of levels usually including hot starts, warm starts and cold starts. Hot starts relate to the case when the power station has been generating no electricity for a short amount of time, whereas cold starts are when the power station has been switched off for a much longer period of time. Each time a start is made, the number of times remaining at which a start can be made is reduced.

In practice, power stations rarely stop generating unless they are undergoing maintenance. The owner of the generating plant will usually have a number of obligations, not just tolling deals. Therefore, it is feasible for tolling deal contracts to include the case where the power station does not generate, even although this will rarely happen in real life.

Tolling deals demonstrate characteristics of options on the difference between fuel and power prices, which also have a number of other restrictions on them. Therefore, they can also be thought of as spark spread options which have a number of constraints which must be satisfied. Spark spread options are described in Section 3.2.

## 8.3 Deterministic Models

### 8.3.1 Intuitive Approach

An intuitive approach can be used to give an indication of the value which can be placed on a tolling deal for given power and gas forward curves. This type of model is useful as it is easily understood and intuitively appealing. The ease of understanding makes it more accessible to people with no prior knowledge of details relating to the area. It is usual for tolling deals to state the levels at which generation can take place. In this model, it can be assumed, for simplicity, that the power plant can run at one of two levels, either full load or the minimum stable generation level. The solution of the model gives indications of the generation level which should be set in each period. The number of starts are not included in this model as this would increase the complexity considerably.

This approach works in the following way:

1. Forward curves of fuel and power prices are obtained from the market. These are a good approximation to the behaviour of the such prices.
2. The cost in £/MWh of generating via the tolling contract is calculated.
3. For each day in the time period, this cost is subtracted from the power price to give the expected value of producing one unit of power on that day. If the difference is positive, then the power plant should run at full load on that day. Otherwise, if the difference is negative, then the power plant should run at minimum stable generation.
4. The mark to market value on each day is calculated by discounting this expected value by a factor of  $e^{-r(t-t_0)/n}$ , where  $r$  is the risk free rate of interest and  $t$  is the present time period,  $t_0$  is the valuation date and  $n$  is the number of days in the year. This gives a true representation of the value of the tolling deal on the valuation date.
5. The mark to market values for each day are multiplied by the level at which the plant should run on that day to give the expected value for that day.
6. The expected value of the tolling deal is calculated by summing the expected values for each day over the duration of the contract.



### 8.3.2 Linear Programming Approach

A linear program was constructed to model the same problem as the intuitive approach. The power plant can generate at any level between two limits; full load and the minimum stable generation level. This model works by calculating the mark to market value on each day in the time period. The model aims to maximise the value of the tolling deal by finding the optimal generating level for each day. The model uses given forward curves for gas and power prices and calculates the mark to market value on each day.

The model is formulated as follows:

$$\begin{aligned} & \max \sum_{t=1}^T e^{-r(t-t_0)/n} (s_t - ap_t)x_t \\ & \text{subject to:} \\ & \quad \text{MSG} \leq x_t \leq \text{FL} \\ & \quad \forall t = 1, \dots, T. \end{aligned}$$

The following notation is used:

- $x_t$  — amount of power produced at time  $t$
- $s_t$  — selling price per unit of power at time  $t$
- $p_t$  — buying price per unit of gas at time  $t$
- $a$  — conversion factor based on efficiency of plant and p/therm to £/MWh

MSG denotes the minimum stable generation level and FL is full load.

The decision variable in the above problem is the amount of power produced in period  $t$ ,  $x_t$ . The above model is deterministic in that the values of all variables are known with certainty over the entire time period. In deterministic models, the uncertain prices can be approximated in order to give known values. In the energy industries, uncertain prices can be approximated using values obtained from the forward curve. In this model, both the cost of buying one unit of gas,  $p_t$ , and the cost of selling one unit of power,  $s_t$ , have been approximated in this way.

### 8.3.3 Linear Programming Approach (With Integer Constraints)

The above linear programming model can be adapted to incorporate a limit on the number of starts that can be made. This requires the inclusion of integer constraints. This allows the situation where the power plant is switched off to be modelled. The generating plant can now operate at any level between full load and the minimum stable generation level, or can be switched off. The value of the tolling deal can then be obtained, subject to the constraints on the number of starts that can be made. The level at which the power plant should be generating on each day is also given.

The model is formulated as:

$$\max \sum_{t=1}^T e^{-r(t-t_0)/n} (s_t - a p_t) x_t$$

subject to:

$$\alpha_t \text{MSG} \leq x_t \leq \alpha_t \text{FL or } x_t = 0$$

$$T - \sum_{t=1}^T \alpha_t \leq g,$$

where:

$$\alpha_t = \begin{cases} 0 & \text{if } x_t = 0 \\ 1 & \text{if } x_t > 0 \end{cases}.$$

The maximum number of starts permitted per year is denoted by  $g$ .  $g$  is always less than  $T$ .  $\alpha_t$  is an integer variable which equals zero if no power is produced at time  $t$  and one otherwise. The decision variable in this problem is the amount of power produced,  $x_t$ , on each day in the time period. The number of starts have been modelled by assuming that every day on which no power is produced accounts for one start. By this logic, if the plant is switched off for three days in a row, then this would account for three starts. This is a valid assumption since it is much harder and more expensive to start up a power plant when it has been switched off for a longer period of time.

### 8.3.4 Case Study

The following data are used to represent a typical tolling deal:

$$r = 6\%$$

$$\text{MSG} = 390 \text{ MWh}$$

$$\text{FL} = 680 \text{ MWh}$$

$$\text{Efficiency of plant} = 55\%$$

These values can be substituted into the model given above.

A conversion factor is needed since gas is priced in p/therm and electricity is in £/MWh. In order to achieve consistency in the model, all amounts and prices should be converted into £/MWh. Also, the efficiency of the plant needs to be taken into account. The conversion from p/therm to £/MWh for a plant with 55% efficiency is:

$$a = 0.341212 \times \frac{100}{55} = 0.620385,$$

where 0.341212 is the standard conversion factor from p/therm to £/MWh. This conversion factor is used in the construction of the models.

The models were constructed, again using Visual Basic for Applications, and solved using actual forward curves. These models are again complex due to the number of decision variables and constraints present. The objective function values for each model using the same curve are:

$$\text{Intuitive Approach} = 1,749,797$$

$$\text{Linear Programming Approach} = 1,749,797$$

$$\text{Linear Program With Integer Constraints Approach} = 1,930,010.$$

The results are shown graphically in Figures 8.1, 8.2 and 8.3. This shows that it is optimal to generate at either the minimum stable generation level or at full load for the intuitive and linear programming approaches. In the linear programming with integer constraints approach, it is optimal to generate at any one of three levels; off, minimum stable generation or full load. As is expected, the generating plant will generate at a higher level when the power price is larger than the gas

price and at a lower level when the gas price is higher.

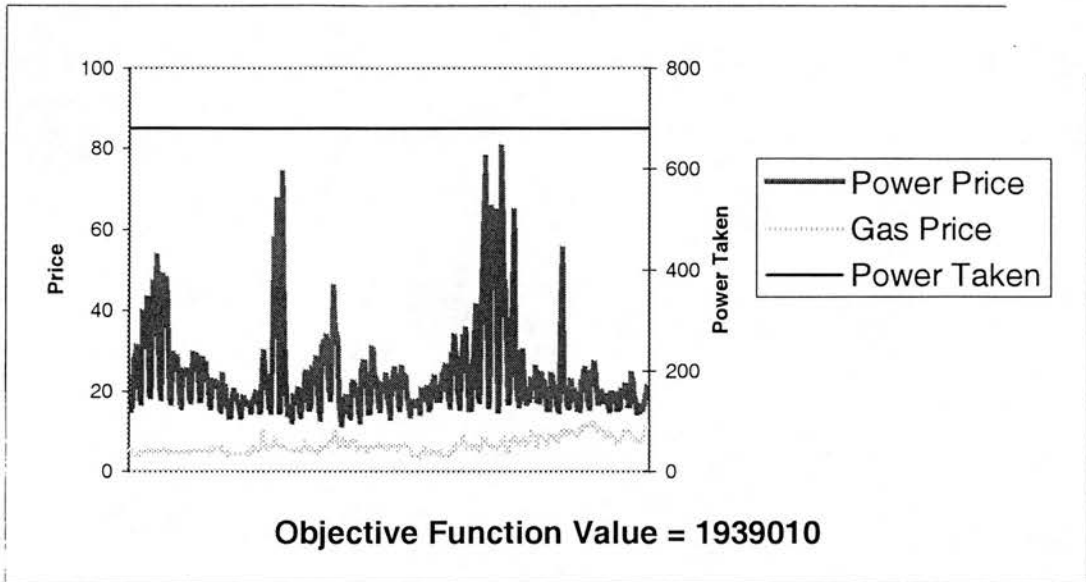


Figure 8.1: Results from Intuitive Approach

## 8.4 Stochastic Models

Models are now constructed to incorporate stochastic variables. The stochastic variables in this problem are fuel and power prices and so the spot price tree must incorporate both. It is relatively easy to construct a spot price tree for two variables which are uncorrelated. This can be achieved by constructing a tree for each separate variable and combining the resulting two-dimensional trees to create one with three dimensions. The resulting three dimensions are time, variable one and variable two. The probabilities on the branches of the three-dimensional tree are the product of the probabilities on the corresponding branches of the two-dimensional trees.

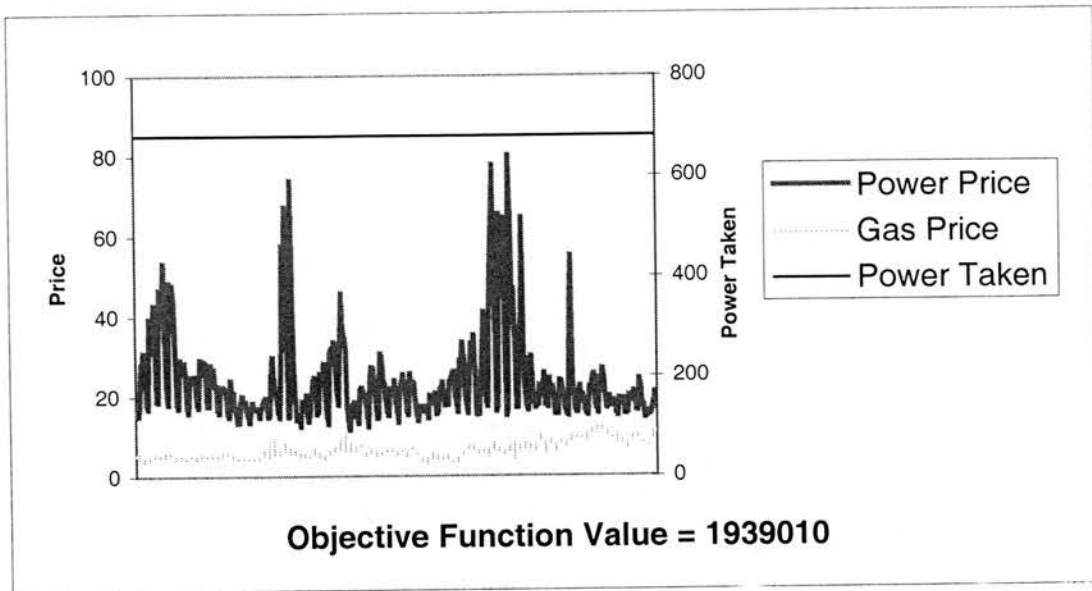


Figure 8.2: Results from Linear Programming Approach

Various methods exist for constructing a spot price tree for two variables. Boyle [11], has extended the Cox et al. model, described in Section 4.3, to value an option which is a function of two underlying state variables. These state variables may be correlated. The algorithm described assumes that the joint density of the two underlying assets is a bivariate lognormal distribution. The binomial tree used in the Cox et al. model is replaced with a tree which has five possible jumps at each node. This is thought to achieve the most efficient algorithm.

Cho and Lee [16], extend the method described by Boyle [11], to produce a three jump process model to price contingent claims whose underlying variables follow geometric Brownian motion. The method incorporates skewness, as well as the mean and variance of the lognormal distribution. Advantages of this approach include the improved accuracy which is obtained by using the skewness information. The efficiency of the procedure is enhanced by determining

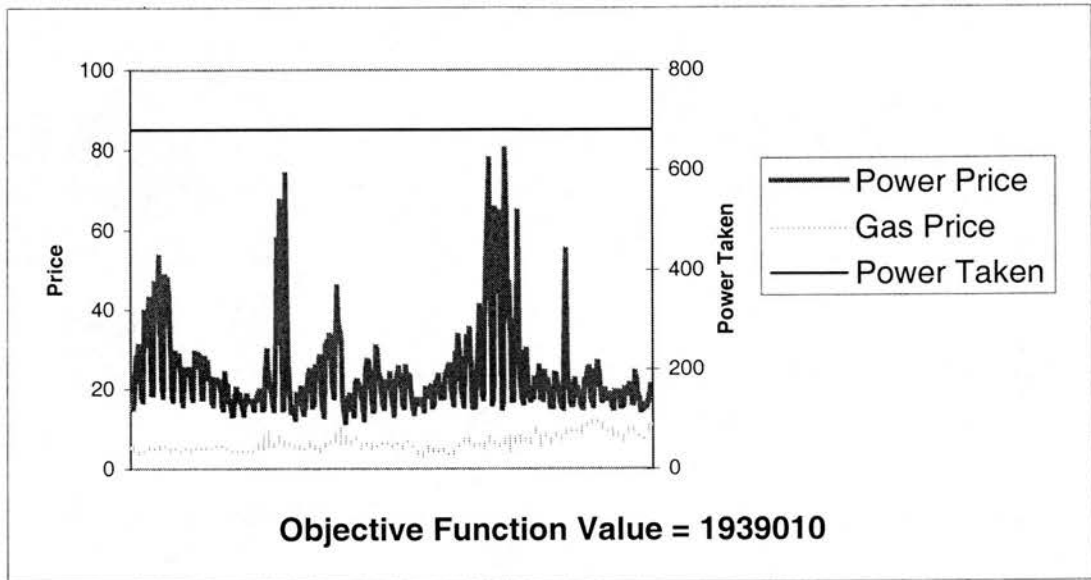


Figure 8.3: Results from Linear Programming with Integer Constraints Approach parameters directly within the model.

Hull [36], describes three methods for calculating a spot price tree for two correlated variables:

- **Transforming Variables**

The stochastic processes underlying the two correlated variables can be transformed to eliminate the correlation. Separate binomial trees can then be constructed for each new process. As there is now no correlation between the variables, separate binomial trees may be constructed for each and combined.

- **Changing the Geometry of the Tree**

Rubinstein [55], suggests a method for building a binomial pyramid for two correlated variables which directly incorporates the correlation into the process. This method advances with a shape similar to a pyramid, rather

than a tree structure. This method does not readily extend to trinomial models.

- **Adjusting the Probabilities**

A tree is constructed for each variable assuming there is no correlation between them. The trees are combined and the probabilities on each of the branches are adjusted to reflect the correlation.

It was decided to use the third of these approaches to adapt the method described earlier in Section 4.3.1. This approach fits easily with those methods used for gas storage in Section 6.5.1 and take or pay contracts in Section 7.5.1.

### 8.4.1 Spot Price Tree Construction for Correlated Assets

The approach can be used in conjunction with the method described by Clewlow and Strickland, [19]. The initial processes for the fuel and power prices,  $s_1$  and  $s_2$ , are defined to be:

$$\begin{aligned} dx_1(t) &= [\theta_1(t) - \alpha_1 x_1(t)]dt + \sigma_1 dz_1(t) \\ dx_2(t) &= [\theta_2(t) - \alpha_2 x_2(t)]dt + \sigma_2 dz_2(t), \end{aligned}$$

where  $x_i(t) = \ln s_i(t)$ ,  $\alpha_i$  is the rate of mean reversion of the process and  $\sigma_i$  is the volatility. The same basic procedure for constructing the tree is followed. This method is described in Section 4.3.1. By assuming that  $\theta_i(t) = 0$  and the initial value of  $x_i$  is zero, the processes become:

$$\begin{aligned} dx_1^*(t) &= -\alpha_1 x_1^*(t)dt + \sigma_1 dz_1(t) \\ dx_2^*(t) &= -\alpha_2 x_2^*(t)dt + \sigma_2 dz_2(t). \end{aligned}$$

The volatility of each of the processes can be defined by using the two parameter volatility term structure given in Clewlow and Strickland [19]. The volatilities are:

$$\begin{aligned} \sigma_1(t) &= \sigma_{s_1} e^{-\alpha_1(T-t)}, \\ \sigma_2(t) &= \sigma_{s_2} e^{-\alpha_2(T-t)}. \end{aligned}$$

A spot price tree is constructed for the fuel price and another is constructed for the power price. The method described by Hull and White [35], and outlined

in Section 4.3.1 can be used for this purpose. Each tree was constructed for a time period of a year with daily resolution time steps:

$$\Delta t = \frac{1}{365}.$$

The tree is constructed with constant price steps:

$$\Delta x^* = \sqrt{\frac{3\sigma_i^2(1 - e^{-2\alpha\Delta t})}{2\alpha}},$$

where  $\sigma_i$  represents the volatility of each process.

Putney [52], describes the instantaneous correlation between the two variables by means of a three factor curve:

$$\rho(T - t) = \rho_\infty + (\rho_0 - \rho_\infty)e^{-\kappa(T-t)}.$$

A constant correlation can be achieved by setting  $\rho_0 = \rho_\infty$  and  $\kappa = 1$ .

The two trees can be combined, creating a three-dimensional tree with nine branches at each node. In order to reduce the size of the tree, a maximum number of price nodes,  $j_{\max}$ , which can occur at each time period is defined. This can be obtained from the following formula:

$$j_{\max} = \left\lceil \frac{-0.184}{e^{-\alpha\Delta t} - 1} \right\rceil + 1 = 23.$$

$j_{\max}$  is dependent upon the rate of mean reversion,  $\alpha$ . As  $\alpha$  may differ for each price process,  $x_1$  and  $x_2$ ,  $j_{\max}$  must be worked out for each individual spot tree. The smallest value should then be chosen for the maximum number of price nodes when constructing the combined tree. This allows both spot trees to change branching pattern at the same point in time and avoids problems when the two are combined.

The probabilities associated with the upper, middle and lower branches in the first tree are defined to be  $p_u$ ,  $p_m$  and  $p_d$  respectively. Similarly, the probabilities associated with the upper, middle and lower branches in the second tree are denoted by  $q_u$ ,  $q_m$  and  $q_d$ . Assuming that the variables are uncorrelated, the probability associated with any one of the nine branches in the combined tree is the product of both probabilities associated with the corresponding branches in



the uncorrelated trees:

		Tree 1		
		Down	Mid	Up
Tree 2	Down	$p_d q_d$	$p_m q_d$	$p_u q_d$
	Mid	$p_d q_m$	$p_m q_m$	$p_u q_m$
	Up	$p_d q_u$	$p_m q_u$	$p_u q_u$

The probabilities can then be adjusted to reflect the correlation between the variables. Hull and White [35], suggest the following method for achieving this. If the correlation between the variables is positive, the probabilities become:

		Tree 1		
		Down	Mid	Up
Tree 2	Down	$p_d q_d + 5\varepsilon$	$p_m q_d - 4\varepsilon$	$p_u q_d - \varepsilon$
	Mid	$p_d q_m - 4\varepsilon$	$p_m q_m + 8\varepsilon$	$p_u q_m - 4\varepsilon$
	Up	$p_d q_u - \varepsilon$	$p_m q_u - 4\varepsilon$	$p_u q_u + 5\varepsilon$

The sum of the adjustments in each row and column is zero. As a result, the adjustments do not change the mean and the standard deviation of the processes for  $x_1$  and  $x_2$ . The adjustments shown above, have the effect of inducing a correlation between the variables of  $36\varepsilon$ . Therefore, the value of  $\varepsilon$  should be taken to be  $\frac{1}{36}\rho$ . The values of the adjustments were chosen as they have the effect that as the limit  $\Delta t$  tends towards zero, the probabilities tend towards:

$$p_u = q_u = \frac{1}{6}, p_m = q_m = \frac{2}{3} \text{ and } p_d = q_d = \frac{1}{6}.$$

This is consistent with the probability formulae defined in Section 4.3.1. A similar process, which is described by Hull and White [35], can be used if the correlation is negative.

One drawback of this approach is that some probabilities may become negative at a few nodes. In order to overcome this, the value of  $\varepsilon$  can be changed at any node where the probabilities are negative. Hull and White [35], suggest that the value in this case should be the maximum  $\varepsilon$  for which the probabilities remain positive. For clarity, a representation of the two spot trees and the combined tree is given in Figure 8.4. This diagram shows that the two spot price trees can be

combined to create a tree with nine branches at each node. This allows every possible combination of the two spot price trees to be represented. In this way, a spot price tree can be constructed for specified, correlated fuel and power forward curves.

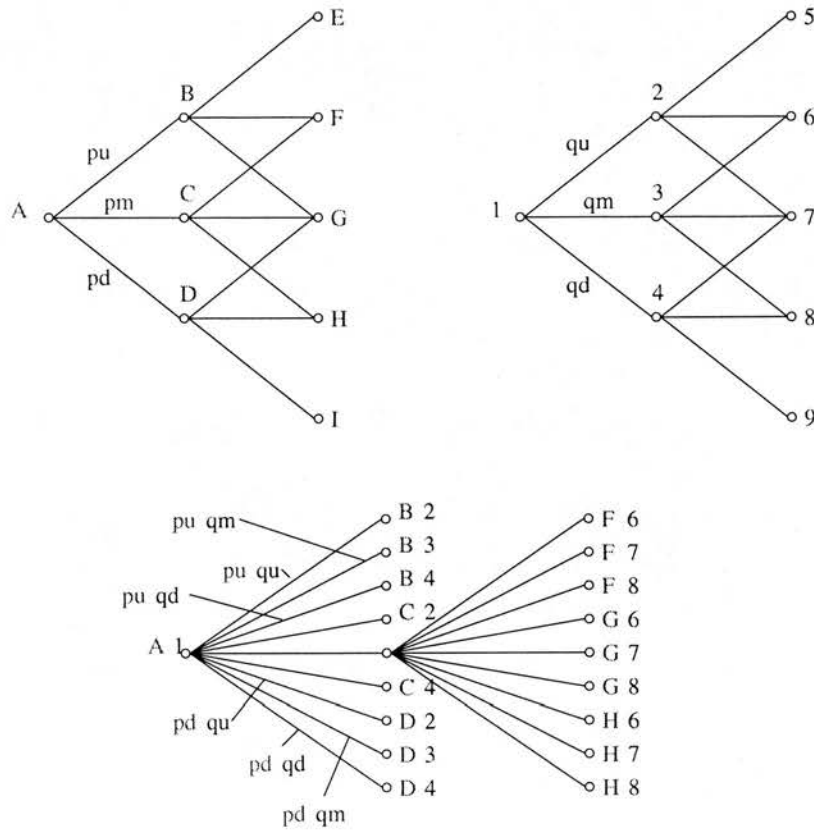


Figure 8.4: Combining the Spot Price Trees

**Tree Validation Example**

As shown in Section 6.5.1, the values obtained from the tree building process can be checked by using the tree to value an option. In this case, the values produced by the tree can be used to price a spark spread option. This is an option on the difference between fuel and power price. When the strike price of such an option is zero, the Margrabe model, detailed in Section 3.5.3, can be used to determine the value.

The Margrabe model is used to price options to exchange one risky asset for another and is defined to be:

$$c = e^{-r(T-t)} [F_1 N(d_1) - F_2 N(d_2)],$$

where:

$$d_1 = \frac{\ln(F_1/F_2) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

$$\sigma = \left[ \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2) dt \right]^{1/2}.$$

$F_1$  and  $\sigma_1$  denote the forward price and volatility of the power and  $F_2$  and  $\sigma_2$  denote the forward price and volatility of the fuel.

The following values for the constants were assumed:

$r = 0.06$  (risk free rate of interest)

$T = 365/365$  (day on which the option expires)

$t = 1/365$  (day on which option is valued)

$F_1 = 19.15$  (forward price of power for day on which option expires)

$F_2 = 18.14$  (forward price of fuel for day on which option expires).

The tree can be used to value a spark spread call option with a strike price of zero. This is an option giving the right to convert one unit of fuel into one unit of power at a cost equal to the strike price. The probabilities on each branch can be used to calculate the expected value of the option, which can then be discounted back to the first period. At the end of each branch in the last period of the tree, the spot price for fuel is subtracted from the spot price for power. The expected value can be calculated by working backwards. This process can be continued until the first node of the tree is reached. The value must be discounted at each stage in order to give the value of the option on the valuation date. In this way, the value of the option can be derived.

The result of using the tree to value a single European spark spread call option with a strike price of zero can be compared with the value obtained from the Margrabe model to price the same option. This comparison will validate

the values produced by the tree building procedure. The Margrabe model is commonly used in energy markets to perform this type of valuation. The results are:

$$\begin{aligned} \text{Tree Value} &= 0.117701 \\ \text{Margrabe Value} &= 0.120021 \\ \text{Percentage Difference} &= 1.971\%. \end{aligned}$$

This shows that there is little difference between the values obtained from each approach. The Margrabe model breaks down when a non-zero strike price is introduced. This is because Margrabe assumes that the power and fuel prices are both lognormally distributed. Introducing a strike price in effect creates a new variable  $F_2 + K$ , where  $F_2$  is the price of fuel. This variable must also be lognormally distributed in order for the formulae to hold. As  $K$  increases, this becomes less true and the Margrabe model breaks down. As the strike price is increased, the results from the tree will diverge significantly from those given by the Margrabe formulae.

The results of this comparison are shown in the following table:

Strike	Tree Value	Margrabe Value	Percentage Difference
0	0.117701	0.120021	1.971%
1	0.074548	0.088752	19.054%
2	0.050745	0.065424	28.926%
3	0.033355	0.048097	44.199%
4	0.021970	0.035277	60.572%
5	0.014610	0.025824	76.751%
6	0.009745	0.018873	93.675%
7	0.006520	0.013775	111.284%
8	0.004377	0.010043	129.451%
9	0.002941	0.007316	148.730%
10	0.001991	0.005327	167.542%

These results are shown graphically in Figure 8.5. This shows that as the strike price increases, the difference between the value obtained from the tree and the value given by the Margrabe model increases. As the Margrabe model breaks

down when the strike price increases, it can be assumed that the tree gives a more accurate value. As financial contracts will usually have a strike price which is greater than zero, it is more appropriate to use the tree valuation method in this case.

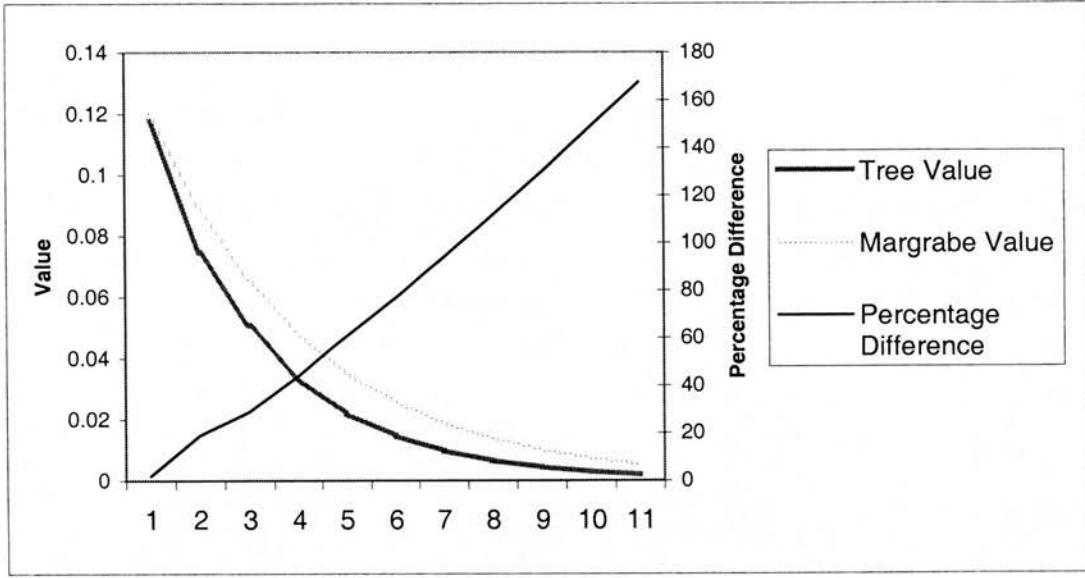


Figure 8.5: Single European Option

### 8.4.2 Stochastic Dynamic Programming Approach

The above spot price tree, which was constructed for correlated fuel and power prices, can be incorporated into a stochastic dynamic programming model. The aim is to develop a model to try to place a value on a tolling deal. It was decided to incorporate the following features into the model:

- Non-zero strike price.
- Cost of switching between different generating levels.
- Different efficiencies when running at different generating levels.

- A limit on the total volume produced to satisfy environmental constraints.
- A limit on the number of starts that can be made.

A non-zero strike price and the cost of switching between different levels can be incorporated directly into the formulation. Different efficiencies at different generating levels reflect what happens in reality. The efficiency of power plants increases as the generation level increases, i.e. the marginal amount of power produced increases when running at higher levels. It is not necessary to consider the efficiency of the power plant when it has been switched off. These efficiencies can be directly incorporated into the formulation. The fuel prices can be converted to take account of the efficiencies. The conversion to take account of these efficiencies is  $\frac{F_2}{e}$  where  $F_2$  is the price of the fuel and  $e$  is the efficiency.

The plant can generate at any level between the minimum stable generation and full load, or it can be switched off. Levels between off and the minimum stable generation level are not possible. It is assumed, for simplicity, that it is always optimal to run at either the minimum stable generation level or at full load, provided the plant is not switched off. This is a reasonable assumption as a similar property of take or pay contracts has been proven by Mavroidis [44].

Therefore, there are three possible operating levels; off, minimum stable generation (MSG) and full load (FL). At each node in the tree, the model compares the profit which can be obtained from running at one of these levels with the profit which can be obtained from the others. The greatest of these values is chosen, and the operating level at that node is determined.

Environmental constraints are imposed by the Environment Agency and overseen by the Government. An example of such a constraint could be a limit on the amount of sulphur dioxide produced by coal-fired power stations or the amount of carbon dioxide produced by gas-fired power stations. The amount of harmful gases produced varies, depending upon the level at which the plant has been generating and also on the quality of the fuel which has been introduced. When generating at full load, a coal-fired generating plant could produce 90

tonnes of SO<sub>2</sub> per day compared with 55 tonnes per day at the minimum stable generation level. These values are from an actual power station.

These environmental constraints can be incorporated into the dynamic program by creating an additional dimension. This dimension can be thought of as the number of days remaining on which the plant can generate at full load in order to satisfy the environmental constraint. This can be achieved by calculating the maximum number of days,  $k_{\max}$ , on which the plant can generate at full load, whilst ensuring that the amount of sulphur dioxide produced is below the stated level. The following formula can be used:

$$k_{\max} = \left\lceil \frac{\text{MAQ of SO}_2 - (\text{ndays} \times \text{DQ of SO}_2 \text{ at MSG})}{\text{DQ of SO}_2 \text{ at FL} - \text{DQ of SO}_2 \text{ at MSG}} \right\rceil.$$

If it is decided to run the plant at full load on a specific day, then there will be one less available day on which to run at full load for the remainder of the time period. The structure of this problem lends itself to a backwards recursion approach for a stochastic dynamic programming problem.

All dynamic programs have a stage and a state as described in Section 4.2.4. The stage in this problem is the time period,  $t$ . The state consists of two dimensions; the price level,  $j$ , and the number of days remaining on which the plant can run at full load,  $k$ . The maximum expected value of the tolling deal in time periods  $t, \dots, T$  given that the current price level is  $j$  and there are  $k$  days remaining on which the plant can generate at full load is denoted by  $f_t(j, k)$ . The aim is to find  $\max_{0 \leq k \leq k_{\max}} f_1(1, k)$ , the maximum expected value of the tolling deal on day 1 at price level 1. There is only one possible price level on the first day.

The stochastic dynamic program uses a backward recursion and can be formulated as follows:

$$f_t(j, k) = \max \begin{cases} e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k) \\ r_t^j(\text{MSG}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k) \\ r_t^j(\text{FL}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k - 1) \end{cases}$$

$$f_t(j, k) = 0, \forall t, j \text{ and } k.$$

The following notation is used:

- $m$  — index of price level at time  $t + 1$   
 $r_t^j(x)$  — profit from generating at level  $x$  at time  $t$  and price level  $j$   
 $p_t(j, m)$  — transition probability of going from price level  $j$  at time  $t$  to price level  $m$  at time  $t + 1$

The above stochastic dynamic programming formulation can now be extended to incorporate a limited number of starts. This is an important feature of tolling deals. In order to model this constraint, another two dimensions are added to the formulation. The first dimension represents the number of starts remaining in the time period and is denoted by  $n$ .  $n$  runs from 0 to the maximum number of starts available. The second represents the level at which the plant is generating. This can take one of three possible values, corresponding to the case when the power station is off, when it is running at minimum stable generation or when it is running at full load. This dimension is necessary in order to keep track of changes in the state.

The stochastic dynamic program can be formulated to include both the environmental constraint and a limit of the number of starts as follows:



$$f_t(j, k, n, \text{OFF}) = \max \begin{cases} e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{OFF}) \\ r_t^j(\text{MSG}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n-1, \text{MSG}) \\ r_t^j(\text{FL}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n-1, \text{FL}) \end{cases}$$

$$f_t(j, k, n, \text{MSG}) = \max \begin{cases} e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{OFF}) \\ r_t^j(\text{MSG}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{MSG}) \\ r_t^j(\text{FL}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{FL}) \end{cases}$$

$$f_t(j, k, n, \text{FL}) = \max \begin{cases} e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{OFF}) \\ r_t^j(\text{MSG}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{MSG}) \\ r_t^j(\text{FL}) + e^{-r\Delta t} \sum_m p_t(j, m) f_{t+1}(m, k, n, \text{FL}) \end{cases}$$

$$f_{T+1}(j, k, n, \text{OFF}) = 0, \forall j, k \text{ and } n$$

$$f_{T+1}(j, k, n, \text{MSG}) = 0, \forall j, k \text{ and } n$$

$$f_{T+1}(j, k, n, \text{FL}) = 0, \forall j, k \text{ and } n.$$

The following notation is used:

- $j, m$  — indices of price levels
- $f_t(j, k, n, x)$  — maximum expected value of tolling deal in time periods  $t, \dots, T$  given current price level is  $j$ , there are  $k$  days remaining on which the plant can run at full load, there are  $n$  starts remaining and the plant is currently generating at level  $x$
- $r_t^j(x)$  — profit from generating at level  $x$  at time  $t$  and price level  $j$
- $p_t(j, m)$  — transition probability of going from price level  $j$  at time  $t$  to price level  $m$  at time  $t+1$

The reward function,  $r_t^j(x)$  is given by the following formula:

$$r_t^j(x) = (s_t(j) - c_t(j))x - d_t(x),$$

where  $s_t(j)$  is the value from the spot price tree at time  $t$  and price level  $j$ ,  $c_t(j)$  is the contract price at time  $t$  and price level  $j$  and  $d_t(x)$  is the cost of generating  $x$  units at time  $t$ .

It is necessary to have three optimisations at every point, in order to keep track of the level at which the plant is generating. This means that three different problems must be solved at each stage. This allows the constraint on the number of starts to be incorporated. The aim is to find:

$$\begin{aligned} & \max_{0 \leq k \leq k_{\max}, 0 \leq n \leq n_{\max}} f_1(1, k, n, \text{OFF}), \\ & \max_{0 \leq k \leq k_{\max}, 0 \leq n \leq n_{\max}} f_1(1, k, n, \text{MSG}), \\ & \max_{0 \leq k \leq k_{\max}, 0 \leq n \leq n_{\max}} f_1(1, k, n, \text{FL}). \end{aligned}$$

This will give the maximum expected values when starting in each of the three states.

### 8.4.3 Case Study

The constants incorporated into the deterministic models in Section 8.3.4 were again used in the construction of the stochastic models.

The stochastic dynamic programming model was constructed in stages, increasing in complexity at each one. The first stage was to formulate a simple model which has the option at every time period of running at the minimum stable generation level or at full load. The strike price is set equal to zero in order to allow comparisons to be made. This stochastic model can again be checked for accuracy by setting the volatility equal to zero. This prevents the price from fluctuating at each time period and is equivalent to a deterministic dynamic programming model. The result should be approximately equal to the result from the linear programming model, given in Section 8.3.2, of the same problem. The following results were obtained:

Linear Programming Approach = 1,749,797  
Dynamic Programming Approach = 1,749,803  
Percentage Difference = 0.00038%.

The next stage was to add different efficiencies into the formulation for when the power plant was running at the minimum stable generation level and at full load. These efficiencies were obtained for an actual power station:

Efficiency at MSG = 51.5%  
Efficiency at FL = 54.1%

These efficiencies cannot be incorporated into a linear program. Incorporation would lead to non-linear constraints with much increased solution times. This is not the case with the intuitive model, in which they can be readily included. With the volatility of the stochastic dynamic program equal to 0%, the results of the comparison are:

Intuitive Approach = 3,134,004  
Dynamic Programming Approach = 3,132,100  
Percentage Difference = 0.06%.

The model was then adapted to include the environmental constraint. The problem was formulated for a coal-fired power station which has a limit on the amount of sulphur dioxide which is released in the year. The power station produces 90 tonnes of SO<sub>2</sub> per day when running at full load and 55 tonnes per day when running at the minimum stable generation level. The generating plant cannot produce more than 21,700 tonnes of SO<sub>2</sub> per year.

The stochastic dynamic programming model can be checked against the linear program by omitting the efficiencies from the formulation and adapting it to include the environmental constraint. The volatility remains equal to zero, creating a dynamic program. The results of this comparison are:

Linear Programming Approach = 1,332,005  
Dynamic Programming Approach = 1,332,010  
Percentage Difference = 0.00037%.

These results are sufficiently close to assume that the stochastic dynamic program is formulated correctly. The final stage in the development of the model is the inclusion of the constraint on the number of starts that can be made. Due to the limitations on the size of the problem that can be solved using Visual Basic for Applications, the entire problem cannot be implemented. It is possible, however, to construct the model by omitting the environmental constraints. As the model is complicated, a check can be made by setting the strike price to zero, omitting the efficiencies and restricting the plant to running at either the minimum stable generation level or full load.

The resulting model can act as a dynamic program when the volatility is set to zero. The value obtained can be compared with the value from solving the linear program (with integer constraints). The results from this comparison are:

Linear Program (with Integer Constraints) Value = 1,930,010  
Dynamic Program Value = 1,945,240  
Percentage Difference = 0.78%.

Again, it can be assumed that the model is formulated correctly. The volatility of the stochastic dynamic programs can now be set to 100%. This allows the full spot price tree to be used when valuing tolling deals. Both full stochastic dynamic programming models can then be solved. The results from valuing each type of tolling deal are:

Model which Incorporates Environmental Constraints = 2,749,970  
Model which Includes a Limit on the Number of Starts = 2,194,594.

The models developed in this chapter provide methods by which to value various types of tolling deal. As there are many constraints which can be present in such contracts, the inclusion of all of them produces a model which is extremely difficult to solve. Therefore, the models can be adapted to include the constraints which are present in a certain contract or those deemed to be most important. The models provide indications of the value which can be placed on tolling deals and the decisions which should be made. Again, the strong mathematical foundations are likely to inspire confidence in the results produced.



# Chapter 9

## Conclusions and Further Research

### 9.1 Introduction

The main aims of the thesis are to consider modelling developments arising out of the recent changes which have taken place in the energy industries and to apply mathematical programming techniques to solve the resulting problems. This chapter identifies the major areas which have been covered in this thesis. A summary of each of the chapters is given and conclusions are made. A discussion of various ways that the research can be extended concludes the thesis.

### 9.2 Conclusions

This thesis has highlighted various areas which are important after the restructuring of the energy industries. These areas were identified in conjunction with Innogy plc. Mathematical programming methods for valuing financial contracts are proposed. The techniques used can also be adapted to model other types of financial contract. The interest in valuing contracts using these techniques is relatively recent and is due to the recent restructuring of the industries. An increase in research in the area of valuation of contracts and assets has been observed as market participants need to develop new techniques to assist in the new situations.

A general model is detailed in Chapter 5 which describes the position which market participants now find themselves in. The model aims to maximise the value resulting from the coordination of a portfolio of contracts and assets. This model was seen to be too large and too complex to apply to this situation, as it requires the optimisation of more than one contract and asset. Therefore, there was seen to be a need to investigate each of the individual contracts and assets. The main areas were identified to be: gas storage, take or pay contracts and tolling deals. Together, these areas represent the majority of all traded complex contracts in energy industries.

Mathematical programming models were constructed for each of these areas in a number of stages. Each approach begins with a simple model which gradually increases in complexity. This allows one stage to be used as a check for the following stages, resulting in a validated model for the problem. This is necessary in order to demonstrate that each of the models remains accurate as it becomes more complex. Different features were incorporated into the models to capture elements of the real-world problem which can not be included in simpler models. The complex models should then give different values. These values are not necessarily higher as the added features may have a detrimental effect. Mathematical programming methods were chosen as they have been extensively applied to energy industries in the past.

Accurate price data are required in order to place a representative value on the contract. Prices are not known with certainty on any day in the future, therefore they can be modelled as stochastic variables. The value of these uncertain variables can be incorporated into the models by means of a scenario tree which has been built by adapting the methods described by Hull and White [34], and Clewlow and Strickland [19]. The stochastic model to value a tolling deal requires a scenario tree which includes two correlated variables. This was achieved by adapting a technique described by Hull and White [35].

In Chapter 2, the reasons behind the recent restructuring of the energy industries are described, together with details of the changes this restructuring has brought. These changes have altered the focus of market participants, who

must now develop new methods and techniques to model different contracts and situations. Generators no longer have an assured market; they must compete for a share of an increasingly competitive market.

The proposals of the European Commission for a fully competitive and liberal trans-European energy network and market are described. These changes will permit any company to buy and sell energy in any other country belonging to the European Union. This demonstrates the direction in which the industries may be headed in the future. Companies may wish to ensure that their models and techniques can be easily adapted in order to apply them to other European countries. A background on the electricity and gas industries in England and Wales is given. The industries in England and Wales were selected as this market is highly developed and this is the area in which the research will be applied. The recent changes in the energy industries have led to a need for the development of models to value new types of contract that have been and are being introduced.

In Chapter 3, financial terms used throughout the remainder of the thesis are defined. This leads to a description of common financial models: the Black Scholes model, the Black model and the Margrabe model. These models can be applied to place a value on simple financial contracts, such as options and forwards. These models are used to check the values produced by the trees which are constructed to incorporate a variety of prices into the stochastic models later in the thesis.

Various characteristics which are present when examining a series of energy prices are described. These include large jumps, mean reversion and seasonality. Energy prices can be modelled using a stochastic process. Various methods for modelling such processes are detailed. This chapter aims to give a background of financial terms and models which are commonly found in the energy industries.

In Chapter 4, a description of mathematical programming solution techniques used in the remainder of the thesis is given. Deterministic approaches described, in which the values of all variables are known with certainty throughout the time period, can be modelled using linear, integer, non-linear and dynamic



programming. Stochastic models include variables with uncertain values and can be solved using techniques such as stochastic linear and stochastic dynamic programming.

Uncertain variables can be incorporated into such models by means of a scenario tree. There are a number of techniques for generating these scenario trees. Two different methods are described which are commonly used for this purpose. The first of these approaches uses a given market forward curve to generate a tree of spot prices which is consistent with the values present in the curve. This technique is consistent with the majority of pricing techniques currently used in energy markets.

The second approach involves using statistics which have been calculated for a historical data set. The values at the nodes of the tree, together with the probabilities for each branch, are determined by solving a non-linear program to minimise the square distance between the calculated statistics and these variables. This technique requires a large amount of data in order to accurately predict these values. This chapter aims to give an overview of the solution techniques which are used in the remainder of the thesis. It is written so as to be accessible to people with little knowledge of these techniques.

Chapter 5, contains a description of literature describing the ways in which mathematical programming techniques have been applied in the energy industries. The literature describes situations which have arisen after the recent restructuring of the energy industries. It can be broadly divided into two main areas: the optimisation of profits which can be obtained by market participants in a pool-type industry structure and the optimisation of a portfolio of contracts and assets. Many countries have adopted the pool-type industry structure and research into this situation can be adapted and applied to each. Portfolio optimisation is important since the majority of market participants must now coordinate their assets and contracts in order to meet their obligations and maximise all possible profits.

A general stochastic linear program is presented within this chapter, which aims to model the situation in which a participant is trading in an industry with a pool structure. A portfolio of contracts and assets are held which are to be coordinated and optimised. The assets could take the form of power stations or gas storage facilities described in Chapter 6, whereas the contracts could be take or pay contracts described in Chapter 7, or tolling deals given in Chapter 8. This model was initially based on ideas presented by Fleten et al. [28] and Mulvey and Ziemba [46]. The model aims to maximise the profits resulting from the portfolio in order to meet demand. This model is large and reasonably complex and has not been implemented since more benefits can be obtained from investigating individual contracts and assets.

The gas storage problem, described in Chapter 6 is one such asset. This chapter places a value on an amount of storage which has been booked in a gas storage facility. The model was constructed in a number of stages, allowing validation at each stage to take place. The model building process begins with an intuitive model on a basic level. This leads to a linear programming approach. The sensitivity of these models to different scenarios lead to the development of a stochastic dynamic programming model.

Stochastic prices are incorporated by means of a spot price tree which is used in conjunction with a stochastic dynamic programming approach. The resulting model can be checked by setting the volatility to zero. This allows no variation in gas prices to occur and therefore the price on each day is equal to that in the forward price curve. This results in a dynamic programming model which gives an answer which is close to the value of the linear programming model. The models developed give indications of the days on which gas should be injected or extracted from the storage facility in order to obtain the maximum expected value from the facility. The models were solved using values for an actual storage facility. The stochastic dynamic program was shown to give a higher value.

Chapter 7 describes various properties of take or pay contracts, which account for approximately 80% of all traded complex contracts. These contracts give the holder the right to vary the amount of gas taken on a particular day within

certain limits. Bounds are also specified on the annual amount which can be taken. This chapter develops a model, which includes stochastic prices, to place a value on this type of contract. The model was constructed in a number of stages in order to allow validation, in a similar way to that constructed for the gas storage problem in the previous chapter.

Intuitive and linear programming approaches are described first. A semi-analytic linear programming model was developed to try to take account of the optionality present in the take or pay contract. This approach was shown to neglect some of this optionality which leads to the development of a stochastic dynamic programming model. The stochastic variables are incorporated into the stochastic dynamic programming model using the tree approach described in Section 4.3.1. The models were solved using values which describe a typical take or pay contract. When the results were compared, the stochastic dynamic programming model was shown to give a higher value.

Tolling deals are described in Chapter 8. This type of contract is becoming more common in the energy industries. It allows the holder to convert the price of fuel into the price of power, subject to certain constraints. The aim was to develop a stochastic model of this situation which maximises the value of the contract when a suggested amount of power is taken at each stage. Tolling deals are linked to power stations, therefore in this type of contract the amount to be taken corresponds to the level at which the power station should generate. Models to value this type of contract were again constructed in stages to allow each stage to be checked for accuracy before proceeding. The approach begins with an intuitive model, which leads to a linear programming model. Both the intuitive and linear programming approaches neglect any constraints that are to be modelled. The linear program was adapted to incorporate a limit on the number of starts which can be made. This is achieved by including constraints featuring integer variables.

Uncertain variables were incorporated into a stochastic model. For this type of contract, two sets of energy prices are required to be taken into account; those for fuel and those for energy. This leads to two sets of stochastic variables which

are correlated. The procedure for constructing the spot price tree needs to be adapted in order to take the correlation into account. This can be achieved by adapting a technique described by Hull and White [35]. The resulting spot price tree can be incorporated into the stochastic dynamic program in a similar way as in previous chapters.

A number of factors were deemed to be important when modelling tolling deals. The stochastic dynamic programming model incorporates the following:

- Different efficiencies at different operating levels.
- Environmental limits on the amount of harmful gases produced.
- A limit on the number of starts which the power station can make.

Implementation of the model required these factors to be included in stages to enable each resulting model to be checked for accuracy. This chapter results in two fully implemented tolling deal models with stochastic prices which have been validated at a number of stages.

The stochastic dynamic programming models presented in Chapters 6, 7 and 8 give detailed models by which to value contracts and assets for a variety of situations. These models provide a way of estimating the value of certain contracts and assets based on strong mathematical theory. These models are a definite improvement on models currently being employed and represent an appropriate method for valuing a variety of different situations. The inclusion of stochastic prices into the formulation allows the model to take account of a variety of possible price paths. This provides a more accurate representation of future prices than a single scenario.

### 9.3 Further Research

The models developed can be extended to incorporate a number of other features. Those incorporated will depend upon which are deemed to be the most important at the time. Examples of factors which could be included are:

- **Maintenance Days.**  
Power plants and storage facilities have days on which they are shut down or not available in order to allow maintenance or repairs to be carried out.
- **Unplanned Outages.**  
Breakdowns or problems can lead to the power plant being shut down unexpectedly. Allowances should be made for this type of issue.
- **Underdeliveries and Overdeliveries.**  
Although a certain amount of commodity is taken on a certain day, it is possible that more or less commodity will be delivered. This leads to problems with storage or unsatisfied demand.

Each of the models described relies heavily on accurate price data. In order to improve these models, the underlying price process can be examined to determine whether a different process can represent the market in a more accurate way. The spot price tree construction method can be adapted to incorporate this new process. Both the take or pay and the tolling models assume, for simplicity, that the price of buying and selling energy is equal and can be represented by the mid price. This was necessary to reduce the number of variables in each of the models in order to eliminate a portion of the complexity, but both bid and offer prices can be incorporated in order to give greater accuracy.

Finally, the techniques described here can be adapted to model and value other types of complex contract which may arise, not just in the energy industries, but also in other commodity markets.

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