Some Problems in the Invariant Theory of Parabolic Geometries

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Declaration

The material contained within this work is original, except where explicitly mentioned to the contrary. This thesis has been composed by myself.

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Abstract

The methods of Bailey, Eastwood and Graham for the parabolic invariant theory of conformal geometry are adapted to study the conformal polynomial invariants in the jets of differential forms, with analogous results being obtained. The methods of Bailey and Gover are then used to give the 'exceptional invariants'. These methods are extended to a different problem—that of the polynomial invariants in the jets of curves at a point, yielding complete results for a particular class of invariants.

A construction was given by Graham, Jenne, Mason and Sparling of a set of conformally invariant, linear differential operators with leading term a power of the Laplacian, on general conformal manifolds. Their method involves the use of the 'ambient metric' construction. We give an alternative construction of most of these operators, using an invariant operator on the 'tractor bundle,' and describe the relationship between the tractor bundle and the ambient construction. We also relate these ideas to methods used by Wünsch to find some conformally invariant powers of the Laplacian.

We introduce another parabolic geometry, not appearing previously in the literature, which we call contact-projective geometry. The flat model is $\mathrm{Sp}(2n+2,\mathbb{R})$ acting on $\mathbb{P}^{2n+1}$. The invariants of positively homogeneous functions on the flat model are studied, using methods similar to those of the conformal case. We suggest a curved version of this geometry and describe the form of a tractor bundle—a vector bundle with connection and a skew-symmetric bilinear form; and an ambient space—an affine manifold of one higher dimension equipped with a symplectic form.

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Chapter 1

Introduction

1.1 Invariants in Geometry

In this thesis, we will be looking at the following kinds of invariant theory problems:

- Looking for all G-equivariant differential operators between sections of homogeneous bundles on a homogeneous space G/H, where G is a Lie group and H is a closed Lie subgroup.

- Finding polynomial maps in the jets of curves on G/H which either simply rescale or are invariant under the G-action.

- Looking for all local scalar invariants of a geometric structure whose flat model is such a homogeneous space.

- Looking for "curved analogues" of the invariant differential operators mentioned above i.e. invariant operators on a curved geometric structure with the same leading terms as the invariant operators on the homogeneous space and which reduce to the same operator in the flat case.

The flat models of the geometries we will be looking at all have G a semi-simple Lie group and H a parabolic Lie subgroup. However, to illustrate each of these invariant theory problems, we will show how they can be solved in the simpler case of Euclidean space, which is the flat model for Riemannian geometry. In this case, we have $\mathbb{R}^n = G/H$ where G is the group of Euclidean motions and $H = O(n)$.

1.1.1 Invariant Differential Operators

Consider the problem of finding all Euclidean invariant differential operators (not necessarily linear) from functions to functions.

Proposition 1.1 *A G-invariant differential operator on functions on $\mathbb{R}^n$ is equivalent to an H-invariant polynomial in the jets of functions at the origin.*

Proof. First, note that an invariant differential operator is equivalent to a G-invariant polynomial on the total space of the jet bundle. Clearly, such a G-invariant polynomial defines, by restriction, an H-invariant polynomial in jets at the origin. To complete the proof, we need to show that an H-invariant polynomial on jets at the origin defines a G-invariant polynomial on the total space of the jet bundle.

Suppose f is a genuine function on $\mathbb{R}^n$, rather than just a jet. Since G is transitive, for any point, $x \in \mathbb{R}^n$, there exists $g \in G$ such that $x = g \cdot 0$. Then for an H-invariant I we may define an operator $\tilde{I}$ by setting $\tilde{I}f(x) = I(g^{-1} \cdot f)$. If $\hat{g} \cdot 0 = g \cdot 0$, then $g^{-1}\hat{g} \in H$, so

$$I(\hat{g}^{-1} \cdot f) = I((g^{-1}\hat{g})^{-1}g^{-1} \cdot f) = (g^{-1}\hat{g})^{-1} \cdot I(g^{-1} \cdot f) = I(g^{-1} \cdot f),$$

thus $\tilde{I}f$ is well-defined. Similarly, one can check that $\tilde{I}$ is invariant under the G-action. $\square$

We find these H-invariant polynomials as follows. Taking the jet of a function, f, at the origin gives us a list of symmetric tensors,

$$(f, \nabla f, \nabla^{(2)} f, \nabla^{(3)} f, \ldots)\big|_{x=0}, \qquad \nabla^{(k)} f \in \bigodot^k \mathbb{R}^{n*}$$

where $\nabla^{(r)} f$ denotes $\overbrace{\partial_i \ldots \partial_k}^{r} f$, and we write ∂_i for $\frac{\partial}{\partial x^i}$. The group $H = O(n)$, acts upon each tensor in this list in the usual way. Our problem is therefore reduced to that of finding the $O(n)$ invariant polynomials on the module whose elements are such lists of tensors.

The solution to this problem is then given by Weyl's theory for the orthogonal group (see §3.1 and [We])—every invariant polynomial in the jet of a function, f, is a linear combination of complete contractions of derivatives of f with the metric tensor (for the notion of complete contractions, see chapter 3). In particular, we see that every invariant linear differential operator from functions to functions is a linear combination of powers of the Laplacian.

1.1.2 Invariants of Curves

We look for all Euclidean invariant polynomials in the jets of curves in $\mathbb{R}^n$. Again, the transitivity of G allows us to consider only H-invariants at the origin. We can assume that a curve, $x(t)$, is parametrised such that $x(0) = 0$. Then taking the jet of such a curve gives a list of vectors,

$$(x_1, x_2, x_3, \ldots),$$

where $x_r = \left.\frac{d^r x}{dt^r}\right|_{t=0}$. The group $O(n)$ acts on each vector in this list in the usual way, and so Weyl's theory tells us that every invariant is a polynomial in the dot products $\{x_r \cdot x_s\}$.

1.1.3 Riemannian Invariants

We look for the local invariants of a Riemannian structure i.e. polynomials in the components of the metric tensor, together with its inverse and derivatives which are invariant under a change of coordinates. The first step in solving this problem is to reduce to an algebraic problem in the invariant theory of $O(n)$. We do this as follows:

1. As we are only working locally and as we have translation invariance, we need only look at invariants in the jets of metrics and their inverses at the origin in $\mathbb{R}^n$, under the action of jets of origin preserving changes of coordinates.

2. By using normal coordinates, the problem is reduced to that of the invariants of jets at the origin of metrics in normal coordinates and their inverses, under the (non-linear) action of the orthogonal group.

3. The final stage is non-trivial, and involves finding a one to one (non-linear) correspondence between the jets of metrics in normal coordinates and the Riemann curvature tensor and its covariant derivatives.

One thereby obtains the usual action of $O(n)$ on each term of a list of tensors,

$$(R, \nabla R, \nabla^{(2)}R, \nabla^{(3)}R, \ldots)$$

where $\nabla^{(r)}R$ denotes the r-th covariant derivative of the Riemann curvature tensor, evaluated at the origin. The invariants of the module of such lists of tensors are then given by Weyl's theory—all such invariants are linear combinations of complete contractions of the Riemann curvature tensor and its covariant derivatives with the metric tensor.

1.1.4 Some Invariant Operators in Riemannian Geometry

Given one of the differential operators of §1.1.1, we look for an invariant differential operator on a Riemannian manifold with the same leading order term, which reduces to our original operator in the flat case. Here, we can construct such operators simply by replacing the coordinate derivatives in the original expression with the Levi-Civita connection. For example, the Laplacian constructed from the Levi-Civita connection is a curved analogue of the usual Laplacian on $\mathbb{R}^n$. Note that these curved analogues are not unique—adding any scalar Riemannian structural invariant yields another curved analogue.

1.2 Parabolic Geometries

A *parabolic geometry* is one for which the flat model is a homogeneous space, G/P, where G is a semi-simple Lie group and P is a parabolic subgroup. For the definition of a parabolic subgroup, see for example, [FH]. In this thesis, we will only be looking at the following examples of parabolic geometries:

- A *conformal structure* on a manifold, $\mathcal{M}^n$, is an equivalence class of Riemannian metrics, $[g_{ij}]$, where $g_{ij} \sim \hat{g}_{ij}$ if $\hat{g}_{ij} = \Omega^2 g_{ij}$, for some nowhere vanishing smooth function Ω. Let G denote the identity connected component of $O(n+1,1)$, which preserves a symmetric bilinear form, $\tilde{g}$, on $\mathbb{R}^{n+2}$, and let P be the subgroup of G preserving a chosen null vector, $e_0 \in \mathbb{R}^{n+2}$, up to positive scale. Then $S^n = G/P$ is the flat model for conformal geometry. (See chapter 2).

- A *projective structure* on a manifold is an equivalence class of torsion free affine connections, where two connections are equivalent if they have the same geodesics, considered as unparametrised curves. Let G denote the group $PSL(n+1,\mathbb{R})$, which acts on $\mathbb{R}^{n+1}$ in the usual way, and let P be the subgroup preserving some point $e_0 \in \mathbb{R}^{n+1} \setminus \{0\}$ up to scale. Then $\mathbb{P}^n = G/P$ is the flat model for projective structures. (See chapter 6).

- A (strictly pseudoconvex) *CR structure* is the structure that arises as the natural geometry of the boundary of a smooth strictly pseudoconvex domain in $\mathbb{C}^n$. (See, e.g. [F] for more details). The flat model is $S^{2n+1} = G/P$ where $G = SU(n+1,1) \subset SL(n+2,\mathbb{C})$ and P is the subgroup of G preserving some null vector in $\mathbb{C}^{n+2}$ up to complex scale.

- A *contact-projective* structure on a manifold, $\mathcal{M}^{2n+1}$, is a projective structure, together with a "compatible" contact structure (see definition 7.3). The flat model is $\mathbb{P}^{2n+1} = G/P$, where $G = Sp(2n+2,\mathbb{R})$ preserves a non-degenerate skew-symmetric form on $\mathbb{R}^{2n+2}$, and P is the subgroup preserving some $e_0 \neq 0$ in $\mathbb{R}^{2n+2}$ up to scale.

In the examples for the Euclidean/Riemannian case, we were led to problems in the invariant theory of the orthogonal group. Similarly, in the case of parabolic geometries, one is led to a problem in the invariant theory of the parabolic, P. The first difficulty one encounters, in this case, is that it is not usually obvious how to construct *any* invariants. Even when one is able to construct some invariants, it is not easy to prove that one has all the invariants. The reason for the increase in the level of difficulty is that we no longer have complete reducibility of finite dimensional representations.

1.3 History

There is a long history associated to the study of differential invariants—see, for example, [T], for projective and conformal invariants. The branch of developments to which this thesis belongs originates in Fefferman's paper, [F], of 1979. Fefferman showed that the coefficients of the asymptotic expansion of the Bergman kernel of a strictly pseudoconvex domain in $\mathbb{C}^n$ are invariants of the CR structure on the boundary, and was therefore led to trying to list these invariants.

In order to construct CR invariants, Fefferman proceeded as follows. To a CR structure, he was able to associate a manifold of 1 higher dimension, with a formally defined Kähler-Lorentz metric. (This is analogous to the Fefferman-Graham ambient metric construction which we will describe in §2.3). This construction is obstructed at finite order in all dimensions. The problem of finding invariants was reduced, non-trivially, to the problem of finding the invariant polynomials on a space of lists of tensors under the action of the parabolic subgroup, P, of $SU(n+1,1)$; these tensors being the derivatives at e_0 of a homogeneous function on $\mathbb{C}^{n+2}$. (See §7.6 for the description of an analogous module). Since the P action on each tensor is the restriction of the G action, one can construct invariants as linear combinations of complete contractions of these tensors with the preserved Hermitian form. The problem is to find whether all invariants arise in this way. Fefferman was able to make some progress in solving this problem.

Fefferman studied the invariant theory of a similar module under the action of a parabolic subgroup of $O(n+1,1)$ as a model problem, and obtained nearly complete results. Eastwood and Graham study this problem in [EGm] and show that it has applications in the conformally invariant differential operators on sections of certain homogeneous bundles on S^n (see also §3.3). However, although they made some progress, complete results in the invariant theory remained unforthcoming.

A breakthrough came when Gover ([Go1]) introduced new methods to completely solve the invariant theory of an analogous module under the action of a parabolic subgroup of $SL(n, \mathbb{R})$. Using Gover's methods, together with some new techniques, Bailey, Eastwood and Graham ([BEGm]) were able to solve the above model problem and Fefferman's original CR problem.

These methods can also be applied to the study the structural invariants of conformal and projective structures. A construction of invariants of conformal structures is given by Fefferman and Graham in [FGm1]. This is achieved by associating to a manifold, $\mathcal{M}^n$ with conformal structure, an $n+2$-dimensional manifold with a formally defined pseudo-Riemannian metric (see also §2.3). This construction is obstructed at finite order in even dimensions. The invariants of this Riemannian structure then yield invariants of the conformal structure. It was conjectured in [FGm1] that all invariants (in even dimensions, those below the order of obstruction) can be constructed in this way. The reduction of this problem to a problem in the invariant theory of a parabolic subgroup of $O(n+1, 1)$ is carried out in [FGm2]. This invariant theory problem is solved in [BEGm].

A complete account of the structural invariants problem in the projective case is given in [Go3].

1.4 Conformal Invariants

In chapter 2, we give a proof of the relationship between two constructions associated to conformal structures. In addition to the Fefferman-Graham ambient metric construction described above, there is another construction which enables one to find invariants of conformal structures, due essentially to T.Y.Thomas [T]. The tractor bundle on $\mathcal{M}^n$ is an $n+2$-dimensional vector bundle, equipped with a metric and connection, along with a D-operator—an invariant extension of the connection to the bundle (see §2.2 and [BEGo]). In even dimensions, although the ambient construction is obstructed at finite order, the tractor bundle remains well defined.

It has been observed by Graham (as mentioned in [BEGo]) that the conformal tractor bundle is essentially the tangent bundle of the ambient metric construction and that the tractor D-operator is closely related to the Levi-Civita connection on the ambient construction. Since no proof appears in the literature, we provide a proof of this fact in chapter 2. Also in this chapter, we describe some relationship (see §2.5.2) between the D-operator and methods used by Wünsch ([Wü]) for constructing conformally invariant differential operators.

1.4.1 Invariants of Curves

In chapter 4, we look at the problem of finding all polynomials in the jets of curves on S^n, the flat model for conformal geometry, which simply rescale under the G-action. This is a type of invariant theory problem that has not, to our knowledge, previously been studied in the case of parabolic geometries. As in §1.1.2, we need only consider polynomials in the jets of curves at $[e_0]$ under the action of P. We assume any curve, $x(t)$, has $x(0) = [e_0]$. Let σ_q denote the one dimensional representation of P, where an element, $p \in P$ such that $pe_0 = \lambda e_0$, acts by λ^{-q}. We are thus looking for P-equivariant polynomial maps from the space of jets of curves at $[e_0]$ to some σ_q.

Our method of constructing invariants is as follows. To any regular curve, $x(t)$, in S^n, we associate a unique lifting, $X(t)$, to the preserved null cone in $\mathbb{R}^{n+2}$ with unit tangent. The jet of $X(t)$ at $t = 0$ is a list of vectors,

$$(X_0, X_1, X_2, \ldots),$$

where $X_r = \frac{d^r X}{dt^r}\big|_{t=0} \in \mathbb{R}^{n+2}$. The group, P, acts on each vector in this list in the usual way and so any complete contraction formed from the X_r with the preserved metric tensor, $\tilde{g}$, and volume form, $\tilde{\epsilon}$, will be invariant under the P-action. Such a contraction will not, in general, be polynomial in the components of the derivatives of $x(t)$ and hence, not an invariant. However, linear combinations of complete contractions of the forms

$$u^q.\mathrm{contr}(\tilde{g} \otimes \ldots \otimes \tilde{g} \otimes X_{r_1} \otimes \ldots \otimes X_{r_d}) \tag{1.1}$$

$$\text{and} \qquad u^q.\mathrm{contr}(\tilde{\epsilon} \otimes \tilde{g} \otimes \ldots \otimes \tilde{g} \otimes X_{r_1} \otimes \ldots \otimes X_{r_d}), \tag{1.2}$$

where $u = (\dot{x}(0) \cdot \dot{x}(0))^{\frac{1}{2}}$, will be polynomial, for large enough q, and hence invariants. We say a function, f, on the space of jets is *time homogeneous of degree* k if $f(x(\lambda t)) = \lambda^k f(x(t))$ for $\lambda \in \mathbb{R}$. Since every invariant is the sum of invariant time homogeneous parts, we need consider only time homogeneous invariants. We also decompose invariants into *odd* and *even* parts, depending on the behaviour under orientation reversal. Our main result is the following:

Theorem 1.2

- *A spanning set for the vector space of even invariants, time homogeneous of degree k, taking values in σ_q, with $3q \geq 2k$ is given by those complete contractions of the form (1.1) with the appropriate homogeneity degree, with q even.*

- *A spanning set for the vector space of odd invariants, time homogeneous of degree k, taking values in σ_q, with $3q \geq 2k - n(n-2)$ is given by those complete contractions of the form (1.2) with the appropriate homogeneity degree, with $q + n$ even.*

1.4.2 Differential Invariants of Curves

In §4.4, we consider the case of (not necessarily polynomial) differential invariants of generic curves in S^n. Green, [Gn], shows how one can use Cartan's method of moving frames to give a set of generators for the algebra of these invariants. In $\mathbb{R}^3$, the Serret-Frenet formulae give the speed, curvature and torsion as invariants, and the algebra of invariants is generated by these invariants and their derivatives. We give an analogue of the Serret-Frenet formulae for flat conformal geometry, which gives a direct method for the computation of these moving frames, and hence, invariants. Using the method introduced in [BEGo] of associating a list of sections of the tractor bundle to a regular curve on a conformal manifold, we describe the construction of curved analogues of both these invariants and the polynomial invariants above.

1.4.3 Invariants of Differential Forms

In chapter 5, we use the methods of [BEGm] to study conformally invariant differential operators on closed k-forms on S^n. We consider the case $1 \leq k \leq n-1$—the case $k = 0$ is covered in [BEGm]. Since P fixes $[e_0]$ and since d is an invariant operator on k-forms of weight 0, we can define the P-modules,

$$\mathcal{J}_k = \{ \text{ jets at } [e_0] \text{ of closed } k\text{-forms on } S^n, \text{ of conformal weight } 0 \ \}.$$

Rather than studying this module directly, we will study the invariant theory of the P-module,

$$J_k = \left\{ \begin{array}{l} \text{jets at } e_0 \text{ of functions on } W \text{ taking values in } \bigwedge^k W^*, \text{ homogeneous} \\ \text{of degree } -k, \text{ satisfying } df = 0,\ X \lrcorner f = 0 \text{ and } \Delta f = 0. \end{array} \right\},$$

where X denotes the position vector and W denotes the standard representation, $\mathbb{R}^{n+2}$, of G, with other notation as defined in §5.1. The reason that this is of interest is the following:

Theorem 1.3 *For $1 \leq k \leq n-1$ and n odd, $J_k \cong \mathcal{J}_k$ as P-modules.*

An *invariant* of J_k is a P-equivariant polynomial map, $I : J_k \to \sigma_q$. Any complete contraction of the derivatives in $\mathbb{R}^{n+2}$ of an element, f, of J_k at e_0 with $\tilde{g}$ and $\tilde{\epsilon}$ will be an invariant. We define a *Weyl invariant* to be a linear combination of such contractions each taking values in the same σ_q. An invariant which can not be written as a Weyl invariant is said to be an *exceptional invariant.* Any invariant is the sum of invariant homogeneous parts, so we need only consider homogeneous invariants. The total homogeneity degree will be denoted by d. We are able to find a complete set of generators for the invariants of J_k:

Theorem 1.4

- *Every even invariant of J_k is a Weyl invariant.*
- *There are no odd invariants of J_k of degree $d < n/k$*
  Every invariant of degree n/k is exceptional—up to scale, there is at most one such invariant.
  Every odd invariant of degree $d \geq \frac{n+1}{k}$ is a Weyl invariant.

1.4.4 Conformally Invariant Powers of the Laplacian

In section 2.6, we give explicit formulae for some curved analogues of powers of the Laplacian in conformal geometry, in terms of the tractor D-operator. Using the ambient metric construction, [GmJMS] gives an algorithmic method of constructing conformally invariant linear differential operators with the m-th power of the Laplacian as the leading term, acting on functions of conformal weight $m - \frac{1}{2}n$, providing either n is odd or $n \geq 2m$. This result seems likely to be sharp—it is shown in [Gm] that if $n = 4$ there is no such curved analogue of the cube of the Laplacian. In theorem 2.22, we give a formula for a curved analogue of the m-th power of the Laplacian for n odd or $n > 2m$, using the tractor D-operator.

1.5 Contact-Projective Structures

There are a number of other parabolic geometries which have been studied, in addition to those already described, such as the "almost Hermitian symmetric" structures which appear in [Ba]. In chapter 7, we introduce a further example, that of contact-projective structures, which, to our knowledge, has not previously been studied. The flat model is as described in §1.2.

We describe some features of the curved geometry and show that the tractor bundle and ambient construction can be described in terms of extra structure on the corresponding constructions from the projective case.

We then go on to study the invariant theory of the following module, using the methods of [BEGm]: let

$$\mathcal{F}_k = \left\{ \begin{array}{l} \text{Jets at } e_0 \text{ of functions, } f, \text{ on } \mathbb{R}^{2n+2} \text{ which are positively} \\ \text{homogeneous of degree } k \text{ and which vanish to order } k+1 \text{ at } e_0. \end{array} \right\}$$

In this case, Weyl invariants are linear combinations of complete contractions of the derivatives of f at e_0 with the preserved symplectic form, each transforming by the same character of P.

Theorem 1.5 *Every invariant of degree $d < 2n$ or $d > 2n+1$ is a Weyl invariant.*

The cases of degree $d = 2n, 2n+1$ remain open—although it is clear that non-zero Weyl invariants do exist in these degrees, we are able to give an example of an exceptional invariant.

The module, $\mathcal{F}_k$, is the module of jets of sections of a homogeneous line bundle on $\mathbb{P}^{2n+1}$, modulo the kernel of an invariant differential operator. Thus an invariant of $\mathcal{F}_k$ is an $\mathrm{Sp}(2n+2, \mathbb{R})$-invariant differential operator depending only on derivatives of such sections lying in the complement to this kernel. The above theorem gives a construction for almost all of these operators.

Chapter 2

Conformal Geometry

A conformal manifold is a smooth manifold $\mathcal{M}^n$, with an equivalence class of Riemannian metrics $[g_{ij}]$, (the conformal structure), where $g_{ij} \sim \hat{g}_{ij}$ if and only if $g_{ij} = \Omega^2 \hat{g}_{ij}$, where Ω is a smooth positive valued function. We will assume that the dimension $n \geq 3$. (Note that we use the abstract index notation, as in [PR], throughout).

We start off with the flat model for conformal geometry, i.e. a model for those conformal structures which have a Euclidean metric in the conformal class. We then introduce some notation for curved conformal structures, and give a brief introduction to two structures associated to curved conformal structures: the conformal tractor bundle—an $n+2$-dimensional vector bundle with metric and connection (see [BEGo]); and the Fefferman-Graham ambient metric construction—a formal power series construction of an $n+2$-dimensional pseudo-Riemannian manifold (see [FGm1]) – and explain the relationship between the two.

We then introduce the methods of Wünsch and Günther (see [GüWü]) for the study of conformally invariant differential operators, and show some relationship with the tractor calculus. Finally, in §2.6, we give a formula using the tractor D-operator for a curved analogue of the k-th power of the Laplacian for n odd or $1 \leq k < n/2$.

2.1 The Flat Model

We describe the flat model for conformal geometry (see, for example, [PR]). Take $\mathbb{R}^{n+2}$ with coordinates

$$X^I = \begin{pmatrix} X^0 \\ X^i \\ X^\infty \end{pmatrix} \qquad i = 1, \ldots, n. \tag{2.1}$$

For any fixed, positive definite, bilinear form, (g_{ij}), on $\mathbb{R}^n$, we define the metric, $\tilde{g}_{IJ}$ on $\mathbb{R}^{n+2}$ with signature $(n+1,1)$, as having the matrix given in block form by

$$(\tilde{g}_{IJ}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & g_{ij} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The null cone of $\tilde{g}_{IJ}$, is $\mathcal{Q} = \{X^I : \tilde{g}_{IJ}X^IX^J = 0\}$ (we use the summation convention throughout). S^n is the space of generators of $\mathcal{Q}$, and we use as coordinates $\mathbb{R}^n \to S^n$,

$$x^i \mapsto \left[\begin{pmatrix} 1 \\ x^i \\ -\frac{1}{2}x^2 \end{pmatrix}\right]$$

where x^2 denotes $x^i x_i$. To tie this in with the conformal class of g_{ij}, notice that, if we use as coordinates for $\mathcal{Q}$, away from the origin,

$$(t, x^i) \mapsto t\begin{pmatrix} 1 \\ x^i \\ -\frac{1}{2}x^2 \end{pmatrix}, \tag{2.2}$$

the restriction of $\tilde{g}_{IJ}dX^IdX^J$ is $t^2 g_{ij}dx^idx^j$.

Let G denote the identity connected component of $O(\tilde{g})$. From the above discussion, we see that the usual action of G on $\mathbb{R}^{n+2}$ preserves $\mathcal{Q}$ and $[g_{ij}]$ and thus acts on S^n by conformal automorphisms. In fact, G acts transitively, and is the full group of conformal motions of S^n (see, e.g. [E]). Fix a point, e_0, with coordinates $e_0^I = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{Q}$, and let P be the parabolic subgroup of G given by $P = \{p \in G : pe_0 = \lambda e_0,\ \lambda > 0\}$. Explicitly,

$$P = \left\{ \begin{pmatrix} \lambda & r_j & t \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix} : \lambda > 0,\ m^i{}_j \in SO(g),\ t = -\frac{1}{2\lambda} r_j r^j,\ s^i = -\frac{1}{\lambda} m^{ij} r_j \right\}.$$

S^n can be identified with G/P. P acts on S^n as the group of all orientation preserving conformal transformations of $\mathcal{Q}$ fixing $[e_0] \in S^n$.

2.1.1 Notation for Curved Conformal Structures

For curved conformal structures, we fix a metric, g_{ij}, and denote its Levi-Civita connection by ∇_i. We define the Riemann curvature tensor, $R_{ij}{}^k{}_l$, by

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) U^k = R_{ij}{}^k{}_l U^l$$

We define the Ricci tensor, R_{ij}, by $R_{jl} = R_{kj}{}^k{}_l$, and the scalar curvature, $R = R_k{}^k$. The rho-tensor, P_{ij}, is a trace adjusted multiple of the Ricci tensor, and is given by

$$P_{ij} = \frac{1}{n-2} \left(R_{ij} - \frac{R}{2(n-1)} g_{ij} \right).$$

The trace free part of the Riemann curvature tensor is the conformally invariant Weyl tensor, which can be expressed (see [Ei1], [BEGo] etc.) as

$$W_{ijkl} = R_{ijkl} - 2g_{k[i}P_{j]l} - 2g_{l[j}P_{i]k}.$$

Two other tensors which will be of interest are the Cotton-York tensor, C_{ijk}, and the Bach tensor, B_{ij}, which are given (see, e.g., [FGm1]) by

$$\begin{aligned} C_{jkl} &= \nabla_l P_{jk} - \nabla_k P_{jl} \\ \text{and} \qquad B_{jk} &= \nabla^l C_{jkl} - P^{il} W_{ijkl} \end{aligned}$$

When $n = 3$, the Weyl tensor vanishes, and the Cotton-York tensor is conformally invariant. When $n = 4$, the Bach tensor is conformally invariant.

2.2 The Conformal Tractor Bundle

One construction which will be of use to us is the conformal tractor bundle. This is based on the work of T.Y.Thomas (see, for example, [T]) which was developed in a modern form in [BEGo], whose presentation we will follow. The idea is to associate to an n-dimensional conformal manifold an $(n+2)$-dimensional vector bundle, with metric and connection. This bundle is related to another construction—one can construct a principal P-bundle as a sub-bundle of the bundle of 2-frames and define the Cartan conformal connection by means of a naturally defined form taking values in the Lie algebra of G (see [Ko] for further details). The tractor bundle is an associated vector bundle of this principal bundle, but we will define it directly.

Let $\mathcal{M}$ be an n-dimensional manifold with conformal structure, and let $\mathcal{E}^i$ denote the tangent bundle. We can describe a conformal structure by a global tensor field, g_{ij}, taking values in a line bundle, which we denote $\mathcal{E}[2]$. We choose a square root, $\mathcal{E}[1]$, for this bundle, and we say functions and tensors taking values in $\mathcal{E}[w]$ and $\mathcal{E}^i[w] = \mathcal{E}^i \otimes \mathcal{E}[w]$ etc. have conformal weight w.

A *conformal scale* is a nowhere vanishing local section, τ, of $\mathcal{E}[1]$. This defines a metric $\tau^{-2}g_{ij}$ in the conformal class. We choose a conformal scale, and work with the Levi-Civita connection, ∇, of this chosen metric. If we choose a new scale, $\Omega^{-1}\tau$, where Ω is a nowhere vanishing smooth function, the metric is rescaled by Ω^2, and the Levi-Civita connection, $\widehat{\nabla}$, of this new metric is given (see, e.g., [PR]), by

$$\begin{aligned} \widehat{\nabla}_i f &= \nabla_i f + w\Upsilon_i f \\ \widehat{\nabla}_i U^j &= \nabla_i U^j + (w+1)\Upsilon_i U^j - U_i \Upsilon^j + U^k \Upsilon_k \delta_i{}^j \\ \widehat{\nabla}_i \omega_j &= \nabla_i \omega_j + (w-1)\Upsilon_i \omega_j - \Upsilon_j \omega_i + \Upsilon^k \omega_k g_{ij} \end{aligned} \tag{2.3}$$

where $\Upsilon_i = \Omega^{-1}\nabla_i\Omega$, and f, U^i and ω_i are sections of $\mathcal{E}[w]$, $\mathcal{E}^i[w]$ and $\mathcal{E}_i[w]$ respectively. We will also need to know how the rho-tensor transforms under this rescaling :

$$\widehat{P}_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j - \tfrac{1}{2}\Upsilon^k \Upsilon_k g_{ij}. \tag{2.4}$$

We will use "hats" in this manner throughout to denote transformed quantities.

2.2.1 The Tractor Bundle

The tractor bundle, $\mathcal{E}^I$, for any given choice of conformal scale is identified with the direct sum

$$\mathcal{E}^I = \mathcal{E}[1] \oplus \mathcal{E}^i[-1] \oplus \mathcal{E}[-1],$$

and under change of scale, this identification transforms according to

$$\begin{pmatrix} \hat{U}^0 \\ \hat{U}^i \\ \hat{U}^\infty \end{pmatrix} = \begin{pmatrix} U^0 - \Upsilon_j U^j - \frac{1}{2}\Upsilon^2 U^\infty \\ U^i + \Upsilon^i U^\infty \\ U^\infty \end{pmatrix} \tag{2.5}$$

where U^0, U^i and U^∞ are sections of $\mathcal{E}[-1]$, $\mathcal{E}^i[-1]$ and $\mathcal{E}[1]$ respectively. One can easily check that such transformations obey a group law, so that $\mathcal{E}^I$ is well defined. For an alternative definition of the tractor bundle in terms of jets, see [BEGo].

2.2.2 The Tractor Metric

The tractor bundle $\mathcal{E}^I$ has a natural, non-degenerate, symmetric form, g_{IJ} which is defined invariantly by

$$g_{IJ}U^I V^J = g_{ij}U^i V^j + U^0 V^\infty + U^\infty V^0, \text{ where } U^I = \begin{pmatrix} U^0 \\ U^i \\ U^\infty \end{pmatrix}, \ V^I = \begin{pmatrix} V^0 \\ V^i \\ V^\infty \end{pmatrix}.$$

2.2.3 Projecting Parts

We adopt the notation suggested by Buchdahl whereby the existence of a short exact sequence

$$0 \to C \to A \to B \to 0$$

is indicated by writing $A = B+C$. This "+" is not commutative, but is associative, and so we can write, for example, $A = B + C + D$ without ambiguity.

From the above transformation law, U^∞ is conformally invariant. If U^∞ is zero, then U^i is conformally invariant, and so on. Thus, we can write

$$\mathcal{E}^I = \mathcal{E}[1] + \mathcal{E}^i[-1] + \mathcal{E}[-1],$$

and similarly for tensor powers of $\mathcal{E}^I$. For example,

$$\mathcal{E}^{IJ} = \mathcal{E}[2] \;+\; \begin{array}{c} \mathcal{E}^i \\ \oplus \\ \mathcal{E}^j \end{array} \;+\; \begin{array}{c} \mathcal{E} \\ \oplus \\ \mathcal{E}^{ij}[-2] \\ \oplus \\ \mathcal{E} \end{array} \;+\; \begin{array}{c} \mathcal{E}^i[-2] \\ \oplus \\ \mathcal{E}^j[-2] \end{array} \;+\; \mathcal{E}[-2]\,.$$

For U^0, U^i and U^∞ as above, we call U^∞ the primary part, U^i the secondary part and so on, similarly for tensor powers. The first non-zero part of a given 'tractor' is referred to as the projecting part.

There is a preferred section, X^I, of $\mathcal{E}^I[1]$, which in any choice of conformal scale is given by $X_I = (0\ 0\ 1)$. Clearly, $X^I X_I = 0$. As $U^\infty = U^I X_I$, this section gives, by contraction, a map, $\mathcal{E}^I \to \mathcal{E}[1]$, which maps a section, U^I, of $\mathcal{E}^I$ to its primary part. This section also supplies us with the invariant injection, $\mathcal{E}[-1] \to \mathcal{E}^I$, given by $U^0 \mapsto U^0 X^I$.

2.2.4 The Tractor Connection

The tractor bundle has a connection, which in any choice of conformal scale is defined by

$$\nabla_j \begin{pmatrix} U^0 \\ U^i \\ U^\infty \end{pmatrix} = \begin{pmatrix} \nabla_j U^0 - P_{ji} U^i \\ \nabla_j U^i + \delta_j{}^i U^0 + P_j{}^i U^\infty \\ \nabla_j U^\infty - U_j \end{pmatrix}. \tag{2.6}$$

We use ∇_i to denote the tractor connection on $\mathcal{E}^I$ and the induced connection on tensor powers of $\mathcal{E}^I$ as well as the Levi-Civita connection on conformally weighted tensor fields. One can check, from equations (2.3) and (2.4), that the tractor connection is conformally invariant on unweighted tensor powers of $\mathcal{E}^I$ and preserves the tractor metric i.e. $\nabla_i g_{JK} = 0$.

2.2.5 Tractor curvature

The tractor curvature, $\Omega_{ij}{}^{K}{}_{L}$, is defined by $(\nabla_i\nabla_j - \nabla_j\nabla_i)U^K = \Omega_{ij}{}^{K}{}_{L}U^L$. In a given scale, $\Omega_{ij}{}^{K}{}_{L}$ can be written in block matrix form as

$$\begin{pmatrix} 0 & C_{lij} & 0 \\ 0 & W_{ij}{}^{k}{}_{l} & -C^{k}{}_{ij} \\ 0 & 0 & 0 \end{pmatrix}$$

It is pointed out in [BEGo] that a conformal manifold is locally equivalent to the flat model if and only if the tractor curvature vanishes.

2.2.6 The D-operator

Let Δ denote the Laplacian, $\nabla^i\nabla_i$. The operator $D_I : \mathcal{E}[w] \to \mathcal{E}_I[w-1]$ is defined by

$$D^I f = \begin{pmatrix} -(\Delta + wP)f \\ (n+2w-2)\nabla^i f \\ w(n+2w-2)f \end{pmatrix} \tag{2.7}$$

and is conformally invariant. This operator can be applied to conformally weighted sections of any tensor power of $\mathcal{E}^I$, by using the tractor connection and Δ formed from the tractor connection, and remains invariant in this case.

2.3 The Ambient Metric Construction

In this section, we outline, briefly, the Fefferman-Graham ambient metric construction, which is a useful tool in the study of conformal invariants (see [FGm1]). Versions of this construction also appear in the work of Thomas ([T]) and Schouten and Haantjes ([SH]). The idea is to associate to an n-dimensional conformal manifold, an $n+2$-dimensional pseudo-Riemannian manifold, defined in terms of formal power series. It is useful to bear in mind the flat model for conformal geometry, which is the result of the construction in the flat case.

2.3.1 The Ambient Space

Let $\mathcal{M}$ be an n-dimensional manifold with conformal structure $\bar{g}_{ij}$ taking values in the line bundle $\mathcal{E}[2]$.

Definition 2.1 We denote the total space of the line bundle, $\mathcal{E}[-1]$, by $\mathcal{Q}$, with projection $\pi : \mathcal{Q} \to \mathcal{M}$. The *ambient space*, $\widetilde{\mathcal{M}}$, on which we will define the ambient metric, is given by $\widetilde{\mathcal{M}} = \mathcal{Q} \times I$, where $I = (-1, 1) \subset \mathbb{R}$. We identify $\mathcal{Q}$ with its image under the inclusion map $p \mapsto (p, 0)$, where $p \in \mathcal{Q}$.

We fix a local section, τ, of $\mathcal{E}[-1]$ or, equivalently, a metric, $g_{ij} = \tau^2 \bar{g}_{ij}$, for $\mathcal{M}$. Any other local section is of the form $t\tau$, where $t \in C^\infty(M)$. We use coordinates (t, x^i) for $\mathcal{Q}$, where (x^i), $i = 1 \dots n$, is a coordinate system for $\mathcal{M}$. On $\mathcal{Q}$, we can define a tautological, symmetric 2-tensor g_0, defined for $(x, t) \in \mathcal{Q}$ and $X, Y \in T_{(x,t)}\mathcal{Q}$ by

$$g_0(X, Y) = t^2 g(\pi_* X, \pi_* Y).$$

Note that, since the vertical vector is orthogonal to everything, g_0 is degenerate.

On $\mathcal{Q}$, for $s > 0$, we have dilations $\delta_s : \mathcal{Q} \to \mathcal{Q}$, defined by $\delta_s(x, t) = (x, st)$, which gives us a notion of homogeneity. For example, as $\delta_s^* g_0 = s^2 g_0$, we see that g_0 is homogeneous of degree 2. The dilations δ_s can be extended in a natural way to $\widetilde{\mathcal{M}}$, thus giving a notion of homogeneity on this space.

2.3.2 The Ambient Metric

The aim is to find a metric, $\tilde{g}$, defined as a formal power series on $\widetilde{\mathcal{M}}$ with signature $(n + 1, 1)$, which is homogeneous of degree 2, has g_0 as its restriction to $\mathcal{Q}$, and is Ricci flat. We expect this to be unique only up to formal diffeomorphisms of $\widetilde{\mathcal{M}}$ which fix $\mathcal{Q}$ and commute with dilations. This is done locally, on a coordinate patch in $\mathcal{M}$, and on a small neighbourhood of $\mathcal{Q}$ in $\widetilde{\mathcal{M}}$. The result of the analysis is the following theorem from [FGml]

Theorem 2.2 *If n is odd, one can find such a metric $\tilde{g}$ defined as a formal power series, unique up to diffeomorphisms as above. If $\bar{g}_{ij}$ has a real analytic representative, this series converges and $\tilde{g}$ actually exists in a neighbourhood of $\mathcal{Q}$.*

If n is even, there exist conformal structures for which no such solution exists. However, there exists a metric, defined as a formal power series solution, which is unique up to addition of order $n/2$ terms, and up to diffeomorphisms as above, for which all components of the Ricci tensor along $\mathcal{Q}$ vanish to order $(n-4)/2$ and all components tangential to $\mathcal{Q}$ vanish to order $(n-2)/2$. □

We have local coordinates (t, x^i) for $\mathcal{Q}$ such that $g_0 = t^2 g_{ij}(x)dx^i dx^j$. As shown in [FGm1], we can choose a function, ρ, on $\widetilde{\mathcal{M}}$, homogeneous of degree 0, which vanishes on $\mathcal{Q}$ and we can extend to coordinates, (t, x^i, ρ), for $\widetilde{\mathcal{M}}$, such that $\tilde{g}_{ij}$ has the form

$$\tilde{g} = t^2 g_{ij}(x,\rho)dx^i dx^j + 2\rho dt^2 + 2t dt d\rho, \tag{2.8}$$

where $g_{ij}(x,\rho)$ is defined as a formal power series with $g_{ij}(x,0) = g_{ij}(x)$. Of course, in even dimensions, the above holds only up to finite order. In terms of these coordinates, the vanishing of the Ricci curvature of $\tilde{g}$ is given by

$$\rho g''_{ij} - \rho g^{kl} g'_{ik} g'_{jl} + \frac{1}{2}\rho g^{kl} g'_{kl} g'_{ij} + \frac{(2-n)}{2} g'_{ij} - \frac{1}{2} g^{kl} g'_{kl} g_{ij} + \mathrm{Ric}(g)_{ij} = 0 \tag{2.9}$$

where the prime denotes differentiation with respect to ρ, and $\mathrm{Ric}(g)_{ij}$ denotes the Ricci curvature operator in the x variables alone acting on $g_{ij}(x,\rho)$ for fixed ρ.

2.3.3 The Curvature of the Ambient Metric

Let $x^0 = t$ and $x^\infty = \rho$. Fefferman and Graham found that the Riemann curvature tensor, $\tilde{R}$, of $\tilde{g}$ has as components on $\mathcal{Q}$, at $(x,t) \in \mathcal{Q}$

- $\tilde{R}_{IJK0} = 0$
- $\tilde{R}_{ijkl} = t^2 W_{ijkl}$
- $\tilde{R}_{ijk\infty} = t^2 C_{kij}$
- $\tilde{R}_{\infty ij\infty} = \frac{t^2}{(n-4)} B_{ij}$, for $n \neq 4$

where W, C and B are calculated at $x \in \mathcal{M}$ in terms of the representative, g_{ij}, of the conformal structure, and other components are given by the usual symmetries of the curvature tensor. When $n = 4$, the components $\tilde{R}_{\infty ij\infty}$ are undetermined. Note that $\tilde{R}_{ijKL} = t^2\Omega_{ijKL}$, where Ω is the tractor curvature.

Let $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{\mathcal{M}}$. Note that, as a consequence of the Bianchi identity $\widetilde{\nabla}_{[I}\tilde{R}_{JK]LM}$ and Ricci flatness, we have

$$\widetilde{\nabla}^I R_{IJKL} = 0. \tag{2.10}$$

Definition 2.3 We denote by X the vector field on $\widetilde{\mathcal{M}}$ which is the infinitesimal generator of the dilation, δ_s, so

$$Xf(p) = \frac{d}{ds}f(\delta_s p)\Big|_{(s=0)} \qquad \text{for } p \in \widetilde{\mathcal{M}},\ f \in C^\infty(\widetilde{\mathcal{M}})$$

In the coordinates (t, x^i, ρ), $X = t\frac{\partial}{\partial t}$. For future use, we note that

$$\begin{aligned} \widetilde{\nabla}_I X_J &= \tilde{g}_{IJ}, & (2.11)\\ \tilde{R}_{IJKL}X^L &= 0, & (2.12) \end{aligned}$$

and that $X^I X_I = 2t^2\rho$ vanishes on $\mathcal{Q}$.

2.4 The Tractor Bundle and the Tangent Bundle of The Ambient Space

It has been pointed out by Graham (see, for example, [BEGo]) that tractors are essentially tangent vectors to the ambient space, $\widetilde{\mathcal{M}}$, and that D_I is related to its Levi-Civita connection. We prove these facts here, as the proofs do not appear in the literature.

Definition 2.4 Denote by $\widetilde{\mathcal{M}}^I(k)$ the sheaf of vector fields on $\widetilde{\mathcal{M}}$ which are homogeneous of degree k, with tensor powers denoted by, for example, $\widetilde{\mathcal{M}}^{IJ}(k)$.

Proposition 2.5 *There is a canonical isomorphism,*

$$\mathcal{E}^{\overbrace{IJ\ldots K}^{k}}[w+k] \cong \widetilde{\mathcal{M}}^{IJ\ldots K}(w)\Big|_{\mathcal{Q}}.$$

Proof. Firstly, we note that sections of $\mathcal{E}[k]$ are equivalent to k-homogeneous functions on $\mathcal{Q}$—if $u(x)$ is a section of $\mathcal{E}[k]$, then $t^k u(x)$ is a homogeneous function on $\mathcal{Q}$ and vice versa.

As seen earlier, given a metric, g_{ij}, in the conformal class, one can take coordinates (t, x^i, ρ) such that the ambient metric has the form (2.8). If, however, we take a conformally related metric

$$\hat{g}_{ij} = \Omega^2 g_{ij}$$

as our starting point, we calculate that the corresponding coordinates, $(\hat{t}, \hat{x}^i, \hat{\rho})$, are given by

$$\begin{aligned} \hat{t} &= \Omega^{-1} t \left(1 - \tfrac{1}{2}\Upsilon^2 \rho\right) &&+O(\rho^2) \\ \hat{x}^i &= x^i + \rho \Upsilon^i &&+O(\rho^2) \\ \hat{\rho} &= \Omega^2 \rho &&+O(\rho^2) \end{aligned} \tag{2.13}$$

where $\Upsilon_i = \Omega^{-1} \nabla_i \Omega$. As a basis for $T\widetilde{\mathcal{M}}$, we use

$$(e_0, e_i, e_\infty) := \left(t\frac{\partial}{\partial t}, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \rho}\right). \tag{2.14}$$

From the change of coordinates, (2.13), we see that

$$\begin{aligned} t\tfrac{\partial}{\partial t} &= \hat{t}\tfrac{\partial}{\partial \hat{t}} &&+O(\rho) \\ \tfrac{\partial}{\partial x^i} &= \tfrac{\partial}{\partial \hat{x}^i} - \Upsilon_i \hat{t}\tfrac{\partial}{\partial \hat{t}} &&+O(\rho) \\ \tfrac{\partial}{\partial \rho} &= \Omega^2 \tfrac{\partial}{\partial \hat{\rho}} + \Upsilon^i \tfrac{\partial}{\partial \hat{x}^i} - \tfrac{1}{2}\Upsilon^2 \hat{t}\tfrac{\partial}{\partial \hat{t}} &&+O(\rho). \end{aligned}$$

For U a vector field, homogeneous of degree 0, we see that the components of U with respect to the bases (e_0, e_i, e_∞) and $(\tilde{e}_0, \tilde{e}_i, \tilde{e}_\infty)$ are homogeneous of degree 0 and are related by

$$\begin{pmatrix} \hat{U}^0 \\ \hat{U}^i \\ \hat{U}^\infty \end{pmatrix} = \begin{pmatrix} U^0 - \Upsilon_j U^j - \frac{1}{2}\Upsilon^2 U^\infty \\ U^i + \Upsilon^i U^\infty \\ \Omega^2 U^\infty \end{pmatrix}.$$

Comparing this with our definition of the tractor bundle, we have shown that there is a canonical isomorphism

$$\mathcal{E}^I[1] \cong \widetilde{\mathcal{M}}^I(0)\Big|_{\mathcal{Q}}.$$

The result follows by taking tensor products, together with the equivalence between k-homogeneous functions on $\mathcal{Q}$ and sections of $\mathcal{E}[k]$. □

Proposition 2.6 *Let $\tilde{f}^{J...K}$ be a section of $\overbrace{\widetilde{\mathcal{M}}^{J...K}}^{k}(w-k)$. Using the coordinates, (t, x^i, ρ), for $\widetilde{\mathcal{M}}$ and basis, (2.14), for $T\widetilde{\mathcal{M}}$, we have $\tilde{f}^{J...K}|_{\mathcal{Q}} = t^{w-k} f^{J...K}$, where $f^{J...K}$ is a section of $\mathcal{E}^{J...K}[w]$. Then*

$$t^{k-w}\left\{(n+2w-2)\widetilde{\nabla}_I \tilde{f}^{J...K} - X_I \widetilde{\Delta} \tilde{f}^{J...K}\right\}\Big|_{\mathcal{Q}} = D_I f^{J...K}.$$

Proof. Given $\tilde{g}$ as in (2.8), one can calculate that the Levi-Civita connection has Christoffel symbols, $\tilde{\Gamma}^K_{IJ}$, given by

$$\tilde{\Gamma}^K_{0J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^k{}_j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{\Gamma}^K_{\infty J} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} g^{kl} g'_{jl} & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.15}$$

$$\tilde{\Gamma}^K_{iJ} = \begin{pmatrix} 0 & -\frac{1}{2} g'_{ij} & 0 \\ \delta^k{}_i & \Gamma^k_{ij}(x,\rho) & \frac{1}{2} g^{kl} g'_{il} \\ 0 & \rho g'_{ij} - g_{ij} & 0 \end{pmatrix}, \tag{2.16}$$

where $\Gamma^k_{ij}(x,\rho)$ denotes the Christoffel symbols of $g_{ij}(x,\rho)$ for fixed ρ.

Define the tensor $T_I{}^{J...K}$ by

$$T_I{}^{J...K} := \widetilde{\nabla}_I \tilde{f}^{J...K} = \partial_I \tilde{f}^{J...K} + \tilde{\Gamma}^J_{IL} \tilde{f}^{L...K} + \ldots + \tilde{\Gamma}^K_{IL} \tilde{f}^{J...L} \tag{2.17}$$

Since $(X_I) = t^2(2\rho \;\; 0 \;\; 1)$ and $f^{J...K}$ is a section of $\mathcal{E}^{J...K}[w]$, we see from (2.7) that we need to show

$$\{T_0{}^{J...K}\}|_{\mathcal{Q}} = w t^{w-k} f^{J...K} \tag{2.18}$$

$$\{T_i{}^{J...K}\}|_{\mathcal{Q}} = t^{w-k} \nabla_i f^{J...K} \tag{2.19}$$

$$\left\{(n+2w-2) T_\infty{}^{J...K} - \widetilde{\Delta} \tilde{f}^{J...K}\right\}\Big|_{\mathcal{Q}} = -t^{w-k}(\Delta + wP) f^{J...K} \tag{2.20}$$

where ∇_i is the tractor connection, and Δ is the Laplacian formed from the tractor connection. To prove (2.18), from (2.15) we have

$$\begin{aligned} \{T_0{}^{J...K}\}|_{\mathcal{Q}} &= \left\{t\frac{\partial}{\partial t}(\tilde{f}^{J...K}) + \delta^J{}_L \tilde{f}^{L...K} + \ldots + \delta^K{}_L \tilde{f}^{J...L}\right\}\Big|_{\mathcal{Q}} \\ &= (w-k) t^{w-k} f^{J...K} + \underbrace{t^{w-k} f^{J...K} + \ldots + t^{w-k} f^{J...K}}_{k} \\ &= w t^{w-k} f^{J...K} \end{aligned}$$

From (2.9) one finds that

$$g'_{ij}(x,0) = 2P_{ij}(x). \tag{2.21}$$

Substituting in (2.16) we get

$$\left.\tilde{\Gamma}^{K}_{iJ}\right|_{Q} = \begin{pmatrix} 0 & -P_{ij} & 0 \\ \delta^{k}{}_{i} & \Gamma^{k}_{ij} & P^{k}{}_{i} \\ 0 & -g_{ij} & 0 \end{pmatrix},$$

which, by comparing with (2.6), proves (2.19).

Finally, by calculating $\tilde{\Delta}\tilde{f}$, one finds that

$$t^{k-w}\left.\left\{\tilde{\Delta}\tilde{f}^{J\ldots K} - (n+2w-2)T_{\infty}{}^{J\ldots K}\right\}\right|_{Q}$$

is $(\Delta + wP)f^{J\ldots K}$, thus completing the proof. □

2.5 Invariant Differential Operators

2.5.1 The Methods of Wünsch and Günther

In this subsection, we introduce some notation, following the presentation of [Wü], along with some propositions on the behaviour of differential operators. The reader is referred to [Wü] and [GüWü] for further information and for the proofs of these propositions.

Definition 2.7 Let $\mathfrak{D}$ be the set of linear differential operators acting on some space of tensor fields of a certain type, $\mathcal{T}$, whose coefficients are polynomials in g_{ij}, g^{ij} and the partial derivatives of g_{ij}.

Definition 2.8 We say an operator $D(g) \in \mathfrak{D}$ has *conformal weight* w if under the change of metric,

$$\hat{g}_{ij} = \lambda^2 g_{ij}, \qquad \lambda \text{ constant},$$

D transforms according to

$$D(\hat{g}) = \lambda^{w} D(g).$$

Let $\mathfrak{D}(w)$ denote the subset of $\mathfrak{D}$ consisting of those operators of conformal weight w.

Definition 2.9 We say $D(g) \in \mathfrak{D}(w)$ is *conformally invariant* on $\mathcal{T}$ if there exists some w_0 such that under a rescaling of the metric

$$\hat{g}_{ij} = \Omega^2 g_{ij} \tag{2.22}$$

for Ω a smooth positive function, D transforms according to

$$D(\hat{g})[\Omega^{w_0} u] = \Omega^{w+w_0} D(g)[u], \qquad \text{for all } u \in \mathcal{T}.$$

Proposition 2.10 *Let $D(g) \in \mathfrak{D}(w)$, then under the rescaling (2.22),*

$$D(\hat{g})[\Omega^{w_0} u] = \Omega^{w+w_0} \left\{ D(g) + \sum_{k=1}^{m} P_k(w_0, g, \Upsilon) \right\} [u], \qquad u \in \mathcal{T},$$

where each $P_k(w_0, g, \Upsilon)$ is homogeneous of degree k in Υ_i and its derivatives, and $\Upsilon_i = \Omega^{-1} \nabla_i \Omega$.

Wünsch shows that to show conformal invariance, it is sufficient to consider invariance under infinitesimal transformations, giving the following:

Proposition 2.11 *$D(g) \in \mathfrak{D}(w)$ is conformally invariant, for a particular w_0, if and only if*

$$P_1(w_0, g, \Upsilon)[u] = 0, \qquad u \in \mathcal{T}.$$

Definition 2.12 Let $\mathfrak{d}(w)$ denote those elements of $\mathfrak{D}(w)$ for which $P_k(w_0, g, \Upsilon)$, $k = 1, \dots m$, contain no derivatives of Υ_i, and let $\mathfrak{d} = \bigcup \mathfrak{d}(w)$.

Proposition 2.13 *$D(g) \in \mathfrak{d}$ if and only if $P_1(w_0, g, \Upsilon)$ has the form*

$$P_1(w_0, g, \Upsilon) = \Upsilon_i Q^i_{w_0}(g)$$

Definition 2.14 From the above, we can define the operator, $\mathfrak{X}^i_{w_0}$, acting on operators in $\mathfrak{d}$, by

$$P_1(w_0, g, \Upsilon) = \Upsilon_i \mathfrak{X}^i_{w_0} D(g).$$

Example 2.15 Under change of scale, $\widetilde{\nabla}_i f = \nabla_i f + w_0 \Upsilon_i f$, for functions of weight w_0. Thus we have $P_1(w_0, g, \Upsilon) = w_0 \Upsilon_i$, and so $\mathfrak{X}^j_{w_0} \nabla_i = w_0 \delta^j{}_i$.

We will usually omit the dependence on w_0 from the notation. Note that $D(g) \in \mathfrak{D}(w)$ is conformally invariant if and only if $D \in \mathfrak{d}$ and $\mathfrak{X}^i D = 0$.

Definition 2.16 Define the *conformal covariant derivative*, $\overset{c}{\nabla}$, acting on elements of $\mathfrak{d}$ by

$$\overset{c}{\nabla}_i D := \nabla_i D + P_{ij} \mathfrak{X}^j D.$$

Example 2.17 As $\mathfrak{X}^i f = 0$, we find $\overset{c}{\nabla}_i f = \nabla_i f$. From example 2.15, we have $\mathfrak{X}^j_{w_0} \overset{c}{\nabla}_i = w_0 \delta^j{}_i$, so that $\overset{c}{\Delta} := \overset{c}{\nabla}{}^i \overset{c}{\nabla}_i = (\Delta + w_0 P) f$.

Proposition 2.18 *The operators defined above have the following properties :*

(i) $\mathfrak{X}^i,\ \overset{c}{\nabla}_i : \mathfrak{d}(w) \to \mathfrak{d}(w)$

(ii) $\mathfrak{X}^i$ *and* $\overset{c}{\nabla}_i$ *are linear, obey a Leibniz rule and commute with contractions.*

Several other useful properties are given in [Wü]. The conformal covariant derivative is essentially a form of derivative which gives a simpler transformation law, enabling one to find conformally invariant differential operators. For example, on functions of weight w_0, $\mathfrak{X}^i \overset{c}{\Delta} = (n + 2w_0 - 2)\overset{c}{\nabla}{}^i$, and so $\overset{c}{\Delta}$ is conformally invariant on functions of weight $1 - \frac{1}{2}n$. Also using these methods, Wünsch finds that

$$(\overset{c}{\Delta})^3 - \frac{16}{n-4} B^{ij} \overset{c}{\nabla}_i \overset{c}{\nabla}_j \tag{2.23}$$

defines, for $n \neq 4$, a conformally invariant differential operator on functions of conformal weight $3 - \frac{1}{2}n$, with leading term Δ^3. See corollary 2.21 for an alternative proof.

2.5.2 The Tractor D_I-Operator and the Conformal Covariant Derivative.

From examples 2.15 and 2.17, we see that, when acting on conformally weighted functions, the operators $\overset{c}{\nabla}{}^i$ and $\overset{c}{\Delta}$ occur as components of the D^I operator, (2.7), and that the forms of $\mathfrak{X}^j$ follow from the change of scale formula, (2.5). More generally, we have the following:

Proposition 2.19 *Let f be a function of conformal weight w. Then*

(i) $(-1)^k(\overset{c}{\Delta})^k f$ *is the* $00\ldots0$*-component of* $\overbrace{D^I D^J \ldots D^K}^{k} f$,

(ii) $\overset{c}{\nabla}{}^i\overset{c}{\nabla}{}^j\ldots\overset{c}{\nabla}{}^k f$ *is a multiple of the* $ij\ldots k$*-component of* $D^I D^J \ldots D^K f$ *up to addition of trace terms.*

One can prove this by induction. The result is true for $k = 1$, as noted above. The inductive step is simply a matter of following the definitions through. For example, one finds $\mathfrak{X}^l(\overset{c}{\Delta})^k$ by looking at the transformation of $D^I D^J \ldots D^K f$ under change of scale. It is then easy to see that $(-1)^k\overset{c}{\nabla}{}^l(\overset{c}{\Delta})^k f$ is the $0\ldots0$-component of $\nabla^l D^I D^J \ldots D^K f$, where ∇^l is the tractor connection. □

2.6 Powers of the Laplacian and the Tractor D-operator

In 3.13, we will outline the method of [EGm] for finding, for flat conformal geometry, a conformally invariant differential operator on functions of conformal weight $-\frac{1}{2}n + k$ with the k-th power of the Laplacian as the leading order term.

In [GmJMS], it is shown that one find curved analogues of powers of the Laplacian by applying similar methods on the ambient metric construction. As the ambient metric is only defined up to finite order in even dimensions, this gives a curved analogue of Δ^k for $k \in \{1, 2, \ldots\}$ in odd dimensions, but only for $1 \leq k \leq \frac{1}{2}n$ in even dimensions. This result seems likely to be sharp—it is shown in [Gm] that that, for $n = 4$, there is no conformally invariant linear differential operator with leading term Δ^3.

The main result of this section is theorem 2.22, in which we give a simple formula in terms of the tractor D-operator for curved analogues of Δ^k with $k \in \{1, 2, \ldots\}$ for n odd, and $1 \leq k \leq \frac{1}{2}n - 1$ for n even.

2.6.1 The Flat Case

Recall that, for a particular choice of conformal scale, the D-operator, $D^I : \mathcal{E}[w] \to \mathcal{E}^I[w-1]$, is given by

$$D^I f = \begin{pmatrix} -(\Delta + wP)f \\ (n + 2w - 2)\nabla^i f \\ w(n + 2w - 2)f \end{pmatrix} \tag{2.24}$$

and similarly for tensor powers of $\mathcal{E}^I[w]$, by replacing the Levi-Civita connection with the tractor connection.

If $w = -\frac{1}{2}n + 1$, then $D^I f = -X^I(\Delta + wP)f$, and $(\Delta + wP)f$, being the projecting part, must be conformally invariant.

If f has weight $w = -\frac{1}{2}n + k$, then, as $\overbrace{D^J \ldots D^K}^{k-1} f$ is a section of $\mathcal{E}^{J \ldots K}[-\frac{1}{2}n + 1]$, we see that

$$D^I D^J \ldots D^K f = -X^I \left(\Delta + \left(1 - \tfrac{1}{2}n\right) P\right) D^J \ldots D^K f.$$

It is easy to show that, in the flat case, $D^I D^J \ldots D^K f$ is symmetric over all indices (see also proposition 2.20), and so $D^I D^J \ldots D^K f$ takes the form $X^I X^J \ldots X^K \psi$ for some ψ which, being the projecting part, must be conformally invariant. From the formulae (2.24) and (2.6), one can see that ψ has leading term $(-1)^k \Delta^k f$, and hence we have recovered the conformally invariant powers of the Laplacian. In terms of a flat metric, ψ is clearly just $(-1)^k \Delta^k f$.

2.6.2 Curved Case

As is noted in [BEGo], one can obtain curved analogues of Δ and Δ^2 in the same way.

For f of weight $-\frac{1}{2}n+1$, the projecting part of $D^I f$ is $-(\Delta+(1-\frac{1}{2}n)P)f$, which gives us the conformally invariant Laplacian. Again, $D^I D^J f$ is symmetric, and so for f of weight $-\frac{1}{2}n+2$, the projecting part of $D^I D^J f$ is the 00-component, which turns out to be a curved analogue of the square of the Laplacian.

However, $D^I D^J D^K f$ is not symmetric, and so one cannot simply set $w = -\frac{1}{2}n+3$ and read off the projecting part as the analogue of Δ^3. In fact, we have the following :

Proposition 2.20 *For $T^{AB\ldots C}$ a section of $\mathcal{E}^{AB\ldots C}[w]$ and $n \neq 4$,*

$$\begin{aligned}
(D_I D_J - D_J D_I)T^{AB\ldots C} = &\\
&(n+2w-4)(n+2w-2)\left(\tilde{R}_{IJ}{}^{A}{}_{K}T^{KB\ldots C}+\ldots+\tilde{R}_{IJ}{}^{C}{}_{K}T^{AB\ldots K}\right)\\
&+2X_I\left(\tilde{R}_{JK}{}^{A}{}_{L}D^K T^{LB\ldots C}+\ldots+\tilde{R}_{JK}{}^{C}{}_{L}D^K T^{AB\ldots L}\right)\\
&-2X_J\left(\tilde{R}_{IK}{}^{A}{}_{L}D^K T^{LB\ldots C}+\ldots+\tilde{R}_{IK}{}^{C}{}_{L}D^K T^{AB\ldots L}\right)
\end{aligned}$$

where $\tilde{R}$ is the tractor form of the restriction to $\mathcal{Q}$ of the curvature tensor of the ambient metric (see 2.3.3).

Proof. Either by direct calculation, or one can use the correspondence given in proposition 2.6, as follows. We abuse notation by omitting the restrictions to $\mathcal{Q}$, and omit the powers of t which should appear. Using equation (2.11) we obtain

$$\begin{aligned}
D_I D_J T^{AB\ldots C} &= \left((n+2w-4)\tilde{\nabla}_I - X_I\tilde{\Delta}\right)\left((n+2w-2)\tilde{\nabla}_J - X_J\tilde{\Delta}\right)T^{AB\ldots C}\\
&= (n+2w-4)(n+2w-2)\tilde{\nabla}_I\tilde{\nabla}_J T^{AB\ldots C} + X_I X_J\tilde{\Delta}^2 T^{AB\ldots C}\\
&\quad -X_I\left((n+2w-2)\tilde{\Delta}\tilde{\nabla}_J T^{AB\ldots C} - 2\tilde{\nabla}_J\tilde{\Delta}T^{AB\ldots C}\right)\\
&\quad -(n+2w-4)X_J\tilde{\nabla}_I\tilde{\Delta}T^{AB\ldots C} - (n+2w-4)\tilde{g}_{IJ}\tilde{\Delta}T^{AB\ldots C}
\end{aligned} \tag{2.25}$$

From equation (2.10), and since $\tilde{R}_{IJKL}$ is trace free, we find

$$\tilde{\Delta}\tilde{\nabla}_J T^{AB\ldots C} = \tilde{\nabla}^K\tilde{\nabla}_J\tilde{\nabla}_K T^{AB\ldots C} + \tilde{R}_{KJ}{}^{A}{}_{L}T^{LB\ldots C}+\ldots+\tilde{R}_{KJ}{}^{C}{}_{L}T^{AB\ldots L}, \tag{2.26}$$

$$\tilde{\nabla}_J\tilde{\Delta}T^{AB\ldots C} = \tilde{\nabla}_K\tilde{\nabla}_J\tilde{\nabla}^K T^{AB\ldots C} + \tilde{R}_{JK}{}^{A}{}_{L}T^{LB\ldots C}+\ldots+\tilde{R}_{JK}{}^{C}{}_{L}T^{AB\ldots L}. \tag{2.27}$$

From equations (2.25), (2.26) and (2.27), we see that

$$\begin{aligned}(D_I D_J - D_J D_I)T^{AB\ldots C} = &\\ &(n+2w-4)(n+2w-2)\left(\tilde{R}_{IJ}{}^A{}_K T^{KB\ldots C} + \ldots + \tilde{R}_{IJ}{}^C{}_K T^{AB\ldots K}\right)\\ &+2(n+2w-2)X_I\left(\tilde{R}_{JK}{}^A{}_L \tilde{\nabla}^K T^{LB\ldots C} + \ldots + \tilde{R}_{JK}{}^C{}_L \tilde{\nabla}^K T^{AB\ldots L}\right)\\ &-2(n+2w-2)X_J\left(\tilde{R}_{IK}{}^A{}_L \tilde{\nabla}^K T^{LB\ldots C} + \ldots + \tilde{R}_{IK}{}^C{}_L \tilde{\nabla}^K T^{AB\ldots L}\right)\end{aligned}$$

and the result follows, from equation (2.12) and proposition 2.6. □

Corollary 2.21 *For f weight $-\frac{1}{2}n+3$, and $n \neq 4$, one sees that*

$$D^I D^J D^K f + 2X^I \tilde{R}^{JLMK} D_L D_M f$$

is totally symmetric. Thus the only non-zero component is the 000-component, which gives a curved analogue of Δ^3. By proposition 2.19, we see that this is the operator (2.23) of Wünsch.

There is another way in which the tractor calculus can be used to calculate curved analogues of the Laplacian:

Theorem 2.22 *Let f be a section of $\mathcal{E}[-\frac{1}{2}n+k]$. Then*

$$\underbrace{D_C \ldots D_B}_{k-1}(\Delta - \tfrac{1}{2}(n-2)P)D^B \ldots D^C f = (k-1)!(n-4)(n-6)\ldots(n-2k)\Delta^k f$$
$$+\text{lower order terms,} \qquad (2.28)$$

where the Laplacian on the left hand side is formed from the tractor connection. Hence for $n \notin \{4, 6, \ldots, 2k\}$, we have a formula for a curved analogue of the k-th power of the Laplacian.

Proof. As $D^A D^B \ldots D^C f = -X^A(\Delta - \frac{1}{2}(n-2)P)D^B \ldots D^C f)$, we can define a section, $T^{B\ldots C}$, of $\mathcal{E}^{B\ldots C}[-\frac{1}{2}n-1]$ by

$$T^{B\ldots C} = \left(\Delta - \tfrac{1}{2}(n-2)P\right) D^B \ldots D^C f.$$

Now $T^{B\ldots C}$ has as components

$$T^{\overbrace{0\ldots0}^{p}\overbrace{c\ldots d}^{q}\overbrace{\infty\ldots\infty}^{r}}, \qquad p+q+r = k-1. \qquad (2.29)$$

It is easy to see from (2.7), that the maximum number of derivatives of f occurring in the expression for such a component is $2p+q+2$.

However, we know that in the flat case, the only non-zero component is the $0\ldots0$-component, and thus for $q+r\neq 0$, the highest derivative occurring in such a component of $T^{B\ldots C}$ involves a curvature term, which implies that number of derivatives is strictly less than $2p+q+2$, unless $q+r=0$.

For a section, $U_K = (\sigma \quad \mu_k \quad \rho)$, of $\mathcal{E}_K[\tilde{w}]$, one can calculate

$$D_K U^K = (n+2\tilde{w})(n+\tilde{w}-1)\rho + (n+2\tilde{w})\nabla_k\mu^k - \Delta\sigma + (n+\tilde{w}-1)P\rho$$

From this, we see that the maximum number of derivatives acting on the component (2.29) in the expression for $D_C\ldots D_B T^{B\ldots C}$ is $q+2r$. The only term with $2k$ derivatives of f therefore comes from the $T^{0\ldots0}$ component and so $\Delta^k f$ appears in the expression $D_C\ldots D_B(\Delta - \frac{1}{2}(n-2)P)D^B\ldots D^C f$ with coefficient

$$(-1)^{k-1}\prod_{\tilde{w}=-n/2-1}^{-n/2-k+1}(n+2\tilde{w})(n+\tilde{w}-1)$$

and no higher order terms appear. □

Chapter 3

Parabolic Invariant Theory for Conformal Geometry

The first section of this chapter reviews Weyl's invariant theory for the orthogonal groups (see [We]), together with some extensions of this, from [BEGm]. The second section gives a rough idea of the plan for attacking an invariant theory problem in the conformal case. The final section describes two invariant theory problems which were studied in [BEGm].

First, some notation. Suppose we have a vector space, V and its dual V^* There is an invariant pairing of vectors and covectors given by *contraction*—if $u^i \in V$ and $v_j \in V^*$, we can form the scalar $u^i v_i$. Similarly, if we have, for example, tensors u^{ij} and ${v_{ijk}}^l$, we can form tensors such as

$$u^{ij}{v_{ijk}}^l, \qquad u^{ij}{v_{ikl}}^l \qquad \text{and} \qquad u^{ij}{v_{ijk}}^k.$$

Such contractions may, of course, involve more than two tensors. If the result of such a contraction is a scalar, as in the last case above, then we call this a *complete contraction* otherwise, a *partial contraction.* We will denote, for example, a complete contraction formed from the tensors u and v, by

$$\mathrm{contr}(u \otimes v),$$

Note that there may be many such contractions.

3.1 Weyl's Orthogonal Invariant Theory

Let g be a positive-definite, symmetric quadratic form on $\mathbb{R}^n$, and let $O(g)$ and $SO(g)$ denote the corresponding orthogonal and special orthogonal groups, respectively. Let $\epsilon \in \bigwedge^n \mathbb{R}^n$ denote the volume form on $\mathbb{R}^n$ corresponding to g, and let ϵ^* denote the dual volume form. We denote the m-th symmetric tensor product of a vector space, V, by $\bigodot^m V$ and we denote by $\bigodot_0^m V$ the subspace consisting of symmetric tensors which are trace-free with respect to g.

3.1.1 Invariants of Vectors

Let $(u^{(1)}, u^{(2)}, \ldots, u^{(m)})$ be a collection of vectors in $\mathbb{R}^n$. We denote by $u^{(i)} \cdot u^{(j)}$ the inner product given by g, and denote by $\left|u^{(i_1)}\ u^{(i_2)} \ldots u^{(i_n)}\right|$ a complete contraction of ϵ^* with $u^{(i_1)}, \ldots, u^{(i_n)}$ in the obvious way. The groups $SO(g)$ and $O(g)$ act on vectors in $\mathbb{R}^n$ in the usual way.

Definition 3.1 An $SO(g)$-*invariant* is a polynomial in the components of the $u^{(i)}$ which is invariant under the $SO(g)$-action. Similarly for $O(g)$-invariants.

As $O(g)$ and $SO(g)$ act linearly, any invariant is the sum of invariant homogeneous parts, so we need only consider homogeneous polynomials. We use d to denote the total homogeneity degree.

3.1.2 Weyl's First Main Theorem

We would like a list of generators for the algebra of invariants. This is given by 'Weyl's first main theorem' (see [We]) :

Theorem 3.2 *A set of generators for the $O(g)$-invariants is given by*

$$u^{(i)} \cdot u^{(j)}, \qquad 1 \leq i \leq j \leq m.$$

Any $SO(g)$ invariant, I, can be expressed uniquely in form $I = I_{odd} + I_{even}$, where I_{odd} changes sign under change of orientation while I_{even} is invariant. The generators for the $SO(g)$ invariants are given by

$$u^{(i)} \cdot u^{(j)}, \qquad 1 \leq i \leq j \leq m$$

$$\text{and} \qquad \left|u^{(i_1)}\, u^{(i_2)} \ldots u^{(i_n)}\right|, \qquad 1 \leq i_1 < i_2 < \ldots < i_n \leq m.$$

Generators of the first type are even invariants, while those of the second type are odd. □

Any complete contraction containing two occurrences of ϵ can always be rewritten in terms of g's, as

$$\left|u^{(i_1)} \ldots u^{(i_n)}\right| \cdot \left|u^{(j_1)} \ldots u^{(j_n)}\right| = \begin{vmatrix} (u^{(i_1)} \cdot u^{(j_1)}) & \cdots & (u^{(i_1)} \cdot u^{(j_n)}) \\ \vdots & \ddots & \vdots \\ (u^{(i_n)} \cdot u^{(j_1)}) & \cdots & (u^{(i_n)} \cdot u^{(j_n)}) \end{vmatrix}. \qquad (3.1)$$

Thus we can assume ϵ appears only in odd invariants, with exactly one appearance in each term in this case.

3.1.3 Weyl's Second Main Theorem

Weyl's second main theorem (theorem 3.3) tells us the relationships between the invariants. Here, we only consider the $O(g)$-invariants. We need to distinguish between a polynomial as a formal expression in the variables $(u^{(i)} \cdot u^{(j)})$ and as a polynomial on components of vectors in $\mathbb{R}^n$ obtained on substitution of the $u^{(l)}$. For example, an expression such as $(u^{(1)} \cdot u^{(1)})(u^{(2)} \cdot u^{(2)}) - (u^{(1)} \cdot u^{(2)})^2$ does not vanish identically, i.e. is not the zero polynomial in the $u^{(i)} \cdot u^{(j)}$'s, but, for $n = 1$, vanishes on substitution of any vectors $u^{(1)}$ and $u^{(2)}$.

Theorem 3.3 *Those polynomials in the $(u^{(i)} \cdot u^{(j)})$ which vanish on substitution form an ideal, generated by polynomials of the form*

$$\begin{vmatrix} (u^{(i_1)} \cdot u^{(j_1)}) & \cdots & (u^{(i_1)} \cdot u^{(j_{n+1})}) \\ \vdots & \ddots & \vdots \\ (u^{(i_{n+1})} \cdot u^{(j_1)}) & \cdots & (u^{(i_{n+1})} \cdot u^{(j_{n+1})}) \end{vmatrix}, \qquad \begin{array}{l} 1 \leq i_1 < i_2 < \ldots < i_{n+1} \leq m \\ 1 \leq j_1 < j_2 < \ldots < j_{n+1} \leq m \end{array}$$

□

Corollary 3.4

- *If $m \leq n$, then any polynomial in the $(u^{(i)} \cdot u^{(j)})$ which vanishes on substitution vanishes identically.*

- *If $m = n+1$, then any polynomial in the $(u^{(i)} \cdot u^{(j)})$ which vanishes on substitution and has degree 1 in at least one of the $u^{(l)}$ vanishes identically.*

Proof. If $m \leq n$, then there are no generators of the above form. If $m = n+1$, the only generator of the relations is of degree 2 in each $u^{(l)}$. □

3.1.4 Invariants of Tensors

The above results can be generalized to a list, $(u^{(1)}, \ldots, u^{(m)})$, where each $u^{(i)} \in \bigotimes^{k_i} \mathbb{R}^{n*}$, for some k_i. Invariants are defined in the same way and Weyl's theory gives us the following :

Theorem 3.5 *Even and odd $SO(g)$ invariants can be written as linear combinations of complete contractions of the forms*

$$\operatorname{contr}\left(g^{-1} \otimes \ldots \otimes g^{-1} \otimes u^{(i_1)} \otimes \ldots \otimes u^{(i_d)}\right) \tag{3.2}$$

$$\text{and} \qquad \operatorname{contr}\left(\epsilon \otimes g^{-1} \otimes \ldots \otimes g^{-1} \otimes u^{(i_1)} \otimes \ldots \otimes u^{(i_d)}\right) \tag{3.3}$$

respectively. Any $O(g)$ invariant can be written as a linear combination of complete contractions of the form (3.2) □

For more details of the following, including a proof of theorem 3.6, see appendices A and B of [BEGm]. We now suppose that each $u^{(i)}$ has symmetry given by a Young tableaux with s_i rows, and that, for some K, the tensors $u^{(1)}, \ldots, u^{(K)}$ are completely trace-free. We say that a linear combination of partial contractions is *allowable* if it contains no internal contractions of a completely trace free tensor, and *vanishes formally* if it vanishes identically or is forced to vanish by the Young symmetries of the $u^{(l)}$ (for a precise statement of this last definition, see [BEGm], pp.543-545). For example, if $u^{(1)}$ is a rank-1 tensor and $u^{(2)}$ is anti-symmetric, rank-2, then $u^{(1)}_a u^{(1)}_b u^{(2)}_{cd}\, g^{ac}_{\;\;\cdot} g^{bd}$ vanishes formally.

Theorem 3.6 *Let $(u^{(1)}, \ldots, u^{(m)})$ be a list of tensors as above. Let I be a linear combination of allowable formal partial contractions of g and the $u^{(l)}$, of degree d_i in $u^{(i)}$ and formally taking values in the space of symmetric m-tensors. Then if I vanishes on substitution in dimension n and*

$$\sum_{i=1}^{m} d_i s_i \leq n-1$$

then I vanishes formally. □

3.2 Parabolic Invariant Theory

3.2.1 Preliminaries

Let $W = \mathbb{R}^{n+2}$ with coordinates (2.1), let $\tilde{\epsilon} \in \bigwedge^{n+2} W$ denote the standard volume form on W, and define e_0, G, P and $\mathcal{Q}$ as in section 2.1. We define

$$\mathcal{E}(k) := \{\text{jets at } e_0 \text{ of functions homogeneous of degree } k \text{ on } W\},$$

$$\mathcal{F}(k) := \{\text{jets at } e_0 \text{ of functions homogeneous of degree k on } \mathcal{Q}\}.$$

Similarly, we denote, for example, by $\mathcal{E}^I$ the space of jets at e_0 of functions homogeneous of degree 0 on W taking values in W. G acts on the space of homogeneous functions by $(h \cdot f)(x) = f(h^{-1}x)$, for f a homogeneous function and $h \in G$. Since P preserves $\mathcal{Q}$ and the ray through e_0, there is an induced action of P on $\mathcal{E}(k)$ and $\mathcal{F}(k)$. The P-action is however, somewhat complicated. We define σ_k to be the one dimensional representation of P where the element,

$$\begin{pmatrix} \lambda & r_j & t \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \in P,$$

acts by λ^{-k}. There is a P-module homomorphism Eval: $\mathcal{E}(k) \to \sigma_k$ given by evaluation at e_0. Regarding the coordinate functions X^I as an element of $\mathcal{E}^I(1)$, we define $e \in W \otimes \sigma_1$ by the evaluation at e_0 of X^I, which we write $e = \text{Eval}(X^I)$. By regarding differential operators as acting on formal power series, we can apply them to spaces of jets.

3.2.2 Invariant Theory

Let $\mathcal{T}$ be a P-module given by the space of jets of some class of functions taking values in $\bigotimes^k W^*$, which vanish on any contraction with X^I. For example, these functions may have Young symmetry, and be in the kernel of some conformally invariant differential operators. The P-action on a scalar valued function, f is given by $(p \cdot f)(x) = f(p^{-1}x)$, for $p \in P$, with the obvious extension of this to tensor valued functions.

Definition 3.7 An *invariant* of $\mathcal{T}$ is a P-equivariant, polynomial map $I : \mathcal{T} \to \sigma_q$ for some q.

Since P acts by linear transformations, any invariant is the sum of invariant homogeneous parts, thus it suffices to consider only homogeneous invariants. We will denote by d the homogeneity degree of any invariant under consideration.

The first step in analysing the invariants of the module, $\mathcal{T}$, is to obtain a suitable algebraic description of the modules. Given a function, $f_{IJ\ldots K}$, we can define a list of tensors, $T^{(l)}$, for $l = 0, 1, 2, \ldots$, by

$$T^{(l)}_{\underbrace{IJ\ldots K}_{k}\,,\,\underbrace{AB\ldots C}_{l}} = \mathrm{Eval}\left(\partial_A \partial_B \ldots \partial_C f_{IJ\ldots K}\right),$$

where we use the comma to seperate off the first k indices. We thus hope to obtain

$$\mathcal{T} = \left\{(T^{(0)}, T^{(1)}, \ldots) \text{ such that the } T^{(l)} \text{ satisfy some set of conditions.}\right\}$$

Among these conditions, we suppose that, for $l = 1, 2, \ldots$, any contraction of e with $T^{(l)}$ can be expressed in terms of $T^{(l-1)}$, and any contraction of e with $T^{(0)}$ vanishes (the *linking conditions*).

As the P-action on elements of $\mathcal{T}$ is given by the usual action of P on tensors, one can construct invariants by taking complete contractions involving the tensors $\tilde{\epsilon}$, $\tilde{g}^{-1}$, e, e^* and the $T^{(l)}$. (We use, for example, e^* to denote the covector with coordinates $e^*_I = \tilde{g}_{IJ} e^J$). Any two occurrences of $\tilde{\epsilon}$ can be rewritten in terms of $\tilde{g}^{-1}$'s and any e contracted into one of the T's can be removed using the linking conditions, so we need only consider contractions containing at most one $\tilde{\epsilon}$ or

$\tilde{\epsilon}_0 := e^* \lrcorner\, \tilde{\epsilon}$ and no other occurrences of e's, so we are left with complete contractions of the form

$$\begin{aligned} &\operatorname{contr}(T^{(l_1)} \otimes \dots T^{(l_d)} \otimes \tilde{g}^{-1} \otimes \dots \otimes \tilde{g}^{-1}) \\ &\operatorname{contr}(\tilde{\epsilon} \otimes T^{(l_1)} \otimes \dots T^{(l_d)} \otimes \tilde{g}^{-1} \otimes \dots \otimes \tilde{g}^{-1}) \\ &\operatorname{contr}(\tilde{\epsilon}_0 \otimes T^{(l_1)} \otimes \dots T^{(l_d)} \otimes \tilde{g}^{-1} \otimes \dots \otimes \tilde{g}^{-1}). \end{aligned} \tag{3.4}$$

Definition 3.8 A *Weyl invariant* is an invariant which can be written as the linear combination of complete contractions of the above form, where each contraction takes values in the same σ_q. Any invariant which is not Weyl, is said to be an *exceptional* invariant.

The aim is to find the extent to which all invariants are Weyl invariants. In [BEGm], two methods are given by which we can express an invariant as a Weyl invariant, one which can be used for invariants of high homogeneity degree, and one for invariants of low degree. In either case, we first need to describe our invariant in an intermediate form.

Definition 3.9 We say that an invariant, I, of $\mathcal{T}$ is a *weak Weyl invariant* if there exists $m \in \mathbb{N}$ and a map

$$C : \mathcal{T} \to \bigodot_0^m W \otimes \sigma_{m+q},$$

which can be written as a linear combination of partial contractions of the tensors $T^{(l)}$, e, $\tilde{g}^{-1}$, $\tilde{\epsilon}$ and $\tilde{\epsilon}_0$, such that

$$C = \underbrace{e \otimes \dots \otimes e}_{m} \otimes I.$$

For high degree invariants, (see, by way of illustration, subsection 5.3.1), one constructs P-equivariant maps, $\tilde{C} : \mathcal{T} \to \mathcal{F}^{\overbrace{IJ\dots K}^{m}}(q+m)$ and $\tilde{I} : \mathcal{T} \to \mathcal{F}(q)$ with $\operatorname{Eval}(\tilde{C}) = C$, $\operatorname{Eval}(\tilde{I}) = I$ and

$$\tilde{C}^{IJ\dots K} = X^I X^J \dots X^K \tilde{I}.$$

Then one hopes to constrain m so that, for large enough d, one can apply a version of the tractor D-operator to both sides of the above equation and evaluate at e_0 to realise I as a Weyl invariant.

For low degree invariants, (see, for example, theorem 5.20), one hopes to be able to use theorem 3.6 to "cancel" e's from C to leave I as a Weyl invariant.

3.3 The modules $\mathcal{H}_k$ and $\mathcal{K}$

In this section, we describe the modules, $\mathcal{H}_k$ and $\mathcal{K}$, for which the invariant theory is studied in [BEGm] and state the results.

3.3.1 $\mathcal{H}_k$

Let $k \in \mathbb{Z}$, with $k \geq 0$. Define

$$\mathcal{H}(k) := \{f \in \mathcal{E}(k) : \widetilde{\Delta} f = 0\}.$$

Since P preserves the metric, there is an induced P-action on $\mathcal{H}(k)$. We say that a function satisfying $\widetilde{\Delta} f = 0$ is *harmonic*. We regard $\bigodot_0^k W^*$ as the homogeneous degree k, harmonic polynomials on W, giving an inclusion $\bigodot_0^k W^* \hookrightarrow \mathcal{H}(k)$. We define the P-module $\mathcal{H}_k$, for $k \geq 0$, by

$$\mathcal{H}_k := \mathcal{H}(k)/\bigodot_0^k W^*,$$

i.e. the space of those elements of $\mathcal{H}(k)$ which vanish to order $k+1$. As P-modules, $\mathcal{H}(k) \cong \mathcal{H}_k \oplus \bigodot_0^k W^*$. It is shown in [EGm] that the invariant theory of this module is related to the invariant theory of homogeneous functions on $\mathcal{Q}$, or equivalently, conformally weighted functions on S^n :

Theorem 3.10 *If n is odd then $\mathcal{F}(k) \cong \mathcal{H}(k)$ as P-modules.*

There is a family of conformally invariant linear differential operators acting on $\mathcal{E}[k]$ (see p. 19), which, in the coordinates above, are just the trace-free parts of iterated gradients. For a function, u, on $\mathbb{R}^n$, denote by $\nabla^{(l)} u$ the tensor with components given by $\underbrace{\partial_a \ldots \partial_d}_{l} u$, and denote the trace free part (with respect to g_{ij}) by $\nabla_0^{(l)} u$. Then $\nabla_0^{(k+1)}$ is the formula for a conformally invariant differential operator on $\mathcal{E}[k]$, and thus on $\mathcal{F}(k)$. (In another conformal scale, the formula will also contain lower order terms.)

For $k \geq 0$, it turns out that the sub-module $\ker\nabla_0^{(k+1)}$ of $\mathcal{F}(k)$ is isomorphic (as a P-module) to $\bigodot_0^k W^*$, thus, in odd dimensions,

$$\mathcal{H}_k \cong \mathcal{F}(k)/\ker\nabla_0^{(k+1)}.$$

Proof of theorem 3.10. We reproduce the proof of [EGm], as this contains several ideas which we will need later.

Let f be a function, homogeneous of degree k on $\mathcal{Q}$, and let f_0 be an arbitrary homogeneous extension to W. The general extension of f to W is of the form $\tilde{f} = f_0 + Qh_0$, where $Q = X^I X_I$ is the defining function of the null cone, $\mathcal{Q}$, and h_0 is homogeneous of degree $k-2$. The aim is to find a unique extension, $\tilde{f}$, of f, which satisfies $\widetilde{\Delta}\tilde{f} = 0$.

Let $E = X^I\partial_I$ denote the Euler field, so that for a homogeneous degree k function, f, we have $Ef = kf$. It is a simple calculation to show that $[\widetilde{\Delta}, Q] = 2(n+2E+2)$. Now

$$\begin{aligned}
\widetilde{\Delta}\tilde{f} &= \widetilde{\Delta}f_0 + \widetilde{\Delta}(Qh_0) \\
&= \widetilde{\Delta}f_0 + [\widetilde{\Delta}, Q]h_0 + Q\widetilde{\Delta}h_0 \\
&= \widetilde{\Delta}f_0 + 2(n+2E+2)h_0 + Q\widetilde{\Delta}h_0.
\end{aligned}$$

We see that, as h_0 is homogeneous, degree $k-2$, for $(n+2k-2) \neq 0$, h_0 is uniquely defined on $\mathcal{Q}$, by

$$h_0|_{\mathcal{Q}} = \frac{-\widetilde{\Delta}f_0}{2(n+2k-2)}.$$

However, if $(n+2k-2) = 0$, there is an obstruction given by $\widetilde{\Delta}f_0$.

We continue the proof by induction. Suppose we have found an extension f_{m-1} of f such that $\widetilde{\Delta}f_{m-1} = 0 \mod Q^{m-1}$. The general such extension is of the form $f_m = f_{m-1} + Q^m h_{m-1}$, where h_{m-1} is homogeneous of degree $k-2m$. Using the formula $[\widetilde{\Delta}, Q^m] = 2mQ^{m-1}(n+2E+2m)$, (see corollary 5.3), we find

$$\widetilde{\Delta}f_m = \widetilde{\Delta}f_{m-1} + 2mQ^{m-1}(n+2E+2m)h_{m-1} + Q^m\widetilde{\Delta}h_{m-1}.$$

So for $(n + 2k - 2m) \neq 0$, h_{m-1} is uniquely determined on $\mathcal{Q}$ by

$$h_{m-1}|_{\mathcal{Q}} = \frac{-\widetilde{\Delta} f_{m-1}}{2m(n + 2k - 2m)Q^{m-1}}.$$

In this way, for n odd, we can find a unique extension, $\tilde{f}$, of f to W, defined as a formal power series off $\mathcal{Q}$, for which $\widetilde{\Delta}\tilde{f} = 0$. As the Taylor expansion of $\tilde{f}$ at e_0 is determined by that of f, we have an induced map, $\mathcal{F}(k) \to \mathcal{H}(k)$.

This map is injective, since if $\tilde{f}$ vanishes to infinite order at e_0, then by restriction to $\mathcal{Q}$, so does f. The map is also surjective, since any such $\tilde{f}$ is the extension of its restriction to $\mathcal{Q}$, as the extension is unique. Finally, since the P-action clearly commutes with taking the extension, this map is a P-module isomorphism.
□

Remark 3.11 Note that the bulk of this proof can be written in terms of the lie algebra $\mathfrak{sl}(2)$. If we write

$$x = -\frac{1}{4}Q, \qquad y = \widetilde{\Delta}, \qquad h = E + \frac{n+2}{2},$$

then we find that x, y and h satisfy

$$[x, y] = h, \qquad [h, x] = 2x, \qquad [h, y] = -2y$$

i.e. they satisfy the commutator relations of the standard generators of $\mathfrak{sl}(2)$, and the argument can be written in terms of these commutators.

Remark 3.12 In the first step of the proof, we found an extension of f, unique up to addition of second order terms, given by

$$f_1 = f_0 - \frac{Q\widetilde{\Delta} f_0}{2(n + 2k - 2)} \mod Q^2.$$

We define the operator $D_I : \mathcal{F}(s) \to \mathcal{F}_I(s-1)$ by

$$\begin{aligned} D_I f &= (n + 2s - 2)\partial_I f_1|_{\mathcal{Q}} \\ &= \left((n + 2s - 2)\partial_I f_0 - X_I \tilde{\Delta} f_0\right)\Big|_{\mathcal{Q}} \end{aligned}$$

Clearly, this definition is independent of the choice of extension, f_0. The definition can also be extended to tensor valued functions to give an operator D_I : $\mathcal{F}^{AB\ldots E}(s) \to \mathcal{F}_I{}^{AB\ldots E}(s-1)$. This operator is the action on jets of the tractor D_I operator for the flat model.

Remark 3.13 As in [EGm], we can recover the conformally invariant powers of the Laplacian as the obstruction to finding the harmonic extension as follows. If we change to coordinates

$$t = X^0, \qquad x^i = \frac{X^i}{X^0}, \qquad \rho = \frac{Q}{2(X^0)^2},$$

so that

$$\begin{pmatrix} X^0 \\ X^i \\ X^\infty \end{pmatrix} = t \begin{pmatrix} 1 \\ x^i \\ \rho - \frac{x^2}{2} \end{pmatrix},$$

(cf. the coordinates of (2.2)) we can again write a homogeneous degree k function, f, on $\mathcal{Q}$ as $f = t^k u(x)$, where u represents a function of conformal weight k in a given conformal scale. A general extension of f is of the form $f_0 = t^k u_0(x, \rho)$, where $u_0(x, 0) = u(x)$. As before, we look at the problem of finding a harmonic extension for f. In terms of these coordinates,

$$\widetilde{\Delta} f_0 = t^{k-2} \{(n + 2k - 2)u_0' - 2\rho u_0'' + \Delta u_0\} \tag{3.5}$$

where prime denotes differentiation with respect to ρ. Let $f_1 = t^k u_1(x, \rho)$ denote our first approximation. Then, using equation (2.21), we see that the condition $\widetilde{\Delta} f_1|_{\mathcal{Q}} = 0$, for $(n + 2k - 2) \neq 0$, uniquely determines $u_1'(x, \rho)$ on $\mathcal{Q}$ by

$$u_1'(x, \rho)|_{\mathcal{Q}} = \frac{-\Delta u}{(n + 2k - 2)},$$

but if $k = 1 - \frac{n}{2}$, the obstruction is $t^{-1-\frac{n}{2}} \Delta u$. When $k = m - \frac{n}{2}$, by differentiating equation (3.5) $m - 1$ times with respect to ρ, and restricting to $\mathcal{Q}$, the obstruction is found, inductively, to be $t^{-m-\frac{n}{2}} \Delta^m u$. Clearly, this construction gives conformally invariant differential operators from $\mathcal{E}[m - \frac{n}{2}]$ to $\mathcal{E}[-m - \frac{n}{2}]$ which, in terms of our chosen conformal scale, are simply the powers of the Laplacian. If we change conformal scale, lower order terms appear in the formulae. □

3.3.2 $\mathcal{K}$

We define an invariant, I, of a conformal structure, $[g]$, of weight w, to be a polynomial in the variables $\nabla^{(l)} g_{ij}$ and $(\det g_{ij})^{-1}$ such that I is a Riemannian invariant (i.e. does not depend on the choice of coordinate system used to represent and differentiate g) which satisfies $I(\Omega^2 g) = \Omega^w I(g)$, for smooth positive functions, Ω.

We can construct such invariants using the ambient metric construction. Denote by $\widetilde{\nabla}^{(l)}u$ the tensor with components given by $\underbrace{\widetilde{\nabla}_A \ldots \widetilde{\nabla}_D}_{l} u$, where $\widetilde{\nabla}$ denotes the Levi-Civita connection of the ambient metric, $\tilde{g}$. To every contraction of the form

$$\mathrm{contr}(\widetilde{\nabla}^{(l_1)}\tilde{R} \otimes \ldots \otimes \widetilde{\nabla}^{(l_d)}\tilde{R})$$

(up to the order of obstruction) is associated an invariant of the conformal structure obtained by restricting to the metric bundle $\mathcal{Q}$ and pulling back to $\mathcal{M}$ by some choice of representative, g, of the conformal structure. In [FGm1], Fefferman and Graham conjecture that every invariant below the order of obstruction is a linear combination of invariants of this type.

In [FGm2], it is shown, using a "conformal normal form", that any invariant can be expressed as a polynomial in the components of $\tilde{R}$ and its derivatives. One can Taylor expand a representative of the conformal structure in Riemann normal coordinates at some fixed point, and normalise by demanding that to all orders of covariant derivative, $\nabla_{(a} \ldots \nabla_b R_{ij)} = 0$. There is a P-action on the space of such infinite order conformal structures, and the conformally equivalent normal forms are the orbits of this action.

The map from such normalised structures to components of covariant derivatives of $\tilde{R}$ turns out to be a bijection with polynomial inverse. The P-action on the covariant derivatives of $\tilde{R}$ is then a rescaling of the usual action of P on tensor powers of $\mathbb{R}^{n+2}$, and the problem is reduced to studying the invariant polynomials on this space.

The invariant theory for this problem follows from that for the following linearised problem: in terms of the flat model, we define $\mathcal{K}$ to be the space of jets at e_0 of functions, f, homogeneous of degree -2 on W, taking values in $\bigotimes^4 W^*$ such that f is trace free with respect to $\tilde{g}$, and

$$\begin{aligned} f_{IJKL} &= f_{[IJ][KL]}, \\ f_{[IJK]L} &= 0, \\ X^I f_{IJKL} &= 0, \\ \partial_{[I} f_{JK]LM} &= 0. \end{aligned} \tag{3.6}$$

One can easily show that a function satisfying the above conditions also satisfies

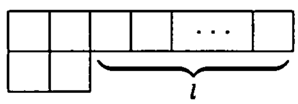
$$\partial^I f_{IJKL} = 0 \qquad \text{and} \qquad \widetilde{\Delta} f_{IJKL} = 0. \tag{3.7}$$

3.3.3 Results for $\mathcal{H}_k$ and $\mathcal{K}$

Denote by $W^{(l)}$, for $l \geq 0$, the submodule of $\bigotimes^{l+4} W^*$ consisting of totally trace-free tensors with the symmetries

$$\begin{aligned} T_{IJKL,AB...D} &= T_{[IJ][KL],(AB...D)}, \\ T_{[IJK]L,AB...D} &= 0, \\ T_{IJ[KL,A]B...D} &= 0, \end{aligned}$$

where the comma is used to separate the first four indices. $W^{(l)}$ is the irreducible representation of G with the Young diagram



In [BEGm] it is shown that, as P-modules,

$$\mathcal{H}_k = \left\{ \begin{array}{l} (T^{(k+1)}, T^{(k+2)}, \ldots) : T^{(l)} \in \bigodot_0^l W^* \otimes \sigma_{k-l}, \\ e \lrcorner T^{(l+1)} = (k-1)T^{(l)}, \text{ for } l > k,\ e \lrcorner T^{(k+1)} = 0 \end{array} \right\},$$

and that

$$\mathcal{K} = \left\{ \begin{array}{l} (T^{(0)}, T^{(1)}, \ldots) : T^{(l)} \in W^{(l)} \otimes \sigma_{-l-2},\ e^L T^{(0)}_{IJKL} = 0, \\ \text{and for } l \geq 0,\ e^L T^{(l+1)}_{IJKL,AB...D} = -(l+1) T^{(l)}_{IJK(A,B...D)} \end{array} \right\}.$$

For the module, $\mathcal{K}$, it follows that we also have the linking condition

$$e^A T^{(l+1)}_{IJKL,AB...D} = -(l+2) T^{(l)}_{IJKL,B...D} \qquad \text{for } l \geq 0.$$

The result of the invariant theory for $\mathcal{H}_k$ and $\mathcal{K}$ is the following theorem from [BEGm].

Theorem 3.14 *Let $d_{\mathcal{T}}$ denote n for $\mathcal{H}_k$ and $n/2$ for $\mathcal{K}$. Then*

- *Every invariant is a weak Weyl invariant.*
- *Every even invariant is a Weyl invariant.*
- *There are no odd invariants of degree $d < d_{\mathcal{T}}$.*
  Every odd invariant of degree $d_{\mathcal{T}}$ is exceptional.
  Every odd invariant of degree $d > d_{\mathcal{T}}$ is a Weyl invariant. $\square$

In [BGo], a set of generators for the exceptional invariants of $\mathcal{H}_k$ and $\mathcal{K}$ are given. For an illustration of the methods used, see 5.3.3.

Chapter 4

Conformal invariants of curves

Let $x(t)$ be a curve in S^n. For fixed c, one can consider the jet of $x(t)$ at $t = c$. In this chapter we attempt to list the polynomials (with real coefficients) in the components of such a jet which are conformally invariant. Since no such invariant polynomial can involve components of the 0-jet, we need only consider those conformal transformations fixing $x(c)$. We will use the flat model for conformal geometry and the notation of section 2.1. We can, without loss of generality, consider $x(t)$ in S^n such that $x(0) = [e_0]$, under the action of the subgroup P of G which fixes $[e_0]$ and we look for the polynomials in the jet of $x(t)$ at $t = 0$ which simply rescale under the P-action.

4.1 Preliminaries

Let $x(t)$ be a curve in S^n, with $x(0) = [e_0]$. Using coordinates $\mathbb{R}^n \to S^n$,

$$x^i \mapsto \begin{pmatrix} 1 \\ x^i \\ -\frac{1}{2}x^2 \end{pmatrix},$$

the set of jets of such curves is isomorphic to

$$J = \{(x_1, x_2, \ldots) : x_r \in \mathbb{R}^n\} \tag{4.1}$$

the isomorphism being given by $x_r^i = \frac{d^r}{dt^r} x^i(t)|_{t=0}$.

The P-action on S^n is induced from the usual P-action on $\mathbb{R}^{n+2}$ as follows :

Let $p = \begin{pmatrix} \lambda & r_j & c \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \in P$. Then

$$\left[\begin{pmatrix} \lambda & r_j & c \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ x^i \\ -\frac{1}{2}x^2 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ \frac{m^i{}_j x^j - \frac{1}{2}x^2 s^i}{\lambda + r_j x^j - \frac{1}{2}cx^2} \\ -\frac{x^2}{2\lambda(\lambda + r_j x^j - \frac{1}{2}cx^2)} \end{pmatrix}\right]$$

So we see that

$$(p.x)^i = \frac{m^i{}_j x^j - \frac{1}{2}x^2 s^i}{\lambda + r_j x^j - \frac{1}{2}cx^2}$$

when this exists. The P-action on the jet at $t = 0$ is obviously quite complicated, being non-linear in the components of the x_r (one sees that this action is, in fact, polynomial in the components of the x_r), but the action of the Levi factor, L, is reasonably straightforward, and is given by

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix} \cdot (x_1, x_2, \ldots) = (\lambda^{-1} m x_1, \lambda^{-1} m x_2, \ldots).$$

Due to the complicated nature of the P-action on J, the only invariants that we can easily find are polynomials in $x_1 \cdot x_1$. Various powers of this invariant turn up regularly in what follows, so we define $u = (x_1 \cdot x_1)^{1/2}$.The next section gives us an alternative description of J which will make the construction of invariants easier.

4.2 Isomorphism Theorem

Definition 4.1 Let $x(t)$ be a curve in S^n. A *lifting* of $x(t)$ is a curve $X(t)$ in $\mathcal{Q}$ such that $[X(t)] = x(t)$.

Proposition 4.2 *Let $x(t)$ be a curve in S^n with $x(0) = [e_0]$, defined on some neighbourhood of 0. If $\dot{x}(t) \neq 0$ then there exists a unique lifting, $X(t)$, satisfying*

$\dot{X}(t) \cdot \dot{X}(t) = 1$. *It is given in coordinates by*

$$X^I(t) = (\dot{x}(t) \cdot \dot{x}(t))^{-1/2} \begin{pmatrix} 1 \\ x^i(t) \\ \frac{-1}{2} x^2(t) \end{pmatrix} \tag{4.2}$$

We will refer to this lifting as the preferred lifting. □

We can regard the jet of this preferred lifting as being an element of the set, $\{(X_0, X_1, X_2, \ldots) : X_s \in \mathbb{R}^{n+2}\}$, by setting $X_r^I = \frac{d^r}{dt^r} X^I(t)|_{t=0}$. One can easily check that, for such a jet, the following hold :

$$\begin{gathered} X_0 = u^{-1} e_0, \quad X_0 \cdot X_1 = 0, \quad X_0 \cdot X_2 = -1, \quad X_0 \cdot X_3 = 0, \\ X_1 \cdot X_1 = 1, \quad X_1 \cdot X_2 = 0 \end{gathered} \tag{4.3}$$

for some $u > 0$, and for $k = 1, 2, 3, \ldots$

$$\sum_{i=1}^{k} \binom{k-1}{i-1} X_i \cdot X_{k+2-i} = 0 \quad \text{and} \quad \sum_{i=1}^{k} \binom{k-1}{i-1} X_{i-1} \cdot X_{k+3-i} = 0 \tag{4.4}$$

Definition 4.3 Let $J_0 = \{(x_1, x_2, x_3, \ldots) \in J : x_1 = 0\}$, let

$\mathcal{J} = \{(X_0, X_1, X_2, \ldots) : X_r \in \mathbb{R}^{n+2}$, and equations (4.3) and (4.4) are satisfied$\}$,

and let $\phi : J \setminus J_0 \to \mathcal{J}$ be the map on jets induced by the construction above.

Note that $J \setminus J_0$ and $\mathcal{J}$ are not P-modules, as they are not vector spaces. The main result of this section is the following :

Theorem 4.4 *The map*

$$\phi : J \setminus J_0 \to \mathcal{J}$$

is a bijection and the induced P-action on $\mathcal{J}$ is given by

$$p \cdot (X_0, X_1, \ldots) = (p \cdot X_0, p \cdot X_1, \ldots),$$

where $p \in P$, $(X_0, X_1, \ldots) \in \mathcal{J}$ and the action on the right hand side is the usual action of P on $\mathbb{R}^{n+2}$.

Proof. We prove the last part first. For a curve, $x(t)$, with lifting $X(t)$, it is clear that, for $p \in P$ acting on $\mathbb{R}^{n+2}$ in the usual way, $(p \cdot X)(t)$ is a lifting of $(p \cdot x)(t)$. Since this lifting has the required properties, it must be the preferred lifting of $(p \cdot x)(t)$, and the action on $\mathcal{J}$ follows.

Suppose we have a curve, $x(t)$, such that $x(0) = [e_0]$ and $\dot{x}(0) \neq 0$. In coordinates, let $x^i_r = \frac{d^r}{dt^r} x^i(t)\big|_{t=0}$ for $r = 1, 2, \ldots$, and let $f(t) = (\dot{x}(t) \cdot \dot{x}(t))^{-1/2}$. We define $X^I(t)$ as before and let $X^I_r = \frac{d^r}{dt^r} X^I(t)\big|_{t=0}$. From the definition, we see that $X^0_r = \frac{d^r}{dt^r} f(t)\big|_{t=0}$ and from this we get

$$
\begin{aligned}
X^i_r &= \left.\frac{d^r}{dt^r}\left(f x^i\right)\right|_{t=0} \\
&= \left.\sum_{s=1}^{r} \binom{r}{s} \left(\frac{d^s}{dt^s} x^i\right) \left(\frac{d^{r-s}}{dt^{r-s}} f\right)\right|_{t=0} \\
&= u^{-1} x^i_r + \sum_{s=1}^{r-1} \binom{r}{s} x^i_s X^0_{r-s} \qquad \text{for } r = 1, 2, \ldots.
\end{aligned} \tag{4.5}
$$

Or alternatively, we can turn this around, so that if we are given $(\tilde{X}_0, \tilde{X}_1, \ldots) \in \mathcal{J}$, we have

$$
x^i_r = \tilde{u}\left(\tilde{X}^i_r - \sum_{s=1}^{r-1} \binom{r}{s} \tilde{X}^0_{r-s} x^i_s\right) \tag{4.6}
$$

for $r = 1, 2, \ldots$, and where $\tilde{u} = (\tilde{X}^0_0)^{-1}$.

We have already seen that $\mathrm{Im}\phi \subset \mathcal{J}$. From (4.6) one sees that ϕ is injective, so to prove that $\mathrm{Im}\phi = \mathcal{J}$ it is sufficient to prove that $\phi : J \setminus J_0 \to \mathcal{J}$ is a surjection.

Now suppose we are given $(\tilde{X}_0, \tilde{X}_1, \ldots) \in \mathcal{J}$. Since equations (4.3) are satisfied, $\tilde{X}^0_0 = \tilde{u}^{-1}$ for some $\tilde{u} > 0$, so we can use equation (4.6) to define $(x_1, x_2, \ldots)$. Let $\phi(x_1, x_2, \ldots) = (X_0, X_1, \ldots) \in \mathcal{J}$. We use equations (4.3), (4.4), (4.5) and (4.6) to show that $(X_0, X_1, \ldots) = (\tilde{X}_0, \tilde{X}_1, \ldots)$ and hence that ϕ is surjective.

Our proof of this is by induction. We will only prove the general step, although we list the features of the start of the induction, some of which will be needed for the general argument. When we refer to one of equations (4.3) in the following list, we will mean that the result on the right hand side follows from the fact that both $(X_0, X_1, \ldots)$ and $(\tilde{X}_0, \tilde{X}_1, \ldots)$ satisfy this equation.

$$\begin{array}{llrl}
X_0 = u^{-1}e_0 & \Rightarrow & X_0^i & = \tilde{X}_0^i, X_0^\infty = \tilde{X}_0^\infty \\
X_0 \cdot X_1 = 0 & \Rightarrow & X_1^\infty & = \tilde{X}_1^\infty = 0 \\
X_1 \cdot X_1 = 1 \text{ with (4.6), (4.5)} & \Rightarrow & X_0^0 & = \tilde{X}_0^0 = u^{-1} \\
 & & \text{and } X_1^i & = \tilde{X}_1^i \\
X_0 \cdot X_2 = -1 & \Rightarrow & X_2^\infty & = \tilde{X}_2^\infty = -u \\
X_1 \cdot X_2 = 0 \text{ with (4.6), (4.5)} & \Rightarrow & X_1^0 & = \tilde{X}_1^0 \\
 & & \text{and } X_2^i & = \tilde{X}_2^i \\
X_0 \cdot X_3 = 0 & \Rightarrow & X_3^\infty & = \tilde{X}_3^\infty = 0
\end{array}$$

The proof of these is very similar to the general step. As our inductive hypothesis, we assume that for $k \geq 3$ we have the following :

$$\begin{array}{rll}
X_r^0 & = \tilde{X}_r^0 & \text{for } r = 1, 2, \ldots, k-2, \\
X_r^i & = \tilde{X}_r^i & \text{for } r = 1, 2, \ldots, k-1, \\
\text{and } X_r^\infty & = \tilde{X}_r^\infty & \text{for } r = 1, 2, \ldots, k.
\end{array}$$

Under this hypothesis, it follows from equations (4.5) and (4.6), together with the above results, that

$$X_k^i = \tilde{X}_k^i + ukX_1^i(X_{k-1}^0 - \tilde{X}_{k-1}^0). \tag{4.7}$$

Then from (4.7), the above results and our inductive hypothesis, one finds that

$$\begin{array}{rll}
X_1 \cdot X_k & = \tilde{X}_1 \cdot \tilde{X}_k + ku(X_{k-1}^0 - \tilde{X}_{k-1}^0), & \\
X_2 \cdot X_{k-1} & = \tilde{X}_2 \cdot \tilde{X}_{k-1} - u(X_{k-1}^0 - \tilde{X}_{k-1}^0), & \\
\text{and} \quad X_s \cdot X_{k-s+1} & = \tilde{X}_s \cdot \tilde{X}_{k-s+1} & \text{for } s = 3, \ldots, k-2.
\end{array}$$

When we substitute these into $\sum_{s=1}^{k-1} \binom{k-2}{s-1} X_s \cdot X_{k-s+1} = 0$ we find that this gives

$$\sum_{s=1}^{k-1} \binom{k-2}{s-1} \tilde{X}_s \cdot \tilde{X}_{k-s+1} + u(X_{k-1}^0 - \tilde{X}_{k-1}^0) = 0.$$

So we have $X_{k-1}^0 = \tilde{X}_{k-1}^0$ from which, with (4.7), we see that $X_k^i = \tilde{X}_k^i$. The result, $X_{k+1}^\infty = \tilde{X}_{k+1}^\infty$, follows immediately from $X_0 \cdot X_{k+1} = u^{-1}X_{k+1}^\infty$ and the second equation of (4.4). □

4.3 Invariant Theory

In this section we show that the construction given in the previous section can be used to give a set of generators for a particular class of invariants.

Definition 4.5 An *invariant* is a P-equivariant map $I : J \to \sigma_q$ which is a polynomial with real coefficients in the components of the x_r.

Recall that the P-action on $\mathcal{J}$ is polynomial, so that this definition makes sense. Any P-invariant is, by restriction, also an invariant of the reductive subgroup L. According to Weyl's theory, odd and even invariants can be written respectively as linear combinations of complete contractions of the form

$$\begin{aligned} &\operatorname{contr}(g \otimes \ldots \otimes g \otimes x_{r_1} \otimes \ldots \otimes x_{r_d}) \\ \text{and} \qquad &\operatorname{contr}(\epsilon \otimes g \otimes \ldots \otimes g \otimes x_{r_1} \otimes \ldots \otimes x_{r_d}). \end{aligned} \tag{4.8}$$

The first tool we need is the following lemma, which tells us that although $\mathcal{J}$ is not isomorphic to the whole of J, we retain enough information to construct invariants.

Lemma 4.6 *If $I : J \to \sigma_q$ is a polynomial map and the restriction of I to $J \setminus J_0$ is P-equivariant, then $I : J \to \sigma_q$ is an invariant.*

Proof. Let $J^k \subset J$ denote the set of k–jets, and J_0^k denote the corresponding subset of J_0. For each $p \in P$, the maps I and $p \cdot I$ are polynomials, and are therefore, by restriction, maps $I : J^k \to \sigma_q$ and $p \cdot I : J^k \to \sigma_q$, for some k. Hence $\lambda^{-q} I - p \cdot I : J^k \to \sigma_q$ is a polynomial map, which is zero on $J^k \setminus J_0^k$. By continuity, it must be zero on the whole of J^k, and hence on J. $\square$

To use the set $\mathcal{J}$ to find invariants, it is helpful to know what some of the X_r look like in terms of the elements of J. One can calculate, for example, that

$$X_0 = \begin{pmatrix} u^{-1} \\ 0 \\ 0 \end{pmatrix}, X_1 = \begin{pmatrix} -\frac{1}{u^3}(x_1 \cdot x_2) \\ \frac{1}{u} x_1 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} * \\ \frac{1}{u} x_2 - \frac{2}{u^3}(x_1 \cdot x_2) x_1 \\ -u \end{pmatrix},$$

where $*$ denotes a term of the form u^{-5} multiplied by a linear combination of terms of the first type in (4.8).

Since P preserves $\tilde{g}$ and $\tilde{\epsilon}$, it is clear from the P-action on $\mathcal{J}$ that any linear combination of complete contractions of the forms

$$\begin{aligned} &\operatorname{contr}(\tilde{g} \otimes \ldots \otimes \tilde{g} \otimes X_{r_1} \otimes \ldots \otimes X_{r_d}) \\ \text{and} \quad &\operatorname{contr}(\tilde{\epsilon} \otimes \tilde{g} \otimes \ldots \otimes \tilde{g} \otimes X_{r_1} \otimes \ldots \otimes X_{r_d}) \end{aligned} \tag{4.9}$$

is invariant under the P-action, but one sees from the the form of the X_r, that such a contraction will only be polynomial in the components of J, and hence (by lemma 4.6) an invariant, when multiplied by a sufficiently large power of u. Clearly, an invariant arising from a contraction of the first type will be even, and an invariant arising from a contraction of the second type will be odd. We now show that every invariant can be expressed as $I = u^q I'$, for some $q \in \mathbb{Z}^+$, where I' is a linear combination of complete contractions of the above type, before returning to the problem of finding which of the $u^q I'$ are polynomial and hence invariants.

Theorem 4.7 *If $I : J \to \sigma_q$ is an invariant, then $I = u^q I'$, where $q \in \mathbb{Z}^+$ and I' is a linear combination of complete contractions of the form (4.9).*

Proof. We start by proving the theorem for even invariants, with the following lemmata:

Lemma 4.8 *Any even invariant $I : J \to \sigma_q$ can be written as a linear combination of complete contractions in the quantities*

$$\tilde{g}, \qquad u, \qquad X_r^I \qquad \textit{and} \qquad X_s^0.$$

Proof. We know already that any even invariant can be written as a linear combination of complete contractions of the first type in (4.8). We can use equation (4.6) to rewrite I in terms of u, g, X_r^i and X_s^0.

Since $g_{ij}X_r^iX_s^j = \tilde{g}_{IJ}X_r^IX_s^J - X_r^0X_s^\infty - X_r^\infty X_s^0$, we can rewrite I in terms of u, $\tilde{g}$, X_r^I, X_r^0 and X_r^∞. Finally, replacing X_r^∞ by $uX_r \cdot X_0$, we have written I in the desired form. □

Lemma 4.9 *Let $I : J \to \sigma_q$ be an even invariant. Then there exists $m \in \mathbb{N}$ and a map*

$$C : J \to \bigodot_0^m \mathbb{R}^{n+2} \otimes \sigma_{m+q}$$

given by a linear combination of partial contractions of the quantities $X_r, \tilde{g}, \tilde{g}^{-1}$ and u such that the component

$$C^{0\ldots0} = I,$$

with all the other components being zero.

Proof. We first express I as in lemma 4.8. Let m be the maximum number of uncontracted X_r^0 components that appear in any term. Since $X_0^0 = u^{-1}$, we can, by multiplying through by an appropriate number of uX_0^0's, write I so that each term has this number of (uncontracted) X_r^0 components. (By considering the L-action on $I, u, X_r \cdot X_s$ and X_r^0, it is also easy to see that the power of u occurring in each term is $q + m$.) If we now replace each uncontracted X_r^0 in this expression by X_r with a free upper case index, we obtain an expression which has I as its $0\ldots0$ component. We obtain an expression for C by taking the symmetric,trace free part of this expression, which leaves the $0\ldots0$ component unaffected (see also [BEGm], proof of lemma 2.3). We now have a map $C : J \to \bigodot_0^m \mathbb{R}^{n+2} \otimes \sigma_{m+q}$ with $C^{0\ldots0} = I$ invariant, and we define a P-equivariant map $\theta : J \to \bigodot_0^m \mathbb{R}^{n+2} \otimes \sigma_q$ by

$$\theta := u^{-m}C - X_0 \otimes \ldots \otimes X_0 I$$

which has the property that its $0\ldots0$ component vanishes. We need to prove that θ actually vanishes identically.

For some vector space, V, we have the L-direct sum decomposition

$$(\bigodot_0^m \mathbb{R}^{n+2} \otimes \sigma_q) \otimes \mathbb{C} = (\sigma_{q-m} \oplus_L V) \otimes \mathbb{C}$$

and it is clear that $\mathrm{Im}\theta \subset V$. But $\bigodot_0^m \mathbb{R}^{n+2} \otimes \mathbb{C}$ is an irreducible module for the complexified Lie algebra, $\mathfrak{g}_{\mathbb{C}}$, and $\sigma_{-m} \otimes \mathbb{C} \hookrightarrow \bigodot_0^m \mathbb{R}^{n+2} \otimes \mathbb{C}$ is its highest weight space. Since $\mathfrak{p}_{\mathbb{C}}$ contains all the raising operators of $\mathfrak{g}_{\mathbb{C}}$, any $\mathfrak{p}_{\mathbb{C}}$-submodule of an irreducible $\mathfrak{g}_{\mathbb{C}}$-module not containing the highest weight space must be trivial, since otherwise, any non-zero vector could be raised by elements of $\mathfrak{p}_{\mathbb{C}}$ to a non-zero highest weight vector. The P-module obtained by taking the linear span of elements of $\mathrm{Im}\theta$, $\langle \mathrm{Im}\theta \rangle \otimes \mathbb{C}$, is seen to be such a submodule of $(\bigodot_0^m \mathbb{R}^{n+2} \otimes \sigma_q) \otimes \mathbb{C}$ and so θ must vanish. □

To complete the proof of theorem 4.7 for even invariants, we find that given C as in lemma 4.9, since $X_2^\infty = -u$, and since the expression for each term in C contains u^{q+m},

$$I = (-u)^{-m} \tilde{g}_{I_1 J_1} \ldots \tilde{g}_{I_m J_m} C^{I_1 \ldots I_m} X_2^{J_1} \ldots X_2^{J_m}$$

is an expression for I of the required form. □

Proof of theorem 4.7 for odd invariants. The proof is analogous to that for even invariants. We start with our invariant expressed in terms of contractions of the second type in (4.8). Let $(\tilde{\epsilon}_0)_{J \ldots K} = u X_0^I \tilde{\epsilon}_{IJ \ldots K}$. Then $(\tilde{\epsilon}_0)_{\infty J \ldots K}$ is zero if any of $J \ldots K$ takes the value 0 or ∞, and agrees with $\epsilon_{j \ldots k}$ otherwise. So we can replace any occurrence of ϵ by $\tilde{\epsilon}_0$ and continue as in the case of even invariants, but including an $\tilde{\epsilon}_0$ in each term. The concluding highest weight argument is unchanged. □

We now return to the problem of finding which of the $I = u^q I'$ are polynomial, where I' is a linear combination of complete contractions of the form 4.9. The first step is the following:

Lemma 4.10

(i) $u^{2p} X_r \cdot X_s$ *is polynomial whenever* $p \geq (r+s+2)$ *and is therefore an invariant for such* p.

(ii) $u^{2p+n} |X_{r_1} \ldots X_{r_{n+2}}|$ *is polynomial whenever* $p \geq \left(\sum_{i=1}^{n+2} r_i \right) - 2 - \frac{1}{2} n(n+1)$ *and is therefore an invariant for such* p.

Proof. One simply looks at the most negative powers of u occurring in the expression for X_r and follow this through the expressions for $X_r \cdot X_s$ and $|X_{r_1} \ldots X_{r_{n+2}}|$ to find the maximum number of u's that might be needed. □

To get a better grip on the power of u that is needed, we introduce the notion of time homogeneity.

Definition 4.11 Treating an element of J as though it were an actual curve, we say a function, f, on J is *time homogeneous of degree* k if for $x(t) \in J$, $\lambda \in \mathbb{R}$, we have

$$f(x(\lambda t)) = \lambda^k f(x(t)).$$

In terms of jets, this is $f((\lambda x_1, \lambda^2 x_2, \lambda^3 x_3, \ldots)) = \lambda^k f((x_1, x_2, x_3, \ldots))$. The degree, k, is the total number of time derivatives occurring in the expression for f. For example, u^n is time homogeneous of degree n.

Lemma 4.12 *Any invariant is the sum of time homogeneous parts, each of which is separately invariant.*

Proof. Since the P-action commutes with such a reparametrisation, if $q(x)$ is a time homogeneous polynomial of degree k, then $q(p \cdot x)$ is also time homogeneous of degree k, so the time homogeneous parts of an invariant must be invariant. □

Thus we need only consider time homogeneous invariants. Henceforth, invariants under consideration will be assumed to be time homogeneous. We can now give a list of all time homogeneous invariants satisfying certain conditions.

Theorem 4.13

(i) Let $I = u^q I'$ be time homogeneous of degree k, where I' is a linear combination of complete contractions of the first type in 4.9. If $3q \geq 2k$ and q is even, then I is polynomial and hence an even invariant. Such invariants span the vector space of even invariants satisfying these conditions.

(ii) Let $I = u^q I'$ be time homogeneous of degree k, where I' is a linear combination of complete contractions of the second type in 4.9. If $3q \geq 2k - n(n-2)$ and $q+n$ is even, then I is an odd invariant. Such invariants span the vector space of odd invariants satisfying these conditions.

Proof. These results follow from theorem 4.7 and lemma 4.12, together with the fact that each component of X_r is time homogeneous of degree $r-1$. □

Example 4.14 The following are invariants :

$$u^4 X_2 \cdot X_2 = 2u^2(x_1 \cdot x_3) + 3u^2(x_2 \cdot x_2) - 6(x_1 \cdot x_2)^2$$

and, when n=2, $$u^4 |X_0\ X_1\ X_2\ X_3| = u^2 |x_3\ x_1| + 3(x_1 \cdot x_2)|x_1\ x_2|.$$

We have proved that we can list every time homogeneous odd or even invariant satisfying certain inequalities. However, there are invariants that do not satisfy these inequalities. There are two obvious ways to find examples of such invariants.

Firstly, suppose $n \geq 3$, so that $n(n-2) > 0$. Then we can find odd invariants $I : J \to \sigma_q$, which are time homogeneous of degree k, with $3q < 2k$. Multiplying any two such invariants together will give an even invariant which does not satisfy the above inequality.

Example 4.15 Let $n = 3$. Then $u^7\,|X_0\ X_1\ X_2\ X_3\ X_4|$ is an odd invariant with $q = 7$ and $k = 12$, and

$$u^{14}\,|X_0\ X_1\ X_2\ X_3\ X_4|^2 = \begin{vmatrix} (X_0 \cdot X_0) & \cdots & (X_0 \cdot X_4) \\ \vdots & \ddots & \vdots \\ (X_4 \cdot X_0) & \cdots & (X_4 \cdot X_4) \end{vmatrix}$$

(from equation (3.1)) is an even invariant with $q = 14$ and $k = 24$, so $3q \not\geq 2k$.

Secondly, suppose we have two independent even invariants $I_1, I_2 : J \to \sigma_q$ which are time homogeneous of degree k, and $3q = 2k$. It is easy to see that the terms in each invariant containing no powers of u^2 must be the same up to scalar multiple. By taking a suitable linear combination of I_1 and I_2, then, we get an invariant with each term containing u^2. Dividing by u^2 gives an invariant that is not on our list.

Example 4.16

$$u^4(4X_3 \cdot X_3 - 3(X_2 \cdot X_2)^2) =$$
$$4u^2(x_3 \cdot x_3) - 24(x_1 \cdot x_2)(x_2 \cdot x_3) + 12(x_2 \cdot x_2)(x_1 \cdot x_3) + 9(x_2 \cdot x_2)^2$$

is an invariant with $k = 8$ and $q = 4$.

We do not know how to list these other invariants.

4.4 Conformal Differential Invariants of Curves

The idea of an invariant as given above is distinct from, although related to, the usual idea of a conformal differential invariant of a curve, the difference being that we have only been interested in those invariants which are polynomial. We will again use the flat model for conformal geometry, but we will assume that the preserved metric on $\mathbb{R}^{n+2}$ has the form

$$\tilde{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_n & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Definition 4.17 Let J^m denote the set of m-jets at $t = 0$ of curves, $x(t)$, in S^n such that $x(0) = [e_0]$. A *differential invariant* is a P-invariant smooth map $D : J^m \to \mathbb{R}$ for some m.

In [Gn], it is shown that on a dense subset of suitably non-degenerate curves in S^n there are n fundamentally independent conformal differential invariants. The algebra of differential invariants is the algebra generated by these n invariants and their derivatives.

These invariants can be found using Cartan's method of moving frames. For any lifting of the curve, h say, to G, the pullback of the Maurer-Cartan form on G, $h^{-1}dh$, is an element of the Lie algebra $\mathfrak{g}$ of G. To each suitable map to $S^n = G/P$, one associates a canonical lifting, $\tilde{h}$, to G such that $\tilde{h}^{-1}d\tilde{h}$ takes values in a particular subspace of $\mathfrak{g}$. The components of $\tilde{h}^{-1}d\tilde{h}$ are the independent differential invariants. See [Gn] for more details of this method in general.

We will show how to find such a lifting to G by using the lifting of the curve to $\mathbb{R}^{n+2}$ as in equation (4.2) and then using an analogue of the Serret-Frenet formulae.

4.4.1 The Serret-Frenet Formulae

Before we study the conformal case, we see how this method works in the familiar case of invariants of curves in $\mathbb{R}^3$ under Euclidean transformations, by using the Serret-Frenet formulae.

Recall that for a curve, $y_0(t)$ in $\mathbb{R}^3$, one can find $y_1(t)$, $y_2(t)$ and $y_3(t)$ (the unit tangent, unit normal and unit binormal) such that y_1, y_2 and y_3 are orthonormal with respect to the standard inner product, by setting

$$\begin{aligned} vy_1 &= \dot{y}_0, && \text{where } v = |\dot{y}_0|, \\ \kappa y_2 &= \dot{y}_1, && \text{where } \kappa = |\dot{y}_1|, \\ \tau y_3 &= \dot{y}_2 + \kappa y_1, && \text{where } \tau = |\dot{y}_2 + \kappa y_1|. \end{aligned}$$

Euclidean space, $\mathbb{R}^3$, is the homogeneous space, E/H, where E is the group of Euclidean motions and H is the stabiliser of the origin. We regard E as being the matrix group with elements of the form

$$\tilde{A} = \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \qquad \text{where } A \in O(3),\ b \in \mathbb{R}^3,$$

acting by $\tilde{A} \cdot x = Ax + b$, for $x \in \mathbb{R}^3$. The subgroup, H, consists of those elements with $b = 0$. Let m be the matrix with columns y_1, y_2 and y_3. As $y_i \cdot y_j = \delta_i^{\ j}$ for $i, j = 1, 2, 3$, we have $m^t m = I_3$, so that $m \in O(3)$, and

$$M = \begin{pmatrix} m & y_0 \\ 0 & 1 \end{pmatrix}$$

gives a canonical lifting of y_0 to E. As $\dot{y}_3 = -\tau y_2$, the pull-back of the Maurer-Cartan form has the form

$$M^{-1}dM = \begin{pmatrix} 0 & -\kappa & 0 & v \\ \kappa & 0 & -\tau & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} dt.$$

The components of this matrix, v, κ and τ (the velocity, curvature and torsion), are the required invariants.

4.4.2 Conformal Analogue of the Serret-Frenet formulae

Proposition 4.18 *Let $x(t)$ be a curve in S^n and let $X(t)$ be the preferred lifting to Q. If $X(t)$ together with its first $n+1$ derivatives (regarded as vectors in $T_{X(t)}\mathbb{R}^{n+2}$) form a basis for $T_{X(t)}\mathbb{R}^{n+2}$ in some neighbourhood of $t = 0$, then we can define*

$$\begin{aligned}
Y_0 &= X(t), & & \\
Y_1 &= \dot{Y}_0, & & \\
Y_2 &= -(\dot{Y}_1 + \tau Y_0), & &\text{where } \tau = \tfrac{1}{2}\dot{Y}_1 \cdot \dot{Y}_1, \\
\kappa_1 Y_3 &= -\dot{Y}_2 + \tau Y_1, & &\text{where } \kappa_1 = |\dot{Y}_2 - \tau Y_1|, \\
\kappa_2 Y_4 &= \dot{Y}_3 - \kappa_1 Y_0, & &\text{where } \kappa_2 = |\dot{Y}_3 - \kappa_1 Y_0|, \\
\kappa_3 Y_5 &= \dot{Y}_4 + \kappa_2 Y_3, & &\text{where } \kappa_3 = |\dot{Y}_4 + \kappa_2 Y_3|, \\
&\vdots & &\ \ \vdots \qquad \vdots \\
\kappa_{n-1} Y_{n+1} &= \dot{Y}_n + \kappa_{n-2} Y_{n-1}, & &\text{where } \kappa_{n-1} = |\dot{Y}_n + \kappa_{n-2} Y_{n-1}|.
\end{aligned}$$

Then $Y_0 \cdot Y_2 = 1$ and $Y_i \cdot Y_i = 1$, for $i = 1, 3, 4, \ldots n+1$, with the remaining dot products vanishing, so $\{Y_0, \ldots, Y_{n+1}\}$ is an orthonormal basis for $T_{X(t)}\mathbb{R}^{n+2}$, with respect to $\tilde{g}$.

Corollary 4.19 *Let h be the matrix with columns Y_0, Y_1, Y_3, Y_4,..., Y_{n+1}, Y_2, i.e.*

$$h = (Y_0\ Y_1\ Y_3\ Y_4\ \ldots Y_{n+1}\ Y_2).$$

Then h satisfies $h^t\tilde{g}h = \tilde{g}$ so that $h \in G$. The pullback of the Maurer-Cartan form, $h^{-1}dh$, is

$$\begin{pmatrix}
0 & -\tau & \kappa_1 & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & \tau \\
0 & 0 & 0 & -\kappa_2 & \cdots & 0 & 0 & -\kappa_1 \\
0 & 0 & \kappa_2 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & -\kappa_{n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & \kappa_{n-1} & 0 & 0 \\
0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0
\end{pmatrix} dt,$$

and by evaluating τ,κ_1,...,κ_{n-1} at $t = 0$ we obtain our differential invariants.

4.5 Curves in Curved Conformal Geometry

In curved conformal geometries, by using the conformal tractor bundle, we can carry out analogous constructions to those given earlier to yield curved analogues of the invariants discussed earlier in this chapter. We begin by giving the description of conformal circles from the point of view of [BEGo], which introduces the idea of associating tractors to a curve.

Let $\mathcal{M}$ be a manifold with a conformal structure, and let $x(t)$ be a curve in $\mathcal{M}$, parametrised such that $\nabla_i t \neq 0$ along $x(t)$. Choose a tangent vector $\dot{x}^i$ along $x(t)$ such that $\dot{x}^i \nabla_i t = 1$, and let $v = (\dot{x}_i \dot{x}^i)^{\frac{1}{2}}$. We denote $\dot{x}^i \nabla_i$ by $\frac{d}{dt}$ and $\frac{d\dot{x}^i}{dt}$ is denoted $\ddot{x}^i$.

4.5.1 Conformal Circles

Conformal circles are the analogue for conformal geometry of geodesics in Riemannian geometry (see e.g. [BE], [E] and [BEGo]). A curve, $x(t)$, is a projectively parametrised conformal circle if

$$\frac{d^3 x^j}{dt^3} = 3\frac{\dot{x} \cdot \ddot{x}}{\dot{x}^2}\ddot{x}^j - \frac{3\ddot{x}^2}{2\dot{x}^2}\dot{x}^j + \dot{x}^2 \dot{x}^i P^j_{\ i} - 2P_{kl}\dot{x}^k \dot{x}^l \dot{x}^j.$$

An alternative description is given in [BEGo]: in the notation of section 2.2, the *velocity tractor* and *acceleration tractor* are defined by

$$U^I = \frac{d}{dt}\left(v^{-1} X^I\right) \qquad \text{and} \qquad A^I = \frac{d}{dt} U^I,$$

respectively. (Here, $\frac{d}{dt}$ denotes $\dot{x}^i \nabla_i$, where ∇_i is the tractor connection). As v and X^I both have conformal weight 1, $v^{-1}X^I$ has conformal weight 0, and so U^I and A^I are conformally invariant. The condition $A^I A_I = 0$ can be regarded as a condition on the parametrisation and, as such, gives a preferred family of parametrisations for any curve. A curve for which $A^I A_I = 0$ is said to be *projectively parametrised.* In [BEGo], it is shown that a curve is a projectively parametrised conformal circle if and only if

$$A^I A_I = 0 \qquad \text{and} \qquad \frac{dA^I}{dt} = 0.$$

4.5.2 Curved Analogues

In order to construct invariants, we associate a list of tractors to each regular curve. Define for $r = 0, 1, 2, \ldots$,

$$Z_r^I = \frac{d^r}{dt^r}\left(v^{-1}X^I\right)\Big|_{t=0.}$$

As above, Z_r^I is conformally invariant for each r. We can construct polynomial invariants by taking a linear combination of complete contractions of the form (4.9) satisfying the conditions of theorem 4.13 and then replacing the X_r^I's by the Z_r^I's and using the tractor metric etc. It is easy to check that in the flat case, $(Z_0^I, Z_1^I, Z_2^I, \ldots)$ is the element of $\mathcal{J}$ given by the lifting of the curve, $x(t)$, as in section 4.2. In this case, the above construction gives the same invariants as previously.

Similarly, one can construct analogues of the invariants of the previous section, by letting $Y_0 = v^{-1}X^I$, and continuing as before, using the same formulae for $Y_1, \ldots Y_{n+1}$ and $\tau, \kappa_1, \ldots, \kappa_{n-1}$, but using the d/dt given as above. This again gives the same invariants in the flat case. Note that $\tau = 0$ for projectively parametrised curves.

Chapter 5

Conformal Invariants of Differential Forms

In this chapter we will be considering jets of closed differential k-forms at $[e_0]$ in S^n, $(1 \leq k < n)$, and looking for the polynomials in the components of such a jet which are invariant under the action of P. Such polynomials define scalar valued G-equivariant differential operators on k-forms.

In the first section, we introduce the operators we will be using in the second section, and list the commutators of these which we will need. Secondly, we find a P-module isomorphism between the set, J_k, of such jets, and the set of jets of tensor valued functions on $\mathbb{R}^{n+2}$ satisfying certain conditions. The proof is related to the isomorphism theorem 3.10, and in a similar way, we are only partially successful, obtaining an isomorphism only in odd dimensions.

In the third section, the methods of [BEGm] and [Go2] are applied to the modules so obtained, to find which invariants of these modules are Weyl invariants. Finally, we complete the picture by describing the exceptional invariants, using the methods of [BGo].

In this chapter, we take $W = \mathbb{R}^{n+2}$ as the standard representation of P etc., and where square brackets are used to denote skew symmetrisation, the normalization will be slightly different from the usual notation. For an element, $T^{I \ldots K}$, of $\bigwedge^k W$, we normalize by $T^{[IJ \ldots K]} = kT^{IJ \ldots K}$. Similarly for symmetrisation.

5.1 Operators and Commutators

Recall that the operators used in the proof of theorem 3.10 could be thought of as the standard generators of the Lie algebra, $\mathfrak{sl}_2$. In this section, we will introduce the operators which appear in the proof of our isomorphism theorem and present the table of commutators/anti-commutators which we will need.

Let $f_{JK\ldots L}$ be a tensor valued function on W taking values in $\bigwedge^k W^*$, which is homogeneous of degree w. We define the operators Q, E, V, δ, d, $X\lrcorner$, $X\wedge$ and Δ as follows :

$$\begin{aligned}
(Qf)_{JK\ldots L} &= X^I X_I f_{JK\ldots L} \\
(Ef)_{JK\ldots L} &= X^I \partial_I f_{JK\ldots L} = w f_{JK\ldots L} \\
(Vf)_{JK\ldots L} &= k f_{JK\ldots L} \\
(\delta f)_{K\ldots L} &= \partial^J f_{JK\ldots L} \\
(df)_{IJ\ldots L} &= \partial_{[I} f_{JK\ldots L]} \\
(X\lrcorner\, f)_{K\ldots L} &= X^J f_{JK\ldots L} \\
(X\wedge f)_{IJ\ldots L} &= X_{[I} f_{JK\ldots L]} \\
(\Delta f)_{JK\ldots L} &= \partial^I \partial_I f_{JK\ldots L}
\end{aligned}$$

$E = X^I \partial_I$ is the Euler field, and we regard $Q = X^I X_I$ as the defining function of the null cone $\mathcal{Q}$. We set

$$h_1 = E + \tfrac{n+2}{2}, \qquad h_2 = -\tfrac{1}{2}(E+V),$$

$$\begin{aligned}
x_1 &= -\tfrac{1}{4}Q, & x_2 &= d, & x_3 &= \tfrac{1}{2}X\wedge, \\
y_1 &= \Delta, & y_2 &= -\tfrac{1}{2}X\lrcorner\,, & y_3 &= \delta.
\end{aligned}$$

Notice that h_1, x_1 and y_1 are the generators of $\mathfrak{sl}_2$, as in theorem 3.10. One can calculate the following table of commutators and anti-commutators :

Table 5.1

$A \backslash B$	h_1	h_2	x_1	y_1	x_2	y_2	x_3	y_3	
h_1	0	0	$2x_1$	$-2y_1$	$-x_2$	y_2	x_3	$-y_3$	
h_2		0	$-x_1$	y_1	0	0	$-x_3$	y_3	$[A,B]$
x_1			0	h_1	x_3	0	0	$-y_2$	
y_1				0	0	$-y_3$	x_2	0	
x_2					0	h_2	0	y_1	
y_2						0	$-x_1$	0	$AB+BA$
x_3							0	h_1+h_2	
y_3								0	

Lemma 5.2 *Let x, y and h be the standard generators of $\mathfrak{sl}_2$. The following hold :*

(i) $[y, x^m] = -mx^{m-1}(h + m - 1)$.

(ii) $[y^m, x^m] = \sum_{j=1}^{m}(-1)^j \binom{m}{j}^2 j! x^{m-j} y^{m-j}(h + j - 1)(h + j - 2)\ldots(h)$.

Corollary 5.3 *By substituting $x = -\frac{1}{4}Q$, $y = \Delta$ and $h = E + \frac{n+2}{2}$, we find :*

(i) $[\Delta, Q^m] = 2mQ^{m-1}(n + 2E + 2m)$,

(ii) $[\Delta^m, Q^m] =$
$- \sum_{j=1}^{m} 2^j \binom{m}{j}^2 Q^{m-j}\Delta^{m-j}(n + 2E + 2j)(n + 2E + 2j - 2)\ldots(n + 2E + 2)$.

5.2 Isomorphism Theorem

Let u be a given k-form on S^n, for $0 < k < n$. By pulling this back to $\mathcal{Q}$ and choosing an arbitrary homogeneous formal power series extension off $\mathcal{Q}$, we define a function, f, homogeneous of degree $-k$, on W taking values in $\bigwedge^k W^*$ such that

$$f_{IJ\ldots K} dX^I \wedge dX^J \wedge \ldots \wedge dX^K\big|_{T^*\mathcal{Q}} = \pi^* u,$$

$$(X \lrcorner f)|_{T^*\mathcal{Q}} = 0 \qquad \text{and} \qquad (df)|_{T^*\mathcal{Q}} = 0. \tag{5.1}$$

By, for example $h_{IJ\ldots K}|_{T^*\mathcal{Q}}$ where h is a skew tensor valued function, we mean that the domain of the function is restricted to $\mathcal{Q}$ and also that the corresponding k-form, $h_{IJ\ldots K} dX^I \wedge dX^J \wedge \ldots \wedge dX^K$, is restricted to $T^*\mathcal{Q}$.

Of course, there will be many such functions for a particular k-form, u. Our aim is to find, by choosing a suitable formal power series extension off $\mathcal{Q}$, a function $\tilde{f}$ satisfying

$$(X \lrcorner \tilde{f}) = 0, \qquad (d\tilde{f}) = 0 \qquad \text{and} \qquad (\delta \tilde{f}) = 0 \tag{5.2}$$

to all orders, in addition to the first condition in (5.1).

Remark 5.4 For a function, $\tilde{f}$, satisfying $(d\tilde{f}) = 0$ and $(X \lrcorner \tilde{f}) = 0$, the condition, $(\delta \tilde{f}) = 0$, is equivalent to $\Delta \tilde{f} = 0$, since $\delta = [\Delta, X \lrcorner]$ and $\Delta = d\delta + \delta d$.

Definition 5.5 We say that two functions taking values in $\bigwedge^s W^*$ are equal *mod* Q^m , if $g = h + Q^{m-1} X \wedge a + Q^m b$ where a and b are suitable (skew-symmetric) tensor valued functions.

Then, since $X_I Y^I = 0$ on $\mathcal{Q}$, for any vector $Y \in T\mathcal{Q}$, $f = 0$ mod Q is equivalent to $f|_{T^*\mathcal{Q}} = 0$. The above discussion motivates the following :

Definition 5.6 For a closed k-form, u, on S^n, and $m \geq 2$, we say that a function, f, on W is a *good extension (of u) of order m*, if f is homogeneous of degree $-k$ and takes values in $\bigwedge^k W^*$ such that

(i) $f_{IJ\ldots K}dX^I \wedge dX^J \wedge \ldots \wedge dX^K\big|_{T^*Q} = \pi^* u$,

(ii) $X \lrcorner f = 0 \bmod Q^m$,

(iii) $df = 0 \bmod Q^m$,

(iv) $\delta f = 0 \bmod Q^{m-1}$.

We regard a function satisfying (5.1) as being a good extension of order 1.

Proposition 5.7 *Suppose u is a closed k-form on S^n, f is a good extension of u of order m, for $m \in \mathbb{N}$, and suppose that $(n-2k-2m+6)$, $(n-2k-2m+4)$, $\ldots$,$(n-2k-4m)$ are non-zero. Then there exist tensor valued functions a and b such that $f + Q^{m-1}X \wedge a + Q^m b$ is a good extension of order $m+1$. This extension is unique (mod Q^{m+1}).*

Proof. We will consider the case $k > 1$ and $m > 1$, the cases $k = 1$ and $m = 1$ being slight simplifications of this. Since f is a good extension of order m, there exist skew, tensor valued functions, c, e, r, s, v and w such that :

(i) $X \lrcorner f = Q^{m-1}X \wedge r + Q^m s$,

(ii) $df = Q^{m-1}X \wedge c + Q^m e$,

(iii) $\delta f = Q^{m-2}X \wedge v + Q^{m-1}w$.

Before we find a and b, we give a lemma listing some relationships between c, e, r, s, v and w. The proof of this lemma can be found in appendix A

Lemma 5.8 *For $(n-2k-2m+6)$, $(n-2k-2m+4)$, $\ldots$,$(n-2k-4m) \neq 0$, we have*

$$X \wedge r + X \wedge (X \lrcorner s) = 0, \tag{5.3}$$

$$(n-2k+2)X \wedge v + 2(m-1)X \wedge (X \lrcorner w) = 0 \bmod Q^2, \tag{5.4}$$

$$dc = 2me \bmod \mathrm{Q}, \tag{5.5}$$

$$2(m-1)X \wedge r = X \wedge v \bmod Q^2, \tag{5.6}$$

$$2ms = dr + X \lrcorner c \bmod \mathrm{Q}. \tag{5.7}$$

□

We continue, now, with the proof of proposition 5.7. If f is a good extension of order m, then the general good extension of order m is of the form

$$\tilde{f} = f + Q^{m-1} X \wedge a + Q^m b.$$

We see then that

(i') $X \lrcorner \tilde{f} = Q^{m-1}(X \wedge r - X \wedge (X \lrcorner a)) + Q^m(s + a + X \lrcorner b)$

(ii') $d\tilde{f} = Q^{m-1}(X \wedge c - X \wedge (da) + 2mX \wedge b) + Q^m(db + e)$

(iii') $\delta \tilde{f} = Q^{m-2}(X \wedge v - 2(m-1)X \wedge (X \lrcorner a)) + Q^{m-1}(w + (n-2k-2)a + 2mX \lrcorner b)$ mod Q^m

We see that if $\tilde{f}$ is to be a good extension of order $m+1$, we need the following:

1. $X \wedge r = X \wedge (X \lrcorner a) \bmod Q^2$
2. $s + a + X \lrcorner b = 0 \bmod Q$
3. $X \wedge c - X \wedge (da) + 2mX \wedge b = 0 \bmod Q^2$
4. $db + e = 0 \bmod Q$
5. $X \wedge v - 2(m-1)X \wedge (X \lrcorner a) = 0 \bmod Q^2$
6. $w + (n + 2k - 2)a + 2mX \lrcorner b = 0 \bmod Q$

In particular, from 2 and 6, we need $w + (n - 2k - 2m + 2)a - 2ms = 0$, so for $(n - 2k - 2m + 2) \neq 0$, a is uniquely determined (mod Q) by

$$a = \frac{2ms - w}{n - 2k - 2m + 2}. \tag{5.8}$$

From 3 we see that b is uniquely determined (mod Q) by

$$b = \frac{1}{2m}(da - c). \tag{5.9}$$

To complete the proof, we need to show that a and b satisfy 1,2,4 and 5. Since 1 and (5.6) imply 5, we need only show that 1,2 and 4 are satisfied.

To see that 4 is satisfied, we deduce from (5.9) that $db = -\frac{1}{2m} dc \bmod Q$, and 4 follows from (5.5).

For 1, we see from (5.8) that

$$\begin{aligned} X \wedge (X \lrcorner a) &= \frac{1}{n-2k-2m+2}(2mX \wedge (X \lrcorner s) - X \wedge (X \lrcorner w)) \bmod Q^2 \\ &= \frac{1}{n-2k-2m+2}\left(-2mX \wedge r + \frac{(n-2k+2)}{2(m-1)} X \wedge v\right) \bmod Q^2, \\ &\quad \text{(by (5.3) and (5.4))} \\ &= X \wedge r \bmod Q^2, \qquad \text{by (5.6).} \end{aligned}$$

For 2, we see from (5.9) that

$$\begin{aligned} s + a + X \lrcorner b &= s + a + \frac{1}{2m} X \lrcorner (da) - \frac{1}{2m} X \lrcorner c \bmod Q \\ &= s + a + \frac{1}{2m}(E+V)a - \frac{1}{2m} d(X \lrcorner a) - \frac{1}{2m} X \lrcorner c \bmod Q \end{aligned}$$

Applying lemma A.1 to 1, we get $d(X \lrcorner a) = dr \bmod Q^2$ for $(n-2k-2m+6) \neq 0$. Also we have $(E+V)a = -2ma$, and so we have

$$\begin{aligned} s + a + X \lrcorner b &= \frac{1}{2m}(2ms - X \lrcorner c - dr) \bmod Q \\ &= 0 \bmod Q \qquad \text{from (5.7).} \end{aligned}$$

□

Definition 5.9 Let

$$\mathcal{J}_k = \{\text{jets at } [e_0] \text{ closed k-forms on } S^n, \text{ of conformal weight } 0\},$$

$$J_k = \left\{ \begin{array}{l} \text{jets at } e_0 \text{ of functions on } W \text{ taking values in } \bigwedge^k W^*, \text{ homogeneous} \\ \text{of degree } -k, \text{ satisfying } df = 0,\ X \lrcorner f = 0 \text{ and } \Delta f = 0. \end{array} \right\}$$

The group, P, acts on a tensor valued function, u, on S^n by $(p \cdot u)(x) = \hat{u}(p^{-1} \cdot x)$, where $x \in S^n$, and $\hat{u}$ is obtained from u by the usual action of P on tensors on S^n. P acts on a tensor valued function, f, on W by $(p \cdot f)(X) = \hat{f}(p^{-1} \cdot X)$, for $X \in W$, where $\hat{f}$ is obtained similarly. From the definitions of the P-actions on tensors on S^n and W, it is clear that the P-action on a k-form, u, and a good extension, f, must be compatible. These P-actions give induced actions on J_k and $\mathcal{J}_k$.

Theorem 5.10 *There is a P-module isomorphism*

$$J_k \cong \mathcal{J}_k$$

for $1 \leq k \leq n-1$ if n is odd, or for $\frac{1}{2}n + 2 < k \leq n-1$ if n is even.

Proof. Given any closed k-form, u, on S^n, of conformal weight 0, we can choose a good extension of order 1. By repeatedly applying proposition 5.7, we see that we can find an extension of order m if

$$0 \notin \{n - 2k - 4m + 4, n - 2k - 4m + 6, \ldots, n - 2k + 4\}.$$

If n is even and $k \leq \frac{1}{2}n+2$, then we can, in general, find a good extension of order m only for $m < \frac{1}{4}(n - 2k + 4)$. However, if n is odd or if $n - 2k + 4 < 0$, then we are able to obtain a unique function, $\tilde{f}$, satisfying (5.2) to all orders. As the Taylor expansion of $\tilde{f}$ at e_0 is determined by that of u, we have an induced map $\mathcal{J}_k \to J_k$.

This map is injective, since if $\tilde{f}$ vanishes to infinite order at e_0, then by restriction to $\mathcal{Q}$, so does u. The map is also surjective, since any such $\tilde{f}$ is the extension of its restriction to $\mathcal{Q}$, as the extension is unique. Finally, since the P-action commutes with taking the extension, this map is a P-module isomorphism. $\square$

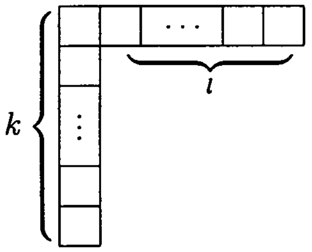
5.3 Invariant Theory

In this section we study the module J_k using the methods of [BEGm].

For $l \geq 0$ and $1 \leq k \leq n-1$, we denote by $W_k^{(l)}$ the G-submodule of $\bigotimes^{l+k} W^*$ consisting of totally trace free tensors with symmetries given by

$$\begin{aligned} T_{[IJ\ldots K],AB\ldots D} &= k\, T_{IJ\ldots K,AB\ldots D}, \\ T_{IJ\ldots K,(AB\ldots D)} &= l\, T_{IJ\ldots K,AB\ldots D}, \\ \text{and} \qquad T_{[IJ\ldots K,A]B\ldots D} &= 0, \end{aligned}$$

where we use the comma to separate the first k indices. Notice that $W_k^{(l)}$ is the irreducible representation of G with the Young diagram



Proposition 5.11 *As P-modules,*

$$J_k = \left\{ \begin{array}{l} (T^{(0)}, T^{(1)}, \ldots) : T^{(l)} \in W_k^{(l)} \otimes \sigma_{-l-k},\ e^L T^{(l+1)}_{IJ\ldots KL,AB\ldots D} = -T^{(l)}_{IJ\ldots K(A,B\ldots D)} \\ \textit{for } l \geq 0, \textit{ and } e^L T^{(0)}_{IJ\ldots KL} = 0 \end{array} \right\}$$

Such a list of tensors also satisfies, for $l \geq 0$,

$$e^A T^{(l+1)}_{IJ\ldots KL,AB\ldots D} = -(l+k) T^{(l)}_{IJ\ldots KL,B\ldots D} \tag{5.10}$$

Proof. Given a function, f, which is homogeneous of degree $-k$, taking values in $W^{(0)}$, satisfying equations (5.2), we define

$$T_{IJ\ldots K,AB\ldots D} = \mathrm{Eval}(\partial_A \partial_B \ldots \partial_D f_{IJ\ldots K}). \tag{5.11}$$

Recall that equations (5.2) also imply that $\Delta f = 0$, so that equations (5.2) give the $T^{(l)}$ the required symmetries and the trace free conditions.

The evaluation of $X^I f_{IJ...K} = 0$ at e_0 gives $e^L T^{(0)}_{IJ...KL} = 0$, and differentiating the former equation $(l+1)$-times gives the linking condition, $e^L T^{(l+1)}_{IJ...KL,AB...D} = -T^{(l)}_{IJ...K(A,B...D)}$. Conversely, given such a set of T's, equation (5.11) defines a homogeneous jet satisfying equations (5.2), so we have a (P-module) isomorphism.

The last statement of the proposition then follows immediately from Euler's equation for homogeneous functions. □

We define invariants of J_k, Weyl invariants and exceptional invariants as in definitions 3.7 and 3.8. For the first two contractions of (3.4), $q = -\sum_{i+1}^{d}(l_i + k)$, and for the third, $q = 1 - \sum_{i+1}^{d}(l_i + k)$.

Proposition 5.12 *The tensors $u^{(l)}$, for $l \geq 0$, defined by*

$$u^{(l)}_{i...j,a...d} = T^{(l)}_{i...j,a...d} \tag{5.12}$$

form a complete set of components for elements of J_k. They have the symmetries

$$\begin{aligned} u_{[i...j],ab...d} &= k u_{i...j,ab...d}, \\ u_{i...j,(ab...d)} &= l u_{i...j,ab...d} \\ \text{and} \qquad u_{[i...j,a]b...d} &= 0 \end{aligned} \tag{5.13}$$

but are otherwise unrestricted.

Proof. A general tensor $T \in W^{(l)}_k$ has as components

$$T_{\underbrace{0...0}_{p'}\underbrace{i...j}_{q'}\underbrace{\infty...\infty}_{r'}\underbrace{0...0}_{p}\underbrace{a...b}_{q}\underbrace{\infty...\infty}_{r}}, \qquad p' + q' + r' = k \text{ and } p + q + r = l. \tag{5.14}$$

(We can assume that $p' = 0$ or 1 and that $r' = 0$ or 1, for components not forced to vanish by the skew symmetry of the first k indices.) In the following, $C_1, \ldots C_5$ are unimportant, non-zero constants, depending on p, q, r, p', q' and r'.
Using the linking condition, (5.10), we find that

$$\begin{aligned} T_{\underbrace{0...0}_{p'}\underbrace{i...j}_{q'}\underbrace{\infty...\infty}_{r'}\underbrace{0...0}_{p}\underbrace{a...b}_{q}\underbrace{\infty...\infty}_{r}} &= C_1 T_{\underbrace{0...0}_{p'}\underbrace{i...j}_{q'}\underbrace{\infty...\infty}_{r'}\underbrace{0...0}_{r}\underbrace{a...b}_{q}\underbrace{\infty...\infty}_{r}} \\ &= C_2 T_{\underbrace{0...0}_{p'}\underbrace{i...j}_{q'}\underbrace{\infty...\infty}_{r'}\underbrace{a...b}_{q}\underbrace{{}^{c}c...{}^{d}d}_{2r}} \end{aligned}$$

since T is trace free, and since for $\psi \in \bigotimes^2 W^*$,

$$\tilde{g}^{IJ}\psi_{IJ} = g^{ij}\psi_{ij} + \psi_{\infty 0} + \psi_{0\infty}. \tag{5.15}$$

Then using $e^L T^{(l+1)}_{IJ\dots KL;AB\dots D} = -T^{(l)}_{IJ\dots K(A,B\dots D)}$, we find that

$$T_{\underbrace{\infty\dots\infty}_{r'}\underbrace{i\dots j}_{q'}0,\underbrace{a\dots b}_{q}\underbrace{c\dots d}_{2r}}{}^{c\ \ d} = C_3 T_{\underbrace{\infty\dots\infty}_{r'}\underbrace{i\dots j(a,}_{(q'+1)}\underbrace{\dots b)}_{q-1}\underbrace{c\dots d}_{2r}}{}^{c\ \ d} \tag{5.16}$$

Finally, from (5.10),

$$\begin{aligned}
T_{\underbrace{i\dots j}_{k-1}\infty,\underbrace{a\dots d}_{q-1}} &= C_4 T_{\underbrace{i\dots j}_{k-1}\infty,0\underbrace{a\dots d}_{q-1}0} \\
&= C_4(-T_{\underbrace{i\dots j}_{k-1}0,\infty\underbrace{a\dots d}_{q-1}0} - T_{\underbrace{i\dots j}_{k-1}e,}{}^{e}{}_{\underbrace{a\dots d}_{q-1}}) \\
&= C_5 T_{\underbrace{i\dots j}_{k-1}0,\underbrace{a\dots d}_{q-1}}{}^{e}{}_{e} - C_4 T_{\underbrace{i\dots j}_{k-1}e,}{}^{e}{}_{\underbrace{a\dots d}_{q-1}} \\
&= C_5 T_{i\dots j(a,\underbrace{b\dots d}_{q-2})}{}^{e}{}_{e} - C_4 T_{\underbrace{i\dots j}_{k-1}e,}{}^{e}{}_{\underbrace{a\dots d}_{q-1}}
\end{aligned}$$

So we see that such a component as in (5.14) can be written as the sum of terms of the form $\mathrm{tr}^{r+r'}(u^{(q+2r+r'-p')})$, where $\mathrm{tr}^m(u)$ denotes a tensor obtained from u by taking m traces. □

A P-invariant is, by restriction, an invariant of the reductive subgroup L. A typical element of L, $h = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$, acts on each $u^{(l)}$ by $h \cdot u^{(l)} = \lambda^{l+k} m \cdot u^{(l)}$ where m acts by the usual action of $SO(g)$ on covariant tensors.

By Weyl's theory, any L-invariant, I, can be uniquely written as $I = I_{even} + I_{odd}$ where I_{even} and I_{odd} are linear combinations of complete contractions of the forms (3.2) and (3.3) respectively. A P-invariant is said to be odd or even if it is so considered as an L-invariant. We consider odd and even invariants separately.

Theorem 5.13 *There are no non-zero odd invariants of degree $d < n/k$. Any odd invariants of degree $d = n/k$ are exceptional.*

Proof. Any odd invariant is a linear combination of complete contractions of the form (3.3), but at most k indices of ϵ can be contracted into each u if the result is to be non-zero, so if $kd < n$, such a contraction vanishes.

Similarly, since odd Weyl invariants are given by linear combinations of complete contractions of the second and third types in (3.4), there are no non-zero odd Weyl invariants of degree $d < \frac{n+1}{k}$. □

Proposition 5.14 *Every invariant, $I : J_k \to \sigma_q$, is a weak Weyl invariant. i.e. there exists $m \in \mathbb{N}$ and a map*

$$C : J_k \to \bigodot_0^m W \otimes \sigma_{m+q}$$

given by a linear combination of partial contractions of the quantities $T_{I\ldots K,A\ldots B}$, $\tilde{g}^{-1}$, e, and for odd invariants, $\tilde{\epsilon}_0$, such that the only non-zero component is

$$C^{0\ldots 0} = I.$$

The proof will follow after two lemmata (cf. proof of proposition 2.1 of [BEGm]).

Lemma 5.15 *Let $I : J_k \to \sigma_q$ be an even invariant. Then we can express I as a linear combination of complete contractions of the quantities*

$$\tilde{g}^{IJ}, \qquad T_{I\ldots J\infty,A\ldots D\infty\ldots\infty} \qquad \text{and} \qquad T_{I\ldots K,A\ldots D\infty\ldots\infty}$$

Proof. First, express I as a linear combination of terms of the form (3.2). We can replace the u's with the corresponding T's using (5.12). The g contractions can then be replaced by $\tilde{g}$ contractions using (5.15). The invariant, I, can be written in the desired form by eliminating the uncontracted 0-components of the T's using (5.16) and (5.10). □

Lemma 5.16 *Let $I : J_k \to \sigma_q$ be an odd invariant. Then we can express I as a linear combination of complete contractions of the quantities*

$$\tilde{g}^{IJ}, \qquad (\tilde{\epsilon}_0)^{IJ\ldots K}, \qquad T_{I\ldots J\infty,A\ldots D\infty\ldots\infty} \qquad \text{and} \qquad T_{I\ldots K,A\ldots D\infty\ldots\infty}$$

Proof. We substitute $(\tilde{\epsilon}_0)_{\infty}^{j\ldots k}$ for $\epsilon^{j\ldots k}$ in a contraction of the form (3.3), as in the proof of theorem 4.7 for odd invariants, then continue as in the proof for even invariants, but including $\tilde{\epsilon}_0$ in each term. □

Proof of proposition 5.14. We have an expression for I from 5.15 or 5.16 with only the ∞-components of the T's uncontracted. Let m be the maximum number of free ∞-components that occur in any term. Since $e_\infty = 1$, with other components zero, we can put enough e_∞'s into each term so that each term has m free ∞-components. We now replace each ∞ by a free, upper case index, to give a function taking values in $\bigotimes^m W^* \otimes \sigma_{m+q}$ whose $\infty \dots \infty$-component is I. Since $\tilde{g}_{\infty\infty} = 0$, the expression obtained by symmetrising and removing traces has the same $\infty \dots \infty$-component. By raising the indices of this expression, we obtain a function $C^{IJ\dots K}$ taking values in $\bigodot_0^m W \otimes \sigma_{q+m}$, with $C^{0\dots 0} = I$. The highest weight argument of 4.9 gives us that the remaining components are zero. □

Lemma 5.17 *We can take the value of m in proposition 5.14 to satisfy $m \leq -kd - q$ for even invariants and $m \leq 1 - kd - q$ for odd invariants.*

Proof. Let C be as in proposition 5.14. If $C^{IJ\dots K}$ can be expressed as $e^{(I}\psi^{J\dots K)}$ for some ψ, then we cancel e. We assume we have cancelled as much as possible.

Suppose I is an even invariant. We can assume that C contains a term involving only $\tilde{g}^{IJ}$ and the $T^{(l)}$. As each $T^{(l)}$ takes values in $W_k^{(l)} \otimes \sigma_j$, where $j \leq -k$ and since C has degree d in the T's, we see that C must take values in $\bigodot_0^m W \otimes \sigma_j$ with $j \leq -kd$. Since C takes values in $\bigodot_0^m W \otimes \sigma_{q+m}$, we have $q + m \leq -kd$.

If I is an odd invariant, each term contains $\tilde{\epsilon}_0$, which takes values in $\bigwedge^{n+1} W \otimes \sigma_1$, weakening the above inequality by 1. □

5.3.1 Invariants of Degree $d \geq n/k$

We revert to regarding an element of J_k as the jet of an homogeneous function on W satisfying equations (5.2), rather than using our algebraic description. For any invariant, I, we can construct C as in 5.14. In the expression for C, we can replace e^I by X^I, $T_{IJ\dots K,AB\dots D}$ by $\partial_A \partial_B \dots \partial_D f_{IJ\dots K}$, and, since $\tilde{g}_{IJ}$, $\tilde{\epsilon}$ and X^I define jets, this gives a P-equivariant map

$$\hat{C} : J_k \to \mathcal{E}^{IJ\dots K}(q + m)$$

such that $\mathrm{Eval}(\hat{C}) = C$. Let $\tilde{C} : J_k \to \mathcal{F}^{IJ\ldots K}(q+m)$ be the map induced by the restriction of this map to $\mathcal{Q}$.

Proposition 5.18 *Let $I : J_k \to \sigma_q$ be an invariant, with $\tilde{C}$ constructed as above. There exists a unique P-equivariant map, $\tilde{I} : J_k \to \mathcal{F}(q)$, such that* $\mathrm{Eval}(\tilde{I}) = I$ *and*

$$\tilde{C}^{IJ\ldots K} = X^I X^J \ldots X^K \tilde{I}. \tag{5.17}$$

Proof. Suppose f is an actual homogeneous function defined on some neighbourhood in $\mathcal{Q}$ of e_0. For x near e_0, we can write $x = \hat{h}e_0$ for some $\hat{h} \in G$. If a function, $\tilde{I}$, satisfying $\mathrm{Eval}\tilde{I} = I$ is to respect the G-action, i.e. $h \cdot (\tilde{I}f) = \tilde{I}(h \cdot f)$ for $h \in G$, then

$$(\tilde{I}f)|_x = (\tilde{I}f)|_{\hat{h}e_0} = \hat{h}^{-1} \cdot (\tilde{I}f)|_{e_0} = \tilde{I}(\hat{h}^{-1} \cdot f)|_{e_0} = I(\hat{h}^{-1} \cdot f) \tag{5.18}$$

so that $\tilde{I}$ is uniquely defined by our requirements. One can check with a similar calculation that $\tilde{I}$ defined by (5.18) is well defined and homogeneous. This defines $\tilde{I}$ as a P-equivariant mapping on J_k by taking jets at e_0.

$\tilde{C}^{IJ\ldots K}$ and $X^I X^J \ldots X^K \tilde{I}$ both define P-equivariant mappings into $\mathcal{F}^{IJ\ldots K}(q+m)$, which give $e \otimes \ldots \otimes e \otimes I$ when evaluated at e_0, and so must be equal, by uniqueness. $\square$

Theorem 5.19 *Every even invariant of degree $d \geq n/k$ and every odd invariant of degree $d \geq \frac{n+1}{k}$ is a Weyl invariant.*

Proof. For any invariant I, define C as in proposition 5.14, and $\tilde{C}$ and $\tilde{I}$ as in 5.18. We show that repeatedly applying the D-operator of remark 3.12 to both sides of (5.17) gives I as a Weyl invariant. Recall that for $n + 2s \neq 2$,
$D_I : \mathcal{F}^{J\ldots K}(s) \to \mathcal{F}_I{}^{J\ldots K}(s-1)$ is given by

$$D_I f = \Big((n+2s-2)\partial_I f - X_I \tilde{\Delta} f\Big)\Big|_{\mathcal{Q}}$$

where an arbitrary homogeneous extension off $\mathcal{Q}$ of f is used. One can calculate that for $(X^I f^{J\ldots K}) \in \mathcal{F}^{IJ\ldots K}(r)$,

$$D_I(X^I f^{J\ldots K}) = (n+2r)(n+r-1) f^{J\ldots K}. \tag{5.19}$$

We assume that m satisfies the inequality of lemma 5.17, which, together with our assumption about the degree, gives $m+q \leq -n$. We apply D repeatedly to both sides of (5.17). Since we are applying D_I to a jet of homogeneity $s \leq m+q \leq -n$, so that $(n+2s)$ and $(n+s-1)$ are non-zero, it follows from (5.19) that

$$D_I \dots D_K (X^I \dots X^K \tilde{I})$$

gives a (non-zero) multiple of $\tilde{I}$, which when evaluated at e_0 gives a multiple of I. $\mathrm{Eval}(D_I \dots D_K C^{I \dots K})$ thus realises I as a Weyl invariant. □

5.3.2 Invariants of Degree $d < n/k$

Theorem 5.20 *Every even invariant of degree $d < n/k$ is a Weyl invariant. If the invariant is a polynomial in the components of $T^{(0)},\dots,T^{(l)}$, then it may be written as a linear combination of complete contractions using only $T^{(0)},\dots,T^{(l)}$.*

Proof. (cf. proof of theorem 6.1 of [BEGm]) Given an invariant, I, we can find C as in proposition 5.14, where C is a linear combination of partial contractions of $\tilde{g}$, e, $T^{(0)},\dots,T^{(l)}$.

Using the linking conditions, we can eliminate any e's contracted with any $T^{(k)}$, hence we may assume that each e in the expression for C has a free index. As in lemma 5.17, we may cancel any e's which appear in each term. If $m=0$ after cancellation, then we are done. If $m>0$, we show that we can cancel another e to reduce m inductively to 0.

Suppose $m>0$. Then there are terms containing no e's. Let $F^{I \dots K}$ be the sum of all such terms. We need only show that F vanishes on substitution of the $T^{(k)}$. Since $e^i = 0$, we see that $F^{ij\dots k}$ vanishes upon substitution of any choice of tensors $T^{(0)},\dots,T^{(l)}$. In particular, we can construct such a list of tensors from a list of tensors $u^{(0)}, \dots, u^{(l)}$ with lower case indices, which have symmetries as in (5.13) and are trace free with respect to g, as follows.

We set $T_{ij\dots k,a\dots b} = u_{ij\dots k,a\dots b}$, set the ∞-components of each T to zero, and determine the 0-components of the T's according to the linking conditions.

Then $F^{i...k}$ is a linear combination of partial contractions of $u^{(0)},\ldots,u^{(l)}$ which as a formal expression is identical to $F^{IJ...K}$ if we replace lower case indices by upper case indices, u's by T's and g by $\tilde{g}$.

We can assume $F^{I...K}$ is allowable in the sense that there are no internal contractions of the T's, as each $T^{(k)}$ is trace free. We deduce that $F^{i...k}$ is an allowable contraction of the u's, which vanishes on substitution of any choice of $u^{(0)},\ldots,u^{(l)}$. Theorem 3.6 with $s_i = k$ then gives that $F^{i...k}$ vanishes formally for $d < n/k$. Thus $F^{I...K}$ vanishes formally, and, in particular, vanishes on substitution of the T's. □

5.3.3 Exceptional Invariants

It is shown in theorem 5.13, theorem 5.19 and theorem 5.20 that all odd invariants of degree n/k are exceptional and that these are the only exceptional invariants. So we only consider odd invariants of degree n/k, where n/k is an integer. We use the methods of [BGo] and we revert to the standard normalisation when denoting skew-symmetrisation of indices by square brackets etc.

Choose a point $B \in W^*$, such that $B(e_0) \neq 0$, and set $\xi = \frac{B}{B(X)}$, so that $\xi(X) = 1$. We define a jet of homogeneity 0, taking values in $\bigwedge^n W$ by

$$\eta = \xi \lrcorner \tilde{\epsilon}_0,$$

where we use $\tilde{\epsilon}_0$ to denote the jet $X^* \lrcorner \tilde{\epsilon}$. Although we also use $\tilde{\epsilon}_0$ to denote $e^* \lrcorner \tilde{\epsilon}$, context should prevent any confusion. One can show that

$$\tilde{\epsilon}_0|_{\mathcal{Q}} = (X \wedge \eta)|_{\mathcal{Q}}. \tag{5.20}$$

Choosing a different point, $\tilde{B} \in W^*$, will lead to a different jet, $\tilde{\eta}$. One can show that there exists a jet ρ taking values in $\bigwedge^{n-1} W$ such that

$$\tilde{\eta}|_{\mathcal{Q}} = (\eta + X \wedge \rho)|_{\mathcal{Q}} \tag{5.21}$$

Let $\bar{\eta} = \mathrm{Eval}(\eta)$.

Proposition 5.21 *The map $I : J_k \to \sigma_{-n}$ defined by*

$$I = \mathrm{contr}(\bar{\eta} \otimes \overbrace{T^{(0)} \otimes \ldots \otimes T^{(0)}}^{n/k})$$

is an invariant. We call this the basic exceptional invariant.

Proof. Follows immediately from (5.21), since $X \lrcorner T^{(0)} = 0$ □

Lemma 5.22 *If k is odd, then $I = 0$.*

Proof. Follows trivially from permuting the indices of η. For example, if $n = 2$ and k=1, then

$$\begin{aligned} I &= \mathrm{contr}(\bar{\eta}^{JK} T_J^{(0)} T_K^{(0)}) \\ &= -\mathrm{contr}(\bar{\eta}^{KJ} T_J^{(0)} T_K^{(0)}) \\ &= -I \end{aligned}$$

and so I must vanish. □

Theorem 5.23

- *If $n \not\equiv 0 \bmod k$, or if k is odd (and hence if n is odd), then J_k has no exceptional invariants.*
- *If $n \equiv 0 \bmod k$, and n is even, then every exceptional invariant is a scalar multiple of the basic exceptional invariant.*

The proof appears after the following lemma:

Lemma 5.24 *Let I be a non-zero exceptional invariant and let $C : J_k \to \bigodot_0^m W \otimes \sigma_{q+m}$ be as in proposition 5.14. Then m satisfies $m = 1 - n - q$.*

Proof. We have that $m \leq 1 - n - q$. If $m < 1 - n - q$, then we could express I as a Weyl invariant, as in the proof of theorem 5.19, but we have shown that there is no non-zero odd Weyl invariant of degree n/k. □

Proof of theorem 5.23. Let $\tilde{C}$ be as in proposition 5.18, and we assume that any X's contracted into f and its derivatives have been eliminated using $X \lrcorner f = 0$ and Euler's equation.

We write $\tilde{C}$ as $\tilde{C} = \tilde{C}_0 + \tilde{C}_x$, where $\tilde{C}_0$ is the sum of the terms containing no free X's, and assume that $\tilde{C}_0$ and $\tilde{C}_x$ are both separately symmetric, by symmetrising as necessary. Since $\tilde{C}$ has homogeneity $1-n$, $\tilde{\epsilon}_0$ has homogeneity 1, f has homogeneity $-k$ and as $\tilde{C}$ has degree n/k, $\tilde{C}_0$ must contain $\tilde{\epsilon}_0$ with all but one index contracted into n/k lots of f, with no derivatives occurring. Notice that m must equal 1.

Let $\tilde{C}_\eta$ be obtained by replacing $(\tilde{\epsilon}_0)^{JK\ldots L}$ by $X^{[J}\eta^{K\ldots L]}$ in the expression for $\tilde{C}_0$. From equation (5.20), we see that, on $\mathcal{Q}$, $\tilde{C}_0 = \tilde{C}_\eta$. Since $X \lrcorner f = 0$, the X must contribute a free index in each non zero term, hence

$$\tilde{C}^J_\eta = X^J \tilde{E}_\eta,$$

where $\tilde{E}_\eta$ is a contraction of the form $\mathrm{contr}(\eta \otimes f \otimes \ldots \otimes f)$. Since $\tilde{C}_x$ has an X in each term, $\tilde{C}^J_x = X^J \tilde{E}_x$, for some $\tilde{E}_x$. Taking equation (5.17) and cancelling X's we get

$$\tilde{I} = \tilde{E}_\eta + \tilde{E}_x.$$

Evaluation at e_0 gives I as the sum of two terms, the first of which is of the form we require, and the second of which is a Weyl invariant and hence zero. □

Chapter 6

Projective Structures

We begin this chapter by reviewing the geometry of projective structures following [Ei2]. As in the conformal case, we then describe the flat model and go on to look at the projective tractor bundle, following the presentation of [BEGo]. Finally, we describe the ambient construction of [B]—associated to an n-dimensional projective manifold is a Ricci flat affine manifold of dimension $n+1$. The reason for studying projective structures is the generalisation to contact-projective structures, which follows in chapter 7.

6.1 Projective Structures

Definition 6.1 Let $\mathcal{M}^n$ be an n-dimensional manifold (we will assume $n \geq 2$). A *projective structure* on $\mathcal{M}$ is an equivalence class of torsion free, affine connections which have the same geodesics, considered as unparametrised curves.

If ∇_i and $\widehat{\nabla}_i$ are two connections in such an equivalence class, then the two are related by a transformation of the form

$$\begin{aligned} \widehat{\nabla}_i U^j &= \nabla_i U^j + \Upsilon_i U^j + U^k \Upsilon_k \delta_i{}^j \\ \widehat{\nabla}_i \omega_j &= \nabla_i \omega_j - \Upsilon_i \omega_j - \Upsilon_j \omega_i \end{aligned} \tag{6.1}$$

for some field Υ_i. We will again use "hats" to denote corresponding quantities following a transformation of the above form.

Locally, we choose a line bundle $\mathcal{E}(1)$ whose $(-n-1)^{st}$ power, $\mathcal{E}(-n-1)$, is identified with the canonical bundle of $\mathcal{M}$. Sections of, for example, $\mathcal{E}^i(w) = \mathcal{E}^i \otimes \mathcal{E}(w)$ are said to have *projective weight* w. The line bundles $\mathcal{E}(w)$ have an induced connection, which under change of scale transforms according to

$$\widehat{\nabla}_i f = \nabla_i f + w\Upsilon_i f.$$

A nowhere vanishing section of $\mathcal{E}(1)$ is known as a *projective scale*, and associated to such a section, τ, is a connection ∇_i, in the equivalence class, such that $\nabla_i\tau = 0$. The induced connection on $\mathcal{E}(w)$ is flat for connections defined on this way. Henceforth, we will consider only such connections. Choosing another projctive scale, $\hat{\tau} = \Omega^{-1}\tau$ generates a transformation of the form (6.1), where $\Upsilon_i = \Omega^{-1}\nabla_i\Omega$.

For a choice of projective scale as above, we define the curvature tensor, $R_{ij}{}^k{}_l$, by

$$(\nabla_i\nabla_j - \nabla_j\nabla_i)U^k = R_{ij}{}^k{}_l U^l.$$

The curvature can be expressed uniquely in the form

$$R_{ij}{}^k{}_l = W_{ij}{}^k{}_l + \delta_i{}^k P_{jl} - \delta_j{}^k P_{il}, \tag{6.2}$$

where the *Weyl tensor*, $W_{ij}{}^k{}_l$, is trace free and $(n-1)P_{ij} = R_{ij}$. The Bianchi identities, $R_{[ij}{}^k{}_{l]} = 0$ and $\nabla_{[i}R_{jk]}{}^l{}_m = 0$ give us that

$$\begin{aligned} P_{[ij]} &= 0, \\ \text{and} \qquad \nabla_k W_{ij}{}^k{}_l &= 2(n-2)\nabla_{[i}P_{j]l} \end{aligned} \tag{6.3}$$

Under a transformation of the form (6.1), $W_{ij}{}^k{}_l$ is invariant, while for P_{ij}, we have

$$\widehat{P}_{ij} = P_{ij} - \nabla_i\Upsilon_j + \Upsilon_i\Upsilon_j.$$

6.2 The Flat Model

Consider $\mathbb{R}^{n+1}$ equipped with the standard volume form, ϵ. The flat model for projective geometry is simply the projective space $\mathbb{P}^n$, which is $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where $X \sim X'$ if $X = \lambda X'$ for some $\lambda \neq 0$. The geodesics are the straight lines in $\mathbb{P}^n$. To check that we have a projective structure, we need only check that these are the geodesics of an equivalence class of torsion-free affine connections. We see this as follows. As coordinates for $\mathbb{R}^{n+1}$, we use

$$\begin{pmatrix} X^0 \\ X^i \end{pmatrix}, \qquad \text{for } i = 1, \ldots, n.$$

We use as a coordinate chart, $\mathbb{R}^n \to \mathbb{P}^n$,

$$x^i \mapsto \left[\begin{pmatrix} 1 \\ x^i \end{pmatrix} \right].$$

The straight lines in $\mathbb{P}^n$ are the geodesics of the standard connection on this embedded $\mathbb{R}^n$. This connection is torsion-free, so the equivalence class of torsion-free connections with these geodesics is a projective structure.

Let G denote the group $\mathrm{PSL}(n+1, \mathbb{R})$. (Recall that for n odd, $\mathrm{PSL}(n+1, \mathbb{R}) = \mathrm{SL}(n+1, \mathbb{R})/\pm I$, and for n even, $\mathrm{PSL}(n+1, \mathbb{R}) = \mathrm{SL}(n+1, \mathbb{R})$.) G acts simply and transitively on $\mathbb{P}^n$ in the obvious way. Fix a point, $e_0 \in \mathbb{R}^{n+1}$ with coordinates

$$e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $P = \{p \in G : pe_0 = \lambda e_0, \text{ for some } \lambda \neq 0\}$, so that $\mathbb{P}^n$ is the homogeneous space G/P. Explicitly, for n even, P is the subgroup of G consisting of elements of the form

$$\begin{pmatrix} \lambda & r_j \\ 0 & m^i{}_j \end{pmatrix},$$

and for n odd, P is isomorphic to the group of elements of this form with $\lambda > 0$.

6.3 The Projective Tractor Bundle

Definition 6.2 The *co-tractor bundle,* $\mathcal{E}_I$ is $\mathcal{J}^1\mathcal{E}(1)$, the first jet bundle of $\mathcal{E}(1)$.

$\mathcal{E}_I$ has composition series $\mathcal{E}_I = \mathcal{E}(1) + \mathcal{E}_i(1)$, with the transformation under change of scale given by

$$(\hat{U}_0 \quad \hat{U}_i) = (U_0 \quad U_i + \Upsilon_i U_0) \tag{6.4}$$

where U_i and U_0 are sections of $\mathcal{E}_i(1)$ and $\mathcal{E}(1)$ respectively. The *tractor bundle,* the dual of the co-tractor bundle, is $\mathcal{E}^I = \mathcal{E}^i(-1) + \mathcal{E}(-1)$, with transformation under change of scale given by

$$\begin{pmatrix} \hat{V}^0 \\ \hat{V}^i \end{pmatrix} = \begin{pmatrix} V^0 - \Upsilon_j V^j \\ V^i \end{pmatrix}.$$

We define primary and secondary parts in the same way as in the conformal case. The preferred sections in this case are the sections X^I of $\mathcal{E}^I(1)$ and Y_I^j of $\mathcal{E}_I^j(-1)$. X^I defines the projection $\mathcal{E}_I \to \mathcal{E}(1)$ and the injection $\mathcal{E}(-1) \to \mathcal{E}^I$, while Y_I^j defines the projection $\mathcal{E}^I \to \mathcal{E}^i(-1)$ and the injection $\mathcal{E}(1) \to \mathcal{E}_I$ in the obvious way. In a chosen scale, Y_I^j is represented by $(0 \quad \delta_i{}^j)$ and X^I is represented by

$$X^I = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One easily sees that $X^I Y_I^j = 0$. The tractor connection on $\mathcal{E}_I$ is given by

$$\nabla_i(U_0 \quad U_j) = (\nabla_i U_0 - U_i \quad \nabla_i U_j + P_{ij} U_0) \tag{6.5}$$

and that on $\mathcal{E}^I$ is given by

$$\nabla_i \begin{pmatrix} V^0 \\ V^j \end{pmatrix} = \begin{pmatrix} \nabla_i V^0 - P_{ik} V^k \\ \nabla_i V^j + V^0 \delta_i{}^j \end{pmatrix}. \tag{6.6}$$

The tractor curvature, $\Omega_{ij}{}^K{}_L U^L$, defined by $(\nabla_i \nabla_j - \nabla_j \nabla_i) U^K = \Omega_{ij}{}^K{}_L U^L$, in block matrix form, is

$$\begin{pmatrix} 0 & -2\nabla_{[i} P_{j]l} \\ 0 & W_{ij}{}^k{}_l \end{pmatrix}.$$

Again, the tractor curvature vanishes if and only if $\mathcal{M}$ is locally equivalent to the flat model. From (6.3), one sees that, for $n \geq 3$, the tractor curvature vanishes if and only if $W_{ij}{}^{k}{}_{l} = 0$.

The operator $D_I : \mathcal{E}(w) \to \mathcal{E}_I(w-1)$ is defined by

$$D_I f = (wf \quad \nabla_i f).$$

This definition is invariant, even if f has tractor indices.

6.4 The Ambient Construction

The ambient construction in the projective case is essentially due to T.Y.Thomas, (e.g. [T]), and has been described in a modern form by T.N.Bailey, [B], whose presentation we follow.

Definition 6.3 A *special affine manifold* is a manifold with a volume form, $\epsilon_{I\ldots L}$ and a torsion-free connection, $\widetilde{\nabla}_I$, with $\widetilde{\nabla}_I \epsilon_{J\ldots L} = 0$.

Definition 6.4 A *Killing vector* on a special affine manifold is a vector field, V^I satisfying

$$\mathcal{L}_V(\widetilde{\nabla}_I U^J) = \widetilde{\nabla}_I \mathcal{L}_V(U^J) \tag{6.7}$$

for all vector fields, U^J, where $\mathcal{L}_V$ denotes the Lie derivative.

Lemma 6.5 *A vector field, V^I, is a Killing vector if and only if $V^I \tilde{R}_{IJ}{}^{K}{}_{L} + \widetilde{\nabla}_J \widetilde{\nabla}_L V^K = 0$, where $\tilde{R}_{IJ}{}^{K}{}_{L}$ denotes the curvature of $\widetilde{\nabla}_I$.*

Proof. Expanding the left hand side of (6.7), we obtain

$$\mathcal{L}_V(\widetilde{\nabla}_I U^J) = V^K \widetilde{\nabla}_K \widetilde{\nabla}_I U^J - \left(\widetilde{\nabla}_I U^K\right)\left(\widetilde{\nabla}_K V^J\right) + \left(\widetilde{\nabla}_K U^J\right)\left(\widetilde{\nabla}_I V^K\right),$$

while the right hand side gives

$$\begin{aligned}\widetilde{\nabla}_I \mathcal{L}_V(U^J) &= \widetilde{\nabla}_I \left(V^K \widetilde{\nabla}_K U^J - U^K \widetilde{\nabla}_K V^J\right)\\ &= \left(\widetilde{\nabla}_I V^K\right)\left(\widetilde{\nabla}_K U^J\right) + V^K \widetilde{\nabla}_I \widetilde{\nabla}_K U^J - \left(\widetilde{\nabla}_I U^K\right)\left(\widetilde{\nabla}_K V^J\right) - U^K \widetilde{\nabla}_I \widetilde{\nabla}_K V^J.\end{aligned}$$

Comparing these, we see that V^I is a killing vector if and only if

$$V^I \tilde{R}_{IJ}{}^K{}_L U^L + U^L \widetilde{\nabla}_J \widetilde{\nabla}_L V^K = 0$$

for each vector field, U^I, from which the result follows. □

Definition 6.6 An *Euler field*, is a vector field, X^I, which, in addition to being a Killing vector, satisfies

$$\widetilde{\nabla}_I X^J = \delta_I{}^J.$$

Thus X^I is an Euler field if and only if $\widetilde{\nabla}_I X^J = \delta_I{}^J$ and $X^I \tilde{R}_{IJ}{}^K{}_L = 0$. Note that an Euler field could have zeros. We will, henceforth, work locally, on neighbourhoods containing no zeros of the Euler field.

Proposition 6.7 *Let $\widetilde{\mathcal{M}}$ be a special affine manifold with an Euler field, X^I. Then the quotient of $\widetilde{\mathcal{M}}$ by X^I has a natural projective structure.*

Proof. Assuming X^I has no zeros since we are working locally, let $\mathcal{M}$ denote the n-manifold of integral curves of X^I. Then a vector field U^i on $\mathcal{M}$ can be represented by a vector U^I on $\widetilde{\mathcal{M}}$ such that $\mathcal{L}_X U^I = 0$, modulo terms of the form fX^I such that $X^I \widetilde{\nabla}_I f = 0$. Note that the condition $\mathcal{L}_X U^I = 0$ is equivalent to $X^J \widetilde{\nabla}_J U^I = U^I$.

Similarly, we represent a covector field U_i on $\mathcal{M}$ by a field U_I such that $X^J \widetilde{\nabla}_J U_I = -U_I$ and $X^I U_I = 0$.

The line bundles $\mathcal{E}(w)$ are defined by representing their sections by functions on $\widetilde{\mathcal{M}}$ such that $X^I \widetilde{\nabla}_I f = wf$. Let $U^i_{(1)}, \ldots, U^i_{(n)}$ be n vector fields on $\mathcal{M}$ represented by $U^I_{(1)}, \ldots, U^I_{(n)}$. Then

$$f = \epsilon_{IJ\ldots L} X^I U^J_{(1)} \ldots U^L_{(n)}$$

satisfies $X^I \widetilde{\nabla}_I f = (n+1)f$ and so $\mathcal{E}(-n-1)$ is canonically isomorphic to the canonical bundle of $\mathcal{M}$.

We can choose a vector field, ω_I, on $\mathcal{M}$ satisfying

$$X^I \omega_I = 1 \qquad \text{and} \qquad \widetilde{\nabla}_{[I}\omega_{J]} = 0,$$

and define a connection ∇_i where $\nabla_i U^j$, $\nabla_i V_j$ and $\nabla_i f$ are represented by

$$\widetilde{\nabla}_I U^J - \omega_I U^J - U^K \omega_K \delta_I{}^J$$
$$\widetilde{\nabla}_I V_J + \omega_I V_J + \omega_J V_I$$
$$\widetilde{\nabla}_I f - w f \omega_I$$

respectively, where f is a section of $\mathcal{E}(w)$. One can easily check that this is consistent.

Since $\widetilde{\nabla}_{[I}\omega_{J]} = 0$, the connection ∇_i is flat on $\mathcal{E}(w)$ and so must be defined by a projective scale. Any other such ω_I is of the form $\hat{\omega}_I = \omega_I - \Upsilon_I$, where $X^I \Upsilon_I = 0$ and $\nabla_{[I}\Upsilon_{J]} = 0$. The covector Υ_i represented by Υ_I thus gives the change in connection generated by a change of projective scale. □

Theorem 6.8 *Let $\mathcal{M}^n$ be a manifold with a projective structure and let $\widetilde{\mathcal{M}}$ be the total space of $\mathcal{E}(-1)$ (with the zero section deleted). Then $\widetilde{\mathcal{M}}$ carries a natural special affine structure with Euler field and is Ricci flat (i.e. the curvature of $\widetilde{\nabla}_I$ satisfies $\tilde{R}_{IJ}{}^I{}_L$=0).*

Proof. Let $x^1, \ldots, x^n$ be coordinates on $\mathcal{M}$. A choice of projective scale is a nowhere vanishing section, τ, of $\mathcal{E}(1)$, and any nowhere vanishing section of $\mathcal{E}(-1)$ can be written as $t(x)\tau^{-1}$. We use t as a coordinate on $\widetilde{\mathcal{M}}$. Let ∂_0 denote $\frac{\partial}{\partial t}$ and ∂_i denote $\frac{\partial}{\partial x^i}$. If the connection on $\mathcal{M}$ associated to τ is $\nabla_i U^j = \partial_i U^j + \Gamma^j_{ik} U^k$, then one can check that in terms of the basis (∂_0, ∂_i), the invariant connection on $\widetilde{\mathcal{M}}$ is given by

$$\widetilde{\nabla}_I U^J = \partial_I U^J + \tilde{\Gamma}^J_{IK} U^K$$

where

$$\tilde{\Gamma}^0_{IK} = \begin{pmatrix} 0 & 0 \\ 0 & -tP_{ik} \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma}^j_{IK} = \begin{pmatrix} 0 & \frac{1}{t}\delta_k{}^j \\ \frac{1}{t}\delta_i{}^j & \Gamma^j_{ik} \end{pmatrix}. \tag{6.8}$$

The curvature, $\tilde{R}_{IJ}{}^K{}_L$, of $\widetilde{\nabla}_I$ has $\tilde{R}_{0J}{}^K{}_L = 0$ and

$$\tilde{R}_{ij}{}^K{}_L = \begin{pmatrix} 0 & -2t\nabla_{[i}P_{j]l} \\ 0 & W_{ij}{}^k{}_l \end{pmatrix}.$$

Then it is easy to check that $\widetilde{\nabla}_i$ is Ricci flat, the Euler field, X, is $t\partial_0$ and the preserved volume form is $\epsilon = t^n dt \wedge dx^1 \wedge \ldots \wedge dx^n$. □

It is noted in [BEGo] that the projective tractor bundle is the tangent space of $\widetilde{\mathcal{M}}$ and that the D-operator is the connection on this space. Using the above, one can check this in a similar manner to the corresponding result in the conformal case.

Proposition 6.9 *Manifolds with projective structures are equivalent to Ricci flat special affine manifolds with Euler fields in the following sense:*

- *Given a projective structure, the quotient of the ambient construction by the Euler field can be canonically identified with the original projective structure.*
- *A Ricci flat special affine manifold with Euler field can be canonically identified with a subset of the manifold obtained by performing the ambient construction on the quotient projective structure.*

Proof. Calculation. □

Chapter 7

Contact-Projective Structures

After recalling some facts about contact structures, we define a contact-projective structure as being a projective structure with a 'compatible' contact structure and describe some features of the geometry. The flat model for such a structure is $\mathrm{Sp}(2n, \mathbb{R})$ modulo a parabolic subgroup. After checking that the definition is not so strong as to prevent curved contact-projective structures, we describe the tractor bundle and ambient construction, which are inherited from the projective case. We shall see that the projective tractor bundle is given a further splitting and acquires a skew bilinear form. Similarly, the affine structure associated to a projective structure acquires a covariantly constant symplectic form.

The remainder of the chapter studies an invariant theory problem, using methods similar to those of both [BEGo] for conformal geometry and [Go2] for projective geometry.

Our motivation for studying contact-projective structures is as follows. Firstly, they are closely analogous to conformal structures and projective structures, which both appear widely in geometry. Indeed, we would suggest that contact-projective geometry is associated with the symplectic group in the same way that conformal and projective geometry are associated with the pseudo-orthogonal groups and special linear groups, respectively. In view of these relationships, we expect that contact-projective structures may appear in applications, although we have as yet been unable to find any. Finally, we see that the invariant theory methods of [BEGm] and [Go2] etc. can be adapted to study problems in the invariant theory of a parabolic subgroup of $Sp(2n+2,\mathbb{R})$, with applications in finding invariant differential operators under the action of $Sp(2n+2,\mathbb{R})$ on $\mathbb{P}^{2n+1}$.

7.1 Contact Structures

Definition 7.1 A *contact form* on a $2n+1$ dimensional manifold, $\mathcal{M}$, is a 1-form, θ, non-degenerate in the sense that, at each point,

$$\theta \wedge (d\theta)^n \neq 0,$$

where $(\mathrm{d}\theta)^n$ denotes the n-th exterior power.

Notice that if θ is a contact form and Ω is a nowhere vanishing function on $\mathcal{M}$, then $\Omega\theta$ is also a contact form.

Definition 7.2 A *contact structure* is an equivalence class of contact forms, $[\theta]$, where $\theta \sim \hat{\theta}$ if $\hat{\theta} = \Omega\theta$ for some nowhere vanishing function, Ω. We will often regard a contact structure as being a line bundle valued 1-form.

7.1.1 The Contact Distribution

A contact structure defines a $2n$-dimensional distribution, given for any θ in the contact structure by

$$\mathcal{D} = \{U \in T\mathcal{M} \ : \ \theta(U) = 0\},$$

known as the *contact distribution.* A consequence of the non-degeneracy of the contact form is that $\mathrm{d}\theta$ provides an isomorphism between $\mathcal{D}$ and its dual.

In addition, each contact form, θ defines a unique vector field, $Y \in T\mathcal{M}$, known as the *characteristic vector field,* by insisting that $\theta(Y) = 1$ and $Y \lrcorner\, \mathrm{d}\theta = 0$.

7.2 The Flat Model

Let W denote $\mathbb{R}^{2n+2}$ with coordinates, $X^I = \begin{pmatrix} X^0 \\ X^i \\ X^\infty \end{pmatrix}$, $i = 1, \ldots, 2n$, and let

$$e_0^I = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad (Q_{ij}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad \text{and} \quad (\tilde{Q}_{IJ}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & (Q_{ij}) & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

where I_n denotes the identity on $\mathbb{R}^n$. We write Sp(2n,$\mathbb{R}$) for the group preserving Q and $\mathrm{Sp}(2n+2, \mathbb{R})$ for the group preserving $\tilde{Q}$. Let $G = \mathrm{Sp}(2n+2, \mathbb{R}) / \pm I_{2n+2}$. Define $P = \{p \in G \ : \ pe_0 = \lambda e_0 \text{ for some } \lambda \neq 0\}$. Explicitly, P is isomorphic to

$$\left\{ \begin{pmatrix} \lambda & r_j & t \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix} : \lambda > 0, \ m \in \mathrm{Sp}(2n, \mathbb{R}), \ r_j = \lambda s^i Q_{ik} m^k{}_j \right\}$$

Define a 1-form $\tilde{\alpha}$ on W by $\tilde{\alpha} = \tilde{Q}_{IJ} X^I dX^J$. On $\mathbb{P}^{2n+1} = G/P$, the restriction of $\tilde{\alpha}$ is a contact structure, while on W, $d\tilde{\alpha}$ is a symplectic form.

7.3 Curved Contact-Projective Structures

Suppose we have a projective structure, $[\nabla_{i'}]$, on $\mathcal{M}^{2n+1}$, where indices of the form i', j', k' and so on, run through $1, \ldots, 2n, \infty$, while indices such as i, j, k etc. will run through $1, \ldots, 2n$. We will sometimes write e.g. θ_i for the corresponding components of $\theta_{i'}$. We will again assume that we are only using connections which are defined by nowhere vanishing sections of $\mathcal{E}(1)$, where $\mathcal{E}(-2n-2)$ is identified locally with the canonical bundle, κ. Now suppose we also have a contact structure on $\mathcal{M}$, which we regard as a 1-form, Θ, taking values in some line bundle, L. Then $\Theta \wedge (d\Theta)^n$ gives a trivialisation of $L^{n+1} \otimes \kappa$, and so we have $L = \mathcal{E}(2)$. Thus a nowhere vanishing section, τ, of $\mathcal{E}(1)$ also defines a contact form, $\theta_{i'} = \tau^{-2}\Theta_{i'}$.

Definition 7.3 A *contact-projective structure* is a contact structure, $\Theta_{i'}$, together with a projective structure, $[\nabla_{i'}]$, which are compatible in the sense that, for each choice of scale, we have

$$\nabla_{(i'}\theta_{j')} = 0 \qquad \text{and} \qquad W_{i'j'}{}^{k'}{}_{l'}\theta_{k'} = 0. \tag{7.1}$$

Suppose that we have a geodesic for which the tangent at some point lies in the distribution, $\mathcal{D}$. Then the first of these equations is sufficient (although not necessary) to ensure that the tangent remains in $\mathcal{D}$—if $U^{i'}$ is a tangent to a geodesic, so that $U^{i'}\nabla_{i'}U^{j'} \propto U^{j'}$ and if $U^{i'}\theta_{i'} = 0$ at some point, then $\nabla_{(i'}\theta_{j')} = 0$ ensures that $U^{i'}\nabla_{i'}(U^{j'}\theta_{j'}) = 0$.

Lemma 7.4 *Given that $\nabla_{(i'}\theta_{j')} = 0$, the condition $W_{i'j'}{}^{k'}{}_{l'}\theta_{k'} = 0$ is equivalent to*

$$\nabla_{i'}\nabla_{j'}\theta_{k'} = \theta_{j'}P_{k'i'} - \theta_{k'}P_{j'i'}. \tag{7.2}$$

Proof. It follows easily from $\nabla_{(i'}\theta_{j')} = 0$ and the Bianchi identity $R_{[i'j'}{}^{k'}{}_{l']} = 0$, that

$$\nabla_{i'}\nabla_{j'}\theta_{k'} = R_{j'k'}{}^{l'}{}_{i'}\theta_{l'}.$$

Then the result follows upon substituting for R from (6.2). □

7.3.1 Existence of Curved Structures

A priori, it is, of course, possible that the equations, (7.1), for the compatibility of a contact structure and a projective structure, force any contact-projective structure to be locally isomorphic to the flat model. We prove the following :

Theorem 7.5 *Given a contact form, $\theta_{i'}$, there exists, locally, a connection $\nabla_{i'}$ satsifying (7.1) for which the Weyl curvature has non-zero components. Then it follows (see §6.3) that the contact-projective structure defined by $\theta_{i'}$ and $\nabla_{i'}$ can not be locally isomorphic to the flat model.*

Suppose we have a contact form, $\theta_{i'}$ on $\mathcal{M}$. It follows from a classical theorem of Darboux (see e.g. [Bl] and references therein) that, about every point of $\mathcal{M}$, there exist coordinates $(x^1, \dots, x^{2n}, x^\infty)$ such that

$$\theta = dx^\infty - \sum_{i=1}^{n} x^{n+i} dx^i.$$

Then a basis $(e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}, e_\infty)$ for $T\mathcal{M}$ can be taken to be

$$\left(\frac{\partial}{\partial x^1} + x^{n+1}\frac{\partial}{\partial x^\infty}, \dots, \frac{\partial}{\partial x^n} + x^{2n}\frac{\partial}{\partial x^\infty}, \frac{\partial}{\partial x^{n+1}}, \dots, \frac{\partial}{\partial x^{2n}}, \frac{\partial}{\partial x^\infty} \right).$$

Note that the non-zero commutators of these vectors are

$$[e_i, e_{n+i}] = -e_\infty, \qquad i = 1, \dots, n. \tag{7.3}$$

Then we write the connection $\nabla_{i'}$ as

$$\nabla_{i'} U^{j'} = e_{i'}(U^{j'}) + \Gamma^{j'}_{i'k'} U^{k'}.$$

The condition $\nabla_{(i'}\theta_{j')} = 0$ is that $\Gamma^\infty_{(i'j')} = 0$, while, the condition that $\nabla_{i'}$ be torsion-free is easily seen to be that

$$[e_{i'}, e_{j'}] = 2\Gamma^{k'}_{[i'j']} e_{k'}.$$

Together with (7.3) this gives

$$\Gamma^k_{[i'j']} = 0 \qquad \text{and} \qquad (\Gamma^\infty_{i'j'}) = \frac{1}{2}\begin{pmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the components $\Gamma^\infty_{i'j'}$ are forced on us, to prove theorem 7.5 we only need to show the following:

Proposition 7.6 *Let $\Gamma^\infty_{i'j'}$ be as above. If we choose the components of $\Gamma^k_{i'j'}$ to be zero apart from*

$$\Gamma^n_{\infty\tilde{n}} = \Gamma^n_{\tilde{n}\infty} = \frac{1}{2n+2}, \qquad \Gamma^n_{n\tilde{n}} = \Gamma^n_{\tilde{n}n} = \frac{1}{2} \qquad \text{and} \qquad \Gamma^{\tilde{n}}_{\tilde{n}\tilde{n}} = -\frac{1}{2}$$

where we write $\tilde{n}$ for $2n$, then $W_{i'j'}{}^{k'}{}_{l'}\theta_{k'} = W_{i'j'}{}^{\infty}{}_{l'} = 0$ but W has non-zero components.

Proof. The curvature of $\nabla_{i'}$ is given by

$$R_{i'j'}{}^{k'}{}_{l'} = e_{i'}\Gamma^{k'}_{j'l'} - e_{j'}\Gamma^{k'}_{i'l'} + \Gamma^{k'}_{i'm'}\Gamma^{m'}_{j'l'} - \Gamma^{k'}_{j'm'}\Gamma^{m'}_{i'l'} - 2\Gamma^{m'}_{[i'j']}\Gamma^{k'}_{m'l'},$$

from which it is a straightforward but tedious task to calculate that

$$P_{\tilde{n}\tilde{n}} = -\frac{1}{4(n+1)}, \qquad R_{i'j'}{}^{\infty}{}_{l'} = \begin{cases} \frac{1}{4(n+1)} & i' = 2n,\ j' = \infty,\ l' = 2n \\ -\frac{1}{4(n+1)} & i' = \infty,\ j' = 2n,\ l' = 2n \end{cases}$$

with all other components of $P_{i'j'}$ and $R_{i'j'}{}^{\infty}{}_{l'}$ being zero. Then together with (6.2) this gives $W_{i'j'}{}^{\infty}{}_{l'} = 0$.

To complete the proof we simply need to exhibit a non-zero component of $W_{i'j'}{}^{k'}{}_{l'}$. It is easy to check that for $n \neq 1$,

$$W_{\tilde{n}1}{}^{n}{}_{n+1} = R_{\tilde{n}1}{}^{n}{}_{n+1} = -\frac{1}{n+1}$$

and for $n = 1$,

$$W_{2\infty}{}^{1}{}_{2} = R_{2\infty}{}^{1}{}_{2} = \frac{1}{4}$$

□

7.3.2 The Splittings of $T\mathcal{M}$ and $T^*\mathcal{M}$

Let $(e_1, \ldots, e_{2n})$ be a basis for $\mathcal{D}$. We use $(e_1, \ldots, e_{2n}, e_\infty = Y)$ as a basis for $T\mathcal{M}$. Let $(e^1, \ldots, e^{2n}, e^\infty = \theta)$ be the dual basis for $T^*\mathcal{M}$.

Since $\nabla_{(i'}\theta_{j')} = 0$, $d\theta$ has components $\nabla_{i'}\theta_{j'}$. In terms of this basis, we have

$$Y^{i'} = \begin{pmatrix} Y^i \\ Y^\infty \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \nabla_{i'}\theta_{j'} = \begin{pmatrix} Q_{ij} & 0 \\ 0 & 0 \end{pmatrix},$$

where Q_{ij} is skew, and provides the isomorphism between $\mathcal{D}$ and its dual. Let Q^{ij} be the inverse of Q_{ij}, defined by $Q^{ij}Q_{ik} = \delta_k{}^j$. Unprimed lower case indices are raised and lowered according to

$$V_j = V^i Q_{ij} \qquad \text{and} \qquad V^i = Q^{ij} V_j$$

Under change of scale, $\hat{\tau} = \Omega^{-1}\tau$, we have the change of connection as in (6.1) and $\hat{\theta}_{i'} = \Omega^2\theta_{i'}$, so

$$\widehat{\nabla}_{i'}\hat{\theta}_{j'} = \Omega^2\left(\nabla_{i'}\theta_{j'} + \Upsilon_{i'}\theta_{j'} - \Upsilon_{j'}\theta_{i'}\right).$$

We can write this as

$$\widehat{\nabla}_{i'}\hat{\theta}_{j'} = \Omega^2\begin{pmatrix} Q_{ij} & -\Upsilon_j \\ \Upsilon_i & 0 \end{pmatrix},$$

from which it is easy to see that

$$\hat{Y}^{i'} = \Omega^{-2}\begin{pmatrix} \Upsilon^i \\ 1 \end{pmatrix}.$$

If we now change to the basis $(e_1, \ldots, e_{2n}, \hat{Y})$ for $T\mathcal{M}$ with the corresponding dual basis for $T^*\mathcal{M}$, we find that the components of vectors and co-vectors are transformed according to

$$\begin{pmatrix} \hat{V}^i \\ \hat{V}^\infty \end{pmatrix} = \begin{pmatrix} V^i - \Upsilon^i V^\infty \\ \Omega^2 V^\infty \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{U}_i & \hat{U}_\infty \end{pmatrix} = \begin{pmatrix} U_i & \Omega^{-2}(U_\infty + \Upsilon^i U_i) \end{pmatrix} \tag{7.4}$$

In other words, for each choice of projective scale, we have the splittings

$$\mathcal{E}^{i'} = \mathcal{E}(2) \oplus \mathcal{E}^i \qquad \text{and} \qquad \mathcal{E}_{i'} = \mathcal{E}(-2) \oplus \mathcal{E}_i, \tag{7.5}$$

with the transformation under change of scale following from (7.4).

7.3.3 The Connections on $\mathcal{D}$ and $T\mathcal{M}$

One can represent a vector field , U^i, in $\mathcal{D}$ by a field $U^{i'}$ on $T\mathcal{M}$ satisfying $\theta_{i'}U^{i'} = 0$. Similarly, one can represent a co-vector field, U_i, in $\mathcal{D}^*$ by $U_{i'}$ modulo $\theta_{i'}$. We define a connection, $\underline{\nabla}_i$ on $\mathcal{D}$ by representing $\underline{\nabla}_i U^j$ and $\underline{\nabla}_i U_j$ respectively by

$$\begin{aligned} &\nabla_{i'}U^{j'} + Y^{j'}U^{k'}(\nabla_{i'}\theta_{k'}), \\ &\nabla_{i'}U_{j'} - Y^{k'}U_{k'}(\nabla_{i'}\theta_{j'}). \end{aligned} \tag{7.6}$$

One can check that this is consistent. Note that $\underline{\nabla}_i Q_{jk}$ is represented by $\nabla_{i'}\nabla_{j'}\theta_{k'}$ and so it follows from (7.2) that

$$\underline{\nabla}_i Q_{jk} = 0.$$

Under change of projective scale $\hat{\tau} = \Omega^{-1}\tau$, we have

$$\begin{aligned} \widehat{\underline{\nabla}}_i U^j &= \underline{\nabla}_i U^j + \Upsilon_i U^j - U_i \Upsilon^j + U^k \Upsilon_k \delta_i{}^j, \\ \text{and} \quad \widehat{\underline{\nabla}}_i U_j &= \underline{\nabla}_i U_j - \Upsilon_i U_j - \Upsilon_j U_i - U_k \Upsilon^k Q_{ij}, \end{aligned}$$

where $\Upsilon_i = \Omega^{-1}\underline{\nabla}_i \Omega$.

Theorem 7.7 *The connection $\nabla_{i'}$ can be expressed in terms of $\underline{\nabla}_i$, $\underline{R}_{ij}{}^k{}_l$ and $\underline{\nabla}_\infty$, where $\underline{R}_{ij}{}^k{}_l$ denotes the curvature of $\underline{\nabla}_i$ and $\underline{\nabla}_\infty$ denotes the vector field, Y.*

The proof consists of lemmata 7.8 and 7.9 and propositions 7.10,7.11 and 7.12.

Lemma 7.8 *$\nabla_{i'} Y^{j'}$ takes the form*

$$(\nabla_{i'} Y^{j'}) = \begin{pmatrix} -P^j{}_i & -P^j{}_\infty \\ 0 & 0 \end{pmatrix}.$$

Proof. Firstly, from the definition of $Y^{i'}$, we have

$$\theta_{j'} \nabla_{i'} Y^{j'} = \nabla_{i'}(\theta_{j'} Y^{j'}) - Y^{j'} \nabla_{i'} \theta_{j'} = 0.$$

Secondly, since $0 = \nabla_{k'}\left(Y^{i'} \nabla_{i'} \theta_{j'}\right) = Y^{i'} R_{i'j'}{}^{l'}{}_{k'} \theta_{l'} + (\nabla_{i'}\theta_{j'})(\nabla_{k'} Y^{i'})$, it follows that

$$(\nabla_{i'}\theta_{j'})(\nabla_{k'} Y^{i'}) = -P_{j'k'} + \theta_{j'} Y^{i'} P_{i'k'}.$$

The 'jk''-component of this is $Q_{ij} \nabla_{k'} Y^i = -P_{jk'}$, from which the result follows. □

Lemma 7.9 *The traces of R_{ijkl} are as follows:*

$$R_{ij}{}^i{}_l = (2n-1)P_{jl}, \qquad R_i{}^i{}_{kl} = 2P_{kl}, \qquad R_{ij}{}^k{}_k = 0 \qquad \textit{and} \qquad R_{ijl}{}^i = P_{lj}.$$

Proof. Applying $\nabla_{i'}$ to $\nabla_{j'} \nabla_{k'} \theta_{l'} = 2\theta_{[k'} P_{l']j'}$, skewing over i' and j' and projecting onto $\mathcal{D}$ gives

$$R_{ijkl} - R_{ijlk} = Q_{ik} P_{lj} - Q_{il} P_{kj} - Q_{jk} P_{li} + Q_{jl} P_{ki}.$$

This, together with the Bianchi identity (projected onto $\mathcal{D}$), $R_{[ij}{}^k{}_{l]} = 0$, easily gives the result. □

Proposition 7.10 *Let $\underline{R}_{ij}{}^k{}_l$ denote the curvature of $\underline{\nabla}_i$, and let $\underline{R}_{jl} = \underline{R}_{ij}{}^i{}_l$. Then*

$$\nabla_{i'}U^{j'} = \begin{pmatrix} \underline{\nabla}_i U^j - P^j_{\ i}U^\infty & \underline{\nabla}_\infty U^j + \frac{1}{2}(R^j{}_k - \underline{R}^j{}_k)U^k - P^j_{\ \infty}U^\infty \\ \underline{\nabla}_i U^\infty + U_i & \underline{\nabla}_\infty U^\infty \end{pmatrix},$$

and

$$\nabla_{i'}U_{j'} = \begin{pmatrix} \underline{\nabla}_i U_j + U_\infty Q_{ij} & \underline{\nabla}_i U_\infty + P^k_{\ i}U_k \\ \underline{\nabla}_\infty U_j + \frac{1}{2}(\underline{R}^k{}_j - R^k{}_j)U_k & \underline{\nabla}_\infty U_\infty + P^k_{\ \infty}U_k \end{pmatrix},$$

Proof. We need prove only the first of these, as the second then follows. Note that, for a function, f, on $\mathcal{M}$, $\underline{\nabla}_i\underline{\nabla}_j f$ is represented by $\nabla_{i'}\nabla_{j'}f - Y^{k'}(\nabla_{k'}f)(\nabla_{i'}\theta_{j'})$, then, since $\nabla_{i'}$ is torsion-free, it follows that

$$\left(\underline{\nabla}_i\underline{\nabla}_j - \underline{\nabla}_j\underline{\nabla}_i\right) f = -2Q_{ij}(\underline{\nabla}_\infty f). \tag{7.7}$$

By using the representation of (7.6) and applying lemma 7.8, we find that $\underline{\nabla}_i\underline{\nabla}_j U^k$ is given in terms of the components of $\nabla_{i'}\nabla_{j'}U^{k'}$ by

$$\underline{\nabla}_i\underline{\nabla}_j U^k = \nabla_i\nabla_j U^k + P_i{}^k U_j - Q_{ij}(\nabla_\infty U^k).$$

Taking the skew part, bearing (7.7) in mind, we obtain

$$\underline{R}_{ij}{}^k{}_l U^l - 2Q_{ij}\underline{\nabla}_\infty U^k = \left(R_{ij}{}^k{}_l - P_i{}^k Q_{jl} + P_j{}^k Q_{il}\right)U^l - 2Q_{ij}(\nabla_\infty U^k).$$

Contracting over the indices i and k, using the first statement of lemma 7.9, then multiplying by Q^{-1} and rearranging gives

$$\nabla_\infty U^j = \underline{\nabla}_\infty U^j + \frac{1}{2}(R^j{}_k - \underline{R}^j{}_k)U^k. \tag{7.8}$$

From $\theta_{j'}\nabla_{i'}U^{j'} = \nabla_{i'}(\theta_{j'}U^{j'}) - U^{j'}(\nabla_{i'}\theta_{j'})$ together with (7.8) and the definition of $\underline{\nabla}_i$ we see that

$$\nabla_{i'}\begin{pmatrix} U^j \\ 0 \end{pmatrix} = \begin{pmatrix} \underline{\nabla}_i U^j & \underline{\nabla}_\infty U^j + \frac{1}{2}(R^j{}_k - \underline{R}^j{}_k)U^k \\ U_i & 0 \end{pmatrix}.$$

Finally, it follows from lemma 7.8 that

$$\nabla_{i'}\begin{pmatrix} 0 \\ U^\infty \end{pmatrix} = \begin{pmatrix} -P^j_{\ i}U^\infty & -P^j_{\ \infty}U^\infty \\ \underline{\nabla}_i U^\infty & \underline{\nabla}_\infty U^\infty \end{pmatrix}.$$

□

Proposition 7.11

$$P_{j\infty} = -\frac{1}{2n+1}\underline{\nabla}_i P^i{}_j \qquad \textit{and} \qquad P_{\infty\infty} = \frac{1}{2n}\left(P^{ij}P_{ij} + \frac{1}{2n+1}\underline{\nabla}_i\underline{\nabla}_j P^{ij}\right).$$

Proof. By calculating the 'ijk' components of $\nabla_{i'}\nabla_{j'}Y^{k'}$, using 7.8 and 7.10, and skewing over i and j, one finds

$$R_{ij}{}^k{}_\infty = -2\underline{\nabla}_{[i}P_{j]}{}^k - 2P^k{}_\infty Q_{ij}.$$

Contracting over the i and k indices leads to the first result. The second result follows from a similar calculation. □

Proposition 7.12 *The curvature $\underline{R}_{ijkl}$ of $\underline{\nabla}_i$ can be expressed as*

$$\underline{R}_{ijkl} = W_{ijkl} + Q_{ij}(\underline{R}_{kl} - R_{kl}) - 2Q_{k[i}P_{j]l} - 2Q_{l[i}P_{j]k}$$

and W_{ijkl} is totally trace-free with respect to Q^{ij}. It follows, by representation theory, that W_{ijkl} necessarily vanishes if $n = 1$.

Proof. The form of $\underline{R}_{ijkl}$ is given by a simple calculation, using proposition 7.10, so we just need to check that W_{ijkl} is trace-free. It follows from (6.2) that $W_{ijkl} = R_{ijkl} - Q_{ik}P_{jl} + Q_{jk}P_{il}$, from which it is easy to check using lemma 7.9 that W_{ijkl} is trace-free. □

7.4 The Contact-Projective Tractor Bundle

Since we have a projective structure, we have a tractor bundle and connection as in the previous chapter. The splitting (7.5) of $\mathcal{E}^{i'}$ means that, for any projective scale, we can identify the tractor bundle with the direct sum

$$\mathcal{E}^I = \mathcal{E}(1) \oplus \mathcal{E}^i(-1) \oplus \mathcal{E}(-1)$$

with the transformation under change of scale given by

$$\begin{pmatrix} \hat{V}^0 \\ \hat{V}^i \\ \hat{V}^\infty \end{pmatrix} = \begin{pmatrix} V^0 - \Upsilon_j V^j - \Upsilon_\infty V^\infty \\ V^i - \Upsilon^i V^\infty \\ V^\infty \end{pmatrix}, \tag{7.9}$$

where V^0, V^i and V^∞ are sections of $\mathcal{E}(-1)$, $\mathcal{E}^i(-1)$ and $\mathcal{E}(1)$ respectively. Similarly, the splitting of $\mathcal{E}_{i'}$ gives the identification, for any projective scale

$$\mathcal{E}_I = \mathcal{E}(1) \oplus \mathcal{E}_i(1) \oplus \mathcal{E}(-1),$$

with the transformation law

$$\begin{pmatrix}\hat{U}_0 & \hat{U}_i & \hat{U}_\infty\end{pmatrix} = \begin{pmatrix}U_0 & U_i + \Upsilon_i U_0 & U_\infty + \Upsilon^i U_i + \Upsilon_\infty U_0\end{pmatrix}, \tag{7.10}$$

with U_0, U_i and U_∞ being sections of $\mathcal{E}(-1)$, $\mathcal{E}_i(1)$ and $\mathcal{E}(1)$ respectively.

7.4.1 The Skew Form on $\mathcal{E}^I$

The contact structure $\Theta_{i'}$ defines a section of $\mathcal{E}_I(1)$, which in any projective scale is given by

$$\theta_I = (0 \quad \theta_{i'}),$$

and $D_I\theta_J$ gives a section $\tilde{Q}_{IJ}$ of $\mathcal{E}_{[IJ]}$, of the form

$$\tilde{Q}_{IJ} := D_I\theta_J = \begin{pmatrix} 0 & \theta_{j'} \\ -\theta_{i'} & \nabla_{i'}\theta_{j'} \end{pmatrix}.$$

It follows from (7.1) and (7.2) that $\nabla_{i'}\tilde{Q}_{JK} = 0$.

In terms of our splitting,

$$\theta_I = (0 \;\; 0 \;\; 1) \qquad \text{and} \qquad \tilde{Q}_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & Q_{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

It is easy to check, using (7.9) and (7.10), that $\tilde{Q}_{IJ}$ provides a canonical isomorphism between $\mathcal{E}^I$ and $\mathcal{E}_I$. Note that $\theta_I = X^I\tilde{Q}_{IJ}$.

7.4.2 The Tractor Connection

From proposition 7.10 we see that the tractor connection on $\mathcal{E}^I$ given in (6.5) and (6.6) can be written in the form

$$\nabla_i \begin{pmatrix} V^0 \\ V^j \\ V^\infty \end{pmatrix} = \begin{pmatrix} \underline{\nabla}_j V^0 - P_{jk}V^k - P_{j\infty}V^\infty \\ \underline{\nabla}_i V^j - P^j{}_i V^\infty + V^0\delta_i{}^j \\ \underline{\nabla}_i V^\infty + V_i \end{pmatrix}$$

and

$$\nabla_\infty \begin{pmatrix} V^0 \\ V^j \\ V^\infty \end{pmatrix} = \begin{pmatrix} \underline{\nabla}_\infty V^0 - P_{\infty k} V^k - P_{\infty\infty} V^\infty \\ \underline{\nabla}_\infty V^j + \frac{1}{2}(R^j{}_k - \underline{R}^j{}_k) V^k - P^j{}_\infty V^\infty \\ \underline{\nabla}_\infty V^\infty + V^0 \end{pmatrix}.$$

The connection on $\mathcal{E}_I$ follows using the isomorphism given by $\tilde{Q}$.

7.4.3 Tractor Curvature

The tractor curvature is inherited from the projective case and therefore vanishes if and only if $W_{i'j'}{}^{k'}{}_{l'} = 0$. Since $\nabla_{i'} \tilde{Q}_{JK} = 0$, it follows that $\Omega_{i'j'KL} = \Omega_{i'j'LK}$, and so the curvature, $\Omega_{i'j'}{}^K{}_L$, is given in block form by

$$\begin{pmatrix} 0 & -2\nabla_{[i'} P_{j']l} & -2\nabla_{[i'} P_{j']\infty} \\ 0 & W_{i'j'}{}^k{}_l & -2\nabla_{[i'} P_{j']}^{\;k} \\ 0 & 0 & 0 \end{pmatrix}.$$

7.4.4 The Characterisation of a Contact-Projective Structure by its Tractor Bundle

We have shown that a contact-projective structure gives rise to a tractor bundle with connection and a $\tilde{Q}_{IJ}$. Conversely, suppose we are given a manifold, $\mathcal{M}$, with a projective tractor bundle and connection along with $\tilde{Q}_{IJ}$, a flat, non-degenerate section of $\mathcal{E}_{[IJ]}$. In a given projective scale, we can write $\tilde{Q}_{IJ}$ in the form

$$\tilde{Q}_{IJ} = \begin{pmatrix} 0 & \theta_{j'} \\ -\theta_{i'} & A_{i'j'} \end{pmatrix},$$

for some $\theta_{j'}$ and some skew tensor $A_{i'j'}$. Then

$$\nabla_{i'} \tilde{Q}_{JK} = \begin{pmatrix} 0 & \nabla_{i'}\theta_{k'} - A_{i'k'} \\ -\nabla_{i'}\theta_{j'} - A_{j'i'} & \nabla_{i'} A_{j'k'} + P_{i'j'}\theta_{k'} - P_{i'k'}\theta_{j'} \end{pmatrix} = 0,$$

so that $\nabla_{i'}\theta_{j'} = A_{i'j'}$, $\nabla_{(i'}\theta_{j')} = 0$ and then $W_{i'j'}{}^{k'}{}_{l'}\theta_{k'} = 0$, from (7.2). Since we have a projective tractor bundle with connection, we have a projective structure on $\mathcal{M}$, and for any projective scale, θ_j, the projecting part of $X^I Q_{IJ}$ is a compatible contact form. Thus the existence of a bundle of the above type is equivalent to having a contact-projective structure.

7.5 The Ambient Construction

Theorem 7.13 *Let $\mathcal{M}^n$ be a manifold with a contact-projective structure and let $\widetilde{\mathcal{M}}$ be the total space of $\mathcal{E}(-1)$ (with the zero section deleted). Then $\widetilde{\mathcal{M}}$ carries a natural special affine structure with an Euler field and covariantly constant symplectic form and is Ricci flat.*

Proof. Given a projective structure, we define $\widetilde{\mathcal{M}}$ to be the affine manifold given by theorem 6.8 and use the coordinates given therein, determined by some choice of projective scale. Such a choice of scale also determines a contact form, $\theta = \theta_{i'} dx^{i'}$. Let $\tilde{\Theta}$ denote the 1-form on $\widetilde{\mathcal{M}}$ defined by

$$\tilde{\Theta}(V) = t^2\theta(\pi_* V), \qquad \text{for } V \in T^*\widetilde{\mathcal{M}}$$

where $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ is the natural projection. In terms of the standard basis, $\tilde{\Theta}_I = (0 \;\; t^2\theta_{i'})$. Then $\tilde{Q} := d\tilde{\Theta}$ is a symplectic form on $\widetilde{\mathcal{M}}$. In fact, from (6.8) and (7.1), we see that

$$\tilde{Q}_{IJ} = \widetilde{\nabla}_I \tilde{\Theta}_J = \begin{pmatrix} 0 & t\theta_{j'} \\ -t\theta_{i'} & t^2\nabla_{i'}\theta_{j'} \end{pmatrix}.$$

Again using (6.8) and (7.1), one can check that $\widetilde{\nabla}_I \tilde{Q}_{JK} = 0$. □

Theorem 7.14 *Let $\widetilde{\mathcal{M}}$ be a Ricci flat affine manifold with an Euler field, X^I, and a covariantly constant symplectic form, $\tilde{Q}_{IJ}$. Then the quotient of $\widetilde{\mathcal{M}}$ by X^I has a natural contact-projective structure.*

Proof. Applying proposition 6.7 gives us a projective structure on the quotient, $\mathcal{M}$. Now define $\Theta_J = X^I \tilde{Q}_{IJ}$. It is easy to check that $\widetilde{\nabla}_I \Theta_J = \tilde{Q}_{IJ}$. Then, since $X^J \Theta_J = 0$ and $X^I \widetilde{\nabla}_I \Theta_J = \Theta_J$, we see that Θ_I represents a covector field on $\mathcal{M}$ of weight 2. Note that $\Theta \wedge (d\Theta)^n$ is given by $X \lrcorner (\tilde{Q})^{n+1}$, which is nowhere vanishing since $\tilde{Q}$ is a symplectic form and we are working in a neighbourhood where X^I has no zeros, and so Θ_I represents a contact structure. One can check as in §7.4.4 that $\widetilde{\nabla}_I \tilde{Q}_{JK} = 0$ imposes the compatibility conditions. □

Proposition 7.15 *A manifold with a contact-projective structure is equivalent to a Ricci flat affine manifold with Euler field and covariantly constant symplectic form (in the sense of proposition 6.9).*

Proof. Follows essentially from proposition 6.9. In addition, one has to check only that on performing the construction of theorem 7.14 and then that of theorem 7.13 one ends up with the same symplectic form. This is easily seen from the proofs of theorems 7.13 and 7.14. □

7.6 Parabolic Invariant Theory

Let W, G, $\tilde{Q}$, e_0 and the parabolic subgroup, P, of G be as in §7.2. We denote by σ_w the 1-dimensional representation of P where the element

$$\begin{pmatrix} \lambda & r_j & t \\ 0 & m^i{}_j & s^i \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$$

acts by λ^{-w}. A Levi factor, L, of P is given by elements of the above form with $s^i = 0$ (hence $r_j = 0$) and $t = 0$. Let

$$\mathcal{F}(k) = \{\text{jets at } e_0 \text{ of functions on } W\text{, positively homogeneous of degree } k\}\,,$$

with, for example, $\mathcal{F}^I(k)$ defined in the obvious way. G acts on the space of positively homogeneous functions by $(g \cdot f)(x) = f(g^{-1}x)$ where $g \in G$. Since P preserves the ray through e_0, we have an induced action of P on $\mathcal{F}(k)$. Paralleling the conformal case, we have a P-module isomorphism given by evaluation at e_0, and a preferred element $e \in W \otimes \sigma_1$ given by the evaluation of the coordinate functions, X^I, at e_0, which we write e=Eval(X^I). The coordinate derivative $\partial/\partial X^I$ defines a P-equivariant map

$$\partial_I : \mathcal{F}^{JK\ldots M}(k) \to \mathcal{F}_I{}^{JK\ldots M}(k-1).$$

For $k \in \{0, 1, 2, \ldots\}$, $\bigodot^k W^*$ may be regarded as the polynomials on W which are homogeneous of degree k. This gives us an inclusion $\bigodot^k W^* \to \mathcal{F}(k)$. As will become apparent from the following algebraic description of $\mathcal{F}(k)$, there exists a natural complement, $\mathcal{F}_k$, so that as P-modules,

$$\mathcal{F}(k) \cong \mathcal{F}_k \oplus \bigodot^k W^*.$$

Proposition 7.16 *As P-modules,*

$$\mathcal{F}(k) = \left\{ \begin{array}{c} (T^{(0)}, T^{(1)}, \dots) \; : \; T^{(l)} \in \bigodot^l W^* \otimes \sigma_{k-l}, \\ e \lrcorner T^{(l+1)} = (k-l)T^{(l)}, \; l = 0, 1, 2, \dots \end{array} \right\}$$

and if $k \in \{0, 1, 2, \dots\}$, then

$$\mathcal{F}_k = \left\{ \begin{array}{l} (T^{(k+1)}, T^{(k+2)}, \dots) \; : \; T^{(l)} \in \bigodot^l W^* \otimes \sigma_{k-l}, \; e \lrcorner T^{(k+1)} = 0 \\ \text{and for } l > k, \; e \lrcorner T^{(l+1)} = (k-l)T^{(l)}. \end{array} \right\}$$

Proof. Given $f \in \mathcal{F}(k)$, one can define $T^{(l)} \in \bigodot^l W^* \otimes \sigma_{k-l}$ by

$$T^{(l)}_{\underbrace{IJ \dots K}_{l}} := \mathrm{Eval}(\partial_I \partial_J \dots \partial_K f).$$

Then it follows from Euler's equation for homogeneous functions that

$$e \lrcorner T^{(l+1)} = (k-l)T^{(l)}. \tag{7.11}$$

For $f \in \mathcal{F}_k$, $T^{(0)} = \dots = T^{(k)} = 0$. Conversely, such a list of tensors is easily seen to give an element of $\mathcal{F}(k)$. □

The definition of an invariant of the module $\mathcal{F}_k$ parallels the conformal case in the obvious way. Again, we consider only homogeneous invariants, denoting the total homogeneity degree by d.

It follows from Weyl's theory (see appendix B) that we can construct invariants from complete contractions of e, $\tilde{Q}^{-1}$ and the $T^{(l)}$. From the linking conditions, we see that we can write such an invariant without any occurrences of e.

Definition 7.17 A *Weyl invariant* is a linear combination of complete contractions of the form

$$\mathrm{contr}\left(\tilde{Q}^{-1} \otimes \dots \otimes \tilde{Q}^{-1} \otimes T^{(l_1)} \otimes \dots \otimes T^{(l_d)}\right), \tag{7.12}$$

all taking values in the same σ_q. An invariant which can not be written in this form is an *exceptional invariant.*

Definition 7.18 We say an invariant, $I : \mathcal{F}_k \to \sigma_q$, is a *weak Weyl invariant* if there exists $m \in \mathbb{N}$ and a map

$$C : \mathcal{F}_k \to \bigodot^m W \otimes \sigma_{m+q},$$

given by a linear combination of partial contractions of the tensors $T^{(l)}$, e and $\tilde{Q}^{-1}$ such that

$$C = \underbrace{e \otimes \ldots \otimes e}_{m} \otimes I$$

Proposition 7.19 *Every invariant $I : \mathcal{F}_k \to \sigma_q$ is a weak Weyl invariant.*

The proof of proposition 7.19 appears after the following lemma.

Lemma 7.20 *An invariant, I, can be expressed as a linear combination of complete contractions of the quantities*

$$\tilde{Q}^{IJ} \qquad \textit{and} \qquad T_{\underbrace{I \ldots K}_{r} \underbrace{\infty \ldots \infty}_{s}}, \ \textit{where } r + s \geq k + 1.$$

Proof. Applying Weyl's theory for L and removing '0' indices using (7.11), we see that any invariant can be written as a linear combination of complete contractions over the lower case indices of

$$Q^{ij} \qquad \text{and} \qquad T_{\underbrace{i \ldots k}_{r} \underbrace{\infty \ldots \infty}_{s}}, \ \text{where } r + s \geq k + 1.$$

Then since $Q^{ij}\psi_{ij} = \tilde{Q}^{IJ}\psi_{IJ} - \psi_{0\infty} + \psi_{\infty 0}$, for $\psi \in \bigotimes^2 W^*$, we can rewrite Q contractions as $\tilde{Q}$ contractions then eliminate '0'-indices using (7.11). $\square$

Proof of proposition 7.19. Let m be the maximum number of ∞'s in the expression for I given by lemma 7.20. Replace the ∞'s by free upper case indices and add e^*'s if necessary to bring the number of free indices in each term to m. C is then obtained by symmetrisation. Clearly, $C_{\infty \ldots \infty} = I$. One can view C as a map into $\bigodot^m W \otimes \sigma_{q+m}$ by raising indices. It is sufficient to show that

$$C - e \otimes \ldots \otimes eI : \mathcal{F}_k \to \bigodot^m W \otimes \sigma_{q+m}$$

vanishes. This follows by the highest weight argument of lemma 4.9, since $\bigodot^m W \otimes \mathbb{C}$ is an irreducible representation of $\mathfrak{g}_{\mathbb{C}}$, the complexification of the Lie algebra of G. $\square$

7.6.1 Invariants of Degree $d > 2n + 1$

Lemma 7.21 *We can take the integer, m, which occurs in the proof of proposition 7.19 to satisfy $m \leq -d - q$*

Proof. Since each $T^{(l)}$ takes values in $\bigodot^l W^* \otimes \sigma_j$ with $j < 0$ and C is a contraction of d such tensors and takes values in $\bigodot^m W^* \otimes \sigma_{q+m}$, we must have $q + m \leq -d$. □

Proposition 7.22 *There exists a map $\tilde{C} : \mathcal{F}_k \to \mathcal{F}^{\overbrace{IJ\ldots K}^{m}}(q+m)$ with the property that* $\mathrm{Eval}(\tilde{C}) = C$ *and a P-equivariant map $\tilde{I} : \mathcal{F}_k \to \mathcal{F}(q)$ with* $\mathrm{Eval}(\tilde{I}) = I$ *such that*

$$\tilde{C}^{\overbrace{IJ\ldots K}^{m}} = X^I X^J \ldots X^K \tilde{I}. \tag{7.13}$$

Proof. The proof is analogous to that of proposition 5.18. □

Lemma 7.23 *Let $f \in \mathcal{F}(r)$. Then*

$$\partial_I (X^I f) = (2n + 2 + r) f,$$

with the result also holding for homogeneous tensor valued functions.

Proof. Simple calculation. □

Theorem 7.24 *Let $I : \mathcal{F}_k \to \sigma_q$ be an invariant of degree $d > 2n + 1$. Then I is a Weyl invariant.*

Proof. Take $\tilde{C} : \mathcal{F}_k \to \mathcal{F}^{IJ\ldots K}(q + m)$ as given by proposition 7.22 and apply $\partial_I \partial_J \ldots \partial_K$ to both sides of equation (7.13). On evaluation at e_0, the left hand side gives a Weyl invariant. From lemma 7.23, the right hand side, on evaluation at e_0, gives a non-zero multiple of I, providing $(2n+2+r) \neq 0$, where r runs from q to $q + m - 1$. From lemma 7.21, we see that this can be achieved if $d > 2n + 1$. □

7.6.2 Invariants of Degree $d < 2n$

Theorem 7.25 *Let $I : \mathcal{F}_k \to \sigma_q$ be an invariant of degree $d < 2n$, polynomial in the components of $T^{(k+1)}, \ldots, T^{(l)}$. Then I is a Weyl invariant and can be written as a linear combination of complete contractions using only $T^{(k+1)}, \ldots, T^{(l)}$.*

Proof. Define $F^{IJ\ldots K}$ and construct $T^{(k+1)}, \ldots, T^{(l)}$ from a list of symmetric tensors $u^{(k+1)}, \ldots, u^{(l)}$ with lower case indices, as in the proof of theorem 5.20. Then $F^{i\ldots k}$ vanishes on substitution and is a linear combination of partial contractions of $u^{(0)}, \ldots, u^{(l)}$ which as a formal expression is identical to $F^{IJ\ldots K}$ if we replace lower case indices by upper case indices, u's by T's and Q by $\tilde{Q}$. Applying theorem B.8 gives that $F^{ij\ldots k}$ vanishes formally and the result follows in an identical manner to that of theorem 5.20. □

7.6.3 Invariants of Degree $d = 2n, 2n+1$

It is clear that non-zero Weyl invariants do exist in degrees $2n$ and $2n+1$. One might, however, expect exceptional invariants to arise analogously to the conformal case in degree $d = 2n$ and to the projective case in degree $d = 2n+1$ (see [BGo] and [Go2]). We give an example of a degree $2n+1$ invariant which, at least for $n = 1$, is exceptional. For $k = 1$,

$$\begin{vmatrix} T_{ij} & T_{i\infty} \\ T_{\infty j} & T_{\infty\infty\infty} \end{vmatrix}$$

is easily seen to be an invariant with $d = 2n+1$ and $q = -2n-3$. In the case $n = 1$, the independent, non-zero Weyl invariants for $k = 1$, $q = -5$ and $d = 3$ are

$$I_1 = T_{IJ} T_{KL} T^{IJKL}$$

and

$$I_2 = T_{IJ} T_{KL}{}^{I} T^{KLJ}.$$

Expanding these two Weyl invariants shows that for $n = 1$, the above invariant is not a linear combination of Weyl invariants and is therefore exceptional.

7.6.4 Applications

An invariant of $\mathcal{F}(k)$ defines a scalar valued (non-linear) differential operator on functions of projective weight k which is invariant under the action of $Sp(2n+2,\mathbb{R})$ on $\mathbb{P}^{2n+1}$. The sub-module $\bigodot^k W^*$ of $\mathcal{F}(k)$ corresponds to the kernel of an invariant differential operator. Thus the invariants found above define invariant differential operators on functions of projective weight k, depending only on derivatives of the function lying in a space complementary to this kernel in the space of jets. Theorems 7.24 and 7.25 say how all of these invariants can be constructed, with the exception of those in degrees $2n$ and $2n+1$. (See also [BEGm] §12).

The ambient construction and the tractor bundle both supply ways of finding invariants of curved contact-projective structures. The problem is to find whether all invariants arise in this way. It should be possible to apply techniques similar to those used in the conformal, projective and CR cases ([FGm2], [Go3] and [F], respectively) to reduce this problem to that of a problem in the invariant theory of the subgroup, P, of $Sp(2n+2,\mathbb{R})$, which could then be studied using methods used in the above invariant theory.

Appendix A

Calculations

A.1 Proof of Lemma 5.8

We break lemma 5.8 into several lemmata, and give brief sketches of the proofs.

Lemma A.1 *Let ψ be a skew tensor, with $E\psi = w'\psi$ and $V\psi = k'\psi$, and suppose $X \wedge \psi = 0$ mod Q^2. If $(n + k' - w' + 2) \neq 0$, then $d\psi = 0$ mod Q.*

Proof. $X \wedge \psi = 0 \bmod Q^2 \Rightarrow \delta(X \wedge \psi) = 0 \bmod Q$. Since $\delta X \wedge + X \wedge \delta = (n + E - V + 2)$, we have

$$(n + k' - w' + 2)\psi - X \wedge (\delta\psi) = 0 \bmod Q.$$

Since $dX\wedge = 0$, the result follows by taking d of the above equation. □

Lemma A.2 *If $(n - 2k - 2m + 4) \neq 0$, r and s satisfy*

$$\begin{aligned} X \wedge r + X \wedge (X \lrcorner\, s) &= 0 \\ \text{and} \qquad dr + d(X \lrcorner\, s) &= 0 \text{ mod Q.} \end{aligned}$$

Proof. As f is skew, $X \lrcorner\, (X \lrcorner\, f) = 0$. By applying $X \lrcorner$ to equation (i), taking $X\wedge$ of the result and cancelling some Q's we obtain the first result. The second follows from the first upon application of lemma A.1. □

Lemma A.3 *v and w satisfy*

$$(n-2k+2)X \wedge v + 2(m-1)X \wedge (X \lrcorner w) = 0 \bmod Q^2$$

Proof. As in the previous lemma, $\delta(\delta f) = 0$, so we apply δ to equation (iii), take $X\wedge$ of the result and cancel some Q's. $\square$

Lemma A.4 *If $(n-2k-2m)$, $(n-2k-2m-2)$,...,$(n-2k-4m) \neq 0$, then c and e satisfy*

$$dc = 2me \bmod \mathrm{Q}.$$

Proof. We take Δ^{m-1} of equation (ii), and use the second part of corollary 5.3 to obtain

$$\begin{aligned} d(\Delta^{m-1}f) &= C_{m,n,k}\{2(n-2k-2m)X \wedge c \\ &\quad +Q(2m(n-2k-4m+2)e + 2(m-1)dc)\} \bmod Q^2, \end{aligned}$$

where $C_{m,n,k} = -2^{m-2}(m-1)!(n-2k-2m-2)(n-2k-2m-4)\ldots(n-2k-4m+4)$.

Now suppose we have f', c' and e' such that c' and e' have the same eigenvalues under E and V as c and e, satisfying

$$df' = X \wedge c' + Qe' \bmod Q^2. \tag{A.1}$$

Applying Δ to (A.1) gives

$$d\Delta f' = 2dc' + 2(n-2k-4m)e' \bmod Q$$

(from which we see that $de' = 0 \bmod Q$ if $(n-2k-4m) \neq 0$).

But applying $d\delta$ to (A.1) and using $de' = 0 \bmod Q$ gives

$$d\Delta f' = (n-2k-2m+2)dc' - 4me' \bmod Q.$$

From these last two equations, we deduce that $(n-2k-2m)dc' = 2(n-2k-2m)e'$. Substituting $c' = 2C_{m,n,k}(n-2k-2m)c$ and $e' = 2mC_{m,n,k}(n-2k-4m+2)e + 2(m-1)dc$ gives, for $C_{m,n,k} \neq 0$,

$$(n-2k-2m)(n-2k-4m+2)dc = 2m(n-2k-2m)(n-2k-4m+2)e \bmod Q$$

and the result follows. $\square$

Lemma A.5 *If* $(n-2k-2m+4) \neq 0$, *then* r *and* v *satisfy*

$$2(m-1)X \wedge r = X \wedge v \bmod Q^2$$

Proof. As $\delta X\lrcorner + X\lrcorner\,\delta = 0$, we take δ of (i) and add this to $X\lrcorner$ of (iii) which, after cancelling some Q's gives

$$X \wedge (X\lrcorner\, v) - Q((n-2k-2m+6)r + 2X\lrcorner\, s + v + X\lrcorner\, w) = 0 \bmod Q^2.$$

If we take $X\wedge$ of this equation and apply lemmata A.2 and A.3 we obtain

$$2(m-1)(n-2k-2m+4)X \wedge r = (n-2k-2m+4)X \wedge v \bmod Q^2$$

and the result follows. □

Lemma A.6 *If* $(n-2k-2m+2)$ *and* $(n-2k-2m+4) \neq 0$, *then* r, s *and* c *satisfy*

$$2ms = dr + X\lrcorner\, c \bmod \mathrm{Q}.$$

Proof. We have $dX\lrcorner\, f + X\lrcorner\, df = (E+V)f = 0$. So we apply d to equation (i) and add this to $X\lrcorner$ of equation (ii). Taking δ of the resulting equation and applying the second part of lemma A.2, we get

$$(n-2k-2m+2)(2ms - dr - X\lrcorner\, c) = 0 \bmod Q$$

and the result follows. □

Appendix B

Symplectic Invariant Theory

B.1 Weyl's Theory for the Symplectic Groups

We first recall Weyl's invariant theory for the symplectic group (see [We]). We again have a first main theorem (theorem B.2)—giving a list of generators for the invariants; and a second main theorem (theorem B.3)—giving a list of relations. We then give some extensions to this which we need in §7.6, in the manner of appendices A and B of [BEGm].

Let $(v^{(1)}, v^{(2)}, \ldots, v^{(M)})$ be a collection of vectors in $\mathbb{R}^{2n}$ and define Q_{ij} and $\mathrm{Sp}(2n, \mathbb{R})$ be as before. We denote by $[v^{(i)}, v^{(j)}]$ the skew product given by Q. $\mathrm{Sp}(2n, \mathbb{R})$ acts on vectors in $\mathbb{R}^{2n}$ in the usual way. Note that the irreducible representations of $\mathrm{Sp}(2n, \mathbb{C})$ are the spaces of tensors with Young symmetry which are trace-free with respect to Q (see e.g. [FH]).

Definition B.1 An $\mathrm{Sp}(2n, \mathbb{R})$-*invariant* is a polynomial in the components of the $v^{(i)}$ which is invariant under the $\mathrm{Sp}(2n, \mathbb{R})$ action.

Theorem B.2 *The generators for the algebra of invariants are given by*

$$[v^{(i)}, v^{(j)}], \qquad 1 \leq i < j \leq M.$$

□

Theorem B.3 *Those invariants which vanish on substitution form an ideal, generated by polynomials of the form*

$$\begin{array}{l}
\sum \pm[x^{(0)}, y^{(0)}][x^{(1)}, x^{(2)}] \dots [x^{(2n-1)}, x^{(2n)}] \\
\sum \pm[x^{(0)}, y^{(0)}][x^{(1)}, y^{(1)}][x^{(2)}, y^{(2)}][x^{(3)}, x^{(4)}] \dots [x^{(2n-1)}, x^{(2n)}] \\
\quad\vdots \qquad\qquad\qquad\qquad \vdots \\
\sum \pm[x^{(0)}, y^{(0)}][x^{(1)}, y^{(1)}] \dots [x^{(2n)}, y^{(2n)}]
\end{array}$$

where $(x^{(0)}, \dots, x^{(2n)})$ *and* $(y^{(0)}, \dots, y^{(2n)})$ *are arbitrary* $2n+1$ *element subsets of* $(u^{(1)}, u^{(2)}, \dots, u^{(M)})$, *and the sum runs over all permutations of* $(x^{(0)}, \dots, x^{(2n)})$.

□

Corollary B.4 *If* $M < 2n+1$, *then any polynomial in the* $[v^{(i)}, v^{(j)}]$ *which vanishes on substitution vanishes identically.* □

Following from theorem B.2, we have

Theorem B.5 *Let* $(u^{(1)}, \dots, u^{(M)})$ *be a collection of tensors in* $\mathbb{R}^{2n*}$. *Then any* $Sp(2n, \mathbb{R})$ *invariant can be written as a linear combination of complete contractions of the form*

$$\operatorname{contr}\left(Q^{-1} \otimes \dots \otimes Q^{-1} \otimes u^{(i_1)} \otimes \dots \otimes u^{(i_d)}\right). \tag{B.1}$$

□

Now suppose that each $u^{(i)}$ is symmetric. We say that a linear combination of partial or complete contractions is *allowable* if it contains no internal contractions and *vanishes formally* if it vanishes identically or if it necessarily vanishes because of the skew-symmetry of Q. In particular, an allowable linear combination of contractions vanishes formally if the linear combination of contractions obtained by replacing each occurrence of Q_{ij} by $Q_{ij} - Q_{ji}$ vanishes identically.

Proposition B.6 *Let* $(u^{(1)}, \dots, u^{(d)})$ *be a collection of* d *symmetric tensors on* $\mathbb{R}^{2n}$ *of ranks* $r_1, \dots, r_d$ *and let* I *be a linear combination of formal complete contractions of the* $u^{(l)}$ *and* Q, *of degree 1 in each* $u^{(l)}$. *If* I *vanishes on substitution and* $d \le 2n$ *then* I *vanishes formally.*

Proof. Let $(v^{(1)}, \ldots, v^{(d)})$ be a collection of d vectors on $\mathbb{R}^{2n}$. By substituting

$$u^{(l)} = \otimes^{r_l} v^{(l)},$$

we construct a polynomial, J, in the $[v^{(j)}, v^{(k)}]$. Clearly, J vanishes formally if and only if I vanishes formally. If I vanishes on substitution, so must J, then from corollary B.4, if $d \leq 2n$, then J and hence I must vanish formally. □

Proposition B.7 *Let $(u^{(1)}, \ldots, u^{(M)})$ be a collection of symmetric tensors and let I be a linear combination of formal complete contractions of the $u^{(l)}$ with Q of degree d. If I vanishes on substitution in dimension n and $d \leq 2n$ then I vanishes formally.*

Proof. Let I be of degree d_l in each $u^{(l)}$. Taking the complete polarization (see, for example, [We]) gives a linear combination of formal complete contractions of a collection of d symmetric tensors

$$\left(u^{(l,k)} \ : \ l = 1, \ldots, M \ ; \ k = 1, \ldots, d_l\right),$$

which is of degree 1 in each $u^{(l,k)}$, which vanishes formally if and only if I vanishes formally. Since this linear combination vanishes on substitution if I vanishes on substitution, the result follows from applying proposition B.6. □

Theorem B.8 *Let $(u^{(1)}, \ldots u^{(m)})$ be a list of tensors as above. Let I be a linear combination of allowable formal partial contractions of Q and the $u^{(l)}$, of degree d and formally taking values in the space of symmetric m-tensors. Then if I vanishes on substitution in dimension n and $d \leq 2n - 1$, then I vanishes formally.* □

Proof. Suppose we have a linear combination, I, of formal partial contractions of degree d in a collection $(u^{(1)}, \ldots, u^{(M)})$ of symmetric tensors, taking values in the space of symmetric m-tensors. We add a symmetric m-tensor $u^{(M+1)}$ to our collection and form a linear combination of formal complete contractions of the $u^{(l)}$ and Q by contracting I with $u^{(M+1)}$. Applying proposition B.7, we find that this linear combination vanishes formally if I vanishes on substitution and $d \leq 2n - 1$. It follows that I vanishes formally. □

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